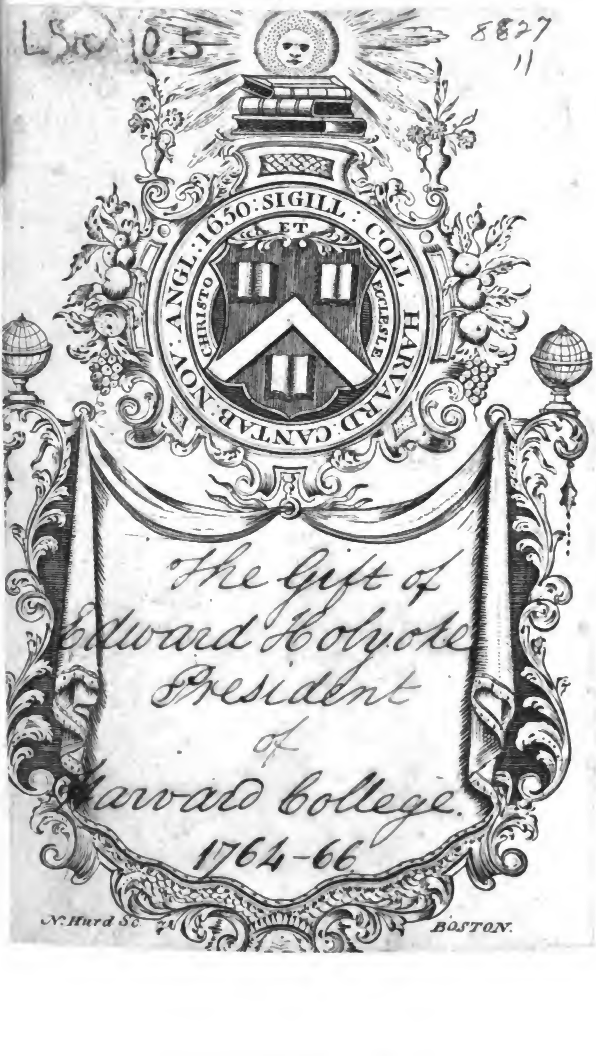
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Miscellanea Curiosa.

BEING A

COLLECTION

Of some of the Principal

PHÆNOMENA

IN

NATURE,

Accounted for by the Greatest Philosophers of this Age.

TOGETHER

With several DISCOURSES read before the ROYAL SOCIETY, for the Advancement of Physical and Mathematical Knowledge.

VOL. II.

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Tabella Poteſtatum.

		s	t	u	w	x	y
$lz^7 =$	$la^7 +$	$7la^6e +$	$21la^5ee +$	$35la^4e +$	$35la^3e^4 +$	$21la^2e^5 +$	$7lae^6 + le^7$
$kz^6 =$	$ka^6 +$	$6ka^5e +$	$15ka^4ee +$	$20ka^3e +$	$15ka^2e^4 +$	$6kae^5 +$	ke^6
$hz^5 =$	$ha^5 +$	$5ha^4e +$	$10ha^3ee +$	$10ha^2e +$	$5hae^4 +$	he^5	
$gz^4 =$	$ga^4 +$	$4ga^3e +$	$6ga^2ee +$	$4gae +$	ge^4		
$fz^3 =$	$fa^3 +$	$3fa^2e +$	$3faee +$	fe			
$dz^2 =$	$da^2 +$	$2dae +$	dee				
$cz =$	$ca +$	ce					

This refers to Page 80 in this Volume.

ADVERTISEMENT.

There is in the Press and will Speedily be publish'd, A Collection of the Travels, Antiquities and Natural Histories of Countries, as they have been deliver'd in to the *Royal Society*; collected from the Philosophical Transactions into one Volume in Octavo.

Miscellanea Curiosa.

PART II.

A Calculation of the Credibility of Humane Testimony.

M*Oral Certitude Absolute*, is that in which the Mind of Man entirely acquiesces, requiring no further Assurance: As if one in whom I absolutely confide, shall bring me word of 1200*l.* accruing to me by Gift, or a Ship's Arrival, and for which therefore I would not give the least valuable Consideration to be Ensur'd.

Moral Certitude Incompleat, has its several Degrees to be estimated by the Proportion it bears to the *Absolute*. As if one in whom I have that degree of Confidence, as that I would not give above One in Six to be ensur'd of the truth of what he says, shall inform me, as above, concerning 1200*l.* I may then reckon that I have as good as the Absolute Certainty of a 1000*l.* or five sixths of Absolute Certainty for the whole Summ.

The *Credibility* of any *Reporter* is to be rated (1) by bis *Integrity* or Fidelity; and (2) by his *Ability*: and a double *Ability* is to be consider-ed; both that of *Apprehending* what is deliver'd, and also of *Retaining* it afterwards, till it be transmitted.

'What follows concerning the Degrees of 'Credibility, is divided into *Four Propositions*. 'The *Two First*, respect the *Reporters* of the 'Narrative; as they either Transmit *Successive-'ly*, or Attest *Concurrently*: the *Third*, the *Sub-'ject* of it; as it may consist of several *Articles*: 'and the *Fourth*, *joins* those three Considerati-'ons together, exemplifying them in *Oral* and 'in *Written* Tradition.

PROP. I.

Concerning the Credibility of a Report, made by Single Successive *Reporters, who are equally Credible.*

Let their Reports have, each of them, five Sixths of Certainty; and let the first Reporter give me a Certainty of 1000 *l.* in 1200 *l.* it is plain, that the Second Reporter, who delivers that Report, will give me the Certainty but of $\frac{5}{6}$*ths* of that 1000 *l.* or the $\frac{5}{6}$*th* of $\frac{5}{6}$*ths* of the full Certainty of the whole 1200 *l.*. And so a Third Reporter, who has it from the second, will transmit to me but $\frac{5}{6}$*ths* of that Degree of Certainty, the Second would have deliver'd me, *&c.*

That is, if, *a*, be put for the Share of Assurance a single Reporter gives me; and, *c*, for that which is wanting to make that Assurance compleat; and I therefore suppos'd to have

a

$\frac{a}{a+c}$ of Certainty from the First Reporter; I shall have from the Second, $\frac{aa}{\overline{a+c}^2}$; from the Third, $\frac{a^3}{\overline{a+c}^3}$.

And accordingly, if, *a*, be = 100, and *c*=6, (the number of Pounds that an 100*l.* put out to Intereſt, brings in at the Year's end;) and conſequently my Share of Certainty from One Reporter, be $=\frac{100}{106}$; which is the preſent value of any Summ to be paid a Year hence: The Proportion of Certainty coming to me from a Second, will be $\frac{100}{106}$ multiplyed by $\frac{100}{106}$ (which is the preſent Value of Mony to be paid after two Years;) and that from a Third-hand Reporter, $=\frac{100}{106}$: thrice multiplied into it ſelf; (the Value of Mony payable at the end of Three Years) &c.

Corollary.

And therefore, as at the Rate of 6 *per Cent.* Intereſt, the preſent Value of any Summ payable after Twelve Years, is but half the Summ So if the Probability or proportion of Certitude tranſmitted by each Reporter, be $\frac{100}{106}$; the Proportion of Certainty after Twelve ſuch Tranſmiſſions, will be but as a half; and it will grow by that Time an equal Lay, whether the Report be true or no. In the ſame manner, if the Proportion of Certainty be ſet at $\frac{100}{101}$ it will come to half from the 70*th* Hand: And if at $\frac{1000}{1001}$, from the 695*th*.

PROP. II.

Concerning Concurrent *Testifications.*

If Two Concurrent Reporters have, each of them, as $\frac{5}{6}$*ths* of Certainty; they will both give me an Assurance of $\frac{35}{36}$*ths*, or of 35 to one: If Three; an Assurance of $\frac{215}{216}$, or of 215 to one.

For if one of them gives a Certainty for 1200 *l.* as of $\frac{5}{6}$*ths*, there remains but an Assurance of $\frac{1}{6}$*th*, or of 200 *l.* wanting to me, for the whole. And towards that the Second Attester contributes, according to his Proportion of Credibility: That is to $\frac{5}{6}$*ths* of Certainty beforehand, he adds $\frac{5}{6}$*ths* of the $\frac{1}{6}$*th* which was wanting: So that there is now wanting but $\frac{1}{6}$*th* of a $\frac{1}{6}$*th*, that is $\frac{1}{36}$*th*; and consequently I have, from them both, $\frac{35}{36}$*ths* of Certainty. So from Three, $\frac{215}{216}$, &c.

That is, if the first Witness gives me $\frac{a}{a+c}$ of Certainty, and there is wanting of it $\frac{c}{a+c}$; the Second Attester will add $\frac{a}{a+c}$ of that $\frac{c}{a+c}$; and consequently leave nothing wanting but $\frac{c}{a+c}$ of that $\frac{a}{a+c}$, $=\frac{c^2}{\overline{a+c}^2}$. And in like manner the third Attester adds his $\frac{a}{a+c}$ of that $\frac{c^2}{\overline{a+c}^2}$, and leaves wanting only $\frac{c^3}{\overline{a+c}^3}$. &c.

Corol-

Corollary.

Hence it follows, that if a ſingle Witneſs ſhould be only ſo far Credible, as to give me the Half of a full Certainty; the Second of the ſame Credibility, would (joined with the firſt) give me $\frac{3}{4}$*ths*; a Third, $\frac{7}{8}$*ths*, *&c.* So that the Coatteſtation of a Tenth, would give me $\frac{1023}{1024}$*ths* of Certainty; and the Coatteſtation of a Twentieth, $\frac{1096999}{1097000}$ or above Two Millions to one, *&c.*

PROP. III.

Concerning the Credit of a Reporter for a Particular Article *of that Narrative, for the whole of which he is Credible in a certain Degree.*

Let there be Six Particulars of a Narrative equally remarkable: If he to whom the Report is given, has $\frac{5}{6}$*ths* of Certainty for the whole, or Summ, of them; he has 35 to one, againſt the Failure in any One certain Particular.

For he has Five to One, there will be no Failure at all. And if there be, he has yet another Five to One, that it falls not upon that ſingle Particular of the Six. That is, he has $\frac{5}{6}$*ths* of Certainty for the whole: and of the $\frac{1}{6}$*th* wanting, he has likewiſe $\frac{5}{6}$*ths*, or $\frac{5}{36}$*ths* of the whole more; and therefore that there will be no Failure in that ſingle Particular, he has $\frac{5}{6}$*ths* and $\frac{5}{36}$*ths* of Certainty, or $\frac{35}{36}$ of it.

In General, if $\frac{a}{a+c}$ be the Proportion of Certainty for the whole; and $\frac{m}{m+n}$ be the chance of the reſt of the particular Articles *m*, againſt ſome

ſome one, or more of them *n*; there will be nothing wanting to an abſolute Certitude, againſt the not failing in Article, or Articles, *n*, but only $\frac{nc}{m+n \times 1+c}$

PROP. IV.

Concerning the Truth of either Oral *or* Written *Tradition, (in Whole, or in Part,)* Succeſſively *tranſmitted, and alſo* Coatteſted *by ſeveral Succeſſions of Tranſmittents.*

(1) Suppoſing the Tranſmiſſion of an *Oral* and *Narrative* to be ſo performed by a Succeſſion of Single Men, or joined in Companies, as that each Tranſmiſſion, after the *Narrative* has been kept for Twenty Years, impairs the Credit of it a *th* part; and that conſequently at the Twelfth Hand, or at the end of 240 Years, its certainty is reduced to a Half; and there grows then an even Lay (*by the Corollary of the ſecond Propoſition*) againſt the Truth of the Relation: Yet if we further ſuppoſe, that the ſame Relation is Coatteſted by Nine other ſeveral Succeſſions, tranſmitting alike each of them; the Credibility of it when they are all found to agree, will (*by the Corollary of the firſt Propoſition*) be as $\frac{1023}{1024}$ of Certainty, or above a Thouſand to one; and if we ſuppoſe a Coatteſtation of Nineteen, the Credibility of it will be, as above Two Millions to One.

(2) In Oral Tradition as a Single Man is ſubject to much Caſuality, ſo a Company of Men cannot be ſo eaſily ſuppos'd to join; and therefore the Credibility of $\frac{1}{100}$ *ths*, or about

$\frac{19}{20}$*ths*,

$\frac{19}{20}$*ths*, may possibly be judged too high a Degree for an Oral Conveyance, to the distance of Twenty Years. But in *Written Tradition*, the Chances against the Truth or Conservation of a single Writing, are far less; and several Copies may also be easily suppos'd to concur; and those since the Invention of Printing exactly the same: several also distinct Successions of such Copies may be as well suppos'd, taken by different Hands, and preserv'd in different Places or Languages.

And therefore if Oral Tradition by any one Man or Company of Men might be suppos'd to be Credible, after Twenty Years at $\frac{19}{20}$*ths* of Certainty; or but $\frac{9}{10}$*ths*; or $\frac{4}{5}$*ths*: a Written Tradition may be well imagin'd to continue, *by the Joint Copies that may be taken* of it for one Place, (like the several Copies of the same Impression) during the space of a 100, if not 200 Years; and to be then Credible at $\frac{100}{101}$*ths* of Certainty, or at the Proportion of a Hundred to one. And then, seeing that the Successive Transmissions of this $\frac{100}{101}$ of Certainty, will not diminish it to a Half, until it passes the Sixty ninth Hand; (for it will be near Seventy Years before the Rebate of Mony, at that Interest, will sink it to half:) It is plain, that written Tradition, if preserv'd but by a single Succession of Copies, will not lose half of its full Certainty, until 70 times a Hundred (if not two Hundred) Years are past; that is, Seven Thousand, if not Fourteen thousand Years; and further, that, if it be likewise preserv'd by Concurrent Successions of such Copies, its Credibility at that Distance may be even encreas'd, and grow far more certain from the several agreeing Deliveries at the end of Seventy Succeſſions, than it would be at the very firſt from either of the Single Hands.

(3) Laſtly, in ſtating the Proportions of Credibility for any Part or Parts of a Copy, it may be obſerv'd; that in an Original not very long, good Odds may be laid by a careful Hand, that the Copy ſhall not have ſo much as a Literal Fault: But in one of greater Length, that there may be greater Odds againſt any Material Error, and ſuch as ſhall alter the Senſe; greater yet, that the Senſe ſhall not be alter'd in any conſiderable Point; and ſtill greater, if there be many of thoſe Points, that the Error lights not upon ſuch a ſingle Article; as in the Third Propoſition.

A Letter

A Letter from the Reverend Dr. Wallis, Professor *of* Geometry *in the University of* Oxford, *and* Fellow *of the* Royal Society, *London, to Mr.* Richard Norris, *concerning the Collection of* Secants; *and the true Division of the Meridians in the Sea-Chart.*

AN old enquiry, (about the Sum or Aggregate of *Secants*) having been of late moved a-new; I have thought fit to trace it from its Original: with such solution as seems proper to it: Beginning first with the general Preparation; and then applying it to the Particular Case.

General Preparation.

1. Because Curve lines are not so easily managed as Straight lines: the Ancients, when they were to consider of Figures terminated (at least on one side) by a Curve line (Convex or Concave) as *AFKE*, Fig. 1. 2. Tab 1. did oft make use of some such expedient as this following, (but diversly varied as occasion requir'd.) Namely,

2. By Parallel Straight lines, as *A F*, *B G*, *C H*, &c. (at equal or unequal distances, as there was occasion,) they parted it into so many

ny Segments as they thought fit; (or supposed it to be so parted.)

3. These Segments were *so many wanting one*, as was the number of those Parallels.

4. To each of these Parallels, wanting one; they fitted Parallelograms, of such breadths as were the Intervals (equal or unequal) between each of them (respectively) and the next following. Which formed an Adscribed Figure made up of those Parallelograms.

5. And, if they began with the Greatest (and therefore neglected the least) such Figure was Circumscribed, (as Fig. 1.) and therefore Bigger than the Curvilinear proposed.

6. If with the Least (neglecting the greatest;) the Figure was Inscribed (as Fig, 2.) and therefore Less than that proposed.

7. But, as the number of Segments was increased, (and thereby their breadths diminished;) the difference of the Circumscribed from the Inscribed (and therefore of either from that proposed) did continually decrease, so as at last to be less than any assigned.

8. On which they grounded their Method of Exhaustions.

9. In cases wherein the Breadth of the Parallelograms, or Intervals of the Parallels, is not to be considered, but their length only; (or, which is much the same, where the Intervals are all the same, and each reputed = 1.) *Archimedes* (instead of Inscribed and Circumscribed Figures) used to say, *All except the Greatest*, and *All except the Least*. As Prop. 11. Lin. Spiral.

Particular Case.

10. Though it be well known, that, in the Terrestrial Globe, all the Meridians meet at the

the Pole, (as *E P. E P*, Fig 3.) whereby the Parallels to the Equator, as they be nearer to the Pole, do continually decrease.

11. And hereby a degree of Longitude in such Parallels, is less than a degree of Longitude in the Equator, or a degree of Latitude.

12. And that, in such proportion, as is the Co-sine of Latitude (which is the semidiamiter of such Parallel,) to the Radius of the Globe, or of the Equator.

13. Yet hath it been thought fit (for some reasons) to represent these Meridians, in the Sea Chart, by Parallel straight lines; as *E P*, *E P*.

14, Whereby, each Parallel to the Equator (as *L A*) was represented in the Sea-Chart, (as *la*,) as equal to the Equator *E E*: and a degree of Longitude therein, as large as in the Equator.

15. By this means, each degree of Longitude in such Parallels, was increased, beyond its just proportion, at such rate as the Equator (or its Radius) is greater than such Parallel, (or the Radius thereof.)

16. But, in the Old Sea-Charts, the degrees of Latitude were yet represented (as they are in themselves) equal to each other; and, to those of the Equator.

17. Hereby, amongst many other Inconveniencies, (as Mr. *Edward Wright* observes, in his *Correction of Errors in Navigation*, first published in the Year 1599,) the representation of Places remote from the Equator, was so distorted, in those Charts, as that (for instance) an *Island* in the Latitude of 60 degrees, (where the Radius of the Parallel is but half so great as that of the Equator) would have its Length (from

(from Eaſt to Weſt) in compariſon of its Breadth (from North to South) repreſented in a double proportion of what indeed it is.

18. For rectifying this in ſome meaſure (and of ſome other inconveniences) Mr. *Wright* adviſeth; that (the Meridians remaining Parallel, as before) the degrees of Latitude, remote from the Equator, ſhould at each Parallel, be protracted in like proportion with thoſe of Longitude.

19. That is; As the Co-Sine of Latitude, (which is the Semi-diameter of the Parallel) to the Radius of the Globe, (which is that of the Equator:) ſo ſhould be a degree of Latitude, (which is every where equal to a degree of Longitude in the Equator,) to ſuch a degree of Latitude ſo protracted (at ſuch diſtance from the Equator;) and ſo to be repreſented in the Chart.

20. That is, every where, in ſuch proportion as is the reſpective Secant (for ſuch Latitude) to the Radius. For, as the Co-ſine, to the Radius; ſo is the Radius to the Secant (of the ſame Arch or Angle;) as Fig. 4. Σ. R : : R. *ſ*.

21. So that (by this means) the poſition of each Parallel in the Chart, ſhould be at ſuch diſtance from the Equator, compared with ſo many *Equinoctial* Degrees or Minutes, (as are thoſe of Latitude,) as are all the Secants (taken at equal diſtances in the Arch) to ſo many times the Radius.

22. Which is equivalent, (as Mr. *Wright* there notes) to the Projection of the *Spherical* ſurface (ſuppoſing the Ey at the Center) on the concave ſurface of a Cylinder, erected at right Angles to the Plain of the Equator.

23. And the division of Meridians, represented by the surface of a Cylinder erected (on the Arch of Latitude) at right Angles, to the Plain of the Meridian (or a portion thereof.) The Altitude of such Projection, (or portion of such Cylindrick surface) being (at each point of such Circular base) equal to the secant (of Latitude) answering to such point. As Fig. 5.

24. This Projection (or portion of the Cylindrick surface) if expanded into a Plain, will be the same with a Plain Figure, whose base is equal to a Quadrantal Arch extended (or a portion thereof) on which (as ordinates) are erected Perpendiculars equal to the Secants, answering to the respective points of the Arch so- extended: The least of which (answering to the *Equinoctial*) is equal to the Radius; and the rest continually increasing, till (at the Pole) it be infinite. As at Fig. 6.

25. So that, as *E R ʃ L.* (a Figure of Secants erected at right Angles on *E L*, the Arch of Latitude extended,) to *E R R L*, (a rectangle on the same base, who's altitude *E R* is equal to the Radius;) so is *E L* (an Arch of the Equator equal to that of Latitude,) to the distance of such Parallel, (in the Chart) from the Equator.

26. For finding this distance, answering to each degree and Minute of Latitude; Mr. *Wright* (as the most obvious way) adds all the Secants (as they are found calculated in the Trigonometrical Canon) from the beginning, to the degree or Minute of Latitude proposed.

27. The sum of all which, except the Greatest, (answering to the Figure Inscribed) is too Little: The sum of all except the Least, (an-

swering

ſwering to the Circumſcribed,) is too Great, (which is that he follows:) And it would be nearer to the Truth than either, if (omitting all theſe) we take the intermediates; for Min. $\frac{1}{2}$, $1\frac{1}{2}$, $2\frac{1}{2}$, $3\frac{1}{2}$, *&c.* or (the doubles of theſe) Min. 1, 3, 5, 7, *&c.* Which yet (becauſe on the Convex ſide of the Curve) would be ſomewhat too Little.

28. But any of theſe ways are exact enough for the uſe intended, as creating no ſenſible difference in the Chart.

29. If we would be more exact; Mr. *Oughtred* directs (and ſo had Mr. *Wright* done before him) to divide the Arch into parts yet ſmaller than Minutes, and calculate Secants ſuiting thereunto.

30. Since the Arithmetick of Infinites introduced, and (in purſuance thereof) the Doctrine of Infinite ſeries (for ſuch caſes as would not, without them, come to a determinate proportion;) Methods have been found for ſquaring ſome ſuch Figures; and (particularly(the Exterior Hyperbola (in a way of continual approach) by the help of an Infinite ſeries. As, in the *Philoſophical Tranſactions*, Numb. 38, (for the Month of *Auguſt*, 1668,) And my Book, *De Motu*, Cap. 5. Prop. 31.

31. In Imitation whereof, it hath been deſired (I find) by ſome, that a like Quadrature for this Figure of Secants (by an Infinite ſeries fitted thereunto) might be found.

32. In order to which, put we for the Radius of a Circle, R; the right Sine of an Arch or Angle, S, the Verſed Sine; V, the Co-Sine (or Sine of the Complement) $\Sigma = R\text{-}V = \surd:$ Rq-Sq: the Secant, *ſ*; the Tangent, T. Fig. 4.

33. Then

33. Then is, Σ. R :: R. ſ. That is, Σ) R² ($S=\frac{R^2}{\Sigma}$; the Secant.

34. And Σ. S :: R.T. That is, Σ) S R ($T=\frac{SR}{}$ the Tangent.

35. Now, if we ſuppoſe the Radius C P, Fig. 7. divided into equal Parts, (and each of them $=\frac{1}{00}R$;) and, on theſe, to be erected the Co-Sines of Latitude L A:

36. Then are the Sines of Latitude in *Arithmetick* Progreſſion.

37. And the Secants anſwering thereunto, $L.ſ=\frac{R^2}{\Sigma}$.

38. But theſe Secants, (anſwering to right Sines in *Arithmetical* Progreſſion) are not thoſe that ſtand at equal diſtance on the Quadrantal Arch extended, Fig. 6.

39. But ſtanding at unequal diſtances (on the ſame extended Arch;) Namely, on thoſe points thereof, whoſe right Sines (whilſt it was a Curve) are in *Arithmetical* Progreſſion. As Fig. 8.

40. To find therefore the magnitude of *R E L ſ*, Fig. 6. Which is the ſame with that of Fig. 8. (ſuppoſing *E L* of the ſame length in both; however the number of Secants therein may be unequal:) we are to conſider the Secants, tho' at unequal diſtances: Fig. 8. to be the ſame with thoſe at equal diſtances in Fig. 7. anſwering to Sines in *Arithmetical* Progreſſion.

41. Now theſe Intervals, (or portions of the baſe) in Fig. 8. are the ſame with the intercepted Arches (or portions of the Arch) in Fig. 7. For this baſe is but tbat Arch extended.

42. And

42. And these Arches (in parts infinitely small) are to be reputed equivalent to the portions of their respective Tangents intercepted between the same ordinates. As in Fig. 7. 9.

43. That is, equivalent to the portions of the Tangents of Latitude.

44. And these portions of Tangents are, to the Equal intervals in the base, as the Tangent (of Latitude) to its Sine.

45. To find therefore the true Magnitude of the Parallelograms (or segments of the Figure;) we must either protract the equal segments of the base, Fig. 7. (in such proportion as is the respective Tangent to the Sine) to make them equal to those of Fig. 8.

46. Or else (which is equivalent) retaining the equal intervals of Fig. 7. protract the Secants in the same proportion. (For, either way the Intercepted Rectangles or Parallelograms will be equally encreased) As *L M* Fig. 9.

47. Namely; As the Sine (of Latiude) to its Tangent; so is the Secant to a Fourth; which is to stand (on the Radius equally divided) instead of that Secant.

$$S.\ \frac{SR}{\Sigma}\ (::\ \Sigma\ .\ R)::\ \frac{R^2}{\Sigma},\ \frac{R^3}{\Sigma 2 = R2 - S2} = L\,M,\ \text{Fig. 9.}$$

48. Which therefore are as the Ordinates in (what I call *Arith. Infin. Prop.* 104) *Reciproca Secundinorum*: supposing Σ^2 to be squares in the order of Secundanes.

R²

$R^2 - S^2).\ R^3 \left(R, + \frac{S^2}{R}, + \frac{S^4}{R^3}, + \right.$

49. This because of $\frac{R^3 - S^2 R}{+ S^2 R}$

$\Sigma^2 = R^2 - S^2$, & the Sines (S, in Arithmetical Progression) is reduced (by division) into this Infinite Series.

$R + \frac{S^2}{R} + \frac{S^4}{R^3} + \frac{S^6}{R^5}$, &c.

$+ S^2 \quad R - \frac{S^4}{R}$

$+ \frac{S^4}{R}$

$+ \frac{S^4}{R} - \frac{S^6}{R^3}$

$+ \frac{S^6}{R^3}$

50. That is, (putting $R = 1$.)
$1 + S^2 + S^4 + S^6$, &c.

51. Then (according to the *Arithmetick* of Infinites) we are to interpret S, successively, by 1 *S*, 2 *S*, 3 *S*, &c. till we come to S, the greatest. Which therefore represents the number of All.

52. And because the first Member doth represent a Series of Equals; the second of Secundans; the third, of Quartans, *&c.* Therefore the first Member is to be multiplied by S; the second, by $\frac{1}{3}$ S; the third, by $\frac{1}{5}$ S; the fourth, by $\frac{1}{7}$ S; &c.

53. Which makes the Aggregate,
$S + \frac{1}{3} S^3 + \frac{1}{5} S^5 + \frac{1}{7} S^7 + \frac{1}{9} S^9$, &c. = E C L M, Fig. 9.

54. This (because S is always less than R = 1) may be so far continued, till some power of S become so small as that it (and all which follow it) may be safely neglected.

55. Now (to fit this to the Sea-Chart, according to Mr. *Wright*'s design:) Having the

proposed Parallel (of Latitude) given; we are to find (by the Trigonometrical Canon) the Sine of such Latitude; and take, equal to it, C L=S. And, by this, find the magnitude of E C L M, Fig. 9; that is, of R E L *s*, Fig. 8. that is, of R E L *s*, Fig. 6. And then, as R R L E (or so many times the Radius,) to R E L *s* (the Aggregate of all the Secants;) so must be a like Arch of the Equator (equal to the Latitude proposed,) to the distance of such Parallel, (representing the Latitude in the Chart) from the Equator. Which is the thing required.

56. The same may be obtained, in like manner, by taking the Versed Sines in *Arithmetical* Progression. For if the right Sines (as here) beginning at the Equator, be in *Arithmetical* Progression, as 1, 2, 3, *&c.* Then will the Versed Sines, beginning at the Pole, (as being their complements to the Radius) be so also.

The Collection of Tangents.

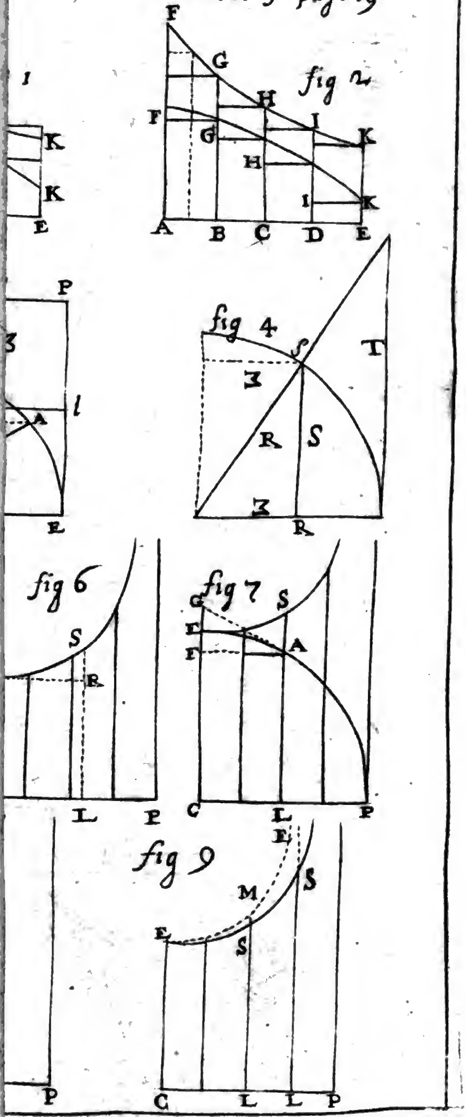
57. The same may be applied in like manner, (though that be not the present business,) to the Aggregate of Tangents, (answering to the Arch divided into equal parts.)

58. For, those answering to the Radius so divided, are $\frac{SR}{\Sigma}$; (taking S in *Arithmetical* Progression.)

59. And then, inlarging the Base, (as in Fig. 8.) or the Tangent (as in Fig. 9.) in the proportion of the Tangent to the Sine.

S.

Plate j page 19



$$S.\ \frac{SR}{\Sigma}\ (::\ \Sigma . R)::\frac{SR}{\Sigma}.\ \frac{SR2}{\Sigma 2}=\frac{SR2}{R2 - S2}.$$

60. We have (by Division) this Series,

$$R^2 - S^2)\,SR^2\,(S, + \frac{S3}{R2}, + \frac{S5}{R4}, +$$

$$S + \frac{S3}{R2} + \frac{S5}{R4} + \frac{S7}{R6} + \frac{S9}{R8}\ \&c.$$

$$\begin{array}{r} SR2 - S3 \\ \hline S3 \\ + S3 - \frac{S5}{R2} \\ \hline + \frac{S5}{R2} \\ + \frac{S5}{R2} - \frac{S7}{R4} \\ \hline + \frac{S7}{R4} \end{array}$$

61. That is (putting R=1)

$S + S^3 + S^5 + S^7 + S^9$, &c.

62. Which (multiplying the respective members by $\frac{1}{2}S$, $\frac{1}{4}S$, $\frac{1}{6}S$, $\frac{1}{8}S$, $\frac{1}{10}S$, &c.) becomes

$\frac{1}{2}S^2 + \frac{1}{4}S^4 + \frac{1}{6}S^6 + \frac{1}{8}S^3 + \frac{1}{10}S^{10}$, &c.

Which is the Aggregate of Tangents to the Arch, whose right Sine is S.

63. And this method may be a pattern for the like process in other cases of like nature.

An easie Demonstration of the Analogy of the Logarithmick Tangents to the Meridian Line or sum of the Secants; with various Methods for computing the same to the utmost Exactness, by E. Halley.

IT is now near 100 Years since our Worthy Countryman, Mr. *Edward Wright*, publish-ed his *Correction of Errors in Navigation*, a Book well deserving the perusal of all such as design to use the Sea. Therein he considers the Course of a Ship on the Globe, stearing obliquely to the Meridian; and having shewn, that the *Departure* from the Meridian, is in all cases less than the *Difference* of *Longitude*, in the *ratio* of *Radius* to the *secant* of the *Latitude*, he concludes, That the sum of the *Secants* of each point of the *Quadrant* being added successively would exhibit a line divided into Spaces, such as the intervals of the parallels of Latitude ought to be in a true Sea-Chart, whereon the Meridians are made parallel Lines, and the *Rhombs* or Oblique Courses represented by right Lines. This is commonly known by the name of the *Meridian Line*, which tho' it generally be called *Mercator*'s, was yet undoubtedly Mr. *Wright*'s Invention, (as he has made it appear in his Preface.) And the Table thereof is to be met with in most Books treating of Navigation, computed with sufficient exactness for the purpose.

It

It was first discovered by Chance, and as far as I can learn, first publish'd by Mr. *Henry Bond*, as an addition to *Norwood's Epitome of Navigation*, about 50 Years since, that the *Meridian Line was analogous to a Scale of Logarithmick Tangents of half the Complements of the Latitudes.* The difficulty to prove the truth of this Proposition, seemed such to Mr. *Mercator*, the Author of *Logarithmotechnia*, that he proposed to wager a good sum of Mony, against whoso would fairly undertake it, that he should not demonstrate either, that it was true or false: And about that time Mr. *John Collins*, holding a Correspondence with all the Eminent Mathematicians of the Age, did excite them to this enquiry

The first that demonstrated the said *Analogy*, was the excellent Mr. *James Gregory in his Exercitationes Geometricæ*, published *Anno* 1668. which he did, not without a long train of Consequences and Complication of Proportions, whereby the evidence of the Demonstration is in a great measure lost, and the Reader wearied before he attain it. Nor with less work and *apparatus* hath the celebrated Dr. *Barrow*, in his *Geometrical Lectures* (Lect. XI. App. 1.) proved, that the *Sum* of all the *Secants* of any arch is analogous to the *Logarithm* of the *ratio* of *Radius* + *Sine* to *Rad.*−*Sine*, or, which is all one, that the *Meridional parts* answering to any degree of Latitude, are as the *Logarithms* of the *rationes* of the *Versed Sines* of the distances from both the *Poles*. Since which the incomparable Dr. *Wallis* (on occasion of a Paralogism committed by one Mr. *Norris* in this matter) has more fully and clearly handled this Argument, as may be seen in Num. 176. of the

the *Transactions.* But neither Dr. *Wallis*, nor Dr. *Barrow*, in their said Treatises, have any where touched upon the aforesaid relation of the *Meridian-line* to the *Logarithmick Tangent*; nor hath any one, that I know of, yet discovered the Rule for computing independently the interval of the *Meridional parts* answering to any two given Latitudes.

Wherefore having attained, as I conceive, a very facile and natural demonstration of the said Analogy, and having found out the Rule for exhibiting the *difference of Meredional parts*, between any two parallels of Latitude, without finding both the Numbers whereof they are the difference: I hope I may be entituled to a share in the Improvements of this useful part of Geometry. And first, let us demonstrate the following *Proposition.*

The Meridian Line is a Scale of Logarithmick Tangents of the half Complements of the Latitudes.

For this Demonstration, it is requisite to premise these four *Lemmata.*

Lemma. I. In the *Stereographick Projection* of the Sphere upon the plain of the Equinoctial, the distances from the Center, which in this case is the Pole, are laid down by the Tangents of half those distances, that is, of half the Complements of the Latitudes. This is evident from *Eucl.* 3. 20.

Lem. II. In the *Stereographick Projection*, the Angles under which the Circles intersect each other, are in all cases equal to the Spherical Angles they represent: Which is perhaps as valuable a property of this *Projection*, as that of all the Circles of the Sphere thereon appear-

ing

ing Circles: But this not being vulgarly known, muſt not be aſſumed without a *Demonſtration.*

Let *EBPL* in Fig. 1. Tab. 2. be any great circle of the Sphere, *E* the Eye placed in its Circumference, *C* its Center, *P* any point thereof, and let *F C O* be ſuppoſed a plain erected at right Angles to the Circle *E B P L*, on which *F C O* we deſign the Sphere to be projected. Draw *E P* croſſing the Plain *F CO* in *p*, and *p* ſhall be the Point *P* projected. To the point *P* draw the Tangent A P G and on any point thereof, as A, erect a perpendicular *AD*, at right angles to the plane *E B P L*, and draw the lines *PD*, *AC*, *DC*: and the Angle *APD* ſhall be equal to the Spherical Angle contained between the plains *A P C*, *D P C.* Draw alſo *A E*, *D E*, interſecting the plain *FCO* in the points *a* and *d*; and joyn *a d*, *p d*: I ſay the Triangle *a d p* is ſimular to the triangle *ADP*. And the Angle *a p d* equal to the Augle *APD*. Draw *PL*, *A K*, parallel to *FO*, and by reaſon of the parallels, *a p* will be to *a d* as AK to AD: But (by *Eucl.* 3. 32.) in the triangle AKP, the angle AKP = LPE is alſo equal to APK = EPG, wherefore the ſides AK, AP, are equal, and 'twill be as *a p* to *a d* ſo AP to AD. Whence the angles DAP, *d a p* being right, the angle APD will be equal to the angle *a p d*; that is, the Spherical Angle is equal to that on the Projection, and that in all Caſes. *Which was to be proved.*

This *Lemma* I lately received from Mr. *Ab. de Moivre*, though I ſince underſtand from Dr. *Hook*, that he long ago produced the ſame thing before the *Society*. However the demonſtration and the reſt of the Diſcourſe, is my own.

Lemma III. On the *Globe*, the *Rumb Lines* make equal angles with every Meridian, and by the aforegoing *Lemma*, they must likewise make equal angles with the Meridians in the *Stereographick Projection* on the plain of the *Equator*: They are therefore, in that Projection, *Proportional Spirals* about the Pole Point.

Lem. IV. In the *Proportional Spiral* (Fig.2.) it is a known property, that the angles BPC, or the arches BD, are *Exponents* of the *rationes* of *BP* to *PC*: for if the arch BD be divided into innumerable equal parts, right lines drawn from them to the Center P, shall divide the Curve BccC, into an infinity of proportionals; and all the lines Pc shall be an infinity of proportionals between PB and PC, whose number is equal to all the points d, d, in the arch BD: Whence and by what I have deliver'd in the next ensuing Discourse it follows, that as BD to Bd, or as the angle BPC to the angle BPc, so is the Logarithm of the *ratio* of PB to PC, to the Logarithm of the *ratio* of PB to Pc.

From these *Lemmata* our Proposition is very clearly demonstrated: For by the first, PB, Pc, PC are the *Tangents* of half the *Complements* of the *Latitudes* in the *Stereographick Projection*: and by the last of them, the differences of Longitude, or angles at the Pole between them, are Logarithms of the *rationes* of those *Tangents* one to the other. But the *Nautical Meridian Line*, is no other than a Table of the Longitudes, answering to each minute of Latitude, on the Rhumb-line, making an angle of 45 degrees with the Meridian Wherefore the Meridian Line is no other than a Scale of Logarithmick Tangents of the half Complements

ments of the Latitudes. *Quod erat demonstrandum.*

Coroll. 1. Becaufe that in every point of any *Rhum Line*, the difference of Latitude is to the *Departure*, as the *Radius* to the *Tangent* of the angle that Rhumb makes makes with the Meridian; and thofe equal *Departures* are every where to the differences of Longitude, as the Radius to the Secant of the Latitude; it follows, that the differences of Longitude are, on any Rhumb, Logarithms of the fame Tangents, but of a differing *Species*; being proportioned to one another as are the Tangents of the angles made with the Meridian.

Coroll. 2. Hence any Scale of Logarithm Tangents, (as thofe of the Vulgar Tables made after *Briggs*'s form; or thofe made to *Napier*'s, or any other form whatfoever) is a Table of the differences of Longitude, to the feveral Latitudes, upon fome determinate Rhumb or other: And therefore, as the Tangent of the angle of fuch *Rhumb*, to the Tangent of any other *Rhumb*: So the difference of the Logarithms of any two Tangents, to the difference of Longitude, on the propofed *Rhumb*, intercepted between the two Latitudes, of whofe half Complements you took the Logarithm Tangents.

And fince we have a very compleat Table of *Logarithm Tangents* of *Briggs*'s form, publifhed by *Vlacq*, *Anno* 1633, in his *Canon Magnus Triangulorum Logarithmicus*, computed to ten Decimal places of the Logarithm, and to every ten Seconds of the Quadrant (which feems to be more than fufficient for the niceft Calculator) I thought fit to enquire the Oblique angle, with which that Rhumb Line croffes the Meridian,

ridian, whereon the ſaid Canon of *Valcq* preciſely anſwers to the differences of Longitude, putting Unity for one minute thereof, as in the Common Meridian Line. Now, the *momentary augment* or *fluxion* of the Tangent Line at 45 degrees, is exactly double to the *fluxion* of the arch of the Circle, (as may eaſily be proved) and the Tangent of 45 being equal to *Radius*, the *fluxion* alſo of the Logarithm Tangent will be double to that of the arch, if the Logarithm be of *Napier*'s form: but for *Briggs*'s form, it will be as the ſame doubled arch, multiplied into 0, 43429, &c. or divided by 2, 30258, &c. Yet this muſt be underſtood only of the addition of an indiviſible arch, for it ceaſes to be true, if the arch have any determinate magnitude.

Hence it appears, that if one minute be ſuppoſed Unity, the length of the arch of one minute being ,000290888208665721596154, &c. in parts of the Radius, the proportion will be as Unity to 2,908882, &c. ſo Radius to the Tangent of 71° 1′ 42″ whoſe Logarithm is 10. 46372611720718325204, &c. and under that angle is the Meridian interſected by that Rhumb Line, on which the *differences* of *Napier*'s Logarithm Tangents of the half Complements of the Latitudes are the true differences of Longitude, eſtimated in minutes and parts, taking the firſt Four Figures for Integers. But for *Vlacq*'s Tables, we muſt ſay.

As .2302585, &c. to 2908882, &c. So Radius to 1,2633114387424456921 2, &c. which is the Tangent of 51° 38′ 9″, and its Logarithm 10,1015104285077209411 62, &c. wherefore in the Rhumb Line, which makes an angle of 51° 38′ 9″ with the Meridian, *Vlacq*'s Logarithm Tan-

Tangents are the true differences of Longitude. And this compared with our second *Corollary* may suffice for the use of the Tables already computed.

But if a Table of Logarithm Tangents be made by extraction of the root of the Infiniteth power, whose Index is the length of the arch you put for Unity, (as for minutes the ,000 2908882th,&c. power) which we will call *a*; such a Scale of Tangents shall be the true Meridian Line, or sum of all the Secants taken infinitely many. Here the Reader is desired to have recourse to my little Treatise of *Logarithms*, in the ensuing Discourse that I may not need to repeat it. By what is there delivered,it will follow, that putting *t* for the excess or defect of any Tangent above or under the *Radius* or *Tangent* of 45; the Logarithm of the *ratio* of *Radius* to such Tangent will be

$$\frac{1}{m} \text{ into } t - \tfrac{1}{2}tt + \tfrac{1}{3}ttt - \tfrac{1}{4}tttt + \tfrac{1}{5}t^5, \&c.$$

when the arch is greater than 45^{gr}, or

$$\frac{1}{m} \text{ into } t + \tfrac{1}{2}tt + \tfrac{1}{3}t^3 + \tfrac{1}{4}t^4 + \tfrac{1}{5}t^5, \&c.$$

when it is less than 45*gr*. And by the same doctrine putting T for the Tangent of any *arch*, and *t* for the difference thereof from the Tangent of another arch, the Logarithm of their *ratio* will be

$$\frac{1}{m} \text{ into } \frac{t}{T} + \frac{tt}{2TT} + \frac{t^3}{3T^3} + \frac{t^4}{4T^4} + \frac{t^5}{5T^5}, \&c.$$

when T is the greater Term, or

$$\frac{1}{m} \text{ into } \frac{t}{T} - \frac{tt}{2TT} + \frac{t^3}{3T^3} + \frac{t^4}{4T^4} + \frac{t^5}{5T^5}, \&c$$

when *T* is the lesser Term:

And if *m* be supposed ,0002908882, &c. = the *a*, its reciprocal $\frac{r}{a}$ will be, 3437,74677078493 02526, &c. which multiplied into the aforesaid *Series*, shall give precisely the difference of Meridional parts, between the two Latitudes, to whose half complements the assumed Tangents belong. Nor is it material from whether Pole you estimate the Complements, whether the elevated or depressed; the Tangents being to one another in the same *ratio* as their Complements, but inverted.

In the same Discourse I also shewed, that the *Series* might be made to converge twice as swift, all the even powers being omitted: and putting τ for the sum of the two Tangents, the same Logarithm would be

$$\frac{2}{m} \text{ or } \frac{2r}{a} \text{ into } \frac{t}{\tau} + \frac{t^3}{3\tau^3} + \frac{t^5}{5\tau^5} + \frac{t^7}{7\tau^7} + \frac{t^9}{9\tau^9}, \&c.$$

but the *ratio* of τ to t, or of the sum of two Tangents to their difference, is the same as that of the *sine* of the sum of the arches, to the *sine* of their difference. Wherefore, if S be put for the *sine* Complement of the Middle Latitude, and *s* for the *sine* of half the difference of Latitudes, the same *Series* will be

$$\frac{2r}{a} \text{ into } \frac{s}{S} + \frac{s^3}{3S^3} + \frac{s^5}{5S^5} \frac{s^7}{7S^7} + \frac{s^9}{9S^9}, \&c.$$

wherein, as the differences of Latitude are smaller, fewer steps will suffice. And if the Equator be put for the middle Latitude, and consequently $S=R$, and *s* to the *sine* of the Latitude, the Meridional parts reckoned from the Equator will be

$\frac{s}{a}$

$$\frac{s}{a} + \frac{s^3}{3rra} + \frac{s^5}{5r^4a} + \frac{s^7}{7r^6a}, \text{ \&c.}$$

which is coincident with Dr. *Wallis*'s ſolution in Numb. 176. of the *Philoſophical Tranſactions*. And this ſame *Series* being half the Logarithm of the *ratio* of *R*+*s* to *R*—*s*, that is, of the *Verſed-ſines* of the diſtances from both Poles, does agree with what Dr. *Barrow* had ſhewn in his XIth. *Lecture*.

The ſame *ratio* of *τ* to *t* may be expreſſed alſo by that of the *Sum of the Co-ſines of the two Latitudes, to the ſine of their difference* : As likewiſe by that of *the Sine of the Sum of the two Latitudes, to the difference of their Co-ſines* : Or by that of *the Verſed-ſine of the Sum of the Co-latitudes, to the difference of the ſines of the Latitudes:* Or as *the ſame difference of the ſines of the Latitudes, to the Verſed-ſine of the difference of the Latitudes* ; all which are in the ſame *ratio* of the Co-ſine of the middle Latitude, to the Sine of half the difference of the Latitudes. As it were eaſie to demonſtrate, if the Reader were not ſuppoſed capable to do it himſelf, upon a bare inſpection of a Scheme duly repreſenting theſe Lines.

This variety of Expreſſion of the ſame *ratio* I thought not fit to be omitted, becauſe by help of the rationality of the Sine of 30*gr.* in all caſes where the Sum or difference of the Latitudes is 30*gr.* 60*gr.* 90*gr.* 120*gr.* or 150 degrees, ſome one of them will exhibit a ſimple *ſeries*, wherein great part of the Labour will be ſaved : And beſides I am willing to give the Reader his choice which of theſe equippolent methods to make uſe of ; but for his exerciſe ſhall

ſhall leave the proſecution of them, and the *compendia* ariſing therefrom, to his own Induſtry. Contenting my ſelf to conſider only the former, which for all uſes ſeems the moſt convenient, whether we deſign to make the whole Meridian Line, or any part thereof, *viz.*

$$\frac{2r}{a} \text{ into } \frac{s}{S} + \frac{s^3}{3S^3} + \frac{s^5}{5S^5} + \frac{s^7}{7S^7} + \frac{s^9}{9S^9} \text{ \&c.}$$

Wherein a is the length of any Arch which you deſign ſhall be the Integer or Unity in your Meridional Parts, (whether it be a Minute, League, or Degree, or any other,) S the Coſine of the Middle Latitude, and s the Sine of half the difference of Latitudes; but the Secants being the Reciprocals of the Co-ſines, $\frac{s}{S}$ will be equal to $\frac{\int s}{rr}$ putting $\int$ for the Secant of the Middle Latitude; and $\frac{2r}{a}$ into $\frac{s}{S}$ will be $= \frac{2\int s}{ar}$ This multiplied by $\frac{ss}{3SS}$ that is by $\frac{\int\int ss}{3rrrr}$, will give the ſecond ſtep: and that again by $\frac{3\int\int ss}{5rrrr}$, the third ſtep; and ſo forward, till you have compleated as many places as you deſire. But the ſquares of the *Sines* being in the ſame *ratio* with the *Verſed-ſines* of the double Arches, we may inſtead of $\frac{ss}{3SS}$ aſſume for our Multiplicator $\frac{v}{3V}$, or the Verſed-ſine of the

the difference of the Latitudes, divided by thrice the Versed-sine of the sum of the Co-latitudes, &c. which is the utmost *Compendium* I can think of for this purpose, and the same *series* will become,

$$\frac{2sr}{aS}\text{ into } 1 + \frac{v}{3V} + \frac{v^2}{5V^2} + \frac{v^3}{7V^3} + \frac{v^4}{9V^4}$$

Hereby we are enabled to estimate the default of the method of making the Meridian line, by the continued addition of the Secants of æquidifferent Arches, which as the difference of those Arches are smaller, does still nearer and nearer approach the Truth. If we assume, as Mr. *Wright* did, the Arch of one minute to be Unity, and one minute to be the common difference of a rank of Arches: It will be in all cases, as the Arch of one Minute, to its Chord :: So the Secant of the middle Latitude, to the first step of our *series*. This by reason of the near equality between *a* and 2 *s*, which are to one another in the *ratio* of Unity to 1—0, 0000000035256645773, &c. will not differ from the Secant *ſ* but in the ninth Figure; being less than it in that proportion. The next step being $+ \frac{2ſ^3 s^3}{3ar^5}$ will be equal to the Cube of the Secant of the middle Latitude multiplied into $\frac{2sss}{3arr} = 0,000000007051329087 15$;

which therefore unless the Secant exceed *ten times Radius*, can never amount to 1 in the fifth place. These two steps suffice to make the Meridian Line, or Logarithm Tangent to far more places than any Tables of Natural Se-

cases,

cants yet extant, are computed to; but if the third ſtep be required, it will be found to be $+ s^5$ into $\frac{2 s^5}{5 ar^4} = 0,0000000000000000089498$; By all which it appears, that Mr. *Wright*'s Table does no where exceed the true Meridian Parts by fully half a Minute: which ſmall difference ariſes by his having added continually the Secants of 1′, 2′, 3′, &c. inſtead of 0½′, 1½, 2′½, 3′½, &c. But as it is, it is abundantly ſufficient for *Nautical Uſes*. That in Sir *Jonas Moor*'s *New Syſtem of the Mathematicks*, is much nearer the Truth, but the difference from *Wright* is ſcarce ſenſible till you exceed thoſe Latitudes where Navigation ceaſes to be practicable, the one exceeding the Truth by about half a Minute, the other being a very ſmall matter deficient therefrom.

For an Example eaſie to be imitated by whoſo pleaſes, I have added the true Meridional Parts to the firſt and laſt Minutes of the Quadrant; not ſo much that there is any occaſion for ſuch occurrancy, as to ſhew that I have obtained, and laid down herein, the full *Doctrine* of theſe *ſpiral Rhumbs*, which are of ſo great concern in the Art of *Navigation*.

The firſt Minute is,	1.00000001410265862178
The Second,	2,00000005641063806707
The Laſt, or 89° 59′ is	30374,963431141422864з

and not 32348, 5279 as Mr. *Wright* has it; by adding the *Secants* of every whole Minute: Nor 30249,8 as Mr. *Oughtred*'s Rule makes it, by adding the *Secants* of every other half Minute. Nor 30364,3 as Sir *Jonas Moor* had concluded it it by I know not what Method, tho' in the reft of his Table he follows *Oughtred.*

And this may fuffice to fhew how to derive the true Meridian Line from the Sines, Tangents, or Secants fuppofed ready made; but we are not deftitute of a Method for deducing the fame independently, from the Arch it felf. If the Latitude from the Equator be eftimated by the length of its Arch *A*, Radius being Unity, and the Arch put for an *Integer* be *a*, as before; the Meridional parts anfwering to that Latitude, will be

$\frac{1}{a}$ into $A + \frac{1}{6}A^3 + \frac{1}{24}A^5 +$ [illegible] A^7 or $\frac{61}{5040}A^7 +$ [illegible] A^9 or $\frac{1385}{362880}A^9$, &c.

which converges much fwifter than any of the former *Series*, and befides has the advantage of *A* encreafing in Arithmetical progreffion, which would be of great eafe, if any fhould undertake *de novo* to make the *Logarithm Tangents*, or the Meridian Line to many more places than now we have them. The Logarithm Tangent to the Arch of $45 + \frac{1}{2}A$ being no other than the aforefaid Series $A + \frac{1}{6}A^3 + \frac{1}{24}A^5$, &c. in *Napeir*'s form, or the fame multiplied into 0,43429, &c. for *Brigg*'s.

But becaufe all thefe *Series* toward the latter end of the Quadrant do converge exceeding flowly, fo as to render this Method almoft ufelefs, or at leaft very tedious: It will be convenient to apply fome other Arts, by affuming the Secants of fome intermediate Latitudes; and you may for *s* or the Sine of α the Arch of half the difference of Latitudes, fubftitute $\alpha - \frac{1}{6}\alpha^3 + \frac{1}{120}\alpha^5 - \frac{1}{5040}\alpha^7 + \frac{1}{362880}\alpha^9$, &c. according

ing to Mr. *Newton*'s Rule for giving the Sine from the Arch: And if α be no more than a Degree, a very few steps will suffice for all the accuracy that can be desired.

And if α be commensurable to a, that is, if it be a certain number of those Arches with which you make your *Integer*, then will $\frac{\alpha}{a}$ be that number: which if we call n, the parts of the Meridional Line will be found to be,

$$\frac{sn}{r} \text{ into } \left\{ \begin{array}{l} 1 + \frac{ss\alpha\alpha}{3r^4} + \frac{s^4\alpha^4}{5r^8} + \frac{s^6\alpha^6}{7r^{12}}, \&c. \\ \quad - \frac{\alpha\alpha}{6rr} - \frac{ss\alpha^4}{6r^6} - \frac{s^4\alpha^6}{6r^{10}}, \&c. \\ \qquad + \frac{1\alpha^4}{120r^4} + \frac{13s^2\alpha^6}{360r^8}, \&c. \\ \qquad\quad - \frac{1\alpha^6}{5040r^6}, \&c. \end{array} \right.$$

In this, the first two steps are generally sufficient for Nautical uses, especially when neither of the Latitudes exceed 60 degrees, and the difference of Latitudes doth not pass 30 degrees.

But I am sensible I have already said too much for the Learned, tho' too little for the Learner; to such I can recommend no better Treatise, than Dr. *Wallis*'s precedent Discourse, wherein he has with his usual brevity, and that perspecuity peculiar to himself, handled this Subject from the first Principles, which here for the most part we suppose known.

I need not shew how, by regressive work, to find the Latitudes from the Meridional Parts, the Method being sufficiently obvious. I shall only conclude with the proposal of a Problem which

which remains to make this Doctrine compleat, and that is this.

A Ship sails from a given Latitude, and having run a certain number of Leagues, has alter'd her Longitude by a given angle, it is required to find the Course steared. The solution hereof would be very acceptable, if not to the Publick, at least to the Author of this Tract, being likely to open some further Light into the Mysteries of Geometry.

To conclude, I shall only add, That Unity being Radius; the *Co-sine* of the Arch *A*, according to the same Rules of Mr. *Newton*, will be

$$1 - \tfrac{1}{2}A^2 + \tfrac{1}{24}A^4 - \tfrac{1}{720}A^6 + \tfrac{1}{40320}A^8 - \tfrac{1}{3628800}A^{10}, \&c.$$

from which and the former *Series* exhibiting the *Sine* by the *Arch*, by division, it is easie to conclude, that the *Natural Tangent* of the Arch *A*, is

$$A + \tfrac{1}{3}A^3 + \tfrac{2}{15}A^5 + \tfrac{17}{315}A^7 + \tfrac{62}{2835}A^9, \&c.$$

and the *Natural Secant* to the same Arch

$$1 + \tfrac{1}{2}A^2 + \tfrac{5}{24}A^4 + \tfrac{61}{720}A^6 + \tfrac{277}{8064}A^8, \&c.$$

and from the Arithmetick of Infinites, the Number of these Secants being the Arch A, it follows, that the sum Total of all the Infinite Secants on that Arch, is

$$A + \tfrac{1}{6}A^3 + \tfrac{1}{24}A^5 + \tfrac{61}{5040}A^7 + \tfrac{277}{72576}A^9, \&c.$$

the

the which, by what foregoes, is the *Logarithm Tangent* of *Napeir's* form, for the Arch of 45gr. $+\frac{1}{2}$ *A*, as before.

And Collecting the Infinite Sum of all the *Natural Tangents* on the said Arch A, there will arise

$$\frac{1}{2}AA + \frac{1}{12}A^4 + \frac{1}{45}A^6 + \frac{17}{2520}A^8 + \frac{31}{14175}A^{10}, \text{ \&c.}$$

which will be found to be the Logaritm of the Secant of the same Arch *A*.

A most compendious and facile Method for Constructing the Logarithms, exemplified and demonstrated from the Nature of Numbers, without any regard to the Hyperbola, with a speedy Method for finding the Number from the Logarithm given. By E. Halley.

THE Invention of the Logarithms is justly esteemed one of the most Useful Discoveries in the Art of Numbers, and accordingly has had an Universal Reception and Applause; and the great Geometricians of this Age, have not been wanting to cultivate this Subject with all the Accuracy and Subtilty a matter of that consequence doth require; and they have demonstrated several very admirable Properties of these Artificial Numbers, which have rendred their Construction much more facile than by those operose Methods at first used by their truly Noble Inventor, the Lord *Napeir*, and our worthy Country-man Mr. *Briggs*.

But notwithstanding all their Eedeavours, I find very few of those who make constant use of Logarithms, to have attained an adequate Notion of them; to know how to make or examine them; or to understand the extent of uſe of them: Contenting themſelves with the Tables of them as they find them, without daring to queſtion them, or caring to know how to rectifie them, ſhould they be found amiſs; being I ſuppoſe under the apprehenſion of ſome great difficulty therein. For the ſake of ſuch the following Tract is principally intended, but not without hopes however to produce ſomething that may be acceptable to the moſt knowing in theſe matters.

But firſt, it may be requiſite to premiſe a definition of Logarithms, in order to render the enſuing Diſcourſe more clear, the rather becauſe the old one *Numerorum proportionalium æqui differentes comites*, ſeems too ſcanty to define them fully. They may more properly be ſaid to be *Numeri Rationum Exponentes*: Wherein we conſider *ratio* as a *Quantitas ſui generis*, beginning from the *ratio* of *equality*, or 1 to 1=0; being Affirmative when the *ratio* is increaſing, as of Unity to a greater Number, but Negative when decreaſing; and theſe *rationes* we ſuppoſe to be meaſured by the Number of *ratiunculæ* contained in each. Now theſe *ratiunculæ* are ſo to be underſtood as in a continued Scale of Proportionals infinite in Number between the two terms of the *ratio*, which infinite Number of mean Proportionals is to that infinite Number of the like and equal *ratiunculæ* between any other two terms, as the Logarithm of the one *ratio* is to the Logarithm of the other. Thus, if there be ſuppoſed between 1 and 10 an infinite Scale of mean Proportionals, whoſe Number is 100000, &c. *in infinitum*; between 1 and 2 there ſhall be 30102, &c. of ſuch Proportionals, and between

1 and 3 there will be 47712 *&c.* of them; which Numbers therefore are the Logarithms of the *rationes* of 1 to 10, 1 to 2, and 1 to 3; and not so properly to be called the Logarithms of 10, 2 and 3.

But if instead of supposing the Logarithms composed of a number of equal *Ratiunculæ*, proportional to each *ratio*, we shall take the *ratio* of Unity to any number to consist always of the same infinite number of *Ratiunculæ*, their magnitude, in this case, will be as their number in the former; wherefore if between Unity and any Number proposed, there be taken any infinity of mean Proportionals, the infinitely little augment or decrement of the first of those means from Unity, will be a *ratiuncula*, that is, the *momentum* or *Fluxion* of the *ratio* of Unity to the said Number: And seeing that in these continual Proportionals all the *ratiunculæ* are equal, their Sum, or the whole *ratio* will be as the said *momentum* is directly; that is, the Logarithm of each *ratio* will be as the Fluxion thereof. Wherefore if the Root of any infinite Power be extracted out of any Number, the *differentiola* of the said Root from Unity, shall be as the Logarithm of that Number. So that Logarithms thus produced may be of as many forms as you please to assume infinite *Indices* of the Power whose Root you seek: as if the *Index* be supposed 100000*&c.* infinitely, the Roots shall be the Logarithms invented by the Lord *Napeir*; but if the said *Index* were 2302585, *&c.* Mr. *Briggs*'s Logarithms would immediately be produced. And if you please to stop at any number of Figures, and not to continue them on, it will

 suffice

suffice to assume an *Index* of a Figure or two more than your intended Logarithm is to have, as Mr. *Briggs* did, who to have his Logarithms true to 14 places, by continual extraction of the Square Root, at last came to have the Root of the 140737488355328*th* Power; but how operose that Extraction was, will be easily judged by whoso shall undertake to examine his *Calculus*.

Now, though the Notion of an Infinite Power may seem very strange, and to those that know the difficulty of the Extraction of the Roots of High Powers, perhaps impracticable; yet by the help of that admirable Invention of Mr. *Newton*, whereby he determines the *Unciæ* or Numbers prefix'd to the Members composing Powers (on which chiefly depends the Doctrine of Series) the Infinity of the Index contributes to render the Expression much more easie: For if the Infinite Power to be resolved be put (after Mr. *Newton*'s Method) $\overline{p+pq}$, $\overline{p+pq}|^{\frac{1}{m}}$ or $\overline{1+q}|^{\frac{1}{m}}$, instead of $1+\frac{1}{m}q+\frac{1-m}{2mm}qq+\frac{1-3m+2mm}{6m^3}q^3+\frac{1-6m+11mm-6m^3}{24m^4}q^4$ &c. (which is the Root when m is finite) becomes

$$1+\frac{1}{m}q-\frac{1}{2m}qq+\frac{1}{3m}q^3+\frac{1}{4m}q^4+\frac{1}{5m}q^5, \text{ \&c.}$$

$m\,m$ being *infinite* infinite, and consequently whatever is divided thereby vanishing. Hence it follows that $\frac{1}{m}$ multiplied into $q-\frac{1}{2}qq+\frac{1}{3}qqq-\frac{1}{4}q^4+\frac{1}{5}q^5$ &c. is the augment of the first of our mean Proportionals between Unity and $1+q$, and is therefore the Logarithm of the *ratio of* 1 to $1+q$; and whereas the Infinite Index

dex m may be taken at pleasure, the several Scales of Logarithms to such *Indices* will be as $\frac{1}{m}$ or reciprocally as the *Indices*. And if the Index be taken 10000, &c. as in the case of *Napeir's* Logarithms, they will be simply $q - \frac{1}{2}qq + \frac{1}{3}qqq - \frac{1}{4}q^4 + \frac{1}{5}q^5 - \frac{1}{6}q^6$ &c.

Again, if the Logarithm of a decreasing *ratio* be sought, the infinite Root of $1-q$ or $\overline{1-q}|^{\frac{1}{m}}$ is $1 - \frac{1}{m}q - \frac{1}{2m}q^2 - \frac{1}{3m}q^3 - \frac{1}{4m}q^4 - \frac{1}{5m}q^5 - \frac{1}{6m}q^6$ &c. whence the decrement of the first of our infinite Number of Proportionals will be $\frac{1}{m}$ into $q + \frac{1}{2}qq + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + \frac{1}{6}q^6$ &c. which therefore will be as the Logarithm of the *ratio* of Unity to $1-q$. But if m be put 10000, &c. then the said Logarithm will be $q + \frac{1}{2}qq + \frac{1}{3}q^3 + \frac{1}{4}q^4 + \frac{1}{5}q^5 + \frac{1}{6}q^6$, &c.

Hence the terms of any *ratio*, being a and b, q becomes $\frac{b-a}{a}$ or the difference divided by the lesser term, when 'tis an increasing *ratio*; or $\frac{b-a}{b}$ when 'tis decreasing, or as b to a. Whence the Logarithm of the same *ratio* may be doubly exprest, for putting x for the difference of the terms a and b, it will be either

$$\frac{1}{m} \text{ into } \frac{x}{b} + \frac{x^2}{2bb} + \frac{x^3}{3b^3} + \frac{x^4}{4b^4} + \frac{x^5}{5b^5} + \frac{x^6}{6b^6} \text{ \&c. or}$$

$$\frac{1}{m} \text{ into } \frac{x}{a} - \frac{x^2}{2aa} + \frac{x^3}{3a^3} - \frac{x^4}{4a^4} + \frac{x^5}{5a^5} - \frac{x^6}{6a^6} \text{ \&c.}$$

But

But if the *ratio* of *a* to *b* be supposed divided into two parts, *viz.* into the *ratio* of *a* to the Arithmetical Mean between the terms, and the *ratio* of the said Arithmetical Mean to the other term *b*, then will the Sum of the Logarithms of those two *rationes* be the Logarithm of the *ratio* of *a* to *b*; and substituting $\frac{1}{2}z$ instead of $\frac{1}{2}a + \frac{1}{2}b$ the said Arithmetical Mean, the Logarithms of those *rationes* will be by the foregoing Rule,

$$\frac{1}{m} \text{ in } \frac{x}{z} + \frac{xx}{2zz} + \frac{x^3}{3z^3} + \frac{x^4}{4z^4} + \frac{x^5}{5z^5} + \frac{x^6}{6z^6} \text{ \&c. and}$$

$$\frac{1}{m} \text{ in } \frac{x}{z} - \frac{xx}{2zz} + \frac{x^3}{3z^3} - \frac{x^4}{4z^4} + \frac{x^5}{5z^5} - \frac{x^6}{6z^6} \text{ \&c.}$$

the Sum whereof $\frac{1}{m}$ in $\frac{2x}{z} * + \frac{2x^3}{3z^3} * + \frac{2x^5}{5z^5} * \frac{2x^7}{7z^7}$ &c. will be the Logarithm of the *ratio* of *a* to *b*, whose difference is *x* and Sum *z*. And this *Series* converges twice as swift as the former, and therefore is more proper for the Practice of making Logarithms: Which it performs with that expedition, that where *x* the difference is but the hundredth part of the Sum, the first step $\frac{2x}{z}$ suffices to seven places of the Logarithm, and the second step to twelve: But if *Briggs*'s first Twenty Chiliads of Logarithms be supposed made, as he has very carefully computed them to fourteen places, the first step alone, is capable to give the Logarithm of any intermediate Number true to all the places of those Tables.

After the same manner may the difference of the said two Logarithms be very fitly applied

plied to find the Logarithms of Prime Numbers, having the Logarithms of the two next Numbers above and below them: For the difference of the *ratio* of a to $\frac{1}{2}z$ and of $\frac{1}{2}z$ to b is the *ratio* of ab to $\frac{1}{4}zz$, and the half of that *ratio*, is that of $\sqrt{ab}$ to $\frac{1}{2}z$, or of the Geometrical Mean to the Arithmetical. And consequently the Logarithm thereof will be the half difference of the Logarithms of those *rationes*, *viz.*

$$\frac{1}{m} \text{ into } \frac{xx}{2zz} + \frac{x^4}{4z^4} + \frac{x^6}{6z^6} + \frac{x^8}{8z^8} \text{ \&c.}$$

Which is a Theorem of good dispatch to find the Logarithm of $\frac{1}{2}z$. But the same is yet much more advantageously performed by a Rule derived from the foregoing, and beyond which, in my Opinion, nothing better can be hoped. For the *ratio* of ab to $\frac{1}{4}zz$ or $\frac{1}{4}aa + \frac{1}{2}ab + \frac{1}{4}bb$, has the difference of its terms $\frac{1}{4}aa - \frac{1}{2}ab + \frac{1}{4}bb$, or the Square of $\frac{1}{2}a - \frac{1}{2}b = \frac{1}{4}xx$, which in the present case of finding the Logarithms of Prime Numbers, is always Unity, and calling the Sum of the terms $\frac{1}{4}zz + ab = yy$, the Logarithm of the *ratio* of $\sqrt{ab}$ to $\frac{1}{2}a + \frac{1}{2}b$ or $\frac{1}{2}z$ will be found

$$\frac{1}{m} \text{ in } \frac{1}{yy} + \frac{1}{3y^6} + \frac{1}{5y^{10}} + \frac{1}{7y^{14}} + \frac{1}{9y^{18}} \text{ \&c.}$$

which converges very much faster than any Theorem hitherto published for this purpose.

Here note $\frac{1}{m}$ is all along applied to adapt these

Rules

Rules to all ſorts of Logarithms. If *m* be 10000 &c. it may be neglected, and you will have *Napeir*'s Logarithms, as was hinted before; but if you deſire *Briggs*'s Logarithms, which are now generally received, you muſt divide your Series by

2,302585092994045684017991454684364207
601101488628772976033328

or multiply it by the reciprocal thereof, *viz.*

0,434294481903251827651128918916605082294397005803666566114454

But to ſave ſo operoſe a Multiplication (which is more than all the reſt of the Work) it is expedient to divide this Multiplicator by the Powers of *z* or *y* continually, according to the Direction of the Theorem, eſpecially where *x* is ſmall and Integer, reſerving the proper Quotes to be added together, when you have produced your Logarithm to as many Figures as you deſire: Of which Method I will give a Specimen.

If the Curioſity of any Gentleman that has leiſure, would prompt him to undertake to do the Logarithms of all Prime Numbers under 100000, to 25 or 30 Figures, I dare aſſure him, that the facility of this Method will invite him thereto; nor can any thing more eaſie be deſired. And to encourage him, I here give the Logarithms of the firſt Prime Numbers under 20 to 60 places, computed by the accurate Pen of Mr. *Abraham Sharp*, (from whoſe Induſtry and Capacity the World may in time expect great Performances) as they were communicated to me by our common Friend Mr. *Euclid Speidall.*

Numb.

Numb.	Logarithm.
2	0,30102999566398119521373889472449 3026768189881462108541310427
3	0,47712125471966243729502790325511 5309200128864190695864829866
7	0,84509804001425683071221625859263 6193483572396323965406503835
11	1,04139268515822504075019997124302 4241706702190466453094596539
13	1,11394335230683776920654189502624 62545611189005053673288598083
17	1,23044892137827302854016989432833 7030007567378425046397380368
19	1,27875360095282896153633347575692 9317951129337394497598906819

The next Prime Number is 23, which I will take for an Example of the foregoing Doctrin; and by the first Rules, the Logarithm of the *ratio* of 22 to 23, will be found to be either

$$\frac{1}{22}-\frac{1}{968}+\frac{1}{31944}-\frac{1}{937024}+\frac{1}{25768160}\ \&c. \text{ or}$$

$$\frac{1}{23}+\frac{1}{1058}+\frac{1}{36501}+\frac{1}{1119364}+\frac{1}{32181715}$$

&c.

As likewise that of the *ratio* of 23 to 24 by a like Procefs.

$$\frac{1}{23}-\frac{1}{1058}+\frac{1}{36501}-\frac{1}{1119364}+\frac{1}{32181715}\ \&c.$$

or,

$$\frac{1}{24}+\frac{1}{1152}+\frac{1}{41472}+\frac{1}{1327104}+\frac{1}{39813120}$$

&c.

And

And this is the Result of the Doctrine of *Mercator*, as improved by the Learned Dr. *Wallis*. But by the second Theorem, *viz.* $\frac{2x}{z} + \frac{2x^3}{3z^3} + \frac{2x^5}{5z^5}$ &c. The same Logarithms are obtained by fewer steps. To wit,

$$\frac{2}{45} + \frac{2}{273375} + \frac{2}{922640625} + \frac{2}{2615686171875}$$

&c. and

$$\frac{2}{47} + \frac{2}{311469} + \frac{2}{1146725035} + \frac{2}{3546361843241} \text{ \&c.}$$

which was invented and demonstrated in the Hyperbolick Spaces Analogous to the Logarithms, by the Excellent Mr. *James Gregory*, in his *Exercitationes Geometricæ*, and since further prosecuted by the aforesaid Mr. *Speidall*, in a late Treatise, in *English*, by him published on this Subject. But the Demonstration as I conceive, was never till now perfected without the consideration of the Hyperbola, which in a matter purely Arithmetical as this is, cannot be so properly applied. But what follows I think I may more justly claim as my own, *viz.* That the Logarithm of the *ratio* of the Geometrical Mean to the Arithmetical between 22 and 24, or of $\surd$ 528 to 23 will be found to be either.

$$\frac{1}{1058} + \frac{1}{1119364} + \frac{1}{888215334} + \frac{1}{626487882248}$$

&c. or

$$\frac{1}{1057} + \frac{1}{3542796579} + \frac{1}{6596765584852 85} \text{ \&c.}$$

All

All theise *Series* being to be multiplied into 0,4342944819 &c. if you design to make the Logarithm of *Briggs*. But with great Advantage in respect of the Work, the said 4342944819, &c. is divided by 1057 and the Quotient thereof again divided by three times the Square of 1057, and that Quotient again by $\frac{1}{3}$ of that Square, and that Quotient by $\frac{7}{5}$ thereof, and so forth, till you have as many Figures of your Logarithm as you desire. As for Example; the Logarithm of the Geometrical Mean, between 22 and 24, is found by the Logarithms of 2, 3 and 11 to be

1057)43429 &c.

3 in 1117249)41087 &c.

$\frac{1}{3}$ in 1117249)12258 &c.

$\frac{7}{5}$ in 1117249)65832 &c.

$\frac{9}{7}$ in 1117249)42088 &c.

1.36131696126690612945009172669805

(41087462810146814347315886368

(12258521544181829460074

(6583235184376175

(4208829765

(2930

Summa.

1.36172783601759287886777711225117

Which is the Logarithm of 23 to thirty two places, and obtained by five Divisions with very small *Divisors*; all which is much less Work than simply multiplying the *Series* into the said Multiplicator 43429, &c.

Before I pass on to the converse of this Problem, or to shew how to find the Number appertaining

pertaining to a Logarithm aſſigned, it will be requiſite to advertiſe the Reader, that there is a ſmall miſtake in the aforeſaid Mr. *James Gregory*'s *Vera Quadratura Circuli & Hyperbolæ*, publiſhed at *Padua Anno* 1667. wherein he applies his Quadrature of the Hyperbola to the making the Logarithms; In *pag*. 48. he gives the Computation of the Lord *Napeir*'s Logarithm of 10, to five and twenty places, and finds it 2302585092994045624017870 inſtead of 2302585092994045684017991, erring in the eighteenth Figure, as I was aſſured upon my own Examination of the Number I here give you, and by compariſon thereof with the ſame wrought by another hand, agreeing therewith to 57 of the 60 places. Being deſirous to be ſitisfied how this difference aroſe, I took the no ſmall trouble of Examining Mr. *Gregory*'s Work, and at length found, that in the inſcribed *Polygon* of 512 Sides, , in the eighteenth Figure, was a 0 inſtead of 9, which being rectified, and the ſubſequent Work corrected therefrom, the reſult did agree to a Unite with our Number. And this I propoſe not to Cavil at an eaſie miſtake in managing of ſo vaſt Numbers, eſpecially by a Hand that has ſo well deſerved of the Mathematical Sciences, but to ſhew the exact coincidence of two ſo very differing Methods to make Logarithms, which might otherwiſe have been queſtioned.

From the Logarithm given to find what *ratio* it expreſſes, is a Problem that has not been ſo much conſidered as the former, but which is ſolved with the like eaſe, and demonſtrated

ſtrated by a like Proceſs, from the ſame general Theorem of Mr. *Newton*: For as the Logarithm of the *ratio* of 1 to $1+q$ was proved to be $\overline{1+q}|^{\frac{1}{m}}-1$, and that of the *ratio* of 1 to $1-q$ to be $1-\overline{1-q}|^{\frac{1}{m}}$: ſo the Logarithm, which we will from henceforth call *L*, being given, $1+L$, will be equal to $\overline{1+q}|^{\frac{1}{m}}$ in the one caſe; and $1-L$ will be equal to $\overline{1-q}|^{\frac{1}{m}}$ in the other: Conſequently $\overline{1+L}|^{m}$ will be equal to $1+q$, and $\overline{1-L}|^{m}$ to $1-q$; that is, according to Mr. *Newton*'s ſaid Rule, $1+mL+\frac{1}{2}m^2L^2+\frac{1}{6}m^3L^3+\frac{1}{24}m^4L^4+\frac{1}{120}m^5L^5$ &c. will be $=1+q$, and $1-mL+\frac{1}{2}m^2L^2-\frac{1}{6}m^3L^2+\frac{1}{24}m^4L^4-\frac{1}{120}m^5L^5$ &c. will be equal to $1-q$, *m* being any infinite Index whatſoever, which is a full and general Propoſition from the Logarithm given to find the Number, be the *Species* of Logarithm what it will. But if *Napeir*'s Logarithm be given, the Multiplication by *m* is ſaved (which Multiplication is indeed no other than the reducing the other *Species* to his) and the *Series* will be more ſimple, *viz.* $1+L+\frac{1}{2}LL+\frac{1}{6}L^3+\frac{1}{24}L^4+\frac{1}{120}L^5$ &c. or $1-L+\frac{1}{2}LL-\frac{1}{6}L^3+\frac{1}{24}L^4-\frac{1}{120}L^5$ &c. This *Series*, eſpecially in great Numbers converges ſo ſlowly, that it were to be wiſhed it could be contracted.

If one term of the *ratio*, whereof L is the Logarithm, be given, the other term will be easily had by the same Rule: For if L were *Napeir*'s Logarithm of the *ratio* of *a* the lesser to *b* the greater term, *b* would be the Product of *a* into $1+L+\frac{1}{2}LL+\frac{1}{6}LLL$ &c. $=a+aL+\frac{1}{2}aLL+\frac{1}{6}aL^3$ &c. But if *b* were given, *a* would be $=b-bL+\frac{1}{2}bLL-\frac{1}{6}bL^3$ &c. Whence, by the help of the *Chiliads*, the Number appertaining to any Logarithm will be exactly had to the utmost extent of the Tables. If you seek the nearest next Logarithm, whether greater or lesser, and call its Number *a* if lesser, or *b* if greater than the given L, and the difference thereof from the said nearest Logarithm you call *l*; it will follow, that the Number answering to the Logarithm L will be either *a* into $1+l+\frac{1}{2}ll+\frac{1}{6}lll+\frac{1}{24}l^4+\frac{1}{120}l^5$ &c. or else *b* into $1-l+\frac{1}{2}ll-\frac{1}{6}lll+\frac{1}{24}l^4-\frac{1}{120}l^5$ &c. wherein as *l* is less, the *Series* will converge the swifter. And if the first 20000 Logarithms be given to fourteen Places, there is rarely occasion for the three first steps of this *Series* to find the Number to as many places. But for *Vlacq*'s great Canon of 100000 Logarithms, which is made but to ten places, there is scarce ever need for more than the first step *a+al* or *a+mal* in one case, or else *b−bl* or *b−mbl* in the other, to have the Number true to as many Figures as those Logarithms consist of.

If future Industry shall ever produce Logarithmick Tables to many more places than now we have them; the aforesaid Theorems will be of more use to reduce the correspondent Natural Numbers to all the places thereof. In order to make the first *Chiliad* serve all

Uses,

Uses, I was desirous to contract this *Series*, wherein all the powers of l are present, into one, wherein each alternate Power might be wanting; but found it neither so simple or uniform as the other. Yet the first step thereof is, I conceive, most commodious for Practice, and withal exact enough for Numbers not exceeding fourteen places, such as are Mr. *Brigs*'s large Table of Logarithms; and therefore I recommend it to common Use.

It is thus: $a + \frac{al}{1 - \frac{1}{2}l}$ or $b - \frac{bl}{1 + \frac{1}{2}l}$ will be the Number answering to the Logarithm given, differing from the Truth by but one half of the third step of the former *Series*. But that which renders it yet more eligible, is, that with equal facility, it serves for *Brigg*'s or any other sort of Logarithm, with the only variation of writing $\frac{1}{m}$ instead of 1, that is, $a + \frac{al}{\frac{1}{m} - \frac{1}{2}l}$ and $b - \frac{bl}{\frac{1}{m} + \frac{1}{2}l}$, or $\frac{\frac{1}{m}a + \frac{1}{2}la}{\frac{1}{m} - \frac{1}{2}l}$ and $\frac{\frac{1}{m}b - \frac{1}{2}lb}{\frac{1}{m} + \frac{1}{2}l}$, which are easily resolv'd into Analogies, *viz.*

As 43429 &c. $- \frac{1}{2}l$ to 43429 $+ \frac{1}{2}l$: : So is a	to the Number sought.
or As 43429 &c. $+ \frac{1}{2}l$ to 43429 $- \frac{1}{2}l$: : So is b	

If more steps of this *Series* be desired, it will be found as follows, $a + \frac{al}{1-\frac{1}{2}l} - \frac{\frac{1}{12}al^3}{1-l} + \frac{\frac{1}{33}al^5}{1-2l}$ &c. as may easily be demonstrated by working out the Divisions in each step, and collecting the Quotes, whose Sum will be found to agreee with our former *Series*.

Thus I hope, I have cleared up the Doctrine of Logarithms, and shewn their Construction and Use independant from the *Hyperbola*, whose Affections have hitherto been made use of for this purpose, though this be a matter purely Arithmetical, nor properly demonstrable from the Principles of Geometry. Nor have I been obliged to have recourse to the Method of Indivisibles, or the Arithmetick of Infinites, the whole being no other than an easie Corollary to Mr. *Newton's* General Theorem for forming Roots and Powers.

A SOLUTION,

Given by Mr. *John Collins*, of a Chorographical Problem, Propoſed by *Richard Townley*, Eſq;

PROBLEM.

The Diſtances of three Objects in the ſame Plain being given, as *A*, *B*, *C*; The Angles made at a fourth Place in the ſame Plain as at *S*, are obſerved: The Diſtances from the Place of Obſervation to the reſpective Objects, are required.

The Problem hath ſix *Caſes*.

Caſe 1. IF the Station be taken without the Triangle made by the Objects but in one of the ſides thereof produced, as at *S* in the *9th* Figure; find the Angle *A C B*; then in the Triangle *A C S* all the Angles and the ſide *A C* are known, whence *either or both* the Diſtances *S A* or *S C* may be found.

Caſe 2. If the Station be in one of the Sides of the Triangle, as in the 10*th* Figure at S, then having the three ſides *AC*, *CB*, *BA* given, find the Angle *C A B*; then again in the Triangle *S A B*, all the Angles, and the ſide *AB*, are known; whence may be found either *A S*, or *S B*, *Geometrically*, if you make the Angle *C A D* equal to the obſerved Angle *C S B*, and draw *B S* parallel to *D A*, you determine the Point of Station S.

Caſe 3. If the three Objects lie in a right Line as *A C B* (ſuppoſe it done) and that a Circle paſſeth through the Station *S*, and the two exterior Objects *A B*, then is the Angle *A B D* equal to the obſerved Angle *A S C* (by 21 of the 3d Book of *Euclid*) as inſiſting on the ſame Arch *A D*: And the Angle *B A D* in like manner equal to the obſerved Angle *C S B*: By this means, the point *D* is determined. Join *D C*, and produce the ſame, thn a Circle paſſing through Points *A B D*, interſects *D C*, produced at *S*, the place of Station.

Calculation.

In the Triangle *A B D*, all the Angles and the ſide *A B* are known, whence may be found the ſide *A D*.

Then in the Triangle *C A D* the two ſides *C A* and *A D* are known and their contained Angle *C A D* is known; whence may be found the Angles *C D A* and *A C D*, the complement whereof to a Semicircle is the Angle *S C A*: in which Triangle the Angles are now all known and the ſide *A C*: whence may be found either of the Diſtances, *S C* or *S A*.

Caſe 4.

Case 4. If the Station be without the Triangle, made by the Objects, the sum of the Angles observed is less than four right Angles. The Construction is the same as in the last Case, and the Calculation likewise; saving that you must make one Operation more, having the three Sides, *A C*, *C B*, *B A*, thereby find the Angle *C A B*, which add to the Angle *E A D*, then you have the two sides, *viz.* *A C*, being one of the Distances, and *A D*, (found as in the former Case) with their contained Angle *C A D*, given to find the Angles *C D A*, and *A C D*, the Complement whereof to a Semicircle, is the Angle *S C A*: Now in the Triangle *S C A*, the Angle at *C* being found, and at *S* observed, and given by Supposition, the other at *A* is likewise known, as being the complementof the two former to a Semicircle, and the side *A C* given; hence the Distance *C S* or *A S* may be found.

Case 5. If the place of Station be at some Point *within* the plain of the Triangle, made by the three Objects, the Construction and Calculation is the same as in the last, saving only that instead of the observed Angle *A S C*, the Angle *A B D* is equal to the Complement thereof to a Semicircle, to wit, it is equal to the Angle *A S D*; both of them insisting on the same Arch *A D*: And in like manner the Angle *B A D* is equal to the Angle *D S B*, which is the Complement of the observed *C S B*; and in this Case, the sum of the three Angles observed, is equal to four right Angles.

In these three latter Cases no use is made of the Angle observed between the two Objects, as *A* and *B*, that are made the Base-line of the Construction; yet the same is of ready use for finding the third Distance, or last side sought, as in the fourth Scheme, in the Triangle *S A B*, there is given the Distance *A B*, its opposite Angle equal to the sum of the two observed Angles, and the Angle *S A B* attained, as in the fourth Case: Hence the third Side or last Distance *S B* may be found.

And here it may be noted, that the three Angles *C A S*, *A S B*, *S B C*, are together equal to the Angle *A C B*, for the two Angles *C S B*, and *C B S*, are equal to *E C B*, as being the Complement of *S C B* to two right Angles; and the like in the Triangle on the other side. *Ergo, &c.*

Case 6. If the three Objects be *A*, *B*, *C*, and the Station at *S*, as before, it may happen, according to the former Constructions, that the Points *C* and *D* may fall close together, and so a right Line joining them may be produced with uncertainty; in such case the Circle may be conceived to pass through the place of Station at *S*, and any two of the Objects (as in the sixth Scheme) through *B* and *C*; wherein making the Angle *D B C* equal to the observed Angle *A S C*, and *B C D* equal to the Complement to 180 degrees of both the observed Angles in *D S B* thereby the Point *D* is determined, through which, and the points *C B*, the Circle is to be described, and joyning *D A*, (produced, when need requireth) where it intersects

tersects the Circle, as at *s*, is the place of Station sought.

This Problem may be of good Use for the due Situation of Sands or Rocks, that are within sight of three Places upon Land, whose distances are well known; or for *Chorographical* Uses, *&c.* Especially now there is a Method of observing Angles nicely accurate by aid of the *Telescope*; and was therefore thought fit to be now Publish'd though it be a competent time since it was delivered in Writing.

The

The Solutions of three Chorographic Problems, by a Member of the Philosophical Society *of* Oxford.

THE three following Problems may occur at Sea, in finding the distance and position of *Rocks*, *Sands*, &c. from the Sea Shoar; or in the Surveying of the Sea Coast; When only two Objects, whose distance from each other is known, can be seen at one Station; but especially they may be useful to one that would make a *Map* of a Country by a Series of Triangles derived from one or more measured Bases; which is the most exact way of finding the bearing and distance of Places from each other, and thence their true Longitude and Latitude; and may consequently occur to one that would in that manner measure a Degree on the Earth.

The first Problem (Fig. 3 and 4.)

There are two Objects, *B* and *C*, whose distance *B C* is known; and there are two stations at *A* and *E*, where the Objects *B C* being visible, and the Stations one from another, the Angles *B A C*, *B A E*, *A E B*, *A E C*, are known by Observation, (which may be made with an ordinary Serveying *Semicircle*, or *Crostaff*; or if the Objects be beyond

beyond the view of the naked Eye, with a *Telescopick Quadrant*) to find the distances or lines *AB*, *AC*, *AE*, *EC*.

Construction.

In each of the Triangles *BAE*, *CAE*, two Angles at *A*, *E*, being known, the third is also known: then take any line $\alpha\epsilon$ at pleasure, on which constitute the Triangles $\beta\alpha\epsilon$, $\alpha\epsilon\gamma$ respectively equiangular to the Triangles *BAE*, *AEC*; join $\beta\gamma$. Then upon *BC* constitute the Triangles *BCA*, *BCE*, equiangular to the correspondent triangles $\beta\gamma\alpha$, $\beta\gamma\epsilon$, join *AE*, and the thing is manifestly done.

The Calculation.

Assuming $\alpha\epsilon$ of any number of parts, in triangles $\alpha\beta\epsilon$, $\alpha\gamma\epsilon$, the angles being given, the sides $\alpha\beta$, $\alpha\gamma$, $\epsilon\beta$, $\epsilon\gamma$ may be found by Trignometry: Then in the Triangle $\beta\alpha\gamma$, having the angle $\beta\alpha\gamma$, and the legs $\alpha\beta$, $\alpha\gamma$, we may find $\beta\gamma$. Then $\beta\gamma$. *BC*:: $\beta\alpha$. *BA*:: $\beta\epsilon$. *BE*:: $\gamma\alpha$. *CA*:: $\gamma\epsilon$. *CF*.

The second Problem (Fig. 5 and 6.)

Three Objects *B*, *C*, *D*, are given, or (which is the same) the sides, and consequently angles of the triangle *BCD* are given; also there are two points or stations *A*, *E*, such, that at *A* may be seen the three points *BCE*, but not *D*; and at the station *E* may be seen *A*, *C*, *D*, but not *B*, that is the angles *BAC*, *BAE*, *AEC*, *AED*, (and consequently *EAC*, *AEC*, are known by observation: to find the lines *AB*, *AC*, *AE* *EC*, *ED*.

Construction.

Take any line $\alpha\varepsilon$ at pleasure, and at its extremities make the angles $\varepsilon\alpha\gamma$, $\varepsilon\alpha\beta$, $\alpha\varepsilon\gamma$, $\alpha\varepsilon\delta$, equal to the correspondent observed angles *EAC*, *EAB*, *AEC*, *AED*. Produce $\beta\alpha$, $\delta\varepsilon$, till they meet in φ, join $\varphi\gamma$; then upon *CB* describe (according to 33. 3. *Eucl.*) a segment of a circle that may contain an angle$=\gamma\varphi\beta$; and upon *CD* describe a segment of a circle capable of an angle$=\gamma\varphi\delta$; suppose *F* the common section of these two circles; join *FB*, *FC*, *FD*; then from the point *C*, draw forth the lines *CA*, *CE*, so that the angle *FCA* may be$=\varphi\gamma\alpha$, and *FCE*$=\varphi\gamma\varepsilon$; so *A*, *E*, the common Sections of *CA*, *CE*, with *FB*, *FD*, will be the points required, from whence the rest is easily deduced.

The Calculation.

Assuming $\alpha\varepsilon$ of any number, in the triangles $\alpha\gamma\varepsilon$, $\alpha\varphi\varepsilon$, all the angles being given, with the side $\alpha\varepsilon$ assum'd, the sides $\alpha\gamma$, $\varepsilon\gamma$, $\alpha\varphi$, $\varepsilon\varphi$, will be known; then in the triangle $\gamma\alpha\varphi$, the angle $\gamma\alpha\varphi$, with the legs $\alpha\gamma$, $\alpha\varphi$, being known, the angles $\alpha\varphi\gamma$, $\alpha\gamma\varphi$, with the side $\varphi\gamma$ will be known: then as for the rest of the work in the other figure, the triangle *BCD* having all its sides and angles known, and the angles *BFC*, *BFD*, being equal to the found $\beta\varphi\gamma$, $\beta\varphi\delta$; how to find *FB*, *FC*, *FD* by *Calculation* (and also *Protraction*) is shewn by Mr. *Collins* in the precedent Discourse, as to all its cases, which may therefore supersede my shewing any other way.

But

But here it must be noted, that if the sum of the observed angles, *B A E*, *A E D*, is 180 degrees: then *A B* and *E D* cannot meet, because they are parallel, and consequently the given Solution cannot take place; for which reason I here subjoin another.

Another Solution.

Upon *B C* (Fig. 7.) describe a segment *B A C* of a circle, so that the angle of the segment may be equal to the observed ∠ $\beta \alpha \gamma$, (which as above quoted is shewn 33. 3. *Euclid.*) and upon *CD* describe a segment *C E D* of a circle capable of an angle equal to the observed *CED*; from *C* draw the diameters of these circles *C G* *C H*; then upon *C G* describe a segment of a circle *G F C*, capable of an angle equal to the observed ∠ *A E C*; likewise upon *C H* describe a circle's segment *C F H*, capable of an angle equal to the observed *C A E*: suppose *F* the common Section of the two last circles *H F C*, *G F C*, join *F H*, cutting the circle *H E C* in *E*, join also *F G*, cutting the circle *G A C* in *A*: I say that *A*, *E*, are the points required.

Demonstration.

For the ∠ *B A C* is $=\beta \alpha \gamma$ by construction of the segment, also the angles *C E H*, *C A G*, are right, because each exists in a semicircle: therefore a circle being described upon *C F* as a diameter, will pass through *E*, *A*; Therefore the angle *C A E*=∠ *C F E*=*C F H*=(by construction) to the observed angle $\gamma \alpha \epsilon$. In like manner the ∠ *C E A*=*C F A*=*C FG*= observ'd angle $\gamma \epsilon \alpha$.

In the stations *A*, *E*, fall in a right line with the point *C*; the lines *G A*, *H E* being parallel,

lel, cannot meet: but in this case the Problem is indeterminate and capable of infinite Solutions. For as before upon *CG* describe a Segment of a circle capable of the observed $L\gamma\epsilon\alpha$, and upon *C H*, describe a Segment capable of the observed $\gamma\alpha\epsilon$: then through *C*, draw a line any way cutting the circles in *A*, *E*, these points will answer the question.

The third Problem.

Four points *B*, *C*,*D*,*F*, (Fig. 8.) or the 4 sides of a quadrilateral, with the angles comprehended are given; also there are two stations *A* and *E* such, that at *A*, only *B C E* are visible, and at *E* only *A D F*, that is, the angles *B A C*, *B A E*, *A E D*, *D E F* are given: to find the places of the two poiuts *A E*, and consequently, the lengths of the lines *AB*, *AC*, *AE*, *E D*, *E F*.

Construction.

Upon *B C* (by 33. 3. *Eucl.*) describe a segment of a circle, that may contain an angle equal to the observed angle *BAC*, then from *C* draw the Chord *C M*, or a line cutting the circle in *M*; so that the Angle *BCM* may be equal to the supplement of the observed angle *B A E*, i. e. its residue to 180 degrees. In like manner on *D F* describe a segment of a circle, capable of an angle equal to the observ'd *D E F*, and from *D* draw the Chord *D N*, so that the angle *F D N* may be equal to the supplement of the observ'd angle *A E F*, join *M N*, cutting the two circles in *A*, *E*: I say *A*, *E*, are the two points requir'd.

Demon-

Demonſtration.

Join *AB*, *AC*, *ED*, *EF*, then is the ∠ *MAB* = ∠ *BCM* (by 21. 3. *Eucl.*) = ſupplement of the obſerv'd ∠ *BAE* by conſtrution, therefore the conſtructed ∠ *BAE*, is equal to that which was obſerved. Alſo the ∠ *BAC* of the ſegment is the conſtruction of the Segment, equal to the obſerv'd ∠ *BAC*. In like manner the conſtructed angles *AEF*, and *DEF*, are equal to the correſpondent obſerved angles *AEF*, *DEF*, therefore *AE* are the points requir'd.

The Calculation.

In the Triangle *BCM*, the ∠ *BCM* (= ſupplement of *BAE*) and ∠ *BMC* (= *BAC*) are given, with the ſide *BC*; thence *MC* may be found; in like manner *DN* in the △ *DNF* may be found. But the ∠ *MCD*) = *BCD* -- *BCM*) is known, with its legs *MC*, *CD*, therefore its Baſe *MD*, and ∠ *MDC*, may be known. Therefore the ∠ *MDN* (= *CDF* -- *CDM* -- *FDN*) is known, with its legs *MD*, *DN*; thence *MN* with the angles *DMN*, *DNM*, will be known. Then the ∠ *CMA* (= (∠ *DMC* + *DMN*) is known, with the ∠ *MAC* = *MAB* + *BAC*) and *MC* before found; therefore *MA* and *AC* will be known. In like manner in the triangle *EDN*, the angles *E*, *N*, with the ſide *DN* being known, the ſides *EN*, *ED*, will be known; therefore *AE* (= *MN* -- *MA* -- *EN*) is known. Alſo in the triangle *ABC*, the ∠ *A* with its ſides *BC*, *CA*, being known, the ſide *AB*, will be known, with the ∠ *BCA*; ſo in the triangle *EFD*, the ∠ *E* with the ſides, *ED*, *DF* being known, *EF* will be found, with the ∠ *EDF*. Laſtly, in the triangle

gle *ACD*, the ∠ *ACD* (=*BCD*--*BCA*) with its legs *AC*, *CD* being known, the side *AD*, will be known; and in like manner *EC* in the triangle *EDC*.

Note, that in this Problem, as also in the first and second, if the two stations fall in a right line with either of the given Objects: the *locus* of *A*, or *E*, being a circle, the particular point of *A*, or *E*, cannot be determined from the things given.

As to the other cases of this third Problem, wherein *A* and *E*, may shift places, *i.e.* only *DFE*, may be visible at *A*, and only *A*, *B*, *C*, at *E*; or wherein *B*, *D*, *E*, may be visible at *A*, and only *C*, *F*, *A*, at *E*; or wherein *A* may be of one side of the quadrilateral, and *E* on the other; or one of the stations within the quadrilateral, and the other without it: I shall for brevity sake omit the Figures, and diversity of the Sines + and -- in the calculation, and presume that the Surveyor will easily direct himself in those cases, by what has been said.

The solution of this third Problem is general, and serves also for both the precedent. For suppose *CD*, the same point in the last figure, and it gives the solution of the second Problem: but if *BC* be suppos'd the same points with *D*, *F*, by proceeding as in the last, you may directly solve the first Problem.

An Arithmetical Paradox, concerning the Chances of Lotteries; by the Honourable Francis Roberts, *Esq; Fellow of the* R. S.

AS ſome Truths (like the *Axioms* of *Geometry* and *Metaphyſicks*) are ſelf-evident at the firſt View, ſo there are others no leſs certain in their Foundation, that have a very different Aſpect, and without a ſtrict and careful Examination, rather ſeem repugnant.

We may find Inſtances of this kind in moſt Sciences.

In *Geometry*, That a Body of an infinite Length, may yet have but a finite Magnitude.

In *Geography*, That if *Antwerp* be due Eaſt to *London*, for that reaſon *London* cannot be Weſt to *Antwerp*.

In *Aſtronomy*, That at the *Barbadoes* (and other Places between the Line and Tropick) the Sun, part of the Year, comes twice in a Morning to ſome Points of the Compaſs.

In *Hydroſtaticks*, That a hollow Cone (ſtanding upon its Baſis) being fill'd with Water, the Water ſhall preſs the bottom with three times the Weight, as if the ſame Water was frozen to Ice; and Figures might be contriv'd to make it preſs a hundred times as much.

Theſe Speculations, as they are generally pleaſant, ſo they may alſo be of good uſe to warn us of the Miſtakes we are liable to, by careleſs and ſuperficial Reaſoning.

I ſhall add one Inſtance in *Arithmetick*, which perhaps may ſeem as great a Paradox as any of the former.

There are two Lotteries, at either of which a Gameſter paying a Shilling for a Lot or Throw; The firſt Lottery upon a juſt Computation of the Odds, has 3 to 1 of the Gameſter; the Second Lottery, but 2 to one; nevertheleſs, the Gameſter has the very ſame diſadvantage (and no more) in playing at the Firſt Lottery, as the Second.

It looks very like a Contradiction, that the Diſadvantage ſhould be no greater in playing againſt 3 to 1, than 2 to 1, but it may thus be reſolv'd.

Let the	*1ſt.*	Lottery conſiſt of	3	Blanks &	3	Prizes of	16 pence	a-piece.
	2d		4		2		2 ſhilling	

In the firſt Lottery the Gameſter hazards a Shilling to win a Groat, and the Chances being equal, it is evident there is 3 to one againſt him.

In the Second Lottery, the Gameſter ventures a Shilling againſt a Shilling, and the Lots being 4 to 2, his Diſadvantage is 2 to 1.

And a Lot at either of them being truly worth juſt 8 Pence, (*viz.* the 6th part of 3 times 16 Pence, or twice 2 Shillings) the Diſadvantage muſt be the very ſame in both Caſes, that is, the Gameſter pays a Shilling for a Lot that is worth but 8 Pence.

The

The Method of finding this Anſwer being ſomewhat out of the common Road, I ſhall here add it, and thereby infinite Solutions of the ſame kind may be diſcovered.

1*ſt*. Lottery.

Let a=the number of Blanks
b=the number of Prizes.
r=the Value of a Prize.

2*d*. Lottery.

Let m=the number of Blanks.
n=the number of Prizes.
s=the value of a Prize.

1 = to what you pay for a Lot, *viz*. a Shilling.

So the Lottery has its Chances for 1, and the Gameſter his for $r-1$. Now the true Odds conſiſting of the compounded Proportion of the Chances and the Values, *viz*. $\frac{a}{b}$ and $\frac{1}{r-1}$, the Share of the Lottery will be a, and that of the Gameſter $rb-b$. Therefore as the preſent caſe ſtands, the firſt Lottery muſt be $a=3rb-3b$, and by the like reaſoning, the ſecond Lottery will be $m=2sn-2n$. Now the Value of a Lot being the Sum of the Prizes divided by the number of Lots, (which muſt be equal in both Lotteries) it yields

$$\frac{rb}{a+b}=\frac{sn}{m+n}$$

a	1	$a = 3rb - 3b$
b	2	$m = 2sn - 2n$
r $\left.\right\}$ $= ?$	3	$\frac{rb}{a+b} = \frac{sn}{m+n}$
m	4	$(*)$
n	5	$(*)$
s	6	$(*)$
$q = ?$		
$7 * \overline{a+b}$	7	Let $\frac{rb}{a+b} = q$
8×3	8	$rb = qa + qb$
$1 + 3b$	9	$3rb = aqa + aqb$
9, 10	10	$3rb = a + 3b$
	11	$3qa + 3qb = a + 3b$
Scope		
11, 12	12	If $a = 0$ to avoid negative Numbers.
$13 \div 3b$	13	$3b = 3qb$
12, 14	14	$q = 1$
	15	$q > 1$ makes $a < 0$, $q < 1$ makes $a > 0$
Scope		
11, 16	16	If $b = 0$
$17 \div 3a$	17	$3qa = a$
16, 18	18	$q = \frac{1}{3}$
	19	$q < \frac{1}{3}$ makes $b < 0$ $q > \frac{1}{3}$ makes $b > 0$
3, 7,		
$20 * \overline{m+n}$	20	$\frac{sn}{m+n} = q$
21×2	21	$sn = qm + qn$
$2 + 2n$	22	$2sn = 2qm + 2qn$
22, 23	23	$2sn = m + 2n$
	24	$2qm + 2qn = m + 2n$
Scope	25	If $m = 0$
24, 25	26	$2qn = 2n$
$26 \div 2n$	27	$q = 1$
25, 27	28	$q > 1$ makes $m < 0$ $q < 1$ makes $m > 0$

Scope

Scope		
24, 29	29	If $n = o$
24, 29	30	$2qm = m$
$30 \div 2m$	31	$q = \frac{1}{2}$
29, 31	32	$q < \frac{1}{2}$ makes $n < o$ $q > \frac{1}{2}$ $n > o$
15,19,28,32	33	that $a\,b\,m\,n$ may be $> o$, q must be $> \frac{1}{2} < 1$
33, 4 (*)	34	Let therefore $Q =$
7 34.	35	$\frac{r\,b}{a+b} = \frac{2}{3}$
35 *, 10	36	$3rb = 2a + 2b = a + 3b$
36 —	37	$a = b$
20, 34.	38	$\frac{s\,n}{m+n} = \frac{2}{3}$
38 *	39	$3sn = 2m + 2n$
39 * $\overline{2}$	40	$6sn = 4m + 4n$
23 * $\overline{3}$	41	$6sn = 3m + 6n$
40, 41	42	$4m + 4n = 3m + 6n$
42 —	43	$m = 2n$
$1 \div 37$	44	$1 = 3r - 3$
44 + 3	45	$3r = 4$
$2 \div n$, 43	46	$2 = 2s - 2$
46 + 2	47	$2s = 4$
5 (*)	48	Let $A = 3$
37, 48	49	$B = 3$
$45 \div 3$	50	$R = \frac{4}{3}$, id est, 16 Pence.
6 (*)	51	Let $M = 4$
43, 51	52	$N = 2$
$47 \div 2$	53	$S = 2$ 2 Shillings.

A New, Exact and Easie Method, of finding the Roots of any Equations Generally, and that without any previous Reduction. By Edm. Halley.

THE principal uſe of the *Analytick Art*, is to bring Mathematical Problems to Equations, and to exhibit thoſe Equations in the moſt ſimple Terms that can be. But this Art would juſtly ſeem in ſome degree defective, and not *ſufficiently Analytical*, if there were not ſome Methods,by the help of which, the Roots (be they Lines or Numbers, might be gotten from the Equations that are found, and ſo the Problems in that reſpect be ſolved. The Ancients ſcarce knew any thing in theſe Matters, beyond *Quadratick Equations.* And what they writ of the *Geometrick Construction* of ſolid Problems, by the help of the *Parabola*, *Ciſſoid*, or any other Curve, were only particular things deſign'd for ſome particular Caſes. But as to *Numerical Extraction*, there is every where a profound Silence; ſo that whatever we perform now in this kind, is entirely owing to the Inventions of the Moderns.

And firſt of all, that great Diſcoverer and Reſtorer of the Modern Algebra, *Francis Vieta*, about 100 Years ſince, ſhew'd a general Method

thod for extracting the Roots of any Equation, which he publish'd under the Title of, *A Numerical Resolution of Powers*, &c. *Harriot*, *Oughtred*, and others, as well of our own Country, as Foreigners, ought to acknowledge whatsoever they have written upon this Subject, as taken from *Vieta*. But what the Sagacity of Mr. *Newton*'s Genius has perform'd in this business, we may rather conjecture (than be fully assur'd of) from that short Specimen given by Dr. *Wallis* in the 94th Chapter of his *Algebra*. And we must be forc'd to expect it, till his great Modesty shall yield to the Intreaties of his Friends, and suffer those curious Discoveries to see the Light.

Not long since (*viz. A. D.* 1690) that excellent Person M. *Joseph Raphson*, F. R. S. publish'd his *Universal Analysis of Equations*, and illustrated his Method by plenty of Examples; by all which he has given Indications of a Mathematical Genius, from which the greatest things may be expected.

By his Example, M. *de Lagney* an ingenious Professor of Mathematicks at *Paris*, was encourag'd to attempt the same Argument; but he being almost altogether taken up in extracting the Roots of pure Powers (especially the Cubick) adds but little about affected Equations, and that pretty much perplex'd too, and not sufficiently demonstrated. Yet he gives two very compendious Rules for the Approximation of a Cubical Root; one a Rational, and the other an Irrational one, Ex. gr. that the side of the Cube $aaa+b$, is between

$$a + \frac{ab}{3aaa+b}, \ \& \ \sqrt{\tfrac{1}{4}aa + \frac{b}{3a}} + \tfrac{1}{2}a.$$

And the root of the 5th Power $a^5 + b$, he makes

$= \frac{1}{2}a + \sqrt{\sqrt{\frac{1}{4}a^4 + \frac{b}{5a}} - \frac{1}{4}aa}$ (where Note, that 'tis $\frac{1}{4}aa$, not $\frac{1}{2}aa$, as 'tis erroneouſly Printed in the French Book.) Theſe Rules were communicated to me by a Friend, I having not ſeen the Book; but having by tryal found the goodneſs of them, and admiring the Compendium, I was willing to find out the Demonſtration. Which having done, I preſently found that the ſame Method might be accommodated to the Reſolution of all ſorts of Equations. And I was the rather inclin'd to improve theſe Rules, becauſe I ſaw that the whole thing might be Explain'd in a *Synopſis*; and that by this means, at every repeated ſtep of the Calculus, the Figures already found in the Root, would be at leaſt Trebled, which all other ways, are encreaſed but in an equal Number with the given ones. Now, the foremention'd Rules are eaſily demonſtrated from the Geneſis of the Cube, and the 5th Power. For, ſuppoſing the ſide of any Cube $= a + e$, the Cube ariſing from thence, is $aaa + 3aae + 3aee + eee$. And conſequently, if we ſuppoſe aaa the next leſs Cube, to any given Non-cubick Number, then eee will be leſs than Unity, and the remainder b, will $=$ the other Members of the Cube, $3aae + 3aee + eee$. Whence rejecting eee upon the account of its ſmallneſs, we have $b = 3aae + 3aee$. And ſince

aaa

aae is much greater than aee, the quantity $\frac{b}{3aa}$ will not much exceed e; so that putting $e = \frac{b}{3aa}$ then the quantity $\frac{b}{3aa+3ae}$ (to which e is nearly equal) will be found

$$= \frac{b}{3aa + \frac{3ab}{3aa}}, \quad \text{or} \quad \frac{b}{3aa+\frac{b}{a}} \quad \text{that is} \quad \frac{ab}{3aa+b} = e$$

And so the side of the Cube $aaa+b$ will be $a+\frac{ab}{3aa+b}$, which is the *Rational Formula* of M. *de Lagney*. But now, if aaa were the next greater Cubick Number to that given, the side of the Cube $aaa-b$, will after the same manner be found to be $a-\frac{ab}{3aa-b}$. And this easy and expeditious Approximation to the Cubick Root, is only (a very small matter) erroneous in point of *defect*, the quantity e, the remainder of the Root thus found, coming something less than really 'tis.

As for the *Irrationale Formula*, 'tis deriv'd from the same Principle, *viz.* $b = 3aae+3aee$, or $\frac{b}{3a} = ae+ee$, and so $\sqrt{\frac{1}{4}aa+\frac{b}{3a}} = \frac{1}{2}a+e$, and $\sqrt{\frac{1}{4}aa+\frac{b}{3a}}+\frac{1}{2}a = a+e$, the Root sought. Also the side of the Cube $aaa-b$, after the same manner, will be found to be $\frac{1}{2}a+\sqrt{\frac{1}{4}aa-\frac{b}{3a}}$.

And

And this *Formula* comes ſomething nearer to the Scope, being erroneous in point of *exceſs*, as the other was in *defect*, and is more accommodated to the ends of Practice, ſince the Reſtitution of the Calculus, is nothing elſe but the continual addition or ſubſtraction of the Quantity $\frac{aee}{3a}$ according as the quantity e can be known. So that we ſhould rather write $\sqrt{\frac{1}{4}a+\frac{b-eee}{3a}}+\frac{1}{2}a$, in the former caſe, and in the latter, $\frac{1}{2}a+\sqrt{\frac{1}{4}aa+\frac{eee-b}{3a}}$. But by either of the two *Formula*'s, the Figures already known in the Root to be extracted, are at leaſt Tripled; which I conclude will be very grateful to all the Students in Arithmetick; and I congratulate the Inventor upon the account of his Diſcovery.

But that the uſe of theſe Rules may be the better perceiv'd, I think it proper to ſubjoin an Example or two. Let it be propos'd to find the ſide of the double Cube, or $aaa+b=2$. Here $a=1$, and $\frac{b}{3a}=\frac{1}{3}$, &ſo $\frac{1}{2}+\sqrt{\frac{7}{12}}$, or 1,26, be found to be the true ſide nearly. Now, the Cube of 1,26, is 2,000376, and ſo 0,63 + $\sqrt{,3969-\frac{,000376}{3,78}}$, or 0,63 + $\sqrt{,3968005291005291}$ = 1,259921049895−; which in 13 Figures, gives the ſide of the double Cube, with very little trouble, *viz.* By one only diviſion, and the extraction of the ſquare Root; when as by the common way of working, how much

much pains it would have cost, the Skilful very well know. This Calculus a Man may continue as far as he pleases, by encreasing the Square by the addition of the quantity $\frac{eee}{3a}$; which Correction, in this case will give, but the encrease of Unity in the 14th Figure of the Root.

Exemp. 2*d.* Let it be propos'd to find the sides of a Cube equal to that English Measure commonly call'd a Gallon, which contains 231 solid Ounces. The next less Cube is 216, whose side $6 = a$, and the remainder $15 = b$; and so for the first Approximation, we have $3 + \sqrt{9 + \frac{5}{6}} =$ the Root. And since $\sqrt{9{,}8333\ldots}$ is $3{,}1358\ldots$, 'tis plain *that* $6{,}358 = a + e$. Now, *let* $6{,}1358 = a$; and we shall then have for its Cube 231,000853894712, & according to the Rule, $3{,}0679 + \sqrt{9{,}41201041 - \frac{{,}000858394712}{18{,}4070}}$ is most accurately equal to the side of the given Cube, which within the space of an Hour, I determin'd by Calculation to be 6.1357924 3966195897, which is exact in the 18th Figure, defective in the 19th. And this *Formula* is deservedly preferable to the *Rationale*, upon the account of the great Divisor, which is not to be manag'd without a great deal of Labour; whereas the extraction of the square Root, proceeds much more easily, as manifold Experience has taught me.

But the Rule for the Root of a pure Sursolid, or the 5th Power, is of something a higher Enquiry, and does much more perfectly yet, do do the busineſs; for it does at leaſt Quintuple the given Figures in the Root, neither is the Calculus very large or operoſe. Tho' the Author no where ſhews his method of Invention, or any Demonſtration, altho' it ſeems to be very much wanting; eſpecially ſince all things are not right in the printed Book, which may eaſily deceive the Unskilful. Now the 5th power of the ſide $a+e$ is compos'd of theſe Members, $a^5+5a^4e+10a^3e^2+10a^2e^3+5ae^4+e^5=a^5+b$; from whence $b=5a^4e+10a^3e^2+10a^2e^3+5ae^4$, rejecting e^5 becauſe of its ſmalneſs. Whence $\frac{b}{5a}=a^3e+2a^2e^2+2ae^3+e^4$, and adding on both ſides $\frac{1}{4}a^4$, we ſhall have $\sqrt{\frac{1}{4}a^4+\frac{b}{5a}}=\sqrt{\frac{1}{4}a^4+a^3e+2a^2e^2+2ae^3+e^4}=\frac{1}{2}aa+ae+ee$. Then ſubſtracting $\frac{1}{4}aa$ from both ſides, $\frac{1}{2}a+e$ will $=\sqrt{\sqrt{\frac{1}{4}a^4+\frac{b}{5a}}-\frac{1}{4}aa}$; to which if $\frac{1}{2}a$ be added, then will $a+e=\frac{1}{2}a+\sqrt{\sqrt{\frac{1}{4}a^4+\frac{b}{5a}}-\frac{1}{4}aa}$ = the root of the Power a^5+b. But if it had a^5-b (the quantity a being too great) the Rule would have been thus, $\frac{1}{2}a+\sqrt{\sqrt{\frac{1}{4}a^4-\frac{b}{5a}}-\frac{1}{4}aa}$.

And this Rule approaches wonderfully, ſo that there is hardly any need of Reſtitution.

But while I conſidered theſe things with my ſelf, I light upon a General Method for the *Formula*'s of all Powers whatſoever, and (which being handſome and conciſe enough) I thought I would not conceal from the Publick.

These Formula's, (as well the *Rational*, as the *Irrational* ones) are thus.

$$\sqrt{aa+b} = \sqrt{aa+b}, \text{ or } a + \frac{ab}{2aa+\frac{1}{2}b}.$$

$$\sqrt[3]{a^3+b} = \frac{1}{2}a + \sqrt{\frac{1}{4}aa + \frac{b}{3a}} \text{ or } a + \frac{ab}{3aaa+b}$$

$$\sqrt[4]{a^4+b} = \frac{1}{3}a + \sqrt{\frac{1}{9}aa + \frac{b}{6aa}}, \text{ or } a + \frac{ab}{4a^4+\frac{3}{2}b}$$

$$\sqrt[5]{a^5+b} = \frac{3}{4}a + \sqrt{\frac{9}{16}aa + \frac{b}{10a^3}}, \text{ or } a + \frac{ab}{5a^5+2b}.$$

$$\sqrt[6]{a^6+b} = \frac{2}{5}a + \sqrt{\frac{4}{25}aa + \frac{b}{15a^4}}, \text{ or } a + \frac{ab}{6a^6+\frac{5}{2}b}.$$

$$\sqrt[7]{a^7+b} = \frac{5}{6}a + \sqrt{\frac{25}{36}aa + \frac{b}{21a^5}}, \text{ or } a + \frac{ab}{7a^7+3b}.$$

And so also of the other higher Powers. But if a were assumed bigger, than the Root sought (which is done with some advantage, as often as the Power to be Resolved, is much nearer, the Power of the *next greater* whole Number, than of the *next less*) in this case, *Mutatis Mutandis*, we shall have the same Expressions of the Roots, *viz.*

$\sqrt{aa}$

$$\sqrt{aa-b} = \sqrt{aa-b}, \text{ or } a - \frac{ab}{2aa-\frac{1}{2}b}.$$

$$\sqrt[3]{a^3-b} = \tfrac{1}{2} + \sqrt{\tfrac{1}{4}aa - \frac{b}{3a}}, \text{ or } a - \frac{ab}{3a^3-b}.$$

$$\sqrt[4]{a^4-b} = \tfrac{2}{3}a + \sqrt{\tfrac{1}{9}aa - \frac{b}{6aa}}, \text{ or } a - \frac{ab}{4a^4-\frac{3}{2}b}.$$

$$\sqrt[5]{a^5-b} = \tfrac{3}{4}a + \sqrt{\tfrac{1}{16}aa - \frac{b}{10a^3}}, \text{ or, } a - \frac{ab}{5a^5-2b}.$$

$$\sqrt[6]{a^6-b} = \tfrac{4}{5}a + \sqrt{\tfrac{1}{25}aa - \frac{b}{15a^4}}, \text{ or } a - \frac{ab}{6a^6-\frac{5}{2}b}.$$

$$\sqrt[7]{a^7-b} = \tfrac{5}{6}a + \sqrt{\tfrac{1}{36}aa - \frac{b}{21a^5}}, \text{ or } a - \frac{ab}{7a^7-3b}.$$

And within these two Terms, the true Root is ever found, being something nearer to the *Irrational* than the *Rational* Expression. But the quantity e found by the *Irrational Formula*, is always too great, as the Quotient resulting from the Rational Formula, is always too little. And consequently, if we have $+b$, the *Irrational Formula* gives the Root something greater than it should be, and the *Rational* something less. But contrary wise if it be $-b$.

And

And thus much may suffice to be said, concerning the extraction of the Roots of pure Powers; which notwithstanding, for common Uses, may be had much more easily by the help of the Logarithms. But when a Root is to be determin'd very accurately, and the Logarithmick Tables will not reach so far, then we must necessarily have recourse to these, or such like Methods. Farther; the Invention and Contemplation of these Formulæ, leading me to a certain Universal Rule, for affected Equations (which I hope will be of use to all the Students in *Algebra* and *Geometry*) I was willing here to give some account of this Discovery, which I will do with all the perspecuity I can. I had given at N°. 188 of the *Transactions*, a very easy and general construction of all affected Equations, not exceeding the Biquadratick Power; from which time I had a very great desire of of doing the same in Numbers. But quickly after, Mr. *Raphson* seem'd in great measure to have satisfy'd this Desire, till Mr. *Lagney* by what he had perform'd in his Book, intimated that the thing might be done more compendiously yet. Now, my Method is thus.

Let z the root of any Equation, be imagin'd to be compos'd of the parts $a+$ or $-e$, of which, let a be assum'd as near z as is possible; which is notwithstanding not *necessary*, but only *commodious*. Then from the Quantity $a+e$ or $a-e$, let there be form'd all the Powers of z, found in the Equation, and the Numerical Co-efficients be respectively affix'd to them: Then let the Power to be *resolv'd*, be substracted from the sum of the given Parts (in the

the first Column where e is not found) which they call the *Homogeneum Comparationis*, and let the difference be $\pm b$. In the next place, take the sum of all the Co-efficients of e in the second Column, to which put $= s$. Lastly, in the third Column let there be put down the sum of all the Co-efficients of $e\,e$, which sum call t. Then will the Root z stand thus in the *Rational Formula*, *viz.* $z = a + \frac{s\,b}{s\,s \pm t\,b}$; and thus in the *Irrational Formula*, *viz.*

$z = a \mp \frac{\frac{1}{2}s \pm \sqrt{\frac{1}{4}ss \mp bt}}{t}$; which perhaps it may be worth while to Illustrate by some Examples. And instead of an *Instrument*, let this *Table* serve, which shews the Genesis of the several Powers of $a \pm e$, and if need be, may easily be continued farther; which for its use I may rightly call a *General Analytical Speculum*. The forementioned Powers arising from a continual Multiplication by $a + e$ ($= z$) come out thus with their adjoyned Co-efficients. *See the Table.* But now, if it be $a - e = z$, the Table is compos'd of the same Members, only the odd Powers of e, as e, e^3, e^5, e^7 are Negative, and the even Powers, as e^2, e^4, e^6, Affirmative. Also let the sum of the Co-efficients of the side e, be $= s$; the sum of the Co-efficients of the Square $ee = t$, the sum of the Co-efficint of $e^3 = u$; of $e^4 = w$; of $e^5 = x$; of $e^6 = y$, &c. But now, since e is supposed only a small part of the Root that is to be enquir'd, all the Powers of e, will be much less than the correspondent Powers of a, and so far the

the firſt Hypotheſis; all the ſuperior ones may be rejected; and forming a new Equation, by ſubſtituting $a+e=z$, we ſhall have (as was ſaid) $\pm b = \pm se \pm tee$. The following Examples will make this more clear.

Example I.

Let the Equation $z^4 - 3z^2 + 75z = 10000$, be propos'd. For the firſt Hypotheſis, let $a=10$, and ſo we have this Equation,

$$z^4 = +a^4 \quad 4a^3e + 6a^2e^2 \quad 4ae^3 + e^4$$
$$-dz^2 = -da^2 \quad dae - de^2$$
$$+cz = +ca \quad ce$$
$$= +10000 \quad 4000e + 600ee \quad 40e^3 + e^4$$
$$-\ 300 \quad 60e - 3ee$$
$$+\ 750 \quad 75e$$
$$-10000$$

$$+450 - \quad 4015e + 597ee - 40e^3 + e^4 = 0$$
$$s \qquad t \qquad u$$

The Signs + and − with reſpect to the Quantities e and e^3, are left as Doubtful, till it be known whether e be Negative or Affirmative; which thing creates ſome difficulty, ſince that in Equations that have ſeveral Roots, the *Homogenea Comparationis* (as they term them) are oftentimes encreaſed by the minute quantity a, and on the contrary, *that* being increaſed, *they* are diminiſhed. But the Sign of e is determin'd from the Sign of the Quantity b. For taking away the *Reſolvend* from the *Homogeneal* formed of a; the Sign of se (and conſequently of the prevailing Parts in the compoſition of it) will always be contrary

trary to the Sign of the difference b. Whence 'twill be plain, whether it must be $+e$, or $-e$; and consequently whether a be taken greater or less than the *True Root*. Now the quantity e is $=\frac{1}{2}s-\sqrt{\frac{\frac{1}{4}ss-bt}{t}}$, when b and t have the same Sign, but when the Signs are different, e is $=\sqrt{\frac{\frac{1}{4}ss+bt}{t}}-\frac{1}{2}s$. But after it is found that it will be $-e$, let the Powers e, e^3, and e^5, &c. in the affirmative Members of the Equation be made Negative, and in the Negative be made Affirmative; that is, let them be written with the contrary Sign. On the other hand, if it be $+e$ (let those forementio'd Powers) be made Affirmative in the Affirmative, and Negative in the Negative Members of the Equation.

Now we have in this Example of ours, 10450 instead of the Resolvend 10000, or $b=+450$, whence it's plain, that a is taken greater than the Truth, and consequently, that 'tis $-e$. Hence the Equation comes to be, $10450-4015e+597ee-4e^3+e^4=10000$. That is, $450-4015e+597ee=0$; and so $450=4015e-597ee$, or $b=se-tee$, whose Root $e=\frac{1}{2}s-\sqrt{\frac{\frac{1}{4}ss-bt}{t}}$ or $\frac{s}{2t}-\sqrt{\frac{ss}{4tt}-\frac{b}{t}}$; that is in the present case, $e=\frac{2007\frac{1}{2}-\sqrt{3761406\frac{1}{4}}}{597}$, from whence we have the Root sought, 9, 886, which is near the Truth. But then substituting this for a second

cond Supposition, there comes $a+e=z$, most accurately 9, 886260393649 5 .... scarce exceeding the Truth by 2 in the last Figure, *viz.* when $\frac{\sqrt{\frac{1}{4}ss+bt}-\frac{1}{2}s}{t}=e$. And this (if need be) may be yet much farther verified, by substracting (if it be $+e$) the quantity $\frac{\frac{1}{2}ue^3+\frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss+tb}}$, from the Root before found; or (if it be $-e$) by adding $\frac{\frac{1}{2}ue^3-\frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss-tb}}$ to that Root. Which Compendium is so much the more Valuable, in that sometimes from the first Supposition alone, but always from the second, a Man may continue the Calculus (keeping the same Coefficients) as far as he pleases. It may be noted, that the fore-mentioned Equation, has also a Negative Root, *viz.* $z=10,26$ .... which any one that has a mind, may determin more accurately.

Example II.

Suppose $z^3-17z+54z=350$, and let $a=10$. Then according to the prescript of the Rule,

$$+z^3=a^3+3a^2e+3ae^2+e^3$$
$$-dz^2=da^2-2dae-de^2$$
$$+cz=ca+ce.$$

 That

That is,
$$\begin{array}{llll} & b & s & t \\ +1000 & +300e & +30e^2 & +e^3 \\ -1700 & -340e & -17e^2 & \\ +540 & +54e & & \\ -350 & & & \end{array}$$

Or, $-510+14e+13ee+e^3=0$. Now, since we have -510, it is plain, that a is assumed less than the Truth, and consequently that e is Affirmative. And from (the Equation) $510=14e+13e^2$, comes $e=\frac{\sqrt{bt+\frac{1}{4}ss}-\frac{1}{2}s}{t}=\frac{\sqrt{6679}-7}{13}$. Whence $z=15,7\ldots$, which is too much, because of a taken wide; therefore Secondly, let $a=15$, and by the like way of Reasoning, we shall find $e=\frac{\frac{1}{2}s-\sqrt{\frac{1}{4}ss-tb}}{t}=\frac{109\frac{1}{2}-\sqrt{11710_4}}{28}$, and consequently $z=14,954068$. If the Operation were to be repeated the third time, the Root will be found conformable to the Truth as far as the 25th Figure; but he that is contented with fewer, by writing $tb+te^3$ instead of tb, or substracting or adding $\frac{\frac{1}{2}e^3}{\sqrt{\frac{1}{4}ss+tb}}$ to the Root before found, will presently obtain his end. Note, the Equation proposed, is not explicable by any other Root, because the *Resolvend* 350, is greater than the Cube of $\frac{17}{3}$, or $\frac{d}{3}$.

Example

Example III.

Let us take the Equation $z^4-80z^3+1998z^2-14937z+5000=0$, which Dr. *Wallis* uses Cap. 62 of his *Algebra*, in the Resolution of a very difficult Arithmetical Problem where by *Vieta*'s Method he has obtain'd the root most accurately; and Mr. *Raphson* brings it also as an Example of his Method, Page 25, 26. Now this Equation is of the form, which may have several Affirmative Roots, and (which increases the difficulty) the *Coefficients* are very great in respect of the *Resolvend* given.

But that it may be the easier manag'd, let it be divided, and acccording to the known Rules of *Pointing*, let $-z^4+8z^3-20z^2+15z=0{,}5$ (where the quantity z is $\frac{1}{10}$ of z in the Equation proposed) and for the first Supposition, let $a=1$. Then $+{}^{2}-5e-2e^2+4e^3-e^4-0{,}5=0$; that is, $1\frac{1}{2}=5e+2ee$; hence $e=\frac{\sqrt{\frac{1}{4}ss+bt}-\frac{1}{2}s}{t}$ is $=\frac{\sqrt{37}-5}{4}$, and so $z=1{,}27$; Whence 'tis manifest that 12, 7 is near the true Root of the Equation proposed. Now Secondly, let us suppose $z=12{,}7$, and then according to the directions of the Table of Powers, there arises

b	s	t	u	
$-26014{,}4641$	$-8193{,}532e$	$-967{,}74e^2$	$-50{,}8e^3$	$-e^4$
$+163870{,}640$	$+38709{,}60e$	$+3048\ e^2$	$+80\ e^3$	
$-322257{,}42$	$-50749{,}2\ e$	$-1998\ e^2$		
$+189699{,}9$	$+14937{,}\ e$			
$+\ 5000.$				

That

That is, $+298,6559-5296\,132\,e+82,26\,e^2+29,2\,e^3-e^4=0$; And so $-298,6559=-5296,132\,e+82,26\,ee$, whose Root e (according to the Rule) $=\frac{\frac{1}{2}s-\sqrt{\frac{1}{4}ss-bt}}{t}$ comes to

$$\frac{2648,066-\sqrt{6987686,106022}}{82,26}=$$

$,0564408033\,1\ldots=e$ less than the Truth. But that it may be corrected, 'tis to be consider'd

that $\frac{\frac{1}{2}ue^3-\frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss-bt}}$, or $\frac{,0026201\ldots\ldots}{2643,423\ldots\ldots}$ is $,00000099$, and consequently e corrected, is $=056470448$. And if you desire yet more Figures of the Root, from the e corrected let there be made $tue^3-te^4=0,43105602423\ldots$, and

$\frac{\frac{1}{2}s-\sqrt{\frac{1}{4}ss-bt-tue^3+te^4}}{t}$ or which is all one,

$$\frac{2648,066-\sqrt{6987685,67496597577\ldots}}{82,26}=$$

$,056441794480744\,02=e$; whence $a+e=z$ the Root is most accurately $12,756441794480744\,02\ldots$ as Dr. *Wallis* found in the forementioned Place; where it may be observ'd, that the repetition of the *Calculus* does ever triple the true Figures in the assumed a, which the first correction, or $\frac{\frac{1}{2}ue^3-\frac{1}{2}e^4}{\sqrt{\frac{1}{4}ss-bt}}$, does quintuple; which is also commodiously done by the *Logarithms*. But the other Correction after

after the first, does also double the number of Figures, so that it renders the *assumed* altogether Seven-fold; yet the first Correction is abundantly sufficient for Arithmetical uses, for the most part.

But as to what is said concerning the number of Places rightly taken in the Root, I would have understood so, that when *a* is but $\frac{1}{10}$ part distant from the true Root, then the first Figure is rightly assumed; if it be within $\frac{1}{100}$ part, then the two first Figures are rightly assumed; if within $\frac{1}{1000}$, and then the three first are so; which consequently manag'd according to our Rule, do presently become nine Figures.

It remains now that I add something concerning our *Rational Formula*, viz. $e = \frac{sb}{ss \pm tb}$ which seems expeditious enough, and is not much Inferior to the former since it will triple the given Number of Places. Now having formed an Equation from $a + e = z$, as before, it will presently appear, whether *a* be taken greater or less than the Truth; since se ought always to have a Sign contrary to the Sign of the difference of the *Resolvend*, and its *Homogeneal* produced from *a*. Then supposing $+b+se+a-tee=o$, the Divisor is $ss-tb$, as often as *t* and *b* have the same Signs; but it is $ss+bt$, when they have different ones. But it seems most commodious for Practice, to write the Theorem thus, $e = \frac{b}{s} \pm \frac{tb}{s}$ since this way the thing is done by one Multiplication and two Divisions, which otherwise would require three Multiplications, and one Division.

Let us take now one Example of this Method, from the Root (of the forementioned Equation) 12,7 ...., where

$$\underset{+b}{298,6559} \underset{-s}{-5296,132e} \underset{+t}{+82,26ee} \underset{+u}{+29,2e^3} - e^4 = o,$$ and so $\frac{b}{s} - \frac{tb}{s} = e$; that is, let it be as s to t, so b to $\frac{tb}{s} = 5296,132)298,$ 6559 into 82,26 (4,63875 ... wherefore the Divisor is $s - \frac{tb}{s} = 5291,49325 \ldots\ldots)$ 298, 6559 (0,056441 ..... $= e$, that is, to five true Figures, added to the Root that was taken. But this *Formula* cannot be corrected, as the foregoing *Irrational* one was; and so if more Figures of the Root are desired, 'tis the best to make a new Supposition, and repeat the *Calculus* again: And then a new Quotient, tripling the known Figures of the Root, will abundantly satisfie even the most Scrupulous.

A Differ-

A Dissertation concerning the Construction of Solid Problems, or Equations of the third or fourth Power, by the help of one (given) Parabola and a Circle.

By Edmund Halley.

HOw all Equations (that involve the third or fourth Power of the unknown Quantity) may be constructed by the help of any given *Parabola* and a *Circle*, the Famous M. *Des Cartes* has shewn and clearly demonstrated in the Third Book of his *Geometry*. But he first of all orders the second Term of the Equation (if it be there) to be thrown out, and then by the Rule there delivered, to find the Roots of the Equation so reduced.

And since that Operation seems too Laborious, some thought fit to invent a like Construction, without any previous Reduction. Amongst whom *Francis a Schooten* has offer'd a Method (for constructing cubical Equations howsoever affected) which might have been called very easie and simple; if (by unfolding the Principle from whence he deduced his Rule) he had better consulted his Reader's Memory, which he burdens with very many and perplex'd Cautions. But lately our Famous Countriman, Mr. *Thomas Baker*, in a whole Treatise written upon these Constru-

ctions,

ctions, has comprehended not only all Cubical, but alſo Biquadratical Equations of every kind, under one General Rule, which he has demonſtrated, and abundantly Illuſtrated with Examples through all Caſes; and moreover at the Cloſe, propos'd a way, by which that General Rule might be Inveſtigated. But he does not ſhew the very Method, by the help of which (as I ſuſpect) he obtain'd his *Univerſal Geometrical Clavis*, or at leaſt might have obtain'd it with much more eaſe. And ſince this Rule of *Baker*'s is no leſs perplex'd with Cautions about the Signs + and − than *Schooten*'s is, ſo that a Perſon can hardly perform thoſe Conſtructions aright, without he has the Book by him; I thought that it wou'd not be either Unpleaſant or Unprofitable to young Students, to explain the Foundations of both Rules, and by ſome emendation of the Method once more, to afford as much light as I cou'd in ſo difficult a Matter. *Carteſius*'s Conſtruction (which does very eaſily diſcover the Roots of all Cubick or Biquadratick Equations, where the ſecond Term is wanting) may be ſuppos'd as known. Yet ſince 'tis the main bottom, on which all that follows does depend; that this Diſſertation may not ſeem to want a principal Part, I'll here add the Rule taken out of his Geometry, altering ſome few things (as I think) for the better.

The ſecond Term being out of the Equation; all cubical Equations, are reduced to this Form, $z^3 * . apz. aaq = o$; and Biquadratical ones to this Form, $z^4 . * . apzz. aaqz. a^2r = o$, where a denotes the *Latus Rectum* of any given Parabola, which is uſed in the Con-

Construction. Or else taking a for Unity, those Equations are reduced to these Forms, *viz.* Cub. $z^3 . * p z . q = o$, and Biquadr. $z^4 * . pzz . qz . r . = o$. Now the *Parabola* FAG, *Fig.* 9 being given, whose Axis is ACDKL, and Parameter $= a$ or 1; let AC be taken $= \frac{1}{2} a$, and be set off always from the Vertex *A*, towards the inner parts of the Figure. Then take CD $\frac{1}{2} p$, in that Line AC, continued towards *C*, if it be $-p$ in the Equation, or towards the contrary Point, if it be $+p$. Farther, from the point *D* (or from the point *C*, if the quantity *p* be not in the Equation) Let DE (erected perpendicular to the Axis) be made $= \frac{1}{2} q$ which is to be set to the right hand if it be $-q$, but to the other side of the Axis if it be $+q$. And then a Circle described on the Center E, which the Radius AE (if the Equation be but a *Cubical* one) will intersect the *Parabola* in as many Points (*viz.* F, G, G,) as the Equation has *True* Roots, of which the Affirmative ones, as GK, shall on the right side of the Axis, and the Negative ones as FL, on the *Left*. But if the Equation be a Biquadratical one, then the Radius of the Circle AE, by adding (if it be $-r$) or Substracting (if it be $+r$) from the Square of it, the Rect-angle $a\,x\,r$, or the *content* under the Parameter, and the given Quantity r; which is very easily done Geometrically. And the Intersections of this Circle with the Parabola, will give (letting fall Perpendiculars from thence to the Axis) all the *true* Roots of the Biquadratical Equation; the *Affirmative* ones being on the *Right* side of the Axis, and the *Negative* ones, on the *Left*. The demonmonstration of all which I leave to *Cartesius* the Inventor. Let it be Noted, that I endeavour here that the *Affirmative* Roots, may always be had on the *Right* side of the Axis, to avoid the Confusion that will necessarily arise from a multitude of Cautions, where the reason of them is not evident.

Having premised these things, in order to make way for the construction of these Equations, even when the second Term is found in them, we are to consider the Rule it self for taking away the second Term, and reducing the Equation to another, such as might be constructed by the foregoing Method. Now all Cubick Equations of this Classis, are reduced to this form, $z^3\ bzz.\ apz.\ aaq = o$, or to this, $z^3.\ bz^2.\ *.\ aaq = o$. Biquadratick ones may be reduc'd to this, $z^4.\ bz^3.\ apz^2.\ aaqz.\ a^3 r = o$, or this, z^4, or this $z^4.\ bz^3.\ *.\ aaqz.\ a^3 r = o$, or this, $z^4.\ bz^3.\ apz^2.\ *.\ a^3 r = o$, or lastly, to this Form, $z^4.\ bz^3\ *.\ *.\ a^3 r = o$. From all which there arises a great Variety, according as the Signs $+$ or $-$ are diversly connected together; and hence the General Rule serving all these cases, is rendred very obscure and difficult, unless (manag'd by the help of the following Method) it be cleared up and delivered from those Intricacies.

The second Term in Biquadratical Equations, is taken away by putting $x = z + \frac{1}{4}b$, if it be $+b$ in the Equation; or $x = z - \frac{1}{4}b$, if it be $-b$. Hence $x - \frac{1}{4}b$ in the first case, and $x + \frac{1}{4}b$ in the second, is $= z$; and so in any Equation proposed, substituting instead of z, its Equal, there will come forth a new Equation, wanting the second Term, all whose Roots x do do exceed, or come ſhort of the *ſought* Root z, by the given difference $\frac{1}{4}b$. But ſince in things of this kind, Examples do more than Precepts, let us propoſe one or two Equations to be conſtructed.

Example I.

$z^4 + bz^3 - apzz - aaqz + a^3r = 0.$

put $x - \frac{1}{4}b = z$, and then will

$x^2 - \frac{1}{2}bx + \frac{1}{16}bb = z^2.$

$x^3 - \frac{3}{4}x^2b + \frac{3}{16}xb^2 - \frac{1}{64}b^3 = z^3.$ and

$x^4 - bx^3 + \frac{3}{8}b^2x^2 - \frac{1}{16}b^3x + \frac{1}{256}b^4 = z^4$

Hence it follows, that

$$x^4 - bx^3 + \frac{3}{8}b^2x^2 - \frac{1}{16}b^3x + \frac{1}{256}b^4 = z^4.$$
$$bx^3 - \frac{3}{4}b^2x^2 + \frac{3}{16}b^3x - \frac{1}{64}b^4 = +bz^3$$
$$-apx^2 + \frac{1}{2}apbx - \frac{1}{16}apb^2 = -apz^2$$
$$-a^2qx + \frac{1}{4}a^2qb = -a^2qz$$
$$+a^3r$$

The Sum of all theſe is a new Equation wanting the ſecond Term, and which conſequently may be conſtructed by *Cartes*'s Rule, by taking inſtead of $\frac{1}{2}p$, half the Coefficient of the third Term, divided by a or the Parameter, that is $-\frac{3}{16}\frac{bb}{a} - \frac{1}{2}p$; and inſtead of $\frac{1}{2}q$, half the Coefficient of the fourth Term, divided by aa, that is, $+\frac{1}{16}\frac{b^3}{a^2} + \frac{1}{4}\frac{pb}{a} - \frac{1}{2}q$. The

Mem-

Members of which that have the Sign + are to be set off to the *left Hand* from the Axis, and those that have the Sign − to the *Right*; in order to find the Center of the Circle required for Construction, whose Intersections with the Parabola (letting fall perpendiculars to the Axis) may give all the true Roots x, namely, the *Affirmative* ones on the Right side of the Axis, and the *Negative* ones on the *Left*. But now, when $x - \frac{1}{2}b = z$, then drawing a Line Parallel to the Axis on the right side of it, and at the distance of $\frac{1}{2}b$, the Perpendiculars terminated on this Parallel, will denote all the enquired Roots z, the *Affirmative* ones on the right side, and the *Negative* ones on the Left. As for what relates to the Radius of the Circle, it is had, by adding the Negative, or taking away the Affirmative parts of the fifth Term divided by aa, from the Square of the Line AE, drawn from the Center *E* found, to *A* the Vortex of the Parabola; which is mostly done, by taking instead of AE, the Line EO which is terminated at O the Intersection of the Parabola, and the fore-mentioned Parallel; for the Square of this comprehends all the parts of the fifth Term, brought into the new Equation upon the casting out of the second Term, as is easily proved: And it remains only, that the Square of EO be increased, if it be $-r$, in the Equation, or diminish'd, if it be $+r$, by the addition or substraction of the Rectangle $a\,r$; from whence the Radius of the Circle desired, is compos'd. This Method of investigating M. *Baker*'s central Rule, is easie and free from all Cautions; and the difference arises only ly from hence, that *I determin the center of the Circle, by the Axis, and he by a Parallel to the Axis; and that I always have four Affirmative Roots on the right side the Axis, which he has sometimes on the right side, and sometimes on the left.*

As for cubical Equations, they are to be reduc'd to Biquadratical ones, before they can be constructed by the same General Rule; which is done by multiplying the Equation propos'd by its Root z, whence arises a Biquadratick Equation, in which the last Term or r, is wanting. Wherefore taking away the second Term, and finding the Center E, the line EO is the Radius of the Circle; *viz.* When ar is $= o$, and the whole fifth Term in the new Equation, arises from the taking away of the second Term. Let this Equation be propos'd to be constructed.

Example II.

$Z^3 - bz^2 + apz + aaq = o$,

which multiply'd into z, becomes

$z^4\ b z^3 + apz^2 + aaqz = o.$

To take away the second Term, put

$x + \frac{1}{4} b = z$, and then will

$$\begin{array}{l} x^4 + bx^3 + \frac{3}{8} bbx^2 + \frac{1}{16} b^3 x + \frac{1}{256} b^4 = + z^4 \\ \quad - bx^3 - \frac{3}{4} b^2 x^2 - \frac{3}{16} b^3 x - \frac{1}{64} b^4 = - bz^3 \\ \qquad + apx^2 + \frac{1}{2} abpx + \frac{1}{16} apb^2 = + apz^2 \\ \qquad\qquad + aaqx + \frac{1}{4} aaqb = + aaqz. \end{array}$$

Now in this new Equation, the half Coefficient (of the third Term) divided by a, *viz.* $-\frac{3bb}{16a} + \frac{1}{2} p$, is to be used instead of $\frac{1}{2} p$; and the

the half Coefficient of the (fourth Term) divided by aa, the Square of the *Latus Rectum*, *viz.* $-\frac{b^3}{16a^2}+\frac{pb}{4a}+\frac{1}{2}q$, is instead of $\frac{1}{2}q$ in *Cartesius*'s Construction, from whence the Center E is determin'd. Then drawing a Parallel to the Axis, at the distance $\frac{1}{4}b$, to the left side (because of $z = x + \frac{1}{4}b$) whose Intersection with the Parabola, let be O; a Circle described on the Center E with the Radius EO, will cut or touch the Parabola in as many Points as the Equation has true Roots, which Roots, or z, are the Perpendiculars let fall from those Points upon the Parallel to the Axis, the Affirmative ones to the Right side, and the Negatives to the Left. If the third or fourth Term, or both, be wanting in the Equation, there's no difference at all (of the Method of investigating the Central Rule) to be observ'd. But the Quantity p or q being wanting, those parts of the Lines CD and DE (in some manner deduced from that Quantity) will be wanting too, and we are to proceed with the other Coefficients of the third and fourth Term in the new Equation, according to the way prescrib'd in the foregoing Examples.

Hitherto we have consider'd Mr. *Baker*'s General Method, than which none more Easie and Expeditious is to be expected, using either a Parabola, or any other Curve for a Construction, *viz.* when the Equation rises to the Biquadratick Power. For while I am writing of this, 'tis my good Luck to hit upon a certain *Geometrick Effection* of the central Rule, which is Expeditious beyond Hope, and will abun-

abundantly satisfy those that are curious in these Matters.

(*Fig.* 10.) Having describ'd the Parabola NAM, whose Vertex is *A*, Axis ABC, and Parameter *a*; let the Equation be reduced to this Form, $z^4 . bz^3 . apz^2 . aaqz . a^3r . = o$; or if it be only a Cubical one, to this, $z^3 . bz_2 . apz . aaq . = o$. Then at the distance BD$=\frac{1}{2}b$, let DH be drawn parallel to the Axis (to the Left Hand if it be $-b$, and to the Right, if it be $+b$) meeting the Parabola in the point *D*, from whence let fall BD perpendicular to the Axis. In the Line AB continued towards *B*, make BK$=\frac{1}{2}a$, and draw the Line DK interminate on either side. Farther, take KC$=$2AB, always in the Axis produced beyond *K*; and if the quantity *p* has the *Sign* $-$, take towards the same parts, CE$=\frac{1}{2}p$, but towards the contrary part, if it be $+p$. Then at the point *E* (but at the point *C* if the quantity be wanting) erect EF perpendicular to the Axis, meeting (if need be) the Line DK produced, in the point *F*, which point is the Center of the Circle required, if the quantity *q* be wanting. But if *q* be in the Equation, then we must take in the Line FE (if need be) produced the length of FG$=\frac{1}{2}q$, which place to the Left Hand if it be $+q$, but to the Right if it be $-q$; and then the point *G* will be the Center of the Circle required for the Construction, and the Radius of it, will be the Line GD, if the quantity *r* be wanting, that is, if the Equation be only a Cubical one; the Square of which same Line (in Biquadratick Equations) is to be encreased by the addition of the Rectangle under r and the Latus Rectum, if it be $-r$, or to be diminished by the same Rectangle if it be $+r$. The Circle thus describ'd, and Perpendiculars let fall from its Intersections with the Parabola, to the Line D H, those that are at the Left Hand as N O will always be the Negative Roots of the Equation, and those at the Right, the Affirmative.

Cubick Equations are otherwise (and something more simply) constructed according to *Schooten's* Rule, in which also the Roots respect the Axis. But because the Inventor himself does neither explain the Investigation nor Demonstration, it will not be amiss to shew the Foundation of it here, and at the same time render the Geometrick Construction more Elegant, and rid it of those *Cautions* in which 'tis involv'd.

This Rule is deriv'd from hence, that every Cubick Equation may be reduced to a Biquadratick one, in which the second Term is wanting. Which is done, by multiplying the Equation proposed into $z-b=o$, if it be $+b$ in the Equation, or into $z+b=o$, if it be $-b$; and the new Equation thus form'd will have the same Roots with the Cubical one, and moreover another Equal to $-b$, if it be $-b$ in the Equation, or contrariwise.

Let the Equation $z^3 - z^2 b + apz + aaq = o$, be proposed to be constructed; multiply this into $z+b$, and it makes

$$\begin{array}{l} z^4 - z^3 b + apz^2 + aaqz \\ \quad + z^3 b - bbz^2 + abpz + aaqb \end{array}$$

Here now the second Term is wanting, and the

the Coefficient of the third Term $-bb+ap$, gives $-\frac{bb}{2a}+\frac{1}{2}p$, in the room of $\frac{1}{2}p$ or CD in *Cartesius*'s Construction; and from half the Coefficient of the fourth Term is made $\frac{1}{2}q+\frac{bp}{2a}$, instead of $\frac{1}{2}q$ or DE, and so the Center of the Circle sought is determin'd. Also because one of the Roots of the new Equation, *viz.* $+b$ is given, a point in the Circumference will be given too, and consequently the Radius. Lastly, Having describ'd the Circle, Perpendiculars let fall from its Intersections with the Parabola, to the Axis, will give the Roots of the Equation, both Affirmative and Negative, in the same manner as before.

Now the Center of the Circle is found by a most easy Construction, and which is to be preferr'd to all others, in Cubick Equations.

Fig. 11. Let *A* be the Vertex, and AF the Axis of the describ'd Parabola AMD; at a distance equal to *b* let DK be drawn parallel to the Axis, to the Right Hand if it be $+b$ in the Equation, and to the Left, if it be $-b$; which Line suppose to meet the Parabola in the point *D*. Upon the Centers *D* and *A*, and with equal Radij, describe on both sides two occult Arches, intersecting one another, and thro' those points of Intersection draw the interminate Line BC which cuts the imaginary Line AD in the middle and at right Angles, and meets the

Axis in *E*. From *E* set off EF $= \frac{1}{2} p$, downwards, if it be $- p$ in the Equation, but upwards towards *A* if it be $+ p$; then at the point *F* (or *E*, if *p* be wanting) erect the Perpendicular FG, meeting the Line BC in *G*, and in GF produced take GH $= \frac{1}{2} q$, to the Right Hand, if it be $- q$ in the Equation, but to the Left, if $+ q$. Then will the point *H* be the Center, and HD the Radius of the Circle sought, which (letting fall Perpendiculars to the Axis from its Intersections with the Parabola) will shew all the Roots (as LM) of the Equation. And how this Construction follows from what went before, is evident enough of it self, so that there is no need of insisting any farther upon the Demonstration of it.

A Dis-

A Discourse concerning the Number of Roots, in Solid and Biquadratical Equations, as also of the Limits of them.

By E. Halley.

HAving in the precedent Discourse shewn a Method, by which solid Problems however affected, might be constructed after a most simple and easy manner, by the help of one given Parabola and a Circle; towards the latter end a certain pleasant Speculation offer'd it self, namely, that from these Constructions, the *Number* of Roots in any Equation, with their *Limits* and *Sines*, would easily follow and be determin'd. Upon which account, I promis'd that I would quickly write a short Dissertation concerning this Subject, in which I was perswaded I should perform something not unprofitable nor ungrateful (if not to the Geometers of the first, yet at least) to those of the second Rank.

But coming to look nearer into the Business, I found I was imprudently fallen in among some of the profound Difficulties of Geometry, and destin'd to handle the same Things, that formerly employ'd the Pains of two Illustrious Men, *Harriot* and *Cartes*;

in which they either of them (by a like Fate tho' in a different way) committed a Paralogism, perhaps the only one in all their Geometrical Writings; as shall be afterwards prov'd. Wherefore being sensible, as well of the Difficulty, as the Excellency of the Subject, I resolv'd to apply to it strenuously, that I might not be thought unable to perform my Promises, and that so noble part of Geometry, and so little cultivated, might not lie any longer wrapt up in Darkness, but be render'd plain and intelligible by these few Lucubrations of mine. But first the Reader must take notice, that while he sets to the Reading of this, he ought to have the forementioned Dissertation (*No.* 188.) at hand by him, and to understand the Constructions there delivered very well; because those things that follow do chiefly depend upon them, neither are they to be here repeated again.

It is plain from *Cartesius*, and what was there said, that both in *Cubick* and *Biquadratick* Equations, the Roots may be expounded by Perpendiculars let fall, upon the Axis or given Diameter of the given Parabola, from the Intersections of that Curve with a Circle. And whereas when a Circle intersects a Parabola, it must necessarily do so, either in four or in two points; it's manifest, that in *Biquadraticks* there must always be, either two or four true Roots, *Affirmative* or *Negative*; as also if the Circle happens to touch it, in which case the equality of two Roots of the *same Sign*, is concluded. But in *Cubick* Equations, because one of the Intersections

tersections is requir'd to the Construction, therefore either but one, or the three remaining Roots, do denote one or three; as in the case of Contact; whence its plain, that there are found two equal Roots, and that the Problem from whence the Equation results, is really *Plain.*

Therefore all *Cubick* Equations however affected, are explicable by one, or by three Roots, which are always *possible*, that is, if we admit *Negative* Roots for *true* ones. So Biquadraticks whose last Term r is affected with the Sign $-$, are explicable by two or four; but if it be $+r$ in the Equation, and it be so great that $\sqrt{GDq - ar}$ (See *Fig.* 10.) be less than that the Circle describ'd with that Radius and on the Center G, can touch the Parabole in any point; the given Equation is altogether impossible, nor is it explicable by any Affirmative or Negative Root; but more of this in the following Pages.

Now since there is so great a difference between the Cases of Cubick and Biquadratick Equations, that they cannot be comprehended together, we will first of all handle the Cubicks, and then the others. The Cubicks are constructed by an infinite Number of Circles in a given Parabola; but the Biquadraticks by one alone *(at least by these Methods)* and that because, putting $z - e$ (or any Indeterminate) equal to nothing, the Cubick Equation is reduced to a Biqudratick having the same Roots with the Cubick, and besides that, another Root equal to e; whence it comes to pass that the Cubick Equation may be constructed by as many different Circles, as you can imagine Quantities *e*, that is, an infinite Number. But among all these, that which I gave before, is the easiest. Yet there is another not much inferior to this, which seems better accommodated to the designs of determining the *Number* of the Roots, and their *Limits*; and which arises from the taking away of the second Term, by putting after the common way $x = z + \text{or} - \frac{1}{3}$ of the Coefficient of the second Term. Now this way is thus: The Parabola ABY (*Fig.* 12.) being given, whose Vertex is *A*, its Axis AE, and *Latus Rectum a*, let the Equation be reduced to the usual Form, *viz.* $z^3 . bz^2 . apz . aaq. = o.$ Then at the distance of $\frac{1}{3}b$ let there be drawn BK (parallel to the Axis, to the Right Hand if it $+ b$, otherwise to the Left) which meets the Parabola in *B*; and let the Line DP interminate on both sides, be erected perpendicular to the suppos'd Line AB, meeting the Axis in the point *G*. From the point *B*, let fall the Perpendicular BC, and let GE be always made equal to AC, and be set off towards the lower parts. From *E* set off $EH = \frac{1}{2}p$, upwards if it be $+ p$ in the Equation, but downwards if $- p$; and from the point *H*, (or *E*, if the quantity *p* be wanting) let the Perpendicular HQ be drawn out, meeting the interminate Line DP in *O*. Lastly, in the interminate Line HQ, make $OR = \frac{1}{2}q$, from *O* to the Right Hand, if it be $- q$, but to the Left, if $+ q$. Then a Circle describ'd on the Center *R* with the Radius RA, will cut the Parabola in as many points, as the Equation

Equation propos'd has Roots, and they will be the Perpendiculars ZY, let fall from the Interſections *Y*, to the Line B K parallel to the Axis; of which thoſe that are to the *Right* Hand of the Line BK, are the *Affirmative* ones, and thoſe to the *Left*, the *Negative.*

The conveniency of this Conſtruction, lies in this, that 'tis perform'd by a Circle paſſing thro' the Vertex, in the ſame manner as if the ſecond Term had been wanting. And therefore to determine the Number of the Roots, 'tis ſufficient to know the Properties of the *Place*, or that Curve Line which diſtinguiſhes the Spaces, in which if the Center of the Circle (that paſſes thro' the Vertex of the Parabola) be placed, the Circumference of it ſhall interſect the Parabola either in one or in three other points: That is, to define the Nature of that Curve, in which, fall the Centers of all the Circles paſſing thro' the Vertex, and then touching the Parabola. Now this *Locus*, is that very *Paraboloid*, which the celebrated Dr. *Wallis* calls the *Semicubical*, in which the Cubes of the Ordinates are as the Squares of the correſpondent Abſciſſes. The *Latus Rectum* of which, is $\frac{27}{8}$ of the *Latus Rectum* of the given Parabola, and its Vertex the point U (*Fig.* 12.) the Line AU being half the *Latus Rectum* of the ſame Parabola. That is, if we put unity for the *Latus Rectum* of the given Parabola, then $\frac{8}{27}$ of the Cube of the ordinate applicate, will = the Square of the intercepted Diameter; or the Cube of $\frac{2}{3}$ VH = the Square of AR, *viz.* if *R* be the Center

Center of the Circle that passes thro' the Vertex of the Parabola, and touches the same afterwards.

This is that Curve which our Countryman Mr. *Neil* (the first of all Mortals) demonstrated to be equal to a given right Line, and by that means obtain'd a Reputation among the principal Geometricians. Its properties have been curiously enquired into, by Dr. *Wallis*, (at the end of his Book of the *Cissoid)* and *Hugenius (Prop.* 8 & 9. of his Tract of the *Evolution* of *Curve Lines)* and others, whose Writings the Reader may consult. This Curve describ'd on either side of the Axis of the Parabola (*viz.* VNL, VPX) comprehends a Space, in which if the Center of the Circle (which passes thro' the Vertex *A)* be placed, it will cut the Parabola in three other points. But the Spaces more remote from the Axis, do afford Centers for Circles that will cut the Parabola but in one point besides the Vertex.

These things well understood, we are now prepar'd to determine the Number of the Roots. And first of all, let the second Term be wanting, and let the *Latus Rectum* $= 1$, or AV $= \frac{1}{2}$. In the Construction VH is $= \frac{1}{2} p$, HR $= \frac{1}{2} q$; and since if it be $+ p$ in the Equation, $\frac{1}{2} p$ is to be set off from *V* towards the upper parts; the Center of the Circle is always found without the Space LVX, and therefore is explicable by one Root only, which is Affirmative if it be $- q$, Negative if $+ q$; and these Roots may be investigated by *Cardan*'s Rules. But if it be $- p$, then UH $= \frac{1}{2} p$, is set off towards the lower parts; and it is possible

possible that HR may fall between the Axis and the Curve UX or UL, *viz.* if the Cube of $\frac{1}{3}$ UH or of p, be greater than the Square of $\frac{1}{2}q$; that is, if $\frac{1}{27}p^3$ be greater than $\frac{1}{4}q^2$; in which case there are three Roots, two Negative, if it be $-q$, and one Affirmative equal to the sum of the others; but if it be $+q$, then there are two Affirmative ones, and one Negative. But if $\frac{1}{27}p^3$ be less than $\frac{1}{4}q^2$, then there is but one Root, Affirmative if it be $-q$, Negative, if $+q$. All which things are taught by those that have handled this part of Geometry.

Now let all the Terms be in, and first let there be proposed, as an Example, this Equation, $z^3 - z^2 b + zp - q = o$, to which *Fig.* 12. serves. In the Construction of this, we have $BC = \frac{1}{3}b$, $UG = \frac{1}{2}AC = \frac{1}{8}b^2$, $UE = \frac{1}{16}b^2$, $UH = \frac{1}{6}b^2 - \frac{1}{2}p$, $GH = \frac{1}{9}b^2 - \frac{1}{2}p$, or $\frac{1}{2}p - \frac{1}{9}b^2$. Hence $HO = \frac{1}{27}b^3 - \frac{1}{6}bp$, or $\frac{1}{6}bp - \frac{1}{27}b^3$, and HR (that is the distance of the Center of the Circle *R* from the Axis) is ever the difference between $\frac{1}{6}bp$ and $\frac{1}{27}b^3 + \frac{1}{2}q$, which Expressions if they are equal, then the Center falls in the Axis: If $\frac{1}{6}bp$ be greater than $\frac{1}{27}b^3 + \frac{1}{2}q$, then it falls to the Left Hand of the Axis, if less, then to the Right. If therefore the square Root of the Cube of $\frac{2}{3}$ UH (that is of $\frac{1}{9}b^2 - \frac{1}{3}p$, or putting $\frac{1}{9}b^2 - \frac{1}{3}p = d$, if $\sqrt{ddd}$) be greater than HR, that is the difference between $\frac{1}{27}b^3 + \frac{1}{2}q$ and $\frac{1}{6}bp$; the Center *R* will be found within the Space NPU circumscrib'd by the Paraboloids UPX, UNL, and the interminate right Line DNP; and so the Circle will cut the Parabola in three points Y, Y, Y, posited to the Right Hand

Hand of the Line BK, and so the Equation will have three Roots. But the Center being without this Space NUP, it is explicable but by one Affirmative Root. Here it may be noted by the by, that the Right Line DP may touch the Paraboloid UPX in the point *P*, EP being $\frac{1}{27} b^3$; but will cut the other Paraboloid UNL in the point *N*, so that letting fall NF perpendicular to the Axis, UF is $\frac{1}{4}$ EU, or $\frac{1}{24} b^2$, and NF $\frac{1}{108} b^3$. But UW (which being perpendicularly applied to the Axis at the point *U*, meets DP in *W*) is $= \frac{1}{54} b^3$, or $\frac{1}{2}$ EP.

Hence we may safely conclude, that if in the Equation either p be greater than $\frac{1}{3} b^2$, or q greater than $\frac{1}{27} b^3$, that there will be found but one Root, and that an Affirmative one. *Carte*'s Rule therefore (*Page* 70. Edit. *Amsterd.* 1659.) is not true, in which he determines that there are always as many *true* Roots, as there are changes of the Sines + and — in the Equation: *Schooten* in his Commentaries vainly endeavouring the defence of this Mistake. Also *Prop.* 5. Sect. 5. of our Country-man *Harriot*'s *Ars Analytice* (as also *Prob.* 18. of *Vieta*'s *Numer. Potest. Resol.*) is hardly found; since from the Limitations which they have there set down, that must agree to the whole Parallelogram PIUW, which we have prov'd does agree only to the Space NUP; that is to afford a Center to the Circle intersecting the Parabola in three other points besides the Vertex.

But the quantity q or the last Term (b and p being given, so that p be less than

$\frac{1}{3} b^2$)

$\frac{1}{3}b^2$) is exactly limitted from the foregoing Equation $\sqrt{ddd} = \frac{1}{27}b^3 + \frac{1}{2}q \backsim \frac{1}{6}bp$; *viz.* when the Circle touches the Parabola. Therefore $\frac{1}{2}q$ ought to be less than $\frac{1}{6}bp - \frac{1}{27}b^3 + \sqrt{d^3}$; but if p be greater than $\frac{1}{4}b^2$, also $\frac{1}{2}q$ ought to be bigger than $\frac{1}{6}bp - \frac{1}{27}b^3 - \sqrt{d^3}$, that the Center may not fall in the little Space NUW. And with these Conditions the Equation will always be explicable by three Roots; otherwise but by one. But whether there be three or one, they are always Affirmative ones, because of the position of the Center *R* to the Right Hand of the Line DP.

And this is the most difficult Case; so that those that well understand what has gone before, will without any trouble take what comes after. Now let the Equation $z^3 - bz^2 + pz + q = o$, be given. Here (that there may be three Roots had) the Center of the Circle ought to be found somewhere within the Space PNΔ, determin'd by the right Lines PN, PΔ, and the Curve of the Parabola NΔ; wherefore since EF is $= \frac{1}{8}bb$, p ought to be less than $\frac{1}{4}bb$. Now for the determination of the quantity q, d being $= \frac{1}{9}b^2 - \frac{1}{3}p$ as before, $\sqrt{d^3} + \frac{1}{27}b^3 - \frac{1}{6}bp$ ought always to be greater than $\frac{1}{2}q$, that so the Center of the Circle may be posited in the forementioned Space PNΔ; which when 'tis so, such an Equation has two Affirmative Roots, and one Negative. But if p be greater than $\frac{1}{3}bb$, or $\frac{1}{2}q$ greater than $\sqrt{d^3} + \frac{1}{27}b^3 - \frac{1}{6}bp$; it is explicable but by one (and that a Negative) Root.

Let

Let the Equation $z^3 = bz^2 - pz - q = o$, be propoſed in the next place. That this Equation may have three Roots, the Center of the Circle muſt be found ſomewhere in the indefinite Space between the right Line D P D and the Curve of the Paraboloid PX. The quantity p is not here liable to Limitations; but $\frac{1}{2} q$ ought always to be leſs than $\sqrt{d^3 - \frac{1}{27} b^3 - \frac{1}{6} bp}$, ſuppoſing d to be $= \frac{1}{9} b^2 + \frac{1}{3} p$. By this means, there are two Negative Roots afforded, and one Affirmative; but otherwiſe, if $\frac{1}{2} q$ be greater than $\sqrt{d^3 - \frac{1}{27} b^3 - \frac{1}{6} bp}$, the Equation is explicable by one only (Affirmative) Root.

Fourthly, Let the Equation $z^3 - bz - pz + q = o$, be propoſed, which has two Affirmative Roots, and one Negative, if the Center of the Circle be found in the indefinite Space between the right Lines P△, PD, and the Curve of the Paraboloid △L; that is, (putting $d = \frac{1}{9} bb + \frac{1}{3} p$) if $\frac{1}{2} q$ be leſs than $\sqrt{d^3 + \frac{1}{27} b^3 + \frac{1}{6} bp}$; but if $\frac{1}{2} q$ be greater than this quantity, there is but one (Negative) Root.

But the four remaining Equations in which we have $+ b$, do not differ from thoſe that have been mention'd already, *as to the Limitation of the Number of the Roots*, if the Sign of the laſt Term be changed, keeping the Sign of the third Term. But then them that were the Affirmative Roots in the former, will be the Negative ones here, and contrariwiſe.

Thus in the Equation $z^3 - bz^2 + pz - q = o$, the Affirmative Roots were either one or three; but in this Equation $z^3 + bz^2 + pz + q = o$;

$+ q = o$, there is either one or three Negative Roots, under the very same Conditions; but no Affirmative Root at all. So also in the Equation $z^3 + bz^2 + pz - q = o$, there are two Negatives and one Affirmative, if p be less than $\frac{1}{3} bb$, and $\frac{1}{2} q$ less than $\sqrt{d^3} + \frac{1}{27} b^3 - \frac{1}{6} bp$; even as in the Equation $z^3 - bz^2 + pz + q = o$, there were two Affirmatives and one Negative: But the quantities p and q exceeding those prescrib'd Measures, there is *here* only one Affirmative Root, which *there* was a Negative one. In like manner, in the Equation $z^3 + bz^2 - pz + q = o$, there are either two Affirmatives and one Negative, or one Negative only

Lastly, For the same reasons in the Equation $z^3 + bz^2 - pz - q = o$, there are two Negatives and one Affirmative, or one Affirmative only, for which, in the Equation $z^3 - bz^2 - pz + q = o$, there were two Affirmatives and one Negative, or one Negative alone; *viz.* as $\frac{1}{2} q$ is either greater or less than $\sqrt{d^2} + \frac{1}{27} b^3 + \frac{1}{6} bp$.

If the third Term (or pz) be wanting, the Center R always falls in the Line IPEΔ, wherefore if it be $z^3 - bz^2 . * . - q$ or $z^3 + bz^2 . * . + q$, there can be but one Root, Affirmative if it be $- b$, Negative, if $+ b$. But if it be $z^3 - bz . * . + q$ or $z^3 + bz^2 . * . - q$, there may be two Affirmatives and one Negative in the former, or one Affirmative and two Negatives in the latter, the Center falling in the Line PΔ between P and Δ, that is if $\frac{1}{4} q$ be less than $\frac{1}{27} b^3$; for if it be greater, there can be but one Negative in the former, or one Affirmative in the latter.

Hitherto

Hitherto we have obtain'd the Number of the Roots in Cubick Equations, it remains that we add somewhat concerning the quantity of the Roots. And here it is first of all to be noted, that every Equation having three Roots, may be expeditiously enough resolv'd by the help of the Table of Sines, that is by the Trisection of an Angle, by putting $\sqrt{\frac{4}{9}b^2 - \frac{4}{3}p}$ or $\sqrt{4d}$ = the Radius of the Circle, if it be + p in the Equation; or $\sqrt{\frac{4}{9}b^2 + \frac{4}{3}p}$, if $-p$; and the Angle to be Trisected, that which has its Sine (in the Table of Sines) $\frac{\frac{1}{27}b^3 + \frac{1}{6}bp + \frac{1}{2}q}{\sqrt{d^3}}$. This Angle being found, the Sine of its third part, as also the Sine of the third part of its Complement to a Semi-circle, and their Sum, will be given from the Table of Sines. Now these Sines are to be multiplied into the Radius $\sqrt{\frac{4}{9}b^2 + \frac{4}{3}p}$, and thus will be obtain'd the quantities (*y*& *y*& *y*& in the *Fig.*) the Sum or Difference of which and $\frac{1}{3}b$, as the case requires, will give the true Roots of the Equation. All these things are deduced from *Cartes*'s Discoveries. But that I may comprehend all the Cases, with as much Brevity as is possible; I say, that the Center *R*, in the first *Formula* of Equations, falling in the Space UGP, the two Intersections *Y*, *Y*, fall between *A* and *B*, and consequently either of the lesser Roots is less than $\frac{1}{3}b$, but the third and greater always exceeds $\frac{1}{3}b$, but is exceeded by *b*. But if the Center falls

falls in the space G N U, there are two greater than $\frac{1}{3}b$, but less than $\frac{2}{3}b$, but the third is $b -$ the two others, and consequently less than $\frac{1}{3}b$; but using the Limitation of the Quantity p, the Roots are included in narrower Bounds. For the greatest Root is less than $\sqrt{\frac{4}{9}b^2 - \frac{1}{4}p} + \frac{1}{3}b$, but greater than $\sqrt{\frac{1}{4}b^2 - p} + \frac{1}{2}b$; but when $\frac{1}{4}bb$ is less than p, that *Limit* becomes $\sqrt{\frac{1}{9}b^2 - \frac{1}{3}p} + \frac{1}{3}b$. The *mean* Root is always less than $\sqrt{\frac{1}{4}b^2 - p} + \frac{1}{2}b$, but greater than $\frac{1}{3}b - \sqrt{\frac{1}{9}b^2 - \frac{1}{3}p}$; but the least Root never exceeds this *Limit*, but vanishes with the Quantity q.

In the second *Formula*, according to the prescrib'd Laws, there are two Affirmative and one Negative Root; and the Center falling in the Space G P E, one of the Affirmatives is greater, and the other less than $\frac{1}{3}b$, but the greater exceeds not b; but the Negative cannot be greater than $\sqrt{\frac{1}{3}bb} - \frac{1}{3}b$, and it is the difference of b and the Sum of the Affirmative Roots. But the Center being posited in the Space E N G Δ, either of the Affirmatives is greater than $\frac{1}{3}b$, but less than $\sqrt{\frac{1}{3}bb} + \frac{1}{3}b$; but the Negative is ever less than $\frac{1}{3}b$. But the nearer Limits (from the Quantity p given) are $\sqrt{\frac{1}{4}bb - p} + \frac{1}{2}b$, of the greatest Affirmative Root; than which it is always less, as also greater than $\sqrt{\frac{1}{9}bb-\frac{1}{3}p}+\frac{1}{3}b$; yet the other Affirmative Root (which is diminiſh'd with the Quantity q) is leſs than this *Limit*. But the Negative Root is always leſs than $\sqrt{\frac{4}{9}bb+\frac{4}{3}p}-\frac{1}{3}b$, and the Quantity q being wanting, vaniſhes.

In the third *Formula*, there are two Negatives and one Affirmative. In this, as in the fourth, the Roots are not limitted by the Quantity b. But the Affirmative Root is ever leſs than $\sqrt{\frac{4}{9}bb+\frac{4}{3}p}+\frac{1}{3}b$, yet greater than $\sqrt{p+\frac{1}{4}bb}+\frac{1}{2}b$; and the greateſt of the Negatives is always greater than $\sqrt{\frac{1}{9}bb+\frac{1}{3}p}-\frac{1}{3}b$, but leſs than $\sqrt{p+\frac{1}{4}bb}-\frac{1}{2}b$. But the leſs of the Negatives is always leſſen'd with the leſſen'd Qantity q.

In the fourth *Formula*, the Center falling within the Space L △ P D; if there be two Affirmative and one Negative Root, the greateſt of the Affirmative Roots cannot be greater than $\sqrt{p+\frac{1}{4}bb}+\frac{1}{2}b$, nor leſs than $\sqrt{\frac{1}{9}bb+\frac{4}{3}p}+\frac{1}{3}b$. But the leſs Negative is leſs than $\sqrt{\frac{4}{9}bb+\frac{4}{3}p}-\frac{1}{3}b$, and greater than $\sqrt{p+\frac{1}{4}bb}-\frac{1}{2}b$. But 'tis to be noted here, that the Negative Roots are every where mark'd with the Affirmative Sine, becauſe theſe are the Affirmative Roots of thoſe four Equations, in which is found $+b$, and q is is affected with the contrary Sine; as I intimated above.

The Demonstration of all these things follows from hence, that where-ever the Center of the Circle *R* falls upon the Curve Lines U P X or U △ L, the Circumference of it touches the Parabola in a Point whose distance from the Axis is $\sqrt{\frac{2}{3}}$ V H, and cuts it on the other side the Axis at the distance of $2\sqrt{\frac{2}{3}}$ U H; but when the Center falls on the Line D P D, one of the Roots is $= o$, and consequently the Cubick Equation is reduced to a Quadratick one, or to $z^2 - bz + p = o$, the Roots of which give the *Limits* when the Quantity q vanishes; and by how much the less q becomes, by so much the nearer do the Roots approach to these Limits. The Equation is also Quadratical, when the Center falls in the Axis; that is, when $\frac{1}{2}q = \frac{1}{6}bp - \frac{1}{27}b^3$, in the first *Formula*; or $\frac{1}{2}q = \frac{1}{27}b^3 - \frac{1}{6}bp$, in the second; in the third 'tis impossible; but in the fourth, when $\frac{1}{2}q = \frac{1}{27}b^3 + \frac{1}{6}bp$; in which case the less of the Affirmative Roots is $\frac{1}{3}b$, and the greater $\sqrt{\frac{1}{3}bb + p} + \frac{1}{3}b$, but the Negative $\sqrt{\frac{1}{3}bb + p} - \frac{1}{3}b$. In the first *Formula*, the Roots are $\frac{1}{3}b$, and $\frac{1}{3}b \pm \sqrt{\frac{1}{3}bb - p}$. But in the second, the Affirmatives are $\frac{1}{3}b$, and $\sqrt{\frac{1}{3}bb - p} + \frac{1}{3}b$, but the Negative $\sqrt{\frac{1}{3}bb - p} - \frac{1}{3}b$. And these things may seem to suffice in Cubicks; but because of the excellent use of the Method, by which, by the help of the *Table* *of Sines*, the Roots of these Equations are found; I thought convenient to add an Example or two, by which the *Compendium* of that Practise may be rendred manifest. Let the Equation $z^3 - 39z^2 + 479z - 1881 = 0$, be propos'd; and the Roots z are sought. Here $\sqrt{\frac{1}{9}bb - \frac{1}{3}p} = \sqrt{9\frac{1}{3}} = \sqrt{d}$, whose double $\sqrt{37\frac{1}{3}}$ is the Radius of the Circle; also

$$\frac{\frac{1}{27}b^3 + \frac{1}{2}q - \frac{1}{6}bp}{\sqrt{d^3}} = \frac{2197 + 940\frac{1}{2} - 3113\frac{1}{2}}{9\frac{1}{3}\sqrt{9\frac{1}{3}}},$$

or $\dfrac{24}{9\frac{1}{3}\sqrt{9\frac{1}{3}}}$ is the *Tabular Sine* of the Angle; that is, making a Division by the help of the Logarithms, *Log.* 9.9251560, to which corresponds an Angle of 57°. 19′. 11½″. The third part of this is 19°. 6′. 24″. and of the Complement, is 40°. 53′. 36″. The Sines give the Logs, 9.514983 and 9.816011; which multiplied into the Radius $\sqrt{37\frac{1}{3}}$, produce Y& and Y&, Log. 0.301030 = 2, and Log. 0.601059 = 4, but the third Y& is equal to the Sum of them, or 6. And therefore the Roots are $13 - 4 = 9$; $13 - 2 = 11$, $13 + 6 = 19$; of which several ones the foremention'd Equation is compos'd. Where 'tis to be noted that the two lesser Roots, do not exceed $\frac{1}{3}b$ or 13, because the Center *R* in the Construction falls on the right hand of the Axis; that is, $\frac{1}{6}bp$ is less than $\frac{1}{27}b^3 + \frac{1}{2}q$.

For

For another Example, let us enquire out the Roots of the Equation $x^3 - 15x^2 - 229x - 525 = o$. Here $\sqrt{\frac{1}{9}bb + \frac{1}{3}p} = \sqrt{101\frac{1}{3}} = \sqrt{d}$, and the Radius of the Circle $= \sqrt{405\frac{1}{3}}$. Also $\frac{\frac{1}{27}b^3 + \frac{1}{6}bp + \frac{1}{2}q}{\sqrt{ddd}} = \frac{125 + 572\frac{1}{2} + 262\frac{1}{2}}{101\frac{1}{3}\sqrt{101\frac{1}{3}}}$ the Tabular Sine of an Arch, whose Log. 9.9736426, and the Arch it self 70°. 14′. 22″. The third part of it, is 23°. 24′. 47″½, and of the Complement, is 36°. 35′. 12″½, whose Log. Sines are 9.599183, and 9.775275, to which adding the Log. $\sqrt{405\frac{1}{3}}$, we have the Log. 0.903089=8, and Log. 1.079181=12, the Sum of which is equal to 20. Hence we conclude that $20 + \frac{1}{3}b$ or 25, is equal to the Affirmative Root, and 8 and $12 - \frac{1}{3}b$, that is 8 and 7 equal to the Negative Roots. But if the Equation had been $x^3 + 15x^2 - 229x - 525 = o$, then 8 & 7 had been the Affirmative Roots, and 25 the Negative. As for the other Cubicks which are explicable by one only Root, they are to be resolv'd by *Cardan*'s Rules, after the second Term is taken away; neither do I see how the business can be done with less Calculation.

But if this Root be desir'd to be expressed in the Terms of the Quantities b, p, q, I say that in the first *Formula* it is, $\frac{1}{3}b +$ or $-$ the Sum or Difference of the Cubick Roots of $\sqrt{\frac{1}{4}qq - \frac{1}{108}p^2b^2 + \frac{1}{27}b^3q - \frac{1}{6}bpq + \frac{1}{27}p^3}$ $\pm \frac{1}{27}b^3 + \frac{1}{2}q - \frac{1}{6}bp$ (*viz.* + if $\frac{1}{27}b^3 + \frac{1}{2}q$ be greater than $\frac{1}{6}bp$, otherwise —) the Sum, when $\frac{1}{3}bb$ is greater than p, the difference when less. And in the other *Formula*, the Root is always compos'd of the same parts, only the Sines + and — being varied, as they will easily perceive that are willing to make the Tryals.

But these Roots are readily enough found by the help of the Log. Table of *versed Sines*; viz. if the Coefficients are *surd* or *broken* Numbers, and the Roots not to be expressed in Numbers, as most commonly it happens.

Now this is the Rule. In the first and second *Formula*, if $\frac{1}{3}bb$ be less than p, let $\frac{1}{3}p - \frac{1}{9}bb = d$, and putting the difference between $\frac{1}{6}bp$, and $\frac{1}{27}b3 + \frac{1}{2}q$ (that is H R) in the first *Formula*, and the difference between $\frac{1}{6}bp + \frac{1}{2}q$ and $\frac{1}{27}b3$ (in the second *Formula*) *Radius*, let the Angle, whose Tangent is $d\sqrt{d}$, be found. Then, as the *Co-sine* of this Angle, to the *versed Sine* of the same, so the *Difference* made *Radius*, to a fourth Quantity, the Cube Root of which will be had by taking the $\frac{1}{3}$ of its Log. Then dividing $\frac{1}{3}p - \frac{1}{9}bb$ by this Cube Root, let the Divisor be substracted from the Quotient, the Remainder will be the Quantity Y& at Fig. 1. The *Sum* of this Remainder and $\frac{1}{3}b$ will be the Root sought, if the Center falls on the Right Hand of the Axis; otherwise their *Difference* will be the Root. But if $\frac{1}{3}bb$ be greater than p, making H R Radius, let $d\sqrt{d}$ (or the distance of the Paraboloid from the Axis) be the Sine of some Arch; let the versed

verſed Sine of this be multiplied into Radius or $\frac{1}{6}bp - \frac{1}{27}b3 \pm \frac{1}{2}q$, and taking $\frac{1}{3}$ of the Log. of the Product, its Cubick Root will be obtained, by which let $\frac{1}{9}bb - \frac{1}{3}p$ be divided. I ſay, that the *Sum* of the Quotient and Diviſor, after the ſame manner added to or taken from $\frac{1}{3}b$, will give the Root ſought. And the like for the third and fourth *Formula*, unleſs that $\frac{1}{27}b3 + \frac{1}{6}bp \pm \frac{1}{2}q$ is to be taken for Radius, and $\frac{1}{9}bb + \frac{1}{3}p$ into $\sqrt{\frac{1}{9}bb + \frac{1}{3}p}$, or $d\sqrt{d}$, for the Sine. But theſe Rules will be perhaps better underſtood by Examples.

Suppoſe the Cubick Equation $z3 - 17z^2 + 54z - 350 = o$, and let the Root z be ſought. Here $\frac{1}{3}bb$ is is greater than p, but q is bigger than the Cube of $\frac{1}{3}b$, and therefore 'tis explicable by one Affirmative Root only. Now $\frac{288}{9} - \frac{54}{3}$ is d, and $\frac{127}{9}\sqrt{\frac{127}{9}}$ is to be taken for the Sine, to the Radius $\frac{4913}{27} + 175 - 153$, that is $\frac{5527}{27}$; and the Arch agreeing thereto is 15°. 30′. 49″. The Log. verſed Sine of this 8.5362376, added to the Log. of the Radius 2.3095913, makes 0.8457889, the 3d part of which 0.2819276, is the Log. of the Cube Root 1.91394, by which, as a Diviſor, dividing $\frac{127}{9}$ or d, the Quotient is 7.37281. The Sum of the Quotient and Diviſor encreaſed by the addition of $\frac{1}{3}b$, is the Root ſought, *viz.* 14.9534, &c.

Having thus diſpatch'd *Cubick* Equations, let us proceed to *Biquadratical* ones. Theſe have always either none, or 2, or 4 true

Roots, the determination of which depends partly on the *Coefficients*, partly on the *Sine* and *Magnitude* of the absolute Number given. A general Construction for all these (and that easy I conceive enough) I have delivered at N° 188, which I suppose the Reader to be acquainted with; but yet the Figure relating to that Matter, I think proper to bring hither, (*Fig.* 2.) In the Construction of the Equation $z^4 - bz^3 + pz^2 - qz + r = 0$, let $BD = \frac{1}{4}b$, $AB = \frac{1}{16}bb$, $BK = \frac{1}{2}$ or $\frac{1}{2}$ the *Parameter*, $KC = 2\,AB = \frac{1}{8}bb$, $KE = \frac{1}{8}bb - \frac{1}{2}p$, $AE - \frac{1}{2} = \frac{3}{16}bb - \frac{1}{2}p$, $FE = \frac{1}{16}b^3 - \frac{1}{4}bp$, and $EG = \frac{1}{16}b^3 - \frac{1}{4}bp + \frac{1}{2}q$. Which done, a Circle on the Center G with the Radius $\sqrt{GD^2 - r}$, will interfect the Parabola, either in none, or 2 or 4 Points, from whence Perpendiculars let fall on DH, will give all the Roots z. But that there may be 4, 'tis evident that the Center of the Circle ought to be found somewhere within a space from any Point of which, three Perpendiculars may be let fall upon the Curve of the Parabola; and also that the Radius is less than the greatest of those Perpendiculars, and greater than the middle one. But that if the Center be posited without this space, so that there can be but one Perpendicular let fall upon the Parabola, and the Radius greater than it, or if it be less than the middle one of the 3 Perpendiculars, but greater than the least of them; then there can be but two Roots only. But there is no Root at all, when the Radius $\sqrt{GD^2 - r}$ is is less than the least of the 3, or than the one as often as there is but one. Now it remains for us to inquire of what kind this Space is, by what Limits 'tis distinguished, and under what Conditions the Radius of the Circle is less or greater than the foremention'd Perpendiculars. And first of all, we must shew how a Perpendicular is to be let fall upon the Parabola. Let (*Fig.* 3.) ABC be a Parabola, AE its Axis, AV $\frac{1}{2}$ the Parameter, G the point from whence the Perpendicular is to be let fall. Let GE be drawn perpendicular to the Axis, and VE be bisected in F, and erecting the Perpendicular FH on the same side of the Axis, let $FH = \frac{1}{4} GE$; I say that a Circle describ'd on the Center H, with the Radius HA, will intersect the Parabola in three points, or one, z, the right Lines GZ drawn to which, will be perpendicular to the Curve of the Parabola. But now that there may be 3 such Intersections, the Center H ought to be so posited, as that it may be within the space included by the Paraboloids (in *Fig.* 1.) that is, that FH may be less than $\sqrt{\frac{8}{27} FV^3}$, or FH^2 less than the Cube of $\frac{2}{3}$ VF; and so $GE = 4 FH$ will be less than $4\sqrt{\frac{8}{27} VF^3}$, that is, the square of GE will be less than $\frac{16}{27} VE^3$. Therefore these Limits coincide with two Paraboloids of the same kind with those which were used in Cubical Equations, but whose Parameter is twice less, *viz.* $\frac{27}{16}$ of the Parameter of the Parabola, that is $\frac{27}{8}$ of AV. And therefore it is that very Curve

Curve Line, by the Evolution of which the Parabola is describ'd (as *Hugenius* has demonstrated) and which, the Line DF *(Fig. 2.)* which is perpendicular to the Parabola in the point D, is always a Tangent to. But the point P (that is, that in which the right Line DF touches the Paraboloid) is the Center of a Circle, which (being describ'd with the Radius DP) coincides with the Parabola in the point D, or has the same *Curvature* with it, as is manifest.

Having therefore describ'd such Paraboloids UXP, VN△ *(Fig. 2.)* on either side the Axis, 'tis clear, that unless the Center of the Circle be placed within these Limits, it cannot intersect the Parabola in more than two points. From whence we may determine, under what conditions, the Coefficients of the intermediate Terms are restrained, in Biquadratick Equations, that so there may be four Roots. And at first sight 'tis plain that p cannot be greater than $\frac{1}{8}\,bb$, (viz. in those Forms where 'tis $+p$) nor q than $\frac{1}{16}\,b^3$. But in General, $\frac{1}{16}\,b^3 \mp \frac{1}{4}\,pb \mp \frac{1}{2}\,q$, that is EG the distance of the Center from the Axis, ought to be less than $EH = 4\sqrt{\frac{1}{27}\,VE^3}$, that is (because $VE = \frac{3}{16}\,bb \mp \frac{1}{2}\,p$) than $\frac{1}{4}\,bb \mp \frac{2}{3}\,p\,\sqrt{\frac{1}{16}\,b2 +}$ or $-\frac{1}{6}\,p$, the Sines + and − being left doubtful, that so they may be varied according to the nature of any Equation; as was shewn above in Cubicks. Neither would I be offensively tedious to the Learned on the one hand, nor deprive Learners on the other, of the Exercise

ercise and Pleasure, of sending out these things by themselves. As for the Limitation of the least Term r, it cannot be found with the same easiness, and that because, to let fall a Perpendicular upon the Curve of a Parabola, is a *solid Probleme*, and which cannot be resolv'd without the solution of a Cubick Equation. Therefore first of all let the second Term be wanting, or if there, let it be taken away, so that the Equation may have this Form $z4.\ *.\ pz^2.\ qz.\ r. = o.$ And if it be $-r$, it is always explicable by two or four Roots; but that there may be four, the Center of the Circle ought to be posited within the forementon'd Paraboloids, or that it may be $-p$, and qq may be less than $\frac{8}{27}\ p3$ or the Cube of $\frac{2}{3}\ p$. Then let the Roots of this Equation $y3.\ *.\ \frac{1}{2}\ py.\ \frac{1}{4}\ q = o$, be gotten, the Quantities p and q having the same Sines as in the Biquadratick. And these Roots are found expeditiously enough by the help of the Table of Sines. But having found those three y (which are ordinately applied to the Axis of the Parabola from the points, where the Perpendiculars to the Curve of it do fall, *viz.* Y Z in *Fig.* 3.) than $pyy - 3y4$ of the *lesser y* will denote the greatest Quantity of r, if it be $-r$, than which if r be less, the Equation will have four Roots, otherwise but two. But if it be $+ r$, it ought to be less than $3y4 - pyy$ of the *middle y*, for if it be greater, it can have but two Roots; at least, if r be less than $3y4 - pyy$ of the *greatest y*. But if it be greater than this, the Equation is not explicable by any true Root at all.

These

These same *Limits*, are otherwise expressed by the Quantity q, viz. $\frac{1}{2}qy - y4$ in the first case, $y4 - \frac{1}{2}qy$ in the second, and $y4 + \frac{1}{2}qy$ in the third. But it may be, that the two lesser Quanties y may not be far different from one another, whence it comes to pass that both of the Perpendiculars are greater than the right Line G A, *viz.* when qq is greater than $\frac{4}{27}p^3$, but less than $\frac{8}{27}p^3$; the Center falling within the space contain'd between the Paraboloids of *Fig.* 1. and 2. In this case, if it be $+ r$, there can be but two Roots, $y4 + \frac{1}{2}qy$ of the *greatest* y being greater than r; otherwise none. But if $\frac{1}{2}qy - y4$ of the *least* y be greater than r mark'd with the Sine $-$, but r be greater than $\frac{1}{2}qy - y4$ of the *mean* y, then there will be four Roots; but two only, if r be found greater than the former, or less than the latter. But if in the Equation it be $+ p$, or if it be $- p$ and qq be greater than $\frac{8}{27}p^3$, the Equation $y^3. *. \frac{1}{2}py. \frac{1}{4}q.$ is explicable by only one Root y; that is, there can be but one Perpendicular only let fall from the Center of the Circle. Whence it may be certainly concluded that there can be but two Roots only in the given Equation, the Sum of which, if it be $- r$, is increas'd with the Quantity r; but if it be $+ r$, the Quantity y being obtain'd, that Quantity r ought to be less than $y4 + \frac{1}{2}qy$, for if it be greater, the Equation propos'd is absurd and impossible.

'Twould be both tedious and needless to run over all Equations of this kind, since 'tis evident (from what has been already said) to those that are attentive, which are Negative

gative and which Affirmative, and that the Limits of these Roots are deriv'd from the found Quantities y. But for an Example (which any one may imitate in the rest of of the Cases) let it be propos'd to discover the *Limits* or *Conditions*, under which, there may be four Affirmative Roots in a Biquadratical Equation. Now this will be as often as the Center of the Circle G is posited in the space UPK (*Fig. 2.*) and also $+r$ or the Radius of the Circle is less than GD. Whence 'tis plain, that the Equation here concern'd is of this Form, $z^4 - bz^3 + pz^2 - qz + r = o$; and that p cannot be greater than $\frac{3}{8}bb$, nor $\frac{1}{4}pb$ (in this case) than $\frac{1}{16}b^3 + \frac{1}{2}q$; again, 'tis necessary that $\frac{1}{4}bb - \frac{2}{3}p$ in $\sqrt{\frac{1}{16}bb - \frac{1}{6}p}$ should be greater than $\frac{1}{16}b^3 + \frac{1}{2}q - \frac{1}{4}bp$; and from these *Limits*, it will be manifest that the Center is contain'd within the space UPK. But in order to the determination of the Quantity r, this Cubick Equation must be first solv'd, $y^3 * - \frac{3}{16}b^2 - \frac{1}{2}py = \frac{1}{32}b^3 + \frac{1}{4}q - \frac{1}{8}pb$; and so will be obtain'd the Points upon which fall the Perpendiculars from the Center to the Curve of the Parabola. Now having found the three Values of this y; the Quantity r ought to be less than $\frac{3}{256}b^4 + \frac{1}{4}bq - \frac{1}{16}bbp + 3y^4 - \frac{3}{8}b^2y^2 + pyy$ of the *middle* y, but greater than $\frac{3}{256}b^4 + \frac{1}{4}bq - \frac{1}{16}bbp + 3y^4 - \frac{3}{8}b^2y^2 + pyy$ of the *least* y. But if r exceed these Limits, there can be but two Roots obtain'd. Lastly, if $\frac{3}{256}b^4 + \frac{1}{4}bp - \frac{1}{16}bbp + 3y^4 - \frac{3}{8}bby + pyy$ of the *greatest* y, be greater than r, then the Equation propos'd is im-

impossible. It happens also that there are four Affirmative Roots, when the Center G is posited in the little space UTS, *viz.* drawing RTS perpendicular upon the middle of the supposed Line AD. But this comes to pass when p is greater than $\frac{1}{16}bb$, and $\frac{1}{4}bb - \frac{2}{3}p\sqrt{\frac{1}{16}bb - \frac{1}{6}p}$ greater than $\frac{1}{8}pb - \frac{1}{128}b3 - \frac{1}{2}q$. In which case always two, sometimes three of the Roots are greater than $\frac{1}{4}b$.

But 'tis to be noted here that that *Limit* produced from the *least y*, is sometimes Negative, or less than nothing; *viz.* as often as the greatest of the three Perpendiculars is greater than GD *(Fig. 2.)* If this happens, the Quantity $+r$ may be diminish'd to nothing from the Limit prescrib'd, by the *middle y*. But the defect of a *Limit* from the *least y*, shews how great $-r$ may be in the Equation, if there be three Affirmative Roots and one Negative one; which if it exceeds, there can be but two, one Affirmatsve and the other Negative. And all these things are demonstrated from hence, that *the forementioned Limits of the Quantity r, are the differences of the Squares of the Line GD, and the Perpendiculars to the Curve of the Parabola.*

But because of the perplexing Cautions arising from the diversity of Sines with these Equations, 'tis better always to take away the second Term, and then to inquire out the number of Roots and the Sines, according to the Rules already deliver'd; especially if those Quantities y are not much different from one another. But of these four Affirmative Roots,

two

two are always leſs than $\frac{1}{4}b$, and two greater, *viz.* if DG be leſs than AG, or $\frac{1}{4}pb$ than $\frac{3}{64}b^3 + q$. But three are always leſs than $\frac{1}{4}b$, as often as the *mean Perpendicular* (or that found from the *mean y*) is greater than AG, or $\frac{3}{8}bby$ greater than $3y^3 - pyy$ of the ſame *mean y*. The fourth and greateſt Root is greater than the greateſt $y + \frac{1}{4}b$; and 'tis equal to the difference of b and the Sum of the other three Roots, and therefore is leſs than b.

But 'tis now time to have done with this Matter. Perhaps thoſe that more perfectly underſtand the Nature of the Parabola, may be able to do all theſe things after a more compendious manner. But there is ſome cauſe to doubt, whether all theſe Quantities b. p, q. r. can be rightly determin'd without the Reſolution of a Cubick Equation, or no. For whatſoever is done in Plain Equations in this Matter, exhibits, not the *true Limits*, but ſome Approximations only.

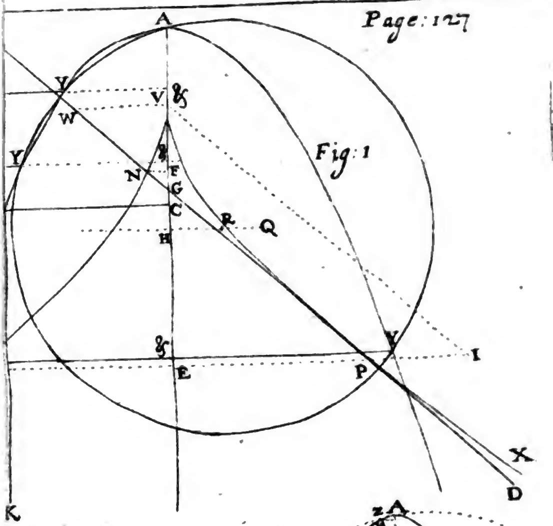
Some

Some Illuſtrious Specimens of the Doctrine of Fluxions; or Examples by which is clearly ſhewn the Uſe and Excellency of that Method in ſolving Geometrical Problems.

By Ab. De Moivre.

YOU have here alſo the Method which I promis'd, concerning the Quadratures of Curvilineal Figures, the Dimenſion of the Solids generated by the Relation of a Plane (and of the Surfaces) the Rectification of Curve Lines, and the Calculation of the Center of Gravity. I know theſe Points have been already handled by ſeveral very learned Men——— But I hope this Attempt of mine will nevertheleſs not altogether diſpleaſe, if (eſpecially) I have had the good Luck to find a ſhorter and more expeditious way to theſe things, than what is commonly known.

But before I proceed farther, I would have it obſerv'd, that I make uſe here, of what the celebrated Mr. *Newton* has demonſtrated, *Page* 251, 252, 253. *Princ. Phil.* concerning *the Momentaneous Increments or Decrements, of Quantities that Increaſe or Decreaſe*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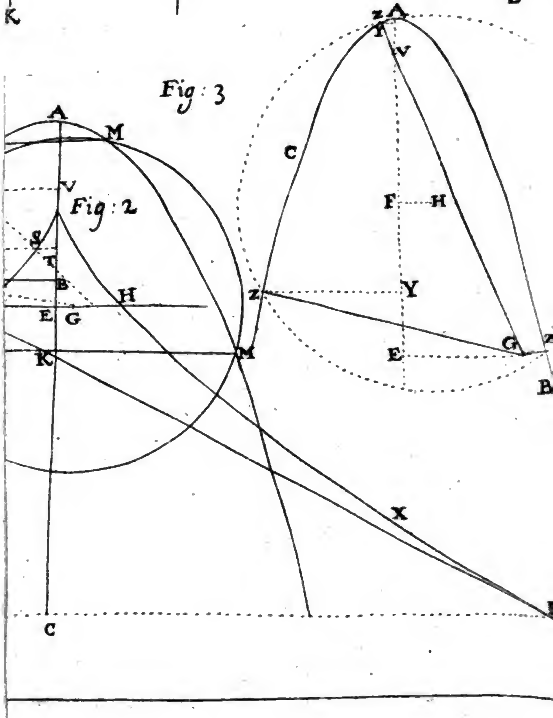
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creaſe by a continual Flux; Eſpecially, that the Momentane of any Power, as $A^{\frac{n}{m}}$ is $\frac{n}{m}\,a\,A^{\frac{n-1}{m}}$. Farther; the Fluxion $\frac{n}{m}\,a\,A^{\frac{n-1}{m}}$ being given, the flowing Quantity $A^{\frac{n}{m}}$ may be found; Firſt, By ſtriking *a* out of the Fluxion; Secondly, By encreaſing the *Index* of the Fluxion by Unity; Thirdly, By dividing the Fluxion by the Index thus increaſed by Unity. In the following Diſcourſe, we ſhall expreſs *Abſciſſe* of any Curve by x, its Fluxion by $\dot{x}$, the *ordinate* by y, and its Fluxion by $\dot{y}$. Theſe things ſuppos'd; that we may come to the *Quadrature of Curves*, Firſt, Take the value of the ordinate applicate, by the help of the Equation expreſſing the Nature of the Curve. Secondly, Let this Value be multiplied by the Fluxion of the Abſciſſe; for the Product ariſing is the Fluxion of the *Area*. Thirdly, Having the Fluxion of the Area, let the flowing Quantity be found, and ſo we ſhall have the Area ſought. *Ex. gr.* Let the Equation $x^m = y^n$ be propos'd, which expreſſes the Nature of all ſorts of Paraboloids. The Value of y is $x^{\frac{m}{n}}$, which multiplied by

$\dot{x}$, gives $x^{\frac{m}{n}}\dot{x}$ for the Fluxion of the Area, and consequently the Area sought is $\frac{n}{m+n}x^{\frac{m+1}{n}}$, or (substituting y instead of $x^{\frac{m}{n}}$) $\frac{n}{m+n}xy$.

Again, suppose a Curve, whose Equation is $x^4+aaxx=yy$ (which is the first of the Excellent Mr. *Craig*'s Examples) putting $y=x\sqrt{xx+aa}$, the Fluxion of the Area will be $\dot{x}x\sqrt{xx+aa}$. Which Expression involving a *surd* Quantity, let us suppose $\sqrt{xx+aa}=z$, then will $xx+aa=z^2$, and consequently $x\dot{x}=z\dot{z}$; and substituting $z\dot{z}$ and z for $x\dot{x}$ and $\sqrt{xx+aa}$, the Fluxion thus freed from Surds, will be $z^2\dot{z}$; which reduced to its Original $\frac{1}{3}z^3$ and putting $\sqrt{xx+aa}$ for z, we have $\frac{1}{3}\overline{xx+aa}\sqrt{xx+aa}$ for the Area sought.

But to shew more effectually how easily these Quadratures are perform'd, I shall add one Example more. Let the Equation of the Curve be $\frac{x^2}{x+a}=y^2$, therefore $y=\frac{x}{\sqrt{x+a}}$, and therefore $\frac{\dot{x}x}{\sqrt{x+a}}$ is the Fluxion of the Area.

Area. Put $\sqrt{x+a} = z$, hence $x = zz - a$, and $\dot{x} = 2z\dot{z}$. Therefore $\frac{x\dot{x}}{\sqrt{x+a}} = 2z^2\dot{z} - 2a\dot{z}$, and consequently $\frac{2}{3}z^3 - 2az$, or $\overline{\frac{2}{3} - \frac{4}{3}a}\,\sqrt{x+a}$ will be the Area sought.

But it often happens that we meet with some Curves (such as the Circle and Hyperbola) which are of such a Nature, that 'tis in vain to attempt the freeing the Fluxions of them from *Surds*. And then reducing the Value of the ordinate into an infinite Series, and multiplying the several Terms of the Series into the Fluxion of the Absciffe (as before) let the Fluent of each of those Terms be found, and so there will arise a new Series, which will exhibit the Quadrature of the Curve.

This Method is with the same ease applied *to the Mensuration of the Solids* generated by the Rotation of a Plane; *viz.* taking for their Fluxions, the Product of the Fluxion of the Absciffe into the circular *Basis*. Let the Proportion of a Square to the inscrib'd Circle be $\frac{n}{1}$. The Equation expressing the Nature of a Circle is $yy = dx - xx$; therefore $4\,\frac{dx\dot{x} - x^2\dot{x}}{n}$ is the Fluxion of a Portion

 of

of the Sphere, and consequently the Portion it self is $4\overline{\frac{1}{2}dx^2 - \frac{1}{3}x^3}$, and the circumscrib'd

n

Cylinder is $4\overline{dx^2 - x^3}$. Therefore the Por-

n

tion of the Sphere is to the circumscrib'd Cylinder, is as $\frac{1}{2}d - \frac{1}{3}x$ to $d - x$.

The *Rectification of Curve Lines* will be obtain'd, if we consider the Fluxion of the Curve as a Hypothenuse of a Rectangular Triangle, whose sides are the Fluxions of the Ordinate and Abscisse. But in the Expression of this Hypothenuse, care must be taken that only one of the Fluxions be remaining, as also only one of the indeterminate Quantities, *viz.* that whose Fluxion is retain'd. Some Examples will render this clear.

(Fig. 1.*)* The right Sine CB being given, to find the Arch AC. Let AB $= x$. CB $= y$. OA $= r$. CE the Fluxion of the Abscisse, ED the Fluxion of the Ordinate, CD the Fluxion of the Arch CA. From the Property of the Circle $2rx - xx = yy$, whence $2r\dot{x} - 2x\dot{x} = 2y\dot{y}$, and therefore $\dot{x} = \frac{y\dot{y}}{r - x}$. But

$$CD^q = \dot{y}\dot{y} + \dot{x}\dot{x} = \dot{y}\dot{y} + \frac{y^2\dot{y}\dot{y}}{rr - 2rx + xx} = \dot{y}\dot{y} + \frac{y^2\dot{y}\dot{y}}{rr - yy} = \frac{rr\dot{y}\dot{y}}{rr - yy};$$

therefore $CD = \frac{r\dot{y}}{\sqrt{rr - yy}}$

=

$= \frac{1}{\sqrt{rr - yy}} \times r\dot{y} = r\dot{y} \times \overline{rr - yy}^{\,-\frac{1}{2}}$. And consequently if $\overline{rr - yy}^{\,-\frac{1}{2}}$ be thrown into an infinite Series, and the several Members of it be multiplied into $r\dot{y}$, and then the flowing Quantity of each be taken, we shall have the length of the Arch AC. After the same manner, giving the versed Sine, the same Arch may be found. For resuming the Equation found above $2r\dot{x} - 2x\dot{x} = 2y\dot{y}$, we have $\dot{y} = \frac{r\dot{x} - x\dot{x}}{yy}$, but $CD^q = \dot{x}\dot{x} + \dot{y}\dot{y} = \dot{x}\dot{x} + \frac{rr\dot{x}\dot{x} - 2rx\dot{x}\dot{x} + x^2\dot{x}\dot{x}}{yy} = \dot{x}\dot{x} + \frac{rr\dot{x}\dot{x} - 2rx\dot{x}\dot{x} + x^2\dot{x}\dot{x}}{2rx - xx}$, that is, (reducing all to the same Denominator, and expunging contradictory Terms) $\frac{rr\dot{x}\dot{x}}{2rx - xx}$, whence $CD = \frac{r\dot{x}}{\sqrt{2rx - xx}}$, and consequently the length of the Arch A C may be easily found from what is said already.

 The

The Fluxion of the Curve Line is sometimes more easily found by comparing the two similar Triangles CED, CBO, for this Proportion arises, CB : CO :: CD, that is for the Circle $\sqrt{2rx - xx} : r :: \dot{x} : \frac{r\dot{x}}{\sqrt{2rx - xx}}$.

The Curve of the Cycloid may be determin'd by the same Method too. Let *(Fig. 2.)* ALK be a Semicycloid, whose generating Circle is ADL. Having any point as B in the Diameter AL, draw BI parallel to the Base LK meeting the Peripheus of the Circle in the point D; compleat the Rectangle AEIB, and draw FH parallel to EI and infinitely near to it, as also BI cutting FH in G, and the Curve AK in H. Put AL $= d$. AB (= EI) $= x$. GH $= \dot{x}$. It is known that the right Line BG is every where equal to the Sum of the Arch AD and the right Sine BD; whence 'tis manifest, that the Fluxion IG is also the Aggregate of the Fluxions of the Arch AD and the right Sine BD. But the Fluxion of the Arch AD was found $= \frac{\frac{1}{2} d\dot{x}}{\sqrt{dx - xx}}$, and the Fluxion of the right Sine BD will be found to be $\frac{d\dot{x} - 2x\dot{x}}{2\sqrt{dx - xx}}$; therefore IG $= \frac{d\dot{x} - x\dot{x}}{\sqrt{dx - xx}}$, and therefore

IH:

$IH^q (= IG^q + GH^q) = \frac{dd\dot{x}\dot{x} - dx\dot{x}\dot{x}}{dx - xx}$; from

whence $IH = \frac{\dot{x}\sqrt{dd - dx}}{\sqrt{dx - xx}} = \frac{\dot{x}\sqrt{d}}{\sqrt{x}} = d^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}$,

and consequently $AI = 2d^{\frac{1}{2}} x^{\frac{1}{2}} = 2\sqrt{dx} = 2AD$. This Conclusion may also very easily be deduc'd from the known Property of the Tangent. For since the little part of it, as IH, is always parallel to the Chord AD, the Triangles IGH, ABD are similar, whence AB : AD :: GH : IH, that is, $x : \sqrt{dx} :: \dot{x} : \frac{\dot{x}\sqrt{dx}}{x}$, therefore $IH = \frac{\dot{x}\sqrt{dx}}{x} = d^{\frac{1}{2}} x^{-\frac{1}{2}} \dot{x}$. By the help of the Fluxion IH also, we may investigate the Area of the Cycloid. The Fluxion of the Area AEI, is the Rectangle $EIG = \frac{dx\dot{x} - x^2\dot{x}}{\sqrt{dx - xx}} = \dot{x}\sqrt{dx - xx}$. But the Fluxion of the Portion ABD is the same; therefore the Area AEI and the correspondent Portion (of the Circle) ABD, are always equal.

Let AB (*Fig.* 3.) be the Curve of the Parabola, whose Axis is AF, Parameter a; let $AE = x$, $EB = y$, $AB = z$, $BD = \dot{x}$, $DC = \dot{y}$, $BC = \dot{z}$. The Equation expressing the Nature of the Parabola, being $ax = yy$, we have $a\dot{x} = 2y\dot{y}$, whence $\dot{x} = \frac{2y\dot{y}}{a}$; but $BC^q = BD^q + CD^q$, that is $\dot{z}\dot{z} = \dot{x}\dot{x} + \dot{y}\dot{y} = \frac{4y^2\dot{y}\dot{y}}{aa} + \dot{y}\dot{y} = \frac{4y^2\dot{y}\dot{y} + aa\dot{y}\dot{y}}{aa}$, and therefore $\dot{z} = \dot{y}\frac{\sqrt{4y^2 + aa}}{a} = \dot{y}\frac{\sqrt{y^2 + \frac{1}{4}aa}}{\frac{1}{2}a}$. If now by this Expression $\dot{y}\frac{\sqrt{y^2 + \frac{1}{4}aa}}{\frac{1}{2}a}$ be thrown into an infinite Series, the Curve AB will easily be known. It appears farther, that giving an *Hyperbolical Space*, this Curve is also given, and *vice versa*. For $\frac{1}{2}a\dot{z} = \dot{y}\sqrt{y^2 + \frac{1}{4}aa}$, and consequently $\frac{1}{2}az$ is the Space whose Fluxion is $\dot{y}\sqrt{y^2 + \frac{1}{4}aa}$. But such a Space is no other than the Exteriour (Equilateral) Hyperbola ABEG, whose Semiaxis $AB = \frac{1}{2}a$, its Absciffe $AE = y$, and its Ordinate $EG = x$.

For

For the Mensuration of *a surface describ'd by the Conversion of a Curve round its Axis*; we are to assume for the Fluxion of it, a Cylindrick Superficies, whose Altitude is the Fluxion of the Curve, and whose distance from the Axis is the Ordinate Applicate corresponding to that Fluxion. *Ex. gr.* Let AC be the Arch of a Circle, which turning round the Axis AD, generates a spherical Superficies, which we would measure. Now DC the Fluxion of the Arch is already found to be $= \frac{r\dot{x}}{\sqrt{2rx-xx}}$, which if we multiply by the Periphery belonging to the Radius BC, that is, by $\frac{c}{r}\sqrt{2rx-xx}$ (putting $\frac{c}{r}$ the Ratio of the Circumference to the Radius) we shall have $c\dot{x}$ for the Fluxion of the spherical Superficies, and consequently that Superficies it self, is cx.

As for *Centers of Gravity*; having gotten the Fluxion of the Solid or Surface, and multiplied the same into its distance from the *Vertex*, the flowing Quantity must be found, which divided by the Solid or Surface it self, the Quotient will shew the distance of the Center of Gravity from the Vertex. Thus to find the Center of Gravity of all the Paraboloids; their Fluxion is thus generally expressed

expressed $x^{\frac{m}{n}}\dot{x}$, which multiplied by x, makes $x^{\frac{m+1}{n}}\dot{x}$, the flowing Quantity of which, *viz.* $\frac{n}{m+2n}x^{\frac{m+2}{n}}$ divided by the Area of the Paraboloid $\frac{n}{m+n}x^{\frac{m+1}{n}}$, gives $\frac{m+2n}{m+2n}x$, the distance of the Center of Gravity from the Vertex.

The Center of Gravity of a Portion of a Sphere, is found after the same manner. For its Fluxion $4\frac{\overline{dx\dot{x}-x^{2}\dot{x}}}{n}$ multiplied into x, makes $4\frac{\overline{dx^{2}\dot{x}-x^{3}\dot{x}}}{n}$, whose flowing Quantity $4\frac{\overline{\frac{1}{3}dx^{3}-\frac{1}{4}x^{4}}}{n}$, divided by the solid Content of the Portion, *viz.* $4\frac{\overline{\frac{1}{2}dxx-\frac{1}{3}x^{3}}}{n}$, gives $\frac{\frac{1}{3}d-\frac{1}{4}x}{\frac{1}{2}d-\frac{1}{3}x}x$, or $\frac{4d-3x}{6d-4x}x$, the distance of the Center of Gravity from the Vertex.

My

My design here was not to be large, and pursue all the Difficulties that may occur. 'Tis sufficient to have made a beginning, and led the Way to those greater Things.

A Me-

A Method of Squaring some sorts of Curves, or Reducing them to more simple Curves.

By A. De Moivre, *R.S.S.*

LET *A* be the Area of a Curve, whose Abscisse is x, and Ordinate Applicate $x^{m}\sqrt{dx - xx}$. Let *B* be the Area of a Curve, whose Absciss is the same with the former, and its Ordinate $x^{m-n}\sqrt{dx - xx}$. Put $\sqrt{dx - xx} = y$. Then will $A = d^{n}B$ into $\frac{2m+1}{2m+4}$ into $\frac{2m-1}{2m+2}$ into $\frac{2m-3}{2m}$ into $\frac{2m-5}{2m-2}$ &c. $=$ P

$$-\frac{1}{m+2}x^{m-1}y^{3} = -Q$$

$$-\frac{d}{m+1}\text{ into }\frac{2m+1}{2m+4}x^{m-2}y^{3} = -R$$

$$-\frac{d^{2}}{m}\text{ into }\frac{2m+1}{2m+4}\text{ into }\frac{2m-1}{2m+2}x^{m-3}y^{3} = -S$$

$$-\frac{d^{3}}{m-1}\text{ into }\frac{2m+1}{2m+4}\text{ into }\frac{2m-1}{2m+2}\text{ into }\frac{2m-3}{2m}$$

$$x^{m-4}y^{3} = -T, \&c.$$

Where we are to Note, 1. That n is suppos'd to be an Integer and Affirmative Number. 2. That the Quantity $d^{n}B$, in the Series

Series expressed by P, is to be multiplied into as many Terms as there are Unities in n. 3. That as many of the following Series expressed by $-Q$, $-R$, $-S$, $-T$, &c. are to be taken, as there are Unities in n. Which to illustrate by an Example or two: If $n=1$, the I say that $A = d^nB$ into $\frac{2m+1}{2m+4} - \frac{1}{m+2} x^{m-1} y^3$; and if $n=2$, then $A = d^nB$ into $\frac{2m+1}{2m+4}$ into $\frac{2m-2}{2m+1} - \frac{1}{m+2} x^{m-1} y^3 - \frac{d}{m+1}$ into $\frac{2m+1}{2m+4} x^{m-2} y^3$. 4. That if y be put $=\sqrt{dx - xx}$, then A will $= Q - R + S - T$, &c. $\pm P$.

COROL. I.

If m be put $=$ to any Term of the following Series $-\frac{1}{2}, \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$, &c. then the Quadrature of the Curve whose Ordinate is $x^m\sqrt{dx - xx}$, or $x^m\sqrt{dx + xx}$, will be expressed in finite Terms, and be found by our Series. To illustrate which by an Example or two; Let it be required to find the Area of a Curve whose Ordinate is $x^{-\frac{1}{2}}\sqrt{dx - xx}$. Let us imagine this Curve to be compar'd with another Curve whose Ordinate is $x^{-\frac{3}{2}}\sqrt{dx - xx}$. Now because in this case $n=1$, therefore will $A = d^nB$ into $\frac{2m+1}{2m+4} - \frac{1}{m+2} x^{m-1} y^3$; but $m = -\frac{1}{2}$, therefore

fore $2m+1=o$, and therefore $A=\frac{1}{m+2}$ $x^{m-1}y^3=-\frac{2y^3}{3\sqrt{x^3}}$.

It is here to be obferv'd, that the Area thus found, is fometimes deficient from the true Area, by a *given* Quantity, or exceeds it by that fame given Quantity. And in order to find that *Defect* or *Excefs*, let the Area found be fuppos'd to be encreas'd or diminifh'd, by a given Quantity q, and then putting $x=o$, let the Area increas'd or diminifh'd, be fuppos'd $=o$. Thus in the prefent cafe, we fhall find $q=\frac{2}{3}d\sqrt{d}$, and confequently $A=\frac{2}{3}d\sqrt{d}-\frac{2y^3}{3\sqrt{x^3}}$.

COROL. II.

If n be put equal to any Term of the following Series, 3, 4, 5, 6, 7, &c. then the Quadrature of the Curve whofe Ordinate is $x^{-n}\sqrt{dx-xx}$, or $x^{-n}\sqrt{dx+xx}$, is expreffed in finite Terms, and is found by our Series.

Let the Area of the Curve be to be found, whofe Ordinate is $x^{-3}\sqrt{dx-xx}$. Suppofe it to be compared with the Area of a Circle, which call *A*. Then will $m=o$, $n=3$, and fo $A=P-Q-R-S$. But fince, in the Denominator of the third Term by which d^nB is multiplied, there is found

2 *m*,

$2m$, a Quantity infinitely small, or rather nothing; the Quantity express'd by P is Infinite; and for the same reason the Quantity express'd by $-S$ is Infinite, and so the Quantities A, $-Q$, $-R$, do vanish. Therefore $P=S$, and dividing the Equation by $\frac{2m+1}{2m+4}$ into $\frac{2m-1}{2m+2}$, we have d^nB into $\frac{2m-3}{2m}$ $= \frac{dd}{m} x^{m-3} y^3$, or dB into $\frac{2m-3}{2} = ddx^{m-3} y^3$; and putting o and 3 for m and n, there will be dB into $-\frac{3}{2} = \frac{y^3}{x^3}$, or $B = -\frac{2y^3}{3x^3}$.

COROL. III.

If m be put equal to any Term of the following Series, $-2, -1, 0, 1, 2, 3, 4, 5,$ &c. the Quadrature of the Curve, whose Ordinate is $x^m \sqrt{dx - xx}$, depends upon the Quadrature of the Circle. But the Area of the Curve, whose Ordinate is $x^m \sqrt{dx + xx}$, depends upon the Quadrature of the Hyperbola; and the relation of that Curve to the Circle of the Hyperbola, is exhibited by our Series in finite Terms.

COROL.

COROL. IV.

If m be expounded by any other Term, different from them already mention'd; the Curve whose Ordinate is $x^m\sqrt{dx - xx}$, or $x^m\sqrt{dx + xx}$, is neither exactly squar'd, nor does it depend upon the Circle or Hyperbola, but is reduced to a more simple Curve by our Series.

THEOR. II.

Let A be the Area of a Curve whose Abscisse is x, and Ordinate $\frac{x^m}{\sqrt{dx - xx}}$. Let B be the Area of a Curve whose Abscisse is the same with the former, but the Ordinate $\frac{x^{m-n}}{\sqrt{dx - xx}}$. Let $\sqrt{dx - xx} = y$. Then will

A =

$A = d^m B$ into $\frac{2m-1}{2m}$ into $\frac{2m-3}{2m-2}$ into $\frac{2m-5}{2m-4}$ into $\frac{2m-7}{2m-6}$ &c. $= P$.

$-\frac{1}{m} x^{m-1} y = -Q.$

$-\frac{d}{m-1}$ into $\frac{2m-1}{2m} x^{m-2} y = -R.$

$-\frac{dd}{m-2}$ into $\frac{2m-1}{2m}$ into $\frac{2m-3}{2m-2} x^{m-3} y = -S.$

$-\frac{d^3}{m-3}$ into $\frac{2m-1}{2m}$ into $\frac{2m-3}{2m-2}$ into $\frac{2m-5}{2m-4} x^{m-4} y = -T$, &c.

The Observations to the first *Theorem*, take place here also, as in what follows.

COROL. I.

If m be put equal to any Term of the following Series, $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \frac{9}{2}$, &c. the Quadrature of the Curve whose Ordinate is $\frac{x^m}{\sqrt{dx - xx}}$, or $\frac{x^m}{\sqrt{dx + xx}}$, is expressed in finite Terms, and exhibited by this Series.

COROL. II.

If n be put equal to any Term of the following Series, 1, 2, 3, 4, 5, 6, 7, &c. Every Curve whose Ordinate is $\frac{x^{-n}}{\sqrt{dx - xx}}$, or $\frac{x^{-n}}{\sqrt{dx + xx}}$, is squared by this Series in finite Terms.

COROL. III.

If m be expounded by any Term of the following Series, 0, 1, 2, 3, 4, 5, 6, 7, &c. the Curve whose Ordinate is $\frac{x^{m}}{\sqrt{dx - xx}}$, depends upon the Quadrature of the Circle. But the Curve whose Ordinate is $\frac{x^{m}}{\sqrt{dx + xx}}$, depends upon the Quadrature of the Hyperbola.

For if on the Center C, (*Fig.* 20.) the Diameter AB $= d$, the Circle AEB be described, and AD be taken $= x$, also erecting the Perpendicular DE, the Line CE be drawn. Then the Se-

Sector AEC divided by $\frac{1}{8}dd$ is equal to the Area of the Curve whose Ordinate is

$$\frac{x^0}{\sqrt{dx-xx}}.$$

After the same manner, if on the Center C, and the Transverse Axis AB$=d$, the Equilateral Hyperbola AE be described, and taking AD$=x$, and erecting DE at right Angles, and joining CE, the Sector ACE divided by $\frac{1}{8}dd$ is equal to the Area of the Curve whose Ordinate is

$$\frac{x^0}{\sqrt{dx+xx}}.$$

COROL. IV.

If m be equal to any Term, that does not fall into the foregoing Limitations, then the Curve whose Ordinate is $\frac{x^m}{\sqrt{dx\pm xx}}$, is neither exactly squar'd, nor does it depend upon the Circle or Hyperbola, but is reduced to a more simple Curve.

THEOR. III.

Let A be the Area of a Curve whose Abscisse is x, and its Ordinate Applicate $x^{m}\sqrt{rr-xx}$; let B be the Area of a Curve whose Abscisse is also x, and its Ordinate $x^{m-2n}\sqrt{rr-xx}$.

Let $\sqrt{rr-xx}=y$. Then will $A=r^{n2}B$ into $\frac{m-1}{m+1}$ into $\frac{m-3}{m}$ into $\frac{m-5}{m-2}$ into $\frac{m-7}{m-4}$ &c.$=P$.

$-\frac{1}{m+2}x^{m-1}y^{3}=-Q.$

$-\frac{rr}{m}$ into $\frac{m-1}{m+2}x^{m-3}y^{3}=-R.$

$-\frac{r^{4}}{m-2}$ into $\frac{m-2}{m+2}$ into $\frac{m-3}{m}x^{m-5}y^{3}=-S.$

&c.

COROL. I.

If m be expounded by any Term of the following Series, 1, 3, 5, 7, 9, &c. the Quadrature of the Curve whose Ordinate is $x^{m}\sqrt{rr-xx}$, or $x^{m}\sqrt{rr+xx}$, is had in finite Terms, and that by the help of this Theorem.

COROL.

COROL. II.

If n be expounded by any Term of the following Series, 2, 3, 4, 5, 6, &c. then the Curve whose Ordinate is $x^{-2n}\sqrt{rr-xx}$, or $x^{-2n}\sqrt{rr+xx}$, is exactly squar'd by this Theorem.

COROL. III.

If m be expounded by any Term of the following Series, $-2, 0, 2, 4, 6, 8$, &c. then the Quadrature of the Curve whose Ordinate is $x^{m}\sqrt{rr-xx}$, depends upon the Circle, but the Quadrature of the Curve whose Ordinate is $x^{m}\sqrt{rr+xx}$, depends upon the Hyperbola.

COROL. IV.

If m be expounded by any Term different from those already taken notice of; then the Curve whose Ordinate is $x^{m}\sqrt{rr-xx}$, or $x^{m}\sqrt{rr+xx}$, depends neither upon the Circle nor the Hyperbola, but is reduced to a more simple Curve.

THEOR. IV.

Let A be the Area of a Curve whose Abscisse is x, and whose Ordinate is $\frac{x^m}{\sqrt{rr-xx}}$; let B be the Area of a Curve whose Abscisse is also x, and its Ordinate $\frac{x^{m-2n}}{\sqrt{rr-xx}}$. Then will $A = r^{2n}B$ into $\frac{m-1}{m}$ into $\frac{m-3}{m-4}$ into $\frac{m-5}{m-4}$ into $\frac{m-7}{m-6}$ &c. $= P$.

$- \frac{1}{m} x^{m-1} y = - Q.$

$- \frac{rr}{m-2}$ into $\frac{m-1}{m} x^{m-3} y = - R.$

$- \frac{r^4}{m-4}$ into $\frac{m-1}{m}$ into $\frac{m-3}{m-2} x^{m-5} y = - S.$

$- \frac{r^6}{m-6}$ into $\frac{m-1}{m}$ into $\frac{m-3}{m-2}$ into $\frac{m-5}{m-4} x^{m-7} y$ $= - T.$ &c.

COROL.

COROL. I.

If m be expounded by any Term of the following Series, 1, 3, 5, 7, 9, &c. the Quadrature of the Curve, whose Ordinate is $\frac{x^m}{\sqrt{rr+xx}}$, is obtained in finite Terms by this Theorem.

COROL. II.

If n be expounded by any Term of the following Series, 1, 2, 3, 4, 5, 6, &c. the Curve, whose Ordinate is $\frac{x^{-2n}}{\sqrt{rr-xx}}$ or $\frac{x^{-2n}}{\sqrt{rr+xx}}$, is squar'd exactly by this Theorem.

COROL. III.

If m be expounded by any Term of the following Series, 0, 2, 4, 6, 8, 10, &c. the Quadrature of the Curve, whose Ordinate is is $\frac{x}{\sqrt{rr-xx}}$, depends upon the Quadrature of the Circle. For if on the Center C, and the Radius CA$=r$, the Circle AEG be deſcrib'd, and taking CD$=x$, DE be erected perpendicular to CD, and CE be drawn; then the Sector CAE divided by $\frac{1}{2}rr$, is equal to the Area of the Curve, whoſe Ordinate is $\frac{x^0}{\sqrt{rr-xx}}$. In like manner, *(Fig.*21.*)* if on the Center C, and the Semitranſverſe Axis CA$=r$, the Equilateral Hyperbola EAM be deſcrib'd, then drawing CF perpendicular to CA, equal to x, FE parallel to the Axis till it meets the Hyperbola in E, and join CE; then the Hyperbolical Sector ACE divided by $\frac{1}{2}rr$, is equal to the Area of the Curve, whoſe Ordinate is $\frac{x^0}{\sqrt{rr+xx}}$.

COROL. IV.

If m be expounded by any Term different from the foregoing, then the Curve, whoſe Ordinate is $\frac{x^m}{\sqrt{rx\pm xx}}$, is neither ex-

actly

actly squar'd, nor does it depend upon the Circle or the Hpperbola, but is reduced to a more simple Curve.

THEOR. V.

Let A be the Area of a Curve, whose Abscisse is x, and its Ordinate $\frac{x^m}{d-x}$; let B be the Area of a Curve, whose Abscisse also is x, and its Ordinate $\frac{x^{m-n}}{d-x}$. Then will A=

$$d^n B - \frac{x^m}{m} - \frac{dx^{m-1}}{m-1} - \frac{ddx^{m-2}}{m-2} \&c.$$

Let the Ordinate be $\frac{x^m}{d+x}$, and then the Area A will $= \frac{x^m}{m} - \frac{dx^{m-1}}{m-1} + \frac{ddx^{m-2}}{m-2} \&c. \pm d^n B$.

COROL.

If m be expounded by any Term of the following Series, 1, 2, 3, 4, 5, 6, &c. the Quadrature of the Curve, whose Ordinate is $\frac{x^m}{d-x}$ or $\frac{x^m}{d+x}$, depends upon the Quadra-

ture

ture of the Hyperbola. For (See *Fig.* 22.) drawing DE, EF at right Angles, take EG $=d$, and draw GH at right Angles to EF, and equal to it. Within the Asymptotes DE, EF, let an Hyperbola be describ'd, passing through the point H; which done, take GK$=x$, towards E in the first Case, and towards F in the second; and draw the Ordinate KL. Then the Area HGKL divided by dd, is equal to the Area of the Curve, whose Ordinate is $\frac{x^0}{d-x}$ or $\frac{x^0}{d+x}$. Hence the Solid generated by a Portion of the *Cissoid*, while it turns about the Diameter of the Generating Circle, is exhibited in finite Terms, supposing the Quadrature of the *Hyperbola*.

THEOR. VI.

Let A be the Area of a Curve, whose Absciſſe is x, and Ordinate $\frac{x^m}{rr+xx}$; let B be the Area of a Curve, whose Absciſſe is also x, and Ordinate $\frac{x^{m-2n}}{rr+xx}$. Then will the

$$\text{Area A} = \frac{x^{m-1}}{m-1} - \frac{rrx^{m-3}}{m-3} + \frac{r^4x^{m-5}}{m-5} \text{ \&c.}$$

$\pm r^{2n}$B.

COROL.

COROL.

If m be expounded by any Term of the following Series, 0, 2, 4, 6, 8, &c. the Quadrature of the Curve, whoſe Ordinate is $\frac{x^m}{rr+xx}$, depends upon the Rectification of the Arch of a Circle. For deſcribing the Circle AEG on the Center C, with the Radius $CA=r$, draw the Tangent $AK=x$, and join CK meeting the Periphery in E; then the Arch AE divided by rr is equal to the Area of the Curve, whoſe Ordinate is $\frac{x^0}{rr+xx}$.

General Corollaries *to theſe ſix* Theorems.

EVery Mechanick Curve (whoſe Quadrature depends upon any one of that Infinite Number of Curves, whoſe Ordinates have the following Forms, $x^m\sqrt{dx\pm xx}$, $\frac{x^m}{\sqrt{dx\pm xx}}$, $x^m\sqrt{rr\pm xx}$, $x^m\sqrt{rr\mp xx}$, $\frac{x^m}{d\pm x}$, $\frac{x^m}{rr+xx}$) may be ſquar'd by theſe Series. It may ſuffice to illuſtrate this by an Example.

Sup-

Supposing the Cube of the Arch of a Circle (corresponding to the versed Sine) to be the Ordinate of a Curve, whose Abscisse is the same versed Sine; let it be requir'd to find the Area of this Curve.

Let the Abscisse be x, the Circular Arch v; then the Fluxion of the Area is $v^3\dot{x}$. Let the Area be v^3x-q. Therefore $v^3\dot{x}+3v^2\dot{v}x-\dot{q}=v^3\dot{x}$, whence $\dot{q}=3v^2\dot{v}x$. But $\dot{v}=\frac{d\dot{x}}{2\sqrt{dx-xx}}$, therefore $\dot{q}=\frac{3dv^2x\dot{x}}{2\sqrt{dx-xx}}$. But (by *Theor.* 2.) $\frac{x\dot{x}}{\sqrt{dx-xx}}=\frac{d\dot{x}}{2\sqrt{dx-xx}}-\dot{y}=\dot{v}-\dot{y}$; and consequently $\dot{q}=\frac{3}{2}dv^2\dot{v}-\frac{3}{2}dv^2\dot{y}$, therefore $q=\frac{1}{2}dv^3-s:dv^2\dot{y}$. Therefore now we are come to this, that the Fluent, of the Expression $\frac{3}{2}dv^2\dot{y}$, is to be found.

Let this Fluent be $\frac{3}{2}dv^2y-r$.

Therefore $\frac{3}{2}dv^2\dot{y}+3dv\dot{v}y-\dot{r}=\frac{3}{2}dv^2\dot{y}$.

And consequently $\dot{r}=3dv\dot{v}y=\frac{3}{2}dd\dot{v}x$.
Let $r=\frac{3}{2}ddvx-s$.

Therefore $\frac{3}{2}dd\dot{v}x=\frac{3}{2}dd\dot{v}x+\frac{3}{2}ddx\dot{v}-\dot{s}$.

And

And consequently $\dot{s} = \frac{1}{2}dd x\dot{v} = \frac{3d^3 x\dot{x}}{4\sqrt{dx - xx}}$ $= \frac{3}{4}d^3\dot{v} - \frac{3}{4}d^3\dot{y}$ (by *Theor.* 2.) Therefore now $s = \frac{3}{4}d^3 v - \frac{3}{4}d^3 y$. And consequently the Area sought, is $= v^3 x - \frac{1}{2}dv^3 + \frac{1}{2}dv^2 y - \frac{1}{2}ddvx + \frac{3}{4}d^3 v - \frac{3}{4}d^3 y$.

Since the Solids and Surfaces generated by the Rotations of Curve Lines, as also the Lengths of Curves, and the Centers of Gravity of all these, do depend upon the Quadratures of Curves; 'tis plain, that these are easily obtain'd too, if they depend upon the foremention'd Curves.

After I had compos'd these Theorems, and shewn them to the Celebrated Mr. *Newton*, (as the supream Judge in all Matters of this Nature) he was pleas'd to give me a sight of some Papers of his, by which I find that he has a long time been Master of a Method, by which any Trinomial Equation (expressing the Nature of a Curve) being given, that Curve is either squar'd, or reduced to a more simple one.

And 'twere to be wish'd, that he thought fit to communicate to the Publick, not only those Things which he has relating to these Matters, but others also of his Noble Inventions, which are not a small Number neither. And I believe this is not my Wish alone, but that of the whole Learned World besides.

I make no question but those Learned Persons (whose Writings in the *Acta Eruditorum* and

and otherwhere, have tended ſo much to the Advancement of Mathematicks) have Methods not unlike to this of mine; and therefore I aſcribe no more to my ſelf in this Matter, than only that I found out theſe Theorems, not knowing whether any Body elſe had done ſo before or no; and reduced them into ſo eaſie a Form, that the whole Calculus relating to them, might be taken in, as it were, at one View.

But before I make an end of Writing, I think it improper, if (having not had an Opportunity ſooner) I make ſome little reply to the Famous Mr. *Leibnitz*'s Animadverſions upon my Series for finding the Root of an Infinite Equation.

That Excellent Perſon thinks this Series not to be General enough, as not reaching the Caſes where z and y are multiplied into one another; upon which account he ſubſtitutes another Series in the room of it, which he aſſerts is infinitely more General. But that which led him into this ſmall Miſtake, I gueſs to be this, that he took the Quantities a, b, c, d, &c. for *given* Quantities, whereas they were to be us'd indifferently, either for *given* or *indeterminate* ones.

But I ſhall add one Example to ſhew that my Series extends to all Caſes. Let [illegible] Equation be [illegible]$z - z^3 = y^3$.

In our T[illegible] let $a = ny$; $b = o$ [illegible]
$h = o$, $i = 1$ [illegible]her let $g = yy$, [illegible]

Then in [illegible]ſe will $z =$ [illegible]

$+ \frac{12y^{11}}{n^{10}}$ [illegible]

Two Problems; viz. *concerning the Solid of Least Resistance, and the Curve of Swiftest Descent.*

Solv'd by J. Craig.

LEMMA.

TO find the Proportion between the Resistance made to the Rectangular Triangle AIG, and that made to the circumscrib'd Rectangle AIGg, while each moves in a Fluid, in the direction of the Line IA, from I towards X.

From any point B let there be drawn BC perpendicular to AG, Bb parallel to AI, and BM perpendicular to AI. Then in bB take $bH = \frac{CM^2}{BC}$ and $bE = BC$; and thro' the points *H* and *E*, draw the Lines HA, EA, which being produced cut Gg in *K* and *F*. I say the Resistance of the Triangle AIG is to the Resistance of the Rectangle AIGg, as the Area of the Triangle AKG, to the Area of the Triangle AFg. And also, that the Resistance upon any part of the Line AG, is to the Resistance upon the correspondent part of the Line Ag

Ag (*ex. gr.* upon AB and Ab) as the Area AHB to the Area AEB. The Demonſtration of which depends upon a General Theorem, which I deduced very eaſily from *Prop.* 35. Newt. *p.* 324.

COROL. I.

Let BG, bg, be infinitely ſmall parts of the Lines AG, Ag, and let bB be produced to *L*; I ſay, that the Reſiſtance upon BG (which call *e*) is to the Reſiſtance upon bg (which call *E*) as $GL^2 : GB^2$.

For $e : E :: KHgb : FEgb$; that is, $e : E :: bg \times bH : bg \times bE$ (by the foregoing *Lemma*) therefore $e : E :: bH : bE$; that is, $e : E :: \frac{CM^2}{BC} : BC :: CM^2 : BC^2$. But $CM^2 : BC^2 :: GL^2 : GB^2$ (becauſe of the ſimilar Triangles BMC, GLB.) Therefore $e : E :: GL^2 : GB^2$. Q: E: D.

COROL. II.

The Reſiſtance upon the infinitely ſmall part GB, is $= \frac{GL^3}{GB^2}$. For if all the infinitely ſmall parts in the Line Ag (as bg) be ſuppos'd equal, then the Reſiſtance upon bg, may be expreſs'd by bg, that is $E = bg$, and ſo $E = GL$. Therefore (by *Cor.* 1.) $e : GL :: GL^2 : GB^2$, whence $e = \frac{GL^3}{GB^2}$. Q: E: D.

COROL.

COROL. III.

Let *r* be the Radius, and *c* the Circumference of any Circle. I say, that the Resistance upon the Conick Surface generated by the Rotation of the Lineola GB about AI, is equal to the Product of $\frac{c \times BM}{r}$ into $\frac{GL^3}{GB^2}$. For the Resistance upon that Conick Surface, is equal to all the Resistances upon the Lineola GB, that is, to all the *e*; that is, equal to the Circumference of the Circle (whose Radius is BM) multiplied into *e*; that is, the Resistance upon that Conick Surface, is equal to $\frac{c \times BM}{r} \times e$, and consequently (by *Corol.* 2.) equal to $\frac{c \times BM}{r} \times \frac{GL^3}{GB^2}$.

Q. E. D.

PROB. I.

To find a Curve Line, by the Rotation of which a Round Solid shall be generated, that, while 'tis moved in a Fluid Medium, in the Direction of its Axis, shall meet with the least Resistance.

(Fig. 24.)

Let OG, GB, be two infinitely small Particles in the Curve sought, which rould about its Axis, will produce the Solid of least Resistance. Draw BM, GP, perpendicular to AQ, also BL, GN, parallel to AQ, and ON, parallel to BM.

Now $\frac{c \times BM \times GL^3}{r \times GB^2}$, is the Resistance upon the Surface generated by the Rotation of the Lineola GB about AQ, and $\frac{c \times GP \times ON^3}{r \times GO^2}$, is the Resistance upon the Surface generated OG, in like manner (by *Cor.* 3.) And the Sum of both these Resistances must be a *Minimum*, viz.

$$\frac{c \times BM \times GL^3}{r \times GB^2} + \frac{c \times GP \times ON^3}{r \times OG^2} = \text{a } Minimum.$$

And consequently in the Line RS (so parallel to AQ that ON = GL) the point *G* is to be sought, such, that this may happen; which, supposing the points *O* and *B* to be fix'd, will be easily found by the common Method, *de Maximis & Minimis*. And prosecuting the Calculus, we shall come at last to this Equation, $\frac{BM \times BL}{BG^4} = \frac{GP \times NG}{OG^4}$; whence 'tis plain that $\frac{BM \times BL}{BG^4}$ = a constant Quantity. So that if the Absciss AM = x, and the Ordinate BM = y, then will BL = dx, LG = dy (which I have suppos'd pos'd constant every where in this Calculus) and consequently $BG^2 = dx^2 + dy^2$, whence

$$\frac{ydx}{dxdx + dydy^2} = \text{a constant Quantity.}$$ Let a be any constant Quantity, and consequently (to observe the Law of Homogeneals) we have

$$\frac{ydx}{dxdx + dydy^2} = \frac{a}{dy^3},$$ as has been found by the Illustrious *L' Hospitall*, and the celebrated *Jo. Bernoulli*.

PROB. II.

To find the Line of Swiftest Descent.

(Fig. 25.)

Let BC, CD, be two infinitely small Particles in the Curve sought. Now this Curve ought to be of such a Nature, that, supposing a Body to have fallen from the Horizontal Line AQ, it may pass from *B* to *D* in the *shortest Time.* Therefore we are to find out the Point *C* (in the Line RS drawn in such a manner parallel to AQ, that the differences of the Ordinates GC, DE, may be equal) such that this may come to pass.

Now the Velocity in C is $\sqrt{LC}$, and that in D is $\sqrt{QD}$; therefore $\frac{BC}{\sqrt{LC}}$ is the Time

 of

of Descent thro' BC, $\frac{CD}{\sqrt{QD}}$ is the Time of Descent thro' CD (by *Prop.* 54. *pag.* 158. *Newt.*) Therefore the point C ought to be such that $\frac{BC}{\sqrt{LC}} + \frac{CD}{\sqrt{QD}}$ may be a *Minimum*. Supposing the points B and D to be fix'd, let the constant Quantities GC = DE = m, LC = b, QD = p; the *Indeterminate* Quantities BG = u, CE = z; whence

$$\frac{\sqrt{m^2 + u^2}}{\sqrt{b}} + \frac{\sqrt{m^2 + z^2}}{\sqrt{p}} = \text{a } Minimum.$$

Therefore

$$\frac{udu}{\sqrt{b}\sqrt{m^2 + u^2}} + \frac{zdz}{\sqrt{p}\sqrt{m^2 + z^2}} = o.$$

But $du = -dz$ (because $v + z$ is constant) therefore

$$\frac{u}{\sqrt{b}\sqrt{m^2 + u^2}} = \frac{z}{\sqrt{p}\sqrt{m^2 + z^2}};$$

whence 'tis manifest that $\frac{u}{\sqrt{b}\sqrt{m^2 + u^2}}$ = a constant Quantity. Now let the Abscisse AL = x, the Ordinate LC = y, and to BG = dx, GC

GC $= dy$, BC $= \sqrt{dx^2 + dy^2}$, and let a be any constant Quantity. Then shall

$$\frac{dx}{\sqrt{y}\sqrt{dx^2 + dy^2}} = \frac{1}{\sqrt{a}},$$ whence $dx\sqrt{a} = \sqrt{y} \times \sqrt{dx^2 + dy^2}$. But now in all Curves, 'tis $dx :: \sqrt{dx^2 + dy^2} ::$ as the Subtangent, to the Tangent. Therefore the Nature of the Curve sought is such, that its Subtangent, is to the Tangent, as $\sqrt{a} : \sqrt{y}$, which that it is a Property of the *Cycloid*, is known to all, that know that the Tangent of the Cycloid, is parallel to the Chord of the Conterminal Arch, in the Generating Circle, whose Diameter is a, and whose Vertex is downwards.

And with the like ease, I can find the Curve of the *Swiftest Descent*, in any other Hypothesis of Gravity.

The Quadrature of the Logarithmical Curve.

By J. Craig.

(Fig. 26.)

LET ONF be the Logarithmical Curve, whose Asymptote is AR, in which let such a point *A* be taken, as that the first Ordinate AO may be equal to the Subtangent or Unity. 'Tis requir'd to find the Area of the Curvilineal Space AONM comprehended under the two Ordinates AO, MN, the Abscisse AM, and the Curve ON. From *O* draw OE parallel to AM and cutting MN in *E*; I say, that the Rectangle under the Segments ME, EN, is equal to the Space sought. *Demonstration.* Let the Ordinate MN = Z, Subtangent AO or ME = s; and to the Axis AR let another Curve HGE be constructed, whose Equation shall be $2sz = x^2$, its Ordinate GM being $= x$. I say, that this Curve is the *Quadratix* of the Logarithmical Curve (according to the Principles of my Method) *viz.* its Subnormal is respectively equal to the Ordinate of this, as is plain from the Calculus of that Method. Therefore (according to what I have shewn in another place) if to the point *G* we draw GC perpendicular and equal to GM, as also HD parallel to GC, and meeting the Lines GM, CM, in B and D; then will the

the Trapezium GBDC $=$ AONM. But GBDC $= \text{GMC} - \text{BMD} = \frac{1}{2}x^2 - \frac{1}{2}\text{BM}^q = \text{SZ} - \frac{1}{2}\text{HA}^q$; but HA $= \sqrt{2}\,\text{AO}^q$ from the Nature of the Curve HGQ, therefore GBDC $= \text{SZ} - \text{AO}^q = \overline{\text{AO} \times \text{MN}} - \text{AO}^q = \text{AO} \times \overline{\text{MN} - \text{AO}} = \text{ME} \times \overline{\text{MN} - \text{ME}} = \text{ME} \times \text{EN}$. Therefore alſo AONM $= \text{ME} \times \text{EN}$. Q. E. D.

When I applied my Method to theſe ſort of Figures, I found that a Miſtake had ſome way or other crept into M. *Bernoulli's* Calculus. For in his moſt excellent Tract of the *Principles of the Differential Calculus*, he aſſigns to the Figure whoſe Equation is $a^z = y^y$, this for its Quadrature, *viz.* $\frac{2yyly - yy}{la}$; whereas the Area of that Figure is $\frac{2yyly - yy}{4la}$; where y denotes the Abſciſſe and z the Ordinate.

A Theorem concerning the Proportion of the Time that a heavy Body spends in descending thro' a right Line joining two given Points, to the (shortest) *Time, in which it passes from the one to the other of these Points, by the Arch of a Cycloid.*

THEOREM.

(Fig. 27.)

IN the Cycloid AVD, whose Basis AD is parallel to the Horizon, and the Vertex V turn'd downwards, if from A be drawn the right Line AB meeting the Cycloid in any point as B, from whence is drawn BC perpendicular to the Curve of the Cycloid in B, and AC be let fall perpendicular to BC from the point A: Then the Time that a Body at rest in A, spends in descending thro' AB (by the force of its Gravity) is to the Time that it spends in falling thro' the Curve AVB, as AB to AC.

Thro' B draw BL parallel to the Axis of the Cycloid VE, and BK parallel to the Basis AD, meeting the Axis in G, and the Circle (whose Diameter is EV) in F and H, and the Cycloid it self in K. Draw the right

right Line EF, which from the Nature of the Cycloid is parallel to BC; whence BM is = EF, and EM = BF = the Arch VF from the Nature of the Cycloid; and conſequently AM is = the Arch EHVF.

By *Propoſition* 25. Part II. *Horolog. Oſcillat. Hugen.* the Time in which a Body at reſt in A deſcribes the Cycloidal Arch AV, is to the Time of Deſcent thro' EV, as the half Circumference to the Diameter.

And (by the laſt *Propoſition* of the forementio'd *Part*) the Time of Deſcent thro' VB, after the Deſcent thro' AV (which is equal to the Time of Deſcent thro' KV, after the Deſcent thro' AK) is to the Time of Deſcent thro' AV, as the Arch VF, to the Semicircumference; and conſequently to the Time of Deſcent thro' EV, as the Arch FV, to the Diameter. Wherefore the Time of deſcribing the Curve AVB, is to the Time of Deſcent thro' EV, as the Arch EHVF, to the Diameter EV. But the Time of Deſcent thro' EV, is to the Time of Deſcent thro' LB or EG, as EV to EF. Therefore *(by Equality)* the Time of deſcribing AVB, is to the Time of Deſcent thro' LB, as the Arch EHVF, to the Subtenſe EF, that is, as AM to MB. Again, the Time of Deſcent thro' LB, is to the Time of Deſcent thro' AB, as LB to AB. Therefore the Time of deſcribing AVB, is to the Time of Deſcent thro' AB, in the Ratio compounded of AM to BM, and LB to BA, and conſequently is equal to the Ratio of AM × LB to MB × BA.

But

But AM × LB = MB × AC; and therefore the Time in which a Body at reſt in A, ſhall deſcribe the Cycloidal Arch AVB, is to the Time of deſcribing the right Line AB, as MB × AC to MB × BA; that is, as AC to AB. Q: E: D.

And the Demonſtration will proceed in like manner, if the point B be between A and V.

An

An Extract of a Letter from the Reverend Dr. John Wallis, *to* Richard Waller, *Esq; Secretary to the Royal Society, concerning the Spaces in the* Cycloid, *which are perfectly Quadrable.*

Oxford, *August* 22. 1695.

SIR,

I Find it is thought by most, that there is no other part of the *Semicycloid* Figure (adjacent to the Curve) that is capable of being Geometrically Squared, but these two, *viz.*

1. The Segment AbV, *(Fig. 28.)* taking $AV = \frac{1}{4}A\alpha$, (which was first observ'd by Sir *Christopher Wren*, and after him by *Hugenius* and others) and it is $= \frac{1}{2}sR = \frac{1}{4}R^2\surd 3$.

2. The Trilinear AdD (taking dD, in the Parallel dDC, passing through the Center C,) which is $= R^2$.

But it is otherwise (as I have shewed in my Treatise, *De Cycloide*, and that, *De Motu*; the Figures of which latter I retain here, so far as they concern this Occasion;) there being

being other Portions of it, equally capable of Quadrature.

In order to which, I there shew (*De Motu*, Cap. 5. *Prop.* 20. *A.* p.802,803,804.) that not only the *Cycloid* is Triple to the Circle Generant, (which was known before) but that the *respective Parts* of that are Triple to those of this: Which is the Foundation on which I build my whole Process concerning the *Cycloid* in both Treatises, (and which is not pretended, that I know of, to have been observ'd or known by any Body before me:) That is, $b\beta\alpha A$ (*Fig.* 28.) Triple to the Sector $B\alpha A$ (taking $b\beta$ parallel to $B\alpha$) where-ever, in the Curve $A\tau$, we take the point b.

I then shew, that the *Cycloid* is a Figure compounded of these two; the Semicircle $AD\alpha$, and the Trilinear $AD\alpha\tau bA$, lying between the two Curves $AD\alpha$ and $Ad\tau$, (and therefore, to Square any part of these, is the same as to Square the respective part of the *Cycloid*.

I shew farther (*Ibidem*, pag. 804.) that this Trilinear is but a distorted Figure (by reason of the Semicircle thrust in between it and its Axis) which being restored to its due Position (by taking out the Semicircle into a different Figure, (as *Fig.* 29.) and thrusting the Lines bB home to the Axis, so as that BV be the same point) is the same with $A\tau\alpha$, (*Fig.* 30.) (the Parallelograms $b\beta\alpha B$ being set upright, which in the *Cycloid* stand sloping; and the Circular Arches $b\beta$, (*Fig.* 28.) becoming streight Lines (in *Fig.* 30.) and the Lines bB being, in both, equal to the respe-

ctive

ctive Arches *B A*, every where ;) which therefore I call *Trilineum Restitutum* (the Trilinear restored to its due Position, which Figure I do not find that any before me has consider'd :) So that to Square any part of this, is the same as to Square the respective part of the *Cycloid*, (or of the Trilinear in the *Cycloid* :) That which in the *Cycloid* lies between two Arches of the Circle Generant in different Positions, answering to that which, in the restored Figure, lies between the respective streight Lines.

And therefore $A\,d\,D\,A, = \tau\,d\,\delta\,\tau$, (*Fig.* 28.) $= A\,d\,D\,A = \tau\,d\,\delta\,\tau$, (*Fig.* 30.) $= R^2$. And $A\,b\,k\,d\,A$, $\tau\,b\,k\,\delta\,\tau$, (*Fig.* 28.) $= A\,b\,k\,d\,A$, $\tau\,b\,k\,\delta\,\tau$, (*Fig.* 30) $= s\,R$. And $b\,k\,d$ (*Fig.* 28.) $= b\,k\,d$, (*Fig.* 30.) $= R^2 - s\,R$, Ibid. *Cap.* 17. *B. pag.* 756. Where, if *b* be taken above *d k D C*, (passing through the Center *C*) these Figures are within the *Cycloid*, and within the restored Figure ; but without them, if *b* be taken below that Line, and adjacent to the Curve $A\,b\,\tau$, in both Cases.

By *R*, I understand the Radius of the Circle Generant ; and by *s*, the Right Sine of the Arch *B A*, whose versed Sine is *V A*.

And, where-ever in my whole Discourse of the *Cycloid*, or the restored Trilinear (which is a Figure of Arches, and a Figure of versed Sines) the Arch *a* is no Ingredient in the designation ; such part or portion of them is capable of being Geometrically squared. But when I exclude *a*, I do therein exclude *P* (for that is an Arch also) and $f = a + s$, and $e = a - s$, because *a* is therein included.

Mr.

Mr. *Caſwell*, (not being aware that I had ſquared theſe Figures) had done the ſame by a Method of his own, (which he ſhewed me lately) which I would have inſerted here, but that he thought it not neceſſary; and inſtead thereof, hath given me the Quadrature of a Portion of the *Epicycloid* (which you will receive with this) and, I think, it is purely new.

The

The Quadrature of a Portion of the Epicycloid.

By Mr. Caswell.

(Fig. 31.)

SUppose *DPV* to be half of an exterior Epicycloid, *VB* its Axis, *V* the Vertex, *VLB* half of the Generant Circle, *E* its Center; *DB* the Base, *C* its Center: Bisect the Arc of the Semicircle *VB* in *L*, and on the Center *C* through *L* draw a Circle cutting the Epicycloid in *P*: Then I say the Curvilinear Triangle *VLP* will be $= BEq$ in $\frac{CE}{CB}$; that is, the Square of the Semidiameter of the Generant Circle will be to the Curvilinear Triangle *VLP*, as *CB* the Semidiameter of the Base, to *CE*; which *CE* in the exterior Epicycloid is the Sum of the Semidiameters of the Base and Generant, but in the interior Epicycloid *Dpu*, 'tis the difference of the said Semidiameters.

COROLLARY.

In the interior Epicycloid, if *CE* is $\frac{1}{2}$ *CB*, the Epicycloid then degenerating into a right Line, the Quadrature of the Triangle *lup* will be in effect the same with the Quadrature of *Hippocrates Chius.*

COROL.

COROL. II.

If the Semidiameter of the Base is supposed infinite, the Epicycloid then being the common Cycloid, the Area of the said Triangle will be equal to the Square of the Radius of the Generant, and so it falls in with that Theorem which *Lalovera* found, and calls *Mirabile*.

Though I do not think the abovesaid Quadrature can easily be deduced from what has been yet published of the Epicycloid, I have not added the Demonstration; but think it enough to name a general Proposition from whence I deduced it, *viz.* The Segments of the Generant Circle are to the Correspondent Segments of the Epicycloid, as CB to $2CE + CB$. For Example, suppose FmG the Position of part of the Generant when the point F of the exterior Cycloid was design'd, then the Segment $FmGn$ is to the Segment $DFnG$ :: as CB to $2CE + CB$.

And consequently the whole Epicycloid to the whole Generant in the same Proportion: Which is the only Case demonstrated by Mousieur *De la Hire*.

It follows also that in the Vulgar Cycloid, its Segments are triple of the Correspondent Sectors of the Generant, which was first shewn by Dr. *Wallis*.

A General

A General Proposition shewing the Dimension of the Areas in all those kinds of Curves which are describ'd by the Equable Revolution of a Circle upon any Basis, either a Rectilineal or a Circular one.

By Edm. Halley.

'TIS known that the *Primary Cycloid*, as also the *Prolate*, and the *Contracted* one (which they call *Trochoids)* have been largely handled by the Celebrated Dr. *Wallis* and others, and their Properties are now common enough; so that there's scarce any thing new left to be discover'd concerning them. But the famous M. *De Lattire* in a late Treatise, having shewn some of the Properties of the *Primary Epicycloid*, the most Ingenious Mr. *Caswell* did upon that occasion not only demonstrate that the Mensuration of the whole Epicycloidal Space, obtain'd also in the parts of the same, but also gave a perfect Quadrature of the Curvilineal Space UPL. But while I was enquiring after the Demonstration of this Quadrature, which is not very obvious, nor as yet given by the Inventour, I light upon the following general

Propoſition, by the help of which all ſorts of Curvilineal Spaces, as well of the Cycloidal as Epicycloidal kind, as well the whole Spaces as the parts, are meaſur'd. And further, not only the Spaces VPL, but alſo innumerable others, QTP and VQTL, are demonſtrated to be capable of an exact Quadrature; and this not only in the *Primary* Epicycloids, but but alſo in the *Prolate* and contracted ones.

The Propoſition is as follows.

Propoſition.

The Area of any Cycloid or Epicycloid, either Primary, Prolate, or Contracted, is to the Area of the Generating Circle, and alſo the Areas of the generated parts in thoſe Curves, are to the Areas of the Analogous Segments of the Circle; as the Sum of twice the Velocity of the Center, and the Velocity of the Circular Motion, to the Velocity of the Circular Motion.

Demonſtration.

(Fig. 32.)

Let YPQRSUB be any Epicycloid deſcrib'd by the Revolution of the Circle ULB, upon the Circular Baſis YMNB. Let the Center of the Generating Circle be in *c*, and drawing cMK, let the Circle ſtand upon the Baſis in the point M, and let the deſcribing point be S. Now diſtinguiſhing the Motions, let the point S firſt of all be carried by the Circular Motion into R, ſo that the Arch

SM

SM is increaſed by the indiviſible Particle RS. Next ſuppoſe the Center c to be tranſferr'd to C; by which Motion the Segment RSM being brought into the Poſition QTN, the point Q will touch the Curve. 'Tis plain that the Triangle RSM is the Momentum or Fluxion of the Segment of the Circle, and that the Trapezium QSMN is the Fluxion of the Curvilineal Space generated in the ſame time. And ſince SM, RM, QM, are ſuppos'd to differ but by a *point* from one another, let the little Area QSMN be conceiv'd to conſiſt of the three Sectors RMS, RMQ, MQN; and ſo the little Area RMS to be to the little Area QSMN, as the Angle RSM to the Sum of the three Angles RMS+RMQ+MQN. But the Angles RMQ+MQN, are equal to the Angles MCN + MKN, or to the Angle cMC; becauſe of the Lines RM, QN, inclin'd to one another in an Angle equal to MKN, and becauſe of the Angle MQN equal to $\frac{1}{2}$ MCN (by *Eucl.* 3. 20.) conſequently the Angle RMS is to the Angles RMS + cMC, that is (by the ſame *Propoſition* mention'd) the Arch $\frac{1}{2}$ RS to the two Arches Cc + $\frac{1}{2}$ RS, or RS to 2Cc + RS, as the little Area RSM, to the little Area QSMN, or as the Momentum of the Circular Segment QTN, to the Momentum of the Epicycloidal Segment QSYMN generated in the ſame time. And ſince theſe Momenta are ever in that ſame Ratio, where-ever the point Q be taken, 'tis manifeſt that the Areas QTN, QSMYN themſelves, generated from theſe Momenta, have alſo the ſame conſtant Ratio, *viz.* of

 the

the Velocity of the Circular Motion RS, to double the Velocity of the Center, adding the Circular Motion, or 2Cc + RS: As also the Area UBZ to the Area UBN, and consequently the Semicircle ULB to the Curvilineal Space UQYNB. Wherefore the *Proposition* is manifest.

And there is no other difference in the manner of demonstrating, if the generating Circle moves upon the Concave side of the Arch, except only that the Angle cMC, in this case, is the *difference* of the Angles MCN, MKN. But if the Basis were a right Line, then MKN vanishing, and RM, QN, being parallel, the Construction will be easier. I forbear drawing Corollaries from this Proposition, since they are obvious. But now in all these Curves, the Portions that are Analogous to those Portions which Doctor *Wallis* has found capable of a perfect Quadrature in the Primary Cycloid, are here also equally squarable; which easily follows from what has been said.

Upon the Center K, thro' the point Q, draw the Circular Arch QZ, and draw ZB cutting off the Segment ZLB = the Segment QTN. Then bisect the Semicircle UB in L, and thro' the point L and on the Center K, describe the Arch PL cutting the Epicycloid in P, the generating Circle in T, and the Chords QN, ZB, in y and X. Let the Arch VZ = a, its Sine = s, the Radius of the generating Circle = r, the Radius of the Base = R, and the Arch CE or the Motion of the Center = m. It is plain that the Sector CKE, is to the Space XyNB, as the Square

Square of KE, to the difference of the Squares of KL and KB, or as RR + 2Rr + rr, to 2Rr + 2rr, that is, as R + r, to 2r, or KE to B[illegible] And consequently the Rectangle BE × CE or rm is equal to the Space XyNB. But the Space VZB is eq[illegible] to the Rectangle $\frac{1}{2}$ ar + $\frac{1}{2}$ sr, and so according to our *Proposition* it will be as *a* to 2*m*, so $\frac{1}{2}$ ar + $\frac{1}{2}$ sr, to $\frac{mar + msr}{a}$ equal to the Curvilineal Space QUZLBNQ. From hence substract the Space XyNB = rm, and there remains the Space QUZXy = $\frac{mrs}{a}$. And since the Spaces ZXL, QyT, are equal, the Space QULTQ shall also be equal to $\frac{mrs}{a}$. Therefore when *a* to *m*, or the Circular Motion is to the Progressive Motion of the Center, in a given Ratio, there will be a perfect Quadrature of the Curvilineal Spaces QULTQ. And the whole Space UPL, will be to the Square of the Radius BE, in the same Ratio (m to a) of the Motions, that is in every Primary Epicycloid, in the Proportion of the Radii, KE, KB, which is Mr. *Caswell*'s Proposition.

But the lesser Spaces QULTQ will be to one another, as the Sines of the Arches UZ; and the Triangular Spaces QTP, by the same Argument, will be as the versed Sines of the Arches QT or ZL, and consequently are also squar'd. After the same manner it will be prov'd, that the Spaces pλτ, pLu, pλτ, are ever to the Square of the Radius BE

 (in

(in all theſe Figures) in the aforeſaid Ratio of *m* to *a*; and their Portions pqt, as the verſed Sines of the intercepted Arches qt; but the remaining Segments as qtϒΛ, qtϒλ, &c. will be as the right Sines of the Compliments of the ſame Arches qt.

But the Ratio of the Velocities, *m* to *a*, is compounded of the Ratio of the Radii KB, BE, and the Ratio of the Angles CKE, VEZ, equably deſcrib'd together; and conſequently giving that Ratio of the Angles, all the forementioned Epicycloidal Spaces will be ſquar'd alſo.

I can eaſily draw Tangents to all theſe Curves, as alſo I ſeem to my ſelf to have gotten their Rectifications, from ſome Areas Analogous to them; which may give occaſion to a more particular handling of this Family of Curves another time.

A

A Method of Raising an infinite Multinomial to any given Power, or Extracting any given Root of the same.

By Mr. A. De Moivre.

'TIS about two Years since, that considering Mr. *Newton*'s Theorem for Raising a Binomial to any given Power, or Extracting any Root of the same; I enquir'd, whether what he had done for a Binomial, could not be done for an infinite Multinomial. I soon found the thing was possible, and effected it, as you may see in the following Paper; I design in a little time to shew the Uses it may be applied to: In the mean while, those that are already vers'd in the Doctrine of Infinite Series, and have seen what Applications Mr. *Newton* has made of his Theorem, may of themselves derive several Uses from this.

I suppose that the Infinite Number Multinomial is $az + bzz + cz^3 + dz^4 + ez^5$, &c. m is the Index of the Power, to which this Multinomial ought to be Rais'd, or if you will, 'tis the Index of the Root which is to be Extracted: I say that this Power or Root of the Multinomial, is such a Series as I have exprest.

For the underſtanding of it, it is only neceſſary to conſider all the Terms by which the ſame Power of z is multiplied; in order thereto I diſtinguiſh two things in each of theſe Terms; Firſt, The Product of certain Powers of the Quantities, a, b, c, d, &c. Secondly, The *Unciæ* (as *Oughtred* calls 'em) prefixt to theſe Products. To find all the Products belonging to the ſame Power of z, to that Product, for inſtance, whoſe Index is $m+r$ (where r may denote any integer Number) I divide theſe Products into ſeveral *Claſſes*; thoſe which immediately after ſome certain Power of a (by which all theſe Products begin) have b, I call *Products* of the firſt *Claſſis*; For *Example*, $a^{m-4} b^3 e$ is a Product of the firſt *Claſſis*, becauſe b immediately follows a^{m-4}; thoſe which immediately after ſome *Power* of a have c, I call Products of the ſecond *Claſſis*, ſo $a^{m-3} ccd$ is a Product of the ſecond *Claſſis*; thoſe which immediately after ſome Power of a have d, I call Products of the third *Claſſis*, and ſo of the reſt.

This being done, I multiply all the Products belonging to z^{m+r-1} (which precedes immediately z^{m+r}) by b and divide 'em all by a; Secondly, I multiply by c and divide by a, all the Products belonging to z^{m+r-2}, except thoſe of the firſt *Claſſis*; Thirdly, I multiply by d and divide by a all the Products belonging to z^{m+r-3}, except thoſe of the firſt and ſecond *Claſſis*; Fourthly, I multiply by e and divide by a all the Terms belonging

longing to z^{m+r-4}, except those of the first, second, and third *Classis*, and so on, till I meet twice with the same Term. Lastly, I add to all these Terms the Product of a^{m-1} into the Letter whose *Exponent* is $r+1$.

Here I must take notice that by the Exponent of a Letter, I mean the Number which expresses what Place the Letter has in the *Alphabet*, so three is the Exponent of the Letter *c* because the Letter *c* is the third in the *Alphabet*.

It is evident that by this Rule, you may easily find all the Products belonging to the several Powers of *z*, if you have but the Product belonging to z^m, *viz.* a^m.

To find the *Unciæ* which ought to be prefixt to every Product, I consider the Sum of Units contain'd in the Indices of the Letters which compose it (the Index of *a* excepted) I write as many Terms of the Series $m \times m-1 \times m-2 \times m-3$, &c. as there are *Units* in the Sum of these Indices, this Series is to be the Numerator of a Fraction, whose Denominator is the Product of the several Series $1 \times 2 \times 3 \times 4 \times 5$, &c. $1 \times 2 \times 3 \times 4 \times 5$, &c. $1 \times 2 \times 3 \times 4 \times 5 \times 6$, &c. the first of which contains as many Terms as there are *Units* in the Index of *b*, the second as many as there are *Units* in the Index of *c*, the third as many as there are *Units* in the Index of *d*, the fourth as many as there are *Units* in the Index of *e*, &c.

De-

Demonſtration.

To raiſe the Series $az + bzz + cz^3 + dz^4$, &c. to any Power whatſoever, write ſo many Series equal to it as there are *Units* in the Index of the Power demanded. Now it is evident that when theſe Series are ſo multiplied, there are ſeveral Products in which there is the ſame Power of z, thus if the Series $az + bzz + cz^3 + dz^4$, &c. is rais'd to its Cube, you have the Products b^3z^6, $abcz^6$, $aadz^6$, in which you find the ſame Power z^6 Therefore let us conſider what is the Condition that can make ſome Products to contain the ſame Power of z, the firſt thing that will appear in relation to it, is that in any Product whatſoever, the Index of z is the Sum of the particular Indices of z in the multiplying Terms (this follows from the Nature of Indices) thus b^3z^6 is the Product of bz^2, bz^2, bz^2, and the Sum of the Indices in the multiplying Terms, is $2 + 2 + 2 = 6$; $abcz^6$ in the Product of az, bzz, cz^3, and the Sum of them Indices of z in the multiplying Terms is $1 + 2 + 3 = 6$ $aadz^6$ is the Product of az, az, dz^4, and the Sum of the Indices of z in the multiplying Terms is $1 + 1 + 4 = 6$; the next thing that appears is, that the Index of z in the multiplying Terms is the ſame with the Exponent of the Letter to to which z is join'd, from which two Conſiderations it follows, that, *To have all the Products belonging to a certain Power of* z, *you muſt find all the Products where the Sum of the Exponents*

ponents of the Letters which compose 'em shall always be the same with the Index of that Power. Now this is the Method I use to find easily all the Products belonging to the same Power of z, Let $m+r$ be the Index of that Power, I consider that the Sum of the Exponents of the Letters which compose these Products must exceed by one those which belong to m^{m+r-1}, now because the Excess of the Exponent of the Letter b above the Exponent of the Letter a, is one, it follows that if each of the Products belonging to z^{m+r-1} is multiplied by b, and divided by a, you will have Products the Sum of whose Exponents will be $m+r$; Likewise the Sum of the Exponents of the Letters which compose the Products belonging to z^{m+r} exceeds by two the Sum of the Exponents of the Letters which compose the Products belonging to z^{m+r-2}; Now because the Exponent of the Letter a is less by two than the Exponent of the Letter c, it follows, that if each Product belonging to z^{m+r-2} is multiplied by c and divided by a, you will have other Products, the Sum of whose Exponents is still $m+r$; Now if all the Products belonging to z^{m+r-2} were multiplied by c and divided by a, you would have some Products that would be the same as some of those found before, therefore you must except out of 'em those that I have call'd Products of the first *Classis*; what I have said shows why all the Products belonging to z^{m+r-3}, except those of the first and second *Classis* must be be multiplied by *d* and divided by *a*: Laſtly you ſee the Reaſon why to all theſe Products is added the Product of a^{m-1} by the Letter whoſe Exponent is $r+1$; 'Tis becauſe the Sum of the Exponents is ſtill $m+r$.

As for what relates to the *Uncia*; obſerve that when you multiply $az+bzz+cz^3+dz^4$, &c. by $az+bzz+cz^3+dz^4$, &c. each Letter *a*, *b*, *c*, *d*, &c. of the ſecond Series is multiplied by each of the Letters *a*, *b*, *c*, *d*, &c. of the firſt Series; Thus the Letter *a* of the ſecond Series is multiplied by the Letter *b* of the firſt, and the Letter *b* of the ſecond Series is multiplied by the Letter *a* of the firſt; therefore you have the two Planes, *ab*, *ab* or 2*ab*; for the ſame reaſon you have 2*ac*, 2*ad*, &c. Therefore you muſt prefix to each Plane of thoſe that compoſe the Square of the infinite Series $az+bzz+cz^3$, &c. the Number which expreſſes how many ways the Letters of each Plane may be changed; likewiſe if you multiply the Product of the two preceeding Series by $az+bzz+cz^3$, &c. each Letter *a*, *b*, *c*, *d*, of the third Series is multiplied by each of the Planes form'd by the Product of the firſt and ſecond Series; Thus the Letter *a* is multiplied by the Planes *bc* and *cb*; the Letter *b* is multiplied by *ac* and *ca*; the Letter *c* is multiplied by *ab* and *ba*; therefore you have the ſix Solids, *abc*, *acb*, *bac*, *bca*, *cab*, *cba*, or 6*abc*; Therefore you muſt prefix to each Solid whereof the Cube of the infinite Series is compos'd, the Number which expreſſes how many ways the Letters of each Solid may be changed. And gene-

generally, *You must prefix to any Product where-of any Power of the infinite Series* $az + bzz + cz^3$, &c. *is compos'd the Number which expresses how many ways the Letters of each Product may be changed.*

Now to find how many ways the Letters of any Product, for instances $a^{m-n} b^h c^p d^r$ may be changed; this is the Rule which is commonly given: Write as many Terms of the Series $1 \times 2 \times 3 \times 4 \times 5$, &c. as there are Units in the Sum of the Indices, *viz.* $m - n + h + p + r$, let this Series be the Numerator of a Fraction whose Denominator shall be the Product of the Series $1 \times 2 \times 3 \times 4 \times 5$, &c. $1 \times 2 \times 3 \times 4 \times 5$, &c. $1 \times 2 \times 3 \times 4 \times 5 \times 6$, &c. $1 \times 2 \times 3 \times 4 \times 5$, &c. whereof the first is to contain as many Terms, as there are Units in the first Index $m - n$; the second as many as there are Units in the second Index h; the third as many as there are Units in the third Index p; the fourth as many as there are Units in the fourth Indexs r. But the Numerator and Denominator of this Fraction have a common Divisor, *viz.* the Series $1 \times 2 \times 3 \times 4 \times 5$, &c. continued to so many Terms as there are Units in the first Index $m - n$; therefore let both this Numerator and Denominator be divided by this common Divisor, then this new Numerator will begin with $m - n + 1$, whereas t'other began with 1, and will contain so many Terms as there are Units in $h + p + r$, that is, so many as there are Units in the Sum of all the Indices, excepting the first; as for the new Denominator, it will be the Product of three

three Series only, that is, of so many as their Indices, excepting the first. But if it happens withal, that n be equal to $b+p+r$ as it always happens in our *Theorem*, then the Numerator beginning by $m-n+1$, and being continued to so many Terms as there are Units in $b+p+r$ or n, the last Term will be m necessarily, so if you invert the Series and make that the first Term which was the last, the Numerator will be $m \times m-1 \times m-2 \times m-3$, &c. continued to so many Terms as there are Units in the Sum of the Indices of each Product, exceping the first Index. There remains but one thing to demonstrate, which is, that, what I have said of Powers whose Index is an Integer, may be adapted to Roots, or Powers whose Index is a Fraction; but it appears at first sight why it should be so: For, the same Reason which makes me consider Roots under the Notion of Powers, will make me conclude, that whatever is said of one may be said of t'other; however, I think sometime to give a more formal Demonstration of it.

A

A Method of Extracting the Root of an Infinite Equation.

By A. De Moivre, *F.R.S.*

THEOREM.

IF $az + bzz + cz^3 + dz^4 + ez^5 + fz^6$, &c. $= gy + hyy + iy^3 + ky^4 + ly^5 + my^6$, &c. then will

$$z \text{ be } = \frac{g}{a}y + \frac{h - bAA}{a}y^2$$

$$+ \frac{i - 2bAB - cA^3}{a}y^3$$

$$+ \frac{k - bBB - 2bAC - 3cAAB - dA^4}{a}y^4$$

$$+ \frac{l - 2bBC - 2bAD - 3cABB - 3cAAC - 4dA^3B - eA^5}{a}y^5$$

$$+ \frac{m - 2bBD - bCC - 2bAE - cB^3 - 6cABC - 3cAAD - 6dAABB - 4dA^3C - 5eA^4B - fA^6}{a}y^6, \text{ \&c.}$$

For the underſtanding of this Series, and in order to continue it as far as we pleaſe; it is to be obſerv'd, 1. That every Capital Letter is equal to the Coefficient of each pre-

preceding Term; thus the Letter B is equal to the Coefficient $\frac{h-bAA}{a}$. 2. That the Denominator of each Coefficient is always a. 3. That the first Member of each Numerator, is always a Coefficient of the Series $gy+hyy+iy^3$, &c. *viz.* the first Numerator begins with the first Coefficient g, the second Numerator with the second Coefficient h, and so on. 4. That in every Member after the first, the Sum of the Exponents of the Capital Letters, is always equal to the Index of the Power to which this Member belongs: Thus considering the Coefficient $\frac{k-bBB-2bAC-3cAAB-dA^4}{n}$, which belongs to the Power y^4, we shall see that in every Member bBB, $2bAC$, $3CAAB$, dA^4, the Sum of the Exponents of the Capital Letters is 4, (where I must take notice, that by the Exponent of a Letter, I mean the Number which expresses what Place it has in the Alphabet; thus 4 is the Exponent of the Letter D) hence I derive this Rule for finding the Capital Letters of all the Members that belong to any Power; *Combine the Capital Letters as often as you can make the Sum of their Exponents equal to the Index of the Power to which they belong.* 5. That the Exponents of the small Letters, which are written before the Capitals, express how many Capitals there is in each Member. 6. That the Numerical Figures or *Unciæ* that occur in these Members, express the Number of Permuta-

tions

tions which the Capital Letters of every Member are capable of.

For the Demonstration of this; suppose $z = Ay + Byy + Cy^3 + Dy^4$, &c. Substitute this Series in the room of z, and the Powers of this Series, in the room of the Powers of z; there will arise a new Series; then take the Coefficients which belong to the several Powers of y, in this new Series, and make them equal to the corresponding Coefficients of the Series $gy + hyy + iy^3$, &c. and the Coefficients A, B, C, D, &c. will be found such as I have determin'd them.

But if any one desires to be satisfied, that the Law by which the Coefficients are form'd, will always hold, I'll desire 'em to have recourse to the Theorem I have given for raising an infinite Series to any Power, or extracting any Root of the same; for if they make use of it, for taking successively the Powers of $Ay + Byy + Cy^3$, &c. they will see that it must of necessity be so. I might have made the Theorem I give here, much more General than it is; for I might have suppos'd, $az^m + bz^{m+1} + cz^{m+2}$ &c. $= gy^m + hy^{m+1} + iy^{m+2}$ &c. then all the Powers of the Series $Ay + Byy + Cy^3$, &c. design'd by the universal Indices, must have been taken successively; but those who will please to try this, may easily do it, by means of the *Theorem for raising an infinite Series to any Power*, &c.

This *Theorem* may be applied to what is called the Reversion of Series, such as finding the Number from its Logarithm given; the Sine from the Arc; the Ordinate of an

Ellipse from an Area given to be cut from any Point in the Axis: But to make a particular Application of it, I'll suppose we have this Problem to solve; *viz.* The Chord of an Arc being given, to find the Chord of another Arc, that shall be to the first as n to 1. Let y be the Chord given, z the Chord required; now the Arc belonging to the Chord y is, $y + \frac{y^3}{6dd} + \frac{3y^5}{40d^4} + \frac{5y^7}{112d^6}$ &c. and the Arc belonging to the Chord z is $z + \frac{z^3}{6dd} + \frac{3z^5}{40d^4} + \frac{5z^7}{112d^6}$ &c. the first of these Arcs is to the second as 1 to n; therefore multiplying the Extreams and Means together, we shall have this Equation:

$$z + \frac{z^3}{6dd} + \frac{3z^5}{40d^4} + \frac{5z^7}{112d^6} \text{ \&c.} = ny + \frac{ny^3}{6dd} + \frac{3ny^5}{40d^4} + \frac{5ny^7}{112d^6} \text{ \&c.}$$

Compare these two Series with the two Series of the *Theorem*, and you will find $a=1$, $b=0$, $c=\frac{1}{6dd}$, $d=0$, $e=\frac{3}{40d^4}$, $f=0$, &c. $g=n$, $h=0$, $i=\frac{n}{6dd}$, $k=0$, $l=\frac{3n}{40d^4}$, $m=0$, &c. hence z will be $=ny + \frac{n-n^3}{6dd}y^3$ &c. or $ny + \frac{1-nn}{2 \times 3dd}yyA$, &c. Supposing A

to

to denote the whole preceding Term, which will be the same Series as Mr. *Newton* has first found.

By the same Method, this general Problem may be solv'd; the Absciffe corresponding to a certain Area in any Curve being given, to find the Absciffe, whose corresponding Area shall be to the first in a given Ratio.

The Logarithmick Series might also be found without borrowing any other Idea, than that Logarithms are the Indices of Powers: Let the Number, whose Logarithm we inquire, be $1+z$, suppose its Log. to be $az+bzz+cz^3$, &c. Let there be another Number $1+y$; thereof its Logarithm will be $ay+byy+cy^3$, &c. Now if $1+z = \overline{1+y}|^n$, it follows, that $az+bzz+cz^3$, &c. $ay+byy+cy^3$, &c. :: n, 1. that is, $az+bzz+cz^3$, &c. $= nay+nbyy+ncy^3$, &c. Therefore we may find a Value of z exprest by the Powers of y; again, since $1+z = \overline{1+y}|^n$, therefore $z = \overline{1+y}|^n - 1$, that is, $z = ny + \frac{n}{1} \times \frac{n-1}{2} yy + \frac{n}{1} \times \frac{n-1}{2} \times \frac{n-2}{3} y^3$, &c. Therefore z is doubly exprest by the Powers of y. Compare these two Values together, and the Coefficients a, b, c, &c. will be determin'd, except the first a which may be taken at pleasure, and gives accordingly, all the different Species of Logarithms.

An Experiment of the Refraction of the Air made at the Command of the Royal Society, March 28. 1699.

By J. Lowthorp, *A. M.*

WE took a Cylinder of Caſt-Braſs (*Fig.* 33.) ABCD, and cut one end of it CD perpendicular to the Axis *ax*, the other end AB enclin'd to it at an Angle of about 27°. 30′. and therefore the Perpendicular to this enclining plain, *pc*, and the Axis of the Cylinder *ax* comprehended an Angle *pca* of about 62*o*. 30*d*. Theſe ends were groun'd very true upon a Glaſs-Grinder's Braſs-Tool, and each of them was compaſt about with a narrow Ferule of thin Braſs *bbbb*. Into the upper ſide of the Cylinder at E was ſolder'd the Braſs-Pipe EF, and into the under ſide at 6 the other Braſs-Pipe GH; the former of theſe Pipes being about 3 Inches long, and the latter 6 Inches. Upon the Plate *ddd* were fixt to two other Plates LL perpendicular to it and parallel to each other. Each of theſe two Plates had an Arch of a Circle (equal to the Circumference of the Cylinder) cut out of its upper Edge, ſo that when the Pipe GH was let through a hole near the middle of the Plate *ddd*, the Cylinder fell into the Arches; and being faſten'd there with Soder, the Axis *ax* laid parallel to the Plate *ddd* and about an Inch and half

half above it. The perpendicular End of the Cylinder DC was clos'd with an Object Glass of a seventy sixth Foot Telescope *oo*; and the other end AB, with a well polish'd flat Glass *ff*; which was carefully chosen to transmit the Object distinct enough notwithstanding its Obliquity to the Visual Rays. The Ferules were well fill'd with Cement round about the Edges of the Glass, and they laid flat and every where touch'd the smooth Ends of the Cylinder, that they might firmly resist the pressure of the excluded Air.

Instead of a Cistern (as in the Torricellian Experiment) we made use of the Inverted Siphon of Brass *(Fig. 34.)* MNO, soder'd to the Plate *ggg*. One of the sides MN stood perpendicular to the Plate, and the other side NO enclin'd to it, and was supported near the upper end O with a little prop *kk*.

We then plac'd the Cylinder (as in *Fig.* 35.) upon a Table which was well fasten'd to a firm Floor; the Pipe GH was let through a hole, and the Axis laid almost parallel to the sides of the Table, and the Plate *ddd* was nail'd down to it. The Tube of the Telescope *ff*, with the Eye-glass, was apply'd to the Object Glass, and a Hair fix'd within it at the common Focus of both Glasses in the Axis of the Cylinder continu'd, x. Upon the Floor (under the Cylinder) we nail'd the Plate *ggg* with the Inverted Siphon upon it, and join'd M to H by the Insertion of the Glass Tube T. The Joints were very carefully clos'd with Cement: And then they were cover'd over with pieces of a Bladder and wrapt hard with strong Thread. There was also a Bladder ty'd below each Joint at *m*, and when it was fill'd with Water it was ty'd above it at *n*; so that no Air could come to the Cement, or insinuate it self through its Pores or Sissures if any happen'd to be left unclos'd.

It is not (I think) an unnecessary trouble, that in this account of the Apparatus I have mention'd so many minute Circumstances, for we found it difficult enough to exclude the Air, and almost impossible to discover the very little holes through which so subtil a Fluid would freely enter and possess the Spaces deserted by the subsiding Mercury. But with all this Precaution the Experiment succeeded at last, as I wish'd, after this manner.

We plac'd the Object *a* (which was a black Thread sliding in a little Frame over a piece of white Paper) in the Axis of the Cylinder *cx* continu'd to it, we filled the Pipes and Cylinder with Mercury; and having stopt the uppermost Pipe at F with the little Iron-stopple K and clos'd it at the other Joints, we let the Mercury run out gently at O into the Bladder *v*, till it remain'd suspended at the usual height (as in the Barometre) leaving the space above it between the Glasses *oo* and *ff* void of Air. We then found the Object, which before appear'd in the Axis at x, rais'd considerably above it; and we reduc'd it to appear at *x* by removing it from *a* to *x*. The Axis therefore, of the visual Ray *xa*, (which was also the Axis of the Cylinder) *xa*, falling perpendicularly on the void space in the Cylinder past through it without any Re-

fraction;

fraction: But emerging obliquely into the Air, it was refracted towards the Perpendicular *pc*, and there receiv'd a new Direction to *x*. And therefore the ſpace *ax* ſubſtended the Angle of Refraction *acx*; which we meaſur'd and found as follows.

	Inches	Depths
The height of the Object above the Axis of viſual-Ray *ax* the unrefracted — — —	0,	425
The Diſtance of the Object from the Refracting Plain, &c. about 51 Feet or — —	612	
Therefore the Angle of Refraction *acx* was —	0. 2′.	23″.
The Angle of Emerſion *pca* (by the Conſtruction of the Cylinder) was — — — —	62.	30.
Therefore the Angle of Incidence *pcx*= (=*pca* + *acx*. — — —	62. 27.	37.

And therefore univerſally (according to the known Laws of Refraction)

The Sines of the Angles of Incidence being —	100000
The Sines of the Angles Emerſion are — —	100036
And the Refractive Power of the Denſe Air —	36

By the Refractive Power of a pellucid Body, I mean that property in it whereby the Oblique Rays of Light are diverted from their direct Courſe; and which is meaſur'd by by the proportional Differences always observ'd between the Sines of the Angles of Incidence and Emersion.

This Property is not always proportional to the Density (at least not to the Gravity) of the Refracting Medium. For the Refractive Power of Glass to that of Water, is as 55 to 34, whereas its Gravity, is as 87 to 34; that is, the Squares of their Refractive Powers are (very near) as their respective Gravities. And there are some Fluids which though lighter than Water yet have a greater power of Refraction; thus the Refractive Power of Spirit of Wine (according to Dr. *Hook*'s Experiments, *Microg.* p. 220.) is to that of Water, as 36 to 33, and its Gravity reciprocally, as 33 to 36, or $36\frac{1}{2}$. But the Refractive Powers of Air and Water seem to observe the simple Proportion of their Gravities, directly; as I have compar'd them in the following Table. The Numbers there expressing the Refraction of Water are taken from the Mean of * Nine Observations at so many several Angles of Incidence, made *Jan.* 25. $164\frac{0}{1}$. by Mr. *Gascoigne* the ingenious First Inventor of the Micrometer, and the ways of measuring Angles by Telescopes, and those of Air are produc'd by the Experiment above related.

* *I am indebted for them to Mr.* Flamsteed, *who had cover'd them with his Observations, and several Passages relating to them, from his Letters to Mr.* Crabtree, *which were happily preserv'd in the Time of our Civil War by Sir* Jonas Moor, *and Mr.* Christopher Towneley; *and are now in the Hands of Mr.* Richard Towneley *of* Towneley *in* Lancashire, *by whom they were imparted to him.*

The

	Water.	*Air.*
The (assum'd) Sines of the Angles of Incidence through — —	100000 :	100000
The Sines of the correspondent Angles of Emersion out of —	134400.	100036
The Refractive Power of — — — — —	34400	36
The Specifick Gravity (if as 900 to 1 at the time of the Experiment) of or (if as 850 to 1) of — — —	34400	38 40

From hence it seems very probable that their Respective Densities and Refractive Powers are in a just Simple Proportion: And if this should be confirm'd by succeeding Experiments, made at different Angles of Incidence, and with Cylinders continuing exhausted through several Changes of the Air, it would be more than probable that the Refractive Powers of the Atmosphere are every where, at all heights above the Earth, in proportion to its Densities and Expansions. And here it would be no difficult matter to trace the Light through it, thereby to terminate the Shadow of the Earth; and (together with proper Expedients for measuring the Quantity of Light illuminating an Opaque Body) to examine at what distances the Moon must be from the Earth to suffer Eclipses of the observ'd Duration. This Limitation is considerable enough in Astronomy, abundantly to recompense the Trouble of prosecuting such a new Experiment.

A

A Discourse concerning a Method of Discovering the true Moment of the Sun's Ingress into the Tropical Sines.

By E. Halley.

IT may perhaps pass for a Paradox, if not seem extravagant, if I should assert that it is an easier matter to be assur'd of the Moments of the Tropicks, or of the Times of the *Sun*'s Entrance into *Cancer* and *Capricorn*, than it is to observe the true Times of the Equinoctials or Ingress into *Aries* and *Libra*. I know the Opinion both of Ancient and Modern Astronomers to the contrary; *Ptolemy* says expressly, Τὰς τῶν τροπῶν τηρήσεις δυσδιακρίτους εἶναι; And *Ricciolus* begins his Chapter of the Solstitial Observations with these words, *Merito* Snellius, *in notis ad observationes* Hassiacas, *pronunciavit, Herculei esse laboris vitare in* Solstitiis *observandis errorem quadrantis diei*, and this because of the exceeding slowness of the change of the Sun's Declination on the day of the Tropick, being not a quarter of a Minute in twenty four Hours. This indeed would make it very difficult, nor would any Instruments suffice to do it, were the Moment of the Tropick to be determin'd from one single Observation. But by three subsequent Observations made

made near the Tropick, at proper Intervals of Time, I hereby design to shew a Method to find the Moment of the Tropicks capable of all the Exactness the most Accurate can desire; and that without any consideration of the Parallax of the Sun, of the Refractions of the Air, of the greatest Obliquity of the Ecliptick, or Latitude of the Place: All which are requir'd to ascertain the Times of the Equinoctials from Observation, and which being faultily assum'd, have occasion'd an Error of near three Hours in the Times of the Equinoctials deduced from the Tables of the Noble *Tycho Brahe* and *Kepler*, the Vernal being so much later, and the Autumnial so much earlier than by the *Calculus* of those Famous Authors.

Now before we proceed, it will be necessary to premise the following *Lemmata*, serving to demonstrate this Method, *viz.*

1. That the Motion of the Sun in the Ecliptick, about the Time of the Tropicks, is so nearly equable, that the difference from Equality is not sensible, from five days before the Tropick, to five days after: And the difference arising from the little Inequality that there is, never amounts to above $\frac{1}{4}$ of a single Second in the Declination, and this by reason of the nearness of the *Apogæon* of the *Sun* to the Tropick of *Cancer.*

2. That for five Degrees before and after the Tropicks, the differences whereby the Sun falls short of the Tropicks, are as the versed Sines of the Sun's distance in Longitude from the Tropicks, which versed Sines in Arches under five Degrees, are beyond the utmost nicety

nicety of Senſe, as the Squares of thoſe Arches. From theſe two follow a third:

3. That for five days before and after the Tropicks, the Declination of the Sun falls ſhort of the utmoſt Tropical Declination, by Spaces which are in duplicate Proportion, or as the Squares of the Times by which the Sun is wanting of or paſt the Moment of the Tropick.

Hence it is evident that if the Shadows of the Sun, either in the Meridian or any other Azimuth, be carefully obſerv'd about the Time of the Tropicks, the Spaces whereby the Tropical ſhade falls ſhort of, or exceeds thoſe at other Times, are always proportionable to the Squares of the Intervals of Time between thoſe Obſervations and the true Time of the Tropick, and conſequently if the Line, on which the Limits of the ſhade is taken, be made the Axis, and the correſpondent Times from the Tropick expounded by Lines, be erected on their reſpective Points in the Axis as Ordinates, the Extremities of thoſe Lines ſhall touch the Curve of a Parabola; as may be ſeen in the Figure: Where a, b, c, e, being ſuppoſed Points obſerved, the Lines a B, b C, c A, e F, are reſpectively proportional to the Times of each Obſervation before or after the Tropical Moment in *Cancer*.

This premiſed, we ſhall be able to bring the Problem of finding the true Time of the Tropick by three Obſervations, to this Geometrical one; having three Points in a Parabola A, B, C, or A, F, C given, together with the direction of the Axis, to find the Di-

Distance of those Points from the Axis. Of this there are two Cases, the one when the Time of the second Observation B is precisely in the middle Time between A and C: In this Case putting t for the whole Time between A and C, we shall have A c the Interval of the remotest Observation A from the Tropick by the following Analogy.

As 2 a c—b c to 2 a c—$\frac{1}{2}$ b c : : So is $\frac{1}{2}$ t or A E : to A c the Time of the remotest Observation A from the Tropick.

But the other Case when the middle Observation is not exactly in the middle between the other two Times, as at F, is something more operose, and the whole Time from A to C being put =t, and from A to F = s, c e = c, and b c = b, the Theorem will stand thus $\frac{ttc - bss}{2tc - 2bs}$ = A c at the Time sought.

To illustrate this Method of Calculation it may perhaps be requisite to give an Example or two for the sake of those Astronomers that are less instructed in the Geometrical part of their Art.

Anno 1500. *Bernard Walther*, in the Month of *June*, at *Nuremburg*, observ'd the Chord of the distance of the Sun from the *Zenith* by a large Parallactick Instrument of *Ptolemy*, as follows:

June 2.	45467.		*June* 8.	44975.
June 9.	44934.	and	*June* 12.	44883.
June 16.	44990.		*June* 16.	44990.

In

In both which Cases the middle Time is exactly in the middle between the Extreams, and therefore in the former three, a c=533, b c=477 and t, the Time between being 14 days, by the first Rule, the Time of the Tropick will be found by this Proportion, as 589 to 827 $\frac{1}{2}$ :: So $\frac{1}{2}$ t or 7 days to 9 days 20$^{h.}$ 2'. whence the Tropick, *Anno* 1500. is concluded to have fallen *June* 11$^{d.}$ 20$^{h.}$ 2'. In the latter three, a c is = 107, and b c = 15, and the whole Interval of Time is 8 days = to t; whence as 199 : to 206 $\frac{1}{2}$ :: so is 4 days to 4$^{d.}$ 3$^{h.}$ 37'. which taken from the 16th day at Noon, leaves 11$^{d.}$ 20$^{h.}$ 23'. for the Time of the Tropick, agreeing with the former to the third part of an hour.

Again, *Anno* 1636. *Gassendus* at *Marseilles*, observ'd the Summer Solstice by a *Gnomon* of 55 Foot high, in order to determine the Proportion of the *Gnomon* to the Solstitial shade, and he hath left us these Observations, which may serve as an Example for the second Rule.

June 19. *St. N.* shadow 31766 parts, whereof the *Gnomon* was 89428.

June 20.	31753
June 21.	31751
June 22.	31759

These being divided into two Sets of three Observations each, *viz.* the 19$^{th.}$ 20$^{th.}$ and 22$^{th.}$ and the 19$^{th.}$ 21$^{th.}$ and 22$^{th.}$ we shall have in the first three c = 13 and b = 7, t = 3 days, s = 1, and in the second c = 15 and

and b = 7, t = 3 and s = 2. Whence, according to the Rule, the 19th day at Noon the Sun wanted of the Tropick a Time proportionate to one day, as ttc — ssb to 2tc — 2bs, that is, as 110 to 64 in the first Set, or 107 to 62 in the second Set; that is, $1^{d.}\ 17^{h.}\ 15'$. in the first, or $1^{d.}\ 17^{h.}\ 25'$. in the second Set: So that we may conclude the Moment of the Tropick to have been *June* $10^{d.}\ 17^{h.}\ 20'$. in the Meridian of *Marselles.*

Now that these two Tropical Times thus obtain'd, will be found to confirm each others Exactness from their near Agreement, appears by the Interval of Time between them; *viz.* $1^{d.}\ 2^{h.}\ 30'$. less than 136 *Julian* Years: whereof $1^{d.}\ 1^{h.}\ 8'$. arises from the defect of the length of the Tropical Year from the *Julian*, and the rest from the Progression of the Sun's *Apogæon* in that Time; so that no two Observations made by the same Observer in the same Place, can better answer each other, and that without any the least Artifice or Force in the management of them.

What were the Methods used by the Ancients to conclude the hour of the Tropicks, *Ptolemy* has no where delivered; but it were to have been wished that they had been aware of this, that so we might have been more certain of the Moments of the Tropicks we have receiv'd from them, which would have been of singular use to determine the Question, Whether the Sun's *Apogæon* be fixt in the Starry Heaven; or if it move, What is the true Motion thereof? It is certain, that if we take the Account of of *Ptolemy*, the Tropick said to be observ'd by *Euctemon* and *Meton*, *Junii* 27. *manè*, *Anno* 432. *ante Christum*, can no ways be reconcil'd without supposing the Observation made the next day, or *June* 28th in the Morning. And *Ptolemy*'s own Tropick observ'd in the third Year of *Antoninus*, *Anno Christi* 140. was certainly on the 23th and not the 24th day of *June*; as will appear to those that shall duly consider and compare them with the length of the Year deduced from the diligent and concordant Observations of those two great Astronomical *Genii*, *Hipparchus* and *Albatâni*; establish'd and confirm'd by the Concurrence of all the Modern Accuracy. For these Observations give the length of the Tropical Year, such as to anticipate the *Julian* Account only one day in 300 Years; but we are now secure that the said Period of the Sun's Revolution does anticipate very nearly three days in 400 Years; so that the *Tables* of *Ptolemy* founded on that Supposition, do err about a whole day in the Sun's Place, for every 240 Years. Which principal Error in so Fundamental a Point, does vitiate the whole Superstructure of the *Almagest*, and serves to convict its Author of want of Diligence, or Fidelity, or both.

But to return to our Method, the great Advantage we have hereby, is, that any very high Building serves for an Instrument, or the Top of any high Tower or Steeple, or even any high Wall whatsoever, that may be sufficient to intercept the Sun, and cast a true shade: Nor is the Position of the Plane

on

on which you take the ſhade, or that of the Line therein, on which you meaſure the Receſs of the Sun from the Tropick, very material; but in what way ſoever you diſcover it, the ſaid Receſs will be always in the ſame Proportion, by reaſon of the ſmalneſs of the Angle, which is not ſix Minutes in the firſt five days: Nor need you enquire the height or diſtance of your Building, provided it be very great, ſo as to make the Spaces you meaſure large and fair. But it is convenient that the Plane on which you take the ſhade be not far from Perpendicular to the Sun, at leaſt not very Oblique, and that the Wall which caſts the ſhade, be ſtraight and ſmooth at Top, and its Direction nearly Eaſt and Weſt, for Reaſons that will be well underſtood by a Reader skilful in the Doctrine of the Sphere. And it will be requiſite to take the Extream greateſt or leaſt Deviation of the ſhadow of the Wall, becauſe the ſhade continues for a good Time at a ſtand, without alteration, which will give the Obſerver leiſure to be aſſur'd of what he does, and not be ſurpriz'd by the quick tranſient Motion of the ſhade of a ſingle Point at ſuch a diſtance. The principal Objection is, that the *Penumbra* or Partile ſhade of the Sun, is in its Extreams very difficult to diſtinguiſh from the true ſhade, which will render this Obſervation hard to determine nicely. But if the Sun be tranſmitted through a *Teleſcope*, after the manner us'd to take his *Species* in a *Solar Eclipſe*, and the upper half of the Object-glaſs be cut off by a Paper paſted thereon, and the exact upper Limb of the Sun be ſeen juſt Emerging out of, or rather contingint the *Species* of the Wall, (the Poſition of the *Teleſcope* being regulated by a fine Hair extended in the *Focus* of the Eye-glaſs) I am aſſur'd that the Limit of the ſhade may be obtain'd to the utmoſt Exactneſs: And of this I deſign to give a *Specimen* by an Obſervation to be made in *June* next, by the help of the high Wall of St. *Paul*'s Church, *London*, of which ſome following Tranſaction may give an Account. In the mean time what I have premis'd may ſuffice to ſet others at work, where ſuch or higher Buildings are to be met with. I ſhall only Advertiſe, that the Winter-Tropick by this Method may be more certainly obtain'd than the *Summer*'s, by reaſon that the ſame *Gnomon* does afford a much larger *Radius* for this manner of Obſervation.

A

A Scale of the Degrees of HEAT.

The Signs and Descriptions of the several Degrees of Heat.

0.		THE Warmth of the *Winter Air* when Water begins to freeze. This is known accurately by placing a Thermometer in Snow preſs'd cloſe together at the Time of a Thaw.
0. 1. 2.		The Warmths of the Winter Air.
2. 3. 4.		The Warmths of the Air in *Spring* and *Autumn*.
4. 5. 6.		The Heat of the Air in *Summer*.
6.		The Heat of the Air at *Noon* in the Month of *July*.
12.	1.	The greateſt Heat that a Thermometer acquires, by the contact of a *Humane Body*; which is much the ſame with that of a *Bird* brooding upon its *Eggs*.
$14\frac{3}{11}$.	$1\frac{1}{4}$.	The nearly greateſt Heat of a *Bath*, that a Perſon holding his Hand ſteady and immoveable in the ſame, can endure for ſome time.
17.	$1\frac{1}{2}$.	The greateſt Heat of a Bath, that a Perſon holding his Hand ſteady and immoveable in the

 ſame,

		same, can endure for some time.
$20\frac{2}{11}$.	$1\frac{3}{4}$.	The Heat of a Bath by which melted Wax swimming upon it, begins to grow stiff, and lose its Transparency.
24.	2.	The Heat of a Bath by which Wax swimming upon it, is melted and preserv'd in a State of Fluidity, without Ebullution.
$28\frac{6}{11}$.	$2\frac{1}{4}$.	The middle Degree of Heat, between that by which Wax is melted, and that which makes Water boil.
34.	$2\frac{1}{2}$.	The Heat by which Water is made to boil vehemently; and a Mixture of 2 parts of *Lead*, 3 of *Tin*, and 5 of *Bismuth*, cooling, begins to harden.
		Water begins to boil with a Heat of 33 parts, and by boiling, hardly conceives a greater Heat than that of 34 parts.
		But Iron growing cool, when it has a Heat of 35 or 36 parts, ceases to make any Ebullition when warm Water falls drop by drop upon it; as it does also with a Heat of 37 parts, when cold Water falls on it in the like manner.
$40\frac{4}{11}$.	$2\frac{3}{4}$.	The least Heat, by which a Mixture of 1 part of *Lead*, 4 of *Tin*, and 5 of *Bismuth* is liquefied and preserv'd in a State of Fluidity.
48.	3.	The least Heat, by which a Mixture of equal parts of *Tin* and *Bismuth* is liquefied. This Mixture grow-

growing cool, when it has a Heat of 47 parts, is coagulated.

57. $3\frac{1}{4}$. The Heat, by which a Mixture of 2 parts of *Tin* and 1 of *Bismuth* is liquefied; as also of 3 parts of *Tin* and 2 of *Lead*. But a Mixture of 5 parts of *Tin* and 2 of *Bismuth* (cooling) does with this Degree of Heat become hard: And the same come to pass in a Mixture of equal parts of *Lead* and *Bismuth*.

68. $3\frac{1}{2}$. The least Heat by which a Mixture of 1 part of *Bismuth* and 8 of *Tin* is liquefied. *Tin* by it self is fus'd with a Heat of 72 parts, and growing cold, hardens with a Heat of 70 parts.

81. $3\frac{3}{4}$. The Heat by which *Bismuth* is fus'd, as also a Mixture of 4 parts of *Lead* and 1 of *Tin*. But a Mixture of 5 parts of *Lead* and 1 of *Tin* when fus'd, and growing cold, hardens with this Degree of Heat.

96. 4. The least Heat by which *Lead* is melted. *Lead* melts with a Heat of 96 or 97 parts, and growing cold, hardens with a Heat of 95 parts.

114. $4\frac{1}{4}$. The Heat with which *Fiery Bodies* (growing cool) wholly cease shining in the Night; as also, that Heat with which (growing warm) they first begin to shine in the the Darkness of the Night, but with a faint and feeble Light, such as can scarce be discern'd.

This Heat liquefies a Mixture of equal parts of *Tin* and *Regulus Martis*; and a Mixture of 7 parts of *Bismuth* and 4 of the same *Regulus* (growing cool) hardens with the same Degree of Heat.

136. $4\frac{1}{2}$. The Heat by which *Fiery Bodies* do in the dark Night appear bright and shining, but not in the Twilight. A Mixture of 2 parts of *Regulus Martis* and 1 of *Bismuth*, as also of 5 parts of *Regulus Martis* and 1 of *Tin*, growing cool, will at this Degree of Heat become hard. The *Regulus* by it self, hardens with a Heat of 146 parts.

161. $4\frac{3}{4}$. The Heat by which *Fiery Bodies*, in the Twilight, a little before the Sun's rising or after his setting, do shine discernably; but not at all in the clear Day-light, or at least very obscurely.

192. 5. The Heat of a small Culinary Fire made of Sea-Coal, burning freely by it self without the help Bellows. The same is the Heat of Iron, as Red-hot as it can be made in such a Fire. The Heat of a small Culinary Fire made of Wood, is some little matter greater, *viz.* about 200 or 210 parts. And the Heat of a large Fire is still greater, especially if it be blown up by the Bellows.

In

In the firſt Column of this Table are the ſeveral Degrees of Heat, going on in an Arithmetical Progreſſion, beginning with that Degree of Heat, which there is in the Air in Froſty Weather, when Water makes the *firſt Advances towards Freezing*; beginning the Account from this, as the loweſt Degree of Heat, or common *Terminus* of Heat and Cold) and ſuppoſing the external Heat of a *Humane Body* to be rated at 12 parts. In the ſecond Column are the Degrees of Heat in a Geometrical Proportion, ſo that the ſecond Degree is double the firſt, the third double the ſecond, and ſo on; the firſt Degree being that external Heat of a *Humane Body*, proportion'd to the Senſe. But now 'tis manifeſt from this Table that the Heat of *Boiling Water* is almoſt 3 times greater than that of a *Humane Body*; and that the Heat of melted *Tin* is 6 times, of melted *Lead* 8 times, of melted *Regulus* 12 times, and of ordinary Culinary Fire 16 or 17 times greater than the forememtion'd Heat of a Humane Body.

This Table was made by the help of a Thermometer and Red-hot Iron. By the Thermometer I found the Meaſure of all the Degrees of Heat as far as that by which Tin is melted; and by the hot Iron I found the Meaſure of the reſt. For the Heat which hot Iron does communicate to cold Bodies contiguous to it in a given time, (that is the Heat which the Iron it ſelf loſes) is as the whole Heat of the Iron. And therefore if the Times of Refrigeration are taken equal, the Degrees of Heat shall be in a Geometrical Proportion, and consequently may easily be found by a Table of Logarithms. First of all therefore I found by a Thermometer made of Linseed Oil, that if when the Instrument was placed in melting Snow, the Oil occupied a Space of 10000 parts, the same Oil rarified by a Heat of the first Degree (that is by that of a Humane Body) would extend to 10256 parts; and by the Heat of Water beginning to boil, to 10705 parts; and by the Heat of Water boiling vehemently, to 10725 parts; and by the Heat of melted Tin (cooling, and beginning to be of the Consistence of an Amalgama) to 11516 parts; and by the Heat of the same Tin when 'tis quite harden'd, to 11496 parts. Therefore the Oil was rarified in the proportion of 40 to 39, by the Heat of a Humane Body; and in the proportion of 15 to 14, by the Heat of boiling Water; and in the proportion of 15 to 13, by the Heat of the melted Tin, beginning to come to the Consistence of an Amalgama; and in the proportion of 23 to 20, by the Heat of the same Tin quite hardned.

The Rarefaction of Air with an equal Degree of Heat, was 10 times greater than that of Oil; and the Rarefaction of Oil nearly 15 times greater than that of Spirit of Wine. Now these things thus found, supposing the Degrees of Heat in the Oil to be proportional to its Rarefaction, and the Heat of a Humane Body to be 12 parts; from hence the Heat of Water when it begins to boil,

comes

comes to be 33 parts, and when it boils venemently, 34 parts; and the Heat of melted Tin beginning to come to the Consistence of an Amalgama, 72 parts; and the Heat of the same, when in cooling 'tis come to downright Hardness, 70 parts. And having determin'd these Things, in order to find out the rest, I heated a piece of Iron 'till it was Red-hot enough, and taking it out of the Fire with a pair of Tongs that were also Red-hot, and I it in a cool place, where the Wind blew constantly. Then putting upon it little pieces of Metals and various other liquable Bodies, I observ'd the times of Refrigeration, 'till all those melted parts having quite lost their Fluidity, became hal'd and solid again, and the Heat of the Iron was equal to that of a Humane Body. Then supposing the Excesses, of the Heats of the Iron and the liquefied Particles approaching to Induration, above the Heat of the Atmosphere founded by the Thermometer, to be in a Geometrick Progression, when the Times are in an Arithmetick one; by this means all the Degrees of Heat were discover'd. But 'tis to be observ'd that I plac'd the Iron not in a serene and quiet Air, but in a Wind blowing uniformly, so that the Air which was heated by the Iron might always be carried away by the Wind, and a cold Air with an uniform Motion might succeed in the place of it. For thus, equal parts of the Air were heated in equal times, and acquired a Heat proportional to that of the Iron. But the Degrees

Degrees of Heat found by this Method had the same Proportion among themselves, that those had which were found by the Thermometer; and therefore the Assumption, that the Rarefactions of the Oil were proportional to the Degrees of Heat, was a just and true one.

The

The Properties of the Catenaria.

By David Gregory, *M. D. Savilian Professor of Astronomy, and* F. R. S.

PROP. I. PROBLEM.

To find the Relation of the Fluxion of the Axis, to the Fluxion of the Ordinate in the Catenaria.

LET FAD be a *Catena* hanging on the Extremites F and D, the lowest point of which (or the Vertex of the Curve) is A, the Axis AD perpendicular to the Horizon, and the Ordinate BD parallel to the same: We are to find the Relation between Bb or Dδ, and dδ; supposing the point *b* infinitely near to B, and bd parallel to BD, as also Dδ to BA.

From the Principles of *Mechanicks*, 'tis plain that three *Powers* which are in *Equilibrio*, are in proportion to one another, as three right Lines parallel to their respective Directions (or inclin'd in any given Angle to them) and terminated at their mutual Intersection.

And

And consequently if Dd expounds the absolute Gravity of the Particle Dd (as it will be in a *Catena* equally thick) then dδ will represent that part of the Gravity which acts perpendicularly upon Dd, and by which it comes to pass that dD (being by the flexibility of the Chain moveable about *d)* endeavours to bring it self into a *Vertical* Position. And therefore if δd (or the Fluxion of the Ordinate BD) be *Constant*, the Action of the Gravity exerted perpendicularly upon the correspondent parts of the *Catena* Dd, will also be constant, or every where the same. Let this Action or Force be expounded by *a*.

Farther; From the above cited *Proposition* in Mechanicks, Dδ or the Fluxion of the Axis AB, will expound the Force to be exerted in the direction dD, which is equivalent to the former *Endeavour* of Dd (by which it tends to bring it self into a *Vertical* Position) and is sufficient to hinder it.

But this force arises from the *Linea Gravis* DA pulling with the direction dD, and is consequently (all the rest continuing as before) proportional to that Line DA. Therefore δd, the Fluxion of the Ordinate, is to δD, the Fluxion of the Abscisse, as the constant right Line *a*, to the Curve DA. Q: E: F.

COROL.

If the right Line DT touches the *Catenaria*, and meets the Axis AB produc'd in T, then will DB : BT : : (dδ : δD : :) a : DA Curve.

PROP.

PROP. II. THEOREM.

(Fig. 34.) *If upon the Perpendicular* AB *as an Axis, and the Vertex* A, *an Equilateral Hyperbola* AH *be describ'd, whose Semiaxis* AC = a; *as also upon the same Axis and Vertex, a Parabola* AP *whose Parameter is quadruple the Axis of the Hyperbola, and the Ordinate of the Hyperbola* HB *be always produc'd till* HF *be equal to the Curve* AP: *I say then, that (making* BD *and* BF, *equal) the Curve* FAD, *in which the Points* F, D, *are posited, is the* Catenaria.

Put AB $= x$; then Bb $= \dot{x}$, and BH $= \sqrt{2ax + xx}$; whence (from the Method of Fluxions) the Fluxion of BH, that is $mh = \frac{a\dot{x} + x\dot{x}}{\sqrt{2ax + xx}}$. Again, since the Parabola AP has for its Parameter $8a$, BP shall $= \sqrt{8ax}$. Whence the Fluxion of BP, that is $np = \frac{2a\dot{x}}{\sqrt{2ax}}$. Wherefore the Fluxion of the Curve

AP

AP $(=\text{Pp}=\sqrt{vp^q + \text{Pn}^q}) = \sqrt{\frac{4a^2\dot{x}^2}{2ax} + \dot{x}^2}$

$=\frac{\sqrt{2a\dot{x}^2 + x\dot{x}^2}}{x}$; which is equal to $\frac{2a\dot{x} + x\dot{x}}{\sqrt{2ax + xx}}$, as appears by multiplying both Numerator and Denominator into $\sqrt{2a + x}$. And since HF is every where = AP, the Fluxion of HF that is $mh + sf$, shall $= \frac{2a\dot{x} + x\dot{x}}{\sqrt{2ax + xx}}$. But we have hitherto found $mh = \frac{a\dot{x} + x\dot{x}}{\sqrt{2ax + xx}}$. Therefore sf (the Fluxion of BF the Ordinate in the *Catenaria*) $= \frac{a\dot{x}}{\sqrt{2ax + xx}}$; and consequently the Fluxion of the Curve AF (that is, $\text{Ff} = \sqrt{sf^q + \text{Fs}^q} = \sqrt{\frac{a^2\dot{x}^2}{2ax + xx} + \dot{x}^2}$) is $= \frac{a\dot{x} + x\dot{x}}{\sqrt{2ax + xx}}$, the Flowing Quantity of

which

which was shewn but now to be $\sqrt{2ax+xx}$. And therefore $AF=\sqrt{2ax+xx}$. And 'tis plain, that the Fluxion of the Ordinate BF, or $\frac{a\dot{x}}{\sqrt{2ax+xx}}$, is to $\dot{x}$ the Fluxion of the Abscisse AB, as the constant Quantity a, to the Curve AF; which was the Property of the *Catenaria* found above. Therefore the points of the *Catenaria* are rightly determin'd by the foregoing Construction. Q: E: D.

COROL. I.

It is manifest from the Construction, that BF the Ordinate in the *Catenaria*, is equal to the Parabolick Curve AP, taking away BH, the correspondent Ordinate, of the conterminal Hyperbola AH.

COROL. II.

'Tis plain from the Demonstration, that the Curve of the *Catenaria* AF, is equal to BH the correspondent Ordinate of the conterminal Equilateral Hyperbola. For since the Fluxions of these Lines are equal, and the Lines themselves do *arise* together, it is manifest that they are always, and every where equal. Whence, giving the *Catena*, AC or a will be given also, as being equal to the Semiaxis

miaxis of the Equilateral Hyperbola, whose Vertex is A, and whose Ordinate belonging the Abscisse AB, is equal to the *Catena* AD.

COROL. III.

All the *Catenaria* are similar to one another; as being generated from the similar Construction of Similar, and similarly posited Figures. From whence it follows, that two right Lines similarly inclin'd to the Horizon, carried thro' the Vertices of the *Catenaria*, will cut off similar Figures, and proportional to the Lines cutting off the Portions of the *Catenaria*.

COROL. IV.

If the *Catena* QAD be suspended on the points Q and D, which are unequally high, the part FAD of the Curve remains the same as if it were suspended by the points F and D, which are equally high. For it is no matter, whether the point be fix'd to the Vertical Plane or not.

COROL. V.

If the force of the *Catena* drawing in the Direction dD, be divided (as is commonly known) into the force as dδ acting with an Horizontal Direction, and the force as δD with a perpendicular Direction: Then it follows, that the force (in the end of the *Catena*)

tena) of approaching directly to the Axis, is to the force of descending perpendicularly in the same (or that part of the sustaining force that acts in the direction BD, is to that part that acts in the direction Dδ) as the Semiaxis of the Conterminal Hyperbola AH, to DA the length of the *Catena* to the Vertex. Whence, the *Catena* being given, this Ratio is also given. And in the same *Catena*, suspended with different degrees of Laxity, that Horizontal force, is as the Axis of the Conterminal Hyperbola; since DA remains the same, if the Extremities be equally high.

COROL. VI.

The *Catena* placed in an Inverted Position in a Vertical Plane, maintains its Figure and does not fall down; and so makes a fine Arch or *Fornix.* That is, very small hard slippery Spheres, dispos'd in the Inverted *Catenaria*, will form an Arch, no part of which will be thrust inwards or outwards by the rest, but (the lowest Points continuing unmov'd) it is preserv'd by vertue of its Figure. For since the Position of the Points of the *Catenaria*, and the Inclination of the parts to the Horizon, is the same, whether it be in the Position FAD, or in an Inverted Position, provided the Curve be in a Plane that is perpendicular to the Horizon, it is evident that it preserves its Figure unchanged, equally in one Position as the other. And on the other hand, the *Catenaria* are the only Genuine Arches or *Fornixes.* And an Arch of any other Figure, is for this reaſon only, ſuſtain'd, becauſe a *Catenaria* is included in the thickneſs of it. For if it were very thin, and conſiſted of parts that were ſlippery, it would not be ſuſtain'd. From the foregoing fifth *Corol.* it may be gather'd with what force an Arch thruſts the Walls outwards, that it ſtands upon; for this is the very ſame with that part of the force (ſuſtaining the *Catena)* which draws with the Horizontal direction. All other Matters requir'd in the ſtrength and firmneſs of Walls, that have Arches ſet upon them, are Geometrically determin'd from this Theory; which are the principal *Things* in *Building.*

COROL. VII.

If inſtead of Gravity, any other force were ſuppos'd acting in like manner upon a flexible Line, the ſame Curve would be produced. *Ex. gr.* Suppoſe a Wind blowing equably, and in directions parallel to a given right Line, the Line thus inflated by the Wind, would be the ſame with the *Catenaria.* For ſince all things that were conſider'd in Gravity, obtain in this other force, 'tis plain that the ſame Curve will be produced.

PROP.

PROP. III. THEOREM.

(Fig. 35.) *The Hyperbola* AH *continuing as before, if through* A *be drawn the right Line* GAL *perpendicular to the Axis* AB, *and the Curve* KR *be describ'd of such a Nature, that* BK *be a third proportional to* BH *and* AC, *and to the right Line* AC *be applied the Rectangle* AV *equal to the Interminate Space* ABKRLA; *then shall the Point* F *(the Concourse of the right Lines* HB, VG*) be in the* Catenaria.

For by Construction $BK = \frac{a^2}{\sqrt{2ax + x^2}}$, wherefore the Fluxion of the Space ABKRLA $(= BKkb = BK \times Bb) = \frac{a^2 \dot{x}}{\sqrt{2ax + x^2}}$.
And since $BF = \frac{\text{Space ABKRLA}}{AC}$, and AC is given, the Fluxion of BF shall $= \frac{\text{the Fluxion of the Space ABKRLA}}{AC} =$

$\frac{a\dot{x}}{\sqrt{2ax+x^2}}$. But in the Construction of the foregoing *Proposition*, the Fluxion of the Ordinate BF, was $= \frac{a\dot{x}}{\sqrt{2ax+x^2}}$. Therefore this Construction amounts to the same with that, and consequently the point F is in the *Catenaria*. Q: E: D.

COROL.

As in the foregoing *Proposition*, the *Catenaria* is describ'd from the length of the Parabolical Curve given; so in this, the description of it depends upon the Quadrature of the Space in which $x^2y^2 = a^4 - 2axy^2$. For BK or $y = \frac{a^2}{\sqrt{2ax+x^2}}$.

PROP.

PROP. IV. THEOREM.

(Fig. 36.) *The Space* AGF *contain'd under the* Catenaria AF, *and the right Lines* FG, AG, *parallel to* AB, BF, *is equal to the Rectangle under the Semiaxis* AC, *and* DH *the difference of the Ordinates in the* Hyperbola *and* Catenaria.

For $\dot{DH}$ ($= \dot{BH} = \dot{BD} =$ by *Proposition* II. $\frac{a\dot{x} + x\dot{x}}{\sqrt{2ax + x^2}} - \frac{a\dot{x}}{\sqrt{2ax + x^2}} =$) $\frac{x\dot{x}}{\sqrt{2ax + x^2}}$. Wherefore the Fluxion of the Rectangle under the given Line AC and DH ($= \frac{ax\dot{x}}{\sqrt{2ax + x^2}} = x \times \frac{a\dot{x}}{\sqrt{2ax + x^2}} = fs \times FG$) $=$ the Fluxion of the Space AFG. And since these Figures do *arise* both together, it follows that the Rectangle under AC and DH is equal to the Space AGF. Q: E: D.

COROL.

Hence it follows that the Space FAD comprehended under the *Catenaria* and Horizontal Line FD, is equal to the Rectangle under FD and BA, less the Rectangle under either Axis of the Hyperbola AH, and DH the excess of the right Line BH or the Curve AD, above the Ordinate BD.

PROP. V. THEOREM.

(Fig. 36.) *If to the right Line* AL *be applied the Rectangle* LE, *equal to the Hyperbolical Space* ALH, *then* E *will be the Center of Gravity of the* Cate*naria* AFD.

Let the Curve FA be conceiv'd to be librated upon the Axis GL. Then (from the Doctrine of Centers of Gravity) it is manifest that the *Momentum* of the ponderating Curve FA is expounded by the Superficies of an upright Cylinder erected upon FA, and cut off by a Plane, passing through GL, and making an Angle of 45° with the Plane of the Curve. And the Fluxion of this Superficies or $\dot{FA} \times FG$, is equal to the Fluxion of the Space ALH or $\dot{BH} \times HL$; because $\dot{FA}$, $\dot{BH}$, as also FG and HL, are equal. And

And consequently (since they *arise* together) the said Superficies of the upright Cylinder is equal to the Hyperbolical Space ALH. Which therefore divided by the Pondus it self AF, or its equal the right Line AL, gives the right Line AE, for the distance of the Center of Gravity from the Axis of Libration GL. So that the point E is the Center of Gravity of the Curve FAD, lying equally on both sides the Axis. Q: E: D.

COROL. I.

The Spaces ABHL, BAH, and AFG, are in Arithmetick Proportion. For the Fluxion of the Space ALH is $\left(= \frac{ax+xx}{\sqrt{2ax+x^2}} \times \dot{x} = \frac{\overline{ax+x^2}\times\dot{x}}{\sqrt{2ax+x^2}} = \frac{\overline{2ax+x^2-ax}\times\dot{x}}{\sqrt{2ax=x^2}} = x\sqrt{2ax+xx} - \frac{ax\dot{x}}{\sqrt{2ax+x^2}}\right)$ = to the Fluxion of the Space BAH less the Fluxion of the Space AGF, by *Proposition* IV. And since these three Figures do *arise* together, it follows that BAH — AGF = (ALH =) BL — BAH. Wherefore 2BAH = BL + AGF.

 Whence

Whence 'tis plain that the Spaces BL, BAH, and AGF, are in Arithmetical Proportion.

COROL. II.

The Center of Gravity of the *Catenaria*, is the *Loweſt* of all thoſe Lines that have the ſame *Termini*, and are of the ſame length. For a heavy Body will deſcend as far as it can. And ſince the Figure it ſelf deſcends as much as its Center of Gravity deſcends; 'tis manifeſt that a flexible heavy Line, will diſpoſe it ſelf in ſuch a manner, as that its Center of Gravity may be lower, than if it aſſum'd any other Figure. And from this one Property of ſuch a Line, all the reſt may eaſily be deduced.

COROL. III.

If there be upright Cylinders erected upon any ſort of Curves, that are of the ſame length, and have the ſame *Termini* D and F, with the *Catenaria* FAD; and theſe Cylinders be cut by a Plane paſſing through DF; then the greateſt of all theſe Superficies ſhall be that which ſtands upon the *Catenaria*. For theſe Superficies (if the Angle contain'd under the Planes be half a right one) divided by the Curves (which in the preſent Caſe are all of the ſame length) give the diſtances of the Centers of Gravity from the right Line DF. And ſince this diſtance is greateſt in the *Catenaria* (becauſe of the greateſt Deſcent of the Center of Gravity) therefore

therefore the Cylindrick Superficies shall there also be greatest. Lastly, Because the same is to be said of Cylindrick Superficies cut off by a Plane that make any Angle with the Plane of the Basis, as is wheu the said Angle is half a right one; the Truth of what was asserted is evident universally.

LEMMA.

(Fig. 37.) *Any Curve as* AFQ, *describ'd by the* Evolution *of another Curve* KU, *if upon any Ordinate, as* FB *(at right Angles to the Axis* AB*) be let fall perpendicularly* UR, *from the correspondent Point* U *in the Curve* KU; *then (the Fluxion of the Axis* AB *continuing the same) shall the* Fluxion *of the Fluxion of the Ordinate* BF, *the Fluxion of the Curve* AF, *and the right Line* FR *be continual Proportionals.*

Let the Lineola Ff be produc'd 'till it meets the next Ordinate wφ in *o*. And because by the Hypothesis Fs = fw, also shall *of* = Ff, and consequently oφ shall be the Fluxion of fs, that is the *Fluxion* of the Fluxion of the Ordinate. Farther, the Triangles oφf, fFR, are Equiangular, because oφf = its Alternate fFR, and foφ = (Ffr =) FfR, because their difference Rfr is as nothing in respect of either of them, since Rr is nothing in respect of fr. And therefore oφ : φf : : fF : FR; but φf = fF, since they differ

fer but by the Fluxion of either. Therefore also o φ : f F : : f F : F R. Q: E: D.

PROP. VI. PROBLEM.

(Fig. 37.) *To find the Curve* KV *by the Evolution of which the* Catenaria *is describ'd.*

Let (as before) AB=x, BF=y. Then by *Proposition* II. $\dot{y} = \frac{a\dot{x}}{\sqrt{2ax+xx}}$, or $2ax\dot{y}^2 + xx\dot{y}^2 = a^2\dot{x}^2$. Wherefore (by the *Newtonian* Method which now generally obtains) $2a\dot{x}\dot{y}^2 + 4ax\dot{y}\ddot{y} + 2x\dot{x}\dot{y}^2 + 2xx\dot{y}\ddot{y}$ (= $2a\dot{x}\ddot{x}$, which because of $\ddot{x} = o$, since the constant $\dot{x}$ has no Fluxion, is) = o. Therefore $\ddot{y} =$

$$\left(\frac{-a\dot{x}\dot{y} - x\dot{x}\dot{y}}{2ax+xx} =\right) \frac{\overline{a+x}}{2ax+xx} \times \frac{a\dot{x}^2}{\sqrt{2ax+xx}},$$

putting instead of $\dot{y}$ its Value $\frac{a\dot{x}}{\sqrt{2ax+xx}}$.

(For the Sign − before the Quantity $\ddot{y}$, denotes

notes only the place of the point R, with reſpect to F, to be oppoſite to the place of the point F, with reſpect to B, when the Curve AFQ is concave towards the Axis AB.) And Ff (by *Prop.* II.) $= \frac{\overline{a+x} \times \dot{x}}{\sqrt{2ax+xx}}$. Wherefore (by the foregoing *Lemma*) FR $=$

$$\left(\frac{\overline{Ff}^{q}}{\ddot{y}} = \frac{\overline{a+x}^{2} \times \dot{x}^{2}}{2ax+xx} \times \frac{\overline{2ax+xx} \times \sqrt{2ax+xx}}{\overline{a+x} \times a\dot{x}^{2}} = \right) \frac{\overline{a+x} \times \sqrt{2ax+xx}}{a}.$$

Again, becauſe of the Rectangular Triangles Fsf, FRU, having the Angles fFs, UFR, equal to one another (becauſe UFs is the Complement of either to a right Angle) we have Fs : sf :: FR : UR, or $x : \frac{a\dot{x}}{\sqrt{2ax+xx}} :: \frac{\overline{a+x} \times \sqrt{2ax+xx}}{a} :$ UR, which therefore is $= a+x$. Therefore the Nature of the Curve KU is ſuch, that if AB $= x$, FR ſhall $= \frac{\overline{a+x} \times \sqrt{2ax+xx}}{a}$, and UR $= a+x$. Q: E: I.

COROL.

COROL. I.

AC : CB :: BH : BH : FR. For this is the Property of the right Line FR, that was found just now.

COROL. II.

The right Line CB is = the right Line BI or UR. For each is $= a+x$.

COROL. III.

The *Evolvent* Line UF is a third Proportional to AC and CB. For because of the similar Triangles fFs, UFR, it is sF : Ff :: FR : UF, or $x : \frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+xx}} :: \frac{\overline{a+x} \times \sqrt{2ax+xx}}{a} :$ UF, which is therefore $= \frac{\overline{a+x}^2}{a}$. Whence $a : a+x :: a+x :$ UF, which is the Radius of the Circle that has the same Curvature with the *Catenaria* at the point F.

COROL.

COROL. IV.

When the point F is in A, or when the Vertex is describ'd by the Evolution, that is, when $x = o$, then the Value of the Evoluent Line (or the Radius of the Curvature) UF (which in this Case coincides with KA) *viz.* $\frac{\overline{a+x}^2}{a}$, becomes only a. That is, the point K where the Curve UK meets the Axis, is as much above the Vertex of the *Catena* A, as C is below it. Whence the Diameter of a Circle that has the same Curvature with the *Catena* at the Vertex, is equal to the Axis of the Conterminal Hyperbola AH. And consequently the *Catena* AD and the Hyperbola AH, have the same Curvature in the Vertex A. For it is known that the foremention'd Circle has the same Curvature with the Equilateral Hyperbola AH, in the Vertex A. But this follows also from the Property of the *Catenaria*, demonstrated at *Proposition* II. For the *Nascent* FH or (AP = the *Nascent* BP =) $\sqrt{8ax}$, is double the *Nascent* BH or ($\sqrt{2ax+xx}$, that is, xx vanishing, when x is very small) $\sqrt{2ax}$. And therefore the same point is as well in the Nascent *Hyperbola*, as in the Nascent *Catenaria*; that is, the one is coincident with the other at their first *arising*, and consequently these Curves have the same Curvature at the Vertex A.

COROL.

COROL. V.

The Curve KU is a third Proportional to the right Line AC, and the Curve AF or the right Line AL. For from the Property of the Evolution, KU = (UKA—KA=VF—KA = $\frac{\overline{a+x}^2}{a} - a = \frac{a^2+2ax-x^2}{a} - a =$) $\frac{2ax+xx}{a}$. And therefore, $a : \sqrt{2ax+xx} :: \sqrt{2ax+xx} :$ KU. But (by *Cor.* II. *Prop.* II.) $\sqrt{2ax+xx}$ = AF. Wherefore AC : AF :: AF : KV.

COROL. VI.

The right Line KI is double of AB. For since BI = (BC =) CA+AB, also AI shall = CA+2AB. But AK=AC (by *Cor.* IV. of this *Proposition.*) Therefore KI=2AB.

COROL. VII.

The Rectangle AC×BR is = to double the Hyperbolical Space BAH. For FR×AC = $(\frac{\overline{a+x} \times \sqrt{2ax+xx}}{a} \times a = \overline{a+x} \times \sqrt{2ax+xx} =$ $x \times \sqrt{2ax+xx} + a \times \sqrt{2ax+xx}$ = AB × BH +AC

+ACxBH=) ABxBH+ACxBD+ACxDH. Wherefore FRxAC—BDxAC (that is, BRxAC) =ABxBH+ACxDH. But (by *Proposition* IV.) ACxDH=Space AGF. Therefore BRxAC= (ABHL+AGF= by *Cor.* I. *Proposition* V.) 2BAH.

PROP. VII. THEOREM.

(Fig. 37.) *If in the Logarithmical Curve* LAG *(whose Subtangent* HS, *given, is equal to the Line* a, *determin'd as at* Cor. II. Prop. II.) *be taken the point* A, *whose distance* AC *from the Assymptote* HP, *is equal to the Subtangent* HS; *and from the points* H, *and* P *(taken at Liberty in the Assymptote, and equally distant from the point* C) *be erected the Lines* HL, PG, *Ordinates to the Logarithmical Curve, the half Sum of which is equal to* HD *or* PF: *Then the points* D *and* F, *shall be posited in the Curve of the* Catenaria, *corresponding to the right Line* AC.

Let AB be put $=x$, and consequently CB or DH the half Sum of the Ordinates HL, PG, will $=a+x$. Let the half difference of them be put $=y$; whence HL $=a+x+y$, and PG $=a+x-y$. And since from the Nature of the Logarithmical Curve, CA is a mean Proportional between them, aa shall $=$

aa

$aa + 2ax + xx - yy$, whence $y = \sqrt{2ax + xx}$. Consequently HL $= a + x + \sqrt{2ax + xx}$, and PG $= a + x - \sqrt{2ax + xx}$. Wherefore the Fluxion of HL, or *lm*, is

$$\frac{a\dot{x} + x\dot{x} + \dot{x}\sqrt{2ax + xx}}{\sqrt{2ax + xx}}.$$

And because of the similar Triangles *lm*L, LHS, 'tis LH : *HS* :: *lm* : *m*L; whence *m*L or $d\mathcal{S}$ the Fluxion of BD, is $= \frac{a\dot{x}}{\sqrt{2ax + xx}}$. That is, the Curve AD, generated after the foregoing manner, from the Logarithmical Curve, is of such a Nature, that if the Axis be x, and its Fluxion $\dot{x}$, the Fluxion of the Ordinate BD is $\frac{a\dot{x}}{\sqrt{2ax + xx}}$. But this is the Property of the *Catenaria* corresponding to the right Line *a*, as was demonstrated at *Prop.* I. Therefore the Curve FAD describ'd as above, is this very *Catenaria* it self. Q: E: D.

C O-

COROLLARIES.

COROL. I.

As the *Catenaria* is deſcrib'd by the help of the Logarithms, ſo on the other hand, by the help of the *Catenaria* (a Curve produced by Nature it ſelf) the Logarithm of any given *Number*, or rather of any given *Ratio*, may be found. As if, putting CA=1, whoſe Log.=0; the Log. of the Number CQ, or of the Ratio between CA and CQ, were ſought. Let CV be a third Proportional to CQ and CA, and CB the half Sum of CQ and CV; then an Ordinate to the *Catenaria* from the point B, *viz.* BD, will be the Log. ſought.

COROL. II.

Vice verſa, if giving the Log. CH or CP, the correſpondent Number HL or PG, or the Ratio of HL to CA, or PG to CA, be ſought. From H or D erect a Perpendicular meeting the *Catena* in D or F, and in the Horizontal Line AR, take CR=HD or PF, or CB. And then will AR be the half difference of the ſought Lines LH, GP, as HD or CR, is (from the above demonſtrated Property of the *Catenaria)* their half Sum. For in three Quantities Geometrically Proportional, ſuch as are HL, CA, PG, the Square of the half Sum of the Extreams leſſen'd by

R the

the Square of the middle Term, is equal to the Square of the half difference of the Extreams. And consequently CR+AR, and CR—AR, are the Numbers HL or GP, agreeing to the given Log. CH or CP.

COROL. III.

It is plain from the Demonstration, that as HD the half Sum of the Logarithmical Ordinates HL, PG, being applied at right Angles to CH, is an Ordinate to the *Catenaria*; so also the half difference of the same HL, PG, applied at right Angles to CA in B, is an Ordinate to the Equilateral Hyperbola, whose Center is C, and its Vertex A; and consequently (by *Cor.* II. *Prop.* II.) = the *Catena* AD. For $y = \sqrt{2ax+xx}$; and since it was shewn in the foregoing *Corol.* that AR is also the half difference of HL and PG; 'tis plain that AR is = the Portion of the *Catenaria* AD. From whence by the way, we may observe a Method, how, from the *Catena* AD given, to find C the Center of the Conterminal Hyperbola, or the point in the Asymptote of the Logarithmical Curve GL. For taking AR= the *Catena* AD, and joining the points B, R, from the middle of BR erect a Line perpendicular to it, which will meet BA the Axis of the *Catena* produced, in the point C, sought. Which is evident, since thus CR will = CB.

COROL.

COROL. IV.

Hence also it follows that if the Angle BDT be equal to ACR, the right Line DT touches the *Catenaria* in D. For then it will be (in the similar Triangles DBT, CAR) DB : BT :: CA : AR, or CA : Curve AD which is = AR. And consequently DT touches the *Catenaria*, by *Corol. Prop.* I.

COROL. V.

It follows also that the Space ACHD = the Rectangle CA × AR. For because (by *Prop.* IV.) AYD = CA × (AD — BD, = AR — AY, by *Cor.* III. of this *Prop.* =) YR; the thing is manifest. And since CA is given, 'tis plain that the Space ACHD is as the Curve AD, and the Fluxion of the former Hd, as the Fluxion of the Latter Dd.

COROL. VI.

If through the point K where CR cuts HD, we draw KZ parallel to PH, meeting AC in Z, and take $CE = \frac{BC + CZ}{2}$; then will E be the Center of Gravity of the Curve FAD. Imagine an upright Cylindrick Superficies erected upon FAD, and to be cut by a Plane passing through PH, and making an Angle of 45 with the Plane of the Curve FAD. This Superficies, will expound the

Momentum of the Curve FAD librated on the Axis PH; and its Fluxion is $DH \times D\dot{d} + PF \times F\dot{f} = 2BC \times \dot{AD} = \overline{2a+2x} \times \frac{a\dot{x}+x\dot{x}}{\sqrt{2ax+xx}} = \frac{2a^2\dot{x}+4ax\dot{x}+2xx\dot{x}}{\sqrt{2ax+xx}} = \frac{a^2\dot{x}}{\sqrt{2ax+xx}} + \frac{a^2\dot{x}+ax\dot{x}}{\sqrt{2ax+xx}} + \frac{3ax\dot{x}+2x^2\dot{x}}{\sqrt{2ax+xx}}$; the Fluent of which, $a \times BD + a\sqrt{2ax+xx} + x\sqrt{2ax+xx} = CA \times BD + CB \times AD$. Wherefore $CA \times BD + CB \times AD =$ (because it *arises* together with it) to the forementionʼd Cylindrick Superficies = the *Momentum* of the Curve FAD with respect to the Axis of Libration PH. Whence the distance of the Center of Gravity of the Curve FAD from the point C, is $\frac{CA \times BD + CB \times AD}{2AD}$, or $\frac{1}{2}\frac{CA \times BD}{AD} + \frac{1}{2}CB$. Farther, because of ZK parallel to AR, ʼtis AD : BD :: (AR : ZK ::) CA : CZ, whence $CZ = \frac{CA \times BD}{AD}$, and therefore CE which by Construction is $= \frac{1}{2}BC + \frac{1}{2}CZ$, shall $= \frac{1}{2}\frac{CA \times BD}{AD} + \frac{1}{2}BC$. That is, the Center of Gravity of the Curve FAD, and the point E determinʼd determin'd by this Construction, are equally distant from the point C. But they are also posited in the same right Line, and towards the same parts, and therefore they coincide with one another. This Coincidence of the point E as determin'd above, with the Center of Gravity as found at *Prop.* V. may be thus *synthetically* shewn. By *Cor.* I. *Prop.* V. $2BAX = AYD + BA \times AR$. Whence $AH + 2BAX = (ACHD + BA \times AR =$ by *Cor.* foregoing) $AR \times CA + BA \times AR$; that is, $BD \times AC + 2BAX = AR \times CB$; or $BD \times AC = AR \times CB - 2BAX$. Whence $BD \times AC + AD \times BC = (AD \times BC + AR \times CB - 2BAX = 2AD \times BC - 2BAX =) \; 2AD \times AC + 2AD \times AB - 2BAX$. And dividing by 2AD, we have $\frac{1}{2} \frac{BD \times AC}{AD} + \frac{1}{2} BC = (AC + \frac{AB \times AD - BAX}{AD} =) \; CA + \frac{ARX}{AR}$. But $\frac{ARX}{AR}$ is the distance of the Center of Gravity of the *Catena* from the Vertex A, determin'd at *Prop.* V. and consequently, according to the 5th *Proposition* $CA + \frac{ARX}{AR}$ is the distance of the point E from C; now $\frac{1}{2} \frac{BD \times AC}{AD} + \frac{1}{2} BC$, is the distance of the point E also from the same point C according to this *Cor.* Whence 'tis manifest that these two Determinations of the point E amount to the same; because $CA + \frac{ARX}{AR} = \frac{1}{2} \frac{BD \times AC}{AD} + \frac{1}{2} BC$.

COROL. VII.

The Center of Gravity of the Space PFADH, is in I the middle point of the right Line CE. For since the Center of Gravity of the Fluxion of AD, or Dd, and Ff, is twice as far distant from PH, as is the Center of Gravity of the Fluxion of ACHD, or DHhd, and FPpf; and Dd+Ff×AC is = DdhH+FfPp; 'tis plain that E, the Center of Gravity of the Fluent FAD, is twice as far distant from PH, as I, the Center of Gravity of the Fluent PFADH. But this may be yet shewn otherwise according to the Method us'd before. Imagine an upright Cylinder to be erected upon the Figure PFADH, and to be cut off by a Plane passing through PH, and making an Angle of 45 with the Plane of the Basis. This Solid will expound the *Momentum* of the Figure PFADH librated on the Axis PH. And the Fluxion of this Solid or *Momentum* (viz. the Solids erected on the Basis PFfp, and HDdh) is produced, by multiplying the *Momentum* of the Fluxion, or the Fluxion of the *Momentum*, into $\frac{1}{2}$ AC given. For by *Cor.* V. of this *Proposition* HDdh=Dd×AC. Wherefore the Fluent *Momentum* it self, is produced by multiplying the *Momentum* of the Curve FAD with respect to the Axis PH (as determin'd at *Cor.* foregoing) *viz.* CA×BD+CB×AD into $\frac{1}{2}$ AC; which will therefore be $\frac{1}{2}$ AC×AC×BD+$\frac{1}{2}$ AC×CB×AD. And consequently if this be divided by the librated

librated Figure PFADH ($=2CA\times AD$, by *Cor.* V. of this *Proposition)* there will arise (for the distance of the Center of Gravity of the Figure PFADH from the Axis PH) $\frac{1}{4}\frac{CA\times BD}{AD}+\frac{1}{4}CB$; which is $=\frac{1}{2}CE$ determin'd above.

COROL. VIII.

If through the point N where DT the Tangent to the *Catenaria* in D, cuts the Line AR, be drawn a Parallel to BC, meeting in O a parallel to AR drawn through E; then will O be the Center of Gravity of the Curve AD. For by *Cor.* 6. the Center of Gravity of the Curve AD is in the right Line EO. But it shall be demonstrated to be in the right Line NO; and consequently that O it self shall be the point. Let DA be conceiv'd to be librated upon the Axis HL; then the *Momentum* of this is the Curve DA multiplied into the distance of the Center of Gravity from HL. And consequently its Fluxion $=$ DA $\times$ Hh (Hh being the Fluxion of the distance of the Axis of Libration from the Center of Gravity) $=\sqrt{2ax+xx}\times\frac{a\dot{x}}{\sqrt{2ax+x^2}}=a\dot{x}$. And therefore the *Momentum* of the Curve DA, with respect to the Axis HL, is $=ax$. And consequently the distance of the Center of Gravity from the same Axis, is ax divided

 by

by AD, or $\frac{AC \times DY}{AR}$. But because DT touches the *Catenaria*, by *Cor.* 4. of this *Proposition*, the Angle BDT, or DNY=ACR, and the Angles at A and Y are right ones, therefore in the Equiangular Triangles RAC, DYN, 'tis RA: AC:: DY: YN; whence $YN = \frac{AC \times DY}{RA}$, that is YN is the distance of the Center of Gravity of the *Catena* AD from the Axis HL; or that Center is in the right Line NO.

COROL. IX.

If through the point I be drawn a right Line parallel to AR, meeting ON produc'd in W; then W shall be the Center of Gravity of the Space ACHD. For by *Cor.* 7. the Center of Gravity of the Space ACHD, is in the right Line IW, but it shall be demonstrated also that 'tis in NW, and consequently W is the point. For (after the same manner as in *Cor.* foregoing) the Fluxion of the *Momentum* of the Space ACHD ponderating upon the Axis HL, will be shewn to be

$$(ACHD \times Hh = AC \times AD \times Hh =)\ a\sqrt{2ax + xx} \times \frac{a\dot{x}}{\sqrt{2ax + xx}} = a^2\dot{x}.$$

And consequently the *Momentum* of the Space ACHD, with respect to the Axis HL, is the Fluent of this Fluxion,

Fluxion, $a^2\dot{x}$, that is, $a^2\dot{x}$. This therefore divided by the Space ACHD, or $a \times \sqrt{2ax+xx}$, gives the distance of the Center of Gravity (of the Space ACHD) from the Axis HL, which is $= \frac{ax}{\sqrt{2ax+xx}} = \frac{AC \times DY}{AR}$. And therefore the Center of Gravity of the Space ACHD, is in the Line NW. And from these two last *Corollaries*, is found the Center of Gravity of any Portion of the *Catena*, though not reaching the Vertex A, or also of any Space comprehended under any Portion of the *Catenaria*, and any other right Lines besides those aforesaid.

COROL. X.

Hence are measur'd the Surfaces and Solids generated by the Rotation of the *Catena* (or a Space comprehended under it, and a right Line) about any given Axis. For a Figure generated by such a Rotation, is (as is vulgarly known) equal to the generating Figure multiplied into the Periphery describ'd by the Center of Gravity in the Rotation, which Periphery is given, since the Radius or Distance of the Center of Gravity from the given Axis, is given. Thus if the *Catena* AD roul'd about the Axis AB, then

then $\frac{\pi}{\rho}$ AN is the Periphery describ'd by the Center of Gravity O ($\frac{\pi}{\rho}$ denoting the Ratio of the Periphery of a Circle to the Radius) and consequently the Surface generated by the Rotation of the *Catena* AD $= (\frac{\pi}{\rho} \times AN \times AD =) \frac{\pi}{\rho} \times AN \times AR$. That is a Circle, the Square of whose Radius is double the Rectangle RAN, will = the Surface generated by the Rotation of the *Catena* AD about the Axis AB. After the same manner the Solid generated by the Rotation of the Space ACHD about AC, may be shewn to be equal to a Cylinder, whose Basis is the forementioned Circle, and its Altitude=AC. Thus also the Surfaces and Solids produced by the Rotation of these Figures about any other given Axis, are measur'd. For giving the Center of Gravity, they are easily discover'd.

Of

Of the Quadratures of Geometrically irrational Figures.

By J. Craig.

LET ACF (*Fig.* 38.) be a Semicircle, whose Diameter is AF, ADE a Geometrically irrational Curve, whose Ordinate BD cuts the Semicircle in C. The Quantities may be noted thus; The Diameter AF $= 2a$, the Abscisse AB $= y$, the Arc AC $= v$, the Ordinate BD $= z$: And let $z = rvy^n$ a General Equation expressing the Nature of the Geometrically irrational Curves ADE, in which r denotes any given and determin'd Quantity, and n an indefinite Exponent of the indetermin'd Quantity y. I say the Area,

$$ABD = \frac{rvy^{n+1}}{n+1} - qv + \sqrt{2ay - yy} \times \frac{ra}{\overline{n+1}|^2} y^n + \frac{2nra^2 + ra^2}{n \times \overline{n+1}|^2} y^{n-1} + \frac{aA \times \overline{2n-1}}{n-1} y^{n+2} + \frac{aB \times \overline{2n-3}}{n-2} y^{n-3} + \frac{aC \times \overline{2n-5}}{n-3} y^{n-4} + \frac{aD \times \overline{2n-7}}{n-4} y^{n-5}$$

$aE \times \overline{2n-9}$

$$+\frac{aE\times 2n-9}{n-5}y^{n-6}+\&c.$$

In this Infinite Series, these things are to be taken notice of: (1.) That the Capital Letters A, B, C, D, E, &c. denote the Coefficients of the Terms immediately preceeding them, *viz.* $A=\frac{2nraa+raa}{n\times n+1\times n+1}$ $B=\frac{aA\times 2n-1}{n-1}$, $C=\frac{aB\times 2n-3}{n-2}$, and so on. (2.) That if the Exponent n be an Integer and Positive, or equal to nothing, or if $2n$ be an odd Number, then the Quadrature of the Space ABD may be exhibited by a finite Quantity: The Series in these Cases breaking off. (3.) That q denotes the Term last breaking off. (4.) That all those Figures in which the Series is broke off have one Geometrically Quadrable Portion very easily assignable from the Series it self, *viz.* if you make the Abscisse $y=\overline{r}^{\frac{1}{n+1}}+\overline{nq+q}^{\frac{1}{n+1}}$; there will arise a Geometrically Quadrable Area answering to this Abscisse. (5.) That only the Irrational Terms $\sqrt{2ay-yy}$ is to be multitiplied into the Terms following it.

Example

Example I.

Let $z = v$, becauſe in this Caſe $r = 1$, $n = o$, therefore $\frac{ra}{\overline{n+1}|^2} y^n$ is the Term laſt breaking off, wherefore $q = a$, whence ABD $= vy - av + a\sqrt{2ay - y2}$: And conſequently if (by Note 4.) you take the Abſciſſe $y = a$, that is, if the Ordinate paſs through the Center of the Circle, there will ariſe a Geometrically Quadrable Portion fitting it, *viz.* Area $= a^2$, that is, the Square of the Radius.

Example II.

Let $z = \frac{vy}{a}$. Becauſe in this Caſe $r = \frac{1}{a}$, $n = 1$, therefore $\frac{2na^2 + ra^2}{n \times \overline{n+1}|^2} y^{n-1}$ is the Term laſt breaking off, wherefore $q = \frac{3a}{4}$; whence ABD $= \frac{vy^2}{2a} - \frac{3av}{4} + \frac{\overline{y+3a}}{4} \sqrt{2ay - y^2}$, and conſequently, if (by Note 4.) you take $y = \sqrt{\frac{3aa}{2}}$, there will ariſe a Geometrically Quadrable

Quadrable Area fitting this Absciffe, *viz.*

$$\text{Area} = \sqrt{:\sqrt{6a^4} - \frac{3a^2}{2} \times \sqrt{\frac{3a^2}{32}} + \frac{3a}{4}}.$$

Example III.

Let $z = \frac{vy^2}{aa}$; In this Case $r = \frac{1}{aa}$, $n = z$, therefore $\frac{aA \times 2n - 1}{n - 1} y^{n-2}$ is the Term last breaking off, therefore $q = \frac{5a}{6}$; whence by Infinite Series, will ABD =

$$\frac{6vy^3 - 15a^3v + 2ay^2 + 5a^2y + 15^3 a\sqrt{2ay - y^2}}{18a^2};$$

And consequently, if (by Note 4.) you take $y = \sqrt[3]{\frac{5a^3}{2}}$, there will arise a Geometrically Quadrable Area fitting this Absciffe, *viz.*

$$\text{Area} = \frac{2ay^2 + 5a^2y + 15a^2}{18a} \times \sqrt{2ay - y^2}.$$

Secondly, Let ACF (*Fig.* 39.) be a *Parabola*, AE its Axis, A the Vertex, and (B *a*) the *Latus Rectum*. And let ADG be a Geometrically irrational Curve, whose Ordinate BD cuts the *Parabola* in C. Let the Absciffe AB = *y*, AB $= y$, the Ordinate BD $= z$, the Arc of the *Parabola* AC $= v$. And let the General Equation expressing the Nature of Infinite irrational Curves be this, $Z = rvy^n$, in which r denotes a given and determinate Quantity, and n an indefinite Exponent of the indetermin'd Quantity y. I say the Area

$$ABD = \frac{ry^{n+1} \times v}{n+1} - qv + \sqrt{2ay + y^2} \, x -$$

$$\frac{r}{n+2 \times n+1} y^{n+1} - \frac{ra}{n+2 \times \overline{n+1}|^2} y^n +$$

$$\frac{ra^2 \times 2n+1}{n \times n+2 \times \overline{n+1}|^2} y^{n-1} - \frac{aA \times 2n-1}{n-1} y^{n-2} +$$

$$\frac{aB \times 2n-3}{n-2} y^{n-3} - \frac{aC \times 2n-5}{n-3} y^{n-4} + \&c.$$

In this Series 'tis to be noted: (1.) That the Capital Letters A, B, C, *&c.* denote the Coefficients of the Term preceding them. (2.) That if the Exponent n be an Integer and Positive, or equal to nothing, or if $2n$ be an odd Number, then the Quadrature may be exhibited by a finite Number of Terms; the Series in these Cases breaking off. (3.) That $+q$ is equal to the Term last breaking off. (4.) That of the Terms multiplying the Quantity $\sqrt{2ay + y^2}$, the last breaking off is to be doubl'd. (5.) That all those Figures in which n is an Integer, Positive and an odd Number, or more generally, all those Figures in which the last Term breaking off has

has an Affirmative Sign or $+$, have one Geometrically Quadrable Portion, and assignable from the Series it self, by taking the Abscisse as in the fourth Note of the preceding Series.

Example I.

Let $z = v$, because in this Case $r = 1$, $n = 0$, therefore the Term last breaking off is $-\frac{ra}{n+2 \times \overline{n+1}|^2} y^n$, whence $+q = \frac{a}{2}$ (by Note 3.) and because in this Case $-\frac{a}{2}$ is the last Term to be multiplied into $\sqrt{2ay - y^2}$, therefore $ABD = vy + \frac{av}{2} + \sqrt{2ay + y^2} \times \overline{-\frac{1}{2}y - a}$.

Example II.

Let $z = \frac{vy}{a}$, because in this Case $r = \frac{1}{a}$, $n = 1$, therefore the Term last breaking off is $\frac{a^2 \times 2n+1}{n \times n+2 \times \overline{n+1}|^2} y^{n-1} = \frac{a}{4}$, whence $q = \frac{1}{4}a$, and $\frac{1}{2}a$ is the last Term to be multiplied by $\sqrt{2ay+y^2}$, therefore

ABD

$$ABD = \frac{vy^2}{a^2} - \frac{av}{4} + \sqrt{2ay + y^2}\,x - \frac{y^2}{6a} -$$

$\frac{7}{12} + \frac{a}{2}$. And if you take $y = \sqrt{\frac{1}{2}aa}$, there will arise a Geometrically Quadrable Area fitting this Abscisse, *viz.* Area $= \frac{1}{12}$

$$\sqrt{:\sqrt{2a^4} + \frac{a^2}{2} \times 5a - \sqrt{\tfrac{1}{2}a^2}:}$$

I have other Theorems of this Nature, for Figures depending on the Circle and Parabola; but these two may suffice as a Specimen to shew the Use of my Method publish'd in my *Treatise of Quadratures*, in determining the Quadratures of Irrational Figures, for which there has been no Method (as far as I know) as yet made Publick.

That the Reader may the more easily come at the Invention of these and the like Theorems, I shall subjoin another, and more hereafter, if need be.

Let therefore (*Fig.* 40.) ACF be a Semicircle, ADE a Geometrically Irrational Curve, whose Ordinate BD cuts the Semicircle in C. Let the Quantities be denoted as before, *viz.* the Diameter AF $= 2a$, the Abscisse AB $= y$, the Arc AC $= v$, the Ordinate BD $= z$; and let $z = rv^2y^n$, an Equation expressing the Nature of the Curves ADE, in which r denotes any given and determin'd Quantity, and n an indefinite Exponent of the indetermin'd Quantity y. I say the Area

$$ABD = \frac{rv^2 y^{n+1}}{n-1} \quad qv^2 + v\sqrt{2ay - y^2} \times$$

$$\frac{2ra}{\overline{n+1}|^2} y^n + \frac{2ra^2 \times \overline{2n+1}}{n \times \overline{n+1}|^2} y^{n-1} +$$

$$\frac{aA \times \overline{2n-1}}{n-1} y^{n-2} + \frac{aB \times \overline{2n-3}}{n-2} y^{n-3} +$$

$$\frac{aC \times \overline{2n-5}}{n-3} y^{n-4} + \frac{aD \times \overline{2n-7}}{n-4} y^{n-5} +$$

$$\frac{aE \times \overline{2n-9}}{n-5} y^{n-6} \&c. - \frac{2ra^2}{\overline{n+1}|^3} y^{n+1} -$$

$$\frac{2ra^3 \times \overline{2n+1}}{n^2 \times \overline{n+1}|^2} y^n - \frac{a^2 A \times \overline{2n-1}}{\overline{n-1}|^2} y^{n-1} -$$

$$\frac{a^2 B \times \overline{2n-3}}{\overline{n-2}|^2} y^{n-2} - \frac{a^2 C \times \overline{2n-5}}{\overline{n-3}|^2} y^{n-3} \&c.$$

In this Theorem these Things are to be taken notice of; (1.) That 'tis made up of two Infinite Series, the former of which (connected by the Sign +) is multiplied into $v\sqrt{2ay-y^2}$; but the Terms of the latter (affected by the Sign —) are Absolute. (2.) That in the former Series, the Capital Letters, A, B, C, *&c.* denote the Coefficients of the Terms respectively preceding them; and in the latter have the same Values as in the former

former. (3.) That the Quadrature may be express'd by a finite Quantity, when n is a positive Integer, or equal to nothing, or if $2n$ be an odd Number; for in these Cases each Series is broke off. (4.) That $2q$ is equal to the last Term breaking off, of the former Series.

Example I.

Let $z = \frac{v^2}{a}$. Because in this Case $n = o$, $r = \frac{1}{2}$, therefore shall the Area ABD $= \frac{yv^2}{a}$

$v^2 + 2v\sqrt{2ay - y^2} - 2ay$.

COROLLARY.

The whole Figure AFE is equal to twice the Square, whose side is ACF, less the Square of the Diameter.

Example II.

Let $z = \frac{yv^2}{a^2}$, because in this Case $n = 1$, $r = \frac{1}{a^2}$, therefore shall the Area ABD =

 y^2v^2

$$\frac{y^2v^2}{2a^2} - \frac{1}{4}v^2 + v\sqrt{2ay - y^2} \times \overline{\frac{y}{2a} + \frac{3}{2}} - \frac{1}{4}y^2 - \frac{3ay}{a^2}.$$

Example III.

Let $z = \frac{y^2v^2}{a^3}$, because in this Case $n = 2$, $r = \frac{1}{a^3}$, therefore shall the Area ABD $= \frac{y^3v^2}{3a^3}$

$$- \frac{5}{6}v^2 + v\sqrt{2ay - y^2} \times \overline{\frac{2y^2}{9a^2} + \frac{5y}{9a} + \frac{5}{3}} - \frac{2y^3}{27a} - \frac{57^2}{18} - \frac{5ay}{3}.$$

While I was writing this, I receiv'd the late Months of the *Lipsick* Acts, in which I read, with a deal of pleasure, several excellent things for promoting Geometry; and among them some Remarks of Mr. *Leibnitz*, and Mr. *J. Bernouilli*, upon my Method of Quadratures. In the *Acts* of *April*, An. 1695. Mr. *Leibnitz* informs us that he has a Method somewhat like ours; and truly, I mightily Congratulate my self, that any thing of mine could have the least likeness to the Thoughts of so great a Geometer. But whereas he says his own is much more General, and shorter than mine; I make no doubt of that. It were to be wish'd, he would

would no longer suppress this Method of his, and several other things he has, especially relating to his *Differential Calculus*, but rather, as soon as his Leisure permits, publish them for the Good of the Commonwealth of Learning. In the mean while we hope the Illustrious Marquess *De l'Hospital* will speedily make publick what is necessary to perfect that *Calculus*, in the latter part of that excellent Work of his, which (in the Preface to the former part) he informs us, he has compos'd upon the *Integral Calculus*. We expect also, with some Impatience, that other Section, in which that Noble Author promises he will shew the Use of his *Calculus* in *Physicks* and *Mechanicks*. For whatever he has publish'd, as well those Specimen to be found scatter'd in the *Lipsick* Acts, and elsewhere, as that excellent Book of his (Intitul'd, *Analyse des Infiniment petits)* cause us to expect great Things from that Noble Marquess.

Whereas the Ingenious Mr. *J. Bernouilli* has thought fit (in the Acts of *February* and *August*, An. 1695.) to say my Method is not General, I freely confess it, as that Sagacious Person might easily perceive in the Course of my Examples. In a Matter so Intricate I took what Steps I could; and if deter'd with the length and difficulty of the Journey, I then made no farther Progress. I might fairly make a Step where I please, since my Application to these Mathematical Studies is only by the by. Mr. *Bernouilli* has partly hinted where my Method is at a

Stand, though he seems not to have taken up the whole Matter. In the mean while I acknowledge my self highly oblig'd, that he has honour'd my Treatise with his Animadversions; but much more so, that he was willing to free me of my Mistakes, with so much Candor and Humanity.

Con-

Concerning the apparent Magnitude of the Sun and Moon, or the aprent distance of two Stars when nigh the Horizon, and when higher elevated.

I Do not design so much to establish any thing of my own that may be satisfactory in solving this admirable appearance, as to detect the Errors of those that have offered at a Solution thereof, and have come short (as I conceive) of being satisfactory; that thereby I may again set the minds of Philosophers on work, and rouse them up to enquire anew after this surprizing *Phænomenon*. That I may do this the more effectually, I shall briefly declare the Matter of Fact, and then proceed to the Reason thereof, given by several, and to their Confutations.

First therefore it is well known that the mean apparent Magnitude of the Moon is 30 *m.* 30 *s.* we will take it *Numero Rotundo* to *de* 30, that is, an Arch of a great Circle in the Heavens of 30 Minuts is covered by her Diameter; and this we'll suppose to be her apparent Diameter, at a full Moon in the midst of Winter, and when she's in the Meridian, and at her greatest Northern Latitude, and consequently the utmost that she can be elevated in our *Horizon*: 'Tis as well known alſo that when ſhe is in this poſture, being looked upon by the naked Eye ſhe appears (that we may accommodate all to ſenſible Meaſures) to be *Magnitudinis Pedalis*, about a foot broad. But the ſame Moon being looked upon juſt as ſhe riſes, ſhe appears to be three or four foot broad, and yet if with an Inſtrument we take her Diameter, both in one poſture and t'other, we ſhall find that ſtill ſhe ſhall be but 30 Minutes; the ſeveral ways of trying this I will not mention, they being as various as are the Methods of taking the Moons apparent Diameter, common enough among the *Aſtronomers*; neither will I inſiſt upon the truth of the Matter of Fact, for that I think cannot reaſonably be queſtioned, after ſo many trials and ſo many experiments thereof, faithfully recorded by undoubted Witneſſes; and it would be very unreaſonable to imagine that ſo many Authors ſhould rack their Brains for ſolving an appearance wherein they were not certain of the matter of Fact. But becauſe of *Nullius in Verba*, I can aſſert that I have accurately try'd it my ſelf, and I have ſo found it: One of the ways I proceeded was thus; I took a very good Teleſcope of about 6 foot long, in the inward *Focus* of whoſe Eye-Glaſs I apply'd a very fine Lattice made of the ſingle hairs of a Man's Head; then looking with this at the Moon when ſhe was juſt riſen and looked extraordinary big, I obſerved what number of the ſquares of the Lattice were occupy'd by her Body; then obſerving her again, when more elevated and free from all extravagant Greatneſs, I ſtill found

found the ſame ſquares of the Lattice poſſeſſed by her. This way is equivalent to that now more uſed, of taking her Diameter by Mr. *Townly*'s *Micrometers*; but I have alſo tried and found the ſame thing by an accurate Sextant, taking the diſtance of the Moons oppoſite Limbs.

Now this *Phænomenon* affords two things to be conſidered, firſt why the Moon (I ſtill name the Moon as being an Object more adapted for our ſight, for the ſame thing holds in the Sun) ſhould ſeem bigger about the *Horizon*, then when more elevated; and ſecondly, ſhe appearing bigger, how comes it to paſs that her Diameter being taken, it is no greater than when ſhe appears leſs. But the Diſquiſition concerning this latter being likely to comprehend the former, I ſhall not divide my Diſcourſe into two Branches, but proceed in the Method propoſed. Only I deſire it may be noted, that I ſuppoſe the *Horizontal* and *Meridional* Moon to be found both of the ſame Angle, whereas in truth the the Meridional Moon (tho' appearing leſs) ſhall be found of the greater Angle: which increaſeth the Wonder. But this proceeding from the different diſtances that one and the other is looked at (the Meridional Moon being nigher us by almoſt a Semidiameter of the Earth) and conſequently eaſily ſolved that way; I have therefore choſen to put between them a plain equality, for avoiding Confuſion and Intricacy in Diſcourſe.

Wherefore let us hear what the Ingenious of theſe latter days can ſay to this appearance. And firſt we find the Celebrated *Des-Cartes* attributing

tributing this appearance rather to a deceived Judgment than to any Natural Affection of the Organ or Medium of ſence; for the Moon (ſays he) being nigh the *Horizon*, we have a better opportunity and advantage of making an Eſtimate of her, by comparing her with the various Objects that incur the ſight, in its way towards her; ſo that tho' we imagine ſhe looks bigger yet 'tis a meer deceit; for we only think ſo, becauſe ſhe ſeems nigher the tops of Trees or Chimneys or Houſes or a ſpace of Ground, to which we can compare her, and eſtimate her thereby; but when we bring her to the Teſt of an Inſtrument that cannot be deluded or impoſed upon by theſe appearances, then we find our Eſtimate wrong, and our Senſes deceived. Theſe Thoughts, methinks, are much below the accuſtomed accuracy of the noble *Des Cartes*; for certainly if it be ſo, I may at any time increaſe the apparent bigneſs of the Moon, tho' in the Meridian; for it would be only by getting behind a Cluſter of Chimneys, a Ridg of a Hill, or the top of Houſes, and comparing her to them in that poſture, as well as in the *Horizon*; beſides if the Moon be look'd at juſt as ſhe is Riſing from an *Horizon* determined by a ſmooth Sea, and which has no more Variety of Objects to compare her to, than the pure Air; yet ſhe will ſeem big, as if lookt at over the rugged top of an uneven Town or rocky Country. Moreover, all variety of adjoining Objects may be taken off, by looking through an empty Tube, and yet the deluded imagination is not at all helped thereby. I come next to the ſolution hereof

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given by the famous *Thomas Hobbs*; and for this we ſhall ſtand in need of *Figure* 41. wherein, ſays he, let the point G be the Center of the Earth, and F the Eye on the ſurface of the Earth ; on the ſame Center G let there be ſtruck the two Arches, E H determining the Atmoſphere, and A D to repreſent that blue ſurface in which we imagine the fixed Stars ; and let F D be the *Horizon*. Divide the Arch A D into three equal parts by the lines B F, C F, it is manifeſt that the Angle AFB is greater than the Angle BFC, and this again greater than the Angle CFD. Wherefore ſays he, to make the Angle CFD equal to the Angle CF D, the Arch CD muſt be greater than the Arch CB ; and conſequently, that the Moon may in the *Horizon* appear under the ſame Angle as when elevated, ſhe muſt cover a greater Arch, and therefore ſeem greater ; that is, the Moon in the Meridian appearing under the Angle BFC, that ſhe may appear under an equal Angle in the *Horizon*, as ſuppoſe CFD, 'tis neceſſary the Arch CD ſhould be greater than CB ; and conſequently tho' ſhe appear to ſubtend a greater Arch when in the *Horizon* then when elevated, yet ſhe appears under the ſame Angle. And all this without Refraction. The Geometry of this Figure is moſt certainly true and demonſtrable. At this I quarrel not ; but it makes no more in our preſent Difficulty than if nothing had been ſaid ; for the Philoſopher has here made a Figure of his own, and from thence he argues as confidently, as if Nature would accommodate her ſelf to his Scheme, and he not

not oblig'd to accommodate his Scheme to Nature; for here he has made the Circle GF representing the Earth very large in proportion to the Circle AD; and then indeed taking the point F in the Earth's surface, and by lines from thence dividing the Angle AFD into what ever equal parts the intercepted Arches AB, BC, CD, shall be unequal. But if he had considered, that the Earth is as it were a point in respect of the Sphere of the fix'd Stars, nay the very annual Orbit of the Earth is almost if not altogether imperceptible (saving the truth of Mr. *Hook's* Attempt) he would have found that the Lines FB, FC, FD, must be all conceived as drawn from the point G, and then equal Angles will intercept equal Arches, and equal Arches equal Angles: And so it happens (at least beyond the possibility of discovery of sense) to the Eye on the surface of the Earth. And besides he should have considered, that all Observations Astronomical are performed as from the Center of the Earth, and therefore it is that they keep such a stir about a Parallax; so that his drawing his lines so far from G as F is, and to another concentrick Circle so nigh as AD, deceived him in this Point.

The famous *Gassendus* has written 4 large Epistles on this Subject, the substance of all which is, that the Moon being nigh the *Horizon* and looked at through a more foggy Air, casts a weaker Light, and consequently forces not the Eye so much as when brighter; and therefore the Pupil does more inlarge it self, thereby transmitting a larger Projection

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on the *Retina*. In this Opinion I do find he is not alone, for in the Journals *des Scavans* this Disquisition being again revived by a French *Abbe*, he therein follows this Sentiment of *Gassendus*. It was first published in the 3d Conference presented to the Dauphin in *August* 1672. but by reason of an Objection moved by Father *Pardye*, it was fain to be re-published with some additions and amendments in *Octob.* 1672. The addition was, that this contracting and enlarging of the Pupil causeth a different shape in the Eye; an open Pupil making the *Crystalline* flatter and the Eye longer, and the narrower Pupil shortning the Eye, and making the *Crystalline* more convex, the first attends our looking at Objects which are remote or which we think so; the latter accompanies the viewing Objects nigh at end. Likewise an open Pupil and flat *Crystalline* attends Objects of a more sedate Light, whilst Objects of more forcible Rays require a greater Convexity and narrow Pupil. From these Positions the *Abbe* endeavoured to give an account of our *Phænomenon* as follows. When the Moon is nigh the *Horizon*, by comparison with interposed Objects, we are apt to imagine her much farther from us then when more elevated, and therefore (says he) we order our Eyes as for viewing an Object farther from us; that is, we something enlarge the Pupil, and thereby make the *Crystalline* more flat; moreover the duskishness of the Moon in that posture does not so much strain the sight; and consequently the Pupil will be more large, and the *Crystalline* more flat: Hence a larger Image shall

shall be projected on the Fund of the Eye, and therefore the Moon shall appear larger. And this disposition of the Eye that magnifies her, magnifies also the divisions of our fore-mentioned Lattice, and consequently she by her Body shall possess no more of the divisions than when she seems less. These two fore-mentioned accidents, *viz.* the Moons imaginary distance and duskishness, gradually vanishing as she rises, a different *Species* is hereby introduced in the Eye, and consequently she seems gradually less and less, 'till again she approaches nigh the *Horizon.* These two Opinions of *Gassendus* and the *Abbe* being so nigh a-kin, I shall consider them both together, and first I assert that a wider or narrower Aperture increases not, neither diminishes the projection on the *Retina.* I know *Honoratus Faber* in his *Synopsis Optica* endeavours to prove the clear contrary to this my Assertion, and that after this manner. *Fig.* 42. A B is an Object, E F the greater aperture of the Pupil, admitting the projection K I on the *Retina*, whereas the lesser aperture CD admits only the projection GH; but GH is less than KI, wherefore a lesser aperture diminishes the projection. I admire that any Man that undertook (as *Honoratus Faber)* to write of Opticks more accurately than all that went before him, should be guilty of so very gross an Error; and I do more admire that the celebrated *Gassendus*, and with him the noble *Hevelius* should be of the same Opinion: For tho' the 'foresaid Figure and Demonstration hold most certainly true in direct projections, as in a dark Room with

with a plain hole; yet it will not hold in Projections made by Refraction, as it is in those on the *Retina* in the Eye, by means of the *Crystalline* and other Coats and Humours of the Eye. For a Demonstration of this observe *Fig.* 43. wherein let A B be a remote Object, and E F the *Crystalline* at its large aperture, projecting the Image IM on the *Retina*. Let then CD be the lesser aperture of the Pupil before the *Crystalline*: I say the Image IM shall be projected as large as before, for the Cone of Rays EAF consists partly of the Cone of Rays CAD, therefore where the former EAF is projected, the latter CAD, as being a part of the former, shall be projected also. So that no more is effected by this narrow Aperture, but that the sides of the radiating Cones are intercepted, and consequently the Point I shall be affected with less light, but it shall still be in the same place: What is said of that Cone and that Point may be said of all other Cones and other Points of the Object. From hence appears first, the Invalidity of the Account given of the Moons appearance by *Gassendus* from this Reason. 2dly, The Reason appears why a Telescopes greater or lesser Aperture, makes no difference in the Angle it receives; for imagine EF to be an Object-glass of a Telescope, and 'tis plain. 3dly, 'Tis evident why a greater or less Aperture on a Telescope should make the Objects appear Lighter or Darker, for thereby more or less Rays are admitted to determine on the Projection of each Point. But all this by the by. And this is sufficient for a Confutation of of *Gaſſendus* and *Faber :* But our forementioned *Abbe* ſuperadds to a greater or leſſer Aperture of the Pupil, as a neceſſary Conſequent, a greater and leſſer Convexity of the *Cryſtalline*, as alſo a lengthening and ſhortening the Tube of the Eye: And this I muſt confeſs would do ſomething if we find it true *in* our Caſe; and this let us try. Firſt, ſays he, the duskiſhneſs of the Moon nigh the *Horizon* admits the Pupil to enlarge it ſelf, the *Cryſtalline* to flatten, and the Eye to lengthen. But what if we change our Object, and inſtead of the Moon take the diſtance between ſome of the fixt Stars; as ſuppoſe thoſe of *Orions* Girdle) we ſhall find the ſame *Phænomenon* in them, and yet I hope neither he nor *Gaſſendus* will aſſert, that they at one time ſtrain the Eye more than at another, or that at any time their *fulgur* ſtrains the Eye at all; if he do, let him take Stars of the leſſer Magnitudes, nay even thoſe that can but juſt be perceived, and then he will be convinced: Or let him conſider whether this will hold in looking at the Sun through very dark Glaſſes, which render the Sight thereof as inoffenſive to the Eye, as that of a green Field. But perhaps he will then ſay that this other Reaſon holds, which is 2dly, That the greater imaginary diſtance at which we think the Moon near the *Horizon*, than when more elevated, makes us Contemplate her as if really ſhe was ſo, *viz.* with ample Pupils, &c. but this I have ſufficiently overthrown in my Remarks againſt *Des Cartes*; therefore I paſs it over, only ſubjoining that if there were any thing in this Surmiſe, my-thinks the *Ho-*

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rizontal Moon ſhould be fancied nigher to us than farther from us; for if we are for trying natural Thoughts, let us take Children to determine the Matter, who are apt to think that could they go to the edge of that ſpace that bounds their Sight, they ſhould be able (as they call it) to touch the Sky; and conſequently the Moon ſeems then rather nigher to us than farther from us.

After I had writ thus far I accidentally caſt my Eye upon *Riccioli*'s Treatiſe of Refraction, at the end of his 2*d.* Volume of the *Almageſt*, *Lib.* 10. *Sect.* 6. *Cap.* 1. *Queſt.* 13. wherin he ſpeaks of our preſent Difficulty; but to my wonder I find him aſſert, that he and Father *Grimaldi* had often taken the *Horizontal* Sun and Moons Diameter by a Sextant, when to the naked Eye they appeared very large; (*Grimaldus* directing his Sight to the left edge, and *Ricciolus* to the right,) and that even by the Inſtrument they always found the Diameters greater than when more elevated, the Sun often ſubtending an Angle of almoſt a Degree, and frequently 45 Minutes, the Moon alſo 38 or 40 Minutes. This is down right contrary to the matter of Fact which I have before alledged, and directly repugnant to the matter of Fact aſſerted by the *French Abbe* in the forecited Journal. Whether of us be in the right I leave to accurate Experiment to determine, and ſubmit the whole to the deciſion of the *Illuſtrious Royal Society*. Only give me leave to add one word againſt *Riccioli*, for had his Experiments been accurately proſecuted, he ſhould have tryed them when the *Horizontal* Moon had look'd ten times more large in Diameter than ordinary; and then if it be true, that even by an Instrument she will be found proportionally broader than really she should subtend an Angle of 300 Minutes, or 5 Degrees: for very often I have seen the Moon when she appeared 10 times broader than ordinary, which the small addition of 8 or 10 Minutes to her usual Diameter will never Cause.

Lastly, as an Apology for my reviving this disquisition to that Noble Company of *English* Philosophers, I shall only imitate the words of the forementioned *Abbe*'s Letter. *Pour la Raison de cette Apparence, & de la tromperie de nos Sens, je la tiens plus Difficile a trouver, que les plus grands Equations d' Algebre, & quand vous y aurez bien pense, vous m' Obligerez de m' en dire vostre Sentement*, &c.

After which I have only to subscribe my self an unworthy Member, and an humble Servant and Admirer of that Illustrious Company.

Dublin
March 10. 8$\frac{6}{7}$.

William Molyneux.

The

The Sentiments of the Reverend and Learned Dr. John Wallis R. S. Soc. *upon the aforeſaid Appearance, communicated in a Letter to the Publiſher.*

AS to the laſt Inquiry (concerning which, you ſay, the Royal Society would be glad to know my Opinion ;) about the apparent Magnitude of the Sun near the *Horizon*, greater than when conſiderably high :

The Inquiry is Ancient : And, I remember, I diſcourſed it near forty Years ago with Mr. *Foſter*, then Profeſſor of Aſtronomy in *Greſham College.* Who did then aſſure me (from his own Obſervation, I ſuppoſe, for I have never examined it my ſelf,) that the apparent Magnitude taken by Inſtrument (however the Fancy may apprehend it) is not greater at the *Horizon*, than when higher. And Mr. *Caſwel* (when your Letter was communicated to our company here) affirmed the ſame.

And (though I have not my ſelf made the Obſervation) I do not doubt but the thing is ſo. For it is agreed, That Refraction near the *Horizon*, though (as to appearance) it alter the *Altitude* of the thing ſeen ; yet it alters not the *Azimuth* at all.

And it must needs be so. For, since this equally respects all points of the *Horizon*; let the Refraction be what it will, the whole *Horizon* can be but a Circle: So that there is no room for the breadth of a thing (as to the Angle at the Eye) to be made greater, whatever its Tallness may (the Refraction not equally affecting all parts in the Circles of *Altitude*.) Nor is there any reason why this should rather thrust the other, than that the other thrust this, out of place.

Whereas, in the *Altitude*, it is otherwise: For while what is near the *Horizon* is inlarged, that which is further off is thereby contracted: which as to the *Azimuth* or *Horizontal* Position cannot be.

In Spectacles indeed it is otherwise; for they represent the Object every way enlarged; and do thereby hide the adjacent parts. But in Refraction by Vapours, supposing all parts of the *Horizon* equally affected by them, one part cannot be expanded in breadth (whatever it may be as to the heighth) without thrusting out another (for the whole *Horizon* can be but a Circle) and, why one part rather than another?

Unless we would say (as perhaps we may, if there shall appear a necessity for it) That the Rays of a lucid Body do expand themselves every way to the prejudice of the parts adjacent, by covering them.

But supposing (which I am apt to believe, till the contrary shall be evinced by Experiment) that the Sun or Moon's apparent Diameter taken by Instrument near the *Horizon*, is the same as taken in a higher Position, (I mean

mean its *Horizontal* Diameter, or that parallel to the *Horizon*; for the erect Diameter, in a Circle Perpendicular to the *Horizon*, may by the Refraction be varied, and thereby made, not greater, but leſs than when higher; as hath been noted in the Name of *Sol Ellipticus* at the *Horizon.*) Suppoſing, I ſay, that the Sun's apparent Diameter *Horizontal*, taken by Inſtrument, is the ſame near the *Horizon*, as in a higher Poſition, I take its Imaginary greatneſs which is fanſied near *Horizon*, to be only a deception of the Eye; or rather the Imagination from the Eye.

For ſure it is, that the Imagination doth not eſtimate the greatneſs of the Object ſeen, only by the Angle which it makes at the Eye; but, by this compared with the ſuppoſed diſtance.

True it is that, *Cæteris paribus*, we judge that to be the greater Object, which makes at the Eye the greater Angle: But not ſo if apprehended at different Diſtances.

For if through a Caſement (or leſſer aperture) we ſee a Houſe at 100 Yards diſtance; this Houſe (though ſeen under a leſs Angel) doth not to us ſeem leſs than the Caſement through which we ſee it, (or this greater than that, becauſe it makes at the Eye the greater Angle:) But the Imagination makes a comparative Eſtimate from the Angle and Diſtance jointly conſidered.

So that, if two things ſeen under the ſame or equal Angles, if to one of them there be ought which gives the apprehenſion of a greater Diſtance, that to the Imagination will appear greater.

Now sure it is, that one great advantage for Estimating of a thing seen, is, from the variety of intermediate Objects between the Eye and the thing seen. For then the Imagination must allow room for all these things.

Hence it is that if we see a thing over two Hills, between which there lies a great Valley unseen, it will appear much nearer than if we see the Valley also: And it will appear as just beyond the first Hill. And if we move forward to the top of the nearest Hill (that so the Valley may be seen) it will then appear much further than before it did.

And on this account it is, that the Sun setting, appears to us as if it were but just beyond the utmost of our visible *Horizon*; because all between that and the Sun is not seen. And, upon the same account, the Heaven it self seems Contiguous to the visible *Horizon*.

Now when the Sun or Moon is near the *Horizon*, there is a prospect of Hills, and Vallies, and Plains and Woods, and Rivers, and variety of Fields, and Inclosures, between it and us: which present to our Imagination a great Distance capable of receiving all these. Or, if it so chance that (in some Position) these Intermediates are not actually seen: Yet having been accustomed to see them, the Memory suggests to us a view as large as is the visible *Horizon*.

But when the Sun or Moon is in a higher Position; we see nothing between us and them (unless perhaps some Clouds) and therefore nothing to present to our Imagination so great a Distance as the other is. And

And therefore, though both be ſeen under the ſame Angle, they do not appear (to the Imagination) of the ſame bigneſs, becauſe not both fanſied at the ſame Diſtances: But that near the *Horizon* is judged bigger (becauſe ſuppoſed farther off) than the ſame when at a greater *Altitude*.

'Tis true, that as to ſmall and middling Diſtances (beſides this Eſtimate from Intermediates) the Eye hath a means within it ſelf to make ſome Eſtimate of the Diſtance. As, when we already know the bigneſs of a thing ſeen, to which we have been accuſtomed; as a Man, a Tree, a Houſe or the like: If ſuch thing appear to us under a ſmall Angle, and indiſtinct, and faintly coloured; the Imagination doth allow ſuch Diſtance, as to make ſuch a thing ſo to appear. And, if this, thro' a Proſpective Glaſs, be repeſented to us under a bigger Angle, and more diſtinct: It is accordingly apprehended as ſo much nearer.

But the caſe is otherwiſe, when we do not by the known bigneſs, judge the Diſtance; but, by the ſuppoſed Diſtance, judge of the bigneſs; as in the Caſe before us.

And accordingly, different Perſons, according to different fancied Diſtances, judge very differently. As, if two Stars be ſhewed to ignorant Perſons, and you ask how far they ſeem to be aſunder: one perhaps will ſay a Foot; another a Yard, or more: And one ſhall ſay, the Sun appears to him as big as a Buſhel; another, as big as a *Holland* Cheeſe: Each eſtimating according to the fancied Diſtance.

Again; in our two Eyes (when the Object is ſeen by both) there is yet another means of eſtimating how far off it is. (And it is this by which we judge of Diſtances.) Namely, there are, from the ſame Object, two different viſual Cones, terminated at the two Eyes: Whoſe two Axes contain, *at the* Object, different Angles, according to different Diſtances: An accuter Angle at a great Diſtance, and more obtuſe when nearer.

Now, that ſuch Object may be *ſeen by* both Eyes, clearly; it is requiſite that the Eyes be put into ſuch a Poſition, as that the Sight of each Eye receive the reſpective Axe at right Angles. Which requires a different Poſition of the two Eyes, according to the different Diſtance of the Object.

As will manifeſtly appear; if we look, with attention, on a Finger (or other ſmall Object) at two or three Inches diſtance from the Eye; and then upon another like Object at three or four Yards beyond it: (and this alternately ſeveral times. For 'twill be manifeſt, that while we look intently on the one, we do not ſee the other (or but confuſedly) though both be juſt before us. And, as we change our view, from the one to the other, we manifeſtly feel a Motion of the Eyes (by their Muſcles) from one poſture to another.

And according to the different poſture in the Eyes, requiſite to a clear Viſion by both, we eſtimate the Diſtance of the Object from us.

And hence it is, that they who have loſt the Sight of one Eye, are at a great diſadvantage,

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as to eſtimating Diſtances, from what they could do while they had the uſe of both.

But now when the Diſtance grows ſo great, as that the Poſition of theſe viſual Axes become Parallel, or ſo near to Parallel, as not to be diſtinguiſhable from it: This advantage is loſt, and we can thenceforth only conclude, that it is far off; but not how far.

Hence it is, that our view can make no diſtinction of the Moon's Diſtance, from that of the other Planets, or even of the fixed Stars: But they ſeem to us as equally remote from us; though we otherwiſe know their Diſtances from us to be vaſtly different. Becauſe the Parallax (as I may ſo call it) from the different Poſition of the two Eyes, is quite loſt, and undiſcernable, in Diſtances much leſs than the leaſt of theſe.

And ſo, of the fixed Stars among themſelves: Which, though they ſeem equally remote from us; many (for ought we know) be at Diſtances vaſtly different. Nor can we tell, which of them is neareſt: (unleſs perhaps we may reaſonably gueſs, thoſe to be neareſt, which ſeem biggeſt.) Becauſe, here not only the Parallax from the Diſtance of the two Eyes; and that from the Earths Semidiameter; but even that from the Semidiameter of the Earths great Orb, is quite loſt; and none remaining, whereby to eſtimate their Diſtance from us.

But (to return to our caſe in hand;) tho' as to ſmall Diſtances, we may make ſome eſtimate from the known *Magnitude* of the Object: And, as to middling diſtances, from the Parallax (as I may call it) ariſing from the

the interval of the two Eyes: Yet even this latter will hardly reach beyond, if ſo far as the viſible *Horizon*: and all beyond it, is loſt.

So that, there being nothing left to aſſiſt the fancy in eſtimating ſo great a diſtance, but only the intermediate Objects: Where theſe intermediates appear to the Eye, (as, when the Sun or Moon are near the *Horizon*:) the diſtance is fancied greater, than where they appear not, (as when farther from it:) and conſequently (though both under the ſame or equal Angles) that near the *Horizon* is fancied the greater. And this I judge to be the true reaſon of that appearance.

You will excuſe (I hope) what excurſion I have made; becauſe though ſome of them might have been ſpared, as to the preſent caſe; yet they are not impertinent to the buſineſs of Viſion; and the eſtimate to be thence made, of *Magnitudes* and Diſtances, by the Imagination.

The Sun's Eclipſe *May 1ſt.* was here obſerved about $\frac{1}{2}$ a Digit; between one and two a Clock after noon.

A Demonstration of an Error committed by common Surveyors in comparing of Surveys taken at long Intervals of Time arising from the Variation of the Magnetick Needle, by William Molyneux. *Esq;* F. R. S.

THE Variation of the Magnetick Needle is so commonly known, that I need not insist much on the Explication thereof; 'tis certain that the true Solar Meridian, and the Meridian shewn by an Needle, agree but in a very few places of the World; and this too, but for a little time (if a moment) together. The Difference between the true Meridian and Magnetick Meridian perpetually varying and changing in all Places and at all Times; sometimes to the Eastward, and sometimes to the Westward.

On which account 'tis impossible to compare two Surveys of the same place, taken at distant times, by Magnetick Instruments, (such as the *Circumferentor*, by which the *Down Survey*, or Sir *William Petty's* Survey of *Ireland* was taken) without due allowance be made for this Variation. To which purpose we ought to know the Difference between the Magnetick Meridian and true Meridian at at that time of the *Down* Survey, and the ſaid difference at the time when we make a new Survey to compare with the *Down* Survey.

But here I would not be underſtood as if I propoſed hereby to ſhew, that a Map of the ſame place, taken by Magnetick Inſtruments at never ſo diſtant times, ſhould not at one time give the ſame *Figure* and *Contents* as at another time This certainly it will do moſt exactly, the variation of the Needle having nothing to do either in the *Shape* or *Contents* of the Survey. All that is affected thereby, is, the Bearings of the Lines run by the Chain, and the Boundaries between Neighbours. And how this may cauſe a conſiderable Error (unleſs due allowance be made for it) is what I ſhall prove moſt fully.

In order to which, let us ſuppoſe that about the Year 1657. (at which time the *Down* Survey was taken) the Magnetick Meridian and true Meridian did agree at *Dublin*, or pretty nigh all over *Ireland*; that is to ſay, that there was no Variation. And indeed by Experiment it was at that time found, as I am well aſſur'd, that at *Dublin* it was hardly half a Degree.

Let us ſuppoſe that in the year, 1695. the Variation was 7 Degrees from the North to the Weſtward; that it was really ſo, I believe I am pretty well aſſured, from an Experiment thereof made by my ſelf with all diligence. But this is not material, let us now only ſuppoſe it.

Let

Let *AB* represent the Survey of two Town-Lands, one in the possession of *A*, and t'other in the possession of *B*, which we call *A* Town-Land and *B* Town-Land, taken by the *Down*-Survey, *Anno* 1657. when there was no Variation.

Let the Line *NS* running through the Point *P* be the true Meridian, and consequently the Magnetick Meridian also at that time, because of the supposed no Variation, and let this Line *NS* be also the Boundary between the two Town-Lands *A* and *B*.

In the year 1695. when the Variation is 7 Degrees from the North to the Westward, *B* having a *Map* of the *Down* Survey, and being suspicious that his Neighbour *A* had incroached on him by a Ditch *PQ*, imploys a Surveyor to inquire into the matter: The Surveyor finds by his *Map* that the Boundary between *B* and his Neighbour *A* run from the Point *P* through a Meadow directly according to the Magnetick Meridian *SPN*; but observing the Ditch *P Q* cast up much to the Eastward of the present Magnetick Meridian, he concludes that *A* has incroached on *B*, and that the Ditch ought to have been cast up alongst the Line *Pq*, the Angle *Q P q* being an Angle of 7 Degrees, that is the present Variation of the Needle; and the Line *Pq* the present Magnetick Meridian: For which Variation, not making any allowance, he positively determines that *B* has all the Land in the Triangle *Q P q*, more than he ought to have; and that his Ditch ought to run alongst the Line *P q*.

'Tis

'Tis true indeed, if the Surveyor go the whole ſurround of the Lands *A* and *B*, he will find their Figure and Contents exactly agreeable to the Map here expreſſed. But then the Bearings of the Lines are all 7 Degrees different from the Bearings in the Map, and they will run in and out upon the adjacent Neighbouring Lands, and cauſe endleſs Differences between their Poſſeſſors; as is manifeſt from the Figure: wherein the prickt Lines repreſent the Diſagreement in the Bearings of the Lines, protracted from the Point *P*; and we ſee *A* incroaching on his Neighbours on the Weſtward, as he incroaches on *B*, and *B*'s Eaſtward Neighbours incroaching on him, and ſo forward and clear round. Whereas, by a due allowance for the Variation of the Needle, all this Confuſion and Diſagreement is avoided, and every thing hits right.

Thus for inſtance in the Caſe before us, knowing that the Magnetick Variation has cauſed the preſent Magnetick Meridian to fall in the Line *n q P s*, 7 Degrees from the North to the Weſtward; to reduce this to the Magnetick Meridian at the time of the Down Survey, I muſt make the Meridian of my Map to fall 7 Degrees to the Eaſtward of my Magnetick Meridian; as we ſee the Line *P Q* falls 7 Degrees to the Eaſtward of the Line *P q*.

What is here ſaid on ſuppoſition that the Magnet had no Variation at the time of the firſt Survey taken, and that it had 7 Degrees variation Weſtward at the time of the ſecond Survey, may eaſily be accommodated

to

to the ſuppoſal of any other Variations at the firſt and ſecond Surveys, *Mutatis mutandis*, for knowing the Variations we know their Difference; and if we know their Difference, this gives us the Angle *Q P q*, by which we reduce them to each other.

The beſt way therefore to make Maps invariable, conſtant and everlaſting, were for the Surveyors, who uſe Megnetick Inſtruments to make always allowance for the Magnetick Variation, and to protract and lay down their Plats by the true Meridian. This the wary Sailer is fully convinced of: and therefore in Steering his Courſe, he conſtantly allows for the preſent Vaiiation, which he obſerves by the *Azimuth* Compaſs, or elſe he would miſs his appointed Harbour oftner then he would hit it: For no two Points on the Globe keep the ſame Bearing to each other by the Magnetick Meridian for any time together. And though the Variation be ſlow, yet in a long Courſe, or in times pretty diſtant, it may cauſe vaſt Errors, unleſs allowed for. Thus for inſtance, ſuppoſe in the year 1660. a Sailor had ſteered from the Land's end of *England* to Cape *Finiſter* in *Spain*, by his Magnetick Compaſs a direct South Courſe; and that at that time there were no Variation. Afterwards *Anno* 1700. when there was (ſuppoſe) 7 Degrees of Variation from the North to the Weſtward, another Sailor intending to make the ſame Paſſage, ſteers directly the ſame Southerly Courſe by his Magnetick Compaſs: I ſay, this laſt Seaman will be carried far into the *Bay of Biſcay* to the Eaſtward, and will miſs of of his deſired Port by many Leagues; but if in his Courſe he hath allowed for this Variation, and inſtead of ſailing a direct Southerly Courſe by his Compaſs, he had ſteer'd 7 Degrees from the South to the Weſtward, he had hit his Point. Whether theſe be the true Bearings of theſe two Places, it matters not: We go on to the Suppoſition that they are.

Perhaps it may be objected, That Surveys may be taken without Magnetick Inſtruments, and that therefore this Error ariſing from the Megnetick Variation, and Change of the Bearings of Lines, may be avoided. To which I anſwer, firſt, That granting a Survey may be taken *without* Magnetick Inſtruments, this is nothing againſt what we have laid down relating to Surveys that are taken *with* Magnetick Inſtruments, as the Down Survey actually was, and moſt Surveys at preſent actually are taken therewith. Secondly, Though a Survey may be taken truly without Megnetick Inſtruments, ſo as to ſhew the exact Angles and Lines of the Plat, and conſequently the true Contents, yet this will not give the true Bearings of the Lines, or ſhew my Poſition in relation to my Neighbours, or the other parts of the Country. This muſt be ſupply'd by the Magnet, or ſomething equivalent thereto, as finding a true Meridian Line on your Land by Celeſtial Obſervations. And I doubt not but the ancient *Egyptians*, before the diſcovery of the Magnet were forced to ſome ſuch Expedient in their Surveys and Applotments of Lands between Neighbour and Neighbour, after the Inundations of the *Nile*, which, we are told

told, gave the first Original to Geometry and Surveying. Absolute Necessity and Use having introduced these, as Delight and Diversion introduced Astronomy amongst the *Chaldeans*.

And this brings me to another Objection which may be made against the Instance before laid down: It may be said, That certainly the Surveyor which *B* imployed was very ignorant, who would choose to judge of the Line *P Q*, rather by its bearing than by determining the Point *Q*, by measuring from *H* and *G*. To this I answer, What if both the Points *H* and *G* were vanish'd since the Down Survey was taken? What if the whole face of the Country were chang'd, save only the Point *P* and the Line *P Q*? How shall the Surveyor then judge of the Line *P Q* but by its bearing? That this is no extravagant Supposition, we have an Example in *Egypt* above-mentioned, where the *Nile* lays all flat before it, and so uniformly covers all with Mud, that there is no distinction. In such a Case your bearing must certainly help you out, there is no other way.

But I answer secondly, To say that the Surveyor might have determin'd the Point *Q* by admeasurement from *G* and *H*, or any other adjoining noted Points, as from *F*, *K*, *I*, &c. 'tis very true; But then 'tis against our Supposition. I am upon shewing an Error that arises from judging of the Line *P Q by Magnetick bearing*, and to tell me that this might be avoided by another way, is to say nothing. I my self shew how it may be avoided by allowing for the Variation; but still it is an Error, till it be avoided.

But thirdly, if *B*'s Surveyor do not allow for the Variation of the Needle, he will never exactly determin even the Points *G*, *F*, *H*, *K*, &c. or any other Points in the Plat; but instead thereof will fall on the Points *g*, *h*, *f*, *k*.

From what has been laid down, we may see the absolute necessity of allowing for the Variation of the Magnet, in comparing old Surveys with new ones; for want of which great Disputes may arise between neighbouring Proprietors of Lands: and it were to be wish'd that our Honourable and Learned Judges would take this Matter into their Consideration whenever any Business of this kind comes before them. Hitherto an absolute Acquiescence in the Down Survey, without any of the fore-mention'd Allowance, has been agreed upon as a standing Rule in our Courts of Judicature in *Ireland*; but that many Men may be injured thereby, I suppose is manifest from what foregoes.

I have only this to add, That least I be thought herein to strike at the Truth or Exactness of the Down Survey, 'tis not at all the intention of this Paper, but rather to confirm it, by shewing which way Men ought to Examine it truly, and not by the common ways used by them, which rather confound it, and all that claim under it.

See the Table Fig. 44.

Al-

Although this Paper was chiefly deſigned for the ending of Conteſts in the Kingdom of *Ireland* about the intereſts of ſome of thoſe whoſe Lands are Neighbouring, and have been ſurveyed by Magnetick Inſtruments, yet conſidering its univerſal Uſe, it was thought it would be very grateful to the Curious to publiſh it here.

A Propoſal concerning the Parallax of the fixed Stars, in Reference to the Earths Annual Orb. In ſeveral Letters of May *the* 2d. June 29. *and* July 20.1693. *from Dr.* John Wallis *to* William Molineux *Eſq;*

SIR,

I Am obliged to you for two Books which you have been pleaſed to ſend me, that of your *Sciothericum Teleſcopicum*, and that of *Dioptricks*; which you have performed ſo well, that I have not been better ſatisfied with any that I have ſeen of that Subject. I ſhould not ſo long have neglected to return my Thanks for them, but that I thought a Letter of bare Thanks to be too empty, unleſs I had ſomewhat elſe to ſend with it.

You will, I hope, give me leave (though I have not the opportunity of being perſonally known to you) to ſuggeſt a Speculation, which hath been in my Thoughts theſe Forty Years or more; but I have not had the opportunity of reducing it to Practice, as being not ſo well ſtored with neceſſary Inſtruments of that kind, nor much exerciſed to Teleſcopick Obſervations. And though I have many Years ſince ſuggeſted it to others, yet neither

ther have they had leisure of convenience of putting it in Practice.

It is concerning the Parallax of the fixed Stars, as to the Earths Annual Orb.

Galileo complains of it a great while since (in his *Systema Cosmicum*) as a thing not attempted to be observed with such diligence as he could wish, and I doubt we have the same cause of complaining still. I know that Dr. *Hook* and Mr. *Flamstead* have attempted somewhat that way, but have desisted before they came to any thing of Certainty. What hath been done to that purpose abroad I know not.

Galileo hath suggested divers things considerable in order to it.

As to the times of Observation; That it should be when the Sun or Earth are in the Tropicks, or as near thereto as may be: Because at those times, if any, will be the greatest difference observable in their meridional Altitude.

As to the Stars to be observed, That they should be such as are as near as may be to the Pole of the Ecliptick: For such as are in the Plain of the Ecliptick, or near unto it, though they may be sometime nearer, sometime farther from us, (which might somewhat alter their apparent Magnitude, if it were so much as to be observable) yet it would little or nothing alter the Parallactick Angle, as *Galilelo* doth there demonstrate.

He notes also, that in a business so nice, the ordinary Instruments of Observation (though pretty large) would be insufficient (he doubts) for this purpoſe, and doth propoſe, that by the ſide of ſome Edifice or Mountain, at ſome Miles diſtance, the ſetting of ſome noted Star (as that of *Lucida Lyræ*) might be obſerved at thoſe different times of the Year, which might be equivalent to an Inſtrument whoſe Radius were ſo large.

Which were a good Expedient if Practicable; but I doubt the Denſity of our Atmoſphere is ſo great, as that it will be hard to diſcern a Star juſt at the Horizon, or even within ſome few Degrees of it: And that the Refraction would be there ſo great, and ſo uncertain, as not to comply with ſo curious an Obſervation.

That which occurred to my Thoughts upon theſe Conſiderations, was to this purpoſe: That ſome Circumpolar Stars (nearer to the Pole of the Equator than is your Zenith, and not far from the Pole of the Zodiack) ſhould be made choice of for this purpoſe. And in caſe the Meridinal Altitude be diſcernably different at different times, ſo will alſo be their utmoſt Eaſt and Weſt Azimuth, which may be better obſerved than their Riſing or Setting: And this will be not obnoxious to the Refraction, as is the Meridional Altitude; (for though the Refraction do affect the Altitude, yet not the Azimuth at all); and we may here have choice of Stars for the purpoſe; which in Obſervations from the bottom of a Well we cannot have; being there confined to thoſe only which paſs very near our Zenith, though very ſmall Stars.

I would then take it for granted, as a thing at least very probable, that the fixed Stars are not at all (as was wont to be supposed) at the same distance from us; but the distance of some, vastly greater than of others; and consequently, though as to the more remote, the Parallax may be undiscernable; it may perhaps be discernable in those that are nearer to us.

And those we may reasonably guess (tho' we are not sure of it) to be nearest to us, which to us do appear biggest and brightest, as are those of the First and Second Magnitude; and there are at least of the Second Magnitude, pretty many not far from the Pole of the Ecliptick, (as that in particular, in the Shoulder of the lesser Bear): And in case we fail in one, we may try again and again on some other; which may chance to be nearer to us than what we try first. And Stars of this bigness may be discerned by a moderate Telescope, even in the day-time; especially when we know just where to look for them.

The manner of Observation I conceive, may be thus: Having first pitched upon the Star we mean to observe, and having then considered (which is not hard to do) where such Star is to be seen in its greatest East or West Azimuth; it may be then convenient to fix very firm and steadily on some Tower, Steeple, or other high Edifice (in a convenient situation) a good Telescopick Object-glass in such position, as may be proper for viewing that Star. And at a due distance from it near the Ground, build on purpose (if already there be not any) some little Stone Wall, or like Place, on which to fix the Eye-glass, so as to answer that Object-glass: And having so adjusted it, as through both *to* see that Star in its desired Station, (which may best be done while the Star is to be seen by Night in such situation, near the time *of one* of the Solstices), let it be there fixed so firmly, as not to be disturbed, (and the place so secured, as that none come to disorder it), and care be taken so to defend both the Glasses, as not to be endangered by Wind and Weather. In which contrivance I am beholden to Mr. *John Caswel* M. A. of *Hart-hall* in *Oxford*, for his Advice and Assistance; with whom I have many Years since, communicated the whole matter.

This Glass being once fixed (and a Micrometer fitted to it, so as to have its Threds perpendicular to the Horizon, to avoid any inconvenience which might arise from diversity of Refraction if any be) the Star may then be viewed from time to time (for the following Year or longer) to see if any change of Azimuth can be observed.

This I thought fit to recommend to your Consideration, who do so well understand Telescopes, and the managery of them; not knowing any who is more likely to reduce it to Practice. If you shall think fit to give your self the trouble of attempting the Experiment, and that it succeed well, it will be a noble Observation, and worth the Labour: And, if it should miscarry, the charge I hope would not be great.

But

But when I ſuggeſt (as a convenient Star for this purpoſe) the ſhoulder of the leſſer Bear (as being the neareſt to the Pole of the Zodiack of any Star that is of the firſt or ſecond Magnitude), I do not confine you to that Star; but (without retracting that) ſuggeſt another; namely, the middle Star, in the Tail of the great Bear, which (tho' ſomewhat farther from the Pole of the Zodiack) is a brighter Star than the other, and may be nearer to us.

But I do it principally upon this Conſideration: namely, That there is adhering to it a very ſmall Star, (which the *Arabs* call *Alcor*, of which they have a Proverbial ſaying, when they would deſcribe a ſharp-ſighted Man; That he can *diſcern the Rider on the middle Horſe of the Wayn*; and of one who pretends to ſee ſmall things but over-look much greater; *Vidit Alcor at non Lunam plenam*): Which *Hevelius* in his Obſervations, finds to be diſtant from it about 9 Minutes, and 5 or 10 Seconds: So that beſides the advantage of diſcovering the Parallax of the greater Star, if diſcernable. Their difference of Parallax of that and of the leſſer Star (being both within the reach of a Micrometer) may do our Work as well. For if that of the greater Star be diſcernable, but that of the leſſer be either not diſcernable, or leſs diſcernable. Their different diſtances from each other at different times of the the Year, may, perhaps (without farther *Apparatas*) be diſcerned by a good Teleſcope of a competent length, furniſhed with a Micrometer, if carefully pre-

preserved from being disordered in the Intervals of the Observations; and discover at once, both, that there is a Parallax, and that the fixed Stars are at different distances from us, wherein, that I be not mistaken, my meaning is not, that the Instrument or Micrometer should be removed for the observing of the lesser Star; but that (*when* the Azimuth of the greater Star is taken) by a Micrometer (consisting of divers fine Threads parallel and transverse) may (at the same time) be observed the Distance of the two Stars, each from other, in that Position (both being at once within the reach of the Micrometer ;) which distance (the Instrument remaining unmoved) if it be found (at different times of the Year) not to be the same ; this will prove, that there is a *different* Parallax of these two Stars.

This latter part of the Observation (of their different distances at different times) I suggest, as more easily practicable though not so nice as the former. For it may be done I think, without any further *Apparatus* there than a good Telescope, of ordinary form, furnished with a Micrometer, (this being carefully kept unvaried during the Interval of these Observations. And if this part only of the Observation (without the other) be pursued ; it matters not though the two Observations (near the two Solstices) be, one at the Eastern, the other at the Western Azimuth (whereby both may be taken in the Night-time,) for the distance must (at both Azimuths) be the same, if after observing the Azimuth of the greater Star it be ne-

ces-

ceſſary to move the Micrometer for meaſuring its diſtance from Alcor that may be done another Night (and it is not neceſſary to be done at one Obſervation) for that diſtance, and cannot be diſcernably varied in a Night or two.

I ſhall give you no farther trouble at preſent, but ſubſcribe my ſelf, Sir,

Yours, &c.

A

A Discourse on this PROBLEM;

Why Bodies dissolved in Menstrua Specifically lighter than themselves, swim therein?

By Mr. WILLIAM MOLYNEUX, of *Dublin*, Member of the *Royal Society*.

THE Liberty of Philosophising being now universally granted between all Men, I am sure that a difference in Opinion will be no breach of affection between two intirely Loving Brothers: And therefore I shall take the freedom to propose my own Thoughts in a matter wherein my Brother Mr. *Thomas Molyneux* hath appeared publickly in the *Novelles de la Republique des Letres, Mois d'Aout* 1684. *Art* 4. and *Mois de Janvier* 1685. *Art* 7. The Problem proposed is, *Why Bodies dissolved float in Liquors lighter than themselves?* as for Example: Mercury dissolved in strong Spirit of Nitre swims therein, tho' each small Particle of Mercury, be far heavier than so much of the Liquor whose place it occupies. This, says he, cannot be solved by the prime Law of Hydrostaticks, which is, that a Body which is an equal quantity is heavier than a like quantity of Liquor, sinks in that Liquor;

quor; thus a Cubick Inch of Iron being heavier than a Cubick Inch of *Aqua-Fortis*, and each Particle (how ſmall ſoever) of Iron being heavier than a like Particle of *Aqua-Fortis*; Iron being put into *Aqua-Fortis* ſhould ſink, and yet we find, that Iron being diſſolved in a convenient quantity of *Aqua-Fortis* floats therein, and does not fall to the Bottom. The Reaſon which my Brother gives for this is, That the internal Motion of the Parts of the Liquor, does keep up the Particles of the diſſolved Solid, for they being ſo every Minute, are movable by the leaſt force imaginable, and the Action of the Particles of the *Menſtruum*, is ſufficient to drive the Atomes of the diſſolved ſolid Body from place to place; and conſequently, notwithſtanding their Gravity, they do not ſink in the Liquor lighter than themſelves. As a Proof of this in the 7*th* Article of *Janvier* 1685. he offers an Experiment known in *Chymiſtry*, that a *Menſtruum* over a digeſting Fire (as the *Chymiſt* ſpeaks) will diſſolve a greater quantity of a Body put into it, than when 'tis off the Fire, and if it be taken off the Fire, and ſuffered to cool, a great Portion will precipitate of that which was perfectly diſſolved, whilſt the *Menſtruum* continued hot. For, *ſays he*, the Particles of the *Menſtruum* acquire a more violent agitation by the Fire, and are therefore able to raiſe and keep up a greater Quantity of the diſſolved Body, or hereby they are able to reſiſt a greater Gravity.

It has been objected againſt this Notion, that the common Experiment of precipitation,

tion, by mixing an *Alkaly* with an *Acid* ſeems to contradict this; for thereby the Fluidity of the *Menſtruum* is not taken away, and conſequently, the internal Agitation of its Parts is not diminiſhed, and yet thereupon, the Particles of the diſſolved Body precipitate all to the Bottom. To this he anſwers in the forecited Article of *January*, that all Mixtures of different Liquors introduce in each a different Conformation of Pores, and therefore the Infuſion of a new Liquor, drives the inſeſible Parts of the diſſolved Body from their Places, and forces them to ſtrike againſt each other, and cling together, and ſo becoming more big and heavier than formerly, the internal Agitation of the Liquor is no longer able to move and ſuſtain them, and conſequently they fall to the Bottom.

This, as fairly and ſhortly as I can propoſe it, is his Sentiment of this Phænomenon.

But I conceive an other Account may be given of this Appearance, and that the foreſaid Law of *Hydroſtaticks* is a little deficient. 'Tis true indeed, if we conſider only the ſpecifick Gravity of a Liquor, and the ſpecifick Gravity of a ſolid Particle floating therein, the forementioned Rule is exact; but in ſinking there is requiſite a ſeparation of the Parts of the Liquor by the ſinking Body; and there being a natural Inclination in the Parts of all Liquors to Union ariſing from an Agreement or Congruity of their Parts, there is a reſiſtance therein to any thing that ſeparates this Conjunction: Now unleſs a Body have weight enough to overcome this Congruity or Union of Parts, ſuch a Body will

will float in a Liquor specifically lighter than it self. But that a heavy Body, as *Mercury* or *Iron* may have its Parts reduced to that Minuteness, that their Gravity or Tendency downwards, is not strong enough to separate the Cohesion or Union of the Parts of a Liquor, will be manifest, if we consider, that the Resistance made by the *Medium* to a falling Body, is according to the Superficies of the Body; but as the Body decreases in Bulk, its Superficies does not proportionably decrease, thus a Sphere of an Inch Diameter, has not eight times less Superficies than a Sphere of two Inches Diameter, tho' it have eight times less Bulk, and consequently passing through a *Medium*, as suppose Air or Water, the Sphere of an Inch Diameter is, proportionably to its Bulk, more resisted, than a Sphere of two Inches Diameter in proportion to its Bulk, and hence it will come to pass, that at last a Body may be reduced to that Minuteness, that its Gravity pressing downwards (which is according to its Bulk) may be less than the resistance of the *Medium*, which operates on the Surface of the Body; seeing as I said before, the Surfaces of Bodies do not decrease so fast as their Bulks, these decreasing in a *Triplicate*, but those in a *Duplicate Ratio* of the Bodies Diameters.

This Account does not at all oppose the Experiment of a *Menstruum* over the Fire, being able to dissolve or sustain a greater Quantity of a heavy Body; for the Reason of this, as 'tis given by my Brother, does not Contradict my Notion. The Account likewise

wise, that He gives of Chymical Precipitation agrees very well with what I propose: So that of these I shall say no more.

But because in the beginning of my *Discourse*, I say that the forementioned Law of *Hydrostaticks* is a little defective, I desire to explain my self a little further in that *Point*. In Weights falling through the Air, were Gravity only consider'd, the Proportions of their Descents would be exactly as *Galileo* has demonstrated; but it is allow'd by all, *that* the Resistance of the Air, not being consider'd in those Demonstrations, they are not Mathematically true in Practise, but that really there is something of that proportion hindred by the Airs Resistance. Now, what is this less than to say, that the Resistance of the Air takes off some of the Operation of Gravity, or is able to withstand or oppose part of its Action? And if so, what shall we say were an Iron Sphere let through a Medium of Water? Surely the Proportions of its descents would be much more *disturbed* herein, as Water is much more Solid and difficult to be separated or passed through than Air, and consequently we must needs grant, that more of the Operation of Gravity, is taken off or resisted by this Opposition of the Water, than that of the Air. And if so, surely there may be a certain degree of Gravity, that may be quite taken off by the resistance of the Water: Were a Pistol Bullet let fall through the Air, it would descend imperceptibly nigh the Proportions that *Galileo* has assigned, but were a single grain of Sand so let fall, it would be much hindred

in

in its Course, and the half of this Grain would be more obstructed; what shall we then say of the ten thousandth part, or of a part the ten thousand millioneth of this, and again of the Infinits Subdivisions of that, 'till at last we come to a part that would be wholly resisted, or kept up; such as I conceive the minute Particles of a Body dissolved in a *Menstruum*?

On this account 'tis, I say, that the fore-mentioned Principle of *Hydrostaticks* is a little defective; for it considers not the natural Congruity of the Parts of a Liquor, whereby they desire, as 'twere, to unite and keep together, just as we see two drops of Water on a dry Board being brought together do jump and coalesce, and therefore Liquors have an innate power of resisting a certain degree of force that would separate them; such as I suppose the degree of Gravity, in the most minute Particles of a Body dissolved in a *Menstruum*.

The fore-mentioned Rule holds true to the most nice Sense in great Bodies, but in those that are by many Millions of Divisions smaller, it seems to fail.

This, in short, is my Conjecture in this matter, which I propose, as my Brother did his, with all submission imaginable, and thereby to give occasion to others to enquire into the Causes of this appearance, rather than to publish my own Sentiments as the undoubted Solution thereof.

But this I must acknowledge, that the internal motion of the parts of a Liquor seems so very agreeable to truth, and explicates so many Phænomena easily and plainly, that I would not be thought to deny it. Neither would I be thought wholly to reject my Brothers Solution of this Problem; for certainly that Motion (whatsoever it is) in a *Menstruum*, which is able to dissolve such a solid Body as Iron, that is, which is able to disturb the close and strong Cohesion of the Parts of Iron, may very well be supposed sufficient to disturb or keep up these parts from resting in the bottom of the Vessel, wherein the solution was made: And certainly no better account can possibly be given of such Solutions, than by supposing such an internal motion in the parts of the *Menstruum* insinuating themselves into the solid Body, and loosening its parts. And tho' it may be objected, that in the parts of Water there may be supposed as violent an internal motion, as in the parts of *Aqua-Fortis*, and yet we see Water will not dissolve Iron as *Aqua-Fortis* does, and common Bees-Wax is disturbed by neither of them, I leave the nice enquiry after this Point to others, *viz.* What kind of Motion and peculiar Conformation of parts is requisite both in the *Menstruum* and in the dissolved Body, that a Solution may result from their Commixture.

Some Reflections on the foregoing Paper by Mr. T. M.

What my Brother has laid down in this Discourse, I think does most undeniably evince that the received Law of *Hydrostaticks* is somewhat defective. For Liquors, tho' they are

are Fluid yet they are Bodies, and therefore conſiſt of parts united; which Union, tho' it be eaſily deſtroy'd, yet of neceſſity it requires ſome degree of Force for the effecting it; nor is it more manifeſt, if rightly conſidered, that a Flint requires Force for the ſeparation of its parts, than that Fluids do for theirs. But however, I imagine, this Property ought not to be relied upon as the ſole Cauſe of this appearance, to which my Brother has apply'd it; nay perhaps does not ſo much as concur the leaſt in the producing this effect; my Reaſon in ſhort is this: Whatever is of ſufficient Power to raiſe the minute Particles of a *heavy Body* in a light Fluid, is certainly a ſufficient cauſe to keep them in that ſtate: Now my Suppoſition may give ſome account of this, what my Brother ſays, never can; for he muſt neceſſarily ſuppoſe them firſt raiſed; and then he gives the reaſon of their not ſinking: Whereas 'tis not to be queſtioned but that that Force which raiſed them, is the ſame which keeps them from falling to the bottom.

But theſe Conjectures (for I eſteem them no more) I leave to the Conſideration of thoſe that deſire to enquire further in this Matter.

Of the weight of a cubic foot of divers grains, &c. try'd in a Vessel of well-season'd Oak, whose concave was an exact cubit foot. By the direction of the Philosophical Society at Oxford.

THE following Bodies were poured gently into the Vessel, and those in the 12 first Experiments were weighed in scales turning with 2 ounces, but the last 7 were weigh'd in scales turning with one ounce. The pounds and ounces here mentioned are Avoirdupois.

	℔.	℥.
1. A foot of Wheat (worth 6 *s.* a Bushel) weigh'd of Avoirdupois weight.	47.	8.
2. *Wheat* of the best sort (worth 6 *s.* 4 *d.* a Bushel)	48.	4.
3. The same sort of *Wheat* measured a second time.	48.	2.
Both sorts were red *Lammas Wheat* of the last year.		
4. White *Oats* of the last year.	29.	8.
The best sort of *Oats* were 2 *d.* in a Bushel better than these.		
5. Blue *Pease* (of the last year) and much worm-eaten.	49.	12.
6. White *Pease* of the last year but one.	50.	8.

7. *Barley*

		℔.	℥.
7.	*Barley* of the laſt year (the beſt ſort ſells for 1 *s.* 6 *d.* in a Quarter more than this.	41.	2.
8.	*Malt* of the laſt years Barley, made 2 Months before.	30.	4.
9.	Field-*Beans* of the laſt year but one.	50.	8.
10.	Wheaten *Meal* (unſifted).	31.	0.
11.	Rye *Meal* (unſifted.)	28.	4.
12.	Pump *Water.*	62.	8.
13.	Bay *Salt.*	54.	1.
14.	White Sea-*Salt.*	43.	12.
15.	Sand.	85.	4.
16.	*Newcaſtle Coal.*	67.	12.
17.	Pit-Coal from *Wednesbury*, 63. but this is very uncertain in the filling the Interſtices betwixt the greater pieces.	63.	0.
18.	Gravel.	109.	5.
19.	Wood-Aſhes.	58.	5.

A further Liſt of the Specifick Gravities *of Bodies, being in proportion as the following numbers.*

PUmp-water	1000
Fir dry	546
Elm dry	600
Cedar dry	613
Walnut-tree dry	631
Crab-tree meanly dry	765
Aſh meanly dry, and of the out-ſide lax part of the Tree	734
Aſhmore dry, but about the Heart	845
Maple-tree	755

Yew of a Knot or Root 16 years old	760
Beech meanly dry	854
Oak very dry, almoſt Worm eaten	753
Oak of the out-ſide ſappy part, fell'd a year ſince	870
Oak dry, but of a very ſound cloſe texture	929
The ſame tried another time	932
Logwood	913
Claret	993
Moil Cyder not clear	1017
Sea-water ſetled clear	1028
College plain Ale the ſame	1028
Urine	1030
Milk	1031
Box the ſame	1031
Redwood the ſame	1031
Sack	1033
Beer Vinegar	1034
Pitch	1150
Pit-Coal of Stafford-ſhire	1240
Speckled wood of Virginia	1313
Lignum Vitæ	1327
Stone-bottle	1777
Ivory	1826
Alabaſter	1872
Brick	1979
Heddington-ſtone, the ſoft lax kind	2029
Burford-ſtone, an old dry piece	2049
Paving-ſtone a hard ſort from about Blaidon	2460
Flint	2542
Glaſs of a quart Bottle	2666
Black Italian Marble	2704
White Italian Marble tried twice	2707

White

White Italian Marble of another sort of a visibly closer texture	2718
Block-tin	7321
Copper	8843
Lead	11345
Quick-*silver*	14019
Quick-silver	13593

The last Experiment was tried with another quantity of Quick-silver, which had been used in Water in the preceding Experiment: However I rather trust the last, for that I found a small mistake (tho' here in the calculation allowed for) in the weight of the Glass containing the Quick-silver in the trial before.

The Solids here mentioned were examined *hydrostatically* by weighing them in Air and Water; but the Fluids, by weighing an equal portion of each in a Glass holding about a quart. The numbers shew the proportion of gravity of equal portions of these Bodies; but if of these Bodies we take portions equally heavy, their magnitudes will be reciprocally proportional to their correspondent numbers, *e.g.* a cubic foot of water is to a cubic foot of Alabaster in gravity as 1000 to 1872; but a pound weight of water, is to a pound weight Albaster in magnitude as 1872 to 1000. So that knowing by the former Table the weight of a cubic foot of Water, and by this, the proportion in gravity betwixt Water and Alabaster, we may by the Rule of Three find the weight of a cubic foot of Alabaster, and so of any other of these Bodies; or we may know their magnitude by knowing their gravity. So that an irregular piece or quantity of theſe Bodies being offered, 'tis but weighing them, and we may know their juſt magnitude without further trouble.

Obſervations of the Comparative, Intenſive or Specific Gravities of various Bodies. Made by Mr. J. C.

PUmp-water,	1000
Cork,	237
Saſſafras Wood,	482
Juniper Wood (dry)	556
Plum-tree, (dry)	663
Maſtic,	849
Santalum Citrinum,	809
Santalum album,	1041
Santalum rubrum,	1128
Ebony,	1177
Lignum Rhodium,	1125
Lignum Aſphaltum,	1179
Aloes,	1177
Succinum pellucidum,	1065
Succinum pingue,	1087
Jet,	1238
The top part of a Rhinocero's horn,	1242
The top part of an Ox horn,	1840
The (Blade) bone of an Ox,	1656
An human Calculus,	1240
Another Calculus humanus,	1433
Another Calculus,	1664
Brimſtone, ſuch as commonly ſold,	1811
Borax,	1720
A ſpotted factitious Marble,	1822
A Gally-Pot,	1928

Oyſter-

Oyſter-ſhell,	2092
Murex-ſhell,	2590
Lapis manati,	2270
Selenitis,	2322
Wood petrefied in *Lough-Neagh*,	2341
Onyx-ſtone,	2510
Turcois-ſtone,	2508
Engliſh Agat	2512
Grammatias lapis,	2515
A Cornelian,	2568
Corallachates,	2605
Talc.	2657
Coral,	2689
Hyacinth (ſpurious)	2631
Jaſper (ſpurious)	2666
A pellucid Pibble,	2641
Rock Cryſtal,	2659
Cryſtallum Diſdiaclaſticum,	2704
A red Paſte,	2842
Lapis Nephriticus,	2894
Lapis Amiantus from *Wales*,	2913
Lapis Lazuli	3054
An Hone,	3288
Sardachates,	3598
A Granat,	3978
A Golden Marcaſite,	4589
A blue Slate with ſhining Particles,	3500
A mineral Stone, yielding 1 part in 160 Metal,	2650
The Metal thence extracted,	8500
The (reputed) Silver Ore of *Wales*,	7464
The Metal thence extracted,	11087
Biſmuth,	9859
Spelter,	7065
Spelter Soder,	8362
Iron of a Key,	7643

Steel,

Steel,	7852
Cast Brass,	8100
Wrought Brass,	8280
Hammer'd Brass,	8349
A false Guinea,	9075
A true Guinea,	18888
Sterling Silver,	10535
A brass Half-Crown,	9468
Electrum, a British Coin,	12071
A Gold Coin of *Barbary*,	17548
A Gold Medal from *Morocco*,	18420
A *Mentz* Gold Ducat,	18261
A Gold Coin of *Alexanders*,	18893
A Gold Medal of Queen *Mary*,	19100
A Gold Medal of Queen *Elizabeth*,	19125
A Medal esteem'd to be near fine Gold,	19636

A

A Letter of Dr. Wallis *to* Dr. Sloane, *concerning the Generation of Hail, and of Thunder and Lightning, and the Effects thereof.*

Oxon. July 26. 1697.

S I R,

I Thank you for the Tranſactions of *June* which you ſent me; wherein I am well pleaſed with Mr. *Halley*'s Remarks on the Torricellian Experiment at the top of *Snowdon-hill* in *Wales*, at the height of 1240 yards perpendicular. Where the height of that Quickſilver in the Baroſcope was 3 Inches and $\frac{8}{10}$ leſs than below at the Sea-ſide; which is an Obſervation of good uſe; and would have been more ſo, had he had the leiſure to make like Obſervations at ſeveral other perpendicular heights in the Aſcent. For from ſuch comparative Obſervations we are to make an Eſtimate, at what proportion the height of the Quickſilver doth decreaſe in reference to the height of the place. I mean whether in the ſame Proportion, or the Duplicate, Sub-duplicate, or how otherwiſe Complicate thereof. From whence we may make a Judgment of the height of the Atmoſphere; if at leaſt it have a determinate height. I did once attempt (a great while ſince) a Computation of it; but wanted a ſufficient number of *Data* to proceed upon.

But

But that which is most surprizing in those Transactions is, the prodigious *Hails* there mentioned; which happen'd at many Places, on different Days, and all within the compass of less than six Weeks. I have been told of the like in other Places about the same time, in *Lincolnshire*, *Hampshire*, and elsewhere; whether or no on the same Days which you mention, I cannot tell; nor can I give a particular Account of them. But it would be kind in those who can, to give you like Accounts thereof with those you have Published, for a like publick Information.

I find it is thought very strange, what should cause so sudden a Congelation of Hail-stones to so great a bigness before they fell. And it is indeed very strange. But it is not necessary that the whole bigness be attained before they begin to fall, but the freezing may continue during the Fall, to increase the Bulk. For I remember that (many Years since) I observed here at *Oxford* a strange shower of Hail, wherein (besides the formed Stones that fell on the Ground, there did hang on the Trees a great deal in the Form of Icicles (a Foot or more in length) so many and heavy, as to break off some Boughs with their weight; and I was then told, that in some places great Branches of Trees were so broken off; which must needs be from the continuing to freeze during the fall.

And truly the Generation of Hail in general, is a thing which deserves to be farther inquired into, than (I think) hath been yet done. I find Mr. *Halley* (in his Narration) ascri-

ascribing it to *Vapour disposing the Aqueous Parts so to congeal.* And not unlikely.

If I may interpose my Opinion, you may take it thus:

Thunder and Lightning are so very like the Effects of fired Gun-powder, that we may reasonably judge them to proceed from like Causes. The violent Explosion of Gun-powder, attended with the Noise and Flash, is so like that of Thunder and Lightning, as if they differed only as Natural and Artificial; as if Thunder and Lightning were a kind of natural Gun-powder, and this a kind of artificial Thunder and Lightning.

Now the principal Ingredients in Gun-powder are, Nitre and Sulphur (the Admistion of Charcole being chiefly to keep the Parts separate for the better kindling of it.) So that if we suppose in the Air, a convenient mixture of Nitrous and Sulphorous Vapours, and those by Accident to take Fire; such Explosion may well follow, with such Noise and Light, as in the firing of Gun-powder. And being once kindled, it will run on from Place to Place as the Vapour leads it, as in a Train of Gun-powder, with like Effects.

This Explosion, if high in the Air, and far from us, will do no Mischief, or not considerable; like a parcel of Gun-powder fired in the open Air, where is nothing near to be hurt by it: But if near, to us (or among us) it may kill Men or Cattle, tear Trees, fire Gunpowder, break Houses, or the like; as Gun-powder would do in like Circumstances.

Now

Now this nearness or farness may be estimated by the Distance of Time between seeing the Flash of Lightning, and hearing the Noise of the Thunder. For though in their Generation, they be simultaneous; yet (Light moving faster than Sound) they come to us successively. I have observ'd that, commonly, the Noise is about Seven or Eight *Seconds* after the Flash (that is, about half a quarter of a Minute); but sometimes much sooner, in a Second or Two, or less than so, and almost immediately upon the Flash. And at such time, the Explosion must needs be very near us, or even amongst us. And, in such Cases, I have (more than once) presaged the Expectation of Mischief, and it hath proved accordingly, in the Destruction of Men or Cattel, and the like. (As once at *Oxford*; when, within half an Hour after such Presage, I heard of one killed at *Medly*, hard by, and others endangered; and another time at *Towcester*, when within a few Hours after, we heard of Five Persons kill'd at *Everton*, about Four or Five Miles from us, and others wounded; beside other Hurt done.)

Now, that there is in Lightning a Sulphorous Vapour, is manifest from the Sulphorous Smell which attends it, especially when Hurt is done; and even where no Hurt is done, from the Lightning it self, more or less discernable. And a sultry Heat in the Air, is commonly a Fore-runner of Lightning soon after.

And that there is also a Nitrous Vapour with it, we may reasonably judge, because we do not know of any Body so liable to a suddain and violent Explosion.

Now

Now these Materials being admitted, it remains to be considered, how they may be kindled in order to such Explosion. As to which,I have been told from Chymists (though I have not seen it tried) That a Mixture of Sulphur, Filings of Steel, with the Admission of a little Water, will not only cause a great Effervescence, but will of it self break forth into an actual Fire.

So that there wants only some Chalybeat or Vitriolick Vapour (or somewhat equivalent) to produce the whole Effect (there being no want of Aqueous Matter in the Clouds.)

And there is no doubt, but that amongst the various *Effluvia* from the Earth, there may be copious Supplies of Matter for such Mixtions.

And 'tis known, that Hay, if laid up too Green, will not only heat, but take Fire of it self.

And while we are discoursing of this, it may suggest somewhat as to the Generation of Hail which is very oft an attendant of Thunder and Lightning. 'Tis well known, in our artificial Congelations, that a Mixture of Snow and Nitre (or even common Salt) will cause a present and very suddain Congelation of Water. And the same in Clouds may cause that of Hail-stones. And the rather, beeause (not only in those prodigiously great, but in common Hail-stones) there seems somewhat like Snow rather than Ice, in the midst of them.

And, as to those in Particular (of which we are now speaking) so very large (as to weigh Half a Pound, or Three Quarters of a Pound) sup-

ſuppoſing them to fall from ſo great a Height, as 'tis manifeſt they did by tne Violence of their Fall: 'Tis very poſſible, that though their firſt Concretion, upon their ſuddain Congelation, might be but moderately great, as in other Hail; yet, in their long Deſcent, if the *Medium* through which they *fall were* alike inclined to Congelation, they might receive a great Acceſſion to their Bulk, and divers of them incorporate into one Like as in thoſe Icicles before mentioned.

Theſe have been my Thoughts, occaſioned by the Conſideration of the ſurprizing Greatneſs of theſe Hail-ſtones, with the great Thunder and Lightning which did attend theſe Storms.

Yours, &c.

THE

A SYNOPSIS OF THE *Aſtronomy* of *Comets.*

THE ancient *Egyptians* and *Chaldeans* (if we may credit *Diodorus Siculus*) by a long Courſe of Obſervations, were able to predict the *Apparitions* of Comets. But ſince they are alſo ſaid, by the Help of the ſame Arts, to have prognoſticated Earthquakes and Tempeſts, 'tis paſt all Doubt, that their Knowledge in theſe Matters, was the Reſult rather of meer *Aſtrological Calculation*, than of any *Aſtronomical Theories* of the Cœleſtial Motions. And the *Greeks*, who were the Conquerors of both thoſe People, ſcarce found any other ſort of Learning amongſt them, than this. So that 'tis to the *Greeks* themſelves as the Inventors (and eſpecially to the Great *Hipparchus*) that we owe this *Aſtronomy*, which is now improv'd to ſuch a Heigth. But yet, amongſt theſe, the Opinion of *Ariſtotle* (who wou'd have Comets to be nothing elſe, but Sublunary Vapours, or Airy Meteors) prevail'd so far, that this most difficult Part of the Astronomical Science lay altogether neglected; for no Body thought it worth while to take Notice of, or write about, the Wandring uncertain Motions of what they esteemed Vapours floating in the *Æther*; whence it came to pass, that nothing certain, concerning the Motion of Comets, can be found transmitted from them to us.

But *Seneca* the *Philosopher*, having consider'd the *Phænomena* of Two remarkable Comets of his Time, made no Scruple to place them amongst the *Cælestial* Bodies; believing them to be *Stars* of equal Duration with the World; tho' he owns their Motions to be govern'd by Laws not as then known or found out. And at last (which was no untrue or vain Prediction) he foretells, that there should be Ages sometime hereafter, to whom Time and Diligence shou'd unfold all these Mysteries, and who shou'd wonder that the Ancients cou'd be ignorant of them, after some lucky Interpreter of Nature had shewn, *in what Parts of the Heavens the Comets wander'd, what, and how great they were.* Yet almost all the Astronomers differ'd from this Opinion of *Seneca*; neither did *Seneca* himself think fit to set down those *Phænomena* of the Motion, by which he was enabled to maintain his Opinion: Nor the *Times* of those Appearances, which might be of use to Posterity, in order to the Determining these Things. And indeed, upon the Turning over very many Histories of Comets, I find nothing at all that can be of Service in this Affair, before, *A. D.* 1337. at which time *Nicephorus Gregoras*, a *Constantinopolitan Historian* and *Astronomer*, did pretty

pretty accurately deſcribe the *Path* of a Comet amongſt the Fix'd Stars, but was too laxe as to the Account of the *Time*; ſo that this moſt doubtful and uncertain Comet, only deſerves to be inſerted in our Catalogue, for the ſake of its appearing near 400 Years ago.

Then the next of our Comets was in the Year 1472, which being the ſwifteſt of all, and neareſt to the Earth, was obſerv'd by *Regiomantanus*. This Comet (ſo frightful upon the Account both of the Magnitude of its Body, and the Tail) mov'd Forty Degrees of a great Circle in the Heavens, in the Space of one Day; and was the firſt, of which any proper Obſervations are come down to us. But all thoſe that conſider'd Comets, until the Time of *Ticho Brahe* (that great Reſtorer of Aſtronomy) believ'd them to be below the Moon, and ſo took but little Notice of them, reckoning them no other than Vapours.

But in the Year 1577, (*Ticho* ſeriouſly purſuing the Study of the Stars, and having gotten large Inſtruments for the Performing Cœleſtial Menſurations, with far greater Care and Certainty, than the Ancients cou'd ever hope for) there appear'd a very remarkable Comet; to the Obſervation of which, *Ticho* vigorouſly applied himſelf; and found by many juſt and faithful Trials, that it had not a *Diurnal Parallax* that was at all perceptible: And conſequently was not only no Aireal Vapour, but alſo much higher than the Moon; nay, might be plac'd amongſt the Planets for any thing that appear'd to the Contrary; the cavilling Oppoſition made by ſome of the

School-men in the mean time, being to no Purpose.

Next to *Ticho*, came the Sagacious *Kepler*. He having the Advantage of *Ticho*'s Labours and Observations, found out the *true Physical* System of the World, and vastly improv'd the Astronomical Science.

For he demonstrated that all the Planets perform their Revolutions in *Elliptick Orbits*, *whose Plains pass thro' the Center of the Sun*, observing this Law, That the *Area's* (*of the Elliptick Sectors, taken at the Center of the Sun, which he proved to be in the common Focus of these Ellipses*) *are always proportional to the Times, in which the correspondent Elliptical Arches are describ'd.* He discover'd also, That *the Distances of the Planets from the Sun are in the* Sesquialtera Ratio *of the Periodical Times*, or (which is all one) That *the Cubes of the Distances are as the Squares of the Times.* This great Astronomer had the Opportunity of observing Two Comets, one of which was a very remarkable one. And from the Observations of these (which afforded sufficient Indications of an *Annual Parallax*) he concluded, That *the Comets mov'd freely thro' the Planetary Orbs, with a Motion not much different from a Rectilinear one*; *but of what Kind he cou'd not then precisely determine.* Next, *Hevelius* (a Noble Emulator of *Ticho Brahe*) following in *Kepler*'s Steps, embraced the same Hypothesis of the *Rectilinear Motion of Comets*, himself accurately observing many of them. Yet, he complain'd, that his Calculations did not perfectly agree to the Matter of Fact in the Heavens: And was aware, that *the Path of a Comet was bent into a Curve Line towards the Sun.*

Sun. At length, came that prodigious Comet of the Year 1680. which deſcending (as it were) from an infinite Diſtance *Perpendicularly* towards the Sun, aroſe from him again with as great a Velocity.

This Comet, (which was ſeen for Four Months continually) by the very remarkable and peculiar Curvity of its Orbit (above all others) gave the fitteſt Occaſion for inveſtigating the *Theory of the Motion.* And the *Royal Obſervatories* at *Paris* and *Greenwich* having been for ſome time founded, and committed to the Care of moſt *excellent Aſtronomers*, the *apparent* Motion of this Comet was moſt accurately (perhaps as far as Humane Skill cou'd go) obſerv'd by Mrs. *Caſſini* and *Flamſteed.*

Not long after, that *Great Geometrician*, the *Illuſtrious Newton*, writing his *Mathematical Principles of Natural Philoſophy*, demonſtrated not only that what *Kepler* had found, did neceſſarily obtain in the *Planetary Syſtem*; but alſo, that all the *Phænomena* of Comets wou'd naturally follow from the ſame Principles; which he abundantly illuſtrated by the Example of the aforeſaid Comet of the Year 1680. ſhewing, at the ſame time, a Method of Delineating the Orbits of Comets Geometrically; wherein he (not without the higheſt Admiration of all Men) ſolv'd a Problem, whoſe Intricacy render'd it worthy of himſelf. This Comet he prov'd to move round the Sun in a Parabolical Orb, and to deſcribe Area's (taken at the Center of the Sun) proportional to the Times.

Wherefore (following the Steps of so *Great a Man*) I have attempted to bring the same Method to *Arithmetical Calculation*; and that with desired Success. For, having collected all the Observations of Comets I could, I fram'd this Table, the Result of a prodigious deal of Calculation, which, tho' but small in Bulk, will be no unacceptable Present to Astronomers. For these Numbers are capable of Representing all that has been yet observ'd about the Motion of Comets, by the Help only of the following *General Table*; in the making of which I spar'd no Labour, that it might come forth perfect, as a Thing consecrated to Posterity, and to last as long as *Astronomy* it self.

The

The Astronomical Elements of the Motions in a Parabolick Orb of all the Comets that have been hitherto duly observ'd.

Com. An	Nodus Ascend. gr. ′ ″	Inclin. Orbitæ. gr. ′ ″	Perihelion. gr. ′ ″	Distan. Perihelia à Sole.	Log. Dist. Perihelia à Sole.	Temp. equat. Perihelii. d. h. ′	Perihelion à Nodo. gr. ′ ″	
1337	♊ 24.21. 0	32.11. 0	♉ 7.59. 0	40666	9.609236	*June* 2. 6.25	46.22. 0	Retrog.
1472	♑ 11.46.20	5.20. 0	♉ 15.33.30	54273	9.734584	*Feb.* 28 22.23	123.47.10	Retrog.
1531	♉ 19.25. 0	17.56. 0	♒ 1.39. 0	56700	9.753583	*Aug.* 24.21.18½	107.46. 0	Retrog.
1532	♊ 20.27. 0	32.36. 0	♋ 21. 7. 0	50910	9.706803	*Oct.* 19 22.12	30.40. 0	Direct.
1556	♍ 25.42. 0	32. 6.30	♑ 8.50. 0	46390	9.666424	*Apr.* 21.20. 3	103. 8. 0	Direct.
1577	♈ 25.52. 0	74.32.45	♌ 9.22. 0	18342	9.263447	*Oct.* 26 18.45	103.30. 0	Retrog.
1580	♈ 18.57.20	64.40. 0	♋ 19. 5.50	59628	9.775450	*Nov.* 28 15.00	90. 8.30	Direct.
1585	♉ 7.42.30	6. 4. 0	♈ 8.51. 0	109358	0.038850	*Sept.* 27 19.20	28.51.30	Direct.
1590	♍ 15.30.40	29.40.40	♏ 6.54.30	57661	9.760882	*Jan.* 29. 3.45	51.23.50	Retrog.
1596	♒ 12.12.30	55.12. 0	♏ 18.16. 0	51293	9.710058	*Juli* 31.19.55	83.56.30	Retrog.
1607	♉ 20.21. 0	17. 2. 0	♒ 2.16. 0	58680	9.768490	*Oct.* 16. 3.50	108.05. 0	Retrog.
1618	♊ 16. 1. 0	37.34. 0	♈ 2.14. 0	37975	9.579498	*Oct.* 29.12.23	73.47. 0	Direct.
1652	♊ 28.10. 0	79.28. 0	♈ 28.18.40	84750	9.928140	*Nov.* 2.15.40	59.51.20	Direct.
1661	♊ 22.30.30	32.35.50	♋ 25.58.40	44851	9.651772	*Jan.* 16.23.41	33.28.10	Direct.
1664	♊ 21.14. 0	21.18.30	♌ 10.41.25	102575½	0.011044	*Nov.* 24.11.52	49.27.25	Retrog.
1665	♏ 18.02. 0	76.05. 0	♊ 11.54.30	10649	9.027309	*Apr.* 14. 5.15½	156. 7.30	Retrog.
1672	♑ 27.30.30	83.22.10	♉ 16.59.30	69739	9.843476	*Feb.* 20. 8.37	109.29. 0	Direct.
1677	♏ 26.49.10	79.03.15	♌ 17.37. 5	28059	9.448072	*Apr.* 26.00.37½	99.12. 5	Retrog.
1680	♑ 2. 2. 0	60.56. 0	♐ 22.39.30	00612½	7.787106	*Dec.* 8.00. 6	9.22.30	Direct.
1682	♉ 21.16.30	17.56. 0	♒ 2.52.45	58328	9.765877	*Sept.* 4.07.39	108.23.45	Retrog.
1683	♍ 23.23. 0	83.11. 0	♊ 25.29.30	56020	9.748343	*Juli* 3. 2.50	87.53.30	Retrog.
1684	♐ 28.15. 0	65.48.40	♏ 28.52. 0	96015	9.982339	*Mai* 29.10.16	29.23.00	Direct.
1686	♓ 20.34.40	31.21.40	♊ 17.00.30	32500	9.511883	*Sept.* 6.14.33	86.25.50	Direct.
1698	♐ 27.44.15	11.46. 0	♑ 00.51.15	69129	9.839660	*Oct.* 8.16.57	3. 7. 0	Retrog.

This Table needs little Explication, since 'tis plain enough from the Titles, what the Numbers mean. Only it may be observ'd, that the *Perihelium* Distances, are estimated in such Parts, as the Middle Distance of the Earth from the Sun, contains 100000.

A General Table for Calculating the Motions of Comets in a Parabolical Orbit.

Med. mot.	Ang. à peribelio.	Logar. pro dist. à Sole.	Med. mot.	Ang. à peribelio.	Logar. pro dist. à Sole.
o	gr. ′ ″		o	gr. ′ ″	
1	1.31.40	0.000077	31	42.55.06	0.062400
2	3. 3.15	0.000309	32	44. 3.20	0.065838
3	4.34.43	0.000694	33	45.10.29	0.069319
4	6. 6. 0	0.001231	34	46.16.35	0.072839
5	7.37. 1	0.001921	35	47.21.36	0.076396
6	9. 7.43	0.002759	36	48.25.33	0.079984
7	10.38. 2	0.003745	37	49.28.27	0.083600
8	12. 7.54	0.004876	38	50.30.19	0.087244
9	13.37.17	0.006151	39	51.31. 8	0.090910
10	15. 6. 7	0.007564	40	52.30.56	0.094596
11	16.34.20	0.009115	41	53.29.44	0.098300
12	18. 1.54	0.010798	42	54.27.32	0.102019
13	19.28.47	0.012609	43	55.24.21	0.105752
14	20.54.54	0.014550	44	56.20.12	0.109490
15	22.20.14	0.016607	45	57.15. 6	0.113240
16	23.44.44	0.018783	46	58. 9. 3	0.116995
17	25. 8.22	0.021072	47	59. 2. 4	0.120756
18	26.31. 8	0.023470	48	59.54.11	0.124518
19	27.52.5[illegible]	0.025969	49	60.45.25	0.128278
20	29.13.47	0.028570	50	61.35.4[illegible]	0.132035
21	30.33.4[illegible]	0.031263	51	62.25.14	0.135792
22	31.52.3[illegible]	0.034045	52	63.13.5[illegible]	0.139544
23	33.10.23	0.036916	53	64. 1.40	0.143291
24	34.27.12	0.039864	54	64.48.38	0.147029
25	35.42.59	0.042892	55	65.34.50	0.150762
26	36.57.41	0.045989	56	66.20.13	0.154482
27	38.11.20	0.049154	57	67.04.50	0.158192
28	39.23.54	0.052382	58	67.48.42	0.161890
29	40.35.22	0.055668	59	68.31.50	0.165578
30	41.45.47	0.059009	60	69.14.16	0.169254

Med. mot.	Angul. à peribelio.	Logar. pro dist. à Sole.	Med. mot.	Ang. à peribelio.	Logar. pro dist à Sole.
o	gr. ′ ″		o	gr. ′ ″	
61	69.55.58	0.172914	91	86.20.34	0.274176
62	70.36.56	0.176557	92	86.46.20	0.277239
63	71.17.16	0.180188	93	87.11.43	0.280284
64	71.56.56	0.183803	94	87.36.45	0.283306
65	72.35.57	0.187404	95	88.01.27	0.286308
66	73.14.15	0.190978	96	88.25.49	0.289293
67	73.51.59	0.194540	97	88.49.48	0.292252
68	74.29. 6	0.198085	98	89.13.32	0.295201
69	75.05.38	0.201614	99	89.36.54	0.298122
70	75.41.35	0.205122	100	90.00.00	0.301030
71	76.16.56	0.208612	102	90 45.14	0 306782
72	76.51.43	0.212080	104	91.29.18	0.312469
73	77.25.57	0.215529	106	92.12.14	0.318060
74	77.59.41	0.218963	108	92.54. 4	0.323587
75	78.32.54	0 222378	110	93.34.52	0 329042
76	79. 5.35	0.225769	112	94.14 40	0.334424
77	79.37.45	0.229142	114	94.53.30	0.339736
78	80. 9.23	0.232488	116	95.31.22	0.344979
79	80.40.34	0.235809	118	96. 8 22	0 350153
80	81.11.16	0.239127	120	96.44.30	0.355262
81	81.41.31	0.242416	122	97.19.48	0.360306
82	82.11.19	0.245684	124	97.54.17	0.365284
83	82.40.40	0.248933	126	98.28.00	0.370200
84	83. 9.34	0.252159	128	99.00.57	0.375052
85	83.38. 4	0.255366	130	99,33.11	0.379842
86	84. 6. 8	0.258552	132	100. 4.43	0.384576
87	84.33.49	0.261720	134	100.35.45	0 389252
88	85. 1. 5	0.264865	136	101. 5.48	0.393868
89	85.27.58	0.267989	138	101.35.22	0.398428
90	85.54.27	0.271092	140	102 4.19	0.402930

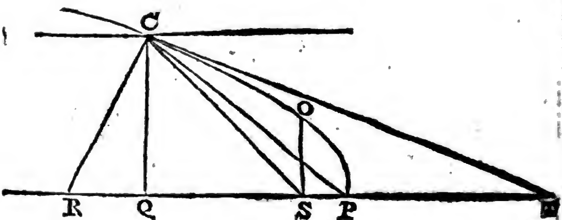
Med.

Med. mot.	*Ang. à peribelio.*	*Logar. pro. dist. à Sole.*	*Med. mot.*	*Ang. à per ibelio*	*Logar. pro dist. à Sole.*
o	gr. ′ ″		o	gr. ′ ″	
142	102.32.41	0.407380	204	113.37.25	0.523406
144	103.00.31	0.411784	208	114. 9.52	0.529705
146	103.27.47	0.416132	212	114.41.23	0.535886
148	103.54.31	0.420430	216	115.12.02	0.541958
150	104.20.43	0.424676	220	115.41.51	0.547922
152	104.46.22	0.428866	224	116.10.52	0.553782
154	105.11.33	0.433012	228	116.39. 7	0.559538
156	105.36.16	0.437110	232	117. 6.38	0.565199
158	106.00.32	0.441164	236	117.33.27	0.570762
160	106.24.23	0.445178	240	117.59.35	0.576233
162	106.47.47	0.449144	244	118.25. 5	0.581616
164	107.10.44	0.453060	248	118.49.57	0.586912
166	107.33.17	0.456936	252	119.14.14	0.592122
168	107.55.27	0.460772	256	119.37.56	0.597252
170	108.17.14	0.464208	260	120. 1. 6	0.602301
172	108.38.37	0.468318	264	120.23.44	0.607274
174	108.59.39	0.472030	268	120.45.52	0.612174
176	109.20.20	0.475705	272	121. 7.30	0.616998
178	109.40.40	0.479340	276	121.28.39	0.621750
180	110.00.40	0.482937	280	121.49.22	0.626438
182	110.20.20	0.486498	284	122. 9.38	0.631056
184	110.39.41	0.490022	288	122.29.28	0.635608
186	110.58.44	0.493512	292	122.48.54	0.640098
188	111.17.28	0.496965	296	123. 7.57	0.644525
190	111.35.55	0.500384	300	123.26.36	0.648893
192	111.54.05	0.503769	310	124.11.40	0.659559
194	112.11.58	0.507121	320	124.54.36	0.669880
196	112.29.34	0.510441	330	125.35.34	0.679876
198	112.46.55	0.513729	340	126.14.44	0.689568
200	113. 4.00	0.516934	350	126.52.12	0.698970

Med. mot.	*Ang. à peribelio.*	*Logar. pro dist. à Sole.*	*Med. mot.*	*Ang. à peribelio.*	*Logar. pro dist. à Sole.*
o	gr. ' "		o	gr. ' "	
360	127.28. 6	0.708104	820	141.49.24	0.970836
370	128. 2.33	0.716976	840	142.10.00	0.978397
380	128.35.38	0.725606	860	142.29.56	0.985771
390	129. 7.27	0.734006	880	142.49.10	0.992970
400	129.38. 4	0.742186	900	143. 7.48	0.100000
410	130. 7.34	0.750160	920	143.25.51	1.006871
420	130.36. 2	0.757930	940	143 43 21	1.013586
430	131. 3.30	0.765516	960	144.00.18	1.020155
440	131.30. 2	0.772918	980	144.16.46	1.026583
450	131.55.41	0.780148	1000	144.32.46	1.032876
460	132.20.30	0.787216	1500	149 26. 8	1.158188
470	132.44.32	0.794122	2000	152.26.15	1.246058
480	133. 7.50	0.800882	2500	154.32.20	1.313703
490	133.30.25	0.807494	3000	156. 7.27	1.368678
500	133.52.20	0.813969	3500	157.22.49	1.414974
520	134.34.18	0.826522	4000	158.24.36	1 454950
540	135.14. 0	0.838600	4500	159.16 36	1 490125
560	135.51.28	0.850187	5000	160. 1 12	1.521521
580	136.27. 6	0.861369	5500	160.40. 5	1.549874
600	137.00.57	0.872155	6000	161.14 24	1 575718
620	137.33.13	0.882575	6500	161 45.00	1 599460
640	138. 3.58	0.892649	7000	162.12.34	1 621417
660	138.33.21	0.902401	7500	162.37.34	1.641838
680	139. 1.29	0.911866	8000	163.00.23	1.660922
700	139.28.25	0.921012	8 500	163 21.20	1.678834
720	139.54.16	0.929907	9000	163 40.42	1 695708
740	140.19. 5	0.938549	9500	163.58.38	1 711662
760	140.42.56	0.946951	10000	164.15.20	1.726784
780	141.05.55	0.955124	50000	170.52. 0	2.197960
800	141.28. 3	0.963082	100000	172.45.44	2.399655

The Construction and Use of the general Table.

As the Planets move in *Elliptick* Orbs, so do the Comets in *Parabolick* ones, having the Sun in their common *Focus*, and describe equal *Area's* in equal *Times*. But now because all *Parabola's* are *similar* to one another, therefore if any determinate Part of the *Area* of a given *Parabola*, be divided into any Number of Parts at Liberty, there will be a *like* Division made in all *Parabola's*, under the same Angles, and the Distances will be proportional: And consequently this one Table of ours will serve for all Comets. Now, the Manner of the Calculation of this Table is thus: In the *Fig.*



Let S be the *Sun*, POC the Orbit of a Comet, P the *Perihelion*, O the Place where the Comet is 90 *gr*. distant from the Sun, C any other Place. Draw the Right Lines CP, CS, and make ST, SR, equal to CS; and then having drawn the Right Lines CR, CT, (whereof the one is a Tangent, and the other a Perpendicular to the Curve) let fall CQ perpendicular to the *Axis* PSR.

Now,

Now, any *Area*, as COPS, being given, 'tis requir'd to find the Angle CSP, and the Distance CS. From the Nature of the *Parabola* RQ is ever = $\frac{1}{2}$ the Parameter of the *Axis*, and consequently if the Parameter be put = 2, then RQ = 1. Let CQ=z; then PQ shall = $\frac{1}{2}zz$, and the Parabolick Segment COP=$\frac{1}{12}zzz$. But the Triangle CSP will = $\frac{1}{4}z$, and so the Mixtilineal *Area* COPS=$\frac{1}{12}z^3+\frac{1}{4}z=a$, whence $z^3+3z=12a$. Wherefore resolving this Cubical Equation, z or the Ordinate CQ will be known. Now, let the *Area* OPS be propos'd to be divided into 100 Parts; this *Area* is $\frac{1}{12}$ of the Square of the Parameter, and consequently $12a$ is = that Square = 4. If therefore the Roots of these Equations z^3+3z = 0,04 : 0,08 : 0,12 : 0, 16, *&c.* be successively extracted, there will be obtain'd so many z or Ordinates CQ respectively, and the *Area* SOP will be divided into 100 Parts. And in like manner is the *Calculus* to be continued beyond the Place O. Now the Root of this Equation (since RQ is=1) is the Tabular Tangent of the Angle CRQ, or $\frac{1}{2}$ the Angle CSP, and so the Angle CSP is given. And RC, the Secant of the same Angle CRQ, is a mean Proportional between RQ or Unity, and RT, which is the Double of SC, as is plain from the *Conicks*. But if SP be put =1, and so the *Latus Rectum* = 4 (as in our Table) then RT will be the Distance sought, *viz.* the Double of SC in the former *Parabola*. After this manner therefore, I compos'd the foregoing Table, which serves to represent the Motions of all Comets: For hitherto there has been none observ'd, but comes within the Laws of the *Parabola*.

It remains now, that we give the Rules for the Calculation, and shew the Way of determining the Place of a Comet seen, by these Numbers. *The Velocity of a Comet moving in a* Parabola, *is every where to the Velocity of a Planet describing a Circle about the Sun, at the same Distance from the Sun,* as $\sqrt{2}$ to 1. as appears from *Cor.* 7. *Prop.* 16. *Lib.* 1. of the *Princip. Phil. Nat. Math.* If therefore a Comet in its *Perihelium* were suppos'd to be as far distant from the Sun as the Earth is, then the Diurnal *Area* which the Comet wou'd describe, wou'd be to the Diurnal *Area* of the Earth, as $\sqrt{2}$ to 1. And consequently, the Time of the Annual Revolution, is to the Time in which such a Comet wou'd describe a Quadrant of its Orbit from the *Perihelium*, as 3.14159, *&c.* (that is the *Area* of the Circle) to $\sqrt{\frac{8}{9}}$. Therefore the Comet wou'd describe that Quadrant in 109 Days, 14 Hours, 46 Minutes; and so that Parabolick *Area* (Analogous to the *Area* POS) being divided into 100 Parts, to each Day there wou'd be alotted 0.912280. of those Parts; the Log. of which, *viz.* 9.960128, is to be kept for continual Use. *But then the Times in which a Comet, at a greater or less Distance, wou'd* describe similar Quadrants, *are as the Times of the Revolutions in Circles,* that is, in the *Sesquiplicate Ratio* of the Distances: And so the *Diurnal Area's*, estimated in *Centesimal* Parts of the Quadrant (which Parts we put for Measures of the mean Motion, like Degrees) are in each, in the *Subsesquialtera* Proportion *of the Distance from the Sun in the* Perihelion.

These

These necessary Things premis'd, let it be propos'd to compute the *apparent Place* of any one of the mention'd Comets, for any *Given Time*. Therefore,

1. *Let the Sun's Place be had, and the Log. of its Distance from the Earth.*

2. *Let the Difference between the Time of the* Perihelion, *and the Time given, be gotten, in Days and Decimal Parts of Days. To the Log. of this Number, let there be added the constant Log.* 9.960128, *and the Complement Arithmetical of the* ½ *of the Log. of the Distance in the* Perihelium *from the Sun: The Sum will be the Log. of the Mean Motion, to be sought in the first Column of the General Table.*

3. *With the Mean Motion let there be taken the correspondent Angle from the* Perihelium, *in the Table, and the Log. for the Distance from the Sun: Then in Comets that are Direct, add, and in Retrograde ones substract; if the Time be after the* Perihelium, *the Angle thus found, to or from the Place of the* Perihelion; *or in Direct Comets, substract; and in Retrograde ones, add; if the Time be before the* Perihelion, *the foresaid Angle to or from the Place of the* Perihelion; *and so we shall have the Place of the Comet in its Orbit. And to the Log. found for the Distance, let there be added the Log. of the Distance in the* Perihelion, *and the Sum will be the Log. of the true Distance of the Comet from the Sun.*

4. *The Place of the Node, together with the Place of the Comet in its Orbit, being given, let the Distance of the Comet from the Node be found; then, the Inclination of the Plane being given, there will be given also (from the common Rules of Trigonometry) the Comet's Place reduced to the Ecliptick, the Inclination or Heliocentrick Latitude, and the Log. of the* curtate *Distance.*

5. *From*

5. *From these Things given (by the very same Rules that we find the Planets Places, from the Sun's Place and Distance given) we may obtain the* Apparent *or* Geocentrick *Place of the Comet, together with the Apparent Latitude.* And this it may be worth while to illustrate by an Example or two.

EXAMPLE I.

Let it be requir'd to find the Place of the Comet of the Year 166$\frac{4}{5}$, March 1^{d}, 7^{h}, 00', P. *M.* London. *That is.* 96^{d}, 19^{h}, 8', *after the* Perihelion, *which happen'd* Novemb. 24^{d}, 11^{h}, 52'.

Log. Dist. Perihel.		0. 011044
Log. Sesquialt.		0. 016566
Comp. Arith.		9 983434
		9. 960128
Log. Temp.		1. 985862
Log. Med. Mot.		1. 929424
Medius Motus		85.001
Perihel.	♌	10. 41. 25
Ang. Corresp.		83. 38. 05—
Comet. in Orb.	♉	17. 3. 20
Ascend. Nod.	♊	21. 14. 00
Com. à Nodo		34. 10. 40
Red. ad Eclip.		32 19. 05
Com. Helioc.	♉	18 54. 55
Incl. Bor.		11. 46. 50
Log. pro dist.		0. 255369
Log. Perihel.		0. 011044
Co-sin. Incl.		9. 990754
Log. dist. Curt.		0. 257167
Log. dist. ☉		9. 997918
☉	♓	21. 44. 45
Com. Visus	♈	29. 18. 30
Lat. Visa		8. 36. 15

EXAM-

EXAMPLE II.

Let it be requir'd to find the Place of the Comet of the Year 1683, July 23°, 13h, 35', *P.M.* London: *Or*, 13h, 40' *Equat. Time. That is*, 21d, 10h, 50' *after the* Perihelion.

Log. dist. Perihel.	9. 748343
Log. Sesquialt.	9. 622514
Comp. Arith.	0. 377486
	9. 950128
Log. Temp.	1. 310723
Log. Med. Mot.	1. 648337
Medius Motus	44. 498
Perihel. ♊	25. 29. 30
Ang. Corresp.	56. 47. 20—
Comet. in Orb. ♈	28. 42. 10
Nod. Descend. ♓	23. 23. 00
Com. à Nodo	35. 19. 10
Red. ad Eclip.	4. 48. 30
Com. Helioc. ♓	28. 11. 30
Incl. Bor.	35. 2. 00
Log. pro dist.	0. 111336
Log. Perihel.	9. 748343
Co-sin. Incl.	9. 913187
Log. dist. Curt.	9. 772866
Log. dist. ☉.	0. 006104
☉ Locus ♌	10. 41. 25
Com. Visus ♋	5. 11. 50
Lat. Bor.	28. 52. 00

At the Instant of Time specified in the first Example, 'twas observ'd (at *London*) that the Comet applied to the Second Star of *Aries*; so that it was found to be 9' more Northerly,

 and

and 3′ to the East, according to Mr. *Hook*'s Observation. But at that of the Second Example, I my self (near *London*, with the same Instruments whereby I formerly observ'd the Southern Constellations) found the Place of the Comet to be ♋, 5°, 11′ ½, and 28°, 52′ North Latitude, which agreed *exactly* with the Observation made at *Greenwich* almost the very same Moment.

As for the Comet of the Year 1680, which came almost to the very *Sun* it self (being in its *Perihelion*, not above ⅙ of the *Semi-Diameter* of the Sun distant from the Surface of it) since the *Latus Rectum* is so very small, could hardly be contained within the Limits of the General Table, because of the excessive Velocity of the *Mean Motion.* Therefore in this Comet, the best Way is (after the *Mean Motion* is found) to get from thence (by the Help of the foregoing Equation $z^3 + 3z = \frac{4}{100}$ of the *Mean Motion*) the Tangent of Half the Angle from the *Perihelion*, together with the Log. for the Distance from the Sun. Which Things being given, we are to proceed by the same Rules, as in the rest.

After this Manner therefore, the Astronomical Reader may examine these Numbers, which I have calculated, with all imaginable Care, from the Observations I have met with. And I have not thought fit to make them publick before they have been duly examin'd, and made as accurate as 'twas possible, by the Study of many Years. I have publish'd this Specimen of Cometical Astronomy, as a *Prodromus* of a designed future Work, lest, happening

to

to die, theſe Papers might be loſt, which every Man is not capable to retrieve, by reaſon of the great Difficulty of the Calculation. Now, it may not be amiſs to put the Reader in mind, That our Five firſt Comets, (the Third and Fourth obſerv'd by *Peter Apian*, the Fifth by *Paulus Fabricius*) as alſo the Tenth ſeen by *Mæſtlin*, if I miſtake not, in the Year 1596. are not ſo certain as the reſt; for the Obſervations were made neither with fit Inſtruments, nor due Care, and upon that Account are diſagreeing with themſelves, and can by no means be reconcil'd with a regular Computation. The Comet which appear'd in the Year 1684. was only taken Notice of by *Blanchinus*, who obſerved at *Rome:* And the laſt, which appear'd in the Year 1698. was ſeen only by thoſe at *Paris*, who determin'd its Courſe in a very uncommon Way. This Comet was very obſcure; and, altho' it mov'd ſwift, and came near enough our Earth; yet we, who are wont to be curious enough in theſe Matters, ſaw nothing of it. For want of Obſervations I have left out of the foregoing Catalogue, thoſe Two remarkable Comets which have appear'd in this our Age, one in *November*, in the Year 1689 the other in *February* in the Year 1702. For they directing their Courſestowards the Southern Parts of the World, and being ſcarce conſpicuous here in *Europe*, met with no Obſervers capable of the Buſineſs. But, if any one ſhall bring from *India*, or the Southern Parts, an accurate Series of requiſite Obſervations, I will willingly fall to work again, and undergo the Fatigue of repreſenting their Orbits in Numbers, as I have done the reſt.

By comparing together the Accounts of the Motions of theſe Comets, 'tis apparent, their Orbits are diſpos'd in no manner of Order; nor can they, as the Planets are, be comprehended within a *Zodiack*, but move indifferently every Way, as well Retrograde as Direct; from whence it *is clear,* they are not carry'd about or mov'd in *Vortices.* Moreover, the Diſtances in their *Perihelium's* are ſometimes greater, ſometimes leſs; which makes me ſuſpect, there may be a far greater Number of them, which moving in Regions more remote from the Sun, become very obſcure; and wanting Tails, paſs by us unſeen:

Hitherto I have conſider'd the Orbits of Comets as exactly *Parabolick*; upon which Suppoſition it wou'd follow, that Comets being impell'd towards the Sun by a Centripetal Force, deſcend as from Spaces infinitely diſtant, and by their Falls acquire ſuch a Velocity, as that they may again run off into the remoteſt Parts of the Univerſe, moving upwards with ſuch a perpetual Tendency, as never to return again to the Sun. But *ſince* they appear frequently enough, and ſince none of them can be found to move with an Hyperbolick Motion, or a Motion ſwifter than what the a Comet might acquire by its Gravity to the Sun, 'tis highly probable they rather move in very Excentrick Orbits, and make their Returns after long Periods of Time: For ſo their Number will be determinate, and, perhaps, not ſo very great. Beſides, the Space between the Sun and the fix'd Stars is ſo immenſe, that there is Room enough for a Comet to revolve, tho' the Period of its Revolution be vaſtly long.

Now,

Now, the *Latus Rectum* of an *Ellipſis*, is to the *Latus Rectum* of a *Parabola*, which has the ſame Diſtance in its *Perihelium*; as the Diſtance in the *Aphelium* in the *Ellipſis*, is to the whole *Axis* of the *Ellipſis*. And the Velocities are in a Subduplicate *Ratio* of the ſame: Wherefore in very Excentrick Orbits this *Ratio* comes very near to a *Ratio* of Equality; and the very ſmall Difference which happens on Account of the greater Velocity in the *Parabola*, is eaſily compenſated in determining the Situation of the Orbit. The principal Uſe therefore of this Table of the Elements of their Motions, and that which induced me to conſtruct it, is, That whenever a new Comet ſhall appear, we may be able to know, by comparing together the Elements, whether it be any of thoſe which has appear'd before, and conſequently to determine its Period, and the *Axis* of its Orbit, and to foretell its Return. And, indeed, there are many Things which make me believe that the Comet which *Apian* obſerv'd in the Year 1531. was the ſame with that which *Kepler* and *Longomontanus* took Notice of and deſcrib'd in the Year 1607. and which I my ſelf have ſeen return, and obſerv'd in the Year 1682. All the Elements agree, and nothing ſeems to contradict this my Opinion, beſides the Inequality of the Periodick Revolutions: Which Inequality is not ſo great neither, as that it may not be owing to Phyſical Cauſes. For the Motion of *Saturn* is ſo diſturbed by the reſt of the Planets, eſpecially *Jupiter*, that the Periodick Time of that Planet is uncertain for ſome whole Days together. How much more therefore will a Comet be ſubject to ſuch like Errors, which riſes almost Four times higher than *Saturn*, and whose Velocity, tho' encreased but a very little, would be sufficient to change its Orbit, from an *Elliptical* to a Parabolical one. This, moreover, confirms me in my Opinion of its being the same; that in the Year 1456. in the Summer time, a Comet was seen passing Retrograde between the Earth and the Sun, much after the same Manner: Which, tho' no Body made Observations upon it, yet from its Period, and the Manner of its Transit, I cannot think different from those I have just now mention'd. Hence I dare venture to foretell, That it will return again in the Year 1758. And, if it should then return, we shall have no Reason to doubt but the rest must return too: Therefore Astronomers have a large Field to exercise themselves in for many Ages, before they will be able to know the Number of these many and great Bodies revolving about the common Center of the Sun; and reduce their Motions to certain Rules. I thought, indeed, that the Comet which appear'd in the Year 1532. might be the same with that observ'd by *Hevelius* in the Year 1661. But *Apian*'s Observations, which are the only ones we have concerning the first of these Comets, are too rude and unskilful, for any thing of Certainty to be drawn from them, in so nice a Matter. I design to treat of all these Things in a larger Volume, and contribute my utmost for the Promotion of this Part of Astronomy, if it shall please God to continue my Life and Health.

In the mean time, those that desire to know how to construct Geometrically the Orb of a Comet, by Three accurate Observations given, may

may find it at the End of the Third Book of Sir *Isaac Newton's* Principles of Natural Philosophy, entituled *De Systemate Mundi*, in the Words of its renowned Inventor. Which have since been more fully explained by my very worthy Collegue Dr. *Gregory*, in his Learned Work of *Astronomia Physica & Geometrica.*

One Thing more perhaps it may not be improper or unpleasant to advertise the Astronomical Reader; That some of these Comets have their Nodes so very near the Annual Orb of the Earth, that if it shall so happen, that the Earth be found in the Parts of her Orb next the Node of such a Comet, whilst the Comet passes by; as the apparent Motion of the Comet will be incredibly swift, so its *Parallax* will become very sensible; and the Proportion thereof to that of the Sun will be given. Wherefore such Transits of Comets do afford us the very best Means, though they seldom happen, to determine the Distance of the Sun and Earth: Which hitherto has only been attempted by *Mars* in his Opposition to the Sun; or else *Venus* in *Perigæo*; whose *Parallaxes* though triple to that of the Sun, are scarce any ways to be perceived by our Instruments; whence we are still in great Uncertainty in that Affair. This use of Comets was the ingenious Thought of that excellent Geometrician Mr. *Nicolas Facio.* Now the Comet of 1472, had a *Parallax* above Twenty times greater than the Sun's. And if the Comet of 1618, had come down, about the Middle of *March*, to his descending Node: Or if that of 1684, had arrived a little sooner at its ascending Node; they would have been yet much nearer the Earth,

Earth, and conſequently have had more notable *Parallaxes.* But hitherto none has threaten'd the Earth with a nearer Appulſe, than that of 1680. For by Calculation I find, that *Novemb.* 11°, 1h, 6', *P. M.* that Comet was not above the Semi-diameter of the the Sun to the Northwards of the Way of *the* Earth. At which Time, had the Earth been *there*, the Comet would have had a *Parallax* equal to that of the Moon, as I take it. This is ſpoken to Aſtronomers: But what *might be* the Conſequences of ſo near an Appulſe; or of a Contact; or, laſtly, of a Shock of the Cœleſtial Bodies, (which is by no means impoſſible to come to paſs) I leave to be diſcuſs'd by the Studious of Phyſical Matters.

A

A

Geometrical Dissertation

Concerning the

RAINBOW;

IN

Which (by a direct Method) is shewn how to find the Diameter of each *Bow*, the Proportion of the Refraction being given: Together with the Solution of the *Inverse* Problem, or how to fend the *Ratio* of the Refraction, the Diameter of the *Iris* being given.

By EDM. HALLEY, *F. R. S.*

ALL the Writers of Natural History, have particularly described the Rainbow (a Meteor so remarkable for its fair Colours) and given an Account of the Causes of it. And the Ancient *Mythologists*, from its wonderful Form and Appearance, thought

thought fit to give it the Title of *Thaumantis*, or the *Child of Wonder*; and placing it in the Number of the *Goddeſſes*, attributed to it the Office of a *Meſſenger* between the *Cœleſtials* and mortal Men; which Fable, perhaps, owes its Original to *Geneſis*, Ch. 9. *V. 13.*

Thoſe that attentively conſider'd the *Phænomena* of the Rainbow, always found, that the Sun's Rays reflected by a Watery Cloud, came to the Eye under a certain Angle; from whence aroſe the Arch, or Circular Figure of it. But as for the Cauſe and Reaſon of the Colours, as alſo of the Magnitude of the Angle, by which we conſtantly find it diſtant from the Point *oppoſite to the Sun*; theſe were Things, that a long while, and very greatly perlex'd, as well the *Moderns*, as Ancients. Neither did they do any thing to the Purpoſe herein, till the Famous Monſieur *Des Cartes* making uſe of the Mathematical Sciences, ſhew'd by ſeveral Examples, that more ſtrict and cloſe Methods of Reaſoning might and ought to obtain, even in our Management of thoſe *Phyſical Speculations*. Amongſt other things (tho' it muſt be own'd that herein he had ſome Light, from the Learned *Antonio de Dominis*, Arch-biſhop of *Spalato*) he explain'd the Theory of the Rainbow. And having diſcover'd the Laws of Refraction, he clearly demonſtrated, that the *Primary Iris* was nothing elſe, but the Sun's Image reflexed from the Concave Surface of innumerable Spherical Drops of Rain; and that with this Condition, that thoſe Rays that were parallel at their Incidence, were not loſt or diſſipated by the *Reflexion*, and the *Two Refractions* (one at the

In-

Ingress, and the other at the *Egress*) but fell (and that alſo parallel) on the Eye. That the Rays were ting'd with Colours by thoſe Refractions, after the ſame manner as we ſee they are by a *Glaſs Priſme.* That the *Secundary Iris* is produced, after the ſame manner, by the Rays that fall more obliquely, only here are Two Reflexions, before the Sun's Rays (which when refracted a Second time proceed parallel to the Eye) emerge out of the Drops of Water. Further, that the Magnitude of each *Iris* depends upon the Degrees of the Refraction, which is different according to the Nature of each tranſparent *Solid* or *Liquid*.

And ſuppoſing the Proportion of the Sines of the Angles of Incidence to the Sines of the refracted Angles, to be in Water, as 250, to 187, he determin'd the Semi-Diameter of each *Iris*, agreeably to Obſervations, *viz.* that of the *Primary Iris*, 41°. 30′. and that of the *Secundary*, 51°. 54′. By which he did not ſo much confirm the *Theory* it ſelf, which was demonſtrated from other Principles, as the *Truth of the fore-mention'd aſſumed Proportion*, (*viz.* that of the Refraction.) But for theſe Things, the Reader may conſult the 8th Chapter of *Cartes*'s *Meteors*, whither I refer him.

But now *Cartes* (who uſed an indirect and tentative Method in determining theſe Angles) did not ſeem clearly to apprehend the Eaſineſs of the Problem he had propoſed to himſelf. And becauſe none (that I know of) ſince him, has handled the ſame Argument more fully; and alſo ſince ſome have miſunderſtood what *Cartes* did, committing very great *Paralogiſms*, in ſome Books (ſince his time) which particularly

larly pretended to explain the *Phænomena* of the Rainbow; I was willing to supply what I thought was wanting in this Doctrine, and from the *Proportion of the Refraction given*, Geometrically to determine the Angle of its Distance from the *Point opposite to the Sun:* Or contrarywise, from the *Iris given*, to determine the refractive Power of the Liquid.

What the Celebrated Mr. *Newton* has done upon this Head, the Reader will find (with much greater Advantage) in his Book of Light and Colours, when he shall think fit to bestow those excellent Discoveries upon the Publick.

But to proceed. 'Tis plain from what *Cartes* has demonstrated, that the *Primary Iris* is form'd by such Rays of the Sun, where the Excess of Two *refracted Angles*, above one Angle of *Incidence*, is the *Greatest of all such Excesses possible.* And that the *Secondary Iris* is form'd by those Rays only, where the Excess of Three refracted Angles, above one of *Incidence*, is in like manner the *Greatest*. And so we may go on to a 3d, 4th, or any other *Iris*, which are form'd, where the Rays emerge after 3, 4, or more Reflexions. But these can never be seen in the Heavens, because of the Sun's Light which is still more and more debilitated by each Reflexion and Refraction: Whence it comes to pass also, that the *Secondary Iris*, is painted with Colours, so much fainter than the Primary one. But in all these the *general Rule* is, that *the Excess of* 4, *or* 5, *or more refracted Angles*, (*viz.* the Number of Reflexions being increased by Unity) *above one Angle of Incidence, is of all the* Greatest.

Now

Now this *greatest Excess* doubled, is always the Distance of the *Iris* from the *Point opposite to the Sun*, when the Number of Reflexions is *uneven*. But if that Number be *even*, then the Double of that *greatest Angle*, is the Distance of the *Iris* from the *Sun it self*, viz. in the 2d, 4th, 6th, &c. *Iris*. All these Things are either purely *Cartesius*'s, or else easily follow from his Writings in the foremention'd Place.

But now to obtain those *greatest Excesses*, having the Refraction of any Liquor given; 'tis to be observ'd, that the Excess of Two refracted Angles, above one of Incidence, is there the *Greatest*, where the *momentaneous Increment* of the Angle of Incidence is exactly double of the *momentaneous Increment* of the refracted Angle. And that the Excess of Three refracted Angles is there the *Greatest*, where the Increment of the Angle of Incidence is triple the Increment of the refracted Angle; and so of the rest. And this is sufficiently evident of it self: But as for the Angles, we may obtain them by the Help of the following *Lemma*, which must therefore be demonstrated.

LEMMA.

The Legs of any plain Triangle continuing; if the Vertical Angle be augmented or diminish'd, by an Angle less than any Angle assign'd; the Momenta *or* Instantaneous Mutations *of the Angles at the Base, are to one another* reciprocally, *as the Segments of the Base.*

At *Fig.* 1. *Plate* 3 suppose the Triangle ABC, whose Vertex is *A*, its Legs AB, AC, and Base BC, upon which let fall the Perpendicular AD. Then let the Angle BAC be increased by the *Indivisible Momentum* CA *c*, and let the Lines B *c d*, *c* D be drawn, which differ, in Imagination only, from the Lines BCD, CD. I say, that the *Momentum* of the Angle ABC (*viz.* CB*c*) is to the *Momentum* of the Angle ACB or ACD, as CD to BD, that is reciprocally as the Segments of the Base.

DEMONSTRATION.

Because the Angle ACD, is the Sum of the Angles ABC, BAC, its *Momentum* also shall equal the Sum of the *Momenta* of those Angles; that is, it shall equal CA *c* + CB*c*. But CA *c* = CD *c*, since, because of the right Angle at D, the Points A, D, C, *c*, are all in the Arch of a Circle, whose Diameter is AC: By *Eucl.* 3.9. And consequently the Sum of the Angles CB *c*, CD *c* (that is the Angle D *c d*) shall be the *Momentum* of the Angle ACD or ACB.

But

But those Angles CB *c*, D *c d*, being indefinitely small, are to one another as their opposite sides, that is, as *c* D or CD to BD, that is as the Segments of the Base reciprocally. Q: E: D.

If each of the Angles B and C be *Acute*, the *Lemma* will still (*mutatis mutandis*) be demonstrated after the same Manner.

COROLLARY.

Hence it follows that the *Momenta* of the Angles at the Base, are to one another *directly*, as the Tangents of those Angles.

By the Help of this *Lemma*, I will be easie to find the Diameter of any *Iris* whatsoever; and that either by *Calculation*, or a *Geometrical Construction*. For taking any right Line, as CA (*Fig.* 2.) let it be divided first of all in D, so that CA, may be to CD, in the *Ratio* of the Refraction in Water, which is as 250 to 187, or more accurately, as 529 to 396. Then let CA be divided so in E, that CE may be to AE, as Unity to the Number of Reflexions, a Ray of the Sun (fit to produce the *Iris* proposed) undergoes: And upon the Diameter AE describing the Semi-Circle ABE, on the Center C with the *Radius* CD describe the Arch BD, meeting the Semi-Circle ABE in the Point B. Lastly, Drawing the Right Lines CB, AB, let CF be let fall perpendicular upon AB produced, and EB parallel thereto. I say then, that CBF is the Angle of Incidence, and CAB the Refracted Angle that we enquire after, and which will produce the *Iris* propos'd.

DE-

DEMONSTRATION.

Because the Triangles ACF, AEB are similar, it will be AF: BF :: AC: EC; that is, as the Number of Reflexions encreas'd by Unity to Unity (by the *Construction*) and consequently the *Momentum* of the Angle CBF, will be to the *Momentum* of the Angle CAF, in the same Proportion (by the foregoing *Lemma.*) But the Sine of the Angle CBF, is to the Sine of the Angle CAF, in the Proportion of the Sides CA, CB, that is, in the Proportion of the Refraction given (also by the *Construction.*) Therefore CAF is the Refracted Angle, corresponding to the Angle of Incidence CBF; and their *Momenta* are in the *Ratio* propos'd, wherefore they are the Angles sought. Q. E. D.

And now, multiplying the Refracted Angle by the Number of the Reflexions encreas'd by Unity, and from the Product substracting the Angle of Incidence, we shall have half the Distance of the *Iris* from the Sun, if the Number of Reflexions be *even*, or from the Point opposite to the Sun, if that Number be *uneven*, as we have shewn already. Hence we may exhibit (by a *Construction* concise and eloquent enough) the Incidencies of all the Orders of *Iris's*, in any Liquor whose Refraction is known. For if the Line AC (*FIG.* 2.) be divided into Two equal Parts at E, into Three equal Parts at *e*, into Four at ε, into Five at *n*, &c. And on the Diameter AE, A*e*, Aε, A*n*, be describ'd, the Semi-Circles ABE, A*be*, Aβε, Aν*n*, which are all intersected in the Points B, *b*, β, *v*, by the Arch DB*b*β*v*, describ'd scrib'd on the Center C with the Radius CD, which is to AC, in the given Proportion of the Refraction: I say then that the Lines AB, Ab, Aβ, Av, will make with the Line AC, the Angles CAB, CAb, CAβ, CAv, equal to the Refracted Angles; and with the respective Rayes CB, Cb, Cβ, Cv, they will make Angles equal to the Angles of Incidence that are required; *viz.* ABC (or rather its complement to a Semicircle) for the *Primary Iris*, AbC, for the *Secondary*, AβC, for a *Third Iris*, AvC, for a *Fourth*, If any one has a mind, to find these Angles by an accurate Calculation, 'twill follow from the same Principle, that putting the Radius=1, and the Ratio of the Refraction as r to s, the Sine of Incidence will be $\sqrt{\frac{4}{3} - \frac{1rr}{3ss}}$, and the Sine of the Refracted Angle $\sqrt{\frac{4ss}{3rr} - \frac{1}{3}}$, from which Angles proceeds the *Primary Iris*. For the *Secondary* the Sine of Incidence will be $\sqrt{\frac{9}{8} - \frac{1rr}{8ss}}$, and the Sine of the Refracted Angle $\sqrt{\frac{9ss}{8rr} - \frac{1}{8}}$. For a *Third Iris*, the Sine of Incidence will be $\sqrt{\frac{16}{15} - \frac{1rr}{15ss}}$, and the Sine of the Refracted Angle $\sqrt{\frac{16ss}{15rr} - \frac{1}{15}}$. For a

Fourth Iris, the Sine of Incidence will be $\sqrt{\frac{25}{24} - \frac{1rr}{24ss}}$, and the Sine of the Refracted Angle $\sqrt{\frac{25ss}{24rr} - \frac{1}{24}}$: and in like manner of the rest. Farther, 'twill be found by *Calculation*, that (taking *Cartes*'s Proportion) the *Primary Iris* is distant 41°. 30′. from the Point opposite to the Sun; *the Secondary*, 51°. 55′. from the same. The *Third*, 40°. 20′. and the *Fourth*, 45°. 33′. from the Sun it self; which *Iris's* perhaps were hardly ever seen for the reasons before mentioned.

And thus much may suffice concerning the Magnitude of the *Irides*, in the perspicuous Drops of a Fluid, whose Refractive Power is known. It remains that nothing be said concerning the *Colours*, which this Phenomenon presents, with the orders of them in each sort of *Iris*, according to all the possible Variations of the Refraction.

And here we *must know* especially, that the Acute and Sagacious Mr. *Newton*, has found by most clear Experiments, that *the Rays of Light are not Simple and Uniform, as they issue out of the Luminous Body, but the pure white Light which we see, consists of Corpuscles of all kinds of Colours, mix'd and hurried with a violent Motion, one amongst another.* And that the diversity of the Colours of things arises, according to the various Dispositions those Objects have, to Refract or Reflect this or that peculiar kind of Light.

The Proof of which is manifest from Refractions, in which these Species are separated from one another, and the *Blue* or *Purple* Light, (even in the same Diaphanous Body) is more Refracted than the *Yellow* or *Red*. But let the Reader consult this incomparable Person's Letters (N°. 80. and the following of the *Philosophical Transactions*) from which Specimen he will be able to judge, how nobly this Argument of *Light* will be managed by him.

To my purpose 'tis sufficient, that all kinds of *Blue* Light, are something more refracted than *Red*, from which difference arises the Latitude of the *Irides*, which is hardly to be determined by Observation, because of the uncertain Limits of the Colours. But by how much the Proportion between CA and CD, is *of greater Inequality*, or by how much the Refraction is greater, so much the greater is the distance of any *Iris* from the Sun, and consequently those borders that are remoter from the Sun, shine with a *Purple* Colour, but those that are nearer, with an intense *Red*.

This may always be seen in the *Primary Iris*, which vanishes in the part opposite to the Sun, if the Sine of Incidence be to the Sine of the Refracted Angle, as CA to CE, or as 2 to 1. But if that Ratio be greater, there can be no *Primary Iris* seen at all.

As for the *Secondary Iris*, 'tis to be noted, that this vanishes into a Point, in the part opposite to the Sun, when the Ratio of the Refraction is as 1 to $\sqrt{\frac{nn1}{3}} + \sqrt{\frac{4}{27}}$, or as 1 to 0,847487... and from thence it returns back to the Sum it self, where it vanishes, if the said Ratio be as 3 to 1, or as CA to Cc.

But in the Ratio's *between* theſe (ſuch as we have in all Fluids known to us, except the Air) by how much the greater is the Ratio, by ſo much is the *Iris* more diſtant from the Point oppoſite to the Sun, or rather from the Sun it ſelf, reckoning the Arch beyond a Semicircle. And conſequently the Colours will ſeem to be in a different order from the *Primary Iris*, in theſe *returnings*, unleſs the diſtance of the *Irides* from the Sun, be taken in *this Sence*; which is alſo every where to be obſerved in the reſt.

The *Third Iris* is confuſed in the part oppoſite to the Sun, the Ratio of the Refraction being as 1 to ,91855--- from thence it returns back to the Sun in the Ratio of 1 to ,68250--- whence again, the order of the Colours being reſtored, in the Ratio of 4 to 1, or CA to C e, it terminates in the part oppoſite to the Sun. The Fourth *Iris* beginning from the Sun, in a Ratio of Equality, paſſes on to the oppoſite Point, in the Ratio of 1 to ,94895--- and thence returns back to the Sun, if the Proportion be as 5 to 4; hence again, it diſperſes to the Point oppoſite to the Sun in the Ratio of 1 to ,56337---, within which compaſs are included the Refractions of all Fluids that are known. Laſtly, The Ratio being as 5 to 1, or CA to C n, it vaniſhes in the very Sun it ſelf; the Colours being every where *inverted* to the ſight in its *return* to the Sun, as they were *erect* in its *egreſs* from it. Hence, in watery Clouds, the *Firſt* and *Fourth* Iris ſhew *deep red* Colours turned towards the Sun; but the *Second* and *Third give Purple*. But perhaps I may ſeem too tedious in in theſe Deſcriptions, the *Rainbow* it ſelf being no more than a Momentary *Phantaſm*.

But whence 'tis that the different Refractive Power of Fluids ariſes, is a Problem of the greateſt Moment, and to be rank'd amongſt the Secrets of Nature, not yet obvious either to our Sences or our Reaſonings. For *pure* Water amongſt all Fluids, does leaſt of all Refract the Rays of Light. When 'tis *Tinctur'd with Salts* diſſolved in it, according to its weight and the quantity of Salt, it increaſes the Refractions. And *Corroſive Spirits* (which are much heavier than Water) do alſo much more Refract the Rays of Light: Nor is it any wonder, ſince being denſer Bodies, they may eaſily be conceived ſo much the more to obſtruct the paſſage of the Rays. But why there ſhould be ſo great a Refraction in *Burning Spirits* and *Oils*, eſpecially in *Spirit of Turpentine, or of Wine*, ſince they are Fluids extreamly *Light* in compariſon of Water, and conſiſt very much of ſubtle Ætherial Particles, does not ſo eaſily appear; but ſeems to require (in order to the Explication of it) a more thorough knowledge of the Nature and Texture of Light.

But from the *diſtance* (of the *Iris* from the Sun) *given*, to find the Ratio of the Refraction, is a thing that will give thoſe that are curious, an occaſion of finding the Refraction of any Fluid, accurately and with little trouble. For if a ſmall drop of any tranſparent Fluid, be ſuppoſed to hang at the bottom of a ſmall Glaſs Tube, and the Sun being near the Horizon and ſhining ſtrongly, it be obſerved under what Angle (with the *Point* *opposite to the Sun)* the Colours of the *Iris* be seen in the drop, then the Proportion sought will be obtained with a little Calculation. It is a Cubical Equation, explicable by one *only* Root, by which, from the *Primary Iris given*, the Ratio is computed, *viz.* $T^3 - 3T^2t - 4rrt = 0$, where T is the Tangent of the Angle *of Incidence requisite*, t the Tangent of $\frac{1}{2}$ the distance of the Iris from the Point opposite to the Sun, to the Radius $r=1$. Whence (according to *Cardanus*'s Rules) arises this Theorem, *viz.* *From the Cube of* t *substract the Product of* 2tr *into the Excess of the Secant of the same Arch above the Radius; the difference shall be the lesser Cube. The Sum of the same, adding* 4trr, *will be the greater Cube. The Sum of the sides of both Cubes, and of* t, *will be equal to the Tangent of the Angle of Incidence, and the half of that, will be the Tangent of the Refracted Angle.* From whence the Ratio sought is manifest.

For an Example of this. In a drop of *Oil of Turpentine*, the distance of the Primary Iris, from the Point opposite to the *Sun, is* observed to be 25°. 40′. 'Tis required to find the Ratio of the Refraction.

t=Tang. 12°. 50′.	=	0,2278063
s=Sec. of the same.	=	1,0256197
ttt	=	0,011822217
$\overline{s-r} \times 2tr$	=	0,01167265

The Difference is the lesser Cube 0,00014952
whose side 0,0530773

The Sum 0,02349482
4trr 0,91122525

Greater

Greater Cube 0, 93472007, whoſe ſide 0, 9777486
t 0, 2278063
T=Tang. Incid. 51°. 32′. 1, 2586322
$\frac{1}{2}$ T=Tang. Refr. 52°. 11′. 0, 6293161

Laſtly, As $\sqrt{TT+4} : \sqrt{TT+1} :: r : s ::$ 1 : 168026. Which Proportion comes very near to that, which Experience ſhews to be in Glaſs and moſt pellucid Solids. The Diamond indeed, exceeds all tranſparent Bodies, not only in reſpect of its hardneſs and value, but alſo its Refractive Power, the Proportion here being as 5 : 2, nearly, or more accurately as 100 : 41. But of this, perhaps more in another place.

While I was writing theſe things, that skillful Geometrician Mr. *De Moivre*, at my requeſt, found a like Equation for determining the Ratio, from the Semidiameter of the *Secondary Iris, given.* By which, the Ratio is indeed ſomething more exactly determined, but that Equation being a Biquadratical one, the Calculation is not ſo eaſily performed. This Equation is $T^4 + \frac{8}{4} T^3 t - 2 T^2 r^2 - \frac{1}{3} r^4 = 0$; where *T* is the Tangent of the Refracted Angle, *t*, the Tangent of $\frac{1}{2}$ the diſtance of the Iris from the Point oppoſite to the Sun, to the Radius r=1. And this Equation is of that Form, as to be always explicable, by an Affirmative and one Negative Root, the one and the leſs of which, is the Tangent of the Refracted Angle, in the *Regreſs* to the Sun, *viz.* when the *Purple Colours* are nearer to the Sun. The greater Root is the Tangent of the Refracted Angle in an Iris

Iris *going out from the Sun*, viz. in a Fluid of *a less Ratio*. In *Oil of Turpentine*, the distance of this Iris from the Point opposite to the Sun, is observed to be 81°. 30′. whence the curious Reader may find out the Roots, 0,80822-- and—2,98131-- the Tangents of the Refracted Angles. Hence is computed the *Ratio of greater Inequality*, as 1 to 0,67995-- such as is in Oil of Turpentine. But from the greater Root comes forth the *lesser Ratio*, as 1 to 0,9540 nearly, such as would be in a Fluid, exhibiting a Secondary Iris of the same Diameter, but which (after the manner of the Primary one) should look towards the Sun with the *Red Colours*.

If any one has a mind to find these Roots by a Geometrical Construction, any Parabola being given, it is done with so much ease, that I need not repeat what I have already offered upon that Head *Philosophical Transactions*, N. 188.

Each of these Equations is deduced from what has been said before, and also from the Rules for the Tangents of the Double and Treble Arch; the bare hint of which, may be instead of a Demonstration even to those that are but meanly vers'd in these things.

This Discourse being already in the *Press*, there came to my hands (by the means of a Friend) a certain Book, whose Title was *Thaumantiadis Thaumasia*, Printed at *Norimberg* 1699, under the Superintendency of *M. Sturmius.* In which the skillful Author seems to have laid together whatever is to be found

of

of this Argument, as well amongſt the *Modern* as the *Ancient* Writers; ſubjoyning and illuſtrating *Cartes*'s, *Eckard*'s, *Honoratus Faber*'s, *and Mariott*'s Calculus. From whence it is plain, that the reſt added very little or nothing to *Cartes*'s Inventions, building upon the ſame Ungeometrical and Tentative Methods of Calculation. But that the Judicious Reader may be ſenſible, *what things I have performed, in the Doctrine of the Iris*, I would have him read the forementioned Book, and compare it with this Diſcourſe; leſt in putting out theſe things, I ſhould ſeem only to have made an unpleaſing Repetition of what had been done before. And of *what vaſt uſe in Aſtronomical Matters, this Lemma of ours may be*, ſhall be ſhewn upon ſome other occaſion.

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