A

RUDIMENTARY TREATISE

ON

ANALYTICAL GEOMETRY

AND

CONIC SECTIONS.

BY

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PREFACE.

In compiling this work the Author has read nearly all the best writers on the subject,—Bourdon, Hymers, O'Brien, Salmon, Young, and Woolhouse, all containing valuable information.

In an elementary treatise like the present, it is impossible to go into the subject in such a manner as O'Brien and Salmon have done in their able and extensive works; yet it is to be hoped that the student will find this work an useful introduction to the more elaborate writings of those two eminent mathematicians, as well as to contain almost, if not all, the properties of the Conic Sections most in use.

Mr. Gompertz, one of the most eminent of our English mathematicians, has just published a valuable tract on Porisms, which he calls "Hints on Porisms, in a Letter to T. S. Davies, Esq., F.R.S., F.S.A.," and in which he solves the following beautiful problem connected with the subject before us:—"*A circle and a parabola cutting each other being given in position and magnitude, to find the points of intersection.*"

It is announced that Professor Davies is about to publish a treatise on this subject himself. This will be a work which those interested in Porisms will look forward to with the greatest interest, as Mr. Davies is undoubtedly inferior to no living geometer on these subjects.

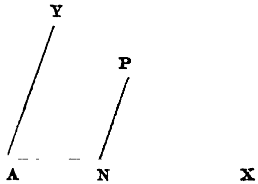
The student will find many examples fully worked out in this treatise, to some of which there are no figures given, but so clearly indicated as to give little trouble in forming them.

ANALYTICAL GEOMETRY.

SECTION I.—STRAIGHT LINE.

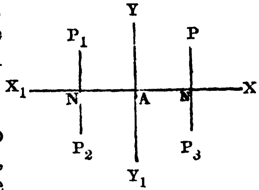
(1.) THE position of a point in a plane surface is determined by referring it to what are termed its axes of co-ordinates, as follows:—

Let there be taken in a plane two indefinite right lines AY, AX, inclined to one another at a given angle YAX. Now, if a point P be taken in the plane, it is evident that if PN, a line parallel to AY, be drawn, the position of the point P will be *determined* if AN and PN be given, since no other than *one* point can fulfil the conditions required by the relation of AN and PN. Now, AN and NP are termed the co-ordinates of the point P, AY and AX are called respectively the axis of ordinates and the axis of abscissæ, and from their relation to one another co-ordinatal axes. And since in using the co-ordinates of a point the abscissa is usually denoted by x, and the ordinate by y, AY and AX are respectively termed the axis of y and the axis of x. The point A is termed the *origin*. In general the co-ordinate axes are taken perpendicular to one another; they are then called rectangular axes, and the co-ordinates are termed rectangular co-ordinates.



(2.) We have said that a point P was *determined* by the co-ordinates AN and NP. This is not strictly true, for the conditions imposed by the *magnitude* of the co-ordinates would be fulfilled also by another point, as shall be shewn.

(3.) Produce the axes of AY and AX, as in the figure, so that the plane surface may be portioned out into four quarters. Now, suppose it is required to *locate* the point whose co-ordinates are a and b. Let $AN = a$, $NP = b$; then P is one point fulfilling the required conditions. But AN_1 may be also taken $= a$, and $N_1P_1 = b$, $N_1P_2 = b$, $NP_3 = b$, and we shall therefore have *four* points equally satisfying the original condition, viz., that the co-ordinates should be a and b. But it is evident that if such ambiguity attend our determination of the position of a point, there will often arise inconvenience. This defect algebraic geometry supplies in the following manner:—A being the origin, AN is equal to $+a$, $NP = +b$. But the abscissa of a point situated *on* the axis of y, is $= 0$; so that at the termination of the first quarter of space the abscissa is $= 0$. And, similarly, a line measured in the direction AN_1, which is equal in magnitude to AN, must be analytically expressed by $-(a)$, to show that we have passed the limit of the *first* quarter, and come into the *second* quarter of space. In a similar manner, it will be easy to conclude that, if the ordinates be measured *below* the axis of x, they must likewise be affected with a different sign. The student should compare this with the change of sign in $\sin\theta$, $\cos\theta$, $\tan\theta$, &c., in passing through the four quadrants.



(4.) It is evident that, conformable to these remarks, we shall have the following results:—

$\left.\begin{array}{l}+a\\+b\end{array}\right\}$ determine the point P

$\left.\begin{array}{l}-a\\+b\end{array}\right\}$ .... P_1

$\left.\begin{array}{l}-a\\-b\end{array}\right\}$ .... P_2

$\left.\begin{array}{l}+a\\-b\end{array}\right\}$ .... P_3.

We thus see that the position of a point is determined by means of its co-ordinates.

(5.) To complete the theory of points, we shall now proceed to determine an analytical expression for the distance between two given points.

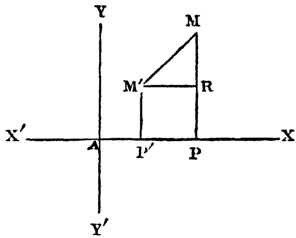
To find the distance between two given points referred to rectangular axes.

Let the co-ordinates of M and $M' = (x'y')$, $(x''y'')$, and let $M'M = D$ be the distance; draw M′R parallel to the axis of x, then in the right-angled triangle M′RM,

$$\text{we have } MM'^2 = MR^2 + M'R^2,$$

$$\text{but } MR = MP - RP = y' - y'', \;\; M'R = PP' = x' - x'',$$

$$\therefore \; MM'^2 = D^2 = (y' - y'')^2 + (x' - x'')^2,$$



$$\therefore \; D = \sqrt{(y' - y'')^2 + (x' - x'')^2}\,*.$$

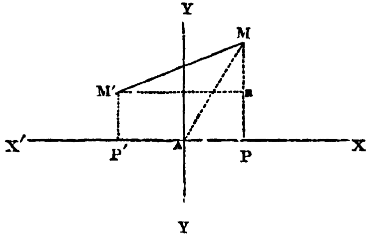
This formula is general, and holds equally in the case where the points are situated in an opposite direction; with respect to one of the axes, it is only necessary to change the sign which corresponds to the change of position.

Thus, for example, to obtain the distance of two points, one of which, M, is in the angle YAX, and the other, M′, is in the angle YAX′, we must change the sign of x'', which gives

* If the angle MM′R, or the angle which the right line MM makes with the axis of $x = \alpha$, we have $\frac{MR}{M'R} = \tan \alpha$; but $MR = y' - y''$, and $M'R = x' - x''$,

$$\therefore \; \tan \alpha = \frac{y' - y''}{x' - x''}.$$

$$D = \sqrt{(y' - y'')^2 + (x' + x'')^2},$$



for $M'M^2 = MR^2 + M'R^2$, and $MR = y' - y''$, as before,

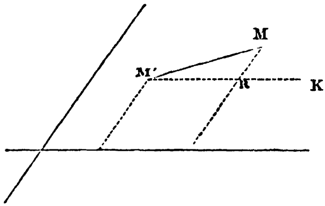
$$\text{but } M'R = AP + AP' = x' + x'',$$

$$\therefore D = \sqrt{(y' - y'')^2 + (x' + x'')^2}.$$

If one of the points M be at the origin, then $x' = 0$, $y' = 0$, and the above becomes

$$D = \sqrt{y^2 + x^2}.$$

When the axes are oblique, the triangle MM′R is oblique; then, by Hann's Trigonometry, page 56,



$$M'M^2 = MR^2 + M'R^2 - 2\,MR\,.\,M'R\cos MRM';$$

but $MR = y' - y''$; $M'R = x' - x''$; and $\cos MRM' = -\cos MRK = -\cos\beta$ (β being the angle MRK, which is the same as the angle that the axes make with each other),

$$\therefore D^2 = (y'-y'')^2 + (x'-x'')^2 + 2\,(y'-y'')\,(x'-x'')\cos\beta,$$

$$\text{or } D = \sqrt{(y'-y'')^2 + (x'-x'')^2 + 2(y'-y'')\,(x'-x'')\cos\beta}.$$

SECTION II.

(6.) Curves are divided into orders or degrees corresponding to the sum of the exponents of x and y, and the term which contains the highest sum of exponents shows the order of the curve. Thus, $y = x + a$ is an equation of the first degree; $y^2 + xy + b^2 = 0$, $y^2 - x^2 + xy - c^2 = 0$, are equations of the second degree; and $y^3 + x^2 + x^2y + cxy^2 + bxy + cx + fy + d$ is an equation of the third degree.

Equation of the first degree.

(7.) The general form of this equation is $Ax + By + C = 0$, the constants A, B, C, being either positive or negative.

This equation may be put under the form $y = -\frac{A}{B}x - \frac{C}{B}$,

$$\text{or } y = mx + h \text{ if } m = -\frac{A}{B},\ h = -\frac{C}{B},$$

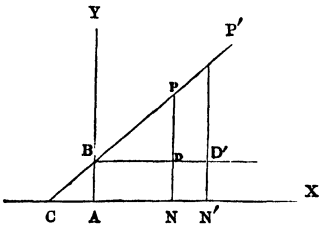
m and h being unrestricted as to sign.

So that we have two forms of the equation of the first degree,

$$Ax + By + C = 0 \ \dots (1), \quad \text{or } y = mx + h \ \dots (2),$$

both equally general.

(8.) To find the equation to a straight line.



Referring the line to the rectangular axes AX and AY,

let $AN = x$, $PN = y$, $AB = h$.

By similar triangles, $\frac{PD}{BD} = \frac{P'D'}{BD'} =$ a constant ratio,

$$\text{or } \frac{PN - DN}{BD} = \frac{PN - AB}{AN} = \text{a constant ratio,}$$

$$\therefore \frac{y-h}{x} = \text{a constant quantity;}$$

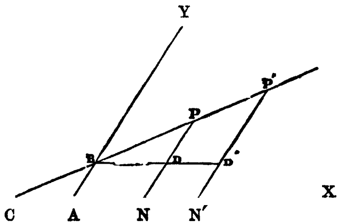
but $\frac{PD}{BD} = \frac{y-h}{x} = \tan PBD = \tan PCX$, which call m,

$$\text{and we have } \frac{y-h}{x} = m,$$

$y - h = mx$, or $y = mx + h$, the equation to the straight line.

(9.) For oblique co-ordinates, the ratio $\frac{PD}{BD}$ is constant, as before, but in this case we have

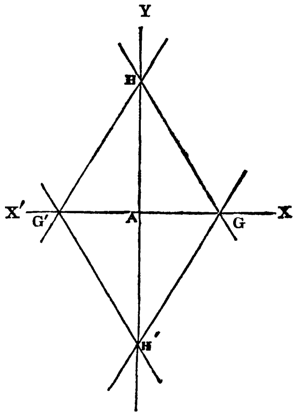
$$\frac{PD}{BD} = \frac{\sin PBD}{\sin BPD} = \frac{\sin PCX}{\sin PBY}.$$



If YAX, the inclination of the axes, $= \beta$, and PCX, the angle which the line PC makes with the axis of x, $= \alpha$, the above becomes

$$\frac{PD}{BD} = m = \frac{\sin \alpha}{\sin(\beta - \alpha)}.$$

(10.) To determine the equation to a right line in terms of the intercepts on the axes*.



The general equation to a right line is $Ax + By + C = 0$,

let $AG = a$ and $AH = b$;

when $x = 0$, $By + C = 0$,

$$\therefore\ y \text{ or } AH = -\frac{C}{B};$$

when $y = 0$, $Ax + C = 0$,

$$\therefore\ x \text{ or } AG = -\frac{C}{A}.$$

Divide the general equation by C,

$$\frac{A}{C}x + \frac{B}{C}y + 1 = 0,$$

$$\text{but } \frac{A}{C} = -\frac{1}{AG} = -\frac{1}{a} \text{ and } \frac{B}{C} = -\frac{1}{AH} = -\frac{1}{b},$$

$$\therefore\ -\frac{1}{a}x - \frac{1}{b}y + 1 = 0,$$

or $\frac{x}{a} + \frac{y}{b} = 1$; this is the equation to HG, which cuts off the intercepts AH and AG, which are positive;

* The equation to the line G′H is evidently

$$y = + mx + h.$$

The equation to the line H′G, cutting the axis of Y below the origin, is

$$y = mx - h.$$

The equation to HG, having m negative, is

$$y = - mx + h.$$

The equation to the line G′H′, which cuts the axis of x to the left of the origin, and the axis of y below it, having both m and h negative, is

$$y = - mx - h.$$

$\frac{x}{a} - \frac{y}{b} = 1$ is the equation to the line GH′, which cuts off a positive intercept on the axis of x, and a negative intercept on the axis of y;

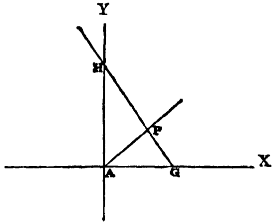
$\frac{y}{b} - \frac{x}{a}$ is the equation to the line HG′, which cuts off a negative intercept on the axis of x, and a positive intercept on the axis of y;

$$\frac{x}{a} + \frac{y}{b} = -1 \text{ is the equation G′H′.}$$

(11.) Let the angle

$$\text{PAG} = \alpha, \quad \text{AG} = a,$$
$$\text{AH} = b,$$

the intercepts of the axes, and let p = perpendicular AP,



$$a \cos \alpha = p, \quad \therefore a = \frac{p}{\cos \alpha},$$

$$b \cos(90 - \alpha) = p, \quad \therefore b = \frac{p}{\cos(90 - \alpha)} = \frac{p}{\sin \alpha};$$

these substituted in the equation $\frac{x}{a} + \frac{y}{b} = 1$,

give $x \cos \alpha + y \sin \alpha = p$.

(12.) Find the equation to a straight line passing through a given point.

Let $x'y'$ be the co-ordinates of the given point; then, as the general equation for every point in the line is

$$y = mx + h \text{ .... } (1);$$

therefore, at the given point,

$$y' = mx' + h,$$

$$\therefore h = y' - mx' \text{ .... } (2).$$

Substitute this value of h in (1), and we have

$$y - y' = m(x - x'), \text{ or } y = m(x - x') + y',$$
$$\text{or } y = mx + y' - mx';$$

we have here determined h, but having no other given condition we cannot determine m, which, therefore, remains indeterminate, as it ought, for any number of lines may be drawn through one point.

(13.) To find the equation to a line passing through two given points.

Let M and M′ be the two given points whose co-ordinates are x', y', and x'', y''.

The equation will be of the form,

$$y = mx + h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and since each of the points M and M′ is in the line,

$$y' = mx' + h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

$$y'' = mx'' + h \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3).$$

Subtract (3) from (2), and we have,

$$y' - y'' = m(x' - x''), \quad \therefore\ m = \frac{y' - y''}{x' - x''}.$$

Substituting this value in equation (2),

$$h = y' - \frac{y' - y''}{x' - x''} \cdot x' = \frac{x'y'' - y'x''}{x' - x''};$$

these values of m and h, substituted in equation (1) give,

$$y = \frac{y' - y''}{x' - x''} \cdot x + \frac{x'y'' - y'x''}{x' - x''} \quad \ldots\ldots (4),$$

for the required equation.

Or thus:—subtract (2) from (1), and we have,

$$y - y' = m(x - x'),$$

an equation which still contains the unknown quantity m.

Subtracting (3) from (2) we obtain,

$$y' - y'' = m(x' - x''),$$

$$\therefore\ m = \frac{y' - y''}{x' - x''}.$$

The value of m substituted in the preceding equation gives

$$y - y' = \frac{y' - y''}{x' - x''}(x - x') \text{ .... } (5),$$

an equation which contains only given quantities.

The identity of the equations (4) and (5) may be easily shewn: we find, from equation (5),

$$y = y' + \frac{y' - y'}{x' - x''} \cdot x - \frac{y' - y''}{x' - x''} \cdot x';$$

which by reduction becomes,

$$y = \frac{y' - y''}{x' - x''} \cdot x + \frac{x'y'' - y'x''}{x' - x''}.$$

The second method is evidently much more simple and elegant than the first.

(14.) To draw through a given point a right line parallel to a line given in position.

Let x', y' be the co-ordinates of the point M, through which we can draw a line DH parallel to a given right line BL.

The equation of the first right line being

$$y = mx + h \text{ .... } (1),$$

that of the required line will be of the form

$$y = m'x + h' \text{ .... } (2),$$

m' and h' being the two constants which we must determine.

Since DH passes through the point M, the equation is

$$y' = m'x' + h' \text{ .... } (3).$$

Subtracting (2) from (3), we have $y - y' = m'(x - x')$; and because the lines are parallel, $m' = m$,

$$\therefore\ y - y' = m(x - x'),$$

which is the equation of the required line, and it differs only from the above equation by the ordinate at the origin, which is in this case $y' - mx'$.

(15.) Two right lines being given to find their point of intersection and also their angle of intersection.

Let $\left.\begin{array}{l} y = mx + h \\ y = m'x + h' \end{array}\right\}$ be the equations to the two lines.

At the point of intersection the co-ordinates are the same for both lines; hence, at this point,

$$mx + h = m'x + h'$$

$$m'x - mx = h - h'$$

$$x = \frac{h - h'}{m' - m};$$

this substituted in the first equation gives

$$y = m \cdot \frac{h - h'}{m' - m} + h, \text{ or } y = \frac{hm' - mh'}{m' - m},$$

therefore $x = \dfrac{h - h'}{m' - m}$, $y = \dfrac{m'h - mh'}{m' - m}$, are the co-ordinates of the point of intersection.

If $m' = m$, we have $x = \dfrac{h - h'}{0}$, $y = \dfrac{m(h - h')}{0}$; that is, these values are infinite, as they ought to be, as the lines are parallel.

To find the angle of intersection. See figure, page 13.

Taking the axes rectangular and putting V for the angle of intersection; in the triangle EMG,

$$MGX = EMG + MEG,$$

$$\therefore \text{ EMG, or } V = MGX - MEG = \alpha' - \alpha,$$

α' and α being the angles that the two right lines form with the axis of x. Then by Hann's Trigonometry, p. 31, we have

$$\tan(\alpha' - \alpha), \text{ or } \tan V = \frac{\tan\alpha' - \tan\alpha}{1 + \tan\alpha' \tan\alpha} \quad \cdots\cdots\cdots \text{(1)},$$

and since the axes are rectangular, $\tan\alpha' = m'$, and $\tan\alpha = m$,

$$\therefore \tan V = \frac{m' - m}{1 + m'm} \quad \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \text{(2)}.$$

When the axes are oblique, we have

$$\frac{\sin \alpha'}{\sin (\beta - \alpha')} = m', \quad \frac{\sin \alpha}{\sin (\beta - \alpha)} = m.$$

From the second relation,

$$\sin \alpha = m \sin (\beta - \alpha), \text{ or}$$

$$\sin \alpha = m \sin \beta \cos \alpha - m \sin \alpha \cos \beta.$$

Dividing by $\cos \alpha$,

$$\tan \alpha = m \sin \beta - m \tan \alpha \cos \beta,$$

$$\therefore \tan \alpha = \frac{m \sin \beta}{1 + m \cos \beta}.$$

In the same manner,

$$\tan \alpha' = \frac{m' \sin \beta}{1 + m' \cos \beta};$$

substituting these in (1) we have,

$$\tan V = \frac{\dfrac{m' \sin \beta}{1 + m' \cos \beta} - \dfrac{m \sin \beta}{1 + m \cos \beta}}{1 + \dfrac{m m' \sin^2 \beta}{(1 + m' \cos \beta)(1 + m \cos \beta)}},$$

reducing and observing that $\sin^2 \beta + \cos^2 \beta = 1$,

$$\tan V = \frac{(m' - m) \sin \beta}{1 + mm' + (m + m') \cos \beta} \quad \ldots\ldots\ldots\ldots\ldots \quad (3).$$

If we suppose $\beta = 90$, $\sin \beta = 1$, and $\cos \beta = 0$, we have

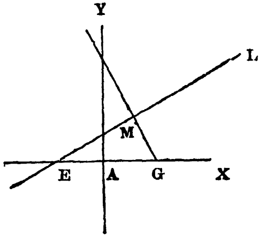
$$\tan V = \frac{m' - m}{1 + mm'}, \text{ as before;}$$

$$\text{when } V = 90, \; 1 + mm' = 0, \; \therefore m' = -\frac{1}{m};$$

when the axes are oblique $1 + m.m' + (m + m') \cos \beta = 0$,

$$\therefore m' = -\frac{1 + m \cos \beta}{m + \cos \beta}.$$

(16.) The relation $1 + m \cdot m' = 0$ can be shewn from the figure. The triangle EMG is right-angled at M; the two angles, MEG and MGE, are complements of each other,



$$\therefore \tan MGE = \cot MEG$$

$$= \frac{1}{\tan MEG},$$

but $\tan MGX$, or $m' = -\tan MGE$, $\tan MEG = m$,

$$\therefore m' = -\frac{1}{m}, \text{ or } mm' + 1 = 0.$$

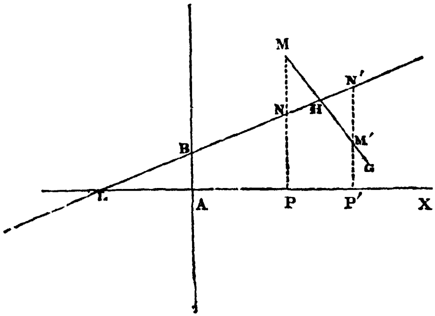
(17.) From a given point to draw a perpendicular to a given line, and find the length of this perpendicular.

Let BL be the given right line, and MG the perpendicular on BL, let x' and y' be the co-ordinates of the point M, and let the equation to the right line BL be

$$y = mx + h \text{ .... } (1),$$

since the right line MG passes through the point x', y', its equation (12) will be of the form

$$y - y' = m'(x - x');$$



but since these lines are to be perpendicular to each other, we have $mm' + 1 = 0$, or

$$m' = -\frac{1}{m};$$

therefore, the preceding equation becomes

$$y - y' = -\frac{1}{m}(x - x')^* \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

This is the equation to the perpendicular MG. Now, to find an expression for the length of the perpendicular MH, we have x', y', the co-ordinates of the point M; and if we can find those of the point H, we have only to substitute these four co-ordinates in the expression for the distance between the given points (art. 5), and we shall have the distance MH.

As the point H is the intersection of BL and MG, we must eliminate x, y by the equations (1) and (2); thus, we must find the values of $x - x'$, $y - y'$, considered as unknown; in order to do this we must put equation (1) in the form

$$y - y' = m(x - x') - y' + mx' + h \quad \ldots\ldots\ldots\ldots \quad (3),$$

$$y - y' = -\frac{1}{m}(x - x') \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2).$$

Subtract, and we have

$$0 = \left(m + \frac{1}{m}\right)(x - x') - y' + mx' + h,$$

* If we take the general equation $Ax + By + C = 0$, then the equation to a right line perpendicular to it is $A(y - y') = B(x - x')$, for

$$y = -\frac{A}{B}x - \frac{C}{B},$$

$$\therefore\ y = mx + h, \text{ as } m = -\frac{A}{B}, \text{ and } h = -\frac{C}{B};$$

substituting this value for m, $y - y' = -\frac{1}{m}(x - x')$ becomes

$$y - y' = \frac{B}{A}(x - x'), \text{ or } A(y - y') = B(x - x').$$

$$\therefore\ x - x' = \frac{y' - mx' - h}{m + \frac{1}{m}} = \frac{m\,(y' - mx' - h)}{m^2 + 1};$$

this value being substituted in (2), gives

$$y - y' = -\frac{1}{m} \cdot \frac{m\,(y' - mx' - h)}{m^2 + 1} = -\frac{(y' - mx' - h)}{m^2 + 1}.$$

Substituting these values of $x - x'$, $y - y'$, in the expression for D, we have, putting P for the perpendicular,

$$P = \sqrt{\frac{m^2\,(y' - mx' - h)^2 + (y' - mx' - h)^2}{(m^2 + 1)^2}}$$

$$= \frac{\sqrt{(m^2+1)(y'-mx'-h)^2}}{(m^2 + 1)} = \frac{(y'-mx'-h)\sqrt{m^2 + 1}}{m^2 + 1},$$

$$\therefore\ P = \frac{\pm\,(y' - mx' - h)}{\sqrt{m^2 + 1}}$$

for the length MH.

(18.) The double sign may be thus explained: y' represents the ordinate MP, and $mx' + h$ represents the ordinate NP of BL, corresponding to x' or AP; hence,

$$MN = MP - PN = y' - mx' - h.$$

Now, this distance may be either positive or negative, that is, it may be either greater than nothing or less than nothing, according as the point M is above or below BL; for example, if the point was in M′, we should have

$$M'N' = N'P' - M'P' = mx' + h - y'.$$

When the point M is above the right line BL, in which case $y - mx' - h > 0$, we have

$$P = \frac{y' - mx' - h}{\sqrt{m^2 + 1}};$$

and if the point M is placed below the right line BL, then we have the condition $y' - mx' - h < 0$,

$$\therefore\ P = \frac{mx' + h - y'}{\sqrt{m^2 + 1}}.$$

We can obtain both these results geometrically; in fact, MP and MH being respectively perpendicular to AP and BL, it follows that the angle NMH equals the angle BLX $= \alpha$. Now, in the right-angled triangle NMH, we have

$$\text{MH} = \text{MN} \cos \alpha = \frac{\text{MN}}{\sec \alpha} = \frac{\text{MN}}{\sqrt{1 + \tan^2 \alpha}},$$

but $\text{MN} = \text{MP} - \text{NP} = y' - mx' - h$, and $\tan \alpha = m$,

$$\therefore \text{ MH} = \text{P} = \frac{y' - mx' - h}{\sqrt{m^2 + 1}}.*$$

If the point M was below BL, we should have

$$\text{M}'\text{N}' = mx' + h - y',$$

$$\therefore \text{ P} = \frac{mx' + h - y'}{\sqrt{m^2 + 1}}.$$

(19.) If the point from which we wish to let fall the perpendiculars be at the origin, we have in this case $x' = 0$, $y' = 0$, then

$$\text{P} = \mp \frac{h}{\sqrt{m^2 + 1}};$$

this result is positive or negative, according as the point B is above or below the origin.

(20.) Through a given point to draw a straight line which shall make a given angle with a given straight line.

Let the given line be represented by the equation $y = mx + h$, then, because the required line passes through the given point x', y', its equation will be of the form $y - y' = m'(x - x')$, and since these right lines ought to form an angle whose tangent is t, we have (art. 15.)

* This is sometimes given $\text{P} = \dfrac{\text{A}x' + \text{B}y' + \text{C}}{\sqrt{\text{A}^2 + \text{B}^2}}$, which is found by putting $-\dfrac{\text{A}}{\text{B}}$ for m, and $-\dfrac{\text{C}}{\text{B}}$ for h.

$$\frac{m' - m}{1 + mm'} = t, \text{ or } \frac{m - m'}{1 + mm'} = t;$$

either of these equations expresses the tangent of the given angle.

We can comprehend these two expressions in one, thus

$$\frac{m' - m}{1 + mm'} = \pm t, \quad \therefore m' = \frac{m \pm t}{1 \mp mt},$$

this value substituted in the above equation for m', gives the equation of the required line.

$$y - y' = \frac{m \pm t}{1 \mp mt}(x - x').$$

Examples on the Straight Line.

1. The equation $(2y + x)(3y - x) = 0$ represents two straight lines inclined to one another at an angle of 135°.

The equation is at once resolved into the two

$$\left.\begin{aligned} 2y + x &= 0 \\ \text{and } 3y - x &= 0 \end{aligned}\right\},$$

$$\left.\begin{aligned} &\text{or (1) } \ldots\ldots\ldots\ldots y = -\frac{1}{2}x \\ &\text{and (2) } \ldots\ldots\ldots\ldots y = \frac{1}{3}x \end{aligned}\right\},$$

each of which represents a straight line passing through the origin, since $x = 0$, gives $y = 0$.

Now, the coefficient of x is the tangent of the angle which the line makes with the axis of x.

Hence, if θ_1 be the angle which (1) makes with axis, and θ_2 be the angle which (2) makes with the same,

$$\left.\begin{aligned} \tan\theta_1 &= -\frac{1}{2} \\ \text{and } \tan\theta_2 &= \frac{1}{3} \end{aligned}\right\}.$$

Now, if ϕ be the angle between the lines (1) and (2), $\phi = \theta_1 - \theta_2$,

$$\therefore \tan\phi = \frac{\tan\theta_1 - \tan\theta_2}{1 + \tan\theta_1 \tan\theta_2} = \frac{-\frac{1}{2} - \frac{1}{3}}{1 - \frac{1}{2}\cdot\frac{1}{3}} = -\frac{\frac{5}{6}}{\frac{5}{6}} = -1,$$

$$\therefore \phi = 135°.$$

2. Find the equation to a straight line which bisects the angle between the lines

$$5y - 2x = 0 \;\ldots\ldots\; (1), \text{ and } 3y + 4x = 12 \;\ldots\ldots\; (2).$$

Let (x, y) be any point in the required line; then, since this line bisects the angle between (1) and (2), the perpendiculars from (x, y) on these must be equal.

Substituting in the formula $P = \dfrac{Ax + By + C}{\sqrt{A^2 + B^2}}$, we get the perpendicular on (1)

$$= \frac{-2x + 5y}{\sqrt{4 + 25}},$$

$$\text{and that on (2)} = \frac{4x + 3y - 12}{\sqrt{16 + 9}},$$

$$\therefore \frac{5y - 2x}{\sqrt{29}} = \frac{4x + 3y - 12}{5} \text{ is equation required,}$$

i. e. $(25 - 3\sqrt{29})y - (10 + 4\sqrt{29})x + 12\sqrt{29} = 0.$

But $\sqrt{29} = 5{\cdot}38$, $\therefore$ equation becomes

$$8{\cdot}86y - 31{\cdot}52x + 64{\cdot}56 = 0.$$

3. Find the angle between the lines whose equations are

$$\left.\begin{array}{r} 1 + 3x + 2y = 0 \\ \text{and } 3 + 2x - 3y = 0 \end{array}\right\}.$$

These equations become

$$y = -\frac{3}{2}x - \frac{1}{2} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\; (1),$$

$$\text{and } y = \frac{2}{3}x + 1 \text{ .... } (2),$$

.herefore, if θ_1 be the angle which (1) makes with the axis of x, and θ_2 the angle which (2) makes with the same,

$$\tan\theta_1 = -\frac{3}{2}, \text{ and } \tan\theta_2 = \frac{2}{3},$$

$$\therefore \tan\theta_1 . \tan\theta_2 = -1;$$

hence the lines are at right angles to each other.

4. Find the length of the perpendicular from the point $x = 3$, $y = 5$, upon the line $7x - 3y = 9$.

The length of a perpendicular from a point x, y, upon the line $Ax + By + C = 0$, is

$$\frac{Ax + By + C}{\sqrt{A^2 + B^2}}.$$

In the present example $x = 3$, $y = 5$, $A = 7$, $B = -3$, and $C = -9$,

$$\therefore p = \frac{21 - 15 - 9}{\sqrt{49 + 9}} = -\frac{3}{\sqrt{58}},$$

or, independently of the sign, the length of the perpendicular equals $\dfrac{3}{\sqrt{58}}$.

5. The following equation represents two straight lines at right angles to each other.

$$2y^2 - 3xy - 2x^2 - y + 2x = 0 \text{ ....} (1),$$

$$\therefore 2y(y - 2x) + x(y - 2x) - (y - 2x) = 0,$$

$$\therefore (2y + x - 1)(y - 2x) = 0.$$

Hence, equation (1) is resolved into the two

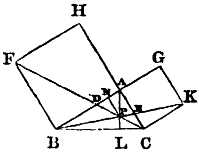
$$\left.\begin{array}{r} 2y + x - 1 = 0 \\ \text{and} \qquad y - 2x = 0 \end{array}\right\},$$

which evidently represent two lines at right angles to each other, for they are each equations of the first degree, and therefore represent straight lines; and since the coefficients of x divided by those of y in the two equations when multiplied together $= -1$, the lines are at right angles to each other.

6. In the figure to Euclid, book i. prop. 47, prove that the the lines FC, KB, and AL meet in one point.

Take A for the origin, AG and AH for co-ordinate axes.

Now, equation



to FC is $\dfrac{x}{AD} + \dfrac{y}{AC} = -1,$

and to KB is $\dfrac{x}{AB} + \dfrac{y}{AE} = -1,$

and AD : AC :: FG : GC :: AB : AB + AC,

$$\therefore\ AD = \frac{AB\,.\,AC}{AB + AC}, \text{ and } \frac{1}{AD} = \frac{1}{AC} + \frac{1}{AB};$$

similarly, AE : AB :: HK : BH :: AC : AB + AC,

$$\therefore\ AE = \frac{AB\,.\,AC}{AB + AC}, \text{ and } \frac{1}{AE} = \frac{1}{AC} + \frac{1}{AB},$$

$\therefore$ equation to FC becomes $x\left(\dfrac{1}{AC} + \dfrac{1}{AB}\right) + \dfrac{y}{AC} = -1,$

and to KB ... $\dfrac{x}{AB} + y\left(\dfrac{1}{AC} + \dfrac{1}{AB}\right) = -1,$

$\therefore$ for P their intersection $\dfrac{x}{AC} - \dfrac{y}{AB} = 0,$

$$\therefore\ \frac{AN}{NP} = \frac{AC}{AB}, \quad \therefore \text{ ANP is similar to ABC,}$$

and $\therefore$ P is in AL, that is, the perpendicular from A on BC.—

7. In the equation $ay^2 + bxy + dy + ex = 0$, find the relations among the coefficients, that it may represent two straight lines.

The condition is, that the equation may be resolvable into two factors of the first degree, and since we observe that if $x = 0$, $y = 0$, and that the coefficient of $x^2 = 0$, we assume the equation to become

$$(\mathrm{A}x + \mathrm{B}y)(\mathrm{C}y + 1) = 0,$$

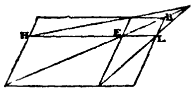
$$\therefore\ \mathrm{AC}xy + \mathrm{BC}y^2 + \mathrm{A}x + \mathrm{B}y = 0;$$

by comparing this with the original equation we get the relation amongst the coefficients—viz.,

$$\left.\begin{aligned} a &= \mathrm{BC} \\ b &= \mathrm{AC} \\ d &= \mathrm{B} \\ e &= \mathrm{A} \end{aligned}\right\},$$

$$\therefore\ ae = \mathrm{A}\,.\,\mathrm{B}\,.\,\mathrm{C} = bd \text{ is the relation required.}$$

8. ABCD is a parallelogram, AE and ED parallelograms about its diameter CB. Show that the diagonals of these, viz., HF and GL, meet CB in the same point.



Take E for the origin, EF and EL for co-ordinate axes.

let $\mathrm{HE} = a$, $\mathrm{EF} = b$, $\mathrm{EL} = a'$, and $\mathrm{EG} = b'$.

Then equation to HF is $-\dfrac{x}{a} + \dfrac{y}{b} = 1$,

... GL ... $\dfrac{x}{a'} - \dfrac{y}{b'} = 1$,

and ... CB ... $x = \dfrac{a'}{b}\,y = \dfrac{a}{b'}\,y$;

therefore, for intersection of CB and HF,

$$x\left(\frac{1}{a'} - \frac{1}{a}\right) = 1,\ \therefore\ x = \frac{aa'}{a - a'},$$

$$\text{and } y = \frac{bx}{a'} = \frac{ab}{a - a'};$$

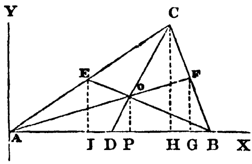
and for intersection of CB and GL,

$$x\left(\frac{1}{a'}-\frac{1}{a}\right)=1, \quad \therefore\ x=\frac{a\,a'}{a-a'},$$

$$\text{and } y=\frac{a\,b}{a-a'};$$

hence, HF and GL intersect CB in the same point.

9. Prove that right lines drawn from the three angles of a triangle to the middle of the opposite sides intersect in the same point.



Let the axis of x coincide with the base of the triangle, and the axis of y be AY, the axes being rectangular. We must now determine the equations of the lines AF, BE, CD, but before we do this we must find the co-ordinates of the points A, B, C, D, E, F.

For the co-ordinates of the point A ... $(y=0,\ x=0)$.

If AB $=c$, then the co-ordinates of the point B .... $(y=0,\ x=c)$.

For the point C .... $(y=y',\ x=x')$.

Now, as D, E, F, are in the middle of AB, AC, BC, we have

$$\text{AD}=\frac{\text{AB}}{2}=\frac{c}{2},\quad \text{EI}=\frac{\text{CH}}{2}=\frac{y'}{2},\quad \text{AI}=\frac{x'}{2},$$

$$\text{FG}=\frac{\text{CH}}{2}=\frac{y'}{2},\quad \text{GH}=\frac{\text{BH}}{2}=\frac{c-x'}{2},$$

$$\therefore\ \text{AG}=x'+\frac{c-x'}{2}=\frac{c+x'}{2};$$

this gives the co-ordinates of the points.

$$\text{D} \ldots\ldots\ldots\ldots\ldots\ldots \left(y=0,\quad x=\frac{c}{2}\right),$$

E .... $\left(y = \frac{y'}{2}, \quad x = \frac{x'}{2}\right)$,

F .... $\left(y = \frac{y'}{2}, \quad x = \frac{c + x'}{2}\right)$.

Since each of the lines AF, BE, CD, passes through two points, and the general equation to a right line through two points is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x'),$$

substituting for x', y', y'', x'', the corresponding values in the figure under consideration, we shall obtain y and x for each particular line.

Since AF passes through the two points

$$(0, 0)\ \frac{y'}{2}, \quad \left(\frac{c + x'}{2}\right),$$

$$\text{its equation is } y - 0 = \frac{0 - \frac{y'}{2}}{0 - \frac{c + x'}{2}}(x - 0),$$

$$\therefore\ y = \frac{y'}{c + x'}\,x \text{ is the equation to AF } \ldots\ldots (1);$$

and, since BE passes through $(0, c)\ \left(\frac{y'}{2}, \quad \frac{x'}{2}\right)$,

$$\text{its equation is } y - 0 = \frac{0 - \frac{y'}{2}}{c - \frac{x'}{2}}(x - c),$$

$$\text{or } y = \frac{y'}{-2c + x'}(x - c) \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

And since CD passes through $(y', x')\ \left(0, \ \frac{c}{2}\right)$, its equation is $(y - y') = \dfrac{y'}{x' - \dfrac{c}{2}}(x - x')$,

$$\text{or } y = y' + \frac{y'}{\left(x' - \dfrac{c}{2}\right)}(x - x'),$$

$$y = \frac{y'x' - \dfrac{y'c}{2} + y'x - y'x'}{x' - \dfrac{c}{2}}.$$

$$= \frac{y'x - \dfrac{y'c}{2}}{\left(x' - \dfrac{c}{2}\right)} = \frac{y'(2x - c)}{2x' - c} \text{ .... } (3);$$

and from these $x = \dfrac{c + x'}{3}$, $y = \dfrac{y'}{3}$,

Or having the co-ordinates of AF, BE, CD, we may proceed as follows:—

Since AF passes through the origin, its equation is of the form $y = ax$, and since it passes through the point $\left(\dfrac{y'}{2}, \dfrac{c + x'}{2}\right)$, we have the relation

$$\frac{y'}{2} = a\left(\frac{c + x'}{2}\right), \therefore a = \frac{y'}{c + x'},$$

$$\therefore \text{ the equation of AF is } y = \frac{y'}{c + x'}x \text{ ...... } (1).$$

The right line BE passing through the point B, or (c, o) its equation is of the form $y = a'(x - c)$. In the êquation $y - y' = a(x - x')$, we must put 0 for y', and c for x', and a for a', but as this line passes through the point E, or

$\left(\dfrac{y'}{2}, \dfrac{x'}{2}\right)$, we have $\dfrac{y'}{2} = a'\left(\dfrac{x'}{2} - c\right)$, $\therefore a' = \dfrac{y'}{x' - 2c}$;

hence the equation to the line BE is

$$y = \frac{y'}{x' - 2c}(x - c) \quad \ldots\ldots\ldots\ldots\ldots \quad (2.)$$

In the same manner for CD,

$$y = \frac{y'}{2x' - c}(2x - c) \quad \ldots\ldots\ldots\ldots\ldots \quad (3.)$$

From equations (1) and (2),

$$\frac{y'}{c + x'} . x = \frac{y'}{x' - 2c}(x - c),$$

$$x'y' . x - 2cy' . x = cy' . x - c^2y' + x'y' . x - cx'y',$$

or by reduction,

$$3cy' . x = cy'(c + x'), \quad \therefore \; x = \frac{c + x'}{3},$$

substitute this value in (1), and we have $y = \frac{y'}{3}$.

From equations (1) and (3) we have,

$$\frac{y'}{c + x'} . x = \frac{y'}{2x' - c}(2x - c)$$

$$2x'y' . x - cy' . x = 2cy' . x - c^2y' + 2x'y' . x - cx'y',$$

$$\text{or } 3cy' . x = cy'(c + x'), \quad \therefore \; x = \frac{c + x'}{3};$$

and, consequently, $y = \frac{y'}{3}$.

Hence we see that the co-ordinates of the point of intersection of the two right lines AF and BE are identical with those of the point of intersection of AF and CD; thus the three lines intersect in the same point.

In reflecting upon the preceding analysis, it is easy to see that the procedure is the same whatever be the inclination of the axes.

However, they become more simple when we take **AB** for the axis of x, and take for the axis of ordinates a straight line AY parallel to **CD**, which we can do when the line **CD** is given in position.

In that case, it is plain that the abscissa x' of the point $C = AD$, or $\frac{c}{2}$, where $c = 2x'$, and the equations (1), (2), (3), will be as follows:—

the equation to the straight line **AP**, $y = \frac{y'}{3x'}x$, *

the equation of the straight line **BE**, $y = -\frac{y'}{3x'}(x - 2x')$,

the equation of the straight line **CD**, $x = x'$,

(since the last line is parallel AY.)

These things being premised, combining the last equation with the first, there results $y = \frac{y'}{3}$; and combining it with the second, there results again $y = \frac{y'}{3}$.

Thus the co-ordinates of the point of intersection of **CD**, **AE**, and of **CD**, **BE**, are

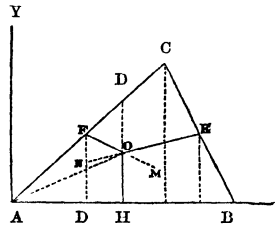
$$x = x', \quad y = \frac{y'}{3} = \frac{CD}{3}.$$

This proves that the choice of axes influences the simplicity of calculation in resolving questions by Analytical Geometry.

10. To prove that the perpendiculars drawn from the angular points of a triangle cut one another in the same point.

* Since the line AF passes through the origin and makes an angle with axis of abscissæ $= \tan^{-1}\frac{y'}{3x'}$, and since the line BE makes an angle with the same axis $= \tan^{-1} - \frac{y'}{3x'}$, and cuts the axis at a distance $2x'$ from the origin, its equation is $y = -\frac{y'}{3x'}(x - 2x')$.

Referring to rectangular axes, since we have to consider the relation of perpendiculars from two sides, take for the axis of abscissæ the line AB, and for the axis of ordinates a line perpendicular to AX, at A.



Let $c = $ AB, x', y' the co-ordinates of the point C, we shall have for the equation of CD, $x = x'$ (1), since it is parallel to the axis of y.

Having found the equations of AP and BE, we must commence to find that of the lines CB, AC, to which they are perpendicular.

Since the line CB passes through two points (y', x'), and (o, c) its equation is $y - y' = a(x - x')$, a having for its value $\frac{y'}{x' - c}$ when we put $y' = 0$, $x' = c$.

That of the line AC passing through the origin is $y = mx$, m being $= \frac{y'}{x'}$, since the point C is on that line.

Those things being done, as AF passes through the origin, it has an equation $y = a'x$; and, as it is perpendicular to CB, we deduce the equation $aa' + 1 = 0$, where $a' = -\frac{1}{a} = \frac{c - x'}{y'}$, since the equation of AF is $y = \frac{c - x'}{y'} x$. (2).

The line BE passing through the point B, where the co-ordinates are o, c, has for its equation $y = m'(x - c)$.

But since it is perpendicular to AC, there results the relation $mm' + 1 = 0$, where $m' = -\frac{1}{m} = -\frac{x'}{y'}$.

Lastly, therefore, the equation BE is

$$y = \frac{-x'}{y'}(x - c) \text{ .... } (3).$$

Now, if we combine the equations (1) and (2),

we find for $x = x'$, $y = \frac{(c - x)x'}{y'}$.

Combining the equations (1) and (3) for $x = x'$, we get

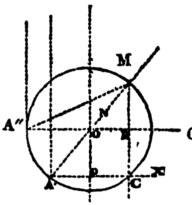
$$y = \frac{-x'}{y'}(x' - c) = \frac{(c - x')x'}{y'},$$

therefore, the co-ordinates of the point of intersection of CD, AF, are the same as those of the point of intersection of CD, BE; wherefore we conclude that these three lines cut one another in the same point.

Section III.—Circle.

Equation to the Circle.

(21.) Let there be a circle having the radius r, the centre of which is O. Let us refer to the axes AX, AY, then if α, β, be the co-ordinates of the centre x, y, the co-ordinates of a point M, in the circumference, we shall have, evidently,



$$(x - \alpha)^2 + (y - \beta)^2 = r^2 \ . \ (1).$$

This relation (1) characterizes equally every point in the circumference, since M may be assumed anywhere.

Let N be a point taken without or within the circumference, x and y being the co-ordinates of that point, we have

$$(x - \alpha)^2 + (y - \beta)^2 = \text{ON}^2 \ldots\ldots\ldots\ldots\ldots (2).$$

But it is clear that ON > or < OM, accordingly as the point is on the exterior or interior of the circumference, whence there necessarily results

$$(x - \alpha)^2 + (y - \beta)^2 > \text{ or } < r^2,$$

as the point is without or within the circumference.

Thus it is evident that the relation (1) does not hold, unless the point is on the circumference.

The constants α, β, r, are the co-ordinates of the centre of the circle, and the radius, whence the circle is completely determined when these constants are given.

The general equation admits of modification according as we take the origin.

If A, a point in the circumference, be taken as the origin, we have $\alpha^2 + \beta^2 = r^2$.

From the form (1)

$$x^2 - 2\alpha x + \alpha^2 + y^2 - 2\beta y + \beta^2 = r^2,$$

$$\therefore \text{ since } (\alpha^2 + \beta^2) = r^2,$$

$$\text{we have } x^2 - 2\alpha x + y^2 - 2\beta y = 0 \ldots\ldots (3).$$

If 0, the centre, be the origin, the equation is (since α and $\beta = 0$)

$$x^2 + y^2 = r^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4).$$

If the origin be at A″, the extremity of the diameter which coincides with the axis of x, we have

$$x^2 + y^2 = 2rx \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (5).$$

If in (3) we make $y = 0$, there results $x^2 - 2\alpha x = 0$, $\therefore x(x - 2\alpha) = 0$, $\therefore y = 0$ is satisfied by either $x = 0$ or $x = 2\alpha$, which shows that $x = 0$; $y = 0$ is placed on the circumference, as the value 0 of y corresponds to $x = 2\alpha$ as well as to $x = 0$; it follows that the circumference cuts the axis of x at a second point, C, where A′C is double of A′D, therefore the chord A′C is cut into two equal parts by the perpendicular from the centre of the circle, a well-known geometrical property of the circle.

Taking (5), $y^2 = 2rx - x^2 = x(2r - x)$.

$$\text{Now, } y = \text{MR}, \quad x = \text{A}''\text{R}, \quad 2r - x = \text{GR},$$

$$\therefore \text{MR}^2 = \text{A}''\text{R} \times \text{GR},$$

$$\therefore \text{A}''\text{R} : \text{MR} :: \text{MR} : \text{GR};$$

that is, the perpendicular let fall from a point in the circumference on a diameter, is a mean proportional between the two segments. The equation may be put $y^2 + x^2 = 2rx$, but if we draw the chord A″M,

$$\text{MR}^2 + \text{A}''\text{R}^2 = \text{A}''\text{M}^2, \quad 2r = \text{A}''\text{G}, \quad x = \text{A}''\text{R},$$

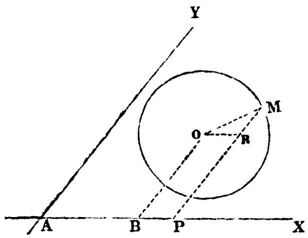
$$\text{therefore, } A''M^2 = A''G \cdot A''R,$$

$$\text{that is, } A''G : A''M :: A''M : A''R,$$

which proves that the chord drawn from one of the extremities of a diameter is a mean proportional between the diameter and a segment formed by the perpendicular let fall from the extremity of the chord on that diameter.

This is another well-known geometrical theorem.

The equation to the circle is much more complicated when the axes are oblique.



We have for the equation to the circle referred to oblique axes (in the same way as at article 5),

$$(x - \alpha)^2 + (y - \beta)^2 + 2(x - \alpha)(y - \beta)\cos\phi = r^2,$$

where ϕ is the inclination of the axes.

(22.) To determine the conditions which are to be fulfilled in order that two circumferences of circles may cut one another, or *touch* at a single point.

Let oo' be the centres of the two circles, rr' their respective radii, and let $oo' = d$, the distance between their centres.

Take the line through the centres for the axis of x, and for the axis of y the perpendicular OY, on the point O.

The circle, of which the radius is r, will have for its equation

$$y^2 + x^2 = r^2 \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

the other circle, the co-ordinates of its centre being o and d, has for its equation

$$y^2 + (x - d)^2 = r'^2 \ldots\ldots\ldots\ldots\ldots\ldots (2).$$

These things being premised, to express where the two circumferences cut, and to obtain the points of intersection, we must eliminate x and y by combining (1) and (2).

This gives

$$x = \frac{r^2 - r'^2 + d^2}{2d} \text{ .... } (3),$$

$$y = \pm \frac{1}{2d} \sqrt{4d^2r^2 - (r^2 - r'^2 + d^2)^2} \text{ .... } (4).$$

We now can proceed to the discussion of the point proposed for consideration.

If the quantity (4) under the radical sign is positive, in that case the values of x and y are real, and the circumferences have two common points.

Proving that the two points of intersection are on the same abscissa, OP, having the two ordinates equal, but of opposite signs.

Therefore, at every *cutting* of the two circles the line joining their centres is perpendicular to the common chord, and bisects it.

However, since y may be real or imaginary (x being always real), we will transform the above expression.

The quantity (4) under the radical is evidently

$$= (2dr + r^2 + d^2 - r'^2)(2dr - r^2 - d^2 + r'^2)$$

$$= \{(r + d)^2 - r'^2\}\{r'^2 - (r - d)^2\} =$$

$$(r + r' + d)(r - r' + d)(r + r' - d)(r' - r + d),$$

$$\therefore y = \pm \frac{1}{2d} \sqrt{(r + r' - d)(r - r' + d)(r + r' + d)(r' - r + d)};$$

under that form, as the third factor $r + r' + d$, under the radical sign, is necessarily positive, y will be real, according as the three other factors are positive, or one of the two positive, and the other two negative.

But this last circumstance cannot exist, since, if one of the three factors is negative, the other two are necessarily positive. For example, suppose $r + d - r'$ is $< o$,

$$\therefore\ r + d < r', \quad \therefore\ r < r' \text{ and } d < r',$$
$$\therefore\ r' - r + d \text{ and } r' - d + r \text{ are positive,}$$

and, *à fortiori*, the three factors cannot all be negative at once.

Therefore, the only case is where the three factors are positive at once, and in that case y is real; that is to say, the two circles cut one another when the sum of each two of the quantities r, r', d is $>$ the third.

Now, if one of the three factors is $(-)$, and the other two $(+)$, in that case y is imaginary, *and the point is not a point of intersection*. Therefore, the two circumferences have not a common point.

But suppose that any one of the factors is $= 0$, it gives, therefore, $y = \pm 0$, and the two circumferences have only *one* common point, which is necessarily on the line joining the centres, where the ordinate is zero; therefore, two circles *touch* when the distance of the centres is equal to the sum or difference of the radii.

Perhaps no example than this more strikingly shews the advantages resulting from the application of Algebra to Geometry.

(23.) To find the equation to the tangent to a circle.

Let us at first consider a line to pass through two points whose co-ordinates are (x', y''), $(x'' y'')$, the equation to this line is, by art. 13,

$$y - y' = \frac{y' - y''}{x' - x''}(x - x') \qquad \text{(1)},$$

and if we combine this with the two equations

$$x'^2 + y'^2 = r^2 \qquad \text{(2)},$$
$$x''^2 + y''^2 = r^2 \qquad \text{(3)},$$

these two latter equations hold because each point (x', y'), (x'', y''), are in the circumference of the circle.

Subtract (3) from (2), and we have

$$y'^2 - y''^2 + x'^2 - x''^2 = 0 \qquad \text{(4)};$$

but since the difference of the squares of any two quantities is equal to the product of the sum and difference of the same quantities, we have

$$(y' + y'')\,(y' - y'') + (x' + x'')\,(x' - x'') = 0;$$

$$\therefore \frac{y' - y''}{x' - x''} = -\frac{x' + x''}{y' + y''}.$$

If this value be substituted in (1) it becomes

$$y - y' = -\frac{x' + x''}{y' + y''}(x - x') \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (5).$$

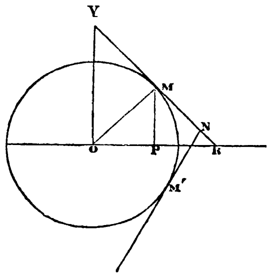
When the two points merge into one, the secant becomes a tangent, and we have $x' = x''$, and $y' = y''$, which gives

$$y - y'' = -\frac{x''}{y''}(x - x''), \text{ and } x''^2 + y''^2 = r^2.$$

By means of these two equations we obtain the equation to the tangent in a more simple form. Clearing the former of fractions, we get

$$y\,y'' + x\,x'' = x''^2 + y''^2, \text{ or } y\,y'' + x\,x'' = r^2,$$

since by the latter, $x''^2 + y''^2 = r^2$.



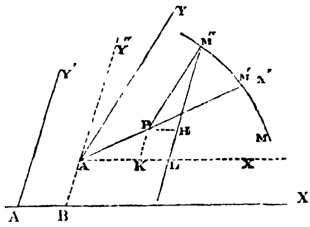
This latter form is easily remembered, from its analogy to the equation of the circle. If in this latter form of the tangent we make $y = 0$, then $x\,x'' = r^2$, or $x = \frac{r^2}{x''}$, which is the abscissa OR of the point where the tangent meets the axis of x'; also we have PR = OR — OP; or,

$$\text{PR} = \frac{r^2}{x''} - x'' = \frac{r^2 - x''^2}{x''}.$$

This line PR is called the subtangent,

Transformation of Co-ordinates.

(24.) The most general problem of transformation is as follows:—an equation being given between x and y, referred to any whatsoever axes, to find the equation when we pass to two new axes.



Let AY, AX, be two axes to which the curve MM'M'' is referred by means of some equation $F(x, y) = 0$; A'X', A'Y', two new axes.

Let AP, PM $= x$, and y, respectively, be the old co-ordinates, and x', y', or A'P', P'M', the new ones.

We seek now to express x', y', in terms of x, y, or *vice versâ*.

Let A'X'' and P'H be drawn parallel to AX, A'Y'', and P'K parallel to AY produce A'Y'' to B.

Let AB $= a$, A'B $= b$, X'A'X'' $= \alpha$, Y'A'X'' $= \alpha'$, and Y''A'X'' $= \theta$.

a and b are known quantities, the co-ordinates of the new origin.

As to the angle θ, it is always given (*à priori*), since it is the angle which the old axes make with each other.

We shall now proceed to give the principal cases.

COR. 1. The most simple of all cases is that in which the two new axes are parallel to the old ones; that is to say, the axes continue in the same direction, the origin alone being different. Then,

$$\left.\begin{cases} x = x' + a \\ y = y' + b \end{cases}\right\} \ldots\ldots (1),$$

$$\text{AP or } x = \text{AB} + \text{BP} = a + \text{A'K} + \text{P'H},$$

$$\text{PM or } y = \text{A'B} + \text{ML} = b + \text{P'K} + \text{MH};$$

thus, these being reduced, we can determine A'K, P'K, P'H, and MH.

But, in the triangles $A'P'K$, $MP'H$, we have, by Hann Trigonometry, page 56,

(1.) $A'K : A'P' :: \sin A'P'K : \sin A'KP'$;

or because $A'P'K = P'A'Y'' = \beta - \alpha$, and

$A'KP' = A'LM = \pi - MLX'' = \pi - \beta$,

$$A'K : x' :: \sin(\beta - \alpha) : \sin\beta, \quad \therefore\ A'K = \frac{x'\sin(\beta - \alpha)}{\sin\beta}.$$

(2.) $P'K : A'P' :: P'A'K : \sin A'KP'$, $\therefore\ P'K = \dfrac{x'\sin\alpha}{\sin\beta}$

(3.) $P'H : MP' :: \sin P'MH : \sin P'HM$,

or because $P'MH = Y'A'Y'' = (\beta - \alpha')$, and

$P'HM = A'LM = \pi - \beta$,

$$P'H : y' :: \sin(\beta - \alpha') : \sin\beta, \quad \therefore\ P'H = \frac{y'\sin(\beta - \alpha')}{\sin\beta}.$$

(4.) $MH : MP' :: \sin MP'H : \sin P'HM$, $\therefore\ MH = \dfrac{y'\sin\alpha'}{\sin\beta}$,

therefore, putting these values in the expressions for x, y we obtain

$$\left.\begin{aligned} x &= \frac{x'\sin(\beta - \alpha) + y'\sin(\beta - \alpha')}{\sin\beta} - a \\ y &= \frac{x'\sin\alpha + y'\sin\alpha'}{\sin\beta} + b \end{aligned}\right\} \quad (2).$$

These are the formulas most general in the transformation of co-ordinates of which it is easiest to deduce particular corresponding formulas to all the positions of the new origin and to the different directions of the new axes with respect to the old ones, in giving to a, b, these suitable values, positive or negative, and to the angles α and α' all values from o up to π.

2. To pass from a system of rectangular to a system of oblique axes.

In this case, let $\theta = \frac{\pi}{2}$, $\therefore \sin\theta = 1$, $\sin(\theta - \alpha) = \cos\alpha$, and $\sin(\theta - \alpha') = \cos\alpha'$; consequently, we have

$$\left.\begin{array}{l} x = x'\cos\alpha + y'\cos\alpha' + a \\ y = x'\sin\alpha + y'\sin\alpha' + b \end{array}\right\} \cdots\cdots\cdots\cdots (3).$$

3. To pass from one rectangular system to another.

In that case, $\theta = \frac{\pi}{2}$, $\quad \alpha' = \frac{\pi}{2} + \alpha$,

$$\sin\theta = 1, \quad \sin(\theta - \alpha) = \cos\alpha,$$

$$\sin(\theta - \alpha') = \sin\left(\frac{\pi}{2} - \frac{\pi}{2} - \alpha\right) = -\sin\alpha,$$

$$\sin\alpha' = \sin\left(\frac{\pi}{2} + \alpha\right) = \cos\alpha,$$

$$\therefore \left.\begin{array}{l} x = x'\cos\alpha - y'\sin\alpha + a \\ y = y'\cos\alpha + x'\sin\alpha + b \end{array}\right\} \cdots\cdots\cdots\cdots (4).$$

This is a very useful transformation.

4. To change from an oblique to a rectangular system.

In the general formula $\alpha' = \frac{\pi}{2} + \alpha$, $\sin\alpha' = \cos\alpha$,

$$\sin(\theta - \alpha') = \sin\left(\theta - \frac{\pi}{2} - \alpha\right) = -\cos(\theta - \alpha),$$

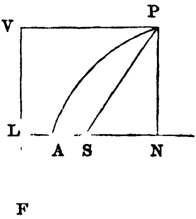
$$\left.\begin{array}{l} \therefore x = \dfrac{x'\sin(\theta - \alpha) - y'\cos(\theta - \alpha)}{\sin\theta} + a \\ \quad y = \dfrac{x'\sin\alpha + y'\cos\alpha}{\sin\theta} + b \end{array}\right\} \cdots (5).$$

If the angle α be negative, then its sine will be negative, but the cosine will remain positive; we have the following modification, which will be found useful;

$$\left.\begin{array}{l} x = \dfrac{x'\sin\alpha' - y'\cos\alpha'}{\sin(\alpha' - \alpha)} \\ y = \dfrac{y'\cos\alpha - x'\sin\alpha}{\sin(\alpha' - \alpha)} \end{array}\right\} \cdots\cdots\cdots\cdots (6).$$

SECTION IV.—PARABOLA.

(25.) If there be taken the fixed line VF, and a fixed point S, then if SP = VP, the perpendicular distance of P from the line VF, the locus of P, is the *parabola*. Through S draw the perpendicular LS, and bisect it in A, then it is evident that A is a point in the curve.



To find the equation to the parabola. Referring to rectangular co-ordinates, let A be the origin, $AN = x$, $NP = y$, $LA = AS = a$;

then $VP = LN = SP$,

$$\therefore\ LN^2 = (x + a)^2 = S'P^2 = y^2 + (x - a)^2, \quad \therefore\ y^2 = 4ax.$$

If S be the origin, then it is evident that $y^2 = 4a(x + a)$, for in that case $SN = x$.

If L be the origin, then $y^2 = 4a(x - a)$, since LN is then $= x$.

Polar Equations.

If S be the pole, $ASP = \theta$,

$$\text{then } r = SP = LN = 2a - 2r\cos\theta, \quad \therefore\ r = \frac{2a}{1 + \cos\theta},$$

$$\text{or } = \frac{a}{\sin^2\frac{\theta}{2}}. \quad \text{(Hann's Trigonometry, page 27.)}$$

It is evident that since $4a$ may be made to equal any constant quantity, the equation to the parabola may be written $y^2 = px$. If $x = a$, then the corresponding value of y is $2a$, therefore the double ordinate through the point is $= 4a$. This double ordinate is generally termed the *parameter*, or *latus rectum*.

PROPERTIES OF THE PARABOLA.

Properties deduced from the Tangent.

(26.) The equation to a straight line passing through (y', x'), (y'', x''), is

$$y - y' = \frac{y' - y''}{x' - x''}(x - x') \ldots\ldots\ldots\ldots\ldots (1),$$

and by the property of the parabola

$$y'^2 = 4ax' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (2),$$

$$\text{and } y''^2 = 4ax'' \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3);$$

therefore, subtracting (3) from (2),

$$(y' + y'')(y' - y'') = 4a(x' - x''),$$

$$\therefore \frac{y' - y''}{x' - x''} = \frac{4a}{y' + y''},$$

hence, by substituting (1), becomes

$$y - y' = \frac{4a}{y' + y''}(x - x').$$

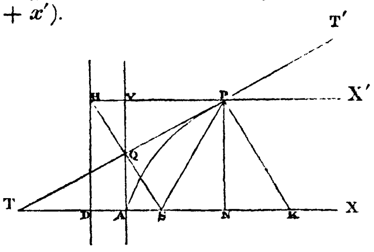
Now, let the points (x', y'), (x'', y''), approach indefinitely near to one another, then the secant joining these two points becomes a tangent, and its equation is

$$y - y' = \frac{2a}{y'}(x - x') \quad \ldots\ldots\ldots\ldots\ldots \quad (4).$$

This equation may be put under a different form for multiplying by y' and transposing,

$$\begin{aligned} yy' &= 2ax - 2ax' + y'^2 \\ &= 2ax - 2ax' + 4ax' \\ &= 2a(x + x') \quad \ldots\ldots\ldots\ldots \quad (5), \end{aligned}$$

an equation closely analogous to the equation of the curve, and only differing from it in this, that y^2 and $2x$ are replaced by yy' and $(x + x')$.



In the annexed figure, suppose P to be a point in the curve $AN = x$, $NP = y$, PT a tangent to the curve at P, cutting the axis of x at T. Then NT is called the *subtangent*, and if PK, perpendicular to PT, be drawn cutting the axis of x at K, PK is termed a *normal*, NK the *subnormal*.

Now, from (5), when $y = 0$, $x = -x'$, therefore, $NT = 2x'$, therefore, the subtangent is equal to twice the abscissa, a very useful and important property.

Since the normal is at right angles to the tangent from (4),

(6), $y - y' = \frac{-y'}{2a}(x - x')$ is the equation to the normal.

To find an expression for the length of the subnormal NK, put $y = 0$ in (6), $\therefore x - x' = 2a = NK$, a *constant*

length of tangent is $\sqrt{NP^2 + NT^2} = 2\sqrt{ax' + x'^2}$, (7),

length of normal is $\sqrt{NP^2 + NK^2} = 2\sqrt{ax' + a^2}$, (8).

We now proceed to the investigation of some properties of the parabola.

When $x' = a$, that is, when the tangent passes through the extremity of the parameter, the tangent and normal are equal, and also the subtangent and subnormal,

$$\text{equation of tangent is then } y - 2a = (x - a),$$

$$\therefore y = (x + a).$$

Therefore, any ordinate to the focal tangent = rad. vector of the point where the ordinate cuts the curve, so that the focal tangent cuts off from the tangent through the vertex a part equal to the distance of the vertex from the focus.

Draw the line HP perpendicular upon the directrix.

Then $SP = HP = DN = TS$, and HTSP is a parallelogram,

$$\cos HPT = \frac{HP^2 + PT^2 - ST^2}{2\,HP \cdot PT} = \frac{PT^2}{2 \cdot HP \cdot PT} = \frac{PT^2}{2\,SP\ PT},$$

$$= \frac{PT}{2\,SP}.$$

The triangle HPS being isosceles, PT bisects the angle HPS, and it also bisects HS at right angles in Q; the point Q is always on the axis of ordinates;

$$\therefore \angle HPT = \angle SPT,$$

therefore, the tangent at any point bisects the angle made by two straight lines, one drawn to the focus, and the other perpendicular to the directrix.

$$ST = AS + AT = a + x,$$

$$SK = SN + NK = x - a + 2a = x + a,$$

$$\therefore ST = SK, \text{ but } ST = SP,$$

$$\therefore SP = ST = SK.$$

From P draw PX′ parallel to the axis

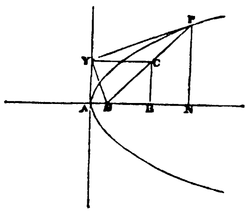
$$T'PX' = PTS = SPT,$$

$$KPX' = SPK.$$

The radius vector and the diameter at the point of contact are equally inclined to the tangent.

These are some of the principal parabolic properties, others will be referred to afterwards.

(27.) The focus S and any point P being joined, and a circle described on SP as diameter, this circle shall touch the axis AY.



Through P draw a tangent cutting axis of (y) in Y; through Y draw YC perpendicular to AY, then from equation to tangent

$$yy' = 2a(x + x'),$$

and equation to AY, $x = 0$,

$$\text{we find AY} = y = \frac{2ax'}{y'} = \frac{y'}{2}$$

(since $x'y'$ is a point in parabola),

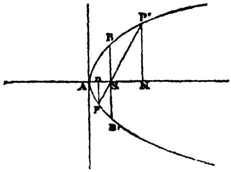
$$\therefore BC = \frac{PN}{2}, \quad \therefore SC = CP,$$

or C is the centre of circle, and $CY = CP$, for the angle

CPY = the angle CYP; but CY is perpendicular to AY, therefore the circle touches AY.

(28.) Any chord Pp is drawn through S, the focus to a parabola; shew that

$$4\,SP\,.\,Sp = L\,(SP + Sp),$$



L being the (latus rectum),

$$SP = AN + AS,$$

$$Sp = An + AS;$$

$$\text{and } SP : Sp :: SN : nS$$

$$:: (AN - AS) : (AS - An)$$

$$:: (SP - 2\,AS) : (2\,AS - Sp),$$

$$\therefore\ SP\,(2AS - Sp) = Sp\,(SP - 2AS),$$

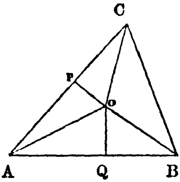
$$\therefore\ 2\,SP\,.\,Sp = 2\,AS\,(SP + Sp).$$

But $4\,AS = BB' = L$,

$$\therefore\ 4\,SP\,.\,Sp = L\,.\,(SP + Sp).$$

(29.) In a plane triangle ABC, if $\tan A \tan \frac{B}{2} = 2$, and AB be fixed, the locus of C is a parabola vertex A and focus B.

Take A for the pole, AB the initial line, AC $= r$. Find O, the centre of inscribed circle, and join OA, OB, OC, and draw perpendiculars OP, OQ, to the sides of triangle.



$$\text{Now, } r = AP + PC = OP\left\{\cot\frac{A}{2} + \cot\frac{C}{2}\right\}$$

$$= OP\left\{\cot\frac{A}{2} + \tan\frac{A+B}{2}\right\},$$

$$\text{and } OP = OQ = QB \tan\frac{B}{2} = (AB - AQ)\tan\frac{B}{2}$$

$$= \left(AB - OQ\cot\frac{A}{2}\right)\tan\frac{B}{2},$$

$$\therefore\ OQ\left(1 + \cot\frac{A}{2}\tan\frac{B}{2}\right) = AB\tan\frac{B}{2},$$

$$\therefore\ OP = OQ = \frac{AB\tan\frac{B}{2}}{1 + \cot\frac{A}{2}\tan\frac{B}{2}},$$

$$\therefore\ r = \frac{AB\tan\frac{B}{2}}{1 + \cot\frac{A}{2}\tan\frac{B}{2}}\left\{\cot\frac{A}{2} + \frac{\tan\frac{A}{2} + \tan\frac{B}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}}\right\}$$

$$= \frac{AB\tan\frac{B}{2}}{1 + \cot\frac{A}{2}\tan\frac{B}{2}}\left(\frac{\cot\frac{A}{2} + \tan\frac{A}{2}}{1 - \tan\frac{A}{2}\tan\frac{B}{2}}\right).$$

But $\tan A \,.\, \tan\frac{B}{2} = 2$,

$$\therefore\ \tan\frac{B}{2} = \frac{1 - \tan^2\frac{A}{2}}{\tan\frac{A}{2}},$$

$$\therefore\ \tan\frac{B}{2}\cot\frac{A}{2} = \frac{1 - \tan^2\frac{A}{2}}{\tan^2\frac{A}{2}},$$

$$\therefore\ 1+\cot\frac{A}{2}\tan\frac{B}{2}=1+\frac{1-\tan^2\frac{A}{2}}{\tan^2\frac{A}{2}}=\cot^2\frac{A}{2},$$

$$\text{and } 1-\tan\frac{A}{2}\tan\frac{B}{2}=1-\left(1-\tan^2\frac{A}{2}\right)=\tan^2\frac{A}{2},$$

$$\therefore\ r=AB\,\frac{1-\tan^2\frac{A}{2}}{\tan\frac{A}{2}}\left(\cot\frac{A}{2}+\tan\frac{A}{2}\right),$$

$$=AB\left(1-\tan^2\frac{A}{2}\right)\left(1+\cot^2\frac{A}{2}\right)$$

$$=AB\cdot\frac{2\cos A}{1+\cos A}\cdot\frac{2}{1-\cos A}.$$

Putting for AB (c), and for A . (θ), we get

$$r=\frac{4c\,.\cos\theta}{1-\cos^2\theta}=\frac{4c\,.\cos\theta}{\sin^2\theta},$$

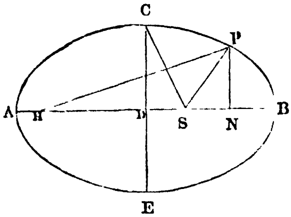
which is the equation to a parabola vertex A and focus B.

Section V.—Ellipse.

(30) An ellipse is a curve in which, if P be any point, SP + PH is a constant quantity, S and H being two fixed points.

Bisect S, H in D, draw a perpendicular through D, and make $DB = DA = a$. With centre S, or H, and radius $= a$, cut the perpendicular in the points C and E. It is evident that A, B, C, E, are points in the curve, and that the curve is divided into four equal and similar parts by the lines CDE, ADB.

Let DS : DA : : e : 1, therefore, $DS = ae$. If $e = 0$, $DS = 0 = DH$, and therefore the points S and H coincide



with D and the ellipse becomes a circle. If $e = 1$, then $DS = DH = DA = DB$, and it is evident that the points of the line AB are the only ones which satisfy the required condition, since $PA + PB > AB$, whenever the point P is above the line AB. It is evident that $e > 1$ is inconsistent with the conditions of construction.

This quantity e is generally termed the eccentricity.

To find the equation of the curve, let D be the origin,

$$DN = x, \ NP = y, \ AD = a, \ CD = b,$$

$$SP^2 = y^2 + SN^2 = y^2 + (x - ae)^2 = y^2 + x^2 + a^2e^2 - 2aex,$$

$$HP^2 = y^2 + HN^2 = y^2 + (x + ae)^2 = y^2 + x^2 + a^2e^2 + 2aex,$$

$$\therefore \ HP^2 - SP^2 = (HP + SP) . (HP - SP) = 4aex,$$

$$\therefore \ HP - SP = 2ex \ \ldots\ldots\ldots\ldots \ (1),$$

$$HP^2 + SP^2 - 2HP . PS = 4e^2x^2,$$

$$\text{and } HP^2 + SP^2 = 2(y^2 + x^2 + a^2e^2),$$

$$\therefore \ 2HP . PS = 2y^2 + (2 - 4e^2)x^2 + 2a^2e^2,$$

$$HP^2 + SP^2 + 2HP . PS = 4\{y^2 + (1 - e^2)x^2 + a^2e^2\}$$

$$\text{but } (HP + SP)^2 = 4a^2,$$

$$\therefore (1.) \ y^2 = (1 - e^2)(a^2 - x^2)$$

is the equation to the ellipse, the origin being at the centre;

$$(1 - e^2) = \frac{DA^2 - DS^2}{DA^2} = \frac{DC^2}{DA^2} = \frac{b}{a}, \ \text{since}$$

$$e = \frac{DS}{DA}, \ \text{and } SC = DA;$$

$$\therefore \ y^2 = \frac{b^2}{a^2}(a^2 - x^2), \ \text{or } \frac{y^2}{b^2} + \frac{x^2}{a^2} = 1,$$

is the equation, a and b being termed respectively the semi-axis major and the semi-axis minor.

If we had taken CD as the axis of x, the equation would have been

$$\frac{x^2}{b^2} + \frac{y^2}{a^2} = 1.$$

If the origin be taken at S, SN $= x$, or NP $= y$, and for x in equation (1) substitute $(x + ae)$,

$$\therefore\ y^2 = (1 - e^2)(a^2 - a^2e^2 - x^2 - 2aex)$$

$$= (1 - e^2)(b^2 - x^2 - 2aex) \quad \ldots\ldots\ldots\ (2).$$

If H be the origin,

$$y^2 = (1 - e^2)(b^2 + x^2 + 2aex) \ldots\ldots\ldots\ (3).$$

If A be the origin,

$$y^2 = (1 - e^2)(2ax - x^2) \ldots\ldots\ldots\ldots\ldots\ (4).$$

Some properties of the ellipse are directly deducible from these equations, from (1),

$$\text{NP}^2 = \frac{\text{CD}^2}{\text{BD}^2} \cdot (\text{BD}^2 - \text{DN}^2)$$

$$= \frac{\text{CD}^2}{\text{BD}^2} \cdot (\text{BD} + \text{DN}) \cdot (\text{BD} - \text{DN}) = \frac{\text{CD}^2}{\text{BD}^2} \cdot \text{AN} \cdot \text{NB},$$

$$\therefore\ \text{NP}^2 \cdot \text{BD}^2 = \text{CD}^2 \cdot \text{AN} \cdot \text{NB} \ \ldots\ldots\ (\text{A}).$$

$$\text{from (4)},\ y^2 = \frac{b^2}{a^2} \cdot (2a - x) \cdot x,$$

$$\therefore\ \text{NP}^2 = \frac{\text{CD}^2}{\text{DB}^2} \cdot \text{BN} \cdot \text{AN},$$

$$\therefore\ \text{NP}^2 \cdot \text{DB}^2 = \text{CD}^2 \cdot \text{AN} \cdot \text{BN} \ldots\ldots\ (\text{B}).$$

We may also deduce the polar equations to the ellipse.

Let D be the pole DN $= x$, NP $= y$, DP $= r$, PDN $= \theta$,

$$\therefore\ x = r\cos\theta,\quad y = r\sin\theta,$$

$$\therefore \text{ from equation (1) } r^2\left(\frac{\sin^2\theta}{b^2} + \frac{\cos^2\theta}{a^2}\right) = 1,$$

$$\therefore\ r^2 = \frac{b^2 a^2}{a^2\sin^2\theta + b^2\cos^2\theta} = \frac{b^2 a^2}{a^2 - (a^2 - b^2)\cos^2\theta},$$

$$\therefore\ r = \frac{b}{\sqrt{1 - e^2\cos^2\theta}}.$$

If S be the pole, SP $= r$, HP $= 2a - r$, PSH $= \theta$, then

$$(2a - r)^2 = r^2 + 4a^2e^2 - 4aer\cos\theta,$$

$$\therefore\ 4a^2 + r^2 - 4ar = 4a^2e^2 + r^2 - 4aer\cos\theta,$$

$$\therefore\ a^2 - ar = a^2e^2 - aer\cos\theta,$$

$$\therefore\ a - r = ae^2 - er\cos\theta,$$

$$r = \frac{a(1 - e^2)}{1 - e\cos\theta}.$$

(31). We shall next proceed to the investigation of the equation to a tangent at any point P in the ellipse, since the principal properties depend upon the tangent.

Let us take a point whose co ordinates are x', y'

Take a contiguous point whose co-ordinates are x'', y''.

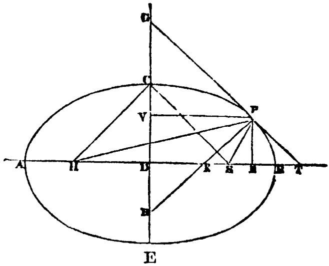
We have, therefore, from (1),

$$y'^2 = (1 - e^2)(a^2 - x'^2)$$

$$y''^2 = (1 - e^2)(a^2 - x''^2),$$

$$\therefore\ y'^2 - y''^2 = -(1 - e^2)(x'^2 - x''^2),$$

$$\therefore\ (y' + y'')(y' - y'') = -(1 - e^2)(x' + x'')(x' - x'').$$



The equation to a line through the two points is (X and Y being the co-ordinates of any point in the line)

$$Y - y' = \frac{y' - y''}{x' - x''}(X - x') = -(1 - e^2)\frac{x' + x''}{y' + y''}(X - x').$$

Now, this line is a secant, but if we suppose the points to be taken indefinitely near one another, ultimately this line becomes a tangent.

Hence, making $x' = x''$, $y' = y''$, we obtain

$$Y - y' = -(1 - e^2)\frac{x'}{y'}(X - x')$$

as the equation to the tangent.

We must now find the values of the subtangent, normal, subnormal, &c.

$$Y - y' = -(1 - e^2)\frac{x'}{y'}(X - x')$$

the equation to the tangent.

Let $Y = 0, \quad \therefore \; -y' + -(1 - e^2)\frac{x'}{y'}(X - x'),$

$$\text{and } X - x' = \frac{1}{1 - e^2} \cdot \frac{y'^2}{x'}$$

$$X = \frac{y'^2 + (1 - e^2)\,x'^2}{(1 - e^2)\,x'} = \frac{(1 - e^2)\,a^2}{(1 - e^2)\,x'} = \frac{a^2}{x'} = DT,$$

$$\therefore\ DT = \frac{BD^2}{DN}.$$

(The student must observe, that in deducing the equation to the curve, we called DN and DP, x and y, respectively, since we were not investigating a relation between any two *particular* co-ordinates, but a relation between each pair. Here, however, P being supposed to be any *particular* point, it is most proper to denote its co-ordinates by x' and y'.)

$$\therefore\ DT \, . \, DN = BD^2, \text{ or } DN : BD :: BD : DT.$$

The subtangent $NT = DT - x' = \dfrac{a^2 - x'^2}{x'}$,

$$\therefore\ NT = \frac{(a + x')\,(a - x')}{ND} = \frac{AN \, . \, NB}{ND},$$

$$\therefore\ NT \, . \, ND = AN \, . \, NB,$$

$$\text{also, } BT = \frac{a^2 - a x'}{x'} = \frac{NB \, . \, DB}{ND},$$

$$\therefore\ BT \, . \, ND = NB \, . \, DB.$$

In the equation to the tangent, make $X = 0$;

$$\therefore\ Y - y' = -(1 - e^2)\frac{x'}{y'}(-x'),\ \therefore\ Y = \frac{y'^2 + (1 - e^2)\,x'^2}{y'}$$

$$= \frac{(1 - e^2)\,.\,a^2}{y'} = \frac{b^2}{y'} = DG,$$

$$\therefore\ DG \, . \, NP = DC^2;$$

$$\text{also, } CG = \frac{b^2}{y'} - b = \frac{b\,.\,(b - y')}{y'} =$$

(if PV be drawn parallel to AB)

$$\frac{CV \, . \, CD}{VD},\quad \therefore\ VD \, . \, CG = CV \, . \, CD$$

A similar relation to BT . ND = NB . DB; in fact, the two relations together may be thus expressed:—

If a tangent be drawn at any point, if, also, perpendiculars from that point be let fall upon the principal diameters, the distances between the extreme points of the diameters, and the points where the tangent meets the diameters, are to the distances between the extreme points and the feet of the perpendiculars as the diameters are to the distance between the feet of the perpendiculars and the centre.

(32) Since the normal is perpendicular to the tangent, and passes through x', y',

$$\mathrm{Y} - y' = \frac{y'(\mathrm{X} - x')}{x'(1 - e^2)} \text{ is its equation.}$$

Making $\mathrm{Y} = 0$, $\quad -y' = \frac{y'}{x'} \cdot \frac{1}{(1 - e^2)}(\mathrm{X} - x')$,

$$\therefore \; -1 = \frac{\mathrm{X} - x'}{(1 - e^2)x'}, \quad \text{and } \mathrm{X} = e^2 x' = \mathrm{DK},$$

$$\therefore \; \frac{\mathrm{DK}}{\mathrm{DN}} = \frac{\mathrm{DS}^2}{\mathrm{DB}^2}, \quad \mathrm{DK} \,.\, \mathrm{DB}^2 = \mathrm{DN} \,.\, \mathrm{DS}^2.$$

Again, the subnormal $\mathrm{NK} = (1 - e^2)x'$

$$= \frac{b^2}{a^2}x' = \frac{\mathrm{DC}^2}{\mathrm{DB}^2} \,.\, \mathrm{DN},$$

$$\therefore \; \mathrm{NK} : \mathrm{DN} :: \mathrm{DC}^2 : \mathrm{DB}^2.$$

Making $\mathrm{X} = 0$, $\quad \mathrm{Y} - y' = \frac{y'}{x'} \cdot \frac{1}{1 - e^2} \cdot -x'$,

$$\therefore \; \mathrm{Y} = \left(1 - \frac{1}{1 - e^2}\right)y' = \frac{-e^2 y'}{0 - e^2} = \frac{-\mathrm{DS}^2 y'}{b^2},$$

$$\therefore \; \mathrm{DR} = \frac{\mathrm{DS}^2 \,.\, \mathrm{PN}}{\mathrm{DC}^2}, \quad \therefore \; \frac{\mathrm{DR}}{\mathrm{PN}} = \frac{\mathrm{DS}^2}{\mathrm{DC}^2},$$

$$RE = b - \frac{e^2 y'}{(1 - e^2)} = b - \frac{DS^2 y'}{b^2},$$

$$\therefore\ RE = b - \frac{(a^2 - b^2) y'}{b^2}$$

$$= b + y' - \frac{a^2 y'}{b^2} = VE - \frac{BD^2 . NP}{DC^2},$$

$$\therefore\ VR = \frac{BD^2 . NP}{DC^2},$$

$$\therefore\ VR : VD :: BD^2 \cdot DC^2.$$

The equations to the tangent and normal may be put under the following form by substituting for $1 - e^2$ its value, viz., $\frac{b^2}{a^2}$,

$$Y - y' = - \frac{b^2}{a^2} \cdot \frac{x'}{y'} (X - x'), \text{ equation to the tangent,}$$

$$\therefore\ a^2 Y y' - a^2 y'^2 = - b^2 X x' + b^2 x'^2,$$

or $a^2 Y y' + b^2 X x' = a^2 y'^2 + b^2 x'^2$, but $a^2 y'^2 + b^2 x'^2 = a^2 b^2$,

$$\therefore\ a^2 Y y' + b^2 X x' = a^2 b^2;$$

$$Y - y' = \frac{a^2}{b^2} \cdot \frac{y'}{x'} (X - x'), \text{ equation to the normal.}$$

(33.) To prove that the normal bisects the angle contained by the focal distances.

Since $DS = DH = ae$, and $DK = e^2 x'$,

$$HK = HD + DK = ae + e^2 x' = e(a + ex'),$$

$$SK = DS - DK = ae - e^2 x' = e(a - ex');$$

but, page 44, $HP + PS = 2a$, and $HP - PS = 2ex'$,

$$\therefore\ HP = a + ex', \text{ and } PS = a - ex';$$

hence, by substituting these values in the above, we have

$$HK = e . HP, \text{ and } SK = e . PS;$$

by division, $\frac{\text{HK}}{\text{SK}} = \frac{e \,.\, \text{HP}}{e \,.\, \text{PS}} = \frac{\text{HP}}{\text{PS}}$,

$$\text{HK} : \text{SK} :: \text{HP} : \text{PS}.$$

hence the angle HPS is bisected by the normal.

Cor. It follows from this that the angle HPG is equal to the angle SPT, that is, the tangent makes equal angles with the focal distances.

Ellipse referred to any System of Conjugates.

(31.) The characteristic property of every system of conjugate diameters is, that each bisects every chord drawn parallel to the other; it follows, that if we refer a curve to any similar system, the new equation ought only to contain the square of the co-ordinates, and a constant quantity.

The equation to the ellipse referred to its centre and principal diameter is

$$a^2 y^2 + b^2 x^2 = a^2 b^2.$$

Substituting, in this equation, for x and y, $x \cos \alpha + y \cos \alpha'$, $x \sin \alpha + y \sin \alpha'$ respectively, there results,

$$(a^2 \sin^2 \alpha' + b^2 \cos^2 \alpha') y^2 + 2 \{a^2 \sin \alpha \sin \alpha' + b^2 \cos \alpha \cos \alpha'\} xy$$
$$+ (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha') x^2 = a^2 b^2;$$

this equation belonging to an oblique system having the same origin.

But by hypothesis the equation ought to contain only the squares of the variables and known quantities, we must therefore put the coefficient of $xy = 0$,

$$\therefore \; a^2 \sin \alpha \sin \alpha' + b^2 \cos \alpha \cos \alpha' = 0 \ldots\ldots\ldots\ldots (1).$$

Hence the above equation is reduced to

$$(a^2 \sin^2 \alpha' + b^2 \cos^2 \alpha') y^2 + (a^2 \sin^2 \alpha + b^2 \cos^2 \alpha) x^2 = a^2 b^2 \ldots (2)$$

Dividing equation (1) by $\cos \alpha \cos \alpha'$, we have

$$a^2 \tan \alpha \tan \alpha' + b^2 = 0, \qquad \therefore \; \tan \alpha \tan \alpha' = -\frac{b^2}{a^2}.$$

Therefore, as we have only one equation to determine the two angles α, α', it follows that the number of systems of conjugate diameters is infinite.

Making successively $y = 0$, $x = 0$, in equation (2), and there results,

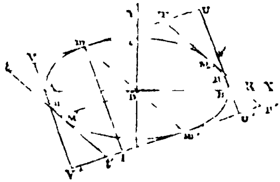
$$\text{for } y = 0 \text{ .... } x^2 = \frac{a^2 b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha};$$

$$\text{and for } x = 0 \text{ .... } y^2 = \frac{a^2 b^2}{a^2 \sin^2 \alpha' + b^2 \cos^2 \alpha'}$$

Which expressions are the squares of the semi-conjugate diameters. Calling these conjugates $2a'$, $2b'$, respectively, we have these relations;—

$$a'^2 = \frac{a^2 b^2}{a^2 \sin^2 \alpha + b^2 \cos^2 \alpha},$$

$$\therefore\ a^2 \sin^2 \alpha + b^2 \cos^2 \alpha = \frac{a^2 b^2}{a'^2},$$



$$\text{and } b'^2 = \frac{a^2 b^2}{a^2 \sin^2 \alpha' + b^2 \cos^2 \alpha'},$$

$$\therefore\ a^2 \sin^2 \alpha' + b^2 \cos^2 \alpha' = \frac{a^2 b^2}{b'^2}.$$

Substituting these values in equation (2), there results

$$a'^2 y^2 + b'^2 x^2 = a'^2 b'^2$$

for the equation to the ellipse, referred to any system of conjugate diameters.

Consequently, the centre being the origin, the equation to the ellipse preserves always the same form for any system of conjugates.

Since for $x = \pm a'$, $y^2 = 0$, and for $y = \pm b'$, $x^2 = 0$, it follows that the two straight lines drawn from the points M M′ $m\,m'$, parallel to the two diameters, are tangents to the curve.

As this equation is of the same form as that betwee rectangular co-ordinates, it is clear that all properties whic are independent of the inclination of the co-ordinates will b equally true for the conjugate diameters as for the princip diameters or axes.

The equation to the tangent is

$$a'^2 Y y'' + b'^2 X x'' = a'^2 b'^2,$$

where (x'', y'') are the co-ordinates of the point of contact.

If in this equation we make $Y = 0$; $x = \frac{a'^2}{x''}$, which is tl value of OR, the point where the tangent meets the axis x; and if from this distance we subtract x'', or OP, we ha the subtangent PR,

$$PR = \frac{a'^2}{x''} - x'' = \frac{a'^2 - x''^2}{x''};$$

this is the same result as obtained at art. 31.

Also as at page 48. The rectangle of the ordinate of tl ellipse at the point of contact and the ordinate of the tange at the centre, is equal to the square of that semi-diamet which is taken for the axis of ordinates.

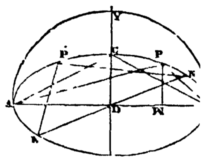
Also, that the rectangle of the abscissa of the point contact, and of the point where the tangent meets the ax of abscissæ, is equal to the square of that semi-diamet which is taken for the axis of abscissæ.

On Supplementary Chords.

(35.) If from the extremities A and B of the major ax two right lines be drawn to any point of the curve, these lin are called supplementary chords.

The equation of the right line BP, passing through the point B, whose co-ordinates are $y = 0$, and $x = a$, is

$$y = m(x - a).$$



The equation of the right line AP, passing through the point $(y = 0, x = -a)$, is

$$y = m'(x + a).$$

Now, let x', y', be the co-ordinates of the point P; and, as this point is in both lines and also in the curve, we have the following relations:—

$$y' = m(x' - a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$y' = m'(x' + a) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$
$$a^2 y'^2 + b^2 x'^2 = a^2 b^2 \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

From (1) $m = \dfrac{y'}{x' - a}$, and from (2) $m' = \dfrac{y'}{x' + a}$,

by multiplication $mm' = \dfrac{y'^2}{x'^2 - a'^2} \ldots\ldots\ldots\ldots (4).$

From (3) we have $-b^2 x'^2 + a^2 b^2 = a^2 y'^2$

$$-b^2(x'^2 - a^2) = a^2 y'^2,$$

$$\therefore \frac{y'^2}{x'^2 - a^2} = -\frac{b^2}{a^2},$$

but from (4) $m\,m' = \dfrac{y'^2}{x'^2 - a^2}$,

$$\therefore mm' = -\frac{b^2}{a^2} \ldots\ldots\ldots\ldots\ldots\ldots (5).$$

The same relation obtains if we draw the supplementary chords from the extremities of any diameter whatever, E, E′; for let x'', y'', be the co-ordinates of the point E, those of the point E′ will be x'', y''; and the equation of the lines drawn from the points E, E′, to any point P′ whatever in the curve will be

$$y - y'' = m(x - x''),$$
$$y + y'' = m'(x + x''),$$

and, as the point P′, or (x', y'), is in both these lines, we have

$$y' - y'' = m(x' - x''),$$
$$y' + y'' = m'(x' + x''),$$
$$\therefore mm' = \frac{y'^2 - y''^2}{x'^2 - x''^2};$$

but the points P′, E, E′, are also in the curve, hence

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2,$$

$$a^2 y''^2 + b^2 x''^2 = a^2 b^2,$$

$$\therefore \frac{y'^2 - y''^2}{x'^2 - y''^2} = -\frac{b^2}{a^2};$$

but from above $\dfrac{y'^2 - y''}{x'^2 - y''^2} = m\,m'$,

$$\therefore m\,m' = -\frac{b^2}{a^2},$$

which is the same result as before.

(36.) To find the angle contained by the supplementary chords, we have

$$\tan P = \frac{m - m'}{1 + m\,m'},$$

where m is the tangent of the angle PBX, and m' the tangent of the angle PAX; we have already shown that

$$m = \frac{y'}{x' - a}, \quad m' = \frac{y'}{x' + a},$$

$$\therefore \frac{m - m'}{1 + m\,m'} = \frac{\dfrac{y'}{x' - a} - \dfrac{y'}{x' + a}}{1 + \dfrac{y'^2}{x'^2 - a^2}} = \frac{2\,a\,y'}{x'^2 - a^2 + y'^2} \ldots (6);$$

and, since the lines meet in the curve, the equation of the curve must be satisfied

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2, \text{ or}$$

$$x'^2 - a^2 = -\frac{a^2 y'^2}{b^2};$$

hence, by substitution, we obtain

$$\frac{m - m'}{1 + mm'} = \frac{2ay'}{-\frac{a^2y'^2}{b^2} + y'^2} = \frac{-2ab^2}{(a^2 - b^2)y'} \quad \text{.........} \quad (7).$$

It is clear from this result, that if we consider any point P of the curve above AB, in which case y' is positive, the tangent of the angle APB is negative; then this angle is necessarily obtuse, as it ought to be, since all the points of the ellipse are within the circumference of a circle described on AB, as a diameter.

As the obtuse angle increases its tangent decreases; hence, when the tangent is least the angle is greatest, and equation (7) will be least when y' is greatest; or when $y' = b$, this value substituted in (7) gives

$$\tan P = \frac{-2ab^2}{b(a^2 - b^2)} = \frac{-2ab}{a^2 - b^2}.$$

On this supposition, the values of m and m' become

$$m = \frac{-b}{a}, \quad m' = \frac{b}{a}, \quad \text{since } y' = b, \text{ gives } x = 0.$$

From the above, it appears that supplementary chords drawn from the extremities of the major axis make the greatest angle when they meet in the point C, the extremity of the minor axis.

(37.) We can deduce some other useful properties by transforming the oblique axes to rectangular axes,

$$x = \frac{x \sin \alpha' - y \cos \alpha'}{\sin(\alpha' - \alpha)},$$

$$y = \frac{y \cos \alpha - x \sin \alpha}{\sin(\alpha' - \alpha)}.$$

Substituting the values in equation (52) page 74, we have

$$(a'^2 \cos^2\alpha + b'^2 \cos^2\alpha)y^2 + (-2a'^2 \sin\alpha \cos\alpha - 2b'^2 \sin\alpha' \cos\alpha')xy$$
$$+ (a'^2 \sin^2\alpha + b'^2 \sin^2\alpha)x^2 = a'^2b'^2 \sin^2(a' - \alpha),$$

which must be the same as $a^2y^2 + b^2x^2 = a^2b^2$, therefore, the following conditions must have place,

$$a'^2 \cos^2\alpha + b'^2 \cos^2\alpha' = a^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (a),$$

$$a'^2 \sin^2\alpha + b'^2 \sin^2\alpha' = b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (b),$$

$$a'^2 \sin\alpha \cos\alpha + b'^2 \sin\alpha' \cos\alpha' = 0 \quad \ldots\ldots\ldots \quad (c),$$

$$a'^2 b'^2 \sin^2(\alpha' - \alpha) = a^2 b^2 \quad \ldots\ldots \quad (d);$$

add (a) and (b), and there results

$$a'^2 + b'^2 = a^2 + b^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (e);$$

or the sum of the squares of any system of conjugate diameters is equal to the sum of the squares of the principal diameters.

Equation (d), multiplied by 4, gives

$$4\,a'b' \sin(\alpha' - \alpha) = 4\,a\,b,$$

and since $\alpha' - \alpha$ is the inclination of the conjugate axes, the left-hand side of the equation expresses the area of a parallelogram whose sides are those axes, and included angle $\alpha' - \alpha$ (see Hann's "Trigonometry," page 60); the right-hand side gives the rectangle of principal axes; hence, every parallelogram circumscribing an ellipse having the sides parallel to a system of conjugate axes, is equal to the rectangle of the principal axes.

Now, since the area of the parallelogram is

$$4\,\mathrm{MD} \,.\, m\,i,$$

$$\text{we have } 4\,\mathrm{MD} \,.\, m\,i = 4\,ab,$$

$$m\,i = \frac{a\,b}{\mathrm{MD}},$$

but $m\,i$ equals a perpendicular on the tangent **T***t* from **D**, which call p, then

$$p = \frac{a\,b}{\mathrm{MD}}, \quad \text{or } p^2 = \frac{a^2 b^2}{\mathrm{MD}^2} = \frac{a^2 b^2}{a^2 + b^2 - a'^2}$$

$$(\text{since } \mathrm{MD}^2 + m\mathrm{D}^2 = a^2 + b^2).$$

$$\begin{aligned}
\text{Now } m\mathrm{D}^2 &= a^2 + b^2 - \mathrm{DM}^2 \\
&= a^2 + b^2 - (x^2 + y^2) \\
&= a^2 + b^2 - \left\{x^2 + \frac{b^2}{a^2}(a^2 - x^2)\right\} \\
&= a^2 + b^2 - x^2 - b^2 + \frac{b^2}{a^2}x^2 \\
&= a^2 - \left(1 - \frac{b^2}{a^2}\right)x^2 \\
&= a^2 - \left(\frac{a^2 - b^2}{a^2}\right)x^2 \\
&= a^2 - e^2x^2 \\
&= (a + ex)(a - ex);
\end{aligned}$$

but by art. 33, $a + ex = \mathrm{HP}$, and $a - ex = \mathrm{SP}$, fig. p. 47.

$$\therefore\ m\mathrm{D}^2 = \mathrm{SP}\,.\,\mathrm{HP},$$

or the rectangle contained by the focal distances of any point of an ellipse, is equal to the square of the corresponding semi-conjugate diameter.

(38.) In the ellipse

$$p^2 = b^2 \cdot \frac{r}{2a - r},$$

where p is the perpendicular from one of the foci on the tangent: the equation to the tangent is

$$y = \frac{-b^2x}{a^2y'}x + \frac{b^2}{y'},$$

perpendicular from the point $(-ae, 0)$ on this

$$= \frac{\dfrac{-b^2x'}{a^2y'} - ae - \dfrac{b^2}{y'}}{\sqrt{1 + \dfrac{b^4x'^2}{a^4y'^2}}}$$

$$= \frac{-b^2\{aex' + a^2\}}{\sqrt{a^4y'^2 + b^4x'^2}}$$

$$= \frac{-b^2a(a + ex')}{\sqrt{a^2b^2\left\{a^2 - x^2 + \frac{b^2}{a^2}x'^2\right\}}}$$

$$= \frac{-b(a + ex')}{\sqrt{a + ax'}\sqrt{a - ex'}}$$

$$= \frac{-b\sqrt{a + ex'}}{\sqrt{a - ex'}},$$

$$\therefore p^2 = \frac{b^2(a + ex')}{a - ex'}$$

$$= b^2 . \frac{\text{HP}}{\text{SP}},$$

$$\therefore p^2 = b^2 . \frac{r}{2a - r}.$$

In the same manner, if we put p' for the perpendicular from the other focus on the tangent, we have

$$p'^2 = b^2\frac{\text{SP}}{\text{HP}},$$

$$\therefore p^2 . p'^2 = b^2 . \frac{\text{SP}}{\text{HP}} . b^2\frac{\text{HP}}{\text{SP}} = b^4,$$

$$p . p' = b^2.$$

That is, the rectangle of the perpendiculars from the foci on the tangent is equal to the square of the semi-axis minor.

(39) If a circle be described on the major axis of an ellipse, and if Y be the ordinate to the circle, and y the ordinate to the ellipse corresponding to the same abscissa, then

$$\frac{y}{\text{Y}} = \frac{b}{a}.$$

By the equation to the circle

$$Y^2 = a^2 - x^2,$$

$$\text{ellipse } y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

Divide the second by the first, and we have

$$\frac{y^2}{Y^2} = \frac{b^2}{a^2}, \quad \text{or } \frac{y}{Y} = \frac{b}{a};$$

that is, the ordinate of an ellipse is to the ordinate of a circle described upon the major axis, as the minor axis is to the major axis.

It is evident that this gives an easy method of describing an ellipse by points.

(40.) If xy, and $x'y'$ be any two points on the conjugate diameters, then

$$\frac{xx'}{a^2} + \frac{yy'}{b^2} = 0,$$

$$\text{for } \frac{y}{x} = \tan\alpha, \text{ and } \frac{y'}{x'} = \tan\alpha',$$

$$\text{but } \tan\alpha \tan\alpha' = -\frac{b^2}{a^2},$$

$$\therefore \frac{y}{x} \cdot \frac{y'}{x'} = -\frac{b^2}{a^2}$$

$$\frac{yy'}{b^2} = -\frac{xx'}{a^2},$$

$$\text{or } \frac{xx'}{a^2} + \frac{yy'}{b^2} = 0.$$

On the Eccentric Angle in the Ellipse.*

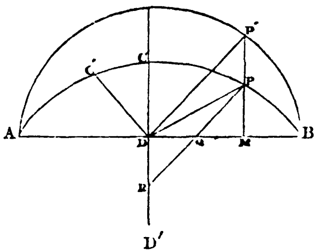
(41.) The co-ordinates of any point xy of an ellipse, may take the form $x = a\cos\phi$, $y = b\sin\phi$.

* O'Brien's "Co-ordinate Geometry," page 111.

If we assume $\frac{x}{a} = \cos\phi$, or, in other words, $\phi = \cos^{-1}\frac{x}{a}$ then, substituting in the equation,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

$$\text{we have } \frac{y^2}{b^2} = 1 - \cos^2\phi = \sin^2\phi,$$



$$\therefore\ y^2 = b^2\sin^2\phi, \text{ or } y = b\sin\phi.$$

Hence, we may assume $x = a\cos\phi$, and $y = b\sin\phi$.

If we describe a circle, AP′B on AB, as a diameter, produce the ordinate MP to meet it in P′, and join P′D; P′DM is the angle whose cosine is $\frac{x}{a}$, for

$$\cos \text{P}'\text{DM} = \frac{\text{DM}}{\text{DP}'} = \frac{\text{DM}}{\text{DB}} = \frac{x}{a};$$

therefore, if we produce the ordinate to meet a circle described on the major axis as a diameter, and draw a line from the point where the ordinate meets the circle to the centre, then ϕ is the angle which that line makes with the major axis.

If PQR be drawn parallel to P′D, meeting CD′ in the point R, and DB in Q, then PR = P′D $= a$, and PQB = P′DB $= \phi$; since $y = b\sin\phi$, we have

$$b\sin\phi = \text{PM} = \text{PQ}\sin\text{PQM} = \text{PQ}\sin\phi,$$

$$\therefore\ \text{PQ} = b.$$

If ϕ and ϕ' be the eccentric angles of the extremities of two conjugate diameters, then $\phi' = \phi + 90°$.

Let (x, y), and (x', y'), be the co-ordinates of the extremities, P and C′ of the two conjugate diameters DP and DC′, ϕ and ϕ' the corresponding eccentric angles,

$$\text{then } x = a\cos\phi,\ y = b\sin\phi,\ x' = a\cos\phi',\ y' = b\sin\phi',$$

$$\text{then, by art. 40, } \cos\phi\cos\phi' + \sin\phi\sin\phi = 0,$$

$$\text{or } \cos(\phi' - \phi) = 0,$$

$$\text{which shews that } \phi' - \phi = 90°, \text{ or } \phi' = \phi + 90°.$$

This property of conjugate diameters, with reference to the eccentric angle, is of great use in problems relating to conjugate diameters.

Cor. Hence

$$x' = a\cos(\phi + 90°) = -a\sin\phi = -\frac{a}{b}y,$$

$$\text{and } y' = b\sin(\phi + 90°) = \quad b\cos\phi = \quad \frac{b}{a}x,$$

which determine the co-ordinates of P and C.

(42.) The properties deduced in art. 37 can be found in a very simple manner, by using the eccentric angle.

Let DP $= r$, C′D $= r'$, PDB $= \theta$, C′DB $= \theta'$,

$$r^2 = x^2 + y^2 = a^2\cos^2\phi + b^2\sin^2\phi \text{ .... } (1);$$

and putting $\phi + 90$ for ϕ, we have

$$r'^2 = a^2\sin^2\phi + b^2\cos^2\phi \text{ .... } (2);$$

add (1) and (2),

$$r^2 + r'^2 = a^2(\sin^2\phi + \cos^2\phi) + b^2(\sin^2\phi + \cos^2\phi)$$

$$= a^2 + b^2 \text{ .... } (4).$$

The area of a parallelogram, whose sides are C′D and PD,

$$= rr'\sin(\theta' - \theta) = rr'(\sin\theta'\cos\theta - \cos\theta'\sin\theta)$$

$$= r\cos\theta \, . \, r'\sin\theta' - r\sin\theta \, . \, r'\cos\theta$$
$$= xy' - x'y$$
$$= ab\sin\phi'\cos\phi - ab\cos\phi'\sin\phi$$
$$= ab(\sin\phi'\cos\phi - \cos\phi'\sin\phi)$$
$$= ab\sin(\phi' - \phi) = ab\{\text{since } \sin(\phi' - \phi) = \sin 90\} = 1.$$

(43.) Given the axes of an ellipse and the inclination of any two conjugate diameters, to determine their length and direction.

In the first place, we shall find a' and b'. By page 57,

$$a'^2 + b'^2 = a^2 + b^2; \quad 2a'b' = \frac{2ab}{\sin \epsilon};$$

by adding and subtracting these two equations, we have

$$(a' + b')^2 = a^2 + b^2 + \frac{2ab}{\sin \epsilon},$$

$$(a' - b')^2 = a^2 + b^2 - \frac{2ab}{\sin \epsilon};$$

$$\therefore a' = \frac{1}{2}\sqrt{a^2 + b^2 + \frac{2ab}{\sin \epsilon}} + \frac{1}{2}\sqrt{a^2 + b^2 - \frac{2ab}{\sin \epsilon}},$$

$$b' = \frac{1}{2}\sqrt{a^2 + b^2 + \frac{2ab}{\sin \epsilon}} - \frac{1}{2}\sqrt{a^2 + b^2 - \frac{2ab}{\sin \epsilon}}.$$

In the equation, page 51, $a^2\tan\alpha\tan\alpha' + b^2 = 0$,

put $\epsilon + \alpha$ for α', and we have

$$a^2\tan\alpha\tan(\epsilon + \alpha) + b^2 = 0,$$

$$\text{but } \tan(\epsilon + \alpha) = \frac{\tan\epsilon + \tan\alpha}{1 - \tan\epsilon\tan\alpha};$$

therefore, by substitution and reduction we have the quadratic

$$a^2\tan^2\alpha + (a^2 - b^2)\tan\epsilon \, . \, \tan\alpha + b^2 = 0,$$

$$\therefore \tan\alpha = -\frac{(a^2-b^2)\tan\mathsf{C}}{2a^2} \pm \frac{1}{2a^2}\sqrt{(a^2-b^2)^2\tan^2\mathsf{C} - 4a^2b^2},$$

this value of $\tan\alpha$ will be real if $(a^2-b^2)^2\tan^2\mathsf{C}$ be greater or at least equal to $4a^2b^2$.

$$\text{If } \tan^2\mathsf{C} = \frac{4a^2b^2}{(a^2-b^2)^2}, \text{ or } \tan\mathsf{C} = \pm\frac{2ab}{a^2-b^2};$$

substituting this value in the $\tan\alpha$, we have

$$\tan\alpha = -\frac{a^2-b^2}{2a^2} \times \pm\frac{2ab}{a^2-b^2} = \pm\frac{b}{a},$$

which, since $\tan\alpha\tan\alpha' = -\frac{b^2}{a^2}$, gives

$$\tan\alpha = -\frac{b}{a}, \quad \tan\alpha' = \frac{b}{a}.$$

(44.) Given two conjugate diameters in an ellipse and their inclination to determine the axes.

Since $a^2 + b^2 = a'^2 + b'^2$, and $2ab = 2a'b'\sin\mathsf{C}$;

by adding and subtracting these equations, we have

$$(a+b)^2 = a'^2 + b'^2 + 2a'b'\sin\mathsf{C},$$
$$(a-b)^2 = a'^2 + b'^2 - 2a'b'\sin\mathsf{C},$$

$$a = \tfrac{1}{2}\sqrt{a'^2+b'^2+2a'b'\sin\mathsf{C}} + \tfrac{1}{2}\sqrt{a'^2+b'^2-2a'b'\sin\mathsf{C}},$$

$$b = \tfrac{1}{2}\sqrt{a'^2+b'^2+2a'b'\sin\mathsf{C}} - \tfrac{1}{2}\sqrt{a'^2+b'^2-2a'b'\sin\mathsf{C}}.$$

These values are always real, for $(a'-b')^2 > 0$, then $a'^2+b'^2 > 2a'b'$, and, *à fortiori*, $a'^2+b'^2 > 2a'b'\sin\mathsf{C}$. We can determine the angle α from the equation

$$\tan\alpha\,.\,\tan(\mathsf{C}+\alpha) = -\frac{b^2}{a^2},$$

as before, in the last article.

(45.) *The Normal in Terms of its Inclination to the Major Axis.*

The equation to the normal is

$$y - y' = \frac{a^2 y'}{b^2 x'}(x - x'),$$

$$\text{or } y = \frac{a^2 y'}{b^2 x'} x + y' - \frac{a^2 y'}{b^2}$$

$$= \frac{a^2 y'}{b^2 x'} x + \frac{b^2 - a^2}{b^2} \cdot y' \;\ldots\ldots\; (1).$$

Let the equation to the normal be

$$y = mx + h \;\ldots\ldots\ldots\ldots\ldots\ldots\; (2),$$

then, comparing (1) and (2),

$$m = \frac{a^2 y'}{b^2 x'}, \text{ and } h = \frac{b^2 - a^2}{b^2} \cdot y' \;\ldots\ldots\; (3),$$

$$m = \frac{a^2}{b^2} \cdot \frac{b}{a} \cdot \frac{\sqrt{a^2 - x'^2}}{x'}$$

$$\frac{b^2 m^2}{a^2} = \frac{a^2 - x'^2}{x'^2},$$

$$\therefore\; x'^2 = \frac{a^4}{a^2 + m^2 b^2},$$

$$\therefore\; a^2 - x'^2 = a^2 - \frac{a^4}{a^2 + m^2 b^2}$$

$$= \frac{a^4 + a^2 m^2 b^2 - a^4}{a^2 + m^2 b^2},$$

$$\therefore\; \sqrt{a^2 - x'^2} = \frac{amb}{\sqrt{a^2 + m^2 b^2}},$$

$$\therefore\ h = \frac{b^2 - a^2}{b^2} \cdot \frac{b}{a} \sqrt{a^2 - x'^2}$$

$$= \frac{b^2 - a^2}{ab} \cdot \frac{amb}{\sqrt{a^2 + m^2 b^2}}$$

$$= -(a^2 - b^2) \cdot \frac{m}{\sqrt{a^2 + m^2 b^2}};$$

$\therefore$ (2) becomes $y = mx - (a^2 - b^2) \cdot \dfrac{m}{\sqrt{a^2 + m^2 b^2}}$,

or $(y - mx)\sqrt{a^2 + m^2 b^2} + m \cdot (a^2 - b^2) = 0$.

The equation to the tangent in terms of the angle that it makes with the major axis, is

$$y = mx \pm \sqrt{m^2 a^2 + b^2}.$$

(46.) In the ellipse, to find the locus of the middle points of a series of parallel chords.

The equation to the ellipse is

$$A^2 y^2 + B^2 x^2 = A^2 B^2 \ \ldots\ldots\ldots\ldots\ldots\ldots\ (1).$$

Let MM′ be any right line whose equation is

$$y = mx + h \ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ (2).$$

Substitute in (1) for y its value from (2),

$$(A^2 m^2 + B^2) x^2 - 2A^2 mh \cdot x + A^2 h^2 - A^2 B^2 = 0,$$

the roots of this equation will give the abscissæ of the two points MM′,

$$x^2 - \frac{2A^2 mh \cdot x}{A^2 m^2 + B^2} + \frac{A^2 h^2 - A^2 B^2}{A^2 m^2 + B^2} = 0;$$

let $x'y'$, and $x''y''$, be the co-ordinates of the two points, and α, β, those of the middle point N,

$$\text{then } \alpha = \frac{x' + x''}{2}, \quad \beta = \frac{y' + y''}{2},$$

since in a quadratic equation the coefficient of the second term taken with a contrary sign is equal to the sum of the roots, we have

$$x' + x'' = -\frac{2A^2mh}{A^2m^2 + B^2},$$

$$\therefore \alpha = -\frac{A^2mh}{A^2m^2 + B^2}.$$

To find the corresponding value of β, we must remark that α and β is a point in the line whose equation is

$$y = mx + h;$$

hence, substituting β for y, and α for x, we obtain

$$\beta = m\alpha + h;$$

substitute the value of α in this equation

$$\beta = -\frac{A^2m^2h}{A^2m^2 + B^2} + h = \frac{+B^2h}{A^2m^2 + B^2}.$$

Dividing β by α, we get

$$\frac{\beta}{\alpha} = \frac{-B^2}{A^2m}.$$

Now, as this result is independent of h, which fixes the position of the chord MM′, it follows that if we put α', β', α'', β'', the co-ordinates of the middle points of the other chords parallel to the first, we shall in like manner find

$$\frac{\beta'}{\alpha'} = \frac{-B^2}{A^2m^2}, \quad \frac{\beta''}{\alpha''} = \frac{-B^3}{A^2m^2},$$

therefore, in general, representing the co-ordinates of the middle points by x and y, we have the relation

$$\frac{y}{x} = \frac{-B^2}{A^2 m}, \text{ or } y = -\frac{B^2}{A^2 m} \cdot x,$$

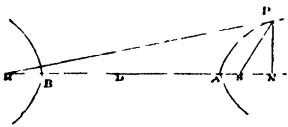
which is evidently a straight line passing through the origin.

Hence, all the diameters of an ellipse pass through the centre; and, conversely, every line which passes through the centre is a diameter, if we put m' for the tangent of the angle which the line forms with the axis of x,

$$m' = \frac{-B^2}{A^2 m}, \quad \therefore m = \frac{-B^2}{A^2 m'}.$$

The same process applies to the hyperbola, by changing $-B^2$ into $+B^2$.

Section VI.—Hyperbola.



(47.) If S and H be two fixed points, and if the difference of HP and SP be a constant quantity $= 2a$ suppose, the locus of P is the hyperbola.

Bisect SH in D, make

$$DB = DA = a;$$

then, A and B are manifestly points in the curve.

Let $\frac{DH}{DB} = e$, a quantity greater than unity; then $DH = ae$.

To find the equation to the curve.

Let D be the origin $DN = x$, $NP = y$;

$$\text{then } HN = ae + x, \quad SN = ae - x,$$

$$HP^2 = y^2 + (ae + x)^2, \; SP^2 = y^2 + (ae - x)^2,$$

$$\therefore HP^2 + SP^2 = 2y^2 + 2a^2e^2 + 2x^2,$$

$$HP^2 - SP^2 = (HP - SP)(HP + SP) = 4aex,$$

$\therefore$ HP + SP $= 2ex$, and HP^2 + SP^2 + 2 HP . PS $= 4e^2x^2$,

$$\therefore\ 2\,\text{HP}\,.\,\text{PS} = (4e^2 - 2)\,x^2 - 2y^2 - 2a^2e^2,$$

nd HP^2 + SP^2 − 2 HP . PS $= 4a^2 = 4(y^2 + a^2e^2) + 4(1 - e^2)x^2$;

herefore the equation to the hyperbola, the point D being he origin, is

$$y^2 = (1 - e^2)\,(a^2 - x^2).$$

This expression, $1 - e^2$, is generally put $= -\dfrac{b^2}{a^2}$, and then

$$\frac{y^2}{b^2} - \frac{x^2}{a^2} = -1, \text{ or } a^2y^2 - b^2x^2 = -a^2b^2 \ldots (1),$$

s the hyperbolic equation.

If we had taken the point A for the origin, y has the same value as in (1), but AN is = DN − DA = DN − a;

$$\therefore\ \frac{y^2}{b^2} - \frac{(x+a)^2}{a^2} = -1,$$

or $y^2 = (1 - e^2)(-2ax - x^2) = (e^2 - 1)(2ax + x^2) \ldots (2)$,

and if we take S the origin, SN = DN − ae,

$$\therefore\ y^2 = (e^2 - 1)\,(2aex + x^2) \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

(48.) We now proceed to the deduction of the polar equations to the hyperbola. If D be the pole, DP $= r$, HDP $= \theta$, then, in equation (1), $x = -r\cos\theta$, $y = r\sin\theta$; and we have

$$r^2\sin^2\theta = (1 - e^2)\,(a^2 - r^2\cos^2\theta),$$

$$\therefore\ r^2\{\sin^2\theta + (1 - e^2)\cos^2\theta\} = a^2\,.\,(1 - e^2),$$

$$\text{and } r^2 = \frac{a^2(1 - e^2)}{1 - e^2\cos^2\theta},$$

$$\therefore\ r = \frac{a\sqrt{e^2 - 1}}{\sqrt{e^2\cos^2\theta - 1}} = \frac{b}{\sqrt{e^2\cos^2\theta - 1}};$$

and if H be the pole, $\angle$PHS $= \theta$, HP $= r$,

$$\text{SP}^2 = (2a - r)^2 = r^2 + 4a^2e^2 - 4aer\cos\theta,$$

$$\therefore\ 4a^2 + r^2 - 4ar = r^2 + 4a^2e^2 - 4aer\cos\theta,$$

$$\therefore\ a^2 - ar = a^2e^2 - aer\cos\theta,\ \ \therefore\ r = \frac{a(1-e^2)}{1 - e\cos\theta}.$$

Now, it is evident that there exists a curve similarly situated with regard to **H**, as **AP** with regard to **S**; these two curves are called opposite hyperbolæ, though, properly speaking, the *hyperbola* is the system compounded of the two.

(49.) We now proceed to the discussion of a few hyperbolic properties; but it will be convenient to first investigate the equation to the tangent, since the most important properties depend upon it.

Let (y', x'), (y'', x''), be two points in the curve,

then, $y'^2 = (1 - e^2)(a^2 - x'^2)$, $\quad y''^2 = (1 - e^2)(a^2 - x''^2)$,

$$\therefore\ y'^2 - y''^2 = (y' + y'')(y' - y'') = (1 - e^2)(x''^2 - x'^2)$$
$$= (e^2 - 1)(x' + x'')(x' - x''),$$

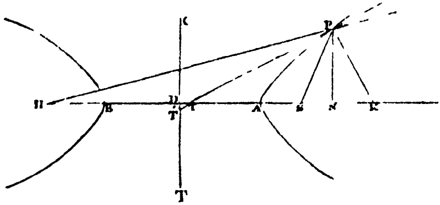
and the equation to a straight line passing through (y', x'), (y'', x''), is, if Y and X be the co-ordinates of the line,

$$\text{Y} - y' = \frac{y' - y''}{x' - x''}(\text{X} - x'),$$

$$\text{or } \text{Y} - y' = \frac{x' + x''}{y' + y''}\,.\,(e^2 - 1)(\text{X} - x').$$

Now, supposing the points (y', x'), (y'', x''), to approach continually, the line ultimately becomes a tangent at the point (y', x'), and its equation is then

$$\text{Y} - y' = \frac{x'}{y'}\,.\,(e^2 - 1)\,.\,(\text{X} - x').$$



We have now to find the subtangent, subnormal, &c.

$$Y - y' = \frac{(e^2 - 1)x'}{y'} . (X - x'),$$

the equation to the tangent at the point (y', x').

If $X = 0$, $Y - y' = \frac{(1 - e^2) x'^2}{y'}$,

$$\therefore Y = \frac{y'^2 + (1 - e^2) x'^2}{y'} = DT' = \frac{(1 - e^2) a^2}{y'} = - \frac{b^2}{y'},$$

this is the point where the tangent meets the conjugate axes below D,

$$\therefore DT' : DC :: DC : PN.$$

Also, if $Y = 0$,

$$- y' = \frac{(e^2 - 1) x'}{y'} (X - x'), \ \therefore X - x' = \frac{y'^2}{(1 - e^2) x'},$$

$$X = \frac{(1 - e^2) x'^2 + y'^2}{(1 - e^2) x'} = \frac{(1 - e^2) a^2}{(1 - e^2) x'} = \frac{a^2}{x'} = TD,$$

$$\therefore TD : DA :: DA : DN,$$

$$DN - DT = TN = \frac{x'^2 - a^2}{x'} = \text{subtangent},$$

another property of the hyperbola.

(50.) The equation to the normal PK now remains to be found.

$Y - y' = \frac{(e^2 - 1) x'}{y'} (X - x')$, is the equation to the tangent; since the normal is perpendicular to the tangent and passes also through y', x', its equation consequently is

$$Y - y' = \frac{y'}{(1 - e^2) x'} (X - x').$$

If in this equation we make $X = 0$,

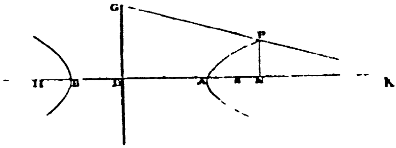
$$Y - y' = \frac{y'}{(1 - e^2) x'} (- x'),$$

$$\therefore\ Y = y' - \frac{y'}{(1-e^2)} = y'\left(\frac{e^2}{e^2-1}\right);$$

so that if the normal were produced to cut DF, or that line produced at a point G, DG would be $= \frac{NP\,e^2}{e^2-1}$, or $\frac{DG}{NP}$ is a constant quantity, equal to

$$\frac{e^2}{e^2-1} = \frac{DS^2}{DS^2-DA^2} = \frac{DS^2}{(DS+DA).(DS-DA)} = \frac{DS^2}{BS.AS};\ *$$

also, if in the normal's equation we make $Y = 0$, we have



$$-y' = \frac{y'}{(1-e^2)x'}(X - x'),$$

$$\therefore\ x' - X' = \frac{(1-e^2)x'y'}{y'} = (1-e^2)x',$$

$$\therefore\ X = e^2x' = NK + ND = KD,$$

$$\therefore\ NK = (e^2-1)x' = \frac{b^2}{a^2}.\ DN \text{ the subnormal.}$$

The properties of the hyperbola just investigated, may be here brought together for classification and reference.

DT : DC : : DC : PN .... (1),

TD : DA : : DA : DN .... (2).

* $\frac{e^2}{e^2-1} = \frac{a^2+b^2}{b^2}$ putting $\frac{a^2+b^2}{a^2}$ for e^2.

$$TN = \frac{DN^2 - AD^2}{DN} \dots\dots\dots\dots (3),$$

$$\frac{DG}{NP} = \frac{DS^2}{BS \cdot AS}, \quad \therefore DG : (NP \cdot DS) :: DS : (BS \cdot AS) \dots \quad (4),$$

$$NK = \frac{CD^2}{AD^2} \cdot DN \dots\dots\dots\dots (5);$$

or, which is the same,

$$DG = \frac{AD^2 + CD^2}{CD^2} \cdot PN.$$

On Directrices and Parameters.

The hyperbola and ellipse have each two directrices, but the parabola has only one.

In the parabola the distance of the directrix from the vertex is $= a$.

In the ellipse and hyperbola the distance of the directrix from the centre $= \frac{a}{e}$.

The parameter, or latus rectum, is the double ordinate drawn through the focus in each of the three curves. In the parabola it is $= 4a$, a being the distance between the vertex and focus. In the ellipse and hyperbola it is $= \frac{2b^2}{a}$, when a and b are the major and minor axes.

(51.) *On the Asymptote to the Hyperbola.*

$$y^2 = (1 - e^2)(a^2 - x^2),$$

$$\therefore y = \sqrt{e^2 - 1} \cdot \left\{x - \frac{a^2}{2 \cdot x}\right\} \text{ \&c. } \dots\dots\dots (1),$$

is the equation to the curve.

From the form of this equation, it seems that there are relations between the curve and the line

$$y = \left(\sqrt{e^2 - 1}\right) x, \text{ which are worth attending to.}$$

It is evident that this line passes through the origin, making an angle $= \tan^{-1} \sqrt{e^2 - 1}$ with the principal diameter;

and since $y^2 = (e^2 - 1)\, x^2$ in the line,

and $y^2 = (e^2 - 1)\, x^2 - (e^2 - 1)\, a^2$ in the curve, it is evident that the lineal ordinate is greater than the hyperbolic corresponding to the same abscissa.

But on considering the equation to the curve when expanded into the series (1) above, it is evident that as x increases the hyperbolic and lineal ordinate approximate nearer and nearer.

So that we may say, that when $x = a$ the line touches the hyperbola, and is therefore an asymptote.

Since in the equation $y = \sqrt{e^2 - 1} \cdot x$, the radical may be taken with either sign, and x may be either positive or negative; there are evidently four asymptotes, all cutting at D, a pair belonging to each hyperbolic branch.

It will be an instructive exercise for the student to endeavour to deduce the asymptotes from the tangential equation

$$Y - y' = \frac{(e^2 - 1) \cdot x'}{y'} (X - x'),$$

considering the asymptote to be a tangent at an infinite distance.

(52) To prove that the normal bisects the exterior angle between the focal distances at any point in the hyperbola.

By art. 47, $\quad HP + SP = 2\,ex,$

and $HP - SP = 2\,a$;

by adding and subtracting we have

$$HP = e\,x + a, \text{ and } SP = e\,x - a,$$

$$SK = DK - DS = e^2 x' - a\,e = e\,(e\,x' - a) = e\,.\,SP,$$

$$HK = DK + DH = e^2 x + a\,e = e\,(e\,x' + a) = e\,.\,HP,$$

$$\therefore \frac{SK}{HK} = \frac{SP}{HP},$$

$\therefore$ **PK** bisects the angle **SPP′**.

From this it is clear that the focal distances make equal angles with the tangent, that is, the angle **SPT** is equal to the angle **HPT**; for **KPT = KPT′**, being right angles, and **SPK = KPP′**, from above; the latter, being taken from the former, leaves the remaining angles **SPT** and **P′PT″** equal, but **P′PT″** is equal to its vertical opposite angle **HPT**,

$$\therefore HPT = SPT.$$

(53.) By proceeding in the same manner as for the ellipse, page 58, we have $p^2 = b^2 \frac{r}{2a + r}$, and therefore we have in the same manner

$$pp' = b^2,$$

or the rectangle of the perpendiculars from the foci on the tangent is equal to the square of the semi-axis minor in the hyperbola as well as in the ellipse.

(54.) When the hyperbola is related to its conjugate diameters, the process is exactly the same as for the ellipse.

The equation referred to its axis is $a^2 y^2 - b^2 x^2 = - a^2 b^2$, and substituting for x and y, as in the ellipse, we get

$$a^2 \sin a \sin a' - b^2 \cos a \cos a' = 0 \ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1),$$

$$\text{and } (a^2 \sin^2 a' - b^2 \cos^2 a')\, y^2$$
$$+ (a^2 \sin^2 a - b^2 \cos^2 a)\, x^2 = - a^2 b^2 \ldots\ldots\ldots\ldots\ldots \quad (2),$$

from (1) we have, dividing by $\cos a \cos a'$,

$$a^2 \tan a \tan a' - b^2 = 0, \text{ or } \tan a \tan a' = \frac{b^2}{a^2} \ldots\ldots\ldots \quad (3),$$

$$\tan a' = \frac{b^2}{a^2 \tan a} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

In equation (2) make $y = 0$ and $x = 0$ successively, and we have

$$x^2 = \frac{- a^2 b^2}{a^2 \sin^2 a - b^2 \cos^2 a}$$

$$y^2 = \frac{- a^2 b^2}{a^2 \sin^2 a' - b^2 \cos^2 a'}.$$

We must observe that, of these two squares, one is positive, and the other is negative, for any value of x corresponding to $y = 0$ is real; but that of y corresponding to $x = 0$ is imaginary; but that they have different signs may be shewn as follows :—multiplying these two equations together, we have

$$x^2 y^2 = \frac{a^4 b^4}{(a^2 \sin^2 \alpha - b^2 \cos^2 \alpha)(a^2 \sin^2 \alpha' - b^2 \cos^2 \alpha')},$$

multiplying out the denominator

$$a^4\sin^2\alpha\sin^2\alpha' + b^4\cos^2\alpha\cos^2\alpha'$$
$$- a^2b^2(\sin^2\alpha'\cos^2\alpha + \sin^2\alpha\cos^2\alpha');$$

but by equation (1), $a^2\sin\alpha\sin\alpha' - b^2\cos\alpha\cos\alpha' = 0$,

$$\therefore\ a^4\sin^2\alpha\sin^2\alpha' + b^4\cos^2\alpha\cos^2\alpha' = 2a^2b^2\sin\alpha\sin\alpha'\cos\alpha\cos\alpha';$$

therefore, the denominator becomes

$$a^2b^2(2\sin\alpha\sin\alpha'\cos\alpha\cos\alpha' - \sin^2\alpha'\cos^2\alpha - \sin^2\alpha\cos^2\alpha')$$
$$= -a^2b^2(\sin\alpha'\cos\alpha - \sin\alpha\cos\alpha')^2 = -a^2b^2\sin^2(\alpha' - \alpha);$$

$$\therefore\ x^2y^2 = \frac{a^4b^4}{-a^2b^2\sin^2(\alpha'-\alpha)} = -\frac{a^2b^2}{\sin^2(\alpha'-\alpha)} \ \ldots\ldots (5).$$

This result, being essentially negative, proves that x^2 and y^2 have different signs.

Taking the axis of x for the transverse diameter in this case, the expression for x^2 is positive, but that of y^2 is negative; putting a'^2 for the first, and $-b'^2$ for the second, we have

$$a'^2 = \frac{-a^2b^2}{a^2\sin^2\alpha - b^2\cos^2\alpha}$$

$$-b'^2 = \frac{-a^2b^2}{a^2\sin^2 a' - b^2\cos^2 a'},$$

$$\therefore\ a^2\sin^2 a - b^2\cos^2 a = -\frac{a^2b^2}{a'^2}$$

$$a^2\sin^2 a' - b^2\cos^2 a' = \frac{a^2b^2}{b'^2};$$

from these values of a'^2 and $-b'^2$, equation (2) becomes

$$a^2y^2 - b'^2x^2 = -a'^2b'^2,$$

the equation to the hyperbola related to oblique axes originating at the centre, and making angles a and a' with the primitive axis of x.

If we put in this equation

$$x = 0, \quad y = \sqrt{-b^2} = \pm b\sqrt{-1}, \text{ and } y = 0, \quad x = \pm a',$$

since the value of y, when $x = 0$, is impossible, the curve does not meet the axis of y, but as $\pm a'$ is the value of x when $y = 0$, the curve meets the axis of x at the extremities of the diameter $2a'$.

Now, if in equation (5) we put a'^2 for x^2 and $-b^2$ for y^2, we obtain the relation

$$-a'^2 b'^2 = \frac{-a^2 b^2}{\sin^2(a' - a')},$$

$\therefore ab \sin(a' - a) = ab$, which is the same as in the ellipse.

(55.) In the general properties of the ellipse, if we put $-b^2$ for b^2, we have the corresponding general properties of the hyperbola.

(56.) In the values of a'^2, b'^2, obtained in art. 54, replacing the expressions for $\sin^2\alpha$, $\cos^2\alpha$, $\sin^2\alpha'$, $\cos^2\alpha'$, by their values in terms of $\tan\alpha$ and $\tan\alpha'$, viz.:—

$$\cos^2 a = \frac{1}{1 + \tan^2 a}, \quad \sin^2 a = \frac{\tan^2 a}{1 + \tan^2 a},$$

$$\cos^2 a' = \frac{1}{1 + \tan^2 a'}, \quad \sin^2 a' = \frac{\tan^2 a'}{1 + \tan^2 a'},$$

we obtain $a'^2 = \dfrac{-a^2 b^2 (1 + \tan^2 a)}{a^2 \tan^2 a - b^2}$ .... (1),

$$-b'^2 = \frac{-a^2 b^2 (1 + \tan^2 a')}{a^2 \tan^2 a' - b^2},$$

or $b'^2 = \dfrac{a^2 b^2 (1 + \tan^2 a')}{a^2 \tan^2 a' - b^2}$ .... (2).

From (1), $\tan^2 a = \dfrac{b^2}{a^2}\left(\dfrac{a'^2 - a^2}{a'^2 + b^2}\right).$

From (2), $\tan^2 a' = \frac{b^2}{a^2}\left(\frac{a^2 + b'^2}{b'^2 - b^2}\right)$;

multiplying these together, we have

$$\tan^2 a \,.\, \tan^2 a' = \frac{b^4}{a^4}\left\{\frac{(a'^2 - a^2)\,(a^2 + b'^2)}{(a'^2 + b^2)\,(b'^2 - b^2)}\right\},$$

$$\text{but } \tan a \tan a' = \frac{b^2}{a^2} \quad \text{(page 75)},$$

$$\text{or } \tan^2 a \tan^2 a' = \frac{b^4}{a^4},$$

$$\therefore \frac{b^4}{a^4}\left\{\frac{(a'^2 - a^2)\,(a^2 + b'^2)}{(a'^2 - b^2)\,(b'^2 - b^2)}\right\} = \frac{b^4}{a^4},$$

$$\text{or } \frac{(a'^2 - a^2)\,(a^2 + b'^2)}{(a'^2 + b^2)\,(b'^2 - b^2)} = 1,$$

$$\therefore (a'^2 - a^2)\,(a^2 + b'^2) = (a'^2 + b^2)\,(b'^2 - b^2),$$

$$a'^2 a^2 - a^4 + a'^2 b'^2 - a^2 b'^2 = a'^2 b'^2 + b^2 b'^2 - a'^2 b^2 - b^4;$$

transposing and reducing,

$$a'^2 (a^2 + b^2) - b'^2 (a^2 + b^2) = a^4 - b^4,$$

$$\text{or } (a'^2 - b'^2)\,(a^2 + b^2) = a^4 - b^4 = (a^2 + b^2)\,(a^2 - b^2),$$

$$\therefore a'^2 - b'^2 = a^2 - b^2.$$

This might have been deduced, as well as the latter part of art. 54, more elegantly, by transforming the oblique ordinates to rectangular, as we have already done in the ellipse, substituting for x and y the values

$$\frac{x \sin' - y \cos a}{\sin (a' - a)}, \quad \frac{y \cos a - x \sin a}{\sin (a' - a)};$$

in the equation $a'^2 y - b'^2 x = - a'^2 b'^2$, we obtain, as in the ellipse,

$$a'^2 \cos^2 \alpha - b'^2 \cos^2 \alpha' = a^2 \quad \dots\dots\dots \quad (1),$$

$$a'^2 \sin^2 \alpha - b'^2 \sin^2 \alpha' = - b^2 \quad \dots\dots\dots \quad (2),$$

$$- a'^2 b'^2 \sin^2 (\alpha' - \alpha) = - a^2 b^2 \quad \dots\dots\dots \quad (3),$$

by adding (1) and (2), we obtain

$$a'^2 - b'^2 = a^2 - b^2 \quad \dots\dots\dots \quad (4).$$

From equation (3), $4\,a'b' \sin(\alpha' - \alpha) = 4\,ab$ ...... (5).

Equation (4) shews that the difference of the squares of any two conjugate diameters is equal to the difference of the square of the principal axes.

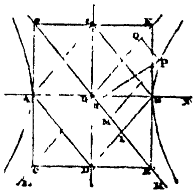
Equation (5) shews that the rectangle described on any system of conjugate diameters, is equal to the rectangle on the axes.

(57.) We have already shewn, that all we have hitherto done for the ellipse will, with slight modifications, apply to the hyperbola; but the converse of this is not true, for there are many properties of the hyperbola which cannot belong to the ellipse; these are the properties relative to the asymptotes. To complete the whole, we shall proceed to find the equation to the hyperbola referred to its asymptotes.

The equation of the hyperbola referred to its axes is

$$a^2 y^2 - b^2 x^2 = - a^2 b^2 \quad \dots\dots\dots \quad (1).$$

We must now pass from a system of rectangular to a system of oblique axes; to do this, we must substitute in the above equation, for x and y, the following values (see art. 24, eq. 3.):—



$$x = x \cos \alpha + y \cos \alpha',$$

$$y = x \sin \alpha + y \sin \alpha',$$

and we have

$$\left.\begin{array}{l} (a^2 \sin^2 \alpha - b^2 \cos^2 \alpha)\, y^2 \\ + (2a^2 \sin \alpha \sin \alpha' - 2b^2 \cos \alpha \cos \alpha')\, xy \\ + (a^2 \sin^2 \alpha - b^2 \cos^2 \alpha)\, x^2 \end{array}\right\} = - a^2 b^2 \quad \dots \quad (2).$$

Here we take for the new axis of x, the asymptote **DK**;

and for the axis of y, the asymptote DL; the angles α and α' are determined by the equations

$$\tan\alpha = -\frac{b}{a}, \quad \tan\alpha' = \frac{b}{a},$$

$$\therefore \cos\alpha = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\alpha' = -\frac{b}{\sqrt{a^2+b^2}},$$

$$\cos\alpha' = \frac{a}{\sqrt{a^2+b^2}}, \quad \sin\alpha' = \frac{a}{\sqrt{a^2+b^2}},$$

$$\cos^2\alpha' = \frac{a^2}{a^2+b^2}, \text{ and } \sin^2\alpha' = \frac{b^2}{a^2+b^2};$$

multiply the latter by a^2, and the former by b^2, then, by subtraction, we have

$$a^2\sin^2\alpha' - b^2\cos^2\alpha' = \frac{a^2b^2 - a^2b^2}{a^2+b^2} = 0.$$

In the same way,

$$a^2\sin^2\alpha - b^2\cos^2\alpha = \frac{a^2b^2 - a^2b^2}{a^2+b^2} = 0.$$

Also,

$$2a^2\sin\alpha\sin\alpha' - 2b^2\cos\alpha\cos\alpha' = -\frac{4a^2b^2}{a^2+b^2}.$$

Hence we see the terms involving x^2 and y^2 vanish, and if we substitute in equation (2) the value of the coefficient of xy, just found, we have

$$xy = \frac{a^2+b^2}{4} \quad \text{.......................} \quad (3).$$

The equation which contains the rectangle of the variables, and the known quantity $\frac{a^2+b^2}{4}$, is called the equation to the hyperbola, referred to its asymptotes as co-ordinate axes.

(58.) If the hyperbola be equilateral, then

$$a = b, \text{ and } xy = \frac{a^2}{2}.$$

(59.) If we wished conversely to determine from equation (2) the angles α and α', so that the terms involving x^2 and y^2 should disappear, we have then the following conditions:—

$$a^2 \sin^2 \alpha' - b^2 \cos^2 \alpha' = 0, \quad a^2 \sin^2 \alpha - b^2 \cos^2 \alpha = 0.$$

From the first of these we have $\tan \alpha' = \pm \frac{b}{a}$,

from the second $\tan \alpha = \pm \frac{b}{a}$.

This shews, that if we take α' for the angle whose tangent is $+\frac{b}{a}$, we must take α for the angle whose tangent is $-\frac{b}{a}$. Therefore, the new axes, with respect to which the equation is brought to the form $xy = k^2$, are the asymptotes of the curve, where k^2 is put for $\frac{a^2 + b^2}{4}$.

It is clear from this equation, or $y = \frac{k^2}{x}$, or $x = \frac{k^2}{y}$, that as we increase x the more is y diminished; and if we suppose x to be greater than any assignable quantity, y will become less than any assignable quantity, and conversely; so that the new axes have the characteristic property of the asymptotes.

Let ϵ be the angle LDK, which the asymptotes make with each other; then, multiplying both sides of the equation $xy = k^2$ by $\sin \epsilon$, we obtain

$$xy \sin \epsilon = k^2 \sin \epsilon.$$

Now, if P be any point in the curve, MP and MD the co-ordinates of this point, parallel to the asymptotes, then the parallelogram DMPQ has for its area

$$\text{DM} \, . \, \text{PH} = xy \sin \epsilon;$$

this is equal to $k^2 \sin \epsilon$, which is a constant quantity, and is independent of the position of the point P.

Therefore, all parallelograms described upon the co-ordinates parallel to the asymptotes are equal. It can be easily shown that this constant area is equal to half the rectangle DBEC contained by the semi-axes, for $\epsilon = 2\alpha'$,

$$\therefore \sin \theta = \sin 2\alpha' = 2 \sin \alpha' \cos \alpha';$$

by art. 57, $\cos \alpha' = \dfrac{a}{\sqrt{a^2 + b^2}}$, $\sin \alpha' = \dfrac{b}{\sqrt{a^2 + b^2}}$,

$$\therefore \sin \theta = 2 \sin \alpha' \cos \alpha' = \frac{2ab}{\sqrt{a^2 + b^2}},$$

$$\therefore xy \sin \theta = \frac{a^2 + b^2}{4} . \sin \theta \text{ becomes}$$

$$xy \sin \theta = \frac{a^2 + b^2}{4} \cdot \frac{2ab}{a^2 + b^2} = \frac{ab}{2}.$$

To the extremities of the minor axis draw the lines BC, BD, AC, AD; the figure thus formed, having its sides parallel to the asymptotes, is four times the area of the figure DIBI′, constructed on the co-ordinates of the point A.

Therefore, if from any number of points in the hyperbola, lines be drawn parallel to the asymptotes, and terminating therein, the parallelograms so formed will all be equal to each other, and to half the rectangle of the principal semi-axes.

(60.) To find the equation of the tangent to any point of the hyperbola referred to its asymptotes.

Let x'', y'', be the co-ordinates of the point P; x', y', the co-ordinates of another point in the curve; the equation to the hyperbola being

$$xy = k^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (1).$$

The secant will be represented by

$$y - y'' = \frac{y' - y''}{x' - x''}(x - x'') \quad \ldots\ldots\ldots \quad (2),$$

and $x'y'$, and $x''y''$, being both points in the curve,

$$x'y' = k^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$x''y'' = k^2 \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4).$$

To find the coefficient $\frac{y' - y''}{x' - x''}$, subtract (4) from (3), and we have $x'y' - x''y'' = 0$; and adding $y''x'$ and subtracting $x'y''$, which are identical, we obtain

$$x'(y' - y'') + y''(x' - x'') = 0,$$

$$\therefore \frac{y' - y''}{x' - x''} = -\frac{y''}{x'};$$

the secant therefore becomes

$$y - y'' = -\frac{y''}{x'}(x - x'').$$

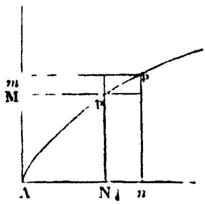
Now, in order that this secant may become a tangent, we must suppose $x' = x''$, and $y' = y''$,

$$\therefore y - y'' = -\frac{y''}{x''}(x - x'')$$

is the equation to the tangent.

$$\text{or } y''x + x''y = 2x'y' = \frac{1}{2}(a^2 + b^2).$$

(61.) To find the area of a parabola.



$$\left.\begin{array}{ll} pn^2 = 4a'mp & \ldots\ldots\ldots (1) \\ PN^2 = 4a'.MP & \ldots\ldots\ldots (2) \end{array}\right\} \text{by the equation to the parabola,}$$

$$\therefore pn^2 - PN^2 = 4a'(mp - MP),$$

$$\text{or } (pn + PN).(pn - PN) = 4a'.Nn,$$

$$\overline{pn} + \overline{PN} \,.\, Mm = \frac{PN^2}{MP} \,.\, Nn, \text{ from (2)},$$

$$\therefore\ MP \,.\, Mm = \frac{PN^2}{pn + PN} \,.\, Nn, \text{ (ultimately,)}$$

$$= \frac{1}{2} PN \,.\, Nn,$$

or the rectangle $P\,m = \frac{1}{2}\,P\,n$;

and so on for any other rectangle; therefore the space Anp is double the space Apm,

$$\text{or } Anpm = 3\,Apm; \quad \therefore\ Apm = \frac{1}{3} Anpm,$$

$$\text{or } Anp = \frac{2}{3} Anpm.$$

Hence, the parabola is $\frac{2}{3}$ of its circumscribing rectangle.

To find the Area of an Ellipse.

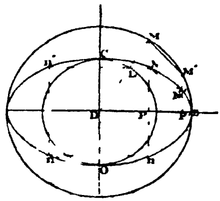
(62.) The area of an ellipse may be found from the constant ratio $\frac{b}{a}$, of the ordinate of an ellipse to that of the circle described upon the major axis.

Conceive any polygon whatever to be inscribed in a circle, of which one of the sides is MM′; from M and M′ let fall the perpendiculars MP and M′P′, these perpendiculars cut the ellipse in N and N′; join NN′, and we thus obtain a polygon inscribed in the ellipse, of which NN′ is one of its sides.

Let Y and Y′ be the co-ordinates of the points M and M′, and y and y' the co-ordinates of the points N and N′, corresponding to the same abscissæ x, x'; the trapeziums MM′PP′, NN′PP′, give

$$MM'PP' = \frac{Y + Y'}{2}(x - x'), \quad NN'PP' = \frac{y + y'}{2}(x - x'),$$

$$\therefore \frac{NN'PP'}{MM'PP'} = \frac{y + y'}{Y + Y'}.$$



Now, since, by art. 39,

$$y = \frac{b}{a} . Y, \quad y' = \frac{b}{a} . Y',$$

$$\therefore \frac{y + y'}{Y + Y'} = \frac{b}{a},$$

$$\therefore \frac{NN'PP'}{MM'PP'} = \frac{b}{a}.$$

In the same manner, each of the trapeziums which compose the polygon inscribed in the ellipse, is to the corresponding trapezium of the polygon inscribed in the circle as b is to a; hence, we conclude that the sum of all the first trapeziums, or the polygon inscribed in the ellipse, is to the second polygon in the same ratio, or

$$\frac{p}{P} = \frac{b}{a};$$

p and P being the polygons respectively. As this relation is true, whatever be the number of the sides of the two polygons, it is true in their limits, which are the areas of the ellipse and circle; hence, if s and S be the respective areas, we have

$$\frac{s}{S} = \frac{b}{a}, \quad \text{or } s = \frac{b}{a} . S;$$

but π represents the ratio of the circumference of a circle to its diameter, or the area of a circle whose radius is 1, πa^2 is the area of a circle whose radius is a, and, consequently,

$$s = \frac{b}{a} . S = \frac{b}{a} \pi a^2 = \pi a b,$$

which equals the area of the ellipse.

For the hyperbola the same relation $s = \frac{b}{a} . S$, obtains, s being the area comprised between the curve and any chord parallel to the second axis, S the corresponding area of an equilateral hyperbola, described upon the first axis. But this does not enable us to find the area of the hyperbola, since we must at first know that of an equilateral hyperbola.

On the Right Line, Circle, and various properties of the Conic Sections.

(63.) In the note, page 16, it is shewn that the length of the perpendicular from a point $x'y'$ on $Ax + By + C = 0$, is

$$\frac{Ax' + By' + C}{\sqrt{A^2 + B^2}};$$

by this we can shew that the length of the perpendicular from $x'y'$ on the line $x\cos\alpha + y\sin\alpha - p = 0$, is

$$x'\cos\alpha + y'\sin\alpha - p;$$

$$\text{for } \frac{A}{\sqrt{A^2 + B^2}}x' + \frac{B}{\sqrt{A^2 + B^2}}y + \frac{C}{\sqrt{A^2 + B^2}}$$

is of the same form, since we may take

$$\frac{A}{\sqrt{A^2 + B^2}} = \cos\alpha, \text{ and } \frac{B}{\sqrt{A^2 + B^2}} = \sin\alpha,$$

as they satisfy the condition $\cos^2\alpha + \sin^2\alpha = 1$.

This is useful in solving a large class of problems.

Given the base and area of a triangle, to find the locus of the vertex*.

Let the equation of the given base be

$$x\cos\alpha + y\sin\alpha - p = 0,$$

and $x\cos\alpha + y\sin\alpha - p$ is the length of the perpendicular from any point x, y, on the base; if the given base $= a$, and given area $= m^2$, the equation of the locus is

$$x\cos\alpha + y\sin\alpha - p = \frac{2m^2}{a},$$

since the perpendicular of any triangle equals twice the area divided by the base.

* See Salmon's "Conic Sections," page 41.

(64.) The equation of a line bisecting the angle contained by the lines

$$x\cos\alpha + y\sin\alpha - p = 0, \text{ and } x\cos\beta + y\sin\beta - p' = 0,$$

$$\text{is } x\cos\alpha + y\sin\alpha - p = x\cos\beta + y\sin\beta - p',$$

which represents a right line passing through the intersection of these two lines, for it is evidently the equation to a right line; and since the co-ordinates of the points of intersection must satisfy the two former equations, they must also satisfy the latter equation.

(65.) The general equation to the circle referred to rectangular axes being

$$(x - \alpha)^2 + (y - \beta)^2 = r^2,$$

$$\text{or } x^2 + y^2 - 2\alpha x - 2\beta y + \alpha^2 + \beta^2 - r^2 = 0 \text{ ...... } (1);$$

every equation of the second degree of the form

$$x^2 + y^2 + Ax + By + C = 0 \text{ .... } (2),$$

having the coefficients of x and y either unity or equal, and not containing the rectangle xy of the variables, is the equation to a circle.

Comparing (1) and (2),

$$-2\alpha = A, \quad -2\beta = B, \text{ and } \alpha^2 + \beta^2 - r^2 = C,$$

$$\alpha = -\frac{A}{2}, \quad \beta = -\frac{B}{2},$$

$$r = \sqrt{\alpha^2 + \beta^2 - C} = \sqrt{\frac{A^2 + B^2}{4} - C},$$

replacing α, β, r, by the above values,

$$\left(x + \frac{A}{2}\right)^2 + \left(y + \frac{B}{2}\right)^2 = \frac{A^2 + B^2}{4} = C.$$

Or by completing the squares in equation (2) we obtain the same result, thus

$$x^2 + Ax + y^2 + By = -C;$$

completing the square in each,

$$x^2 + \text{A}x + \left(\frac{\text{A}}{2}\right)^2 + y^2 + \text{B}x + \left(\frac{\text{B}}{2}\right)^2 = \frac{\text{A}^2}{4} + \frac{\text{B}^2}{4} - \text{C},$$

$$\text{or } + \left(x + \frac{\text{A}}{2}\right)^2 + \left(y + \frac{\text{B}}{2}\right)^2 = \frac{\text{A}^2 + \text{B}^2}{4} - \text{C} \ldots\ldots (3).$$

Examples.

$$\ldots\ldots\ldots\ 2x^2 + 2y^2 - 3x + 4y - 1 = 0.$$

$$x^2 + y^2 - \frac{3x}{2} + 2y = \frac{1}{2};$$

putting the equation in the following form,

$$x^2 - \frac{3x}{2} + y^2 + 2y = \frac{1}{2},$$

complete the squares,

$$x^2 - \frac{3x}{2} + \left(\frac{3}{4}\right)^2 + y^2 + 2y + 1 = \frac{9}{16} + 1 + \frac{1}{2}$$

$$= \frac{9}{16} + \frac{16}{16} + \frac{8}{16} = \frac{33}{16}, \quad \left(x - \frac{3}{4}\right)^2 + (y + 1)^2 = \frac{33}{16},$$

which is the equation to a circle whose radius is $\frac{\sqrt{33}}{4}$, and the co-ordinates of the centre $\frac{3}{4}$ and -1.

Since A, B, C, may be any quantities whatever, we may sometimes have $\frac{\text{A}^2 + \text{B}^2}{4} - \text{C} = 0$, or indeed negative.

If it be equal 0 then the circle is reduced to a point, and if it be negative the radius is imaginary; thus, the equation

$$4x^2 + 4y^2 - 12x - 8y + 13 = 0$$

represents a point whose co-ordinates are $\frac{3}{2}$ and 1, for it can be transformed into

$$\left(x - \frac{3}{2}\right)^2 + (y - 1)^2 = 0.$$

The equation $x^2 + y^2 + 4x - 2y + 7 = 0$,

$$x^2 + 4x + y^2 - 2y = -7,$$

completing the square;

$$x^2 + 4x + 4 + y^2 - 2y + 1 = 4 + 1 - 7 = -2$$

$$(x + 2)^2 + (y - 1)^2 = -2,$$

an equation of which the first member, being the sum of two positive squares, cannot be equal to a negative quantity.

(66.) In the parabola, if PSP′ be a chord through the focus, and $\text{PSN} = \theta$, then, page 37,

$$\text{SP} = \frac{l}{1 - \cos\theta}, \qquad \text{and } \text{SP}' = \frac{l}{1 + \cos\theta},$$

$$\frac{1}{\text{SP}} + \frac{1}{\text{SP}'} = \frac{1 - \cos\theta}{l} + \frac{1 + \cos\theta}{l} = \frac{2}{l},$$

where $l = 2a =$ half the latus rectum.

(67.) In the parabola, the parameter of any diameter = 4 times (distance of the vertex of the diameter from the focus).

The parameter is the double ordinate passing through the focus.

$$\text{SQ} = \frac{2a}{1 + \cos \text{ASQ}}, \quad sq = \frac{2a}{1 + \cos(\pi + qsx)}$$

$$= \frac{2a}{1 + \cos(\pi + \text{ASQ})}$$

$$= \frac{2a}{1 - \cos \text{AS}}$$

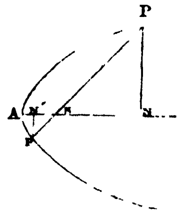
$$\pi = \text{ASP} + 2(\pi - \text{ASQ}), \quad \therefore \text{ASQ} = \frac{\pi}{2} + \frac{\text{ASP}}{2},$$

$$\therefore \text{ASQ} = \frac{\pi}{2} + \frac{\text{ASP}}{2},$$

$$\text{Q}q = \frac{4a}{\cos^2 \frac{\text{ASP}}{2}} = 4\,\text{SP}.$$

(68.) Find the area included between the parabola $y^2 = 4ax$ and the straight line $x = y + a$.

Since $x = a$ gives $y = 0$, we see that the given line passes through the focus of the parabola; and because the coefficient of x is 1, the line makes an angle of $45°$ with the axis of x.



Hence, in the figure,

APP′ is the area required,

and $\angle$ PSN $= 45°$,

$\therefore$ SN = PN, and SN′ = P′N′.

$$\text{Now, APP}' = \text{ASP} + \text{ASP}'$$

$$= \text{APN} - \text{SPN} + \text{AN}'\text{P}' + \text{SP}'\text{N}'$$

$$= \frac{2}{3}(\text{AN}\,.\,\text{PN} + \text{AN}'\,.\,\text{P}'\text{N}') - \frac{1}{2}(\text{PN}^2 - \text{P}'\text{N}'^2).$$

Find x from equations $y^2 = 4ax$

and $x = y + a$,

we get $y^2 = (x - a)^2 = 4ax$, $\therefore x^2 - 6ax + a^2 = 0$

$$\therefore x = 3a \pm 2\sqrt{2}\,.\,a = (3 \pm 2{\cdot}8284)\,a,$$

$$\therefore \text{AN} = (5{\cdot}8284), \quad \text{AN}' = (0{\cdot}1716)\,a,$$

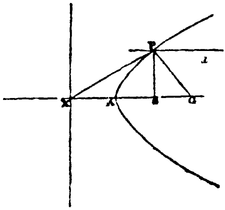
and $y = x - a$, $\therefore$ PN $= 4{\cdot}8284$, P′N′ $= (0{\cdot}8284)\,a$,

$$\therefore \text{AP} = \frac{2}{3} \cdot a^2 \{(5{\cdot}8284)(4{\cdot}8284) + (0{\cdot}1716)(0{\cdot}8284)\}$$

$$= \frac{1}{2} a^2 \{(4{\cdot}8284)^2 - (0{\cdot}8284)^2\}$$

$$= a^2 (18{\cdot}80124 - 11{\cdot}31360) = 7{\cdot}48764 \times a^2. \quad \text{Answer.}$$

(69.) Draw a normal at the extremity of the latus rectum of a parabola whose equation is $y^2 = 4a(x - a)$, and find its distance from the origin of co-ordinates.



If $x = a$, we find from the given equation that $y = 0$; therefore, the curve cuts the axis of x at a distance a from the origin. If $x < a$, then y is impossible; therefore, no portion of the curve lies to the left of $(a, 0)$, that is, A. Hence, A is the vertex of the parabola. Transfer the origin to this point, and we find $y'^2 = 4ax'$,

since $y' = y$ and $x' = x - a$,

therefore X, that is, the old origin, is the foot of the directrix. To draw a normal, then, at the extremity of the latus rectum, take $\text{SG} = \text{SP} = 2a$, and join PG; this is the normal required,

$$\text{for } \angle \text{SPG} = \angle \text{SGP} = \angle \text{GP}x.$$

Join XP, then this is the distance of the normal from the origin, and

$$\text{XP} = \sqrt{\text{SP}^2 + \text{XS}^2} = \sqrt{8\,\text{AS}^2} = 2\sqrt{2} \cdot a.$$

(70.) Find the locus of the intersection of two tangents to a hyperbola which meet one another at right angles.

The equation to the tangent to a hyperbola, in terms of the angle which it makes with the axis of x, is

$$y = mx \mp \sqrt{m^2a^2 - b^2};$$

consequently, for another which intersects this at right angles, we get

$$y = -\frac{1}{m} x \mp \sqrt{\frac{a^2}{m^2} - b^2}$$

Now, this equation becomes

$$my + x = \mp \sqrt{a^2 - m^2b^2},$$

$$\therefore\ m^2y^2 + 2mxy + x^2 = a^2 - m^2b^2 \ \ldots\ldots\ (1);$$

and from the first equation

$$y - mx = \mp \sqrt{m^2a^2 - b^2},$$

$$\therefore\ y^2 - 2mxy + m^2x^2 = m^2a^2 - b^2 \ \ldots\ldots\ldots\ (2);$$

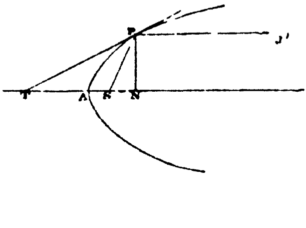
adding (1) and (2), we get

$$(y^2 + x^2)(m^2 + 1) = (a^2 - b^2)(m^2 + 1),$$

$$\therefore\ x^2 + y^2 = a^2 - b^2.$$

Hence, the locus is a circle whose centre is the centre of the hyperbola, and radius $= \sqrt{a^2 - b^2}$.

(71.) Shew that the parameter belonging to any diameter of a parabola varies inversely as the square of the sine of the angle at which the corresponding ordinates are inclined to it.



Let Px' be any diameter, SP its parameter, PT a tangent at its extremity, α the angle at which the ordinates to Px' are inclined, and, consequently, PTS $= \alpha$.

Then we know that it is a property of the parabola that TPS $=$ PTS $= \alpha$,

$$\therefore\ \text{PSN} = 2\alpha.$$

Now, $\dfrac{\text{SN}}{\text{SP}} = \cos 2\alpha = 1 - 2\sin^2\alpha$,

and SN $=$ AN $-$ AS $=$ AN $+$ AS $- 2$AS $=$ SP $- 2$AS,

$$\therefore\ \frac{\text{SP} - 2\text{AS}}{\text{SP}} = 1 - 2\sin^2\alpha, \quad \therefore\ \text{AS} = \text{SP}\sin^2\alpha,$$

or $SP = \frac{AS}{\sin^2\alpha}$, $\therefore$ $SP \propto \frac{1}{\sin^2\alpha}$, since AS is constant.

(72.) If CP′, CQ′, be diameters of an ellipse, making angles α, β, respectively with the axis major, then

$$\frac{CP'^2}{CQ'^2} = \frac{a'^2}{a''^2} = \frac{a^2\sin^2\beta + b^2\cos^2\beta}{a^2\sin^2\alpha + b^2\sin^2\alpha}.$$

Let xy, $x'y'$, be the co-ordinates of P′ and Q′ respectively,

$$\text{then } x = a'\cos\alpha, \text{ and } x' = a''\cos\beta,$$
$$y = a'\sin\alpha, \qquad y' = a''\sin\beta;$$

therefore, substituting in the equation to the ellipse at the points xy, $x'y'$,

$$a^2b^2 = a''^2\{a^2\sin^2\beta + b^2\cos^2\beta\},$$
$$a^2b^2 = a'^2\{a^2\sin^2\alpha + b^2\cos^2\beta\};$$

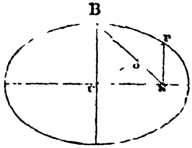
or dividing the former by the latter,

$$1 = \frac{a''^2}{a'^2}\cdot\frac{a^2\sin^2\beta + b^2\cos^2\beta}{a^2\sin^2\alpha + b^2\cos^2\beta},$$

$$\therefore \frac{a'^2}{a''^2} = \frac{a^2\sin^2\beta + b^2\cos^2\beta}{a^2\sin^2\alpha + b^2\cos^2\beta}.$$

(73.) From any point P in an ellipse draw the ordinate PN; join P and the centre C of the ellipse, and also N and the extremity B of the conjugate axis. The locus of their intersection O is required.

Let x', y', be the co-ordinates of P, then



$$y = \frac{y'}{x'}x$$

is equation to PC .... (1),

$$\text{and } y = -\frac{b}{x'}x + b$$

is equation to BN .... (2),

$$x'y = -bx + bx',$$

$$x'(y - b) = - b x,$$

$$x' = - \frac{bx}{y - b}, \quad \therefore \; x'^2 = \frac{b^2 x^2}{(y - b)^2};$$

and from (1), $x'y = xy'$

$$y' = \frac{x'y}{x},$$

$$\therefore \; y'^2 = \frac{x'^2 y^2}{x^2} = \frac{b^2 x^2 y^2}{x^2 (y - b)^2}$$

$$= \frac{b^2 y^2}{(y - b)^2}.$$

Also, since $x'y'$ is a point in the ellipse, we have

$$a^2 y'^2 + b^2 x'^2 = a^2 b^2;$$

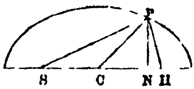
substituting $a^2 y^2 + b^2 x^2 = a^2 (y - b)^2$

$$= a^2 y^2 - 2 a^2 b y + a^2 b^2$$

$= b x^2 + 2 a^2 y = a^2 b^2$, $\therefore$ locus required is a parabola.

(74.) At what point in an ellipse is the angle formed by the two focal distances greatest?

Let P be the required point.



$\angle$ SPH $= \theta$,

CN $= x$. NP $= y$.

Now $\tan \theta = - \tan (\text{PHS} + \text{PSH})$

$$= \frac{\tan \text{PHS} + \tan \text{PSH}}{\tan \text{PHS} \cdot \tan \text{PSH} - 1}$$

$$= \frac{\dfrac{y}{ae - x} + \dfrac{y}{ae + x}}{\dfrac{y^2}{a^2 e^2 - x^2} - 1} = \frac{2 a e y}{y^2 - a^2 e^2 + x^2},$$

which must be a maximum when

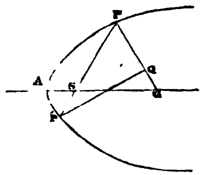
$$y^2 - a^2 e^2 + x^2 = 0;$$

$$i.e., \text{ when } x^2 + y^2 = a^2 e^2 = \text{CH}^2.$$

$$\text{But } x^2 + y^2 = \text{CP}^2 = \text{CH}^2, \qquad \therefore \text{ CP} = \text{CH}.$$

Hence, with centre C and radius CH, describe a circle cutting the ellipse in P, this will then be the required point.

(75.) The area between two normals to a parabola at the extremities of a focal chord and the curve $= \dfrac{20\,a^2}{3 \sin^3 2\theta}$, θ being the angle which one of the normals makes with the axis.



Let PP′ be the chord,

PQ and P′Q the normals, which, since PP′ passes through the focus, are at right angles to one another.

Take x, y, the co-ordinates of P and x', y', those of P′; also the angle PGS $= \theta$, $\quad\therefore\ \angle$ SPG $= \theta$.

Now,

$$\text{area APP}' = \frac{2}{3}\{xy + x'y'\} + \frac{1}{2}\{(a - x')y' - (x - a)y\}$$

$$= \frac{1}{6}\{xy + x'y'\} + \frac{a}{2}\{y + y'\}$$

$$= \frac{1}{24a}\{y^3 + y'^3\} + \frac{a}{2}\{y + y'\},$$

since xy and $x'y'$ are points in the parabola.

But $y = \text{PS} \sin 2\theta$, and $\text{PS} = a + r = 2a + \text{PS} \cos \text{PSG}$
$= 2a - \text{PS} \cos 2\theta$,

$$\therefore \text{ PS} = \frac{2a}{1 + \cos 2\theta} = \frac{a}{\cos^2\theta},$$

$$\text{and } y' = \text{P}'\text{S} \,.\, \sin 2\theta,$$

Likewise, $P'S = 2a + P'S \cos 2\theta$,

$$\therefore \; P'S = \frac{2a}{1 - \cos 2\theta} = \frac{a}{\sin^2\theta},$$

$$\therefore \; y = 2a \tan\theta, \text{ and } y' = 2a \cot\theta,$$

$$\therefore \; APP' = \frac{8a^3}{24a}\{\tan^3\theta + \cot^3\theta\} + a^2\{\tan\theta + \cot\}$$

$$= \frac{a^2}{3}\left\{\frac{\sin^6\theta + \cos^6\theta + 3\sin^2\theta\cos^2\theta}{\sin^3\theta\cos^3\theta}\right\}$$

$$= \frac{a^2}{3}\cdot\left\{\frac{\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta}{\sin^3\theta\cos^3\theta}\right\} = \frac{a^2}{3\sin^3\theta \, . \, \cos^3\theta}.$$

Again,

$$\triangle PQP' = \frac{PQ \, . \, P'Q}{2} = \frac{PP'\cos\theta \;\; PP'\sin\theta}{2} = \frac{PP'^2}{2}\sin\theta\cos\theta$$

$$= \frac{1}{2}\{PS^2 + 2\,PS \, . \, P'S + P'S^2\}\sin\theta \, . \cos\theta$$

$$= \frac{a^2}{2}\left\{\frac{\sin\theta}{\cos^3\theta} + \frac{2}{\sin\theta \, . \, \cos\theta} + \frac{\cos\theta}{\sin^3\theta}\right\}$$

$$= \frac{a^2}{2}\left\{\frac{\sin^4\theta + 2\sin^2\theta\cos^2\theta + \cos^4\theta}{\sin^3\theta \, . \, \cos^3\theta}\right\} + \frac{a^2}{2\sin^3\theta \, . \, \sin^3\theta},$$

therefore, the area

$$APQP' = APP' + PQP' = \left(\frac{a^2}{3} + \frac{a^2}{2}\right)\cdot\frac{1}{\sin^3\theta \, . \, \cos^3\theta}$$

$$= \frac{5a^2}{6\sin^3\theta \, . \, \cos^3\theta} = \frac{20a^2}{3 \, . \, (8\sin^3\theta \, . \, \cos^3\theta)}$$

$$= \frac{20a^2}{3\sin^3 2\theta}.$$

EQUATIONS OF THE SECOND DEGREE*.

(76.) The general form for equations of the second degree, being those in which the ordinates $x\,y$ are involved to the second power, is

$$A x^2 + B y^2 + C x y + a x + b y + c = 0,$$

wherein each of the constants A, B, C, a, b, c, may be either *positive* or *negative*.

Let us, in the first place, transfer the equation to two other rectangular axes parallel to the original ones, and having their origin at a point whose ordinates are $x'y'$; and (24) by substituting $x + x'$ and $y + y'$ for x and y, we shall find the corresponding equation to be

$$\begin{aligned} A(x^2 + 2x'x + x'^2) + B(y^2 + 2y'y + y'^2) \\ + C(xy + y'x + x'y' + x'y') \\ + a(x + x') + b(y + y') + c = 0; \end{aligned}$$

which, arranged for x and y, becomes

$$\begin{aligned} A x^2 + B y^2 + C x y \\ + (2 A x' + C y' + a) x + (2 B y' + C x' + b) y \\ + (A x'^2 + B y'^2 + C x' y' + a x' + b y' + c) = 0. \end{aligned}$$

(77.) The first three coefficients, A, B, C, stand unaffected with the new constants, x', y', by which we observe that they are independent of the position of the origin; and hence the position of the origin of any equation of the second degree depends entirely on the values of the three last coefficients a, b, c.

(78.) We may now assume the values of the two ordinates x', y', at pleasure, since the position of the new origin is entirely arbitrary; and consequently, by the principles of algebra, we may fulfil any two possible conditions which involve them;

* See Woolhouse's "Algebraic Geometry."

let us therefore put the coefficients of x and y each equal to nothing, viz. :—

$$2Ax' + Cy' + a = 0,$$
$$2By' + Cx' + b = 0;$$

and thence

$$x' = \frac{Cb - 2Ba}{4AB - C^2}, \; y' = \frac{Ca - 2Ab}{4AB - C^2};$$

hence also, by substitution, the last term

$$Ax'^2 + By'^2 + Cx'y' + ax' + by' + c = \frac{Cab - Ab^2 - Ba^2}{4AB - C^2} + c;$$

or by assuming

$$Cab - Ab^2 - Ba^2 + c(4AB - C^2) = G,$$

it becomes

$$= \frac{G}{4AB - C^2}$$

The equation is thus transformed into

$$Ax^2 + By^2 + Cxy + \frac{G}{4AB - C^2} = 0 \;\ldots\ldots\ldots\ldots\; (a),$$

in which the fourth and fifth terms are wanting.

(79.) Let us now transfer this equation to two other rectangular axes, inclined at an angle ω with the former, and retaining the same origin; and (24) substituting $x \cos\omega - y \sin\omega$ and $x \sin\omega + y \cos\omega$ for x and y, we get for the corresponding equation

$$A(x^2 \cos^2\omega + y^2 \sin^2\omega - 2xy \cos\omega \sin\omega)$$
$$+ B(x^2 \sin^2\omega + y^2 \cos^2\omega + 2xy \cos\omega \sin\omega)$$
$$+ C\{x^2 \cos\omega \sin\omega - y^2 \cos\omega \sin\omega + xy(\cos^2\omega - \sin^2\omega)\}$$
$$+ \frac{G}{4AB - C^2} = 0,$$

which, arranged for x and y, observing that

$$\cos^2\omega - \sin^2\omega = \cos 2\omega \text{ and } 2\cos\omega\,\sin\omega = \sin 2\omega,$$

becomes

$$(A\cos^2\omega + B\sin^2\omega + C\cos\omega\,\sin\omega)\,x^2$$
$$+ (A\sin^2\omega + B\cos^2\omega - C\cos\omega\,\sin\omega)\,y^2$$
$$+ \{C\cos 2\omega - (A - B)\sin 2\omega\}\,xy$$
$$+ \frac{G}{4AB - C^2} = 0.$$

By taking the value of ω so as to exterminate xy,

$$C\cos 2\omega - (A - B)\sin 2\omega = 0,$$
$$\text{and } \tan 2\omega = \frac{C}{A - B},$$

which reduces the equation to

$$(A\cos^2\omega + B\sin^2\omega + C\cos\omega\,\sin\omega)\,x^2$$
$$+ (A\sin^2\omega + B\cos^2\omega - C\cos\omega\,\sin\omega)\,y^2$$
$$+ \frac{G}{4AB - C^2} = 0;$$

and hence it appears that every line of the second order may be referred to two determinate rectangular axes so that its equation shall be transformed into the above form. By assuming

$$A\cos^2\omega + B\sin^2\omega + C\cos\omega\,\sin\omega = A'',$$
$$A\sin^2\omega + B\cos^2\omega - C\cos\omega\,\sin\omega = B'',$$

it becomes

$$A''x^2 + B''y^2 + \frac{G}{4AB - C^2} = 0 \;\ldots\ldots\ldots\ldots\; (b).$$

(80.) Now if the principal semi-diameters of an ellipse and hyperbola be denoted by a', b', and the former be taken for the axis of x and the origin at the centre, their equations will be as follow:

For the ellipse,

$$\frac{x^2}{a'^2} + \frac{b'^2}{y^2} = 1, \text{ or } b'^2 x^2 + a'^2 y^2 - a'^2 b'^2 = 0;$$

and for the hyperbola,

$$\frac{x^2}{a'^2} - \frac{y^2}{b'^2} = \pm 1, \text{ or } b'^2 x^2 - a'^2 y^2 \mp a'^2 b'^2 = 0\,*,$$

the under sign representing the conjugate hyperbola.

The signs may be all changed if necessary.

By means of these two equations and the foregoing transformed equation, (b), we deduce the following particulars relative to the general equation.

(81.) 1st. When A'', B'', are both negative, and G, $4AB - C^2$ have the same sign, the equation determines an *ellipse;* and when A'', B'', are both of them positive, and G and $4AB - C^2$ have different signs, the locus is also an *ellipse* †.

(82.) 2nd. When A'', B'', are of different signs, and G not $= 0$, the locus is *an hyperbola.*

(83.) 3rd. In each of these cases the squares of the principal semi-diameters are equal to

$$\frac{\pm G}{A''(4AB - C^2)}, \quad \frac{\pm G}{B''(4AB - C^2)},$$

the under sign being for the ellipse, and either sign for the hyperbola.

(84.) 4th. The values of G, A'', B'', are determined from the equations

$$G = Cab - Ab^2 - Ba^2 + c(4AB - C^2) \quad \ldots\ldots\ldots \quad (1),$$

$$\tan 2\omega = \frac{C}{A - B} \quad \ldots\ldots\ldots \quad (2),$$

$$\left.\begin{array}{l} A'' = A\cos^2\omega + B\sin^2\omega + C\cos\omega\sin\omega \\ B'' = A\sin^2\omega + B\cos^2\omega - C\cos\omega\sin\omega \end{array}\right\} \quad \ldots\ldots\ldots \quad (3),$$

* For the immediate values of A'', B'', see article 103.

† The conjugate hyperbolas have the conjugate axis of the other hyperbolas for the transverse, and transverse of the others for the conjugate.

wherein ω is the angle included between the original axis of x and the principal diameter of the curve.

(85.) 5th. The position of the centre of the curve is determined by

$$x' = \frac{Cb - 2Ba}{4AB - C^2}, \; y' = \frac{Ca - 2Ab}{4AB - C^2}.$$

(86.) 6th. When the equation is of the form

$$Ax^2 + By^2 + Cxy + c = 0,$$

wherein the fourth and fifth terms of the general equation are wanting, we have $a = 0$, $b = 0$, and thence $x' = 0$, $y' = 0$, which therefore shews the origin to be at the centre of the curve. This agrees with equation (a), article 78, where the origin is transferred to the centre.

(87.) 7th. By adding the equations (3), article 84, we find

$$A'' + B'' = A + B.$$

Hence we see that, whatever be the position of the axes of co-ordinates, the sum of the coefficients of x^2 and y^2 will be the same.

(88.) 8th. When $G = 0$ and also A'' and B'' of different signs, the general equation defines a straight line.

For in this case the transformed equation (b), article 79, becomes

$$A''x^2 + B''y^2 = 0,$$

which gives

$$\frac{y}{x} = \pm \sqrt{-\frac{A''}{B''}};$$

and this value is real when A'', B'', have different signs.

(89.) 9th. In the two following cases it will be found that no real values of x and y can possibly fulfil the equation (b); and consequently that the equation can have no locus.

First. When G and $4AB - C^2$ are of the same sign, and A'', B'', both of them positive.

Second. When G and $4AB - C^2$ are of different signs, and A'', B'', are both negative.

(90.) 10th. When $G = 0$, and A'', B'', have the same sign, no real values of x and y can satisfy the equation (b), except the particular case of $x = 0$, $y = 0$. In this case, therefore, the locus is the single point corresponding with the new origin $x'\ y'$.

(91.) 11th. It appears that by changing the position of the origin to the centre $x'\ y'$, the equation

$$A x^2 + B y^2 + C x y + a x + b y + c = 0$$

is transformed into the form

$$A x^2 + B y^2 + C x y + h = 0,$$

$$\text{wherein } h = \frac{G}{4AB - C^2}.$$

Also, that by taking two other axes of co-ordinates making an angle ω with these, so that

$$\tan 2\omega = \frac{C}{A - B},$$

the equation

$$A x^2 + B y^2 + C x y + h = 0$$

becomes of the form

$$A'' x^2 + B'' y^2 + h = 0,$$

wherein $A'' + B'' = A + B$ and the constant h is unchanged.

(92.) 12th. Let x'', y'', be the two semi diameters of the curve

$$A x^2 + B y^2 + C x y + h = 0,$$

which coincide with the axes of co-ordinates to which it is referred, and they will be determined by taking first $y = 0$ and then $x = 0$ in the equation, the results being

$$x''^2 = -\frac{h}{A},\ y''^2 = -\frac{h}{B}.$$

Let also a', b', be the principal semi-diameters which coincide with the axes to which the equation

$$A''x^2 + B''y^2 + h = 0$$

appertains; and we similarly have

$$a'^2 = -\frac{h}{A''},\ b'^2 = -\frac{h}{B''}.$$

Hence, as $A'' + B'' = A + B$, we have

$$\frac{1}{x''^2} + \frac{1}{y''^2} = \frac{1}{a'^2} + \frac{1}{b'^2};$$

That is, the sum of the reciprocals of the squares of any two semi-diameters of a curve of the second order, which are perpendicular to each other, is the same; and, in reference to the general equation, is =

$$-\frac{A + B}{h} = -\frac{A + B}{G}(4\,AB - C^2).$$

(93.) When $4\,AB - C^2 = 0$, we have (55) x', y', both of them *infinite*, which shews the centre of the curve to be infinitely remote from the origin. It becomes hence necessary to consider this case separately.

$$\text{Let }\ Ax^2 + By^2 + Cxy + ax + by + c = 0$$

be the general equation, in which $4\,AB - C^2 = 0$.

Then, transferring the origin to a point $x'y'$, the corresponding equation (76) is

$$\begin{aligned} &Ax^2 + By^2 + Cxy \\ &+ (2\,Ax' + Cy' + a)\,x + (2\,By' + Cx' + b)\,y \\ &+ (Ax'^2 + By'^2 + Cx'y' + ax' + by' + c) = 0. \end{aligned}$$

Let $x'y'$ determine some point in the curve, so that

$$Ax'^2 + By'^2 + Cx'y' + ax' + by' + c = 0,$$

and the equation becomes

$$Ax^2 + By^2 + Cxy$$
$$+ (2Ax' + Cy' + a)x + (2By' + Cx' + b)y = 0.$$

But, since $4AB - C^2 = 0$, and $\therefore C = 2\sqrt{AB}$, we have

$$Ax^2 + By^2 + Cxy = (x\sqrt{A} + y\sqrt{B})^2.$$

Hence the reduced equation is equivalent to

$$(x\sqrt{A} + y\sqrt{B})^2$$
$$+ (2Ax' + Cy' + a)x + (2By' + Cx' + b)y = 0.$$

(94.) We shall now, as in article 79, transfer this equation to two other rectangular axes proceeding from the same origin, and making an angle, ω, with the former; and, (eq. 4, p. 36), putting $x\cos\omega - y\sin\omega$ and $x\sin\omega + y\cos\omega$ for x and y, the resulting equation is

$$\{(\cos\omega\sqrt{A} + \sin\omega\sqrt{B})x - (\sin\omega\sqrt{A} - \cos\omega\sqrt{B})y\}^2$$
$$+ \{(2Ax' + Cy' + a)\cos\omega + (2By' + Cx' + b)\sin\omega\}x$$
$$- \{(2Ax' + Cy' + a)\sin\omega - (2By' + Cx' + b)\cos\omega\}y = 0.$$

Let ω satisfy the condition

$$\cos\omega\sqrt{A} + \sin\omega\sqrt{B} = 0,$$

which will give

$$\tan\omega = -\sqrt{\frac{A}{B}},\quad \cos\omega = -\sqrt{\frac{B}{A+B}},\quad \sin\omega = \sqrt{\frac{A}{A+B}},$$

and thence

$$\sin\omega\sqrt{A} - \cos\omega\sqrt{B} = \sqrt{(A+B)};$$

$$(2Ax' + Cy' + a)\cos\omega + (2By' + Cx' + b)\sin\omega$$
$$= -\frac{a\sqrt{B} - b\sqrt{A}}{\sqrt{(A+B)}},$$

and $(2Ax' + Cy' + a)\sin\omega - (2By' + Cx' + b)\cos\omega =$

$$2x'\sqrt{A(A+B)} + 2y'\sqrt{B(A+B)} + \frac{a\sqrt{A}+b\sqrt{B}}{\sqrt{(A+B)}}$$

$$= 2\sqrt{(A+B)}\left\{x'\sqrt{A} + y'\sqrt{B} + \frac{a\sqrt{A}+b\sqrt{B}}{2(A+B)}\right\}.$$

The equation thus becomes

$$(A+B)y^2 - \frac{a\sqrt{B}-b\sqrt{A}}{\sqrt{(A+B)}}x$$

$$-2\sqrt{(A+B)}\left\{x'\sqrt{A} + y'\sqrt{B} + \frac{a\sqrt{A}+b\sqrt{B}}{2(A+B)}\right\}y = 0.$$

(95.) We have (93) assumed $x'y'$ to determine a point in the curve, but not restricted ourselves to any particular point; we may, therefore, take this point where the curve is intersected by a straight line whose equation is

$$x\sqrt{A} + y\sqrt{B} + \frac{a\sqrt{A}+b\sqrt{B}}{2(A+B)} = 0,$$

by means of which we shall have

$$x'\sqrt{A} + y'\sqrt{B} + \frac{a\sqrt{A}+b\sqrt{B}}{2(A+B)} = 0,$$

which reduces the equation to

$$(A+B)y^2 - \frac{a\sqrt{B}-b\sqrt{A}}{\sqrt{(A+B)}}.x = 0,$$

$$\text{or } y^2 - \frac{a\sqrt{B}-b\sqrt{A}}{(A+B)^{\frac{3}{2}}}.x = 0 \text{ ...... } (c).$$

But the equation of a parabola, whose parameter is p, taking the origin at the vertex and the principal axis for the axis of x, is

$$y^2 = px, \text{ or } y^2 - px = 0.$$

Hence the following particulars:—

(96.) 1st. When $a\sqrt{B} - b\sqrt{A}$ *not* $= 0$, the locus is a *parabola* whose parameter is equal to

$$\frac{a\sqrt{B} - b\sqrt{A}}{(A + B)^{\frac{3}{2}}}.$$

(97.) 2nd. According to article 8, the equation

$$x\sqrt{A} + y\sqrt{B} + \frac{a\sqrt{A} + b\sqrt{B}}{2(A + B)} = 0$$

defines a straight line inclined to the original axis of x at an angle whose tangent $= -\sqrt{\frac{A}{B}}$, and which is therefore equal to ω, the inclination to the axis of the curve with the axis of x; this line (95) also passing through the vertex $x'y'$, it must coincide with the axis of the curve. Therefore, the above equation properly represents the *principal diameter* of the curve; by uniting it with the original equation, we may hence find the co-ordinates $x'y'$ of its intersection with the curve, or the vertex.

(98.) 3rd. If $a\sqrt{B} - b\sqrt{A} = 0$, or $a\sqrt{B} = b\sqrt{A}$, the equation (e) gives simply

$$y = 0,$$

which shews the locus in this case to be a straight line corresponding with the new axis of x, the equation of which is given (97).

(99.) 4th. The equation $4AB - C^2 = 0$, giving $C^2 = \pm 2\sqrt{AB}$, the values of the constants A, B, must have the same sign to make C real; that is, they must be either both of them positive or both negative; and hence we may consider them both positive, for, when negative, they can be made so by preliminarily changing all the signs of the original equation. If, under this consideration, C be negative we shall have $C = -2\sqrt{AB}$ instead of $+2\sqrt{AB}$; in this case, the foregoing operations hold good by either substituting $-\sqrt{A}$ instead of $\sqrt{A}$ or $-\sqrt{B}$ for $\sqrt{B}$, or by considering either $\sqrt{A}$ or $\sqrt{B}$ to have a negative value; and tan ω will become hence $= + \sqrt{\frac{A}{B}}$ instead of $- \sqrt{\frac{A}{B}}$.

Thus we see that, when C is *negative*, tan ω is *positive*, and $\omega < \frac{\pi}{2}$; and that, when C is *positive*, tan ω is *negative* and

$$\therefore \omega < \frac{\pi}{2}.$$

The foregoing investigations lead immediately to the solutions of the three following propositions.

(100.) *To express the equations of the principal diameters of a curve of the second order, which is determined by the general equation.*

The co-ordinates of the centre (85) are

$$x' = \frac{Cb - 2Ba}{4AB - C^2},\ y' = \frac{Ca - 2Ab}{4AB - C^2}.$$

Let ω denote the inclination of one of the principal diameters of the curve with the co-ordinate axis of x; and (84)

$$\tan 2\omega = \frac{C}{A - B},$$

from which

$$\tan \omega = \frac{\sec 2\omega - 1}{\tan 2\omega} = \frac{\sqrt{\{(A-B)^2 + C^2\}} - (A - B)}{C}.$$

Now the diameter being inclined to the co-ordinate axis of x at the angle ω, and also passing through the centre $x'\,y'$ of the curve, its equation (13) is

$$y - y' = (x - x') \tan \omega.$$

Hence, by substitution, we have

$$y - \frac{Ca - 2Ab}{4AB - C^2} =$$

$$\left(x - \frac{Cb - 2Ba}{4AB - C^2}\right) \cdot \frac{\sqrt{\{(A-B)^2 + C^2\}} - (A - B)}{C} \ldots (x),$$

for the equation of one of the principal diameters.

The other diameter passing through the centre $x' y'$ perpendicular to this, its equation (15) is

$$y - \frac{Ca - 2Ab}{4AB - C^2} =$$

$$-\left(x - \frac{Cb - 2Ba}{4AB - C^2}\right) \cdot \sqrt{\frac{C}{\{(A-B)^2 + C^2\} - (A-B)}},$$

or, which is the same,

$$y - \frac{Ca - 2Ab}{4AB - C^2} =$$

$$-\left(x - \frac{Cb - 2Ba}{4AB - C^2}\right) \cdot \sqrt{\frac{\{(A-B)^2 + C^2\} + (A-B)}{C}} \ldots (y).*$$

(101.) *Cor.* 1. If the origin of the ordinates be the centre of the curve, its equation (86) will be of the form

$$Ax^2 + By^2 + Cxy + c = 0;$$

and we shall have $a = 0$, $b = 0$. In this case, therefore, the equations of the principal diameters are

$$y = \sqrt{\frac{\{(A-B)^2 + C^2\} - (A-B)}{C}} \cdot x,$$

$$\text{and } y = -\sqrt{\frac{\{(A-B)^2 + C^2\} + (A-B)}{C}} \cdot x.$$

(102.) *Note.* The equation

$$\tan 2\omega = \frac{C}{A - B}$$

* By uniting these equations of the principal diameters with the given equation of the curve we may thence find the positions of the vertices.

applies equally to both diameters. For, if 2ω fulfil this equation, it will also hold good when $2\omega \pm \pi$ is substituted; and ω denoting the inclination of one of the diameters, $\omega \pm \frac{\pi}{2}$ will evidently be that of the other.

From this equation we derive generally

$$\tan\omega = \frac{\sec 2\omega - 1}{\tan 2\omega} = \pm \sqrt{\frac{\{(A-B)^2 + C^2\} - (A-B)}{C}},$$

the upper sign appertaining to one of the axes, and the under sign to the other.

Thus, by making use of the under sign, the equation (x) will become the same as the equation (y), and *vice versâ* ——, because

$$-\sqrt{\frac{\{(A-B)^2+C^2\}-(A-B)}{C}} \cdot \sqrt{\frac{\{(A-B)^2+C^2\}-(A-B)}{C}}$$
$$= -1.$$

When $4AB - C^2 = 0$, see article 97.

(103.) *The equation of a curve of the second order being given to find the values of its principal semi-diameters.*

The squares of the semi-diameters are (83) equal to

$$\frac{\pm G}{A''(4AB - C^2)}, \quad \frac{\pm G}{B''(4AB - C^2)},$$

wherein (84)

$$G = Cab - Ab^2 - Ba^2 + c(4AB - C^2);$$

$$A'' = A\cos^2\omega + B\sin^2\omega + C\cos\omega\sin\omega,$$

$$B'' = A\sin^2\omega + B\cos^2\omega - C\cos\omega\sin\omega,$$

$$\text{and } \tan 2\omega = \frac{C}{A-B}.$$

From the last we deduce

$$\cos^2\omega = \frac{1}{2}\left(1 + \frac{1}{\sec 2\omega}\right) = \frac{1}{2}\left(1 + \frac{A - B}{\sqrt{\{(A-B)^2 + C^2\}}}\right),$$

$$\sin^2\omega = \frac{1}{2}\left(1 - \frac{1}{\sec 2\omega}\right) = \frac{1}{2}\left(1 - \frac{A - B}{\sqrt{\{(A-B)^2 + C^2\}}}\right),$$

$$\cos\omega\sin\omega = \frac{C}{2\sqrt{\{(A-B)^2 + C^2\}}};$$

and hence we get

$$A'' = \frac{A + B + \sqrt{\{(A-B)^2 + C^2\}}}{2},$$

$$B'' = \frac{A + B - \sqrt{\{(A-B)^2 + C^2\}}}{2}.$$

These and the foregoing value of G substituted, the squares of the principal semi-diameters of the curve are found equal to

$$\pm\frac{2\{Cab - Ab^2 - Ba^2 + c(4AB - C^2)\}}{(4AB - C^2)[A + B + \sqrt{\{(A-B)^2 + C^2\}}]},$$

$$\pm\frac{2\{Cab - Ab^2 - Ba^2 + c(4AB - C^2)\}}{(4AB - C^2)[A + B - \sqrt{\{(A-B)^2 + C^2\}}]},$$

the under sign being for the ellipse and either sign for the hyperbola (83).

(104.) When the origin is at the centre of the curve (91) $a = 0$, $b = 0$; and therefore, in this case, the squares of the principal semi-diameters are equal to

$$\frac{\pm 2c}{A + B + \sqrt{\{(A-B)^2 + C^2\}}},\quad \frac{\pm 2c}{A + B - \sqrt{\{(A-B)^2 + C^2\}}}.$$

(105.) *To determine the particular description of a curve of the second order from the immediate relative values of the constants which belong to its equation.*

In (81), (82), and the subsequent articles, the different cases are severally stated, throughout the various relations of A'', B'', G, $4AB - C^2$, &c., where A'', B'', are (84) expressed in terms of the coefficients A, B, C, by means of the arc ω as a subsidiary. It is hence only necessary to transfer the relations of A'', B'', to those of the immediate coefficients A, B, C, which may be easily effected from their values which have already been found (103), viz.:

$$A'' = \frac{A + B + \sqrt{\{(A - B)^2 + C^2\}}}{2},$$

$$B'' = \frac{A + B - \sqrt{\{(A - B)^2 + C^2\}}}{2}.$$

Thus it is evident that, when $(A + B)^2$ is greater than $(A - B)^2 + C^2$, the sign of $A + B$ cannot be affected with either the addition or subtraction of $\sqrt{\{(A - B)^2 + C\}}$, and, consequently, that the values of A'', B'', will both have the same sign with $A + B$. But, when $(A + B)^2$ is greater than $(A - B)^2 + C^2$, we shall have

$$(A + B)^2 - \{(A - B) + C^2\} = 4AB - C^2 \textit{ positive.}$$

Hence, when $4AB - C^2$ is *positive*, A'' and B'' will both of them have the same sign with $A + B$, that is, they will both be positive when $A + B$ is positive, and both negative when $A + B$ is so.

It is also pretty obvious that, when $(A + B)^2$ is less than $(A - B)^2 + C^2$, the values of A'', B'', will have different signs, that is, the one will be positive and the other negative.

In this case we shall have

$$(A + B)^2 - \{(A - B)^2 + C^2\} = 4AB - C^2 \textit{ negative.}$$

Thus we see, when $4AB - C^2$ is *negative*, that A'', B'', are of different signs*.

Again, under the class $4AB - C^2 = 0$, when the value of $a\sqrt{B} - b\sqrt{A} = 0$, we shall have

$$2\sqrt{A}\,(a\sqrt{B} - b\sqrt{A}) = 2a\sqrt{AB} - 2Ab = 0,$$

$$\text{or } Ca - 2Ab = 0.$$

Hence, also, when $a\sqrt{B} - b\sqrt{A}$ not $= 0$, we shall have

$$Ca - 2Ab \text{ not } = 0.$$

By carefully comparing these relations with the articles (51), (52), (58), (59), (60), (66), and (68), we find the different descriptions of the curve to be as in the following arrangement, wherein

$$G = Cab - Ab^2 - Ba^2 + c\,(4AB - C^2).$$

* These relations are also pretty evident from the equations

$$A'' + B'' = A + B,$$
$$4A''B'' = 4AB - C^2.$$

When $4AB - C^2$ is *positive*, and $A + B$ and G are of *different signs*, the locus is an ELLIPSE.

,,	,,	,,	are of *the same signs*,	,,	IMPOSSIBLE.
,,	,,	and $A + B$ and $G = 0$,		,,	a POINT.
,,	is *negative*	,, and G not $= 0$,	,,	,,	an HYPERBOLA.
,,	,,	,, $G = 0$,	,,	,,	a STRAIGHT LINE.
$4AB - C^2 = 0$	,,	and $Ca - 2Ab$ not $= 0$,	,,	,,	a PARABOLA.
,,	,,	$Ca - 2Ab = 0$,	,,	,,	a STRAIGHT LINE.

Note.—When the origin of the axes is at the centre of the curve, and, consequently, the equation is $Ax^2 + By^2 + Cxy + c = 0$, the value of G is simply $c(4AB - C^2)$.

(106.) *Parabola related to its Conjugate Diameters.*

Substitute for x and y respectively, the values

$$a + x\cos\alpha + y\cos\alpha', \text{ and } b + x\sin\alpha + y\sin\alpha';$$

in the equation $y^2 = px$, p being the parameter, and we have

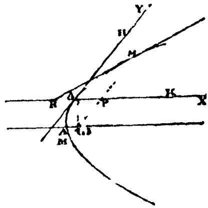
$$b^2 + (x\sin\alpha + y\sin\alpha')^2 + 2b(x\sin\alpha + y\sin\alpha')$$
$$= b^2 + x^2\sin^2\alpha + y^2\sin^2\alpha' + 2xy\sin\alpha\sin\alpha' + 2bx\sin\alpha$$
$$+ 2by\sin\alpha' = p(a + x\cos\alpha + y\cos\alpha'), \text{ or}$$

$$\left.\begin{array}{l} y^2\sin^2\alpha' + x^2\sin^2\alpha + 2xy\sin\alpha\sin\alpha' + b^2 - ap \\ + (2b\sin\alpha - p\cos\alpha)x + (2b\sin\alpha' - p\cos\alpha')y \end{array}\right\} = 0 \ldots (1).$$

That this equation may be of the form $y^2 = kx$, the following conditions must obtain

$$\sin\alpha\sin\alpha' = 0, \quad \sin^2\alpha = 0, \quad 2b\sin\alpha' - p\cos\alpha' = 0,$$

and $b^2 - ap = 0$..................... (2);



when these are fulfilled the equation becomes

$y^2 = \dfrac{p}{\sin^2\alpha'} \cdot x$ .... (3).

As the second of the above conditions establishes the first, it follows that, to determine α, α', a, b, we have only three distinct equations. Thus the number of systems of axes with respect to which the equation preserves the above form is infinite. The relation $\sin\alpha = 0$ shews, besides, that the angle contained by the old and new axis of $x = 0$, or the new axis of x, is parallel to the principal axis. Therefore, in the parabola *all the diameters are parallel to the principal axis*.

In the second place $b^2 - ap = 0$, being that which $y^2 - px$, or $y^2 - px = 0$, becomes when we replace x and y, for the co-ordinates, a, b, of the new origin, we conclude that this origin is on the curve. In giving to a any arbitrary value, we find from the equation $b^2 - pa = 0$, the corresponding value of b, and the point A′ determined by these values represents the new origin.

Lastly, the equation $2\,b \sin \alpha' - p \cos \alpha' = 0$, gives

$$\tan \alpha' = \frac{p}{2\,b} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (4),$$

which is an expression similar to $a = \frac{p}{2\,y''}$, which has been found for the tangent to a parabola, which shews that *the new axis of y is a tangent to the curve.*

By equation (4),

$$\tan^2 a' = \frac{p^2}{4\,b^2}, \qquad \therefore \cos^2 a' = \frac{4\,b^2}{4\,b^2 + p^2},$$

$$\sin^2 a' = \tan^2 a' \cos^2 a' = \frac{p^2}{4\,b^2} \cdot \frac{4\,b^2}{4\,b^2 + p^2}$$

$$= \frac{p^2}{4\,b^2 + p^2} = \frac{p^2}{4\,p\,a + p^2} = \frac{p}{4\,a + p} \text{ (since } b^2 = p\,a),$$

$$\therefore \frac{p}{\sin^2 a'} = 4\,a + p = 4\left(a + \frac{p}{4}\right).$$

Now, if we suppose AG to be the abscissa of the new origin A′ related to the old axes, and we draw the radius vector FA′, we know that this radius vector is expressed by $a + \frac{p}{4}$, therefore $\frac{p}{\sin^2 a'} = 4$ A′F, that is, the parameter of the parabola related to a system of conjugate axes is four times the distance of the focus from the new origin *.

Putting p' for the new parameter, we have the equation $y' = p'x$ for the equation of the parabola related to its diameters.

The equation $y^2 = p'\,x$, or $\frac{y^2}{x} = p'$, shews that, for any system of conjugates, the squares of the ordinates are proportional to the corresponding abscissæ.

* This has been done in a different way at page 89.

Examples.

(1.) Let a circle be described, radius R, about a triangle formed by three tangents to a parabola, whose latus rectum is a, and let x_1, x_2, x_3, be the rectangular abscissæ of the three points of contact, measured from the directrix, then

$$a\mathrm{R}^2 = x_1 x_2 x_3.$$

(2.) Given the base of a triangle and the sum of the tangents of the angles at the base, to determine the locus of the vertex.

(3.) If an ellipse the greatest possible be inscribed within a given triangle, and within the ellipse the greatest triangle, and again within the triangle the greatest ellipse, &c., *ad infinitum*, find the sum of the area of all the triangles and of all the ellipses.

(4.) Given the base of a triangle and the difference of the angles at the base, to determine the locus of the vertex.

(5.) Find the locus of the intersection of the tangent to a parabola, and the perpendicular upon it from the vertex.

(6.) Shew that the locus of the focus of a parabola which shall touch a given straight line, and have a given vertex, is a parabola.

(7.) If any two conic sections have a common focus, their intersections range upon two straight lines passing through the intersection of the directrices.

(8.) Given the base and sum of the sides of a triangle, to find the locus of the centre of the inscribed circle.

(9.) Given the base of a triangle and the sum of the other two sides, to find the locus of the inscribed circle.

(10.) Tangents to a parabola form a given angle with each other; what is the locus of their point of intersection?

(11.) A series of circles being described touching the double branches of the cissoid; the first also touching the directrix, the second the first, the third the second, and so on, it is required to determine the abscissa of the point of contact of the nth circle.

(12.) Let seven points in a conic section be connected in any order by the successive lines a_1, a_2, a_3 ... a_7, forming *any* closed heptagon; let P_{mn} be the intersection of a_m and a_n: then the three pairs of lines $\mathrm{P}_{14}\,\mathrm{P}_{37}$ and $\mathrm{P}_{15}\,\mathrm{P}_{46}$, $\mathrm{P}_{35}\,\mathrm{P}_{46}$ and $\mathrm{P}_{26}\,\mathrm{P}_{37}$, $\mathrm{P}_{14}\,\mathrm{P}_{35}$ and $\mathrm{P}_{15}\,\mathrm{P}_{26}$, will intersect in the same straight line.

(13.) Let

$$y^2 = ax^2 + 2bx + c \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$
$$y^2 = a_1 x^2 + 2b_1 x + c \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (2),$$

be the equations to two conic sections; then, if

$$(b - b_1)^2 = b^2 - ac \pm 2(b^2 - ac)^{\frac{1}{2}} (b_1^2 - a_1 c)^{\frac{1}{2}},$$

an indefinite number of triangles may be inscribed within (1) whose sides shall be tangents to the curve (2).

(14.) If three points be taken in a parabola, the ordinates of which are mp, np, and $-\dfrac{mn}{m+n} \cdot p$, the circle which circumscribes the triangle formed by tangents at these points, will pass through the vertex and focus: if the ordinates be mp, np, and $-\dfrac{mn+1}{m+n} \cdot p$, it will pass through the focus and the intersection of the axis and directrix: if mp, np, and $-\dfrac{mn-1}{m+n} \cdot p$, it will touch the axis at the focus; p being the semi-parameter, and m and n any numbers.

(15.) If two conic sections, whose axes are parallel, intersect one another, the lines joining the points of intersection are equally inclined to the axis.

(16.) Any *seven* points being given, two others can be assigned, such that the *thirty-six* connecting lines of the system shall be tangents *by six together* of *twelve* conic sections. In how many ways can these points be found?

(17.) A conic section is cut in four points by a circle, and two lines, each passing through two of the points of intersection, are made the axes of co-ordinates, their point of meeting being the origin; shew that the equation to the conic section is of the form $x^2 + bxy + y^2 + dx + ey + f = 0$.

(18.) Describe a circle touching three semicircles, the distance of its centre from the diameter $= 2$ (its radius).

(19.) The tangents at the extremity of any focal chord are perpendicular to one another.

(20.) One circle touches another internally, and another circle touches both of them; find the locus of the centre of this latter circle.

(21.) In the parabola, if PT and QT be tangents to it at P and Q, respectively, intersecting in T and PQ, a normal at Q, then will the directrix bisect PT.

(22.) The locus of the feet of the perpendiculars, dropped from the focus upon the tangent to a parabola, is the line that touches the parabola at its vertex.

$$y - y' = \frac{2a}{y'}(x - x'),$$

$$\therefore y = \frac{2a}{y'}x + \frac{2ax'}{y'},$$

$$p = -\sqrt{a^2 + ax'}, \qquad \therefore p^2 = a^2 + ax'.$$

Let $x = o$ in the equation, $yy' = 2a(x + x')$,

$$\text{and } yy' = 2ax',$$

$$\therefore y = \frac{2ax'}{y'},$$

$$\therefore p^2 = \frac{4a^2x'^2}{y'^2} + a^2 = ax' + a^2, \text{ as before,}$$

$\therefore$ the locus is as above.

(23.) From the vertex of a parabola a straight line is drawn, inclined at an angle of 45° to the tangent at any point; find the equation to the curve which is the locus of their intersection.

Let x', y', be the co-ordinates of P, then, since it is in the parabola,

$$y'^2 = 4ax' \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

equation to the tangent at P is $yy' = 2a(x+x') \ldots (2)$,

$$\angle RAT = 45° - RTA,$$

$$\therefore \tan RAT = -\tan RAX = \frac{1 - \frac{2a}{y'}}{1 + \frac{2a}{y'}} = \frac{y' - 2a}{y' + 2a},$$

$$\text{or } \tan RAX = \frac{2a - y'}{2a + y'},$$

$\therefore$ equation to R A, passing through the origin, is

$$y = \frac{2a - y'}{2a + y'} x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (3),$$

$$\text{from (2)}, \; y' = 2a\left(\frac{x + x'}{y}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (4),$$

and from (3), $2ay + yy' = 2ax - xy'$,

$$\text{or } (x + y)\, y' = 2a(x - y),$$

$$\therefore\; y' = 2a\left(\frac{x - y}{x + y}\right) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \quad (5),$$

$$\therefore\; (4) \text{ and } (5)\; \frac{x + x'}{y} = \frac{x - y}{x + y}, \qquad \therefore\; x' = -\frac{x^2 + y^2}{x + y}.$$

By (4), substituting for x', $y' = 2a\left(\dfrac{x - y}{x + y}\right)$.

Substituting these values of x', y', in (1), we obtain

$$a(x - y)^2 + (x + y)(x^2 + y^2) = 0, \text{ the equation required.}$$

(24.) Prove that the length of the longer normal to the ellipse is $\sqrt{a^2 + \dfrac{c^2}{e^2}}$.

(25.) Find the equation to the normal to a parabola under the form $y + 2am = mx - am^3$.

(26.) Determine the locus of a point within a plane triangle. so that the sum of the squares of the straight lines drawn from it to the angular points is constant.

(27.) Given the base of a plane triangle and the difference of the angles at the base; find the curve traced by the vertex.

(28.) Two given unequal circles touch one another externally; shew that the locus of the centre of a circle which always touches the other two is a hyperbola. Find the axes and e, and shew what the hyperbola becomes when the given circles are equal.

(29.) Shew that, by transforming the axes through 30°, the equation

$$ay = 3 \,.\, x^2 + 2\sqrt{3} \,.\, xy + y$$

belongs to the common parabola.

(30.) If, with the co-ordinates of any point in the quadrant of an ellipse as semi-axes, another quadrant be described, with the same centre, the chord of the former will always be a tangent to the latter.

Let $x\,y$ be the co-ordinates of P,

XY those of T.

Then equation to the exterior ellipse is

$$y^2 = \frac{b^2}{a^2}(a^2 - x^2) \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1),$$

and $Y^2 = \frac{y^2}{x^2}(x^2 - X^2)$ is equation to the interior;

substituting from (1) this becomes

$$Y^2 = \frac{b^2}{a^2} \cdot \frac{a^2 - x^2}{x^2}(x^2 - X^2) \ldots\ldots\ldots\ldots (2);$$

equation to A B is

$$Y - b = -\frac{b}{a}X,$$

$$\therefore\ Y = b - \frac{b}{a}, \quad X = \frac{b}{a}(a - X) \ldots\ldots\ldots\ldots (3);$$

combining (3) with (2)

$$(a - X)^2 = \frac{a^2 - x^2}{x^2}(x^2 - X^2)$$

$$x^4 - 2aXx^2 + a^2X^2 = 0,$$

$$(x^2 - aX)^2 = 0,$$

$$x^2 - aX = 0,$$

$$X = \frac{x^2}{a}.$$

Hence the chord meets the quadrant in only one part, and therefore a tangent to it.

(31.) The side of an equilateral hexagon inscribed in an lipse eccentricity e, with two sides parallel to the axis ajor, is to the side of one inscribed in a circle on major is : : $4 - 2e^2 : 4 - e^2$.

Let AP be the side of the hexagon in ellipse, then x and y ing the co-ordinates of P, and $AC = a$.

$$AP = \sqrt{(a-x)^2 + y^2} = PQ,$$

$$= 2x,$$

$$\therefore\ (a-x)^2 + y^2 = 4x^2,$$

$$y^2 = 4x^2 - (a-x)^2 = (1-e^2)(a^2 - x^2) \text{ by the ellipse,}$$

$$(4 - e^2)x^2 + 2ax = (2 - e^2)a^2,$$

$$x^2 + \frac{2a}{4-e^2}x = \frac{2-e^2}{4-e^2}a^2,$$

$$x^2 + \frac{2a}{4-e^2}x + \frac{a^2}{(4-e^2)^2} = \left\{\frac{2-e^2}{4-e^2} + \frac{1}{(4-e^2)^2}\right\}a^2$$

$$= \frac{8 - 4e^2 - 2e^2 + e^4 + 1}{(4-e^2)^2}a^2$$

$$= \frac{9 - 6e^2 + e^4}{(4-e^2)^2}a^2,$$

$$\therefore\ x + \frac{a}{4 - e^2} = \frac{3 - e^2}{4 - e^2}\,a,$$

$$x = \frac{3 - e^2 - 1}{4 - c^2}\,a = \left(\frac{2 - e^2}{4 - e^2}\right)a,$$

$$\text{A P} = 2x = \text{side of hexagon} = 2a\left(\frac{2 - e^2}{4 - e^2}\right),$$

and side of hexagon in the circle on major axis $= a$,

$$\therefore\ \text{A P} : a - 4 - 2e^2 : 4 - e^2.$$

(32.) If ϱ, ϱ', radii vectores, be at right angles to each other from the centre of an ellipse, $\frac{1}{\varrho^2} + \frac{1}{\varrho'^2} = \frac{1}{a^2} + \frac{1}{b^2}$.

$$a^2 y^2 + b^2 x^2 = a^2 b^2,$$

$$\therefore\ \varrho^2 = \frac{a^2 b^2}{a^2 \sin^2\theta + b^2 \cos^2\theta},$$

$$\text{and } \varrho'^2 = \frac{a^2 b^2}{a^2 \cos^2\theta + b^2 \sin^2\theta},$$

$$\frac{1}{\varrho^2} = \frac{a^2 \sin^2\theta + b^2 \cos^2\theta}{a^2 b^2} = \frac{b^2 - (b^2 - a^2)\sin^2\theta}{a^2 b^4},$$

$$\text{also } \frac{1}{\varrho'^2} = \frac{a^2 + (b^2 - a^2)\sin^2\theta}{a^2 b^2},$$

$$\frac{1}{\varrho^2} + \frac{1}{\varrho'^2} = \frac{a\ + b^2}{a^2 b^2} = \frac{1}{a^2} + \frac{1}{b^2}.$$

(33.) If two parabolas have a common axis, a straight line touching the interior, and bounded by the exterior, will be bisected in the point of contact.

(34.) If C be the centre of an ellipse, and in the normal to any point P, PQ be taken equal to the semi-conjugate at P, Q will trace out a circle round C.

(35.) An ellipse has a square described, touching it at the extremities of the minor axis; an ellipse upon the major axis circumscribes this square. This ellipse is dealt with in the same way as before, and these operations are continued till there are $(n + 1)$ ellipses altogether; prove that if the original eccentricity equals $\sqrt{\frac{n}{n+1}}$ the last ellipse becomes a circle.

The side of the square must evidently be $2b$, and if $b_1 =$ semi-minor axis of the first circumscribing ellipse,

$$\therefore\ b^2 = \frac{b_1^2}{a^2}\{a^2 - b^2\},$$

$$a^2 b^2 = b_1^2\{a^2 - b^2\},$$

$$b_1^2 = \frac{a^2 b^2}{a^2 - b^2};$$

$$\text{similarly, } b_2^2 = \frac{a^2 b^2}{a^2 - 2b^2},$$

$$b_n^2 = \frac{a^2 b^2}{a^2 - nb^2}.$$

But if this last ellipse be a circle, $b_n^2 = a^2$,

$$\therefore\ a^2 - nb^2 = b^2,$$

$$(1 + n)\, b^2 = a^2,$$

$$\frac{b^2}{a^2} = \frac{1}{1+n}.$$

$$\text{But } e^2 = 1 - \frac{b^2}{a^2} = 1 - \frac{1}{1+n} = \frac{n}{1+n},$$

$$\therefore\ e = \sqrt{\frac{n}{n+1}}.$$

(36.) If $y^2 - x^2 = a^2$, and $2x'y' = a^2$, represent the same curve referred to rectangular axes with the same origin, find the angle the axis of one system makes with the axis of the other.

Let θ equal the angle required; then, since

$$x = x\cos\theta - y\sin\theta,$$

$$y = x\sin\theta + y\cos\theta;$$

therefore, substituting in the first

$$(x\sin\theta + y\cos\theta)^2 - (x\cos\theta - y\sin\theta)^2 = a^2,$$

$$\text{or } (x^2 - y^2)(\sin^2\theta - \cos^2\theta) + 4xy\sin\theta\cos\theta = a^2,$$

and this is to have the form $a^2 = 2xy$,

$$\therefore \sin^2\theta - \cos^2\theta = 0, \quad 2\sin\theta\cos\theta = 1,$$

$$\text{or } \sin^2\theta = \cos^2\theta, \quad \text{and } \sin\theta . \cos\theta = \frac{1}{2},$$

$$\therefore \theta = \frac{\pi}{4}.$$

(37.) Shew that a conic section is the locus of a point whose distance from a given point is a linear function of co-ordinate of the former point.

$$r = A + Bx + Cy,$$

$$r^2 = x^2 + y^2 = (A + Bx + Cy)^2,$$

an equation of the second degree; therefore, that to a conic section.

(38.) Shew that the angle between two conjugate diameters in an ellipse can never be acute, and the angle between any two conjugate diameters in the hyperbola can never be obtuse.

If γ be the angle between the conjugate diameters,

$$y = mx,$$

$$y = m'x,$$

$$\tan\gamma = \tan(-\alpha + \beta) = \frac{m' - m}{1 + mm'};$$

and this in the ellipse

$$= \frac{-\left(1 + \frac{b^2}{m^2 a^2}\right) m}{1 - \frac{b^2}{a^2}},$$

and is negative; therefore γ is obtuse.

In the hyperbola, $mm' = \frac{b^2}{a^2}$, therefore, m and m' have the same sign or the conjugate diameters lie in the same quadrant; therefore, the angle between them is never obtuse.

(39.) In the hyperbola, if the straight line joining the points of contact of two tangents which intersect in a point (hk) (external) always pass through the focus, find the locus of the point.

The equation to the line joining the points of contact is

$$a^2 ky - b^2 hx = -a^2 b^2,$$

$$x = ae, \quad y = 0;$$

therefore, substituting $-b^2 hae = -a^2 b^2$,

$$\text{or } hae = a^2, \quad h = \frac{a}{e},$$

therefore the locus is the directrix.

(40.) Find the condition that the curve $ax^2 + bxy + cy^2 = d$ may have an asymptote.

(41.) Find the equation to a circle referred to two tangents as co-ordinate axes, and obtain also the equation to the tangent to the circle thus referred.

Let Ax, Ay, be the co-ordinated axes, P any point of the circle whose co-ordinates are x, y, and $c =$ the radius of the circle whose centre is C. Join CP; then

$$(x - c)^2 + (y - c)^2 = c^2,$$
$$\text{or, } x^2 + y^2 - 2c(x + y) + c^2 = 0,$$

is the equation to the circle.

Again, from C draw CV parallel to Ax, meeting the tangent PT at the point P, in V and draw PN parallel to Ay, then

$$\angle \text{PTA} = \angle \text{CPN}, \text{ and } \tan \text{PTA} = -\tan \text{PT}x,$$

$$\therefore \tan \text{PT}x = -\frac{x - c}{y - c},$$

consequently the equation to PT passing through the point P (x, y), is

$$\text{Y} - y = -\frac{x - c}{y - c}(\text{X} - x),$$

$$\text{or, } (\text{Y} - y)(y - c) + (\text{X} - x)(x - c) = 0,$$

which is the equation to the tangent.

(42.) In an ellipse, if α be the vectorial angle of the point of contact referred to the focus, the polar equation to the tangent is

$$a(1 - e^2) = r\{e \cos \theta + \cos(\theta - \alpha)\};$$

hence, shew that the straight line joining the focus and the intersection of two tangents bisects the angle between the radii vectores of their points of contact,

$$\text{ASP} = \alpha.$$

If x be the abscissa of P, the point of contact from the centre of the ellipse,

$$\left.\begin{aligned} \text{SP} &= a - ex \\ \therefore ex &= a - \text{SP} \end{aligned}\right\} \therefore \text{ST} = \frac{a^2}{x} - ae = \frac{ae\,.\,\text{SP}}{a - \text{SP}}.$$

The equation to a straight line is $\frac{1}{r} = A\cos\theta + B\sin\theta$.

If $\theta = 0$, $A = \frac{1}{ST} = \frac{a - SP}{ae \, . \, SP} = (\text{when } \theta = a)\frac{\cos a + e}{a(1-e^2)}$,

from equation $r = SP = \frac{a(1-e^2)}{1 + e\cos\theta}$,

$$\text{and } B = \frac{\sin a}{a(1-e^2)},$$

$$\therefore \frac{1}{r} = \frac{e\cos\theta + \cos(\theta - a)}{a(1-e^2)}.$$

For the point of intersection of two tangents to the points whose vectorial angles are α, β, we have

$$e\cos\theta + \cos(\theta - a) = e\cos\theta + \cos(\theta - \beta),$$

$$\cos(\theta - a) = \cos(\theta - \beta),$$

$$\theta - a = \beta - \theta.$$

$$\theta = \frac{\alpha + \beta}{2}.$$

(43.) Find the relation among the coefficients in the general equation of the second order, that all diameters may be parallel to one another.

(44.) Let lines be drawn from the vertex of a parabola to the angles of a triangle formed by the intersection of three tangents to the curve, make angles with the axis denoted by θ', θ'', θ'''; also let the lines drawn from the vertex to the points of contact make angles with the axis denoted by θ_1, θ_2, θ_3. Then

$$\tan\theta' + \tan\theta'' + \tan\theta''' = \tan\theta_1 + \tan\theta_2 + \tan\theta_3.$$

(45.) Construct the curve whose equation is

$$xy - 2y + x - 1 = 0.$$

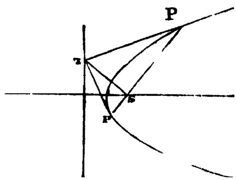
(46.) Find the position of the curve of which the equation is

$$y^2 - 2xy + 3x^2 + 2y - 4x - 3 = 0.$$

(47.) TP and TP′ are two tangents to a parabola, at right angles to each other,

$\text{TP} = b$, and $\text{TP}' = c$:

find the latus rectum



(48.) Given the base of a triangle, and the product of the tangents of the base angles, to find the locus of the vertex.

(49.) Given the base of a triangle, and the difference of the tangents of the angles at the base, to find the locus of the vertex.

(50.) Given the diameter of a parabola, and a tangent through its vertex, to find the locus of the vertex.

(51.) If $(\alpha x + \beta y = \gamma)$ be the equation of any tangent to the ellipse, to determine what relation must hold between α, β, and γ.

(52.) If θ and θ' be the angles which two conjugate semi-diameters r and r' of an ellipse make with the axis major, to shew that

$$\frac{\sin(\theta' + \theta)}{\sin(\theta' - \theta)} = \frac{r^2 - r'^2}{a^2 - b^2}.$$

(53.) The product of the two perpendiculars let fall from any point of the hyperbola upon the asymptotes is invariable.

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Hydraulics.
Hydrodynamics.
Hydrostatic press.
Hydrostatics.
Imperial standard measures of Great Britain; Iron.
Indian Navy, ships of war, and other vessels.
Institution of Civil Engineers, List of Members of the, corrected to March 15, 1852.
Iron balls, weight of cast; bars, angle and T, weight of; castings; experiments; hoop, weight of 10 lineal feet; lock gates; roofs; tubes for locomotive and marine boilers; weights of rolled iron.
Ironmonger's prices.
Just's analysis of Mr. Dixon Robinson's limestone.
Latitudes and longitudes of the principal observatories.
Lead pipes, Table of the weights of.
Leslie (J.), C.E.
Lime, mortar, cements, concrete, &c.
Limestone, analysis of.
Liquids in motion.
Locomotive engines; Table showing the speed of an engine.
Log for a sea-going steamer, form of.
Machines and tools, prices of.
Mahogany, experiments made on the strength of Honduras.
Mallet's experiments on overshot wheels.
Marine boilers; engines.
Masonry and stone-work.
Massachusets railroads.
Mensuration, epitome of.
Metals, lineal expansion of.
Morin's (Col.) experiments.
Motion; motion of water in rivers.
Nails, weight and length.
Navies — of the United States; Indian Navy; Oriental and Peninsular Company; British Navy; of Austria; Denmark; Naples; Spain; France; Germanic Confederation; Holland; Portugal; Prussia; Sardinia; Sweden and Norway; Turkey; Russia Royal West India Mail Company's fleet.
New York, State of, railroads.
Numbers, Table of the fourth and fifth power of.
Paddle-wheel steamers.
Pambour (Count de) and Mr. Parkes' experiments on boilers for the production of steam.
Peacocke's (R. A.) hydraulic experiments.
Pile-driving.
Pitch of wheels. Table to find the diameter of a wheel for a given pitch of teeth.
Plastering.
Playfair (Dr. Lyon).
Preserved timber.
Prices for railways, paid by H. M. Office of Works; smith and founder's work.
Prony's experiments.
Proportions of steam engines and boilers.
Pumping engines; pumping water by steam power.
Rails, chairs, &c., Table of.
Railway, American, statistics; railway and building contractor's prices; carriages.
Rain, Tables of.
Rammell's (T. W.) plan and estimate for a distributing apparatus by fixed pipes and hydrants.
Rennie's (Mr. Geo.) experiments; (the late J.) estimate.
Roads, experiments upon carriages travelling on ordinary roads; influence of the diameter of the wheels; Morin's experiments on the traction of carriages, and the destructive effects which they produce upon roads.
Robinson (Dixon), his experiments and material.
Roofs; covering of roofs.
Ropes, Morin's recent experiments on the stiffness of ropes; tarred ropes; dry white ropes.
Saw-mill.
Screw steamers.
Sewage manures.
Sewers, castings for · their estimates, &c.

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Signs and abbreviations used in arithmetic and mathematical expressions.
Slating.
Sleepers, quantity in cubic feet, &c.
Smeaton's experiments on wind-mills.
Smith and founder's prices.
Specific gravity, Table of.
Steam dredging; Navigation; Tables of the elastic force; Table of Vessels of war, of America; of England; of India; and of several other maritime nations.
Steel, weight of round steel.
Stone, per lb., stone, qr., cwt., and ton, &c., Table of the price.
Stones.
Strength of columns; Materials of construction.
Sugar-mill.
Suspension aqueduct over the Alleghany River; Bridges over ditto.
Table of experiments with H. M. screw steam tender DWARF; of gradients; iron roofs; latent heats; paddle-wheel steamers of H. M. Service and Post-Office Service; pressure of the wind moving at given velocities; prices of galvanized tinned iron tube; specific heats; the cohesive power of bodies; columns, posts, &c., of timber and iron; the comparative strength, size, weight, and price of iron-wire rope (A. Smith's), hempen rope, and iron chain; corresponding velocities with heads of water as high as 50 ft., in feet and decimals; dimensions of the principal parts of marine engines; effects of heat on different metals; elastic force of steam; expansion and density of water; expansion of solids by increasing the temperature; expansion of water by heat; heights corresponding to different velocities, in French metres; lineal expansion of metals; motion of water, and quantities discharged by pipes of different diameters; power of metals, &c.; pressure, &c., of wind-mill sails; principal dimensions of 28 merchant steamers with screw propellers; of steamers with paddle-wheels; pro-... dilatation of metals by heat discharge through thin-lipped orifices; quantities of water, in cubic feet, discharged over a weir per minute, hour, &c.; relative weight and strength of ropes and chains; results of experiments on the friction of unctuous surfaces; scantlings of posts of oak; size and weight of iron laths; weight in lbs. required to crush 1½-inch cubes of stone, and other bodies; weight of a lineal foot of cast-iron pipes, in lbs.; weight of a lineal foot of flat bar iron, in lbs.; weight of a lineal foot of square and round bar iron; weight of a superficial foot of various metals, in lbs.; weight of modules of elasticity of various metals; velocities of paddle-wheels of different diameters, in feet per minute, and British statute miles, per hour; the dimensions, cost, and price per cubic yard, of ten of the principal bridges or viaducts built for railways; the height of the boiling point at different heights;—to find the diameter of a wheel for a given pitch of teeth, &c.
Tables of squares, cubes, square and cube roots.
Teeth of wheels.
Temperature, the relative indications of, by different thermometers.
Thermometers, Table of comparison of different.
Timber for carpentry and joinery purposes; Table of the properties of different kinds of.
Tin plates, Table of the weight of.
Tools and machines, prices of.
Traction, Morin's experiments on.
Tredgold's Rules for Hydraulics, from Eytelwein's Equation.
Turbines, Report on, by Joseph Glynn and others.
Values of different materials.
Water-wheels.
Watson's (H. H.) analysis of limestone from the quarries at Chatburn.
Weight of angle and T iron bars; of woods.
Weights and measures.
West India Royal Mail Company.
Whitelaw's experiments on turbine

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MEASUREMENT.
Of diggers' work—Of brickwork, of facings, &c.
DESIGN OF TILING, and technical terms.
Measurement of Tiling—Example of the mode of keeping the measuring-

ABSTRACTING Bricklayers' and Tilers' work.
Example of bill of Bricklayers' and Tilers' work.
VALUATION of Bricklayers' work, Earthwork, Concrete, &c.
Table of sizes and weights of various articles—Tables of the numbers of bricks or tiles in various works—Valuation of Diggers' and Bricklayers' labour—Table of Constants for said labour.
EXAMPLES OF VALUING.
1. A yard of concrete.—2. A rod of brickwork.—3. A foot of facing.—4. A yard of paving.—5. A square of tiling.
DESIGN, MEASUREMENT, AND VALUATION OF SLATING.

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DESIGN OF CARPENTRY — technical terms, &c.
Brestsummers, an abuse: substitutes for them—Joists, trimmers,

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and right use—Substitutes for girders and quarter-partitions—Quarter-partitions—Roof-framing—Great waste in present common modes of roof-framing—To determine the right mode of subdividing the weight, and the right numbers of bearers for leaded roofs—The same for other roofs—Principle of the truss—Considerations that determine its right pitch—Internal filling or *tracery* of trusses—Collar-beam trusses—Connection of the parts of trusses—Variations on the truss; right limits thereto—To avoid fallacious trussing and roof-framing—Delorme's roofing; its economy on circular plans—Useful property of regular polygonal plans—On combinations of roofing, hips, and valleys—On gutters, their use and abuse—Mansarde or curb-roofs.

Design of Joinery—technical terms, &c.

Modes of finishing and decorating panel-work—Design of doors.

Measurement of Carpenters' and Joiners' work—Abbreviations.

Modes of measuring Carpenters' work—Classification of labour when measured with the timber—Classification of labour and nails when measured separately from the timber.

Examples of Measurement, arch centerings.

Bracketing to sham entablatures, gutters, sound-boarding, chimney-grounds, sham plinths, sham pilasters, floor-boarding, mouldings—Doorcases, doors, doorway linings—Dado or surbase, its best construction—Sashes and sash-frames (examples of measurement)—Shutters, boxings, and other window fittings—Staircases and their fittings.

Abstracting Carpenters' and Joiners' work.

Example of Bill of Carpenters' and Joiners' work.

Valuation of Carpenters' and Joiners' work, Memoranda.

Tables of numbers and weights.

Tables of Constants of Labour.

Roofs, naked floors—Quarter-partitions—Labour on fir, per foot cube—Example of the valuation of deals or battens—Constants of labour on deals, per foot superficial.

Constants of Labour, and of nails, separately.

On battening, weather boarding—Rough boarding, deal floors, batten floors.

Labour and Nails together.

On grounds, skirtings, gutters, doorway-linings—Doors, framed partitions, mouldings—Window-fittings—Shutters, sashes and frames, staircases—Staircase fittings, wall-strings—Dados, sham columns and pilasters.

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Measurement of Stonemason's work.

Example of measuring a spandril step, three methods—Allowance for labour not seen in finished stone—Abbreviations—Specimen of the measuring-book—Stairs—Landings—Steps—Coping—String-courses—Plinths, window-sills, curbs—Columns, entablatures, blockings—Cornices, renaissance niches.

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Table of weight of stone—Table of Constants of Labour—Example of Bill of Masons' work.

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Measurement of Plaster-work.

DESIGNING, MEASURING, AND VALUING ARTIFICERS' WORKS.

Abbreviations — Abstracting of Plasterers' work — Example of Bill of Plasterers' work.

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Memoranda of quantities of materials — Constants of Labour.

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Iron not rightly to be used much more now than in the middle ages—Substitutes for the present extravagant use of iron—Fire-proof (and sanitary) ceiling and flooring—Fire-proof roof-framing in brick and iron — Another method, applicable to hipped roofs — A mode of untrussed roof-framing in iron only — A principle for iron trussed roofing on any plan or scale — Another variation thereof — On the decoration of metallic architecture.

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Properties of numbers.

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Definitions and notation.—2. Addition and subtraction.—3. Multiplication.—4. Division.—5. Involution. — 6. Evolution. — 7. Surds. — Reduction. — Addition, subtraction, and multiplication. —Division, involution, and evolution.—8. Simple equations.— Extermination. — Solution of general problems.—9. Quadratic equations. — 10. Equations in general. — 11. Progression. — Arithmetical progression.—Geometrical progression.—12. Fractional and negative exponents.— 13. Logarithms.—14. Computation of formulæ.

CHAPTER III.—GEOMETRY.

Definitions. — 2. Of angles, and right lines, and their rectangles.

SECT.

— 3. Of triangles. — 4. Of quadrilaterals and polygons.—5. Of the circle, and inscribed and circumscribed figures.—6. Of plans and solids. — 7. Practical geometry.

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1. Weights and measures.—1. Measures of length. — 2. Measures of surface.—3. Measures of solidity and capacity. — 4. Measures of weight. — 5. Angular measure. — 6. Measure of time. — Comparison of English and French weights and measures.
2. Mensuration of superficies.
3. Mensuration of solids.

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1. Definitions and trigonometrical formulæ. — 2. Trigonometrical Tables. — 3. General propositions.—4. Solution of the cases of plane triangles. — Right-angled plane triangles.—5. On the application of trigonometry to measuring heights and distances. —Determination of heights and distances by approximate mechanical methods.

CHAPTER VI.—CONIC SECTIONS.

1. Definitions.—2. Properties of the ellipse.—3. Properties of the hyperbola. — 4. Properties of the parabola.

CHAPTER VII.—PROPERTIES OF CURVES.

1. Definitions.— 2. The conchoid.— 3. The cissoid.—4. The cycloid and epicycloid.—5. The quadratrix.— 6. The catenary.— Relations of Catenarian Curves.

PART II.—MIXED MATHEMATICS.

[CHA]PTER I.—MECHANICS IN GENERAL.

CHAPTER II.—STATICS.

Statical equilibrium.

Centre of gravity.

General application of the princi-[ples of statics to the equilibrium]

of structures.— Equilibrium of piers or abutments. — Pressure of earth against walls.—Thickness of walls. — Equilibrium of polygons. — Stability of arches. — Equilibrium of suspension bridges.

MATHEMATICS FOR PRACTICAL MEN.

SECT.

CHAPTER III.—DYNAMICS.

1. General Definitions.
2. On the general laws of uniform and variable motion. — Motion uniformly accelerated.—Motion of bodies under the action of gravity.— Motion over a fixed pulley, and on inclined planes.
3. Motions about a fixed centre, or axis.—Centres of oscillation and percussion. — Simple and compound pendulums. — Centre of gyration, and the principles of rotation.—Central forces.
4. Percussion or collision of bodies in motion.
5. Mechanical powers. — Levers. — Wheel & axle. — Pulley. — Inclined plane.—Wedge and screw.

CHAPTER IV.—HYDROSTATICS.

1. General Definitions.—2. Pressure and equilibrium of Non-elastic Fluids.—3. Floating Bodies.— 4. Specific gravities. — 5. On capillary attraction.

CHAPTER V.—HYDRODYNAMICS.

1. Motion and effluence of liquids.
2. Motion of water in conduit pipes and open canals, over weirs, &c.—Velocities of rivers.
3. Contrivances to measure the velocity of running waters.

CHAPTER VI.—PNEUMATICS.

1. Weight and equilibrium of air and elastic fluids.
2. Machines for raising water by the pressure of the atmosphere.
3. Force of the wind.

SECT.

CHAPTER VII.—MECHANICAL AGE

1. Water as a mechanical agent.
2. Air as a mechanical agent.— C lomb's experiments.
3. Mechanical agents depending u heat. The Steam Engine Table of Pressure and Temp ture of Steam. — General scription of the mode of ac of the steam engine. — Th of the same. — Descriptio various engines, and formul calculating their power: p cal application.
4. Animal strength as a mecha agent.

CHAPTER VIII.—STRENGTH OF MATERIALS.

1. Results of experiments, and p ciples upon which they sh be practically applied.
2. Strength of materials to r tensile and crushing strain Strength of columns.
3. Elasticity and elongation of b subjected to a crushing or sile strain.
4. On the strength of materials jected to a transverse strai Longitudinal form of bea uniform strength. —Transv strength of other materials t cast iron. — The strength beams according to the ma in which the load is distribu
5. Elasticity of bodies subjected transverse strain.
6. Strength of materials to r torsion.

APPENDIX.

I. Table of Logarithmic Differences.
II. Table of Logarithms of Numbers, from 1 to 100.
III. Table of Logarithms of Numbers, from 100 to 10,000.
IV. Table of Logarithmic Sines, Tangents, Secants, &c.
V. Table of Useful Factors, extending to several places of Decimals.
VI. Table of various Useful Numbers, with their Logarithms.
VII. Table of Diameters, Areas, and Circumferences of Circles, &c.
VIII. Table of Relations of the Arc, Abscissa, Ordinate and Subnormal, in the Catenary.
IX. Tables of the Lengths and Vibrations of Pendulums.
X. Table of Specific Gravities.
XI. Table of Weight of Materials frequently employed in Construction.
XII. Principles of Chronometers.
XIII. Select Mechanical Expedients.
XIV. Observations on the Effect of Old London Bridge on the Tides, &c.
XV. Professor Farish on Isometrical Perspective.