United States Naval Postgraduate School

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Sigurd Hess
Kapitänleutnant, Deutsche Marine
B.S.E.E., Naval Postgraduate School, 1968

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ABSTRACT

A deterministic analysis of the limit cycle oscillations which occur in fixed-point implementations of recursive digital filters due to roundoff and truncation quantization after multiplication operations, is performed. Amplitude bounds, based upon a correlated nonstochastic signal approach and Lyapunov's direct method, as well as an approximate expression for the frequency of zero-input limit cycles, are derived and tested for the two-pole filter. The limit cycles are represented on a successive value phase-plane diagram from which certain symmetry properties are derived. Similar results are developed for other second-order digital filter configurations, and the parallel and cascade forms. The results are extended to include limit cycles under input signal conditions. A basic design relationship between the number of significant digits required for the realization of a filter algorithm with a desired signal-to-noise (limit cycle) ratio is stated.

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I. INTRODUCTION

A. INTRODUCTORY REMARKS

When a digital filter is implemented on a general or special purpose computer, errors due to finite precision in the representation of numbers are unavoidable. The finite arithmetic in the computer generates roundoff or truncation errors which are due to the quantization nonlinearities introduced when implementing a digital filter algorithm. They give rise to nonlinear effects such as limit cycle oscillations as well as approximations in a filter realization. This dissertation is mainly concerned with the type of error which has been called by various names in the literature, such as "deadband effect" [24], "quantizer-induced limit cycle oscillations" [26], or "low-level correlated noise".

The analysis of quantization noise on a statistical basis has been discussed by many authors. This approach is approximate and is based on the assumption that quantization errors occur in a random manner. In this dissertation the generation of spurious signals generated by quantization is analyzed from a correlated, deterministic point of view in an effort to determine: 1) bounds on the limit cycle amplitude, 2) expressions for the limit cycle frequency, and 3) existence conditions.

A digital filter is defined as a time-invariant, discrete or sampled-data system with finite accuracy in the representation of all data and parameter values. More formally, any

time-invariant linear operation on discrete time signals may be classified as a digital filter [1]. Such an operation is defined by the process by which a discrete time output signal is determined from a discrete time input signal.

A digital filter can be viewed as a computer which is programmed to operate on the incoming sequence of numbers in a specified way so as to generate the desired output sequence of numbers. A schematic representation of a low-pass digital filter is depicted in Fig. 1.1. The input $x_2(nT)$ to the filter is a sequence of numbers, equally spaced in time and separated by the sampling time T. If a continuous signal $x_1(t)$ is the input, then an analog to digital (A/D) conversion has to be performed first to generate the required sequence of numbers. Similarly, if a continuous signal is needed as the output, a digital to analog conversion (D/A), by using, for example, a zero-order hold, has to take place. Sampling and processing of data in discrete time is analogous to a filtering operation in continuous time. The synthesis or the development of the discrete time algorithm to meet a filtering specification is amply discussed in the literature and is not considered in this dissertation [4,5,30].

Initially, digital filters were applied to the simulation of analog systems or off-line signal processing of such signals as seismic data. In recent years, digital filters have been used more and more for real time signal processing. By real time, it is implied that digital processing takes place fast enough so that the output of the digital filter is available for direct control or observation in

a larger system. Digital filters are constructed using digital logic components as their basic building blocks and the rapid advance in the development of solid-state devices has made such digital filters practical. The development of large scale circuit integration (LSI) promises to make these systems even more economical.

The advantages of digital filters over their analog counterparts are numerous [1]. Some of the advantages are

a) arbitrary high precision in the computational process,
b) no parameter or component value drifting,
c) flexibility in the processing procedure, which allows the construction of adaptive filters,
d) no necessity for impedance matching,
e) possibility to use time-sharing techniques,
f) easy realization of complex circuits,
g) high reliability,
h) small circuit size,
i) decreasing costs for mass-produced basic building blocks.

One of the inherent limitations of digital filters is related to the fact that all numbers representing either data or filter coefficients are expressed with a finite number of significant digits.

In order to realize digital filters, two distinct problems have to be solved. The first represents the approximation problem, i.e., the filter design required to realize a rational transfer function of finite order which approximates in some sense either a desired frequency response characteristic or a desired time domain response. This approximation problem is not considered here and it is assumed that a desired rational transfer function already exists.

The second problem is concerned with the implementation or synthesis of the filter, i.e., the filter algorithm and a filter configuration which efficiently implement the transfer function. Jackson [2] has discussed these two phases and their interdependence in detail. In this dissertation those special aspects of the implementation phase, pertaining to the generation of limit cycle oscillations are investigated in detail.

Four factors have to be considered when implementing a filter. These are:

a) choice of a numerical algorithm,

b) selection of a specific configuration for the filter,

c) choice of the arithmetic mode, i.e., the number system to be used,

d) specification of the number of significant digits.

Since limit cycle oscillations occur mainly in fixed-point implementations of recursive digital filters, the other possible computational algorithms, such as nonrecursive digital filters and Fast Fourier Transform (FFT) filters are not considered. A recursive digital filter is defined as a filter in which the present output depends on the present input and past inputs and outputs, while for a nonrecursive filter the output depends on past and present inputs only. It should be noted, that most digital filters are of the fixed-point variety because floating-point

arithmetic involves more hardware. Also, most filters are recursive because for the same degree of approximation, recursive filters are generally simpler than nonrecursive forms. A discussion of fixed-point versus floating-point arithmetic, together with considerations of the number of significant digits required for a given precision and signal-to-noise ratio, appears in the next section.

For a given filter transfer function many equivalent configurations, i.e., different arrangements of the arithmetic functions or elements of the filter (such as delays, adders and multipliers) can be devised. Kaiser [3] has shown that a cascade or parallel form composed of first and second-order subfilters is preferable over any direct realization of a higher order digital filter. Thus, a higher order filter is obtained by combining second-order sections. For this reason, second-order filter configurations with the restriction of finite precision in the arithmetic are studied. It should be noted that several different configurations can be derived for the same transfer function. Despite the fact that the configurations have an identical transfer function, their generation of limit cycles and noise due to quantization may be different.

This section is now concluded with a review of those references from the literature which describe several general aspects of digital filtering. Kaiser [4] has reviewed the history of digital filters and presents an extensive bibliography which references work published before 1966.

He discusses various filter design techniques, including nonrecursive filters and the application of the Fast Fourier Transform. A book by Gold and Rader [5] presents a thorough introduction into digital signal processing. The book starts with a development of the basic theory for analysis of linear digital filters, introduces design techniques for digital filters employing the frequency domain and presents some basic concepts of quantization errors. The FFT algorithm is explained and in a separate chapter, written by Stockham, the application of the FFT to implement convolution is described. Oppenheim [6] has edited a collection of 20 important research papers which cover the topics of z-transform theory, digital filter design, nonrecursive digital filters, and the application of the FFT and hardware design for digital filters and FFT implementations. These papers are selected to complement the book by Gold and Rader mentioned above. A comprehensive bibliography of 142 papers and 41 books published before 1970 is included. Some important current research about digital filters is also published in two special issues of the IEEE Transactions on Audio and Electroacoustics [7,8].

B. SOURCES OF ERRORS IN DIGITAL FILTERS

In practical digital filters finite number representation of coefficients and data is required. Theoretically, accuracy could be maintained arbitrarily high, but in the process of implementing a particular filter structure a tradeoff between accuracy, signal-to-noise ratio and overall system

cost has to be performed. Three sources of error have to be considered. They are

a) analog to digital (A/D) conversion errors,
b) errors because of finite representation of the digital filter coefficients,
c) quantization errors, due to rounding off or truncating the result of multiplication of data with filter coefficients.

The first source of error, A/D conversion, is studied extensively by Bennett [9]. Its effect is normally taken into account by placing a noise source at the input of the filter. This noise represents the error generated by the quantization process on the input signal. In this dissertation it is assumed that the sampled data already exists in a form suitable for processing in a digital filter, i.e., each data sample is represented by a finite number of significant digits.

The second source of error, finite representation of filter coefficients, is a deterministic effect. It can be taken into account by recomputing the eigenvalues of the filter with the truncated coefficients. The small changes in the filter coefficients due to finite number representation results in a corresponding change in the eigenvalues. Kaiser [3] has studied the sensitivity of the eigenvalues or pole positions of an n^{th} order digital filter due to coefficient quantization. In this approximate analysis, he concludes that for a direct filter realization, the sensitivity of the pole positions increases with the order n of the equation. This result has been corroborated in work

reported by Knowles and Olcayto [10].

In this paper several numerical results from the simulation of practical filters are stated and compared. Rader and Gold [12] have studied the coefficient quantization problem for second-order digital filters. They conclude that a realization via a pair of coupled first-order sections is less sensitive to coefficient changes than a single second-order form. Mantey [13] has studied the coefficient quantization problem by selecting a state variable representation for the digital filter. His results, as well as the results from the other workers mentioned before, indicate that a digital filter should be realized by a parallel or cascade connection of first or second order subfilters instead of a direct n^{th} order realization.

The third source of error occurs with quantization after arithmetic operations. The two most often employed quantization procedures are roundoff to the nearest integer and truncation. They are described in more detail in Chapter III. The effects of quantization after arithmetic operations can be demonstrated with the example of a first-order digital filter described by the following difference equation[1]:

$$\hat{x}(n) = -a\,\hat{x}(n-1) + u(n). \tag{1.1}$$

---

[1] In this dissertation the simpler notation $\hat{x}(n)$ instead of $\hat{x}(nT)$ will be employed. Some authors also use x_n for $\hat{x}(nT)$. Furthermore, the circumflex is used to designate the results of finite precision arithmetic, i.e., quantized numbers.

Suppose that all numbers $\hat{x}(n)$, a, u(n) are expressed initially with k significant digits and that fixed-point arithmetic is employed for the implementation of the difference equation (1.1). Calculation of the filter response shows that after n iterations $\hat{x}(n)$ is expressed by numbers with (n+1)k significant digits. The foregoing example indicates that the number of significant digits, needed to compute the filter response precisely, increases linearly with each iteration. Any practical filter, realized with k significant digits, has to include quantization after each arithmetic operation to keep the results at a specified finite precision.

The choice of the arithmetic mode influences the quantization noise. Most noise analyses have been carried out for fixed-point arithmetic, because this arithmetic mode is easier to realize than the other modes. If fixed-point arithmetic is employed, the number of significant digits determines the dynamic range of the filter. Since the signal levels in the filter may change drastically from one subsection to the next, scale factors are included at each section to allow use of the full dynamic range available. On the other hand, floating-point arithmetic is more complex, but provides a means of automatic scaling. Since all numbers are represented by a mantissa and an exponent, the exponents represent the scale factors. Block floating-point arithmetic is an intermediate mode, where a single exponent is specified for an entire block of numbers. Weinstein and Oppenheim [14] have performed a comparison of the signal-to-noise ratios

for filters with fixed-point, block floating-point, and floating-point arithmetic. It is shown there that the floating-point arithmetic mode is generally less noisy than the fixed-point mode. As expected, the block floating-point arithmetic mode lies between the other two types as far as quantization noise is concerned.

The analysis of the quantization noise has been attempted from two different points of view. The first results in a stochastic approach. It is based on the necessary assumption that the quantization noise sequences from different multipliers in the filter are statistically independent and uncorrelated with each other and with the processed signal. The quantization noise is described by a uniform probability density function. The assumption leads to acceptably accurate results for most applications with high signal level and sufficient spectral content. Although the simple noise model presented above has been applied successfully in many cases, there exist counterexamples for which the assumptions do not hold. Among these are the limit cycle oscillations which are studied in the next chapters. Ultimately, one must resort to experiment to verify the predictions based on the statistical model. Jackson [2,15,16] has performed a comprehensive study of roundoff noise in digital filters for the fixed-point arithmetic mode using the stochastic approach. His results show excellent agreement between theory and experiment. Using the same stochastic approach Kaneko and Liu [17] have analyzed the quantization noise properties of

floating-point digital filters and Weinstein [18] has reported on the noise properties of block floating-point digital filters.

By contrast, the other approach to the analysis of quantization noise is a deterministic one. Bertram [19] and similarly Slaughter [20] have derived an upper bound on the quantization error in sampled-data control systems which is independent of the forcing function. Johnson [21] and Lack [22] apply Lyapunov's direct method to derive an upper bound on the quantization error in sampled-data control systems which is shown to be tighter than the bound reported by Bertram. The results from Johnson and Lack have been reformulated in Appendix A of this dissertation to apply to fixed-point digital filters. Sandberg [23] has presented an analysis of quantization errors due to roundoff in floating-point recursive filters. The difficulty with the deterministic bounds is that they apply to the situation where all errors add up in the worst possible way. Thus, the deterministic bounds are in general overly pessimistic compared with the bounds from the stochastic approach and with experimental results.

However, in situations where the assumptions of the stochastic approach fail to apply, the deterministic approach is the only method of attack which leads to useful results, This is the case when limit cycles occur which, by their very nature, are generated by quantization error sequences which are highly correlated. The existence of low-level or zero-input correlated noise was first reported by Blackman

[24], who called it the "deadband effect". Betten [25] analyzed the limit cycles using the describing function technique for higher order sampled-data systems and the phase-plane technique for second-order unity feedback control systems with one roundoff quantizer in the forward path of the feedback system. Similarly, results reported by White [26] are derived from the work of Betten. However, these results cannot be generalized to digital filters where several quantizer nonlinearities in the feedback structure complicate the analysis. Jackson [27] linearized the quantized digital filter and derived bounds on the amplitude of zero-input limit cycles which are close to the value which are obtained by experiment. Since his model is not exact, there exist important nontrivial exceptions to the derived bounds. Bonzanigo [28] derived bounds for the amplitude of some simple limit cycles which are essentially the same as those derived by Jackson. Pfundt and Tödtli [29] recognized the existence of limit cycles for the special case of a digital oscillator realized with finite precision arithmetic. However, their major contribution consists of one hardware realization of such oscillators. They do not discuss the theory of limit cycle generation.

C. PREVIEW OF RESULTS

From the remarks of the preceding section it follows that the limit cycle oscillations occuring in fixed-point implementations of recursive digital filters can be analyzed using the deterministic approach. In this section the major

results of the following chapters are previewed.

In Chapter II, a linear model for digital filters is developed. It is shown that the second-order digital filter emerges as the basic building block for the realization of higher order digital filters. Using the state description of digital filters, the existence of 24 canonical forms for second-order digital filters implemented with finite precision arithmetic is presented. Twelve of these forms, to the author's knowledge, have not appeared in the literature before.

In Section II.D, the influence of coefficient accuracy on the response of a second-order digital filter is considered. A new general expression for the shift of the pole locations due to truncation of the filter coefficient of the second-order digital filter is derived. The poles or characteristic values are expressed in polar coordinates to correspond to damping factor and resonant frequency of the filter. Using two examples it is demonstrated that not only sampling too slow, but sampling too fast also may result in an undesirable response.

In Chapter III, limit cycle oscillations caused by quantization after multiplications are investigated using the simple model of a zero-input second-order digital filter with two poles and no zeros. The investigation is performed for magnitude truncation and roundoff quantization. As a new result, it is shown that with magnitude truncation quantization in general no zero-input limit cycles can be sustained. On the other hand, zero input limit cycles of all frequencies

are possible with roundoff quantization. A general matrix formulation of these limit cycles is presented. In general, there exists no known way to evaluate amplitude and frequency of the self-oscillations exactly. However, five amplitude bounds and an approximate expression for the frequency of the limit cycles are derived. Three of the amplitude bounds are new. Two of these three are exact bounds, which is in contrast to the previously known results which are based on an approximate analysis. Chapter III closes with the proofs of three new lemmas which describe some basic symmetry properties of the limit cycles if they are displayed in a specially defined phase-plane diagram, called successive value phase-plane plot.

The conclusions of Chapter III are verified in Chapter IV, where experimental results are reported and compared with the theory. For this purpose three computer programs have been written. The first program is an analysis program for zero-input limit cycles in second-order digital filters employing roundoff quantization. For a choice of values for the filter coefficients, all possible limit cycles are evaluated in a given area of search and displayed in the successive value phase-plane. With the numerical values for the limit cycles available, it is then easy to compare the actual amplitude of the limit cycle with the predicted amplitude obtained from the derived bounds. In this way, limit cycles have been detected which exceed the previously published amplitude bounds considerably.

The second program implements two of the five amplitude bounds derived in Chapter III so that a comparison between the different bounds is possible. One bound is shown to be impractical because it is overly pessimistic. For the remaining four bounds, the region of applicability and their advantages are discussed and compared.

The third program is a simulation of an important special case, the digital oscillator. It is shown that any degree of approximation for a specified sinusoidal oscillation can be achieved by either increasing the amplitude if the quantization step-size is constant, or by decreasing the quantization step-size if the amplitude is constant. In addition, it can be deduced that roundoff is preferable over truncation because a better degree of approximation can be obtained. The existence of constant amplitude limit cycles is contrary to a conclusion reached by Rader and Gold [12] where it is claimed that the output noise of the digital oscillator increases linearly with time. The digital oscillator is another important example where the assumptions of the stochastic approach for the analysis of quantization noise fail to apply.

The results of Chapter III and IV are generalized in Chapter V. The forced response of general digital filters with both poles and zeros is analyzed with regard to possible limit cycle oscillations. First, the forced response of the two-pole filter is investigated for deterministic inputs. As a new result, it is shown that the driven case can be

reduced to a zero-input case if the difference between the response of the quantized digital filter and the corresponding linear digital filter is considered. This difference signal is described by a limit cycle oscillation whose amplitude is estimated by the same bounds which have been derived in Chapter III for the zero-input response. Next, the general second-order digital filter with both zeros and poles in the transfer function of the equivalant linear filter is studied. The zeros are shown not to change the nature of the limit cycle, but to influence the magnitude of the limit cycle amplitude. As a new result it is derived that for specified zeros the magnitude of the limit cycles in the output of the digital filter can be minimized through a proper choice of the filter configuration. Finally, higher order digital filters of the cascade and the parallel form are considered with regard to limit cycles in their output.

In Chapter VI, the derived bounds on the limit cycle amplitude are applied as a design guide to study tradeoffs between the required number of significant digits and the specified signal-to-noise ratio of a digital filter. The chapter concludes with an indication of those problems which remain subject to further research.

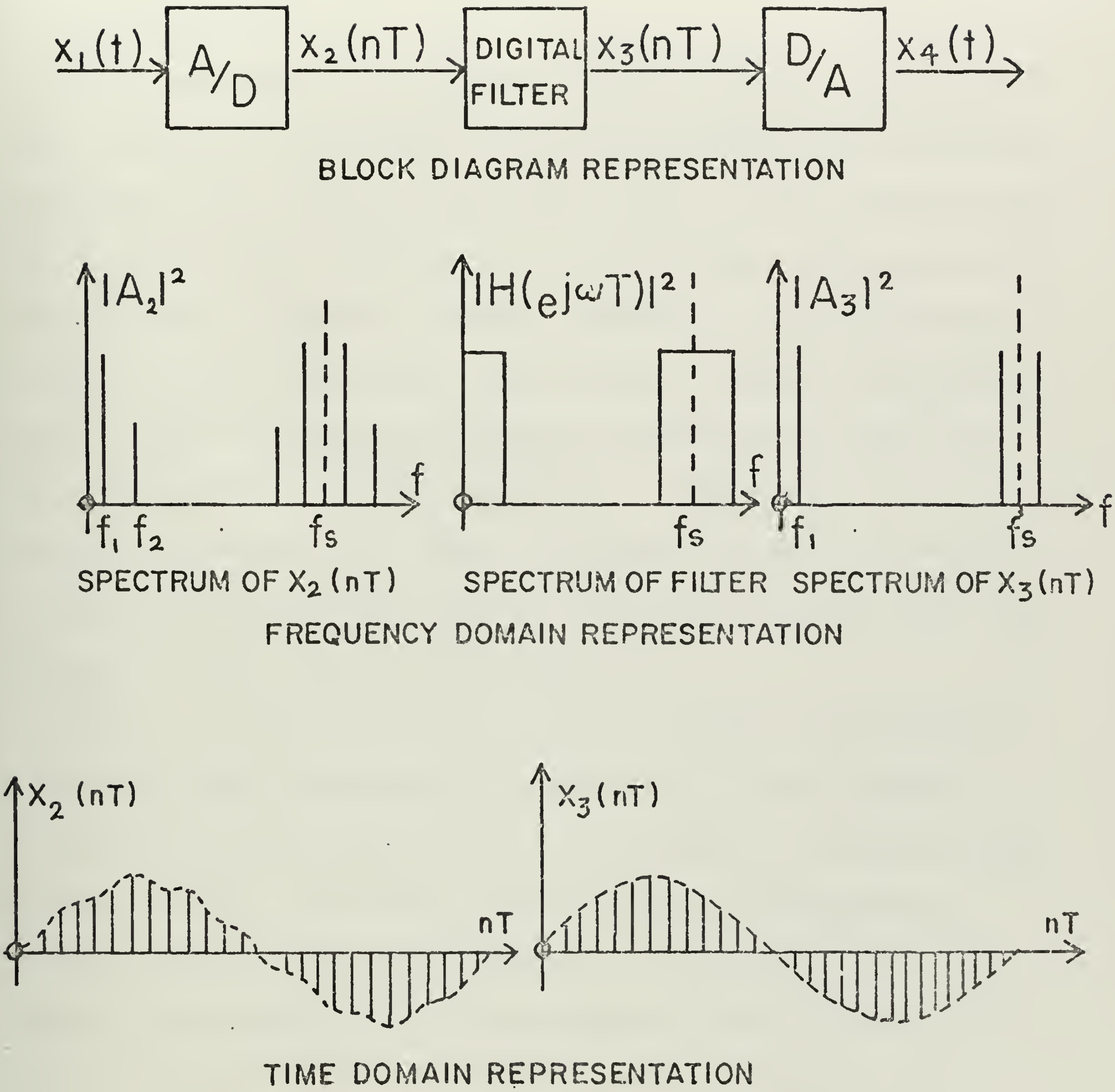


Fig. 1.1: Schematic Representation of a Digital Low-pass Filter.

II. SECOND-ORDER DIGITAL FILTER MODELS WITH FINITE PRECISION ARITHMETIC

A. INTRODUCTION

The realization of a linear digital filter requires perfect arithmetic to perform the necessary additions and multiplications with infinite precision; that is, an infinite number of significant figures must be available. For practical realizations of digital filters, however, finite precision arithmetic has to be used. The linear, infinite precision, digital filter is thus an idealization which can never be fully realized by a digital computer. However, it is important to consider this type of digital filter to develop an understanding of the inherent nonlinear effects which are studied in this and in later chapters.

First, a linear model for digital filters is developed. It is shown that the second-order digital filter emerges as the basic building block for the realization of higher-order digital filters. Starting with the state description of discrete systems, the existence of 24 canonical forms for second-order digital filters implemented with finite precision arithmetic, (i.e., a finite number of significant figures) is presented. The development is based on the concept of transpose configurations of digital filters as devised by Jackson [2]. This method is extended to show the existence of a finite number of canonical forms and twelve new, previously unpublished, canonical forms are derived. The development indicates that 24 canonic forms are possible which have identical transfer functions, even under the

assumption of finite precision arithmetic. However, their error properties are, in general, different.

Second, the influence of coefficient accuracy on the response of a second-order digital filter is considered. A change in a filter coefficient does not affect the linear nature of the digital filter, but simply shifts the pole-zero locations of this filter. A new general expression for the shift of the pole locations, due to truncation of the coefficients of the second-order digital filter, is derived. The poles or characteristic values are expressed in polar coordinates to correspond to damping factor and resonant frequency of the filter. The validity of the result is tested using some examples. The examples show that the positioning of the poles and, thus, the performance of a digital filter, is influenced by the choice of the particular filter structure and the sampling interval. It is generally known that sampling too slow may result in a deteriorated response of a digital filter. On the other hand, it has also been demonstrated before that sampling too fast may also result in an undesired response [4].

B. MODELING OF DISCRETE SYSTEMS

Discrete systems can be implemented using the operations of delay, multiplication and addition. The interconnection of these elements will be represented by a directed graph.

Following Gold and Rader [30], the rule or operator by which a discrete time output signal is determined from a discrete time input signal is expressed as

$$H(z) = \frac{\sum_{i=0}^{N} a_i \; z^{-i}}{1+\sum_{i=1}^{N} b_i z^{-i}} = \frac{X(z)}{U(z)} . \tag{2.1}$$

In (2.1) z^{-1} stands for the delay operator of interval T, and the a_i and b_i are constant coefficients. N determines the order of the system (assuming either a_N or b_N to be unequal to zero). The z-transform calculus, used to describe H(z), plays a useful role in the analysis and synthesis of linear discrete systems similar to the role of the Laplace Transform calculus in the analysis of linear continuous systems. H(z), above, is the transformed equivalent of the following N^{th} order difference equation [30], where x(nT) denotes the output and u(nT) denotes the input:

$$x(nT) = \sum_{i=0}^{N} a_i \; u[(n-i)T] - \sum_{i=1}^{N} b_i x[(n-i)T] \tag{2.2}$$

or in the notation which is used throughout this dissertation

$$x(n) = \sum_{i=0}^{N} a_i \; u(n-i) - \sum_{i=1}^{N} b_i \; x(n-i). \tag{2.3}$$

Since the present value of x(n) depends on (N+1) or fewer values of the input u(n-i) and N or fewer values of the signal x(n-i) the transfer function (2.1), or difference equation (2.3) is of the recursive type. In this dissertation only recursive digital filters are considered because the inherent feedback causes the interesting oscillation phenomena studied in the later chapters.

It should be noted that recursive digital filters are often preferred over nonrecursive types, because the former allow for a simpler realization in that lower order forms provide the same degree of approximation [4].

In the last paragraph, no mention has been made of the approximation problem, i.e., the problem of how the transfer function H(z) is obtained from the specifications for a digital filter. For the purpose of this dissertation, it is assumed that the approximation problem is solved previously and that a particular H(z) is given.

There exist a multitude of forms for realizing linear discrete systems. However, three canonical forms have been defined in the literature (see for example Gold and Rader [5]). These are the direct, the parallel and the cascade forms. No formal definition has been given in the literature for a "canonical form". The following intuitive definition has to be sufficient for the present. A realization is considered canonical if the discrete system is implemented with the minimum number of delays, multipliers and adders. Canonical forms with respect to second-order systems are studied in the next section.

A realization of the direct form is shown in Fig. 2.1. On the other hand, the transfer function H(z) can be partitioned into second-order sections and then be realized in either parallel or cascade form. Second-order sections have been chosen because this is the minimum order for realizing a pair of complex conjugate roots such that the polynomials of the numerator and denominator of the transfer function

have real coefficients. Real roots can then be realized in pairs also, except for the case where N is odd, in which case use of a first-order section becomes necessary.

The parallel form corresponds to a partial fraction expansion of H(z) in the following way (assuming no multiple roots)

$$H(z) = \frac{a_N}{b_N} + \sum_{i=1}^{M} \frac{c_{1i} z^{-1} + c_{oi}}{d_{2i} z^{-2} + d_{1i} z^{-1} + 1} , \qquad (2.4)$$

where $M = [\frac{N+1}{2}]$, indicating the integer part of $(\frac{N+1}{2})$. A configuration of the parallel form is shown in Fig. 2.2. The cascade form corresponds to factorization of H(z) into the product of second-order polynomials, where

$$H(z) = a_o \cdot \prod_{i=1}^{M} \frac{c_{2i} z^{-2} + c_{1i} z^{-1} + 1}{d_{2i} z^{-2} + d_{1i} z^{-1} + 1} . \qquad (2.5)$$

A configuration for the cascade form is shown in Fig. 2.3.

Kaiser [3] has shown that the direct form should be avoided because of coefficient sensitivity, i.e., the effect of changes of the numerical coefficients of the filter causes large variations in the filter response.

Also, Knowles and Edwards [31] have concluded that the direct form is inferior to both the cascade and parallel forms when the effect of roundoff errors after arithmetic operations are considered. In a recent paper by Edwards, Bradley and Knowles [11] the above mentioned conclusions have been tested using the 11th order elliptical bandstop filter. Taking scaling into account to assure the proper

dynamic range for the filter, the ratio of the rms noise level due to roundoff after multiplication for the direct form, to the rms noise of the parallel or the cascade form, was about 10^{12}:1.

Since the direct form is impractical, and higher-order digital filters will be realized as cascade or parallel form, the second-order system (with the first-order system considered a degenerate case) emerges as a basic building block from which all higher order systems can be synthesized. It is for this reason that the study of oscillations in discrete systems will be restricted to the second-order case.

C. STATE SPACE DESCRIPTION OF SECOND-ORDER DIGITAL FILTER MODELS

In this section the state description of discrete systems is used to develop 24 second-order digital filter models under the restriction of finite precision arithmetic. Let a linear discrete system be described by the following set of state equations

$$\begin{aligned} x(n+1) &= A\, x(n) + B\, u(n), \\ y(n) &= C\, x(n) + D\, u(n), \text{ where} \end{aligned} \qquad (2.6)$$

u(n) denotes the input, y(n) denotes the output and x(n) describes the states of the system. A, B, C, D are constant coefficient matrices for the case of a linear, time-invariant system. Equation (2.6) can be expressed by one matrix, called the system matrix S_N, such that

$$\begin{bmatrix} x(n+1) \\ \\ y(n) \end{bmatrix} = S_N \begin{bmatrix} x(n) \\ \\ u(n) \end{bmatrix}. \tag{2.7a}$$

The system is then completely defined by the matrix S_N which is given by

$$S_N = \begin{bmatrix} A & B \\ \\ C & D \end{bmatrix}, \tag{2.7b}$$

where N denotes the order of the system.

The transfer function H(z) for a single-input single-output system is defined to be

$$Y(z) = H(z)\ U(z). \tag{2.8}$$

Then, from (2.6) and (2.8), H(z) can be evaluated as[1]

$$H(z) = C(zI-A)^{-1} B + d. \tag{2.9}$$

In (2.9), I represents the identify matrix. As was pointed out in the previous section, digital filters are usually realized by either cascade or parallel second-order forms, so that x(n) is a 2 x 1 state vector. The most general S-matrix for a second order digital filter is the following (where the subscript N = 2 is dropped, since the second-order case is the only case considered):

$$S = \begin{bmatrix} A & B \\ C & d \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ c_1 & c_2 & d \end{bmatrix}. \tag{2.10}$$

---

[1] In matrix equations lower case letters denote scalars.

Let a filter transfer function with coefficients a, b, c, d, e be given as

$$H(z) = d + \frac{ez^{-2} + cz^{-1}}{bz^{-2} + az^{-1} + 1} \; . \qquad (2.11)$$

From (2.9) - (2.11) the filter coefficients, in terms of the transfer function coefficients of (2.11), are given by

$$a = -(a_{11} + a_{22}), \qquad (2.12a)$$

$$b = a_{11}\, a_{22} - a_{21}\, a_{12}, \qquad (2.12b)$$

$$c = b_1 c_1 + b_2 c_2, \qquad (2.12c)$$

$$e = a_{12} b_2 c_1 + a_{21} b_1 c_2 - a_{11} b_2 c_2 - a_{22} b_1 c_1. \qquad (2.12d)$$

If infinite precision arithmetic is used, many canonical and noncanonical configurations can be found to realize a particular H(z). Since the coefficients a, b, c, e are given by the transfer function as determined from the solution of the approximation problem, it is necessary to find a_{11}, a_{12}, ..., etc., so that a state variable synthesis of the digital filter becomes possible. Any combination of a_{ij}, b_k and c_m, which yields the given values of a, b, c and e is a valid realization.

However, for any practical implementation finite arithmetic multipliers and adders have to be used. Suppose that the filter is realized using binary arithmetic elements with k-bit accuracy, then (2.12a-d) have to be rewritten as

$$a = -(a_{11} + a_{22}), \qquad (2.13a)$$

$$b = [a_{11}a_{22}]_q - [a_{21}a_{12}]_q, \tag{2.13b}$$

$$c = [b_1c_1]_q + [b_2c_2]_q, \tag{2.13c}$$

$$e = [a_{12}b_2c_1]_q + [a_{21}b_1c_2]_q - [a_{11}b_2c_2]_q - $$

$$-[a_{22}b_1c_1]_q. \tag{2.13d}$$

The operation $[\ldots]_q$ denotes quantization (roundoff or truncation) to preserve the finite precision of the results of the multiplication.

Given H(z) with coefficients a, b, c, d and e, a general solution of (2.13a-d) is not obvious for the coefficients a_{ij}, b_k, c_m, in terms of a, b, c, d, e, because there are four equations in eight unknowns, which leaves an infinite variety of choices to be made for four of the eight unknowns. Furthermore, the existence of a solution for (2.13a-d) is not guaranteed at all.

The validity of the latter conclusion can be demonstrated with the simple example of a digital oscillator, realized by two coupled first-order sections. The example is presented in the next paragraph.

Consider a digital oscillator realization with an S-matrix of the form

$$S = \begin{bmatrix} \cos \omega_o T & \sin \omega_o T & 0 \\ -\sin \omega_o T & \cos \omega_o T & 0 \\ 1 & 0 & 0 \end{bmatrix}. \tag{2.14}$$

The z-transformed response of this section to initial conditions $x_1(0) = 0$, $x_2(0) = 1$ is given by

$$Y(z) = \frac{z^{-1} \sin \omega_o T}{z^{-2} - 2z^{-1} \cos \omega_o T + 1} . \tag{2.15}$$

Thus

$$a = -2 \cos \omega_o T,$$

$$b = 1,$$

and the poles of the response have magnitude one (because $b = a_{11} a_{22} - a_{12} a_{21} = 1$), which is the necessary and sufficient condition for oscillation.

The filter coefficients a_{ij} are specified by binary numbers with k bits. Therefore for a specific $\omega_o T$

$$a_{11} = a_{22} = \cos \omega_o T = \frac{p_1}{2^k} \tag{2.16}$$

$$-a_{12} = a_{21} = \sin \omega_o T = \frac{p_2}{2^k} \tag{2.17}$$

where p_1 and p_2 are the decimal representations of a_{11} and a_{12}, both of which must be divisible by 2^k. From (2.15) and (2.12b) it follows that

$$(\frac{p_1}{2^k})^2 + (\frac{p_2}{2^k})^2 = 1. \tag{2.18}$$

This requires that

$$p_1^2 + p_2^2 = 2^{2k}. \tag{2.19}$$

In general, for a given $\omega_o T$, the numbers for p_1 and p_2 which satisfy (2.19) may not exist. Actually, as will be seen later, only a finite set of frequencies can be realized if finite arithmetic is used.

Returning to the set of equations (2.13a-d), it remains a problem for further research to investigate the existence and nature of a general solution for this set of equations. Following a suggestion of Jackson [2], the solution for (2.13a-d) will be simplified by the somewhat restrictive, but intuitively plausible assumption that quantization of product terms can be avoided if the products in (2.13b-d) contain at most one noninteger coefficient. If two terms are contained in the product (for example, see (2.13b)), a necessary condition is that one of these be an integer. If three terms are contained in the product (for example, see (2.13d)) a necessary condition is that two of them be integer or one of them be zero. With this assumption in mind, four integer coefficients have to be selected to be able to evaluate the remaining four coefficients. By inspection of (2.13b) and (2.13c), it can be deduced that two of the a_{ij} coefficients and two of the b_i/c_i coefficients have to be selected as integers.

The resulting sixteen possibilities (for example select a_{11}, a_{12}, b_1, c_1 or a_{11}, a_{21}, b_1, c_1 as integers) are investigated separately by substitution of each set of selected integer coefficients into (2.13d) and application of the following set of rules:

a) An individual product term in (2.13d) is formed without quantization error, if either two multiplier coefficients are integers or one multiplier coefficient is zero.

b) It is impossible to have $b_1 = b_2 = 0$, or $c_1 = c_2 = 0$, because these conditions imply no provision for input and output.

c) Neither a_{12} nor a_{21} are allowed to be zero, because then (2.13b) cannot be solved unambiguously for a_{12} or a_{21}.

d) If a nonzero multiplier coefficient is required to be an integer, then the integer 1 is chosen, because in in this case no muliplier is needed in the practical realization.

As a demonstration of how the above rules are applied, consider the choice of a_{21}, a_{22}, b_1, b_2 as integers. Substituting these integers into (2.13d) shows that the terms $(a_{12}\ b_2\ c_1)$ and $-(a_{11}\ b_2\ c_2)$ contain only one integer coefficient, namely b_2 and by rule (a), it is required that $b_2 = 0$. From rule (b) and (d) this requires that $b_1 = 1$. Since, except for rule (d), no further restrictions exist the remaining integer coefficients a_{21} and a_{22} can either be chosen as $a_{21} = a_{22} = 1$ or $a_{21} = 1$ and $a_{22} = 0$ (the case where $a_{21} = 0$ and $a_{22} = 1$ yields no new result). Substituting the selected coefficients into (2.13 a-d) yields the remaining four coefficients a_{11}, a_{12}, c_1 and c_2 and thus two distinct S-matrices result. They are

$$S_a = \begin{bmatrix} -a & -b & 1 \\ 1 & 0 & 0 \\ c & e & d \end{bmatrix}, \quad S_b = \begin{bmatrix} -(1+a) & -(1+a+b) & 1 \\ 1 & 1 & 0 \\ c & (e+c) & d \end{bmatrix}. \quad (2.20)$$

Another choice is to select a_{21}, a_{22}, b_1, c_2 as integers. However, from (2.13d) it is seen that the term $(a_{12}\ b_2\ c_1)$ contains no integer coefficient at all and by rule (a) the above selection does not lead to a solution.

Working through the remaining 14 possible selections for integer-coefficients, six more S-matrices are obtained. They are

$$S_c = \begin{bmatrix} -a & 1 & c \\ -b & 0 & e \\ 1 & 0 & d \end{bmatrix}, \quad S_d = \begin{bmatrix} -(1+a) & 1 & c \\ -(1+a+b) & 1 & (e+c) \\ 1 & 0 & d \end{bmatrix}, \quad (2.21)$$

$$S_e = \begin{bmatrix} 0 & 1 & 0 \\ -b & -a & 1 \\ e & c & d \end{bmatrix}, \quad S_f = \begin{bmatrix} 1 & 1 & 0 \\ -(1+a+b) & -(1+a) & 1 \\ (e+c) & c & d \end{bmatrix}, \quad (2.22)$$

$$S_g = \begin{bmatrix} 0 & -b & e \\ 1 & -a & c \\ 0 & 1 & d \end{bmatrix}, \quad S_h = \begin{bmatrix} 1 & -(1+a+b) & (e+c) \\ 1 & -(1+a) & c \\ 0 & 1 & d \end{bmatrix}. \quad (2.23)$$

Using linear row and column operations it is possible to transform S_e into S_a, S_g into S_c, S_f into S_b, and S_h into S_d, indicating that only four unique circuit configurations can be derived from the eight S-matrices. Furthermore, $S_c = S_a^T$ and $S_d = S_e^T$. Jackson [2] has shown that for a particular configuration for a digital filter, S_j, a unique "transpose configuration" for the transpose system, S_j^T, is derived from the given configuration for S_j by

a) reversing the direction of the signal flow in the given network, and

b) changing all summation nodes in the given configuration into pickoff nodes in the transpose configuration, and changing all pickoff nodes into summation nodes.

Jackson has also shown that a single S_j matrix and its transpose S_j^T can be realized by twelve different configurations, because the elements of the S_j matrix are not

necessarily in one-to-one correspondence with the multipliers in the flowgraph for the filter realization.

Since there are only two uniquely different S_j matrices, S_a and S_b, and each has 12 different realizations, there are 24 different configurations for a given transfer function $H(z)$, if the restriction of k-digit arithmetic is imposed. These 24 configurations are given in the next paragraph.

Two parallel forms for $d = 0$, and two cascade forms for $d = 1$ are derived from S_a and $S_c = S_a^T$. It should be noted, by comparison of (2.11) with (2.4) and (2.5), that a second-order section becomes a parallel form (compare with (2.4)) when $d = 0$. Similarly, a second-order section becomes a cascade form (compare with (2.5)) when $d = 1$.

The four forms are designated S_{a_1}, S_{a_2}, $S_{a_1}^T$, $S_{a_2}^T$. They are described by the following S_j matrices and drawn in Figs. 2.4-2.5.

$$S_{a_1} = \begin{bmatrix} -a & -b & 1 \\ 1 & 0 & 0 \\ c & e & d=0 \end{bmatrix}. \tag{2.24}$$

$$S_{a_2} = \begin{bmatrix} -a & -b & 1 \\ 1 & 0 & 0 \\ c & e & d=1 \end{bmatrix}. \tag{2.25}$$

The next eight configurations are cascade forms with $d = 1$. Two direct forms are derived from S_{a_3} and $S_{a_3}^T$, where

$$S_{a_3} = \begin{bmatrix} -a & -b & 1 \\ 1 & 0 & 0 \\ (g-a) & (f-b) & 1 \end{bmatrix}. \qquad (2.26)$$

Here, $g = (a+c)$ and $f = (e+b)$. Their configurations are displayed in Figs. 2.6-2.7. The new multipliers g and f are obtained by shifting the first pickoff node in Fig. 2.5 to a position after the first summation node (compare Fig. 2.5 with Fig. 2.6). That a new configuration results is due to the fact that the elements of the S_j matrix are not in one-to-one correspondence with the actual multipliers in the flow graph representing the configuration. The forms for S_{a_4}, $S^T_{a_4}$, S_{a_5}, $S^T_{a_5}$ and S_{a_6}, $S^T_{a_6}$ are shown by Jackson to be of minor importance. They will not be repeated here.

Twelve new configurations which have not appeared in the literature before are now derived from S_b. Two parallel forms, with $d = 0$, and two cascade forms, with $d = 1$, are derived from S_{b_1}, $S^T_{b_1}$, and S_{b_2}, $S^T_{b_2}$.

$$S_{b_1} = \begin{bmatrix} -(1+a) & -(1+a+b) & 1 \\ 1 & 1 & 0 \\ c & (e+c) & d=0 \end{bmatrix}. \qquad (2.27)$$

$$S_{b_2} = \begin{bmatrix} -(1+a) & -(1+a+b) & 1 \\ 1 & 1 & 0 \\ c & (e+c) & d=1 \end{bmatrix}. \qquad (2.28)$$

Their configurations are displayed in Figs. 2.8-2.9. The remaining eight cascade forms require that $d = 1$. Two direct forms are obtained from S_{b_3} and $S_{b_3}^T$, where

$$S_{b_3} = \begin{bmatrix} -(1+a) & -(1+a+b) & 1 \\ 1 & 1 & 0 \\ g-(1+a) & f-(1+a+b) & 1 \end{bmatrix}. \qquad (2.29)$$

Here, $g = (1+a+c)$ and $f = (1+a+b+c+e)$. Their configurations are shown in Figs. 2.10 - 2.11. The other forms are obtained from

$$S_{b_4} = \begin{bmatrix} c-h & (e+c)-r & 1 \\ 1 & 1 & 0 \\ c & (e+c) & 1 \end{bmatrix}, \qquad (2.30)$$

$$S_{b_5} = \begin{bmatrix} c-h & -(1+a+b) & 1 \\ 1 & 1 & 0 \\ c & f-(1+a+b) & 1 \end{bmatrix}, \qquad (2.31)$$

$$S_{b_6} \begin{bmatrix} -(1+a) & (e+c)-r & 1 \\ 1 & 1 & 0 \\ g-(1+a) & (e+c) & 1 \end{bmatrix}, \qquad (2.32)$$

where $h = (1+a) + c$ and $r = (1+a+b) + (e+c)$. Some of their configurations are shown in Figs. 2.12-2.14. As was pointed out earlier, the transpose configurations can be obtained by

reversing the signal flow and replacing summing nodes by pickoff nodes.

It is worth noting that all of the derived 24 configurations have two delays and four multipliers. However, the number of summing nodes and pickoff nodes varies between two and four. This is summarized in the following table:

Summing Node	Pickoff Node	Configuration
2	2	S_{a1}, S^T_{a1}
2	3	S_{a2}, S_{a3}, S_{a4}, S^T_{b1}
3	2	S^T_{a2}, S^T_{a3}, S^T_{a4}, S_{b1}
3	3	S_{a5}, S^T_{a5}, S_{a6}, S^T_{a6}, S_{b4}, S^T_{b4} S_{b2}, S^T_{b2}, S_{b3}, S^T_{b3}
4	3	S_{b5}, S_{b6}
3	4	S^T_{b5}, S^T_{b6}

In a theoretical sense only forms S_{a1} and S^T_{a2} are canonical. In a practical sense a tradeoff between hardware requirements (number of multipliers and adders) and noise performance has to be found.

It remains a subject of further investigation to evaluate the noise performance of the newly derived twelve forms using the methods devised by Jackson [2]. Furthermore it remains to formally define a "canonical form" and relate the order of the difference equation to the necessary number of delays, multipliers and nodes in the actual realization of a digital filter.

D. THE EFFECT OF COEFFICIENT ACCURACY ON THE POLE POSITIONS OF SECOND-ORDER DIGITAL FILTERS

The poles or eigenvalues of a given transfer function H(z) can be evaluated from the characteristic equation of H(z) which is given by

$$z^2 + az + b = (z-z_1)(z-z_2) = 0. \tag{2.33}$$

The roots of the characteristic equation or its eigenvalues are found to be

$$z_1, z_2 = -\frac{a}{2} \pm j\sqrt{b-(\frac{a}{2})^2}. \tag{2.34}$$

This pair of complex roots is expressed in polar coordinates as follows:

$$|z_{1,2}| = r = \sqrt{b}, \tag{2.35}$$

$$\text{Arg } z_{1,2} = \theta = \cos^{-1} (\frac{-a}{2\sqrt{b}}). \tag{2.36}$$

At this point let us digress to find a relation between magnitude and argument of the poles $z_{1,2}$ and the notions of damping and frequency, as they are defined for linear, continuous systems.

For a second-order continuous filter with a damped sinusoidal impulse response of the form $e^{-\alpha t} \sin \omega_o t$, for $t \geq 0$, the roots of the characteristic equation are located at $s = -\alpha \pm j\omega_o$, where s equals the Laplace transform variable.

If this response is sampled at intervals T, the poles of the z-transformed version of the response are given by

$$z^2 - 2ze^{-\alpha T} \cos \omega_o T + e^{-2\alpha T} = 0. \tag{2.37}$$

Comparing (2.37) with (2.33) it follows that

$$a = -2e^{-\alpha T} \cos \omega_o T, \tag{2.38}$$

$$b = e^{-2\alpha T}, \tag{2.39}$$

so that from (2.35) and (2.36)

$$|z_{1,2}| = r = e^{-\alpha T}, \tag{2.40}$$

$$\text{Arg } z_{1,2} = \theta = \omega_o T. \tag{2.41}$$

Thus (2.40-41) can be used to compare the impulse response of the digital filter with the sampled impulse response of a continuous second-order filter. It can be deduced that the polar coordinates r and θ of the pole positions $z_{1,2}$ are a measure of the damping factor α and the resonant frequency ω_o of the equivalent continuous filter. The position of a pair of complex roots which would result in a stable digital filter response is shown in Fig. 2.15. The stability boundary is given in the z-plane by the circle described by $|z| = 1$.

Let us now turn our attention to the effect of coefficient accuracy on the pole positions. Expressions are now developed which show the deviation of r and ω_o from the nominal values if a second-order digital filter is realized with coefficients represented by finite computer word lengths.

Let a_{ij} = nominal filter coefficient,

a_{ij}' = coefficient expressed by a finite computer word.

The deviation from the nominal coefficient is then expressed by

$$\Delta a_{ij} = a_{ij} - a_{ij}'. \tag{2.42}$$

Note that $\Delta a_{ij} = 0$, if $a_{ij} = 0$ or 1. For small changes in the coefficients a_{ij} as given by (2.42) the changes in r and θ depend only upon the A-matrix from (2.10). These changes can be approximated by

$$\Delta r = \frac{\partial r}{\partial a_{11}} \Delta a_{11} + \frac{\partial r}{\partial a_{12}} \Delta a_{12} + \frac{\partial r}{\partial a_{21}} \Delta a_{21} + \frac{\partial r}{\partial a_{22}} \Delta a_{22}, \tag{2.43}$$

and

$$\Delta \theta = \frac{\partial \theta}{\partial a_{11}} \Delta a_{11} + \frac{\partial \theta}{\partial a_{12}} \Delta a_{12} + \frac{\partial \theta}{\partial a_{21}} \Delta a_{21} + \frac{\partial \theta}{\partial a_{22}} \Delta a_{22}. \tag{2.44}$$

The partial derivatives are evaluated using (2.35) and (2.36) which are repeated here for convenience:

$$r = \sqrt{b} = \sqrt{a_{11}\, a_{22} - a_{12}\, a_{21}}\,, \tag{2.45}$$

$$\cos\theta = \cos\omega_o T = \frac{-a}{2\sqrt{b}} = \frac{a_{11} + a_{22}}{2\sqrt{a_{11}a_{22} - a_{21}a_{12}}}\,. \tag{2.46}$$

These partial derivatives are

$$\frac{\partial r}{\partial a_{11}} = \frac{a_{22}}{2r}\,, \tag{2.47a}$$

$$\frac{\partial r}{\partial a_{12}} = -\frac{a_{21}}{2r}\,, \tag{2.47b}$$

$$\frac{\partial r}{\partial a_{21}} = -\frac{a_{12}}{2r}\,, \tag{2.47c}$$

$$\frac{\partial r}{\partial a_{22}} = \frac{a_{11}}{2r}\ , \tag{2.47d}$$

$$\frac{\partial \theta}{\partial a_{11}} = \frac{-a_{11}a_{22} + 2a_{12}a_{21} + a_{22}^2}{4r^3 \sin\theta}\ , \tag{2.48a}$$

$$\frac{\partial \theta}{\partial a_{12}} = -\ \frac{a_{11}a_{21} + a_{22}a_{12}}{4r^3 \sin\theta}\ , \tag{2.48b}$$

$$\frac{\partial \theta}{\partial a_{21}} = -\ \frac{a_{11}a_{12} + a_{22}a_{12}}{4r^3 \sin\theta}\ , \tag{2.48c}$$

$$\frac{\partial \theta}{\partial a_{22}} = \frac{-a_{11}a_{22} + 2a_{12}a_{21} + a_{11}^2}{4r^3 \sin\theta}\ . \tag{2.48d}$$

With these expressions substituted into (2.43) and (2.44) the following general result is obtained:

$$\Delta r = \frac{1}{2r}\ (a_{22}\ \Delta a_{11} - a_{21}\Delta a_{12} - a_{12}\Delta a_{21} + a_{11}\Delta a_{22})\ , \tag{2.49}$$

$$\Delta\theta = \frac{1}{4r^3 \sin\theta}[(-a_{11}a_{22} + 2a_{12}a_{21} + a_{22}^2)\Delta a_{11} -$$

$$-(a_{11}a_{21} - a_{22}a_{21})\Delta a_{12} - (a_{11}a_{12} + a_{22}a_{12})\Delta a_{21} +$$

$$+\ (a_{11}^2 + 2a_{12}a_{21} - a_{11}a_{22})\Delta a_{22}]\ . \tag{2.50}$$

This general expression for the shift in pole positions of the second-order digital filter due to finite accuracy in the representation of the filter coefficients is a new result.

From (2.40-41) it follows that

$$\Delta\alpha = -\frac{\Delta r}{T} e^{\alpha T}, \tag{2.51}$$

$$\Delta\omega_o = \frac{1}{T}\Delta\theta \tag{2.52}$$

The Eqs. (2.51-52) indicate how a change in the pole position of the digital filter is reflected in terms of a corresponding change in damping and resonant frequency of the sampled continuous filter. The significance of the derived results are now tested using two examples:

1. <u>Oscillator Realized by Two Coupled First-Order Difference Equations</u>

   The S-matrix for this type of oscillator is

$$S = \begin{bmatrix} \cos\omega_o T & \sin\omega_o T & 0 \\ -\sin\omega_o T & \cos\omega_o T & 0 \\ 1 & 0 & 0 \end{bmatrix} \tag{2.53}$$

Assuming all Δa_{ij} to be bounded, it is possible to define a Δ such that

$$|\Delta a_{ij}| \leq \Delta. \tag{2.54}$$

The changes in r and θ are bounded by

$$\Delta r \leq \Delta.\ (\sin\omega_o T + \cos\omega_o T), \tag{2.55}$$

$$\Delta\theta \leq \Delta.\ (\cos\omega_o T - \sin\omega_o T). \tag{2.56}$$

From (2.55), it can be seen that through proper choice of the ratio between oscillation frequency ω_o and sampling frequency $\omega_s = \frac{2\pi}{T}$, $\Delta\theta$ can be made almost zero. Thus if

$\Delta\theta \approx 0$ from (2.56), $\omega_o T = \frac{\pi}{4}$ and $\frac{\omega_o}{\omega_s} = \frac{1}{8}$. However, for this ratio the change of damping from the desired value of 1 for an oscillator is a maximum as indicated by (2.55) when $\omega_o T = \frac{\pi}{4}$. Conversely, if $\Delta r \approx 0$, from (2.55), $\omega_o T = \frac{3\pi}{4}$ and $\frac{\omega_o}{\omega_s} = \frac{3}{8}$. However, for this ratio the change of frequency is a maximum as indicated by (2.56).

This result would indicate that, if one desires an oscillator with exact frequency ($\Delta\theta = 0$), then the realization by (2.53) shows a damped response because of finite precision in the coefficients. Alternately, it would indicate that a sinusoidal oscillator is possible ($\Delta r = 0$), but the frequency would not necessarily be at its nominal value. This problem is discussed further in Chapter IV, where it is shown that the effect of quantization errors after multiplication of data samples with coefficients is to introduce nonlinear limit cycles, so that a constant amplitude oscillation ($\Delta r = 0$) can be generated in practice.

2. Oscillator Realized by One Second-Order Difference Equation

$$S = \begin{bmatrix} 2\cos\omega_o T & -1 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}. \qquad (2.57)$$

Using the inequality (2.54) the changes in r and ω_o are then bounded by

$$\Delta r = 0 \text{ (because } a_{22} = 0), \qquad (2.58)$$

$$\Delta\omega_o \leq -\frac{\Delta}{2T \sin \omega_o T} . \qquad (2.59)$$

As can be seen from (2.59) sampling too fast (T and $\sin \omega_o T$ are small) may have a deteriorating effect, because $\Delta\omega_o$ can get intolerably large.

Equation (2.57) is of the same form as the S-matrix (2.24), which has been used previously to derive configurations for second-order filters. For the S-matrix given by (2.24), the changes in r and θ are bounded by

$$\Delta r \leq \Delta \cdot \frac{1}{2\sqrt{b}} = \frac{\Delta}{2r} , \qquad (2.60)$$

$$\Delta\omega_o \leq \Delta \cdot \frac{2b-a}{2Tb\sqrt{4b-a^2}} = \frac{\Delta}{T} \cdot \left(\frac{1}{2r \sin \omega_o T} + \frac{1}{2r^2 \tan \omega_o T}\right) . \qquad (2.61)$$

Again, from (2.61), it can be seen that sampling too fast can have a deteriorating effect on the digital oscillator, or in general on a digital filter. This may become especially critical for narrowband low-pass filters, where both ω_o and T are small.

It is instructive to show the distribution of the actual pole positions in the z-plane for the digital filters of examples 1 and 2. Suppose that a digital filter is realized by fixed-point arithmetic using 3 bits without sign-bit to realize the filter coefficients. Thus there are $2^3 = 8$ possible coefficient values for the a_{ij} terms. The set of realizable pole positions for the digital filters of example 1 and 2 are displayed in Fig. 2.16 [18].

Comparing examples 1 and 2, it can be seen that for a nominal set of pole positions, the effect of finite word length for the a_{ij} is to cause a uniform change in the pole locations over the z-plane for the realization of example 1. This is in contrast to the nonuniform change for example 2, where the possible pole positions are crowded around $\omega_o T = \frac{\Pi}{2}$, which corresponds to $\frac{\omega_o}{\omega_s} = \frac{1}{4}$. If the nominal values for the a_{ij}'s places the pole locations in a region around $\omega_o T \approx 0$ or Π , then the effects of a finite number of digits is to cause a large effect with the realization of example 2.

E. SUMMARY

The purpose of this chaper is twofold. First, some concepts about linear discrete systems were introduced, which are needed for the understanding of later chapters. Second, two new results were derived. The linear model for second-order digital filters was described and 24 canonical circuit representations, under the assumption of k-digit accuracy, were shown to exist. It is important to note that all 24 configurations have identical transfer functions, even under the assumption of k-digit accuracy. However, their error properties are, in general different. Furthermore, a general formula to predict changes in pole positions due to changes from the nominal values of coefficients because of finite representation of numbers was derived.

Some examples were used to demonstrate that with finite arithmetic the linear digital oscillator cannot always be realized to meet a given specification. Also, the sampling frequency, ω_s, effects the pole locations of the digital filter realization.

There remain three problem areas for which further research is necessary. The first is a topological problem. A formal definition for a "canonical form" is needed. Also, a relation between the order of a difference equation and the minimum number of delays, multipliers, summers and pickoff nodes constituting a canonical form should be found. It is well known that the order of the difference equation corresponds to the number of delays and that the number of nonzero and nonunity coefficients in the difference equation corresponds to the number of multipliers needed. However, a similar correspondence is not clear for the necessary number of adders and pickoff nodes.

The second problem concerns the general solution for the set of equations (2.13a-d). Conditions for the existence and the form of a possible solution need to be investigated in more detail.

As a third problem, an investigation of the roundoff noise properties of the twelve newly derived second-order digital filter configurations is necessary.

After modeling of the second-order digital filter under the restriction of k-digit accuracy, and after investigation of the effects of finite precision arithmetic on the pole positions of a digital filter, quantization of products of

data samples with filter coefficients is considered in the next chapter.

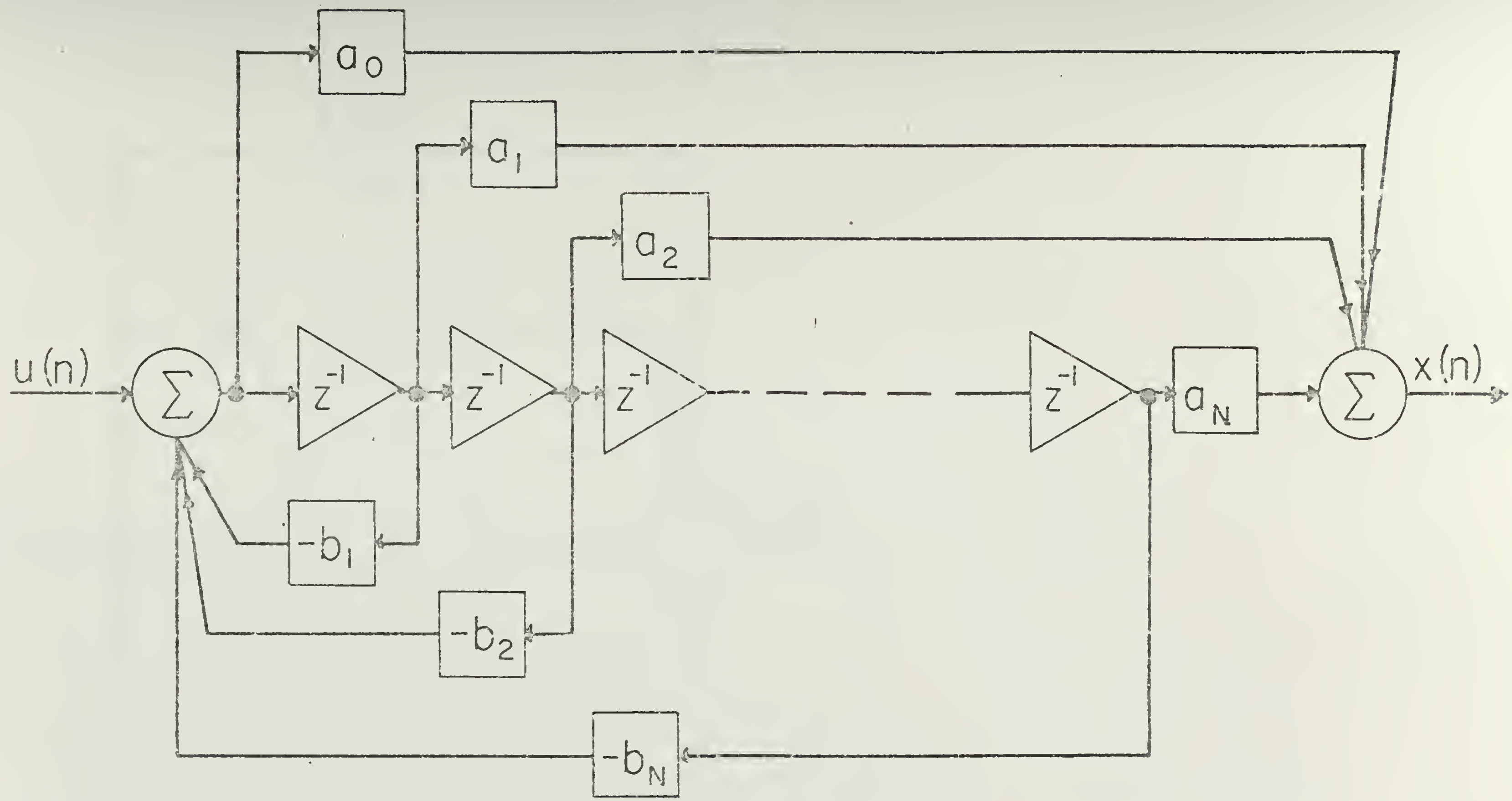


Fig. 2.1: A Digital Filter Realization of the Direct Form.

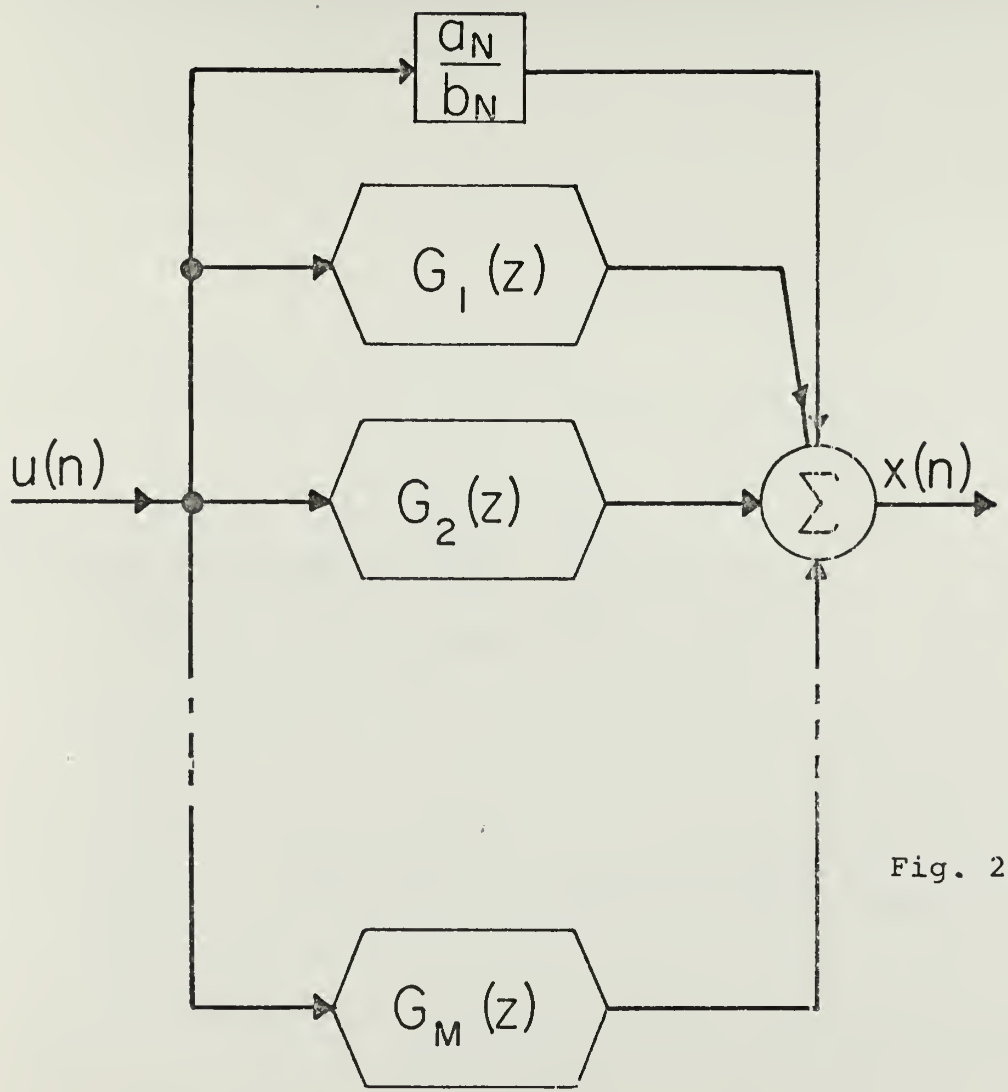


Fig. 2.2: A Digital Filter Realization of the Parallel Form.

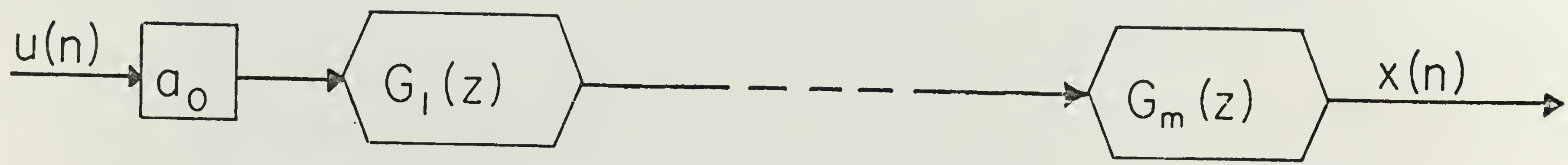


Fig. 2.3: A Digital Filter Realization of the Cascade Form.

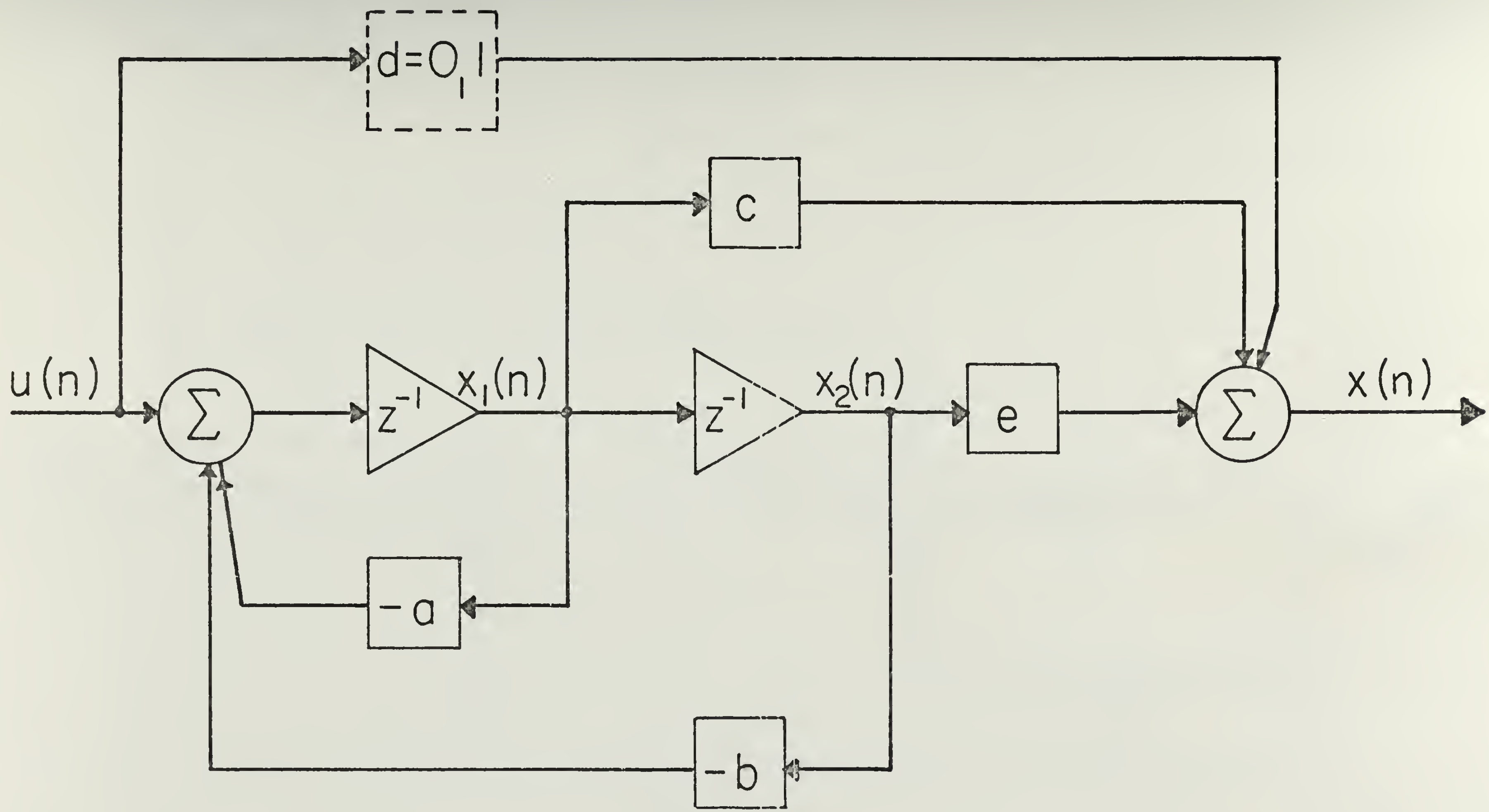


Fig. 2.4: A Realization of the Second-order Digital Filter Using Parallel Form S_{a_1} (d=0) or Cascade Form S_{a_2} (d=1).

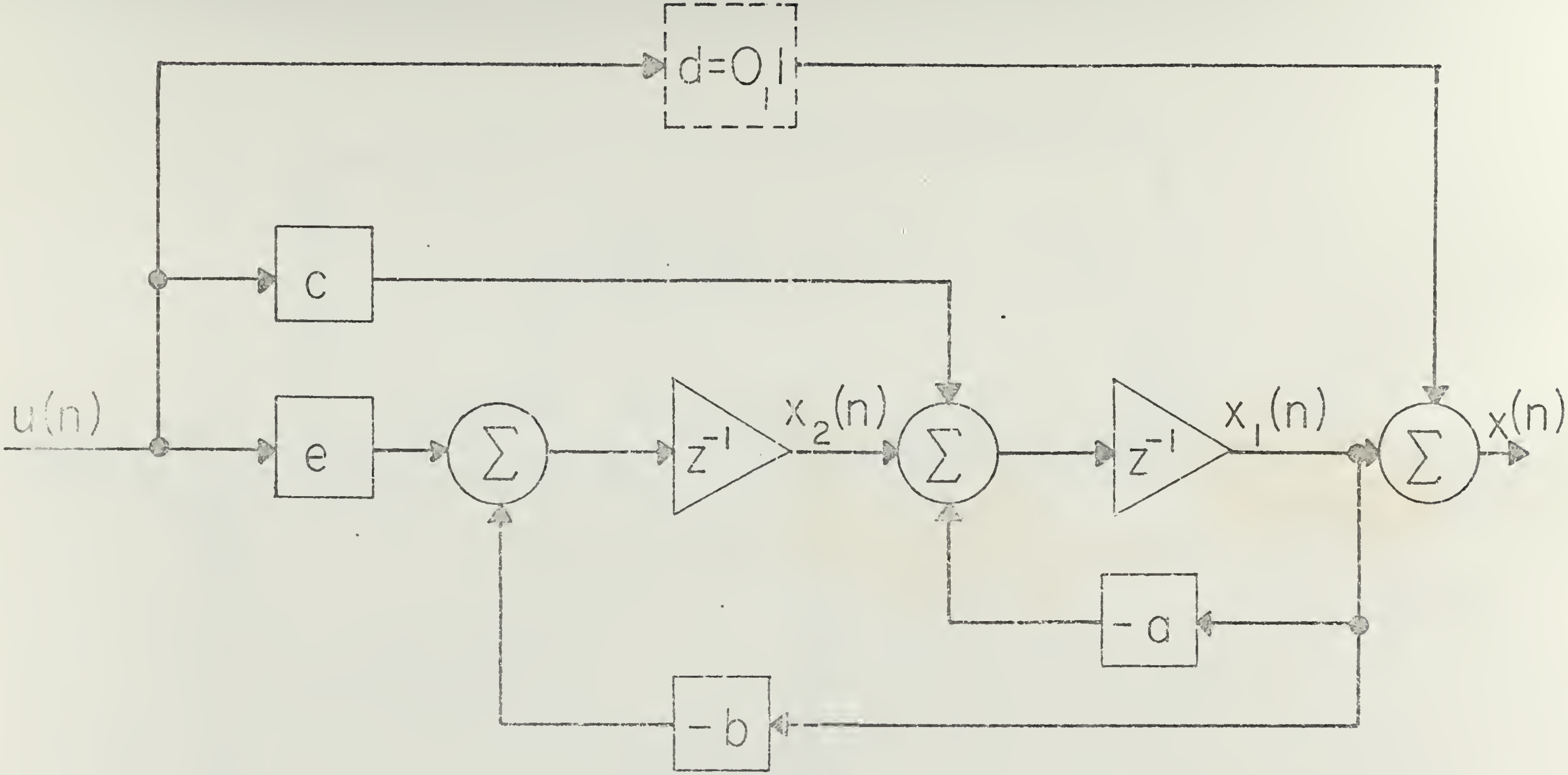


Fig. 2.5: A Realization of the Second-order Digital Filter Using Parallel Form $S_{a_1}^T$ (d=0) and Cascade Form $S_{a_2}^T$ (d=1).

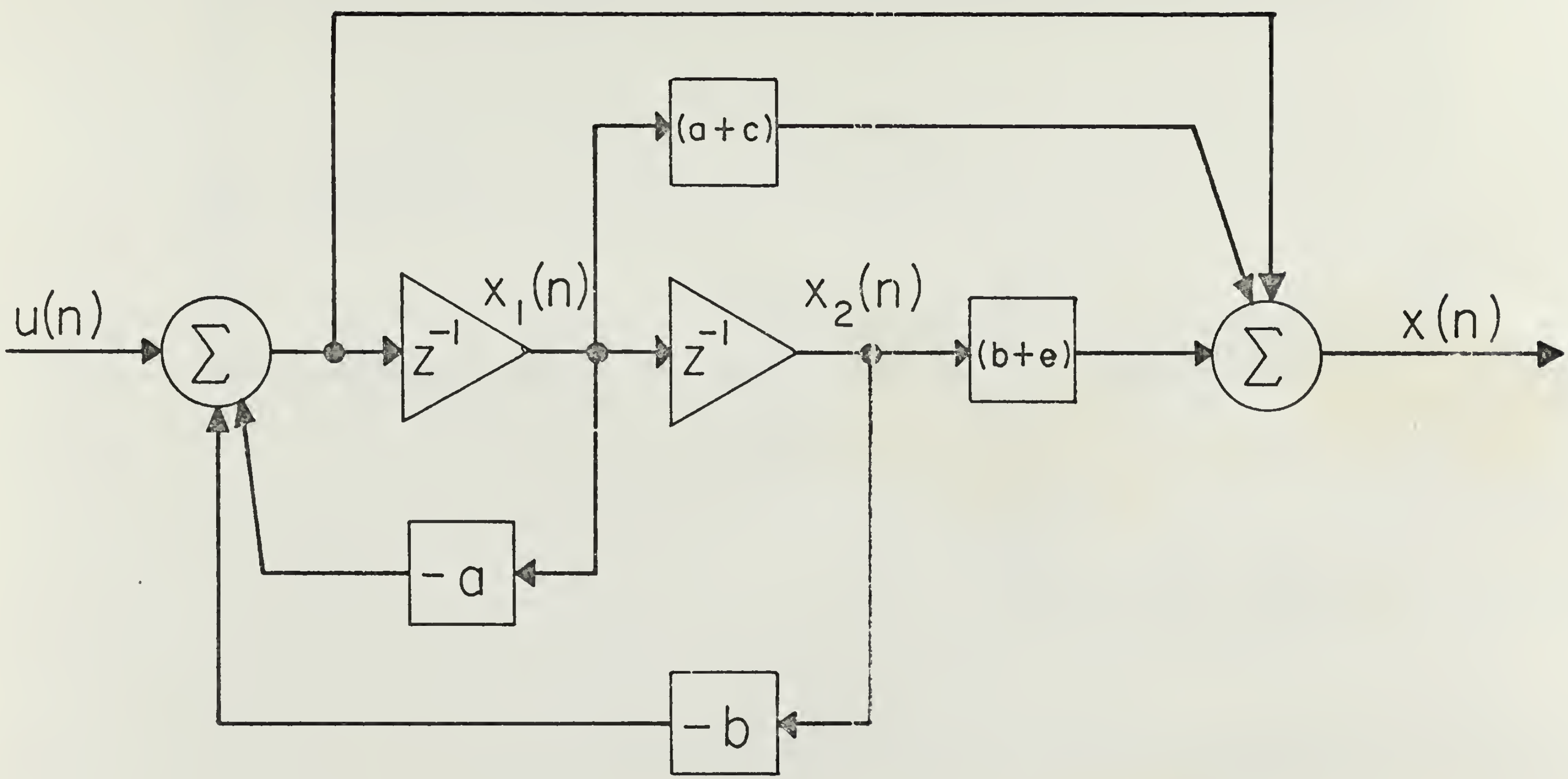


Fig. 2.6: A Realization of the Second-order Digital Filter Using Direct Form S_{a_3}.

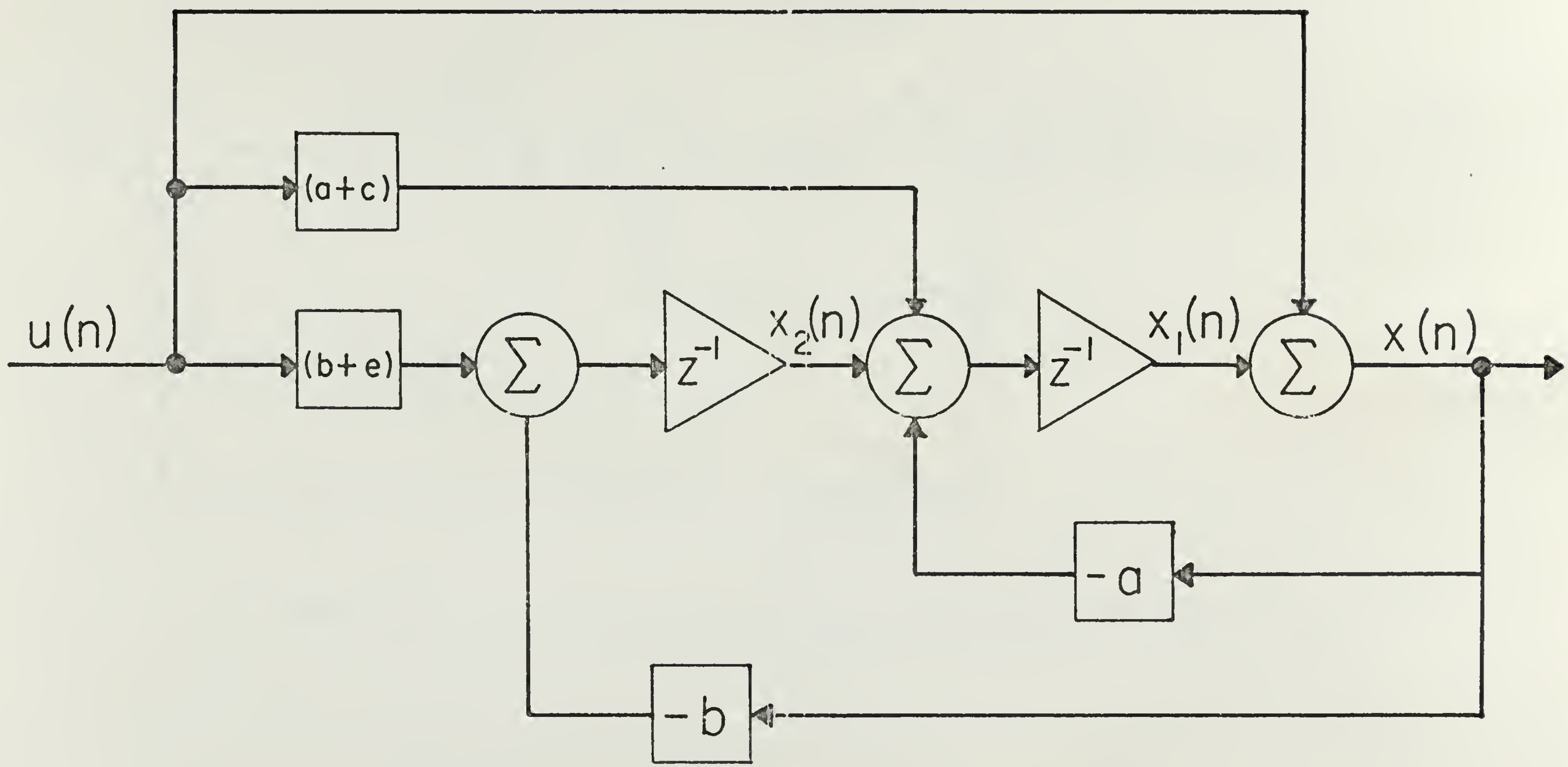


Fig. 2.7: A Realization of the Second-order Digital Filter Using Direct Form S_{a3}^{T}.

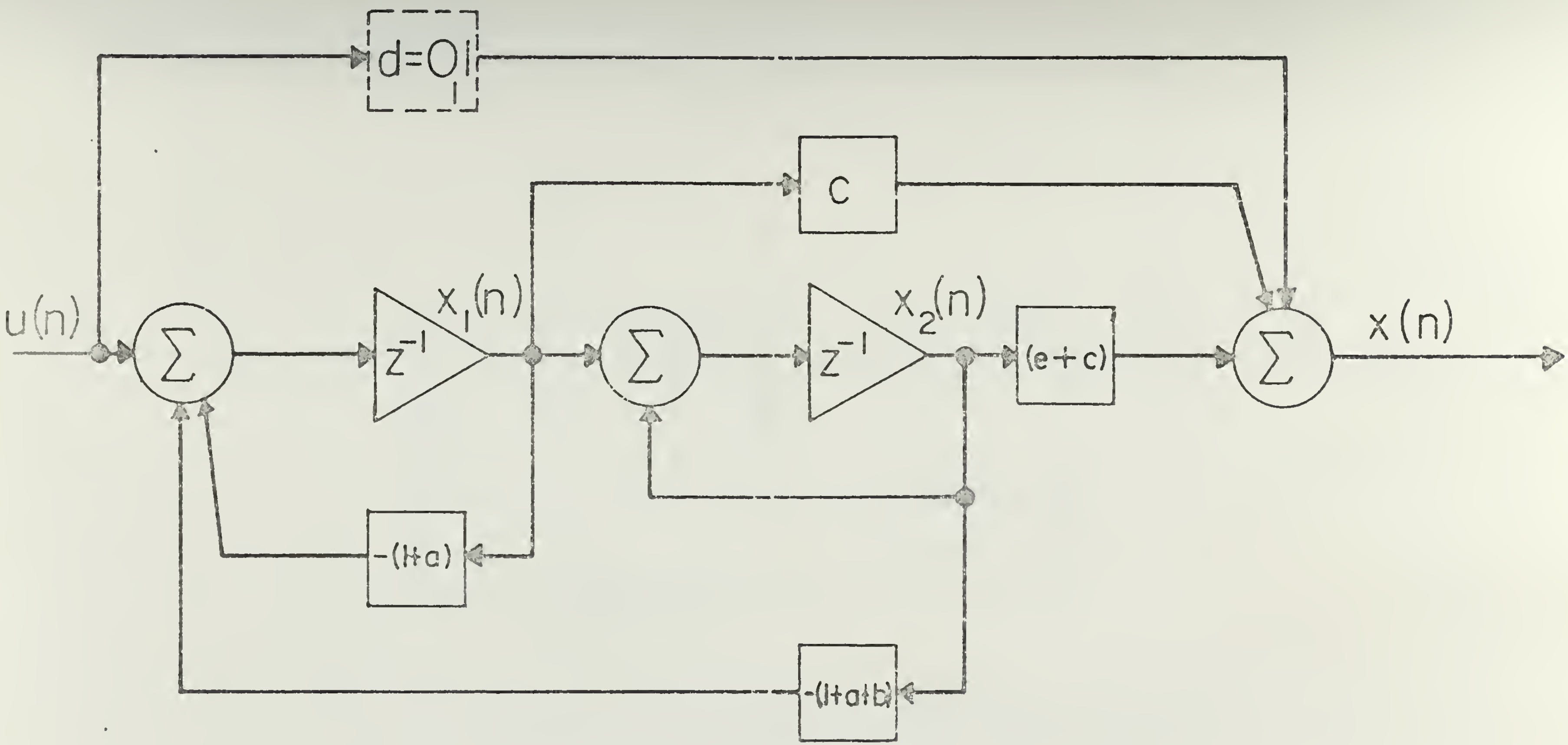


Fig. 2.8: A Realization of the Second-order Digital Filter Using Parallel Form S_{b_1} (d=0) or Cascade Form S_{b_2} (d=1).

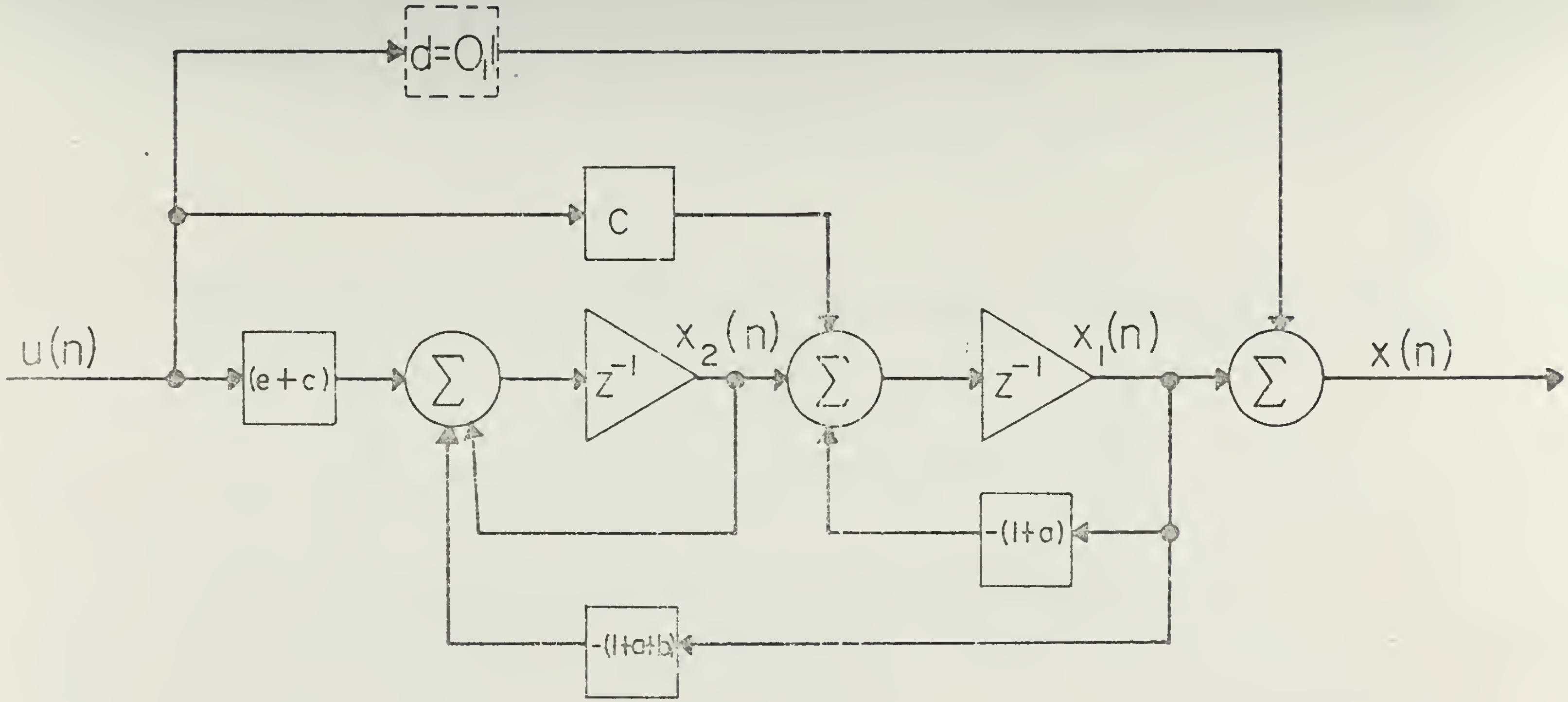


Fig. 2.9: A Realization of the Second-order Digital Filter Using Parallel Form $S_{b_1}^{T}$ (d=0) or Cascade Form $S_{b_2}^{T}$ (d=1).

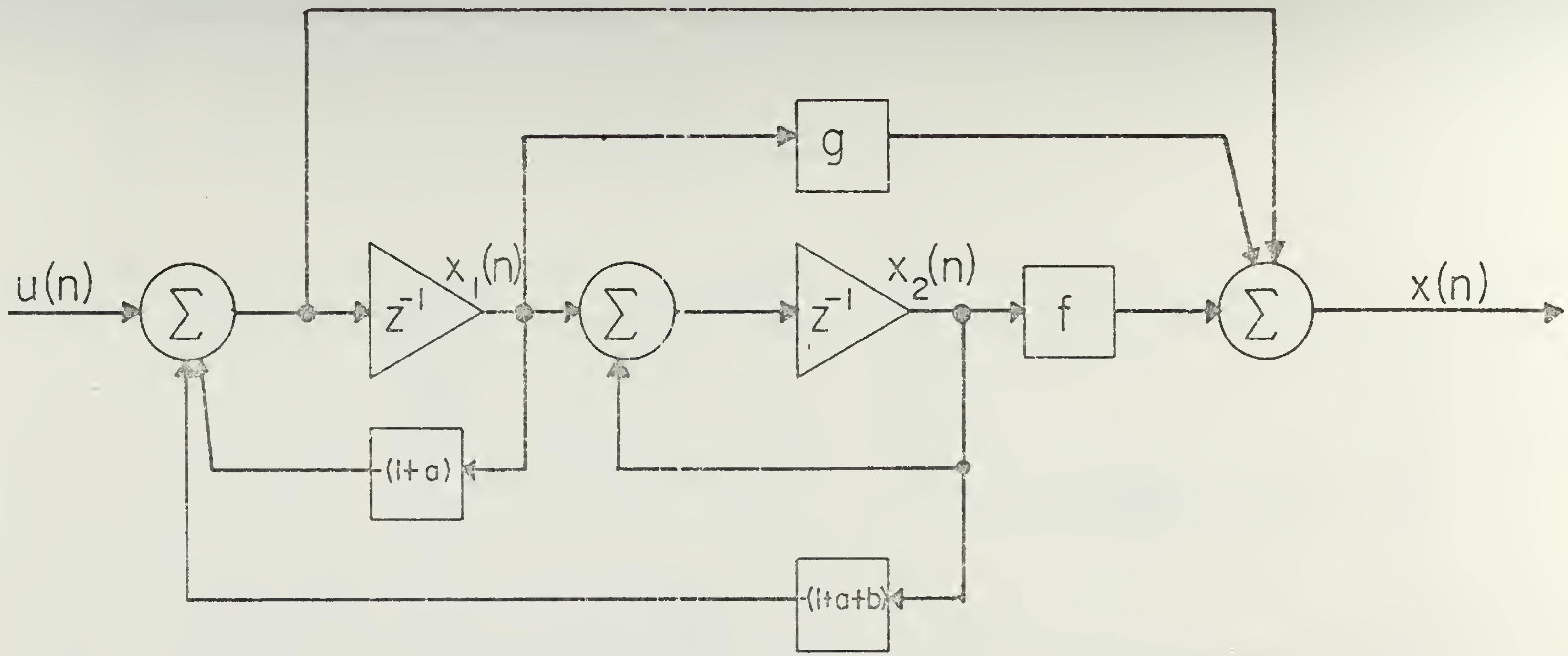


Fig. 2.10: A Realization of the Second-order Digital Filter Using Direct Form S_{b_3}.

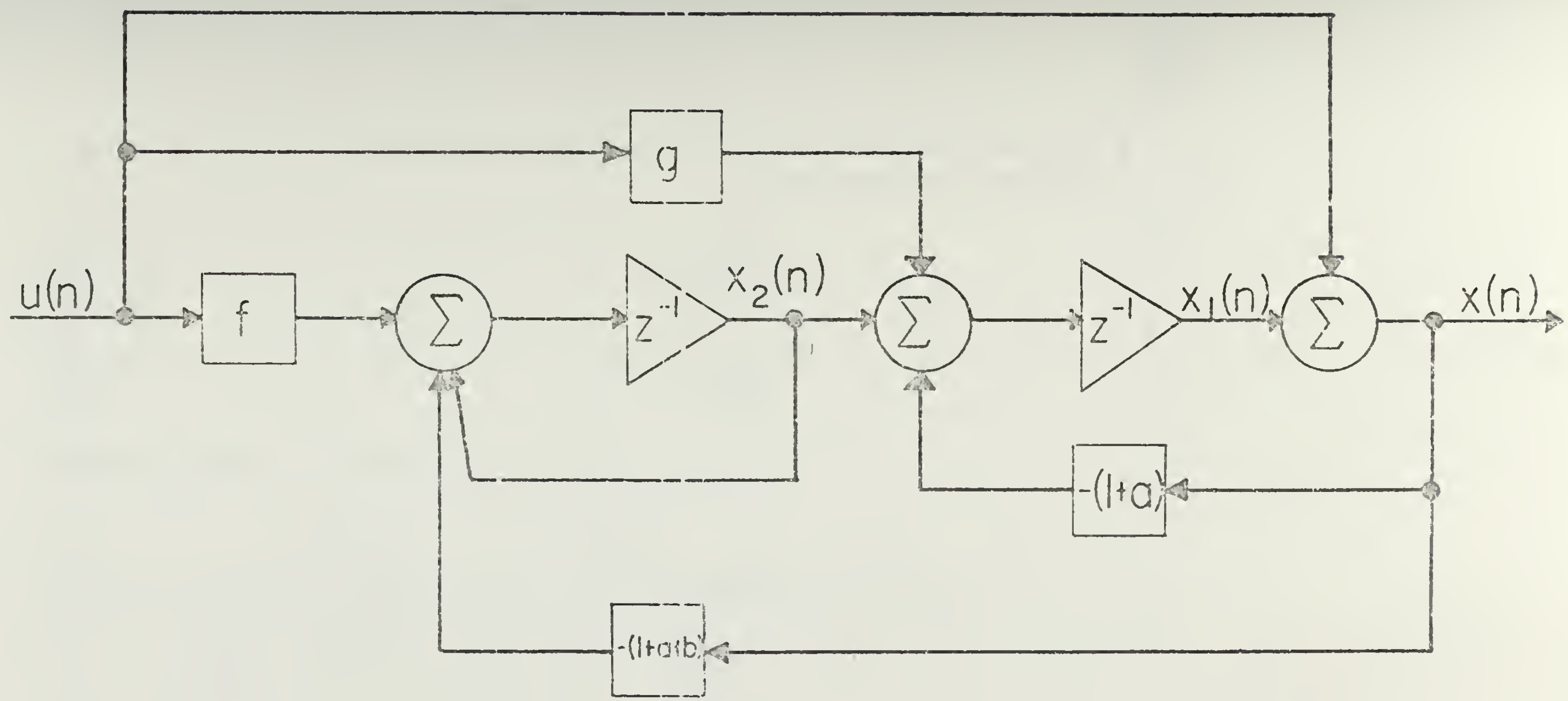


Fig. 2.11: A Realization of the Second-order Digital Filter Using Direct Form $S_{b_3}^T$.

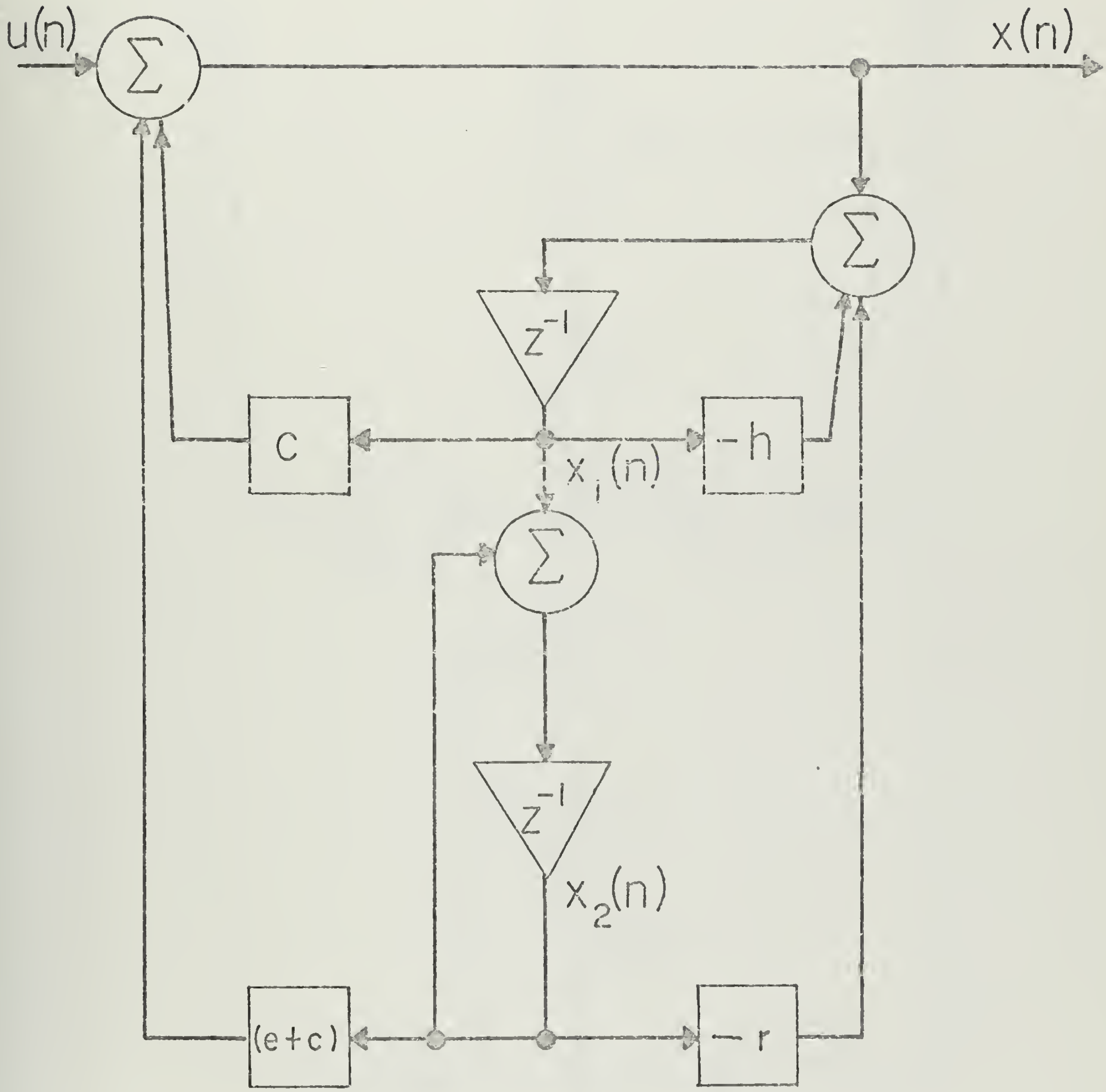


Fig. 2.12: A Realization of the Second-order Digital Filter Using Form S_{34}.

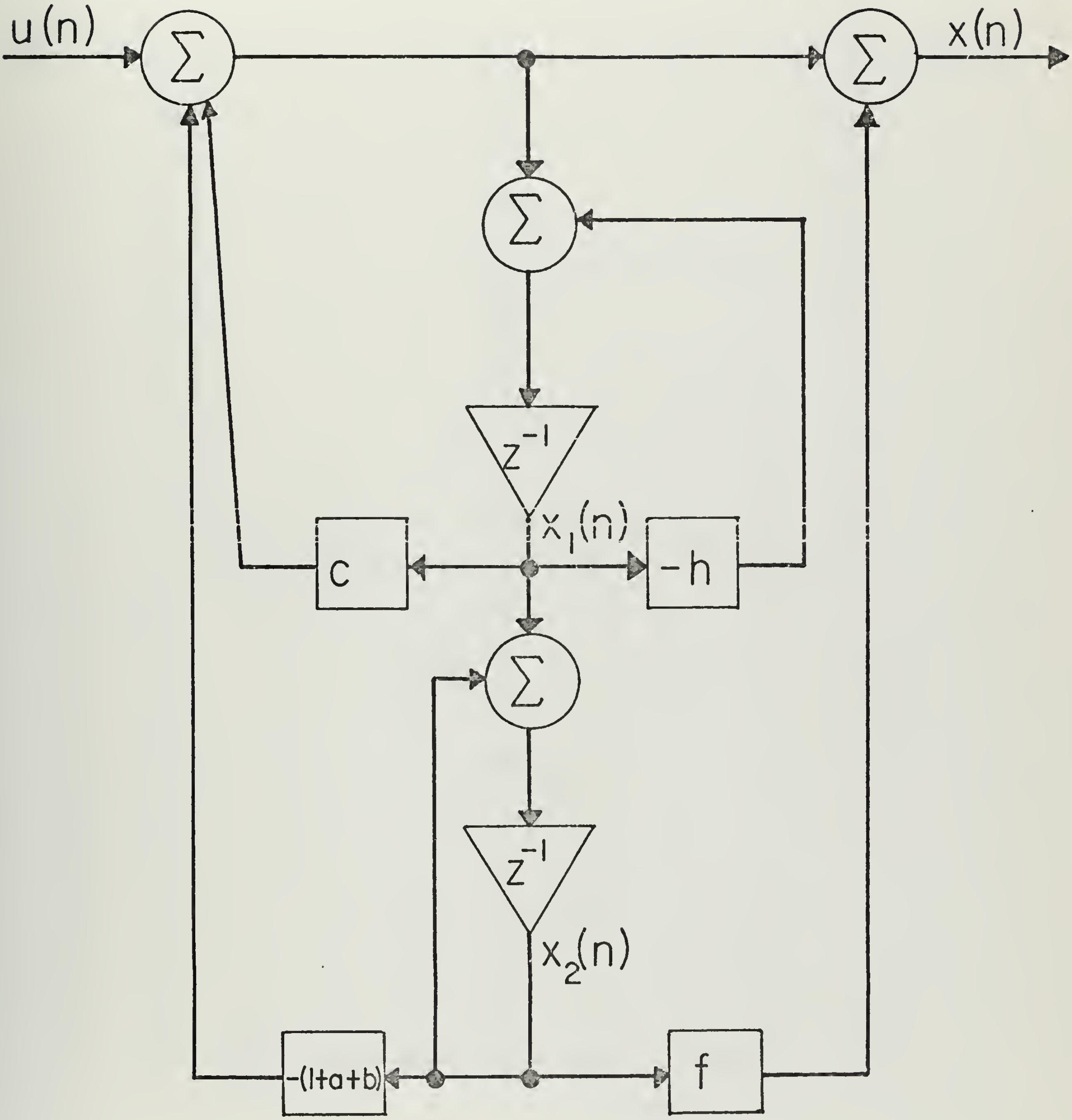


Fig. 2.13: A Realization of the Second-order Digital Filter Using Form S_{b_5}.

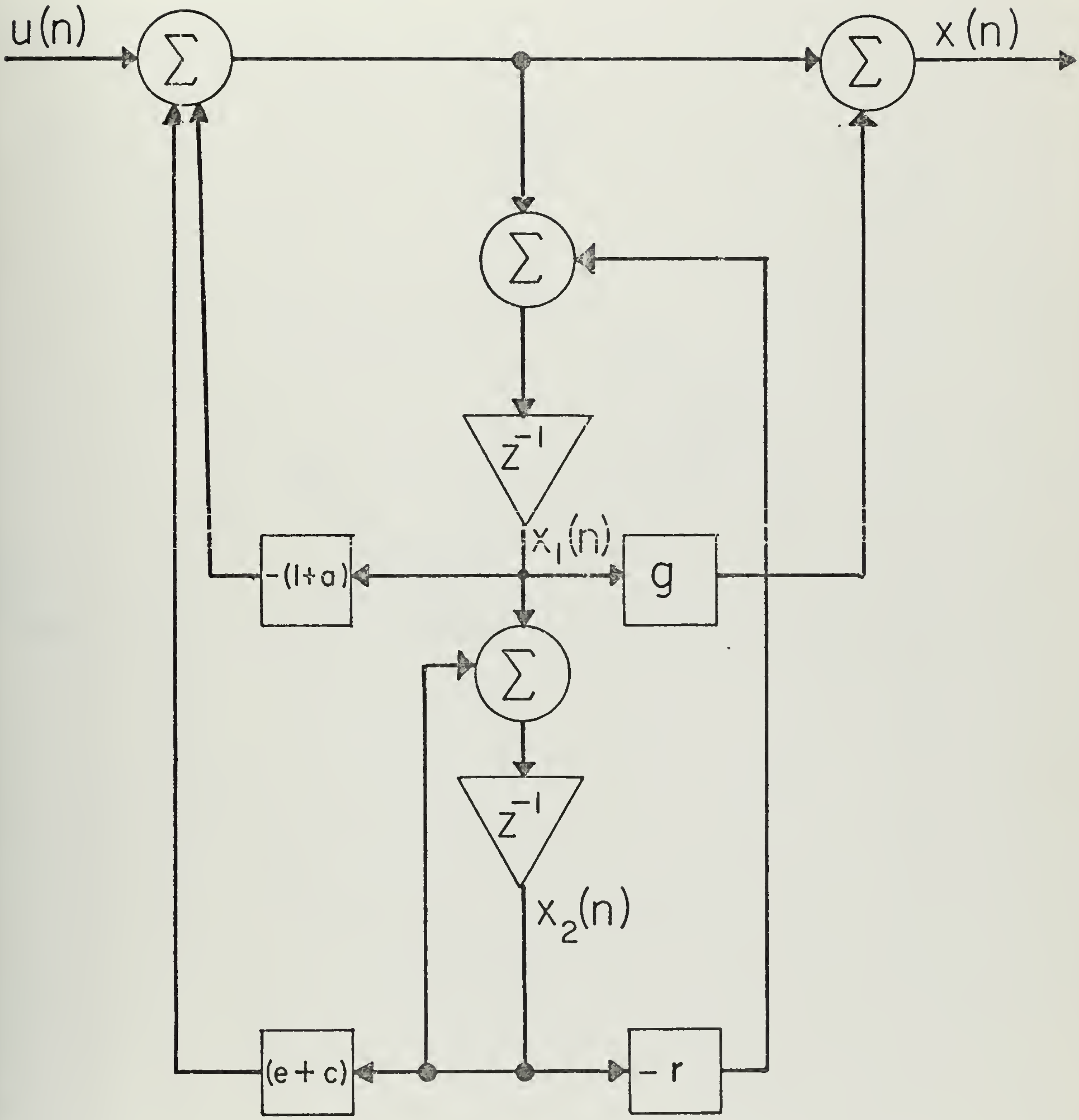


Fig. 2.14: A Realization of the Second-order Digital Filter Using Form S_{b_6}.

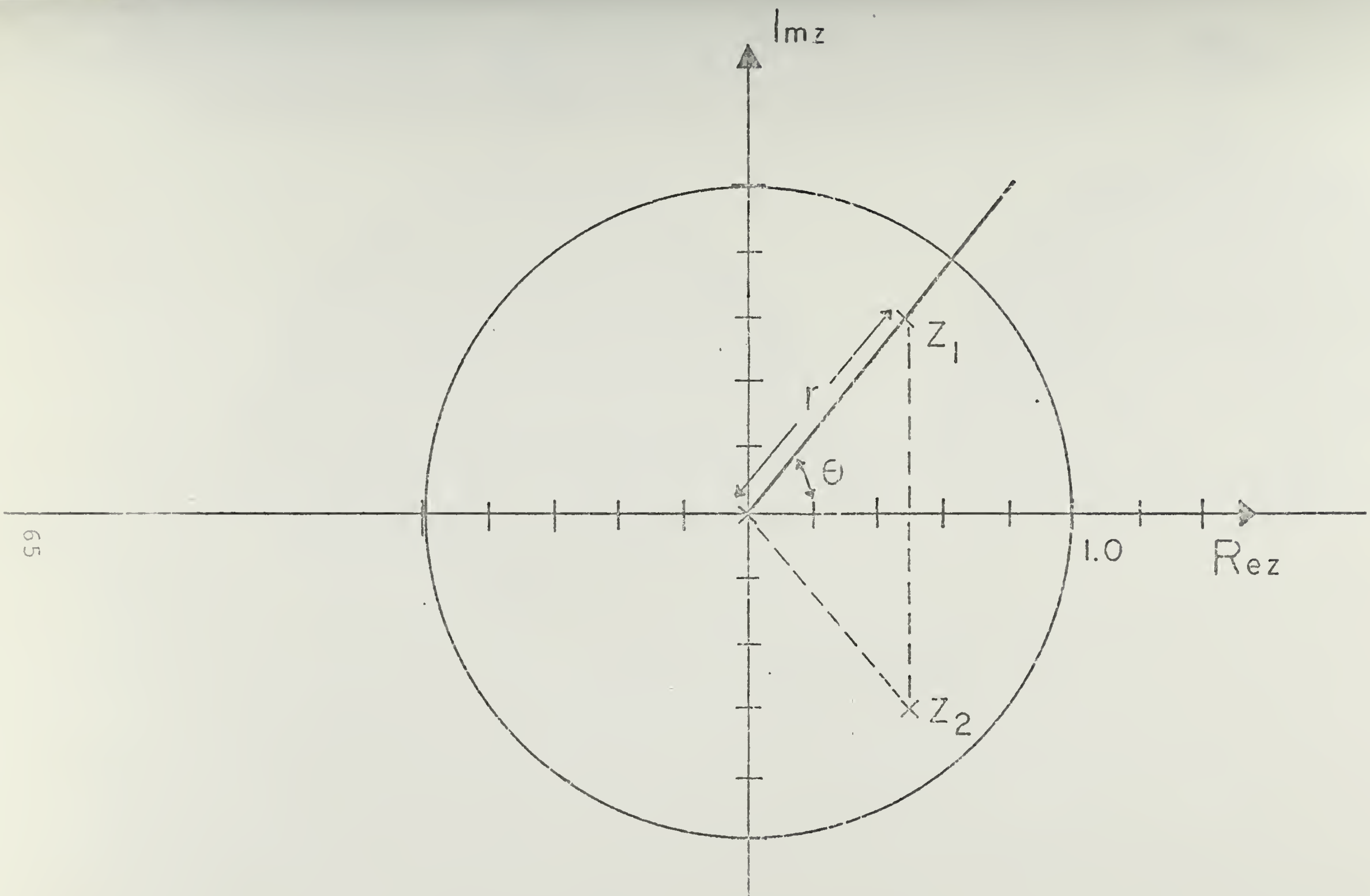
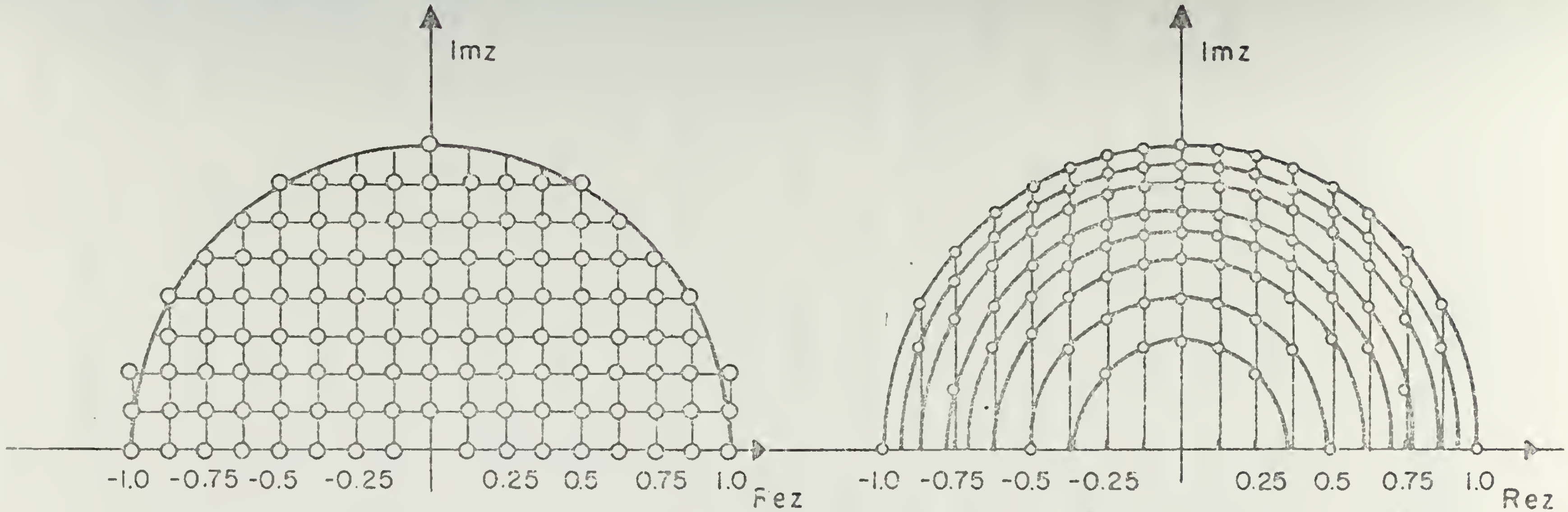


Fig. 2.15: Pole Positions for $H_2(z)$ in the Z-Plane.



POLE POSITIONS FOR COUPLED FORM

(Example 1)

POLE POSITIONS FOR DIRECT FORM

(Example 2)

Fig. 2.16: Grid cf Allowable Pole Positions for $\Delta = 0.125$.

III. ZERO-INPUT LIMIT CYCLE OSCILLATIONS IN DIGITAL FILTERS

A. INTRODUCTION

Initial experiments performed with digital filters showed, to the surprise of the designers, that these filters went into selfsustained oscillations, if spurious input signals or no input at all was applied.

The stability properties of linear first and second-order digital filter subsections are well known. However, the implementation of these systems with digital hardware introduces inherent nonlinearities which tend to make the original stable system unstable. While the finite dynamic range of the implementation assures bounded-input bounded-output stability, an asymptotically stable linear digital filter may, after the introduction of the nonlinearities become marginally stable (the oscillations are bounded) but not asymptotically stable (the natural response does not approach zero).

To be able to analyze the nonlinear effects on the response of digital filters, it is necessary to consider the type of arithmetic used, and the type of nonlinearity introduced into the digital filter through finite precision arithmetic.

One type of nonlinearity is connected with the adders in the digital filter realization. If numbers are added whose sum exceeds the dynamic range of the adder "overflow" occurs,

thereby creating a severe nonlinearity. Limit cycle oscillations due to overflow have been investigated by Ebert, et al [32]. One of their important conclusions is that selfoscillations will not be present if the adder is modified so that it saturates when overflow occurs. For the purpose of the following chapters, it will be assumed that the adders in the digital filter are linear and overflow effects can be neglected.

The other type of nonlinearity is connected with the multipliers in the digital filter. If two numbers are multiplied the product has to be rounded off or truncated in order to preserve the finite representation of all numbers in the digital filter. This quantization of the results of multiplication of data samples with filter coefficients can also cause selfoscillations. For floating-point arithmetic Sandberg [23] has shown that a digital filter realization will be asymptotically stable if the damping of the infinite precision counterpart of the digital filter is sufficiently "large" relative to the number of bits allotted to the mantissa of the data. Under these conditions limit cycle response to a zero-input or to an input sequence that approaches zero is also ruled out. Thus limit cycle oscillations are not a problem when floating-point arithmetic is used. However, as pointed out in Chapter I, digital filters generally employ fixed-point arithmetic for which limit cycle oscillations may occur.

Therefore, in this chapter, selfoscillations caused by quantization after multiplications are investigated. The simple model of a zero-input second-order digital filter with two poles and no zeros is used. This simple model is employed because it allows the study of the natural response of the filter without the distracting influence of zeros in the transfer function.

The chapter begins with the description of the two most often used quantization procedures in Section III.B. These are roundoff and truncation. Their input-output characteristics are presented in this section.

In order to develop some of the analytic techniques to be used in later chapters the effects of quantization are demonstrated with the example of a first-order digital filter. This case of the first-order section with roundoff quantization has already been investigated by Blackman [24]. As a new, additional result it is shown that a first-order digital filter with truncation is always stable. The second-order filter is then investigated.

First, roundoff quantization is considered in Sections D,E,F; then magnitude truncation quantization is investigated in the last section. There exists a fundamental difficulty in the analysis of quantized second-order digital filters. The fact that there is one nonlinearity connected with every multiplier in the feedback paths of the filter complicates any analytical investigation of the nature of the limit cycles. There exists, in general, no known way to evaluate amplitude and frequency of the selfoscillations exactly.

As pointed out in the introductory chapter, a good deal of literature is available where the effects of the quantization nonlinearities are investigated from a statistical point of view. However, the necessary assumptions of statistically independent and uncorrelated quantization noise sources are no longer valid if limit cycles are considered. For this dissertation the effects of the nonlinearities are investigated from the point of view of correlated signals and a deterministic analysis applies.

If selfoscillations are a problem and if it is necessary to keep the magnitude of the limit cycles below a specified signal level, bounds on the amplitude and approximate expressions for the frequency are desirable. Therefore, several amplitude bounds are presented in Section III.D and an approximate expression for the frequency of the limit cycle is presented in Section III.E.

The first amplitude bound is a new result and is derived using Lyapunov functions to estimate the magnitude of the dynamic response of discrete systems with bounded input. The applied analytic technique first appeared in a paper by Johnson [2] devoted to the stability of a class of sampled-data control systems. For the purpose of this dissertation Johnson's technique is modified in Appendix A. The theorem proven in Appendix A is applied in Section III.D of this chapter to estimate the bound on the amplitude of the limit cycles. The importance of this bound rests in the fact that it is obtained without reference to the specific nature of

the roundoff error. It is only necessary to know that the roundoff error is bounded. Furthermore, it is proven that limit cycles always exist if roundoff is employed and the poles of the infinite precision digital filter are inside an annular region in the z-plane which is defined by $\frac{1}{\sqrt{2}} \leq |z| \leq 1.0$.

After the treatment of the Lyapunov bound, a general matrix formulation for the limit cycle and its amplitude bound is presented. These results are new and complement the previously mentioned bound. They also serve to provide a better understanding of the nature of the limit cycle. Both of the mentioned bounds are important because they are absolute bounds. However the Lyapunov bound is pessimistic. This will be demonstrated in the next chapter where numerical values for the two bounds are compared.

A third bound is demonstrated to exist. It is an approximate bound. Its derivation is based on the postulate that roundoff quantization moves the poles of the digital filter onto the unit circle, thus providing a sufficient condition for oscillation. The underlying model has been first reported by Jackson [27], who called it the effective value linear model. The importance of this third bound rests in the fact that the bound is rather tight and easily applicable in a practical sense. Its disadvantage lies in the fact, that the derivation of the bound is based on a sufficient, but not necessary, condition. Examples are stated where the bound obtained from the linear model is exceeded by two or more quantization steps. This is contrary to the

findings reported by Jackson. Therefore, compared with the first two bounds, this amplitude bound is only a rule-of-thumb which is convenient to apply.

In Section III.E some results about the frequency of the limit cycles are developed. Since the frequency is a highly complicated nonlinear function of the argument of the complex conjugate poles of the filter transfer function and the resulting amplitude of the oscillation, it is only possible to formulate an approximate expression for the frequency, based again on the linear model.

A new result about the symmetry of a specially defined successive value phase-plane plot of the limit cycle oscillations is presented in Section III.F. The lemmas proven in this section assert that any limit cycle oscillation from a filter with poles in the right half of the unit circle in the z-plane is equal in magnitude to the response obtained from a filter whose poles are the mirror image of the original poles projected into the left half of the unit circle in the z-plane.

Finally, in Section III.G, magnitude truncation quantization is considered. As a new result it is shown, that no zero-input limit cycles with intermediate frequencies other than $f_o = 0, \frac{1}{2} f_s$ can be sustained.

In summary, then, the results of this chapter describe some theoretical aspectsabout selfoscillations in digital filters due to quantization after multiplications. The conclusions of this chapter are verified in the next chapter, where experimental results are reported and compared.

B. DESCRIPTION OF THE QUANTIZER NONLINEARITIES USING A FIRST-ORDER DIGITAL FILTER

The two most often used quantization procedures are

a) rounding to the nearest integer, and

b) truncation, where the least significant digit of a number represented by sign and magnitude is dropped.

Other quantization procedures are possible. However, they are not used often in practice and are not considered here. First, let us study roundoff quantization.

1. First-order Deadbands (Case of Roundoff).

As an example to develop some of the analytic techniques used in later sections consider a first-order digital filter as shown in Fig. 3.1. For zero-input ($u(n) = 0$) this system can be described by the difference equation

$$x(n) = ax(n-1). \tag{3.1}$$

In Fig. 3.1 the roundoff quantizer Q is inserted into the system. The input-output characteristic of this type of nonlinearity is displayed in Fig. 3.2. Assuming a normalized quantization step-size of $q = 1$, (3.1) has to be rewritten as

$$\hat{x}(n) = [a\,\hat{x}(n-1)]_r = a\,\hat{x}(n-1) \pm 0.5 \mp \delta(n), \tag{3.2}$$

where $[\ldots]_r$ denotes roundoff to the nearest integer, $\hat{x}(n)$ denotes the nonlinear response of the filter and $\delta(n)$ is any number such that $0 \leq \delta(n) < 1.0$. Blackman [24] first reported on the nonlinear effect of roundoff in first-order

filters and called the phenomenon "deadband-effect". The following examples illustrate this.

For $a = 0.9$ and an initial condition of $\hat{x}(0) = 10$ the response is computed from (3.2) as shown below.

n	computed $\hat{x}(n) = 0.9\ \hat{x}(n-1)$	error $\lvert \delta(n) - 0.5 \rvert$	rounded $\hat{x}(n) = [0.9\ \hat{x}(n-1)]_r$
0	-	-	10
1	9.0	0	9
2	8.1	0.1	8
3	7.2	0.2	7
4	6.3	0.3	6
5	5.4	0.4	5
6	4.5	0.5	5

A similar result with changed signs is obtained for the initial condition $\hat{x}(0) = -10$. For the initial condition $\hat{x}(0) = 4$ the following response is calculated.

n	computed $\hat{x}(n) = 0.9\ \hat{x}(n-1)$	error $\lvert 0.5 - \delta(n) \rvert$	rounded $\hat{x}(n) = [0.9\ \hat{x}(n-1)]_r$
0	-	0.4	4
1	3.6	0.4	4
2	3.6	0.4	4

As the examples suggest, the system response for $a = 0.9$ does not go to zero but remains at steady-state values between -5 and +5 depending on the initial conditions used. It was for this reason that Blackman coined the term "deadband-effect" for this kind of response. For $a = 0.9$ the first order deadband is within the limits $\hat{x}(n) = \pm 5$.

The existence of a deadband is defined by the equation

$$|\hat{x}(n)| = |\hat{x}(n-1)|, \tag{3.3}$$

for all n greater than some integer N. Depending on the sign and the magnitude of the filter coefficient a, three possible cases have been considered in the literature to derive the first-order deadbands.

a) <u>$0 < a < 1.0$:</u>

If a is positive one finds by inspection of (3.2) that the sign of $\hat{x}(n)$ is equal to the sign of $\hat{x}(n-1)$ and the existence condition (3.3) for the deadband can be written as

$$\hat{x}(n) = \hat{x}(n-1), \; n > N. \tag{3.4}$$

Thus a zero-frequency limit cycle results. Substituting (3.4) into (3.2) together with $0 \leq |\delta| < 1.0$, the bound on the amplitude is evaluated as

$$|\hat{x}(n)| \leq \frac{0.5}{1-a}. \tag{3.5}$$

b) <u>$-1.0 < a < 0$:</u>

If the coefficient is negative the signs of $\hat{x}(n)$ and $\hat{x}(n-1)$ alternate and the existence condition (3.3) for the deadband can be written as

$$\hat{x}(n) = -\hat{x}(n-1), \; n > N. \tag{3.6}$$

Here, a limit cycle results where the ratio of oscillation frequency f_o to the sampling frequency f_s is $f_o/f_s =$

because two samples are contained in each period of the limit cycle. By the same method as above the amplitude bound is evaluated as

$$|\hat{x}(n)| \leq \frac{0.5}{1+a} = \frac{0.5}{1-|a|}. \tag{3.7}$$

c) <u>$a > 1.0$ or $a < -1.0$:</u>

For these values of the coefficient a the corresponding linear system is unstable. Depending on the initial condition an increasing response or a deadband can exist. If a deadband exists then its amplitude bound, as in the preceding sections, is found to be

$$|\hat{x}(n)| < \frac{0.5}{|a|-1}. \tag{3.8}$$

At this point an interesting observation (first reported by Jackson [2]) can be made. The existence condition (3.3) implies that

$$x(n) = a' \, x(n-1), \tag{3.9}$$

where $a' = \pm 1$. If (3.9) is z-transformed and an initial condition $x(0)$ is assumed, then

$$X(z) = \frac{x(0)}{1-a'z^{-1}}. \tag{3.10}$$

Here $X(z)$ denotes the z-transform of $x(n)$. From (3.10) it can be seen that the limit cycles due to rounding seem to be caused by an effective pole at $a' = \pm 1$. This observation leads to the assumption of an effective value linear model which forms the basis of Jackson's heuristic approach for

derivation of the amplitude bound for the second-order deadbands [27].

2. <u>First-Order Deadbands (Case of Magnitude Truncation).</u>

The input-output characteristic of the magnitude truncation nonlinearity is drawn in Fig. 3.3. Depending on the negative number representation used, there exist two types of truncation. Magnitude truncation results when a one's complement number representation is used. It is the type of truncation considered here.

Value truncation results when a two's complement number representation is used. It is not considered in detail because the results are similar to the ones for rounding with a constant input added as value truncation introduces only a bias of 1/2q for every quantizer. In comparison with the roundoff quantizer the characteristic for magnitude truncation has a deadzone at the origin, which is twice as large as the one for roundoff. From control system theory, it is known that a deadzone stabilizes the system response. Intuitively one would therefore expect that a system with magnitude truncation is "more" stable than the corresponding system with roundoff. On the other hand, since magnitude truncation introduces errors, which can be twice as large as those for roundoff, the latter seems to be preferred.

The first-order filter section including a magnitude truncation quantizer is described by the difference equation

$$\hat{x}(n) = a\,\hat{x}(n-1) \pm \delta(n), \qquad (3.11)$$

where $0 \leq \delta(n) < 1.0$. It will now be proven by contradiction that no stable zero-input limit cycle can be sustained with this kind of system. To be specific, assume $0 < a < 1.0$ and $\hat{x}(n) > 0$, (the proof for the other possibilities follows along similar lines). Suppose a steady-state limit cycle exists. Then, from the existence condition $\hat{x}(n) = \hat{x}(n-1)$ one obtains

$$\hat{x}(n) = \frac{-\delta(n)}{1-a}, \qquad (3.12)$$

For $\delta(n) = 0$, $\hat{x}(n) = 0$. For $0 < \delta(n) < 1.0$, $\hat{x}(n) < 0$. This conclusion is contrary to the hypothesis made above. Therefore, in steady-state $\hat{x}(n) = 0$, which completes the proof. This simple demonstration has, surprisingly, not appeared in the literature. After this introductory study of the first-order digital filter let us turn our attention to the second-order digital filter.

C. MODEL FOR THE ZERO-INPUT SECOND-ORDER DIGITAL FILTER WITH QUANTIZATION

For the study of the natural response (zero-input, initial conditions only) of the second-order digital filter section a configuration with two poles and no zeros is desirable to avoid the distracting influence of the zeros on the response. A survey of the second-order filter configurations from Chapter II shows that the only possible canonical configuration of this kind is the one depicted in Fig. 3.4. That this assumption is not too restrictive will be shown in Chapter V.

The two quantization nonlinearities Q_1 and Q_2, each connected with one of the two multipliers in the feedback paths, are included in Fig. 3.4. The fact that there are two nonlinearities in the loop complicates the analysis of the limit cycle response considerably. The system of Fig. 3.4 is described by the difference equation (where $u(n) = 0$)

$$\hat{x}(n) = -[a\,\hat{x}(n-1)]_q - [b\,\hat{x}(n-2)]_q, \qquad (3.13)$$

where $[\ldots]_q$ denotes quantization of products, either through roundoff ($q = r$) or truncation ($q = t$).

If roundoff quantization is considered, the block diagram of Fig. 3.4 can be rearranged into a form which is needed in the later sections of this chapter. Separating the roundoff quantization nonlinearity into a linear characteristic and a new, sawtooth-shaped nonlinearity as depicted in Fig. 3.6 allows one to perform a block diagram manipulation which results in the diagram shown in Fig. 3.5. The error sequences due to roundoff are designated by ε_a and ε_b and can be viewed as input to the digital filter.

D. BOUNDS ON THE AMPLITUDE OF LIMIT CYCLE OSCILLATIONS IN SECOND-ORDER DIGITAL FILTERS (CASE OF ROUNDOFF)

In this section three different kinds of amplitude bounds for the limit cycles of the system described by (3.13) are presented. Their derivation is based on

a) the use of Lyapunov functions to estimate the region of boundedness of the natural response of (3.13),

b) the use of a special technique assuming that a limit cycle of period qT exists, and

c) the application of an effective value linear model.

1. An Amplitude Bound Using Lyapunov's Direct Method

For roundoff after multiplications the difference equation (3.13) has to be rewritten as

$$\hat{x}(n) = -a\,\hat{x}(n-1) \pm [0.5 - \delta(n-1)]$$
$$-b\,\hat{x}(n-2) \pm [0.5 - \delta(n-2)], \qquad (3.14)$$

where $\delta(n)$ is any number such that $0 \leq |\delta(n)| < 1.0$. The roundoff noise sequences $\varepsilon_a = [0.5 - \delta(n-1)]$ and $\varepsilon_b = [0.5 - \delta(n-2)]$ can be considered as driving functions to

the difference equation (3.14) as indicated in the block diagram of Fig. 3.5. Then, if this kind of input is denoted by $u(n)$, one finds that $|u(n)| \leq 1.0$, because $u(n) = \pm\ [0.5 - \delta(n-1)] \pm [0.5 - \delta(n-1)]$. Rewriting (3.14) in state form, using state variables

$$\hat{x}_1(n) = \hat{x}(n-2),$$

$$\hat{x}_2(n) = \hat{x}(n-1),$$

one obtains from (3.14)

$$\hat{x}(n+1) = \begin{bmatrix} 0 & 1 \\ -b & -a \end{bmatrix} \hat{x}(n) + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(n). \qquad (3.15)$$

Note that $\hat{x}(n+1)$ and $\hat{x}(n)$ in (3.15) are (2×1) state vectors. For $4b > a^2$ and $b < 1.0$, the magnitude of the eigenvalues of the corresponding linear system is less than 1.0 and the homogeneous system is asymptotically stable in the large (ASIL). In addition, the input $u(n)$ of (3.15) is bounded for all $n \geq 0$. Therefore the theorem given in Appendix A is applicable.

Theorem: For the system $\hat{x}(n+1) = A\,\hat{x}(n) + B\,u(n)$, if the homogeneous system is ASIL and has a Lyapunov function $V = x^TQx$ with $\dot{V} = -x^TCx$ and $|u(n)| \leq k_1$ for all $n \geq 0$, then the system is stable and the states are certain to enter a region defined by $\|\hat{x}\| \leq r_2$, where

$$r_2 = k_1\sqrt{\frac{\lambda\ \max(Q)}{\lambda\ \min(Q)}} \cdot \left[\frac{\|A^TQB\|}{\lambda\ \min(C)} + \sqrt{\frac{\|A^TQB\|}{\lambda^2 \min(C)} + \frac{B^TQB}{\min(C)}}\,\right]. \qquad (3.16)$$

Identify the terms of the theorem as follows:

$k_1 = 1.0$, because $|u(n)| \leq 1.0$,

$$A = \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \tag{3.17}$$

λ min/max(Q) = min/max. eigenvalue of the matrix Q, defined by the Lyapunov function

$$V = x^T Qx, \tag{3.18}$$

λ min(C) = min. eigenvalue of the matrix C, defined to be any real, symmetric and positive definite matrix such that $-C = A^T QA - Q$, (3.19)

$\|\hat{x}\|$ = norm of the state vector $\hat{x}$, defined as

$$\max_i |\hat{x}_i| \tag{3.20}$$

$\|A^T QB\|$ = norm of the matrix product $A^T QB$, defined as $\max_i \sum_j a_{ij}$, where a_{ij} are the elements of $A^T QB$. (3.21)

Since the choice for C is arbitrary as long as C is real, symmetric and position definite, let us choose for simplicity

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}. \tag{3.22}$$

Now (3.19) is written as

$$-\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -a \\ 1 & -b \end{bmatrix} \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -a & -b \end{bmatrix} - \begin{bmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{bmatrix}. \tag{3.23a}$$

To solve for the elements q_{ij} of the matrix Q, the following four equations result:

$$-1 = b^2 q_{22} - q_{11}, \tag{3.23b}$$

$$0 = ab\, q_{22} - bq_{21} - q_{12}, \tag{3.23c}$$

$$0 = ab\, q_{22} - bq_{12} - q_{21}, \tag{3.23d}$$

$$-1 = q_{11} - aq_{12} - aq_{21} + (a^2 - 1)q_{22}. \tag{3.23e}$$

Equations (3.23c) and (3.23d) show that

$$q_{12} = q_{21}. \tag{3.24}$$

The remaining solutions for q_{11}, q_{12}, and q_{22} in terms of a and b are

$$q_{11} = 1 + \frac{2b^2(1+b)}{(1-b)[(1+b)^2-a^2]}, \tag{3.25a}$$

$$q_{12} = q_{21} = \frac{2ab}{(1-b)[(1+b)^2-a^2]}, \tag{3.25b}$$

$$q_{22} = \frac{2(1+b)}{(1-b)[(1+b)^2-a^2)]}. \tag{e.25c}$$

Note that Q is real, symmetric and positive define as required, if and only if

$$(1+b) \geq |a|. \tag{3.26}$$

This is just another way to state the range of values for the coefficients a and b for which the homogeneous system of (3.14) or (3.15) is ASIL.

To evaluate the bound (3.17) define

$$w = \|A^T QB\| = \max\left(|bq_{22}|, |q_{12}-aq_{22}|\right), \tag{3.27}$$

and evaluate

$$\lambda \min (c) = 1, \text{ and} \tag{3.28a}$$

$$B^T QB = q_{22}. \tag{3.28b}$$

Then from definition (3.20) the upper bound on the amplitude of the limit cycles for (3.14) can be written as

$$|\hat{x}(n)| \leq \sqrt{\frac{\lambda \max(Q)}{\lambda \min(Q)}} \cdot \left[w + \sqrt{w^2 + q_{22}}\right]. \tag{3.29}$$

The bound (3.29) is pessimistic as can be seen from the numerical values presented in Chapter IV. This may stem from the fact that the choice of the matrix C and thus Q is arbitrary and therefore it is not guaranteed that the "best" upper bound has been found. The latter is a well established disadvantage of Lyapunov's direct method. However the bound is derived without any specific assumption about the roundoff sequences ε_a and ε_b, except for the fact that they are bounded.

It is now shown that limit cycles always exist if roundoff quantization is employed and the filter coefficient b has values such that $b > 0.5$. Previously it has been shown that the system (3.14) is stable. Let us now show by

contradiction that the zero-state cannot be reached for any $(n-1)$ greater than some value N, if $b > 0.5$. Suppose the zero-state is reached and maintained for some $(n-1) \geq N$. Then $\hat{x}(n) = \hat{x}(n-1) = 0$ and $|\hat{x}(n-2)| \geq 1.0$. Substituting this into the difference equation (3.14) one obtains

$$b\hat{x}(n-2) = \pm[0.5 - \delta(n-2)]. \tag{3.30}$$

Since $|b\hat{x}(n-2)| > 0.5$ by hypothesis and $|0.5 - \delta(n-2)| \leq 0.5$, (3.30) cannot be satisfied. This is contrary to the hypothesis.

It follows that the zero-state cannot be reached for $b > 0.5$ and the system is marginally stable. Thus, limit cycles always exist if roundoff is employed and $0.5 < b < 1.0$, which is equivalent to stating that the poles of the z-transformed equivalent of (3.14) are inside an annular region in the z-plane defined as $\frac{1}{\sqrt{2}} < r < 1.0$, where r is the magnitude of the poles. This conclusion has been reached before by Jackson [27], however without the exact proof as outlined in the above paragraph.

2. A General Expression for Zero-Input Limit Cycles.

The periodicity of the limit cycle oscillations can be used to develop a general expression for the limit cycles in matrix form. Let us assume, that a limit cycle of period qT exists. From the definition of a limit cycle and its periodicity it follows that

$$\hat{x}(n) = \hat{x}(n-q),$$

$$\hat{x}(n-1) = \hat{x}(n-q-1), \tag{3.31}$$

and so on. Let us redefine the roundoff sequences from (3.14) as

$$\varepsilon_i = \pm [0.5 - \delta(n-i)] \pm [0.5 - \delta(n-i-1)]. \tag{3.31}$$

Then, from the difference equation (3.14) a set of equations can be written as

$$\begin{aligned}
\hat{x}(n) + a\,\hat{x}(n-1) + b\,\hat{x}(n-2) &= \varepsilon_1, \\
\hat{x}(n-1) + a\,\hat{x}(n-2) + b\,\hat{x}(n-3) &= \varepsilon_2, \\
&\vdots \\
\hat{x}(n-q+2) + a\,\hat{x}(n-q-1) + b\,\hat{x}(n-q) &= \varepsilon_{q-1}, \\
\hat{x}(n-q+1) + a\,\hat{x}(n-q) + b\,\hat{x}(n-q-1) &= \varepsilon_q.
\end{aligned} \tag{3.32}$$

Substituting (3.30) into the system of equations (3.32) and using matrix notation one obtains

$$\begin{bmatrix}
1 & a & b & 0 & 0 & & 0 & 0 & 0 \\
0 & 1 & a & b & 0 & \cdots & 0 & 0 & 0 \\
0 & 0 & 1 & a & b & & 0 & 0 & 0 \\
& & \vdots & & & & \vdots & & \\
0 & 0 & 0 & 0 & 0 & & 1 & a & b \\
b & 0 & 0 & 0 & 0 & \cdots & 0 & 1 & a \\
a & b & 0 & 0 & 0 & & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
\hat{x}(n) \\ \hat{x}(n-1) \\ \hat{x}(n-2) \\ \vdots \\ \hat{x}(n-q+3) \\ \hat{x}(n-q+2) \\ \hat{x}(n-q+1)
\end{bmatrix}
=
\begin{bmatrix}
\varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \vdots \\ \varepsilon_{q-2} \\ \varepsilon_{q-1} \\ \varepsilon_q
\end{bmatrix}. \tag{3.33}$$

In shorter notation (3.33) is rewritten to yield

$$A\hat{x} = \varepsilon, \tag{3.34}$$

where A is a square matrix of dimension q x q and $\hat{x}$ and ε are the vectors constituted by the limit cycle points and the roundoff error sequence respectively.

As a special case, to simplify the algebra involved, consider a symmetric limit cycle whose samples in the first half-period are equal in magnitude but opposite in sign to the samples in the second half-period. This type of limit cycle can be described by

$(\hat{x}_1, \hat{x}_2, \ldots, \hat{x}_{q/2}, \hat{x}_{1+q/2}, \ldots, \hat{x}_q)$, where

$$\begin{aligned} \hat{x}_1 &= - \hat{x}_{1+q/2}, \\ &\vdots \\ \hat{x}_{q/2} &= - \hat{x}_q. \end{aligned} \tag{3.35}$$

Thus, q has to be an even number.

The set of Eqs. (3.33) or (3.34) can be partitioned as follows:

$$
\left[\begin{array}{ccccc|ccccc}
1 & a & b & 0 & & & \cdot & & & \\
0 & 1 & a & b & & & \cdot & & & \\
0 & 0 & 1 & a & \cdots & 0 & 0 & 0 & 0 & \\
0 & 0 & 0 & 1 & & 0 & 0 & 0 & 0 & \\
 & \cdot & & & & b & 0 & 0 & 0 & \cdots \\
 & \cdot & & & & a & b & 0 & 0 & \\
\hline
 & \cdot & & & & 1 & a & b & 0 & \\
 & \cdot & & & & 0 & 1 & a & b & \\
0 & 0 & 0 & 0 & & 0 & 0 & 1 & a & \cdots \\
0 & 0 & 0 & 0 & & 0 & 0 & 0 & 1 & \\
b & 0 & 0 & 0 & \cdots & & \cdot & & & \\
a & b & 0 & 0 & & & \cdot & & &
\end{array}\right]
\left[\begin{array}{c}
\hat{x}_1 \\ \hat{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_{q/2} \\ \hline \hat{x}_{1+q/2} \\ \hat{x}_{2+q/2} \\ \cdot \\ \cdot \\ \cdot \\ \hat{x}_q
\end{array}\right]
=
\left[\begin{array}{c}
\varepsilon_1 \\ \varepsilon_2 \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_{q/2} \\ \hline \varepsilon_{1+q/2} \\ \varepsilon_{2+q/2} \\ \cdot \\ \cdot \\ \cdot \\ \varepsilon_q
\end{array}\right]. \tag{3.36}
$$

In a more compact notation (3.36) can be written as

$$
\begin{bmatrix} B & C \\ C & B \end{bmatrix}
\begin{bmatrix} \hat{x} \\ -\hat{x} \end{bmatrix}
=
\begin{bmatrix} E_1 \\ E_2 \end{bmatrix}. \tag{3.37}
$$

From above it is deduced that $E_1 = -E_2$. Furthermore, instead of solving the system of q equations, it is only necessary to solve the set of q/2 equations.

Instead of (3.34) one now solves

$$
[B-C]\ \hat{x} = E_1, \text{ where} \tag{3.38}
$$

[B-C] is a square matrix of dimension (q/2 by q/2). Note that there is a difference between the matrix A for a limit cycle of length a = m and the submatrix [B-C] for a symmetric limit cycle of length a = 2m, in that the signs of the

three elements in the lower left hand corner of [B-C] are the negative of the terms in the lower left hand corner of A.

From the initial assumption, that a limit cycle (not necessarily symmetric) of length qT exists, it follows that A has to be nonsingular, if and only if at least one $\varepsilon_i \neq 0$. Furthermore, A has to be singular, if and only if all $\varepsilon_i = 0$. The latter condition signifies those values of the coefficients a and b for which a linear response (no roundoff quantization) is possible, whose samples in the period of oscillation are all integers. It is easy to see that this requires that b = 1 and therefore A cannot be singular for b < 1.0.

From (3.31) it is seen that for roundoff quantization $0 \leq |\varepsilon_i| \leq 1.0$, and assuming that not all $\varepsilon_i = 0$, a bound for the $\hat{x}(n-i)$ can be found by solving (3.34) using one of the many methods of solution available.

Using Cramer's rule a solution for $\hat{x}(n-i)$ is found from (3.34) as

$$\hat{x}(n-i) = \frac{1}{\det A} \begin{array}{c} \text{(column } i) \\ \left[\begin{array}{ccccccccc} 1 & a & b & & \varepsilon_1 & & & 0 & 0 \\ 0 & 1 & a & \cdots & \varepsilon_2 & & & 0 & 0 \\ 0 & 0 & 1 & & \varepsilon_3 & & & 0 & 0 \\ & \cdot & & & & & & & \\ & \cdot & & & & & & & \\ & \cdot & & & & & & & \\ b & 0 & 0 & & \varepsilon_{q-1} & & & 1 & a \\ a & b & 0 & \cdots & \varepsilon_q & & \cdots & 0 & 1 \end{array}\right] \end{array} =$$

$$= \frac{\varepsilon_1 \det A_{1i} + \varepsilon_2 \det A_{2i} + \ldots + \varepsilon_q \det A_{qi}}{\det A} \quad . \quad (3.39)$$

The expressions det A_{qi} denote the cofactors of ε_i, that is the determinant with sign $(-1)^{q+i}$ of the matrix formed by deleting the row and column of A which both contain ε_i. Using the bound on ε_i, (3.39) is rewritten to yield a bound on $|\hat{x}(n-i)|$.

$$|\hat{x}(n-i)| = \frac{|\varepsilon_1 \det A_{1i} + \ldots + \varepsilon_q \det A_{qi}|}{|\det A|}$$

$$\leq \frac{\sum_{j=1}^{q} |\det A_{ji}|}{|\det A|} . \tag{3.40}$$

Due to the cyclical nature of the equations (3.33), it is possible to show that the bound from (3.40) has the same numerical value for all i = 0, 1, 2, ..., q-1 and is thus the only bound for the samples contained in the assumed limit cycle.

To show this, it is noted that for every square matrix A with elements a_{ij}

$$\det A = \sum_{\rho} \pm [a_{1p_1} a_{2p_2} a_{3p_3} \cdots a_{qp_q}] . \tag{3.41}$$

The sum is extended over all permutations $\rho = q!$ of the integers 1, 2, 3, ..., q and where a + or - sign is affixed to each product according to whether ρ is an even or an odd permutation. In the expression (3.41) the indices $p_1 p_2 p_3 \ldots p_q$ are the permutations of the integers 1, 2, 3, ..., q.[1]

---

[1] See for example Ayres [33], p.20.

Consider now (3.39) for the q values of $\hat{x}(n-i)$. The matrix A is used to form q new matrices A_{ji} by replacing the i^{th} column of A with the column vector ε. Those matrices are then used to compute the cofactors of ε_j and are of the form

$$A_{ji} = \begin{array}{r} \\ \\ \\ \\ j^{th} \text{ row} \\ \text{deleted} \\ \\ \\ \\ \end{array} \left[\begin{array}{cccc|ccc} 1 & a & b & & & 0 & 0 \\ 0 & 1 & a & \cdots & \cdots & 0 & 0 \\ 0 & 0 & 1 & & & 0 & 0 \\ & \cdot & & & & & \cdot \\ \hdashline & \cdot & & & & & \cdot \\ & \cdot & & & & & \cdot \\ & \cdot & & & & & \cdot \\ b & 0 & 0 & & & 1 & a \\ & & & \cdots & \cdots & & \\ a & b & 0 & & & 0 & 1 \end{array}\right] . \qquad (3.42)$$
$$\begin{array}{c} i^{th} \text{ column} \\ \text{deleted} \end{array}$$

Regardless of which row and column are deleted the matrix A_{ji} will contain the same elements 0, 1, a, b which make up the products in (3.41). Since sign-changes are of no concern the summations

$$\sum_{j=1}^{q} |\det A_{ji}|$$

are equal for all i = 0, 1, 2, ..., q-1 and the bound (3.40) has the same value regardless of i. Thus the bound on the amplitude of a limit cycle with period qT is given by

$$|\hat{x}(n)| \leq \frac{\sum_{j=1}^{q} |\det A_{ji}|}{|\det A|} . \qquad (3.43)$$

Note that this bound is different than the Lyapunov bound, because the period of the limit cycle qT has to be specified. The evaluation of (3.43) gets rather complicated for large q, even if the assumption of symmetry in the limit cycle can be made. Some simple, but important examples will serve to illustrate the use of the developed bound. For $q = 1$, one obtains by inspection

$$\hat{x}(n) = \frac{\varepsilon_o}{1+a+b} \leq \frac{1}{1+a+b} . \tag{3.44}$$

For $q = 2$, (3.33) can be written as

$$\begin{bmatrix} (1+b) & -a \\ a & (1+b) \end{bmatrix} \begin{bmatrix} \hat{x}(n) \\ \hat{x}(n-1) \end{bmatrix} = \begin{bmatrix} \varepsilon_o \\ \varepsilon_1 \end{bmatrix} . \tag{3.45a}$$

It follows that

$$\hat{x}(n) = \frac{\varepsilon_o(1+b) - a\varepsilon_1}{1+2b+b^2-a^2} \quad \text{and}$$

$$|\hat{x}(n)| \leq \frac{|1+b| + |-a|}{|1+a+b| \; |1-a+b|} = \frac{1}{|1-a+b|} . \tag{3.45b}$$

The cases $q = 1, 2$ correspond to limit cycles with frequency $f_o = 0, \frac{1}{2} f_s$. From the knowledge about the signs of a and b, the expressions (3.44) and (3.45) can be combined to yield

$$\hat{x}(n) \leq \frac{1}{1-|a|+b} \tag{3.46}$$

The bound (3.46) has been reported by Jackson [27] and

and Bonzanigo [28] before, however not as the result of a general derivation as presented here. For $q = 4$, (3.33) can be written as

$$\begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ b & 0 & 1 & a \\ a & b & 0 & 1 \end{bmatrix} \begin{bmatrix} \hat{x}(n) \\ \hat{x}(n-1) \\ \hat{x}(n-2) \\ \hat{x}(n-3) \end{bmatrix} = \begin{bmatrix} \varepsilon_o \\ \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \end{bmatrix} . \tag{3.47a}$$

It follows that

$$\hat{x}(n) = \frac{\varepsilon_o(1+a^2b-b^2) - \varepsilon_1(a+ab^2) + \varepsilon_2(a^2-b+b^3) - \varepsilon_3(a^2-2ab)}{1+4a^2b-2b^2-a^4+b^4} . \tag{3.47b}$$

For $a = 0$, one gets

$$\hat{x}(n) \leq \frac{|1-b^2| + |b||b^2-1|}{|1-2b^2+b^4|} = \frac{1}{1-b} . \tag{3.48}$$

The bound (3.48) is larger than what it should be by a factor of 2, because it is assumed that $|\varepsilon_i| \leq 1.0$. However when $a = 0$, $|\varepsilon_i| \leq 0.5$.

Another method of solution for (3.33) is based on the use of a flow graph and application of Mason's gain rule. Since the flow graph demonstrates the continued dependence of each limit cycle point on the previous limit cycle points and the roundoff samples, the flow graph representing (3.33) is included in this chapter as an interesting graphical representation of the generation of limit cycles. The flow graph is shown in Fig. 3.7.

The importance of the derived bound (3.43) lies in the fact that this is an absolute bound. However, the bound is not easy to compute. Numerical values for some representative values of a, b, and q are presented and compared with the other bounds in Chapter IV.

3. The Effective Value Linear Model.

Consider the system depicted in Fig. 3.4 without the quantizers in the feedback loop. The transfer function of this linear system is

$$H(z) = \frac{z^2}{z^2 + az + b} \,. \tag{3.49}$$

From Chapter II, we know that stability (ASIL) is assured, if and only if $0 < b < 1.0$. The case of $b = 1.0$ describes a digital oscillator and is the limiting case between stable and unstable response.

For the nonlinear system (roundoff quantization included) Jackson [27] postulated that limit cycles occur (the system is marginally stable) if, in effect, $b = 1.0$. Define an effective value for b as b' using the difference equation (3.14), which is repeated here for convenience:

$$\hat{x}(n) = -[a\hat{x}(n-1)]_r - [b\hat{x}(n-2)]_r, \tag{3.50a}$$

$$\hat{x}(n) = -[a\hat{x}(n-1)]_r - b'\hat{x}(n-2). \tag{3.50b}$$

Then, by equating (3.50a) and (3.50b), one gets

$$[b\hat{x}(n-2)]_r = b\hat{x}(n-2) \pm [0.5-\delta(n-2)] = b'\hat{x}(n-2). \tag{3.51}$$

The sufficient, but not necessary, condition for a limit cycle response is, according to Jackson, that $b' = 1$. Roundoff effectively moves the poles of the digital filter described by (3.50a) from inside the unit-circle in the z-plane onto the unit-circle.

If $b' = 1$, then from (3.51)

$$\hat{x}(n-2) = \frac{\pm[0.5 - \delta(n-2)]}{1-b} \quad . \tag{3.52}$$

Since $\hat{x}(n-2)$ is equal to $\hat{x}(n)$ delayed by two time-units T and $|0.5 - \delta(n-2)| \leq 0.5$, the bound on $\hat{x}(n)$ is described by

$$|\hat{x}(n)| \leq \frac{0.5}{1-\bar{b}} \quad . \tag{3.53}$$

The consequences of the postulate are now demonstrated with the example of a digital filter with $a = -1.9$, $b = 0.9474$ and initial conditions (0,3). The limit cycle response is computed from (3.50). In addition, the values for ε_a and ε_b are displayed in Fig. 3.8. The bounded nature of ε_a and ε_b is corroborated by Fig. 3.5.

n	computed $\hat{x}(n)$	rounded $\hat{x}(n)$	roundoff ε_a from $[a\hat{x}(n-1)]_r$	roundoff ε_b from $[b\hat{x}(n-2)]_r$
1	-	0	-	-
2	-	3	-	-
3	5.7-0	6-0 = 6	0.3	0
4	11.4-2.8442	11-3 = 8	-0.4	-0.1578
5	15.2-5.6844	15-6 = 9	-0.2	-0.3156
6	17.1-7.5792	17-8 = 9	-0.1	-0.4208
7	17.1-8.5266	17-9 = 8	-0.1	-0.4734
8	15.2-8.5266	15-9 = 6	-0.2	-0.4734
9	11.4-7.5792	11-8 = 3	-0.4	-0.4208
10	5.7-5.6844	6-6 = 0	0.3	-0.3156
11	0-2.8422	0-3 =-3	0	-0.1578
12	- 5.7-0	- 6-0 =-6	-0.3	0

and so on.

The missing samples of the limit cycle for n = 13 to 18 are equal to the negative of the samples for n = 4 to 9. An inspection of the term ε_b reveals that $|\varepsilon_b| = (1-b)\ \hat{x}(n-2)$, which is equivalent to the statement above, that b' = 1.

Returning to (3.53) and noting that the smallest amplitude for a limit cycle is unity, it seems reasonable that no limit cycles are possible if b < 0.5. This statement about the existence of limit cycles has been proven rigorously, without recourse to an effective value linear model in paragraph D.1. The bound (3.53) is easily applicable.

However, since it is based on the assumption that the non-linear system oscillates if b' = 1 (which is a carry-over from linear theory), there exist exceptions from the bound (3.53). Consider another numerical example for a = -1.6, b = 0.9474 and initial conditions (3,9).

n	computed $\hat{x}(n)$	rounded $\hat{x}(n)$	ε_a	ε_b
1	--	3	-	-
2	--	9	-	-
3	14.4- 2.8422	14-3 = 11	-0.4	-0.1578
4	17.6- 8.5266	18-9 = 9	0.4	-0.4734
5	14.4-10.4214	14-10= 4	-0.4	0.4214
6	6.4- 8.5266	6-9 =- 3	-0.4	-0.4734
7	- 4.8- 3.7896	- 5-4 =- 9	-0.2	-0.2104
8	-14.4+ 2.8422	-14+3 =-11	0.4	0.1578
9	-17.6+ 8.5266	-18+9 =- 9	-0.4	0.4734
10	=14.4+10.4214	-14+10=- 4	0.4	-0.4214
11	- 6.4+ 8.5266	- 6+9 = 3	0.4	0.4734
12	4.8+ 3.7896	5+4 = 9	0.2	0.2104
13	14.4- 2.8422	14-3 = 11	-0.4	-0.1578

The values for ε_b are also displayed in Fig. 3.9. An inspection of ε_b reveals that for n = 3, 4, 6, 7, 8, 9, 11, 12, it is still true that $|\varepsilon_b| = (1-b)\ \hat{x}(n-2)$. However, at n = 5, 10, a discontinuous jump occurs and for this reason the effective value linear model is not valid any longer.

For the exceptional class of limit cycle responses

a new bound is now developed. If a discontinuous jump of an ε_b value occurs, then

$$|\varepsilon_b| = (1-b)\ \hat{x}(n-2) - 1. \qquad (3.54)$$

Because $\hat{x}(n-2)$ is a delayed version of $\hat{x}(n)$ and $|\varepsilon_b| \leq 0.5$, (3.54) can be used to evaluate a bound, such that

$$|\hat{x}(n)| \leq \frac{1.5}{1-b}. \qquad (3.55)$$

Experimental data (see Chapter IV and Appendix B) indicates that there exist limit cycles whose amplitudes deviate from the bound (3.53) (but always remain inside the bound (3.55)) as much as 6 quantization step-sizes (i.e., 6 units). Clearly then, the bound (3.53) derived from the effective value linear model is only a rule-of-thumb. The preceding example and the existence of the bound (3.55) is also contrary to a statement made by Jackson [2], that "some of the observed limit cycles for b > 0.9 have exceeded the limits ... as given in (6.4)[1] by ±1 (i.e., by one quantization step), but never more."

The available numerical data indicates that exceptions from the effective value linear model occur for $b \geq \frac{5}{6} = 0.8\hat{3}...$ and for values of a around ±1.5, ±1.0, ±0.5.

E. AN APPROXIMATE EXPRESSION FOR THE FREQUENCY OF LIMIT CYCLE OSCILLATIONS (CASE OF ROUNDOFF)

The limit cycle output of any quantized second-order

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[1] Jackson's formula (6.4) corresponds to (3.53)

system is periodic. The condition for periodicity is given as

$$\hat{x}(n) = \hat{x}(n-q), \text{ for all } q = 1,2,3... \tag{3.56}$$

The period of the limit cycle qT must contain an even number of sign-changes between successive samples. Because of the discrete nature of the response with equally spaced samples the number of sign-changes per limit cycle period qT is adopted to define the frequency f_o of a limit cycle response. Let the number of sign-changes per limit cycle period be 2p. There are two sign-changes per individual oscillation, therefore frequency is defined here as

$$f_o = \frac{p}{qT}, \text{ or} \tag{3.57a}$$

since $f_s = \frac{1}{T}$, one gets

$$\frac{f_o}{f_s} = \frac{p}{q} . \tag{3.57b}$$

The limiting case where all limit cycle points have equal value is defined with p = 0 and q = 1, such that the frequency of the constant case is

$$\frac{f_o}{f_s} = 0. \tag{3.57c}$$

The other limiting case occurs at the Nyquist-frequency, where the samples change sign after every sample (p = 1) and the limit cycle repeats itself after two samples (q = 2). For this case, the frequency is given by

$$\frac{f_o}{f_s} = \frac{1}{2} \cdot \tag{3.57d}$$

From the limiting case (3.57c) and (3.57d), it is concluded that $2p \leq q$, and since p and q are integers, $\frac{f_o}{f_s}$ is always a rational fraction.

Inspection of experimental data shows that the frequency is a highly nonlinear function of the filter coefficients a and b and the amplitude of the particular limit cycle. Only for a few exceptional values of the coefficient a can the frequency f_o be determined analytically. Among these are the cases, where a = -1, 0, +1. The frequencies f_o/f_s for these are $\frac{1}{6}$, $\frac{1}{4}$ and $\frac{1}{3}$ respectively.

In Chapter II, the expression for the frequency of the linear digital filter is stated. Using this expression, an approximate expression for the frequency of the nonlinear system is at hand. This approximation is given by

$$\frac{f_o}{f_s} = \frac{1}{2\pi} \cos^{-1} \left(\frac{-a}{2\sqrt{b}} \right) \tag{3.59}$$

or, if the effective value of b' = 1 is substituted for b, then

$$\frac{f_o}{f_s} = \frac{1}{2\pi} \cos^{-1} \left(-\frac{a}{2} \right). \tag{3.60}$$

How well (3.59) describes the actual frequency of the limit cycles is displayed in Fig. 3.10-12 for values of b 0.5,

0.75 and 0.86 respectively. As can be seen from the graphs the approximation gets better for higher-Q filter sections, i.e., for b close to 1.0. However, it should be noted (see for example Fig. 3.12, a = ± 0.3) that for many values of a two or more oscillation frequencies may exist, depending on the particular set of initial conditions used. This is not surprising for nonlinear systems.

F. PHASE-PLANE PLOTS OF LIMIT CYCLES AND SOME OF THEIR SYMMETRY PROPERTIES

Phase-plane plots are a useful technique for the graphical analysis of second-order differential equations. For the study of difference equations a similar technique is developed here. A special plot (herein referred to as the successive value phase-plane plot) results if successive states of a second-order discrete system are recorded on a cartesian plane with axis x(n) and x(n-1). In the usual phase plane $\dot{x}$ is plotted versus x with time as a parameter. The natural extension of this to discrete time would be to plot the first forward difference $\Delta x(n) = x(n) - x(n-1)$ versus x(n), but experience has shown that meaningful results for the digital filter are obtained only if x(n) and x(n-1) are plotted instead.

It is generally not too useful to employ the successive value phase-plane with axis x(n) and x(n-1) for the analysis of quantized second-order systems because the existence of two nonlinearities complicates the graphical analysis. However, it is instructive to display the limit cycle responses

to exhibit their characteristic features graphically. The results of this analysis are presented in Chapter IV for comparison purposes. In this section some new results are introduced concerning the symmetry of limit cycles. It is shown that the successive value phase-plane plots for two systems with parameters a and -a are identical except for a change of orientation of a symmetric axis. The latter conclusion is also important, because it asserts that any bound on the magnitude of a limit cycle response evaluated with the assumption that $a < 0$, $b > 0$ is equally valid for $a > 0$, $b > 0$. From experimental data it is known that three types of limit cycles exist. They are classified as:

a) Type A: The limit cycle has half-wave symmetry, i.e., half-waves equal each other in magnitude and differ in size. Two samples per cycle are zero. This response is described by

$$[\hat{x}(1), \hat{x}(2), \ldots, \hat{x}(i-1), \hat{x}(i), \hat{x}(i+1), \ldots, \hat{x}(2i-1), \hat{x}(2i)], \tag{3.61}$$

where $\hat{x}(1) = \hat{x}(i+1) = 0$, and

$$\hat{x}(k) = -\hat{x}(i+k), \text{ for } k = 1,2,\ldots, i.$$

b) Type B: The limit cycle has half-wave symmetry, however no zero-samples exist. The response is described by (3.61), except that

$$\hat{x}(1) = \hat{x}(i+1) \neq 0.$$

c) Type C: The limit cycle is unsymmetrical. This response is described by

$$[\hat{x}(1), \hat{x}(2), \ldots, \hat{x}(i), \ldots, \hat{x}(q)], \tag{3.62}$$

where q is always odd.

Three lemmas are now presented about the three types of limit cycles.

Lemma 1: Given a digital filter with roundoff quantization and a defining equation

$$\hat{x}(n) = -[a\,\hat{x}(n-1)]_r - [b\,\hat{x}(n-2)]_r. \tag{3.63}$$

Assume that the digital filter has a limit cycle response of type A. If the sign of the filter parameter a is changed then two new limit cycles are possible, where the two new limit cycles equal either half-cycle of the original limit cycle in magnitude, but differ in sign after every second sample.

Proof: Suppose a limit cycle of type A exists. This response is described by (3.61). Let the sign of the coefficient a be changed and evaluate the filter response for initial conditions $\hat{y}(1) = \hat{x}(1)$, $\hat{y}(2) = -\hat{x}(2)$. From the difference equation above, one obtains successively

$$\hat{y}(3) = -[(-a)(-\hat{x}(2))]_r - [b\,\hat{x}(1)] = \hat{x}(3), \tag{3.64}$$

$$\hat{y}(4) = -[(-a)(\hat{x}(3))]_r - [b(-\hat{x}(2))] = -\hat{x}(4), \tag{3.65}$$

and so on. Thus a new sequence of samples result which has the form

$$[\hat{x}(1),\ -\hat{x}(2),\ \hat{x}(3),\ -\hat{x}(4),\ldots,\ \hat{x}(i-1),\ -\hat{x}(i),\ \hat{x}(i+1),\ldots$$
$$\ldots,\ \hat{x}(2i-1),\ -\hat{x}(2i)]. \tag{3.66}$$

Furthermore for type A limit cycles i is an even number and

the sign of $\hat{x}(i)$ is changed, as are all other samples with even indices. From the conditions stated by (3.61) it is seen that the sequence (3.66) consists of two repetitions of the one new limit cycle

$$[\hat{x}(1), -\hat{x}(2), \hat{x}(3), -\hat{x}(4), \ldots, \hat{x}(i-1), -\hat{x}(i)]. \quad (3.67)$$

Using the same procedure as above for the initial condition $[-\hat{x}(1), \hat{x}(2)]$ another new limit cycle is obtained which has the form

$$[-\hat{x}(1), \hat{x}(2), -\hat{x}(3), \hat{x}(4), \ldots, -\hat{x}(i-1), \hat{x}(i)]. \quad (3.68)$$

It should be noted that the initial conditions $[\hat{x}(1), \hat{x}(2)]$ and $[-\hat{x}(1), -\hat{x}(2)]$ lead in general to completely new limit cycles which bear no resemblance with the sequences given by either (3.61) or (3.67) and (3.68).

<u>Lemma 2</u>: Given the same system as for lemma 1, let the digital filter have a response of type B. If the sign of the filter parameter a is changed, then a new limit cycle is possible, where the new limit cycle equals the original one in magnitude, but differs in sign after every second sample.

<u>Proof</u>: The proof for lemma 2 is identical to the proof for lemma 1, except that only one new limit cycle of the form (3.66) results, because both initial conditions $[\hat{x}(1), -\hat{x}(2)]$ and $[-\hat{x}(1), \hat{x}(2)]$ lead to the same result.

<u>Lemma 3</u>: Given the same system as for lemma 1. Let the digital filter have two responses of type C, where the one is the sign-inverted version of the other.

If the sign of the filter parameter a is changed, then one new limit cycle is possible, where either half-cycle of the new limit cycle sequence equals either of the two original limit cycles in magnitude, but differs in sign after every second sample.

<u>Proof</u>: The two limit cycles of type C are given by (3.62). The first limit cycle and its sign-inverted version are concatenated to yield

$$[\hat{x}(1), \hat{x}(2), \ldots, \hat{x}(q), -\hat{x}(1), -\hat{x}(2), \ldots, -\hat{x}(q)]. \tag{3.69}$$

Applying initial conditions $\hat{y}(1) = \hat{x}(1)$ and $\hat{y}(2) = -\hat{x}(2)$ to the difference equation one obtains successively

$$\hat{y}(3) = -[(-a)\cdot(-\hat{x}(2))]_r - [b\,\hat{x}(1)]_r = \hat{x}(3), \tag{3.70}$$

$$\hat{y}(4) = -[(-a)\cdot\hat{x}(3)]_r - [b(-\hat{x}(2))]_r = -\hat{x}(4), \tag{3.71}$$

and so on.

Since q is odd one gets

$$\hat{y}(q+1) = -[(-a)\hat{x}(q)]_r - [b(-\hat{x}(q-1))]_r = -\hat{x}(1), \tag{3.72}$$

$$\hat{y}(q+2) = -[(-a)(-\hat{x}(1))]_r - [b\,\hat{x}(q)]_r = \hat{x}(2), \tag{3.73}$$

and so on.

A new sequence of samples results which has the form

$$[\hat{x}(1), -\hat{x}(2), \ldots, \hat{x}(q), -\hat{x}(1), \hat{x}(2), \ldots, -\hat{x}(q)]. \tag{3.74}$$

This is the new limit cycle. Application of the initial conditions $-\hat{x}(1)$ and $\hat{x}(2)$ yields the same result as above.

The conclusions of the three lemmas are demonstrated with the following examples:

The limit cycle sequence of type A for $a = -1.8$, $b = 0.9474$ results in two new limit cycles for $a = 1.8$, $b = 0.9474$ of type C as stated in the following table:

n	$\hat{x}(n)=1.8\ \hat{x}(n-1) - 0.9474\ \hat{x}(n-2)$	$\hat{x}(n)=-1.8\ \hat{x}(n-1) - 0.9474\ \hat{x}(n-2)$	$\hat{x}(n)=-1.8\ \hat{x}(n-1) - 9.474\ \hat{x}(n-2)$
0	0	0	0
1	4	-4	4
2	7	7	-7
3	9	-9	9
4	9	9	-9
5	7	-7	7
6	4	4	-4
7	0	0	0
8	-4	-4	4
9	-7	7	-7
10	-9	-9	9
11	-9	9	-9
12	-7	-7	7
13	-4	4	-4

The limit cycle sequence of type B for $a = -1.8$, $b = 0.9474$ results in a new limit cycle of type B for $a = 1.8$, $b = 0.9474$ as given in the following table:

n	$\hat{x}(n) = 1.8\,\hat{x}(n-1)$ $-0.9474\,\hat{x}(n-2)$	$\hat{x}(n) = -1.8\,\hat{x}(n-1)$ $-0.9474\,\hat{x}(n-2)$
0	1	1
1	3	-3
2	4	4
3	4	-4
4	3	3
5	1	-1
6	-1	-1
7	-3	3
8	-4	-4
9	-4	4
10	-3	-3
11	-1	1

The two limit cycle sequences of type C for a = -1.8, b = 0.9484 and its sign-inverted complement result in a new limit cycle for a = 1.8, b = 0.9474 of type B. The original and the new limit cycles are presented in the following table:

n	$\hat{x}(n)=1.8\ \hat{x}(n-1)$ $-0.9474\ x(n-1)$	$\hat{x}(n)=1.8\ \hat{x}(n-1)$ $-0.9474\ \hat{x}(n-2)$	$\hat{x}(n)=-1.8\ \hat{x}(n-1)$ $-0.9474\ \hat{x}(n-2)$
1	1	-1	1
2	5	-5	-5
3	8	-8	8
4	9	-9	-9
5	8	-8	8
6	5	-5	-5
7	1	-1	1
8	-3	3	3
9	-6	6	-6
10	-8	8	8
11	-8	8	-8
12	-6	6	6
13	-3	3	-3
14	1	-1	-1
15	5	-5	5
16	8	-8	-8
17	9	-9	9
18	8	-8	-8
19	5	-5	5
20	1	-1	-1
21	-3	3	-3
22	-6	6	6
23	-8	8	-8
24	-8	8	8
25	-6	6	-6
26	-3	3	3

Many more examples of the presented three types can be constructed from the tables of Appendix B.

If these limit cycles are drawn on the successive value phase-plane as defined in the preceding paragraph, it is seen that the phase-plane plots are identical except for a change in their symmetry axis. The experimental verification of this is delayed until the next chapter.

After the study of limit cycles because of roundoff quantization let us turn our attention to magnitude truncation quantization in second-order digital filters.

G. LIMIT CYCLE OSCILLATIONS IN SECOND-ORDER SYSTEMS (CASE OF MAGNITUDE TRUNCATION).

The second-order system to be studied in this section is the same as the one depicted in Fig. 3.4. However the two nonlinearities in the feedback paths now have the input-output characteristic of the magnitude truncation quantizer as shown in Fig. 3.3.

As might be expected from the result of paragraph B.2, where it has been shown that the first-order section with magnitude truncation is ASIL, little or no limit cycle oscillations can be observed in the second-order case. However, it can be demonstrated that some limit cycles exist. For initial conditions (1, 1) or (0, 1) a limit cycle with $|\hat{x}(n)| = 1$ will always result if $|a| \geq 1.0$.

For complex conjugate poles of the second-order system one predicts with the help of the effective value linear model that no limit cycles with frequencies f_o/f_s between 0 and 1/2 are possible, because in no way can an effective value of b = 1 be obtained from magnitude truncation. The differe

equation for a zero-input digital filter with truncation is given as

$$\hat{x}(n) = -a\ \hat{x}(n-1) \pm \delta(n-1) - b\ \hat{x}(n-2) \pm \delta(n-2), \quad (3.75)$$

where $\delta(n-1)$ or $\delta(n-2)$ are numbers, such that $0 \leq |\delta| < 1.0$. For the purpose of this demonstration suppose that $\hat{x}(n-2) \geq 1$. Since $0 < b < 1.0$ it follows that

$$b\ \hat{x}(n-2) - \delta(n-2) \leq b\ \hat{x}(n-2) < \hat{x}(n-2). \quad (3.76)$$

However, this is contrary to the condition for an effective value of b = 1 which would require that

$$b\ \hat{x}(n-2) - \delta(n-2) = \hat{x}(n-2). \quad (3.77)$$

Therefore, an effective value of b = 1 is never possible for truncation quantization.

The situation is different for pole locations of the effective value linear model which are real. Then self-oscillations are possible. Consider the case where $\hat{x}(n) = \hat{x}(n-1) = \hat{x}(n-2)$. The frequency of this limit cycle is $\frac{f_o}{f_s} = 0$. Applying the above condition to the difference equation (3.75) requires that $a < 0$. Furthermore from (3.75) and assuming that $\hat{x}(n) > 0$ one obtains

$$x(n) = \frac{-\delta(n-1) + \delta(n-2)}{1 + a + b}. \quad (3.78)$$

Assuming that $\hat{x}(n) < 0$ yields a similar result, such that the bound can be written as

$$|\hat{x}(n)| \leq \frac{1}{1 + a + b}. \quad (3.79)$$

Consider the other possible case, where $\hat{x}(n) = -\hat{x}(n-1)$. The frequency of this limit cycle is $\frac{f_o}{f_s} = \frac{1}{2}$. Applying the preceding condition to the difference equation (3.75) requires that a > 0. Following the same steps as in the proof above yields the bound

$$|\hat{x}(n)| \leq \frac{1}{1 - a + b} . \tag{3.80}$$

Since in the first case a < 0 and in the second case a > 0, the expressions (3.79) and (3.80) are identical and the common bound is given by

$$|\hat{x}(n)| \leq \frac{1}{1-|a|+ b} . \tag{3.81}$$

It remains to evaluate the regions of stability for the second-order digital filter with truncation quantization. The smallest possible value for a sample in the limit cycle is one or

$$|\hat{x}(n)| \geq 1. \tag{3.82}$$

Using (3.81) together with (3.82), this requires that

$$|a| \geq b. \tag{3.83}$$

For limit cycle oscillations, it is therefore simply required that

$$|a| \geq 1. \tag{3.84}$$

The condition (3.84) is shown in the parameter plane of Fig. 3.13, which depicts the regions of stable ($|a| < 1.0$)

and unstable ($a \geq 1.0$) response.

Computer simulation for a wide variety of a, b values and representative initial conditions has verified the above stated results. Roundoff quantization seems to be favored in the literature because its quantization errors are smaller by a factor of two if compared with truncation. However, it might be advantageous to use truncation quantization if zero-input limit cycles are a problem.

H. SUMMARY

The purpose of this chapter has been to investigate zero-input limit cycle oscillations in second-order digital filter sections. Since this response depends on the initial conditions only, it is the natural response of these filter sections. The limit cycle oscillations are caused by quantization (either roundoff or truncation) of the results of multiplication of data samples with filter coefficients. The filter sections are assumed to be realized with finite precision, fixed-point arithmetic. The deterministic analysis of the limit cycles is complicated by the fact that there are two quantizer nonlinearities in the filter structure. Since the limit cycles are mostly unwanted noise, it is important for the engineering design of digital filters to provide expressions about bounds on the amplitude and frequency of these limit cycles.

After description of the nonlinearities and the filter models used in this chapter, several new results about limit cycles were presented. It was shown that a general

expression for the generated limit cycles is

$$A\hat{x} = \varepsilon, \tag{3.85}$$

where $\hat{x}$ is a column vector whose q elements are the limit cycle points, ε is the vector representing the roundoff noise sequence and A is a q x q matrix. This equation was used to derive an absolute bound for the amplitude of the limit cycle. Employing Lyapunov functions, another absolute bound on the amplitude was derived. Both bounds are conservative. However, they are absolute bounds, which is in contrast to the bound derived for the effective value linear model.

For complex conjugate poles of the filter transfer function Jackson [27] has shown that the sufficient condition from the effective value linear model yields an amplitude bound, which is given by

$$|x(n)| \leq \frac{0.5}{1-b} . \tag{3.86}$$

This bound is a function of the filter coefficient b only and therefore much easier to apply than the previously presented amplitude bounds. However, it was demonstrated that this bound can be exceeded by several quantization units. The bound should therefore be applied with care. The derived different bounds are evaluated for representative values of the filter coefficients a and b in the next chapter. A detailed comparison of the three bounds is therefore delayed until the next chapter.

As a new result, some lemmas about symmetry properties

of state trajectories in a specially defined successive value phase-plane were introduced. The results are important because they show that any amplitude bound evaluated for specified filter coefficients a and b is equally valid for -a and b.

Finally, it was shown that magnitude truncation quantization cannot sustain a zero-input limit cycle with intermediate frequencies $\frac{f_o}{f_s}$, such that $0 < \frac{f_o}{f_s} < \frac{1}{2}$. However zero-input limit cycles with frequencies $\frac{f_o}{f_s} = 0$ or $\frac{1}{2}$ are possible.

Since it is only possible to state an approximate expression for the frequency of the limit cycles, an important problem remains still open for further research. Frequency is a complicated function of the filter coefficient and the amplitude of the particular limit cycle. It would be useful to, at least, formulate some bounding expressions in order to be able to judge the deviations of the limit cycle frequency from the natural frequency of the corresponding linear filter.

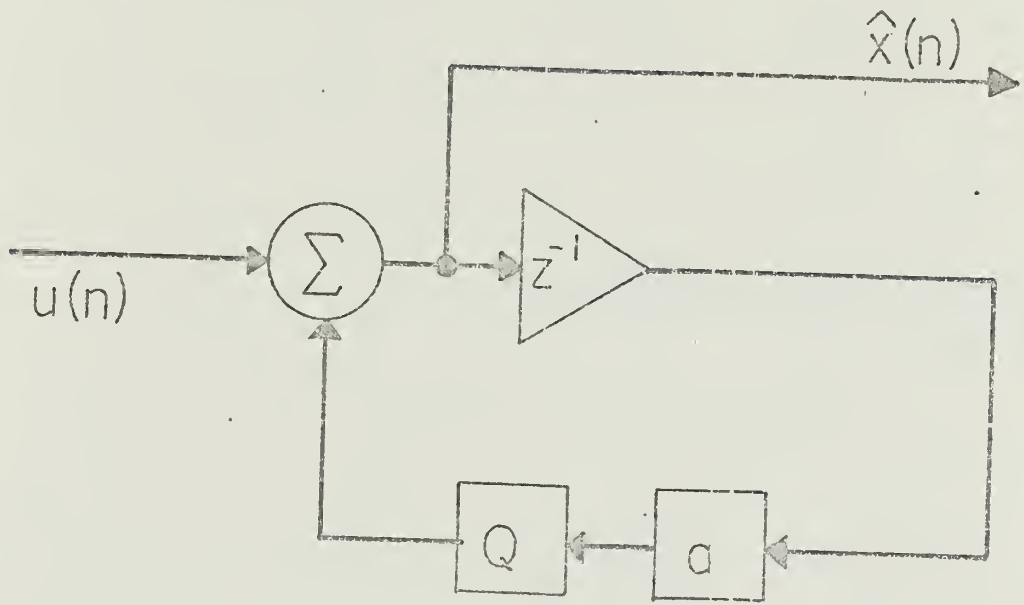

Fig. 3.1: First-order Digital Filter with Quantizer Nonlinearity.

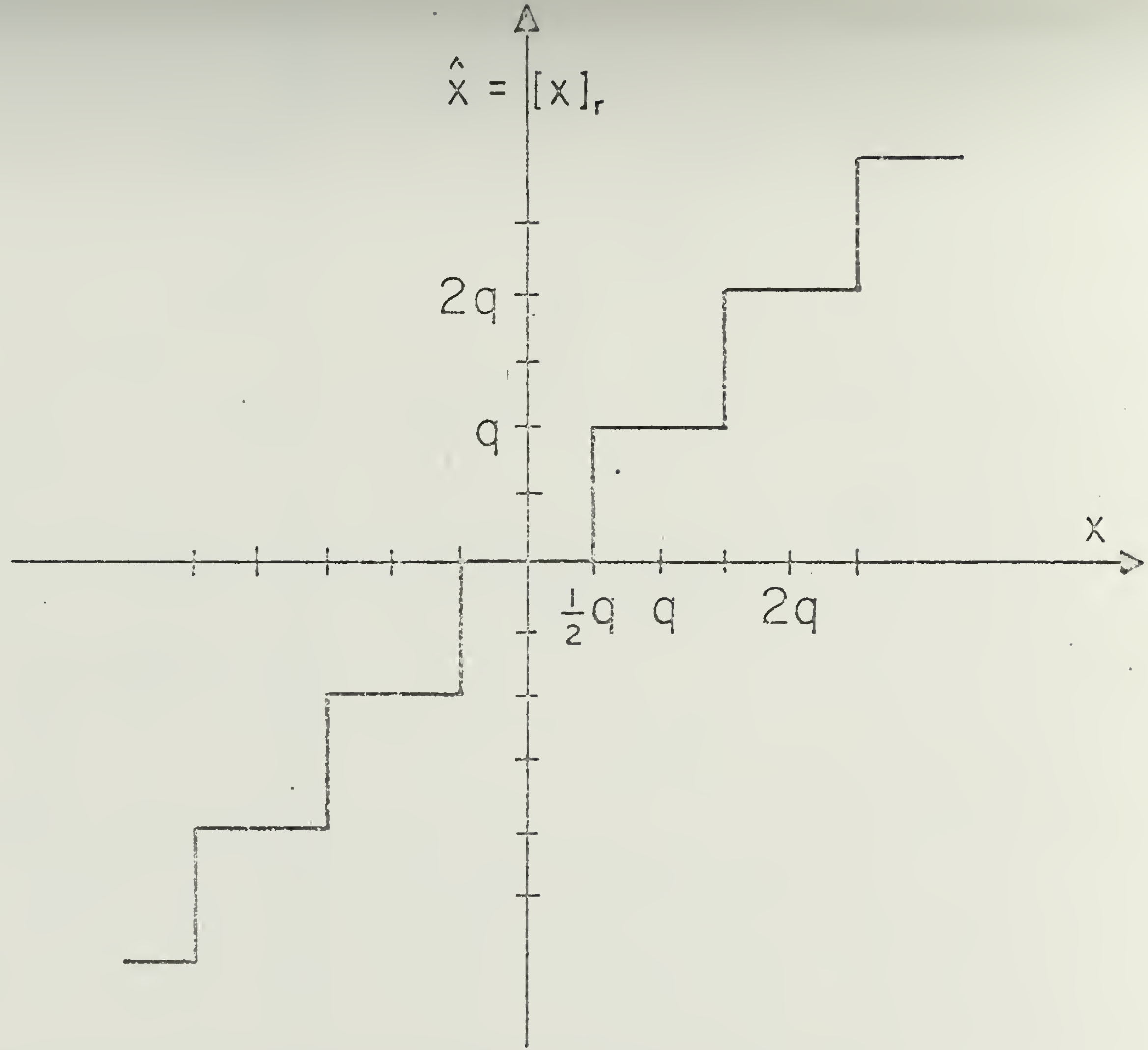

Fig. 3.2: Input-output Characteristic of Round-off Quantizer with Quantization Step-size q.

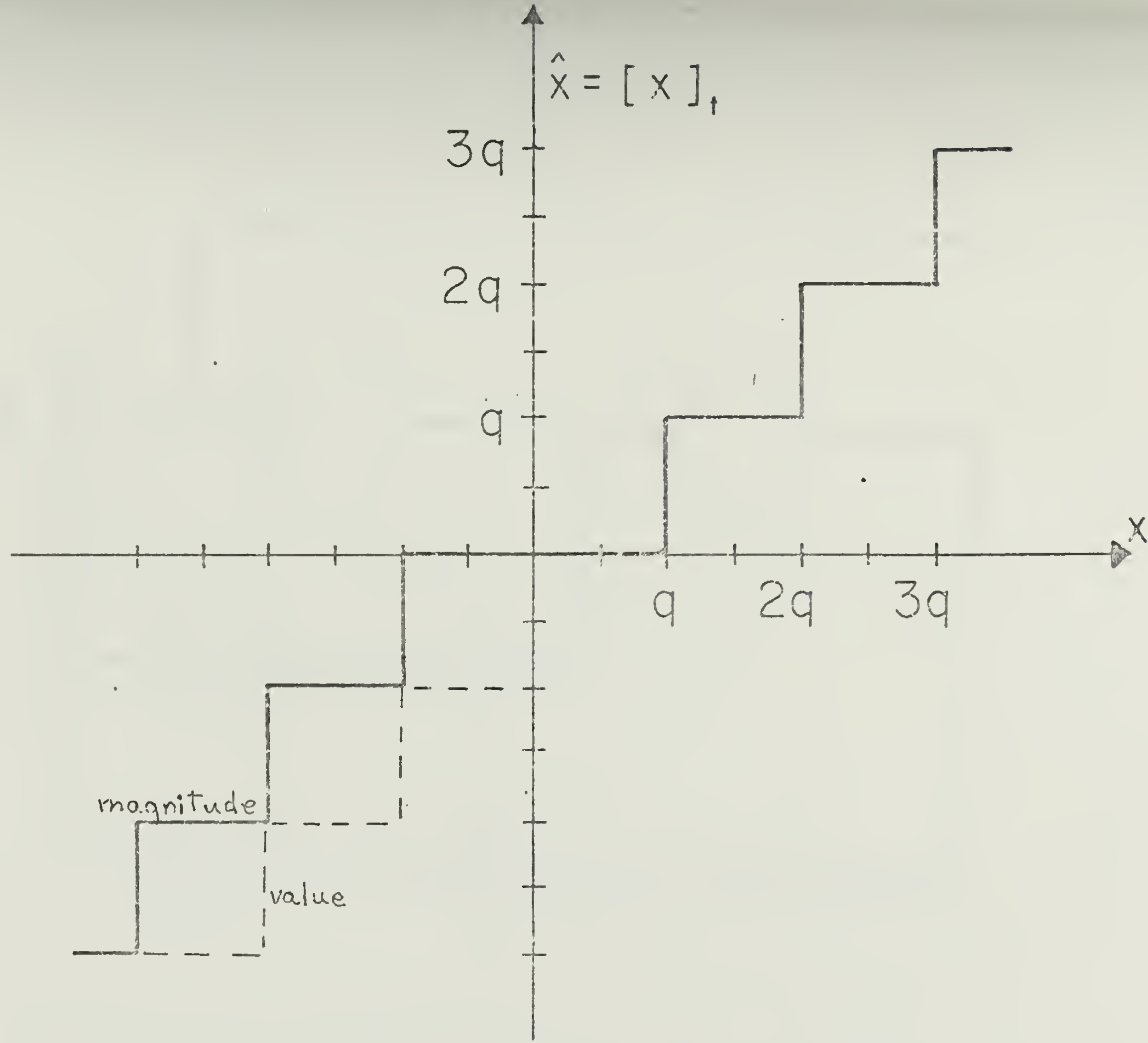

Fig. 3.3: Input-output Characteristic of Truncation-quantizer with Quantization Step-size q.

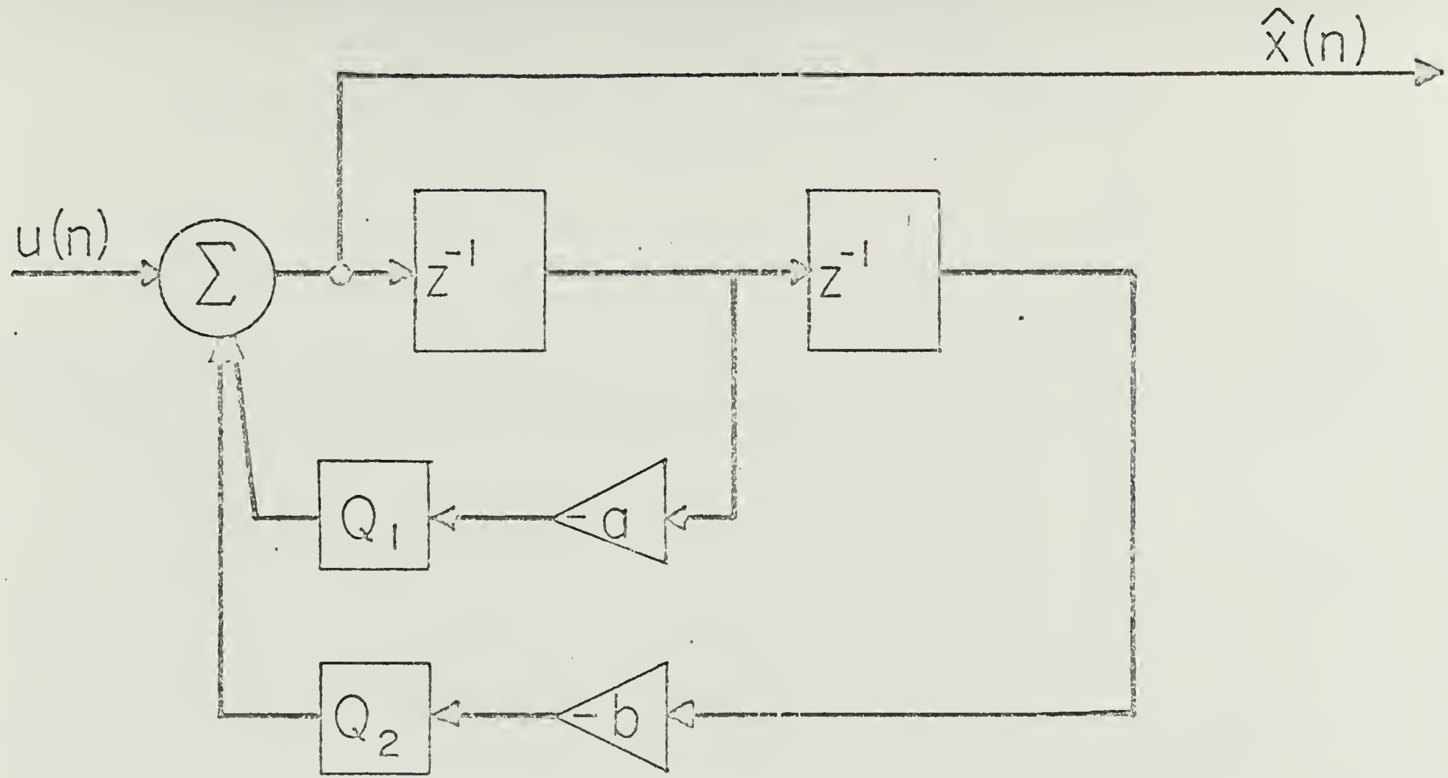

Fig. 3.4: Second-order Digital Filter with Two Poles and Two Quantizer Nonlinearities.

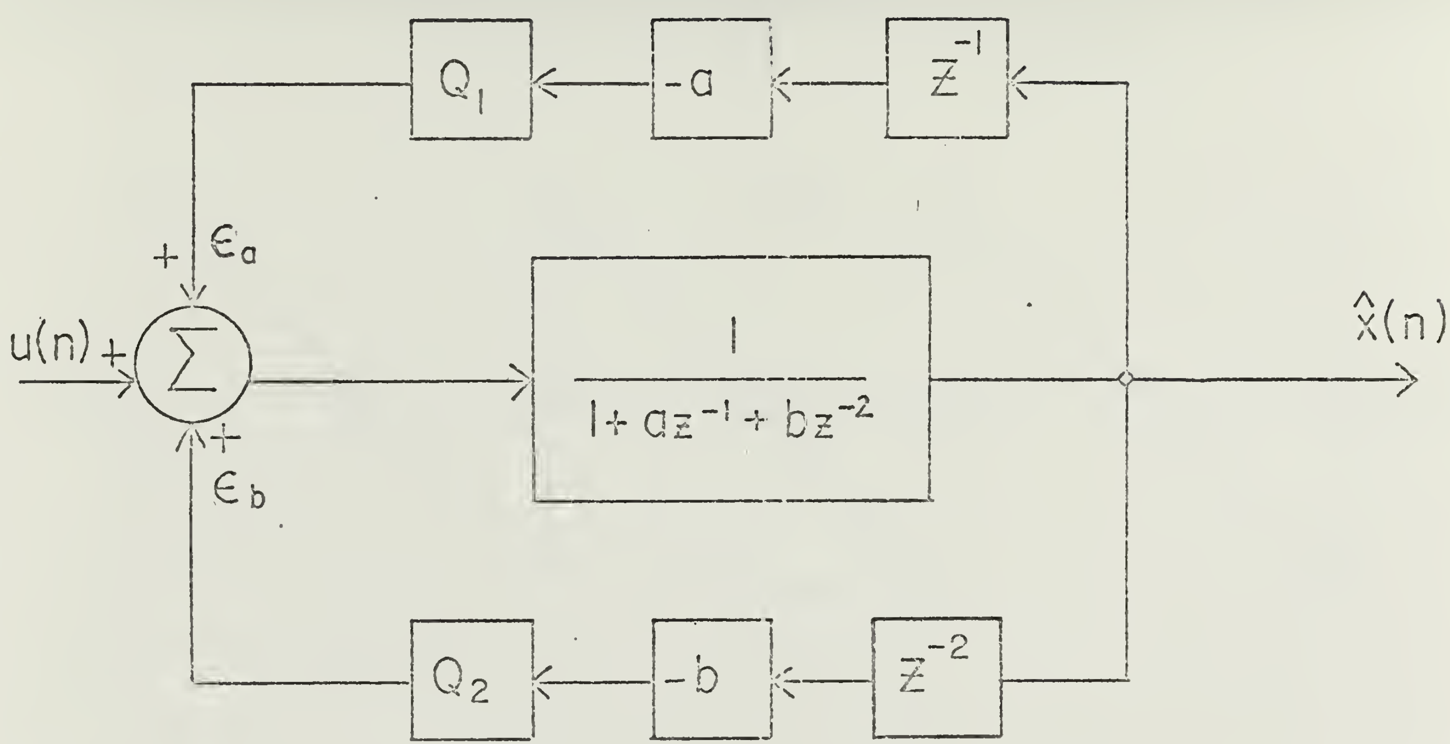

Fig. 3.5: Block Diagram of Second-Order Digital Filter with Two Poles and Sawtooth Nonlinearities.

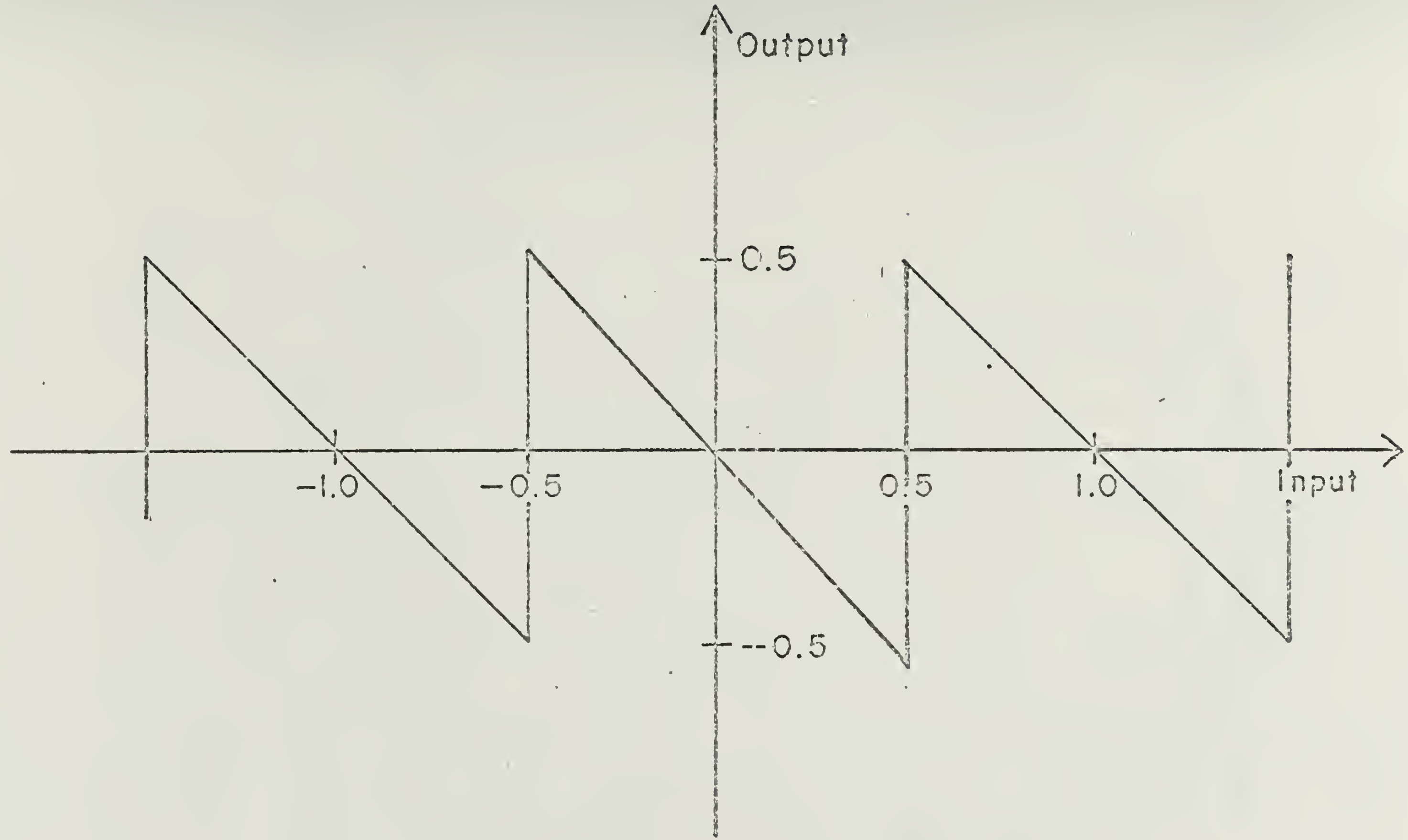

Fig. 3.6: Sawtooth Nonlinearity for Fig. 3.5.

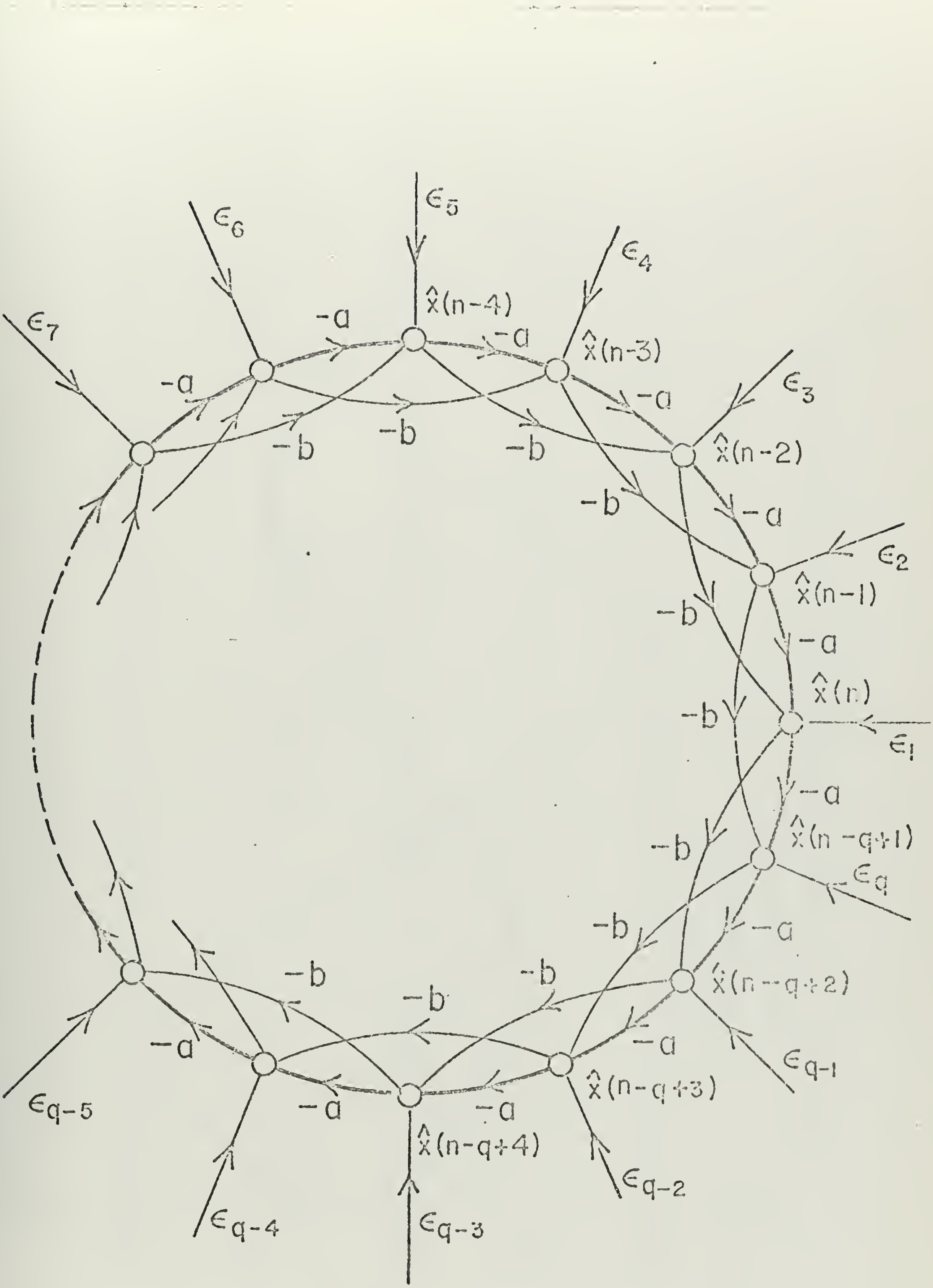

Fig. 3.7: Flow Graph for Limit Cycle Oscillations with Period qT.

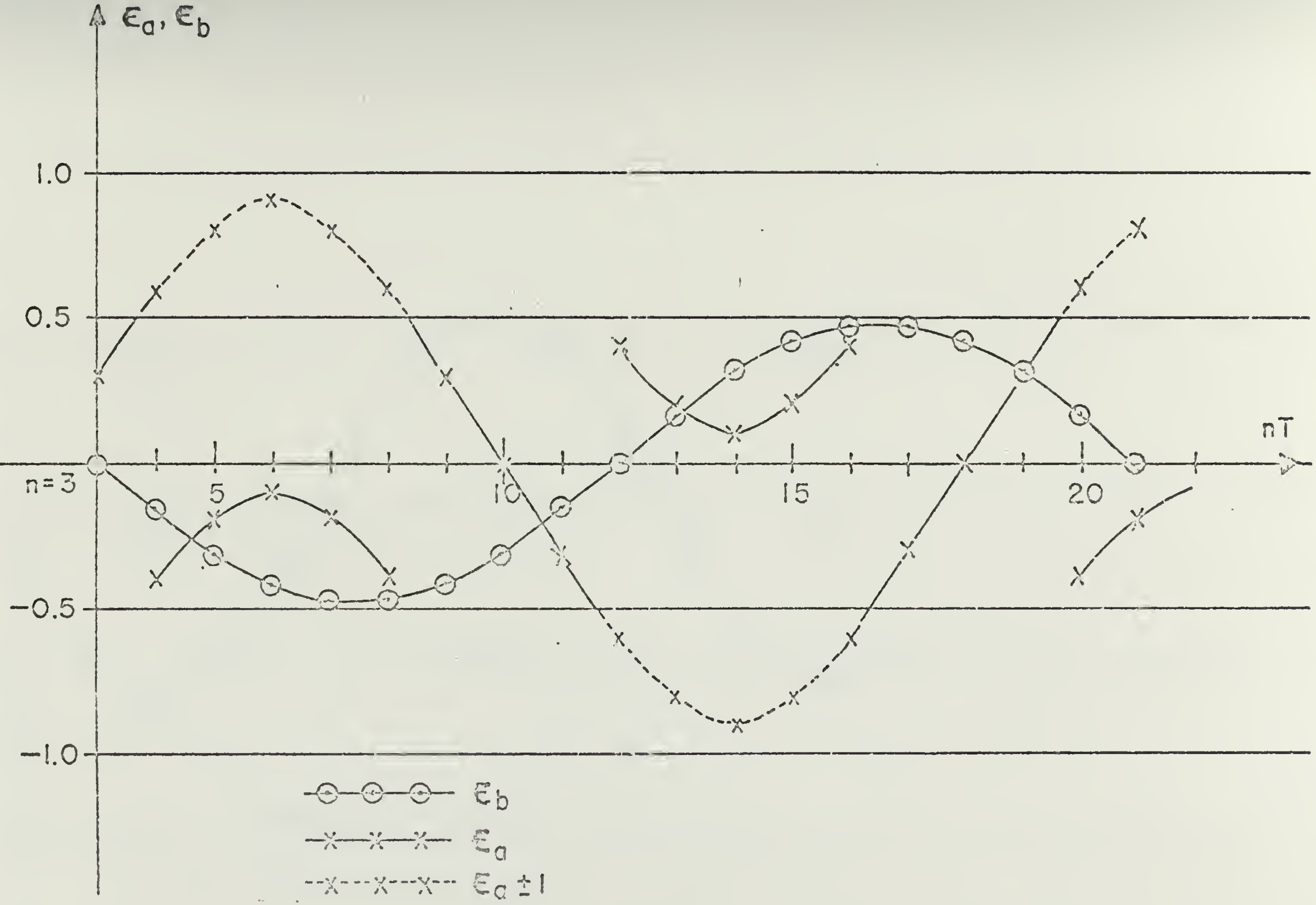

Fig. 3.8; Round-off Sequences ϵ_a, ϵ_b for a = -1.9, b = 0.9474 .

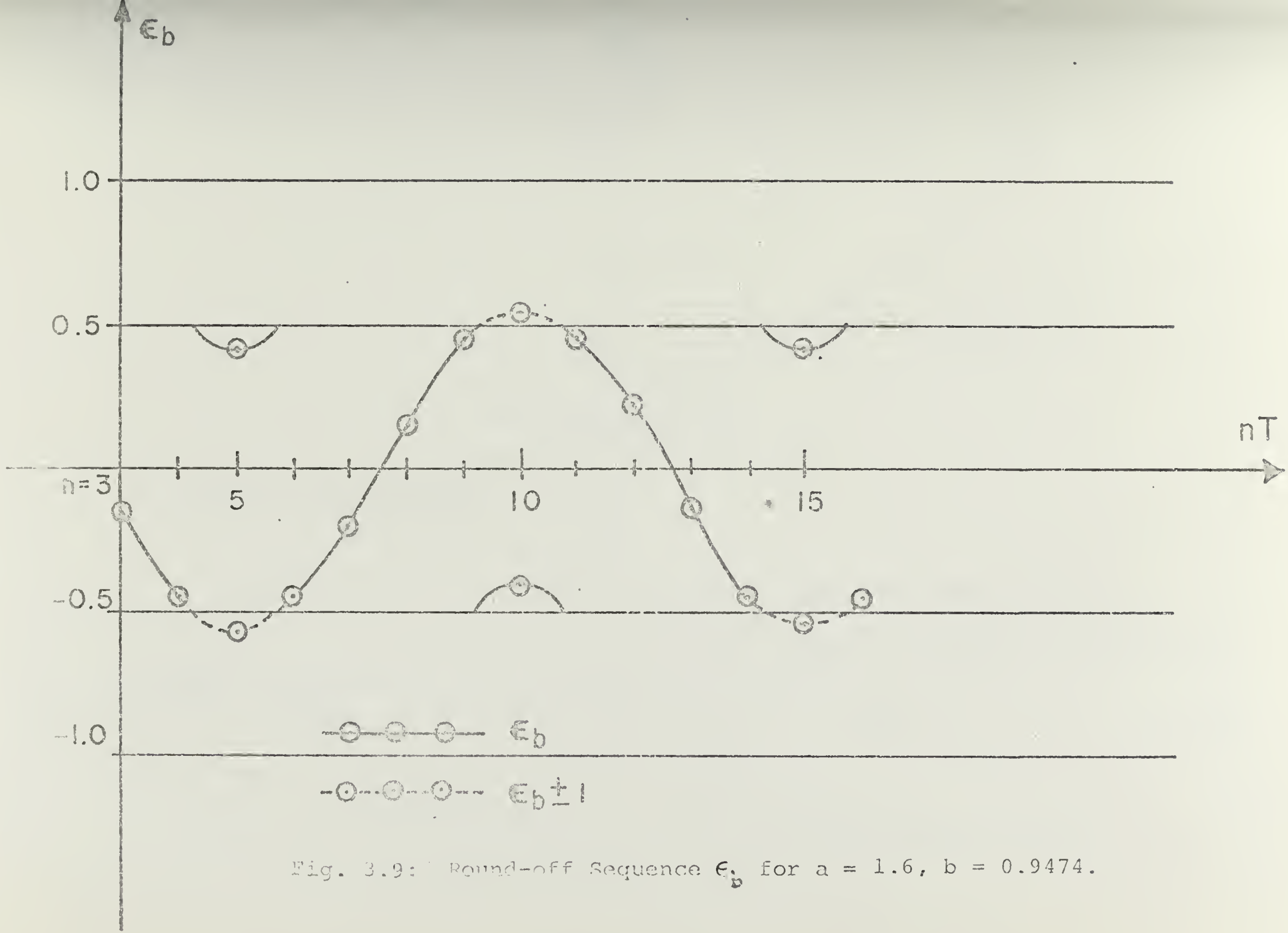

Fig. 3.9: Round-off Sequence ϵ_b for a = 1.6, b = 0.9474.

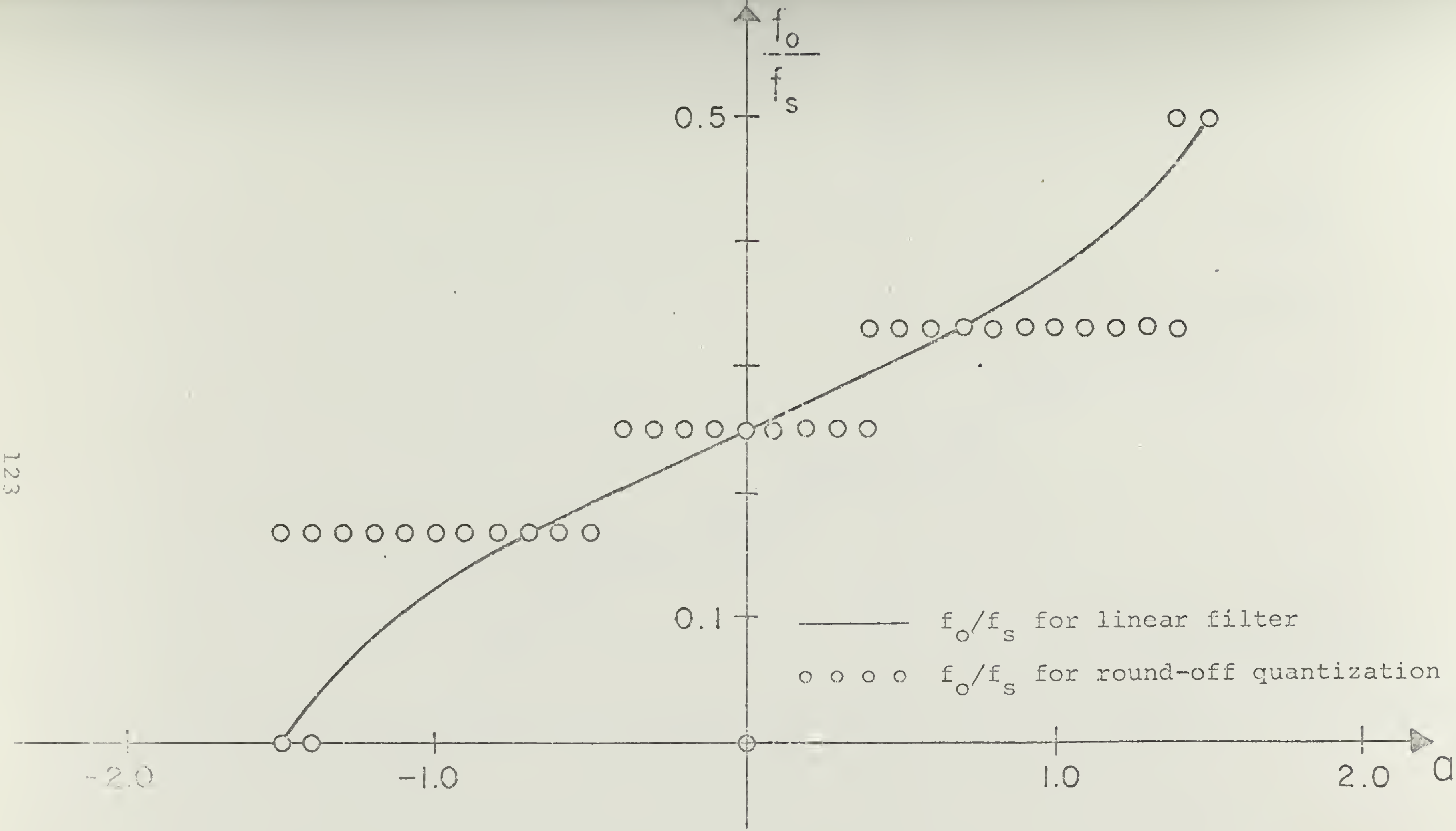

Fig. 3.10: Frequency f_o/f_s vs. a for Zero-input Digital Filter, Parameter b = 0.5.

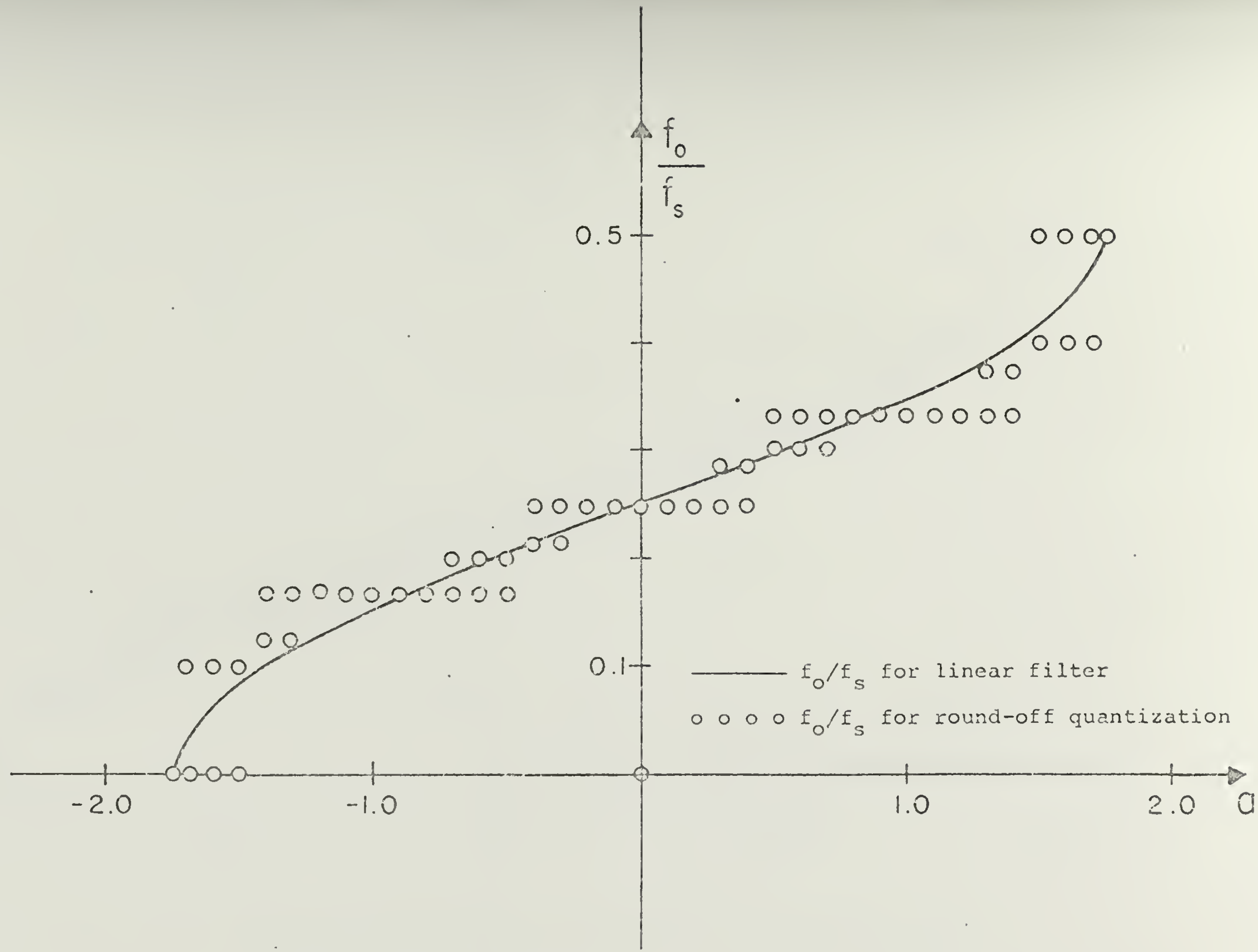

Fig. 3.11: Frequency f_o/f_s vs. a for Zero-input Digital Filter, Parameter b = 0.75.

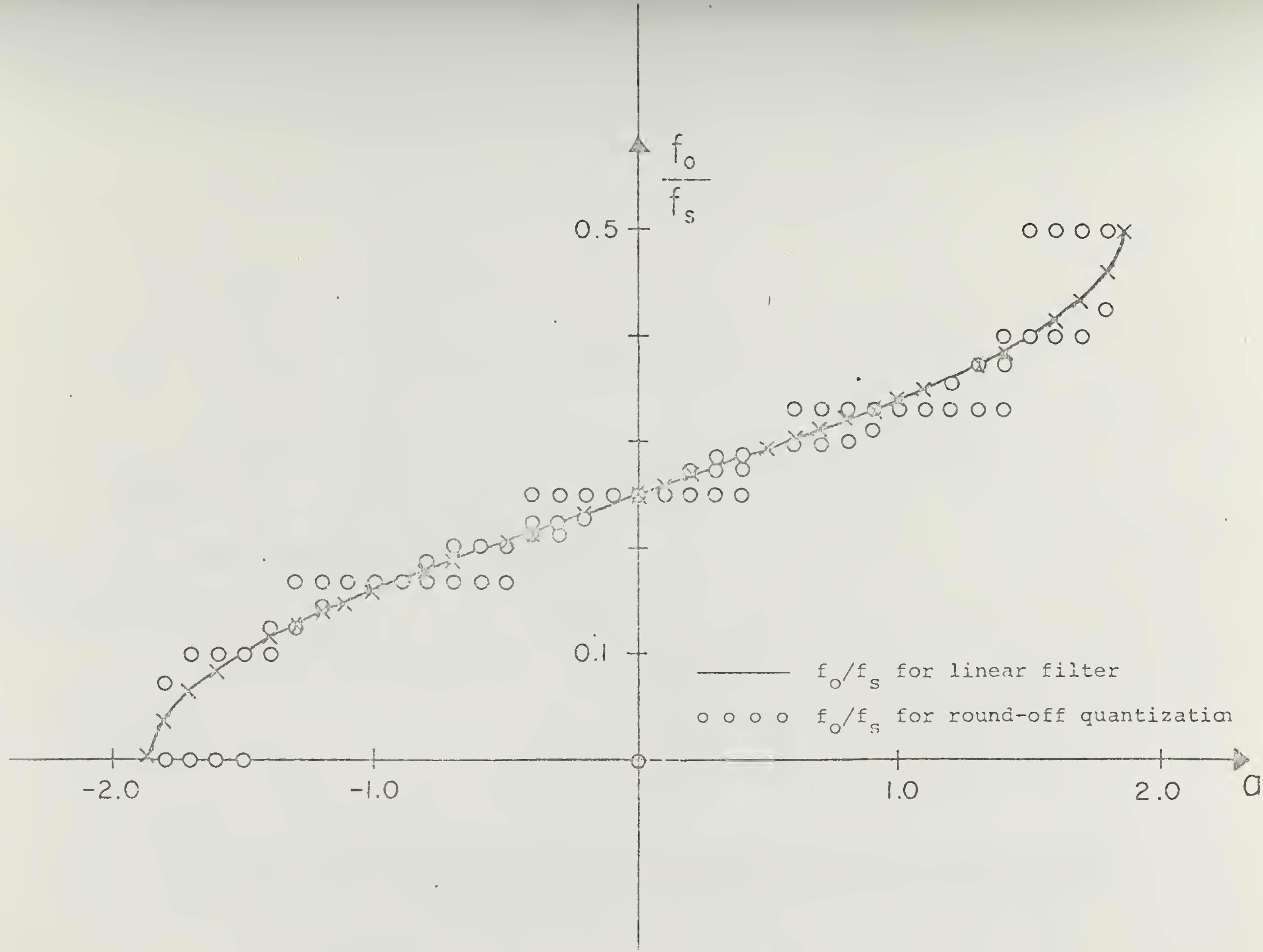

Fig. 3.12: Frequency f_o/f_s vs. a for Zero-input Digital Filter, Parameter b = 0.85.

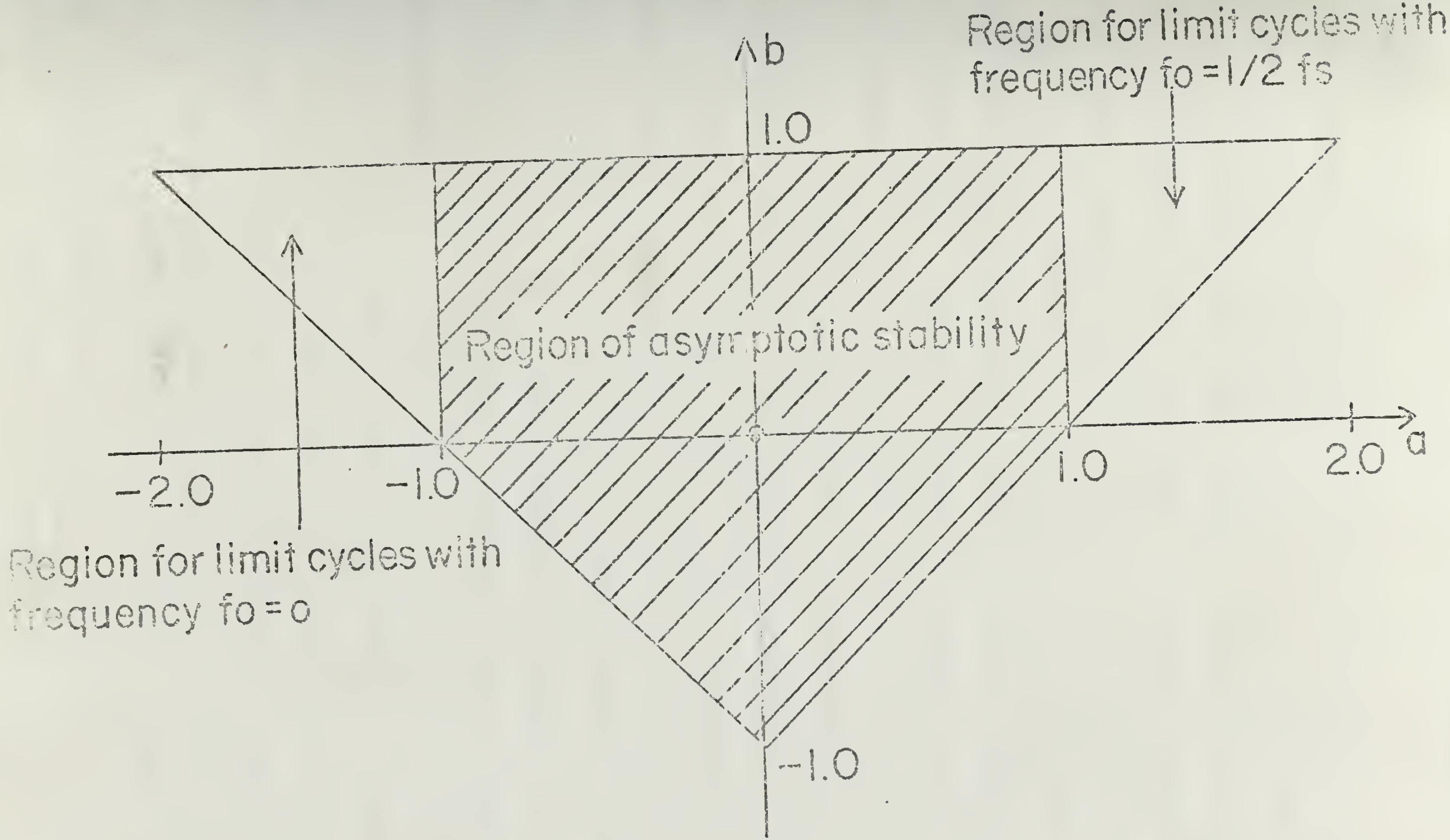

Fig. 3.13: Parameter Plane with Regions of Stability for Second-Order Digital Filter with Truncation.

IV. PRESENTATION OF EXPERIMENTAL RESULTS

A. INTRODUCTION

The analytical results derived in the preceding chapters are now tested and compared. For this purpose three computer programs have been developed. These programs are written in PL/1 and FORTRAN IV and have been run on the IBM 360/67 of the W. R. Church Computer Center of the Naval Postgraduate School.

The first program is an analysis program for zero-input limit cycles in second-order digital filters employing round-off quantization. Given a particular choice for the filter coefficients a and b, all possible limit cycles within a specified area of search are evaluated and displayed in a successive value phase-plane plot. The important feature of this program is that all possible limit cycles are enumerated by solving the filter response for all reasonable choices of initial conditions. With the numerical values for the limit cycles available, it is then possible to compare the actual amplitude of the limit cycle with the predicted amplitude, obtained from the derived bounds. Furthermore, from the resulting successive value phase-plane plots, it can be deduced how closely the limit cycle trajectory fits the elliptical trajectory expected for a nearly sinusoidal response.

The second program implements two of the five amplitude bounds derived in Chapter III, such that a comparison between the different bounds on the basis of numerical results is possible. The five bounds compared in this section are repeated from Chapter III and summarized in the following table.

Eqn. #	Bound	Text eqn. reference	Parameters to be specified	Is the bound exact?	Region of Applicability
(4.1)	$\sqrt{\frac{\lambda\ \max(Q)}{\lambda\ \min(Q)}} \cdot [W + \sqrt{W^2 + q_{22}}]$	(3.29)	a,b	exact	$0 \leq f_o/f_s \leq \frac{1}{2}$
(4.2)	$\sum_{j=1}^{q} \frac{\lvert \det A_{ji} \rvert}{\lvert \det A \rvert}$	(3.43)	a,b,q	exact	$\frac{f_o}{f_s} = \frac{p}{q}$, where $0 \leq 2p \leq q$
(4.3)	$\frac{0.5}{1-b}$ (Jackson)	(3.53)	b	approximate	$0 < \frac{f_o}{f_s} < \frac{1}{2}$
				exact	$\frac{f_o}{f_s} = \frac{1}{3}, \frac{1}{4}, \frac{1}{6}$
(4.4)	$\frac{1.5}{1-b}$	(3.55)	b	approximate	$0 < \frac{f_o}{f_s} < \frac{1}{2}$
(4.5)	$\frac{1}{1-\lvert a \rvert + b}$ (Jackson, Bonzanigo)	(3.46)	a,b	exact	$\frac{f_o}{f_s} = 0, \frac{1}{2}$

The bound (4.1) is shown to be the most pessimistic one. For values of $|a|$, such that $|a|$ approaches (1+b), it is shown that the bound (4.1) gets so pessimistic that it is useless for practical applications. The bound (4.2) has the complication that the period qT of the limit cycle has to be specified as an additional parameter.

The bounds (4.3) and (4.4) are not exact, because they are based on the assumption of an effective value linear model. However the numerical data indicates that (4.3) is exceeded only in some exceptional cases. For those cases the bound (4.4) is valid. A comparison between the actual limit cycle amplitudes and the numbers obtained from (4.4) shows that the bound (4.4) is never exceeded. However its derivation is not exact and an exception of (4.4) may exist.

The bound (4.5) is again an exact bound. It applies to limit cycles with zero-frequency and with the Nyquist frequency ($f_o/f_s = \frac{1}{2}$) only.

The third program is a simulation of an important special case, the digital oscillator, as first considered in example 2 of section II.D. The difference equation for this kind of digital oscillator is given by

$$\hat{x}(n) = -[a\,\hat{x}(n-1)]_q - \hat{x}(n-2). \tag{4.6}$$

The nonlinear Eqn.(4.6) is linearized by assuming that

$$[a\,\hat{x}(n-1)]_q = a'\hat{x}(n-1), \tag{4.7a}$$

and

$$a' = a + \varepsilon. \tag{4.7b}$$

The coefficient a' is approximately constant if ε approaches zero. This is the case if $\hat{x}(n-1)$ is made larger and the quantization step-size is constant or if the quantization step-size is made smaller and $\hat{x}(n-1)$ is kept below a specified constant. Equation (4.6) is rewritten as

$$\hat{x}(n) = -a'\hat{x}(n-1) - \hat{x}(n-2). \tag{4.8}$$

Equation (4.8) is the linearized version of (4.6). Its poles are located on the unit-circle in the z-plane, as given by the characteristic equation

$$z^2 + a'z + 1 = 0. \tag{4.9}$$

From (4.9) it is deduced that (4.6) indeed describes an oscillator.

As a new result bounds on the frequency of the digital oscillator are derived from (4.7b). The result is not exact, however a comparison between the observed frequencies of the limit cycle oscillations and the computed values for the bounds indicates that the bounds are not exceeded for those examples considered in this section.

For a given initial condition, the response of the oscillator defined by (4.6) (either with magnitude truncation or with round-off quantization) is compared with the response expected from an infinite precision linear oscillator. The comparison is done by approximating the limit cycle oscillation with a sinusoidal oscillation employing a least-square criterion. Experiments show that the oscillator with quantization always

exhibits limit cycles with periods which can be as large as several thousand or more samples. Therefore, the average deviation of the limit cycle from the ideal sinusoid is constant over each period of oscillation. This is contrary to a result reported by Rader and Gold [12]. Their formula indicates that the average deviation increases with time. The latter conclusion is not verified by experiment. Their underlying assumption of uncorrelated quantization noise does not hold for the digital oscillator.

Comparison of experimental results with the linear oscillator indicates that a small offset in frequency and amplitude is present. It is verified that the limit cycle oscillations are nearly sinusoidal because the average deviation from the predicted amplitude is small.

For those examples tested, the offset in amplitude and the average deviation from the amplitude are practically constant for a large range of initial conditions. The offset in frequency decreases linearly for an increase of the initial conditions. This important conclusion indicates that any degree of approximation for a specified sinusoidal oscillation can be achieved by either scaling the amplitude if the quantization step-size is constant, or by decreasing the quantization step-size if the amplitude is constant. The rules, deduced from the strictly experimental data, are important if the accuracy of digital generation of sinusoidal oscillations is considered.

Comparison of experimental data obtained for [illegible]

quantization with the experimental data obtained for roundoff quantization shows that roundoff should be preferred over magnitude truncation. With roundoff quantization a better degree of approximation of a sinusoid can be achieved.

B. AN ANALYSIS PROGRAM FOR ZERO-INPUT LIMIT CYCLES.

The analysis program for zero-input limit cycles in second-order digital filters with roundoff quantization is based on the filter configuration presented in Section III.C and depicted in Fig. 3.4. The difference equation representing this model is repeated here for convenience:

$$\hat{x}(n) = -[a\,\hat{x}(n-1)]_r - [b\,\hat{x}(n-2)]_r. \qquad (4.10)$$

To compare the analytical results of Chapter III with the actual limit cycles it is desirable to enumerate all possible limit cycles for a specified set of filter coefficients a and b.

The algorithm for the program is based on the following lemma.

<u>Lemma</u>: Given the difference equation (4.10), the response of this nonlinear difference equation is uniquely specified for any choice of initial conditions.

<u>Proof</u>: Specify any initial condition $\hat{x}(1)$ and $\hat{x}(2)$. The input-output characteristic for a roundoff nonlinearity as depicted in Fig. 3.2 establishes a single-valued relationship between the input values to the nonlinearity and their output values. Thus $\hat{x}(3)$ is uniquely defined by (4.10).

Using $\hat{x}(2)$ and $\hat{x}(3)$ the value for $\hat{x}(4)$ is uniquely

defined by (4.10). Repeating the iterative process to time nT, where n is any integer greater than two, establishes the unique response of (4.10). As an additional observation, it is noted, that uniqueness of solutions for an equation of the type (4.10) is always assured if the nonlinearities define a single-valued relationship between the input and the output.

Consider the result of the lemma in a different context. Let the successive value phase-plane for second-order digital filters be defined in a cartesian coordinate system with the x-axis representing $\hat{x}(n)$ and the y-axis representing $\hat{x}(n-1)$ (compare with Section III.F). Once a state-point is chosen in the phase-plane, the resulting state-trajectory is uniquely specified. A limit cycle is recognized by the observation that the sequence of state-points constituting the limit cycle is represented by a closed curve if the limit cycle state-points are connected by straight lines.

In order to enumerate all possible limit cycles in a specified area of search for a given set of filter coefficients, it seems necessary to apply all possible initial conditions or to start the state-trajectory from all possible state-points in a phase-plane which extends to the largest possible number specified by the dynamic range of the respective filter in either dimension. However, this is neither realistic nor necessary.

After some initial experimentation it has been found that only those initial conditions or state-points have to

be applied to the difference equation which lie inside the square delineated by

$$|\hat{x}(n)| = |\hat{x}(n-1)| = k, \tag{4.11}$$

The limit k is constructed from the bounds (4.3) and (4.5) as

$$k = \max\left[\frac{0.5}{1-b} + \alpha,\ \frac{1}{1-|a|+b}\right]. \tag{4.12}$$

The constant α is a safety factor, which is included to investigate all possible limit cycles. For the computer runs, values of α between 5 and $\frac{1.0}{1-b}$ have been tried. If the initial conditions are chosen as described above, it then remains to count and record the different limit cycles.

Another very important consideration about the experimental implementation of the filter algorithm has to be stated. The solution of the difference equation (4.10) for given filter coefficients a and b and a given state-point $\hat{x}(n-1)$ and $\hat{x}(x-2)$ has to be free of any conversion errors if the decimal numbers a, b, $\hat{x}(n-1)$ and $\hat{x}(n-2)$ are converted to machine numbers as they are represented internally in the computer.

For example, assume that a = -1.2. If binary representation of numbers is employed in the computer, the coefficient cannot be stored as a computer number, because

$$(1.2)_{10} = (1.\overparen{0011}\ldots)_2.$$

Therefore a has to be stored in the computer as

$$a' = a \pm \Delta, \tag{4.13}$$

where Δ = conversion error.

It has been shown in Chapter II, that the result of inexact representation of filter coefficients amounts to a shift in the pole positions of the digital filter. For the present investigation such errors must be avoided. Exact representation of numbers is one of the chief reasons why the analysis program for zero-input limit cycles is written in PL/1 [34]. This computer language allows the use of a decimal arithmetic mode, where numbers are stored and operated on in "packed decimal form"[1] and no conversion errors of the above described type can occur. Another reason for the choice of PL/1 is the relative ease with which the delicate array manipulations for the analysis program can be handled. The analysis program is listed at the end of this dissertation.

A wide variety of values for the filter coefficients a and b have been used for computer runs with the analysis program. For Fig. 4.1 those representative values of a and b are indicated by an X for which the results of the computer simulation are compiled in Appendix B. As an example, the computer results for a = -1.8 and b = 0.937 are displayed in Table 4.1 and Fig. 4.2 at the end of this chapter. In Table 4.1 all possible limit cycles in the area of search

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[1] See reference [34], p. 27, 28.

are enumerated. They are labelled by an identification number and the frequency $F = f_o/f_s$ and the values of the limit cycle points are printed. Furthermore, the approximate frequency $f_o/f_s = 0.006$ (see formula (3.59)) and the amplitude bounds

$$A_1 = \frac{0.5}{1-b} = 7.937, \text{ from formula (4.3)}$$

$$A_2 = \frac{1}{1-|a|+b} = 7.299, \text{ from formula (4.5)},$$

are stated. Limit cycle #1 exceeds the bound given by (4.3), while limit cycles #2 to #5 stay below this bound. The zero-frequency limit cycles #6 to #10 (including the trivial case where $\hat{x}(n) = \hat{x}(n-1) = 0$) stay well inside the bound (4.5), which is expected, because this bound is exact.

The corresponding state-trajectories are displayed in Fig. 4.2. The x-axis represents $\hat{x}(n)$ and its values are labelled in the top line. The y-axis represents $\hat{x}(n-1)$ and its values are labelled in the left column. A state trajectory is constituted by those state points which are designated by the same number. This number is identical with the limit cycle identification number given in Table 4.1.

For example, consider limit cycle #2. Its state trajectory is formed by the 14 state points designated "2", such as (3,0), (5,3), (6,5), (6,6) and so on. The trajectory is very close to an ellipse, which indicates that this limit cycle is nearly sinusoidal in nature. This is not true for limit cycle #1, which shows some deviations from an

elliptical shape at the state points (6,7) and (-6,-7). A close inspection of all the data compiled in Appendix B results in the following conclusions:

a) The amplitudes of limit cycles with frequency $\frac{f_o}{f_s} = 0$ or $\frac{1}{2}$ are well inside the bounds as given by (4.1) and (4.5). The amplitudes of limit cycles with frequencies such that $0 < \frac{f_o}{f_s} < \frac{1}{2}$ are well inside the bounds given by (4.1) and (4.2). This is expected, because these bounds are exact.

b) Most, but not all, amplitudes are below the value given by (4.3). The severest deviations from (4.3) which have been found are 5 and 6 quantization units. The respective limit cycles are exhibited in Table B.15 and B.16 of Appendix B. This result is summarized in the following table:

Value of a and b	Limit cycle Number	Amplitude of Limit Cycle	Bound from (4.3)
$a = -1.4$,	2	23	$[18.519]_t = 18$
$b = 0.973$	3	19	
	4	19	
	6	20	
	7	22	
	8	21	
	9	20	
	10	20	
$a = -1.74$,	2	17	$[11.999]_t = 11$
$b = 0.95833$			

The reported exceptional cases have amplitudes which are

inside the bound given by (4.4). It is worth noting that without the analysis program for zero-input limit cycles, the exceptional limit cycles reported above would probably have been overlooked.

c) A comparison of the successive value phase-plane plots from Figs. B.1-5 with the ones from Figs. B.7-11 shows that they are symmetric around a straight line at an angle of 45° for the former and around a straight line at an angle of 135° for the latter. The distinguishing factor between the two sets of phase-plane plots is, that $a < 0$ in the first case, and $a > 0$ in the second case. For $a = 0$, no symmetry axis can be determined (compare with Fig. B.6). The change in the symmetry axis with the change of sign for the coefficient a has been predicted in section III.F. Inspection of the limit cycle sequences in the corresponding tables B.1-5 and B.7-11 shows that the three lemmas from that section apply.

d) The parameter plane from Fig. 4.1 shows the regions of stability for the digital filter with roundoff quantization. The region for those limit cycles with frequencies, such that $0 < f_o/f_s < \frac{1}{2}$, has been derived in section III.D. The remaining regions for limit cycles with frequencies at zero or at the Nyquist frequency (everything inside the triangle except the shaded region for asymptotic stability) and for asymptotic stability have been derived by Jackson [27]. The boundary of the triangle is obtained if the unit circle in the z-plane is mapped into the parameter plane

with the help of the equations

$$0 < b \leq 1.0, \tag{4.14a}$$

$$1 + b > |a|. \tag{4.14b}$$

In Fig. 4.1, the limit between complex conjugate roots ($a^2 - 4b < 0$) and real roots ($a^2 - 4b > 0$) for the linear filter is shown also.

C. COMPARISON OF THE AMPLITUDE BOUNDS

For comparison of the different amplitude bounds derived in Chapter III, a FORTRAN IV program has been written to compute the bounds given by (4.1) and (4.2) for representative values of a and b. The program is listed at the end of this dissertation. Since the bound given by (4.3), (4.4) and (4.5) can be easily computed by hand, their evaluation is not included in the computer program.

The bound given by (4.1) is based on the application of Lyapunov functions. The formulas derived in section III.D are directly applicable. They are used to compute the bound for values of $b = 0.5$, 0.75, $0.8\hat{3}$ ..., 0.875 and 0.9 and varying values of a, such that $|a| < 1 + b$. The results of this computation are displayed in Fig. 4.3.

The bound given by (4.2) is difficult to compute, because several determinants have to be evaluated. To avoid the awkward evaluation of determinants, a different algorithm than the one given by (4.2) is employed in the computer program. The bound (4.2) has been derived from the matrix

equation (3.34) which is repeated here.

$$A\hat{x} = \varepsilon. \tag{4.15}$$

If this equation is solved for $\hat{x}$, such that

$$\hat{x} = A^{-1}\varepsilon = B\varepsilon, \tag{4.16}$$

the bound on $\hat{x}$ can be written as

$$|\hat{x}| \leq \sum_{i=1}^{q} |b_{ij}| , \tag{4.17}$$

because $|\varepsilon_i| \leq 1.0$ for all $i = 1, 2, \ldots, q$. The elements b_{ij} of (4.17) are the elements of the matrix inverse $B = A^{-1}$ of (4.16). The formula (4.17) is identical to (4.2). Equation (4.17) has been used for the evaluation of the bound instead of (4.2) because matrix inversion requires less computation than the evaluation of determinants. The results of the evaluation of (4.17) for the same values of a and b as used for the evaluation of the previous bound, are displayed in Figs. 4.4-4.8. The difference between the figures results from the different choices for the period of the limit cycle qT, where q is chosen such that q = 4, 5, 6, 7, 8. A closer inspection of the curves of Figs. 4.4-4.8 reveals some very peculiar characteristics. For some values of the coefficient the curves for different values of b merge. For example, in Fig. 4.6, the value of the bound is about equal for values of a around zero. For other values of a the bound peaks, exhibiting mostly two pronounced maxima, which are well apart for the different values of b.

This behaviour is explained if one realizes that limit cycles occur only for those values of a which are around the peaks of the curves exhibited in Figs. 4.4-4.8. That this is true can either be verified with data obtained from the analysis program of the preceding section or by calculating the approximate values of a which correspond to the limit cycle frequencies $\frac{f_o}{f_s}$ and which are possible for a specified period of qT. If q is specified then, from section III.E, the frequencies $\frac{f_o}{f_s} = \frac{p}{q}$ are possible, where

$$2p \leq q, \tag{4.18}$$

and both p and q are integers. Using the approximate expression for the frequency (3.60) the values of a and p/q are related by

$$a = -2 \cos \left(\frac{2\pi p}{q}\right). \tag{4.19}$$

For p = 0 and 2p = q, one obtains a = -2 and a = +2. These values of a are outside the graphs of Figs. 4.4-4.8. However for q = 8 (compare with Fig. 4.8) one obtains the following values of a for which limit cycles can occur:

p	f_o/f_s	a
0	0	-2
1	1/8	-1.4
2	1/4	0
3	3/8	+1.4
4	1/2	+2

These values are in acceptable agreement with the location of the peaks exhibited in Fig. 4.8.

There remains an obvious question to be answered. What meaning does the bound (4.14) have for those values of a which are between the locations of the maxima? The answer is that for these values of a the bound (4.14) has no meaning for the zero-input limit cycles, because there are none. This portion of the curve representing (4.17) is valid for a sequence of the ϵ in (4.15) for which a solution of the $\hat{x}$, according to (4.16) is possible, but this solution does not represent a valid zero-input limit cycle. In other words, if an arbitrary sequence ε (which is not the result of quantization by roundoff) is used as the driving function for the second-order system (4.10) a limit cycle results for a proper choice of the coefficient a. This limit cycle is not a valid zero-input limit cycle.

The different bounds are now compared. This can be done directly from the graphs for the Lyapunov bound (see Fig. 4.3) and the bound defined by (4.2) (See Figs. 4.4-4.8). It should be noted that the scales for the x-axis of the graphs in Fig. 4.3 and Figs. 4.4-4.8 are different, while the scales for the y-axis are equal. The comparison of the graphs shows that the Lyapunov bound is always greater than the bound (4.2) regardless of the choice of q. Both types of bounds generally approach infinity for $|a|$ approaching (1+b), however the Lyapunov bound approaches infinity much faster than does the other bound. It should be noted, however, that Fig. 4.5 (q = 5) and 4.2 (q = 7) do not show the

rapid increase for $a \approx (1+b)$, because for q odd, a value of p, such that $2p = q$, does not exist and therefore no other peaks than the ones on the graph are expected.

The two bounds based on the effective value linear model are given by (4.3) and (4.4). Evaluation of these bounds for $b = 0.5, 0.75, 0.8\hat{3} \ldots, 0.875, 0.9$ yields 1, 2, 3, 4, 5 for the bound (4.3) and 3, 6, 9, 12, 15 for the bound (4.4). Comparing the latter bound with the curves on Figs. 4.4–4.8 shows that the bound (4.4) is below the values for the bound (4.2) in the region of interest for $q = 4, 5, 6$ and above the values for the bound (4.2) for $q = 7,8$. However, from the data of the preceding section one can be reasonably sure (but not certain) that the bound (4.4) will not be exceeded.

In summary then, the following rules and observations are pertinent:

a) The Lyapunov bound from (4.1) is exact, but overly pessimistic. For this reason, the bound seems to be of little value for practical applications.

b) The bound from (4.2) is exact, reasonably close to the observed limit cycle amplitudes, but not easy to compute, especially if q is large.

c) The bound from the effective value linear model (4.3) is easy to compute, but is not exact. This bound and its companion for the exceptional case given by (4.4) are most readily applicable. If an exact bound is needed a check calculation using the bounds (4.1) and (4.2) can be performed.

D. THE DIGITAL OSCILLATOR

In example 1 and 2 of section II.D, two digital oscillator realizations are investigated with respect to coefficient accuracy due to finite representation of numbers. For this section, the digital oscillator defined by the difference equation (4.6) is again considered, this time with respect to quantization errors introduced through roundoff or truncation of the product-term $a\hat{x}(n-1)$. Because $b = 1.0$, the effective values of the poles of the linearized version of (4.6) are always on the unit-circle in the z-plane, regardless whether roundoff or magnitude truncation is employed. It has been discussed earlier that (4.6) indeed describes an oscillator. The oscillator defined by (4.6) has been investigated by Rader and Gold [12]. Assuming that the errors introduced by quantization of the product from $a\hat{x}(n-1)$ are statistically independent and its probability density function is uniform they evaluate the mean squared noise in the output signal caused by roundoff as

$$\sigma^2 = \frac{E_o^2}{12} \cdot \frac{n}{\sin^2(\omega_o T)}, \tag{4.20}$$

where E_o = quantization step-size.

Equation (4.20) indicates that the mean squared noise increases linearly with time nT. It is therefore expected that the sinusoidal output of the digital oscillator gets contaminated by roundoff noise until the output is no longer

recognizable as a sinusoidal signal.

The conclusion above is unreasonable from a physical point of view. Consider the linearized version (4.8) of the difference equation (4.6). Together with (4.7b) it is deduced that the solution of (4.6) must approach the solution of the equivalent linear difference equation (which is a sinusoid) if the quantization step-size and thus ε in (4.7b) decreases to zero. From (4.20), however, it is concluded that the mean squared quantization noise σ^2 increases without bounds regardless how much the quantization step-size is reduced.

In contrast, the simulation of (4.6) shows that for either roundoff or magnitude truncation, limit cycles with periods up to several thousand samples are produced. Since the shape of these limit cycles, except for small offsets in frequency and amplitude and a small deformation of the samples, is essentially sinusoidal and the mean squared noise is independent of the time nT the result of (4.20) is not verified by the available experimental data. For the oscillator defined by (4.6) - as for any other limit cycle oscillation due to quantization - the stochastic approach is not applicable because the roundoff noise is correlated with the output signal and the different noise sources are not statistically independent. Therefore, only a coherent analysis is applicable.

The existence of limit cycles for (4.6) has been

suggested to the author by J. F. Kaiser[1] in a private communication. Recently, Tödtli and Pfundt [29] have reported about limit cycles from digital oscillator realizations. Their work, however, emphasizes the hardware realization of a digital oscillator using logic circuits.

As a new result a heuristic bound on the frequency of the digital oscillator is now derived. If the difference equation (4.6) is linearized and roundoff quantization is assumed, then from (4.7a) and (4.7b) one obtains

$$a\hat{x}(n-1) \pm 0.5 \mp \delta(n-1) = a\hat{x}(n-1) + \varepsilon\hat{x}(n-1). \qquad (4.21)$$

Since $\delta(n-1)$ is bounded by 1.0, a bound on ε is evaluated as

$$|\varepsilon| \leq \frac{0.5}{\hat{x}(n-1)} . \qquad (4.22)$$

For magnitude truncation quantization the bound on ϵ is larger by a factor of two, such that

$$|\varepsilon| \leq \frac{1.0}{\hat{x}(n-1)} . \qquad (4.23)$$

The coefficient a' of (4.7b) of the digital oscillator can now be used together with (3.59) to compute upper and lower bounds on the frequency of the limit cycle oscillations. This estimate is given by

$$\frac{f_o}{f_s} = \frac{1}{2\pi} \cos^{-1} \left(\frac{-a \pm |\varepsilon|}{2}\right). \qquad (4.24)$$

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[1] J. F. Kaiser, Bell Telephone Laboratories, Inc., Murray Hill, New Jersey 07971.

Up to this point the derivation of (4.24) has been exact. However, for the calculation of the bounds on the frequency a choice has to be made about which $\hat{x}(n-1)$ should be used for the calculation of ε from (4.22) or (4.23). If $\hat{x}(n-1) = 0$, no roundoff or truncation occurred and $\varepsilon = 0$. In general $\hat{x}(n-1)$ can have values between 1 and the maximum value of $\hat{x}(n)$. The difference in frequency between the linear oscillator and the quantized oscillator does not depend on the ε evaluated for one specific sample $\hat{x}(n-1)$ but on the average ε evaluated from all the samples in the limit cycle. For this reason $\hat{x}(n-1)$ in (4.22) or (4.23) has been chosen here as the average amplitude of the limit cycle which is estimated by

$$\overline{\hat{x}(n-1)} = \frac{2}{\pi} \cdot A, \tag{4.25}$$

where A is the amplitude of the limit cycle. From (4.22) - (4.25) it follows that the frequency of the quantized digital oscillator is bounded by

$$\frac{1}{2\pi} \cos^{-1} \left(-\frac{a}{2} + \frac{\Delta\pi}{4A}\right) < \left(\frac{f_o}{f_s}\right)_q < \frac{1}{2\pi} \cos^{-1} \left(-\frac{a}{2} - \frac{\Delta\pi}{4A}\right). \tag{4.26}$$

where $\Delta = 0.5$ for roundoff and $\Delta = 1.0$ for magnitude truncation.

The heuristic derivation of the frequency bounds has been verified by experiment. A comparison between the observed frequencies of the limit cycle oscillations and the computed values for the bounds indicates that the bounds are not exceeded for those examples considered in this

section. Next the simulation program for the digital oscillator is considered.

For the purpose of this section the difference equation (4.6) is implemented by a computer program, called Analysis Program for a Digital Oscillator. It is listed at the end of this dissertation. The program is written in FORTRAN IV. To avoid any errors due to number conversion from decimal to binary representation (compare with section IV.B), all arithmetic involved with solving (4.6) is done in an integer mode. This is achieved by scaling all decimal fractions until an integer results. By proper bookkeeping of the scale factors the correct arithmetic result is obtained.

Another important aspect of the computer implementation concerns the variation of the quantization step-size. For programming purposes it is easier to keep the quantization step-size constant and increase the initial conditions and thus the dynamic range of the oscillator response. That this kind of scaling is equivalent to varying the quantization step-size can be seen from the following simple example. Suppose (4.6) is solved for $a = -1.86$, $\hat{x}(1) = 0$ and $\hat{x}(2) = 1$ with roundoff at the second digit after the decimal point. Then $\hat{x}(3) = -[1.86]_r = -1.9$. Now, (4.6) is again solved for the given a and $\hat{x}(1)$, but for $\hat{x}(2) = 10$ and with roundoff at the first digit after the decimal point. Then $\hat{x}(3) = [18.6]_r = -19.0$, and, after scaling down by a factor of 10, the result from the second example equals the result from the first example.

After this introductory description of the program algorithm, the mathematical model used is now presented. Given values for the coefficient and the initial conditions $\hat{x}(1) = 0$ and $\hat{x}(2) = M$, the program performs a comparison between the responses of a linear, infinite precision digital oscillator and a quantized digital oscillator.

The pertinent characteristics of the linear oscillator, amplitude and frequency, are computed from the results of Chapter II as

$$\frac{f_o}{f_s} = \frac{1}{2\pi} \cos^{-1} \left(- \frac{a}{2} \right), \tag{4.27}$$

and

$$A = \frac{M}{\sin\left(\frac{2\pi f_o}{f_s}\right)} \quad . \tag{4.28}$$

To be able to compute frequency and amplitude of the quantized version of the digital oscillator some choices about the necessary approximation for the limit cycles have to be made. Since the resulting limit cycles are nearly sinusoidal a polynomial approximation has been ruled out immediately. A least square approximation has been preferred over a discrete Fourier series expansion because the first is easier to calculate. A discrete Fourier series expansion of the limit cycle is expected to show a broadening of the spectral line which represents this limit cycle. The width of the spectral line may then be a measure of

deviation from the ideal sinusoidal response. This subject has to be left for further research.

The least square approximation for the generated limit cycles is now considered. It is assumed that the samples of the limit cycle with period qT and frequency f_o/f_s

$$[\hat{x}(1) = 0, \hat{x}(2), \ldots, \hat{x}(q)] \tag{4.29}$$

are deformed versions of a sinusoidal response, described by

$$[0, \hat{A} \sin \omega_o T, \hat{A} \sin \omega_o 2T \ldots, \hat{A} \sin \omega_o qT]. \tag{4.30}$$

Then, the amplitude $\hat{A}$ of the sequence (4.30) can be estimated by minimizing the following cost function I, which corresponds to the sum of the squared differences between the members of the sequences (4.29) and (4.30),

$$I = \sum_{i=1}^{q} [\hat{x}(i) - \hat{A} \sin (i-1) \omega_o T]^2. \tag{4.31}$$

I is minimized, if

$$\frac{\partial I}{\partial \hat{A}} = 0, \tag{4.32}$$

and

$$\frac{\partial^2 I}{\partial \hat{A}^2} > 0. \tag{4.33}$$

$$\frac{\partial I}{\partial \hat{A}} = \sum_{i=1}^{q} [2\hat{A} \sin^2 (i-1)\omega_o T - 2\hat{x}(i) \sin(i-1)\omega_o T]. \tag{4.34}$$

$$\frac{\partial^2 I}{\partial \hat{A}^2} = 2 \sum_{i=1}^{q} \sin^2 (i-1)\omega_o T > 0. \tag{4.35}$$

From (4.35) it is seen that I is indeed minimized if the condition (4.32) is applied to (4.34). The estimated amplitude $\hat{A}$ obtained from (4.34) is

$$\hat{A} = \frac{\sum_{i=1}^{q} \hat{x}(i) \sin(i-1)\omega_o T}{\sum_{i=1}^{q} \sin^2(i-1)\omega_o T} . \tag{4.36}$$

The denominator of (4.36) can be simplified[1], such that

$$\hat{A} = \frac{2}{q} \sum_{i=1}^{q} \hat{x}(i) \sin(i-1)\omega_o T. \tag{4.37}$$

As is expected from a least square approximation, $\hat{A}$ is a weighted average. The average deviation from the amplitude, or the measure of deformation, is obtained by computing I from (4.31) and defining the measure of deformation as

$$\delta\hat{A} = \sqrt{\frac{I}{q}} . \tag{4.38}$$

With the numbers from the least square approximation of the limit cycle available, it is now possible to compare the ideal response from the linear digital oscillator with the limit cycle response from the quantized digital oscillator.

For a specified coefficient a and initial conditions

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1 See, for example, Hildebrand [35], p. 273.

$\hat{x}(1) = 0$ and $\hat{x}(2)$ (where $\hat{x}(2)$ varies between 1 and 1200) the absolute differences in frequency and amplitude are computed as

$$\mathrm{FDIFF} = \left| \left(\frac{f_o}{f_s}\right)_{\mathrm{linear}} - \frac{p}{q} \right|, \tag{4.39}$$

$$\mathrm{ADIFF} = |\bar{A} - \hat{A}|. \tag{4.40}$$

For the expression (4.39), $\left(\frac{f_o}{f_s}\right)_{\mathrm{linear}}$ is computed from (4.27), and p and q are integers obtained from the actual limit cycle by counting the number of sign-changes 2p between samples, and the period of the limit cycle q. For the expression (4.40), $\bar{A}$ is computed from (4.28) and $\hat{A}$ is computed from (4.37).

Computer runs have been performed for values of a between -1.10 and -1.90 and initial conditions where $\hat{x}(1) = 0$ and $\hat{x}(2)$ varies from 1 to 1200. For each specified set of numbers a, $\hat{x}(1)$ and $\hat{x}(2)$ the limit cycle response is obtained for roundoff quantization first and for magnitude truncation quantization second. The data is collected in tables B.17-B.32 in Appendix B. For explanatory purposes, the data for a = -1.1 is presented in Tables 4.2 and 4.3 at the end of this chapter. The columns in the tables are labelled as follows:

AMP = amplitude $\bar{A}$ of linear oscillator, evaluated from (4.28)

Q = period of limit cycle,

FDIFF = absolute difference in frequency, defined by (4.39),

ADIFF = absolute difference in amplitude, defined by (4.40),

DELTA = measure of deformation $\delta\hat{A}$, defined by (4.38).

If a limit cycle period qT is larger than q = 5000, the evaluation has been stopped and the entries in the table are arbitrarily set to q = 5000 and zero for the other variables.

A numerical example is now considered to illustrate the use of the tables B.17 - B.32. Suppose the following specifications for a digital oscillator are given:

f_o = oscillator frequency 1000 Hz ± 1Hz,

A = amplitude 1 ± 0.01,

q = 16 samples per oscillation period.

First the sampling frequency f_s is determined from (3.57b),

$$f_s = \frac{f_o q}{p} = \frac{1000 \cdot 16}{1} = 16 \text{ kHz}.$$

Then the oscillator coefficient a is computed using

$$\frac{f_o}{f_s} = \frac{p}{q} = \frac{1}{16}$$

and

$$a = -2 \cos \frac{2\pi p}{q} = -1.85.$$

If the oscillator is realized using roundoff quantization tables B.29 and B.31 apply. FDIFF is computed as

$$FDIFF = \frac{1}{16.10^3} = 0.625 \cdot 10^{-4}.$$

Entering table B.31, ($a = -1.9$), it is found that for $A \approx 1000$, $FDIFF = 0.2 \cdot 10^{-4}$ and the frequency specification is met, even considering a margin of safety. The corresponding value computed from the bound (4.26) is $0.16 \cdot 10^{-3}$. Scaling the amplitude by a factor of 10^{-3}, it is found that the amplitude specification is met, because the average deviation from the amplitude from table B.31 is 4.7 for $A \approx 1000$ or $0.0047 < 0.01$ for $A \approx 1$. For an amplitude of $A = 1$, the required quantization step-size is 10^{-3}. This is obtained from scaling the quantization step-size 1, on which the results in the table are based, by a factor of 1000. This corresponds to 3 significant digits if decimal arithmetic is employed or to $\frac{3}{\log 2} \approx 10$ significant bits if binary arithmetic is used. A check in table B.29 ($a = -1.8$) verifies the preceding approximate calculation. A check in tables B.30 and B.32 for magnitude truncation shows that more bits are needed to meet the frequency specification. For $A \approx 3000$ both the frequency and the amplitude specifications are met. This corresponds to 3.477 significant decimal digits or 12 significant bits if binary arithmetic is employed.

From the preceding discussions it is now possible to draw the following conclusions:

a) The frequency $\frac{f_o}{f_s} = \frac{p}{q}$ of a practical digital oscillator realization can only be realized in discrete steps.

This is due to the fact that the coefficient a, which determines the frequency, is represented by a finite number of digits (compare with Ch. II). In addition, a small offset from the nominal frequency is present due to quantization of product-terms. This offset is tabulated under FDIFF. The offset in frequency decreases approximately linearly with a corresponding increase in amplitude. This decrease is more rapid for roundoff than for magnitude truncation. For $A \approx 1000$ and a quantization step-size of 1, which corresponds to a quantization step-size of 10^{-3} if A is scaled down such that $A \approx 1.0$, the offset in frequency is about 10^{-5} for roundoff and about 10^{-4} for magnitude truncation as obtained from the tables of Appendix B. The corresponding values computed from the bound (4.26) are around 10^{-4} for roundoff and about $0.5 \cdot 10^{-3}$ for magnitude truncation.

b) The offset in amplitude is practically constant over the range of amplitudes considered. The corresponding offset, expressed in percent of the amplitude, decreases by a factor of 10 for every increase of the amplitude by a factor of 10.

c) The deformation of the limit cycle compared with the ideal sinusoidal response is, again, practically constant over the range of amplitudes considered. The small numerical values for the deformation verify the initial choice of a least square approximation for the limit cycle.

d) In summary, and as a consequence of the preceding remarks, it can be concluded that any degree of approximation

to an ideal sinusoidal response can be achieved with the quantized oscillator defined by (4.6). Thus, any elaborate scheme to reduce the deformation of the response (for example, resetting the oscillator as proposed by Gold and Rader [5]) is unnecessary.

E. SUMMARY

The results from three computer programs have been presented to verify the analytic results of the preceding chapters with experimental data.

The analysis program for zero-input limit cycles, i.e., the natural response of a two-pole digital filter, enumerated all possible limit-cycles for specified filter coefficients a and b for the initial conditions studied which have been chosen to cover a broad practical range. The limit cycles are tabulated and plotted on the successive value phase-plane. This data is compiled in Appendix B. The phase-plane plots showed a symmetry axis at 45° or 135°, depending whether $a < 0$ or $a > 0$. The phase-plane plots of the two sets of limit cycles for coefficients $-a$ and b and coefficients $+a$ and b are equal in shape. It is thus concluded that a bound computed for the first set of coefficients is equal to the bound computed for the second set of coefficients. In other words, a change in sign of the coefficient a does not change the value of the respective amplitude bound.

The recorded amplitudes of the limit cycles stay inside the bounds defined by (4.1), (4.2), and (4.5) as expected,

because their derivation is exact. Exceptions for the bound (4.3) are presented. It was concluded that the bound (4.2) should be used if worst-case information about the limit cycle amplitude is needed. The bound (4.4) is most easily applicable. However this bound can only be considered a rule-of-thumb. The numerical comparison of bounds (4.1) and (4.2), using data computed with the comparison program for the above mentioned bounds, shows that the bound (4.1) derived from one Lyapunov function of the digital filter, is primarily of theoretical interest because of its broadness. The results from the digital oscillator simulation program were employed to demonstrate that the stochastic approach for estimating the quantization noise due to round-off or truncation after multiplication as discussed by Rader and Gold [12] fails to agree with experiment. The interpretation of the experimental data shows that only a coherent or deterministic analysis can lead to useful results.

The data also shows that any degree of approximation of an ideal, i.e., infinite precision, digital oscillator can be achieved by reducing the quantization step-size. This is equivalent to specifying more significant digits for data and filter coefficients.

It seems appropriate to comment about the relation between sampling time T and the number of significant digits used to represent sampled data [40]. This was first mentioned in Chapter II, where finite representation of filter coefficients was studied. It was shown there that

sampling too fast may result in an undesirable response.

Consider two neighbouring samples of a function f(nT). Connect the sample values by a straight line with a slope given by

$$\frac{\Delta f(n)}{T} = K = \text{constant}, \qquad (4.41)$$

where $\Delta f(n) = f(n) - f(n-1)$ denotes the first forward difference of the function f(nT). If finite representation of numbers is used, only those changes in the functional values can be distinguished which are larger than the quantization step-size determined by the number of significant digits. Denote the quantization step-size by Δ. It is reasonable to require that

$$\Delta < \Delta f(n)_{max}. \qquad (4.42)$$

For a sinusoidal signal with amplitude A = 1, and $(\frac{\Delta f(n)}{T})_{max} = 1$, it follows that $\Delta < T$ or $f_s < \frac{1}{\Delta}$. From this simple calculation it can be concluded, that it is useless to sample a normalized sinusoid faster than with sampling frequency $f_s = \frac{1}{\Delta}$. The general relation between sampling frequency f_s and quantization step-size Δ is derived from (4.51) and (4.42) as

$$f_s = \frac{K}{\Delta}, \qquad (4.43)$$

where K = maximum slope between samples. The quantization step-size chosen is inversely proportional to the sampling frequency.

As an area for further research reconsider the approximation problem for the limit cycles generated by the digital oscillator. For the treatment of this chapter a least square approximation has been chosen and the numerical results have verified this choice. However, it would be interesting to represent the limit cycles by discrete Fourier series and relate the expected spectral broadening to the deviation of the observed limit cycles from an ideal sinusoidal response.

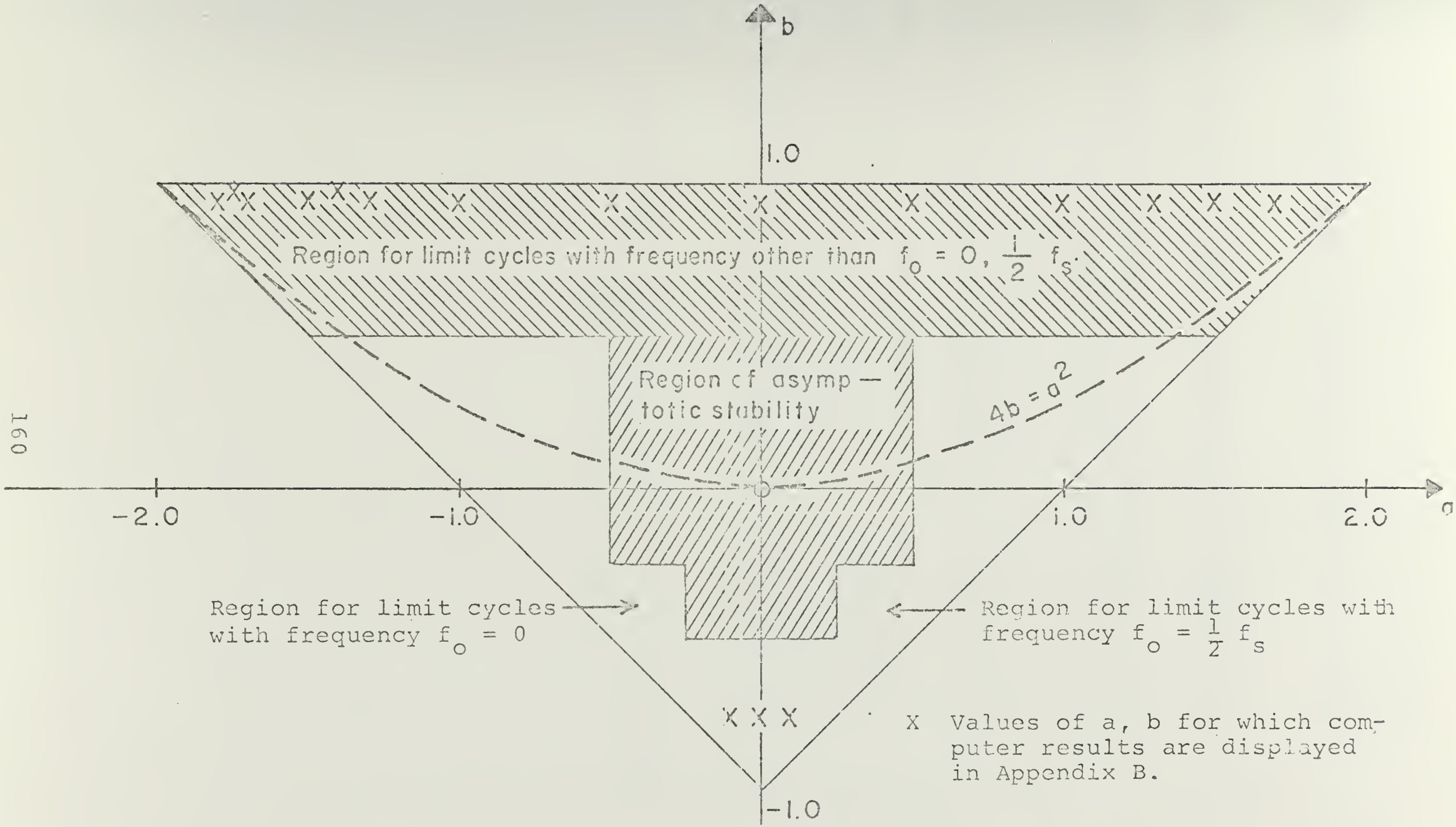


Fig. 4.1: Parameter Plane with Regions of Stability for Second-order Digital Filter with Round-off.

```
LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

A=-1.800000000, B= 0.937000000, APPROXIMATE FREQUENCY F= 0.060006
THE AMPLITUDE BOUNDS ARE A 1=   7.937,A 2=   7.299


LIMIT CYCLE #    1WITH FREQUENCY F= 6.250000E-02 IS
   4   1  -2  -5  -7  -8  -7  -6  -4  -1   2   5   7   8   7   6

LIMIT CYCLE #    2WITH FREQUENCY F= 7.142857E-02 IS
  -6  -5  -3   0   3   5   6   6   5   3   0  -3  -5  -6

LIMIT CYCLE #    3WITH FREQUENCY F= 7.142857E-02 IS
  -5  -4  -2   0   2   4   5   5   4   2   0  -2  -4  -5

LIMIT CYCLE #    4WITH FREQUENCY F= 8.333333E-02 IS
  -4  -3  -1   1   3   4   4   3   1  -1  -3  -4

LIMIT CYCLE #    5WITH FREQUENCY F= 7.142857E-02 IS
  -3  -2  -1   0   1   2   3   3   2   1   0  -1  -2  -3

LIMIT CYCLE #    6WITH FREQUENCY F= 0.000000E+00 IS  -2

LIMIT CYCLE #    7WITH FREQUENCY F= 0.000000E+00 IS  -1

LIMIT CYCLE #    8WITH FREQUENCY F= 0.000000E+00 IS   0

LIMIT CYCLE #    9WITH FREQUENCY F= 0.000000E+00 IS   1

LIMIT CYCLE #   10WITH FREQUENCY F= 0.000000E+00 IS   2
```

Table 4.1: Zero-input Limit Cycle Oscillations for a =-1.8, b = 0.937.

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A=-1.800000000, B= 0.937000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8																		1			
7																	1		1		
6															1	2	2				
5														2	3	3	2	1			
4												1	3	4	4	3					
3											2	4	5	5	4	2					
2											3	5	10	5	3	1					
1									1	4	5	9	5	4							
0								2	3	5	8	5	3	2							
-1								4	5	7	5	4	1								
-2						1	3	5	6	5	3										
-3						2	4	5	5	4	2										
-4						3	4	4	3	1											
-5				1	2	3	3	2													
-6					2	2	1														
-7			1		1																
-8				1																	
-9																					
-10																					

Fig. 4.2: Phase-plane Plot for Digital Filters with a = -1.8, b = 0.937.

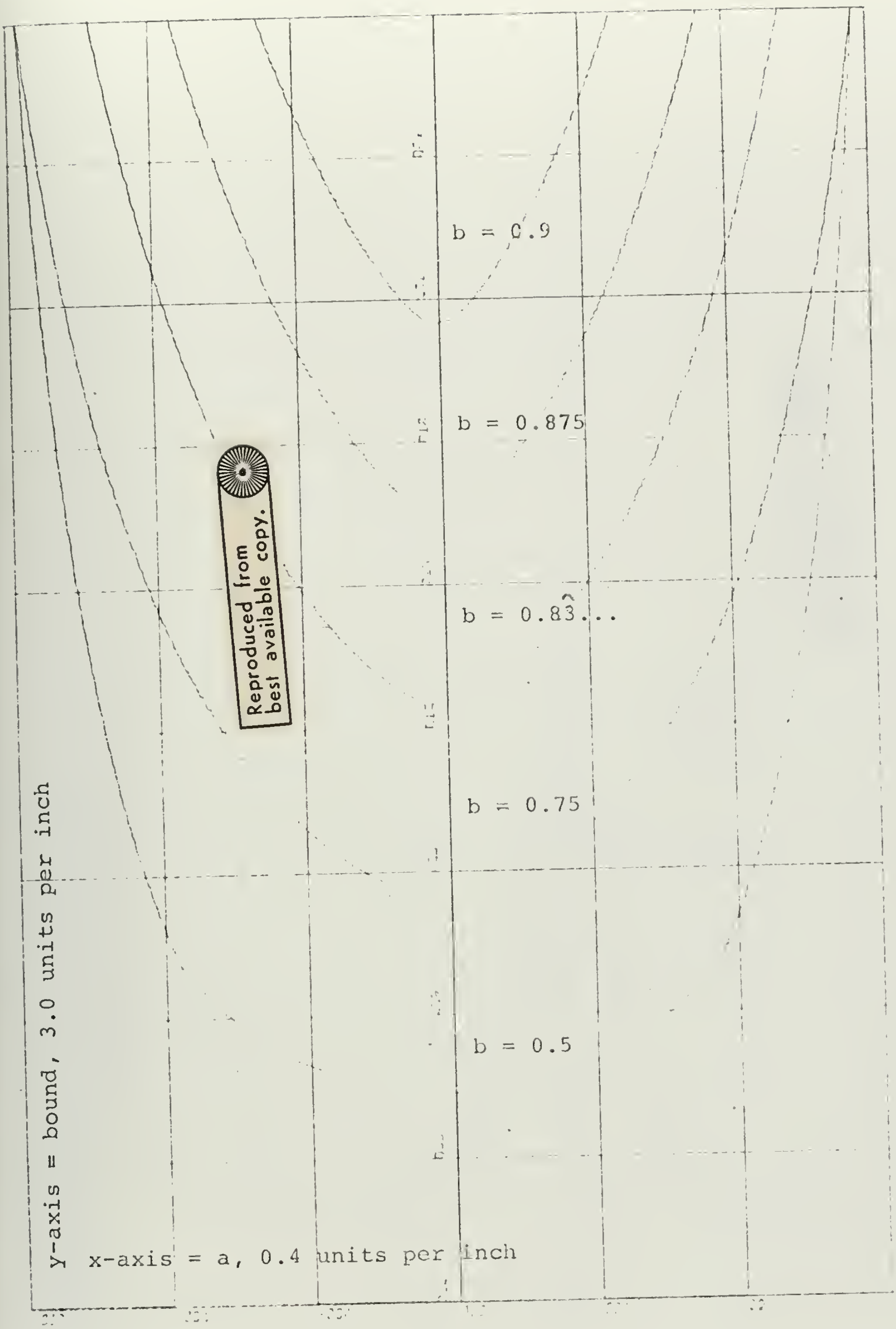
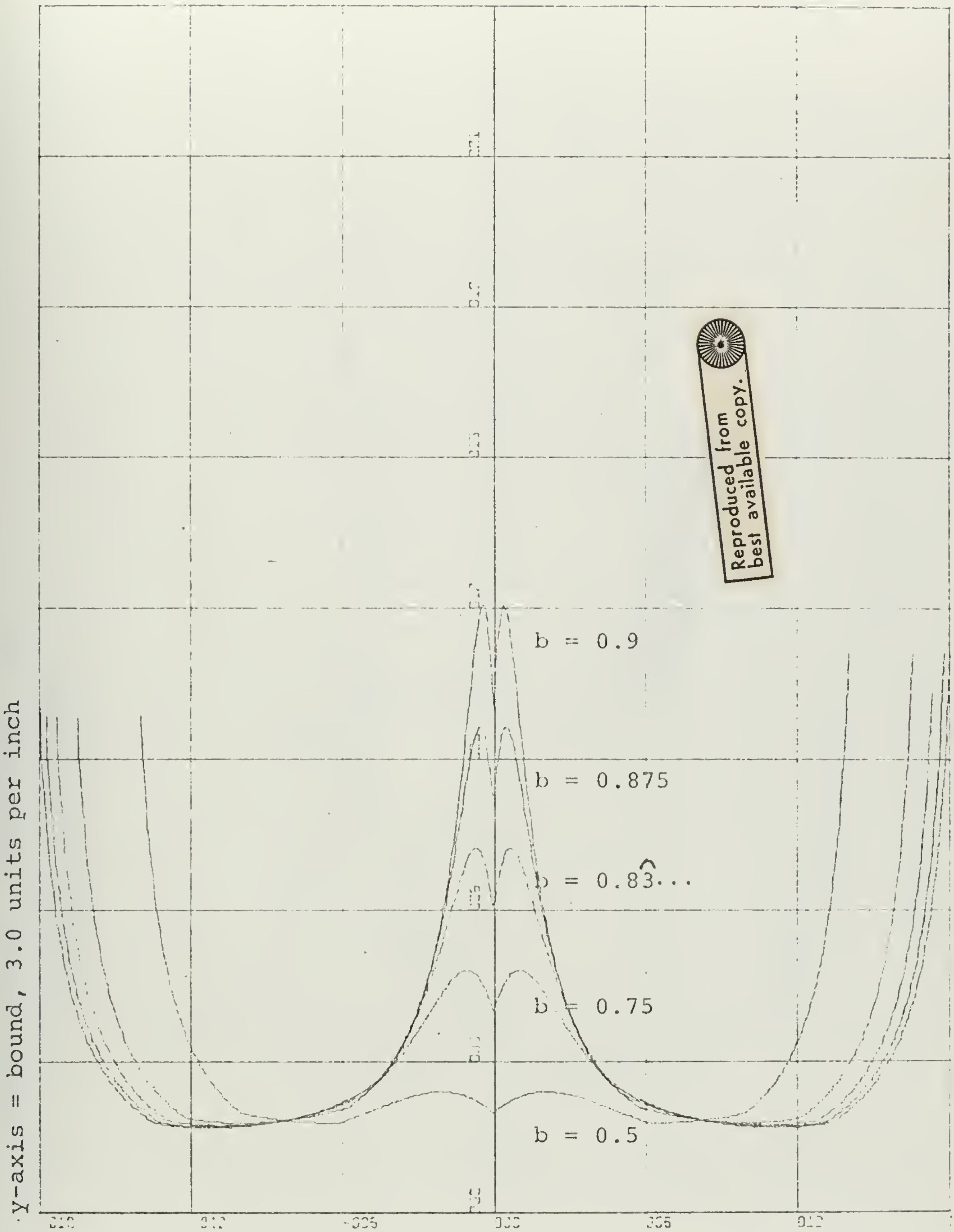


Fig. 4.3: Amplitude Bound (Lyapunov)



x-axis = a, 0.6 units per inch

Fig. 4.4: Amplitude Bound for Period of Limit Cycle q = 4.
(See Eqn. (3.43))

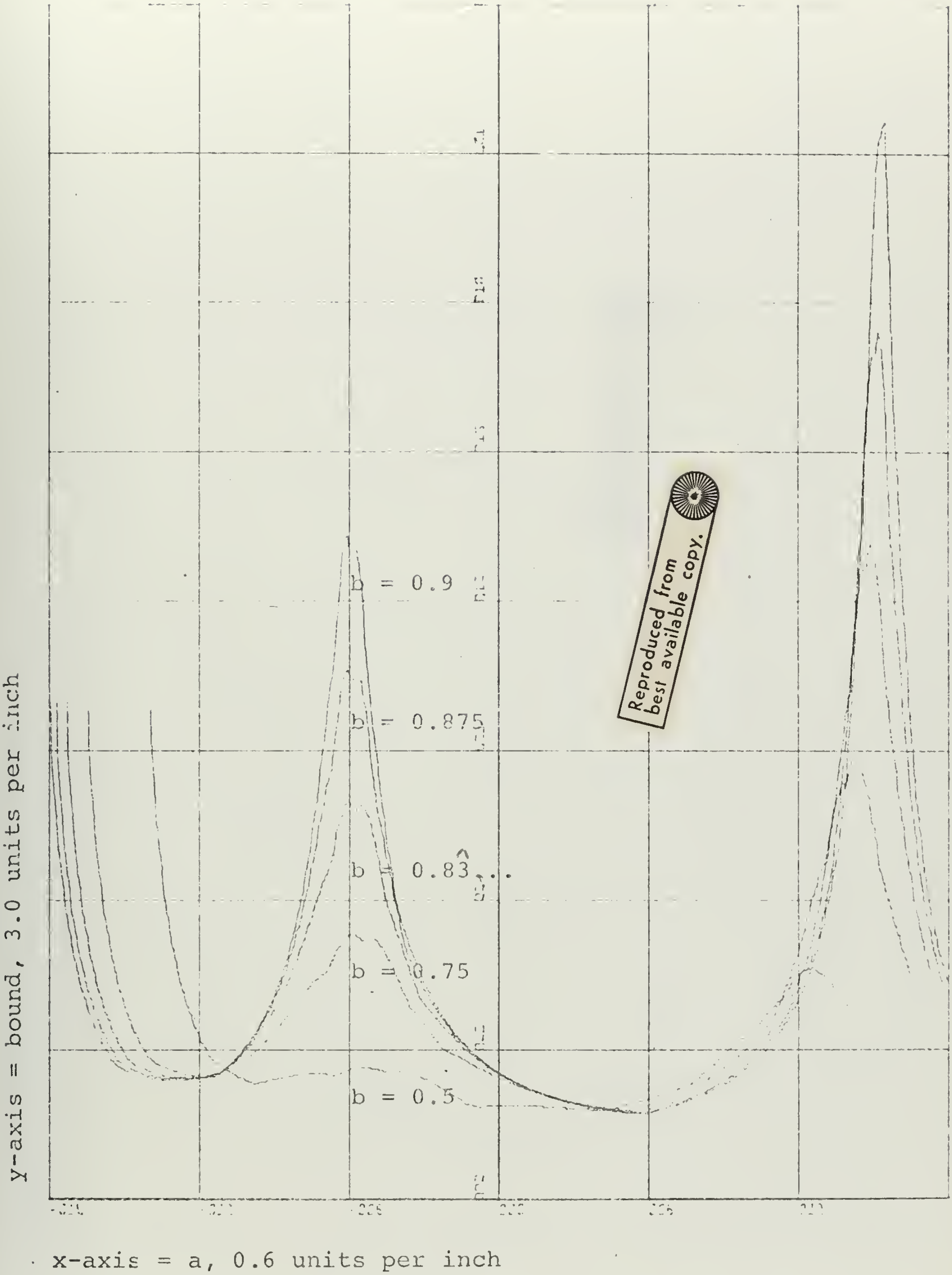


Fig. 4.5: Amplitude Bound for Limit Cycle of Period q = 5.
(See Eqn. (3.43))

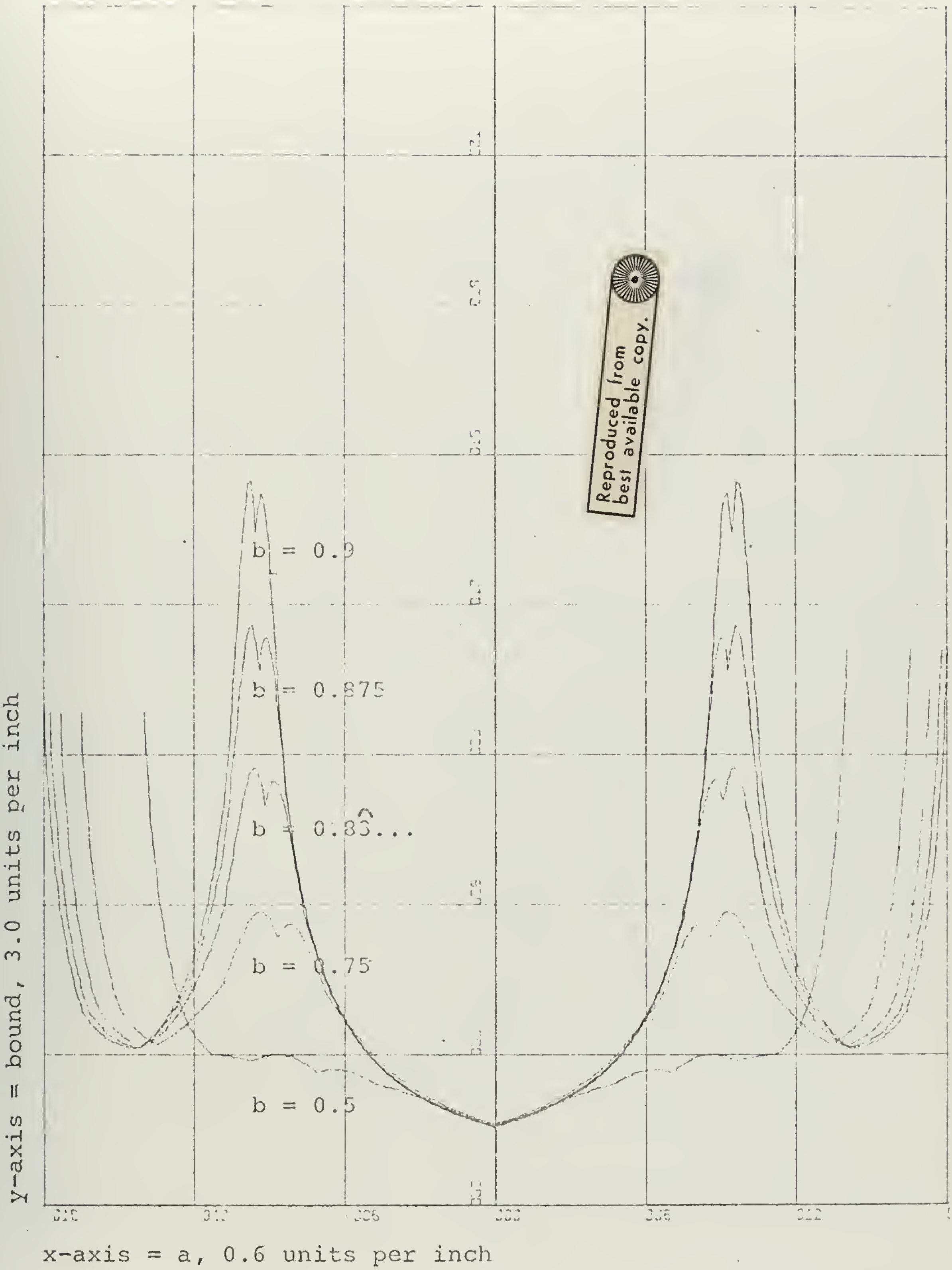


Fig. 4.6: Amplitude Bound for Period of Limit Cycle q = 6.
(See Eqn. (3.43))

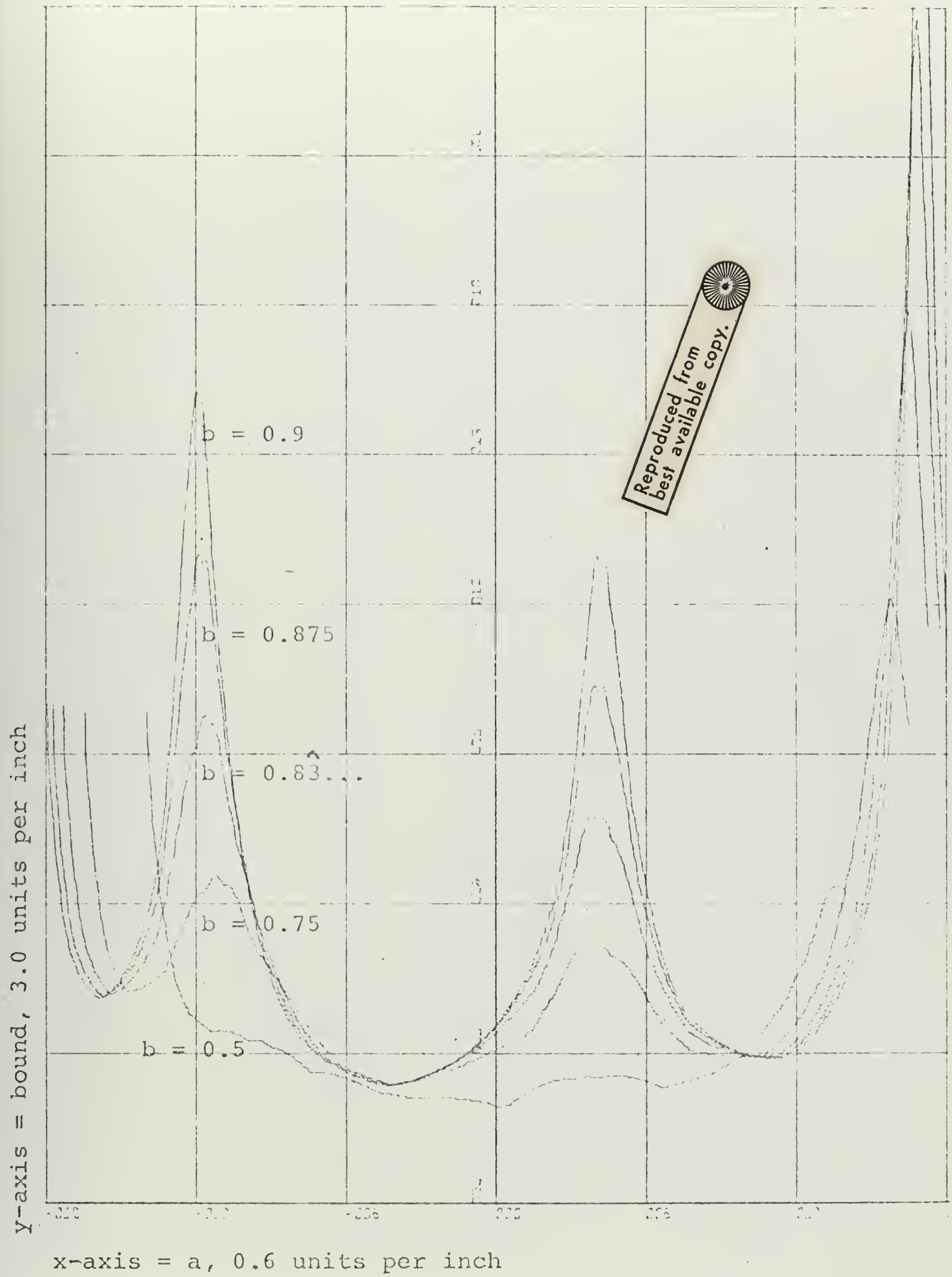
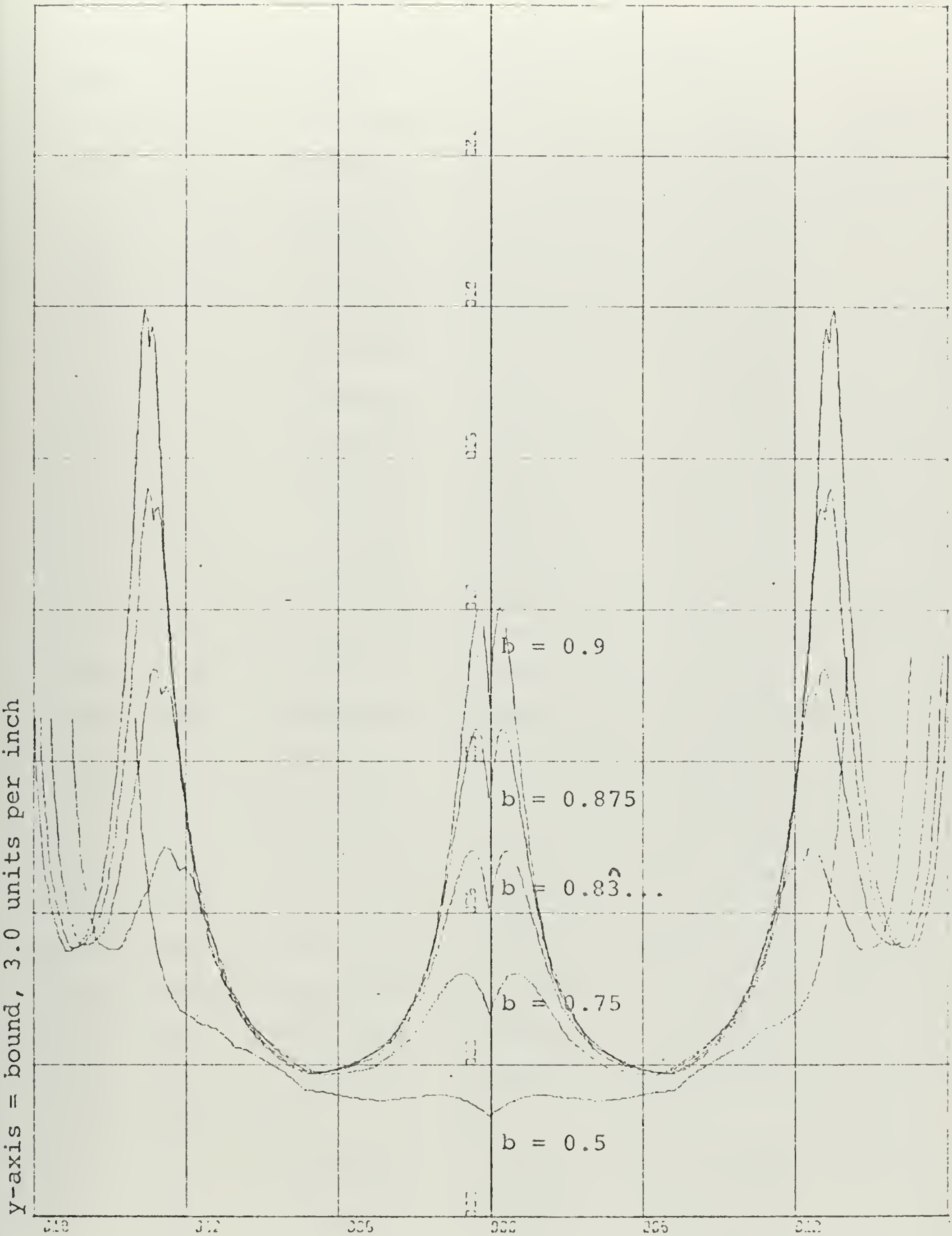


Fig. 4.7: Amplitude Bound for Period of Limit Cycle $c = 7$.
(See Eqn. (3.43))



x-axis = a, 0.6 units per inch

Fig. 4.8: Amplitude Bound for Period of Limit Cycle q = 8. (See Eqn. (3.43))

ROUND-OFF QUANTIZATION ANALYSIS, A =-1.10

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.935E-02	0.954E-06	0.120E-05
2.31	6.	0.935E-02	0.191E-05	0.260E-05
3.46	6.	0.935E-02	0.286E-05	0.379E-05
4.62	6.	0.935E-02	0.572E-05	0.565E-05
6.08	13.	0.347E-02	0.142E 00	0.238E 00
7.29	13.	0.347E-02	0.426E 00	0.275E 00
8.46	58.	0.214E-02	0.740E 00	0.349E 00
9.62	32.	0.106E-02	0.498E 00	0.330E 00
10.78	70.	0.171E-03	0.721E 00	0.315E 00
11.95	19.	0.581E-03	0.487E 00	0.360E 00
23.96	70.	0.171E-03	0.910E 00	0.466E 00
35.99	51.	0.451E-03	0.144E 00	0.332E 00
47.96	172.	0.337E-03	0.114E 00	0.515E 00
59.91	70.	0.171E-03	0.396E 00	0.475E 00
71.89	70.	0.171E-03	0.834E 00	0.483E 00
83.81	286.	0.287E-04	0.100E 00	0.486E 00
95.86	70.	0.171E-03	0.133E 00	0.398E 00
107.84	70.	0.171E-03	0.393E 00	0.332E 00
119.78	159.	0.812E-04	0.451E 00	0.807E 00
239.51	426.	0.369E-04	0.961E 00	0.723E 00
359.23	178.	0.106E-04	0.225E-01	0.637E 00
478.98	2918.	0.144E-04	0.129E 01	0.256E 01
598.72	2918.	0.144E-04	0.472E 01	0.315E 01
718.45	1246.	0.106E-04	0.369E 01	0.233E 01
838.19	89.	0.106E-04	0.408E 00	0.426E 00
0.0	5000.	0.0	0.0	0.0
1077.68	89.	0.106E-04	0.927E 00	0.651E 00
1197.39	1888.	0.459E-05	0.477E 01	0.270E 01
1317.13	3420.	0.399E-05	0.946E 01	0.720E 01
1436.91	178.	0.106E-04	0.146E 01	0.998E 00

Table 4.2: Digital Oscillator Analysis.

TRUNCATION QUANTIZATION ANALYSIS, A =-1.10

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.935E-02	0.954E-06	0.120E-05
2.31	6.	0.935E-02	0.191E-05	0.260E-05
3.46	6.	0.935E-02	0.286E-05	0.379E-05
4.62	6.	0.935E-02	0.572E-05	0.565E-05
5.77	6.	0.935E-02	0.763E-05	0.728E-05
6.93	6.	0.935E-02	0.572E-05	0.781E-05
8.08	6.	0.935E-02	0.134E-04	0.108E-04
9.24	6.	0.935E-02	0.114E-04	0.107E-04
10.39	6.	0.935E-02	0.143E-04	0.127E-04
11.81	56.	0.340E-02	0.393E 00	0.344E 00
23.77	44.	0.178E-02	0.311E 00	0.231E 00
35.71	63.	0.142E-02	0.308E 00	0.516E 00
47.62	63.	0.142E-02	0.889E 00	0.446E 00
59.67	234.	0.806E-03	0.392E 01	0.181E 01
71.67	114.	0.581E-03	0.155E 01	0.829E 00
83.62	114.	0.581E-03	0.204E 01	0.900E 00
95.56	19.	0.581E-03	0.808E 00	0.390E 00
107.51	19.	0.581E-03	0.129E 01	0.536E 00
119.45	38.	0.581E-03	0.298E 00	0.814E 00
239.26	146.	0.220E-03	0.176E 00	0.500E 00
358.97	3372.	0.159E-03	0.610E 00	0.156E 01
478.72	4402.	0.115E-03	0.356E 01	0.445E 01
598.45	972.	0.935E-04	0.757E 00	0.138E 01
718.20	1042.	0.757E-04	0.392E 01	0.152E 01
837.93	1328.	0.656E-04	0.158E 00	0.108E 01
957.67	3514.	0.566E-04	0.135E 01	0.214E 01
1077.42	394.	0.465E-04	0.281E 01	0.131E 01
1197.14	394.	0.465E-04	0.684E-01	0.111E 01
0.0	5000.	0.0	0.0	0.0
0.0	5000.	0.0	0.0	0.0

Table 4.3: Digital Oscillator Analysis.

V. THE FORCED RESPONSE

A. INTRODUCTION

The natural or zero-input response of the quantized digital filter from Fig. 3.4 has been evaluated in the preceding chapters. The linear equivalent of this filter has two poles and no zeros in its transfer function. The forced response of general digital filters with both poles and zeros is now analyzed with regard to possible limit cycle oscillations. This analysis is performed in several steps.

First, the step response of the two-pole digital filter is considered. As a new result it is shown that the forced steady-state response contains two components. The one component is a constant which is determined by the size of the step input and the loop gain of the filter. The other component is a limit cycle oscillation which is related to the zero-input limit cycles described in section IV.B.

Next, the forced response of the same filter type is considered for general, deterministic input signals. As an important new result it is shown that the driven case can be reduced to a zero-input case if the difference between the response of the quantized digital filter and the corresponding linear digital filter is considered. This difference signal is described by a limit cycle oscillation whose amplitude is estimated by the same bounds which have been derived in Chapter III for the zero-input response. In the same context it is shown that both roundoff and truncation quantization lead to limit cycle oscillations.

Using this result, it is concluded that roundoff should be preferred over magnitude truncation because the errors due to magnitude truncation are up to twice as large as the errors due to roundoff. The conclusions of this section are tested by computer simulation of a quantized digital filter with sinusoidal inputs of varying amplitude and frequency. As expected, the amplitude of the limit cycles is independent of the sinusoidal input and remains inside the bounds derived in Chapter III. The frequency of the limit cycles is unpredictable, but seems in most cases to stay inside a band determined on the one end by the resonant frequency of the filter and on the other end by the lowest zero-input limit cycle frequency.

The second-order digital filter section with both zeros and poles in the transfer function of the equivalent linear filter is studied in the following section. This general case is important because all practical filters have zeros in their transfer function. The zeros are shown not to change the nature of the limit cycle. However, they influence the magnitude of the limit cycle amplitude. As a new result it is shown that for specified zeros the magnitude of the limit cycles in the output of the digital filter can be minimized through a proper choice of the filter configuration. If the zeros are located in the right half of the unit circle in the z-plane then the configurations S_{a_1}, S_{a_2}, S_{a_3} are to be preferred over their transpose counterparts $S^T_{a_1}$, $S^T_{a_2}$, $S^T_{a_3}$. If the zeros are in the left half of the unit circle the reverse is true.

Finally, higher order digital filters of the cascade and the parallel form are considered with regard to limit cycles in their outputs. A bound on the amplitude of the limit cycles in the output of the parallel form is stated. A similar result for the filter of the cascade form can be formulated. Each limit cycle output from one filter subsection is the input to the next subsection and is filtered there and in the subsequent sections. To estimate the filtering action of the different stages in the filter cascade to which the limit cycle is input, it is necessary to know the frequency of the limit cycle with sufficient accuracy. This information about the frequency is only available for zero-input limit cycles. For limit cycles which are generated as part of the forced response, it has not been possible to estimate the frequency with reasonable accuracy. This problem remains for future consideration.

B. STEP INPUT TO THE TWO-POLE DIGITAL FILTER

As an introductory example, the step response of the two-pole digital filter from Fig. 3.4 is now considered. It is shown in the later sections that the result obtained for this simple filter configuration can be extended to more complex configurations, including those with zeros in the transfer function. The difference equation defining the digital filter from Fig. 3.4 is repeated here for convenience:

$$\hat{x}(n) = [a\,\hat{x}(n-1)]_q - [b\,\hat{x}(n-1)]_q + u(n). \qquad (5.1)$$

For a step input

$$u(n) = c, \tag{5.2}$$

where c is a signed constant.

The response of the corresponding linear infinite precision digital filter is composed of a decaying sinusoid, superimposed on a constant, whose value is determined by the loop gain $(\frac{1}{1+a+b})$ of the filter. It is therefore reasonable to expect that the step response of the quantized digital filter is composed of a constant amplitude limit cycle superimposed on the above mentioned constant value. That this is indeed the case is shown theoretically and has been verified by computer simulation.

First, the step response of (5.1) is evaluated for roundoff quantization. Then, a similar development is outlined for magnitude truncation quantization. For simplicity, assume that $a < 0$, $b > 0$. The development for $a > 0$, $b > 0$ leads to the same result. Furthermore, assume that the steady-state response of (5.1) contains two separate components, such that for some integer N

$$\hat{x}(n) = A + x_o(n), \quad n \geq N. \tag{5.3}$$

Here the finite precision components of $\hat{x}(n)$ are defined as

A = constant,

$x_o(n)$ = samples of the limit cycle around A.

Substituting (5.3) into (5.1) yields

$$A + x_o(n) = -[a(A+x_o(n-1))]_r - [b(A+x_o(n-2))]_r + c$$

$$= -a[A+x_o(n-1)] \pm (0.5-\delta(n-1))$$

$$-b[A+x_o(n-2)] \pm (0.5-\delta(n-2))+c. \quad (5.4)$$

Separating the constant response from the limit cycle response in (5.4) results in two equations. They are:

$$A = \frac{c}{1+a+b}, \quad (5.5)$$

$$x_o(n) = -a\, x_o(n-1) \pm (0.5- \delta(n-1)) - b\, x_o(n-2)$$

$$\pm (0.5-\delta(n-2)). \quad (5.6)$$

Equation (5.5) is recognized as the steady-state response of the corresponding linear filter. Equation (5.6) defines a limit cycle which is equivalent to the zero-input limit cycles discussed in Chapter III and, therefore, all the amplitude bounds discussed there apply to the limit cycles defined by (5.6). It is important to note, however, that the roundoff sequences $\delta(n)$ are not only a function of $x_o(n)$, but also a function of A, however, they are always bounded by 1.0. The frequency of the limit cycle defined by (5.6) is therefore a nonlinear function of the digital filter coefficients a and b and of the input and output of the filter.

The computer simulation of the digital filter (5.1) has verified these results. The limit cycles which occur,

together with the constant bias (5.5), are essentially but not exactly the same as those which occur for zero-input to the filter.

It has been impossible to predict the limit cycle number, as defined for the zero-input case, for a limit cycle in the output of the filter for a given constant value c of the input u(n). For a variation of the input value c over a range of several hundred units a weak pattern in the repeated occurance of certain limit cycles is visible. However, due to the many exceptions from this pattern, it has been impossible to postulate any law relating the input value c to the occurrence of a particular limit cycle.

The development stated for roundoff quantization is now repeated for magnitude truncation quantization. Using identical assumptions as for roundoff, (5.4) is changed to

$$A + x_o(n) = -a[A + x_o(n-1)] - \delta(n-1)$$

$$-b[A + x_o(n-2)] + \delta(n-2) + c. \qquad (5.7)$$

Separating the constant response from the limit cycle response results in similar equations as derived for roundoff. The nature of the step response of a digital filter employing magnitude truncation is equivalent to the step response of the same filter employing roundoff. The amplitude bounds discussed in Chapter III are again applicable.

C. GENERAL INPUTS TO TWO-POLE DIGITAL FILTERS

The forced response of the two-pole digital fi[illegible]

Fig. 3.4 for a general input signal u(n) is studied in this section. The analysis of the forced response is reduced to the analysis of a zero-input response by subtracting the forced responses of the quantized and the linear digital filter and studying the resulting difference signal. The forced response of the linear digital filter is given by

$$x(n) = -a\ x(n-1) - b\ x(n-2) + u(n). \tag{5.8}$$

The forced response of the quantized digital filter employing roundoff is given by

$$\begin{aligned}\hat{x}(n) = -a\ \hat{x}(n-1) &\pm [0.5 - \delta(n-1)] \\ -b\ \hat{x}(n-2) &\pm [0.5 - \delta(n-2)] + u(n).\end{aligned} \tag{5.9}$$

The difference signal d(n) between $\hat{x}(n)$ and x(n) is defined as

$$d(n) = \hat{x}(n) - x(n). \tag{5.10}$$

Subtracting (5.8) from (5.9), one obtains for d(n)

$$\begin{aligned}d(n) = -a\ d(n-1) &\pm [0.5-\delta(n-1)] \\ -b\ d(n-2) &\pm [0.5 - \delta(n-2)],\end{aligned} \tag{5.11}$$

where $\delta(n-1)$ and $\delta(n-2)$ are numbers, such that $0 \leq |\delta| < 1.0$. Equation (5.11) is identical in form to (3.14), which defined the zero-input limit cycles studied in Chapter III. However, the roundoff sequences $\delta(n-1)$ and $\delta(n-2)$ are not only a function of d(n), but also a function of $\hat{x}$(n)

This can be seen from (5.9). The latter conclusion has no bearing on the amplitude bounds, which were derived for (3.14) in Chapter III. Their derivation did not include any assumption about the specific nature of the roundoff sequences. The only condition used is stated with (3.14) and requires that the roundoff sequences are bounded. Thus, the amplitude bounds derived for (3.14) are directly applicable to the limit cycles defined by (5.11). However, the above mentioned conclusion influences the frequency of the limit cycle because the frequency is now a nonlinear function of the digital filter coefficients a and b, the amplitude of the limit cycle and, additionally, the input signal u(n). For zero-input limit cycles the frequency has been approximated by the expression (3.59), derived from the corresponding linear filter. This approximation is no longer valid for the limit cycles contained in the forced response due to the nonlinear dependence on the input signal u(n).

The foregoing development is now repeated for magnitude truncation quantization. The forced response of the quantized digital filter employing magnitude truncation is given by

$$\begin{aligned}\hat{x}(n) = -a\,\hat{x}(n-1) &\pm \delta(n-1)\\ -b\,\hat{x}(n-2) &\pm \delta(n-2) + u(n).\end{aligned} \tag{5.12}$$

Subtracting (5.8) from (5.12) to obtain the difference signal d(n) as defined by (5.10) yields

$$d(n) = -a\, d(n-1) \pm \delta(n-1)$$

$$-b\, d(n-2) \pm \delta(n-2), \tag{5.13}$$

where $\delta(n-1)$ and $\delta(n-2)$ are numbers, such that $0 < |\delta| < 1.0$. Equation (5.13) is identical in form to (3.68). However, the truncation noise sequences $\delta(n)$ are a function of $\hat{x}(n)$, as can be seen from (5.12).

For the zero-input case (3.68) it has been shown in Chapter III that limit cycles with frequencies f_o/f_s, such that $0 < f_o/f_s < \frac{1}{2}$, cannot exist for magnitude truncation quantization. On the other hand, such limit cycles can exist in the driven case. This is now demonstrated by showing that the coefficient b of (5.13) in general has an effective value of $b = 1$ and limit cycle oscillations can be sustained. Suppose $d(n-2)$ in (5.13) is a negative number and $\hat{x}(n-2)$ in (5.12) is a positive number. The coefficient b has in general values, such that $0 < b < 1.0$. It follows from (5.12) that $\delta(n-2)$ must be a positive number. From (5.13) the effective value for b is defined as b', such that

$$-b\, d(n-2) + \delta(n-2) = -b' d(n-2). \tag{5.14}$$

From (5.14) it is seen that there are many values $0 < b < 1.0$, $|d(n-2)| \geq 1.0$ and $0 \leq \delta(n-2) < 1.0$ which satisfy (5.14) such that $b' = 1$. In other words, from (5.14) and using the bound on $\delta(n-2)$, it is deduced that $b' = 1$ for values of $d(n-2)$ (and therefore, $d(n)$ since $d(n-2)$ is just

delayed version of d(n))such that

$$1.0 \leq |d(n-2)| \leq \frac{1.0}{1-b} . \tag{5.15}$$

If (5.15) is satisfied limit cycles have to be expected. The same conclusion is reached for the case where $d(n-2) > 0$ and $\hat{x}(n-2) < 0$.

That this is indeed the case has been verified by computer simulation. While it has been impossible to predict the frequency of the limit cycle defined by (5.13), the amplitude bounds derived in Chapter III still apply, if a multiplication factor of two is included. This is necessary because the magnitude truncation error sequences are up to twice as large as the corresponding roundoff error sequences. The preceding discussion shows that magnitude truncation quantization offers no advantages over roundoff quantization if suppression of the possible limit cycles is contemplated.

The results of this section were tested for sinusoidal inputs of various amplitudes and frequencies. The linear operation of the digital filter is approximated in the simulation program by employing double precision (16 significant digits), floating-point arithmetic. The difference signal d(n), defined by (5.10), is recorded for n = 1 to 4608. After computation of 512 samples for d(n), it is assumed that the steady-state condition has been reached and the following 4096 samples are then used to compute an estimate of the discrete Fourier Transform for d(n) by employing the Fast Fourier Transform (FFT) algorithm

[36]. The discrete Fourier spectrum of d(n) has been computed because initial experiments indicated that it is impractical to deduce the several possible frequency components of the generated limit cycle by counting the number of sign-changes of the samples and the period of the limit cycle.

For roundoff quantization and one set of filter coefficients a = -1.87, b = 0.95, results concerning the amplitude of the limit cycles are recorded in the following table. The input signal to the digital filter is described by $u(n) = A_i \sin 2\Pi f_i nT$.

A_i	Range of f_o/f_s	Range of $\|d(n)\|_{max}$	Median of $\|d(n)\|_{max}$ for 11,13 or 14 different input frequencies
1	0.000977 to 0.28	2.56 to 12.14	7.1
10	0.000977 to 0.25	1.79 to 15.34	10.0
10	0.038 to 0.051	3.14 to 14.76	8.6
100	0.000977 to 0.25	3.13 to 13.89	9.8
1,000	0.000977 to 0.25	4.38 to 15.01	8.5
1,000	0.036 to 0.050	6.14 to 12.19	10.9
100,000	0.000977 to 0.25	1.78 to 13.90	9.6

As expected, the amplitude of the limit cycles is independent of the input signal. Using the two most easily computable amplitude bounds from Chapter III, (3.53) from the effective value linear model yields

$$|d(n)| \leq 10, \tag{5.16}$$

and (3.55) for the exceptions from the effective va...

linear model yields

$$|d(n)| \leq 30. \tag{5.17}$$

The available data, obtained for various representative values for the filter coefficients a and b, indicates that the experimental and theoretical results for the amplitude of the limit cycles are verified.

The experimental results for magnitude truncation are summarized in the following table:

A_i	Range of f_o/f_s	Range of $\|d(n)\|_{max}$	Median of $\|d(n)\|_{max}$ for 11 different input frequencies
10	0.000977 to 0.25	0.86 to 11.33	9.8
1,000	0.000977 to 0.25	3.20 to 11.39	8.8

The experimentally obtained amplitudes of the limit cycles are well within the computed amplitude bounds, which are 20 (computed from (3.53)) and 60 (computed from 3.55)).

The inspection of the data representing the discrete Fourier spectrum of the difference signal d(n) presents a confusing picture. One or many spectral lines may represent a limit cycle without any apparent pattern concerning the appearance and number of recognizable spectral components. Most, but not all of the frequency components in the spectrum, appear to be within a band defined by

$$\frac{f_r}{f_s} \leq \frac{f_o}{f_s} \leq \frac{f_{\ell c}}{f_s} , \tag{5.18}$$

where

f_r = resonant frequency of the digital filter,

f_o = frequency component in the spectrum representing d(n),

$f_{\ell c}$ = lowest limit cycle frequency from the zero-input case.

Furthermore, if the frequency of the input signal, f_i, and the sampling frequency, f_s, are related by an integer I, such that

$$\frac{f_s}{f_i} = I, \tag{5.19}$$

then a limit cycle containing only one strong spectral component can be expected most of the time (but not always). In addition, harmonics of either the input or the dominant component of the limit cycle can sometimes be observed. It has to be left for further research to develop a better understanding of the frequency of the limit cycles generated by either truncation or roundoff quantization in the presence of a forcing function to the digital filter. Despite the lack of understanding of the frequency of the limit cycles in the forced response, a new result about the amplitude of the limit cycles has been formulated in this section. If limit cycles are considered unwanted noise, then their magnitude can be reduced below a specified threshold value by relating the derived amplitude bounds to the quantization step-size. In other words, using a worst-case design, the necessary number of significant digits can now be

specified so that the limit cycle response is effectively suppressed.

Before this conclusion can be stated more formally, however, it is necessary to extend the results for the two pole filter of Fig. 3.4 to the general case of digital filters with both poles and zeros in the transfer function.

D. THE INFLUENCE OF ZEROS IN THE TRANSFER FUNCTION ON THE LIMIT CYCLES

So far, the natural and the forced response has been studied for the two-pole digital filter as depicted in Fig. 3.4. This restriction arises naturally because the limit cycle oscillations of the quantized digital filter are defined completely by a knowledge of the poles or eigenvalues of the filter. However, all practical digital filter realizations contain zeros in their transfer function. The zeros do not change the nature of the limit cycle. However, they introduce additional gain, which may change the magnitude of the limit cycles.

In this section, the influence of the zeros on the limit cycles of a second-order digital filter section is considered. The treatment is restricted to the six most important and most often used digital filter configurations [2]. They have been derived in Chapter II and are represented by the S-matrices S_{a_1}, $S_{a_1}^T$, S_{a_2}, $S_{a_2}^T$, S_{a_3}, $S_{a_3}^T$. First, consider the configuration S_{a_1}, which is depicted in Fig. 2.4. Define the output of the leftmost summing node as $\hat{y}(n)$. Then, with quantization after multiplication

operations included

$$\hat{y}(n) = -[a\,\hat{y}(n-1)]_q - [b\,\hat{y}(n-2)]_q + u(n). \qquad (5.20)$$

Equation (5.20) is identical to the difference equation considered in the preceding section. It describes the limit cycle oscillations completely.

The output of the digital filter $\hat{x}(n)$ is obtained as

$$\hat{x}(n) = [c\,\hat{y}(n-1)]_q + [e\,\hat{y}(n-2)]_q. \qquad (5.21)$$

If the bound on limit cycles defined by (5.20) is designated by A_1 and roundoff quantization is assumed then the bound A_2 on the limit cycles in the filter output $\hat{x}(n)$ is evaluated from (5.21). Designate $\hat{x}_{\ell c}(n)$ as the limit cycle of the response $\hat{x}(n)$ and $\hat{y}_{\ell c}(n)$ as the limit cycle of the response $\hat{y}(n)$. The limit cycle $\hat{x}_{\ell c}(n)$ has the biggest possible amplitude, if $\hat{y}_{\ell c}(n) = \hat{y}_{\ell c}(n-1) = \hat{y}_{\ell c}(n-2)$. From (5.21) one obtains

$$\hat{x}_{\ell c}(n) = c\,\hat{y}_{\ell c}(n-1) \pm 0.5 \mp \delta(n-1)$$

$$+ e\,\hat{y}_{\ell c}(n-2) \pm 0.5 \mp \delta(n-2), \qquad (5.22a)$$

and

$$|\hat{x}_{\ell c}(n)| \leq |c+e|\,|\hat{y}_{\ell c}(n)| + 1.0. \qquad (5.22b)$$

This can be rewritten as

$$A_2 \leq |c+e|\, A_1 + 1.0. \qquad (5.22c)$$

An identical result is obtained for the configuration S_{a_2} (see Fig. 2.4), because S_{a_1} and S_{a_2} differ only by the value

of the filter coefficient d, which does not enter into the result (5.22c). As can be seen from Fig. 2.4, d = 0 for S_{a_1} and d = 1 for S_{a_2}.

The configuration S_{a_3} is depicted in Fig. 2.6. Following the same steps as for the preceding derivation, it is found that

$$\hat{x}(n) = \hat{y}(n) + [(a+c)\ \hat{y}(n-1)]_q + [(b+e)\ \hat{y}(n-2)]_q. \tag{5.23}$$

If a limit cycle occurs and its amplitude is bounded by A_1, the largest possible contribution of this limit cycle in the filter output is designated A_2 and is bounded by

$$A_2 \leq \ |1+(a+c) + (b+c)|\ A_1 + 1.0. \tag{5.24}$$

Next, the three transpose configurations $S^T_{a_1}$, $S^T_{a_2}$ and $S^T_{a_3}$ are considered. The configuration $S^T_{a_1}$ is depicted in Fig. 2.5. The output of the digital filter is described by the difference equation

$$\hat{x}(n) = -[a\ \hat{x}(n-1)]_q - [b\ \hat{x}(n-2)]_q +$$

$$+[c\ u(n-1)]_q + [e\ u(n-2)]_q. \tag{5.25}$$

Combining the two input terms into one term, designated u'(n), such that

$$u'(n) = [c\ u(n-1)]_q + [e\ u(n-2)]_q, \tag{5.26}$$

(5.25) can be rewritten as

$$\hat{x}(n) = -[a\ \hat{x}(n-1)]_q - [b\ \hat{x}(n-2)]_q + u'(n). \tag{5.27}$$

Equation (5.27) is identical in form to (5.1), which was investigated in the preceding sections. The amplitude bound from Chapter III applies directly to a limit cycle from (5.27). An inspection of Fig. 2.5 for the configuration $S_{a_2}^T$ and Fig. 2.7 for configuration $S_{a_3}^T$ indicates that the conclusion stated above is also applicable for those configurations if the input terms are properly redefined. Redefine $x_1(n)$ for configuration $S_{a_2}^T$ (see Fig. 2.5) as

$$\hat{y}(n) = x_1(n). \tag{5.28}$$

Then it is found from Fig. 2.5 that

$$\hat{y}(n) = -[a\ \hat{y}(n-1)]_q - [b\ \hat{y}(n-2)]_q + [c\ u(n-1)] + + [e\ u(n-2)]. \tag{5.29}$$

The output of the filter is given by

$$\hat{x}(n) = \hat{y}(n) + u(n). \tag{5.30}$$

If the limit cycle amplitude is bounded by A_1, then the contribution A_2 of this limit cycle in the filter output is given by

$$A_2 = A_1. \tag{5.31}$$

Similarly, for configuration $S_{a_3}^T$ (see Fig. 2.7) one obtains

$$\hat{x}(n) = -[a\hat{x}(n-1)]_q - [b\ \hat{x}(n-2)]_q + u(n) + + [(a+c)\ u(n-1)]_q + [(b+e)\ u(n-2)]_q. \tag{5.32}$$

Equation (5.32) can be rewritten into (5.27) if the three input terms are combined into one term u'(n), such that

$$u'(n) = u(n) + [(a+c)\ u(n-1)]_q + [(b+e)\ u(n-2)]_q . \tag{5.33}$$

As has been stated earlier, the amplitude bounds from Chapter III apply directly to a limit cycle from (5.32).

At this point one might well ask what effect the quantization of the input sequences in (5.26) and (5.33) has on the performance of the digital filter. Clearly, it has no influence on the generation of limit cycles. On the other hand, the quantization of the products in (5.26) and (5.33) produces errors, which are best described by a statistical analysis. This has been done successfully by Jackson [2] and others. A comparison of (5.22), (5.24), and (5.31) shows that the size of the limit cycle contribution in the output of the filter can be minimized by a proper choice of configuration.

For example if a Butterworth or Chebyshev low-pass filter[1] is considered, whose zeros occur at $z = -1$, then for configurations S_{a_3} and $S_{a_3}^T$ the coefficients

$$a+c = 2.0, \tag{5.34a}$$

$$b+e = 1.0. \tag{5.34b}$$

From (5.24) and (5.31) it is seen that $S_{a_3}^T$ should be preferred over S_{a_3}. With similar arguments it can be deduced

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[1] For this to be true, it is assumed that the bilinear transformation has been used to obtain $H(z)$ from $H(s)$.

that for zeros in the left half of the unit circle in the z-plane the configurations $S_{a_1}^T$, $S_{a_2}^T$, $S_{a_3}^T$ are generally preferable compared with configurations S_{a_1}, S_{a_2}, S_{a_3}.

If a Butterworth or Chebyshev high-pass filter[1] is considered whose zeros occur at $z = 1$, then an identical treatment as above leads to the conclusion that configuration S_{a_3} should be preferred over $S_{a_3}^T$. For zeros in the right half of the unit circle in the z-plane the configurations S_{a_1}, S_{a_2}, S_{a_3} are generally preferable compared with configurations $S_{a_1}^T$, $S_{a_2}^T$, $S_{a_3}^T$.

In summary, the results of this important section allows one to state an amplitude bound for that part of the output of a general second-order digital filter, which is contributed by the limit cycle oscillations generated in that filter. Depending on the numerical values of the zeros in the transfer function some filter configurations are preferable over others if the magnitude of the generally unwanted limit cycles is to be minimized.

E. HIGHER ORDER DIGITAL FILTERS

It has been shown in Chapter II that higher order digital filters are generally realized by either a cascade or a parallel form which is composed of second-order subsections. These two forms are now investigated with respect to their limit cycle behaviour.

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[1] For this to be true, it is assumed that the bi[illegible]near transformation has been used to obtain $H(z)$ fr[illegible].

First, consider the parallel form. From Fig. 2.2, it is seen that all second-order subsections share a common input and the output is summed over all sections. Thus, each section generates its own limit cycle and all these limit cycles are then summed together to form part of the output. If the bound on the amplitude of the limit cycle generated in section G_i is given by A_i, then a bound A of the sum of all the limit cycles in the overall output is given by

$$A \leq \sum_{i=1}^{M} A_i . \tag{5.35}$$

The situation is more difficult for a cascade form. From Fig. 2.3, it is seen that the output of each section feeds as input into the next section. Therefore, a limit cycle generated in section G_i is input to section G_{i+1} and gets filtered there and in all subsequent sections. If it is assumed that the limit cycle oscillation generated in filter section G_i is practically sinusoidal, then this limit cycle can be approximated by

$$x_o(n) = A_{1i} \sin n\, \omega_i T. \tag{5.36}$$

The magnitude A_{2i} of that portion of the overall output, which is due to the limit cycle oscillation from section G_i is then given by

$$A_{2i} = A_{1i} \prod_{k=i}^{M} \left| H_k(e^{j\omega_i T}) \right| . \tag{5.37}$$

To evaluate (5.37) it is necessary to know the frequency

$\frac{f_i}{f_s}$ for the limit cycle under consideration with sufficient accuracy.

Accurate information about the frequency is only available for zero-input limit cycles. For limit cycles, which are generated as part of the forced response, it has not been possible to estimate the frequency with reasonable accuracy. This has been pointed out in the preceding sections B and C. For the cascade form of digital filters, the magnitude of that portion of the overall output, which is due to limit cycle oscillations from the individual filter subsections can in general not be evaluated with acceptable accuracy.

However, if the passbands of the individual filter sections are well separated, then it is safe to assume that only the limit cycles generated in the last section of a cascade are of significance. This observation suggests that filter sections with equal or nearly equal passbands should not be put in cascade with each other, because the nearly sinusoidal limit cycles of the first stage will most likely be enhanced in the next stage.

F. SUMMARY

The results of the preceding chapters are generalized in this chapter, such that they are applicable as design guides for practical digital filters. First, the step response and then the forced response for general, but deterministic inputs was evaluated for the two-pole di ital

filter. It was shown that the forced responses of the quantized and the linear digital filter differ by a signal, which is described by a limit cycle oscillation. The amplitude of this limit cycle is estimated by the same bounds which have been developed in Chapter III for the zero-input case.

The results for the two-pole filter are then extended to the case of a general second-order digital filter, whose linear equivalent has both zeros and poles in the transfer function. It was shown that the part of the filter output which is due to limit cycles can be minimized by a proper choice of digital filter configurations. If the zeros of the filter are in the right half of the z-plane, the configurations S_{a_1}, S_{a_2}, S_{a_3} are to be preferred over the transpose configurations $S^T_{a_1}$, $S^T_{a_2}$, $S^T_{a_3}$. If the zeros of the filter are in the left half of the z-plane, configurations $S^T_{a_1}$, $S^T_{a_2}$, $S^T_{a_3}$ are to be preferred over configurations S_{a_1}, S_{a_2}, S_{a_3}.

Finally, limit cycles in higher order filters were considered. The parallel form is easy to analyze, because the limit cycles generated in the individual filter subsections are added in the output. The cascade form is more difficult to analyze, because each limit cycle is filtered in the following filter subsections. If the passbands of the cascaded filter subsections are well separated only the limit cycles generated in the last stage are of significance.

There remains an important area for which further search

is necessary. It has been impossible to predict the frequency of the limit cycles which are present in the forced response of the digital filter. This information is needed to evaluate the magnitude of the limit cycle output of a digital filter realized by the cascade form.

VI. CONCLUSIONS AND SUGGESTIONS FOR FURTHER RESEARCH

The purpose of this research has been to contribute to the analytic solution of the phenomenon called limit cycle oscillations in recursive digital filters with finite precision arithmetic. Despite the fact that these limit cycles can become large as a function of the quantization step-size, they can be made arbitrarily small by increasing the number of significant digits of the data. This increase results in an increased dynamic range with a corresponding improvement of the signal-to-noise ratio in the filter output, where the limit cycle signal is the noise.

The existence of limit cycles should be viewed as an important consideration to be included in the design and implementation phases for digital filters. Building a recursive digital filter involves several steps, as follows:

a) The approximation problem.

b) Specification of the number of significant digits for the filter coefficient (this is not necessarily equivalent to step (e) which follows).

c) Selection of a filter configuration.

d) Choice of the arithmetic mode which is most likely to be fixed-point arithmetic if economical special purpose hardware is designed.

e) Specification of the number of significant digits for the data.

These steps are interdependent. Since no optimal procedures have been devised which encompass several of the above steps and would lead to a unique solution, it is

necessary to go through design steps (a) through (e) several times until a satisfactory realization of a digital filter has been formulated. Design decisions, to be made at step (c) and (e), are influenced by consideration of the possible, undesired limit cycles.

As an example, consider the worst-case design of a second-order digital module. For simplicity, the influence of the zeros in the transfer function on the magnitude of the limit cycles is neglected (in practice, the influence of the zeros will most likely result in a reduction of the limit cycles; see Chapter V).

Assuming that intermediate limit cycles with frequencies $\frac{f_o}{f_s}$, such that $0 < \frac{f_o}{f_s} < \frac{1}{2}$, are a problem, the amplitude bound (4.4) is selected as design guide. The signal-to-noise ratio in terms of the number of significant digits k for the data can be written as

$$\left(\frac{S}{N}\right)_{db} \cong 20 \log \frac{A_{signal}}{A_{\ell c}} = 20 \log \frac{10^k(1-b)}{1.5}. \qquad (6.1)$$

For the evaluation of (6.1) it is assumed that the signal is kept at its maximum level of 10^k units, and the amplitude of the limit cycle is bounded by $\frac{1.5}{1-b}$, from (4.4). The number of decimal digits k, needed for a specified signal-to-noise ratio, can be related to the number of binary digits K for a binary realization through the relation

$$2^K = 10^k, \text{ or} \qquad (6.2)$$

$$K = \frac{k}{\log_{10} 2} = \frac{k}{0.30103} \text{ [bit]}. \qquad (6.3)$$

Some numerical examples for the evaluation of (6.1) and (6.3) are stated in the following table:

$(S/N)_{db}$	K[bit] for coefficient b=0.9	K[bit] for coefficient b=0.99
30	8.9	12.2
60	13.9	17.2
90	18.9	22.2
120	23.9	27.2

From (6.1) and (6.3) it can be deduced that, practically speaking, the existence of limit cycles in digital filters is not a limiting factor on their realization, but only an important consideration to be included in the implementation phase of these filters.

The major contributions of this reasearch are now summarized. In Chapter II, the linear model for second-order digital filters was described and 24 canonical circuit configurations, under the assumption of k-digit accuracy, were shown to exist. A general formula to predict changes in pole positions due to changes from the nominal values of coefficients because of finite representation of numbers was derived. Chapter III is the most important chapter of this dissertation. Many new results about zero-input limit cycle oscillations due to roundoff were presented. It was shown that the limit cycles can be expressed by a matrix equation which was then used to derive an absolute bound for the amplitude of the limit cycles. Other

amplitude bounds were derived employing Lyapunov functions and the effective value linear model devised by Jackson [27]. However, it was also demonstrated, that the latter bound can be exceeded by several quantization units and is, therefore, only a convenient to apply rule-of-thumb. The limit cycles were portrayed in a specially defined successive value phase-plane plot. Some symmetry properties of the state trajectories were shown to exist. An approximate expression for the frequency of the limit cycles was derived. Finally, it was shown that magnitude truncation quantization cannot sustain zero-input limit cycles other than with frequencies $f_o/f_s = 0, \frac{1}{2}$. The analytic results of Chapter III were verified in Chapter IV with experimental data obtained from three computer programs. The analysis program for zero-input limit cycles enumerates all possible limit cycles for specified filter coefficients and tabulates and plots them on the successive value phase-plane. The second program evaluates the different amplitude bounds to facilitate a comparison between the bounds. The third program simulates a digital oscillator which can be used for digital function generation. The data shows that any degree of approximation of an ideal sinusoid can be achieved which is contrary to a formula derived by Rader and Gold [12]. The results of the preceding chapters were generalized in Chapter V. The forced response for general, but deterministic inputs was evaluated for the two-pole filter. The results for the two-pole filter were then extended to

to the case of a general second-order digital filter with both zeros and poles in the transfer function. It was shown that the magnitude of the limit cycles can be minimized by a proper choice of the digital filter configuration. Limit cycles in higher order digital filters, either realized by the parallel or the cascade form, were considered. Finally, it was concluded, that magnitude truncation quantization offers no advantages over roundoff quantization because the errors due to magnitude truncation can be twice as large as those due to roundoff.

There remain several unanswered problems which have arisen as a result of this research and should be noted for further investigation. Among them are the following:

a) A topological investigation of canonical second-order digital filters to determine a relation between the minimum number of delays, multipliers and adders and the order of the difference equation of a canonical digital filter.

b) A general definition and derivation of all possible canonical forms for the second-order digital filter under the restriction of k-digit accuracy (see Eqs. (2.13a-d)).

c) An investigation of the roundoff noise properties of the twelve newly derived second-order digital filter configurations S_{b_1} through S_{b_6} and their transposes (see section II.C).

d) An exact analytical expression for the frequency of limit cycles generated in the presence of a driving function.

e) A general relationship between the sampling period T and the number of significant digits required to represent data, i.e., a relationship between the precision of the data if it is to approximate a continuous signal in some specified sense.

f) A spectral analysis of the limit cycles of a digital oscillator, i.e., the frequency spectrum of the limit cycle and a measure of its deviation from a pure sinusoid.

g) An optimum design procedure which starts with the filter specifications and results in an optimum implementation, complete with coefficient truncation and selection of a "best" configuration.

APPENDIX A

AN UPPER BOUND ON THE DYNAMIC RESPONSE OF NONAUTONOMOUS (FORCED) DISCRETE SYSTEMS USING LYAPUNOV'S DIRECT METHOD

A. INTRODUCTION

The region of boundedness of the dynamic response of the discrete system with constant coefficient matrices A and B

$$x(n+1) = A\,x(n) + B\,u(n), \tag{A.1}$$

is studied in this appendix. A bound on the dynamic response of the system (A.1) utilizing Lyapunov's direct method is presented. The method of proof is based on a paper by Johnson [21] in which a bound on the quantization error in sampled-data control systems is derived. Correcting and clarifying comments by Lack [22] and Johnson [22] are used in the proof to improve the bound.

The importance of the theorem which is stated in the next section rests on the fact that an absolute bound on the magnitude of the dynamic response is obtained without reference to the detailed nature of the input to the system which must only be bounded.

B. THEOREM: A BOUND ON THE DYNAMIC RESPONSE OF FORCED DISCRETE SYSTEMS

It is well known [37] that the linear discrete system (A.1) is bounded-input bounded-output (BI-BO) stable, if the homogeneous system

$$x(n+1) = Ax(n) \tag{A.2}$$

is asymptotically stable in the large (ASIL). In this case (A.2) possesses a Lyapunov function $V = x^T Qx$, where Q is a real, symmetric and positive definite matrix and can be found for any real, symmetric, positive definite matrix C from the equation

$$-C = A^T QA - Q. \tag{A.3}$$

If the input to the system (A.1) is bounded by a constant k_1 as

$$|u(n)| \leq k_1, \tag{A.4}$$

then the following theorem which estimates the region of boundedness of the dynamic response can be stated.

Theorem: For the system $x(n+1) = Ax(n) + Bu(n)$, if the homogeneous system is ASIL and has a Lyapunov function $V = x^T Qx$ with $\Delta V = -x^T Cx$ and $|u(n)| \leq k_1$ for all $n \geq 0$, then the system is stable and the states are certain to enter a region defined by

$\|x\| \leq r_2$, where

$$r_2 = k_1 \cdot \sqrt{\frac{\lambda \max(Q)}{\lambda \min(Q)}} \cdot \left[\frac{\|A^T QB\|}{\lambda \min(C)} + \sqrt{\frac{\|A^T QB\|}{\lambda^2 \min(C)} + \frac{B^T QB}{\lambda \min(C)}} \right]. \tag{A.5}$$

Here

$\lambda \min(C)$ = min. eigenvalue of the matrix C,

$\lambda \min/\max\ (Q)$ = min./max. eigenvalues of the matrix Q,

$\|A^TQB\|$ = norm of the matrix product A^TQB,

$\|x\|$ = norm of the state vector x.

C. PROOF OF THE THEOREM:

The system (A.1) is stable as mentioned in section B, and proven in a paper by Kalman and Bertram [37].

Before the rest of the theorem can be proven it is necessary to state a lemma, which is given by LaSalle and Lefschetz [38] for continuous time systems and is here rewritten and proven for discrete-time systems.

Lemma: Consider the discrete system

$$x(n+1) = Ax(n) + Bu(n), \tag{A.6}$$

with V being the Lyapunov function for (A.6). Define sets of points in n-space, where n is the dimension of the system, as

$$M = \{x: \|x\| \leq r_1\},$$

$$M^c = \text{complement of } M = \{x: \|x\| > r_1\}$$

$$M_r = \{x: r_1 < \|x\| < r_2\},$$

$$M_r^c = \text{complement of } M_r + M = \{x: \|x\| \geq r_2\}. \tag{A.7}$$

These regions are depicted in Fig. A.1. If $\Delta V \leq 0$ for all x in M^c and if $V[x(n_2), n_2] > V[x(n_1), n_1]$ for all $n_2 \geq n_1 \geq 0$, all $x(n_1)$ in M and all $x(n_2)$ in M_r^c, then each state of (A.6) which at some time n_o, such that $n_1 > n_o \geq 0$, is in M, can never thereafter leave M_r.

To prove the lemma let $x(n_o)$ be a state of (A.6) which at $n_o \geq 0$ is in M. Assume that at some later time $N > n_o$, $x(N)$ is in M_r^c.

If this is the case, then there exists a n_1, where $n_o < n_1 < N$, such that $x(n)$ is in M^c for all n such that $n_1 \leq n < N$ and that n_1 is the smallest number with this property.

This implies that $x(n_1)$ is in M and therefore by hypothesis $V[x(n_1), n_1] < V[x(N),N]$. But this increase in V is contrary to the hypothesis that $\Delta V < 0$ for all $x(n)$ in M^c. Therefore if the state $x(n_o)$ is in M at some time n_o it can never thereafter leave M_r.

It is important to recognize that if the state of this system is in M at a time n_o, it is not necessarily true that the state of the system remains in M for all subsequent time. The lemma only asserts that the state remains inside the region M_r.

The previous lemma will now be employed to derive the bound on the dynamic response of the system (A.1). Use $V = x^TQx$ as a Lyapunov function for the forced system. Then the change in V is computed from the forward difference as

$$\Delta V = V[x(n+1)] - V[x(n)] =$$

$$= -x^T C x + 2x^T A^T Q B u + u^T B^T Q B u. \qquad (A.8)$$

Since $|u| \leq k_1$ and, for $\|x\|$ sufficiently large, the sign of ΔV will be dominated by the term $-x^T C x$, which is negative definite. It follows, that there exists a region of points in n-space, M^c, within which $\Delta V < 0$. Here, M^c is the complement of M, as defined before by (A.7).

To estimate r_1 it is recognized that $B^T Q B$ is a positive scalar because Q is chosen according to (A.3) as positive definite. To assure that $\Delta V < 0$, u(n) is replaced by its bound k_1 and $2A^T Q B$ is assumed to be positive. Then $\Delta V < 0$ if

$$B^T Q B \, k_1^2 + |2x^T A^T Q B k_1| \leq x^T C x. \qquad (A.9)$$

For a real, symmetric, positive definite matrix, it is known that[1]

$$\min \, (x^T C x) = \lambda_{min}(C) \, \|x\|^2. \qquad (A.10)$$

Substituting (A.10) into (A.9) one gets

$$|2x^T A^T Q B k_1| \leq \lambda_{min}(C) \, \|x\|^2 - B^T Q B k_1^2, \qquad (A.11)$$

which implies the inequality (A.9). Introducing norms of matrices[2], (A.11) is rewritten as

---

[1] See, for example Bellman [39], p. 110-113.

[2] See, for example, Kalman and Bertram [37], p. 102-103.

$$2\,\|x\|\,\|A^TQB\|\,k_1 \leq \lambda_{min}(C)\,\|x\|^2 - B^TQBk_1^2. \tag{A.12}$$

If the equality-sign in (A.12) is used a bound r_1 on the region M in which $\Delta V < 0$, is defined by

$$\lambda_{min}(C)\,r_1^2 - 2k_1\,\|A^TQB\|\,r_1 - B^TQBk_1^2 = 0 \tag{A.13}$$

Solving the quadratic equation (A.13) for the one positive root the bound on the region M is found to be

$$r_1 = k_1\left[\frac{\|A^TQB\|}{\lambda_{min}(C)} + \sqrt{\frac{\|A^TQB\|^2}{\lambda^2_{min}(C)} + \frac{B^TQB}{\lambda_{min}(C)}}\right]. \tag{A.14}$$

In (A.14) only the positive sign of the root can be used, because $r_1 \geq 0$. However, as was pointed out earlier, r_1 cannot be the absolute bound for the dynamic response of system (A.6) because it is not assured that the state remains in M, once it got there. It, therefore, remains to evaluate a correction factor to properly estimate the boundary of the region M_r, which the state will never leave, once it has been in the region M.

Since Q is real, symmetric and positive definite it follows that there exists

$$\max_{x\epsilon M} V = \max_{\|x\|=r_1} [x^TQx] = r_1^2\;\lambda_{max}(Q). \tag{A.15}$$

If r_2 is defined as $\|x\| = r_2$, where

$$\min_{x\epsilon M_r^C} V = \max_{x\epsilon M} V, \text{ then} \tag{A.16}$$

$$r_2^2 \lambda_{min}(Q) = r_1^2 \lambda_{max}(Q), \tag{A.17}$$

From (A.17) it follows that the correction factor to properly estimate the boundary of the region M_r is given by

$$r_2 = r_1 \cdot \sqrt{\frac{\lambda_{max}(Q)}{\lambda_{min}(Q)}} . \tag{A.18}$$

Now the conditions of the lemma are satisfied, because $\Delta\dot{V} < 0$ for all x in M^c and furthermore

$$V[x\varepsilon M_r^c] > \max_{x\varepsilon M} V = r_1^2 \lambda_{max}(Q), \tag{A.19}$$

and each state, which at some time was in $\bar{M}$ can never thereafter leave M_r. Thus r_2 is the desired bound such that

$$\|x\| \leq r_2 = k_1 \cdot \sqrt{\frac{\lambda_{max}(Q)}{\lambda_{min}(Q)}} \cdot \left[\frac{\|A^TQB\|}{\lambda_{min}(C)} + \sqrt{\frac{\|A^TQB\|^2}{\lambda^2_{min}(C)} + \frac{B^TQB}{\lambda_{min}(C)}}\right]. \tag{A.20}$$

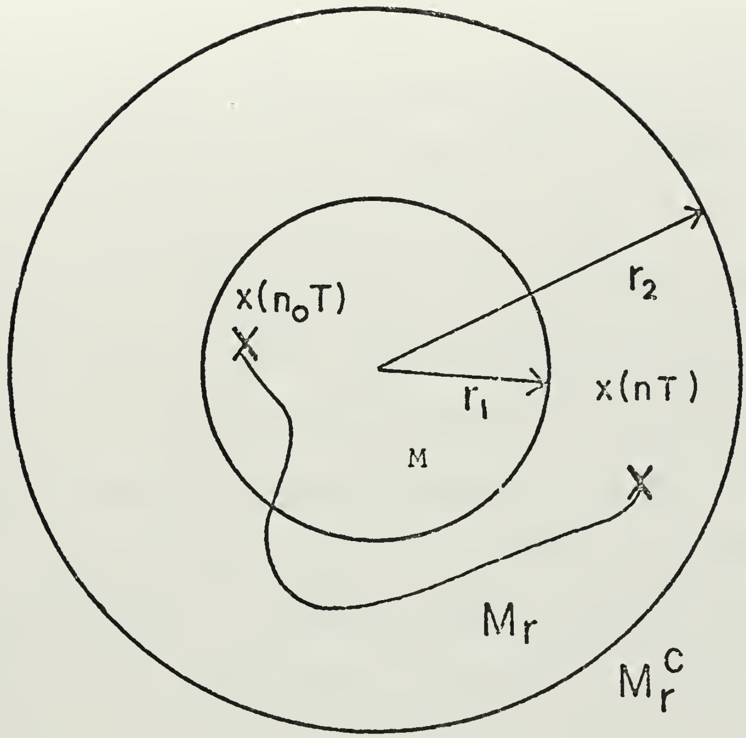

Fig. A.1: Separation of State Space into Regions M, M_r, M_r^c.

APPENDIX B

EXPERIMENTAL DATA COMPILED FROM COMPUTER SIMULATIONS

A. ANALYSIS PROGRAM FOR ZERO-INPUT LIMIT CYCLES

The data compiled in this section applies to the digital filter discussed in Section IV.B. For given values for the filter coefficients a and b, all possible limit cycles within the region of search, are enumerated in the tables B.1 to B.16 and labelled by an identification number. The frequency $F = f_o/f_s$ and the values of the limit cycle points are stated. Furthermore, for each set of numbers for a and b, the values for the approximate frequency (3.59) from the linear model and the amplitude bound (3.46) and (3.53) are printed.

The phase-plane plots for most selections of a and b are displayed in Figs. B.1 to B.14. The x-axis represents $\hat{x}(n)$ and its values are labelled in the top line. The y-axis represents $\hat{x}(n-1)$ and its values are labelled in the left column. A state trajectory is identified by those state-points which are labelled with the same number. This number is the same as the limit cycle identification number given in the corresponding table. For example, consider limit cycle #2 from table B.1. The corresponding state trajectory is displayed in Fig. B.1 and is constituted by the 14 state points labelled "2", such as (1,-3), (5,1), (8,5), (9,8), (8,9), (6,8), (3,6), (-1,3), (-5,-1), (-8,-5), (-9,-8), (-8,-9), (-6,-8), (-3,-6).

Table B.1:

LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

```
A=-1.700000000, B= 0.937000000, APPROXIMATE FREQUENCY F= 0.079402
THE AMPLITUDE BOUNDS ARE A 1=  7.937,A 2=  4.219


LIMIT CYCLE #   1WITH FREQUENCY F= 7.142857E-02 IS
  -8  -6  -3   1   5   8   9   8   6   3  -1  -5  -8  -9

LIMIT CYCLE #   2WITH FREQUENCY F= 8.333333E-02 IS
   5   2  -2  -5  -7  -7  -5  -2   2   5   7   7

LIMIT CYCLE #   3WITH FREQUENCY F= 8.333333E-02 IS
  -6  -4  -1   2   4   5   5   4   2  -1  -4  -6

LIMIT CYCLE #   4WITH FREQUENCY F= 8.333333E-02 IS
  -6  -5  -3   0   3   5   6   5   3   0  -3  -5

LIMIT CYCLE #   5WITH FREQUENCY F= 8.333333E-02 IS
  -5  -4  -2   1   4   6   6   4   1  -2  -4  -5

LIMIT CYCLE #   6WITH FREQUENCY F= 8.333333E-02 IS
  -4  -3  -1   1   3   4   4   3   1  -1  -3  -4

LIMIT CYCLE #   7WITH FREQUENCY F= 1.000000E-01 IS
  -3  -2   0   2   3   3   2   0  -2  -3

LIMIT CYCLE #   8WITH FREQUENCY F= 1.000000E-01 IS
  -2  -1   0   1   2   2   1   0  -1  -2

LIMIT CYCLE #   9WITH FREQUENCY F= 0.000000E+00 IS  -1

LIMIT CYCLE #  10WITH FREQUENCY F= 0.000000E+00 IS   0

LIMIT CYCLE #  11WITH FREQUENCY F= 0.000000E+00 IS   1
```

Fig. B.1:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A=-1.700000000, B= 0.937000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																			1		
8																	1			1	
7																2		2			
6														1	5	4	5				
5													2	4	3	3	4	2	1		
4												5	3	6	6	3	5				
3										1	4	6	7	7	6	4					
2									2	3	7	8	8	7	3	2					
1									5	6	8	11	8	6	5	1					
0								4	7	8	10	8	7	4							
-1						1	3	6	8	9	8	6	3								
-2						2	5	7	8	8	7	5	2								
-3						4	6	7	7	6	4	1									
-4					3	5	6	6	5	3											
-5			1	2	4	5	5	4	2												
-6					3	4	3	1													
-7				2		2															
-8		1			1																
-9			1																		
-10																					

Table B.2:

LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

```
A=-1.500000000, B= 0.937000000, APPROXIMATE FREQUENCY F= 0.108924
THE AMPLITUDE BOUNDS ARE A 1=  7.937,A 2=  2.288


LIMIT CYCLE #   1WITH FREQUENCY F= 1.071429E-01 IS
   1  -3  -6  -6  -3   1   5   7   6   2  -3  -7  -8  -5  -1   3
   6   6   3  -1  -5  -7  -6  -2   3   7   8   5
LIMIT CYCLE #   2WITH FREQUENCY F= 1.111111E-01 IS
  -6  -4   0   4   6   5   2  -2  -5
LIMIT CYCLE #   3WITH FREQUENCY F= 1.111111E-01 IS
  -6  -5  -2   2   5   6   4   0  -4
LIMIT CYCLE #   4WITH FREQUENCY F= 1.000000E-01 IS
  -5  -3   0   3   5   5   3   0  -3  -5
LIMIT CYCLE #   5WITH FREQUENCY F= 1.111111E-01 IS
  -5  -4  -1   2   4   4   2  -1  -4
LIMIT CYCLE #   6WITH FREQUENCY F= 1.111111E-01 IS
  -4  -2   1   4   5   4   1  -2  -4
LIMIT CYCLE #   7WITH FREQUENCY F= 1.000000E-01 IS
  -4  -3  -1   1   3   4   3   1  -1  -3
LIMIT CYCLE #   8WITH FREQUENCY F= 1.000000E-01 IS
  -3  -2   0   2   3   3   2   0  -2  -3
LIMIT CYCLE #   9WITH FREQUENCY F= 1.000000E-01 IS
  -2  -1   0   1   2   2   1   0  -1  -2
LIMIT CYCLE #  10WITH FREQUENCY F= 0.000000E+00 IS  -1
LIMIT CYCLE #  11WITH FREQUENCY F= 0.000000E+00 IS   0
LIMIT CYCLE #  12WITH FREQUENCY F= 0.000000E+00 IS   1
```

Fig. B.2:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A=-1.500000000, B= 0.937000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8																1					
7																	1		1		
6													1	1	3	2	1				
5												1	2	4	6	4	3	1			
4											3	6	5	7	5	6	2				
3										1	4	7	8	8	7	4	1	1			
2								1	2	5	8	9	9	8	5	3					
1								1	6	7	9	12	9	7	6	1					
0							3	4	8	9	11	9	8	4	2						
-1						1	5	7	9	10	9	7	5	1							
-2						2	6	8	9	9	8	6	3	1							
-3				1	1	4	7	8	8	7	4	1									
-4					3	5	6	7	6	5	2										
-5				1	2	4	5	4	3	1											
-6					1	3	2	1	1												
-7			1		1																
-8						1															
-9																					
-10																					

Table B.3:

```
LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

A=-1.300000000, B= 0.937000000, APPROXIMATE FREQUENCY F= 0.132826
THE AMPLITUDE BOUNDS ARE A 1=  7.937,A 2=  1.570


LIMIT CYCLE #   1WITH FREQUENCY F= 1.363636E-01 IS
   0   5   7   4  -2  -7  -7  -2   4   7   5   0  -5  -7  -4   2
   7   7   2  -4  -7  -5

LIMIT CYCLE #   2WITH FREQUENCY F= 1.333333E-01 IS
  -6  -2   3   6   5   1  -4  -6  -4   1   5   6   3  -2  -6

LIMIT CYCLE #   3WITH FREQUENCY F= 1.333333E-01 IS
  -6  -3   2   6   6   2  -3  -6  -5  -1   4   6   4  -1  -5

LIMIT CYCLE #   4WITH FREQUENCY F= 1.250000E-01 IS
  -5  -2   2   5   5   2  -2  -5

LIMIT CYCLE #   5WITH FREQUENCY F= 1.363636E-01 IS
  -5  -3   1   4   4   1  -3  -5  -4   0   4   5   3  -1  -4  -4
  -1   3   5   4   0  -4

LIMIT CYCLE #   6WITH FREQUENCY F= 1.363636E-01 IS
  -4  -2   1   3   3   1  -2  -4  -3   0   3   4   2  -1  -3  -3
  -1   2   4   3   0  -3

LIMIT CYCLE #   7WITH FREQUENCY F= 1.250000E-01 IS
  -3  -2   0   2   3   2   0  -2

LIMIT CYCLE #   8WITH FREQUENCY F= 1.250000E-01 IS
  -2  -1   1   2   2   1  -1  -2

LIMIT CYCLE #   9WITH FREQUENCY F= 1.666667E-01 IS
  -1   0   1   1   0  -1

LIMIT CYCLE #  10WITH FREQUENCY F= 0.000000E+00 IS    0
```

Fig. B.3:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A=-1.30000000, B= 0.937000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8																					
7													1		1	1		1			
6													3	2	3	2	3				
5											1	2	4	5	5	4	2	1			
4									1	3	5	5	6	6	5	5	3	1			
3									2	5	6	6	7	6	6	5	2				
2							1	3	4	6	7	8	8	7	6	4	3	1			
1							2	5	6	8	9	9	8	6	5	2					
0						1	5	6	7	9	10	9	7	6	5	1					
-1						3	5	6	8	9	9	8	6	5	3						
-2				1	2	4	6	7	8	8	7	6	4	2	1						
-3					3	5	6	6	7	6	6	5	3								
-4				1	2	5	5	6	6	5	5	2	1								
-5				1	3	4	5	5	4	3	1										
-6					2	3	2	3	2												
-7				1		1	1		1												
-8																					
-9																					
-10																					

Table B.4:

LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

```
A=-1.000000000, B= 0.937000000, APPROXIMATE FREQUENCY F= 0.163611
THE AMPLITUDE BOUNDS ARE A 1=  7.937,A 2=  1.067



LIMIT CYCLE #   1WITH FREQUENCY F= 1.666667E-01 IS
  -4  -7  -3   4   7   3

LIMIT CYCLE #   2WITH FREQUENCY F= 1.666667E-01 IS
  -3  -7  -4   3   7   4

LIMIT CYCLE #   3WITH FREQUENCY F= 1.666667E-01 IS
  -2  -7  -5   2   7   5

LIMIT CYCLE #   4WITH FREQUENCY F= 1.666667E-01 IS
  -1  -7  -6   1   7   6

LIMIT CYCLE #   5WITH FREQUENCY F= 1.666667E-01 IS
   0  -7  -7   0   7   7

LIMIT CYCLE #   6WITH FREQUENCY F= 1.666667E-01 IS
  -6  -7  -1   6   7   1

LIMIT CYCLE #   7WITH FREQUENCY F= 1.666667E-01 IS
   5   7   2  -5  -7  -2

LIMIT CYCLE #   8WITH FREQUENCY F= 1.666667E-01 IS
  -6   0   6   6   0  -6

LIMIT CYCLE #   9WITH FREQUENCY F= 1.666667E-01 IS
  -6  -1   5   6   1  -5

LIMIT CYCLE #  10WITH FREQUENCY F= 1.666667E-01 IS
  -6  -2   4   6   2  -4

LIMIT CYCLE #  11WITH FREQUENCY F= 1.666667E-01 IS
  -6  -3   3   6   3  -3

LIMIT CYCLE #  12WITH FREQUENCY F= 1.666667E-01 IS
  -6  -4   2   6   4  -2

LIMIT CYCLE #  13WITH FREQUENCY F= 1.666667E-01 IS
  -6  -5   1   6   5  -1

LIMIT CYCLE #  14WITH FREQUENCY F= 1.666667E-01 IS
  -5   0   5   5   0  -5

LIMIT CYCLE #  15WITH FREQUENCY F= 1.666667E-01 IS
  -5  -1   4   5   1  -4

LIMIT CYCLE #  16WITH FREQUENCY F= 1.666667E-01 IS
  -5  -2   3   5   2  -3

LIMIT CYCLE #  17WITH FREQUENCY F= 1.666667E-01 IS
  -5  -3   2   5   3  -2

LIMIT CYCLE #  18WITH FREQUENCY F= 1.666667E-01 IS
  -5  -4   1   5   4  -1

LIMIT CYCLE #  19WITH FREQUENCY F= 1.666667E-01 IS
  -4   0   4   4   0  -4

LIMIT CYCLE #  20WITH FREQUENCY F= 1.666667E-01 IS
  -4  -1   3   4   1  -3

LIMIT CYCLE #  21WITH FREQUENCY F= 1.666667E-01 IS
```

```
  -4   -2    2    4    2   -2

LIMIT CYCLE #   22WITH FREQUENCY F= 1.666667E-01 IS
  -4   -3    1    4    3   -1

LIMIT CYCLE #   23WITH FREQUENCY F= 1.666667E-01 IS
  -3    0    3    3    0   -3

LIMIT CYCLE #   24WITH FREQUENCY F= 1.666667E-01 IS
  -3   -1    2    3    1   -2

LIMIT CYCLE #   25WITH FREQUENCY F= 1.666667E-01 IS
  -3   -2    1    3    2   -1

LIMIT CYCLE #   26WITH FREQUENCY F= 1.666667E-01 IS
  -2    0    2    2    0   -2

LIMIT CYCLE #   27WITH FREQUENCY F= 1.666667E-01 IS
  -2   -1    1    2    1   -1

LIMIT CYCLE #   28WITH FREQUENCY F= 1.666667E-01 IS
  -1    0    1    1    0   -1

LIMIT CYCLE #   29WITH FREQUENCY F= 0.00000CE+C0 IS    0
```

Table B.4: (Continued)

Fig. B.4:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A=-1.000000000, B= 0.937000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8																					
7											5	6	7	1	2	3	4	5			
6										4	8	9	10	11	12	13	8	6			
5									3	13	14	15	16	17	18	14	9	7			
4								2	12	18	19	20	21	22	19	15	10	1			
3							1	11	17	22	23	24	25	23	20	16	11	2			
2						7	10	16	21	25	26	27	26	24	21	17	12	3			
1					6	9	15	20	24	27	28	28	27	25	22	18	13	4			
0				5	8	14	19	23	26	28	29	28	26	23	19	14	8	5			
-1				4	13	18	22	25	27	28	28	27	24	20	15	9	6				
-2				3	12	17	21	24	26	27	26	25	21	16	10	7					
-3				2	11	16	20	23	25	24	23	22	17	11	1						
-4				1	10	15	19	22	21	20	19	18	12	2							
-5				7	9	14	18	17	16	15	14	13	3								
-6				6	8	13	12	11	10	9	8	4									
-7				5	4	3	2	1	7	6	5										
-8																					
-9																					
-10																					

Table B.5:

```
LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

A=-0.500000000, B= 0.937000000, APPROXIMATE FREQUENCY F= 0.208424
THE AMPLITUDE BOUNDS ARE A 1=  7.937,A 2=  0.696



LIMIT CYCLE #   1WITH FREQUENCY F= 2.058824E-01 IS
   6   5  -3  -7  -1   6   4  -4  -6   1   7   3  -5  -6   2   7
   2  -6  -5   3   7   1  -6  -4   4   6  -1  -7  -3   5   6  -2
  -7  -2

LIMIT CYCLE #   2WITH FREQUENCY F= 2.058824E-01 IS
  -6   0   6   3  -4  -5   1   6   2  -5  -5   2   6   1  -5  -4
   3   6   0  -6  -3   4   5  -1  -6  -2   5   5  -2  -6  -1   5
   4  -3

LIMIT CYCLE #   3WITH FREQUENCY F= 2.000000E-01 IS
  -5   0   5   3  -3

LIMIT CYCLE #   4WITH FREQUENCY F= 2.058824E-01 IS
  -5  -1   4   3  -2  -4   0   4   2  -3  -4   1   5   2  -4  -4
   2   5   1  -4  -3   2   4   0  -4  -2   3   4  -1  -5  -2   4
   4  -2

LIMIT CYCLE #   5WITH FREQUENCY F= 2.000000E-01 IS
  -5  -3   3   5   0

LIMIT CYCLE #   6WITH FREQUENCY F= 2.000000E-01 IS
  -4  -1   3   3  -1

LIMIT CYCLE #   7WITH FREQUENCY F= 2.000000E-01 IS
  -3   1   4   1  -3

LIMIT CYCLE #   8WITH FREQUENCY F= 2.000000E-01 IS
  -3   0   3   2  -2

LIMIT CYCLE #   9WITH FREQUENCY F= 2.000000E-01 IS
  -3  -1   2   2  -1

LIMIT CYCLE #  10WITH FREQUENCY F= 2.000000E-01 IS
  -3  -2   2   3   0

LIMIT CYCLE #  11WITH FREQUENCY F= 2.000000E-01 IS
  -2   1   3   1  -2

LIMIT CYCLE #  12WITH FREQUENCY F= 2.000000E-01 IS
  -2   0   2   1  -1

LIMIT CYCLE #  13WITH FREQUENCY F= 2.000000E-01 IS
  -2  -1   1   2   0

LIMIT CYCLE #  14WITH FREQUENCY F= 1.666667E-01 IS
  -1   0   1   1   0  -1

LIMIT CYCLE #  15WITH FREQUENCY F= 0.000000E+00 IS    0
```

Fig. B.5:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A=-0.500000000, B= 0.937000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8																					
7												1	1	1							
6									1	1	2	2	2	2	1	1					
5								1	2	2	5	4	4	3	2	2	1				
4							1	2	4	4	4	7	4	4	4	2	1				
3						1	2	3	4	6	10	11	8	6	4	5	2	1			
2					1	2	4	4	8	9	13	12	9	10	4	4	2	1			
1					1	2	4	7	11	12	14	14	13	11	7	4	2	1			
0					2	5	4	10	13	14	15	14	12	8	4	3	2				
-1				1	2	4	6	9	12	14	14	13	9	6	4	2	1				
-2				1	2	4	4	8	11	13	12	11	10	4	4	2	1				
-3				1	2	3	4	7	10	9	8	7	4	5	2	1					
-4					1	2	4	4	4	6	4	4	4	2	1						
-5					1	2	2	5	4	4	3	2	2	1							
-6						1	1	2	2	2	2	1	1								
-7								1	1	1											
-8																					
-9																					
-10																					

Table B.6:

LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

A= 0.000000000, B= 0.937000000, APPROXIMATE FREQUENCY F= 0.250000
THE AMPLITUDE BOUNDS ARE A 1= 7.937,A 2= 0.516

LIMIT CYCLE # 1WITH FREQUENCY F= 2.500000E-01 IS	-7	7	7	-7
LIMIT CYCLE # 2WITH FREQUENCY F= 2.500000E-01 IS	-7	6	7	-6
LIMIT CYCLE # 3WITH FREQUENCY F= 2.500000E-01 IS	-7	5	7	-5
LIMIT CYCLE # 4WITH FREQUENCY F= 2.500000E-01 IS	-7	4	7	-4
LIMIT CYCLE # 5WITH FREQUENCY F= 2.500000E-01 IS	-7	3	7	-3
LIMIT CYCLE # 6WITH FREQUENCY F= 2.500000E-01 IS	-7	2	7	-2
LIMIT CYCLE # 7WITH FREQUENCY F= 2.500000E-01 IS	-7	1	7	-1
LIMIT CYCLE # 8WITH FREQUENCY F= 2.500000E-01 IS	-7	0	7	0
LIMIT CYCLE # 9WITH FREQUENCY F= 2.500000E-01 IS	-7	-1	7	1
LIMIT CYCLE # 10WITH FREQUENCY F= 2.500000E-01 IS	-7	-2	7	2
LIMIT CYCLE # 11WITH FREQUENCY F= 2.500000E-01 IS	-7	-3	7	3
LIMIT CYCLE # 12WITH FREQUENCY F= 2.500000E-01 IS	-7	-4	7	4
LIMIT CYCLE # 13WITH FREQUENCY F= 2.500000E-01 IS	-7	-5	7	5
LIMIT CYCLE # 14WITH FREQUENCY F= 2.500000E-01 IS	-7	-6	7	6
LIMIT CYCLE # 15WITH FREQUENCY F= 2.500000E-01 IS	-6	6	6	-6
LIMIT CYCLE # 16WITH FREQUENCY F= 2.500000E-01 IS	-6	5	6	-5
LIMIT CYCLE # 17WITH FREQUENCY F= 2.500000E-01 IS	-6	4	6	-4
LIMIT CYCLE # 18WITH FREQUENCY F= 2.500000E-01 IS	-6	3	6	-3
LIMIT CYCLE # 19WITH FREQUENCY F= 2.500000E-01 IS	-6	2	6	-2
LIMIT CYCLE # 20WITH FREQUENCY F= 2.500000E-01 IS	-6	1	6	-1
LIMIT CYCLE # 21WITH FREQUENCY F= 2.500000E-01 IS	-6	0	6	0
LIMIT CYCLE # 22WITH FREQUENCY F= 2.500000E-01 IS	-6	-1	6	1
LIMIT CYCLE # 23WITH FREQUENCY F= 2.500000E-01 IS	-6	-2	6	2
LIMIT CYCLE # 24WITH FREQUENCY F= 2.500000E-01 IS	-6	-3	6	3
LIMIT CYCLE # 25WITH FREQUENCY F= 2.500000E-01 IS	-6	-4	6	4
LIMIT CYCLE # 26WITH FREQUENCY F= 2.500000E-01 IS	-6	-5	6	5
LIMIT CYCLE # 27WITH FREQUENCY F= 2.500000E-01 IS	-5	5	5	-5
LIMIT CYCLE # 28WITH FREQUENCY F= 2.500000E-01 IS	-5	4	5	-4
LIMIT CYCLE # 29WITH FREQUENCY F= 2.500000E-01 IS	-5	3	5	-3
LIMIT CYCLE # 30WITH FREQUENCY F= 2.500000E-01 IS	-5	2	5	-2
LIMIT CYCLE # 31WITH FREQUENCY F= 2.500000E-01 IS	-5	1	5	-1

```
LIMIT CYCLE #  32WITH FREQUENCY F= 2.500000E-01 IS  -5   0   5   0
LIMIT CYCLE #  33WITH FREQUENCY F= 2.500000E-01 IS  -5  -1   5   1
LIMIT CYCLE #  34WITH FREQUENCY F= 2.500000E-01 IS  -5  -2   5   2
LIMIT CYCLE #  35WITH FREQUENCY F= 2.500000E-01 IS  -5  -3   5   3
LIMIT CYCLE #  36WITH FREQUENCY F= 2.500000E-01 IS  -5  -4   5   4
LIMIT CYCLE #  37WITH FREQUENCY F= 2.500000E-01 IS  -4   4   4  -4
LIMIT CYCLE #  38WITH FREQUENCY F= 2.500000E-01 IS  -4   3   4  -3
LIMIT CYCLE #  39WITH FREQUENCY F= 2.500000E-01 IS  -4   2   4  -2
LIMIT CYCLE #  40WITH FREQUENCY F= 2.500000E-01 IS  -4   1   4  -1
LIMIT CYCLE #  41WITH FREQUENCY F= 2.500000E-01 IS  -4   0   4   0
LIMIT CYCLE #  42WITH FREQUENCY F= 2.500000E-01 IS  -4  -1   4   1
LIMIT CYCLE #  43WITH FREQUENCY F= 2.500000E-01 IS  -4  -2   4   2
LIMIT CYCLE #  44WITH FREQUENCY F= 2.500000E-01 IS  -4  -3   4   3
LIMIT CYCLE #  45WITH FREQUENCY F= 2.500000E-01 IS  -3   3   3  -3
LIMIT CYCLE #  46WITH FREQUENCY F= 2.500000E-01 IS  -3   2   3  -2
LIMIT CYCLE #  47WITH FREQUENCY F= 2.500000E-01 IS  -3   1   3  -1
LIMIT CYCLE #  48WITH FREQUENCY F= 2.500000E-01 IS  -3   0   3   0
LIMIT CYCLE #  49WITH FREQUENCY F= 2.500000E-01 IS  -3  -1   3   1
LIMIT CYCLE #  50WITH FREQUENCY F= 2.500000E-01 IS  -3  -2   3   2
LIMIT CYCLE #  51WITH FREQUENCY F= 2.500000E-01 IS  -2   2   2  -2
LIMIT CYCLE #  52WITH FREQUENCY F= 2.500000E-01 IS  -2   1   2  -1
LIMIT CYCLE #  53WITH FREQUENCY F= 2.500000E-01 IS  -2   0   2   0
LIMIT CYCLE #  54WITH FREQUENCY F= 2.500000E-01 IS  -2  -1   2   1
LIMIT CYCLE #  55WITH FREQUENCY F= 2.500000E-01 IS  -1   1   1  -1
LIMIT CYCLE #  56WITH FREQUENCY F= 2.500000E-01 IS  -1   0   1   0
LIMIT CYCLE #  57WITH FREQUENCY F= 0.000000E+00 IS   0
```

Table B.6: (Continued)

Fig. B.6:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A= 0.000000000, B= 0.937000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8																					
7				1	2	3	4	5	6	7	8	9	10	11	12	13	14	1			
6				14	15	16	17	18	19	20	21	22	23	24	25	26	15	2			
5				13	26	27	28	29	30	31	32	33	34	35	36	27	16	3			
4				12	25	36	37	38	39	40	41	42	43	44	37	28	17	4			
3				11	24	35	44	45	46	47	48	49	50	45	38	29	18	5			
2				10	23	34	43	50	51	52	53	54	51	46	39	30	19	6			
1				9	22	33	42	49	54	55	56	55	52	47	40	31	20	7			
0				8	21	32	41	48	53	56	57	56	53	48	41	32	21	8			
-1				7	20	31	40	47	52	55	56	55	54	49	42	33	22	9			
-2				6	19	30	39	46	51	54	53	52	51	50	43	34	23	10			
-3				5	18	29	38	45	50	49	48	47	46	45	44	35	24	11			
-4				4	17	28	37	44	43	42	41	40	39	38	37	36	25	12			
-5				3	16	27	36	35	34	33	32	31	30	29	28	27	26	13			
-6				2	15	26	25	24	23	22	21	20	19	18	17	16	15	14			
-7				1	14	13	12	11	10	9	8	7	6	5	4	3	2	1			
-8																					
-9																					
-10																					

Table B.7:

LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

```
A= 0.500000000, B= 0.937000000, APPROXIMATE FREQUENCY F= 0.291576
THE AMPLITUDE BOUNDS ARE A 1=  7.937,A 2=  0.696



LIMIT CYCLE #   1WITH FREQUENCY F= 2.941176E-01 IS
  -6   4   4  -6  -1   7  -3  -5   6   2  -7   2   6  -5  -3   7
  -1

LIMIT CYCLE #   2WITH FREQUENCY F= 2.941176E-01 IS
  -6   5   3  -7   1   6  -4  -4   6   1  -7   3   5  -6  -2   7
  -2

LIMIT CYCLE #   3WITH FREQUENCY F= 2.941176E-01 IS
  -6   3   4  -5  -1   6  -2  -5   5   2  -6   1   5  -4  -3   6
   0

LIMIT CYCLE #   4WITH FREQUENCY F= 2.941176E-01 IS
  -6   2   5  -5  -2   6  -1  -5   4   3  -6   0   6  -3  -4   5
   1

LIMIT CYCLE #   5WITH FREQUENCY F= 3.000000E-01 IS
  -5   3   3  -5   0   5  -3  -3   5   0

LIMIT CYCLE #   6WITH FREQUENCY F= 2.941176E-01 IS
  -5   2   4  -4  -2   5  -1  -4   3   2  -4   0   4  -2  -3   4
   1

LIMIT CYCLE #   7WITH FREQUENCY F= 2.941176E-01 IS
  -5   1   4  -3  -2   4   0  -4   2   3  -4  -1   5  -2  -4   4
   2

LIMIT CYCLE #   8WITH FREQUENCY F= 3.000000E-01 IS
  -4   1   3  -3  -1   4  -1  -3   3   1

LIMIT CYCLE #   9WITH FREQUENCY F= 3.000000E-01 IS
  -3   2   2  -3   0   3  -2  -2   3   0

LIMIT CYCLE #  10WITH FREQUENCY F= 3.000000E-01 IS
  -3   1   2  -2  -1   3  -1  -2   2   1

LIMIT CYCLE #  11WITH FREQUENCY F= 3.000000E-01 IS
  -2   1   1  -2   0   2  -1  -1   2   0

LIMIT CYCLE #  12WITH FREQUENCY F= 3.333333E-01 IS  -1   1   0

LIMIT CYCLE #  13WITH FREQUENCY F= 3.333333E-01 IS  -1   0   1

LIMIT CYCLE #  14WITH FREQUENCY F= 0.000000E+00 IS   0
```

Fig. B.7:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A= 0.500000000, B= 0.937000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8																					
7								1	2	1											
6						1	2	4	3	4	3	2	1								
5					2	4	3	5	7	6	5	4	3	2							
4					1	3	6	7	6	8	7	6	7	4	1						
3				2	4	5	7	8	9	10	9	8	6	5	3	2					
2				1	3	7	6	9	10	11	11	10	9	7	6	4	1				
1				2	4	6	8	10	11	13	12	11	10	8	7	3	2				
0					3	5	7	9	11	12	14	13	11	9	6	5	4				
-1					1	4	6	8	10	11	13	12	11	10	8	7	3	1			
-2					2	3	7	6	9	10	11	11	10	9	7	6	4	2			
-3						1	4	5	7	8	9	10	9	8	6	5	3	1			
-4							2	3	6	7	6	8	7	6	7	4	2				
-5								1	4	3	5	7	6	5	4	3	1				
-6									2	1	4	3	4	3	1	2					
-7												2	1	2							
-8																					
-9																					
-10																					

Table B.8:

LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

A= 1.000000000, B= 0.937000000, APPROXIMATE FREQUENCY F= 0.336389
THE AMPLITUDE BOUNDS ARE A 1= 7.937,A 2= 1.067

LIMIT CYCLE # 1WITH FREQUENCY F= 3.333333E-01 IS	0	7	-7
LIMIT CYCLE # 2WITH FREQUENCY F= 3.333333E-01 IS	-1	7	-6
LIMIT CYCLE # 3WITH FREQUENCY F= 3.333333E-01 IS	-2	7	-5
LIMIT CYCLE # 4WITH FREQUENCY F= 3.333333E-01 IS	-3	7	-4
LIMIT CYCLE # 5WITH FREQUENCY F= 3.333333E-01 IS	-4	7	-3
LIMIT CYCLE # 6WITH FREQUENCY F= 3.333333E-01 IS	-5	7	-2
LIMIT CYCLE # 7WITH FREQUENCY F= 3.333333E-01 IS	-6	7	-1
LIMIT CYCLE # 8WITH FREQUENCY F= 3.333333E-01 IS	-7	7	0
LIMIT CYCLE # 9WITH FREQUENCY F= 3.333333E-01 IS	1	-7	6
LIMIT CYCLE # 10WITH FREQUENCY F= 3.333333E-01 IS	2	-7	5
LIMIT CYCLE # 11WITH FREQUENCY F= 3.333333E-01 IS	3	-7	4
LIMIT CYCLE # 12WITH FREQUENCY F= 3.333333E-01 IS	4	-7	3
LIMIT CYCLE # 13WITH FREQUENCY F= 3.333333E-01 IS	5	-7	2
LIMIT CYCLE # 14WITH FREQUENCY F= 3.333333E-01 IS	6	-7	1
LIMIT CYCLE # 15WITH FREQUENCY F= 3.333333E-01 IS	-6	6	0
LIMIT CYCLE # 16WITH FREQUENCY F= 3.333333E-01 IS	-6	5	1
LIMIT CYCLE # 17WITH FREQUENCY F= 3.333333E-01 IS	-6	4	2
LIMIT CYCLE # 18WITH FREQUENCY F= 3.333333E-01 IS	-6	3	3
LIMIT CYCLE # 19WITH FREQUENCY F= 3.333333E-01 IS	-6	2	4
LIMIT CYCLE # 20WITH FREQUENCY F= 3.333333E-01 IS	-6	1	5
LIMIT CYCLE # 21WITH FREQUENCY F= 3.333333E-01 IS	-6	0	6
LIMIT CYCLE # 22WITH FREQUENCY F= 3.333333E-01 IS	-5	6	-1
LIMIT CYCLE # 23WITH FREQUENCY F= 3.333333E-01 IS	-5	5	0
LIMIT CYCLE # 24WITH FREQUENCY F= 3.333333E-01 IS	-5	4	1
LIMIT CYCLE # 25WITH FREQUENCY F= 3.333333E-01 IS	-5	3	2
LIMIT CYCLE # 26WITH FREQUENCY F= 3.333333E-01 IS	-5	2	3
LIMIT CYCLE # 27WITH FREQUENCY F= 3.333333E-01 IS	-5	1	4
LIMIT CYCLE # 28WITH FREQUENCY F= 3.333333E-01 IS	-5	0	5
LIMIT CYCLE # 29WITH FREQUENCY F= 3.333333E-01 IS	-5	-1	6
LIMIT CYCLE # 30WITH FREQUENCY F= 3.333333E-01 IS	-4	6	-2
LIMIT CYCLE # 31WITH FREQUENCY F= 3.333333E-01 IS	-4	5	-1

LIMIT CYCLE # 32WITH FREQUENCY F= 3.333333E-01 IS	-4	4	0
LIMIT CYCLE # 33WITH FREQUENCY F= 3.333333E-01 IS	-4	3	1
LIMIT CYCLE # 34WITH FREQUENCY F= 3.333333E-01 IS	-4	2	2
LIMIT CYCLE # 35WITH FREQUENCY F= 3.333333E-01 IS	-4	1	3
LIMIT CYCLE # 36WITH FREQUENCY F= 3.333333E-01 IS	-4	0	4
LIMIT CYCLE # 37WITH FREQUENCY F= 3.333333E-01 IS	-4	-1	5
LIMIT CYCLE # 38WITH FREQUENCY F= 3.333333E-01 IS	-4	-2	6
LIMIT CYCLE # 39WITH FREQUENCY F= 3.333333E-01 IS	-3	6	-3
LIMIT CYCLE # 40WITH FREQUENCY F= 3.333333E-01 IS	-3	5	-2
LIMIT CYCLE # 41WITH FREQUENCY F= 3.333333E-01 IS	-3	4	-1
LIMIT CYCLE # 42WITH FREQUENCY F= 3.333333E-01 IS	-3	3	0
LIMIT CYCLE # 43WITH FREQUENCY F= 3.333333E-01 IS	-3	2	1
LIMIT CYCLE # 44WITH FREQUENCY F= 3.333333E-01 IS	-3	1	2
LIMIT CYCLE # 45WITH FREQUENCY F= 3.333333E-01 IS	-3	0	3
LIMIT CYCLE # 46WITH FREQUENCY F= 3.333333E-01 IS	-3	-1	4
LIMIT CYCLE # 47WITH FREQUENCY F= 3.333333E-01 IS	-3	-2	5
LIMIT CYCLE # 48WITH FREQUENCY F= 3.333333E-01 IS	-2	4	-2
LIMIT CYCLE # 49WITH FREQUENCY F= 3.333333E-01 IS	-2	3	-1
LIMIT CYCLE # 50WITH FREQUENCY F= 3.333333E-01 IS	-2	2	0
LIMIT CYCLE # 51WITH FREQUENCY F= 3.333333E-01 IS	-2	1	1
LIMIT CYCLE # 52WITH FREQUENCY F= 3.333333E-01 IS	-2	0	2
LIMIT CYCLE # 53WITH FREQUENCY F= 3.333333E-01 IS	-2	-1	3
LIMIT CYCLE # 54WITH FREQUENCY F= 3.333333E-01 IS	-1	2	-1
LIMIT CYCLE # 55WITH FREQUENCY F= 3.333333E-01 IS	-1	1	0
LIMIT CYCLE # 56WITH FREQUENCY F= 3.333333E-01 IS	-1	0	1
LIMIT CYCLE # 57WITH FREQUENCY F= 0.000000E+00 IS	0		

Table B.8: (Continued)

Fig. B.8:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A= 1.000000000, B= 0.937000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8																					
7				1	2	3	4	5	6	7	8										
6				14	21	29	38	39	30	22	15	9									
5				13	20	28	37	47	40	31	23	16	10								
4				12	19	27	36	46	48	41	32	24	17	11							
3				11	18	26	35	45	53	49	42	33	25	18	12						
2				10	17	25	34	44	52	54	50	43	34	26	19	13					
1				9	16	24	33	43	51	56	55	51	44	35	27	20	14				
0				8	15	23	32	42	50	55	57	56	52	45	36	28	21	1			
-1					7	22	31	41	49	54	56	55	54	53	46	37	29	2			
-2						6	30	40	48	53	52	51	50	49	48	47	38	3			
-3							5	39	47	46	45	44	43	42	41	40	39	4			
-4								4	38	37	36	35	34	33	32	31	30	5			
-5									3	29	28	27	26	25	24	23	22	6			
-6										2	21	20	19	18	17	16	15	7			
-7											1	14	13	12	11	10	9	8			
-8																					
-9																					
-10																					

Table B.9:

```
LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

A= 1.300000000, B= 0.937000000, APPROXIMATE FREQUENCY F= 0.367174
THE AMPLITUDE BOUNDS ARE A 1=  7.937,A 2=  1.570


LIMIT CYCLE #    1WITH FREQUENCY F= 3.636364E-01 IS
   0  -5   7  -4  -2   7  -7   2   4  -7   5

LIMIT CYCLE #    2WITH FREQUENCY F= 3.636364E-01 IS
   0   5  -7   4   2  -7   7  -2  -4   7  -5

LIMIT CYCLE #    3WITH FREQUENCY F= 3.666667E-01 IS
  -6   6  -2  -3   6  -5   1   4  -6   4   1  -5   6  -3  -2   6
  -6   2   3  -6   5  -1  -4   6  -4  -1   5  -6   3   2

LIMIT CYCLE #    4WITH FREQUENCY F= 3.750000E-01 IS
  -5   5  -2  -2   5  -5   2   2

LIMIT CYCLE #    5WITH FREQUENCY F= 3.636364E-01 IS
  -5   4   0  -4   5  -3  -1   4  -4   1   3

LIMIT CYCLE #    6WITH FREQUENCY F= 3.636364E-01 IS
  -5   3   1  -4   4  -1  -3   5  -4   0   4

LIMIT CYCLE #    7WITH FREQUENCY F= 3.636364E-01 IS
  -4   3   0  -3   4  -2  -1   3  -3   1   2

LIMIT CYCLE #    8WITH FREQUENCY F= 3.636364E-01 IS
  -4   2   1  -3   3  -1  -2   4  -3   0   3

LIMIT CYCLE #    9WITH FREQUENCY F= 3.750000E-01 IS
  -3   2   0  -2   3  -2   0   2

LIMIT CYCLE #   10WITH FREQUENCY F= 3.750000E-01 IS
  -2   2  -1  -1   2  -2   1   1

LIMIT CYCLE #   11WITH FREQUENCY F= 3.333333E-01 IS  -1   1   0

LIMIT CYCLE #   12WITH FREQUENCY F= 3.333333E-01 IS  -1   0   1

LIMIT CYCLE #   13WITH FREQUENCY F= 0.000000E+00 IS   0
```

Fig. B.9:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A= 1.300000000, B= 0.937000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8																					
7				1		2	1		2												
6					3	3	3	3	3												
5				2	3	4	6	5	4	3	1										
4				1	3	6	5	8	7	6	5	3	2								
3					3	5	8	7	9	8	7	6	3								
2				2	3	4	7	9	10	10	9	8	4	3	1						
1						3	6	8	10	12	11	10	7	5	3						
0						1	5	7	9	11	13	12	9	8	6	2					
-1							3	6	8	10	12	11	10	7	5	3					
-2							2	3	4	7	9	10	10	9	8	4	3	1			
-3									3	5	8	7	9	8	7	6	3				
-4									1	3	6	5	8	7	6	5	3	2			
-5											2	3	4	6	5	4	3	1			
-6													3	3	3	3	3				
-7													1		2	1		2			
-8																					
-9																					
-10																					

Table B.10:

LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

```
A= 1.500000000, B= 0.937000000, APPROXIMATE FREQUENCY F= 0.391076
THE AMPLITUDE BOUNDS ARE A 1=  7.937,A 2=  2.288



LIMIT CYCLE #   1WITH FREQUENCY F= 3.928571E-01 IS
   2   3  -7   8  -5   1   3  -6   6  -3  -1   5  -7   6  -2  -3
   7  -8   5  -1  -3   6  -6   3   1  -5   7  -6

LIMIT CYCLE #   2WITH FREQUENCY F= 3.888889E-01 IS
  -6   5  -2  -2   5  -6   4   0  -4   6  -5   2   2  -5   6  -4
   0   4

LIMIT CYCLE #   3WITH FREQUENCY F= 4.000000E-01 IS
  -5   5  -3   0   3

LIMIT CYCLE #   4WITH FREQUENCY F= 3.888889E-01 IS
  -5   4  -1  -2   4  -4   2   1  -4   5  -4   1   2  -4   4  -2
  -1   4

LIMIT CYCLE #   5WITH FREQUENCY F= 4.000000E-01 IS
  -5   3   0  -3   5

LIMIT CYCLE #   6WITH FREQUENCY F= 4.000000E-01 IS
  -4   3  -1  -1   3

LIMIT CYCLE #   7WITH FREQUENCY F= 4.000000E-01 IS
  -3   4  -3   1   1

LIMIT CYCLE #   8WITH FREQUENCY F= 4.000000E-01 IS
  -3   3  -2   0   2

LIMIT CYCLE #   9WITH FREQUENCY F= 4.000000E-01 IS
  -3   2   0  -2   3

LIMIT CYCLE #  10WITH FREQUENCY F= 4.000000E-01 IS
  -2   2  -1   0   1

LIMIT CYCLE #  11WITH FREQUENCY F= 4.000000E-01 IS
  -2   1   0  -1   2

LIMIT CYCLE #  12WITH FREQUENCY F= 5.000000E-01 IS  -1   1

LIMIT CYCLE #  13WITH FREQUENCY F= 0.000000E+00 IS   0
```

Fig. B.10:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A= 1.500000000, B= 0.937000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8						1															
7			1		1																
6					1	2	2	1	1												
5				1	2	5	4	3	2	1											
4					2	4	4	7	4	4	2										
3				1	1	3	6	9	8	6	5	1									
2						2	4	8	11	10	9	4	2	1							
1						1	4	7	10	12	11	7	4	1							
0							2	5	9	11	13	10	8	3	2						
-1								1	4	6	10	12	11	6	4	1					
-2								1	2	4	8	11	10	9	4	2					
-3										1	3	7	9	8	7	5	1	1			
-4											2	4	4	6	4	4	2				
-5												1	2	5	4	3	2	1			
-6													1	1	2	2	1				
-7																	1		1		
-8																1					
-9																					
-10																					

Table B.11:

```
LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

A= 1.700000000, B= 0.937000000, APPROXIMATE FREQUENCY F= 0.420598
THE AMPLITUDE BOUNDS ARE A 1=  7.937,A 2=  4.219



LIMIT CYCLE #   1WITH FREQUENCY F= 4.166667E-01 IS
  -5   2   2  -5   7  -7   5  -2  -2   5  -7   7

LIMIT CYCLE #   2WITH FREQUENCY F= 4.285714E-01 IS
  -8   6  -3  -1   5  -8   9

LIMIT CYCLE #   3WITH FREQUENCY F= 4.285714E-01 IS
   3   1  -5   8  -9   8  -6

LIMIT CYCLE #   4WITH FREQUENCY F= 4.166667E-01 IS
  -6   6  -4   1   2  -4   5  -5   4  -2  -1   4

LIMIT CYCLE #   5WITH FREQUENCY F= 4.166667E-01 IS
  -6   5  -3   0   3  -5   6  -5   3   0  -3   5

LIMIT CYCLE #   6WITH FREQUENCY F= 4.166667E-01 IS
  -6   4  -1  -2   4  -5   5  -4   2   1  -4   6

LIMIT CYCLE #   7WITH FREQUENCY F= 4.166667E-01 IS
  -4   4  -3   1   1  -3   4  -4   3  -1  -1   3

LIMIT CYCLE #   8WITH FREQUENCY F= 4.000000E-01 IS
  -3   3  -2   0   2

LIMIT CYCLE #   9WITH FREQUENCY F= 4.000000E-01 IS
  -3   2   0  -2   3

LIMIT CYCLE #  10WITH FREQUENCY F= 4.000000E-01 IS
  -2   2  -1   0   1

LIMIT CYCLE #  11WITH FREQUENCY F= 4.000000E-01 IS
  -2   1   0  -1   2

LIMIT CYCLE #  12WITH FREQUENCY F= 5.000000E-01 IS  -1   1

LIMIT CYCLE #  13WITH FREQUENCY F= 0.000000E+00 IS   0
```

Fig. B.11:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A= 1.700000000, B= 0.937000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9			2																		
8		3			3																
7				1		1															
6					6	5	4	2													
5			2	1	5	4	6	5	1												
4					4	6	7	7	4	6											
3						5	7	9	8	7	5	3									
2						1	4	8	11	10	9	6	1								
1						3	6	7	10	12	11	7	4								
0								5	9	11	13	10	8	5							
-1									6	7	10	12	11	7	4	2					
-2									1	4	8	11	10	9	6	1					
-3										2	5	7	9	8	7	5					
-4												4	6	7	7	4	6				
-5													1	5	4	6	5	1	3		
-6														3	6	5	4				
-7																1		1			
-8																	2			2	
-9																			3		
-10																					

Table B.12:

LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

A=-0.100000000, B=-0.750000000, APPROXIMATE FREQUENCY F= 0.000000
THE AMPLITUDE BOUNDS ARE A 1= 2.000,A 2= 6.667

```
LIMIT CYCLE #   1WITH FREQUENCY F= 0.000000E+00 IS  -6
LIMIT CYCLE #   2WITH FREQUENCY F= 0.000000E+00 IS  -6  -5
LIMIT CYCLE #   3WITH FREQUENCY F= 0.000000E+00 IS  -2
LIMIT CYCLE #   4WITH FREQUENCY F= 0.000000E+00 IS  -2  -1
LIMIT CYCLE #   5WITH FREQUENCY F= 5.000000E-01 IS  -2   0
LIMIT CYCLE #   6WITH FREQUENCY F= 5.000000E-01 IS  -2   1
LIMIT CYCLE #   7WITH FREQUENCY F= 5.000000E-01 IS  -2   2
LIMIT CYCLE #   8WITH FREQUENCY F= 0.000000E+00 IS  -5
LIMIT CYCLE #   9WITH FREQUENCY F= 5.000000E-01 IS   2  -1
LIMIT CYCLE #  10WITH FREQUENCY F= 0.000000E+00 IS   2   0
LIMIT CYCLE #  11WITH FREQUENCY F= 0.000000E+00 IS   2   1
LIMIT CYCLE #  12WITH FREQUENCY F= 0.000000E+00 IS  -1
LIMIT CYCLE #  13WITH FREQUENCY F= 5.000000E-01 IS  -1   0
LIMIT CYCLE #  14WITH FREQUENCY F= 5.000000E-01 IS  -1   1
LIMIT CYCLE #  15WITH FREQUENCY F= 0.000000E+00 IS   2
LIMIT CYCLE #  16WITH FREQUENCY F= 0.000000E+00 IS   0
LIMIT CYCLE #  17WITH FREQUENCY F= 0.000000E+00 IS   0   1
LIMIT CYCLE #  18WITH FREQUENCY F= 0.000000E+00 IS   1
LIMIT CYCLE #  19WITH FREQUENCY F= 0.000000E+00 IS   5
LIMIT CYCLE #  20WITH FREQUENCY F= 0.000000E+00 IS   5   6
LIMIT CYCLE #  21WITH FREQUENCY F= 0.000000E+00 IS   6
```

Fig. B.12:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A=-0.100000000, B=-0.750000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8																					
7																					
6																20	21				
5																19	20				
4																					
3																					
2									7	9	10	11	15								
1									6	14	17	18	11								
0									5	13	16	17	10								
-1									4	12	13	14	9								
-2									3	4	5	6	7								
-3																					
-4																					
-5					2	8															
-6					1	2															
-7																					
-8																					
-9																					
-10																					

Table B.13:

```
LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

A= 0.000000000, B=-0.750000000, APPROXIMATE FREQUENCY F= 0.000000
THE AMPLITUDE BOUNDS ARE A 1=  2.000,A 2=  4.000



LIMIT CYCLE #   1WITH FREQUENCY F= 0.000000E+00 IS  -2
LIMIT CYCLE #   2WITH FREQUENCY F= 0.000000E+00 IS  -2  -1
LIMIT CYCLE #   3WITH FREQUENCY F= 5.000000E-01 IS  -2   0
LIMIT CYCLE #   4WITH FREQUENCY F= 5.000000E-01 IS  -2   1
LIMIT CYCLE #   5WITH FREQUENCY F= 5.000000E-01 IS  -2   2
LIMIT CYCLE #   6WITH FREQUENCY F= 0.000000E+00 IS  -1
LIMIT CYCLE #   7WITH FREQUENCY F= 5.000000E-01 IS  -1   0
LIMIT CYCLE #   8WITH FREQUENCY F= 5.000000E-01 IS  -1   1
LIMIT CYCLE #   9WITH FREQUENCY F= 5.000000E-01 IS  -1   2
LIMIT CYCLE #  10WITH FREQUENCY F= 0.000000E+00 IS   0
LIMIT CYCLE #  11WITH FREQUENCY F= 0.000000E+00 IS   0   1
LIMIT CYCLE #  12WITH FREQUENCY F= 0.000000E+00 IS   0   2
LIMIT CYCLE #  13WITH FREQUENCY F= 0.000000E+00 IS   1
LIMIT CYCLE #  14WITH FREQUENCY F= 0.000000E+00 IS   1   2
LIMIT CYCLE #  15WITH FREQUENCY F= 0.000000E+00 IS   2
```

Fig. B.13:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A= 0.00000000, B=-0.75000000

	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9
9																			
8																			
7																			
6																			
5																			
4																			
3																			
2								5	9	12	14	15							
1								4	8	11	13	14							
0								3	7	10	11	12							
-1								2	6	7	8	9							
-2								1	2	3	4	5							
-3																			
-4																			
-5																			
-6																			
-7																			
-8																			
-9																			

Table B.14:

LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

A= 0.100000000, B=-0.750000000, APPROXIMATE FREQUENCY F= 0.000000
THE AMPLITUDE BOUNDS ARE A 1= 2.000,A 2= 6.667

LIMIT CYCLE #	1WITH FREQUENCY F=	0.000000E+00	IS	-2	
LIMIT CYCLE #	2WITH FREQUENCY F=	0.000000E+00	IS	-2	-1
LIMIT CYCLE #	3WITH FREQUENCY F=	5.000000E-01	IS	-2	0
LIMIT CYCLE #	4WITH FREQUENCY F=	5.000000E-01	IS	-2	1
LIMIT CYCLE #	5WITH FREQUENCY F=	5.000000E-01	IS	-2	2
LIMIT CYCLE #	6WITH FREQUENCY F=	5.000000E-01	IS	-6	5
LIMIT CYCLE #	7WITH FREQUENCY F=	5.000000E-01	IS	-6	6
LIMIT CYCLE #	8WITH FREQUENCY F=	5.000000E-01	IS	-5	5
LIMIT CYCLE #	9WITH FREQUENCY F=	5.000000E-01	IS	-5	6
LIMIT CYCLE #	10WITH FREQUENCY F=	0.000000E+00	IS	-1	
LIMIT CYCLE #	11WITH FREQUENCY F=	5.000000E-01	IS	-1	0
LIMIT CYCLE #	12WITH FREQUENCY F=	5.000000E-01	IS	-1	1
LIMIT CYCLE #	13WITH FREQUENCY F=	5.000000E-01	IS	-1	2
LIMIT CYCLE #	14WITH FREQUENCY F=	0.000000E+00	IS	0	
LIMIT CYCLE #	15WITH FREQUENCY F=	0.000000E+00	IS	0	1
LIMIT CYCLE #	16WITH FREQUENCY F=	0.000000E+00	IS	0	2
LIMIT CYCLE #	17WITH FREQUENCY F=	0.000000E+00	IS	1	
LIMIT CYCLE #	18WITH FREQUENCY F=	0.000000E+00	IS	1	2
LIMIT CYCLE #	19WITH FREQUENCY F=	0.000000E+00	IS	2	

Fig. B.14:

LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS. X(N-1)
A= 0.100000000, B=-0.750000000

	-10	-9	-8	-7	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	7	8	9	10
10																					
9																					
8																					
7																					
6					7	9															
5					6	8															
4																					
3																					
2									5	13	16	18	19								
1									4	12	15	17	18								
0									3	11	14	15	16								
-1									2	10	11	12	13								
-2									1	2	3	4	5								
-3																					
-4																					
-5																8	9				
-6																6	7				
-7																					
-8																					
-9																					
-10																					

Table B.15:

LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

A=-1.400000000, B= 0.973000000, APPROXIMATE FREQUENCY F= 0.124428
THE AMPLITUDE BOUNDS ARE A 1= 18.519,A 2= 1.745

```
LIMIT CYCLE #   1WITH FREQUENCY F= 1.274510E-01 IS
  -4 -16 -18  -9   5  16  17   8  -6 -16 -16  -6   8  17  16   5
  -9 -18 -16  -4  10  18  15   3 -11 -18 -14  -2  11  17  13   1
 -12 -18 -13   0  13  18  12  -1 -13 -17 -11   2  14  18  11  -3
 -15 -18 -10   4  16  18   9  -5 -16 -17  -8   6  16  16   6  -8
 -17 -16  -5   9  18  16   4 -10 -18 -15  -3  11  18  14   2 -11
 -17 -13  -1  12  18  13   0 -13 -18 -12   1  13  17  11  -2 -14
 -18 -11   3  15  18  10

LIMIT CYCLE #   2WITH FREQUENCY F= 1.250000E-01 IS
 -10 -20 -18  -6  10  20  18   6

LIMIT CYCLE #   3WITH FREQUENCY F= 1.250000E-01 IS
  -6   9  19  18   7  -8 -18 -17

LIMIT CYCLE #   4WITH FREQUENCY F= 1.250000E-01 IS
  14  23  18   3 -14 -23 -18  -3

LIMIT CYCLE #   5WITH FREQUENCY F= 1.250000E-01 IS
  -7   8  18  17   6  -9 -19 -18

LIMIT CYCLE #   6WITH FREQUENCY F= 2.500000E-01 IS -17  -1  17   1

LIMIT CYCLE #   7WITH FREQUENCY F= 1.250000E-01 IS
 -22 -16  -1  15  22  16   1 -15

LIMIT CYCLE #   8WITH FREQUENCY F= 1.250000E-01 IS
 -21 -15  -1  14  21  15   1 -14

LIMIT CYCLE #   9WITH FREQUENCY F= 1.250000E-01 IS
 -20 -11   4  17  20  11  -4 -17

LIMIT CYCLE #  10WITH FREQUENCY F= 1.250000E-01 IS
 -20 -16  -3  12  20  16   3 -12

LIMIT CYCLE #  11WITH FREQUENCY F= 1.250000E-01 IS
 -17  -7   7  17  17   7  -7 -17

LIMIT CYCLE #  12WITH FREQUENCY F= 1.250000E-01 IS
 -17  -9   4  15  17   9  -4 -15

LIMIT CYCLE #  13WITH FREQUENCY F= 1.250000E-01 IS
 -17 -10   3  14  17  10  -3 -14

LIMIT CYCLE #  14WITH FREQUENCY F= 1.250000E-01 IS
 -17 -12   0  12  17  12   0 -12

LIMIT CYCLE #  15WITH FREQUENCY F= 1.250000E-01 IS
 -17 -14  -3  10  17  14   3 -10

LIMIT CYCLE #  16WITH FREQUENCY F= 1.250000E-01 IS
 -17 -15  -4   9  17  15   4  -9

LIMIT CYCLE #  17WITH FREQUENCY F= 1.276596E-01 IS
 -16  -7   6  15  15   6  -7 -16 -15  -5   8  16  14   4  -8 -15
 -13  -3   9  16  13   2 -10 -16 -12  -1  11  16  11  -1 -12 -16
 -10   2  13  16   9  -3 -13 -15  -8   4  14  16   8  -5 -15

LIMIT CYCLE #  18WITH FREQUENCY F= 1.276596E-01 IS
 -16  -8   5  15  16   7  -6 -15 -15  -6   7  16  15   5  -8 -16
 -14  -4   8  15  13   3  -9 -16 -13  -2  10  16  12   1 -11 -16
```

```
 -11    1  12   16   10   -2 -13 -16   -9    3   13   15    8   -4  -14

LIMIT CYCLE #   19WITH FREQUENCY F= 1.250000E-01  IS
 -15  -7    5   14   15    7  -5 -14

LIMIT CYCLE #   20WITH FREQUENCY F= 1.250000E-01  IS
 -15  -9    2   12   15    9  -2 -12

LIMIT CYCLE #   21WITH FREQUENCY F= 1.274510E-01  IS
 -15 -10    1   11   14    9  -1 -10 -13   -8    2   11   13    7   -3  -11
 -12  -6    4   12   13    6  -5 -13 -13   -5    6   13   12    4   -6  -12
 -11  -3    7   13   11    2  -8 -13 -10   -1    9   14   11    1  -10  -15
 -11   0   11   15   10   -1 -11 -14  -9    1   10   13    8   -2  -11  -13
  -7   3   11   12    6   -4 -12 -13  -6    5   13   13    5   -6  -13  -12
  -4   6   12   11    3   -7 -13 -11  -2    8   13   10    1   -9  -14  -11
  -1  10   15   11    0  -11

LIMIT CYCLE #   22WITH FREQUENCY F= 1.250000E-01  IS
 -15 -12   -2    9   15   12   2  -9

LIMIT CYCLE #   23WITH FREQUENCY F= 1.250000E-01  IS
 -15 -14   -5    7   15   14   5  -7

LIMIT CYCLE #   24WITH FREQUENCY F= 1.250000E-01  IS
 -14  -6    6   14   14    6  -6 -14

LIMIT CYCLE #   25WITH FREQUENCY F= 1.250000E-01  IS
 -14  -7    4   13   14    7  -4 -13

LIMIT CYCLE #   26WITH FREQUENCY F= 1.250000E-01  IS
 -14  -8    3   12   14    8  -3 -12

LIMIT CYCLE #   27WITH FREQUENCY F= 1.250000F-01  IS
 -14 -10    0   10   14   10   0 -10

LIMIT CYCLE #   28WITH FREQUENCY F= 1.250000E-01  IS
 -14 -12   -3    8   14   12   3  -8

LIMIT CYCLE #   29WITH FREQUENCY F= 1.250000E-01  IS
 -14 -13   -4    7   14   13   4  -7

LIMIT CYCLE #   30WITH FREQUENCY F= 1.250000E-01  IS
 -13  -9    0    9   13    9   0  -9

LIMIT CYCLE #   31WITH FREQUENCY F= 1.250000E-01  IS
 -12  -5    5   12   12    5  -5 -12

LIMIT CYCLE #   32WITH FREQUENCY F= 1.250000E-01  IS
 -12  -7    2   10   12    7  -2 -10

LIMIT CYCLE #   33WITH FREQUENCY F= 1.250000E-01  IS
 -12  -8    1    9   12    8  -1  -9

LIMIT CYCLE #   34WITH FREQUENCY F= 1.250000E-01  IS
 -12  -9   -1    8   12    9   1  -8

LIMIT CYCLE #   35WITH FREQUENCY F= 1.250000E-01  IS
 -12 -10   -2    7   12   10   2  -7

LIMIT CYCLE #   36WITH FREQUENCY F= 1.285714E-01  IS
 -11  -4    5   11   10    3  -6 -11  -9   -2    6   10    8    1   -7  -11
  -8   0    8   11    7   -1  -8 -10  -6    2    9   11    6   -3  -10  -11
  -5   4   11   11    4   -5 -11 -10  -3    6   11    9    2   -6  -10   -8
  -1   7   11    8    0   -8 -11  -7   1    8   10    6   -2   -9  -11   -6
   3  10   11    5   -4  -11

LIMIT CYCLE #   37WITH FREQUENCY F= 1.250000E-01  IS
 -10  -4    4   10   10    4  -4 -10

LIMIT CYCLE #   38WITH FREQUENCY F= 1.250000E-01  IS
 -10  -5    3    9   10    5  -3  -9

LIMIT CYCLE #   39WITH FREQUENCY F= 1.250000E-01  IS
 -10  -7    0    7   10    7   0  -7
```

Table B.15: (Continued)

```
LIMIT CYCLE #  40WITH FREQUENCY F= 1.250000E-01 IS
 -10  -9  -3   5  10   9   3  -5

LIMIT CYCLE #  41WITH FREQUENCY F= 1.250000E-01 IS
  -9  -4   3   8   8   3  -4  -9

LIMIT CYCLE #  42WITH FREQUENCY F= 1.250000E-01 IS
  -9  -5   2   8   9   5  -2  -8

LIMIT CYCLE #  43WITH FREQUENCY F= 1.250000E-01 IS
  -9  -6   1   7   9   6  -1  -7

LIMIT CYCLE #  44WITH FREQUENCY F= 1.250000E-01 IS
  -9  -7  -1   6   9   7   1  -6

LIMIT CYCLE #  45WITH FREQUENCY F= 1.250000E-01 IS
  -9  -8  -2   5   9   8   2  -5

LIMIT CYCLE #  46WITH FREQUENCY F= 1.250000E-01 IS
  -8  -3   4   9   9   4  -3  -8

LIMIT CYCLE #  47WITH FREQUENCY F= 1.250000E-01 IS
  -8  -4   2   7   8   4  -2  -7

LIMIT CYCLE #  48WITH FREQUENCY F= 1.296296E-01 IS
  -8  -5   1   6   7   4  -1  -5  -6  -3   2   6   6   2  -3  -6
  -5  -1   4   7   6   1  -5  -8  -6   0   6   8   5  -1  -6  -7
  -4   1   5   6   3  -2  -6  -6  -2   3   6   5   1  -4  -7  -6
  -1   5   8   6   0  -6

LIMIT CYCLE #  49WITH FREQUENCY F= 1.250000E-01 IS
  -8  -7  -2   4   8   7   2  -4

LIMIT CYCLE #  50WITH FREQUENCY F= 1.250000E-01 IS
  -7  -3   3   7   7   3  -3  -7

LIMIT CYCLE #  51WITH FREQUENCY F= 1.250000E-01 IS
  -7  -5   0   5   7   5   0  -5

LIMIT CYCLE #  52WITH FREQUENCY F= 1.250000E-01 IS
  -6  -4   0   4   6   4   0  -4

LIMIT CYCLE #  53WITH FREQUENCY F= 1.250000E-01 IS
  -5  -2   2   5   5   2  -2  -5

LIMIT CYCLE #  54WITH FREQUENCY F= 1.250000E-01 IS
  -5  -3   1   4   5   3  -1  -4

LIMIT CYCLE #  55WITH FREQUENCY F= 1.250000E-01 IS
  -5  -4  -1   3   5   4   1  -3

LIMIT CYCLE #  56WITH FREQUENCY F= 1.250000E-01 IS
  -4  -2   1   3   3   1  -2  -4

LIMIT CYCLE #  57WITH FREQUENCY F= 1.250000E-01 IS
  -4  -3   0   3   4   3   0  -3

LIMIT CYCLE #  58WITH FREQUENCY F= 1.250000E-01 IS
  -3  -1   2   4   4   2  -1  -3

LIMIT CYCLE #  59WITH FREQUENCY F= 1.250000E-01 IS
  -3  -2   0   2   3   2   0  -2

LIMIT CYCLE #  60WITH FREQUENCY F= 1.250000E-01 IS
  -2  -1   1   2   2   1  -1  -2

LIMIT CYCLE #  61WITH FREQUENCY F= 1.666667E-01 IS
  -1   0   1   1   0  -1

LIMIT CYCLE #  62WITH FREQUENCY F= 0.000000E+00 IS    0
```

Table B.15: (Continued)

Table B.16:

LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A

```
A=-1.740000000, B= 0.958330000, APPROXIMATE FREQUENCY F= 0.075800
THE AMPLITUDE BOUNDS ARE A 1= 11.999,A 2=  4.580



LIMIT CYCLE #    1WITH FREQUENCY F= 8.333333E-02 IS
   5   0  -5  -9 -11 -10  -6   0   6  10  11   9

LIMIT CYCLE #    2WITH FREQUENCY F= 7.142857E-02 IS
 -17 -15 -10  -3   5  12  16  17  15  10   3  -5 -12 -16

LIMIT CYCLE #    3WITH FREQUENCY F= 8.333333E-02 IS
  -3   3   8  11  11   8   3  -3  -8 -11 -11  -8

LIMIT CYCLE #    4WITH FREQUENCY F= 8.333333E-02 IS
   6   0  -6 -10 -11  -9  -5   0   5   9  11  10

LIMIT CYCLE #    5WITH FREQUENCY F= 8.333333E-02 IS
 -10  -8  -4   1   6   9  10   8   4  -1  -6  -9

LIMIT CYCLE #    6WITH FREQUENCY F= 8.333333E-02 IS
 -10  -9  -6  -1   4   8  10   9   6   1  -4  -8

LIMIT CYCLE #    7WITH FREQUENCY F= 8.333333E-02 IS
  -9  -7  -3   2   6   8   8   6   2  -3  -7  -9

LIMIT CYCLE #    8WITH FREQUENCY F= 8.333333E-02 IS
  -9  -8  -5  -1   3   6   7   6   3  -1  -5  -8

LIMIT CYCLE #    9WITH FREQUENCY F= 8.333333E-02 IS
  -8  -6  -2   3   7   9   9   7   3  -2  -6  -8

LIMIT CYCLE #   10WITH FREQUENCY F= 8.333333E-02 IS
  -8  -7  -4   0   4   7   8   7   4   0  -4  -7

LIMIT CYCLE #   11WITH FREQUENCY F= 8.333333E-02 IS
  -7  -5  -2   2   5   7   7   5   2  -2  -5  -7

LIMIT CYCLE #   12WITH FREQUENCY F= 8.333333E-02 IS
  -7  -6  -3   1   5   8   9   8   5   1  -3  -6

LIMIT CYCLE #   13WITH FREQUENCY F= 8.333333E-02 IS
  -6  -4  -1   2   4   5   5   4   2  -1  -4  -6

LIMIT CYCLE #   14WITH FREQUENCY F= 8.333333E-02 IS
  -6  -5  -3   0   3   5   6   5   3   0  -3  -5

LIMIT CYCLE #   15WITH FREQUENCY F= 8.333333E-02 IS
  -5  -4  -2   1   4   6   6   4   1  -2  -4  -5

LIMIT CYCLE #   16WITH FREQUENCY F= 8.333333E-02 IS
  -4  -3  -1   1   3   4   4   3   1  -1  -3  -4

LIMIT CYCLE #   17WITH FREQUENCY F= 1.000000E-01 IS
  -3  -2   0   2   3   3   2   0  -2  -3

LIMIT CYCLE #   18WITH FREQUENCY F= 1.000000E-01 IS
  -2  -1   0   1   2   2   1   0  -1  -2

LIMIT CYCLE #   19WITH FREQUENCY F= 0.000000E+00 IS   -1

LIMIT CYCLE #   20WITH FREQUENCY F= 0.000000E+00 IS    0

LIMIT CYCLE #   21WITH FREQUENCY F= 0.000000E+00 IS    1
```

B. DIGITAL OSCILLATOR ANALYSIS

The data compiled in this section applies to the digital oscillator discussed in Section IV.D.

Given the coefficient a and initial conditions $\hat{x}(1) = 0$, $\hat{x}(2) = IC$, where IC is given by numbers from 1 to 1200, the resulting values are tabulated in tables B.17 to B.32. The columns are labelled:

AMP = amplitude of linear oscillator response,

Q = period of the limit cycle qT,

FDIFF = $|(f_o/f_s)_{linear} - (f_o/f_s)_q|$, where the two frequencies are the linear and the limit cycle frequency respectively,

ADIFF = $|AMP - \hat{A}|$, where AMP is given above and $\hat{A}$ is the estimated amplitude of the limit cycle,

DELTA = average deviation from the estimated amplitude of the limit cycle $\delta\hat{A}$.

The program allows for a maximum number of 5000 samples per limit cycle. In those cases where the evaluation of the limit cycle has to be terminated at q = 5000, the values for AMP, FDIFF, ADIFF and DELTA are set to zero.

Table B.17: Digital Oscillator Analysis

ROUND-OFF QUANTIZATION ANALYSIS, A =-1.20

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.191E-01	0.954E-06	0.120E-05
2.31	6.	0.191E-01	0.191E-05	0.260E-05
3.84	7.	0.473E-02	0.227E 00	0.159E 00
5.01	34.	0.525E-03	0.548E 00	0.265E 00
6.18	20.	0.242E-02	0.308E 00	0.227E 00
7.42	20.	0.242E-02	0.211E 00	0.334E 00
8.77	34.	0.525E-03	0.553E 00	0.334E 00
10.02	34.	0.525E-03	0.134E 00	0.205E 00
11.19	74.	0.107E-02	0.436E 00	0.324E 00
12.36	20.	0.242E-02	0.193E-01	0.213E 00
25.06	34.	0.525E-03	0.112E 00	0.240E 00
37.47	88.	0.144E-03	0.105E 01	0.590E 00
50.12	34.	0.525E-03	0.632E 00	0.292E 00
62.53	278.	0.102E-03	0.252E 01	0.109E 01
74.99	210.	0.354E-04	0.946E 00	0.457E 00
87.52	183.	0.427E-04	0.250E 00	0.761E 00
100.02	61.	0.427E-04	0.382E-01	0.376E 00
112.46	149.	0.674E-04	0.463E 00	0.649E 00
124.98	210.	0.354E-04	0.102E 01	0.850E 00
249.99	332.	0.674E-05	0.854E 00	0.892E 00
375.00	2568.	0.203E-05	0.289E 01	0.273E 01
499.99	3110.	0.477E-05	0.416E 01	0.361E 01
624.99	1782.	0.334E-05	0.110E 01	0.138E 01
749.98	332.	0.674E-05	0.114E 01	0.105E 01
875.00	393.	0.954E-06	0.396E 00	0.161E 01
999.99	1450.	0.256E-05	0.559E 01	0.360E 01
1125.00	1904.	0.417E-06	0.424E 01	0.184E 01
1250.01	393.	0.954E-06	0.219E 01	0.154E 01
0.0	5000.	0.0	0.0	0.0
1499.99	2961.	0.167E-05	0.184E 01	0.647E 01

Table B.18: Digital Oscillator Analysis

TRUNCATION QUANTIZATION ANALYSIS, A =-1.20

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.191E-01	0.954E-06	0.120E-05
2.31	6.	0.191E-01	0.191E-05	0.260E-05
3.46	6.	0.191E-01	0.286E-05	0.379E-05
4.62	6.	0.191E-01	0.572E-05	0.565E-05
6.08	13.	0.626E-02	0.142E 00	0.238E 00
7.29	13.	0.626E-02	0.426E 00	0.275E 00
8.51	26.	0.626E-02	0.858E 00	0.491E 00
9.72	26.	0.626E-02	0.357E 00	0.491E 00
11.02	46.	0.459E-02	0.329E 00	0.328E 00
12.28	33.	0.393E-02	0.731E 00	0.270E 00
24.82	114.	0.154E-02	0.699E 00	0.342E 00
37.31	74.	0.107E-02	0.175E 01	0.809E 00
49.75	370.	0.107E-02	0.574E-01	0.104E 01
62.33	27.	0.565E-03	0.389E 00	0.439E 00
74.76	344.	0.672E-03	0.630E-01	0.704E 00
87.27	54.	0.565E-03	0.547E 00	0.678E 00
99.74	54.	0.565E-03	0.665E 00	0.584E 00
112.20	27.	0.565E-03	0.192E 01	0.365E 00
124.78	196.	0.376E-03	0.712E 00	0.698E 00
249.83	88.	0.144E-03	0.276E 00	0.666E 00
374.76	2322.	0.134E-03	0.348E 01	0.293E 01
499.77	1185.	0.957E-04	0.825E 01	0.301E 01
624.75	1158.	0.848E-04	0.571E 00	0.166E 01
749.77	1998.	0.640E-04	0.820E 00	0.262E 01
874.78	508.	0.542E-04	0.416E 01	0.134E 01
999.76	1226.	0.510E-04	0.113E 01	0.223E 01
1124.76	3922.	0.451E-04	0.267E 00	0.406E 01
1249.77	2188.	0.398E-04	0.310E 01	0.173E 01
1374.77	630.	0.354E-04	0.213E 01	0.194E 01
1499.77	1741.	0.327E-04	0.447E 01	0.424E 01

Table B.19: Digital Oscillator Analysis

ROUND-OFF QUANTIZATION ANALYSIS, A =-1.30

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.293E-01	0.954E-06	0.120E-05
2.83	8.	0.124E-01	0.858E-01	0.607E-01
3.97	22.	0.102E-02	0.217E 00	0.227E 00
5.29	22.	0.102E-02	0.378E 00	0.327E 00
6.62	22.	0.102E-02	0.584E 00	0.278E 00
7.83	36.	0.150E-02	0.526E 00	0.357E 00
9.26	22.	0.102E-02	0.265E 00	0.265E 00
10.59	22.	0.102E-02	0.520E 00	0.270E 00
11.91	22.	0.102E-02	0.206E 00	0.269E 00
13.23	22.	0.102E-02	0.493E-01	0.136E 00
26.35	226.	0.216E-03	0.165E 01	0.742E 00
39.50	51.	0.130E-03	0.111E 01	0.682E 00
52.67	51.	0.130E-03	0.428E 00	0.300E 00
65.90	124.	0.288E-03	0.473E 00	0.969E 00
78.91	80.	0.116E-03	0.925E 00	0.512E 00
92.06	80.	0.116E-03	0.439E 00	0.860E 00
105.30	932.	0.454E-04	0.317E 01	0.191E 01
118.48	386.	0.787E-04	0.224E 01	0.948E 00
131.61	182.	0.218E-04	0.191E 01	0.932E 00
263.21	910.	0.218E-04	0.997E 00	0.148E 01
394.81	3356.	0.185E-04	0.340E 01	0.267E 01
526.39	1798.	0.960E-05	0.275E 00	0.186E 01
657.98	808.	0.823E-05	0.258E 01	0.210E 01
0.0	5000.	0.0	0.0	0.0
0.0	5000.	0.0	0.0	0.0
1052.75	626.	0.423E-05	0.295E 00	0.138E 01
1184.35	3312.	0.525E-05	0.412E 01	0.210E 01
1315.95	3494.	0.608E-05	0.704E 00	0.409E 01
0.0	5000.	0.0	0.0	0.0
1579.12	626.	0.423E-05	0.263E 01	0.273E 01

Table B.20: Digital Oscillator Analysis

TRUNCATION QUANTIZATION ANALYSIS, A =-1.30

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.293E-01	0.954E-06	0.120E-05
2.31	6.	0.293E-01	0.191E-05	0.260E-05
3.46	6.	0.293E-01	0.286E-05	0.379E-05
4.94	20.	0.126E-01	0.205E 00	0.208E 00
6.27	34.	0.967E-02	0.436E 00	0.362E 00
7.59	62.	0.778E-02	0.759E 00	0.452E 00
8.95	7.	0.547E-02	0.180E 00	0.931E-01
10.33	78.	0.364E-02	0.817E 00	0.453E 00
11.51	7.	0.547E-02	0.123E 00	0.816E-01
12.79	7.	0.547E-02	0.407E 00	0.235E 00
25.96	50.	0.262E-02	0.177E 01	0.811E 00
39.16	72.	0.150E-02	0.160E 00	0.103E 01
52.31	166.	0.117E-02	0.218E 00	0.458E 00
65.47	94.	0.913E-03	0.122E 01	0.798E 00
78.65	210.	0.711E-03	0.106E 01	0.683E 00
91.79	326.	0.652E-03	0.325E 00	0.777E 00
104.94	790.	0.590E-03	0.214E 01	0.207E 01
118.14	428.	0.466E-03	0.411E 01	0.156E 01
131.30	254.	0.411E-03	0.605E 00	0.116E 01
262.99	298.	0.199E-03	0.167E 01	0.927E 00
394.46	778.	0.148E-03	0.489E 01	0.217E 01
0.0	5000.	0.0	0.0	0.0
657.64	3528.	0.872E-04	0.998E 01	0.435E 01
0.0	5000.	0.0	0.0	0.0
920.80	4176.	0.677E-04	0.714E 01	0.525E 01
1052.40	2532.	0.563E-04	0.339E 01	0.302E 01
1183.99	1950.	0.514E-04	0.409E 01	0.502E 01
1315.58	2132.	0.452E-04	0.737E 01	0.356E 01
1447.18	1841.	0.409E-04	0.826E 01	0.363E 01
1578.77	2234.	0.372E-04	0.387E 01	0.271E 01

Table B.21: Digital Oscillator Analysis

ROUND-OFF QUANTIZATION ANALYSIS, A =-1.40

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.401E-01	0.954E-06	0.120E-05
2.83	8.	0.159E-02	0.858E-01	0.607E-01
4.24	8.	0.159E-02	0.121E 00	0.858E-01
5.66	8.	0.159E-02	0.172E 00	0.121E 00
7.07	8.	0.159E-02	0.355E-01	0.251E-01
8.25	54.	0.304E-02	0.122E 01	0.588E 00
9.90	8.	0.159E-02	0.503E-01	0.355E-01
11.07	70.	0.198E-02	0.302E-01	0.408E 00
12.73	8.	0.159E-02	0.136E 00	0.962E-01
14.14	8.	0.159E-02	0.711E-01	0.503E-01
27.99	150.	0.750E-04	0.140E 01	0.585E 00
42.05	87.	0.155E-03	0.226E 01	0.759E 00
56.04	166.	0.857E-04	0.789E 00	0.939E 00
70.05	166.	0.857E-04	0.133E 01	0.839E 00
84.06	166.	0.857E-04	0.493E 00	0.662E 00
98.01	466.	0.178E-04	0.131E 00	0.520E 00
112.03	474.	0.936E-05	0.180E 01	0.147E 01
125.97	150.	0.750E-04	0.574E 00	0.559E 00
140.04	316.	0.936E-05	0.123E 01	0.150E 01
280.07	474.	0.936E-05	0.407E 00	0.138E 01
420.11	158.	0.936E-05	0.295E 01	0.945E 00
560.14	79.	0.936E-05	0.134E 01	0.715E 00
700.18	316.	0.936E-05	0.875E 00	0.198E 01
840.17	4400.	0.715E-06	0.313E 01	0.533E 01
980.25	79.	0.936E-05	0.517E 00	0.529E 00
1120.23	1414.	0.417E-06	0.757E 00	0.277E 01
1260.25	1256.	0.715E-06	0.636E 01	0.213E 01
1400.28	3926.	0.298E-06	0.207E 01	0.323E 01
1540.32	1572.	0.131E-05	0.163E 01	0.167E 01
1680.34	1414.	0.417E-06	0.177E 01	0.150E 01

Table B.22: Digital Oscillator Analysis

TRUNCATION QUANTIZATION ANALYSIS, A =-1.40

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.401E-01	0.954E-06	0.120E-05
2.31	6.	0.401E-01	0.191E-05	0.260E-05
3.84	7.	0.163E-01	0.227E 00	0.159E 00
5.29	22.	0.977E-02	0.378E 00	0.327E 00
6.62	22.	0.977E-02	0.254E 00	0.202E 00
8.07	15.	0.674E-02	0.438E 00	0.422E 00
9.26	22.	0.977E-02	0.596E 00	0.514E 00
10.87	38.	0.499E-02	0.284E 00	0.373E 00
12.11	15.	0.674E-02	0.245E 00	0.264E 00
13.59	38.	0.499E-02	0.734E-02	0.399E 00
27.69	288.	0.188E-02	0.240E 01	0.129E 01
41.67	86.	0.132E-02	0.145E 01	0.704E 00
55.66	478.	0.102E-02	0.171E 01	0.843E 00
69.65	102.	0.859E-03	0.783E 00	0.755E 00
83.67	110.	0.681E-03	0.240E 00	0.726E 00
97.79	236.	0.527E-03	0.541E 01	0.280E 01
111.66	118.	0.527E-03	0.237E 01	0.103E 01
125.72	189.	0.392E-03	0.237E 01	0.115E 01
139.79	134.	0.274E-03	0.129E 01	0.648E 00
279.73	702.	0.189E-03	0.284E 01	0.134E 01
419.74	1665.	0.135E-03	0.449E 01	0.291E 01
559.74	1618.	0.108E-03	0.223E 00	0.195E 01
699.82	150.	0.750E-04	0.204E 01	0.103E 01
839.83	1358.	0.652E-04	0.120E 01	0.264E 01
979.85	4398.	0.568E-04	0.437E 01	0.290E 01
1119.87	1674.	0.511E-04	0.685E 01	0.418E 01
1259.90	4430.	0.449E-04	0.124E 02	0.714E 01
0.0	5000.	0.0	0.0	0.0
1539.97	3072.	0.359E-04	0.164E 01	0.324E 01
0.0	5000.	0.0	0.0	0.0

Table B.23: Digital Oscillator Analysis

ROUND-OFF QUANTIZATION ANALYSIS, A =-1.50

AMP	Q	FDIFF	ADIFF	DELTA
1.70	10.	0.150E-01	0.291E 00	0.127E 00
3.40	10.	0.150E-01	0.180E 00	0.785E-01
5.10	10.	0.150E-01	0.111E 00	0.485E-01
6.22	9.	0.392E-02	0.225E 00	0.125E 00
7.92	46.	0.633E-02	0.649E 00	0.489E 00
9.33	9.	0.392E-02	0.146E 00	0.861E-01
10.89	9.	0.392E-02	0.381E 00	0.181E 00
12.34	98.	0.278E-02	0.574E 00	0.623E 00
13.82	62.	0.212E-02	0.766E 00	0.348E 00
15.56	9.	0.392E-02	0.371E 00	0.163E 00
30.45	114.	0.992E-03	0.670E 00	0.544E 00
45.68	114.	0.992E-03	0.125E 01	0.739E 00
60.67	96.	0.443E-03	0.487E 00	0.338E 00
75.83	288.	0.443E-03	0.769E 00	0.848E 00
90.98	349.	0.414E-03	0.157E 00	0.122E 01
106.04	61.	0.273E-03	0.226E 00	0.493E 00
121.18	61.	0.273E-03	0.755E 00	0.538E 00
136.33	305.	0.273E-03	0.137E 01	0.146E 01
151.41	270.	0.212E-03	0.233E 00	0.923E 00
302.64	557.	0.125E-03	0.214E 00	0.955E 00
453.80	1418.	0.761E-04	0.435E 01	0.265E 01
604.97	1122.	0.535E-04	0.380E 01	0.156E 01
756.21	1409.	0.516E-04	0.497E 01	0.276E 01
907.38	1061.	0.409E-04	0.103E 01	0.132E 01
0.0	5000.	0.0	0.0	0.0
0.0	5000.	0.0	0.0	0.0
1360.96	3974.	0.293E-04	0.100E 02	0.401E 01
1512.15	600.	0.268E-04	0.134E 01	0.225E 01
1663.32	3226.	0.237E-04	0.920E-01	0.269E 01
0.0	5000.	0.0	0.0	0.0

Table B.24: Digital Oscillator Analysis

TRUNCATION QUANTIZATION ANALYSIS, A =-1.50

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.516E-01	0.954E-06	0.120E-05
2.83	8.	0.997E-02	0.858E-01	0.607E-01
4.24	8.	0.997E-02	0.121E 00	0.858E-01
5.88	42.	0.402E-02	0.693E 00	0.327E 00
7.07	8.	0.997E-02	0.355E-01	0.251E-01
8.82	42.	0.402E-02	0.108E 01	0.591E 00
10.29	42.	0.402E-02	0.615E 00	0.365E 00
12.06	26.	0.358E-03	0.297E 00	0.250E 00
13.23	42.	0.402E-02	0.668E 00	0.356E 00
14.91	94.	0.199E-02	0.284E 00	0.425E 00
30.00	198.	0.113E-02	0.102E 00	0.780E 00
45.09	164.	0.827E-03	0.155E 01	0.738E 00
60.32	26.	0.358E-03	0.336E 00	0.264E 00
75.40	26.	0.358E-03	0.122E 00	0.189E 00
90.48	26.	0.358E-03	0.323E 00	0.200E 00
105.56	26.	0.358E-03	0.584E 00	0.495E 00
120.71	451.	0.273E-03	0.388E 01	0.130E 01
135.72	26.	0.358E-03	0.215E 00	0.215E 00
150.96	512.	0.208E-03	0.277E 01	0.106E 01
302.10	660.	0.125E-03	0.579E 01	0.259E 01
453.26	582.	0.935E-04	0.322E 00	0.123E 01
604.45	530.	0.676E-04	0.259E 01	0.968E 00
755.66	2494.	0.494E-04	0.382E 01	0.182E 01
906.85	1825.	0.417E-04	0.242E 00	0.281E 01
1058.00	1208.	0.395E-04	0.147E 01	0.213E 01
0.0	5000.	0.0	0.0	0.0
1360.41	2790.	0.270E-04	0.400E 00	0.375E 01
1511.57	930.	0.270E-04	0.204E 01	0.141E 01
1662.78	3668.	0.224E-04	0.146E 01	0.394E 01
0.0	5000.	0.0	0.0	0.0

Table B.25: Digital Oscillator Analysis

ROUND-OFF QUANTIZATION ANALYSIS, A =-1.60

AMP	Q	FDIFF	ADIFF	DELTA
1.70	10.	0.242E-02	0.291E 00	0.127E 00
3.40	10.	0.242E-02	0.180E 00	0.785E-01
5.10	10.	0.242E-02	0.111E 00	0.485E-01
6.81	10.	0.242E-02	0.359E 00	0.157E 00
8.51	10.	0.242E-02	0.686E-01	0.300E-01
10.21	10.	0.242E-02	0.222E 00	0.970E-01
11.91	10.	0.242E-02	0.248E 00	0.108E 00
13.61	10.	0.242E-02	0.424E-01	0.185E-01
14.78	48.	0.175E-02	0.652E 00	0.243E 00
17.01	10.	0.242E-02	0.137E 00	0.600E-01
33.37	88.	0.144E-03	0.113E 01	0.615E 00
50.00	166.	0.679E-05	0.602E 00	0.812E 00
66.58	39.	0.148E-03	0.103E 01	0.759E 00
83.34	498.	0.679E-05	0.236E 01	0.166E 01
100.01	166.	0.679E-05	0.234E 01	0.851E 00
116.67	166.	0.679E-05	0.777E 00	0.621E 00
133.38	674.	0.426E-04	0.206E 01	0.208E 01
150.01	498.	0.679E-05	0.250E 01	0.165E 01
166.68	830.	0.679E-05	0.747E 00	0.224E 01
333.38	625.	0.165E-04	0.610E 01	0.298E 01
500.03	166.	0.679E-05	0.881E 00	0.845E 00
666.66	3310.	0.477E-06	0.127E 02	0.486E 01
833.33	3310.	0.477E-06	0.698E 00	0.263E 01
1000.00	1904.	0.477E-06	0.346E 00	0.201E 01
1166.73	498.	0.679E-05	0.350E 01	0.227E 01
0.0	5000.	0.0	0.0	0.0
1500.01	3974.	0.715E-06	0.101E 02	0.491E 01
1666.68	4306.	0.119E-05	0.336E 00	0.464E 01
1833.33	3476.	0.119E-06	0.768E 01	0.609E 01
1999.99	1572.	0.894E-06	0.381E 01	0.344E 01

Table B.26: Digital Oscillator Analysis

TRUNCATION QUANTIZATION ANALYSIS, A =-1.60

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.643E-01	0.954E-06	0.120E-05
2.83	8.	0.226E-01	0.858E-01	0.607E-01
4.24	8.	0.226E-01	0.121E 00	0.858E-01
6.22	9.	0.869E-02	0.225E 00	0.125E 00
7.78	9.	0.869E-02	0.302E 00	0.146E 00
9.51	46.	0.628E-02	0.423E 00	0.724E 00
11.23	28.	0.473E-02	0.379E 00	0.403E 00
12.68	46.	0.628E-02	0.743E 00	0.597E 00
14.43	28.	0.473E-02	0.236E 00	0.409E 00
16.18	66.	0.364E-02	0.459E 00	0.467E 00
32.72	86.	0.223E-02	0.424E 00	0.620E 00
49.28	48.	0.175E-02	0.458E 00	0.575E 00
66.10	116.	0.103E-02	0.928E 00	0.134E 01
82.81	378.	0.758E-03	0.377E 00	0.121E 01
99.37	126.	0.758E-03	0.476E 00	0.713E 00
116.16	204.	0.525E-03	0.123E 01	0.997E 00
132.82	243.	0.464E-03	0.952E 00	0.102E 01
149.44	768.	0.448E-03	0.681E 00	0.203E 01
166.15	574.	0.371E-03	0.115E 01	0.143E 01
332.82	692.	0.185E-03	0.175E 01	0.256E 01
499.43	2116.	0.136E-03	0.282E 01	0.218E 01
666.12	1551.	0.981E-04	0.287E 01	0.248E 01
832.77	3444.	0.806E-04	0.378E 01	0.412E 01
999.43	1610.	0.681E-04	0.147E 00	0.218E 01
1166.07	888.	0.610E-04	0.148E 01	0.186E 01
1332.75	810.	0.527E-04	0.306E 01	0.316E 01
1499.46	732.	0.426E-04	0.149E 01	0.195E 01
0.0	5000.	0.0	0.0	0.0
0.0	5000.	0.0	0.0	0.0
1999.44	449.	0.334E-04	0.163E 01	0.159E 01

Table B.27: Digital Oscillator Analysis

ROUND-OFF QUANTIZATION ANALYSIS, A =-1.70

AMP	Q	FDIFF	ADIFF	DELTA
1.70	10.	0.117E-01	0.291E 00	0.127E 00
3.40	10.	0.117E-01	0.180E 00	0.785E-01
6.00	12.	0.497E-02	0.113E 00	0.801E-01
8.00	12.	0.497E-02	0.414E-01	0.293E-01
9.50	34.	0.657E-04	0.233E 00	0.426E 00
11.40	34.	0.657E-04	0.482E 00	0.265E 00
13.16	56.	0.985E-03	0.754E 00	0.477E 00
15.20	34.	0.657E-04	0.937E 00	0.295E 00
17.26	126.	0.999E-03	0.733E 00	0.111E 01
19.00	34.	0.657E-04	0.434E 00	0.453E 00
38.19	57.	0.582E-03	0.237E 00	0.924E 00
56.99	102.	0.657E-04	0.100E 01	0.754E 00
75.98	34.	0.657E-04	0.284E 00	0.662E 00
94.98	238.	0.657E-04	0.159E 01	0.187E 01
113.97	782.	0.657E-04	0.272E 01	0.295E 01
132.97	34.	0.657E-04	0.283E 00	0.557E 00
151.97	102.	0.657E-04	0.169E 00	0.761E 00
170.96	34.	0.657E-04	0.310E 00	0.338E 00
189.96	34.	0.657E-04	0.133E 00	0.302E 00
379.92	34.	0.657E-04	0.221E 00	0.150E 00
569.87	34.	0.657E-04	0.354E 00	0.363E 00
759.41	3228.	0.110E-04	0.476E 01	0.293E 01
0.0	5000.	0.0	0.0	0.0
1139.10	1042.	0.924E-05	0.300E 01	0.322E 01
1328.95	1042.	0.924E-05	0.415E 01	0.368E 01
1518.75	1982.	0.632E-05	0.318E 01	0.451E 01
1708.55	1427.	0.387E-05	0.174E 01	0.414E 01
1898.36	3726.	0.250E-05	0.921E 01	0.428E 01
2088.24	2888.	0.459E-05	0.478E 01	0.397E 01
2278.08	1914.	0.423E-05	0.425E 01	0.307E 01

Table B.28: Digital Oscillator Analysis

TRUNCATION QUANTIZATION ANALYSIS, A =-1.70

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.784E-01	0.954E-06	0.120E-05
3.40	10.	0.117E-01	0.180E 00	0.785E-01
5.10	10.	0.117E-01	0.111E 00	0.485E-01
6.81	10.	0.117E-01	0.359E 00	0.157E 00
8.51	10.	0.117E-01	0.686E-01	0.300E-01
10.80	32.	0.545E-02	0.609E 00	0.409E 00
11.91	10.	0.117E-01	0.248E 00	0.108E 00
14.40	32.	0.545E-02	0.549E 00	0.380E 00
16.20	32.	0.545E-02	0.364E 00	0.376E 00
18.00	32.	0.545E-02	0.260E 00	0.198E 00
37.06	474.	0.242E-02	0.633E 00	0.126E 01
56.13	78.	0.144E-02	0.887E-02	0.453E 00
75.09	302.	0.110E-02	0.314E 00	0.108E 01
94.35	45.	0.588E-03	0.800E 00	0.984E 00
113.00	550.	0.790E-03	0.293E 01	0.209E 01
131.97	236.	0.682E-03	0.559E 00	0.1[illegible]3E 01
151.00	934.	0.564E-03	0.161E 01	0.217E 01
170.01	642.	0.484E-03	0.194E 01	0.147E 01
188.91	214.	0.484E-03	0.224E 01	0.999E 00
378.74	192.	0.241E-03	0.172E 01	0.174E 01
568.57	260.	0.161E-03	0.395E 00	0.140E 01
758.55	1414.	0.101E-03	0.101E 01	0.201E 01
948.36	396.	0.829E-04	0.894E 00	0.201E 01
1138.10	826.	0.768E-04	0.284E 01	0.196E 01
1327.96	679.	0.643E-04	0.608E 01	0.458E 01
1517.76	713.	0.581E-04	0.847E 00	0.282E 01
0.0	5000.	0.0	0.0	0.0
1897.41	781.	0.473E-04	0.137E 01	0.456E 01
2087.29	4992.	0.404E-04	0.120E 02	0.993E 01
2277.10	1132.	0.383E-04	0.166E 01	0.255E 01

Table B.29: Digital Oscillator Analysis

ROUND-OFF QUANTIZATION ANALYSIS, A =-1.80

AMP	Q	FDIFF	ADIFF	DELTA
2.30	14.	0.355E-03	0.508E 00	0.210E 00
4.61	14.	0.355E-03	0.459E 00	0.113E 00
6.91	14.	0.355E-03	0.594E 00	0.166E 00
9.22	14.	0.355E-03	0.861E-01	0.931E-01
11.52	14.	0.355E-03	0.135E 00	0.816E-01
13.83	14.	0.355E-03	0.373E 00	0.132E 00
16.13	14.	0.355E-03	0.324E 00	0.106E 00
18.44	14.	0.355E-03	0.729E 00	0.236E 00
20.74	14.	0.355E-03	0.221E 00	0.567E-01
23.05	14.	0.355E-03	0.270E 00	0.163E 00
46.10	14.	0.355E-03	0.163E-01	0.366E-01
69.14	14.	0.355E-03	0.254E 00	0.129E 00
92.19	14.	0.355E-03	0.325E-01	0.732E-01
115.24	14.	0.355E-03	0.238E 00	0.960E-01
137.50	654.	0.823E-04	0.241E 00	0.210E 01
161.33	14.	0.355E-03	0.222E 00	0.674E-01
183.52	1184.	0.739E-05	0.156E 01	0.398E 01
207.43	14.	0.355E-03	0.205E 00	0.504E-01
229.20	167.	0.731E-04	0.168E 01	0.916E 00
458.91	418.	0.128E-04	0.286E 01	0.228E 01
688.26	404.	0.954E-06	0.342E 01	0.196E 01
917.67	404.	0.954E-06	0.449E-01	0.189E 01
1147.09	404.	0.954E-06	0.212E 01	0.151E 01
1376.40	794.	0.525E-05	0.167E 00	0.167E 01
1605.93	404.	0.954E-06	0.336E 01	0.145E 01
0.0	5000.	0.0	0.0	0.0
0.0	5000.	0.0	0.0	0.0
2294.14	2814.	0.775E-06	0.759E-01	0.600E 01
2523.61	404.	0.954E-06	0.561E 01	0.253E 01
0.0	5000.	0.0	0.0	0.0

Table B.30: Digital Oscillator Analysis

TRUNCATION QUANTIZATION ANALYSIS, A =-1.80

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.949E-01	0.954E-06	0.120E-05
3.40	10.	0.282E-01	0.180E 00	0.785E-01
6.00	12.	0.116E-01	0.113E 00	0.801E-01
8.00	12.	0.116E-01	0.414E-01	0.293E-01
10.51	38.	0.716E-02	0.909E 00	0.479E 00
12.36	62.	0.886E-02	0.690E 00	0.823E 00
15.06	39.	0.514E-02	0.113E 00	0.697E 00
16.81	38.	0.716E-02	0.531E 00	0.290E 00
19.64	66.	0.397E-02	0.769E 00	0.680E 00
21.82	66.	0.397E-02	0.201E 01	0.121E 01
44.87	68.	0.175E-02	0.156E 01	0.991E 00
67.52	232.	0.149E-02	0.133E 01	0.165E 01
90.66	55.	0.944E-03	0.624E 00	0.101E 01
113.53	248.	0.798E-03	0.318E 01	0.258E 01
136.35	262.	0.736E-03	0.157E 01	0.156E 01
159.38	152.	0.585E-03	0.129E 00	0.125E 01
182.42	346.	0.471E-03	0.231E 00	0.254E 01
205.31	180.	0.439E-03	0.372E 01	0.168E 01
228.36	402.	0.356E-03	0.210E 01	0.154E 01
457.64	514.	0.201E-03	0.345E 01	0.227E 01
686.83	139.	0.159E-03	0.615E 00	0.177E 01
916.45	626.	0.102E-03	0.586E-01	0.215E 01
0.0	5000.	0.0	0.0	0.0
1375.35	682.	0.644E-04	0.740E 00	0.379E 01
1604.71	3466.	0.576E-04	0.144E 02	0.424E 01
1834.09	4218.	0.519E-04	0.171E 02	0.812E 01
2063.52	4998.	0.456E-04	0.120E 00	0.527E 01
2292.97	181.	0.401E-04	0.130E 01	0.249E 01
0.0	5000.	0.0	0.0	0.0
2751.76	2938.	0.344E-04	0.608E 01	0.602E 01

Table B.31: Digital Oscillator Analysis

ROUND-OFF QUANTIZATION ANALYSIS, A =-1.90

AMP	Q	FDIFF	ADIFF	DELTA
4.18	26.	0.121E-01	0.108E 01	0.434E 00
5.85	18.	0.501E-02	0.972E 00	0.278E 00
8.77	18.	0.501E-02	0.417E 00	0.861E-01
14.20	22.	0.509E-02	0.147E 00	0.153E 00
16.18	20.	0.541E-03	0.139E 01	0.321E 00
19.42	20.	0.541E-03	0.587E 00	0.152E 00
22.65	20.	0.541E-03	0.517E 00	0.959E-01
25.66	218.	0.826E-04	0.624E 01	0.289E 01
28.19	58.	0.118E-02	0.164E 00	0.493E 00
32.01	178.	0.205E-04	0.425E 01	0.278E 01
64.72	20.	0.541E-03	0.482E 00	0.154E 00
96.46	298.	0.206E-03	0.547E 00	0.172E 01
128.31	654.	0.826E-04	0.455E 01	0.341F 01
161.80	20.	0.541E-03	0.991E-01	0.815E-01
192.91	298.	0.206E-03	0.462E 01	0.222E 01
224.84	258.	0.154E-03	0.293E 01	0.262E 01
256.96	258.	0.154E-03	0.623E 00	0.104E 01
288.69	218.	0.826E-04	0.306E 01	0.143E 01
320.52	1604.	0.426E-04	0.199E 02	0.665E 01
640.88	1564.	0.298E-04	0.204E 02	0.101E 02
961.10	3820.	0.178E-04	0.116E 01	0.474E 01
1281.26	376.	0.941E-05	0.117E 01	0.249E 01
1601.59	3483.	0.102E-04	0.634E 01	0.136E 02
1921.88	376.	0.941E-05	0.350E 01	0.233E 01
0.0	5000.	0.0	0.0	0.0
2562.39	3087.	0.682E-05	0.716E 01	0.998E 01
0.0	5000.	0.0	0.0	0.0
0.0	5000.	0.0	0.0	0.0
0.0	5000.	0.0	0.0	0.0
0.0	5000.	0.0	0.0	0.0

Table B.32: Digital Oscillator Analysis

TRUNCATION QUANTIZATION ANALYSIS, A =-1.90

AMP	Q	FDIFF	ADIFF	DELTA
1.15	6.	0.116E 00	0.954E-06	0.120E-05
3.49	10.	0.495E-01	0.180E 00	0.785E-01
6.91	14.	0.209E-01	0.594E 00	0.166E 00
11.70	18.	0.501E-02	0.125E 01	0.278E 00
13.07	16.	0.120E-01	0.134E 00	0.844E-01
15.68	16.	0.120E-01	0.424E 00	0.168E 00
20.47	18.	0.501E-02	0.829E 00	0.274E 00
23.39	18.	0.501E-02	0.124E 00	0.865E-01
26.31	18.	0.501E-02	0.143E 00	0.540E-01
29.93	166.	0.368E-02	0.630E 01	0.284E 01
61.84	248.	0.188E-02	0.721E 01	0.314E 01
93.17	115.	0.163E-02	0.149E 01	0.104E 01
125.27	406.	0.118E-02	0.250E 01	0.354E 01
157.34	136.	0.929E-03	0.250E 01	0.197E 01
189.47	39.	0.741E-03	0.868E 00	0.764E 00
221.05	39.	0.741E-03	0.307E 00	0.631E 00
253.33	176.	0.595E-03	0.166E 01	0.992E 00
285.62	98.	0.479E-03	0.810E 00	0.264E 01
317.50	804.	0.454E-03	0.322E 01	0.284E 01
637.71	1556.	0.230E-03	0.488E 01	0.460E 01
958.14	730.	0.144E-03	0.598E 01	0.571E 01
1278.19	2586.	0.116E-03	0.480E 01	0.558E 01
1598.48	1422.	0.916E-04	0.194E 01	0.583E 01
1918.75	810.	0.760E-04	0.111E 01	0.363E 01
2238.97	494.	0.660E-04	0.790E 01	0.344E 01
2559.24	2589.	0.574E-04	0.404E 01	0.958E 01
2879.58	2708.	0.495E-04	0.121E 02	0.623E 01
3199.69	850.	0.469E-04	0.268E 01	0.292E 01
3520.09	2234.	0.406E-04	0.268E 01	0.565E 01
3840.21	1206.	0.391E-04	0.905E 00	0.460E 01

Computer Program A: Analysis Program for Zero-input Limit Cycle Oscillations in Digital Filters with Roundoff.

```
ONE: PROCEDURE OPTIONS(MAIN);
     /* EVALUATION OF ALL POSSIBLE LIMIT CYCLES DUE TO
        ROUNDING AFTER MULTIPLICATIONS FOR DIGITAL
        FILTERS WITH TWO POLES */
DECLARE (A,B) DECIMAL FIXED (9,6);
DECLARE (F,F_APPROX,TWO_PI) DECIMAL FLOAT(12);
DECLARE DF_A ENTRY (DECIMAL FIXED(6),DECIMAL FIXED(6),
DECIMAL FIXED(6)) RETURNS(DECIMAL FIXED(6));
OPEN FILE(SYSPRINT) PAGESIZE(75);
     TWO_PI=6.28318537;
     A=-1.94;
     B=0.95833;
     DO LL=1 TO 37;
     A=A+0.1;
NINE:A1=0.5/(1.0-ABS(B));
     A2=1./(1.+B-ABS(A));
     IF (4.0*B-A**2) < 0
          THEN DO;
               F_APPROX=0.0;
               GOTO TEN;
               END;
     IF A>-.01 & A<.01
          THEN F_APPROX=0.25;
          ELSE F_APPROX=ATAN(-SQRT(4.*B-A**2)/A)/TWO_PI;
     IF F_APPROX < 0
          THEN F_APPROX=0.5+F_APPROX;
TEN: PUT EDIT
     ('LIMIT CYCLE OSCILLATIONS OF DIGITAL FILTER, TYPE A')
     (PAGE,LINE(5),COLUMN(20),A);
     PUT EDIT('A=',A,', B=',B,', APPROXIMATE FREQUENCY F=',
     F_APPROX,'THE AMPLITUDE BOUNDS ARE A 1=',A1,',A 2=',
     A2)(LINE(8),COLUMN(20),A,F(12,9),A,F(12,9),A,F(9,6),
     LINE(9),COLUMN(20),A,F(7,3),A,F(7,3));
     PUT SKIP(4);
     A1=A1 + 6.0;
     A2=A2+5.0;
     IF A1>=A2
          THEN II=A1;
          ELSE II=A2;
TWO: BEGIN;
     /* GO THROUGH FILTER RESPONSES FOR ALL POSSIBLE
        INITIAL CONDITIONS */
DECLARE MATRIX(-II:II,-II:II) BINARY FIXED(6);
DECLARE U DECIMAL FIXED(6);
     M=1; U=0;
     ILIM=10*A1;
     /* INITIALIZE THE MATRIX TO -1 */
     DO I=-II TO II;
     DO J=-II TO II;
     MATRIX(I,J)=-1;
     END;
     END;
     DO I=-II TO II;
     /* SUBSCRIPT I DENOTES ROW OF MATRIX OR X(N) */
     DO J=-II TO II;
     /* SUBSCRIPT J DENOTES COLUMN OF MATRIX OF X(N-1) */
     N=1;ISIGN=0;IFLAG=0;
THREE:  /*RUN TRANSIENT RESPONSE FOR ONE INITIAL CONDITION*/
BEGIN;
DECLARE LIMIT(ILIM) BINARY FIXED(15),
     (X1,X2) DECIMAL FIXED(6);
     DO L=1 TO ILIM;      /* INITIALIZE LIMIT TO 0 */
     LIMIT(L)=0;
     END;
     IF MATRIX(I,J)>0
          THEN GOTO SIX;
```

```
      MATRIX(I,J)=0;
      X1=I;
      X2=J;
FOUR:       K=DF_A(X1,X2,U);
      IF ABS(K)>II | ABS(X1)>II
           THEN GOTO FIVE;
      IF MATRIX(K,X1)=-1
           THEN DO;   /*NEW TRANSIENT POINT DETECTED*/
                MATRIX(K,X1)=0;
           FIVE:X2=X1;
                X1=K;
                GOTO FOUR;
                END;
           IF MATRIX(K,X1)=0
                THEN DO; /*NEW LIMIT CYCLE DETECTED*/
                IFLAG=1;
                MATRIX(K,X1)=M;
                LIMIT(N)=K;
                N=N+1;
                IF SIGN(K)=SIGN(X1)
                     THEN GOTO FIVE;
                     ELSE IF SIGN(SIGN(K)+SIGN(X1))=1
                          THEN GOTO FIVE;
                          ELSE DO;
                          /* SIGN-CHANGE OCURRED INDICATING
                             NEW HALF-CYCLE OF OSCILLATION */
                               ISIGN=ISIGN+1;
                               GOTO FIVE;
                               END;
                END;
           /* OLD LIMIT CYCLE OR END OF PRESENT LIMIT CYCLE
              DETECTED */
           DO I2=-II TO II;
           /* RESET ALL MATRIX POINTS EXCEPT LIMIT CYCLE */
           DO J2=-II TO II;
           IF MATRIX(I2,J2)=0
                THEN MATRIX(I2,J2)=-1;
           END;
           END;
           IF IFLAG=0
                THEN GOTO SIX;
           F=ISIGN;
           F_APPROX=2*(N-1);
           F=F/F_APPROX;
           PUT SKIP(2);
           PUT EDIT('LIMIT CYCLE # ',M,'WITH FREQUENCY F=',
           F,' IS')(COLUMN(20),A,F(3),A,E(13,6),A,X(3));
           IF (N-1) <= 4
                THEN PUT EDIT
                ((LIMIT(L) DO L=1 TO N-1))(F(4));
                ELSE DO;
                     N1=0;
                     PUT EDIT(' ')(SKIP,COLUMN(19),A);
                     DO L=1 TO N-1;
                     PUT EDIT(LIMIT(L))(F(4));
                     N1=N1+1;
                     IF N1=16 THEN DO;
                          PUT EDIT(' ')(SKIP,COLUMN(19),A);
                          N1=0;
                          END;
                     END;
                     END;
           M=M+1;
END THREE;
      SIX:END;
      END;
SEVEN:  /* WRITE MATRIX OF SIZE 15 X 15 OR LESS */
BEGIN;
      PUT EDIT('LIMIT CYCLES ARRANGED IN PHASE PLANE X(N) VS.
                (PAGE,SKIP(10),COLUMN(20),A);
      PUT EDIT('A=',A,',  B=',B)
                (SKIP(1),COLUMN(20),A,F(12,8),A,F(12,8));
```

```
      PUT SKIP(5);
      IF II < 15
           THEN I3=II;
           ELSE I3=15;
      DO I2=-I3 TO I3;
      IF I2=-I3
           THEN PUT EDIT(I2)(COLUMN(20),F(3));
           ELSE PUT EDIT(I2)(F(3));
      END;
      PUT SKIP(2);
      DO J2=I3 TO -I3 BY -1;
      PUT EDIT(J2)(COLUMN(17),F(3));
      DO I2=-I3 TO I3;
      IF MATRIX(I2,J2)>0
           THEN PUT EDIT(MATRIX(I2,J2))(F(3));
           ELSE PUT EDIT('   ')(A(3));
      END;
      PUT SKIP(2);
      END;
END SEVEN;
END TWO;
END;
DF_A:PROCEDURE (X1,X2,U) DECIMAL FIXED(6);
     /* FUNCTION PROCEDURE TO SOLVE DIFFERENCE EQUATION
        FOR TWO POLE FILTER WITH ROUND-OFF AFTER
        MULTIPLICATIONS */
DECLARE (X,X1,X2,U) DECIMAL FIXED(6);
     X=ROUND(-A*X1,0) + ROUND(-B*X2,0) + U;
     RETURN(X);
END DF_A;
END ONE;
```

Computer Program B: Evaluation of Two Amplitude Bounds for Zero-input Limit Cycle Oscillations in Digital Filters with Roundoff.

```
C   THE PROGRAM COMPUTES THE NUMERICAL VALUES FOR THE
C   AMPLITUDE BOUNDS, TYPE LYAPUNOV AND TYPE PERIOD QT
C   ASSUMED FOR VALUES OF B = 0.5, 0.75, 0.83, 0.875, 0.9
C   AND VARYING A
C
C
      DIMENSION X(400),Y(400),Z(400),G(10,10),
     *NN(10),MM(10)
      REAL LABEL/4H    /
      REAL*8 ITITLE(12)/'AMPPLITUDE BOUND (LYAPUNOV) FOR ZER
     * LIMIT CYCLES IN DIGITAL FILTERS, S.F.HESS, 0126 '/
        REAL*8 ITAG(12)/'AMPLITUDE BOUND (ASSUME PERIOD QT)
     * LIMIT CYCLES IN DIGITAL FILTERS, S.F.HESS, 0126 '/
C
C   COMPUTE REPRESENTATIVE VALUES FOR A AND B
C
      DELTA=0.01
      WRITE(6,100)
  100 FORMAT(1H1)
      DO 1 I=1,5
      B=(I-0.5)/I
      A=-0.91-B
      WRITE(6,101) B
  101 FORMAT(5X,'B =',F10.6,/)
      DO 2 J=1,400
      X(J)=0
    2 Y(J)=0
      DO 3 J=1,400
      A=A+DELTA
      X(J)=A
C
C
C   COMPUTE AMPLITUDE BOUND(LYAPUNOV) FOR VARYING A
C
C
      BB=1.0 + B
      DENOM=(1.0-B)*(BB**2-A**2)
C
C   COMPUTE ELEMENTS OF Q-MATRIX, Q11, Q12, Q22
C
      Q11=1.0+(2.0*(B**2)*BB)/DENOM
      Q12=(2.0*A*B)/DENOM
      Q22=(2.0*BB)/DENOM
      QSUP=Q11-Q22
      QADD=Q11+Q22
      ROOT=SQRT(QSUB**2 + 4.0*Q12**2)
C
C   COMPUTE MIN AND MAY EIGENVALUES OF MATRIX Q
C
      EMIN=(QADD-ROOT)/2.0
      EMAX=(QADD+ROOT)/2.0
C
C   COMPUTE NORM OF MATRIX-PRODUCT (A TRANSPOSE QB)
C
      W1=ABS(Q22*B)
      W2=ABS(Q12-A*Q22)
      W=AMAX1(W1,W2)
      Y(J)=SQRT(EMAX/EMIN)*(W+SQRT(W**2+Q22))
C
C   DISPLAY 3 VALUES OF THE BOUND AND CALL GRAPH-ROUTINE
C
      IF(J.EQ.1) WRITE(6,102)A,Y(J)
  102 FORMAT(5X,'A =',E13.6,3X,'Y =',E13.6,/)
      IF(A.GT.-0.005.AND.A.LT.0.005) WRITE(6,102)A,Y(J)
      IF(A.GE.0.9+B)GOTO 4
    3 CONTINUE
```

```
    4 WRITE(6,102)A,Y(J)
      IF(I.EQ.1) MODCUR=1
      IF(I.GE.2.AND.I.LE.4) MODCUR=2
      IF(I.EQ.5) MODCUR=3
      CALL DRAW(J,X,Y,MODCUR,0,LABEL,ITITLE,0.4,3.0,0,4,2,
     *2,8,0,1,LAST)
    1 CONTINUE
C
C
C   COMPUTE AMPLITUDE BOUND (PERIOD OF LIMIT CYCLE IS QT)
C   SELECT PARAMETER Q SUCH THAT K = Q, AND K.LE.10
C
C
      TWOPI=6.283185
      DO 5 K=4,8
C
C   COMPUTE REPRESENTATIVE VALUES FOR A AND B
C
      DO 15 I=1,5
      B=(I-0.5)/I
      A=-0.91-B
      WRITE(6,103) K,B
  103 FORMAT(5X,'PERIOD OF LIMIT CYCLE Q =',I3,', B =',
     *F10.6,/,
     *'THUS BOUND IS VALID AROUND FOLLOWING VALUES OF A',/)
      IP=0
   16 IP=IP + 1
      IF(FLOAT(IP).GE.FLOAT(K/2)) GOTO 17
      AA=-2.0*COS(TWOPI*IP/K)
      WRITE(6,106) AA,IP
  106 FORMAT(15X,F10.6,3X,I2)
      GOTO 16
   17 CONTINUE
      DO 12 J=1,400
   12 Z(J)=0
      DO 14 J=1,400
      A=A+DELTA
      X(J)=A
C
C   SET UP MATRIX G DESCRIBING THE LIMIT CYCLE FOR K.GE.4
C   INDEX N DENOTES ROW OF G, INDEX M DENOTES COLUMN OF G
C
      DO 6 N=1,K
      DO 7 M=1,K
    7 G(N,M)=0
    6 CONTINUE
      DO 9 N=1,K
      N1=N+1
      N2=N+2
      IF(N1.LE.K)G(N,N1)=A
      IF(N2.LE.K)G(N,N2)=B
    9 G(N,N)=1.0
      G(K,1)=A
      G(K,2)=B
      G(K-1,1)=B
   10 CONTINUE
C
C   INVERT THE G-MATRIX USING A STANDARD GAUSS-JORDAN METHOD
C   MINV IS A SSP SUBROUTINE AND EXCEPTS ONLY ONE-DIMENSIONAL
C   ARRAYS AS MATRICES
C   SUBROUTINE ARRAY CONVERTS FROM ONE TO TWO DIMENSIONAL
C   ARRAYS AND VICE VERSA
C
      CALL ARRAY(2,K,K,10,10,G,G)
      CALL MINV(G,K,D,NN,MM)
      CALL ARRAY(1,K,K,10,10,G,G)
      IF(D.EQ.0)GOTO 5
C
C   COMPUTE AMPLITUDE BOUND
C
      DO 11 N=1,K
      C=G(1,N)
```

```
   11 Z(J)=Z(J)+ABS(C)
C
C  DISPLAY 3 VALUES OF THE BOUND AND CALL GRAPH-ROUTINE
C
      IF(J.EQ.1)WRITE(6,104)A,Z(J)
  104 FORMAT(5X,'A =',E13.6,3X,'Z =',E13.6,/)
      IF(A.GT.-0.005.AND.A.LT.0.005)WRITE(6,104)A,Z(J)
      IF(A.GE.0.9+B)GOTO 13
   14 CONTINUE
   13 WRITE(6,104)A,Z(J)
      IF(I.EQ.1) MODCUR=1
      IF(I.GE.2.AND.I.LE.4) MODCUR=2
      IF(I.EQ.5) MODCUR=3
      CALL DRAW(J,X,Z,MODCUR,0,LABEL,ITAG,0.6,2.5,0,4,2,
     *2,8,9,1,LAST)
   15 CONTINUE
    5 CONTINUE
      STOP
      END
```

Computer Program C: Analysis Program for a Digital Oscillator.

```
C   DIGITAL OSCILLATOR ANALYSIS FOR ROUND-OFF AND
C   TRUNCATION QUANTIZATION OF THE RESULT OF MULTIPLICATION
C   OF DATA-SAMPLES WITH THE OSCILLATOR COEFFICIENT A
C
C
      DIMENSION IX(5005),SINUS(5005),AAA(2,5,35)
      TWOPI=6.283185
      IA=-8
C
C   COMPUTE OSCILLATOR RESPONSE FOR VALUES OF A=-0.9 TO -1.9
C
      DO 3 J=1,11
      IA=IA-1
      A=IA/10.0
C
C   COMPUTE FREQUENCY OF LINEAR OSCILLATOR
C
      FLIN=ARCOS(-IA/20.0)/TWOPI
      WRITE(6,100)
  100 FORMAT(1H1)
      WRITE(6,101)A,FLIN
  101 FORMAT(5X,'FREQUENCY OF LINEAR OSCILLATOR FOR A=',
     *F5.2,', EQUALS F =',E12.6,/)
      IX(1)=0
      IX(2)=0
C
C   GENERATE INITIAL CONDITIONS OF IX(1)=0 AND IX(2)=IC,
C   WHERE IC VARIES FROM 1 TO 1200 IN STEPS OF 1,10,100
C
      DO 2 I=1,30
      IF(I.LE.10) ISTEP=1
      IF(I.GE.11.AND.I.LE.19)ISTEP=10
      IF(I.GE.20.AND.I.LE.30)ISTEP=100
      IX(2)=IX(2)+ISTEP
      IFLAG=1
C
C   INITIALIZE VARIABLES, LARGEST POSSIBLE NUMBER OF LIMIT
C   CYCLE POINTS IS 5000
C
   10 CONTINUE
      IQ=3
      IP=0
      DO 1 II=1,5000
      IX(II+2)=0
    1 SINUS(II)=0
C
C   GENERATE LIMIT CYCLE WITH INITIAL CONDITIONS IX(1)
C   AND IX(2), COUNT LIMIT CYCLE POINTS IQ
C   AND NUMBER OF HALF-CYCLES IP
C
    4 IZ=IX(IQ-1)
      IX(IQ)=IQUANT(IFLAG,IA,IZ)-IX(IQ-2)
C
C   DETECT SIGN-CHANGES OF IX TO EVALUATE IP
C
      IF(IX(IQ).EQ.0)GOTO 5
      IF(IX(IQ).GT.0.AND.IX(IQ-1).LT.0) GOTO 5
      IF(IX(IQ).LT.0.AND.IX(IQ-1).GT.0) GOTO 5
      GOTO 6
    5 IP=IP+1
    6 CONTINUE
      IF(IX(IQ).EQ.IX(2).AND.IZ.EQ.IX(1)) GOTO 7
      IF(IQ.EQ.5000) GOTO 12
      IQ=IQ+1
      GOTO 4
```

```
C
C   END OF LIMIT CYCLE DETECTED
C
    7 IQ=IQ-2
      AAA(IFLAG,5,I)=FLOAT(IQ)
      FQUANT=IP/(2.0*IQ)
      FDIFF=ABS(FQUANT-FLIN)
      AAA(IFLAG,2,I)=FDIFF
      IF(IFLAG.EQ.1) WRITE(6,102)IP,IQ,FQUANT,FDIFF
      IF(IFLAG.EQ.2) WRITE(6,103)IP,IQ,FQUANT,FDIFF
  102 FORMAT(10X,'ROUND-OFF QUANTIZATION ANALYSIS',/,
     *15X,'P =',I4,', Q =',I5,', FQUANT =',E13.6,
     *', FDIFF =',E13.6,/)
  103 FORMAT(10X,'TRUNCATION QUANTIZATION ANALYSIS',/,
     *15X,'P =',I4,', Q =',I5,', FQUANT =',E13.6,
     *', FDIFF =',E13.6,/)
C
C   COMPUTE ESTIMATE FOR THE AMPLITUDE, CALLED AA
C
      AA=0
      DO 8 K=1,IQ
      SINUS(K)=SIN((K-1)*FQUANT*TWOPI)
    8 AA=AA+IX(K)*SINUS(K)
      AA=(2.0*AA)/IQ
      AMP=IX(2)/SINUS(2)
      ADIFF=ABS(AA-AMP)
      AAA(IFLAG,1,I)=AMP
      AAA(IFLAG,3,I)=ADIFF
      WRITE(6,104)AA,AMP,ADIFF
  104 FORMAT(15X,'ACAP =',E13.6,', AMP =',E13.6,
     *', ADIFF =',E13.6,/)
C
C   COMPUTE AVERAGE DEVIATION FROM AMPLITUDE, CALLED FOM
C
      FOM=0
      DO 9 K=1,IQ
      SINUS(K)=AA*SINUS(K)
    9 FOM=FOM + (IX(K)-SINUS(K))**2
      FOM=SQRT(FOM/IQ)
      AAA(IFLAG,4,I)=FOM
      WRITE(6,105) FOM
  105 FORMAT(15X,'AVG AMPLITUDE DEVIATION =',E13.6,/)
      GOTO 11
   12 WRITE(6,106)
  106 FORMAT(5X,'LIMIT CYCLE STOPPED AT Q = 5000',/)
      DO 14 M=1,4
   14 AAA(IFLAG,M,I)=0.
      AAA(IFLAG,5,I)=5000.
C
C   REPEAT THE CALCULATIONS FOR TRUNCATION (IFLAG=2)
C
   11 IFLAG=IFLAG+1
      IF(IFLAG.EQ.2)GOTO 10
    2 CONTINUE
C
C   WRITE TABLE OF RESULTS FOR AMP,IQ,FDIFF,ADIFF,FOM
C
      IFLAG=1
   13 WRITE(6,100)
      IF(IFLAG.EQ.1)WRITE(6,107)A
  107 FORMAT(/////,18X,
     *'ROUND-OFF QUANTIZATION ANALYSIS, A =',F5.2,/)
      IF(IFLAG.EQ.2)WRITE(6,108)A
  108 FORMAT(/////,18X,
     *'TRUNCATION QUANTIZATION ANALYSIS, A =',F5.2,/)
      WRITE(6,109)
  109 FORMAT(18X,'AMP',5X,'Q',5X,'FDIFF',7X,'ADIFF',7X,
     *'DELTA',/)
      WRITE(6,110)(AAA(IFLAG,1,N),AAA(IFLAG,5,N),
     *AAA(IFLAG,2,N),AAA(IFLAG,3,N),AAA(IFLAG,4,N),N=1,30)
  110 FORMAT(F22.2,F6.0,3E12.3,/)
      IFLAG=IFLAG+1
```

```
      IF(IFLAG.EQ.2) GOTO 13
    3 CONTINUE
      STOP
      END

      FUNCTION IQUANT(IFLAG,IA,IZ)
C
C
C   THE FUNCTION COMPUTES THE QUANTIZED PRODUCT -A*IX(IQ-2)
C
C
      IF(IFLAG.EQ.1)IR=5
      IF(IFLAG.EQ.2)IR=0
      IPROD=-IA*IZ
      IF(IPROD.GE.0)GOTO 1
      IPROD=IPROD-IR
      GOTO 2
    1 IPROD=IPROD+IR
    2 PROD=FLOAT(IPROD)/10.0
      IQUANT=INT(PROD)
      RETURN
      END
```

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13. ABSTRACT

A deterministic analysis of the limit cycle oscillations which occur in fixed-point implementations of recursive digital filters due to roundoff and truncation quantization after multiplication operations, is performed. Amplitude bounds, based upon a correlated nonstochastic signal approach and Lyapunov's direct method, as well as an approximate expression for the frequency of zero-input limit cycles, are derived and tested for the two-pole filter. The limit cycles are represented on a successive value phase-plane diagram from which certain symmetry properties are derived. Similar results are developed for other second-order digital filter configurations, and the parallel and cascade forms. The results are extended to include limit cycles under input signal conditions. A basic design relationship between the number of significant digits required for the realization of a filter algorithm with a desired signal-to-noise (limit cycle) ratio is stated.

14. KEY WORDS	LINK A		LINK B		LINK C	
	ROLE	WT	ROLE	WT	ROLE	WT
Limit Cycle Oscillations						
Digital Oscillator						
Digital Filter						
Quantization						
Roundoff						
Truncation						