Composition of Roofs in Derived Category

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Abstract

In that paper, we prove that the composition of two roofs is another roof by using mapping cone of a morphism of cochain complexes.

Contents

1 Introduction

Assume that $\mathcal{A}$ is an abelian category. P. Aluffi defines a mapping cone $MC(f)$ of a morphism $f: A \to B$ in $C(\mathcal{A})$ and homotopy between two morphisms in that category in [Al]. Also, in [Kr], H. Krause defines triangulated category and the localizing class. After that, he proves that the homotopic category $K(\mathcal{A})$ is triangulated in Section 2.5.

Using this information, we prove that for a given upside down roof in the localization of $K(\mathcal{A})$, we can obtain a regular roof in that category. This allows us to compute the composition of two regular roofs in the localization of homotopic category.

2 Mapping Cone and Homotopy

The collection $C(\mathcal{A})$ consisting of all cochain complexes in an abelian category $\mathcal{A}$ forms an abelian category. It is easy to show that the set of morphisms of that category is an abelian group, finite products and coproducts exist since they exist in $\mathcal{A}$.

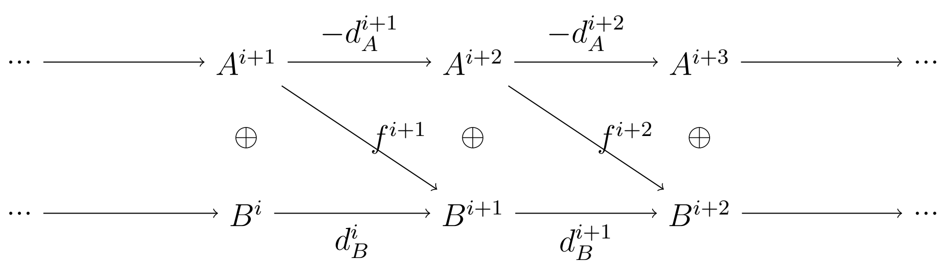
Definition 2.0.1. A morphism f between cochain complexes is quasi isomorphism if it induces an isomorphism in cohomology.

For a given morphism $f: A \to B$ between cochain complexes A and B, we define a mapping cone $MC(f)$ as $MC(f)^i = A[1]^i \oplus B^i = A^{i+1} \oplus B^i$ for all i. Here, we get the morphisms

$$d^i_{MC(f)}: MC(f)^i \to MC(f)^{i+1},$$
$$d^i_{MC(f)}(a,\ b) = (-d^{i+1}_A(a),\ f^{i+1}(a) + d^i_B(b))$$

between those objects.

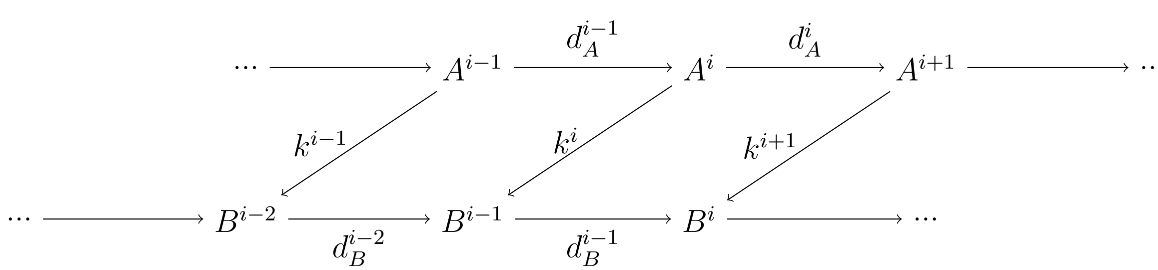
$MC(f)$ is a cochain complex since $d^{i+1}_{MC(f)} \circ d^i_{MC(f)} = 0$.



Definition 2.0.2. A homotopy k between two morphisms of cochain complexes $f,\ g: A \to B$ is a collection of morphisms $k^i: A^i \to B^{i-1}$ such that for all i,

$$g^i - f^i = d^{i-1}_B \circ k^i + k^{i+1} \circ d^i_A.$$

The above morphisms f and g are homotopic if there is a homotopy between them. We use the following diagram to show that homotopy and use the symbol $f \sim g$ to mean there exists a homotopy between the morphisms f and g.



Definition 2.0.3. A morphism $f: A \to B$ is a homotopy equivalence if there is a morphism $g: B \to A$ such that $f \circ g \sim id_B$ and $g \circ f \sim id_A$.

A and B are homotopy equivalent if there is a homotopy equivalence $A \to B$.

Proposition 2.0.1. *[Al] If $f,\ g:\ A \to B$ are homotopic, then $H^\bullet(f) = H^\bullet(g)$.*

Corollary 2.0.1. *If $f:\ A \to B$ is homotopy equivalence, then $H^\bullet(A) \cong H^\bullet(B)$.*

Every homotopy equivalence is a quasi isomorphism, but every quasi isomorphism may not be a homotopy equivalence.

3 Triangulated Categories

Definition 3.0.4. [Kr] Assume that $\mathcal{A}$ is an additive category with an equivalence $\mathcal{F}:\ \mathcal{A} \to \mathcal{A}$. A triangle $(f,\ g,\ h)$ in $\mathcal{A}$ is a sequence of morphisms $X \xrightarrow{f} Y \xrightarrow{g} Z \xrightarrow{h} \mathcal{F}(X)$ for all objects X, Y and Z in $\mathcal{A}$.

A morphism between two triangles $(f_1,\ g_1,\ h_1)$ and $(f_2,\ g_2,\ h_2)$ is a triple $(k_1,\ k_2,\ k_3)$ of morphisms in $\mathcal{A}$ making the following diagram commute.

$$\begin{array}{ccccccc} X & \xrightarrow{f_1} & Y & \xrightarrow{g_1} & Z & \xrightarrow{h_1} & \mathcal{F}(X) \\ {\scriptstyle k_1}\downarrow & & {\scriptstyle k_2}\downarrow & & {\scriptstyle k_3}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(k_1)} \\ X' & \xrightarrow[f_2]{} & Y' & \xrightarrow[g_2]{} & Z' & \xrightarrow[h_2]{} & \mathcal{F}(X') \end{array}$$

The category $\mathcal{A}$ is called pre-triangulated if it has a class of exact triangles satisfying the following conditions.

1. A triangle is exact if it is isomorphic to an exact triangle.
2. For all objects X in $\mathcal{A}$, the triangle $0 \longrightarrow X \xrightarrow{id} X \longrightarrow 0$ is exact.
3. Each morphism $f:\ X \to Y$ can be completed to an exact triangle $(f,\ g,\ h)$.
4. A triangle $(f,\ g,\ h)$ is exact if and only if the triangle $(g,\ h,\ -\mathcal{F}(f))$ is exact.
5. Given two exact triangles $(f_1,\ g_1,\ h_1)$ and $(f_2,\ g_2,\ h_2)$, each pair of maps k_1 and k_2 satisfying $k_2 \circ f_1 = f_2 \circ k_1$ can be completed to a morphism;

$$\begin{array}{ccccccc} X & \xrightarrow{f_1} & Y & \xrightarrow{g_1} & Z & \xrightarrow{h_1} & \mathcal{F}(X) \\ {\scriptstyle k_1}\downarrow & & {\scriptstyle k_2}\downarrow & & {\scriptstyle k_3}\downarrow & & \downarrow{\scriptstyle \mathcal{F}(k_1)} \\ X' & \xrightarrow[f_2]{} & Y' & \xrightarrow[g_2]{} & Z' & \xrightarrow[h_2]{} & \mathcal{F}(X') \end{array}$$

$\mathcal{A}$ is a triangulated category if in addition it satisfies the following axiom.

6. **The Octahedral Axiom:** Given exact triangles $(f_1,\ f_2,\ f_3)$, $(g_1,\ g_2,\ g_3)$ and $(h_1,\ h_2,\ h_3)$ with $h_1 = g_1 \circ f_1$, there exists an exact triangle $(k_1,\ k_2,\ k_3)$ making the following diagram commutative.

$$
\begin{array}{ccccccc}
X & \xrightarrow{f_1} & Y & \xrightarrow{f_2} & U & \xrightarrow{f_3} & \mathcal{F}(X) \\
{\scriptstyle =}\downarrow & & \downarrow{\scriptstyle g_1} & & \downarrow{\scriptstyle k_1} & & \downarrow{\scriptstyle =} \\
X & \xrightarrow{h_1} & Z & \xrightarrow{h_2} & V & \xrightarrow{h_3} & \mathcal{F}(X) \\
 & & \downarrow{\scriptstyle g_2} & & \downarrow{\scriptstyle k_2} & & \downarrow{\scriptstyle \mathcal{F}(f_1)} \\
 & & W & \xrightarrow{=} & W & \xrightarrow{g_3} & \mathcal{F}(Y) \\
 & & \downarrow{\scriptstyle g_3} & & \downarrow{\scriptstyle k_3} & & \\
 & & \mathcal{F}(Y) & \xrightarrow{\mathcal{F}(f_2)} & \mathcal{F}(U) & &
\end{array}
$$

Remark 3.0.1. If $\mathcal{A}$ is a pretriangulated category, then A^{op} is a pretriangulated category, too.

4 The Localization of A Category

Definition 4.0.5. [Kr] Assume that $\mathcal{A}$ is a category and F is a class of maps in $\mathcal{A}$. F is a localizing class if the following conditions are satisfied.

1. If $f,\ g$ are composible maps in F, then $g \circ f$ is in F.

2. The identity map id_A is in F for all $A \in \mathcal{A}$.

3. If $f:\ A \to B$ is in F, then every pair of maps $B' \to B$ and $A \to A''$ in $\mathcal{A}$ can be completed to a pair of commutative diagrams;

$$
\begin{array}{ccc}
A' & \longrightarrow & A \\
\downarrow{\scriptstyle f'} & & \downarrow{\scriptstyle f} \\
B' & \longrightarrow & B
\end{array}
\qquad
\begin{array}{ccc}
A & \longrightarrow & A'' \\
\downarrow{\scriptstyle f} & & \downarrow{\scriptstyle f''} \\
B & \longrightarrow & B''
\end{array}
$$

   such that f' and f'' are in F.

4. If $f,\ g:\ A \to B$ are maps in $\mathcal{A}$, then there is some $h:\ A' \to A$ in F with $f \circ h = g \circ h$ if and only if there is some $k:\ B \to B''$ in F with $k \circ f = k \circ g$.

Definition 4.0.6. [Kr] Assume that $\mathcal{A}$ is a category and F is a class of maps in $\mathcal{A}$. The localization of $\mathcal{A}$ with respect to F is a category $\mathcal{A}[F^{-1}]$ together with a functor $\mathcal{F}:\ \mathcal{A} \to \mathcal{A}[F^{-1}]$ such that $\mathcal{F}(f)$ is an isomorphism for all f in F and any functor $\mathcal{G}:\ \mathcal{A} \to \mathcal{B}$ such that $\mathcal{G}(f)$ is an isomorphism for all f in F factors uniquely through $\mathcal{F}$.

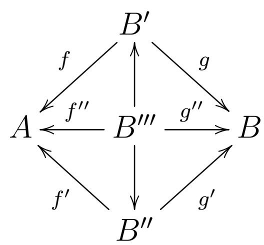
We can always find a localization like that.

Definition 4.0.7. Assume that $\mathcal{A}$ is a category and F is a localizing class. The objects of $\mathcal{A}[F^{-1}]$ are the objects of $\mathcal{A}$. The morphisms $A \to B$ in $\mathcal{A}[F^{-1}]$ are equivalence classes of diagrams $A \xleftarrow{f} B' \xrightarrow{g} B$ with the morphism f in F for all objects A and B in the category $\mathcal{A}[F^{-1}]$. We will call those morphisms as regular roofs.

A pair $(f,\ g)$ is also called a fraction because it is written as $g \circ f^{-1}$ in $\mathcal{A}[F^{-1}]$.

Remark 4.0.2. The functor $\mathcal{F}:\ \mathcal{A} \to \mathcal{A}[F^{-1}]$ sends a map $f:\ A \to B$ to the pair $(id_A,\ f)$.

Definition 4.0.8. $(f,\ g)$ and $(f',\ g')$ are equivalent if there exists a commutative diagram with f'' in F;



5 Composition of Two Roofs

Definition 5.0.9. [Al] Assume that $\mathcal{A}$ is an abelian category. $K(\mathcal{A})$ is a category whose objects are the objects in $C(\mathcal{A})$ and the set of morphisms is

$$Hom_{K(\mathcal{A})}(A,\ B) = Hom_{C(\mathcal{A})}(A,\ B)/\sim$$

where $\sim$ is homotopy relation.

If $f \circ g \sim id$ in $C(\mathcal{A})$, then $f \circ g = id$ in $K(\mathcal{A})$. As a result, homotopy equivalences in $C(\mathcal{A})$ become isomorphisms in $K(\mathcal{A})$ and we say that $K(\mathcal{A})$ is obtained by inverting all homotopy equivalences in $C(\mathcal{A})$. It is an additive category, but not abelian in general since homotopic maps don't have same kernels and cokernels.

In [Kr], H. Krause proves that $K(\mathcal{A})$ is a triangulated category.

Remark 5.0.3. The set of quasi isomorphisms in $K(\mathcal{A})$ for a given abelian category $\mathcal{A}$ forms a localizing class.

Theorem 5.0.2. *[Al] Assume that $\mathcal{A}$ is an abelian category and we have two morphisms $L \xrightarrow{\alpha} \underline{K}$ and $M \xrightarrow{\beta} \underline{K}$ with β is a quasi isomorphism for objects L and M in $K(\mathcal{A})$. Then, there exists a cochain complex K, morphisms $K \to L$ which is quasi isomorphism* *and $K \to M$ in $K(\mathcal{A})$ such that the following diagram commutes.*

$$\begin{array}{ccc} & K & \\ {}^{\gamma_2}\swarrow & & \searrow^{\gamma_1} \\ L & & M \\ {}_{\alpha}\searrow & & \swarrow_{\beta} \\ & \underline{K} & \end{array} \tag{1}$$

Proof. Assume that γ is the composition $L \longrightarrow \underline{K} \longrightarrow MC(\beta)$, $K = MC(\gamma)[-1]$ and $K^i = L^i \oplus M^i \oplus \underline{K}^{i-1}$. We define morphisms $K^i \to L^i$, $(l,\ m,\ \underline{k}) \to l$ and $K^i \to M^i$, $(l,\ m,\ \underline{k}) \to -m$ as in [Al].

We want to prove that L and M are connected by a regular roof as well.

For the rest of the proof, we need to show that the Diagram 1 commutes.

$H^\bullet(M) \cong H^\bullet(\underline{K})$ since β is a quasi isomorphism. This implies that $MC(\beta)$ is exact, so $H^\bullet(MC(\beta)) = 0$.

$$MC(\beta)^i = M[1]^i \oplus \underline{K}^i = M^{i+1} \oplus \underline{K}^i,$$

$$d^i_{MC(\beta)}:\ MC(\beta)^i \to MC(\beta)^{i+1},\ d^i_{MC(\beta)}(m,\ \underline{k}) = (-d^{i+1}_M,\ \beta^{i+1}(m) + d^i_{\underline{K}}(\underline{k})).$$

We define $\gamma^i(l) = (0,\ \alpha^i(l))$ and

$$MC(\gamma)^i = L[1]^i \oplus M^{i+1} \oplus \underline{K}^i = L^{i+1} \oplus M^{i+1} \oplus \underline{K}^i$$

where $d^i_{MC(\gamma)}:\ MC(\gamma)^i \to MC(\gamma)^{i+1}$ with

$$d^i_{MC(\gamma)}(l,\ m,\ \underline{k}) = (-d^{i+1}_L(l),\ \gamma^{i+1}(l) + d^i_{MC(\beta)}(m,\ \underline{k})) = \\ (-d^{i+1}_L(l), -d^{i+1}_M(m),\ \alpha^{i+1}(l) + \beta^{i+1}(m) + d^i_{\underline{K}}(\underline{k})).$$

$$K = MC(\gamma)[-1],$$

$$MC(\gamma)[-1]^i = MC(\gamma)^{i-1} = L^i \oplus M^i \oplus \underline{K}^{i-1} = K^i$$

and $d^i_K = -d^{i-1}_{MC(\gamma)}$ with

$$d^i_K = (d^i_L(l),\ d^i_M(m),\ -\alpha^i(l) - \beta^i(m) - d^{i-1}\underline{K}(\underline{k})).$$

Assume that $h^i:\ K^i \to \underline{K}^{i-1}$ takes $(l,\ m,\ \underline{k})$ to $-\underline{k}$. We need to show that

$$\alpha^i \circ \gamma^i_2 - \beta^i \circ \gamma^i_1 = d^{i-1}_{\underline{K}} \circ h^i + h^{i+1} \circ d^i_K$$

for all $i \in \mathbb{Z}$ which shows that $\alpha \circ \gamma_2$ and $\beta \circ \gamma_1$ are homotopic maps in $K(\mathcal{A})$. This will show that they are same maps.

For all $(l,\ m,\ \underline{k}) \in K^i$,

$$\begin{aligned}(\alpha^i \circ \gamma_2^i - \beta^i \circ \gamma_1^i)(l,\ m,\ \underline{k}) &= \alpha^i(\gamma_2^i(l,\ m,\ \underline{k})) - \beta^i(\gamma_1^i(l,\ m,\ \underline{k})) \\ &= \alpha^i(l) - \beta^i(-m) = \alpha^i(l) + \beta^i(m)\end{aligned}$$

since $\mathcal{A}$ is additive. On the other hand,

$$\begin{aligned}(d_{\underline{K}}^{i-1} \circ h^i + h^{i+1} \circ d_K^i)(l, m, \underline{k}) &= d_{\underline{K}}^{i-1}(h^i(l,\ m,\ \underline{k})) + h^{i+1}(d_N^i(l,\ m,\ \underline{k})) \\ &= d_{\underline{K}}^{i-1}(-k) + \alpha^i(l) + \beta^i(m) + d_{\underline{K}}^{i-1}(\underline{k}) = \alpha^i(l) + \beta(m).\end{aligned}$$

This shows the maps are homotopic and the diagram is commutative.

We need to show that γ_2 is a quasi isomorphism. We have an exact triangle;

$$\begin{array}{ccc} & L & \\ \nearrow & & \searrow \\ L[1] + MC(\beta) & \longleftarrow & MC(\beta) \end{array}$$

This triangle is isomorphic to an exact triangle;

$$\begin{array}{ccc} & MC(\beta) & \\ \nearrow & & \searrow \\ L[1] & \longleftarrow & L[1] + MC(\beta) \end{array}$$

Then, we take its cohomology and the triangle still will be exact.

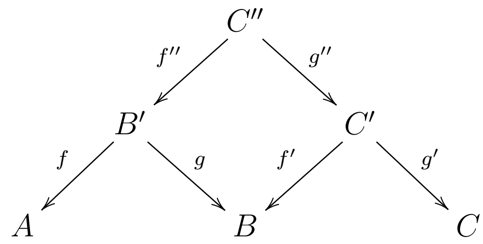
$$\begin{array}{ccc} & H^\bullet(MC(\beta)) & \\ \nearrow & & \searrow \\ H^\bullet(L[1]) & \longleftarrow & H^\bullet(L[1] + MC(\beta)) \end{array}$$

$H^\bullet(MC(\beta)) = 0$, so $H^\bullet(L[1]) \cong H^\bullet(L[1] + MC(\beta))$ and

$$L[1] + MC(\beta) = L[1] + M[1] + \underline{K} = K[1].$$

Consequently, $H^\bullet(L[1]) \cong H^\bullet(K[1])$. This means $H^\bullet(L) \cong H^\bullet(K)$, hence γ_2 is a quasi isomorphism. $\square$

The pair $(f \circ f'',\ g' \circ g'')$ is the composition of two pairs $(f,\ g)$ and $(f',\ g')$ as in the following commutative diagram;



References

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[Kr] H. Krause, **Derived Categories, Resolutions, and Brown Representability**, arxiv: math/ 0511047v3, 2006.