T H E
AMERICAN YOUTH:
B E I N G
A NEW AND COMPLETE COURSE
O F
Introductory Mathematics :
DESIGNED FOR THE USE OF
PRIVATE STUDENTS.

B Y
CONSIDER and JOHN STERRY.

V O L. I.

—————where the mind
In endless growth and infinite ascent,
Rises from state to state, and world to world.
THOMSON.
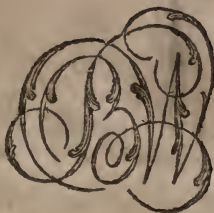
PREFACE.

TIME, *ever big with wonders to be unfolded to the human mind, has ushered in, through a series of the most important events, the rising Empire of America; who hath established her own Independence, and the flame of her liberty has spread itself to the remotest parts of the earth; the effect of which great example has not yet spent its force, but must continue to operate throughout ages, and form a grand ingredient in the active fermentation, and in the history of nations.*

But the great object of true national dignity and grandeur, consists in the cultivation of the human mind, whereby the natural savage barbarity, rudeness and imbecility of human nature are eradicated, and those principles of knowledge and virtue engrafted in the soul, which are the foundation of that knowledge and preeminence of merit, which is the noblest of all distinctions.

As soon as we begin to exist, that savage and imbecile spirit takes root in the soul, and grows as the mind enlarges, till the seeds of knowledge by cultivation do take effectual root, and then like the tender bud it will burst its native bonds, expand and flourish in its own beauty: The veil will then disappear, and an
infinite

infinite diverfity of fcenes, both pleafing and inftructing, will open themfelves to our view. But in order to prepare the mind for thefe pleafing and enlarging views, we muft early employ ourfelves in the ftudy of fomething which is noble and important, whereby our minds may be cultivated aud brought to maturity. " *A juft and perfect acquaintance with the fimple elements of fcience, is a neceffary ftep towards our future progrefs and advancement ; and this, affifted by laborious inveftigation, and habitual enquiry will conftantly lead to eminence and perfection.*"

" *But as the various modes, fituations and circumftances of life are various, fo accident, habit and education, have each their predominante influence, and give to every mind its particular bias.*" *It is, therefore, for this reafon, we particularly admire thofe things which are the moft compatible with our genius and purfuits in life.*

" *Riches and honours are the gifts of fortune, cafually beftowed, or hereditarily received, and are frequently abufed by their poffeffors ; but the fuperiority of wifdom and knowledge, is a pre-eminence of merit, that originates with the man, and is the nobleft of all diftinctions.*"

Since, therefore, the cultivation of knowledge is a thing of the laft importance, too many attempts cannot be made to render it univerfal, and fince youth is the time therefor, we have therefore, " *only to point out to them fome valuable acquifition, and the means of obtaining it. The active principles are immediately put in motion, and the certainty of the conqueft is enfured from a determination to conquer.*" *But of all the fciences cultivated by mankind, none are more ufeful than the*
Mathematics,

Mathematics, to call forth a spirit of enterprise and enquiry. The unbounded variety of their application, which is of universal utility to mankind, first prompts our curiosity to have in possession a treasure of such inestimable value. By their elegant and sublime manner of reasoning, our minds are enlightened and our understanding enlarged, and thereby we acquire a habit of reasoning, an elevation of thought, that determines the mind and fixes it for every other pursuit; and none but those who either from sordid views, or a gross ignorance of what they dispise, will ever think their time misspent, or their labours useless in the pursuit of that, which is the guide of our youth, and the perfection of our reason.

The subject of the present performance, is Arithmetic and Algebra, the foundation of all our mathematical enquiries.

Although a great number of books has been published on the subject of Mathematics, yet few of them are adapted to the capacity of young and tender minds. Where is that simplicity, plainness and brevity, which is absolutely necessary for the young and unassisted beginner? That close and refined reasoning with which those Authors' writings are replete, renders them unfit for learners in general, and entirely useless to those unassisted by a Tutor: They have consulted more the elegance of their diction, and refined demonstrations, than the method of conveying their knowledge to their readers. Others again, in attempting to render their subjects attainable to the weakest minds, have been so prolix and voluminous, as even to discourage a learner at the sight of their works: Thus, we see that writers in general, aim at the extremes, while the true and proper medium is for the most part omited. Propriety therefore, and
compatibility

compatibility ought always to be the grand text, while, simplicity joined with brevity leads the chain of argument.

In all countries, where the sciences are cultivated, local interests have been particularly considered, which must therefore exclude those who neglect the cultivation of the Arts and Sciences, from many advantages of their works.

Taking into consideration the works of those who have gone before us on this subject, the utility of an alteration appeared manifest, while reason and convenience urged the practicability thereof.

In the prosecution of this plan, we have in the first Book of the present Volume, explained the rudiments and application of numbers; beginning with the properties of an unit, we have led the learner by easy and natural gradations to the most remote analogies of the science. In all the calculations relating to money, we have made use of the Federal Money, or Money of the United States, which is not only much more concise than the present practice by pounds, shillings, &c. but it is equally estimable for its simplicity and brevity. The denominations of this money being in a decimal ratio, are therefore above all other numbers, the most natural and easy to be managed, and which must consequently give it a preference to any other method whatever.

The subject of the second Book is Algebra, or the analytic art; which above all others is the most extensive and sublime. It was by this, with the consideration of motion, that one did in some measure do honour to human nature itself, by his almost divine invention*; which succeeding ages will view with pleasing admiration.   *Algebra*

* Alluding to Sir Isaac Newton's invention of Fluxions.

Algebra is a general method of discovering truth in all cases where proper data can be established, with the greatest expedition, elegance and ease.

In delivering the rudiments of this science, we have particularly consulted the ease and accomodation of the learner, by confining every thing within the sphere of the ingenious Student, and therefore, exploding those tedious and complicated explanations, which are commonly to be found in authors on this subject. The leading questions are short and simple, and the method of arguing brief and conspicuous ; which particular, although of the last importance to facilitate the progress of learners, is too much neglected by most writers, and consequently, deter many from becoming acquainted with this interesting and important acquirement. Great attention has been paid to render the doctrine of irrational quantities plain and intelligible, particularly the method of expanding quantities into infinite series, and noting their powers and roots, which is a matter of the last importance in the higher branches of the Mathematics. And finally, through the whole of the following sheets, simplicity and brevity has been our general aim, and at the same time to explode all foreign and provincial customs, and adapt the whole to the practice and convenience of the United States.

Thus far, for the satisfaction of the learner, we have explained the economy of the present performance, we shall now submit it to the candid public, and from the pains we have taken to render the subject useful to learners in general, are not without hopes of its meeting with their approbation.

THE AUTHORS.

Preston (Connecticut) July, 1790.
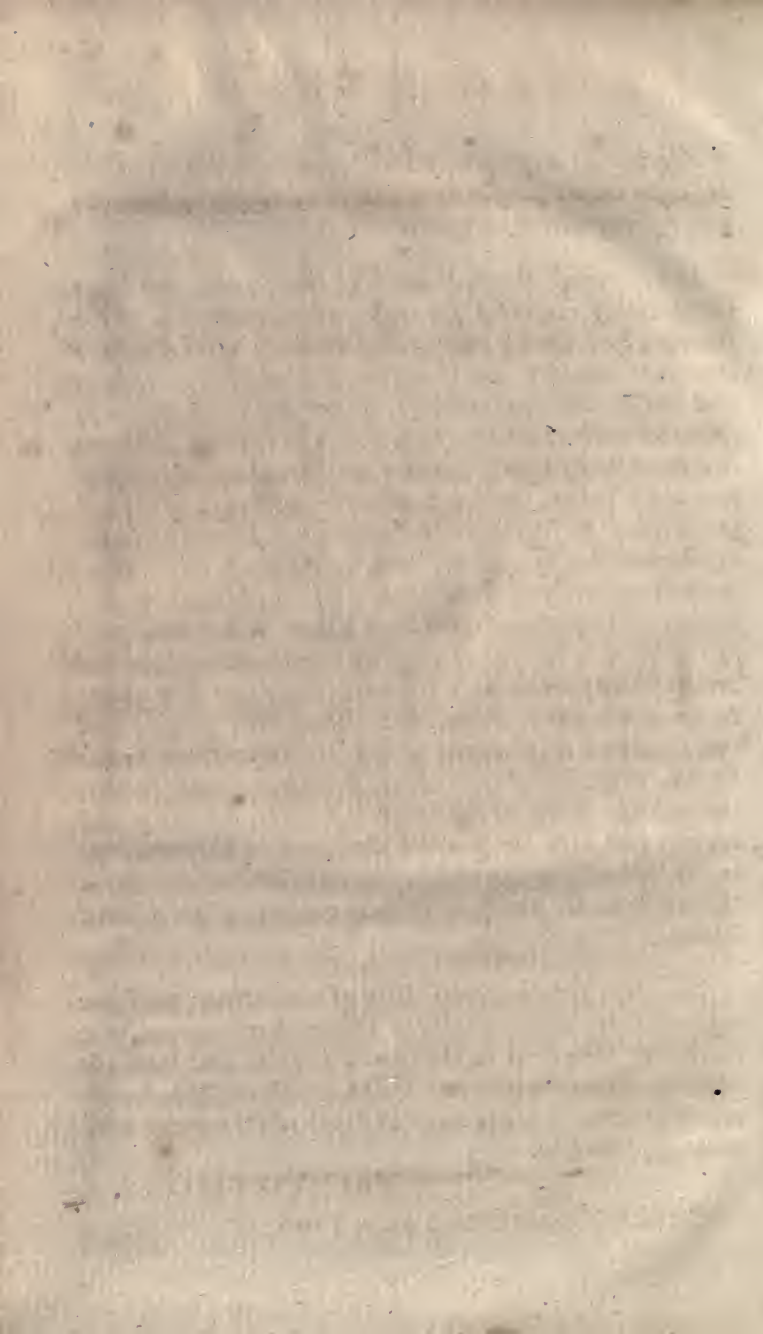
RECOMMENDATIONS.

Extract of a letter from Mr. NATHAN DABOLL, *Teacher of Mathematics and Astronomy, in the Academic School in Plainfield, to the Authors ; dated March* 1, 1787.

GENTLEMEN,

" I HAVE perused the first Volume of your new course of introductory Mathematics, entitled THE AMERICAN YOUTH ; and it appears to me a work well executed, and compatible with its design. You have given your rules and examples in a concise, plain and familiar manner, and consequently well-adapted your matter to the capacities of learners : I therefore esteem it a very valuable performance, and wish you success in its publication, and that it may meet with an encouragement from the public equal to its merit."

From

RECOMMENDATIONS.

GENTLEMEN,

" YOUR Treatise of Arithmetic and Algebra, I have perufed with care and attention, and have the pleafure of affuring you I think it a work of ingenuity and merit. My reading of mathematical books hath been extenfive ; yet I know of no writer who hath treated thefe fubjects in a more fcientific and comprehenfive manner, and at the fame time accommodated his matter fo well to the capacities of learners, as I find to be done in your work. If you publifh it (which I hope you will not fail to do) I have no doubt it will be received into our Schools and Seminaries, as it is high time that Ward, Hammond, and other inferior treatifes now in common ufe, were exploded. For my own part, as a lover of Mathematics, I wifh you all poffible fuccefs, and that you may be encouraged to proceed, and write on the higher and more fublime branches of the Mathematics ; and that a fpirit of emulation may be excited among the Youth of America, to excel in thefe ufeful and exalted, but hitherto much-neglected purfuits."

RECOMMENDATIONS.

GENTLEMEN,

" YOUR Manuscript Treatise on Arithmetic and Algebra, entitled THE AMERICAN YOUTH, has been put into my hands. I have paid particular attention in its perusal. I have heretofore been considerably engaged in the reading and study of authors upon the various branches of Mathematics, though of late I have been more diverted from that pursuit. It has however given me great pleasure and satisfaction to observe the ingenuity, conciseness and perspicuity which appears in your work, notwithstanding the extensive and finished researches demonstrated in all your rules and examples ; yet it appears to me exceedingly well accommodated to the capacity of a learner, and your method through the whole more easy than any I have before seen. If you should publish your book (which my high esteem for mathematical science, and sincere regard for the progress of literature among the youth of our country, induces me earnestly to wish you may) justice obliges me to say, that I am clearly of opinion it will be found more useful among students than any other author now extant upon the subject. I sincerely wish

you

RECOMMENDATIONS.

you fuccefs, and that you may meet with every encouragement which the merit of fo important a work deferves."

<hr />

From the Rev. JOSEPH HUNTINGTON, *D. D. one of the Truftees of Dartmouth College, &c. to the Hon.* JOHN DOUGLAS, *Efq. dated Coventry, May 23, 1788.*

"I HAVE with much pleafure perufed the mathematical compofition in which the two Meffrs. STERRY's are united, and really think it worthy of publication and encouragement : The fcience of Arithmetic and Algebra has hitherto been extended nearly to its bounds, but I efteem this work an excellent piece for the ftudy of youth, to lead them to the knowledge of this ufeful fcience, fince it is more eafy and intelligible to tender capacities than any work of the kind preceding, and this, more efpecially in the moft abftrufe part of the whole fcience, *i. e.* Algebra. I could wifh that you, Sir, and many other gentlemen, eminent for their friendfhip to the liberal fciences, might pay attention to the work I have alluded to."

TABLE of CONTENTS.

PART I.

A

A Supplement to P A R T III.

A Supplement to P A R T I.

B O O K II.

ALTHOUGH the Authors examined the Proof-Sheets, yet the following escaped their Notice.

ERRATA.

PAGE 28, laft line, dele See the Example. P. 34, l. 12. read 69530000. l. 14, r. 720800. P. 35, l. 18, for 3, r. 4. P. 55, l. 4, r. content. l. 24, for 3 qr. 3 na. r. 1 qr. 3 na. l. 26, for 3 qr. 2 na. r. 1. qr. 2 na. P. 67, l. 4, r. 105 dol. P. 74, l. 9, r. 56388. l. 19, r. 15480 yards. P. 77, l. 20, for in, r. is. P. 80, l. 13, for 13, r. 15. P. 85, l. 1, for $20\frac{8}{20} \div 20$, r. $20\frac{8}{20} \div 24$. P. 100, l. 15, for .5, r. 5. P. 106, l. 20, r. preceding. P. 107, l. 27, r. $8 =$. P. 113, l. 12, r. 31.415, &c. P. 132, l. 25, r. numbers. P. 157, l. 13, r. operation. P. 169, l. 13, r. 49 cts. P. 193, l. 7, r. 51 dol. $72\frac{24}{58}$ cts. l. 15, r. fellowfhip. P. 322, l. 5, r. $6 \times 2 \times 6$. P. 245, l. 16, for $+2a$, r. $+2b$. P. 246, for ax, r. az. P. 249, for $\sqrt{aw} - yb$, r. $\sqrt{aw - yb}$. P. 266, l. 16, r. $\overline{x+y}|^2 \div \overline{x+y}|^2$. P. 288, for $a^{\frac{3}{6}}$, r. $a^{\frac{2}{6}}$. P. 353, l. 16, for laft, r. xvi. P. 357, l. 19, r. $v = \frac{}{c}$. P. 379, l. 9, read axiom 8. l. 10, r. axiom 8. l. 18, r. axiom 8. P. 380, l. 1, r. ax. 7. l. 2, r. ax. 9. l. 3, r. ax. 8. l. 5, r. ax. 8.

BOOK I.

OF ARITHMETIC.
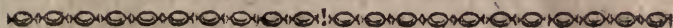
PART I.

ARITHMETIC of WHOLE NUMBERS.

CHAP. I.

Of DEFINITIONS and ILLUSTRATIONS.

ARITHMETIC consists of three parts; two of which are natural, and the third artificial. The first part of natural Arithmetic, is wherein an unit or integer represents one whole quantity, of any kind or species; and is therefore stiled Arithmetic of whole numbers. The second part of natural Arithmetic, is wherein an unit is considered as broken or divided into parts, either even or uneven, which are considered either as pure parts of an unit, or as parts mixed with an unit; and is usually stiled the doctrine of vulgar fractions. The third part, or artificial Arithmetic, is an easy and elegant method of managing fractional, or broken quantities; the operations are nearly similar to those of whole numbers. This part is of general use in the various branches of the Mathematics.

C   THE

THE operations of common Arithmetic in all its parts, are performed by the various ordering and difpofing of ten *Arabic* characters, or numeral figures ; which are thefe following, viz.

one two three four five fix feven eight nine cypher

1   2   3   4   5   6   7   8   9   0

AN unit (by *Euclid*) is that by which every thing that is, is one ; and number is compofed of a multitude of units.

NINE of the aforefaid figures, are compofed of units ; each character reprefenting fo many units put together in one fum, as was intended they fhould denote ; nine of thofe units, being the greateft number which is thought beft for any one character to reprefent ; the laft of the before-mentioned characters, is a cypher, or as fome call it a nothing ; for of itfelf it is nothing ; becaufe, if ever fo many cyphers be added to, or fubtracted from an unit or number, they will neither increafe nor diminifh its value : confequently a cypher of itfelf is no affignable quantity ; but cyphers annexed or prefixed to an unit or number, will increafe, or diminifh that unit or number in a tenfold proportion.

THAT the learner may underftand the following fheets, it is abfolutely neceffary for him to be well acquainted with the following *Algebraic* figns.

SIGNS & NAMES.	SIGNIFICATIONS.
+ Plus, or more,	is the fign of Addition : as 4+6, which denotes that 6 is to be added to 4, and is read thus,———— 4 more 6.
— Minus, or lefs,	is the fign of Subtraction : as 4—2, which fignifies that 2 is to be taken from 4 ; and is read thus, 4 lefs 2.

× into,

× into, or with,	is the sign of Multiplication: thus 4×3 denotes, that 3 is to be multiplied into 4; and is read thus, 4 into, or with, 3.
÷ by,	is the sign of Division: thus 6÷3, is 6 divided by 3, or $\frac{6}{3}$, signifies the same thing; and is read thus, 6 by 3.
= equal,	is the sign of Equality: and whenever this sign is placed between any two quantities, it denotes that those quantities are equal: thus 9=9; that is, 9 equals 9; also 6+4=10, is read 6 more 4 equals 10.
:: so is,	is the sign of Proportionality; and is always placed between the second and third numbers that are in proportion: thus 2:4::4:8.
: to,	is also a sign of Proportion, and is placed between the first and second, third and fourth numbers in proportion: thus 2:4::3:6; is read thus, 2 to 4 so is 3 to 6.
$\overline{4+6}×2$	denotes the sum of 4 & 6 multiplied with 2.
÷	is the sign of continued Proportion.

THE whole doctrine of Number is founded on the five following general rules, to wit, Notation, Addition, Subtraction, Multiplication and Division.

CHAP. II.

Of NOTATION or NUMERATION.

NOTATION or Numeration teaches us, how to exprefs the value of figures ; and confequently to note or write down any propofed number, according to its juft value ; in the operation of which, two things muft be obferved, viz. the order of writing down figures, and the method of valuing each in its proper place, as in the following Table ;

NUMERATION TABLE.												
Hundreds of Thoufands of Millions	Tens of Thoufands of Millions	Thoufands of Millions	Hundreds of Millions	Tens of Millions	Millions	Hundreds of Thoufands	Tens of Thoufands	Thoufands	Hundreds	Tens	Units
3	2	1	9	8	7	6	5	4	3	2	1

Here the order of reckoning begins on the right hand, to wit, at unity, and so on as the table directs. But to make the understanding of this table plain, it is required to express the value of the numeral figures 321. First, beginning at the first figure on the right hand, viz. at 1, which stands in the units' place, where it represents its own simple value, which is an unit, or 1 ; the next to be considered is the figure 2, which stands in the tens' place, representing so many tens, as the figure 2 is composed of units, which are two ; so that the figure 2 standing in the place of tens represents 2 tens, or 20 ; the next figure, 3, stands in the hundreds' place, and signifies as many hundreds as the said figure hath units, viz. 3, that is, three hundred : now, if the whole value of the figures 321 be expressed, the expression will be three hundred twenty-one. Altho the figure 3, is in the last place on the right, or the first on the left, yet when we come to read or express them, we begin with the figure 3 ; because the method of reading figures is the same as that of words. Hence the first figure in numbering, is the first figure on the right hand ; but in reading or expressing the value of numbers, the first figure in the expression is the first figure on the left hand. Again, let it be required to read or express 7645. Here as before, the first figure of the proposed number, to wit. 5, stands in the units' place, and is 5 units, or five, the second figure which is 4, is in the tens' place, and is four tens or 40, the third figure which is 6, in the hundreds' place, is called hundreds, and the fourth figure, which stands in the thousands' place, is for the same reason called thousands ; and the expression for the whole value, beginning as before, is seven thousand six hundred forty-five.

If

IF what has been said concerning notation and valuation of figures, be thoroughly considered, together with the following examples and their answers, the whole business of Numeration will appear plain to the meanest capacity.

EXAMPLES.

What is the value of 56434 ?
Answer. *Fifty-six thousand four hundred thirty-four.*
What is the value of 7843217 ?
Anf. *Seven million eight hundred forty-three thousand two hundred seventeen.*
What is the value of 640036 ?
Anf. *Six hundred forty thousand thirty-six.*
What is 891000002 ?
Anf. *Eight hundred ninety-one million two.*

CHAP. III.

Of ADDITION of SIMPLE WHOLE NUMBERS.

ADDITION is the collecting or putting together several quantities or numbers into one sum, so that their total amount may be known ; and in order to perform the operations of this rule, two things must be carefully observed, which are, First, the right placing or setting each figure in its proper place ; that is, units must stand under units, tens under tens, hundreds under hundreds, and so on, setting each denomination under that of the same value: thus 246 + 25 + 163, being set as directed, will stand

thus,

thus, $\begin{cases} 246 \\ 25 \\ 163 \end{cases}$

THE second thing to be obſerved, is the right col-
lecting or adding together each perpendicular row
of figures, placed as before directed ; which is per-
formed as in the following example, being the ſame
as made uſe of above, viz. 246 + 25 + 163 :

or thus, $\begin{cases} 246 \\ 25 \\ 163 \end{cases}$

Then ſtriking a line beneath the figures, as in the
example ; begin on the right hand at the units' place,
adding together all thoſe figures which ſtand in the
units' place, and if their ſum be under ten, ſet it down
underneath in the units' place ; but if their ſum ex-
ceed ten, ſet down the ſurplus, carrying one to the
next place, viz. the tens' place : or, more generally,
as many tens as the ſum of thoſe units amounts to,
you muſt carry to the next place of figures, to wit,
the tens' place, adding them up with all the figures
that ſtand in that perpendicular line ; and ſo on for
the reſt ; remembering to carry one for every ten of
your aggregate : the whole of which will be illuſ-
trated in the following

EXAMPLE.

Find the ſum of the following numbers, viz.
392 + 466 + 256.

THOSE numbers being placed as the rule directs,
will ſtand

thus, $\begin{cases} 392 \\ 466 \\ 256 \end{cases}$

1114 = ſum required.   Then

Then begin with the bottom figure, in the units'
place ; saying 6 and 6 is 12, and 2 is 14; setting
down 4, carry 1 to the next, or place of tens, saying
5 and 1 that I carry make 6, and 6 is 12, and 9 is
21 ; here because the aggregate or sum total is 21
units (or because it stands in the tens' place) 2 tens
and one unit ; therefore set down 1 and carry 2 to the
next place, saying 2 and 2 that I carry make 4, and
4 is 8, and 3 is 11 ; which being the sum of the last
place of figures in the example, set down the whole.
[See the work at the bottom of the preceding page.]

THE reason of setting down the surplus, or odd
figures, and carrying for the tens, as in the last and
all other examples in addition of simple quantities,
is to shorten the work under consideration ; and to
save the trouble of using superfluous figures. To
exemplify which, let us make use of the foregoing
example, to wit, 392+466+256, which must be
placed

thus,		3	9	2	
		4	6	6	
		2	5	6	
			1	4	*the sum of the row of units*
		2	0	0	*the sum of the row of tens*
		9	0	0	*the sum of the row of hundreds*
	1	1	1	4	*the sum of the whole ;*

then adding up each single row, set down its sum in
its proper place, in the same manner as if there were
but one single row ; supplying the vacant places on
the right hand with cyphers. Hence the result of
this operation is the same as in the former method of
carrying for the tens ; and hence also it appears, that,
adding the cyphers, makes no alteration in the value
of the sum of the other figures.

THE

THE manner of proving your work, flows as a natural confequent, from the following felf-evident propofition, on which the truth of the rule depends, viz. that every whole is equal to all its parts taken together. Wherefore if you divide, or feperate the given numbers into two, or more parcels, according to your propofition ; and by adding together each part fo feperated, if the fum of all thofe parts added together, is equal to the fum total of all the given numbers, found before feperation, your work is right.

THIS method will appear plain by the following example. Suppofe it were required to add together the following numbers, viz. 3489+6725+2324+ 6744 ; which according to the rule of Notation muft ftand thus,

$$3489$$
$$6725$$
$$2324$$
$$6744$$

$$19282 = \textit{fum before feperation.}$$

First part $\begin{cases} 3489 \\ 6725 \end{cases}$ | Second part $\begin{cases} 2324 \\ 6744 \end{cases}$

$$10214 = \textit{fum of firft part.}$$ $$9068 = \textit{fum of fecond part.}$$

The fum of the firft and fecond parts $\begin{cases} 9068 \\ 10214 \end{cases}$

Sum of all the parts 19282

which agrees with the fum total before feperation ; therefore the work is right. But the moft ufual methods of proving Addition, is either by beginning at the top, and reckoning downwards ; which fum, if equal to that found by cafting upwards, the work is right. Or, firft add together all the propofed num-

D   bers

bers into one fum; then feperate the upper number from the reft, by a line, and add together the remaining numbers beneath; placing their fum under the former, or fum total before feperation; which being done, add the fum laft found to the upper line in your example; which fum, if equal to the fum total or firft addition, the work is right: this is the fame in effect, as the firft method of proof, though a little different in mode, as will appear by the following example.

$$34678$$

$$
\begin{array}{r}
24532 \\
12760 \\
53865 \\
21671
\end{array}
$$

147506 = *fum of the whole*
112828 = *fum of all but the upper line*
147506 = 34678 + 112828 = *fum of the whole*:
therefore the work is right.

TAKE the following examples, without their anfwers, for practice.

		2	6538764
3457643	460039	372	875623
4567012	914321	42734	43521
2354123	675422	8173456	6300
1678432	342310	37240	579
		421	84
		2	1

CHAP. IV.

*Of SUBTRACTION of SIMPLE WHOLE NUM-
BERS.*

SUBTRACTION is the taking one number
out of another; whereby the remainder, differ-
ence, or excefs may be known: thus 3 taken out or
from 5, leaves 2, which is the difference between 3
and 5; and is alfo the excefs of 5 above 3.

HENCE it follows, that the number from which
fubtraction is to be made, muft be equal to, or
greater than the fubtrahend, or number to be fub-
tracted; and alfo, that Subtraction is the reverfe of
Addition; for Subtraction is the taking of one num-
ber from another, but Addition is the collecting or
putting them together.

HERE the Notation is the fame as in Addition,
to wit, thofe numbers which are of like value, muft
ftand directly beneath each other; that is, units muft
ftand under units, tens under tens, &c. After having
thus placed your numbers, the lefs beneath the great-
er, you may proceed to fubtract them apart, by ob-
ferving the following

RULE.

BEGIN with the firft figure on the right-hand, which
ftands in the units' place, and fubtract the lower fi-
gure from that which ftands directly over it, of the
fame value; fetting down the remainder (if any) be-
neath in the units' place: If the figure in your fub-
trahend be equal to the figure which ftands directly
over it, you muft fet a cypher for the remainder;
but if the lower, or figure in your fubtrahend, con-
tains more units than your upper figure, you muft
add 10 to the upper figure, or fuppofe it to be fo add-
ed

ed in your mind; then fubtract your lower figure
from your upper fo increafed, fetting down the re-
mainder or difference in its proper place; then pro-
ceed to your next place of figures; now it is fuppof-
ed that the 10 you before added was borrowed from
your next fuperior place of figures, where you muft
pay what you before borrowed, which is performed
as the ufual method is, by calling the lower figure,
ftanding in that place, one more than it really is;
then fubtracting it fo augmented, from your upper
figure, or figure ftanding directly over it, fet down
the difference as before directed; and fo on, from one
place of figures to another, until the whole be complet-
ed; the whole of which, is illuftrated in the following

EXAMPLES.

Suppose, that from 4567, you were to fubtract
3692; which numbers, being placed according to
the rule, will ftand

thus, $\begin{cases} 4567 \\ 3692 \end{cases}$

Here begin with the 2, faying 2 from 7 and there
remains 5, fetting it down as directed; then proceed
to your next place of figures, faying 9 from 6 I can-
not, becaufe my lower figure, to wit, 9, contains
more units than my upper, or figure from which I
would fubtract; therefore I fuppofe 10 to be added to
the upper figure which makes 16; then faying 9 from
16 and there remains 7; then proceed to the next
place, where you muft pay what you have borrowed,
by faying 6 and 1 that I borrowed make 7; then 7
from 5 I cannot, but 7 from $5+10=15$, and there
remains 8; then to the next place, faying 3 and 1 that
I borrowed make 4, 4 from 5 and there remains
1; now there being no more places of figures, fet
down the 1 and the work is done. (See the example.)

THE

THE truth of subtraction is founded on the same self-evident proposition, or axiom, as that of Addition, viz. the whole is equal to all its parts taken together. From which proposition is deduced the following method of proving your work, to wit, by adding the subtrahend, or number to be subtracted, to the remainder : for the number from which subtraction is made, is here considered as the whole, and the subtrahend, as a part of that whole ; consequently if that part be taken from the whole, the remainder will be the other part ; therefore if both parts when added together, be equal to the whole, the work is right.

HENCE it is manifest that subtraction may be proved by subtraction ; for if from

$$67834 \text{ the whole,}$$
$$\text{is taken } 53723 \text{ a part of that whole,}$$

there will remain 14111 the other part ;
and if from 67834 the whole, there is taken the last part 14111

there will remain 53723 the first part, or subtrahend: consequently, &c.

AGAIN, if from 27942 the whole,
$$\text{is taken } 13724 \text{ a part of that whole ;}$$

there will remain 14218 the other part,
$$27942 = \text{sum of the subtrahend}$$
and remainder = the whole.

TAKE the following examples for practice.

From	37654	394076	2876955	7654109
take	28765	123468	423610	347472
Rem.				

CHAP. V.

CHAP. V.

Of SIMPLE MULTIPLICATION.

MULTIPLICATION is a rule by which a given number may be increafed any number of times propofed.

THERE are three requifites in Multiplication : firft, the multiplicand, or number to be multiplied : fecond, the multiplier, which denotes how many times the multiplicand is to be taken ; for by *Euclid*, as many units as there are in the multiplier, fo many times is the multiplicand to be added to itfelf : third, the product, or multiplicand increafed fo many times as there are units in the multiplier.

SUPPOSE for example, that 7 be increafed 4 times ; that is, to multiply 7 into or with 4 ; thefe numbers muft be placed as in Addition,

thus, $\left\{ \begin{array}{l} 7 \ multiplicand \\ 4 \ multiplier \end{array} \right.$

28 *product.*

Now that 4 times 7 make 28, will appear evident by fetting down the multiplicand 4 times, and adding up the whole, as in this, $\left\{ \begin{array}{l} 7 \\ 7 \\ 7 \\ 7 \end{array} \right.$

28 = *fum or product.*

HENCE it is plain, that multiplication is a concife method of Addition.

BUT before you proceed any further on the fubject of multiplication, you muft learn the following Table :——

MUL-

MULTIPLICATION TABLE.

1	2	3	4	5	6	7	8	9	10	11	12
2	4	6	8	10	12	14	16	18	20	22	24
3	6	9	12	15	18	21	24	27	30	33	36
4	8	12	16	20	24	28	32	36	40	44	48
5	10	15	20	25	30	35	40	45	50	55	60
6	12	18	24	30	36	42	48	54	60	66	72
7	14	21	28	35	42	49	56	63	70	77	84
8	16	24	32	40	48	56	64	72	80	88	96
9	18	27	36	45	54	63	72	81	90	99	108
10	20	30	40	50	60	70	80	90	100	110	120
11	22	33	44	55	66	77	88	99	110	121	132
12	24	36	48	60	72	84	96	108	120	132	144

FOR an explanation of the foregoing Table, suppose that it were required to find the product of 3×4. First, look in the left hand column for 3, and right opposite with it in the column under 4 at the top, is 12, the product of 3×4.

AGAIN, to find the product of 9×12. Look for 9 in the left hand column as before, and right opposite to it, under 12 in the upper column, is 108, the product required; and the like is to be understood of all the rest.

HAVING given you this short, but comprehensive idea of the foregoing Table, we shall now proceed to examples,

examples, with this caution, to wit, that in multiplying, care muſt be taken, that the product of the firſt figures, ſtand directly under its multiplier; alſo remembering to carry 1 for every 10 of the product.

EXAMPLES.

It is required to multiply 120×94; which placed as before directed will ſtand thus,

$$
\begin{array}{r}
120 \; \textit{multiplicand} \\
94 \; \textit{multiplier} \\
\hline
480 \\
1080 \\
\hline
11280 \; \textit{product.}
\end{array}
$$

Here you begin with that figure of your multiplier, which ſtands in the units' place, viz. 4, ſaying 4 times 0 is 0, which ſet down directly under the figure you are multiplying with; then ſay 4 times 2 is 8, which ſet under the 9; then 4 times 1 is 4, which alſo place as in the example; and the product of the multiplicand with the firſt figure of your multiplier, is 480: then begin with the next figure of your multiplier, ſaying 9 times 0 is 0, which place under your multiplying figure, then ſay 9 times 2 is 18; here ſet down 8 and carry 1 to the next place, ſaying 9 times 1 is 9, and 1 that I carry makes 10; now this being the product of the laſt place of figures, ſet down the whole, and the product of the multiplicand, with the ſecond figure of your multiplier is 1080, or more properly 10800: then adding up both products, their ſum is 11280, the product required. (See the example above.)

It is required to multiply 2439×421; theſe numbers placed as directed will ſtand

thus,

thus, $\left\{\begin{array}{c}2439\\421\end{array}\right\}$ *factors*

$2439 = product$ of 2439×1
$48780 = product$ of 2439×20
$975600 = product$ of 2439×400

$1026819 = product$ of 2439×421

THE annexing of cyphers, as in the laſt example, is to ſupply the vacant places; and to ſhew the ſeveral products are increaſed in a tenfold proportion, with regard to the places in which your multiplying figures ſtand. Thus the product of the multiplicand with the ſecond figure of your multiplier, is not the product of 2439×2, but the product of 2439×2 tens or 20; which product is 10 times more than it would have been, had the multiplying figure (2) ſtood in the units' place; ſo alſo the annexing of two cyphers, as in the product of the multiplicand with the third figure of the multiplier, to wit, 4, is becauſe that figure ſtands in the hundreds' place; and therefore the product is not 2439×4, but really the product of 2439×400; yet thoſe cyphers may be omitted, by obſerving the direction in the beginning of this chapter, viz. that the firſt figure of the ſeveral products ſtand directly beneath its correſponding figure of the multiplier.

Find the product of 24354×32001

thus, $\left\{\begin{array}{c}24354\\32001\end{array}\right\}$ *factors*

24354
48708
73062

$779352354 = 24354 \times 32001 = product$ *required.*

E   HERE

HERE you may obferve that we pafs the cyphers, taking care only to place the next figure according to the foregoing directions.

WHEN there are cyphers on the right-hand of the multiplicand, or multiplier, or to both, you may multiply the figures as before, neglecting the cyphers, until you have found the product of the digets only; to which annex fo many cyphers as there are in both factors: as in thefe,

$$21200 \Big\} \; factors \qquad\qquad 347650 \Big\} \; factors$$
$$34 \qquad\qquad\qquad\qquad\quad 200$$

$$848 \qquad\qquad\qquad 69530000 = 347650 \times 200$$
$$636$$

$$720820 = 21200 \times 34$$

$$24000000 \Big\} \; factors$$
$$24000000$$

$$96$$
$$48$$

$$576000000000000 = 24000000 \times 24000000$$

IF it be required to multiply any number with 10, 100, 1000, &c. you need only annex to your multiplicand fo many cyphers as are in the multiplier, and the work is done; as in the following,

$$4647 \times 10 = 46470 \qquad\qquad 20 \times 100 = 2000$$
$$5224 \times 1000 = 5224000 \qquad 300 \times 1000 = 300000$$
$$\qquad\qquad\qquad\qquad 26460 \times 10000 = 264600000$$

HERE it may perhaps be ufeful, to acquaint the learner of the method of performing Multiplication by Addition; which in fome cafes will be found ufe-
<div align="right">ful:</div>

ful : the method is as follows : firſt, ſet down the 9 digets, or numeral figures, in a ſmall column made for that purpoſe ; then againſt 1, place the multiplicand, againſt 2, double the multiplicand, againſt 3, three times the multiplicand, and ſo on to the laſt.

Find the product of 2439×421 by Addition.

1	2439=*multiplicand*
2	4878=2 *times do.*
3	7317=3 *do.* *do.*
4	9756=4 *do.* *do.*
5	12195=5 *do.* *do.*
6	14634=6 *do.* *do.*
7	17073=7 *do.* *do.*
8	19512=8 *do.* *do.*
9	21951=9 *do.* *do.*

againſt $\begin{cases} 1 & is & 2439 \\ 2 & & 4878 \\ 4 & & 9756 \end{cases}$

Sum 1026819=2439×421=prod. req.

HERE it is evident, that the foregoing table will ſerve let the multiplier be any number whatever ; for ſuppoſe it were required to find the product of 2439×6734.

OPERATION.

againſt $\begin{cases} 4 & is & 9756 & =2439×4 \\ 3 & & 7317 & =2439×30 \\ 7 & & 17073 & =2439×700 \\ 6 & & 14634 & =2439×6000 \end{cases}$

Sum 16424226=2439×6734=prod.req.

EXAM-

EXAMPLES.

$$691861 \times 26 = 17988386$$
$$346732 \times 652 = 226069264$$
$$7901375 \times 30000 = 237041250000$$
$$129186 \times 98 = 12660228$$
$$76001 \times 1302 = 98953302$$
$$3581 \times 2007 = 7187067$$

THE proof of Multiplication, is beſt done by Diviſion.

CHAP. VI.

Of DIVISION of SIMPLE NUMBERS.

DIVISION is a ſpeedy method of ſubtracting one number from another; to know how many times one number is contained in another; and alſo what remains.

THERE are three requiſites in Diviſion; the diviſor; the dividend, and the quotient; which ſhews how many times the diviſor is contained in the dividend.

WHEN any number meaſures another, the number ſo meaſured, is ſaid to be a multiple of the other: thus, 21 is meaſured by 7, for 7 is contained juſt 3 times in 21; conſequently 21 is a multiple of 7.

ONE number is ſaid to meaſure another, by a third number, when it either multiplies, or is multiplied by the meaſuring number, produces the number meaſured. (See *Euclid's* 7th book, def. 23.)

HENCE it follows, that in Diviſion the quotient muſt be ſuch a number, which if multiplied with the diviſor, will produce the dividend; conſequently

Diviſion

Divifion is the reverfe of Multiplication ; and there-
fore operations in Divifion, muft be performed direct-
ly reverfe of thofe in Multiplication ; that is, the di-
vifor muft be placed firft ; then make a ftroke on the
right-hand of it, and fet down your dividend, on the
right-hand of which, make another ftroke, to feper-
ate the dividend from the quotient ; then begin on
the left-hand, and decreafe the dividend by a re-
peated fubtraction of the products of the divifor and
each quotient figure, as they become known.

EXAMPLES.

REQUIRED to divide 344 by 4 ; the operation of
which will ftand in the following order,

$$
\begin{array}{c}
\textit{dividend} \\
\textit{divifor } 4 \overline{)344}(86 \ \textit{quotient} \\
32 \\
\overline{} \\
24 \\
24 \\
\overline{} \\
00 \\
\overline{}
\end{array}
$$

THE explanation of the above is as follows : firft
enquire how many times your divifor, which confifts
of 1 figure, is contained in the firft figure of your divi-
dend, which is 0 times ; becaufe your divifor (4) is
greater than the firft figure of your dividend (3), as
appears by infpection ; and therefore cannot meafure
it ; for a greater number to meafure a lefs is abfurd ;
therefore you muft increafe the value of the firft
figure of the dividend, by taking the annexed figure
(4) into the expreffion ; which will then be 34 (for
the reafons before given) ; then enquire how many
times your divifor is contained in thofe two figures
of

of the dividend, to wit, 34; which is 8 times, for 8 times 4 is 32, and 32 being the greatest multiple of the divisor that can be made under 34; consequently 8 must be the first figure of the quotient, which place as in the example; then multiplying the quotient figure (8) with your divisor, as in Multiplication, subtract their product from those two figures of the dividend, by which the said quotient figure was obtained; and to the remainder (2) annex the next figure of your dividend (4), and the remainder so increased becomes 24; then enquire how many times 4 is contained in 24, which is 6 times; therefore place 6 in the quotient, and multiply it with your divisor, subtracting their product as before, and the work is done. (See the example page 37.)

Now the quotient obtained in the example is 86; and there being no remainder, shews that 4 is contained in 344, just 86 times.

THE greatest difficulty in division, is when your divisor consists of many places of figures, and does not exactly measure the figures of the dividend with which you compare it: therefore to find the right quotient figure, may be done by considering that the product of the quotient figure with your divisor, must never be greater than that part of the dividend, with which you compare it; nor yet so small, that the number remaining after subtracting the product of the quotient figure and divisor from the aforesaid part of the dividend, shall be greater than the divisor. Therefore by supposing a figure for the quotient, and multiplying it with a figure or two on the left-hand of your divisor, you may easily determine the right quotient figure; which may be obtained by such mental operations, on the second or third trial, at farthest.

By thoroughly observing the foregoing directions, you may proceed to the performance of the following

examples ; wherein we shall prove those operations, performed in the last chapter ; in order to which, we shall begin with the second example ; taking the product of the factors for a dividend, and the multiplier for a divisor ; and proceed as before. (See the operation annexed.)

$$
\begin{array}{r}
dividend \\
divisor\ 421\,)\,1026819(2439\ quotient \\
842 \\
\hline
1848 \\
1684 \\
\hline
1641 \\
1263 \\
\hline
3789 \\
3789 \\
\hline
00
\end{array}
$$

Note, *It will be best to point the figures of the dividend, as they are annexed to the several remainders ; without which you may annex a wrong one.*

HERE you may see the quotient is the same as the multiplicand of the example before quoted ; which proves that the product of $2439 \times 421 = 1026819$.

Required to divide 779352354 by 32001.

OPERATION.

```
32001)779352354(24354=779352354÷32001
      64002
      ─────
      139332
      128004
      ─────
       113283
        96003
       ─────
        172805
        160005
        ─────
         128004
         128004
         ─────
         .....0
         ─────
```

Again, divide 1798836 by 26.

OPERATION.

```
26)1798836(69186=1798836÷26
   156
   ───
    238
    234
    ───
     48
     26
     ──
     223
     208
     ───
      156
      156
      ───
       ─
```

Once

Once more, divide 12660228 by 98.

OPERATION.

```
98)12660228(129186=quotient required.
   98
   ──
   286
   196
   ──
    900
    882
    ──
     182
      98
     ──
      842
      784
      ──
       588
       588
       ──
```

IF there be cyphers annexed to the divifor and dividend, expunge an equal number in both factors: as in the following example.

Divide 694000 by 2000.

OPERATION.

```
2(000)694(000(347=694000÷2000
     6
     ──
     9
     8
     ──
     14
     14
     ──
      0        F        IT
```

It will fometimes happen in Divifion, that the remainder, when augmented by annexing the next figure of the dividend, is lefs than the divifor, and confequently cannot be meafured by it; in which cafe, place 0 in the quotient, and annex the next figure of the dividend to the former number; but if this number be ftill lefs than the divifor, place 0 in the quotient and annex another figure of the dividend; and fo on, in like manner till the faid number be fo increafed, that it may be meafured by the divifor. (See this illuftrated in the following.)

Divide 98953302 by 1302.

OPERATION.

$$1302 \overline{)98953302} (76001 = 98953302 \div 1302$$
$$9114$$
$$\overline{7813}$$
$$7812$$
$$\overline{1302}$$
$$1302$$
$$\overline{\ldots 0}$$

The proof of the remaining examples in Multiplication, are left to the fagacity of the learner.

It is required to divide 32176432 by 3476.

OPERATION.

```
3476)32176432(9256
     31284
     ─────
      8924
      6952
      ────
      19723
      17380
      ─────
       23432
       20856
       ─────
        2576 remainder.
```

HERE follows some examples and their answers without their work.

What is the quotient of 23884044718÷45007 ?
Answer. 530674.

What is the quotient of 34500000÷100000 ?
Answer. 345.

What is the quotient of 244572000÷356 ?
Answer. 687000.

What is the quotient of 1332250÷365 ?
Answer. 3650.

THAT Division is a speedy method of subtraction, as before hinted, may be thus proved. Suppose 18 were to be divided by 6 : first subtract the divisor from the dividend, and the divisor again from that remainder, and so on till nothing remains. (See the operation in the next page.)

OPERATION.

$$
\begin{array}{r}
18 \ \textit{dividend} \\
-6 \ \textit{divisor} \\
\hline
12 \ \textit{remainder} \\
-6 \ \textit{divisor} \\
\hline
6 \ \textit{remainder} \\
-6 \ \textit{divisor} \\
\hline
0
\end{array}
$$

HENCE it is manifest, that the divisor is contained in the dividend, just 3 times; that is, 3 times 6═18: consequently, &c. Q. E. D.

THE next thing to be considered, is the proof of your work, i. e. whether the quotient found is a true one. The method is directly reverse of that used for the proof of Multiplication; for, as the truth of Multiplication is known by Division, so that of Division is known by Multiplication; that is, by multiplying the quotient with the divisor, which product must be equal to the dividend; therefore multiply the quotient with the divisor, and to their product add what remains after division; which aggregate will be equal to the dividend, if the work is right.

THERE is another method of proving Division; which is much shorter than the former, and is no more than adding together the products of the several quotient figures with the divisor, as they stand in your operation; which aggregate, together with the remainder, will be equal to the dividend. (See the following example.)

Required to divide 8765452 by 3463.

OPERATION.

$$3463)8765452(2531$$
$$+6926\cdots=3463\times2000$$

$$18394$$
$$+17315\quad=3463\times500$$

$$10795$$
$$+10389\quad=3463\times30$$

$$4062$$
$$+3463\quad=3463\times1$$

$$+599\ \textit{remainder}$$

$$8765452=\textit{dividend.}$$

Or, $6926000+1731500+103890+3463+599=$ 8765452. Therefore, &c.

A Supplement to Chapter VI.

NOTWITHSTANDING what hath been said on this subject, respecting the division of simple quantities, is universally true; yet there is another method of dividing quantities, which is very ready in practice; and is therefore called Short Division: this method is performed by the following Rules.

RULE I.

ARRANGE the factors as before in Division; then by comparing the divisor with the dividend, you

<div align="right">will</div>

will obtain a quotient figure, which muſt be ſet in
its proper place, under that part of the dividend by
which your diviſor was compared ; valuing ſaid fig-
ure as though there were no other ; alſo obtain the
difference (if any) of the product of the diviſor and
quotient figure, and the aforeſaid part of the divi-
dend ; prefixing that difference in your mind to the
next figure of your dividend ; which forms an expreſ-
ſion for obtaining the next quotient figure, which
muſt be ſet directly under that figure, to which the
difference was prefixed ; and ſo on till the whole
be completed.

EXAMPLES.

Divide 46782 by 3.

Those numbers being placed as directed will
ſtand thus,

$$3 \overline{)46782}$$
$$15594 = 46782 \div 3$$

Again, divide 68432 by 4 :

thus, $4 \overline{)68432}$

17108 = *quotient required.*

Note 1. *If there be a remainder after the laſt quo-*
tient figure is found, ſet it at a little diſtance on the
right-hand of your quotient, making a dot with your
pen, denoting the ſeperation ; as in the following.

Divide 23764 by 5 : thus, $5 \overline{)23764}$

rem.

$$4752 . 4 = \frac{23764}{5}$$

Again,

Again, find the quotient of 73215 ÷ 6:

thus, 6) 73215

12202 . 3 *rem.*

Alſo, divide 43206 by 8:

thus, 8) 432106

54013 . 2 *rem.*

Note 2. *If your diviſor be* 10, *ſeperate the firſt fig-
ure on the right-hand of your dividend for a re-
mainder, and the work is done.*

thus, 10) 76435 (2 *rem.*

Find the quotient of 645384 ÷ 12:

thus, 12) 645384

53782 = *anſwer.*

RULE II.

1. RESOLVE your diviſor into ſeveral parts ſuch,
that their continued product ſhall be equal to the
given diviſor.

2. SUBSTITUTE thoſe parts ſucceſſively as diviſors,
in the following manner, viz. divide the given divi-
dend by one of thoſe parts, now called diviſors, and
the reſulting quotient by another of thoſe diviſors,
and ſo on; the laſt quotient ariſing by ſuch diviſors,
will be the quotient required.

EXAMPLE.

Divide 2904 by 24.

YOUR

Your divisor resolved into parts as above directed will be, either 8 and 3, 6 and 4, or 12 and 2; for 8×3=24, 6×4=24, or 12×2=24; therefore let the parts be 6 and 4; then 2904÷6=484, and 484÷4 =121=quotient required; and if the others be tryed they will equally succeed.

CHAP. VII.

ADDITION of COMPOUND QUANTITIES or NUMBERS.

ADDITION of compound quantities, is the adding together numbers of different denominations, so that their aggregate, or total amount may be known. The operations are performed by the following general

RULE.

1. WRITE down the several denominations so, that all those of the same name may stand directly under each other.

2. BEGIN on the right-hand, at the least of the given denominations, adding together the whole of that denomination, as in Simple Addition; then divide this sum by such a number, as it takes parts to make one of the next greater denomination, placing the remainder (if any) under its own denomination, and carrying the quotient to the said next greater denomination, add them up with the whole of that denomination, then divide as before; and so on, from one denomination to another, until the whole be completed.

SECT.

SECT. I.

ADDITION of TROY WEIGHT.

TROY WEIGHT is that by which gold, filver, jewels, medical compofitions, and all liquors are weighed. It is divided into four denominations, to wit, ℔. pounds, *oz.* ounces. *dwt.* pennyweights and *gr.* grains, according to the following

TABLE.

24*gr.*=1*dwt.* 480*gr.*=20*dwt.*=1*oz.* 7560*gr.*= 240*dwt.*=12 *oz.*=1 ℔.

EXAMPLES.

Find the fum of the following, 14℔. 11*oz.* 16*dwt.* 13*gr.*+19℔. 10*oz.* 17*dwt.* 17*gr.*+17℔. 11*oz.* 12*dwt.* 22*gr.*

THESE numbers being placed, according as the general rule directs, will ftand

	℔.	oz.	dwt.	gr.
	14	11	16	13
thus,	19	10	17	17
	17	11	12	22
	52	10	7	4

=*fum required.*

THEN begin at the leaft denomination, to wit, grains, and adding together all that denomination, we find the fum to be 52: now becaufe 24 grains make one pennyweight, divide 52 by 24, and the quotient will be 2, leaving a remainder of 4, which write under grains, and carry the quotient 2, to the next place, and adding it up with that denomination, we find the fum to be 47, which divide by 20 (becaufe 20 pennyweights make one ounce) and the quotient will

G

be

be 2, leaving a remainder 7, which write in its proper place, and carry the quotient 2, to the next place; this being added up with the denomination, we find the sum to be 34, which divided by 12 quotes 2, and 10 remaining; write this under its own denomination, and carry the quotient 2 to the next place, which added up with that denomination, we find the sum to be 52; and because this is the last denomination, write the whole, and the work is done. Hence we find the sum total to be 52 ℔. 10 oz. 7 dwt. and 4 gr. as was required. (See the example, page 49.)

℔.	oz.	dwt.	gr.		℔.	oz.	dwt.	gr.
37	10	17	19		47	11	19	24
12	7	12	17		27	8	17	20
17	10	17	12		19	7	12	17
18	9	19	23		10	5	15	17

SECT. II.

ADDITION of MONEY.

THIS is to find the aggregate, or sum total of several sums of money.

EVERY nation of the world has a particular method of reckoning their money. Great-Britain makes use of pounds, shillings, pence and farthings; and the United States followed the same method, until the present system of government was established; by which it is enacted, that all the monies of every nation or kingdom, shall be reckoned or estimated in America, in dollars and cents: so that these two species of money are to be made the standard money of the United States.

Note that 100 *cents make one dollar.*

EXAM-

EXAMPLES.

Find the sum of 174*dol.* 17*cts.*+197*dol.* 19*cts.*+ 375*dol.* 92*cts.*+275*dol.* 92*cts.* These being placed according to the general rule, will stand

	dol.	cts.
thus,	174	17
	197	19
	375	92
	275	92
	1023	20 *sum required.*

Note. *Since* 100 *cents make one dollar, we must divide the sum of the cents by* 100 ; *but to divide by* 100 *is no more than to seperate the two right-hand figures of the dividend for a remainder, the rest are the quotient. Therefore, after you have added up the last place of figures in the cents' place, proceed to the dollars' place as though the whole was but one denomination.*

Find the sum of 127*dol.* 19*cts.*+278*dol.* 19*cts.*+ 137*dol.* 19*cts.*+122*dol.* 92*cts.*+127*dol.* 90*cts.*

dol.	cts.
127	19
278	19
137	19
122	92
127	90
793	39 = *sum required.*

dol.

dol.	ets.	dol.	cts.	dol.	cts.
127	17	3787	19	2784	19
172	57	3729	72	1234	27
189	68	4229	91	3456	78

Total

HAVING thus explained the principles, and given a general rule for the Addition of all compounds in whole numbers; we fhall leave the reft to the fagacity of the learner, who with the affiftance of the following tables and examples, will be able to manage any fuch compounds as have relation therewith.

SECT. III.

Of AVOIRDUPOIS WEIGHT.

BY Avoirdupois Weight are weighed, flefh, butter, cheefe, falt; alfo all coarfe and droffy commodities; as grocery wares; likewife pitch, tar, rofin, wax, iron, fteel, copper, brafs, tin, lead, hemp, flax, tobacco, &c.

THE characters in Avoirdupois Weight are *dr. oz. lb. qr. C. T.* that is drachm, ounce, pound, quarter, hundred, tun.

TABLE.

16 *dr.*═1 *oz.* 256*dr.*═16*oz.*═1*lb.* 7168*dr.*═ 448*oz.*═28*lb.*═1*qr.* 28672*dr.*═1792*oz.*═112*lb.* ═4*qr.*═1*C.* 573440 *dr.*═35840 *oz.*═2240 *lb.*═ 80*qr.*═20*C.*═1*T.*

EXAMPLES.

T.	*C.*	*qr.*	*lb.*	*oz.*	*dr.*	*T.*	*C.*	*qr.*	*lb.*	*oz.*	*dr.*
346	12	2	16	10	14	576	19	1	16	12	13
67	16	3	22	8	10	867	4	0	24	14	13
46	10	3	15	12	15	453	6	3	27	3	4

SECT.

S E C T. IV.

Of *APOTHECARIES WEIGHT.*

THE Apothecaries pound and ounce is the fame as the pound and ounce Troy, but differently divided, as in the following

TABLE.

20*gr.*=1Ә. 60*gr.*=3Ә.=1Ʒ. 480*gr.*=24Ә.
=8Ʒ.=1ʒ. 5760*gr.*=288Ә.=96Ʒ.=12ʒ.=1℔.

APOTHECARIES make ufe of thefe weights in the compofition or mixture of their medicines, but fell their drugs by Avoirdupois Weight.

EXAMPLES.

℔.	ʒ.	Ʒ.	Ә.	gr.	℔.	ʒ.	Ʒ.	Ә.	gr.
124	10	4	2	14	266	9	5	1	15
64	8	6	1	16	76	10	4	2	14
30	11	7	0	17	96	11	6	2	10
50	9	3	1	12	10	7	1	1	

S E C T. V.

BY Long Meafure, is eftimated length, where no regard is had to breadth : or in other words, it meafures the diftance of one thing from another : and the ufual method of dividing and fub-dividing of length, is into degrees, leagues, miles, furlongs, poles, yards, feet, inches, and barley-corns, as in the following

TABLE I.

3*bc.*=1*in.* 36*bc.*=12*in.*=1*f.* 108*bc.*=36*in.*=3*f.*
=1*yd.* 594*bc.*=198*in.*=16½*f.*=5½*yd.*=1*p.* 23760*bc.*
=7920*in.*=660*f.*=220*yd.*=40*p.*=1*fur.* 190080*bc.*
=63360*in.*

$=63360$ in.$=5280f.=1760yd.=320p.=8fur.=1m.$
$570240bc.=190080in.=15840f.=5280yd.=960p.=$
$24fur.=3m.=1le.$

TABLE II.

$3bc.=1in.$ $36bc.=12in.=1f.$ $108bc.=36in.=3f.$
$=1yd.$ $1188bc.=396in.=33f.=16yd.=1ch.$ $23760bc.$
$=7820in.=660f.=220yd.=20ch.=1fur.$ $190080bc.$
$=63360in.=5280f.=1760yd.=160ch.=8fur.=1m.$
$570240bc.=190080in.=15840f.=5280yd.=480ch.$
$=24fur.=2m.=1le.$ $11404800bc.=3801600in.=$
$316800f.=105600yd.=9600ch.=480fur.=60m.=$
$20le.=1deg.$

THE length of a degree as laid down in table 2d.
is not to be underſtood as the true one, but the length
of a degree as commonly received and practiſed ; for
the length of the greateſt degree is $70\frac{1}{10}$ miles, and
the leaſt $67\frac{3}{4}$ miles nearly ; a mean degree is there-
fore $68\frac{92}{100}$ miles.

EXAMPLES.

deg.	le.	m.	fur.	ch.	yd.	f.	in.	bc.
120	14	2	6	14	5	2	10	1
87	12	0	7	12	3	1	5	0
90	19	1	5	18	2	2	4	2

deg.	le.	m.	fur.	ch.	yd.	f.	in.	bc.
332	15	1	7	12	8	1	10	2
	19	2	0	14	9	2	9	
		1	6	13	5	0		
			4	10	4			
				9				

S E C T. VI.

Of LAND MEASURE.

THE use of this measure, is to find the area or superficial content of any piece of land in acres, and parts of an acre; which parts are as in the following

TABLE.

9 *sq. f.* = 1 *sq. yd.* 1089 *sq. f.* = 121 *sq. yd.* = 1 *sq. ch.*
10890 *sq. f.* = 1210 *sq. yd.* = 10 *sq. ch.* = 1 *sq. qr.* 43560
sq. f. = 4840 *sq. yd.* = 40 *sq. ch.* = 4 *sq. qr.* = 1 *sq. acre*

EXAMPLES.

ac.	qr.	ch.	yd.	f.		ac.	qr.	ch.	yd.	f.
24	2	8	104	8		92	1	7	100	7
37	3	7	111	7		27	3	7	98	8
47	2	4	90	7		39	0	7	117	7

S E C T. VII.

Of CLOTH MEASURE.

THE divisions of Cloth Measure are as in the following

TABLE.

4 *na.* = 1 *qr.* 16 *na.* = 4 *qr.* = 1 *yd.* Also, 3 *qr.* = 1 *ell*
Flem. 5 *qr.* = 1 *ell Eng.* 6 *qr.* = 1 *ell Fr.*

EXAMPLES.

yd.	qr.	na.		ell Fl.	qr.	na.
226	3	2		327	3	3
74	3	0		39	2	1
362	2	3		500	3	2

---

ell Eng.

ell Eng.	qr.	na.		ell Fr.	qr.	na.
327	4	3		529	5	3
90	3	2		468	2	2
264	2	1		436	4	3
354	3	3		43	3	1

SECT. VIII.

Of DRY MEASURE.

DRY MEASURE is fo called becaufe it meafures all fuch dry commodities as corn, wheat, rye, oats, barley, peas, beans, and all kinds of grafs-feed; alfo all kinds of roots and fruits.

THE ftandard of this meafure is a bufhel of a cylindrical form, of the following dimenfions, viz: $18\frac{1}{2}$ inches in diameter, and 8 inches in altitude; confequently a veffel of fuch form and dimenfions will contain $2150\frac{42}{100}$ cubic inches, which is the content of the Winchefter bufhel: Therefore the quart Dry Meafure, contains $67\frac{2}{10}$ cubic inches nearly; and the divifions are as in the following

TABLE.

67 . 2 cu. in.=1 qrt.   268 . 8 cu. in.=4 qrt.=1 gal.
537 . 6 cu. in.=8 qrt.=2 gal.=1 pc.   2150 . 42 cu. in.
=32 qrt.=8 gal.=4 pc.=1 bufh.

EXAMPLES.

bufh.	pc.	gal.	qrt.	bufh.	pc.	gal.	qrt.	bufh.	pc.	gal.	qrt.
57	3	1	3	37	3	1	1	2	3	1	2
24	0	0	2	19	0	0	0		1	1	
47	2	1	0	33	2	0	3	2	3	1	3

SECT.

S E C T. IX.

Of LIQUID MEASURES.

IN Liquid Meaſures, the gallon is made the ſtandard, and from thence are deduced the other denominations made uſe of in ſuch meaſures. The wine gallon is ſuppoſed to contain 231 cubic inches, conſequently the quart muſt contain $57\frac{3}{4}$ cubic inches; from thence is deduced the following

TABLE of WINE MEASURE.

$57\frac{3}{4}$ *cu. in.*=1 *qrt.* 231 *cu. in.*=4 *qrt.*=1 *gal.* 9702 *cu. in.*=168 *qrt.*=42 *gal.*=1 *tr.* 14553 *cu. in.*=252 *qrt.* =63 *gal.*=$1\frac{1}{2}$ *tr.*=1 *hhd.* 19404 *cu. in.*=336 *qrt.*=84 *gal.*=2 *tr.*=$1\frac{1}{3}$ *hhd.*=1 *pun.* 29106 *cu. in.*=504 *qrt.* =126 *gal.*=3 *tr.*=2 *hhd.*=$1\frac{1}{2}$ *pun.*=1 *bt.* 58212 *cu. in.*=1008 *qrt.*=252 *gal.*=6 *tr.*=4 *hhd.*=3 *pun.*=2 *bt.* 1 *tun.*

EXAMPLES.

tun	hhd.	gal.	qrt.	tun.	hhd.	gal.	qrt.
237	2	62	3	279	2	57	2
234	1	27	0	273	0	39	0
72	2	25	3	99	2	47	3
34	0	59	0	93	1	24	2

Of ALE or BEER MEASURE.

THE gallon of Ale or Beer Meaſure contains 282 cubic inches, as in the following

TABLE.

$70\frac{1}{2}$ *cu. in.*=1 *qrt.* 282 *cu. in.*=4. *qrt.*=1 *gal.* 2397 *cu. in.*=34 *qrt.*=$8\frac{1}{2}$ *gal.*=1 *fir.* 4794 *cu. in.*=68 *qrt.* =17 *gal.*=2 *fir.*=1 *kil.* 9588 *cu. in.*=136 *qrt.*=34 *gal.*

H

gal.=4 *fir.*=2 *kil.*=1 *bar.* 14382 *cu. in.*=204 *qrt.*=
51 *gal.*=6 *fir.*=3 *kil.*=1½ *bar.*=1 *bhd.*

EXAMPLES.

bhd.	kil.	fir.	gal.	qrt.	bhd.	kil.	fir.	gal.	qrt.
79	2	1	7	2	73	2	1	6	3
64	3	0	5	0	97	1	1	7	2
49	1	1	6	2	37	2	1	2	0

S E C T.   X.

Of the MEASURE of TIME.

In the division of Time, a year is made the standard or integer, which is determined by the revolution of some celestial body in its orbit; which body is either the sun or moon. The time measured by the sun's revolution in the ecliptic (or imaginary circle in the heavens, so called by astronomers) from any equinox or soltice to the same again, is 365 days, 5 hours, 48 minutes, 57 seconds, and is called the solar or tropical year.——Although the solar year before mentioned, is the only proper or natural year, yet the civil or Julian year is the one which the different nations of the world make use of in the regulation of civil affairs.

The civil solar year contains 365 days, 6 hours; but in common mathematical computations, the odd hours are generally neglected, and the year taken only for 365 days; from which, the divisions in the following T A B L E are made, wherein a second is considered (as it really is) the least part of time that can be truly measured by any mechanical engine.

60″.=1′.   3600″.=60′.=1 h.   86400″.=1440′.=
24 h.=1 d.   31536000″.=525600′.=8760 h.=365 d.
=1 year.   *EXAM*-

EXAMPLES.

y.	d.	h.	'	"
167	272	14	42	29
234	173	22	58	59
39	290	19	19	19
43	222	22	22	22
99	99	20	57	21

y.	d.	h.	'	"
173	192	10	17	29
346	364	23	59	59
199	170	19	17	16
344	19	10	34	46
79	38	23	43	43

SECT. XI.

Of CIRCULAR MOTION.

WHAT is here meant by Circular Motion, is that of the heavenly bodies in their orbits; which are reckoned in signs, degrees, minutes, and seconds, as in the following

TABLE.

$60'' = 1'$. $3600'' = 60' = 1°$. $108000'' = 1800' = 30° = 1 S$. $1296000'' = 21600' = 360° = 12 S. = great circle$ of the ecliptic.

EXAMPLES.

S.	°	'	"
10	12	30	10
9	11	47	47
8	4	37	4
7	24	42	36

S.	°	'	"
11	13	13	13
8	17	23	43
7	29	44	27
6	19	38	59

Note. *In the Addition of Circular Motion, when the sum of the signs exceed 12, or any multiple of it, write such excess in the place of signs, rejecting the rest.*

Note.

Note. *In order to prevent a misconstruction of the abbreviations, in the nine preceding* TABLES, *we have subjoined the following explanation, viz.* gr. *stands for grains.* Ə *scruples.* ʒ *drachms.* ℥ *ounces.* ℔ *pounds.*—bc. *barley-corns.* in. *inches.* f. *feet.* yd. *yards.* ch. *chains.* p. *poles.* fur. *furlongs.* m. *miles.* le. *leagues.* deg. *degrees.*— ſq. *ſquare.* qr. *quarters.* ac. *acres.*—na. *nails.* Flem. *Flemiſh.* Eng. *Engliſh.* Fr. *French.*—cu. *cubic.*—qrt. *quarts.* gal. *gallons.* pc. *pecks.* buſh. *buſhels.*—tr. *tierces.* hhd. *hogſheads.* pun. *puncheons.* bt. *butts.*—fir. *firkins.* kil. *kilderkins.* bar. *barrels.*—″ *ſeconds.* ′ *minutes.* h. *hours.* d. *days.* y. *years.* ° *degrees.* S. *Signs.*

CHAP. VIII.

SUBTRACTION of COMPOUNDS.

SUBTRACTION of Compounds is the taking one number from another : and is performed by the following general

RULE.

1. RANGE the given denominations according to the directions in the laſt chapter.

2. BEGIN at the ſame place as in Addition, to wit, at the leaſt of the given denominations, ſubtracting the lower number from the upper, as in Simple Subtraction, writing the difference under its own name ; but if the number in the ſubtrahend or under number, be greater than that which ſtands directly over it (as it often happens) you muſt add to your upper number, ſo many units of that denomination as are equal to
one

one of the next greater; from which perform the intended subtraction, writing the difference as before. Then proceed to the next place, where you muſt pay what you before borrowed of this denomination, by adding one to the ſubtrahend; and then perform ſubtraction as before; and ſo on to the laſt place, where the ſubtraction is performed as in ſimple quantities.

EXAMPLES.

From 37 ℔ 10 oz. 17 dwt. 20 gr. take 27 ℔ 11 oz. 19 dwt. 17 gr.

Theſe numbers being placed according to the rule, will ſtand

	℔	oz.	dwt.	gr.
thus, {	37	10	17	20
	27	11	19	17
	9	10	18	3	diff. required.

HERE beginning at the leaſt denomination, to wit, at grains, ſubtract 17 from 20, and there remains 3, which write under its own name; then proceed to the next denomination; but here the under number is the greateſt, and therefore cannot be taken from the upper; wherefore add 20 to the upper number (becauſe 20 pennyweights make one ounce) and the ſum is 37, from which take 19, and their remains 18; or take 19 from 20, and then add 17, and the ſum will be 18, as before. Then proceed to the next place; and here again, the under number is the greateſt, therefore add 1 to 11 for what you before borrowed, and the ſum will be 12, which taken from 22, leaves 10, which write in its proper place, and proceed to the laſt denomination, where paying what you before borrowed, perform the ſubtraction as in whole numbers,

bers,

bers, and the remainder will be 9. Hence we find the whole difference to be 9 pounds, 10 ounces, 18 pennyweights, and 3 grains.

	℔	oz.	dwt.	gr.		dol.	cts.		dol.	cts.
From	27	10	13	17		37	19		78	92
Take	22	8	19	19		21	18		27	75
Rem.	5	1	13	22		16	1		51	17

As the foregoing rule is general, the learner by duly obferving the application of it, to the above examples, may very readily perform the following ones without any further direction.

	T.	C.	qr.	lb.	oz.	dr.		yd.	qr.	na.
From	324	19	3	17	2	15		227	3	2
Take	233	17	2	20	13	14		204	1	3
Rem.										

ell Flem.	qr.	na.	ell Eng.	qr.	na.	ell Fr.	qr.	na.
From 52	1	3	42	4	1	53	3	3
Take 35	2	1	36	2	3	49	5	0
Rem.								

	T.	hhd.	gal.	qrt.		hhd.	kil.	fir.	gal.	qrt.
From	37	3	36	2		33	2	1	7	3
Take	23	1	37	3		27	1	0	4	3
Rem.										

	y.	d.	h.	′	″		℔	℥	з	э	gr.
From	434	320	17	24	42		47	10	7	2	14
Take	329	370	19	47	29		45	8	5	1	17
Rem.											

THE

THE method of proving your work, is the same as that of Simple Subtraction.

C H A P. IX.

MULTIPLICATION and DIVISION of COM-POUNDS.

S E C T. I.

Of MULTIPLICATION.

MULTIPLICATION of Compound Numbers is the multiplying any sum composed of divers denominations, with a simple multiplier, according to the following

R U L E.

BEGIN the operation as in all other compounds, multiplying that denomination with your multiplier, as in Simple Multiplication ; then divide this product by as many units as make one of the next greater denomination, writing the remainder as in Addition ; then note the quotient, and proceed to the next place, and multiply that denomination with your multiplier, to which add the aforesaid quotient ; then divide this product as before, and so on, till you have multiplied your multiplier with every denomination in your multiplicand ; and the result will be the product required.

EXAMPLES.

Multiply 120 ℔ 10 oz. 13 *dwt.* 17 *gr.* with 4.

OPER-

OPERATION.

℔	oz.	dwt.	gr.	
120	10	13	17	*multiplicand*
			4	*multiplier*
483	6	14	20	*product required.*

HERE we begin with 4×17=68; then 68÷24=2, and 20 remaining, which write in its proper place; then 4×13=52, to which add 2, the quotient juſt found, and the ſum will be 54; then 54÷20=2, and 14 remaining, which write in its proper place; then 4×10=40, to which add the laſt quotient 2, and the ſum is 42; now 42÷12=3, and 6 remaining, which write in its proper place. Laſtly, 4×120=480, to which add 3, the laſt found quotient, and the ſum is 483. Hence we find the whole product to be 483 pounds, 6 ounces, 14 pennyweights, and 20 grains.

Multiply 127 *dol.* 17 *cts,* with 6.

OPERATION.

dol.	cts.	
127	17	
	6	
763	2	*product.*

S.	°	′	″		yd.	qr.	na.	
10	13	42	10		4	2	3	
			4				6	
5	24	48	40	*prod.*	28	0	2	*product.*

ell Flem.

ell Fl.	qr.	na.	ell Eng.	qr.	na.	ell Fr.	qr.	na.
17	2	1	10	4	2	13	5	3
		7		12				8
124	0	3	130	4	0	111	4	0 *prod.*

deg.	le.	m.	fur.	p.	f.	in.	bc.
12	10	2	5	10	10	1	2
							4
50	3	1	5	2	7	6	2 *product.*

Note. *You may resolve your multiplier into several
parts, as in Short Division, and if those parts when
multiplied together, do not exactly make the given
multiplier, add as many times the multiplicand to
the product, as the product of the said parts fall
short of the given multiplier; as in these:*

Find the product of 127 *dol.* 19 *cts.* ×15.

HERE the parts of the multiplier are 3 and 5.

Therefore, $\left\{ \begin{array}{cc} dol. & cts. \\ 127 & 19 \end{array} \right.$

$$\begin{array}{cc} & 3 \\ \hline 381 & 57 \\ & 5 \\ \hline 1907 & 85 \end{array}$$ *(because* 3×5=15)=127 *dol.* 19 *cts.* ×15.

Required the product of 197 *dol.* 87 *cts.* ×23.

I   Let

Let the parts be 3 and 7. Therefore,

```
                    dol.  cts.
                    197   87
                            3
            ──────────────────
                    593   61
                            7
            ──────────────────  dol.  cts.
                   4155   27 = 197  87 × 21.
add 2 times 197 dol. 87 cts. or  395  74
            ──────────────────
                   4551    1 = product req.
```

What is the product of 22 ℔ 6 oz. 10 dwt. 12 gr. ×32 ?

Answer. 721 ℔ 4 oz. 16 dwt.

What is the product of 13 yd. 3 qr. 2 na. ×48 ?

Answer. 666 yd.

SECT. II.

DIVISION of COMPOUNDS.

DIVISION being directly the reverse of Multiplication, needs no other explanation than the following examples; only observe, that when any denomination is not exactly measured by the divisor, the remainder must be reduced to the next inferiour denomination, and added to it; then perform the division.

EXAMPLES.

```
  ℔  oz. dwt. gr.      dol. cts.        dol. cts.
2)375 11  13   14    4)347  12        7)784  49
  ───────────────      ─────────        ─────────
  187 11  16   19       86  78          112   7 quo.
  ───────────────      ─────────        ─────────
                                              0 le.
```

```
       o   le. m. fur. ch. yd. f. in.
    4)47  14   2   6  12   5   1   7
```

3165 *dol.*÷6=527 *dol.* 50 *cts.* and 527 *dol.* 50 *cts.*
÷5=105 *dol.* 50 *cts.*

Likewife, 101 *dol.* 50 *cts.*÷5=20 *dol.* 30 *cts.* (*be-cause* 6×5×5=150)=3165÷150

Miscellaneous Questions for the Learner's Practice.

SIR Isaac Newton was born in the year 1642, and died in 1726 : What was his age when he died ?

There are two numbers, the greater 96, and the lefs 45 : What is their fum and difference ?

To find a number such, that 426 taken from it, will leave 127 remainder.

A certain number of merchants in trade, gained 19140 dollars, which being equally divided, a fhare was found to be 4785 dollars : How many merchants were there in that trade ?

What is the quotient of 3276 divided by 3, and by 9 ?

What number is the divifor of 1530320, when the quotient is 470 ?

What is the coft of 51 yards of broadcloth, at 4 *dol.* 10 *cts.* per yard ?

CHAP. X.

REDUCTION.

BY Reduction, numbers composed of different denominations are brought into one, by unfolding the several denominations by the parts that compose them. Or, from any number of homologous parts, to discover the number of certain heterogeneous, or unlike denominations. The former is called Reduction by Multiplication, and the latter Reduction by Division.———Reduction by Multiplication has the following general

RULE.

BEGIN at the greatest denomination mentioned, multiplying it with as many units as one of this denomination contains units of the next inferiour dedenomination ; and to the product add the numbers in the less denomination ; then multiply this sum as before, add as above, and so on (multiplying with as many units as it takes those of the next less denomination to make one of the present), until you have reduced the given parts to the denomination required.

EXAMPLES.

Required the number of cents equal to 1000 dollar.

OPERATION.

```
1000
    100 = number of cents in a dollar.

100000 = number of cents required.
```

Reduce

Reduce 1057 *dol.* 90 *cts.* into cents.

OPERATION.

dol. *cts.*
1,057 90
    100
————————————

105790=*number of cents required.*
————————————

But to reduce the monies of foreign nations, to those of the United States, consult the following

TABLE.

	dol.	*cts.*
Pound Sterling of Great-Britain=	4	44
Livre Tournois of France		18½
Guilder of the United Netherlands		39
Mark Banco of Hamburgh		33⅓
Rix Dollar of Denmark	1	
Rix Dollar of Sweden	1	
Real Plate of Spain		10
Milree of Portugal	1	24
Pound Sterling of Ireland	4	10
Tale of China	1	48
Pagoda of India	1	94
Rupee of Bengal		55½
Mexican Dollar	1	
Crown of France	1	11
Crown of England	1	11

Note. *The gold coins of France, England, Spain, and Portugal, are valued at 89 cents per pennyweight.*

In

In 127 pounds sterling of Great-Britain, how many cents?

Here multiply the pounds with 444.

$$
\begin{array}{r}
127 \\
444 \\
\hline
508 \\
508 \\
508 \\
\hline
56388 \text{ the answer.}
\end{array}
$$

In 274 livres tournois of France, how many cents?

Multiply with 18, and add half the multiplicand to that product.

$$
\begin{array}{r}
274 \\
18 \\
\hline
2192 \\
274 \\
\hline
4932 \\
137 \\
\hline
5069 \text{ the answer.}
\end{array}
$$

In 540 marks banco of Hamburg: how many cents?

Multiply

Multiply with 33, and add one third of the multiplicand to that product.

$$
\begin{array}{r}
540 \\
33 \\
\hline
1620 \\
1620 \\
\hline
17820 \\
180 \\
\hline
18000 \ \textit{the answer.}
\end{array}
$$

In 424 rupees of Bengal : how many cents?

Multiply with 55, and proceed as in the livres tournois of France.

$$
\begin{array}{r}
424 \\
55 \\
\hline
2120 \\
2120 \\
\hline
23320 \\
212 \\
\hline
23532 \ \textit{the answer.}
\end{array}
$$

Note. *In reducing the following species of money to cents, take the following methods.*

For the Guilders of the United Netherlands, *multiply*
	with	39
Real Plate of Spain		10
Milree of Portugal		124
Pound Sterling of Ireland		410

<div align="right">*Tale*</div>

Tale of China	148
Pagoda of India	194
Crown of France	111
Crown of England	111

In 127 ℔, how many ounces, pennyweights and grains?

  127
   12 $=$*number of ounces in* 1 *pound*

  1524 $=$*number of ounces in* 127 *pounds*
   20 $=$*number of pennyweights in* 1 *ounce*

  30480 $=$*number of pennyweights in* 127 *pounds*
    24 $=$*number of grains in* 1 *pennyweight*

 121920
 60960

 731520 $=$*number of grains in* 127 *pounds.*

  ℔. *oz.* *dwt.* *gr.*
In 12 8 12 4 how many grains?
  12

  152 $=$ 12×12+8
   20

  3052 $=$ 152×20+12
   24

 12212
 6104

 73252 $=$ 3052×24+4 $=$*number of grains req.*

           In

In 333 milrees of Portugal : how many cents ?
Anſwer. 41292.

In 555 tales of China : how many cents ?
Anſwer. 82140.

REDUCTION by DIVISION.

THIS method is directly reverſe of the former ; for
where we before multiplied, here we muſt divide
with the ſame number ; and therefore admits of the
following

R U L E.

DIVIDE the numbers in each denomination, by the
number of units that make one of the next ſuperiour
denomination ; and the quotients reſulting, will be the
numbers in the ſeveral denominations required.

EXAMPLES.

In 57200 cents : how many dollars ?
$$1(00)572(00$$

Therefore 572 *dollars is the anſwer.*

In 73252 grains Avoirdupois : how many penny-
weights, ounces, and pounds ?

```
24)73252 | 20|3052
   72     |  ‾‾‾‾‾‾
  ‾‾‾‾    | 12|152 . 12 rem.
  125     |  ‾‾‾‾‾‾
  120     |  12 . 8 rem.
  ‾‾‾‾    |
    52
    48
   ‾‾
    4
```

K   Therefore

Therefore in 73252 grains, there are 3052 penny-weights, 152 ounces, or 12 pounds.

Note. *The several remainders are of the same name of their dividends.*

In 41292 cents: how many milrees of Portugal?
$41292 \div 124 = 333$, *the answer.*

In 82140 cents: how many tales of China?
Answer. 555.

In 56388 cents: how many pounds sterling of England?
Answer. 127.

Note. *In reducing cents into livres tournois of France, you must multiply with 2, and divide that product by 37.——The mark banco of Hamburg, multiply with 3, and divide that product by 100.——The rupee of Bengal, multiply with 2, and divide by 111.*

In 752 nails: how many yards?
Answer. 47 *yards.*
In 15840 ~~barley-corns~~ *yards* : how many miles?
Answer. 3 *miles.*

In 469 gallons: how many hogsheads?
Answer. 7 *hhd.* 38 *gal.*

Miscellaneous Questions.

THE comet of 1680, at its greatest distance from the sun, was 11184768000 miles: now suppose a body had been projected from the sun, with a degree of swiftness equal to that of a cannon ball,

which

which is at the rate of 480 miles per hour : in what time would this body reach the aforesaid comet ; allowing the year to confist of 365 days ?

Anfwer. 2660 *years.*

How many times will a fhip of 97 feet 6 inches long, fail her length, in the diftance of 12800 leagues and 10 yards:

Anfwer. 2079408.

A MERCHANT bought 4 tuns, 15 hundreds, and 24 pounds of fugar, and ordered it to be put up into parcels of 24 pounds, of 20, of 16, of 12, of 8, of 4, of 2, and of each a like number. How many parcels will be made of the fugar ?

Anfwer. 124.

A GENTLEMAN had 15 dollars to pay among his labourers—to every boy he gave 10 cents—to every woman 20 cents, and to every man 45 cents : the number of men, women and boys was the fame. I demand the number of each fort ?

Anfwer. 20.

THERE are five tooth wheels placed in fuch order, that their teeth play directly into each other : the firft wheel contains 500 teeth—the fecond 750—the third 1500—the fourth 2000, and the fifth 3000 : how many times will the fifth wheel turn in 100 turns of the firft ?

Anfwer. 600.

THE velocity of light being at the rate of 10000000 miles per minute, takes up 6 years, 32 days, 5 hours, and 20 minutes in coming from the neareft fixed ftar to the earth : what is the diftance of that ftar ?

Anfwer. 32000000000000.

PART.

PART II.

CONTAINING THE DOCTRINE OF

VULGAR FRACTIONS.

CHAP. I.

DEFINITIONS and ILLUSTRATIONS.

A FRACTION is a broken quantity, or the parts of an unit, which are expreſſed like quantities in diviſion ; to wit, by writing two quantities, one above and the other below a ſmall line ;

thus, $\begin{cases} \frac{3}{4} \begin{array}{l} \textit{numerator} \\ \textit{denominator or diviſor} \end{array} \end{cases}$ or $\begin{cases} \dfrac{1 \times 3}{4} = \dfrac{1}{4} \times 3 \end{cases}$

which is three times the quotient of unity divided by 4 : therefore in all Vulgar Fractions, unity is divided into ſuch parts, as are expreſſed by the denominator ; that is, the denominator expreſſes what kind of parts the unit is divided into, and the numerator the number of thoſe parts.

HENCE it follows, that all Vulgar Fractions whatſoever, repreſent the quotients of quantities, which are to unity, as the numerator to the denominator ; thus, if the fraction be $\frac{3}{4}$, it will be $\frac{3}{4} : 1 :: 3 : 4$; and ſo on for others.

ALL Vulgar Fractions whatſoever, fall under the five following forms, viz. proper, improper, ſingle, compounded, and mixed.

A

A PROPER fraction, is when the numerator is lefs then the denominator: thus $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{7}{12}$, are proper factions.

AN improper fraction, is when the numerator is greater than the denominator: thus $\frac{5}{4}$, $\frac{7}{3}$, and $\frac{10}{5}$, are improper fractions.

A SINGLE fraction, is a fimple expreffion for the parts of an unit: thus $\frac{1}{2}$, $\frac{1}{3}$, and $\frac{3}{4}$, are fingle fractions.

A COMPOUND fraction, is a fraction of a fraction: thus, $\frac{1}{3}$ of $\frac{1}{2}$ and $\frac{2}{3}$ of $\frac{1}{4}$ of $\frac{5}{7}$, are compound fractions.

WHEN whole numbers are joined or connected with fractions, they are fometimes called mixed numbers; as $10\frac{1}{2}$, and $15\frac{9}{8}$.

A MIXED fraction, is when either or both the numerator and denominator, is a mixed number:

thus, $\left\{ \dfrac{12\frac{1}{2}}{17} \right.$ and $\dfrac{17\frac{11}{10}}{42\frac{12}{10}}$, are mixed fractions.

ANY whole number may be expreffed in the form of a Vulgar Fraction, by writing unity, or 1 under it: thus, $120 = \dfrac{120}{1}$ and $52 = \dfrac{52}{1}$ &c.

THE common meafure of two numbers, is any number that will meafure both without a remainder: thus, 3 is the common meafure of 9 and 12; becaufe it meafures 9 by 3, and 12 by 4.

THE greateft common meafure of two numbers, is the greateft number that will meafure both without a remainder: thus, 7 is the greateft common meafure of 21 and 49; becaufe no number greater than 7 can meafure 21 and 49, without a remainder.

ANY number that can be meafured by feveral other numbers, the number meafured, is called their common multiple: thus, 24 is a common multiple of 4 and 6, for $2 \times 12 = 24$, $4 \times 6 = 24$, and $6 \times 4 = 24$: the leaft number that can be meafured in this manner, is

called

called the leaſt common multiple: thus, 12 is the leaſt common multiple of 4 and 6 ; becauſe no number leſs than 12, can be divided by 4 and 6, without a remainder.

A PRIME number is that, which is meaſured only by unity: as 5, 7, 11, 19, &c.

NUMBERS prime to each other are ſuch, as no number except unity will meaſure both without a remainder: thus, 9 and 4 are numbers prime to each other; for although 2 will meaſure 4 without a remainder, yet it cannot divide 9 without a remainder: 3 may meaſure 9, but it cannot meaſure 4: therefore, &c.

A COMPOSED number is that, which ſome certain number meaſures: thus, 6, 8 and 12, are compoſed numbers; for $3 \times 2 = 6$, $4 \times 2 = 8$, and $2 \times 6 = 12$.

CHAP. II.

REDUCTION of VULGAR FRACTIONS.

REDUCTION of Vulgar Fractions, is the changing of one fraction into another of equivalent value; and thereby fitting them for the purpoſe of Addition, Subtraction, &c.

THE whole buſineſs of Reduction, is compriſed in the following Problems.

PROBLEM I.

To find the leaſt common multiple of ſeveral numbers.

RULE.

1. RANGE the numbers in a direct line.

2. FIND what number will divide two or more of them without a remainder; by which divide them, and

and set their quotients together with the undivided numbers, in a line beneath.

3. DIVIDE this line in the same manner as the first; and so on, from line to line, until no number, except unity will divide two of them without a remainder; then the continued product of all the divisors, and the last quotients, will be the least common multiple required.

EXAMPLES.

Find the least common multiple of 4, 8, and 12.

OPERATION.

$$4 \overline{)} \begin{array}{ccc} 4 & 8 & 12 \\ 1 & 2 & 3 \end{array}$$

WHENCE, $4 \times 1 \times 2 \times 3 = 24$, the least common multiple required.

Find the least common multiple of 2, 4, 6, 7 and 20.

OPERATION.

$$\begin{array}{c} 2 \\ 2 \end{array} \bigg| \begin{array}{ccccc} 2 & 4 & 6 & 7 & 20 \\ 1 & 2 & 3 & 7 & 10 \\ 1 & 1 & 3 & 7 & 5 \end{array}$$

WHENCE, $2 \times 2 \times 3 \times 7 \times 5 = 420$, the least common multiple required.

PROBLEM II.

To find the greatest common measure of two or more quantities.

RULE.

1. FIND the greatest common measure of any two of the quantities, by dividing the greater by the less, and the divisor by the remainder; and so on, dividing the last divisor, by the last remainder, till noth-

ing

ing remains; and the laſt diviſor made uſe of, will be the greateſt common meaſure of theſe two quantities.

2. FIND the greateſt common meaſure of any one of the other quantities, and the common meaſure laſt found; and ſo on, from one number to another, thro' the whole; and the laſt common meaſure thus found, will be the greateſt common meaſure required.

EXAMPLES.

Find the greateſt common meaſure of 12 and 15.

OPERATION.

$$12 \overline{)15}(1$$
$$\underline{12}$$

$$3\overline{)12}(4$$
$$\underline{12}$$

HENCE, 3 is the greateſt common meaſure required.

Find the greateſt common meaſure of 12, 18, 26, 36.

OPERATION.

Firſt find the greateſt common meaſure of 12 and 18.

$$\text{thus,}\left\{12\overline{)18}(1\right.$$
$$\underline{12}$$

$$6\overline{)12}(2$$
$$\underline{12}$$

HENCE, the greateſt common meaſure of 12 and 18 is 6.

Again, find the greateſt common meaſure of 6 and 26,

thus

thus, $\left\{ \begin{array}{l} 6)26(4 \\ 24 \end{array} \right.$

$2)6(3$
6

Therefore the greatest common measure is 2.

Lastly, find the greatest common measure of 2 and 36 :

thus, $\left\{ 2 \right) \dfrac{36}{18}$

Consequently the greatest common measure of 12, 18, 26, and 36, is 2 ; which was to be done.

PROBLEM III.

To abbreviate, or reduce a Vulgar Fraction to its least or most simple terms.

RULE.

FIND the greatest common measure of the numerator and denominator, by the last problem ; then divide them by their greatest common measure, and the result will be the terms of the fraction required. Or,

DIVIDE both the numerator and denominator of the given fraction, by such a number, as will divide them without a remainder, and the resulting fraction in the same manner ; and so on, till no number except unity, will divide both without a remainder ; and you will have the fraction required.

EXAMPLES.

Reduce $\dfrac{64}{384}$ to its most simple terms.

L   THE

THE greateſt common meaſure of 64 and 384, is 64. Therefore $64 \div 64 = 1$, and $384 \div 64 = 6$; conſequently $\frac{64}{384} = \frac{1}{6}$, *the fraction required.*

Or, $\frac{64 \div 8}{384 \div 8} = \frac{8}{48}$, and $\frac{8 \div 8}{48 \div 8} = \frac{1}{6}$, *the ſame as before.*

Find the value of $\frac{35}{45}$, in its moſt ſimple terms.

Thus, $\frac{35 \div 5}{45 \div 5} = \frac{7}{9}$, *the fraction required.*

Reduce $\frac{192}{480}$, to its moſt ſimple terms.   *Anſ.* $\frac{2}{5}$

PROBLEM IV.

To write a mixed number, in the form of a Vulgar Fraction.

RULE.

MULTIPLY the whole number with the denominator of the fraction, and to the product add its numerator; then under this, write the ſaid denominator; and you will have the fraction required.

EXAMPLES.

Write $4\frac{1}{2}$, in the form of a Vulgar Fraction.
Thus, $4 \times 2 = 8$, and $8 + 1 = 9$ *the numerator;*

Whence $\frac{9}{2}$ *is the fraction required.*

$12\frac{6}{10} = \frac{12 \times 10 + 6}{10} = \frac{126}{10}$; and $40\frac{20}{100} = \frac{40 \times 100 + 20}{100}$

$= \frac{4020}{100}$; Alſo, $20\frac{17}{20} = \frac{20 \times 20 + 17}{20} = \frac{417}{20}$.

PROB-

PROBLEM V.

To find the value of an improper fraction.

RULE.

DIVIDE the numerator of the given fraction by the denominator; and the quotient will be the value sought.

EXAMPLES.

Find the value of $\frac{120}{12}$.

Thus, $\frac{120}{12} = 120 \div 12 = 10$; $\frac{126}{10} = 126 \div 10 = 12\frac{6}{10}$

$\frac{4020}{100} = 4020 \div 100 = 40\frac{20}{100}$; $\frac{417}{20} = 20\frac{17}{20}$.

PROBLEM VI.

To write a whole number in the form of a Vulgar Fraction, whose denominator is given.

RULE.

MULTIPLY the whole number with the given denominator; and under this product write the said denominator; and you will have the fraction required.

EXAMPLES.

Reduce 40 to its equivalent Vulgar Fraction, whose denominator is 10.

Thus, $40 \times 10 = 400 = numerator$.

Whence, $\frac{400}{10}$ is the fraction required.

Change 304 into its equivalent Vulgar Fraction, having 5 for its denominator.

Thus,

Thus, $\dfrac{304 \times 5}{5} = \dfrac{1520}{5}$ *the fraction required.*

Change 3476 into its equvialent Vulgar Fraction, having 12 for its denominator.

Thus, $\dfrac{3476 \times 12}{12} = \dfrac{41712}{12}$ *the fraction required.*

PROBLEM VII.

To alter or change a Vulgar Fraction into another of equivalent value; whose denominator is given.

RULE.

MULTIPLY the given numerator with the proposed denominator; the product divided by the denominator of the given fraction, will give a new numerator; under which write the proposed denominator; and you will have the fraction required.

EXAMPLES.

Change $\dfrac{1}{2}$ into its equivalent Vulgar Fraction, whose denominator is 20.

Thus, $\dfrac{20 \times 1}{2} = 10$ *the new numerator.*

Therefore, $\dfrac{10}{20}$ *is the fraction required.*

Change $\dfrac{15}{20}$ into its equivalent Vulgar Fraction, having 40 for its denominator.

Thus, $\dfrac{15 \times 40}{20} = 30$: *therefore* $\dfrac{30}{40}$ *is the fraction req.*

Change $\dfrac{17}{20}$ into its equivalent Vulgar Fraction, whose denominator is 24.

Thus,

Thus, $\frac{17 \times 24}{20} = 20\frac{8}{20}$: therefore $\frac{20\frac{8}{20}}{20} = $ *fraction req.*

PROBLEM VIII.

To change a Vulgar Fraction into another of equivalent value, whose numerator is given.

RULE.

MULTIPLY the given denominator with the proposed numerator; and the product divided by the numerator of the given fraction, will give a new denominator; over which write the proposed numerator; and you will have the fraction required.

EXAMPLES.

Change $\frac{5}{10}$ into its equivalent Vulgar Fraction, whose numerator is 20.

Thus, $\frac{10 \times 20}{5} = 40$: therefore, $\frac{20}{40}$, *is the fraction req.*

Change $\frac{7}{9}$ into its equivalent Vulgar Fraction, whose numerator is 8.

Thus, $\frac{9 \times 8}{7} = 10\frac{2}{7}$: therefore, $\frac{8}{10\frac{2}{7}}$, *is the fraction req.*

Change $\frac{24}{27}$ into its equivalent Vulgar Fraction, whose numerator is 37. *Ans.* $\frac{37}{41\frac{15}{24}}$

PROBLEM IX.

To reduce a mixed fraction to simple terms.

RULE.

1. REDUCE the numerator and denominator of the given fraction to improper fractions.

2, MULTIPLY

2. MULTIPLY the numerator of the denominator, into the denominator of the numerator, for a new denominator; and multiply the numerator of the numerator, into the denominator of the denominator, for a new numerator; and you will have the terms of the fraction required.

EXAMPLES.

Reduce $\dfrac{4\frac{1}{4}}{7\frac{1}{3}}$ to simple terms.

First, $\dfrac{4\frac{1}{4}}{7\frac{1}{3}} =$ (by reducing to impr. fract.) $\dfrac{\frac{17}{4}}{\frac{22}{3}} = \dfrac{3 \times 17}{4 \times 22}$

$= \dfrac{51}{88}$ the fraction required.

Reduce $\dfrac{8\frac{1}{2}}{10}$ to simple terms.

Thus, $\dfrac{8\frac{1}{2}}{10} = \dfrac{\frac{17}{2}}{10} = \dfrac{17}{2 \times 10} = \dfrac{17}{20}$; and $\dfrac{12\frac{1}{3}}{16} = \dfrac{\frac{37}{3}}{16} = \dfrac{37}{48}$.

Also, $\dfrac{20}{30\frac{1}{2}} = \dfrac{20}{\frac{61}{2}} = \dfrac{20 \times 2}{61} = \dfrac{40}{61}$; $\dfrac{10}{20\frac{10}{20}} = \dfrac{10}{\frac{410}{20}} = \dfrac{10 \times 20}{410}$

$= \dfrac{200}{410}$; $\dfrac{300}{640\frac{1}{4}} = \dfrac{300}{\frac{2561}{4}} = \dfrac{1200}{2561}$.

PROBLEM X.

To reduce a compound fraction to a simple one of equal value.

RULE.

1. REDUCE all such parts of the given fraction as are whole numbers, mixed numbers, and mixed fractions; according to the foregoing rules; that is, whole and mixed numbers must be reduced to improper fractions, and mixed fractions to simple terms.

2. MUL-

2. MULTIPLY all the numerators continually to-
gether, for a new numerator, and all the denomina-
tors continually together, for a new denominator ;
and the former product written above the latter, will
give the fraction required.

Note. *Any number that is found among the nume-*
rators and denominators, may be struck out of both.

EXAMPLES.

Reduce $\frac{2}{3}$ of $\frac{3}{4}$ of $\frac{5}{6}$, to a simple fraction.

Thus, $\frac{2\times3\times5}{3\times4\times6}=$ (by striking out the 3) $\frac{2\times5}{4\times6}=\frac{10}{24}$ *the*
fraction required.

Reduce $\frac{3}{4}$ of $\frac{7}{9\frac{1}{2}}$, to a simple fraction.

First, $\frac{7}{9\frac{1}{2}}=\frac{14}{19}$; then $3\times14=42$ *the new numerator, and*

$4\times19=76$ *the new denominator* : therefore $\frac{42}{76}$ *is the*
fraction required.

$\frac{1}{2}$ of $\frac{4}{6}$ of $\frac{2}{4\frac{1}{2}}$ of $8=\frac{1}{2}$ of $\frac{4}{6}$ of $\frac{4}{9}$ of $\frac{8}{1}=\frac{128}{108}$.

$\frac{4}{3}$ of $\frac{2}{5}$ of $\frac{1\frac{7}{4}}{2\frac{2}{3}}=\frac{4}{3}$ of $\frac{2}{5}$ of $\frac{51}{88}=\frac{4\times2\times51}{3\times5\times88}=\frac{408}{1320}$.

PROBLEM XI.

To reduce several fractions of different denominators,
to equivalent fractions, having a common denominator.

RULE.

1. REDUCE all fractions to simple terms.

2. MUL-

2. MULTIPLY each numerator into all the denominators except its own, for new numerators.

3. MULTIPLY all the denominators continually together, for a new and common denominator, and this written under the several new numerators, will give the fractions required.

EXAMPLES.

Reduce $\frac{1}{2}$, $\frac{3}{4}$, and $\frac{5}{6}$, to their equivalent fractions, having a common denominator.

First, $\begin{cases} 1 \times 4 \times 6 = 24 \text{ the new numerator for } \frac{1}{2} \\ 2 \times 3 \times 6 = 36 \text{ the new numerator for } \frac{3}{4} \\ 5 \times 2 \times 4 = 40 \text{ the new numerator for } \frac{5}{6} \end{cases}$

Then $2 \times 4 \times 6 = 48$ *the new and common denominator.*

Hence $\frac{24}{48}$, $\frac{36}{48}$, and $\frac{40}{48}$, *are the fractions required.*

$\frac{4}{7}$, $\frac{3}{4}$, and $\frac{1}{9}$, *reduced to a common denominator* $= \frac{4 \times 4 \times 9}{7 \times 4 \times 9}$, $\frac{3 \times 7 \times 9}{7 \times 4 \times 9}$, and $\frac{1 \times 7 \times 4}{7 \times 4 \times 9} = \frac{144}{252}$, $\frac{189}{252}$, and $\frac{28}{252}$.

$\frac{1}{3}$ and $\frac{4}{3}$ of $\frac{2}{5}$ of $\frac{4\frac{1}{4}}{7\frac{1}{3}}$, *reduced to a common denominator* $=$ $\frac{1 \times 1320}{3 \times 1320}$, and $\frac{3 \times 408}{3 \times 1320} = \frac{1320}{3960}$, and $\frac{1224}{3960}$.

$2\frac{2}{7}$, $\frac{2\frac{1}{4}}{4}$, and $\frac{1}{3}$ of 4, *reduced to a common denominator* $=$ $\frac{720}{336}$, $\frac{189}{336}$, and $\frac{448}{336}$.

PROB-

PROBLEM XII.

To reduce several fractions of different denominators, to others of equivalent value, having the least possible common denominator,

RULE.

1. REDUCE all the fractions to simple terms.

2. FIND the least common multiple of all the denominators; and you will have the least common denominator required.

3. DIVIDE the denominator thus found by the denominator of each fraction, and multiply the quotient with its numerator, and you will have new numerators, under which write the common denominator; and you will have the fractions required.

EXAMPLES.

Reduce $\frac{1}{8}$, $\frac{3}{4}$, and $\frac{1}{2}$ to equivalent fractions, that shall have the least possible common denominator.

First, the least common multiple of 8, 4, and 2, is 8:

Then, $\overline{8 \div 8} \times 1 = 1$, *the new numerator for* $\frac{1}{8}$

And, $\overline{8 \div 4} \times 3 = 6$, *the new numerator for* $\frac{3}{4}$

Also, $\overline{8 \div 2} \times 1 = 4$, *the new numerator for* $\frac{1}{2}$.

Hence the fractions required are $\frac{1}{8}$, $\frac{6}{8}$, and $\frac{4}{8}$.

Reduce $\frac{1}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, and $\frac{5}{6}$, to equivalent fractions, having the least possible common denominator.

First, the least common multiple of the 3, 4, 5, and 6, is 60.

Then, $\overline{60 \div 3} \times 1 = 20$, *the new numerator for* $\frac{1}{3}$

And, $\overline{60 \div 4} \times 3 = 45$, *the new numerator for* $\frac{3}{4}$

M   Also,

Also, $\overline{60 \div 5} \times 4 = 48$ *the new numerator for* $\frac{4}{5}$

Lastly, $\overline{60 \div 6} \times 5 = 50$ *the new numerator for* $\frac{5}{6}$.

Hence, $\frac{20}{60}$, $\frac{45}{60}$, $\frac{48}{60}$, and $\frac{50}{60}$, *are the fractions req.*

PROBLEM XIII.

To change the fraction of one denomination to the fraction of a greater one, retaining its same value.

RULE.

CHANGE the given fraction into a compound one, by writing its value in all the intermediate denominations up to the one wherein the value of the fraction is to be expressed; and the value of this compound fraction, will be the fraction required.

EXAMPLES.

Change $\frac{1}{3}$ of a nail, to the fraction of an ell Eng.

First, $\frac{1}{3}$ *of a nail* $= \frac{1}{3}$ *of a quarter, and* $\frac{1}{4} \frac{1}{5}$ *of an ell*:

Therefore, $\frac{1}{3}$ *of a nail* $= \frac{1}{3}$ *of* $\frac{1}{4}$ *of* $\frac{1}{5} = \frac{1}{60}$ *the fraction req.*

2 pennyweights, reduced to the fraction of a pound $= \frac{2}{20}$ of $\frac{1}{12} = \frac{2}{240}$. 3 grains, reduced to fraction of an ounce $= \frac{3}{24}$ of $\frac{1}{20} = \frac{3}{480}$. $\frac{1}{3}$ of a cent, reduced to the fraction of a milree of Portugal $= \frac{1}{3}$ of $\frac{1}{124} = \frac{1}{372}$. 10 cents, reduced to the fraction of a pound sterling of Ireland $= \frac{10}{410} = \frac{1}{41}$. $\frac{7}{8}$ of a cent, reduced to the frac-

tion-

tion of a dollar$=\frac{7}{8}$ of $\frac{1}{100}=\frac{7}{800}$.   1 drachm Avoir-

dupois$=\frac{1}{16}$ of $\frac{1}{16}$ of $\frac{1}{112}$ of $\frac{1}{20}$ of a tun.

PROBLEM XIV.

To change the fraction of one denomination to the frac-
tion of a lefs one, retaining its fame value.

RULE.

MULTIPLY the numerator of the given fraction
into all the intermediate denominations down to the
one wherein the value of the given fraction is to be
expreffed, and under this product, write the given de-
nominator, and you will have the fraction required.

EXAMPLES.

Reduce $\frac{1}{70}$ of an ell Eng. to the fraction of a nail.

Thus, $1 \times 5 \times 4 = 20$ *the numerator*

Therefore, $\frac{20}{70} = \frac{2}{7}$ *is the fraction required.*

Reduce $\frac{3}{1120}$ of a ℔ Troy to the fract. of a grain.

Thus, $\frac{3 \times 12 \times 20 \times 24}{1120} = \frac{17280}{1120}$ *is the fraction required.*

$\frac{2}{1240}$ of a pound Troy, reduced to the fraction of a

pennyweight$=\frac{2 \times 12 \times 20}{1240} = \frac{480}{1240}$; $\frac{8}{17920}$ of an hun-

dred weight, reduced to the fraction of an ounce$=$

8×112-

$$\frac{8\times112\times16}{17920}=\frac{14336}{17920}.$$ $\frac{1}{372}$ of a milree of Portugal,

reduced to the fraction of a cent $=\frac{1\times124}{372}=\frac{124}{372}=\frac{1}{3}.$

PROBLEM XV.

To find the value of a Vulgar Fraction in known parts of the integer.

RULE.

MULTIPLY the numerator of the given fraction with the parts in the next inferiour denomination, and divide the product by the denominator; then if there be any remainder, multiply it with the parts in the next inferiour denomination, and divide by the former divisor, and so on, and the several quotients resulting will exhibit the value sought.

EXAMPLES.

Find the value of $\frac{5}{24}$ of an ounce Troy.

OPERATION.

$$
\begin{array}{r}
5 \\
20 \\
\hline
\end{array}
$$

$$24\overline{)100}(4\ pennyweights.$$
$$\underline{96}$$

$$
\begin{array}{r}
4 \\
24 \\
\hline
\end{array}
$$

$$24\overline{)96}(4\ grains.$$
$$\underline{96}$$

Therefore,

Therefore, $\frac{5}{24}$ of an ounce=4 *dwt.* 4 *gr. the value sought.*

Find the value of $\frac{5}{7}$ of an ounce Troy.

OPERATION.

$$
\begin{array}{c}
5 \\
20 \\
\hline
7)\,100(\\
\quad 14 \quad 2\,rem. \\
\overline{\quad\;\; 24} \\
\hline
7)\,48(\\
\quad 6 \quad 6\,rem. \\
\hline
\end{array}
$$

Therefore 14 *dwt.* 6$\frac{6}{7}$ *gr. is the value sought.*

Find the value of $\frac{6}{7}$ of an hundred weight.

OPERATION.

$$
\begin{array}{c}
6 \\
4 \\
\hline
7)\,24(\\
\quad 3 \quad 3\,rem. \\
\overline{\quad\; 28} \\
\hline
7)\,84(\\
\quad 12 \\
\hline
\end{array}
$$

Therefore 3 *qr.* 12 *lb. is the value sought.*

Find the value of $\frac{1}{41}$ of a pound sterl. of Ireland:

Thus,

Thus, $\frac{1 \times 410}{41} = 10$ *cts. the value sought.*

Find the value of $\frac{2}{97}$ of a pagoda of India.

Thus, $\frac{2 \times 194}{97} = 4$ *cts. the value sought.*

PROBLEM XVI.

To reduce the known parts of an integer to their equivalent Vulgar Fraction.

RULE.

1. REDUCE the given parts to the least denomination mentioned.

2. REDUCE the integer to the same denomination; and the latter written beneath the former, will be the fraction required.

EXAMPLES.

Reduce 3 *dwt.* 7 *gr.* to the fraction of a pound.

OPERATION.

dwt.	gr.		oz.
3	7		12
24			20
79			240
			24
			960
			480
			5760

Therefore, $\frac{79}{5760}$ is the fraction required.

Reduce

Reduce 10 *cts.* to the fraction of a pound sterling of Ireland.

Thus, $\frac{10}{410}$ *is the fraction required.*

$10\frac{8}{10}$ *in.* reduced to the fraction of a foot $=\frac{10\frac{8}{10}}{12}=\frac{9}{10}$.

$5\frac{1}{3}$ *p.* reduced to the fract. of an acre $=\frac{5\frac{1}{3}}{160}=\frac{16}{480}=\frac{1}{30}$.

CHAP. III.

ADDITION, SUBTRACTION, MULTIPLICATION, AND DIVISION OF VULGAR FRACTIONS.

SECT. I.

Of ADDITION of VULGAR FRACTIONS.

RULE.

1. REDUCE all the fractions to a common denominator, by the rule to problem XI of the last chapter: those of different denominations to the same, by the rules to problem XIII or XIV.

2. ADD all the numerators together for a new numerator, under which write the common denominator; and you will have a fraction equal to the sum required.

EXAMPLES.

Find the sum of $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}$.

Thus,

Thus, $\frac{1}{2}+\frac{1}{3}+\frac{1}{4}=$ (by reducing to a common denominator) $\frac{12}{24}+\frac{8}{24}+\frac{6}{24}=\frac{12+8+6}{24}=\frac{26}{24}$ *sum required.*

Required the sum of $2\frac{1}{7}+\frac{2\frac{1}{4}}{4}+\frac{1}{3}$ of 4.

Thus, $2\frac{1}{7}+\frac{2\frac{1}{4}}{4}+\frac{1}{3}$ of $4=\frac{15}{7}+\frac{9}{16}+\frac{4}{3}=$ (by reduction) $\frac{720}{336}+\frac{189}{336}+\frac{448}{336}=\frac{720+189+448}{336}=\frac{1357}{336}$ *sum req.*

Find the sum of $\frac{3}{4}$ of a grain $+\frac{5}{7}$ of an ounce.

First, $\frac{3}{4}$ *of a grain* $=\frac{3}{4}$ *of* $\frac{1}{24}$ *of* $\frac{1}{20}=\frac{3}{1920}$ *of an ounce;* then the fum becomes $\frac{3}{1920}+\frac{5}{7}=\frac{9621}{13440}$ *the fum req.*

SECT. II.

Of SUBTRACTION of VULGAR FRACTIONS.

RULE.

1. PREPARE the fractions as in Addition.
2. SUBTRACT the numerator of one fraction from the numerator of the other, and the refult placed above the common denominator will be the difference required.

EXAMPLES.

From $\frac{1}{3}$ take $\frac{1}{4}$.

Thus, $\frac{1}{3}-\frac{1}{4}=\frac{4}{12}-\frac{3}{12}=\frac{4-3}{12}=\frac{1}{12}$ *the difference req.*

From

From $\frac{4}{5}$ take $\frac{1}{3}$ of $\frac{1}{3}$.

Thus, $\frac{4}{5}-\frac{1}{3}$ of $\frac{1}{3}=\frac{4}{5}-\frac{1}{9}=\frac{36}{45}-\frac{5}{45}=\frac{36-5}{45}=\frac{31}{45}$ the difference required.

$3\frac{1}{3}-\frac{3}{4}$ of $\frac{1}{9}$ of $\frac{2}{3}=\frac{10}{3}-\frac{6}{108}=\frac{1080}{324}-\frac{18}{324}=\frac{1080-18}{324}$
$=\frac{1062}{324}$; $1\frac{1}{4}-\frac{\frac{17}{4}}{\frac{22}{3}}=\frac{5}{4}-\frac{51}{88}=\frac{440}{352}-\frac{204}{352}=\frac{236}{352}$.

From $\frac{5}{7}$ of an ounce take $\frac{3}{4}$ of a grain Troy.

First, $\frac{3}{4}$ of a grain $=\frac{3}{1920}$ of and ounce : Therefore,
$\frac{5}{7}-\frac{3}{1920}=\frac{9589}{13440}$ is the difference required.

SECT. III.

OfMULTIPLICATIONofVULGAR FRACTIONS.

RULE.

1. REDUCE all whole and mixed numbers to improper fractions, mixed fractions to simple terms, and fractions of different denominations to the same.

2. MULTIPLY all the numerators together for a new numerator, and all the denominators together for a new denominator; and you will have the terms of the fraction required.

EXAMPLES.

Required the product of $\frac{1}{3}\times\frac{3}{4}$.

Thus, $\frac{1\times3}{3\times4}=\frac{3}{12}$ the product required.

N

$6\frac{1}{3}\times$

$$6\tfrac{1}{3} \times \tfrac{1}{4} \text{ of } \tfrac{2}{9} = \text{(by reduction)} \ \tfrac{19}{3} \times \tfrac{2}{36} = \tfrac{19 \times 2}{3 \times 36} = \tfrac{38}{108};$$

$$\tfrac{4}{3\tfrac{1}{3}} \times \tfrac{3\tfrac{1}{3}}{4} = \text{(by reduction)} \ \tfrac{12}{10} \times \tfrac{10}{12} = \tfrac{12 \times 10}{10 \times 12} = \tfrac{120}{120} = 1;$$

$$\tfrac{1}{3}\text{lb} \times \tfrac{1}{4}dr. = \tfrac{1}{3} \times \tfrac{1}{4 \times 16 \times 16} = \tfrac{1 \times 1}{3 \times 4 \times 16 \times 16} = \tfrac{1}{3072}\text{lb}$$

SECT. IV.

Of DIVISION of VULGAR FRACTIONS.

RULE.

PREPARE the numbers as in Addition, then multiply the numerator of the divisor into the denominator of the dividend, and the numerator of the dividend into the denominator of the divisor; then the latter written above the former, will give the quotient required.   Or,

INVERT the divisor, that is, write the denominator in the place of the numerator, and the numerator in the place of the denominator; then proceed as in Multiplication, and the result will give the quotient required.

EXAMPLES.

Required the quotient of $\tfrac{1}{2} \div \tfrac{1}{4}$.

Thus, $1 \times 4 = 4$ *the numerator*; and $1 \times 2 = 2$ *the denom.*

Therefore, $\tfrac{4}{2} = 2$ *is the quotient required.*

Or, $\tfrac{4}{1} \times \tfrac{1}{2} = \tfrac{4}{2}$ *the same as before.*

$$\tfrac{2}{3} \div \tfrac{8}{27} = \tfrac{27}{8} \times \tfrac{2}{3} = \tfrac{27 \times 2}{8 \times 3} = \tfrac{54}{24}; \quad \tfrac{4}{7} \div \tfrac{4}{28} = \tfrac{28}{4} \times \tfrac{4}{7} = \tfrac{}{28 \times 4}$$

$$\frac{28\times4}{4\times7}=\frac{112}{28}=4; \quad \frac{1\frac{1}{3}}{8}\div\frac{3\frac{1}{2}}{1}=\text{(by reduction)}\frac{4}{24}\div\frac{7}{2}=$$

$$\frac{4\times2}{7\times24}=\frac{8}{168}; \quad \frac{1}{4}\text{ of }\frac{1}{2}\div\frac{1}{3}\text{ of }1=\frac{1}{8}\div\frac{1}{3}=\frac{3\times1}{1\times8}=\frac{3}{8}.$$

Miscellaneous Questions.

A MAN at hazard won the firſt throw $2\frac{1}{2}$ dollars—the ſecond throw he won as much as he then had in his pocket—the third throw he won 4 dollars, and the fourth throw he won double of all that he then had, at which time he found that he had in all 45 dollars. How many had he at firſt.

Anſwer. **3 dollars.**

THERE is a certain club, whereof $\frac{1}{4}$ are merchants, $\frac{1}{3}$ mathematicians, $\frac{1}{5}$ mechanics, and 13 phyſicians. How many were there in the whole?

Anſwer. **60.**

REQUIRED the difference between three times thirty-three and a third; and three times three and thirty and a third.

Anſwer. **$60\frac{2}{3}$.**

A MAN who was driving ſome ſheep to market, was met by another who demanded the number of ſheep in his drove: the drover to evade a direct anſwer replies, that if I had as many more, and half as many more, and $12\frac{1}{2}$ ſheep, I ſhould have 100. What number had he?

Anſwer. **35.**

PART

PART III.

CONTAINING THE DOCTRINE OF

DECIMAL FRACTIONS.

CHAP. I.

DEFINITIONS and ILLUSTRATIONS.

A DECIMAL Fraction is formed from a proper Vulgar Fraction, by dividing the numerator with cyphers annexed to it, by the denominator; that is, the equivalent Decimal of any Vulgar Fraction is found by multiplying the numerator with 10, 100, or 1000, &c. till it be so increased, that it may be exactly measured by its denominator; and this quotient will be the decimal required:

Thus, $\frac{1}{4} \times 100 = \frac{1 \times 100}{4} = \frac{100}{4} = 25$; and $\frac{1}{2} \times 10 = \frac{1 \times 10}{2} = \frac{10}{2} = 5$; Also, $\frac{3}{4} \times 100 = \frac{3 \times 100}{4} = \frac{300}{4} = 75$:

which quotients are expressed by writing them with a point on the left-hand: Thus, $\frac{1}{4} = .25$, $\frac{1}{2} = .5$, and $\frac{3}{4} = .75$; which are respectively equal to $\frac{25}{100}$, $\frac{5}{10}$, and $\frac{75}{100}$; but these denominators are always omitted, and the numerators written as above, where the point distinguishes them from whole numbers: Thus, $2.3 = 2\frac{3}{10}$, $4.25 = 4\frac{25}{100}$, &c.

HENCE

HENCE it appears that every Decimal Fraction, is equal to a Vulgar one, whose numerator is the decimal, and the denominator unity, with as many cyphers annexed to it as there are places of figures in the numerator: Thus, .1, .44, and .127, are respectively equal to $\frac{1}{10}$, $\frac{44}{100}$, and $\frac{127}{1000}$.

THEREFORE it follows, that in decimals, unity is divided in 10, 100, or 1000, &c. equal parts; and the given decimal represents the number of those parts: Thus, .1 $=\frac{1}{10}$ represents one tenth part of an unit, .44 represents forty-four hundred parts of an unit, &c. Therefore, in decimals, cyphers annexed neither increase nor diminish their value; but cyphers prefixed, diminish their value in a ten fold proportion: Thus, .440$=\frac{440}{1000}=$(by the nature of Division) $\frac{44}{100}$ $=.44$; but .04$=\frac{4}{100}=\frac{1}{10}$ of ($\frac{4}{10}$) .4, and so on for any other decimal.

WHENCE it follows, that the farther any diget or numeral figure stands from the units' place, or decimal point towards the right-hand, the less will be its value, to wit, in a tenfold proportion. Thus in the decimal .1111, the figure next to the decimal point is $\frac{1}{10}$, the second is $\frac{1}{100}$, the third $\frac{1}{1000}$, and the fourth $\frac{1}{10000}$, that is, .1111$=\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\frac{1}{10000}$, which is plainly a series of numbers in geometrical proportion, decreasing by the common divisor 10. Again .0123$=\frac{1}{100}+\frac{2}{1000}+\frac{3}{10000}$; and the like to be understood of all others.

HENCE, the notation of decimals, or the valuation of the several places from unity downwards, is the same among themselves as that of integers or whole numbers; therefore every figure is to be valued according to the distance it stands from unity downwards.

CHAP. II.

ADDITION, SUBTRACTION, MUL-TIPLICATION, and *DIVISION* of *DECIMAL FRACTIONS.*

SECT. I.

Of ADDITION of DECIMALS.

R U L E.

1. WRITE the given decimals in such order, that those places of equal distance from unity or the decimal point, may stand directly under each other.

2. Find their sum as in whole numbers, then distinguish with a point as many places of figures on the right-hand, as are equal to the greatest number found in any given decimal; and you will have the sum required.

EXAMPLES.

Find the sum of .176+.1264+.34+.994

These numbers being placed according to the rule will stand

thus, $\begin{cases} .176 \\ .1264 \\ .34 \\ .994 \end{cases}$

1.6364=*sum required.*

Find the sum of 34.123+6437.27+347.2+1.347634+347634.1.

thus,

thus,
$$\left\{\begin{array}{r} 34.123 \\ 6437.27 \\ 347.2 \\ 1.347634 \\ 347634.1 \end{array}\right.$$

$354454.040634 =$ *sum required.*

Required the sum of $25.124 + 12.247 + 24.3485 + 352.1 + 4578.74$

thus,
$$\left\{\begin{array}{r} 25.124 \\ 12.247 \\ 24.3485 \\ 352.1 \\ 4578.74 \end{array}\right.$$

$4992.5595 =$ *sum required.*

$17.45 + .42 + 345.284 + 34 + 4232.425 = 4629.579$

S E C T. II.

Of SUBTRACTION of DECIMALS.

R U L E.

WRITE down the numbers as in Addition, then subtract the less from the greater as in whole numbers, remembering to point off in the remainder as in Addition; and you will have the difference sought.

EXAMPLES.

Required the difference between 12.19, and 8.9

thus,
$$\left\{\begin{array}{r} 12.19 \\ 8.9 \end{array}\right.$$

$3.29 =$ *difference required.*

Required

Required the difference between 342.364, and 299.2437

$$\text{thus,}\ \begin{cases} 342.364 \\ 299.2437 \end{cases}$$

43.1203 = *difference required.*

2473.0024 — 1999.99998 = 473.00242 ;
2479.3777 — 930.000045 = 1549.377655 ;
9999.8888 — 8888.9999 = 1110.8889

SECT. III.

Of MULTIPLICATION of DECIMALS.

RULE.

WRITE the numbers and multiply them as in common Multiplication ; then diftinguifh with a point as many places of decimals in the product, as are equal to the number in both factors ; and you will have the product required.

Note. *If the number of places in the product, are lefs than the number of decimal places in both factors, you muft fupply the deficiency by prefixing cyphers.*

THAT the number of decimal places in the product, ought to be equal to the number in both factors, may be thus demonftrated.

SUPPOSE .34 were to be multiplied with .27 ; the product of thefe two numbers by common Multiplication is 918 ; but $.34 = \frac{34}{100}$ and $.27 = \frac{27}{100}$; therefore, $.34 \times .27 = \frac{34}{100} \times \frac{27}{100} = \frac{918}{10000} =$ (by the nature of decimal notation) .0918, confifting of as many places of figures as there were in both factors ; and the fame will hold true in any others. *Q. E. D.*

EXAM-

EXAMPLES.

Required the product of 2.438 × .005.

OPERATION.

```
    2.438
     .005
```

.012190 = *product required.*

Required the product of 34.38 × 24.7

OPERATION.

```
    34.38
    24.7
```

```
    24066
    13752
    6876
```

849.186 = *product required.*

Required the product of 384.02 × .01

thus, $\begin{cases} 384.02 \\ \quad .01 \end{cases}$

3.8402 = *product required.*

Required the product of 2.7122 × 3.2121

thus, $\begin{cases} 3.2121 \\ 2.7122 \end{cases}$

```
    64242
    64242
    32121
   224847
   64242
```

8.71185762 = *product required.*

O

In the multiplication of decimals, where the factors confist of a great number of decimal places, the operation becomes very prolix, and befides, a great part of it is entirely ufelefs, fince that four or five places of decimals in the product, is fufficient for common purpofes. Therefore to abridge the work by obtaining the product true to any defigned number of places of decimals, you muft obferve the following

R U L E.

1. WRITE the multiplier inverted, fo that the units' place may ftand under that figure of the multiplicand, to whofe place the product is to be found true.

2. In multiplying with the feveral figures of the multiplier, you muft reject all the figures of the multiplicand, that are to the right-hand of the figure you are multiplying with ; placing the firft figure of the feveral products directly under each other, increafed by adding 1 from 5 to 15, 2 from 15 to 25, &c. of the product of the multiplying figure with the proceeding figure of the multiplicand, when you begin to multiply ; and the fum of all the products will be the product required.

EXAMPLES.

Required the product of 3.2121 × 2.712, to three places of decimals.

3.2121

3.2121
2172

6424 = *product of* 3.212 × 2 .
2248 = *product of* 3.21 × 7, *increased by adding* 1 *for*
  32 = *product of* 3.2 × 1   [*the prod. of* 7 × 2
    6 = *product of* 3 × 2

8.710 = *product required.*

Required the product of 3.24211 × 2.34634, to four places of decimals.

  3.24211
436432

  64842 = 3.2421 × 2
   9726 = 3.242 × 3
   1297 = 3.24 × 4, *increased by adding* 1 *for* 4 × 2
    194 = 3.2 × 6, *increased by adding* 2 *for* 6 × 4
     10 = 3 × 3, *increased by adding* 1 *for* 3 × 2

  7.6069 = *product required.*

Required the product of 2.13214 × 2.21134, to five places of decimals.

2.13214
431122

426428 = 2.13214 × 2
 42643 = 2.1321 × 2, *increased by adding* 1 *for* 2 × 4
  2132 = 2.132 × 1
   213 = 2.13 × 1
    64 = 2.1 × 3, *increased by adding* 1 *for* 3 × 3
     2 = 2 × 4

4.71488 = *product required.*

<div align="right">Required</div>

Required the product of 27.17 × 19.14, in integers only.

27.17
4191
―――
272 = 27.1 × 1, *increased by adding* 1 *for* 1 × 7
244 = 27 × 9, *increased by adding* 1 *for* 9 × 1
  3 = 2 × 1, *increased by adding* 1 *for* 1 × 7
  1 = 4 × 0, *increased by adding* 1 *for* 4 × 2
―――
  520 = *product required.*

SECT. IV.

Of DIVISION of DECIMALS.

In division of decimals, it may at first appear difficult to determine the number of decimal places the quotient must consist of ; but this difficulty will vanish, when we consider that the quotient must be such a number that when multiplied with the divisor will produce the dividend; therefore it follows, that the number of decimal places in the divisor and quotient taken together, must be equal to the number in the dividend, by the nature of Multiplication ; consequently the difference between those in the divisor and dividend, must be equal to the number in the quotient; which affords the following

RULE.

RANGE the numbers and divide them as in common Division, then point off as many places of decimals in the quotient, as are equal to the difference between those in the divisor and dividend ; and you will have the quotient required.

Note

Note 1. *If there are not so many places of figures in the quotient, as are equal to the difference between those in the divisor and dividend, you must supply the defect by prefixing cyphers.*

2. *If the places of figures in the dividend, are less in number than those in the divisor, you must annex cyphers to the dividend.*

EXAMPLES.

Required the quotient of 849.186 divided by 24.7

OPERATION.

```
24.7)849.186(34.38=quotient required.
     741
     ————
     1081
      988
      ————
       938
       741
       ————
       1976
       1976
       ————
          0
```

Note. If the divisor be 10, *or* 100, &c. *the quotient may be found by removing the decimal point in the dividend, as many places towards the left-hand as there are cyphers in the divisor: thus, the quotient of* 1000)2737.45 *is* 2.73745 *and* .0234÷100=.000234.

Required the quotient of .012190÷2.438

OPERATION.

$$2.438 \overline{)\ .012190\ (5}$$
$$12190$$
$$\overline{\qquad}$$
$$0$$

HERE, the quotient found by division is 5 ; but the difference between the decimal places in the divisor and dividend are three ; therefore .005 is the quotient required.

Required the quotient of $2 \div 42$.

OPERATION.

$$42 \overline{)\ 200000\ (.04761\ \&c. =}\ quotient\ required.$$
$$168$$
$$\overline{\qquad}$$
$$320$$
$$294$$
$$\overline{\qquad}$$
$$260$$
$$252$$
$$\overline{\qquad}$$
$$80$$
$$42$$
$$\overline{\qquad}$$
$$38\ \&c.$$

Required the quotient of $165.6995001296 \div 52.7438$

OPER-

OPERATION.

52.7438) 165.6995001296 ($3.141592 = quotient\ req.$
    1582314

$\overline{}$

    746810
    527438

$\overline{}$

    2193720
    2109752

$\overline{}$

     839681
     527438

$\overline{}$

    3122432
    2637190

$\overline{}$

     4852429
     4746942

$\overline{}$

      1054876
      1054876

$\overline{}$
         0

HERE, as in Multiplication, the work may be greatly contracted, by finding the quotient true to any determinate number of decimal places: The method is as follows.

R U L E.

1. RANGE the numbers as in common Division.
2. TAKE the figures of the given divisor, to as many places of decimals as you intend the quotient shall confist of, for your firft divisor, and find a quotient figure by comparing this divisor as in common
                                    Division ;

Division ; then subtract its product with the divisor, from the dividend as usual, calling the remainder a new dividend.

3. REJECT the right hand figure of your former divisor, and call the result a new divisor ; then find a quotient figure by comparing the new divisor and dividend together, and place it in the former quotient, subtracting as before ; and so on, making each remainder a new dividend, and rejecting the right-hand figure of the last divisor for a new one ; also remembering to add for the figures rejected as in Multiplication.

Note 1. *If there are not so many places of decimals in the divisor, as you intend there shall be in the quotient, supply the defect by annexing cyphers.*

2. *You may determine how many places of whole numbers there will be in the quotient, by consider- ing that the first figure of the quotient, is always of the same denomination of that figure of the divi- dend, which stands directly over the units' place of the product of the first quotient figure and divisor.*

EXAMPLES.

Required the quotient of 10.1934÷4.2, to three places of decimals.

4.200)10.1934(2.427 = *quotient required.*
       8400
       ────

   420)1793
       1680
       ────

    42)113
        84
       ────

     4)29
       28
       ────
        1

Required the quotient of 165.6995001296 ÷ 52.7438, to five places of decimals.

52.74380)165.6995001296(3.141592 = *quotient req.*
         15823140

52.7438)746810
        527438
        ──────

52.743)219372
       210975 = 52.743 × 4, *encreased by add-*
                        [*ing 3 for 4 × 8*

52.74)8397
      5274
      ────

52.7)3123
     2637 = 527 × 5, *encreased by adding*
                    [*2 for 5 × 4*

52)486
   473 = 52 × 9, *increased by adding 5*
                [*for 9 × 7*

5)13
  10
  ──
   3   P   Required

Required the quotient of 780.516÷24.3, in integers only.

OPERATION.

24)780.516(32=*quotient required.*
/73=24×3, *increased by adding* 1 *for* 3×3

2)5
  5=2×2, *increased by adding* 1 *for* 2×4

0

C H A P.   III.

Of REDUCTION of DECIMALS.

PROBLEM I.

To reduce a Vulgar Fraction to its equivalent decimal.

RULE.

ANNEX cyphers to the numerator, and divide by the denominator till nothing remains, and the quotient will be the decimal required.

EXAMPLES.

Reduce $\frac{3}{20}$ to its equivalent decimal.

Thus, 20)3.00(.15=*the decimal required.*
20

100
100

0   Reduce

Reduce $\dfrac{18}{20}$ to its equivalent decimal.

Thus, $20\overline{)18.0}$(.9=*the decimal required*.
    180
    ——
    0

Reduce $\dfrac{6}{15}$ to its equivalent decimal.

Thus, $15\overline{)6.0}$(.4=*the decimal required*.
    60
    ——
    0

Required the equivalent decimal of $\dfrac{8}{9}$.

OPERATION.

$9\overline{)8.00000}$(.8888 *&c. ad infinitum.*
    72
    ——
    80
    72
    ——
    80
    72
    ——
    8 *&c.*

HERE, we have what is called a circulating de-
cimal for the quotient, that is, a continual repetition
of the same figure without any possibility of ever
coming to an end, as is evident from the example.
Therefore it follows, that the equivalent decimal of
$\frac{8}{9}$ can never be found in finite terms; but may be
obtained to any degree of exactness you please.

Note.

Note. *When a vulgar fraction is annexed to any number of cents, reduce the fraction to its equivalent decimal, and annex it to the cents, and the whole will become a decimal: Thus,* $37\frac{3}{4}$ *cents* $= .3775$

PROBLEM II.

To reduce numbers of different denominations to their equivalent decimal.

RULE.

REDUCE the given numbers to their equivalent vulgar fraction, by problem xvi of vulgar fractions, then proceed as in the last problem.

EXAMPLES.

REDUCE 3 qr. 2 na. to their equivalent decimal of a yard.

First, 3 qr. 2 na. $= \frac{14}{16}$ of a yard ;

Then $16) 14.000 (.875 = the decimal required.$
$ 128$

$ \overline{}$

$ 120$
$ 112$

$ \overline{}$

$ 80$
$ 80$

$ \overline{}$

$ 0$

$4 h. 30' 10''$, reduced to the decimal of a day $=$ .187615 &c.

8 S. reduced to the decimal of the ecliptic $= .666$ &c. ad infinitum.

$10\frac{8}{10}$ in. reduced to the decimal of a foot $= .9$

$5\frac{1}{2}p$. reduced to the decimal of an acre $=.0333$ *&c.*
ad infinitum.

PROBLEM III.

To find the value of a decimal in known parts of the
integer.

RULE.

1. MULTIPLY the given decimal with the parts
in the next inferiour denomination, and point off as
in common multiplication of decimals; and the
whole numbers will be the value of the given de-
cimal in that denomination.

2. MULTIPLY the remaining decimal with the
parts in the next inferiour denomination, and point
off as before, and fo on, thro all the inferiour deno-
mination, if need be ; and you will have the value
fought.

EXAMPLES.

Find the value of .875 of a yard.

OPERATION.

```
   .875
      4
   ----
  3.500
      4
   ----
```
$qr. na.$
: 2.000 *therefore*, .875 = 3 2, *the value fought.*

Find the value of .426 of a pound troy.

OPER-

OPERATION.

```
      .426
       12
    ───────
     5.112
       20
    ───────
     2.240
       24
    ───────
      960
      480
    ───────
```
                                    lb oz. dwt. gr.
     5.760 *therefore* .426 = 5   2   5.76

Find the value of .75 of a pound sterling of Great-Britain.

OPERATION.

```
       .75
       444
    ───────
       300
       300
       300
    ───────
```
                              £.   cts.   dol. cts.
     333.00 *therefore* .75 = 333 = 3   33.

Find the value of .37752 of a pound sterling of Great-Britain.

                          cts.   dol.   cts.
Thus .37752 × 444 = 167.61888 = 1 6761888.

Note. *There never can be more than two places of*
cents, and where there are other figures annexed,
they are the parts of another cent : thus, in the
last example, the 6761888 cts. is 67 cents, and
.61888 of another.   A

A SUPPLEMENT TO PART III,

CONTAINING THE DOCTRINE OF

CIRCULATING DECIMALS.

CHAP. I.

DEFINITIONS and ILLUSTRATIONS.

A CIRCULATING decimal is generated or produced from a vulgar fraction, whose numerator and denominator are incommensurable to each other ; and therefore if the numerator with cyphers annexed, be divided by the denominator, there will always be a remainder, or the quotient will run on sempiternally ; consequently the true and adequate decimal of every such vulgar fraction, must consist of an infinite number of decimal places, which is therefore not assignable in finite terms, and consequently the true and complete decimal impossible.

NOTWITHSTANDING the equivalent decimal of every vulgar fraction of the kind above described, if actually completed, would then consist of an infinite number of decimal places ; yet from a few of the first, we obtain some certain law by which the figures ever after circulate or return again ; and it is for this reason they are called circulating decimals : the circulating figures are called repetends, of which there are four kinds, viz. single, compound, mixedsingle, and mixed compound.

A SINGLE repetend is a continual repetition of the same figure : Thus .666 &c. and .2222 &c. are single repetends, which are expressed by writing the re-

<div align="right">peating</div>

peating figure with a point over it: thus, for .666 &c.
write .6̇ for .2222 &c. we write .2̇ ; and so on for
others.

A COMPOUND repetend is when the same figures
circulate or return alternately: thus .9595 &c. and
.321321 &c. are compound repetends, which are ex-
pressed by writing the combination of figures that
circulate or return together, with a point over the
first and last figure: thus, instead of .9595 &c. we
write .9̇5̇ for .321321 &c. we write .3̇21̇ ; and so
on for others.

A MIXED single repetend is when one or more fi-
gures occur before the repeating ones: thus .172444̇
&c. and .1942777̇ &c. are mixed single repetends.

A MIXED compound repetend is when several fi-
gures stand before those that circulate alternately:
thus .1724̇74̇7 &c. and .41972̇972̇ &c. are mixed
compound repetends,

THOSE combinations of figures, which circulate
or return together, are called circulates, of which
there are three kinds, viz. similar, dissimilar, similar
and conterminous.

SIMILAR circulates are those that consist of the
same number of repeating figures, beginning either
before or after the decimal point: thus 42.7̇ and 9.1̇9̇
are similar circulates.

DISSIMILAR circulates are those that consist of an
unequal number of repeating figures, beginning at
different places: thus 1.7̇7̇ and 217.4̇ are dissimilar
circulates.

SIMILAR

SIMILAR and conterminous circulates, are those which consist of an equal number of repeating figures, beginning and ending together : thus, 27.4̇7̇ and 4.7̇3̇ are similar and conterminous circulates.

CHAP.   II.

Of REDUCTION of CIRCULATING DECIMALS.

PROBLEM   I.

To reduce a single repetend to its equivalent Vulgar Fraction.

R U L E.

UNDER the given repetend, with as many cyphers annexed to it, as there are places of whole numbers, write as many 9's as there are places of figures in the repetend ; and you will have the Vulgar Fraction required.

THE reason of this rule will appear obvious, when we consider, that .9̇ = 1 ; for $\frac{1}{9}$ = .111 &c. = .1̇ ; consequently .1̇ × 9 = $\frac{1}{9}$ × 9 ; that is, .9̇ = $\frac{9}{9}$ = 1 ; whence it follows, that each figure of the repetend is equal to that figure divided by 9 : thus .3̇ = $\frac{3}{9}$ = $\frac{1}{3}$ .5̇ = $\frac{5}{9}$, &c.

EXAMPLES.

Required the least Vulgar Fraction equivalent to .7̇2̇
Thus,

Thus, $\frac{7\,2}{9\,9} = \frac{8}{1\,1} = $ *fraction required.*

$2\,1.\overset{.}{3} = \dfrac{2\,1\,3\,0\,0}{9\,9\,9}.$ $6\,4\overset{.}{3}.2\overset{.}{5} = \dfrac{6\,4\,3\,2\,5\,0\,0\,0}{9\,9\,9\,9\,9}.$ $1.7\,4\,2\overset{.}{1} =$

$\dfrac{1\,7\,4\,2\,1\,0}{9\,9\,9\,9\,9}.$ $1\,2\,7.0\,0\,0\,2 = \dfrac{1\,2\,7\,0\,0\,0\,2\,0\,0\,0}{9\,9\,9\,9\,9\,9\,9}.$

PROBLEM II.

To reduce a mixed compound repetend to its equivalent Vulgar Fraction.

RULE.

WRITE down as many 9's as there are places of figures in the repetend, to which annex as many cyphers as are equal to the number of occurring places of figures in the finite part, (i. e. the figures occurring before the alternate circulates) for a denominator; then multiply the 9's in the denominator, with the finite part, to which product, add the infinite or circulating part for a numerator; and you will have the fraction required.

Note. *When the circulate begins any where in the integral part, omit the cyphers in the denominator, and annex as many to the numerator as there are places of whole numbers included in the circulate.*

THE reason of this rule will appear plain from the following. Suppose the decimal whose equivalent VulgarFraction is required, to be .5$\overset{.}{3}$: Conceive it to be divided into finite and infinite parts; that is, conceive it to be made of the finite part .5 and the infinite or circulating part .0$\overset{.}{3}$; then .5$\overset{.}{3}$=.5+.0$\overset{.}{3}$; but .$\overset{.}{3}$=$\frac{3}{9}$; consequently .0$\overset{.}{3}$=$\frac{1}{10}$ of $\frac{3}{9}$=$\frac{3}{90}$; wherefore

.5$\overset{.}{3}$

$.5\dot{3} = \frac{5}{10} + \frac{3}{90} = \frac{450}{900} + \frac{30}{900} = \overline{\frac{9 \times 5 + 3}{90}}$, which is the same as the rule.

EXAMPLES.

Required the Vulgar Fraction equivalent to $.47\dot{3}\dot{9}$
First, 9990 = *denominator.*
Then 999 × 4 = 3996 = *product of the 9's in the denominator and finite part*; and 3996 + 739 = 4735 = *numerator.*

Wherefore $\frac{4735}{9990}$ *is the fraction required.*

Required the equivalent Vulgar Fraction of $5.2\dot{7}$:

Thus, $\overline{52 \times 9} + 7 \div 900 = 468 + 7 \div 900 = \frac{475}{900}$ *the fraction required.*

Required the equivalent Vulgar Fraction of $42.\dot{3}$:

Thus, $\overline{990 \times 4} + 230 \div 99 = \frac{4190}{99}$ *the fraction required.*

Required the equivalent Vulgar Fraction of $321.\dot{7}$:

Thus, $\overline{999 \times 3} + 217 \div 999 = 3214 \div 999$; then $\frac{321400}{999} = $ *fraction required.*

PROBLEM III.

To determine whether the decimal equivalent to any Vulgar Fraction be finite, or infinite; and if infinite, to find the number of places of figures that constitute the circulate.

RULE.

1. REDUCE the given fraction to its least terms.
2. DIVIDE the denominator of the resulting fraction

tion by 2, 5 or 10, as often as you can without a remainder, making the refult a divifor, and 999 &c. a dividend, divide till nothing remains, then will the circulate confift of as many places of figures as you ufed places of 9's.

> Note. 1. *The circulate will begin, after as many places of figures as you made divifions of the denominator.*
>
> 2. *In dividing the denominator as above, if the quotient become equal to unity, then the decimal is finite, confifting of as many places of figures as you made divifions of the denominator.*

THE principles on which this rule is inveftigated, may be fhewn in the following manner.

Firft, let it be premifed, that if unity with cyphers annexed, be divided by any prime number, except 2, or 5, the figures in the quotient will begin to repeat when the remainder becomes unity; confequently 999 &c. divided by any prime number, except 2, or 5, will leave no remainder.

Now if the places of figures in the circulate are any number, when the dividend is unity, they will remain the fame, let the dividend be any other number whatever; for it is plain, that if the decimal be multiplied with any number, every circulate will be equally multiplied, and what one is increafed will be carried to another, and fo on through the whole; confequently, the places of figures will remain the fame: But to multiply the decimal or quotient with any number, is the fame thing, as to divide the divifor by the fame number before divifion is made; whence, &c.

EXAMPLES.

EXAMPLES.

Required to know, whether the equivalent decimal, of $\frac{158}{557}$ is infinite or finite, and if infinite, how many places of figures there will be in the circulate.

First, $\frac{158}{557}$ reduced to its leaft terms $=\frac{2}{7}$; then $999999 \div 7 = 142857$, and therefore the decimal is infinite, whofe circulate confifts of 6 places of figures, beginning at the tenth's place.

Required to know whether the equivalent dicimal of $\frac{210}{1120}$ is infinite, or finite; and if infinite, how many places of figures there will be in the circulate.

First, $\frac{210}{1120} =$ (by reducing to its leaft terms) $\frac{3}{16}$; then, $16 \div 2 = 8$, $8 \div 2 = 4$, $4 \div 2 = 2$, and $2 \div 2 = 1$: Confequently the decimal is finite, confifting of 4 places of figures.

Required to know whether the equivalent decimal of $\frac{364}{490}$ is infinite, or finite; and if infinite, to know how many places of figures there will be in the circulate.

First, $\frac{364}{490} =$ (by reducing to its leaft terms) $\frac{52}{70}$; then $70 \div 10 = 7$, and $999999 \div 7 = 142857$: Confequently the dicimal is infinite, and the circulate confifts of 6 places of figures, beginning at the hundredth's place.

PROBLEM IV.

To make diffimilar circulates, fimilar and conterminous.

RULE.

1. Find the leaft poffible common multiple of the feveral numbers expreffing the number of places of figures in the given circulates.

2. Change the given circulates into others, confiſting each of as many places of figures as the leaſt common multiple found as above, and the work will be done.

EXAMPLES.

Make .7̇2̇7̇, .1̇7̇9̇, .1̇2̇ and .1̇9̇ ſimilar and conterminous.

Firſt the leaſt common multiple of 3, 3, 2 and 2, is 6.

Diſſimilar. Similar and conterminous.

$$\text{Then,} \begin{cases} .7̇2̇7̇ = .7̇27727̇ \\ .1̇7̇9̇ = .1̇79179̇ \\ .1̇2̇ = .1̇21212̇ \\ .1̇9̇ = .1̇91919̇ \end{cases}$$

Make 24.3̇, .4̇76̇2̇, 32, .6̇ and .5̇76̇ ſimilar and conterminous.

Diſſimilar. Similar and conterminous.

$$\text{Thus,} \begin{cases} 24.3̇ = 24.3̇33333333333̇ \\ .4̇76̇2̇ = .4̇7624762476̇2̇ \\ 32.6̇ = 32.6̇66666666666̇ \\ .5̇76̇ = .5̇76576576576̇ \end{cases}$$

CHAP.

CHAP. III.

ADDITION, SUBTRACTION, MUL-TIPLICATION AND DIVISION OF CIRCULATING DECIMALS.

SECT. I.

Of ADDITION of CIRCULATING DECIMALS.

RULE.

MAKE the given circulates fimilar and conter-minous, by problem IV, of the laft chapter; then add them together as in common Addition, and becaufe each figure of the circulate is equal to that figure divided by 9, you muft divide the fum of the circulates, by as many places of 9's there are pla-ces of figures in the circulate, and writing the re-mainder (if any) directly beneath the figures of the circulate, carry the above quotient to the next place; then proceed as in common decimals, and you will have the fum required.

Note. *When the remainder confifts of a lefs number of places than the circulate, you muft fupply the defect by prefixing cyphers.*

EXAMPLES.

EXAMPLES.

Required the sum of $3.\dot{3} + 4.27\dot{1} + 3.\dot{7}2\dot{5}$:

      Diſſimilar.   *Similar and conterminous.*

$$\text{Thus,} \begin{cases} 3.\dot{3} & = 3.\dot{3}3\dot{3} \\ 4.27\dot{1} & = 4.2\dot{7}\dot{1} \\ 3.\dot{7}2\dot{5} & = 3.\dot{7}2\dot{5} \end{cases}$$

$$11.3\dot{3}\dot{0} = \textit{ſum required.}$$

Required the sum of $24.3274\dot{2}\dot{5} + 37.27\dot{4} + 27.\dot{3}\dot{5} + 34.2\dot{7}$:

      Diſſimilar.   *Similar and conterminous.*

$$\text{Thus,} \begin{cases} 24.3274\dot{2}\dot{5} & = 24.327\dot{4}2542\dot{5} \\ 37.27\dot{4} & = 37.27\dot{4}4444\dot{4} \\ 27.\dot{3}\dot{5} & = 27.3\dot{5}3535\dot{3} \\ 34.2\dot{7} & = 34.2\dot{7}7777\dot{7} \end{cases}$$

$$123.23\dot{3}1830\dot{0}\dot{1} = \textit{ſum req.}$$

SECT. II.

Of SUBTRACTION of CIRCULATING DECIMALS.

RULE.

PREPARE the given numbers, as in Addition, and then ſubtract them as in common Subtraction, only with this difference, viz. when the circulate to be ſubtracted, is greater than the one from which Subtraction is to be made, you muſt make the right-hand

                    figure

figure of the difference lefs by unity, than as found by common Subtraction. The reafon of this rule will appear plain from the following.

SUPPOSE $1.8\dot{1}$ were to be taken from $2.\dot{7}\dot{2}$; the difference by common Subtraction would be $.91$; but $2.\dot{7}\dot{2} = \frac{270}{99}$ and $1.8\dot{1} = \frac{130}{99}$, then $2.\dot{7}\dot{2} - 1.8\dot{1} = \frac{270}{99} - \frac{180}{99} = \frac{90}{99} = .\dot{9}\dot{0}$; whence, &c.

EXAMPLES.

Required the difference between $6.47\dot{2}\dot{9}$ and $3.\dot{4}\dot{9}$:

Diffimilar.   Similar and conterminous.

Thus, $\begin{cases} 6.47\dot{2}\dot{9} = 6.47\dot{2}9729 \\ 3.49 = 3.494949\dot{4} \end{cases}$

$2.978023\dot{4} =$ difference required.

Required the difference between $4.37\dot{5}\dot{2}$ and $1.121\dot{0}$:

Diffimilar.   Similar and conterminous.

Thus, $\begin{cases} 4.37\dot{5}\dot{2} = 4.375\dot{2}525\dot{2} \\ 1.121\dot{0} = 1.121\dot{0}121\dot{0} \end{cases}$

$3.254240\dot{4}\dot{1} =$ difference required.

S E C T.   III.

Of MULTIPLICATION of CIRCULATING DE-CIMALS.

R. U L E.

INSTEAD of the given circulates, write their equi-valent Vulgar Fractions, and find their product as
R   ufual;

uſual ; then this product thrown into a decimal, will give the product required.

EXAMPLES.

Required the product of $3.\dot{2} \times .\dot{7}$

Firſt, $.3\dot{2} = \frac{29}{90}$ and $.\dot{7} = \frac{7}{9}$; wherefore $.3\dot{2} \times .\dot{7} = \frac{29}{90} \times \frac{7}{9} = \frac{203}{810}$, which thrown into a decimal, is, $.2506173\dot{9} = product\ required.$

Required the product of $1.\dot{8} \times 2.\dot{7}$:

Thus, $1.\dot{8} \times 2.\dot{7} = \frac{17}{9} \times \frac{25}{9} = \frac{425}{81} = 5.24691358\dot{0}$ *the product required.*

Required the product of $.20 \times .\dot{3}\dot{6}$:

Thus, $.20 \times .\dot{3}\dot{6} = \frac{20}{100} \times \frac{36}{99} = \frac{720}{9900} = \frac{8}{110} = .0\dot{7}\dot{2} = product\ required.$

SECT. IV.

Of DIVISION of CIRCULATING DE-CIMALS.

RULE.

Change the given decimals into their equivalent Vulgar Fractions, and find their quotient as uſual ; then this quotient thrown into a decimal, will give the quotient required.

EXAMPLES.

Required the quotient of $.\dot{2}\dot{6}$ divided by $.\dot{3}$:
Firſt, $.\dot{2}\dot{6} = \frac{24}{90}$ and $.\dot{3} = \frac{3}{9}$:

<div align="right">Wherefore,</div>

Wherefore, $.\dot{2}\dot{6} \div .\dot{3} = \frac{24}{90} \div \frac{3}{9} = \frac{216}{270} = .8$ *the quotient required.*

Required the quotient of $.\dot{9} \div .\dot{1}0\dot{8}$:

Thus, $.\dot{9} \div .\dot{1}0\dot{8} = \frac{9}{9} \div \frac{108}{999} = \frac{1}{1} \div \frac{12}{111} = \frac{111}{12} = 9.25$ $=$ *quotient required.*

Required the quotient of $2.\dot{9} \div .\dot{2}\dot{7}$:

Thus, $2.\dot{9} \div .\dot{2}\dot{7} = \frac{27}{9} \div \frac{27}{99} = \frac{99}{9} = 11$ *the quotient required.*

A

A Supplement to Part I,

CONTAINING THE DOCTRINE AND APPLICATION OF RATIOS, OR PROPORTION, EXTRACTION OF ROOTS, &c.

CHAP. I.

Of PROPORTION or ANALOGY.

PROPORTION is a degree of likenefs which quantities bear to each other, by a fimilitude of ratios.

RATIO is the mutual refpect of two quantities of the fame kind ; but they form no Analogy, becaufe there can be no fimilitude of ratios between two quantities, and therefore Analogy confifts of three quantities at leaft, whereof the fecond fupplies the place of two : Thus the refpect of 2 to 6, being compared with 18, it will be, 2:6::6:18.

SECT. I.

Of CONTINUED PROPORTION ARITHMETICAL,

OR

ARITHMETICAL PROGRESSION.

WHEN quantities increafe or decreafe by an equal difference, thofe quantities are in Arithmetical Proportion continued : Thus, the number 1, 2, 3, &c. are a feries of quantities in Arithmetical Proportion

continued,

continued, increasing by unity, or 1, which is called the common difference of the series.

ALSO, the numbers 2, 4, 6, 8, are numbers in Arithmetical Progression, whose common difference is 2; but the numbers 9, 7, 5, 3, 1, are a series of quantities in Arithmetical Progression, decreasing by the common difference, 2.

LEMMA I.

If three numbers are in Arithmetical Progression, the sum of the two extreme numbers will be double the mean or middle number.

THUS, let 1, 3, 5, be the numbers in progression; Then, 1+5, the sum of the two extremes=3+3 the double of the mean. Again, in the numbers 14, 10, 6, the sum of the two extremes are 14+6= 20, and the double of the mean 10+10=20; and the like will hold in any other numbers.

LEMMA II.

If four numbers are in Arithmetical Progression, the sum of the two extremes will be equal to the sum of the two means.

LET the number be 4, 7, 10, 13; then 4+13=17, the sum of the two extremes, and 7+10=17, the sum of the two means: Again, in the numbers 16, 13, 10, 7; 16+7=13+10.

AND since in four numbers as above, the sum of the two extremes, is equal to the sum of the two means, we have no reason to doubt of the like, let the terms be any number whatever: Whence it follows, that in any Arithmetical series, of any assignable number of terms whatever, the sum of any two terms equidistant from the mean, will be equal to

the

the fum of any other two terms, equidiftant from the mean ; as in thefe, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20 ; where $2 + 20 = 4 + 18 = 6 + 16 = 8 + 14 = 10 + 12$: Therefore, &c.

LEMMA III.

In any feries of numbers in Arithmetical Progreffion, the feveral terms are formed or made up by the addition of the common difference to the firft term, fo often repeated, as there are number of terms to the feveral places, except the firft.

LET the feries be, 1, 4, 7, 10, 13, 16, 19, 22, &c. wherein the common difference is 3.

Now $1 + 3 = 4$ *the fecond term,* $1 + 3 + 3 = 7$, *the third term* ; $1 + 3 + 3 + 3 = 10$, *the fourth term* ; $1 + 3 + 3 + 3 + 3 = 13$, *the fifth term* ; and $1 + \overline{3 \times 7} = 22$, *the 8th term,* &c. Confequently the difference of the two extremes, is equal to the common difference multiplied with the number of terms lefs 1 : Thus in the above feries, the common difference is 3, and number of terms 8 ; therefore $\overline{8 - 1} \times 3 = 7 \times 3 = 21 =$ difference of the two extremes.

PROBLEM I.

To find the fum of a feries of numbers in Arithmetical Progreffion.

THERE are feveral ways of deducing a rule for the folution of this problem, but perhaps none more fimple and natural than the following.

LET the feries whofe fum is required, be $2 + 4 + 6 + 8 + 10 + 12$.

Or,

Or,
$$2+2+2+2+2+2$$
$$+ \quad + \quad + \quad + \quad +$$
$$2 \quad 2 \quad 2 \quad 2 \quad 2$$
$$+ \quad + \quad + \quad +$$
$$2 \quad 2 \quad 2 \quad 2$$
$$+ \quad + \quad +$$
$$2 \quad 2 \quad 2$$
$$+ \quad +$$
$$2 \quad 2$$
$$+$$
$$2$$

which is the fame as the former, though differently expreffed : Now under the given feries place the fame inverted and add up the whole.

Thus,

1 term.	2 term.	3 term.	4 term.	5 term.	6 term.
2 +	2 +	2 +	2 +	2 +	2
2	+	+	+	+	+
+	2	2	2	2	2
2	2	+	+	+	+
+	+	2	2	2	2
2	2	2	+	+	+
+	+	+	2	2	2
2	2	2	2	+	+
+	+	+	+	2	2
2	2	2	2	2	+
+	+	+	+	+	2
2	2	2	2	2	2

$$14 + 14 + 14 + 14 + 14 + 14 = \textit{fum.}$$

By this means the terms of the feries are reduced to an equality, to wit, equal to the fum of the firft and laft term ; but the fum above found, is evidently double the fum of the propofed feries : Whence
it

it follows, that the fum of an Arithmetical feries, is equal to half the product of the firft and laft term, with the number of terms; wherefore if the firft term, laft term, and number of terms of an Arithmetical Progreffion be given, the fum of the feries may be found by the following

R U L E.

Multiply the fum of the firft and laft terms, or two extremes, with the number of terms, and half of that product will be the fum required.

EXAMPLES.

Let the firft term of a feries of numbers in Arithmetical Progreffion, = 1, laft term = 37, and number of terms 19; required the fum of the feries.

OPERATION.

Firft, $1 + 37 = 38 = $ *fum of the firft and laft terms:* Then $\overline{38 \times 19} \div 2 = 722 \div 2 = 361$ *the fum required.*

A man bought 20 yards of broad-cloth; for the firft yard he gave 2 dol. and for the laft 80 dol. what did the whole coft?

The fum of the two extremes, is $2 + 80$, then $\overline{2 + 80 \times 20} \div 2 = 820$ dol. *the anfwer.*

A man travelled 12 days, the firft day 4 miles, and the laft day 40 miles; what was the diftance travelled in the 12 days? Anfwer. 264 *miles.*

PROBLEM II.

To find the common difference of an Arithmetical feries, when the two extremes and number of terms are given.   A

A RULE for the folution of this problem, is eafily deduced from the inference to Lemma III ; for fince the difference of the two extremes, is equal to the common difference multiplied with the number of terms lefs 1, it follows, that if that difference, be divided by the number of terms lefs 1, the quotient muft be the common difference of the feries ; whence the following rule is evident.

R U L E.

DIVIDE the difference of the two extremes by the number of terms lefs 1, and the quotient will be the common difference required.

EXAMPLES.

In an Arithmetical feries, there is given the firft term = 3, laft term = 60, and number of terms 20 : Required the common difference.

OPERATION.

The difference of the two extremes, is 60—3 ; therefore(pr.rule) $\frac{60-3}{20-1} = \frac{57}{19} = the\ common\ difference$
required.

Four men differing in their ages by an equal interval : The age of the firft, is 19 years, and the fourth 40 : What are their feveral ages ?

OPERATION.

Firft, find the common difference of their ages : Thus, $\overline{40-19} \div \overline{4-1} = 21 \div 3 = 7$ *years* ; therefore

S   I

19+7=26 years, *the age of the second*, and 26+7=33 years, *the age of the third*; laftly, 33+7=40 years, *the age of the fourth, as given above.*

A man owes a certain debt, to be difcharged at 8 feveral payments; all of which are to be made in Arithmetical Progreffion, the firft payment to be 4 dol. and the laft 32 dol. Query, the whole debt and each payment.

OPERATION.

$$\frac{\overline{32+4} \times 8}{2} = 144 \text{ dol.}$$ *the whole debt*, and $\overline{32-4} \div$

$\overline{8-1} = 4$ dol. *the common difference*; wherefore 4+4 =8 dol. *the fecond payment*, and 8+4=12 dol. *the third payment*; alfo, 12+4=16 dol.; *for the fourth*; moreover 16+4=20 dol. *for the 5th*, in like manner 20+4=24 dol. *for the 6th*, and 24+4=28 dol. *for the 7th*; laftly 28+4=32 dol. *for the laft payment as before.*

PROBLEM III.

To find the number of terms of an Arithmetical feries, when the firft term; laft term and common difference are given.

FROM the laft rule, it is eafy to conceive how a rule for the folution of this problem may be obtained; for fince the difference of the two extremes, divided by the number of terms lefs 1, gives the common difference; it follows, that the difference of the two extremes, divided by the common difference, muft quote the number of terms lefs 1.

Whence is deduced the following

R U L E.

RULE.

DIVIDE the difference of the two extremes, by the common difference, the quotient increased by unity or 1, will be the number of terms.

EXAMPLES.

Given the first term of an Arithmetical series=2, last term=167, and common difference 3, to find the number of terms.

OPERATION.

$$\frac{167-2}{3}+1=\frac{165}{3}+1=55+1=56 \text{ the number of}$$

terms required.

A man bought a quantity of broad-cloth ; for the first yard he gave 6 dol. for the second, 10 dol. and so on, in Arithmetical Progression, to the last yard, for which he gave 246 dol. ; what was the quantity of cloth bought ?

OPERATION.

$$\frac{246-6}{4}+1=\frac{240}{4}+1=61, \text{ the number of yards bought.}$$

A man travels from Boston, to a certain place, in the following manner, viz. the first day 10 miles ; the second day 15 miles, and so on, till a day's journey is 55 miles : In how many days will he perform the whole journey ; also, how many miles is the place he goes to, distant from Boston ?

Answer. He will perform the whole in eleven days. The place distant from Boston, 330 miles.

S E C T.

SECT. II.

Of CONTINUED PROPORTION GEOMETRICAL,

Or

GEOMETRICAL PROGRESSION.

GEOMETRICAL Progreffion continued, differs from Arithmetical Progreffion in this; in Arithmetical Progreffion, each following term of the feries is formed or made up by the Addition or Subtraction of the common difference, (as we have before fhewn): Whereas in Geometrical Progreffion, each fucceffive term of the feries, is produced by the Multiplication or Divifion of the preceeding term, with a common multiplier or divifor: Or in other words, Arithmetical Progreffion, is the effect of a conftant Addition or Subtraction; but Geometrical Progreffion, of a conftant Multiplication or Divifion.

THUS, 2, 4, 8, 16, 32, 64, 128, &c. are a feries of numbers in Geometrical Proportion continued; whofe refpective terms are compofed by the Multiplication of the Ratio or common multiplier, (2): thus, $2 \times 2 = 4$, *the fecond term*, $4 \times 2 = 8$, *the third term*; $8 \times 2 = 16$, *the fourth term*, and fo on.

ALSO, 16, 8, 4, 2, are a feries of numbers in Geometrical Proportion, continually decreafing by the divifion of the Ratio, or common divifor, (2): Thus, $\frac{16}{2} = 8$, *the fecond term*, $\frac{8}{2} = 4$, *the third term*, $\frac{4}{2} = 2$, *the fourth term*, and $\frac{2}{2} = 1$, *the fifth term*.

LEMMA I.

If three numbers are in Geometrical Progreffion, the product of the two extremes, will be equal to the product of the mean with itfelf. LET

LET the numbers be 2, 8, 32; where $2 \times 32 = 64$, and $8 \times 8 = 64$; confequently $2 \times 32 = 8 \times 8$.

LEMMA II.

In any Geometrical Proportion confifting of four terms, the product of the two extremes, is equal to the product of the two means.

IF the numbers are, 2, 8, 32, 128, it will be $2 \times 128 = 8 \times 32$; therefore $2 : 8 :: 32 : 128$.

CONSEQUENTLY, if the product of any two numbers, be equal to the product of any other two numbers, thofe four numbers are proportional.

HENCE it may be eafily underftood, that if any number of terms are in ÷ the product of the two extremes, will be equal to the product of any other two terms, equidiftant from thofe extremes.

LET the feries be 3, 6, 12, 24, 48, 96; where $3 \times 96 = 6 \times 48 = 12 \times 24$.

WHEN numbers are compared together, in order to difcover their relation to each other, the number compared is writen firft, and called the antecedent, and the number by which you compare the other, being written next, is called the confequent : Thus if you would compare 2 with 4, the numbers muft be wrote thus, 2, 4; where 2 is the antecedent, and 4 the confequent : Again in thefe, $3 : 6 :: 6 : 12$; where 3 is antecedent, and 6 its confequent; alfo, 6 the middle term, is an antecedent to 12, its confequent. Therefore in every feries of numbers in Geometrical Proportion continued, all the terms except the laft, are antecedents, and all except the firft are confequents.

THUS in the feries 3, 9, 27, 81, 243, 729, the numbers 3, 9, 27, 81, 243, are all antecedents, and

9, 27, 81, 243, 729, are all confequents; therefore
3 : 9 :: 9 : 27 :: 27 : 81 :: 81 : 243 :: 243 : 729.

THE Ratio is had by dividing any confequent by
its antecedent.

LEMMA III.

*If any numbers are proportional, it will be, as any
one of the antecedents, is to its confequent ; fo is the fum
of all the antecedents, to to the fum of all the confe-
quents, (Vid. Euclid's fifth book, Propofition 12.)*

LET the numbers be thefe, 4, 8, 16, 32, 64, then
4 : 8 :: $\overline{4+8+16+32}$: $\overline{8+16+32+64}$, that is,
4 : 8 :: 60 : 120 ; for $\overline{4 \times 120 = 8 \times 60}$; therefore,
&c.

PROBLEM I.

To find the fum of any Geometrical feries increafing.

SUPPOSE the fum of the following feries, 1, 4, 16,
64, 256, is required : Multiply this feries with the
Ratio, which is 4, and the product will be a new fe-
ries, 4, 16, 64, 256, 1024 : Now it is plain, that
the fum of the produced feries, is as many times the
fum of the former, as the Ratio hath units ; or the
produced feries, is to the propofed, as the Ratio to
unity, or 1 : Subtract the firft feries from the fecond.

Thus, $\left\{ \begin{array}{l} 4, 16, 64, 256, 1024 \\ 1, 4, 16, 64, 256. \end{array} \right.$

—1, * * * * +1024, or, 1024—1,
which is evidently equal to the fum of the firft feries
multiplied with the Ratio, lefs 1, by what has been
faid ; confequently the fame divided by the Ratio,
lefs 1, muft give the fum of the propofed feries ; that

is, $\dfrac{\overline{256 \times 4} - 1}{4 - 1} = \dfrac{\overline{1024} - 1}{4 - 1} = $ *sum of the series re-*

quired.

THEREFORE, when the first term, last term, and Ratio of a Geometrical series are given, we may find the sum of all the terms by the following

R U L E.

MULTIPLY the last term with the Ratio, from which product, subtract the first term, divide the remainder by the Ratio less 1, and the quotient resulting will be the sum of the series.

MR. WARD, in his introduction to the Mathematics, page 78, has given an analytical investigation of a rule for finding the sum of any series in increasing; which is after the manner following.

LET a Geometrical series be given, suppose the following, 2, 4, 8, 16, 32, 64.

Put $x = $ *sum of the series :*

Then, $x - 64 = $ *sum of all the antecedents :*

And $x - 2 = $ *sum of all the consequents :*

Therefore, $2 : 4 :: x - 64 : x - 2$; per Lemma III.

Consequently, $\overline{x - 2} \times 2 = \overline{x - 64} \times 4$;

That is, $2x - 4 = 4x - 256$:

Then, $4x - 2x = 256 - 4$:

Therefore, (by division) $2x - x = 128 - 2$:

Whence, $x = \overline{128 - 2} \div 2 - 1$, *which affords the same rule as that above.*

Or finding the value of x in the equation $4x - 2x = 256 - 4$, *to wit,* $x = \overline{256 - 4} \div \overline{4 - 1}$ *which admits of the following*

R U L E.

RULE.

FROM the product of the second and laſt terms, ſubtract the ſquare of the firſt, divide the remainder by the ſecond term leſs the firſt ; and the quotient will be the ſum of the ſeries.

EXAMPLES.

In a Geometrical ſeries, there is given, the firſt term=3, laſt term=243, and Ratio 3; to find the ſum of the ſeries, per Rule firſt.

OPERATION.

Firſt, 243×3=729=*product of the laſt term with the Ratio ;* then 729—3÷3—1=726÷2 = 363 *the ſum required.*

A man bought a quantity of cloth ; for the firſt yard he gave 2 dol. for the ſecond 4 ; and ſo on, in continued proportion Geometrical to the laſt yard, for which he gave 256 dol. what did the whole coſt ?

Here, is given the firſt, ſecond, and laſt terms, to find the ſum of the ſeries, per Rule ſecond.

256×4—4=1024—4=1020=*product of the ſecond and laſt terms, leſs the ſquare of the firſt* ; then $\frac{1020}{4—2}=\frac{1020}{2}=510 dol.$=*the aforeſaid difference divided by the ſecond term leſs the firſt=ſum that the whole cloth coſt.*

BUT in finding the ſum of the ſeries by the foregoing rules, it is neceſſary to have the laſt term given : therefore the next thing in order, is, to ſhew how the laſt term of the ſeries, when it is not given in the queſtion, may be obtained.

PROBLEM

PROBLEM II.

The first term, Ratio, and number of terms of a Geo-metrical series being given, to find the last term.

I. WHEN the first term and Ratio are alike.

RULE I.

1. WRITE down an Arithmetical series of a convenient number of terms, whose first term, and common difference is unity or 1.

2. WRITE a few of the leading terms of the Geometrical series, under the first terms of the Arithmetical one.

Thus, $\begin{cases} 1, & 2, & 3, & 4, & 5, & \textit{Indices, or exponents.} \\ 2, & 4, & 8, & 16, & 32, & \textit{Geometrical series.} \end{cases}$

3. ADD together any two of the indices, and multiply the terms in the Geometrical series, which belong to those indices, together, and their product will be that term of the Geometrical series, which the sum of those two corresponding indices point out.

4. CONTINUE the addition of the indices, and multiply their corresponding terms, of the Geometrical series, respectively as before, until the sum of the indices is equal to the number of terms, the product answering thereunto, will be the last term required.

II. WHEN the first term is either greater or less than the Ratio, (unity excepted.)

RULE II.

1. WRITE down an Arithmetical series, beginning with a cypher, the common difference, the same as in the last rule.

T

2. PLACE the leading terms of the Geometrical
feries, under the Arithmetical, fo that the cypher
may ftand over the firft term of the Geometrical fe-
ries; then add the indices, and multiply their corref-
ponding terms as before.

3. DIVIDE that product by the firft term, and the
quotient will be that term of the feries, which is de-
nominated by the fum of thofe indices : The reft
the fame as before.

III. WHEN the firft term is unity or 1.

RULE III.

WRITE down the terms, and place their indices as
in the laft rule ; then add the indices, and multiply
the terms which they denominate, together, till the
fum of the indices is one lefs than the number of
terms, and the refult will be the laft term, as requir-
ed.

AN example in each of the foregoing rules, will
make their application eafy.

In a Geometrical feries, there is given, the firft
term=2, Ratio 2, and number of terms 12, to find
the laft term, per rule 1.

OPERATION.

Thus, $\begin{cases} 1, & 2, 3, 4, 5, 6, \textit{Indices.} \\ 2, & 4, 8, 16, 32, 64, \div. \end{cases}$

Here, 4+2=6, *the index of the fixth term ; confe-
quently* 4×16=64, *the fixth term. Again,* 6+6=12,
and 64×64=4096=*twelfth term, as required.*

Suppofe the firft term of a feries in $\div$, is 3, Ratio
2, and number of terms 15 ; required the laft term,
per rule 2.

OPERATION.

OPERATION.

First, $\begin{cases} 0, & 1, & 2, & 3, & 4, & 5, \textit{ Indices.} \\ 3, & 6, & 12, & 24, & 48, & 96, \div. \end{cases}$

Then, $3+5=8$, *and* $24\times96=2304$; *therefore,* $2304\div3=768=eighth\ term.$ *Again,* $3+4=7$, *and* $24\times48=1152$; *therefore,* $1152\div3=384=seventh$

term. Lastly, $7+8=15$; *whence* $\dfrac{384\times768}{3}=98304$

$=15th$, *and last term which was to be done.*

Given first term $=1$, Ratio 4, and number of terms 11, to find the last term, per rule 3.

OPERATION.

Thus, $\begin{cases} 0, & 1, & 2 & 3, & 4, \textit{ Indices.} \\ 1, & 4, & 16, & 64, & 256, \div. \end{cases}$

Then, $4+3+3=10=number\ of\ terms\ less\ one=$ *index to the* 11th *term* ; *therefore,* $256\times64\times64=$ $1048576=11th\ term\ as\ was\ required.$

Miscellaneous Questions.

A MAN hired himself to a farmer, for 28 weeks upon these considerations ; that for the first week to have 1 *ct.* ; for the second 2 *cts.* ; and the third 4 *cts.* ; and so on, in $\div$: What did his 28 weeks wages amount to ?

The last term by the foregoing rules, is, 134217728, which multiplied with the Ratio (2) produces

268435456 ; therefore, $\dfrac{268435456-1}{2-1}=268435455$

cts. $=2684354\ dol.\ 55\ cts.\ the\ answer.$ A

A MAN bought 20 yards of velvet, at the following prices, viz. for the first yard he gave 2 *cts.*; for the second, 4 *cts.*; for the third, 8 *cts.* and so on, in Geometrical Proportion: How much did the whole cost ?

Answer. 20971 *dol.* 50 *cts.*

A MERCHANT sold 24 yards of lace; the first yard for 3 pins, the second for 9, the third for 27; and so on, in triple Proportion Geometrical : Now suppose he afterwards sold his pins 120 for a cent : What did his lace amount to, and what was his gain in the whole, when he gave 50 *cts.* per yard for his lace ?

Ans. { *Lace come to,* 4236443047 *dol.* 20 *cts.* { *Gain in the whole,* 4236443035 *dol.* 20 *cts.*

A THRESHER agreed with a farmer to work for him 25 days, for no other consideration than 2 barley-corns for the first day 8; for the second 32; for the third; and so on, in quadruple proportion Geometrical : How much did his wages amount to, allowing 7680 barley-corns to make one pint, and the barley to be sold for 25 *cts.* per bushel ?

Answer. 381774870 *dol.* 75 *cts.*

SUPPOSE a wheat-corn had been sowed at the creation, and continued to increase in a ten-fold proportion every year, down to the present time ; now allowing 5000 years for the elapse of time : What would be the number of wheat-corns produced ?

Here the first term being 1, the Ratio 10, and the number of terms 5000, it is therefore plain, that the last term will be 1, having as many cyphers annexed, as there are number of terms less one ; consequently its value is 1(4999)0's, where the numeral figures included in the parenthesis, express the number of cyphers annexed to the 1 : Next to find the sum of the series.   *First,*

First, 1(4999) 0's $\times$ 10 $=$ 1(5000) 0's, *then* 1(5000) 0's $-$ 1 $=$ (5000) 9's $=$ *the number of* 9's *therefore*

$$\frac{5000 \ 9's}{10 - 1} = 11111111111111111111 \ \&c. \text{ to } 5000$$

places of figures $=$ *number of wheat-corns produced*; which number far exceeds all human imagination; for the whole space occupied by our solar system, which is at least twenty thousand million of miles in diameter, is by much too small, to contain the aforesaid quantity of wheat: Nay, such a quantity would take up more space, than is contained in the whole heavens on this side the fixed stars. Hence we may learn the great power of progressive numbers, and that small portion of space, necessary to express a number by the help of numeral figures contrived for that purpose, which so far exceeds all our imagination.

CHAP. II.

DISJUNCT PROPORTION,

OR

The RULE of THREE.

WHEN of four numbers, the first has the same Ratio to the second, as the third has to the fourth: Or when the second is the same multiple or quotient of the first, as the fourth is of the third; then are those numbers said to be in Disjunct Proportion.

IF four numbers are proportional directly, as the first to the second; so is the third to the fourth; then will they also be proportional; Inversely, Alternately,

ly, Compoundedly, Dividedly, and Mixtly. (*Vid Book* 11. *Chap.* XII.)

S E C T.   I.

DIRECT PROPORTION,
OR
The RULE of THREE DIRECT.

THIS is sometimes called the golden rule, from the great benefit people in all kinds of business receive from it, as well the farmer and mechanic as the merchant, &c. It consists of four numbers, which are proportional, as the first to the second; so is the third to the fourth, as above: The two first are a supposition, the third a demand, and the fourth the answer. The two suppositions and the demand are always given, and the fourth required.

Let the four numbers be, a, b, c, d. *Then* $a : b :: c : d$, *directly*; *therefore*, $a \times d = b \times c$, *or* $ad = bc$, per Lemma II, of the last Section.

Whence by the nature of division $bc \div a = d$, *that is*, if the product of the second and third terms, be divided by the first, the quotient will be the fourth. Or since the Ratio of the first to the second, is the same as that of the third to the fourth; it follows, that $b \div a \times c = d$, that is, if the second term be divided by the first, and that quotient multiplied into the third, it will produce the fourth.

Now, in order to prepare your numbers for obtaining a fourth proportional, according to the foregoing rules, you must observe the following

R U L E.

R U L E.

WRITE that number which is of the fame name with the number fought, in the middle place, and the other two fo, that the expreffion may read according to the nature of the queftion.

Let the following conditions be expreffed in numbers.

What is the coft of 24lb. of cheefe, when the price of 3lb. is 20 *cts.* ?

Here the middle number muft be coft, becaufe the fourth, or number required, is always of the fame name and denomination of the fecond, by the nature of the proportion : Hence the above conditions in numbers, is,

Thus, 3lb. 20*cts.* 24bl. ; that is, if 3 pounds coft 20 *cts.* what will 24 pounds coft ? Then to find a fourth number, proceed as before directed.

Note. *If the firft and third numbers are not of the fame name, they muft be made fo by the rules of reduction : Alfo, if any of the numbers are compounds, they muft be reduced to the leaft denomination mentioned.*

EXAMPLES.

If 4lb. of cheefe coft 32 *cts.*; what will 320lb. coft at the fame rate ?

OPERATION.

These numbers being placed according to the
rule, will stand thus,

lb. cts. lb.
4 : 32 :: 320

$$32$$
$$\overline{}$$
$$640$$
$$960$$
$$\overline{}$$

4) 10240
1(00).25(60 = 25 *dol.* 60
[*cts.* the *answer.*

Or, 32 ÷ 4 = 8 ; *therefore,* 320 × 8 = 2560 *cts.* = 25
dol. 60 *cts. the same as before.*

What will 6 yards of holland coft, when the price
of 40 yards, is 24 *dol.* 40 *cts.* ?

OPERATION.

yd. *dol. cts.* yd.
As 40 : 24 40 :: 6 *stated.*
Then, 24.40 ÷ 40 = .61, *and* 6 × .61 = 366 *cts.* =
3 *dol.* 66 *cts. the answer.*

Find the value of 100lb. of flax, when the price of
1lb. is 12 *cts* ?

OPERATION.

lb. *cts.* lb.
As 1 : 12 :: 100

$$12$$

12.00 = 12 *dol. the answer.*

What

What is the coſt of 40lb. of cheeſe, when the price of 3lb. is 15 *cts.*

OPERATION.

Firſt, 15÷3=5, *the ratio of the firſt term to the second.*

Then, 40×5=200 *cts.*=2 *dol. the anſwer.*

What is the coſt of 87lb. of tobacco, at $8\frac{1}{2}$ *cts.* per lb. ?

OPERATION.

$$\begin{array}{cccc} \text{lb.} & \text{cts.} & \text{cts.} & \text{lb.} \\ As\ 1 : & 8\frac{1}{2} & =8.5 :: & 87 \\ & & & 8.5 \\ \hline & & & 435 \\ & & & 696 \\ \hline \end{array}$$

$$739.5=739\frac{1}{2}\ cts.=7\ dol.$$
$$[39\frac{1}{2}\ cts.\ the\ anſwer.$$

A goldſmith ſold a tankard for 29 *dol.* 97 *cts.* at the rate of 1 *dol.* 11 *cts.* per oz. : What was the weight of it ?

Anſwer. 27 *oz.*

A man bought ſheep at 1 *dol.* 11 *cts.* per head, to the amount of 51 *dol.* 6 *cts.* : How many ſheep did he buy ? *Anſwer.* 46.

SECT. II.

RECIPROCAL, *or* INVERTED PROPORTION,
OR
The RULE *of* THREE INDIRECT.

THIS kind of proportion, is the reverſe of the former, as to the performance ; for the greater the

U   third

third term is, in respect of the first, the less will be the fourth, in respect of the second; whereas in direct proportion, the greater or less the third term is, in respect of the first, the greater or less will be the fourth term, in respect of the second; but to illustrate the former. If two men can produce a certain effect in 12 days: In how many days would 6 men produce the same? Here it is manifest, that 6 men would produce the effect in less time than 2; and therefore the greater the third term is, the less will be the fourth. Again, if 10 men can produce a certain effect in 6 days: In how many days would 4 men do the same? Here it is evident, that 10 men would produce the effect in less time than 4 men; and therefore the less the third term is, the greater will be the fourth: Consequently, more requires less, and less requires more, in indirect proportion.

HERE the same rule is to be observed, in stating your question, as in the former proportion, and the results in respect of names and denominations are the same also: Then to find a fourth proportional, proceed with the following rules.

RULE I.

MULTIPLY the first and second numbers together, and divide that product by the third; the quotient resulting will be the fourth proportional required.

RULE II.

DIVIDE the second number by the third, and that quotient multiplied into the first, will produce the fourth.

<div align="right">RULE</div>

R U L E III.

DIVIDE the third term by the firft, and the fecond
term by this quotient ; and the refulting quotient
will be the fourth number.

EXAMPLES.

If 5 men can perform a certain piece of work in
days : How long will four men be in doing the
fame ?

OPERATION.

Men.	D.	Men.		Or,
5	8	4		$\frac{4}{8 \div 5} = \frac{40}{4}$
	5			$= 10$ days as before.
4)	40	D.		
	10	$=10$ the an.		

If 20 bufhels of grain, at 50 *cents* per bufhel, will
pay a debt : How many bufhels at 60 *cents* per bufh-
el will pay the fame ?

OPERATION.

OPERATION.

```
      cts.  Bush.  cts.
      50    20    60
      20
      ─────────────
6(0)100(0
     16 4/6
```
Answer. 16 4/6 *bushels.*

If 2 yards of cloth, 1 yard and 3 quarters wide, is sufficient to make a coat; how many yards of 1 yard wide, will make the same?

OPERATION.

```
      y.   q. y.   y.
      1    3 : 2 :: 1
      4            4
   ───────      ──────
      7            4
      2
   ───────
   4)14
      3 2/4 = 3 1/2 yards the answer.
```

A man being defirous to draw off a cafk of brandy into bottles, finds that if he makes ufe of three quart bottles, it will require 60: How many five-pint bottles will it require, to draw off the aforefaid cafk of brandy.   *Answer.* 72 *bottles.*

A man bought a piece of cloth 9 quarters wide, and 11 quarters long: How many yards of 3 quarters cloth will line it?   *Answer.* 8 1/4 *yards.*

If 3 1/2 yards of yard-wide cloth will make a coat: How many yards of 7 quarters cloth, will make the fame?   *Answer.* 2 *yards.*

S E C T.

SECT. III.

COMPOUNDED RATIO.

COMPOUNDED Ratio is when the antecedent and confequent taken together, is compared to the confequent itfelf: thus, $a : b :: c : d$, directly, therefore by compofition; as $a+b : b :: c+d : d$.

Note. *The fame Rule is to be obferved here, as in direct proportion.*

EXAMPLES.

If A can produce a certain effect in 5 days, B can do the fame in 7 days; fet them both about it together, in what time will it be finifhed?

OPERATION.

$$As \; \overline{5+7} : 7 :: 5$$
$$5$$
$$12)35(2 \; days.$$
$$24$$
$$\overline{}$$
$$11$$
$$24$$
$$\overline{}$$
$$44$$
$$22$$
$$\overline{}$$
$$12)264($$
$$22 \; hours. \quad Anf. \; 2 \; days \; 22 \; h.$$

If A in in 5 hours, can make 1000 nails, B in 8 hours, can make 2000: In what time would they jointly make 50000 nails?

Here

Here you muſt firſt find in what time each perſon would make 50000 nails, and then proceed as in the laſt example.

OPERATION.

       n.   h.   n.

As 1000 : 5 :: 50000 : $\overline{50000 \times 5 \div 1000}$ = 250 *bours, the time it would take A to make* 50000 nails.

       n.   h.   n.

As 2000 : 8 :: 5000 : $\overline{50000 \times 8 \div 2000}$ = 200 *bours, the time it would take B to make* 50000 *nails.*

Therefore, as $\overline{250 + 200} : 200 :: 250 : \overline{200 \times 250 \div}$ 450 = 111$\frac{1}{9}$ *bours, the time it would take them jointly to make* 50000 *nails, as was required.*

Note. *From this operation, we have the following general theorem for folving all queſtions of a ſimilar nature, let the perſons or agents employed, be any number whatever.*

THEOREM.

Multiply the joint effect with the time each one would produce his particular effect, and divide the product by the ſaid particular effect ; then multiply all the reſulting quotients together for a dividend, and make the ſum of them a diviſor ; then divide, and the reſulting quotient will be the time required.

<div align="right">S E C T.</div>

S E C T.   IV.

DIVIDED RATIO.

DIVIDED Ratio is when the excefs wherein the an-
tecedent exceeds the confequent, is compared with
the confequent : *Thus, a : b :: c : d, directly ; therefore
by divifion as a—b : b :: c—d : d.*

EXAMPLES.

If A can do a piece of work in 8 days, A and B
can do it in 5 days : In what time can B do the fame
work ?

OPERATION.

d. h.

As 8—5=3 : 5 :: 8 : 5×8÷3=40÷3=13 8, *the
time required.*

Two fhips, one in chafe of the other, the head-
moft fhip is 48 miles diftant from the other, and fails
at the rate of 4 miles per hour, and the fternmoft
fhip at the rate of 7 miles per hour : How long be-
fore the fternmoft fhip will overtake the other ?

OPERATION.

As 7—4=3 : 1 :: 48 : 48×1÷3=16 *hours, the
time required.*

A hare is is 50 leaps before a grey-hound, and
takes 4 leaps to the grey-hound's three ; but 2 of
the grey-hound's leaps are as much as three of the
hare's : How many leaps muft the grey-hound take
to catch the hare ?

*Here you muft firft find how many leaps of the hare,
anfwers to three of the grey-hound's : Thus,* 2 : 3 :: 3

: 3×3÷2=4½=4.5 :   *Then,*

Then, as $4.5 - 4 = .5 : 3 :: 50 : \overline{3 \times 50} \div .5 = 300$ *the anfwer.*

The hour and minute-hand of a clock are exactly together at 12 o'clock; when are they next together?

Here the proportion of the velofities of the hour and minute-hand, is as 1 to 12. Therefore, $12 - 1 = 11 : 1$

$:: 12 : \overline{12 \times 1} \div 11 = 1h. \ 5\frac{5}{11}', \ the\ anfwer.$

If A, B and C, can produce a certain effect in 12 days, A can do it in 30 days and C in 50 days, in what time will B do the fame work?

Firft find the time in which A and C, would produce the effect jointly, by Ratio of compofition. Thus,

$\overline{30 + 50} : 50 :: 30 : \overline{50 \times 30} \div 70 = 21\frac{3}{7} \ days.$ *Then, as* $21\frac{3}{7} - 12 = 9\frac{3}{7} : 12 :: 21\frac{3}{7} : 25\frac{49}{66}$, *the time required.*

There is an ifland 100 miles in circumference, and two footmen, A and B, fet out together, to travel the fame way round it, A travels 15 miles per day, and B 17 miles: When will they come together again?

Firft, find how many miles B muft travel to overtake A, after their departure: Thus, as $17 - 15 = 2 : 17 :: 100 : 850$, *the number of miles B muft travel, which is 50 days journey; therefore they will be together again 50 days after their departure.*

There is three pendulums of unequal lengths; the firft of which vibrates once in 12 feconds, the fecond in 18 feconds, and the third in 24 feconds: Now fuppofing them all to move from a line of conjunction, at the fame moment of time: When will they come into the fame fituation again, and move on together? *Firft*

First, find the time when the two first pendulums will move on together, as in the last example : Thus, 18—12 : 18 :: 1 : 18 × 1 ÷ 6 = 3, *the number of vibrations of the first, which is performed in 36 seconds* = 2 *vibrations of the second. Therefore, after the first has vibrated 3 times, and the second 2, they will move on together again.*

In the next place, we must examine into the situation of the third pendulum, at the conjunction of the two first. In 36 seconds, there is 1.5 vibration of the third pendulum, which is therefore, .5 of a vibration, distant from the conjunction of the other two ; wherefore, .5 : 1 :: 3 : 6, the number of vibrations of the first, at which time, they all come into a line of conjunction, and move on together. Consequently, when the first has made 6 vibrations, the second will have performed 4, and the third three = 24 × 3 = 72 *seconds, the time required.*

If A can do a piece of work in 20 days ; A and B in 13 days ; A and C in 11 days ; and B and C in 10 days : How many days will it take each person to perform the same work ?

OPERATION.

As 20—13 : 13 :: 20 : 37¼ *the time that B would do it.*

As 20—11 : 11 :: 20 : 24⁴⁄₉ *the time that C would do it.*

CHAP. III.

SIMPLE INTEREST.

SIMPLE intereſt is a premium of a certain ſum paid for the loan of money borrowed for a particular term of time, at any rate per cent or hundred, as the borrower and lender ſhall agree.

Thus, if 100 dollars be lent at 6 per cent per annum, the premium for 1 year will be 6 dollars, for 2 years 12 dollars, for 3 years 18 dollars ; and ſo on.

The ſum lent is called the principal, and the premium per 100, the Ratio or rate per cent ; and the amount is the principal and intereſt added together.

All the varieties of ſimple intereſt, are compriſed in the following caſes.

C A S E I.

When the ſum lent, is for any number of years, and the rate per cent, any number of dollars.

R U L E.

Multiply the principal with the number of years, and that product with the Ratio, and divide by 100 ; the quotient reſulting, will be the intereſt required.

EXAMPLES.

Required the intereſt of 700 dollars, for 4 years, at 6 per cent per annum ?

OPERATION.

OPERATION.

$$700$$
$$4$$

$$2800$$
$$6$$

$$1(00)168)00$$

Anfwer. 168 *dollars, the intereſt required.*

Required the intereſt of 3520 dollars, for 7 years, at 6 per cent per annum.

OPERATION.

$$3520$$
$$7$$

$$24640$$
$$6$$

$$1(00)1478(40 = 1478 \; dol. \; 40 \, cts. \; the$$
$$[anſwer.$$

What is the intereſt of 57821 dollars, for 5 years, at 5 per cent per annum ?   *Anſ.* 2891 *dol.* 5 *cts.*

What is the intereſt of 5972 dollars, for 12 years, at 3 per cent per annum ?   *Anſ.* 716 *dol.* 64 *cts.*

CASE II.

When the ſum is lent for years and months ; the Ra-tio the ſame as before.

RULE.

REDUCE the number of months into the decimal of a year, then multiply the principal with the time,
                                                    and

and that product with the Ratio, then divide by 100 and you will have the intereſt required.

Or,

MULTIPLY the principal with the number of years, and take parts of the principal for the reſt part of the time, and add them to the reſt ; then proceed as before directed.

Required the intereſt of 735 dollars, for 5 years, 4 months, at 5 per cent per annum.

OPERATION.

4 months=⅓ of a year, 3)735

$$\begin{array}{r} 5 \\ \hline 3675 \\ 245=735\div3 \\ \hline 3920 \\ 5 \; ratio. \\ \hline \end{array}$$

1(00)196)00=196 *dollars, the* [*intereſt required.*

Required the intereſt of 52374 dollars, for 7 years 8 months, at 6 per cent per annum.

OPERATION.

OPERATION.

8 months $=\frac{2}{3}$ *of a year,* 3)52374
$\qquad\qquad\qquad$ 7

$\qquad\qquad$ 366618
$\qquad\qquad$ 17458 $=\frac{1}{3}$ *of* 52374
$\qquad\qquad$ 17458

$\qquad\qquad$ 401534
$\qquad\qquad\qquad$ 6 $=ratio.$
$\qquad\qquad\qquad\qquad$ *dol. cts.*
1(00)24092)04 $=$ 24092 4, *the in-*
$\qquad\qquad\qquad\qquad\qquad$ [*terest required.*

What is the interest of 32104 dollars, for 4 years, 3. months, at 5 per cent per annum ?
Ans. 6827 *dol.* 10 *cts.*

C A S E   III.

When the Ratio is dollars and parts of a dollar, the rest the same as before.

R U L E.

1. REDUCE the number of months into the decimal of a year, and multiply the principal with the whole time.

2. REDUCE the fractional parts of the Ratio into the decimal of a dollar.

3. MULTIPLY the former result with the latter, and divide by 100, and you will have the interest required : $\qquad$ Or,

MULTIPLY the principal with the number of years, and take parts of the principal for the rest part of the time, and add them to the former product ; then
$\qquad\qquad\qquad\qquad\qquad\qquad$ multiply

multiply this product with the dollar's part of the rate, and take parts of the multiplicand for the reft part of the rate, and add them to the latter product; then divide them by 100, and you will have the intereft required.

EXAMPLES.

Required the intereft of 700 dollars, for 3 years 6 months, at $6\frac{1}{2}$ per cent per annum.

OPERATION.

$$
\begin{array}{r}
700 \\
3.5 = time. \\
\hline
3500 \\
2100 \\
\hline
2450.0 \\
6.5 = ratio. \\
\hline
122500 \\
147000 \\
\hline
\end{array}
$$

          dol. cts.

1(00)159 25.00 = 159 25, *the anfwer.*

Or. 6 *months* = $\frac{1}{2}$ *a year* 2)700

$$
\begin{array}{r}
3 \\
\hline
2100 \\
350 = 700 \div 2 \\
\hline
\end{array}
$$

for the $\frac{1}{2}$ *per cent* 2)2450

$$
\begin{array}{r}
6 \\
\hline
14700 \\
1225 \\
\hline
\end{array}
$$

         dol. cts.

1(00)159(25 = 159 25 *as be-*

fore.   Required

Required the interest of 3520 dollars 17 cents, for 2 years 6 months, at $5\frac{1}{4}$ per cent.

OPERATION.

2)3520.17
            2
――――――――
        704034
        176003.5=3520.17÷2

4)8800.375
            5
――――――――
        44001875
        2200.093
――――――――――――― dol. cts.
1(00)462(01.968=462 1. 968 *the*
                                    [anf.

CASE IV.

When the sum is lent for any number of weeks.

RULE.

REDUCE the number of weeks into the decimal of a year, and proceed as in the last case.

Or,

FIND the interest of the given sum, according to the foregoing rules for one year; then say, as 52, the number of weeks in a year, is to the interest thus found; so is the given number of weeks, to the interest required.

EXAMPLES.

Required the interest of 720 dollars, for 10 weeks, at $5\frac{1}{2}$ per cent per annum.

OPERATION.

OPERATION.

```
     720
     .192 = time nearly.
    ─────────
     1440
     6480
     720
    ─────────
     138.240
       5.5 = ratio.
    ─────────
     691200
     691200
    ───────── dol. cts.
   7.60.3200 = 7  60.32 the answer.
```

```
                 w.  dol. cts.  w.
   Or, 720   As 52 : 39 60 :: 10
        .5.5              10
      ─────           ─────────
       3600       52 )396.00(7.61 = 7 dol. 61 cts.
       3600        )364
                   ─────
 1)00 )39)60.0       320
                     312
                    ─────
                       80
                       52
                    ─────
                       28
```

Note. *The reason why the two methods of operation a-
bove, do not bring out the same answer, is because the
decimal of 10 weeks can never be exactly found; yet the errour arising from any such computation,
will be inconsiderable.*

Required

dol. cts.

Required the interest of 527 2, for 13 weeks, at
$\frac{1}{2}$ per cent per annum.

OPERATION.

$$
\begin{array}{r}
527.2 \\
.25 = time. \\
\hline
26360 \\
10544 \\
\hline
1318.00 \\
5.5 \\
\hline
659000 \\
659000 \\
\hline
\end{array}
$$

1(00)72(49.000 = 72 *dol.* 46 *cts. the anf.*

CASE V.

When the fum is lent for any number of days.

RULE.

REDUCE the days into the decimal of a year, and
proceed as in the laft cafe.

Or,

1. MULTIPLY the given fum with the number of
days, and that product with the Ratio for a dividend.
2. MULTIPLY 365, the number of days in a year,
with 100 for a divifor; then divide, and the quotient
will be the intereft required.

Or,

As 365 days, is to the intereft of the principal for
one year; fo is the time propofed, to the intereft re-
quired.   Y   *EXAMPLES.*

EXAMPLES.

Required the interest of 300 dollars, for 219 days at 6 per cent per annum.

OPERATION.

Or, 219   365

```
                                  Or, 219      365
              300                     300      100
               .6=time,              ─────     ─────
              ─────                   65700     36500
              180.0                      6
                6=ratio,            ───────  ──dol. cts.
              ─────       d. 365(00)3942(00)1080 as before
   1(00)10)80.0=10 80ct. anf. 365
                                    ─────
                                    292.0
                                    2920
                                    ─────
                                      0
```

Required the interest of 1000 dollars, for 35 days, at 6 per cent per annum.

OPERATION.

```
                   d.   dol.  d.
   1000     As 365 : 60 :: 35
      6                    60
   ─────                   ──
   60.00             365)2100 (5 dol. 75 cts.
                        )1825
                        ─────
                        275.0
                        2555
                        ─────
                        1950
                        1825
          dol. cts.     ─────
   Anf. 5 75 123/365     125
```

CASE

CASE VI.

When the principal, Ratio, and interest are given to find the time.

RULE.

1. FIND the interest of the principal for one year, at the given rate.

2. SAY as the interest thus found, is to one year; so is the given interest, to the time required.

EXAMPLES.

Required the time in which 500 dollars will gain 150 dollars, at 6 per cent per annum.

OPERATION.

```
              dol.  y.   dol.
          As 30 : 1 :: 150
  500                    1
    6                 ───
  ───
30.00            30)150(5 = 5 years the time
                    150          [required.
```

Find in what time 700 dollars will gain 159 dol. 25 cts. at 6½ per cent per annum.

OPERATION.

```
              dol. cts. y.  dol. cts.
  700      As 45  50 : 1 :: 159 25
    6.5=time.                 1
  ───                       ───
 3500                                  y.
 4200           45.50)15925(3.5 = 3.5
 ───                  13650       [the an.
45.500
                      2275.0
                      22750
                             Req.
```

—————— dol. cts.

Required the time in which 283 33⅓ will amount to 370 dol. 50 cts. at 6 per cent per annum.

OPERATION.

dol. cts.	dol. y. dol. cts.
283 33⅓	As 17 1 87 16⅔
6	1
—————	—————
17.0000	87 16 ⅔
	3

17×3=51)261.50(5.12 years, the
        255   [answer.
     ————
      65
      51
     ————
    140
    102
    ————
     38

CASE VII.

When the Ratio, time, and amount are given to find the principal.

RULE.

As the amount of 100 dollars, at the rate per cent and time given, is to 100 dollars; so is the given amount, to the principal required.

EXAMPLES.

Required the principal that will amount to 3766 dol. 40 cts. in 7 years, at 6 per cent per annum.

OPERATION.

OPERATION.

$$100$$
$$6$$
$$\overline{}$$
$$6.00$$
$$7$$
$$\overline{}$$
$$42.00$$

      dol.   *dol. cts.*   *dol. cts.*
As 100+42=142 : 100 :: 3766 40 : 2793 23 $\frac{134}{142}$
the answer.

Required the principal that will amount to 868 dollars in 4 years, at 6 per cent per annum.

OPERATION.

  100   100+24=124 : 100 :: 868
   4   100
  ───   ───
  400   124)86800(700
   6   868
  ───   ───
24.00   000

    Therefore 700 *dol. is the principal required.*

Required the principal that will amount to 270 dollars, in 2 years at 6 per cent per annum.

OPERATION.

OPERATION.

| 100 | *As* 100+12=112 : 100 :: 270 |
| 2 | 100 |

```
  200          112)27000(241.07
    6          224
 ─────         ────
12.00           460
                448
                ────
                120
                112
                ────
                8.00
                7 8 4
                ─────
                16
```

Therefore, 241 *dol.* 7$\frac{16}{112}$ *cts. is the principal re-quired.*

Admit I have a legacy of 196 dol. 66$\frac{2}{3}$ cts. to pay, but is not due till the end of 3 years, and the lega-tee being in want of money, desires I would lend him some : What sum must he have to amount to his legacy in 3 years, at 6 per cent per annum ?

Answer. 166 *dol.* 66$\frac{2}{3}$ *cts.*

CASE VIII.

When the principal, amount, and time are given to find the Ratio.

RULE.

1. SUBTRACT the principal from the amount, and the remainder is the interest.

2.

2. SAY as the given principal, is to its interest; so is 100 dollars, to the interest of 100 dollars for the given time.

3. DIVIDE the interest of 100 dollars thus found by the given time, and the quotient will be the ratio required.

EXAMPLES.

Required the rate per cent per annum such, that 1240 dollars may amount to 1400 in 3 years.

OPERATION.

```
1400        1240 : 200 :: 100
1200                  100
────              ──────────
 200        124)0)2000(0(16.12
                   124
                  ─────
                   760
                   744
                  ─────
                    16.0
                    124
                   ────
                    360
                    248
```

dol. cts. 112

Then, 16,12 ÷ 3 = 5.37 *the Ratio required.*

Required

Required the rate per cent per annum, that 100 dollars in 7 years will amount to 135 dollars.

OPERATION.

135
100
_____
35=*interest*,

As 100 : 35 :: 100
         100
         _____

1)00(00(35
      7(35 *dol.*
       5=*ratio req.*

At what Ratio will 3333 dollars 33⅓ cents amount to 4000 dollars in 20 years.   *Answer.* 6 *dol.*

CASE IX.

COMMISSION *or* PROVISION.

THIS is a premium allowed to factors for buying or selling goods, wares, or merchandize, at so much per cent, without any regard to time ; which rate is governed according to the customs of particular places.

THE method of proceeding, is the same as in case III, except no regard is had to time.

EXAMPLES.

If I buy goods for my correspondent in Philadelphia, to the value of 4000 dollars : What may I demand for my commission, at 4½ per cent ?

OPERATION.

OPERATION.

4000
   4.5=*ratio*,

20000
16000

1(00)180(00.0=180 *dol. the answer.*

Required the commiſſion for ſelling 5720 dollars worth of goods, at $2\frac{1}{2}$ per cenr.

OPERATION.

5720
   2.5=*ratio*,

28600
11440

143.000=143 *dol. the anſ.*

My correſpondent ſends me word, that he has dif-burſed goods on my account, to the value of 13333 dollars $33\frac{1}{3}$ cents : What is his commiſſion at $2\frac{1}{2}$ per cent ?   *Anſwer.* 333 *dol.* $33\frac{1}{3}$ *cts.*

CASE X.

BROKERAGE.

BROKERAGE is an allowance of ſo much per cent, made to perſons called brokers, for finding cuſtom-ers, and ſelling to them goods, wares, &c. which be-long to other men.

Z

R U L E.

Find the interest of the given sum, at one per cent ; or which is the same thing ; divide the given sum by 100, and take parts of the quotient, agreeing with the rate per cent. ———

Or,

Reduce the rate per cent to a decimal, and multiply it with the given sum ; then divide by 100, and the quotient will be the answer.

EXAMPLES.

Required the Brokerage of 1000 dollars, at 25 cents per cent.

OPERATION.

1(00)10(00

25 cents = ¼ *of a dollar, therefore,* 10 ÷ 4 = 2 *dollars* 50 cents = *Brokerage of* 1000 *dollars divided by* 4 = *Brokerage required.* ———

Or,
1000
.25 = *ratio,*

———

5000
2000

———

2.5000 = 2 *dol.* 50 *cts. as be-*
[*fore.*

Required the Brokerage of 324 dollars 40 cents, at ⅐ of a dollar per cent.

OPERATION.

OPERATION.

324.40
.20=$\frac{1}{5}$ of a dollar,

1(00)64.8800=64.88 *cts. the Brokerage required.*

What is the Brokerage of 15600 dollars, at 77 cents per cent ? *Anf.* 120 *dol.* 12 *cts.*

CHAP. IV.

COMPOUND INTEREST.

COMPOUND Intereft arifes from the computation of the intereft of any principal added to its intereft, when the payment fhould be made ; which forms a new principal at every time when the payments become due ; and is for this reafon, fometimes called intereft upon intereft.

THUS, if 100 dollars be put to intereft at 6 dollars per cent per annum ; at the end of the firft year, the intereft will be 6 dollars as in fimple intereft, which if added to its principal will be 106 dollars, for a new principal the fecond year, which principal at the end of the fecond year, will amount to 112 dollars 36 cents ; which is 36 cents more than if 100 dollars had been put out at fimple intereft only.

THE Compound Intereft of any fum may be found by the following

RULE.

1. FIND the intereft of the propofed fum for the firft year at the given rate per cent, as in fimple intereft. 2.

2. ADD this interest to its principal, which amount makes the principal for the second year.

3. FIND the interest of the second year's principal, in the same manner as you did the first, and add it to its principal, for the third year's principal, which must be computed as before; and so on, for the time required.

4. SUBTRACT the given principal from the last amount, and the remainder will be the Compound Interest required.   Or,

FIND the amount of one dollar for one year, at the given rate per cent, and multiply it continually with the principal, as many times as the given number of years, and the resulting product will be the amount; from which subtract the principal, and the remainder will be the Compound Interest.

EXAMPLES.

Required the Compound Interest of 100 dollars, for 3 years, at 6 per cent per annum.

OPERATION.

100	100	106	112.36
6	6	6.36	6.7416
6.00	106	112.36	119.1016
	6	6	
	6.36	6.7416	

Then, 119 *dol.* 10.16 *cts.*—100 *dol.* = 19 *dol.* 10.16 *cts.* = *interest required.*

Or,

Or,

As 100 : 106 :: 1 : 1.06 $\left\{\begin{array}{l} 100 \times 1.06 \times 1.06 \times \\ 1.06 = 119.1016 = 119 \\ dol. \ 10.16 \ cts. \end{array}\right.$ *the amount of 1 dollar for 1 year, at 6 per cent.*

Then, 119 *dol.* 10.16 *cts.* — 100 *dol.* = 19 *dol.* 10.16 *cts. the same as before.*

THE following is a Table of the amount of 1 dollar, from 1 to 30 years; for the more ready computing Compound Interest at 6 per cent per annum.

Years.	The amount of 1 dol. at 6 per cent, &c. comp. interest.	Years.	The amount of 1 dol. at 6 per cent, &c. comp. interest.	Years.	The amount of 1 dol. at 6 per cent, &c. comp. interest.
1	1.06	11	1.898298558	21	3.399563600
2	1.1236	12	2.012196471	22	3.603537416
3	1.191016	13	2.132928260	23	3.819749661
4	1.26247696	14	2.260903955	24	4.048934641
5	1.338225577	15	2.396558193	25	4.291870719
6	1.418519112	16	2.540351684	26	4.549382962
7	1.503630259	17	2.692772785	27	4.822345940
8	1.593848074	18	2.854339152	28	5.111686697
9	1.689478959	19	3.025599502	29	5.418387899
10	1.790847696	20	3.207135472	30	5.743491729

BY the above table, the amount of any sum may be computed from 1 to 30 years, by only multiplying the principal with the numbers standing against the number of years in the table, and the product will be the amount required.

EXAMPLES.

Required the amount of 127 dollars, for 7 years, at 6 per cent per annum.

OPERATION.

OPERATION.

Against 7 *in the table is* 1.503630 *&c.*

127

10525410
3007260
1503630
           dol. cts.
190.961010 = 190 96.1 *&c.*

the answer.

Required the Compound Interest of 555 dollars, for 30 years, at 6 per cent per annum.

OPERATION.

Against 30 *in the table is* 5.743491 *&c.*

.555

28717455
28717455
28717455

3187.637505 = *amount.*

Then, 3187 *dol.* 63.7505 *cts.* —555 *dol.* = 2632 *dol.* 63.7505*cts. the interest required.*

C H A P.

CHAP. V.

REBATE or DISCOUNT.

REBATE or Difcount is when any fum of money is due at a certain time to come, and the debtor is ready to make prefent payment, provided he can have allowance made him at a certain rate per cent per annum, which allowance is called the Rebate or Difcount, and the prefent payment, a fum of money, which if put to interest, would amount to the given fum, at the rate per cent and time given.

THE Rebate of any fum is found by the following

RULE.

As the amount of 100 dollars, at the rate per cent and time given, is to the interest of 100 dollars, at the fame rate and time ; fo is the given fum, to the Rebate : And from the given fum fubtract the Rebate, and the remainder will be the prefent payment.

EXAMPLES.

A, hath 100 dollars due to him, to be paid at the end of 2 years ; but his debtor agrees to make prefent payment, provided A will make a Rebate at 6 dollars per cent per annum : Required the Rebate.

OPERATION.

OPERATION.

First, 100
       6 = *ratio,*
       ―――――
       600
         2 = *time,*
       ―――――
    12.00
       Then, as 100 + 12 = 112 12::100
                                12
                      ―――――
              112)1200(10.71 =
              112 [*rebate*
              ―――――
                80.0
                784
              ―――――
                 160
                 112
              ―――――
                  48

Required the Rebate of 720 dollars, for $2\frac{1}{2}$ years, at 6 per cent per annum.

OPERATION.

OPERATION.

Then, as 100 + 15 = 115 : 15 :: 720

```
100                           15
  6                         ────
────                        3600
2)600                        720
  2                         ────
────            115)10800(93.91 = re-
1200                  1035     [bate req.
 300                  ────
────                   450
15.00                  345
                      ────
                      105.0
                      1035
                      ────
                       150
                       115
                      ────
                        35
```

Find what fum ought to be paid down for a debt
of 1000 dollars due 3¼ years hence, difcounting at 5
per cent per annum.

OPERATION.

```
      100
        5
      ────
    2)500
        3
      ────
    1500
     250
      ────
    17.50
```

A a   *As*

As 100+17.50=117.50 : 17.50 :: 1000
                                 1000

117.50)1750000(148.93= *reb*.
      11750

      57500
      47000

    105000
     94000

   11000.0
   105750

     42500
     35250

      7250

And, 1000—148.93=851 *dol.* 7 *cts.* the answer.

Suppose I have a legacy due to me of 4000 dollars, whereof 800 dollars is to be paid in 8 months, and the rest at the end of 16 months : How much ought I to receive for present payment, allowing 6 per cent, &c. discount?   *Answer.* 3732 *dol.* 21 *cts.*

A owes B 15000 dollars, one half of which is to be paid in 4 months, and the rest at the end of 8 months : What ought B to receive in present payment, allowing 6 per cent discount?   *Anf.* 14564 *dol.* 58 *cts.*

CHAP. VI.

EQUATION of PAYMENTS
The COMMON WAY.

EQUATION of payments, is when several sums of money are due at different times, to find a certain time when the whole may be paid without loss to either party.

RULE.

MULTIPLY each payment with its respective time, and divide the sum of the products by the sum of the payments; the quotient resulting, will be the time required.

THIS rule will give the equated time near enough for common practice in matters of this nature; but not accurately true, because the rule is founded on a supposition that the sum of the interests of the debts due before the equated time, computed from the times they become due to that time, is equal to the sum of the interest of the debts, payable after the equated time, computed from that time to their respective terms of payment; that is, the gain made by the debtor's keeping those debts which become due before the equated time, until that time, is equal to the loss sustained by paying those debts at the equated time, which are not due till afterwards; but it is manifest, that the gain made by keeping a debt any time after it is due, is equal to the interest of that debt for that time; but the loss sustained by paying a debt any time before it becomes due, is plainly no more than the rebate of the debt for that time; and since the rebate is always less than the interest of the same

sum,

fum, it follows that the fuppofition is not true, and confequently the rule falfe.

EXAMPLES in equation of payments the common way.

A owes B 100 dollars, whereof 50 dollars is to be paid at the end of 4 months and the reft at the end of 8 months : Required the time when the whole may be paid without lofs to either party.

OPERATION.

Firft, 50×4=200 *the firft payment with its time :*
Secondly, 50×8=400 *the fecond payment with its time :*
Then, 400+200=600 *the fum of the products.*
And, 50+50=100 *the fum of the payments :*
Confequently, 600÷100=6 *months, the time required.*

W owes X 865 dollars, whereof 50 dollars is to be paid prefent, 195 dollars to be paid in 8 months, and the reft at the end of 12 months : Required the equated time to pay the whole.

OPERATION.

50×1=50 *the firft payment with its time :*
195×8=1560 *the fecond payment with its time :*
620×12=7440 *the laft payment with its time :*
And 50+1560+7440=9050 *the fum of the products.*

Confequently, $\frac{9050}{865}$=10 *months* 13 $\frac{755}{865}$ *days the time required.*

P owes a debt to be paid at 5 feveral payments, in the following manner, to wit, $\frac{1}{5}$ in 4 months, $\frac{1}{5}$ at 8 months, $\frac{1}{5}$ at 12 months, $\frac{1}{5}$ at 16 months, and $\frac{1}{5}$ at

20 months : Required the equated time to pay the whole.

OPERATION.

Suppose the debt $=25$ *dollars, one fifth of which is* 5 *dollars, then* $5 \times 4 + 5 \times 8 + 5 \times 12 + 5 \times 16 + 5 \times 20 = 20 + 40 + 60 + 80 + 100 = 300.$ *Therefore,* $\frac{300}{25} = 12$ *months, the time required.*

In the folution of the above queftion, we made ufe of 25 dollars to reprefent the whole debt ; but any other number would have equally fucceeded, as may be thus analytically demonftrated.

Let $x =$ *any fum whatever, to be paid in manner as above.*

Then, $\frac{1}{5} x$ *is* $\frac{x}{5}$, *and* $\frac{x}{5} \times 4 + \frac{x}{5} \times 8 + \frac{x}{5} \times 12 + \frac{x}{5} \times 16 + \frac{x}{5} \times 20 = \frac{4x}{5} + \frac{8x}{5} + \frac{12x}{5} + \frac{16x}{5} + \frac{20x}{5} = \frac{60x}{5} =$ *fum of the products of the feveral payments with their respective times : Therefore,* $\frac{60x}{5} \div x = \frac{60x}{5x} = \frac{12x}{x} = 12$ *months the fame as before.*

$\mathcal{Q}. \ E. \ D.$

CHAP. VII.

BARTER.

BARTER is the exchanging one commodity for another in fuch a manner, that the parties bartering, may neither of them fuftain lofs. Thus, fuppofe A hath 50lb. of ginger, at 30 cents per lb. and would Barter with B for pepper at 70 cents per lb.

lb. What quantity of pepper muſt B give A for his 50 lb. of ginger ?

In the ſolution of this queſtion, and all others of the like nature, you muſt firſt find the value of the given quantity at the given price, and then find how much of the quantity ſought at its price, will amount to the value of the given quantity, and the reſult will be the anſwer to the queſtion.

Thus, in the above queſtion, the quantity given, is 50lb. of ginger, at 30 cents per lb. and the quantity ſought is pepper, at 70 cents per lb. Therefore, as 70 cts. : 1lb. :: 15 dol. (the price of the ginger) : 21¾ lb. the quantity of pepper required. Conſequently in Barter, the method of operation is the ſame as in the rule of three direct.

EXAMPLES.

Required the quantity of flax, at 8 cents per lb. that muſt be given in Barter, for 12 lb. of indigo, at 2 dol. 50 cts. per lb.

OPERATION.

  lb. dol. cts. lb. dol.
Firſt, as 1 : 2 50 :: 12 : 30, *the value of the indigo.*

  cts. lb. dol. lb.
Then, 8 : 1 :: 30 : 375, *the anſwer.*

A hath rum at 70 cents per gallon ready money, but in Barter he muſt have 80 cents ; B hath raiſins at 12 cents per lb. ready money : How many lb. of raiſins muſt A have for 60 gallons of rum.

Here you muſt firſt find what B's raiſins ought to be per lb. in Barter, which muſt be as much more in proportion, as A's price in ready money, is to his price in Barter ;

Barter; which to obtain, say as 70 cts. : 80 cts. :: 12 cts. : 13.71 cts. ⸗ *price of B's raisins per lb. in Barter; then proceed as before directed, and the quantity of raisins that B must give A will be found* ⸗ *350.11lb.*

How much wheat at $91\frac{2}{3}$ cts. per bushel, must be given for 8 cwt. of sugar at $8\frac{1}{3}$ cts. per lb.?

Answer. $81\frac{5}{11}$ *bushels.*

A hath rum at 70 cents per gallon ready money, but in Barter he must have 84 cents ; B hath corn at 50 cents per bushel ready money : How much must B have per bushel in Barter for his corn ; also, how many bushel of corn B must give A for a hogshead of rum containing 120 gallons ?

Answer. B must have $57\frac{1}{7}$ *cts. per bushel in Barter, and must give A* 168 *bushel of corn for the* 120 *gallons of rum.*

D hath 12 cwt. of sugar, which he will sell to H for 8 dollars $33\frac{1}{3}$ cents per cwt. ready money, but in Barter he must have $8\frac{1}{3}$ cents per lb. H hath a horse which he would sell for 90 dollars ready money, but in Barter he must have 20 per cent advance : They Barter, D takes the horse, and H the sugar : Query which is in debt, and how much ?

Answer. H is in debt 3 *dol.* $37\frac{1}{7}$ *cts. ready money.*

CHAP. VIII.

LOSS and GAIN.

LOSS and gain is a rule by which merchants are instructed how to raise or fall in the prices of their goods, so as to gain or loose so much per lb. bag, or barrel, &c.

The

THE operations are performed by the rule of three direct.

EXAMPLES.

Suppofe I buy cheefe at 6 dollars per 100lb. and fell it again at 8 cents per lb. What do I gain in buying and felling 600lb. ?

Here you muft firft find what 600lb. comes to, at 6 dollars per 100lb. and 600lb. at 8 cents per lb. then fubtract one fum from the other, and the refult will be the anfwer.

OPERATION.

Firft, 6×6=36 *dol. the value of* 600*lb. at* 6 *dol. per* 100*lb.*

Then, 600×8 *cts.*= 48 *dol. the price of* 600*lb. at* 8 *cts. per lb.*

And, 48—36=12 *dol. the anfwer.*

When butter coft 7 dollars per firkin of 56lb. To find how it muft be fold per lb. to gain 25 per cent.

OPERATION.

As 56*lb.* : 7 *dol.* :: 1*lb.* : 12½*cts. the price that the butter coft per lb.*

As 100*lb.* : 12½ *cts.* :: 100+25=125 : 15.625 *cts. the anfwer.*

When tea coft 75 cts. per lb. To find how it muft be fold per lb. to gain 25 per cent.

OPERATION.

As 100 : 75 :: 100+25=125 : 93.75 *cts. the anfwer.*

At 12½ *cts. profit in a dollar : How much per cent ?*
As 1 *dol.* : 12.5 *cts.* :: 100 *dol.* : 12½ *per cent the*
answer.

Bought rum at 50 cents per gallon, and paid im-
post, at 8 cents per gallon, and afterwards sold it at
53 cents per gallon : What do I loose in laying out
600 dollars. *Answer.* 86 *dol.* 21 *cts.*

If I buy tallow at 12½ cents per lb. and give 2⅞
cents per lb. to a chandler to make it into candles,
and 14oz. of tallow make a dozen of candles, which
I sell at 19$\frac{52}{72}$ cents per dozen : What do I gain in
buying and selling 180lb. of tallow.

Answer. 12 *dol.* 50 *cts.*

C H A P. IX.

FELLOWSHIP.

FELLOWSHIP is a rule, when several per-
sons as merchants, &c. trade in company with
a joint stock, to ascertain each man's proportional
part of the gain or loss, which arises from the em-
ployment of the joint stock, according to the quan-
tity of goods, sum of money, &c. each man puts in-
to the said stock ; which admits of a two-fold con-
sideration.

S E C T. I.

FELLOWSHIP SINGLE.

SINGLE Fellowship is when all the several stocks
are employed in the common stock, an equal term
of time. Therefore, since the times of the several

stocks

ſtocks employed in the joint ſtock, are all equal ; it
follows, that each partner's ſhare of the gain or loſs,
is as his ſhare of that ſtock : Wherefore it is mani-
feſt ; if I put in ¼ of the whole ſtock, I ought to
have ¼ of the whole gain, or ſuffer ¼ of the whole
loſs : Hence we have the following

R U L E.

MULTIPLY each partner's part of the joint ſtock,
with the whole gain or loſs, and divide the ſeveral
products by the whole ſtock, and the quotients re-
ſulting will be the anſwer to the queſtion. Or, as the
whole ſtock is to the whole gain or loſs ; ſo is each
man's particular part of that ſtock, to his particular
part of the gain or loſs.

EXAMPLES.

Two partners, A and B, conſtitute a joint, ſtock of
300 dollars, whereof A put in 200 dollars, and B
100 dollars, and they trade and gain 150 dollars :
Required each man's part of the gain.

OPERATION.

150	150
200	100

3)00)300)00   3)00)150)00
   100= *A's gain.*   50= *B's gain.*

Or,

As 300 : 150 :: 200 : 150×200÷300=100 *dol.*
A's part of the gain.

As 300 : 150 :: 100 : 150×100÷ 300=50 *dol.* B's
part of the gain :

Or,

Or, 150÷300=.5 *the ratio of the first term to the second :*

Therefore, 200×.5=100 *A's part, and* 100×.5= 50 *B's part as before.* (Vid. Chap. 11.)

Three merchants, A, B, and C, make a joint stock of 2000 dollars, whereof A put in 500 dollars, B 800 dollars, and C 700 dollars; and by trading gain 400 dollars : Required each man's part of the gain?

OPERATION.

First, 400÷2000=.2 *the ratio of the first term to the second.*

$$Therefore, \begin{cases} 500×.2=100 \; dol. \; A's \\ 800×.2=160 ——B's \\ 700×.2=140 ——C's \end{cases} gain.$$

Four merchants enter into partnership, and constitute a joint stock of 60000 dollars, whereof A put in 15000 dollars 24 cents, B 20000 dollars 76 cents, C 21000 dollars, and D 3999 dollars, and in trade they gain 24000 dollars : Required each partner's share of the gain?

OPERATION.

First, 24000÷60000=.4 *the ratio of gain :* There-fore, 15000.24×.4=6000 *dol.* 9.6 *cts. A's part of the gain ; and* 20000.76×.4=8000 *dol.* 30.4 *cts. B's part of the gain ; also,* 21000×.4=8400 *dol. C's part ; lastly,* 3999×.4=1599 *del.* 60 *cts. D's part.*

Six farmers, A, B, C, D, E, and F, hired a farm for 300 dollars ; A paid 20 dollars, B 30, C 40, D 60, E 80, and F 70 dollars ; and they gained 60 dollars : What is each man's part of the gain?

Answer.

Answer. A's 4 dol. B's 6, C's 8, D's 12, E's 16, and F's 14 dol.

SECT. II.

COMPOUND FELLOWSHIP.

THE only difference between Fellowship single and compound, is, that in the latter regard must be had to the time each partner's stock continues in company ; whereas in single Fellowship the times of continuance are all supposed equal, and when the times are equal, the shares of gain or loss, are as their stocks, as we have before shewn : Therefore when the stocks are equal, the shares must be as the times. Consequently, when neither the stocks nor times are equal, the shares must be as their products ; which affords the following

RULE.

1. MULTIPLY each man's stock with the time it is employed, and find the sum of all the products.

2. As the sum of the products thus found, is to the whole gain or loss ; so is the product of each man's stock with its time, to its proportional part of the gain or loss.

Or,

FIND the ratio between the two first terms, and proceed as in the last rule.

EXAMPLES.

Two men, A and B, made a joint stock of 600 dollars, whereof A put in 200 dollars for 2 months, and B put in 400 dollars for 4 months ; at the expir-

ation

ation of which, they find they have loft 200 dollars; Required each man's part of the lofs?

OPERATION.

Firft, 200 × 2 = 400 = *A's ftock with its time:*
And, 400 × 4 = 1600 = *B's ftock with its time:*
Then, 400 + 1600 = 2000 *the fum of the products of each man's ftock, with its time: Therefore, as* 2000 :

200 :: 400 : 200 × 400 ÷ 2000 = 40 *dol. A's part of the*

lofs; and as 2000 : 200 :: 1600 : 200 × 1600 ÷ 2000
= 160 *dol. B's part of the lofs.*

Or, 200 ÷ 2000 = .1 *the ratio of lofs; then,*
400 × .1 = 40 *A's part, and* 1600 × .1 = 160 *B's part, the fame as before.*

Three merchants made a joint ftock of 8000 dollars in the following manner, viz. A put in 1200 dollars for 3 years, B 2000 dollars for 7 years, and C 4800 dollars for 8 years; and at the end thereof, they find they have gained 6720 dollars: Required each man's part of the gain?

OPERATION.

Firft, 1200 × 3 = 3600 = *A's ftock with its time:*
And, 2000 × 7 = 14000 = *B's ftock with its time:*
Alfo, 4800 × 8 = 38400 = *C's ftock with its time:*
Then, 3600 + 14000 + 38400 = 56000 *the fum of the products:*

And, 6720 ÷ 56000 = .12 *the ratio of gain:*
Therefore, 3600 × .12 = 432 *dol. A's part of the gain; and* 14000 × .12 = 1680 *dol. B's part; Alfo,* 38400 × .12 = 4608 *dol. C's part.*

Two

Two merchants, A and B, made a joint ſtock ; A put in at firſt, 300 dollars for 7 months, and 4 months after put in 500 dollars more : B put in at firſt, 700 dollars, and 3 months after put in 200 dollars more. Now at the end of 7 months, they make a ſettlement of their accounts, and find they have gained 1860 dollars : Required each man's part of the gain, according to his ſtock and time ?

Firſt, $300 \times 4 = 1200$ *the product of A's firſt ſtock with its time*, *and* $\overline{300 + 500} \times 3 = 800 \times 3 = 2400$ *the product of A's increaſed ſtock, with the remainder of the time* : *Therefore*, $1200 + 2400 = 3600$ *the product of A's ſtock with the whole time, according to the queſtion*.

Secondly, $700 \times 3 = 2100$ *the product of B's firſt ſtock with its time*, *and* $\overline{700 + 200} \times 4 = 900 \times 4 = 3600$ *the product of B's augmented ſtock, with the remainder of the time* : *Therefore*, $2100 + 3600 = 5700$ *the product of B's whole ſtock, with the whole time*, *and* $3600 + 5700 = 9300$ *the ſum of the products*.

Hence, $1860 \div 9300 = .2$ *the ratio of gain* : *Therefore*, $3600 \times .2 = 720$ *dol.* $= A's$ *part of the gain, and* $5700 \times .2 = 1140 = B's$ *part of the gain*.

Four merchants, A, B, C and D, enter into partnerſhip for 12 months : A put into the common ſtock at firſt, 300 dollars, B 400, C 500, and D 800 dollars, and at the end of four months, A took out 200 dollars, and 3 months after that, he put in 100 dollars more ; B at the end of 2 months took out 200 dollars, and 2 months after that, put in 200 dollars more : C at the end of 6 months, took out 300 dollars, and two months after that, put in 200 dollars more : D at the end of 8 months, took out 400 dollars, and 2 months after that, put in 200 dollars
more :

more : Now at the end of 12 months, they find they have gained 406 dollars : Required each man's part of the gain ?

OPERATION.

First, $300 \times 4 = 1200$ *the product of A's first stock with its time*, and $\overline{300 - 200} \times 3 = 100 \times 3 = 300$ *the product of A's remaining stock for 3 months after the taking out of the 200 dol. Again*, $\overline{100 + 100} \times 5 = 200 \times 5 = 1000$ *the product of A's stock with the remainder of the time according to the question ; then,*

$\overline{1200 + 300 + 1000} = 2500$ *the product of A's stock for the whole time.*

Secondly, to obtain the product of B's stock with its time, proceed as before : Thus $400 \times 2 = 800$; *then,*

$\overline{400 - 200} \times 2 = 200 \times 2 = 400$; *and* $\overline{200 + 200} \times 8 =$ $400 \times 8 = 3200$. *Hence,* $\overline{800 + 400 + 3200} = 4400$ *the product of B's stock with its time.*

Thirdly, $500 \times 6 = 3000$; *then* $\overline{500 - 300} \times 2 = 200$ $\times 2 = 400$; *and* $\overline{200 + 200} \times 4 = 400 \times 4 = 1600$; *wherefore* $3000 + 400 + 1600 = 5000$ *the product of C's stock with its time.*

Fourthly, $800 \times 8 = 6400$; *then* $\overline{800 - 400} \times 2 =$ $400 \times 2 = 800$; *and* $\overline{400 + 200} \times 2 = 600 \times 2 = 1200$; *therefore* $6400 + 800 + 1200 = 8400$ *the product of D's stock with its time.*

Consequently,

Confequently, $2500 + 4400 + 5000 + 8400 = 20300$ *the fum of all the products according to the queftion.*

Therefore, $406 \div 20300 = .02$ *the ratio of gain* ; *and* $2500 \times .02 = 50$ *dol. A's part of the gain* ; *alſo*, $4400 \times .02 = 88$ *dol. B's part* ; *likewiſe*, $5000 \times .02 = 100$ *dol. C's part* ; *laſtly*, $8400 \times .02 = 168$ *dol. D's part.*

CHAP. X.

COMPOUND PROPORTION.

COMPOUND Proportion, is uſed in the ſolution of queſtions that require ſeveral operations in ſimple proportion, whether direct or reciprocal.

For inſtance : Suppoſe a footman performs a journey of 240 miles in 8 days, when the days are 16 hours long : In what time would he perform a journey of 540 miles, when the days are but 12 hours long. This queſtion reſolved by ſimple proportion is thus,

$$\text{m.} \quad \text{d.} \quad \text{m.}$$
$$As\ 240 : 8 :: 540 : \frac{540 \times 8}{240} = 18\ days.$$

THAT is it would require 18 days to perform a journey of 540 miles, when the days are 18 hours long ; but it is required to know how many days it will take to perform the ſaid journey of 540 miles when the days are but 12 hours long ; which is thus :

$$As\ 16\,h. : \overline{540 \times 8 \div 240}\ (18d.) :: 12 : \overline{540 \times 8 \times 12 \div}$$
$$\overline{240 \times 12} = 24\ days,\ by\ inverſe\ proportion.$$

Now

Now from the laſt analogy, is deduced the fol-
lowing rule, for ſtating and working all queſtions in
compound proportion, at one operation.

R U L E.

1. PLACE that term which is of the ſame name of
the term ſought, ſo that it may ſtand in the middle
place :

$$\left\{ \begin{array}{l} * : 8 :: * \\ * :\text{———}:: * \end{array} \right. \begin{array}{l} d. \\ \text{See the aforeſaid queſtion.} \end{array}$$

Thus,

2. WRITE the remaining terms of ſuppoſition, one
above the other in the firſt places, and the terms of
demand in like manner in the third places, ſo that
the firſt and third terms in each row, may be of the
ſame name and denomination :

$$\text{Thus,} \left\{ \begin{array}{lll} m. & d. & m. \\ 240 : & 8 :: & 540 \\ h. & & h, \\ 16 : & \text{———} :: & 12 \end{array} \right.$$

3. HAVING thus ſtated your queſtion, find your
diviſor by comparing the terms in each row : Thus
if the firſt term gives the ſecond, does the third term
require more or leſs ? If more, diſtinguiſh the leſs
extreme with a point over it ; but if the third term
require leſs, point the greater extreme :

$$\text{Thus,} \left\{ \begin{array}{lll} .m. & d. & m. \\ 240 : & 8 :: & 540 \\ h. & & .h. \\ 16 : & \text{———} :: & 12 \end{array} \right.$$

4. MULTIPLY together the terms which are point-
ed for a diviſor, and the remaining terms for a divi-
dend, and the quotient reſulting will be the anſwer :

Thus, $\overline{540 \times 8 \times 16} \div \overline{240 \times 12} = 24$ days as before.

C c   *EXAMPLES.*

EXAMPLES.

If 12 buſhels of corn are ſufficient for a family of
9 perſons 12 months : How many buſhels will be
ſufficient for a family of 16 perſons, 20 months ?

OPERATION.

*Here buſhels are ſought ; therefore the queſtion ſtated
will ſtand*

$$\text{Thus,} \begin{cases} \quad .\text{per. b. per.} \\ 9 : 12 :: 16 \\ \quad . \text{ m.} \qquad \text{m.} \\ 12 : —— :: 20 \end{cases}$$

*Then ſay, if 9 perſons eat 12 buſhels in 12 months,
16 perſons will eat more ; therefore point the leſs ex-
treme, which is 9. Again, ſay, if 12 months require
12 buſhels for 9 perſons, 20 months will require more ;
therefore point the leſs extreme, which is 12.*

Therefore, $\overline{12 \times 20 \times 16} \div \overline{12 \times 9} = 3840 \div 108 = 35\frac{5}{9}$
buſhels, the quantity of corn required.

Note. *If the ſame quantity is found both in the diviſor
and dividend, it may be expunged from both :*

Thus in the above expreſſion, $\overline{12 \times 20 \times 16} \div \overline{12 \times 9}$, *the
12 may be ſtruck out of the diviſor and dividend : thus,*

$\overline{20 \times 16} \div 9 = 35\frac{5}{9}$ *the ſame as before.*

If 15 dollars be the hire of 8 men 5 days : What
time will 40 dollars hire 20 men ?

OEERATION.

OPERATION.

.dol. d. dol.
15 : 5 :: 40
m. m.
8 :——:: 20

Whence $\overline{8 \times 5 \times 40} \div \overline{15 \times 20} = 1600 \div 300 = 5\frac{1}{3}$ *days,*
the time required.

If 200 dollars in 2 years, gain 15 dollars : What
will 150 dollars gain in half a year ?

Thus, $\overline{15 \times 150 \times 26} \div \overline{200 \times 104} = 2\frac{69}{208}$ *dol. the*
answer.

If 1500 lb. of bread serve 400 men 14 days : How
many pounds of bread will serve 140 men 9 days ?

Thus, $\overline{1500 \times 140 \times 9} \div \overline{400 \times 14} = 337$ *lb.* 8oz. *the*
answer.

If 12 Clerks will write 72 sheets of paper in 3
days : How many Clerks will write 140 sheets in 8
days ?

Answer. $\overline{12 \times 3 \times 140} \div \overline{72 \times 8} = 8\frac{432}{576}$ *Clerks.*

If 5000 bricks are sufficient to make a wall 4 feet
high and 5 feet long : How many bricks of the
same size will make 7 feet of wall 2 feet high ?

Answer. 3500.

CHAP. XI.

CONJOINED PROPORTION.

CONJOINED Proportion is when, in a rank of numbers, the first term is compared with the second, and the second term being increased or diminished, is compared with the third, and so on ; from thence to determine the equality of any of the terms : Thus, if $3a = 4b$, and $8b = 12c$, then will $3a = 6c$; because, as $4b : 3a :: 8b : 6a = 12c$, or $3a = 6c$ as before. Again, if $24a = 32b$, $48b = 30c$, and $10c = 9d$, then will $24a = 20c = 18d$; because, as $32b : 24a :: 48b :$

$$\overline{24a \times 48b} \div 32b = 36a, \text{ that is, } 48b = 36a = 30c, \text{ and}$$

$24a = 20c$. Again as $30c : \overline{24a \times 48b} \div 32b :: 10c :$

$$\overline{24a \times 48b \times 10c} \div \overline{32b \times 30c} = 12a = 9d, \text{ or } 24a = 18d.$$
Hence from the foregoing analogy we have the following

RULE.

1. BEGIN with that term whose equality with any other term is required, which call A and write out all the terms up to the one B, by which the aforesaid term is to be compared.

2. MULTIPLY all the alternate numbers together, beginning with the first, for a dividend, and all the remaining ones together for a divisor.

3. Divide, and the quotient will be the answer.

EXAMPLES.

EXAMPLES.

If G in 48 days can produce a certain effect, which will require H 64 days to perform; H can produce an effect in 80 days, which will take L 50 days to perform: Which is the most profitable to hire, G or L, and what is the difference?

OPERATION.

First, 48, 64, 80, are the numbers written out according to the rule:

$\overline{48 \times 80 \div 64} = 60 = 50$ *days of L, that is, 60 days of G are equal to 50 days of L; and therefore it is the most profitable to hire L, to wit, in the proportion of 60 to 50, or as 6 to 5.*

If D in 24 days can do as much as E can in 32 days, E can do as much in 48 days, as F can in 30 days, and F can do as much in 10 days, as G can in 9 days: Which is the most profitable to hire, D, F, or G?

OPERATION.

First, find which is the most profitable to hire, D or F:

Thus, $\overline{24 \times 48 \div 32} = 36 = 30$ *days of F, that is, 36 days of D are equal to 30 days of F; and therefore F is more profitable to hire than D.*

Again, $\overline{24 \times 48 \times 10 \div 32 \times 30} = 12 = 9$ *days of G; that is, 12 days of D are equal to 9 days of G, and therefore G is more profitable to hire than D; and since F is more profitable to hire than D, and G more profitable than F; it follows, that G is the most profitable to hire of the three.*   CHAP.

CHAP. XII.

ALLEGATION.

BY Allegation we are taught how to mix quantities of different quality, fo that any quantity collectively taken, may be of a mean or middle quality ; that is, it fhews us the value of any part of a compofition, made of things all of a different quality.

WE fhall confider Allegation, under the two following general heads, viz. Allegation Medial, and Allegation Alternate.

SECT. I.

ALLEGATION MEDIAL.

THIS is when any number of things are given, and the price of each : To find the price of any quantity of a mixture compounded of the whole.

RULE.

1. MULTIPLY each quantity with its price, and find the fum of all the products.

2. DIVIDE the fum of the products by the fum of all the quantities, and the quotient refulting will be the mean price required.

EXAMPLES.

A man is minded to mix 20 bufhels of wheat, at 100 cents per bufhel, with 10 bufhels of rye, at 50 cents per bufhel : Required the price of a bufhel of this mixture.　　　　　　　　　　*OPERATION.*

OPERATION.

First, 20×100=2000 *cts.* = *price of all the wheat, and* 10×50=500 *cts.* = *price of the rye ; then* 2000+500=2500 *the sum of the products, and* 20+ 10=30 *the sum of the quantities : Therefore,* 2500÷ 30=83⅓ *cts. the price of a bushel, as was required.*

A man would mix 27 bushels of wheat, at 75 cents per bushel, with 40 bushels of rye, at 60 cents per bushel, and 24 bushels of oats, at 24 cents per bushel : Required the price of a bushel of this mixture.

OPERATION.

First, 27×75=1885 *cts.* = *price of the wheat, and* 40×60=2400 *cts.* = *price of the rye, also,* 24×24= 576 *cts.* = *the price of the oats ; then* 1885+2400+ 576=4861 *the sum of the products, and* 27+40+24 =91 *the sum of the quantities.*

Whence 4861 *cts.*÷91= *price of a bushel, as was required.*

A maltster would mix 70 gallons of one sort of beer, worth 12 cents per gallon, with 20 gallons of another sort, worth 24 cents per gallon, and 20 gallons of a third sort, worth 22 cents per gallon : How may this mixture be sold per gallon without gain or loss ?
Answer. 16 *cts.*

Required what a gallon of the following mixture is worth, viz. 60 gallons of malaga, at .5 dollars per gallon, 40 gallons at .7 dollars per gallon, and 12 gallons at .3 dollars per gallon.

Answer. .55 *dol.*

A Goldsmith melts 18 ℔. of gold bullion, of 12 carats fine, with 10 ℔. of 16 carats fine, and 20 ℔. of

10

10 carats fine : How many carats fine is a pound of this mixture. *Anſwer.* 12 *carats.*

> Note. *Goldſmiths ſuppoſe every quantity of gold to conſiſt of* 24 *parts, which they call carats ; but gold is generally mixed with ſome other metals, ſuch as copper, braſs, &c. which is called alloy, and the quality of the gold is eſtimated according to the quantity of alloy in it : Thus if* 20 *carats of pure gold, and* 4 *of alloy are mixed together, the gold is called* 20 *carats fine.*

S E C T.   II.

A L L E G A T I O N   A L T E R N A T E.

ALLEGATION Alternate conſiſts of 3 caſes.

C A S E   I.

When the prices of the ſeveral quantities to be mixed are given, to find what number of each ſort muſt be taken, to compoſe a mixture whoſe mean price ſhall be as given in the queſtion.

R U L E.

1. WRITE all the particular rates or prices directly under each other, and the mean price on the left hand.

Thus, mean price, 4 $\left\{\begin{array}{l} 1 \\ 6 \\ 2 \\ 5 \end{array}\right.$ particular prices.

2. COUPLE or connect the particular prices with lines, ſo that one or more of thoſe greater than the mean price, may be coupled with one or more of thoſe leſs.   Thus,

Thus, 4 $\left\{\begin{array}{l}1\\6\\2\\5\end{array}\right.$ Or thus, 4 $\left\{\begin{array}{l}1\\6\\2\\5\end{array}\right.$

3. WRITE the difference between the mean price and every particular price, directly against the one with which it is coupled.

Thus, 4 $\left\{\begin{array}{l}1\\6\\2\\5\end{array}\right\}\begin{array}{l}1\\2\\2\\3\end{array}$ Or thus, 4 $\left\{\begin{array}{l}1\\6\\2\\5\end{array}\right\}\begin{array}{l}1\\2\\2+1=3\\3+2=5\end{array}$

4. THE difference ſtanding againſt each particular price, is the quantity that muſt be taken of that kind ; and where two or more differences are found ſtanding againſt any one particular price, their ſum is the quantity.

A maltſter has the following ſorts of beer, viz. at 12 cents, 22 cents, and 24 cents per gallon : Required the quantity of each ſort that muſt be taken to make a compoſition worth 20 cents per gallon.

OPERATION.

20 $\left\{\begin{array}{l}12\\22\\24\end{array}\right.\begin{array}{l}2+4=6\\8\\8\end{array}$

Therefore, there muſt be taken 6 gallons at 12 cts. 8 gallons at 22 cts. and 8 gallons at 24 cts. which may be proved by Allegation Medial.

To find how much wheat at 100 cents per buſhel, rye at 75 cents, corn at 40 cents, and oats at 30 cents per buſhel, may be mixed together, ſo that the mixture may be ſold for 50 cents per buſhel, without gain or loſs.

D d   *OPERATION.*

OPERATION.

$$
50 \left\{ \begin{array}{l} 100 \\ 75 \\ 40 \\ 30 \end{array} \right.
\quad
\begin{array}{ll}
\textit{bush.} & \textit{cts.} \\
20 & at\ 100 \\
10 \text{——} 75 \\
25 \text{——} 40 \\
50 \text{——} 30
\end{array}
\left. \begin{array}{l} \\ \\ \\ \\ \end{array} \right\}
\textit{The answer.}
$$

A merchant has coffee worth 12, 15, 16, and 10 cents per lb. and would make a mixture worth 14 cents per lb. What quantity of each sort must be taken ?

OPERATION.

$$
14 \left\{ \begin{array}{l} 12 \\ 15 \\ 16 \\ 10 \end{array} \right.
\quad
\begin{array}{ll}
\textit{lb.} & \textit{cts.} \\
1 & at\ 12 \\
2 \text{——} 15 \\
4 \text{——} 16 \\
2 \text{——} 10
\end{array}
\quad Or, \quad
14 \left\{ \begin{array}{l} 16 \\ 15 \\ 12 \\ 10 \end{array} \right.
\quad
\begin{array}{ll}
\textit{lb.} & \textit{cts.} \\
2 & at\ 16 \\
4 \text{——} 15 \\
2 \text{——} 12 \\
1 \text{——} 10
\end{array}
$$

Proceeding in this manner, by varying the order of linking the particulars, you will discover five more answers to this question, in whole numbers.

How these kind of questions can admit of various answers, is easy to conceive ; for if any two of the particular prices make a balance by their increment and decrement, in respect of the mean price, then will any multiple or quotient of the same, make a balance also : Therefore all numbers which are in the same proportion, equally answer the question. Consequently, there are some questions which will admit of an infinite variety of answers : Hence it is that these questions are sometimes called indeterminate or unlimited problems ; yet by an analytical pro-

cefs

cefs, we can difcover all the poffible anfwers in whole numbers, when thofe anfwers are limited to finite terms. *(Vid. Book* ii, *Chap.* xxiii*.)*

CASE II.

When the quantity of one of the particulars is limit-ed or given, thence to proportion all the others in the compofition by it.

RULE.

1. OBTAIN the difference between the mean price and every particular price, as in the laft rule.

2. As the difference found againft the fimple whofe quantity is given, is to the quantity itfelf; fo is each difference, to its refpectiye quantity of the compofition.

EXAMPLES.

A farmer would mix 12 bufhels of wheat at 72 cents per bufhel, with rye at 48 cents, corn at 36 cents, and barley at 30 cents per bufhel, fo that the whole compofition may be fold for 38 cents per bufhel : Required the quantity of each fort that muft be taken.

OPERATION.

$$38 \begin{cases} 72 \\ 48 \\ 36 \\ 30 \end{cases} \quad \begin{matrix} 8 \\ 2 \\ 10 \\ 34 \end{matrix}$$

Whence, as 8 : 12 :: 2 : 3, *the quantity of rye, and, as* 8 : 12 :: 10 : 15, *the quantity of corn* ; *alfo, as* 8 : 12 :: 34 : 51, *the quantity of barley.*

To

To find how many gallons of frontenaic at 81 cents, claret at 60 cents, and port at 51 cents per gallon, muſt be mixed with 42 gallons of madeira at 90 cents per gallon, ſo that the whole compoſition may be ſold for 72 cents per gallon, without profit or loſs.

$$
Firſt,\ 72 \begin{cases} 90 \text{———} \\ 81 \text{———} \\ 60 \text{———} \\ 51 \text{———} \end{cases} \begin{matrix} 21 \\ 12 \\ 9 \\ 18 \end{matrix}
$$

Then, $42 \div 21 = 2$; therefore, $12 \times 2 = 24$, the quantity of the claret, and $9 \times 2 = 18$, the quantity of the frontinaic; alſo, $18 \times 2 = 36$, the quantity of port.

A tobacconiſt would mix 6 lb. of tobacco worth 6 cents per lb. with another ſort at 11 cents, and a third ſort at 12 cents: What quantity muſt be taken of each ſort, to make a mixture worth 10 cents per lb? *Anſwer. 8 lb. of each ſort.*

CASE III.

When the whole compoſition is equal to a given quantity; that is, when the ſum of all the quantities which make up the compoſition, collectively taken, amount to the given quantity: To find the ſeveral quantities themſelves.

RULE.

1. Link or couple the ſeveral particulars, and find their differences, as in the laſt caſe.

2. As the ſum of the differences, is to the ſum of the whole compoſition or given quantity; ſo is each difference, to its reſpective quantity of the compoſition.

EXAMPLES.

EXAMPLES.

A grocer having fugars at 4 cents, 8 cents, and 12 cents per lb. would make a compofition of 240 lb. worth 10 cents per lb. Required the quantity of each fort that muft be taken.

OPERATION.

$$\text{Firft, } 10 \left\{ \begin{array}{l} 4 \\ 8 \\ 12 \end{array} \right. \begin{array}{l} 2 \\ 2 \\ 6+2=8 \end{array}$$

$$12 = \text{fum of the differences.}$$

Then, as 12 : 240 :: 2 : 40, and, as 12:240 :: 2 : 40; alfo, as 12 : 240 :: 8 : 160. Therefore, there muft be taken, 40 lb. at 4 cts. 40 lb. at 8 cts. and 160 lb. at 12 cts.

A merchant would mix brandy of the following prices, viz. at 60 cents, 72 cents, and 84 cents per gallon, together with water at 0 cents per gallon, fo that a compofition of 846 gallons, may be fold for 48 cents per gallon, without gain or lofs : Required the quantity of each fort that muft be taken.

OPERATION.

$$\text{Firft, } 48 \left\{ \begin{array}{l} 72 \\ 60 \\ 84 \\ 0 \end{array} \right. \begin{array}{l} 48 \\ 48 \\ 48 \\ 24+12+36=72 \end{array}$$

$$216 = \text{fum of the differences.}$$

$$\text{Then, as } 216 : 846 :: \left\{ \begin{array}{l} 48 : 188 \text{ at } 72 \text{ cts.} \\ 48 : 188 \text{ at } 60 \text{ cts.} \\ 48 : 188 \text{ at } 84 \text{ cts.} \\ 72 : 282 \text{ of water.} \end{array} \right.$$

In

In this cafe might be ftarted, a variety of very curious queftions about the fpecific gravities of metals ; but as they would require the knowledge of fome things which are not treated of in this volume, we defift.

CHAP. XIII.

Of POSITION, or the GUESSING RULE.

POSITION is a method of folving queftions, by fuppofing numbers, and then adding them, fubtracting, multiplying, &c. according as the refult or number given in the queftion is produced by addition, fubtraction, multiplication, &c, of the number required.

POSITION is diftinguifhed into two kinds, fingle and double.

SECT. I.

Of SINGLE POSITION.

SINGLE Pofition is when one quantity is required, the propertics of which are given in the queftion.

RULE.

SUPPOSE a number for the quantity required, and multiply or divide it, &c. according as the quantity required was multiplied, divided, &c. then ; as the refult of the fuppofition, is to the fuppofition, fo is the refult given in the queftion, to the number required.

EXAMPLES.

EXAMPLES.

To find such a number, that being divided by 2, 4, and 8, respectively, the sum of the quotients shall be 7.

OPERATION.

Suppose the number to be 24, *then,* $\frac{24}{2} + \frac{24}{4} + \frac{24}{8} =$ 12+6+3=21.

Whence, 21 : 24 :: 7 : $\overline{24 \times 7} \div 21 =$ 8, *the number required.*

For, $\frac{8}{2} + \frac{8}{4} + \frac{8}{8} =$ 4+2+1=7; *therefore, &c.*

A man having a certain sum of money, said one half, one third, and one fourth of it being added together, made 13 dollars : What sum had he ?

Suppose he had 36 *dol. then* $\frac{36}{2} + \frac{36}{3} + \frac{36}{4} =$ 18+12 +9=39, *which ought to be* 13, *by the question.*

Therefore, 39 : 36 :: 13 : 12, *the answer.*

Three men found a purse of dollars, disputed how it should be divided between them. A said he would have one third ; B said he would have one third and one quarter ; well says C, I shall have but 2 dollars left for my part : How many dollars were there in the purse, and how many did each one take ?

Suppose the purse contained 12 *dollars :*
Then, $\frac{12}{3} + \frac{12}{3} + \frac{12}{4} =$ 4+4+3=11 :
And, 12—11=1, *which ought to be* 2.

Wherefore, 1 : 2 :: 12 : 24, *the number of dollars in the purse ; whence,* $\frac{24}{3} =$ 8, *the number of dollars that A took ; and* $\frac{24}{3} + \frac{24}{4} =$ 8+6=14, *the number that B took.*

Delivered to a banker, a certain sum of money, to receive interest for the same, at the annual rate of 6 dollars per cent ; at the end of 7 years, received

for intereſt and principal, 2495 dollars 27$\frac{7}{9}$ cents :
What was the ſum lent ?

Anſwer. 1736 *dol.* 11$\frac{1}{9}$ *cts.*

S E C T. II.

Of D O U B L E   P O S I T I O N.

DOUBLE Poſition is when there are ſeveral un-
known numbers in the queſtion, analogous to each
other ; ſo that when one or more are found, the reſt
may be had, either by addition, ſubtraction, or mul-
tiplication, &c. according as the queſtion requires.

R U L E.

1. ASSUME two convenient numbers, and work
with them as the queſtion directs, finding their re-
ſults.

2. FIND the difference between theſe reſults and
the reſult given in the queſtion, and call thoſe differ-
ences errors, which place under their reſpective ſup-
poſitions.   *Thus,* $\begin{cases} x, \ y, \textit{ ſuppoſitions.} \\ a, \ b, \textit{ errors.} \end{cases}$

3. MULTIPLY the firſt error with the ſecond ſup-
poſition ; and the ſecond error with the firſt ſuppoſi-
tion.   *Thus,* $a \times y$, and $b \times x$.

4. IF the errors are alike, that is, both too great,
or both too ſmall, or more properly, the numbers
from whence they were deduced, are both either
greater or leſs than the true ones, you muſt divide the
difference of the products, by the difference of the
errors, that is, $\overline{a \times y} - \overline{b \times x} \div \overline{a - b}$; but if the errors
are unlike, that is, one too great and the other too
ſmall, divide the ſum of the products by the ſum of
the

the errors : Thus, $\overline{a \times y + b \times x} \div \overline{a + b}$ and the quotient in either cafe, will be the number fought.

EXAMPLES.

A, B, and C, difcourfing of their money : Says B, I have 6 dollars more than A : Says C, I have 7 dollars more than B : Well fays A, the fum of all our money is 100 dollars : How much had each one ?

Suppofe A had 20 *dol. then B muft have* 20+6=26 *dol. and C* 26+7=33 *dol. but* 20+26+33=79, *which fhould be* 100 *by the queftion.*

Therefore, 100—79=21, *the firft error, too fmall.*

Again, fuppofe A had 24 *dol. then B muft have* 24+6 =30, *and C* 30+7=37, *but* 24+30+37=91, *which fhould be* 100. *Therefore,* 100—91=9, *the fecond error, too fmall.*

Whence, 24×21=504=*product of the fecond fuppofition and firft error ;*

And, 20×9=180=*product of the firft fuppofition and fecond error ;*

Wherefore, $\overline{504-180} \div \overline{21-9} =27$ *dol. A's money ;*

Then, 27+6=33 *dol.=B's money, and* 33+7=40 *C's money.*

A man having been to market with hogs, pigs and geefe ; received for them all 190 dollars, for every hog he received 4 dollars, for every pig 75 cents, and for every goofe 25 cents ; there were for every pig two hogs and three geefe : What was the number of each fort ?

Suppofe he had 12 *pigs, then he muft have* 24 *hogs, and* 36 *geefe, by the queftion; and* 12 *pigs at* 75 *cts. each, is* 9 *dol.* 24 *hogs at* 4 *dol. each, is* 96 *dol. and* 36 *geefe at* 25 *cts. each, is* 9 *dol. but* 9+96+9=114, *which fhould*

E e   *be*

be 190: *Therefore*, 190—114=76, *the first error, too small*.

Again, suppose he had 16 *pigs, then he must have* 32 *hogs, and* 48 *geese; and* 16 *pigs at* 75 *cts. is* 12 *dol.* 32 *hogs at* 4 *dol. is* 128 *dol. and* 48 *geese at* 25 *cts. is* 12 *dol. but* 12+128+12=152 *which should be* 190. *Therefore*, 190—152= 38, *the second error, too small*.

Whence we have $\overline{16 \times 76 - 12 \times 38} \div \overline{76 - 38} = 760$ $\div 38 = 20$, *the number of pigs, and* 20×2=40, *the number of hogs; also,* 20×3=60, *the number of geese*.

C H A P. XIV.

CONCERNING PERMUTATION

AND

COMBINATION.

SECT. I.

Of PERMUTATION.

PERMUTATION is the changing or varying the order of things; and is when any number of quantities are given; to find how many ways it is possible to range them, so that no two parcels shall have the same quantities standing in the same place, with respect to each other.

PROBLEM I.

To find all the variations or changes that can be made of any number of things, all different one from another.

FIRST it is evident, that any one thing is capable of one position only, and therefore cannot possibly have any change or variation; but any two quantities; as

a

a and *b*, are capable of change or variation; as *a b*, and *b a*, that is, the number of variations is 1×2. Again, if there be 3 quantities; as *a*, *b*, *c*, their variations are *a b c*, *a c b*, *b a c*, *b c a*, *c a b*, *c b a*; for taking only the two firſt, *a* and *b*, the number of their variations is 1×2; therefore taking in *c*, the number of changes is $1 \times 2 \times 3 = 6$; and ſo on for any number of quantities. Hence we have the following

R U L E.

MULTIPLY together the natural ſeries of numbers, 1, 2, 3, 4, &c. continually, till your multiplier is equal to the number of things propoſed, and the laſt product will be the number of variations required.

EXAMPLES.

In how many different poſitions may a company of 8 perſons ſtand?

Anſwer. $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 = 40320$ *poſitions.*

How many changes may be rung with 12 bells?

Anſwer. $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 \times 11 \times 12 = 479001600$, *the number of changes required.*

P R O B L E M   II.

To find all the poſſible alternations or changes that can be made of any given number of different quantities, by taking any given number of them at a time.

THE manner in which this problem is ſolved, is directly the reverſe of the laſt; for it is manifeſt, that let the number of quantities be ever ſo many, and we take one of them at a time, the number of alternations will be equal to the number of quantities. Therefore

it

it follows, that the operation muſt begin at the number of things propoſed, and then decreaſe by unity, till the number of multiplications are one leſs than the number of things propoſed. Hence we get the following

R U L E.

MULTIPLY continually together, the terms of the ſeries, beginning at the number of things propoſed ; and decreaſing by unity or 1, until the number of multiplications, are one leſs than the number of things to be taken at a time, and the laſt product will be the number of alternations required.

EXAMPLES.

How many different poſitions may a company of 9 men be placed in, taking 3 at a time ?

Here the number of multiplications muſt be 2, and the ſeries 9, 8, 7, 6, &c. Therefore, $9 \times 8 \times 7 = 504$, the number of poſitions required.

How many alternations will the letters $a\ b\ b$ admit of, taking 2 at a time ?

Anſwer. $3 \times 2 = 6$, *the number of alternations requir-*
ed, and the letters will ſtand thus, a b, b a, a b, b a,
b b, b b.

How many alternations or changes can be made with the letters $a\ b\ c\ d$, taken 3 at a time ?

Anſwer. $4 \times 3 \times 2 = 24$, *the number of alternations*
required ; and the letters will ſtand

Thus, $\begin{cases} a\,b\,c, a\,c\,b, b\,a\,c, b\,c\,a, c\,a\,b, c\,b\,a = alter.\ of\ a\,b\,c \\ a\,c\,d, a\,d\,c, c\,a\,d, c\,d\,a, d\,a\,c, d\,c\,a = do.\ of\ a\,c\,d \\ b\,c\,d, b\,d\,c, c\,b\,d, c\,d\,b, d\,c\,b, d\,b\,c = do.\ of\ b\,c\,d \\ d\,a\,b, d\,b\,a, a\,b\,d, a\,d\,b, b\,d\,a, b\,a\,d = do.\ of\ d\,a\,b. \end{cases}$

How

How many alternations or changes can be made with the letters of the word Algebra, taking 4 at a time?

Anſwer. $7 \times 6 \times 5 \times 4 = 840$.

PROBLEM III.

To find all the alternations or changes that can be made of any given number of quantities, which conſiſt of ſeveral of one ſort, and ſeveral of another.

RULE.

1. FIND the product of the ſeries, $1 \times 2 \times 3 \times 4$, &c. to the number of things to be changed, which call your dividend.

2. FIND all the alternations that can be made of each of thoſe things which are of the ſame ſort, by problem 1, and multiply them continually together for your diviſor.

3. DIVIDE, and the quotient reſulting will be the anſwer.

EXAMPLES.

Find all the variations that can be made of the following letters, *a a b c c c.*

OPERATION.

Firſt, $1 \times 2 \times 3 \times 4 \times 5 \times 6 = 720 =$ *number of variations that can be made of 6 different things, and* $1 \times 2 = 2$, *the variations of the a's; alſo,* $1 \times 2 \times 3 = 6$ *the variations of the c's.*

Whence, $720 \div \overline{6 \times 2} = 60$, *the number of variations required.*

Find all the different numbers that can be made of the following numeral figures, 11122777.

OPERATION.

OPERATION.

First, $1 \times 2 \times 3 = 6 =$ *variations of the* 1*'s, and* 1×2 $= 2 =$ *variations of the* 2*'s ; also,* $1 \times 2 \times 3 = 6 =$ *varitions of the* 7*'s.*

Whence, $\overline{1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8} \div \overline{6 \times 2 \times 7}$ $= 40320 \div 72 = 560$, *the answer.*

S E C T. II.

Of COMBINATION.

COMBINATION of quantities, is, when any number of things are given, to find all the different forms in which those quantities can be possibly ordered, and from thence, all the different combinations in those forms, without any regard to the order in which the several quantities stand in those combinations. That is, by combination we determine how many ways it is possible to combine any number of things, so that no two combinations shall have the same things in both. Combinations of the same form, are those that have a like number of quantities which repeat in the same manner in both : Thus, *a a c d*, and *y y x z*, are of the same form; but *a a a b c*, and *s m n r y*, are of different forms.

PROBLEM I.

To find all the different combinations that can be made of any number of quantities all different one from another, by taking any number of them at a time.

THE rule for the solution of this problem, is easily deduced from the rule to Problem II, of permutation. For it is plain, that the number of combina-
tions

tions multiplied with the changes in the number of things taken at a time, gives the number of alternations in the whole. Therefore it follows, that the number of alternations in the whole, divided by the changes in a number of things equal to thofe taken at a time, gives the number of all the different combinations. Hence we have the following

R U L E.

1. FIND all the alternations or changes of the given quantities, taken as many at a time, as are equal to the number of things to be combined at a time ; and call the refult your dividend.

2. FIND all the changes in as many quantities, as are equal to thofe to be taken at a time ; and call the refult your divifor.

3. DIVIDE, and the refulting quotient will be the number of combinations required.

EXAMPLES.

Find all the different combinations that can be made with the following numeral figures, 1, 2, 3, 4, 5, 6, taken 2 at a time.

Here the number of given quantities are 6 ; and the number to be taken at a time are 2 ; therefore, $6 \times 5 = 30 =$ dividend ; and $1 \times 2 = 2 =$ divifor.

Whence $30 \div 2 = 15$, the number of combinations required ; and the figures will ftand as follows :

$$12, \ 13, \ 14, \ 15, \ 16$$
$$23, \ 24, \ 25, \ 26$$
$$34, \ 35, \ 36$$
$$45, \ 46$$
$$56.$$

FIND all the different combinations that can be made, with the following letters, *a b c d b*, taken 3 at a time.

Here the number of quantities are 5, and the number to be taken at a time are 3 ; therefore, $5 \times 4 \times 3 = 60 =$ dividend ; and $1 \times 2 \times 3 = 6 =$ divisor.

Whence, $60 \div 6 = 10$, the number of combinations required : and the letters will stand as follows :

$$a\,b\,c, \quad a\,b\,d, \quad b\,b\,b, \quad a\,c\,d$$
$$a\,c\,b, \quad a\,d\,b, \quad b\,c\,d$$
$$b\,c\,b, \quad b\,a\,b$$
$$c\,a\,b$$

How many different combinations may be made with the following numeral figures, 1, 2, 3, 4, 5, 6, 7, 8, 9, taken 5 at a time ?

Answer. 126 *combinations.*

PROBLEM II.

To find the number of different combinations that may be made from any number of sets, by taking one out of each set and combining them together ; the things in every set being all different one from another.

RULE.

MULTIPLY the number of things in each set continually together, and the product resulting, will be the number of combinations required.

EXAMPLES.

How many different combinations of two letters, may be made of these two sets *a n w* and *s x y* ?
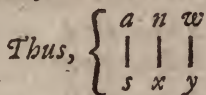
Here

Here the number of things in each set are 3 :

Therefore, $3 \times 3 = 9$, the number of combinations required.

The method of making the combinations, may be shewn in the following manner.

Write down the two sets one beneath the other, and join those letters that are to be combined, with a straight line,

$$Thus, \begin{cases} a & n & w \\ | & | & | \\ s & x & y \end{cases}$$

Then drawing lines from *s* to *a*, from *x* to *n*, and from *y* to *w*, you will have three of the required combinations, to wit, *s a*, *x n*, and *y w*.

Again, let the sets be placed as before :

Thus,
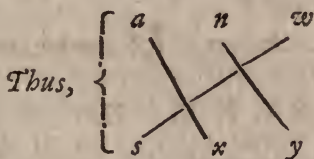
Then joining *s* and *w*, *x* and *a*, and *y* to *n*, we get *s w*, *x a* and *y n*. Once more, place the sets as above.

Thus,
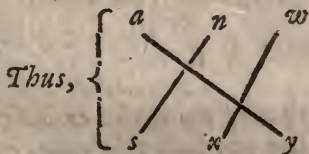
Then joining *s* and *n*, *x* and *w*, and *y* to *a*, we get *s n*, *x w*, and *y a*.

Hence, all the combinations are as follows,

$$\begin{matrix} s\,a, & s\,n, & s\,w, \\ x\,a, & x\,n, & x\,w, \\ y\,a, & y\,n, & y\,w, \end{matrix}$$

F f   Suppose

Suppose there are three flocks of sheep ; in one of which there is 10, and in the other two, 20 each : To find how many ways it is possible to choose 3 sheep, one out of each flock.

Thus, 10 × 20 × 20 = 4000, *the answer.*

PROBLEM III.

To find the number of forms in which any given number of quantities may be combined, by taking any number at a time ; wherein there are several of one sort, and several of another.

RULE.

1. Write the quantities according to the order of the letters.   *Thus, a, a, b, c, d.*

2. Join the first letter to the second, third, fourth, &c. to the last ; and the second letter to the third, fourth, &c. to the last ; also, the third letter to the fourth, fifth, &c. to the last : Proceeding in like manner through the whole, taking care to reject all combinations that have before accrued ; and you will have the combinations of all the twos.

3. Join the first letter to every one of the twos, and the second, third, fourth, &c. in like manner to the last ; and you will have the combinations of all the threes.

Thus, a a a, a a b, a a c, a a d, a b c, a b d, a c d,
 b a a, a b b, b a c, b a d, b b c, b b d, b c d,
 c a a, ——— c c a, ——— c c b, ——— c c d,
 d a a, ——— ——— d d a, ——— ——— d d c,

And proceed in this manner, till the number of things in the combination, are equal to the number to be taken at a time.

Note. *All those combinations which contain more things of the same sort, than are given of the like kind in the question, must be rejected.*   EXAM.

EXAMPLES.

Find all the different forms of combination, that can be made of the letters *a a b b c c*, taken 4 at a time.

OPERATION.

a a, a b, a c, b b, b c, c c = *combinations of the twos.*
a a b, a a c, b b a, b a c, b b c, b c c, a c c, = *combinations of the threes.*

a a b b, a a b c, b b c a, c c a b, a a c c, b b c c, = *combinations of the fours.*

Whence, a a b b, b b c c, a a c c, and a a c b, b b a c, c c a b, are the two forms required.

Find all the different forms of combination that can be made of the following figures, 22334455, taken 3 at a time.

OPERATION.

Thus, 22, 23, 24, 25, 33, 34, 35, 44, 45, 55 = *combinations of the twos.*

223, 224, 225, 234, 235, 245, 233, 334, 335, 345, 244, 344, 445, 255, 355, 455 = *combinations of the threes.*

Whence, 223, 224, 225, 233, 433, 533, 244, 344, 544, 255, 355, 455, *and* 234, 235, 245, 345, *are the forms required.*

THUS far, concerning Permutation and Combination.

CHAP. XV.

Of INVOLUTION.

WHEN any number is multiplied into itself, and that product multiplied with the same number; and so on, it is what is called Involution, and the several products refulting, are called the powers of the multiplying quantity, or root. Thus, $\overline{3\times3}$, $\overline{3\times3\times3}$, $\overline{3\times3\times3\times3}$, &c. are the powers of 3. And generally, $\overline{a\times a}$, $\overline{a\times a\times a}$, and $\overline{a\times a\times a\times a}$ &c. are the powers of a; whose height is denominated by the number of multiplications more one.

HENCE, the 2d power of 10, is $10\times10=100$
the 3d ——— $10\times10\times10=1000$
the 4th ——— $10\times10\times10\times10=10000$.

Therefore it follows, that the powers of any quantity, are a series of numbers in Geometrical Proportion continued, whose first term and ratio is the fame, to wit, the root of the power: Confequently the height of the power at any particular term, will be expreffed by the exponent of that term: As in

thefe, $\dfrac{1}{10}$, $\dfrac{2}{10\times10}$, $\dfrac{3}{10\times10\times10}$, $\dfrac{4\&c.\ Expon.}{10\times10\times10\times10}$, &c. ÷.

HERE it is evident, that the index, or exponent of each term of the Geometrical series, is equal to the number of multiplications of the first term with itself, to that place, more one, and is therefore called the index, or exponent of the power.

Thus, $\begin{cases} 1+1+1+1+1=5. \\ 5\times5\times5\times5\times5=3125=5th\ power\ of\ 5. \end{cases}$
and fo on for others.

WHENCE

WHENCE it follows, that to raise any number to any given power, is no more than to multiply the given number into itself, so often as there are units in the index of the power—1.

EXAMPLES.

Required the 5th power of 9.

OPERATION.

$$9$$
$$9$$

$$81 = 2d\ power\ of\ 9$$
$$9$$

$$729 = 3d\ power\ of\ 9$$
$$9$$

$$6561 = 4th\ power\ of\ 9$$
$$9$$

$$59049 = 5th\ power\ of\ 9,\ as\ requir.$$

Required the 7th power of 8.

Thus, $8 \times 8 \times 8 \times 8 \times 8 \times 8 \times 8 = 2097152 = 7th$ *power of 8.*

CHAP. XVI.

Of EVOLUTION.

EVOLUTION is the converse of Involution; and is when any power is given, to find

the

the number from whence such power was produced, which number (as we before said) is called the root of the power; and the bufinefs of finding it, is called extraction of roots.

ALL powers whatever, are produced by the continual multiplication of their roots into themfelves, as is evident from what has been faid; yet there are many powers which have no finite root, that is, whofe true and adequate root cannot be expreffed in finite terms; but by approximation may be determined to any affigned degree of exactnefs.

THESE powers are called furds, or irrational powers.

PROBLEM I.

To extract the root of the fquare or fecond power of any number.

RULE.

1. PREPARE the given number for extraction, *i. e.* diftinguifh it into periods of two figures each, by beginning at the unit's place and placing a point over the firft, third, fifth, &c. figures of the given number, and if there are decimals, point them in the fame manner, from unity towards the right hand.

2. FIND a number by the help of a table of powers, whofe fquare is equal to, or lefs than the firft period on the left hand, and this number will be the firft figure of the root, which place in the form of a quotient; then fubftract its fquare from the aforefaid period; and to the remainder annex the next period for a dividend.

3. DOUBLE the firft figure of the root for a divifor.

4. FIND fuch a quotient figure, that when annex-

ed

ed to the divisor and the result multiplied with the same number, the product will be equal to, or less than the dividend; and this will be the second figure of the root.

5. To the remainder annex the third period for a new dividend, and add the figure in the root last found to your former divisor for a new one.

6. FIND the third figure of the root as you found the second; and so on, till all be done.

> Note 1. *If there is a remainder after all the periods are annexed, the given number is a surd, and you must approximate to the root, by annexing cyphers two at a time, to the remainder.*
>
> 2. *If the given number consists of integers and decimals, you must ponit off as many places in the root, as there were periods of decimals in the given number.*

EXAMPLES.

Required the square root of 58081.

OPERATION.

$$58081(241$$
$$4$$

1st *divisor*=44) 180
    4 176

2d *divisor*=481) 481
         481
    ───────
         0

Therefore, 241 *is the root required, as may be proved by involution : Thus,* 241×241=58081, *which is the same as the given number : Whence, &c.*

Required

Required the fquare root of 1000.

OPERATION.

$$\dot{1}0\dot{0}0(31.622 \, \&c. = \textit{root required.}$$

    9
    ───
61)100
 1 61
─── ───
626)39.00
  6 3756
─── ───
6322)14400
  2 12644
─── ───
63242)175600
   2 126484
&c. ───
      49116 &c.

Required the fquare root of 105462.5625 :

OPERATION.

OPERATION.

$$105462.5625(324.75 = \text{root requir.}$$
$$9$$

$$62)154$$
$$2 \ 124$$

$$644)3062$$
$$4 \ 2576$$

$$6487)48656$$
$$7 \ 45409$$

$$64945)324725$$
$$324725$$

$$0$$

PROBLEM II.

To extract the square root of a Vulgar Fraction.

RULE.

EXTRACT the root of the numerator, for the nume-
rator of the root ; and the root of the denominator,
for the denominator of the root.

EXAMPLE.

Required the square root of $\frac{225}{1024}$.

OPERATION.

G g

OPERATION.

$$\dot{2}2\dot{5}(15 = \textit{numerator of the root.}$$
$$\begin{array}{r} 1 \\ \hline \end{array}$$

$$\dot{1}0\dot{2}4(32 = \textit{denominator of the root.}$$
$$9$$

$$25)125$$
$$125$$
$$\overline{0}$$

$$62)\ 124$$
$$124$$
$$\overline{}$$

Whence, $\frac{15}{32}$ *is the root required.*

PROBLEM III.

To find the root of the third power or cube, by approximation.

RULE.

1. DISTINGUISH the given number into periods of three figures each, by beginning at the unit's place, and placing a point over the firſt, fourth, ſeventh, figures, &c. and if there are decimals, point them from the unit's place towards the right hand, in the ſame manner.

2. FIND the root of the firſt period on the left hand, by the help of the table of powers, and annex to it, as many cyphers as there are remaining periods, then involve this number to the ſame power as the given number, and call the reſult the ſuppoſed cube; then : As twice the ſuppoſed cube + the given cube ; is to twice the given cube + the ſuppoſed cube ; ſo is the root of the ſuppoſed cube ; to the root required, nearly.

3. IF a greater degree of exactneſs is required, involve the root already found, to the third power, and
call

call the refult the fuppofed cube, with which pro-
ceed as as before, and fo on, to any degree of exact-
nefs.

Note. *When the root is finite, you may fometimes*
fave the trouble of repeating an operation, by in-
creafing the right hand figure of the root found, by
unity.

EXAMPLES.

Find the cube root of 1367631.

OPERATION.

Firft, 1367631 *is the given number prepared for ex-*
traction, the root of whofe firft period (1) *is* 1; *then*
100 × 100 × 100 = 1000000 = *fuppofed cube; and,*

as 1000000 × 2 + 1367631 : 1367631 × 2 + 1000000 ::
100, *i. e.* 3367631 : 3735262 :: 100

$$100$$

$$3367631)373526200(110$$
$$3367631 \quad +1$$

$$3676310 \quad 111 = \text{root requir.}$$
$$3367631$$

$$3086790$$

Required the cube root of 729001101.

Firft, 729001101 *is the given number pointed, and*
the root of the firft period (729) = 9; *therefore* 900 ×
900 × 900 = 729000000 = *fuppofed cube; then,*

as 729000000 × 2 + 729001101 : 729001101 × 2 +
72900000.0 :: 900.

That

That is, 2187001101 : 2187002202 :: 900
$$ 900$$

$$2187001101)1968301981800(900.0004 =$$
$$19683009909 \qquad [\textit{root nearly.}$$

$$9909000000.$$

THE cube root of a Vulgar Fraction, is found by extracting the root of the numerator and denominator.

PROBLEM IV.

To extract the roots of powers in general.

RULE.

1. LET the index of the power whose root is to be extracted, be denoted by n.

2. POINT the given number into periods of as many figures each, as there are units in n, beginning at the unit's place ; and if there are integers and decimals together, let them be pointed both ways from unity.

3. FIND the root of the first period, by the help of the table of powers, and this will be the first figure of the root.

4. SUBTRACT the n power of the first figure of the root, from the first period, and to the remainder annex the first figure of the next period, which result call your dividend.

5 INVOLVE the root now found to the $n-1$ power, and multiply the result with n for your divisor.

6. DIVIDE, and the quotient will be the second figure of the root.

7. INVOLVE all the root now found to the n power, and subtract it (always) from as many periods, as
$$ \text{you}$$

you have found figures of the root : But if the number to be subtracted, is greater than the aforesaid periods, the last figure of the root is too great, which must therefore be diminished, so that the n power of the root now found, may be taken from the aforesaid periods.

8. To the remainder annex the first figure of the next period for a new dividend, then find a new divisor as before; and so on, till the whole be done.

EXAMPLES.

Required the cube root of 61209.566621 :

OPERATION.

Here n $= 3$, *therefore the given number pointed is* 61209.566621, *and the nearest root of the first period* (61) *is* 3, *which is the first figure of the root, the* n *power of which is* $3 \times 3 \times 3 = 27$; *and* $61 - 27 = 34$, *which having the first figure of the next period annexed to it, becomes* $342 = $ *first dividend, and* $3 \times 3 \times 3 = 27 = $ *first divisor: Whence,* $27)342(9 = $ *second figure of the root, and the whole of the root now found is* 39; *therefore,* $39 \times 39 \times 39 = 59319 = $ n *power of* 39, *which being subtracted from the two first periods, leaves* 1890, *and* $18905 = $ *second dividend; also,* $39 \times 39 \times 3 = 4563 = $ *second divisor; whence,* $4563)18905(4 = $ *third figure of the root. Again,* $394 \times 394 \times 394 = 61162984$, *which subtracted from the three first periods, leaves* 46582, *then,* $465826 = $ *third dividend, and* $394 \times 394 \times 3 = 465708 = $ *third divisor; whence,* $465708)465826(1 = $ *fourth and last figure of the root, and because there are two periods of decimals in the given number, the root required is* 39.41; *for* $39.41 \times 39.41 \times 39.41 = 61209.566621 = $ *the number whose root was required: Whence, &c.*

Required

Required the 6th root of 148035889.

OPERATION.

First, extract the square root, and then the cube root of that result will give the root required:

Thus, 148035889(12167
      1

      22) 48
       2 44

        241) 403
          1 241

          2426) 16258
             6 14556

            24327)170289
                  170289

                      ——
                       0

Again, 12167(23=*root required.*
2×2×2=8

2×2×3=12)41
23×23×23=12167

                ——
                 0

The fame at one operation :

          Thus, 148035889(23 *as before.*
2×2×2×2×2×2=64

2×2×2×2×2×3=96)840
23×23×23×23×23×23=148035889

                        ——   In

In extracting the roots of heigher powers, it will be beſt to extract ſquare root out of ſquare root ſucceſſively, as often as the index of the given power is diviſible by 2 : Thus, in the 16th power, the index (16) is diviſible by 2, four times; for $16 \div 2 = 8$, $8 \div 2 = 4$, $4 \div 2 = 2$, and $2 \div 2 = 1$: Whence it follows, that the root of the 16th power may be obtained by four ſeveral extractions of the ſquare root; and the like may be ſhewn of all the even powers.

THE END OF BOOK FIRST.
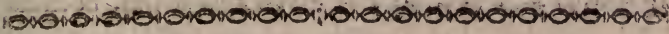
BOOK II.

OF ALGEBRA.

CHAP. I.

Of DEFFINITIONS

AND

ILLUSTRATIONS.

ALGEBRA, one of the moſt important branches of mathematical ſcience, is a method of computation by ſigns and ſymbols, which have been invented and found uſeful for that purpoſe. Its invention is of the higheſt antiquity, and has juſtly challanged the praiſe and admiration of the learned in all ages. Arithmetic is indeed uſeful, and is not to be the leſs valued, becauſe it is allowed to be the moſt clear and evident of the ſciences; yet it is confined in its object, and partial in its application. Geometry for clearneſs of principles, and elegance of demonſtration, no leſs deſerves, than commands our eſteem; but the many beautiful theories, that ariſe from the application of Algebra and Geometry to each other, fully evince the excellency and exten-

ſiveneſs

fivenefs of the former. The doctrine of Fluxions, which is efteemed the fublimity of human fcience, depends on the noble fcience of Algebra for its exiftance and application. In a word, Algebra is juftly efteemed the key to all our mathematical inquiries.

In Algebra, like quantities are thofe which have the fame letters: Thus, ax and ax are like quantities; but ax and dx are unlike quantities.

Given or abfolute numbers, are thofe whofe values are known: Thus, 6, 7, &c. are given numbers, becaufe their refpective values are known; but the quantities x, y, &c. are not given quantities, becaufe their values are not known, and are therefore called unknown quantities.

Simple quantities are fuch as have but one term: Thus, b, axb, and xyz, are fimple whole quantities, and $\dfrac{eb}{b}$ and $\dfrac{ab}{cd}$ are fimple fractional quantities.

Compound quantities are fuch as confift of feveral terms connected by the figns $+$ and $-$: Thus, $a+b+c-d$ and $ax-xy$ are compound whole quantities, and $\dfrac{a}{b}+\dfrac{c}{d}-\dfrac{dx}{b}$; $\dfrac{a+b}{c-d}$ are compound fractional quantities. Compound quantities have fometimes a line drawn over them; as $\overline{a+b+c-d}$.

Co-efficients are numbers prefixed to quantities, denoting how many times the quantity to which they are prefixed, ought to be taken: Thus, $3\,a$ denotes that the quantity a is to be taken 3 times; alfo, na fhews that the quantity a is to be taken as many times as there are units in n: Therfore, co-efficients multiply the quantities to which they are prefixed; and quantities which have no co-efficient prefixed to them, are always underftood to have an unit for their co-efficient: Thus, a is $1a$, x $1x$, &c.

A

A Positive, or an affirmative quantity, is a quantity having the fign + before it ; as + *a* : Alfo, all quantities that have no figns fet before them, as the leading quantity generally hath none, are underftood to have the fign +, and are therefore called pofitive quantities.

When quantities have the fign — before them, they are called negative quantities : As — *a*, — *x* ; and when any quantity is to be diftinguifhed, as a quantity to be fubtracted, the fign — muft be placed immediately before it.

Quantities are faid to have like figns, when they are all + or all —.

Unlike figns is when the figns are + and —.

A quantity confifting of two terms, as, $\overline{a+b}$, is called a binominal ; $\overline{a+b+c}$, a trinominal ; $\overline{a+b+c+d}$, a quadrinominal, &c.

A residual quantity, is the difference of two quntities. Thus, $\overline{a-b}$, is a refidual quantity.

The letters made ufe of to reprefent the unknown quantities, are thofe of the laft part of the alphabet, and the letters of the firft part, reprefent thofe that are known.

The principal figns by which quantities are managed in Algebra, are the following, in addition to thofe made ufe of in the firft book of this treatife.

Signs, and Explanations.

$\sqrt{}$ is the fign of the fquare root.

$\sqrt[3]{}$ —————— of the cube root.

$\sqrt[n]{}$ —————— of the *n* root.

$\pm$ —————— of more or lefs.

$\sqrt{x}$

$\sqrt{x}$ or $x^{\frac{1}{2}}$ denotes the square root of x.

$\sqrt[3]{x}$ or $x^{\frac{1}{3}}$ the cube root of x.

$\sqrt{\overline{a+b}}$ or $\overline{a+b}|^{\frac{1}{2}}$ the square root of $\overline{a+b}$.

$\sqrt[n]{\overline{a+b}}$ or $\overline{a+b}|^{\frac{1}{n}}$ the n root of $\overline{a+b}$.

$\frac{1}{a}$ the reciprocal of a.

$\frac{y}{x}$ the reciprocal of $\frac{x}{y}$.

$a \pm b$ the sum or difference of a and b.

AXIOMS.

1. IF to those quantities that are equal, there be added the same quantity, their sum will be equal.

2. IF from those quantities that are equal, there be taken the same quantity, the remainders will be equal.

3. IF those quantities which are equal, be multiplied with the same quantity, their products will be equal.

4. IF those quantities that are equal, be divided by the same quantity, the quotients will also be equal.

5. Two quantities respectively equal to a third, are equal to each other.

6. EQUAL powers, or roots of equal quantities, are equal to each other.

7. IF to any whole number, there be added any other whole number, the sum will be a whole number.

8. IF from any whole number, there be taken any other whole number, what remains will also be a whole number.

9. If any whole number be multiplied with any other whole number, the product will also be a whole number.

CHAP. II.

ADDITION of WHOLE QUANTITIES.

ADDITION confists of three cafes.

CASE I.

When the quantities are alike, and have like figns.

RULE.

ADD the co-efficients together, and to their fum annex the common quantity, prefixing the common fign.

EXAMPLES.

$2ab$	$-2x$	$-3xy$	$3x-4a$	$3x^2+b$
$2ab$	$-6x$	$-2xy$	$2x-2a$	$2x^2+3b$
$6ab$	$-x$	$-10xy$	$6x-4a$	$6x^2+2b$
$3ab$	$-5x$	$-2xy$	$2x-a$	$1x^2+3b$
$4ab$	$-4x$	$-xy$	$x-a$	$3x^2+b$
$17ab$	$-18x$	$-18xy$	$14x-12a$	$15x^2+10b$

$$2\ av - 3\ xy^2 + 2\ az - b - 4\ w^{\frac{1}{2}} - 6\ a + 3 - 2\ d$$
$$10\ av - xy^2 + 3\ az - 3\ b - 6\ w^{\frac{1}{2}} - 2\ a + 1 - 8\ d$$
$$av - 6\ xy^2 + 9\ az - b - w^{\frac{1}{2}} - a + 0 - d$$

$$13\ av - 10xy^2 + 14az - 5b - 11w^{\frac{1}{2}} - 9a + 4 - 11d\ \textit{fum}.$$

C A S E · II.

When the quantities are alike, but have unlike figns.

R U L E.

1. ADD all the affirmative quantities into one fum by the laft rule, and the negative into another.

2. SUBTRACT their co-efficients, the lefs from the greater, and to their difference, prefix the fign of the greater, annexing the common quantity.

THE reafon of the foregoing rule will appear evident, if you put $a =$ debt due to B, and $- a$ the want of a debt, or a debt due from B; then the balance is evidently equal 0, or $+ a - a = 0$: Whence, &c.

EXAMPLES.

$- ay$	
$+ ay$	$+ 4a - 2y$
$+4 ay$	$+ 3a + 6y$
$-3 ay$	$- 8a + 2y$
$-6 ay$	$-10a - 9y$
$- 10\ ay =$ *fum of the negative.*	$-18a - 11y$
$+ 5\ ay =$ *fum of the affirmative.*	$+ 7a + 8y$
$- 5\ ay =$ *fum required.*	$-11a - 3y$
	$3\ d^2$

$$3 d^2 + 8 d\sqrt{x^2 + yy}$$
$$-4 d^2 - 2 d\sqrt{x^2 + yy}$$
$$-7 d^2 - 8 d\sqrt{x^2 + yy}$$

$$-3 a + 4 d - w^2$$
$$- a - 9 d + w^2$$

$$-8 d^2 - 2 d\sqrt{x^2 + yy}$$

$$-4 a - 5 d \qquad *$$

CASE III.

When the quantities are unlike, and have unlike signs.

RULE.

WRITE the quantities one after another with their proper signs, and they will be the sum required.

Note. *If there be like quantities given, you must collect them by the preceding rules.*

EXAMPLES.

$$-3 a$$
$$+4 b$$
$$-2 c$$
$$-8 d$$

$$7 cy + 99 y$$
$$-4 d + 7 c - y$$

$$7 cy + 99 y - 4 d + 7 c - y = \textit{fum.}$$

$$-3 a + 4 b - 2 c - 8 d = \textit{fum.}$$

$$3 w^2 + 27 y^3$$
$$-2 y^{\frac{1}{2}} - 2 d\varsigma + 2 y^{\frac{1}{2}}$$

$$4 abv + 4\sqrt{av^2}$$
$$- cb^2 + 96 - {}^3\sqrt{a^2} - 4 abv$$

$$3 w^2 + 27 y^3 - 2 dc$$

$$4\sqrt{av^2} - cb^2 - {}^3\sqrt{a^2} + 96$$

CHAP.

CHAP. III.

SUBTRACTION of WHOLE QUAN-TITIES.

ALGEBRAIC Subtraction is performed by the following general

RULE.

CHANGE the signs of the quantities in the subtra-hend (or suppose them in your mind to be changed) then add the quantities with their signs changed, to the number from which subtraction is to be made, by the rules of the last chapter, and their sum will be the remainder required.

THE reason of this rule will appear obvious, when we consider that subtraction is the reverse of addition; and therefore, to subtract an affirmative or negative quantity, is the same thing as to add its opposite kind: Whence, if $- a$ is to be taken from $+ a$, the difference will be $+ 2 a$, for if the remainder $2 a$ be added to the subtrahend $- a$, their sum will be $= a$ $=$ the number from which subtraction was made: Whence, &c.

EXAMPLES.

From	$4 a$	$4 bu - 3 b^2$	$3 zy + 6 a - 5$
Take	$3 a$	$2 bu - 2 b^2$	$4 zy + \ a + 4 + 6$
Remains	a	$2 bu - \ b^2$	$- zy + 5 a - 9 - 6$

$34 \ v^{\frac{i}{r}}$

$$34 \, v^{\frac{1}{3}} + 6 \, bc + 7 \, c^2 b^2 \qquad 3^3 \, \sqrt{} \, aw - yb + 6 \, dy$$
$$-16 \, v^{\frac{1}{3}} + 16 \, -4 \, c^2 b^2 \qquad -2^3 \, \sqrt{} \, aw - yb + 6 \, dy$$
$$\overline{50 \, v^{\frac{1}{3}} + 6 \, bc - 16 + 11 \, c^2 b^2 \quad 5^3 \, \sqrt{} \, aw - yb \quad *}$$

If any doubt arife, refpecting the truth of the peration, add the remainder to the fubtrahend, which fum muft be equal to the other number.

CHAP. IV.

Of MULTIPLICATION.

ALGEBRAIC Multiplication confifts of three cafes.

CASE I.

When both the factors are fimple quantities.

RULE.

MULTIPLY the co-efficients together, and to their roduct annex all the letters in both factors, as in a 'ord; this expreffion being wrote with its proper gn, will give the product required.

Note. *Like figns give +, and unlike figns — for the product.*

I i   *EXAMPLES.*

EXAMPLES.

$+ 3a$	$3abc$	$-21yy$	$-3wyy$
$+ 4v$	$- 6a$	$2y$	$-2a$
$+12av$	$-18aabc$	$-42yyy$	$+6awyy$ product.

CASE II.

When one of the factors is a compound quantity.

RULE.

1. WRITE the compound quantity for the multiplicand, and the simple quantity for the multiplier.

2. OBTAIN the product of the multiplier with every particular term of the multiplicand, by the last rule, and place the terms of the product one after another, with their proper signs, found as in the last rule, and you will have the product required.

EXAMPLES.

$a + b$	$3ab + cd$	$2aa + 2ab + bb$
a	d	$2a$
$aa + ab$	$3abd + cdd$	$4aaa + 4aab + 2abb.$

$au - 4cv + 34$	$27ddd - aaa$
$-3y$	$3w$
$-3auy + 12cvy - 102y$	$81dddw - 3aaaw.$

CASE III.

When both the factors are compound quantities.

RULE.

RULE.

Multiply every particular term of the multiplier, with all the feveral terms of the multiplicand, as in the laft rule, the feveral products collected into one fum by the rules of addition, will give the whole product required.

EXAMPLES.

$$\begin{array}{l} v + y \\ v + y \\ \hline vv + vy \\ + vy + yy \\ \hline vv + 2\,vy + yy \end{array} \qquad \begin{array}{l} a - b \\ a + b \\ \hline aa - ab \\ + ab - bb \\ \hline aa \quad * \quad -bb \end{array} \qquad \begin{array}{l} v - 2z \\ v + 2z \\ \hline vv - 2\,vz \\ + 2\,vz - 4zz \\ \hline vv \quad * \quad -4zz \end{array}$$

$$\begin{array}{l} yy + xx \\ yy - xx \\ \hline yyyy + yyxx \\ - yyxx - xxxx \\ \hline yyyy \quad * \quad - xxxx \end{array} \qquad \begin{array}{l} 2xy + x - 4 \\ 2x - 1 \\ \hline 4xxy + 2xx - 8x \\ - 2xy - x + 4 \\ \hline 4xxy + 2xx - 2xy - 9x + 4 \end{array}$$

That $+ \times -$ or $- \times +$ gives $-$, and $- \times -$ gives $+$ for the product, is demonftrable feveral ways, but none more fimple than the following. Suppofe $a = b$; then $a - b = 0$: Now it is plain, that if this expreffion be multiplied with any number whatever, the product will be $= 0$: Therefore, fuppofe $a - b = 0$, is to be multiplied with $+ n$; now it is manifeft, the firft term of the product $a \times n$ will be pofitive; or $+ na$, becaufe both the factors are pofitive;

tive ; confequently the other term of the product $+ n$ $\times - b$ muft be negative, or $- nb$; for both terms of the product taken together, muft deftroy each other, and their amount $= 0$; that is, $na - nb = 0$: Confequently $+ \times -$, or $- \times +$ gives $-$ for the product.

AGAIN, fuppofe $a - b = 0$, be multiplied with $- n$; the firft term of the product $- n \times a$ will be negative, or $- na$, by what has been proved : Confequently, the other term $- n \times - b$ will be pofitive, or $+ nb$; for both terms taken together muft $= 0$; thus, $- na + nb = 0$: Confequently, $- \times -$ gives $+$ for the product. $\mathcal{Q}. E. D.$

C H A P. V.

Of DIVISION.

DIVISION being the converfe of multiplication ; it follows, that the quotient muft be fuch a quantity, that if multiplied with the divifor, will produce the dividend ; confequently, like figns in divifion give $+$, and unlike figns $-$ for the quotient.

C A S E I.

When the divifor is a fimple quantity.

R U L E.

1. WRITE down the quantities, in form of a vulgar fraction, having the divifor for the denominator.
2. EXPUNGE all thofe quantities in the dividend and divifor, that are alike ; and divide the co-effi-
cients

cients of the quantities by any number that will divide them without a remainder; the refult will be the quotient fought.

EXAMPLES.

$$\frac{8au}{2a} = 4u \text{ the quotient}; \quad \frac{24zy - 4z}{2z} = 12y - 2; \quad \frac{az}{a} = z$$

$$\frac{ab + bd}{-b} = -a - d; \quad \frac{12adz - 8dcz}{-4z} = -3ad + 2dc$$

$$\frac{16bcu}{12c} = \frac{4bu}{3}; \quad \frac{8uzy}{12dcu} = \frac{2yz}{3dc}.$$

IF you divide any quantity by itfelf, the quotient will be unity or 1 : Thus, $\frac{x}{x} = 1$; for if the quotient be multiplied with the divifor, the product will be the dividend ; thus, $x \times 1 = x$: Confequently, if any term of the dividend be like that of your divifor, the quotient of that term will be 1 : As in

Thefe, $\frac{av + bv + v}{v} = a + b + 1$; $\frac{2ab + 2bc - 2}{2} = ab$

$+ bc - 1$; alfo, $\frac{3vyz - 3vyz}{3vyz} = 1 - 1 = 0.$

C A S E II.

When the divifor and dividend are both compound quantities.

R U L E.

1. RANGE the quantities in the divifor and dividend, according to the order of the letters.

2. FIND how often the firft term of the divifor is contained in the firft term of the dividend, and place the refult in the quotient.   3.

3. MULTIPLY the quotient term thus found, with the whole divisor, subtract the product from the dividend, and to the remainder bring down the next term of the dividend ; which forms a new dividend.

4. DIVIDE the first term of your new dividend, by the first term of your divisor, as before ; and so on, until nothing remains, as in common Arithmetic, and you will have the quotient required.

EXAMPLES.

Suppose it is required to divide $2yyy + 8yy + 8y$ by $yy + 2y$; which being ranged as directed in the rule, the operation will stand

$$
\text{Thus, } yy + 2y\,)\,2yyy + 8yy + 8y\,(\,2y + 4
$$
$$
\underline{2yyy + 4yy}
$$
$$
* \quad +4yy + 8y
$$
$$
, \quad +4yy + 8y
$$
$$
\overline{\qquad * \qquad *}
$$

Here the first term of the dividend, which is $2yyy$, being divided by the first term of the divisor yy, the quotient is $2y$; which being placed in the quotient as in vulgar Arithmetic, and multiplied with all the terms of the divisor, the product is $2yyy + 4yy$, which subtracted from the dividend, the remainder is $4yy$, to which annex the next term of the dividend $8y$, the new dividend becomes $4yy + 8y$, and dividing $4yy$ by yy, the quotient is 4; which being annexed to the quotient term before found, and multiplied with every term of the divisor, produces $4yy + 8y$, which subtracted from the last dividend, the remainder is nothing; and having brought down all the terms of the proposed dividend, the work is done; therefore, $2y + 4$ is the true quotient, for $2y + 4 \times yy + 2y = 2yyy + 8yy + 8y =$ the given dividend. Divide

Divide $6\,avv - 3\,av - 2\,vy + 2\,v + 2y - 1$ by $2\,v - 1$.

OPERATION.

$$2v - 1) \; 6\,avv - 3\,av - 4\,vy + 2\,v + 2y - 1 \; (3\,av -$$
$$ 6\,avv - 3\,av \qquad\qquad\qquad\qquad [2y + 1$$
$$\overline{\;\; * \qquad * \;\; -4\,vy}$$
$$\qquad\qquad -4\,vy \qquad\quad +2y$$
$$\overline{\qquad\qquad * \;\; +2\,v \;\; * \;\; -1}$$
$$\qquad\qquad\qquad 2\,v \qquad -1$$
$$\overline{\qquad\qquad\qquad * \qquad\quad *}$$

Divide $vvv - yyy$ by $v - y$.

OPERATION.

$$v - y) \; vvv - yyy \; (vv + vy + yy$$
$$ vvv - vvy$$
$$\overline{\;\; * \; + vvy - yyy}$$
$$\qquad + vvy - vyy$$
$$\overline{\qquad\;\; * \; + vyy - yyy}$$
$$\qquad\qquad + vyy - yyy$$
$$\overline{\qquad\qquad * \qquad *}$$

Divide 1 by $1 - v$

OPERATION.

OPERATION.

$$1 - v) 1 \qquad (1 + v + vv + \&c.$$
$$\underline{1 - v}$$
$$* + v$$
$$\underline{+ v - vv}$$
$$* + vv$$
$$\underline{+ vv - vvv}$$
$$* \quad + vvv$$

In this example, the divisor cannot exactly be found in the dividend, without a remainder; and you have what is called an infinite series for the quotient; that is, if the division could be carried on *ad infinitum*, you would have a series of terms for the quotient, that would come infinitely near to an equality with the true quotient, and therefore might be considered as such; for when ratios from that of equality, are but indefinitely little, or less than can be assigned, they may be considered as equal; but as it is impossible to carry on the division *ad infinitum*, or take in a sufficient number of terms to express the true quotient: Therefore, in general you need only take a few of the leading terms for the quotient, which will be sufficiently near for most purposes. : But more of this in its proper place, since the knowledge of Algebraic fractions, is in most cases, absolutely necessary, in order to obtain an infinite series by division.

CASE III.

When the quantities in the divisor cannot be found in the dividend.

RULE.

RULE.

PLACE the dividend above, and the divisor below a small line, in form of a vulgar fraction; and the expression will be the quotient required.

EXAMPLES.

The quotient of a divided by b, is $\frac{a}{b}$.

The quotient of $21\ bx \div d = \frac{21\ bx}{d}$.

The quotient of $8\ ac + dc \div zx + ab = \frac{8\ ac + dc}{zx + ab}$

CHAP. VI.

INVOLUTION of WHOLE QUAN-TITIES.

INVOLUTION is the raising of powers from quantities called roots, and differs from multiplication in this, viz. that in involution the multiplier is constant, or the same; therefore when any quantity is drawn into itself, and afterwards into that product, and so on, the mode of operation is called involution, and the number produced, the power, whose height is usually denominated by placing numeral figures over the right hand of the root, or quantity to be involved, and are called indices or exponents of the powers which they denominate: Thus, $a^2 = aa$ denominates the square of a, $a^3 =$

aaa

aaa the cube of *a*, a^4 the fourth power of *a*; and generally, a^n the *n* power of *a*.

INVOLUTION of fimple quantities is performed by the following

RULE.

MULTIPLY the index or exponent of the given quantity or root, with the exponent which denominates the power required, making the product the exponent of the power fought.

> Note. *If the quantities to be involved, have co-efficients, the co-efficients muft be involved as in vulgar Arithmetic, to the fame height as the index of the power required denotes.*

EXAMPLES.

The fquare of $u = a^{1 \times 2} = a^2$; *the cube of* $a = a^{1 \times 3} = a^3$; *the fquare of* $a^2 = a^{2 \times 2} = a^4$; *the cube of* $3a^2 = 3 \times 3 \times 3 \times a^{2 \times 3} = 27a^6$; *the 4th power of* $4x^3y^2 = 4 \times 4 \times 4 \times 4 \times x^{3 \times 4} \times y^{2 \times 4} = 256x^{12}y^8$; *the n power of* $x = x^{1 \times n} = x^n$.

IF the quantity propofed to be involved is pofitive, all its powers will be pofitive: Alfo, if the quantity propofed be negative, all its powers whofe exponents are even numbers, will likewife be pofitive; becaufe any even number of multiplications of a negative quantity, gives a pofitive one for the product, fince $- \times -$ gives $+$; confequently $- \times - \times - \times - = + \times +$ for the product; therefore, that power of the negative quantity, only is negative, when its expo-
nent

nent is an odd number: As may be seen in the following form,

$$
-a\;\text{the root,}\;\begin{cases}
\begin{aligned}
&-a \\
&-a \\
\hline
&a^2 = square \\
&-a\;\text{the root} \\
\hline
&-a^3 = cube \\
&-a\;\text{the root} \\
\hline
&a^4 = 4th\;power \\
&-a\;\text{the root} \\
\hline
&-a^5 = 5th\;power.
\end{aligned}
\end{cases}
$$

INVOLUTION of compound quantities, is performed by the following

RULE.

MULTIPLY the root into itself, and then into that product, and so on, until the number of multiplications are one less than the exponent of the power required; the result will be the power sought.

EXAMPLES.

Let the binomial $a + b$ be involved to the 5th power.

OPERATION.

OPERATION.

$a+b$ *the root*
$a+b$

$aa+ab$
$\quad +ab+bb$

$aa+2ab+bb = $ *square*
$a+b$

$aaa+2aab+abb$
$\quad +aab+2abb+bbb$

$aaa+3aab+3abb+bbb = $ *cube*
$a+b$

$aaaa+3aaab+3aabb+abbb$
$\quad +aaab+3aabb+3abbb+bbbb$

$aaaa+4aaab+6aabb+4abbb+bbbb = $ *4th power*
$\quad a+b$

$aaaaa+4aaaab+6aaabb+4aabbb+abbbb$
$\quad +aaaab+4aaabb+6aabbb+4abbbb+bbbbb$

$a^5+5a^4b+10a^3b^2+10a^2b^3+5ab^4+b^5 = $ *5th do.*

Involve

Involve $a - b$ to the 3d power.

OPERATION.

$$a - b$$
$$a - b$$

$$a^2 - ab$$
$$\quad - ab + b^2$$

$$a^2 - 2ab + b^2 = 2d\ power$$
$$a - b$$

$$a^3 - 2a^2b + ab^2$$
$$\quad - a^2b + 2ab^2 - b^3$$

$$a^3 - 3a^2b + 3ab^2 - b^3 = 3d\ power.$$

IT is to be obſerved in the foregoing examples.

1. THAT all the terms in the ſeveral powers, raiſed from the binomial $a + b$, are affirmative.

2. THE terms in the ſeveral powers raiſed from the reſidual $a - b$, have the ſigns $+$ and $-$, alternately; the firſt term being a pure power of a, is conſequently affirmative; the ſecond term hath a negative ſign, and ſo on, alternately; but b is no where found negative, only where its exponent is an odd number; as in $a^3 - 3a^2b + 3ab^2 - b^3$; where the ſecond and fourth terms are negative, becauſe the exponent of b in thoſe terms, is an odd number.

3. THAT the firſt term of any power, either of the binomial or reſidual, hath the exponent of the power: That is, the index of the firſt term, is equal to the index of the power; but in the reſt of the terms following, the exponents of the leading quantity, decreaſe in arithmetical progreſſion, unity or 1, being the common difference; ſo that the quantity a is

never

never found in the laſt term ; but the exponents of b, on the contrary, increaſe in the ſame progreſſion that the exponents of a decreaſe ; that is, the quantity b, is not to be found in the firſt term ; but in the ſecond term, its exponent is unity or 1 ; in the third term 2, and ſo on in the ſaid arithmetical progreſſion, to the laſt term, where its exponent is equal to the exponent of the power.

4. That the number of terms in any power, is one more than the number which denominates that power.

HENCE from the foregoing obſervations it follows.

1. THAT the ſum of the exponents of both quantities in any term, are equal to the exponent of the power in which thoſe terms belong : Thus, the 6th power of $a + b = a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$, where you will pleaſe to obſerve, that the ſum of the exponents of a and b, in any term, are equal to the exponent of the power : Thus in the third term, the exponents of a and b, are 4 and 2, whoſe ſum $= 6 =$ exponent of the power.

2. THE method of writing without a continual involution, the terms in any power of a binomial, or reſidual quantity, without their co-efficients : Thus the terms of the 4th power of $x + y$ without their co-efficients, will ſtand thus : $x^4 + x^3y + x^2y^2 + xy^3 + y^4$; and the terms of the 4th power of $x - y = x^4 - x^3y + x^2y^2 - xy^3 + y^4$.

IN order to find the co-efficients of the ſeveral terms, it is neceſſary to have the co-efficient of one of the terms given : And becauſe the firſt term or leading quantity is a pure power, having its index equal to the index of the given power ; its co-efficient is therefore unity or 1 : Conſequently, you

have

have the co-efficient of the firſt term given ; thence to find the co-efficients of the reſt of the terms by the following

R U L E.

Divide the co-efficient of the preceding term, by the exponent of y in the given term; the quotient multiplied with the exponent of x, in the ſame term, increaſed by 1, will give the co-efficient required.

Or,

Multiply the co-efficient of any term, with the exponent of the leading quantity, in the ſame term; the product divided by the number of terms to that place, will give the co-efficient of the next ſubſequent term.

EXAMPLES.

Given $x^4 + x^3 y + x^2 y^2 + xy^3 + y^4$, to find the co-efficients of the ſeveral terms.

Firſt, the co-efficient of x^4 is 1; thence to find the co-efficient of $x^3 y$: And becauſe the exponent of y in the given term, is unity or 1; then per rule, $\frac{1}{1} \times 3 + 1 = 1 \times 4 = 4$, the co-efficient required : Again, $\frac{4}{2} \times 2 + 1 = \frac{4}{2} \times 3 = \frac{12}{2} = 6$, the co-efficient of the third term ; and $\frac{6}{3} \times 1 + 1 = \frac{6}{3} \times 2 = \frac{12}{3} = 4$, the co-efficient of the fourth term; but the next term hath the exponent of the power, being the laſt term of the 4th power of $x + y$, and conſequently, its co-efficient an unit or 1. Therefore, the co-efficients of the ſeveral terms of the 4th power of $x + y$, are 1, 4, 6, 4, 1.

Hence

HENCE you may obferve, that the co-efficients of the feveral terms increafe, until the exponents of x and y become equal to each other, and then decreafe in the fame order in which they increafed. And generally, the co-efficients of the terms increafe, until the exponents of the two quantities become equal in one term, if the exponent of the power is an even number ; and when the exponent is odd, two of the terms will have equal co-efficients, and then decreafe in the fame order. Therefore, in finding the co-efficients, you need only obtain the co-efficients, until they decreafe ; the reft of the terms having the fame co-efficients decreafing.

THE n power of $x + a = x^n + nx^{n-1}a + n\times\dfrac{n-1}{2}x^{n-2}a^2 + n\times\dfrac{n-1}{2}\times\dfrac{n-2}{3}x^{n-3}a^3$ &c. to $n+1$, terms.

Let $a + b + c$ be involved to the fecond power.

OPERATION.

$$a + b + c$$
$$a + b + c$$

$$a^2 + ab + ac$$
$$\quad + ab + b^2 + bc$$
$$\quad\quad + ca + bc + c^2$$

$$a^2 + 2ab + 2ac + b^2 + 2bc + c^2 = \text{2d power.}$$

CHAP. VII.

Of *MULTIPLICATION* and *DIVISION* of *POWERS* of the *fame ROOT*.

MULTIPLICATION of powers of the fame root, is performed by the following

RULE.

ADD the exponents of the powers together, and make their fum the exponent of the product.

EXAMPLES.

$a^3 \times a^2 = a^{3+2} = a^5$; $6^2 \times 6^3 = 6^{2+3} = 6^5 = 7776$; $6\ x^3 \times 4\ x^4 = 6 \times 4 \times x^{3+4} = 24\ x^7$; $-a^4 \times a^6 = -a^{10}$; *alfo,* $-a^1 \times -a^2 = a^3$; *in like manner,* $\overline{a-b}|^2 \times \overline{a-b}|^6 = \overline{a-b}|^{2+6} = \overline{a-b}|^8$; *and univerfally,* $a^m \times a^n = a^{m+n}$.

DIVISION of powers that have the fame root, is effected by the following

RULE.

FROM the exponent of the dividend, fubtract the exponent of the divifor, and the remainder will be the exponent of the quotient.

L l   *EXAMPLES.*

EXAMPLES.

$\frac{a^5}{a^6} = a^{8-6} = a^2$; $\frac{a^3}{a^2} = a^{3-2} = a^1$; $\frac{a^6 b^4}{a b^2} = a^{6-1}$ $b^{4-2} = a^5 b^2$;

Also, $\frac{\overline{a+x}|^6}{\overline{a+x}|^2} = \overline{a+x}|^{6-2} = \overline{a+x}|^4$; $\frac{\overline{a+b+c}|^{12}}{\overline{a+b+c}|^8}$ $= \overline{a+b+c}|^4$

Hence it follows, that in division of powers which have the same root, if you divide a less power by a greater, the exponent of the quotient will be negative; for we have shewn, that to divide any power of a by a, is to subtract one from the exponent of the power of a: Thus, $\frac{a^2}{a} = a^1$; therefore, $\frac{a}{a} = a^{1-1}$ $= a^\circ$; but $\frac{a}{a} = 1$ by the nature of division; consequently, $a^\circ = 1$ by equality; and therefore, $\frac{1}{a} = \frac{a^\circ}{a}$ $= a^{\circ-1} = a^{-1}$; and $\frac{1}{a^2} = \frac{a^\circ}{a^2} = a^{\circ-2} = a^{-2}$; and so on for any power of $\frac{1}{a}$: Likewise, $\frac{\overline{x+y}|^2}{\overline{x+y}|^2} =$ $\overline{x+y}|^{2-2} = \overline{x+y}|^\circ =$ (because, $\frac{\overline{x \times y}|^2}{\overline{x+y}|^2} = 1$,) 1 ; consequently, $\frac{1}{\overline{x+y}|^1} = \frac{\overline{x+y}|^\circ}{\overline{x+y}|^1} = \overline{x+y}|^{-1}$; therefore, $\frac{1}{\overline{x+y}|^2} = \frac{\overline{x+y}|^\circ}{\overline{x+y}|^2} = \overline{x+y}|^{\circ-2} = \overline{x+y}|^{-2}$; and $\frac{1}{\overline{x+y}|^3}$ $=$

$=\overline{x+y}\,^{-3}$. And generally, $\dfrac{1}{\overline{x+y}\,^{n}} = \dfrac{\overline{x+y}\,^{0}}{\overline{x+y}\,^{n}} = \overline{x+y}\,^{-n}$. Therefore, a^{0}, a^{-1}, a^{-2}, a^{-3}, and, $\overline{x+y}\,^{0}$, $\overline{x+y}\,^{-1}$, $\overline{x+y}\,^{-2}$, $\overline{x+y}\,^{-3}$, and $\overline{x+y}\,^{-n}$ respectively $= 1$, $\dfrac{1}{a}$, $\dfrac{1}{a^{2}}$, $\dfrac{1}{a^{3}}$, 1, $\dfrac{1}{x+y}$, $\dfrac{1}{\overline{x+y}\,^{2}}$, $\dfrac{1}{\overline{x+y}\,^{3}}$, $\dfrac{1}{\overline{x+y}\,^{n}}$, and of which they are pofitive powers.

HENCE the propriety of uſing negative exponents.

THE multiplication, and diviſion of powers which have the ſame root, having negative exponents, is performed by the ſame rule as thoſe powers which have affirmative ones; that is, add the exponents of the factors in multiplication, and in diviſion ſubtract them.

EXAMPLES.

a^{-2} *multiplied with* $a^{-4} = a^{-2-4} = a^{-6}$; $a^{-3} \times a^{-1} = a^{-1-3} = a^{-4}$; $a^{-2} \times a^{2} = a^{-2+2} = a^{0} = 1 = a \div a$, and $^{-4}\sqrt{a} \times {}^{+2}\sqrt{a} = {}^{-2}\sqrt{a}$.

$a^{-6} \div a^{-3} =$ (by the nature of ſubtraction) $a^{-6+3} = a^{-3} = 1 \div a^{3}$; and $a^{-3} \div a^{-6} = a^{-3+6} = a^{3}$; but by the nature of multiplication and diviſion, $a^{-3} \div a^{-6} = a^{-3} \div a^{-3} \times a^{-3} = 1 \div a^{-3} = a^{0} \div a^{-3} = a^{0+3} = a^{3}$; likewiſe, $^{-4}\sqrt{z+y}$

$$^{-4}\sqrt{z+y} \div {}^{-2}\sqrt{z+y} = {}^{-4+2}\sqrt{z+y} = {}^{-2}\sqrt{z+y}.$$

CHAP. VIII.

EVOLUTION of WHOLE QUANTITIES.

EVOLUTION is the unfolding of powers produced by involution; thereby difcovering the roots with which they are compofed, and is therefore the reverfe of involution.

THE rule for evolution of powers, whofe roots are fimple quantities, flows from this confideration; that to involve any fimple quantity to any power, is to multiply the exponent of the quantity, with the exponent of the power; making the product the exponent of the required power; confequently, if the exponent of the power, be divided by the index which denominates the root required, the quotient will be the exponent of the root. Therefore, when the exponent of the power whofe root is required, is not a multiple of the number which denominates the kind of root required; it follows, that the root will be expreffed by a fractional exponent: Thus, the fquare root of $a^5 = a^{\frac{5}{2}}$, and the cube root of $a^4 = a^{\frac{4}{3}}$. Whence, we have the following rule for evolution of fimple quantities.

RULE.

EXTRACT the root of the co-efficient, as in vulgar arithmetic, and divide the exponent of the power,

by

by the index of the root rquired ; making the root of the co-efficient, the co-efficient of the root.

EXAMPLES.

The cube root of $a^9 = a^{\frac{9}{3}} = a^3$: The square root of $4 a^4 = 2 a^{\frac{4}{2}} = 2 a^2$: The cube root of $64 x^9 a^3 = ^3\sqrt{64} \times x^{\frac{9}{3}}a^{\frac{3}{3}} = 4 x^3 a$: The 4th root of $256 a^4 b^{12} = ^4\sqrt{256} \times a^{\frac{4}{4}}b^{\frac{12}{4}} = 4 ab^3$: The cube root of $-27 a^3 = -3a^3$. But the square root of a negative quantity ; as $-x^2$, cannot be affigned, becaufe no even number of multiplications, either of a pofitive or negative quantity, can give a negative one for the product, as was fully explained in chapter VI ; therefore, the fquare root of $-x^2$ is an imaginary quantitiy : And fince the fquare of any negative or pofitive quantity, is always pofitive ; it follows, that the fquare root of x^2 may be $+x$, or $-x$. Therefore, when the number which denominates the root to be extracted, is odd, the fign of the root will the fame as the fign of the power ; and when the number which denominates the root, is even, the fign of the root may be either $+$ or $-$: Thus, the cube root of $-27 a^{15} b^9 = -3 a^5 b^3$, and the 4th root of $16 a^8 x^4 = 2 a^2 x$ or $-2 a^2 x$; the the n power of $x^m = x^{\frac{m}{n}}$

EVOLUTION of compound quantities, requires a different method of proceeding from that of fimple ones.

To extract the fquare root of a compound quantity we have the following

RULE.

R U L E.

1. Range the quantities according to the order of the letters, fo that the firft term fhall have the index of the power.

2. Find the root of the firft term, as in evolution of fimple quantities, and place it in the quotient.

3. Subtract the fquare of the root thus found, from the firft term of the power propofed, and to the remainder bring down the reft of the terms for a dividend.

4. Divide the firft term of the dividend, by double the root, and write the refult in the quotient, for the fecond term of the root,

5. Add the laft term of the quotient to your divifor, and multiply their fum with the faid quotient term, fubtracting the product from the dividend; and fo on, to obtain the next term of the root, by the help of thofe already found, in the fame manner as the fecond term was obtained by the help of the firft.

EXAMPLE.

Extract the fquare root of $a^2 + 2ay + y^2 + 2za + 2yz + z^2$.

The fquare root of the firft term viz. a^2, is a, which being placed in the quotient, is the firft term of the root, (fee the operation annexed) which fquared and fubtracted from the firft term of the propofed power, leaves no remainder; the reft of the terms being brought down for a dividend, the firft term, viz. $2ay$ divided by $2a$ (the double of the root) gives y for the fecond term of the root; which with the divifor, being multiplied with y, and the product fubtracted from the firft terms of the

dividend,

dividend, the remainder is nothing; the remaining terms being brought down as before and divided by the double of the two first terms of the root, gives z for the third term of the root, which added to the divisor and multiplied with z, the product subtracted as before, leaves no remainder: Therefore, the root sought, is

$\overline{a+y+z}$, for $\overline{a+y+z} \times \overline{a+y+z} = a^2 + 2ay + y^2 + 2az + 2yz + z^2$.

OPERATION.

$$a^2 + 2ay + y^2 + 2az + 2yz + z^2 \, (a + y + z = root$$
$$\underline{a^2}$$
$$2a+y)^* \quad + 2ay + y^2 + 2az + 2yz + z^2$$
$$\underline{2ay + y^2}$$
$$\ast \quad \ast$$
$$2a + 2y + z) + 2az + 2yz + z^2$$
$$\underline{2az + 2yz + z^2}$$
$$\ast \quad \ast \quad \ast$$

And universally, to extract any root.

RULE.

1. RANGE the terms of the given power, as in the last rule.

2. EXTRACT the root of the first term as before, and place it in the quotient for the first term of the root.

3. SUBTRACT the power of the root thus found, and to the remainder bring down the next term for a dividend.

4. INVOLVE the root to a dimension lower by unity than the number which denominates the root required, and multiply the result with the index of the

root to be extracted, which product call your divisor.

5. FIND how often the divisor is contained in the dividend, and write the result in the quotient for the second term of the root.

6. INVOLVE the whole of the root thus found, to the dimension of the given power, and subtract the result from the given power; and call the remainder a new dividend.

7. INVOLVE the whole of the root in the same manner as you did the first term, and multiply the result as before for a new divisor.

8. DIVIDE as before, and the result will be the third term of the root; and so on, till the whole be finished.

EXAMPLES.

Required the square root of $16y^6 - 48y^5 + 36y^4 + 96y^2 + 64$.

OPERATION.

$$16y^6 - 48y^5 + 36y^4 - 64y^3 + 96y^2 + 64(4y^3$$
$$16y^6$$

$$4y^3 \times 2 = 8y^3 \overline{\smash{\big)}} - 48y^5 \quad (-6y^2$$

$$16y^6 - 48y^5 + 36y^4 = \overline{4y^3 - 6y^2}\big|^2$$

$$\overline{4y^3 - 6y^2}\big|^1 \times 2 = 8y^3 - 12y^2 \overline{\smash{\big)}} - 64y^3 + 96y^2 + 64(-8$$

$$16y^6 - 48y^5 + 36y^4 - 64y^3 + 96y^2 + 64$$

$$* \quad * \quad * \quad * \quad * \quad *$$

Therefore, $4y^3 - 6y^2 - 8$ *is the root required.*

Required

Required the cube root of $8a^3 + 12a^2b + 6ab^2 + b^3$.

OPERATION.

$$8a^3 + 12a^2b + 6ab^2 + b^3 \ (2a$$
$$8a^3$$
$$\overline{2a|^2 \times 3 = 12a^2)\ 12a^2b\ (b}$$
$$\overline{2a+b|^3 = 8a^3 + 12a^2 + b6ab^2 + b^3}$$
$$* \qquad * \qquad * \qquad *$$

Whence, $2a+b$, *is the root required.*

CHAP. IX.

Of ALGEBRAIC FRACTIONS or BROKEN QUANTITIES.

ALGEBRAIC fractions are formed by the division of quantities incommensurable to each other: Thus, if x is to be divided by y, it will be (by case III, of algebraic division) $\frac{x}{y}$, which is an algebraic fraction; wherein x is the numerator and y the denominator. When fractions are connected with undivided quantities, as $a + \frac{x}{y}$, and $a + \frac{cx+z}{a+b}$, they are called mixed quantities; also, if the denominator is less than the numerator, the fraction is called improper.

THE various operations, necessary in managing algebraic fractions, are comprised in the following problems.

M m   PROB.

PROBLEM I.

To reduce a mixed quantity to an improper fraction of equal value.

RULE.

MULTIPLY the denominator of the fraction with the integral part, to which product add the numerator, and under their sum, subscribe the denominator, for the fraction required.

EXAMPLES.

$$a + \frac{a}{y} = \frac{a \times y + a}{y} = \frac{ya + a}{y}; \quad au + \frac{a+b}{c} = \frac{au \times c + \overline{a+b}}{c} =$$

$$\frac{cau + a + b}{c}; \quad v + z + \frac{v+z}{a-2} = \frac{\overline{v+z} \times \overline{a-2} + \overline{v+z}}{a-2} =$$

$$\frac{av + az - 2v - 2z + v + z}{a-2} = \frac{av + az - v - z}{a-2}.$$

PROBLEM II.

To reduce an improper fraction to a whole or mixed quantity.

RULE.

DIVIDE the numerator by the denominator for the integral part, and write the denominator under the remainder for the fractional part; and you will have the number required.

EXAMPLES.

EXAMPLES.

$$\frac{ac+ab}{c}=a+\frac{ab}{c}; \quad \frac{ay+2y^2}{a+y}=y+\frac{y}{a+y}; \quad \frac{a^2-y^2}{a}=a+$$

$$\frac{-y^2}{a}; \quad \frac{a^2+b^2}{a-b}=a+b+\frac{2b^2}{a-b}.$$

PROBLEM III.

To reduce fractions of different denominations, to fractions of the same value, that shall have a common denominator.

RULE.

1. REDUCE all mixed quantities to improper fractions.

2. MULTIPLY every numerator separately taken, into all the denominators except its own, for the several numerators, and all the denominators together for the common denominator, which being wrote under the several numerators, will give the fractions required.

EXAMPLES.

Reduce $\frac{x}{2}$ and $\frac{y}{4}$ to fractions of the same value, having a common denominator. First, $x \times 4 = 4x$ and $y \times 2 = 2y$ for the numerators : Then, $2 \times 4 = 8$, the common denominator. Therefore, $\frac{4x}{8}$ and $\frac{2y}{8}$ are the fractions required.

Reduce $\frac{v}{y}$, $\frac{z}{v}$, and $\frac{a}{c}$ to equivalent fractions, having a common denominator.

$\mathbf{1} \times v$

$$v \times v \times c = cv^2 \left.\right\}$$
$$z \times y \times c = cyz \left.\right\} = numerators.$$
$$a \times y \times v = ayv \left.\right\}$$

$$c \times v \times y = cvy = \text{common denominator.}$$

Therefore, $\dfrac{cv^2}{cvy}$, $\dfrac{cyz}{cvy}$ and $\dfrac{ayv}{cvy}$ are the fractions re-

quired; which are respectively equal to $\dfrac{v}{y}$, $\dfrac{z}{v}$, $\dfrac{a}{c}$;

for $\dfrac{cv^2}{cvy} =$ (by the nature of division) $\dfrac{v}{y}$; and the like

for the rest. Whence, &c.

Reduce, $\dfrac{a-v}{2v}$, $\dfrac{vb}{2}$, and $\dfrac{ay}{v}$ to a common de-

nominator, retaining their respective values.

$$\overline{a-v} \times 2 \times v = 2av - 2v^2 \left.\right\}$$
$$vb \times 2v \times v = 2v^3 b \left.\right\} = numerators.$$
$$ay \times 2v \times 2 = 4avy \left.\right\}$$

$$2v \times 2 \times v = 4v^2 = \text{common denominator.}$$

Therefore, $\dfrac{2av - 2v^2}{4v^2}$, $\dfrac{2v^3 b}{4v^2}$, and $\dfrac{4avy}{4v^2}$ are the

fractions required.

$a + \dfrac{b}{x}$, $\dfrac{cx}{ba}$, and $\dfrac{bc}{ax}$ reduced to a common denomin-

ator, are $\dfrac{bx^2 a^3 + b^2 a^2 x}{ba^2 x^2}$, $\dfrac{cax^3}{ba^2 x^2}$, and $\dfrac{b^2 acx}{ba^2 x^2}$.

PROBLEM IV.

To find the greatest common measure of algebraic fractions.

RULE.

RULE.

1. RANGE the quantities as in divifion.

2. DIVIDE the greater quantity by the lefs, and the laft divifor by the laft remainder, until nothing remains; taking care to expunge thofe quantities that are common to each divifor; and the laft divifor will be the greateft common meafure required.

EXAMPLES.

Find the greateft common meafure of $\dfrac{va - a^2}{vy^2 - y^2 a}$.

OPERATION.

$$va - a^2) \; vy^2 - y^2 a$$
$$\text{Or, } v - a) \; vy^2 - y^2 a \; (\; y^2$$
$$vy^2 - y^2 a$$

$$* \qquad *$$

Therefore, $v - a$, is the greateft common meafure required.

Find the greateft common meafure of $\dfrac{a^2 - b^2}{a^2 - 2ab + b^2}$.

OPERATION.

OPERATION.

$$a^2 - 2ab + b^2) \; a^2 - b^2 \; (\; 1$$
$$\underline{a^2 - 2ab + b^2}$$
$$\qquad * \quad 2ab - 2b^2) \, a^2 - 2ab + b^2$$

Or, (*by casting out* $2b$) $a - b) \; a^2 - 2ab + b^2 \; (\; a$
$$\underline{a^2 - ab}$$
$$\qquad * - ab + b^2) \; a - b$$
$$\text{Or,} \; a - b) \; a - b \; (\; 1$$
$$\underline{a - b}$$
$$\qquad * \quad *$$

Therefore, $a - b$, *is the greatest common measure required.*

PROBLEM V.

To reduce fractions to their least terms.

RULE.

1. FIND their greatest common measure by the last problem.
2. DIVIDE both terms of the proposed fraction by their greatest common measure, and the quotients will be the respective terms of the fraction, reduced to its least terms.

EXAMPLES.

EXAMPLES.

Reduce $\dfrac{xa + a^2}{xy^2 + y^2 a}$ to its leaſt terms.

First, $xa + a^2$) $xy^2 + y^2 a$

Or, $x+a$) $xy^2 + y^2 a$ (y^2
$$\dfrac{xy^2 + y^2 a}{\text{* *}}$$

Then, $x + a$) $xa + a^2$ ($a =$ numerator.
$$\dfrac{xa + a^2}{\text{* *}}$$

And, $x + a$) $xy^2 + y^2 a$ ($y^2 =$ denominator.
$$\dfrac{xy^2 + y^2 a}{\text{* *}}$$

Therefore, $\dfrac{a}{y^2}$ is the proposed fraction in its leaſt terms.

Reduce $\dfrac{y^4 - x^4}{y^5 - x^2 y^3}$ to its leaſt terms.

First, the greateſt common meaſure is $y^2 - x^2$:

Then, $y^2 - x^2$) $\dfrac{y^4 - x^4}{y^5 - x^2 y^3} = \dfrac{y^2 + x^2}{y^3} = frac. req.$

PROBLEM VI.

To add algebraic fractions.

RULE.

1. PREPARE the given fractions by reduction; that is, mixed quantities muſt be reduced to improp-
er

er fractions, and all fractions to a common denominator.

2. ADD all the numerators together, under which write the common denominator; and you will have the sum required.

FOR, put $\frac{v}{y} = a$, and $\frac{z}{y} = b$; then will $v = ya$ and $z = yb$ by the nature of division; consequently $ya + yb = v + z$, and therefore by division $a + b = \frac{v+z}{y}$. But, $a + b = \frac{v}{y} + \frac{z}{y}$; consequently, $\frac{v}{y} + \frac{z}{y} = \frac{v+z}{y}$; which is the same as the rule.

EXAMPLES.

Given $\frac{u}{6}$, $\frac{u}{6}$ and $\frac{4z}{6}$ to find their sum.

$u + u + 4z = 2u + 4z$ *and* $\frac{2u+4z}{6} =$ *sum requir.*

Having $\frac{u}{2}$, $\frac{3u}{y}$ and $\frac{3}{u}$ given to find their sum.

First, $u \times y \times u = u^2 y$, *and* $3u \times 2 \times u = 6u^2$, *also,* $3 \times 2 \times y = 6y$; *then,* $2 \times y \times u = 2uy$, *and* $u^2 y + 6u^2 + 6y \div 2uy =$ *sum required.*

$\frac{4x}{2a} + x + \frac{2x}{3} = \frac{4x}{2a} + \frac{5x}{3} = \frac{12x + 10\,ax}{6a}$

P R O B L E M   VII.

To subtract one fraction from another.

R U L E.

1. PREPARE the quantities as in the last problem.

2.

2. SUBTRACT the numerator of the subtrahend from the numerator of the other fraction, and write the common denominator under their difference; and you will have the fraction required.

FOR put $\frac{v}{y} = m$ and $\frac{a}{y} = n$; then $v = ym$ and $a = yn$; also, $yn - ym = a - v$ by equality; and dividing the whole by y, it will be $n - m = \frac{a - v}{y}$; but the difference of m and n, is manifestly equal to the difference of $\frac{a}{y}$ and $\frac{v}{y}$; consequently, $\frac{a}{y} - \frac{v}{y} = \frac{a - v}{y}$. Hence, &c.

EXAMPLES.

From $\frac{a}{b}$ take $\frac{cx}{ab}$. *First*, $a \times ab = a^2 b$, *and* $cx \times b = cbx$; *also,* $b \times ab = ab^2$. *Therefore,* $\frac{a^2 b}{ab^2}$ *and* $\frac{cbx}{ab^2}$ *are the fractions reduced; and* $\frac{a^2 b - cbx}{ab^2} =$ *difference required.* From $\frac{c^2 - x^2}{a^2}$ take $\frac{c^2 + x^2}{2}$, *and it will be* $\frac{c^2 - x^2}{a^2} - \frac{c^2 + x^2}{2} = \frac{2c^2 - 2x^2}{2a^2} - \frac{c^2 a^2 + x^2 a^2}{2a^2} = \frac{2c^2 - 2x^2 - c^2 a^2 - x^2 a^2}{2a^2}$. From $-x + \frac{x}{2}$ take $-\frac{3x}{4}$. *The fractions reduced are* $\frac{-4x}{8}$ *and* $\frac{-6x}{8}$. *therefore,* $\frac{-4x + 6x}{8} = \frac{2x}{8} =$ *difference required, by the nature of subtraction.*

N n   PROB.

PROBLEM VIII.

To multiply fractional quantities together.

RULE.

MULTIPLY the numerators together for the numerator of the product, and the denominators together for the denominator of the product; and you will have the product required.

FOR put $\frac{v}{z} = m$ and $\frac{a}{b} = n$; then $v = zm$ and $a = bn$; also, $bn \times zm = a \times v$; that is, $bznm = av$, and dividing by bz, $nm = \frac{av}{bz}$; but $m \times n = \frac{v}{z} \times \frac{a}{b}$; consequently, $\frac{v}{z} \times \frac{a}{b} = \frac{av}{bz}$: Therefore, &c.

EXAMPLES.

$$\frac{3y}{6} \times \frac{4x}{3y} = \frac{12xy}{18y} = \frac{2x}{3}, \text{ or } \tfrac{2}{3}x; \frac{a+b}{2x} \times \frac{a-b}{a-1} = \frac{a^2 - b^2}{2ax - 2x};$$

$$\text{and } a + \frac{d}{c} \times v = \frac{ca + d}{c} \times \frac{v}{1} = \frac{cav + dv}{c}; \text{ also, } \frac{vy}{4} \times$$

$$-v + \frac{6v}{3} = \frac{vy}{4} \times \frac{-3v + 6v}{3} = \frac{vy}{4} \times \frac{3v}{3} = \frac{3v^2 y}{12} = \frac{v^2 y}{4},$$

$$\text{or, } \tfrac{1}{4} v^2 y.$$

PROBLEM IX.

To divide one fraction by another.

RULE.

MULTIPLY the denominator of the divisor, with the numerator of the dividend, for the numerator of
the

the required quotient, and the numerator of the divisor, with the denominator of the dividend, for the denominator of the quotient. Or,

INVERT the terms of the divisor, and proceed as in multiplication.

For put $\dfrac{x}{y} = m$ and $\dfrac{z}{d} = n$; then $x = ym$ and $z = dn$. Multiply $z = dn$ by y, and it will be $yz = ydn$; in like manner, $dx = dym$; therefore, $\dfrac{ydn}{dym} = \dfrac{yz}{dx}$; but $\dfrac{ydn}{ydm} =$ (by division) $\dfrac{n}{m}$, and therefore by restitution $\dfrac{z}{d} \div \dfrac{x}{y} = \dfrac{yz}{dx}$: Consequently, &c.

EXAMPLES.

$\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{a \times d}{c \times d} = \dfrac{ad}{cb}$. Or, $\dfrac{a}{b} \div \dfrac{c}{d} = \dfrac{d}{c} \times \dfrac{a}{b} = \dfrac{ad}{cb}$ as before; $\dfrac{a-u}{v} \div \dfrac{a+u}{v} = \dfrac{a-u \times v}{a+u \times v} = \dfrac{va-uv}{va+uv} = \dfrac{a-u}{a+u}$: Therefore, in division of fractions that have the same denominator, cast off the denominators, and divide the numerator of the dividend, by the numerator of the divisor, for the quotient.

Thus, $\dfrac{4a^2}{3} \div \dfrac{6y^2}{4} \times a = \dfrac{4a^2}{3} \div \dfrac{6y^2 a}{4} = \dfrac{16a^2}{18y^2 a}$; $\dfrac{6a}{3} \div \dfrac{ay}{3} = \dfrac{6a}{ay}$.

PROBLEM X.

To find the powers of fractional quantities.

RULE.

RULE.

1. PREPARE the given fraction, if need be, by the rules of reduction.

2. INVOLVE the numerator to the height of the power proposed, as in involution of whole quantities, for the numerator of the power required.

3. INVOLVE the denominator in like manner, for the denominator of the aforesaid power.

EXAMPLES.

Find the square of $\frac{cy - y}{y + 1}$.

$\overline{cy - y} \times \overline{cy - y} = c^2 y^2 - 2 cy^2 + y^2$, and $\overline{y + 1} \times \overline{y + 1} = y^2 + 2y + 1$; therefore, $\frac{cy^2 - 2 cy^2 + y^2}{y^2 + 2y + 1}$ $=$ power required.

The 4th power of $\frac{za}{zy} = \frac{az \times az \times az \times az}{zy \times zy \times zy \times zy} = \frac{z^4 a^4}{z^4 y^4}$.

PROBLEM XI.

To find the roots of fractional quantities.

RULE.

1. EXTRACT the root of the numerator, by the rules for extracting the roots of whole quantities, for the numerator of the root required.

2. EATRACT the root of the denominator in like manner, for the denominator of the required root.

EXAMPLES.

EXAMPLES.

Find the fquare root of $\dfrac{a^6}{x^8}$.

Here, $a^{6 \div 2} = a^3$, *for the numerator of the root,
and* $x^{8 \div 2} = x^4$ *for the denominator of the root; there-
fore,* $\dfrac{a^3}{x^4}$ *is the root required.* The cube root of $\dfrac{a^3}{z^3 y^6}$
$= \dfrac{a}{zy^2}$. $\sqrt{\dfrac{a^2 b^4}{z^2 c^6}} = \dfrac{ab^2}{zc^3}$.

The fquare root of $\dfrac{x^2 - 4x + 4}{y^2 + 6y + 9} = \dfrac{x - 2}{y + 3}$.

But if the propofed quantity hath not a true root
of the kind required, it muft be diftinguifhed by the
fign of the root : Thus, the fquare root of $\dfrac{a^2 - x^2}{a^2 + x^2}$
$= \sqrt{\dfrac{a^2 - x^2}{a^2 + x^2}}$, *or* $\overline{\dfrac{a^2 - x^2}{a^2 + x^2}}\Big|^{\frac{1}{2}}$.

CHAP. X.

CONCERNING SURDS or IRRA-
TIONAL QUANTITIES.

IF the whole doctrine of furds, with every thing
therein, which might be of ufe, were to be ex-
plained according to the methods ufed by fome
writers on the fubject, it would become very complex,
and by far the moft intricate and dfficult part of all
Algebra ; and neceffarily fwell this volume beyond

its defigned limit: And befides, there are many things in the explanation and management of furd quantities, as was taught by many writers on Algebra, which were then thought neceffary, are now at moft, confidered as ufeful. We fhall therefore, endeavour on the one hand, to avoid all fuch tedious reductions, and complicated explanations, as would ferve rather to puzzle, than inftruct the learner: And on the other hand, not to omit any thing which is neceffary, either in the explanation or management of fuch furds as generally arife in algebraic operations.

A Surd quantity is that which has no exact root: Thus, the fquare root of 5 cannot exactly be found in fiinite terms, but is expreffed by $5^{\frac{1}{2}}$, or $\sqrt{5}$; the cube root of a by $a^{\frac{1}{3}}$, or $\overline{a}|^{\frac{1}{3}} = \sqrt[3]{a}$: The reciprocal of the fquare root of $a + y$, or 1 divided by the fquare root of $a + y$, is expreffed by $\overline{a+y}|^{-\frac{1}{2}} = \frac{1}{\sqrt{a+y}}$.

Therefore, the roots of irrational or furd quantities, may be confidered as powers having fractional exponents; that is, the index fhewing the height of the power, is here placed as the numerator of a fraction, whofe denominator is the radical fign.

S E C T. I.

Of REDUCTION of SURD QUANTITIES.

Reduction of furds has the following problems.

P R O B.

PROBLEM I.

To reduce a rational quantity to the form of an irrational, or surd quantity.

RULE.

INVOLVE the rational quantity to the height of the proposed radical sign, or index shewing the root to be extracted ; the power distinguished by the radical sign, will be the form required.

EXAMPLES.

a, reduced to the form of the $\sqrt[3]{x}$, is $\sqrt[3]{a^{1\times3}} = \sqrt[3]{a^3}$; 6 reduced to the form of the square root of 2, $= \sqrt{6^{1\times2}} = \sqrt{6^2} = \sqrt{36}$.

Reduce $\dfrac{3}{4}$ to the form of a cube root. $\overline{\dfrac{3}{4}}^{1\times3}\Big]^{\frac{1}{3}}$

$= \overline{\dfrac{3^3}{4^3}}\Big]^{\frac{1}{3}} = \sqrt[3]{\frac{27}{64}} =$ form required ; $u+y$ reduced to the

form of a fourth root, is $\sqrt[4]{u+y}^{1\times4} = \sqrt[4]{u+y}^4$ $= \sqrt[4]{u^4 + 4u^3y + 6u^2y^2 + 4uy^3 + y^4}$. Also, $u = \sqrt{u^2}$ $= \sqrt[3]{u^3} = \sqrt[4]{u^4} = \sqrt[5]{u^5} = \sqrt[n]{u^n}$.

PROBLEM II.

To reduce surds of different radical signs to the same.

RULE.

REDUCE the indices of the surds to a common denominator, and the surds will have the same radical sign as required.   *EXAMPLES.*

EXAMPLES.

The $\sqrt[3]{}$ of a, and the $\sqrt{}$ of b, reduced to the same radical sign $= a^{\frac{1\times 2}{6}}$ and $b^{\frac{1\times 3}{6}} = a^{\frac{2}{6}}$ and $b^{\frac{3}{6}}$, or $\sqrt[6]{a^2}$ and $\sqrt[6]{b^3}$. $z^{\frac{1}{4}}$ and $y^{\frac{2}{3}}$ reduced to the same sign $= z^{\frac{1\times 3}{12}}$ and $y^{\frac{2\times 4}{12}} = z^{\frac{3}{12}}$ and $y^{\frac{8}{12}}$, or $\sqrt[12]{z^3}$ and $\sqrt[12]{y^8}$.

$\sqrt[\frac{3}{2}]{ay+by}$ and $\sqrt[\frac{1}{2}]{a+y}$ reduced to a common radical sign, are $\sqrt[\frac{3\times 2}{4}]{ay+by}$, $\sqrt[\frac{1\times 2}{4}]{a+y} = \sqrt[\frac{1}{4}]{\overline{ay+by}|^6}$ $\sqrt[\frac{1}{4}]{\overline{a+y}|^2}$: Also, $y^{-\frac{1}{2}}$ and $y^{-\frac{2}{3}} = y^{-\frac{3}{6}}$ and $y^{-\frac{4}{6}}$

PROBLEM III.

To reduce surds to their most simple terms.

RULE.

1. DIVIDE the quantity under the radical sign, by such a rational divisor, as will quote the greatest rational power contained in the proposed surd without a remainder.

2. EXTRACT the root of the rational power, and place it before the surd, with the sign of multiplication, and the proposed surd will be in its most simple terms.

EXAMPLES.

Reduce $\sqrt{32}$ to its most simple terms.

Here

Here $\frac{32}{2} = 16$ *the greatest rational power contained in*
$\sqrt{32}$; *therefore, the* $\sqrt{32} = \sqrt{16 \times 2} = \sqrt{16} \times \sqrt{2} =$
$4 \times \sqrt{2} = 4\sqrt{2}$. *The* $\sqrt{\frac{27}{28}} = \sqrt{\frac{9 \times 3}{4 \times 7}} = \frac{3}{2} \times \sqrt{\frac{3}{7}}$;
$\sqrt{a^3 x} = \sqrt{a^2 \times ax} = a \times \sqrt{ax} = a\sqrt{ax}$; $\overline{a^3 x - a^2 y}|^{\frac{1}{2}}$
$= \overline{a^2 \times ax - y}|^{\frac{1}{2}} = a \times \overline{ax - y}|^{\frac{1}{2}}$.
$\overline{\frac{64 a^3 x}{54 y}}|^{\frac{1}{3}} = \frac{4a}{3} \times \overline{\frac{x}{2y}}|^{\frac{1}{3}}$.

SECT. II.

Of ADDITION of SURD QUAN-TITIES.

ADDITION of furd or irrational quantities, confifts of the following cafes.

CASE I.

When the propofed furds are of the fame irrational quantity (or can be made fo by reduction) and the radical fign the fame in all.

RULE.

ADD the rational to the rational, and to their fum annex the irrational part with its radical fign.

EXAMPLES.

$3\sqrt{20} + 6\sqrt{20} = \overline{3+6} \times \sqrt{20} = 9\sqrt{20}$; $\sqrt{3a^2 x} +$
$\sqrt{27x} = \sqrt{a^2 \times 3x} + \sqrt{9 \times 3x} = \overline{a+3}\sqrt{3x}$; $\overline{\frac{27}{5}}|^{\frac{1}{2}} +$

O o

$\frac{48}{5}|^{\frac{1}{2}}$

$$\overline{\frac{48}{5}}\Big]^{\frac{1}{2}} = \overline{\frac{9\times3}{5}}\Big]^{\frac{1}{2}} + \overline{\frac{16\times3}{5}}\Big]^{\frac{1}{2}} = 3\times\overline{\frac{3}{5}}\Big]^{\frac{1}{2}} + 4\times\overline{\frac{3}{5}}\Big]^{\frac{1}{2}} = 7\times\overline{\frac{3}{5}}\Big]^{\frac{1}{2}};$$

$$\overline{ax^3-x^3}\Big]^{-\frac{1}{3}} + \overline{8a-8}\Big]^{-\frac{1}{3}} = \overline{x^3\times a-1}\Big]^{-\frac{1}{3}} +$$

$$\overline{3\times a-1}\Big]^{-\frac{1}{3}} = x^{-1}\times\overline{a-1}\Big]^{-\frac{1}{3}} + \tfrac{1}{2}\times\overline{a-1}\Big]^{-\frac{1}{3}} = \frac{1}{x}$$

$$+\tfrac{1}{2}\times\overline{a-1}\Big]^{-\frac{1}{3}} = \frac{x+2}{2x}\times\frac{1}{\overline{a-1}\big]^{\frac{1}{3}}};$$

CASE II.

When the irrational or furd quantity, and the radical fign are not the fame in all.

RULE.

Connect the furds with their proper figns + or —; and you will have the fum required.

Note. *If the fum confifts of two terms, it is called a binomial, or refidual furd, as the fign is + or —.*

EXAMPLES.

$$\sqrt{a} + \sqrt{x} = \sqrt{a} + \sqrt{x} = fum;\ \ ^3\sqrt{16} + \sqrt{27}$$
$$= \ ^3\sqrt{8\times2} + \sqrt{9\times3} = 2\ ^3\sqrt{2} + 3\sqrt{3}:$$
$$\tfrac{3}{4}\times\overline{\frac{81}{16}}\Big]^{\frac{1}{3}} + \tfrac{2}{3}\times\overline{\frac{27}{36}}\Big]^{\frac{1}{2}} = \tfrac{3}{4}\times\overline{\frac{27\times3}{8\times2}}\Big]^{\frac{1}{3}} + \tfrac{2}{3}\times\overline{\frac{9\times3}{36\times1}}\Big]^{\frac{1}{2}} =$$
$$\tfrac{3}{4}\times\tfrac{3}{2}\times\overline{\frac{3}{2}}\Big]^{\frac{1}{3}} + \tfrac{2}{3}\times\tfrac{3}{6}\times 3^{\frac{1}{2}} = \tfrac{9}{8}\times\overline{\frac{3}{2}}\Big]^{\frac{1}{3}} + \tfrac{6}{18}\times 3^{\frac{1}{2}}:$$
$$\sqrt{ax}\ added\ to\ -\sqrt{xy-y^2} = \sqrt{ax} - \sqrt{xy-y^2}.$$

SECT. III.

Of SUBTRACTION of SURD QUAN-TITIES.

CASE

CASE I.

When the radical sign and quantity are the same in all.

RULE.

FIND the difference of the rational parts, to which annex the common irrational or surd quantity, with the sign of multiplication.

EXAMPLES.

$$80^{\frac{1}{2}} - 45^{\frac{1}{2}} = \sqrt{16 \times 5} - \sqrt{9 \times 5} = 4 \times 5^{\frac{1}{2}} - 3 \times 5^{\frac{1}{2}}$$
$$= \overline{4 - 3}\sqrt{5} = \sqrt{5}; \quad {}^3\sqrt{40 a^3 y^2} - {}^3\sqrt{135 y^2} =$$
$${}^3\sqrt{8 a^3} \times \overline{5 y^2} - {}^3\sqrt{27 \times 5 y^2} = 2 a {}^3\sqrt{5 y^2} -$$
$$3 {}^3\sqrt{5 y^2} = \overline{2 a - 3} {}^3\sqrt{5 y^2} :$$
$$\left.\frac{54 y^3}{20}\right|^{\frac{1}{2}} - \left.\frac{150 y^3}{80}\right|^{\frac{1}{2}} = \left.\frac{9 y^2 \times 6 y}{4 \times 5}\right|^{\frac{1}{2}} - \left.\frac{25 y^2 \times 6 y}{16 \times 5}\right|^{\frac{1}{2}} =$$
$$\frac{3 y}{2} \times \left.\frac{6 y}{5}\right|^{\frac{1}{2}} - \frac{5 y}{4} \times \left.\frac{6 y}{5}\right|^{\frac{1}{2}} = \frac{12 y - 10 y}{8} \times \left.\frac{6 y}{5}\right|^{\frac{1}{2}} = \frac{y}{4} \times$$
$$\left.\frac{6 y}{5}\right|^{\frac{1}{2}}; \quad 4 \times \overline{a^2 y - a^2}\Big|^{-\frac{1}{2}} - 2 \times \overline{a^2 y - a^2}\Big|^{-\frac{1}{2}} = -\frac{1}{4}$$
$$\times \overline{y - 1}\Big|^{-\frac{1}{2}}.$$

CASE II.

When the irrational parts are not the same in all.

RULE.

CHANGE the sign of the quantity to be subtracted, the expression connected, is the difference required.

EXAMPLES.

EXAMPLES.

$27^{\frac{1}{2}}$ *subtracted from* $80^{\frac{1}{2}}$, $= \sqrt{16\times5} - \sqrt{9\times3} = 4\sqrt{5} - 3\sqrt{3}$; $48^{\frac{1}{2}} - 16^{\frac{1}{3}} = 48^{\frac{3}{6}} - 16^{\frac{2}{6}} = {}^{6}\sqrt{48^3} - {}^{6}\sqrt{16^2}$; $3^{4}\sqrt{ab} - z$ *subtracted from* ${}^{4}\sqrt{z^2 - y^2}$, $= {}^{4}\sqrt{z^2 - y^2} - 3^{4}\sqrt{ab} - z.$

S E C T. IV.

Of MULTIPLICATION of SURD QUANTITIES.

SURDS being confidered as powers having fractional exponents; it therefore follows, that to multiply one furd with another, is to add their fractional exponents together, making the denominator of their fum the radical fign, and the numerator the index of the root.

HENCE is deduced the following rule for multiplication of furds.

R U L E.

1. REDUCE the indices of the furds to a common denominator.

2. ANNEX the product of the furds, to the product of the rational parts with the fign of multiplication; and it will give the product required.

EXAMPLES.

$$\sqrt[6]{16} \times \sqrt{8} = \sqrt[3]{8 \times 2} \times \sqrt{4 \times 2} = 2\sqrt[3]{2} \times$$
$$2\sqrt{2} = 2 \times 2^{\frac{1}{3}} \times 2 \times 2^{\frac{1}{2}} = 2 \times 2^{\frac{2}{6}} \times 2 \times 2^{\frac{3}{6}} = 4\sqrt[6]{2^3 2^2}$$
$$= 4\sqrt[6]{32}; \ \overline{z^2 + y^2}|^{\frac{1}{2}} \times \overline{z^2 + y^2}|^{\frac{1}{2}} = \overline{z^2 + y^2}|^{\frac{1}{2} + \frac{1}{2}} =$$
$$z^2 + y^2; \ z^{\frac{1}{2}} \times y^{\frac{1}{2}} = \overline{zy}|^{\frac{1}{2}}; \ \overline{a+y}|^{-\frac{1}{2}} \times \overline{a+y}|^{-\frac{1}{2}} =$$
$$\overline{a+y}|^{-\frac{1}{2} - \frac{1}{2}} = \overline{a+y}|^{-\frac{2}{2}} = \overline{a+y}|^{-1}; \ a^{\frac{1}{m}} \times b^{\frac{1}{n}} =$$
$$a^{\frac{n}{nm}} \times b^{\frac{m}{nm}} = \sqrt[nm]{a^n b^m}.$$

SECT. V.

Of DIVISION of SURD QUANTI-
TIES.

RULE.

1. REDUCE the surds to the same index.

2. DIVIDE the rational by the rational, and to the quotient annex the quotient of the surd quantities; and it will be the quotient required.

Note. *If the quantity is the same in both factors, they are divided by subtracting their exponents.*

EXAMPLES.

$$a^{\frac{1}{2}} \div a^{\frac{1}{3}} = a^{\frac{3}{6}} \div a^{\frac{2}{6}} = a^{\frac{3-2}{6}} = a^{\frac{1}{6}} = \sqrt[6]{a}; \ \sqrt{32} \div \sqrt{18}$$
$$= \sqrt{16 \times 2} \div \sqrt{9 \times 2} = 4\sqrt{2} \div 3\sqrt{2} = \frac{4}{3} \times \sqrt{\frac{2}{2}} =$$
$$\frac{4}{3}\sqrt{1} = \sqrt{\frac{16}{9}} = \frac{4}{3}; \ \overline{xa+ya}|^{\frac{1}{2}} \div \sqrt{a} = \frac{\overline{xa+ya}|^{\frac{1}{2}}}{a} =$$

$$\overline{x+y}|^{\frac{1}{2}}$$

$$\overline{x+y}\Big|^{\frac{1}{2}} \; ; \quad x^{\frac{1}{2}} \div y^{\frac{1}{3}} = x^{\frac{3}{6}} \div y^{\frac{2}{6}} = \overline{\frac{x^3}{y^2}}\Big|^{\frac{1}{6}} \; ; \quad \frac{x^{\frac{1}{2}}}{y^{\frac{1}{3}}} \div$$

$$\frac{x^{\frac{1}{3}}}{y^{\frac{1}{2}}} = \frac{x^{\frac{3}{6}}}{y^{\frac{2}{6}}} \div \frac{x^{\frac{2}{6}}}{y^{\frac{3}{6}}} = \frac{y^{\frac{3}{6}} x^{\frac{3}{6}}}{x^{\frac{2}{6}} y^{\frac{2}{6}}} = \overline{\frac{y^3 x^3}{x^2 y^2}}\Big|^{\frac{1}{6}} = \overline{yx}\Big|^{\frac{1}{6}} \; ;$$

$$a^{\frac{1}{n}} \div a^{\frac{1}{m}} = a^{\frac{m-n}{mn}} .$$

S E C T. VI.

Of INVOLUTION of SURD QUAN-TITIES.

THE powers of surds are found by the following

R U L E.

INVOLVE the rational part, as in involution of numbers ; and to the result annex the power of the surd, found by multiplying its exponent with the exponent of the power required.

EXAMPLES.

The square of $\sqrt[3]{6} = 6^{\frac{1}{3} \times 2} = 6^{\frac{2}{3}} = \sqrt[3]{6^2} = \sqrt[3]{36}$. The cube of $\sqrt{3} = 3^{\frac{1}{2} \times 3} = 3^{\frac{3}{2}} = \sqrt{3^3} = \sqrt{27}$. The square of $2\sqrt[3]{x^2} = 2 \times 2 \times x^{\frac{2}{3} \times 2} = 4 \times x^{\frac{4}{3}} = 4\sqrt[3]{x^4}$. The cube of $\sqrt[3]{ax - bx} = \overline{ax - bx}\Big|^{\frac{1}{3} \times 3} = \overline{ax - bx}$: Therefore, when the index of the power required, is equal to, or a multiple of the exponent of the root ; the power of the furd be-

comes

comes rational. *The cube of* $\overline{a-x}\rvert^{-\frac{2}{3}} = \overline{a-x}\rvert^{-\frac{2}{3}} \times 3$ $= \overline{a-x}\rvert^{-\frac{6}{3}} = \overline{a-x}\rvert^{-2}$. *The* n *power of* $y^{\frac{1}{m}} =$ $y^{\frac{1}{m}} \times n = y^{\frac{n}{m}} = {}^m\sqrt{y^n}$.

IF the propofed furd is a binomial 'or refidual one, involve it as in chapter VII.

Thus, the fquare of $\sqrt{6} + 2\sqrt{x} = 6 + 4x + 4\sqrt{6x}$. *(See the operation annexed.)*

OPERATION.

$$\sqrt{6} + 2\sqrt{x}$$
$$\sqrt{6} + 2\sqrt{x}$$

$$6 + 2\sqrt{6}x$$
$$+ 2\sqrt{6}x + 4x$$

$$6 + 4x + 4\sqrt{6}x.$$

S E C T. VII.

Of EVOLUTION of SURDS.

THE powers of furds are found by multiplying their exponents with the index or exponent of the power to which they are to be involved; as we have fhewn; confequently, if thofe exponents be divided by the index of the root to be extracted, the quotient will be the exponent of the root; which gives the following

R U L E.

EXTRACT the root of the rational part, as in common extraction of roots; and annex the root of the furd, found by dividing the index of the furd, by the index of the root required.

EXAMPLES.

EXAMPLES.

The cube root of $\sqrt{a} = a^{\frac{1}{2} \div 3} = a^{\frac{1}{6}} = \sqrt[6]{a}$. The cube root of $\frac{8}{27}\sqrt{3} = \sqrt[3]{\frac{8}{27}} \times 3^{\frac{1}{2} \div 3} = \frac{2}{3} \times 3^{\frac{1}{6}} = \frac{2}{3}\sqrt[6]{3}$. The square root of $\sqrt[3]{\frac{a}{x}} = \sqrt[6]{\frac{a}{x}}$. The square root of $\sqrt[3]{a^3 + b^3} = \overline{a^3 + b^3}\big|^{\frac{1}{3} \div 2} = \overline{a^3 + b^3}\big|^{\frac{1}{6}}$. The cube root of $x^{-\frac{2}{3}} = x^{-\frac{2}{9}} = \frac{1}{\sqrt[9]{x^2}}$. If the proposed surds are binomial, residual, or trinomial, &c. find their roots as in Chap. VIII.

The square root of $x^8 + 6x^4\sqrt{y} + 9y = x^4 + 3\sqrt{y}$.

The n root of $20 + 2\sqrt{x} + z = \overline{20 + 2\sqrt{x} + z}\big|^{\frac{1}{n}}$.

CHAP. XI.

Of INFINITE SERIES.

AN infinite series, is formed from a fraction whose denominator is a compound quantity, by dividing the numerator by the denominator; or the extracting the root of a surd quantity, which if continued in either case, would run on sempiternally; that is, the number of terms in the series would be infinite; but by obtaining a few of the first terms of the series, you will easily perceive, what law the series observe in their progression; by which means you may continue the series by notation as far as you please, without an actual performance of the whole operation at large. PROB.

PROBLEM I.

To find an infinite series by division ; that is, to throw a compound fractional expression into such a series, whose sum, if the number of terms were continued ad infinitum, would be equal to the given fractional expression.

RULE.

DIVIDE the numerator by the denominator until you have 3, 4, 5, or more terms in the quotient.

EXAMPLES.

Throw $\dfrac{1}{y+v}$ into an infinite series.

OPERATION.

OPERATION.

$$y+v) \; 1 \quad (\frac{1}{y} - \frac{v}{y^2} + \frac{v^2}{y^3} - \frac{v^3}{y^4} + \&c.$$

$$1 + \frac{v}{y}$$

$$\overline{ }$$

$$0 - \frac{v}{y}$$

$$- \frac{v}{y} - \frac{v^2}{y^2}$$

$$\overline{ }$$

$$* + \frac{v^2}{y^2}$$

$$+ \frac{v^2}{y^2} + \frac{v^3}{y^3}$$

$$\overline{ }$$

$$* - \frac{v^3}{y^3}$$

$$- \frac{v^3}{y^3} - \frac{v^4}{y^4}$$

$$\overline{ }$$

$$* + \frac{v^4}{y^2}$$

$$\&c.$$

HERE the law of the progreſſion which the ſeries obſerve, is plain ; for each ſucceeding term is produced, by multiplying the preceding one with $-\frac{v}{y}$: Thus, the firſt term of the ſeries is $\frac{1}{y}$, which being multiplied with $-\frac{v}{y}$, gives $-\frac{v}{y^2}$ for the ſe-

cond

cond term, and $-\dfrac{v}{y^2} \times -\dfrac{v}{y} = \dfrac{v^2}{y^3} =$ third term;

also, $\dfrac{v^2}{y^3} \times -\dfrac{v}{y} = -\dfrac{v^3}{y^4}$ the 4th term, which multiplied with $-\dfrac{v}{y}$ will give the 5th term; and so on, multiplying the preceding term by the common ratio $-\dfrac{v}{y}$, you may find any number of terms at pleasure.

But in order to have a converging series, or a series wherein the terms continually decrease, the greatest term of the divisor must stand first in the order of arrangement; for suppose in the above example, that y is very great in respect of v; then will $\dfrac{v}{y^2}$ be very great in respect of $\dfrac{v^2}{y^3}$; so that in this supposition, the terms being multiplied with the powers of v, and divided by those of y; it follows, that each succeeding term is very little in respect of the preceding one, and consequently the series, a converging series. Again, put v for the first term of the divisor (the supposition the same as before) and the series will be $\dfrac{1}{v} - \dfrac{y}{v^2} + \dfrac{y^2}{v^3}$, &c. and since y is very great in respect of v; it follows, that $\dfrac{1}{v}$ is very little in respect of $\dfrac{y}{v^2}$, and $\dfrac{y}{v^2}$ very little in respect of $\dfrac{y^2}{v^3}$; consequently, the series is a diverging one; that is, a series whose terms continually increase, and therefore, the farther you proceed in them, the farther you will be from the truth. Hence, &c.

And

AND since it is impossible to assign an infinite number; it follows, that the number of terms expressing the true value of such a series, is not assignable; yet the taking of a few of the first terms will be sufficient for any practical purpose.

Throw $\dfrac{a^2}{v-d}$ into an infinite series.

OPERATION.

$$v-d)\ a^2 \qquad (\frac{a^2}{v}+\frac{a^2 d}{v^2}+\frac{a^2 d^2}{v^3}+\text{, }\&c.$$

$$a^2 - \frac{a^2 d}{v}$$

$$\ast + \frac{a^2 d}{v}$$

$$+ \frac{a^2 d}{v} - \frac{a^2 d^2}{v^2}$$

$$\ast + \frac{a^2 d^2}{v^2}$$

$$+ \frac{a^2 d^2}{v^2} - \frac{a^2 d^3}{v^3}$$

$$\ast \quad + \frac{a^2 d^3}{v^3}, \&c.$$

HERE, each preceding term, after the first, is multiplied with $\dfrac{d}{v}$, and the product is the next term following; therefore, the law of the progression is manifest.

Throw $\dfrac{1}{1+b^2}$ into an infinite series.

OPERATION.

OPERATION.

$$1 + b^2) 1 \qquad (1 - b^2 + b^4 - b^6, \&c.$$
$$\underline{1 + b^2}$$
$$\overline{0 - b^2}$$
$$\underline{- b^2 - b^4}$$
$$* \quad + b^4$$
$$\underline{+ b^4 + b_6}$$
$$* \quad - b^6$$
$$\underline{- b^6 - b^8}$$
$$* \quad + b^8$$
$$\&c.$$

Here the law of the continuation is the preceding terms multiplied with $- b^2$.

PROBLEM II.

To extract the root of a compound surd in an infinite series; that is, to throw a compound surd quantity into a converging series, whose sum, if the terms were infinitely continued would be equal to the root required.

RULE.

Extract the root of the quantity, as in common algebraic extraction; the operation continued as far as is thought necessary, will give the series required.

EXAMPLES.

Throw $\sqrt{a^2 + y^2}$ into an infinite series.

OPERATION.

OPERATION.

$$a^2 + y^2 \left(a + \frac{y^2}{2a} - \frac{y^4}{8a^3} +, \, \&c. \right.$$

$$\underline{a^2}$$

$$2a + \frac{y^2}{2a} \Big)* + y^2$$

$$+ y^2 + \frac{y^4}{4a^2}$$

$$2a + \frac{y^2}{4a} - \frac{y^4}{8a^3} \Big) - \frac{y^4}{4a^2}$$

$$- \frac{y^4}{4a^2} - \frac{y^6}{8a^4} + \frac{y^8}{64a^6}$$

$$+ \frac{y^6}{8a^4} - \frac{y^8}{64a^6}$$

$$\&c.$$

That is, $\overline{a^2 + x^2}\Big|^{\frac{1}{2}} = a + \frac{x^2}{2a} + \frac{x^4}{8a^3} + \&c.$

Find

Find the value of $\overline{1-x^2}\,|^{\frac{1}{2}}$ in an infinite series.

OPERATION.

$$1-x^2\ (1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}, \&c.$$

$$\underline{1}$$

$$2-\frac{x^2}{2})\ \ 0-x^2$$

$$\underline{-x^2+\frac{x^4}{4}}$$

$$2-x^2-\frac{x^4}{8})\ *\ -\frac{x^4}{4}$$

$$\underline{-\frac{x^4}{4}+\frac{x^6}{8}+\frac{x^8}{64}}$$

$$2-x^2-\frac{x^4}{4}-\frac{x^6}{16})\ *\ -\frac{x^6}{8}-\frac{x^8}{64}$$

$$\underline{-\frac{x^6}{8}+\frac{x^8}{16}+\frac{x^{10}}{64}+\frac{x^{12}}{256}}$$

$$\underline{\qquad * \qquad \&c.}$$

PROBLEM III.

To reduce any surd or fractional quantity into an infinite series, by the celebrated Binomial Theorem, invented by that Prince of Mathematicians, the illustrious Sir ISAAC NEWTON, which is as follows.

Binomial

Binomial Theorem.

$$\overline{P + PQ}|^{\frac{m}{n}} = P^{\frac{m}{n}} + \frac{m}{n}AQ + \frac{m-n}{2n}BQ + \frac{m-2n}{3n}CQ$$
$$+ \frac{m-3n}{4 \cdot n}DQ + \frac{m-4n}{5n}EQ + \frac{m-5n}{6n}FQ + \&c.$$

Wherein it is to be observed, that $P + PQ$ is the quantity whose power is to be thrown into an infinite series; P represents the first term of the proposed quantity; Q the other terms divided by the first; $\frac{m}{n}$ the index of the power, whether it be affirmative or negative: And $A =$ first term of the series; B the second; C the third; D the fourth; E the fifth; F the sixth, &c. that is, the several terms of the series, are $A = P^{\frac{m}{n}}$, $B = \frac{m}{n}AQ$, $C = \frac{m-n}{2n}BQ$, $D = \frac{m-2n}{3n}CQ$, &c.

EXAMPLES.

Reduce $\overline{a^2 + x^2}|^{\frac{1}{2}}$ into an infinite series.

Here $a^2 = P$, $\frac{x^2}{a^2} = Q$, $m = 1$, and $n = 2$:

Therefore, $A = P^{\frac{m}{n}} = a$, $B = \frac{m}{n}AQ = \frac{x^2}{2a}$, $C = \frac{m-n}{2n}BQ = -\frac{x^4}{8a^3}$, $D = \frac{m-3n}{4n}CQ = \frac{x^6}{16a^4}$, &c.

That is, $a + \frac{x^2}{2a} - \frac{x^4}{8a^3} + \frac{x^6}{16a^4}$, &c. *is the series required.*

Expand

Expand $\dfrac{1}{1+z} = \overline{1+z}\vert^{-1}$ into an infinite series.

Here $\overline{z+1}\vert^{-1} = \overline{1+z}\vert^{-\frac{1}{1}}$; *therefore,* $m = -1$, $n = 1$, $P = 1$, *and* $Q = \dfrac{z}{1}$: *Consequently,* $A = \left(P^{\frac{m}{n}}\right)$ 1, $B = \left(\dfrac{m}{n}AQ\right) - z$, $C = \left(\dfrac{m-n}{2n}BQ\right)z^2$, $D = \left(\dfrac{m-2n}{3n}CQ\right) - z^3$, $E = \left(\dfrac{m-3n}{4n}DQ\right)z^4$.

That is, $\overline{1+z}\vert^{-1} = 1 - z + z^2 - z^3 + z^4$, &c.

Find the value of $\dfrac{v}{a+y}$ in an infinite series.

Here $\dfrac{v}{a+y} = v \times \overline{a+y}\vert^{-1}$; *Wherefore,* $P = a$, $Q = \dfrac{y}{a}$, $m = -1$, *and* $n = 1$: *Then,* $A = a^{-1}$, *or* $\dfrac{1}{a}$ $B = -\dfrac{y}{a^2}$, $C = \dfrac{y^2}{a^3}$, $D = -\dfrac{y^3}{a^4}$.

That is, $v \times \overline{a+y}\vert^{-1} = v \times \dfrac{1}{a} - \dfrac{y}{a^2} + \dfrac{y^2}{a^3} - \dfrac{y^3}{a^4}$, &c.

Consequently, $\dfrac{v}{a+y} = \dfrac{v}{a} - \dfrac{vy}{a^2} + \dfrac{vy^2}{a^3} - \dfrac{vy^3}{a^4}$, &c.

PROBLEM IV.

To find the sum of an infinite series, geometrically decreasing.

RULE.

DIVIDE the square of the first term by the difference between the first and second, and the quotient will be the sum required. THUS,

Thus, the sum of the infinite series $v - a + \dfrac{a^2}{v}$

$- \dfrac{a^3}{v^2}$, &c. $= v^2 \div v + a$; and the sum of $v +$

$\dfrac{v^2}{a} + \dfrac{v^3}{a^2}$, &c. $= \dfrac{v^2}{av - v^2 \div a} = av \div \overline{a - v}$; for if v^2

be divided by $v + a$, and av by $a - v$, the quotients will be the series proposed. Therefore, the rule is manifest.

EXAMPLES.

Given $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8}$, &c. *ad infinitum*, to find their sum.

Thus, $1^2 \div 1 - \frac{1}{2} = 2$ *the sum required.*

Given $\frac{6}{10} + \frac{6}{100} + \frac{6}{1000}$, &c. *ad infinitum*, to find their sum.

Thus, $\overline{\dfrac{6}{10}}^2 \div \overline{\dfrac{6}{10} - \dfrac{6}{100}} = \frac{2}{3}$ *the sum required.*

Given $2 - \frac{2}{3} + \frac{2}{9} - \frac{2}{27}$, &c. *ad infinitum*, to find their sum.

Thus, $4 \div 2 + \frac{2}{3} = 1\frac{5}{7} = $ *sum required.*

CHAP. XII.

Of PROPORTION or ANALOGY ALGEBRAICALLY CONSIDERED.

WHEN quantities are compared together with regard to their differences, or quotients, their relations are expressed by their ratios. The relation of quantities, arising from the first comparirison

rifon, is expreffed by an arithmetical ratio, that of the fecond, by a geometrical ratio; and the quantities themfelves are faid to be in arithmetical, or geometrical proportion, as the ratios of their comparifon are arithmetical, or geometrical : Which proportions, together with fuch others as arife from the alternation, converfion, &c. of thofe proportions that are of any confiderable ufe in Mathematics, will be noticed in the following order.

S E C T. I.

Of *ARITHMETICAL PROPORTION.*

WHEN quantities increafe by addition or fubtraction of the fame quantity, thofe quantities are in arithmetical proportion : Thus, a, $a + d$, $a + 2d$, $a + 3d$, &c. or x, $x - d$, $x - 2d$, $x - 3d$, &c. are quantities in arithmetical proportion ; wherein the quantity d, which is continually added or fubtracted, is the common difference of the feries ; therefore, when in any four quantities, the difference between the firft and fecond, is equal to the difference between the third and fourth, thofe quantities are in arithmetical proportion ; as in thefe, y, $y - n$, $y - 2n$, $y - 3n$; where $y - \overline{y - n} = n$; and $\overline{y - 2n} - \overline{y - 3n} = n$. Therefore, &c.

T H E O R E M I.

If three quantities be in arithmetical proportion, the fum of the two extremes will be double the mean.

Thus if a, $a + d$, $a + 2d$ are in arithmetical proportion, then will $a + \overline{a + 2d} = \overline{a + d} + \overline{a + d}$.

THEO.

THEOREM II.

If four quantities be in arithmetical proportion, the sum of the two extremes will be equal to the sum of the two means.

Thus, if a, $a + d$, $a + 2d$, $a + 3d$ are quantities in arithmetical proportion, then will $\overline{a + a + 3d} = \overline{a + d + a + 2d}$.

THEOREM III.

In a series of arithmetical proportionals, the sum of the two extreme terms, is equal to the sum of any two terms equally distant from the extremes.

Let the series be a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$, &c. to z: Under which write the same series with their order inverted; then adding those terms together which stand directly opposite each other, and the sum of any two such terms, will be equal to the sum of the first and last terms, as plainly appears by the following

EXAMPLE.

Proposed series, a, $a + d$, $a + 2d$, $a + 3d$, $a + 4d$, &c. to z
Series inverted, z, $z - d$, $z - 2d$, $z - 3d$, $z - 4d$, &c. to a

$a + z$, $a + z$, $a + z$, $a + z$, $a + z$, &c. = the sum of every two terms.

Now from this example, a rule for finding the sum of all the terms of any arithmetical series, may be easily deduced; for it is plain, that the sum $a + z + a + z + a + z$, &c. or $a + z$, taken as many times as there are number of terms, is double the sum of the series a, $a + d$, $a + 2d$, &c. Consequently, that sum

divided

divided by 2, will be equal to the fum of the feries ;
that is, (puting $n =$ number of terms, and $s =$ fum
of the feries) $\frac{a+z \times n}{2} = \frac{na+nz}{2} = s$: Or in words,
the fum of the firft and laft terms multiplied with
half the number of terms, will give the fum of the
feries.

But in any arithmetical feries, the co-efficient of
the common difference (d) in any term, is 1 lefs
than the number of terms to that place ; confequent-
ly, its co-efficient in the laft term, is equal to the
number of terms lefs 1 ; and therefore, the laft term
$z = a + \overline{n-1} \times d = a + dn - d$. Confequently,
$s = a + a + \overline{dn - d} \times \frac{n}{2} = \frac{na + na + dn^2 - dn}{2}$
$= \frac{2na + dn^2 - dn}{2}$, which is a theorem for finding
the fum of any arithmetical feries, when the firft term,
common difference, and number of terms are given.
And univerfally, puting

$a =$ *firft term of an arithmetical feries,*
$d =$ *common difference,*
$l =$ *laft term,*
$n =$ *number of terms,*
$s =$ *fum of all the feries.*

Then having given any three of thofe five quan-
tities, the reft may be found by the following theo-
rems.

Theorem 1. $\frac{na+nl}{2} = s$. Theorem 2. $\frac{2s}{l+a} = n$.

Theorem 3. $\frac{l-a}{n-1} = d$. Theorem 4. $\frac{2s-na}{n} = l$.

R r

Theorem

Theorem 5. $\dfrac{2s-nl}{n}=a$. Or, $n=\dfrac{l-a}{d}+1$. $l=nd-d+a$. $a=l+d-nd$.

SECT. II.

Of GEOMETRICAL PROPORTION.

WHEN of four quantities, the product of the two extremes is equal to the product of the two means; those quantities are in geometrical proportion: As, a, ar, b, br; where $a \times br = ar \times b$: Also, when quantities increase with a common multiplier, or decrease by a common divisor; as, a, ar, ar^2, ar^3, ar^4, &c. and a, $\dfrac{a}{r}$, $\dfrac{a}{r^2}$, $\dfrac{a}{r^3}$, $\dfrac{a}{r^4}$, &c. those quantities are said to be in geometrical proportion continued, where the common multiplier or divisor r is the common ratio.

THEOREM I.

In any series of quantities in geometrical proportion continued, the first term hath the same ratio to the second, as the second hath to the third, and as the third to the fourth, &c.

THUS, in a, ar, ar^2, ar^3, ar^4, &c. and a, $\dfrac{a}{r}$, $\dfrac{a}{r^2}$, $\dfrac{a}{r^3}$, $\dfrac{a}{r^4}$, &c. $a : ar :: ar : ar^2 :: ar^2 : ar^3 :: ar^3 : ar^4 ::$ &c. and $a : \dfrac{a}{r} :: \dfrac{a}{r} : \dfrac{a}{r^2} :: \dfrac{a}{r^2} : \dfrac{a}{r^3} :: \dfrac{a}{r^3} : \dfrac{a}{r^4} ::$ &c. For, $a \times ar^2 = ar \times ar$, and $a \times ar^4 = ar \times ar^3$; also, $a \times \dfrac{a}{r^2} = \dfrac{a}{r} \times \dfrac{a}{r}$, and so on for the rest.

THEO.

THEOREM II.

In a series of geometrical proportionals continued, the product of the two extremes, is equal to the product of any two terms equally distant from the extremes.

THUS, in the series a, ar, ar^2, ar^3, ar^4, &c. If x be the last term, then will $\frac{x}{r}$ be the last term but one, and $\frac{x}{r^2}$ the last term but two; wherefore, $a \times x = ax$, the product of the two extremes, and $ar \times \frac{x}{r} = \frac{arx}{r} = ax$ the product of the second and last term but one: That is, $a \times x = ar \times \frac{x}{r}$; in like manner, $a \times x = ar^2 \times \frac{x}{r^2}$, and so on for the rest.

THEOREM III.

The sum of any series of quantities in geometrical proportion continued, is obtained by multiplying the last term by the ratio, and dividing the difference between that product and the first term, by the ratio less 1.

THUS, let the series whose sum is required, be $a + ar + ar^2 + ar^3 + ar^4$, which multiplied with r, gives $ar + ar^2 + ar^3 + ar^4 + ar^5$, from which subtract the former.

Thus, $\left\{ \begin{array}{l} ar + ar^2 + ar^3 + ar^4 + ar^5 \\ a + ar + ar^2 + ar^3 + ar^4 \end{array} \right.$

$$-a \quad * \quad * \quad * \quad * \quad + ar^5$$

Now

Now it is plain, that the difference $ar^s - a$ is equal to the fum of the propofed feries multiplied by $r-1$; confequently, the fame divided by $r-1$, will give the fum of the feries required: That is, (puting $s =$ fum)
$$\frac{ar^s - a}{r-1} = s.$$

Or, generally $ar + ar^2 + ar^3 + ar^4$, &c. $+ \frac{x}{r^4} + \frac{x}{r^3} + \frac{x}{r^2} + \frac{x}{r} + x = r \times a + ar + ar^2 \; ar^3$, &c. $+ \frac{x}{r^5} + \frac{x}{r^4} + \frac{x}{r^3} + \frac{x}{r^2} + \frac{x}{r}$. That is, the fum of any geometrical feries wanting the firft term, is equal to the fum of the fame feries wanting the laft term, multiplied with the ratio. Wherefore, $s - a = \overline{s - x} \times r$; that is, $s - a = sr - rx$, and $sr - s = rx - a$: Hence, $s = \frac{rx - a}{r - 1}$. And fince r is not in the firft term of the feries, it follows, that in the laft term, its exponent will be 1 lefs than the number of terms; and therefore, (puting $n =$ number of terms) $x = ar^{n-1}$: Confequently, $s =$ (by writing for x its equal ar^{n-1})
$$\frac{ar^{n-1} \times r - a}{r - 1} = \frac{ar^n - a}{r - 1}:$$ And univerfally, puting

$a =$ *firft term of a geometrical feries*,
$r =$ *ratio*,
$l =$ *laft term*,
$s =$ *fum of the feries*.

THEN having given any three of the aforefaid quantities, the reft may be readily found by the following theorems, which are deduced from the above equation.

Theorem

Theorem 1. $\dfrac{rl-a}{r-1}=s.$

2. $rl+s-sr=a.$

3. $\dfrac{s-a}{s-l}=r.$

4. $\dfrac{sr-s+a}{r}=l.$

THEOREM IV.

If four quantities are proportional, as $a:b::c:d$, then will any of the following forms, also be proportional. viz.

Directly, $a:b::c:d.$

Alternately, $a:c::b:d.$

Inversely, $b:a::d:c.$

Compoundedly, $\overline{a+b}:b::\overline{c+d}:d.$

Dividedly, $a:\overline{b-a}::c:\overline{d-c}.$

Mixtly, $\overline{b+a}:\overline{b-a}::\overline{d+c}:\overline{d-c}.$

SECT. III.

Of HARMONICAL PROPORTION.

HARMONICAL proportion arises from the comparison of musical intervals, or the relation of those numbers which assign the lengths of strings sounding musical notes.

THE most useful part of this proportion in practical Mathematics, is contained in the following theorems.

THEO.

THEOREM I.

If three quantities be in harmonical proportion, the first will be to the third, as the difference between the first and second, to the difference between the second and third.

THUS, if a, b and c, be in harmonical proportion, then, as $a : c :: \overline{b - a} : \overline{c - b}$: Consequently, $ac - ab = cb - ca$, by multiplying means and extremes: From which equation is deduced the following theorems.

Theorem 1. $\dfrac{cb}{2c - b} = a$. Theorem 2. $\dfrac{2ac}{a + c} = b$.

Theorem 3. $\dfrac{ab}{2a - b} = c$.

THEOREM II.

If four quantities be in harmonical proportion, the first will be to the fourth, as the difference between the first and second, is to the difference between the third and fourth.

THUS, if the quantities a, b, c, d, are harmonical proportionals, it will be, $a : d :: b - a : d - c$: Wherefore, $ad - ac = db - da$. From which equation, we get the following theorems.

$$1.\ a = \frac{db}{2d - c}.$$

$$2.\ b = \frac{2da - ac}{d}.$$

$$3.\ c = \frac{2da - db}{a}.$$

4.

$$4. \quad d = \frac{ac}{2a - b}.$$

CHAP. XIII.

Of SIMPLE EQUATIONS.

AN equation is an expreſſion, aſſerting the equali-
ty of two quantities, which are compared to-
gether by writing the quantities with the ſign of
equality between them. Thus, if $x + 3$ is equal to
$2x - 1$, the equation is formed thus, $x + 3 = 2x - 1$:
Alſo, $8 - 3 = 15 - 10$.

A SIMPLE equation, is an equation which in-
volves one unknown quantity, without including
its powers. Thus, $3x - 2 = 2x + 2$ is a ſimple equa-
tion which expreſſes the value of the unknown quan-
tity; when that quantity ſtands alone on one ſide of
the equation, the reſt being on the other ſide, which
if known, we then have a determined value of the
unknown quantity in known terms. And the buſi-
neſs of bringing the unknown quantity to ſtand
alone on one ſide of a ſimple equation, is called re-
duction of ſimple equations : To effect which pur-
poſe, we have the following rules.

RULE I.

ANY quantity may be taken from one ſide of an
equation and placed on the other, if you change its
ſign. Or which is the ſame thing, ſubtract the quan-
tity from both ſides.

FOR,

For, if from thofe quantities which are equal, there be taken the fame quantity, what remains will be equal.

EXAMPLES.

Given $x - 6 = 20$, to find the value of x.

Thus, $x = 20 + 6$, *per rule, and* $x = 26$ *by addition.*

For, -6 *taken from* $x - 6$, *leaves* x, *and* -6 *taken from* 20, *leaves* $20 + 6$, *or* 26, *by the nature of fubtraction. Therefore,* &c.

Given $x + 4 = 30 - 5$, to find the value of x.

Thus, $x = 30 - 5 - 4$ *by tranfpofition* :
Or, $x = 30 - 9 = 21$ *by addition and fubtraction.*
If $x - 3 + 1 = 21$:
Then will $x = 21 + 3 - 1$ *by tranfpofition* :
Or, $x = 23$ *by addition and fubtraction.*

R U L E II.

When the unknown quantity is multiplied with any number, it may be taken away by dividing all the reft of the terms in the equation by it.

For if thofe quantities which are equal, be divided by the fame quantity, their quotients will be equal.

EXAMPLES.

Given $4y - 12 = 2y + 4$, to find the value of y.

Firft, $4y - 2y = 12 + 4$ *by tranfpofition* :
Then, $2y = 16$ *by addition and fubtraction* :
Or, $y = \dfrac{16}{2} = 8$, *per rule.*

If

If $6y + 3 = y + 18$, then will $6y - y = 18 - 3$ by *tranfpofition*; *and* $5y = 15$ *by fubtraction.*

Whence, $y = \dfrac{15}{3} = 5$ *by divifion.*

Let $3x - 10 = 20 - x + 6$, be given to find x.
Firft, $3x + x = 20 + 6 + 10$ *by tranfpofition* :
Or, $4x = 36$, *and therefore,* $x = \dfrac{36}{4} = 9$.

RULE III.

WHEN any part of the equation is divided by any quantity, that quantity may be taken away by multiplying all the reft of the terms by it; which is the fame as to multiply all the terms in the equation by that quantity. And if thofe quantities which are equal, be multiplied with the fame quantity, their products will be equal.

EXAMPLES.

Given, $\dfrac{v}{6} + 2 = 10$, to find the value of v.

Thus, $v + 12 = 60$, *per rule* :
And $v = 60 - 12 = 48$ *by tranfpofition and fubtraction.*

Let $\dfrac{y}{2} + \dfrac{2y}{4} + \dfrac{3}{4} = 16$, be given to find y.

Firft, $\dfrac{16y}{32} + \dfrac{16y}{32} + \dfrac{24}{32} = 16$ *by reduction* :

Then, $\dfrac{32y + 24}{32} = 16$ *by addition* :

And $32y + 24 = 512$ *by multiplication* :

S 2 *Whence,*

Whence, $y = \dfrac{512 - 24}{32} = 15\frac{8}{32}.$

Alfo, if $\dfrac{3y}{2} + 6 = 2y + 4$, then will $3y + 12 = 4y + 8$ *per rule ;*

And $4y - 3y = 12 - 8$ *by tranfpofition.*

Whence, $y = 4.$

R U L E   IV.

If any quantity be found on both fides of the equation, having the fame fign, it may be expunged from both. Alfo, if all the terms of an equation be multiplied with the fame quantity, it may be ftruck out of them all.

EXAMPLES.

If $2x + 4a = x + 4a + 2$; then will $2x = x + 2$ *per rule :*

And $2x - x = 2$; *or,* $x = 2$:

Alfo, if $6x + c = b + c$, then will $6x = b$, *and* $x = \dfrac{b}{6}.$

Moreover, if $\dfrac{3xa}{c} + \dfrac{2xa}{c} - \dfrac{xa}{c} = \dfrac{da}{c}$, then will $3x + 2x - x = d$:

And $4x = d$ *by addition and fubtraction :*

Whence, $x = \dfrac{d}{4}.$

R U L E   V.

If that part of the equation which involves the unknown quantity be a radical expreffion, it may be
made

made free from furds by tranſpoſing the reſt of the terms by the preceding rules, ſo that the furd may ſtand alone on one ſide of the equation : Then take away the radical ſign, and involve the other ſide of the equation to the power whoſe index is equal to the denominator of the radical ſign.

EXAMPLES.

If $\sqrt{x+3} + 4 = 20$:

Then will $\sqrt{x+3} = 20 - 4 = 16$ by tranſpoſi-tion :

And $x + 3 = 16 \times 16 = 256$ by involution :

Or, $x = 256 - 3 = 253$.

And, if $4 + \sqrt{2x+6} = 9$; then will $\sqrt{2x+6} = 9 - 4 = 5$ by tranſpoſition :

And $2x + 6 = 25$ by involution :

Whence, $x = \dfrac{19}{2} = 9\frac{1}{2}$.

In like manner, if $\sqrt[3]{ax} + 3 = 10$; then will $\sqrt[3]{ax} = 10 - 3 = 7$; and $ax = 343$ by involution ; or, $x = \dfrac{343}{a}$.

R U L E VI.

IF both ſides of an equation be a complete power, or can be made ſo by the preceding rules, it may be reduced to more ſimple terms, by extracting the root of both ſides.

EXAMPLES.

Given, $y^2 + 6y + 9 - 57 = 87$, to find the val-ue of y.

Firſt,

First, $y^2 + 6y + 9 = 87 + 57 = 144$ *by tranf.*
Then, $y + 3 = 12$ *by extracting the root:*
Or, $y = 12 - 3 = 9$ *by transposition.*

Given, $9y^2 + 24y + 16 = 4y^2 + 32y + 64$, to find the value of y.

First, $3y + 4 = 2y + 8$ *by extracting the root:*
And $3y - 2y = 8 - 4$ *by transposition:*
That is, $y = 4$.

RULE VII.

ANY analogy may be converted into an equation, by afferting the product of the two extremes equal to the product of the two means.

EXAMPLES.

If $6 + x : 10 :: 4 : 6$; then will $36 + 6x = 40$, *by multiplying means and extremes,* and $6x = 4$; *or,* $x = \frac{4}{6}$.

And, if $\frac{2x}{3} : a :: 10 : 2$; then will $\frac{4x}{3} = 10a$; *and* $4x = 30a$; *or* $x = \frac{30a}{4}$.

And in like manner, if $6 : x - 2 :: 4 : 5$; then will $30 = 4x - 8$:

And $4x = 30 + 8 = 38$; *or,* $x = \frac{38}{4} = 9\frac{2}{4}$.

COROLLARY.

HENCE it follows, that an equation may be turned into an analogy, by dividing either side of it into
two

two such parts, which if multiplied together, would produce the same side again ; making those parts, either the two means or extremes ; then dividing the other side in like manner for the other two terms.

CHAP. XIV.

CONCERNING the extermination of un-known quantities, and reducing those equations which contain them, to a single one.

PROBLEM I.

To exterminate two unknown quantities, or reduce two equations containing them, to a single one.

RULE I.

FIND the value of one of the unknown quantities in each of the given equations, by the rules of the preceding chapter. And puting these two values equal to each other, you will have an equation in-volving only one unknown quantity ; which equa-tion if a simple one, is to be resolved as in the last chapter.

EXAMPLES.

Given, $\left\{ \begin{array}{l} 2x+y=14 \\ 6x-3y=30 \end{array} \right\}$ to find x and y.

From the first equation, we have $x = \dfrac{14-y}{2}$:

And

And from the second, $x = \dfrac{30 + 3y}{6}$:

Therefore, $\dfrac{14 - y}{2} = \dfrac{30 + 3y}{6}$:

And $84 - 6y = 60 + 6y$ *by multiplication :*

Whence, $84 - 60 = 12y$:

Or, $12y = 24$;

And therefore, $y = \dfrac{24}{2} = 2$, *and* $x = \dfrac{14 - y}{2} = (by$ *writing* 2 *for y its equal*$) \dfrac{14 - 2}{2} = 6$.

Given, $\left\{ \begin{array}{l} 3v + y = 22 \\ v : y :: 2 : 5 \end{array} \right\}$ to find v *and* y.

From the first equation, $v = \dfrac{22 - y}{3}$, *and the analogy turned into an equation, gives* $5v = 2y$, *or* $v = \dfrac{2y}{5}$, *and therefore,* $\dfrac{22 - y}{3} = \dfrac{2y}{5}$.

Whence we get, $110 - 5y = 6y$ *by multiplication :*
And $11y = 110$:

Or, $y = \dfrac{110}{11} = 10$:

And $v = \dfrac{2y}{5} = (by \ writing$ 10 *for y its equal*$) \dfrac{20}{5} = 4$.

RULE II.

Find the value of one of the unknown quantities in either of the given equations ; and inftead of the unknown quantity in the other equation, fubftitute its value thus found, and there will arife a new equation having only one unknown quantity, whofe value is to be found as before.

EXAMPLES.

EXAMPLES.

Given, $\left\{ \begin{array}{l} z + y = 10 \\ z - y = 7 \end{array} \right\}$ to find z and y.

From the first equation, we have $z = 10 - y$, *which substituted for z in the second equation,*

Gives $10 - y - y = 7$, or $10 - 2y = 7$:

And $2y = 10 - 7 = 3$:

Or, $y = 1.5$:

Whence, $z = ($ *by writing* 1.5 *for y its equal*$) 10 - 1.5 = 8.5$:

Given, $\left\{ \begin{array}{l} 2z - 2y = 10 \\ 3y + z = 65 \end{array} \right\}$ to find z and y.

From the first equation $z = \dfrac{10 + 2y}{2}$, *and this value substituted in the second equation, gives* $3y + \dfrac{10 + 2y}{2} = 65$:

Or, $6y + 10 + 2y = 130$; *whence,* $8y = 120$:

Or, $y = \dfrac{120}{8} = 15$; *and* $z = \dfrac{10 + 2y}{2} = 5 + y = 5 + 15 = 20$.

RULE III.

IF the unknown quantity is of lower dimension in one of the given equations than in the other; find the value of the unknown quantity in the equation where it is of least dimension, and raise this value to the same height as the unknown quantity in the other equation; or on the contrary. Then compare this value with the value of the unknown quantity

found

found from the other equation ; and you will have a new equation, with which proceed as before.

EXAMPLES.

Given, $\left\{ \begin{array}{l} v+y=10 \\ v^2-y^2=60 \end{array} \right\}$ to find v and y.

From the first equation $v=10-y$;

And therefore, $v^2=\overline{10-y}|^2=100-20y+y^2$:

Then, $100-20y+y^2=60+y^2$ *by rule 1st.*

Whence, $y=2$ *by reduction* :

Or, $100-20y+y^2-y^2=60$ *by rule 2d.*

Whence, $40=20y$:

Or, $y=\dfrac{40}{20}=2$ *as before* ;

And $v=10-y=10-2=8$:

Given, $\left\{ \begin{array}{l} z^2+y^2=25 \\ z^2 : yz :: 4 : 3 \end{array} \right\}$ to find z and y.

The analogy turned into an equation, gives $3z^2=4zy$, *which divided by* z, *gives* $3z=4y$, *or,* $z=\dfrac{4y}{3}$:

Whence, $z^2=\dfrac{16y^2}{9}$:

And therefore, $\dfrac{16y^2}{9}+y^2=25$:

Or, $16y^2+9y^2=225$:

Whence we get, $y^2=9$; *or,* $y=\sqrt{9}=3$:

And $z=\dfrac{4y}{3}=\dfrac{12}{3}=4$.

PROB.

PROBLEM II.

To exterminate any three unknown quantities, x, y, and z, or to reduce three simple equations that involve them, to a single one.

RULE.

FIND the value of x in the three given equations; then compare the firſt value of x with the ſecond, and there will ariſe a new equation involving only y and z. Again compare the firſt, or ſecond value of x with the third, and there will ariſe another equation involving only y and z; then proceed with theſe two equations as directed in the laſt problem.

EXAMPLE.

Given, $\begin{cases} 2x + y + 2z = 15 \\ x + 6y - z = 29 \\ 4x - 2z + 2y = 12 \end{cases}$ to find x, y and z.

From the firſt equation, we have, $x = \dfrac{15 - 2z - y}{2}$

From the ſecond, $x = 29 + z - 6y$:

From the third, $x = \dfrac{12 + 2z - 2y}{4}$:

Whence, $\dfrac{15 - 2z - y}{2} = 29 + z - 6y$:

And $29 + z - 6y = \dfrac{12 + 2z - 2y}{4}$:

From the firſt of theſe equations, we get $15 - 2z - y = 58 + 2z - 12y$; *or,* $11y = 58 - 15 + 4z$:

T t

Whence,

Whence, $y = \dfrac{43 + 4z}{11}$.

From the second, we have $116 + 4z - 24y = 12 + 2z - 2y$:

That is, $22y = 116 - 12 + 2z$; *or,* $y = \dfrac{104 + 2z}{22}$

Consequently, $\dfrac{104 + 2z}{22} = \dfrac{43 + 4z}{11}$:

Whence, $1144 + 22z = 946 + 88z$:

And $88z - 22z = 198$:

That is, $66z = 198$:

Or, $z = \dfrac{198}{66} = 3$:

Whence, $y = \dfrac{43 + 4z}{11} = \dfrac{43 + 12}{11} = 5$, *and* $x = \dfrac{15 - 2z - y}{2} = \dfrac{15 - 6 - 5}{2} = 2$.

AND, nearly in the fame manner, may be exterminated any number of unknown quantities; but there are often much fhorter methods for their extermination, which are beft learned by practice; yet fome of them may be thus generally given.

R U L E.

LET the given equations be multiplied or divided by fuch numbers, or quantities, that by addition, fubtraction, multiplication, divifion, involution or evolution of any two, or more of the equations, one or more of the unknown quantities may vanifh. Then taking the refult and the other equations, and proceed as before, until you have an equation involving

volving only one unknown quantity, whofe value may be found by the foregoing rules.

EXAMPLES.

Given, $\left\{ \begin{array}{l} 2x + 3y = 29 \\ 3x + 2y = 31 \end{array} \right\}$ to find x and y.

Multiply the firft equation with 2, and it will give $4x + 6y = 58$, *and the fecond with 3, gives* $9x + 6y = 93$, *from which fubtract,* $4x + 6y = 58$; *and you will have*

$5x = 35$; *or,* $x = \dfrac{35}{5} = 7$, *and* $3y = 29 - 2x$; *or,* $y = \dfrac{29 - 2x}{3} = \dfrac{29 - 14}{3} = 5.$

Given, $\left\{ \begin{array}{l} 2x + 4y + 3z = 38 \\ 3x + 5y + 6z = 63 \\ 4x + 7y + 12z = 109 \end{array} \right\}$ to find $x, y,$ and z.

From double the firft equation fubtract the fecond, and from double the fecond, fubtract the third, and the refults will be, $\left\{ \begin{array}{l} x + 3y = 13 \\ 2x + 3y = 17. \end{array} \right.$

Again, from the fecond of thefe equations, fubtract the firft, and the refult will be $x = 4$; *and from double the firft fubtract the fecond, and it will give* $3y = 9$; *or,* $y = \dfrac{9}{3} = 3.$ *And from the firft of the given equations, we have* $3z = 38 - 2x - 4y$; *or,* $z = \dfrac{38 - 2x - 4y}{3} = \dfrac{38 - 8 - 12}{3} = 6.$

Miscellaneous Examples.

Given, $\left\{\begin{array}{l} v + y = 12 \\ vy = 32 \end{array}\right\}$ to find v and y.

The first equation involved to a square, gives $v^2 + 2vy + y^2 = 144$; *and* $v^2 - 2vy + y^2 = 16$ *by subtracting* $4vy$ ($= 128$) *from the last equation; or,* $v - y = 4$ *by evolution:*

And therefore, $\overline{v + y} + \overline{v - y} = 12 + 4$:

Or, $2v = 16$; *and* $v = \frac{16}{2} = 8$:

Again, $\overline{v + y} - \overline{v - y} = 12 - 4$:

That is, $2y = 8$; *or,* $y = \frac{8}{2} = 4$.

Given, $\left\{\begin{array}{l} vy = 144 \\ \dfrac{v}{y} = 9 \end{array}\right\}$ to find v and y.

First, $v = 9y$ *by multiplication:*
Consequently, $vy = 9y \times y = 144$:

That is, $9y^2 = 144$; *or,* $y^2 = \frac{144}{9} = 16$.

Whence, $y = \sqrt{16} = 4$; *and* $v = 9y = 36$.

Given, $\left\{\begin{array}{l} v - y = 56 \\ \dfrac{v}{y} = 8 \end{array}\right\}$ to find v and y.

First, $v = 56 + y$; *and therefore,* $\dfrac{56 + y}{y} = 8$;

or, $56 + y = 8y$; *whence,* $y = \frac{56}{7} = 8$, *and* $v = 8y = 64$.

Given,

Given, $\left\{ v + \sqrt{16 + v^2} = \dfrac{32}{\sqrt{16 + v^2}} \right\}$ to find v.

First, $v \times \sqrt{16 + v^2} + \sqrt{16 + v^2} \times \sqrt{16 + v^2} = 32.$

That is, $v\sqrt{16 + v^2} + 16 + v^2 = 32$:

Then, $v\sqrt{16 + v^2} = 16 - v^2$:

And by involution, $v^2 \times \overline{16 + v^2} = \overline{16 - v^2}\big|^2 = 256 - 32v^2 + v^4$:

That is, $16v^2 + v^4 = 256 - 32v^2 + v^4$:

Or, $16v^2 = 256 - 32v^2$; and $16v^2 + 32v^2 = 256$:

Whence, $v^2 = \dfrac{256}{48}$; or, $v = \sqrt{\dfrac{256}{48}} = \sqrt{\dfrac{256 \times 1}{16 \times 3}}$

$= \dfrac{16}{4}\sqrt{\dfrac{1}{3}} = 4\sqrt{\dfrac{1}{3}}.$

Given, $\left\{ \begin{array}{l} x^2 + y^2 = a \\ xy = b \end{array} \right\}$ to find x and y.

First, $x^2 + 2xy + y^2 = a + 2b$:

Then, $x + y = \sqrt{a + 2b}.$

Again, $x^2 - 2xy + y^2 = a - 2b$:

Then, $x - y = \sqrt{a - 2b}$:

Therefore, $\overline{x + y} + \overline{x - y} = \sqrt{a + 2b} + \sqrt{a - 2b}$

That is, $2x = \sqrt{a + 2b} + \sqrt{a - 2b}$:

Or, $x = \dfrac{\sqrt{a + 2b} + \sqrt{a - 2b}}{2}$:

And $\overline{x + y} - \overline{x - y} = 2y = \sqrt{a + 2b} - \sqrt{a - 2b}$

Whence, $y = \dfrac{\sqrt{a + 2b} - \sqrt{a - 2b}}{2}.$

C H A P.

CHAP. XV.

Of the SOLUTION *of a variety of* QUES-
TIONS, *that produce* SIMPLE EQUA-
TIONS.

AFTER forming a clear and diftinct idea of the
queſtion propoſed ; the unknown quantities
muſt be expreſſed by letters, which muſt be ordered in
ſuch a manner, as to expreſs the conditions given in
the queſtion concerning thoſe quantities. Thus, if
the ſum (*s*) of two quantities (*x* and *y*) are requir-
ed ; then is *x* + *y* = *s*, an expreſſion anſwering that
condition. Alſo, if the difference *(d)* of thoſe
quantities is required ; that condition muſt be ex-
preſſed thus, *x* — *y* = *d* (*x* being the greater) Their
product *(p)* is expreſſed thus, *xy* = *p*. Their quo-

tient *(q)* is $\frac{x}{y}$ = *q*. Alſo, the ſum of their ſquares
(a) is expreſſed thus, $x^2 + y^2 = a$, and the differ-
ence of their ſquares *(b)* thus, $x^2 - y^2 = b$.

HAVING expreſſed the unknown quantities in
equations anſwering their relations, or properties, as
given in the queſtion ; you are next to conſider
whether your queſtion is limited or not ; that is, whe-
ther the quantities ſought, are each of them capable
of more known values than one ; which may always
be diſcovered in the following manner. If the
equations that ariſe from expreſſing the conditions of
the queſtion, are in number equal to the quantities
ſought, then is the queſtion truly limited : That is,
each of the quantities ſought, cannot have more val-
ues than one in giving the anſwer : But, if the equa-

tions expressing the conditions of the question, are fewer in number than the quantities sought, then the question is an unlimited one ; that is, the quantities sought, are each of them of an indeterminate value, and consequently, the question proposed, capable of innumerable answers.

AFTER you have discovered that the proposed question is limited ; you must then proceed to exterminate the unknown quantities by the rules already given, or other methods, which you may learn by practice ; to which we now proceed.

1. What number is that, from which if you take 40, the remainder will be 115 ?

Call the number sought v:
Then will $v - 40 = 115$ by the question:
Or, $v = 115 + 40 = 155$ the number sought.

2. What number is that, from which if you take 10, and multiply the remainder with 4, the product will be 30 ?

Call the number sought v:
Then will $v - 10$ be the remainder :
And $v - 10 \times 4 = 30$ by the question :
That is, $4v - 40 = 30$:
Or, $4v = 30 + 40 = 70$; or, $v = \frac{70}{4} = 17\frac{1}{2}$.

3. To find two numbers whose sum is 80, and their difference 16.

Let $v =$ the least of the required numbers :
Then will $v + 16 =$ the greater by the nature of subtraction :
And $v + v + 16 = 80$ by the question :
That is, $2v = 80 - 16 = 64$:
Or, $v = \frac{64}{2} = 32$; and $v + 16 = 32 + 16 = 48$, the greater number required.

4.

4. What number is that, which if multiplied with one third of itſelf, will produce the number ſought?

If you call the number ſought v :

Then will $\dfrac{v}{3}$ be one third part of v :

And $v \times \dfrac{v}{3} = v$ by the queſtion :

That is, $\dfrac{v^2}{3} = v$; or, $v^2 = 3v$, and $v = 3$ the number ſought.

5. Suppoſe the diſtance between Boſton and York, to be 150 miles ; and that a traveller ſets out from Boſton, and travels at the rate of 5 miles an hour ; another ſets out at the ſame time from York, and travels at the rate of 8 miles an hour : It is required to know how far each will travel before they meet.

If you put v for the diſtance that muſt be travelled by the one which ſets out from Boſton, and y the diſtance travelled by the other before they meet :

Then will $v + y = 150$, the diſtance travelled by both, and $v : y :: 5 : 8$ by the queſtion :

That is, $8v = 5y$; or, $v = \dfrac{5y}{8}$.

Alſo, $v = 150 - y$; conſequently, $\dfrac{5y}{8} = 150 - y$:

That is, $5y = 1200 - 8y$:
Whence, $y = 1200 \div 13 = 92\frac{4}{13}$.
And $v = 150 - y = 57\frac{9}{13}$.

6. What fraction is that, if you add 1 to the numerator, the value will be $\frac{1}{2}$; but if you add 1 to the denominator, the value will be $\frac{1}{3}$:

Put $\dfrac{v}{y}$ for the fraction ſought :

Then

Then will, $\dfrac{v+1}{y} = \dfrac{1}{2}$ $\Bigg\}$ by the question.

And $\dfrac{v}{y+1} = \dfrac{1}{3}$

That is, $2v + 2 = y$; or, $v = \dfrac{y-2}{2}$:

And $3v = y + 1$; or, $v = \dfrac{y+1}{3}$:

Consequently, $\dfrac{y-2}{2} = \dfrac{y+1}{3}$:

Or, $3y - 6 = 2y + 2$:

Whence, $y = 8$, the denominator :

And $v = \dfrac{y-2}{2} = \dfrac{8-2}{2} = 3$ the numerator :

Therefore, $\frac{3}{8}$ is the fraction required.

7. What two numbers are those whose sum is 60, and the sum of their squares 2250 ?

Call one of the numbers w, *and the other* y :

Then will, $w + y = 60$ $\Bigg\}$ by the question.

And $w^2 + y^2 = 2250$

First, $w^2 + 2wy + y^2 = 60^2 = 3600$:

And $w^2 + 2wy + y^2 - w^2 + y^2 = 2wy = 1350$:

Therefore, $4wy = 2700$:

Then, $w^2 + 2wy + y^2 - 4wy = w^2 - 2wy + y^2 = 900$:

Whence, $w - y = \sqrt{900} = 30$:

And $2w = 60 + 30 = 90$, or $w = 45$:

And $y = 60 - w = 60 - 45 = 15$.

8. There are three numbers in arithmetical progression, the first added to the second will make 15, and the second added to the third, 21 : What are those numbers ?

U u

Let

Let x, y and *z* reprefent the three numbers :

Then will $x + y = 15$, *the fum of the firft and fe-cond :*

And $y + z = 21$, *the fum of the fecond and third :*

Alfo, $x + z = 2y$ *by the nature of the proportion.*

Whence, $x + y + y + z = 15 + 21 = 36$:

That is, $x + 2y + z = 36$; *or,* $x + z = 36 - 2y$:

But $x + z = 2y$; *therefore,* $2y = 36 - 2y$; *or,* $4y = 36$:

Whence, $y = 9$; *and* $x = 15 - y = 15 - 9 = 6$:

And $z = 21 - y = 21 - 9 = 12$.

9. Two merchants traded in partnerfhip ; the fum of their ftocks was 600 dollars ; one's ftock was in company 8 months, but the other drew out his at the end of 6 months, when they fettled their accounts, and divided the gain equally between them : What was each man's ftock ?

Call one of the ftocks x ; then $600 - x =$ *the other :*

But, $x : 600 - x :: 6 : 8$ *by the queftion :*

Confequently, $8x = 3600 - 6x$; *or,* $14x = 3600$:

Whence, $x = \dfrac{3600}{14} = 257\frac{1}{7}$; *and* $600 - x = 600 - 257\frac{1}{7} = 342\frac{6}{7}$ *the others ftock.*

10. To find three numbers, fuch that if the firft be added to the fecond, their fum will be 12 ; and the fecond added to the third, their fum will be 20 ; alfo, if the firft be added to the third, their fum will be 16.

Call the firft number x, the fecond y, and the third z :

Then will $\left.\begin{array}{c} x + y = 12 \\ \text{And } y + z = 20 \\ \text{Alfo, } x + z = 16 \end{array}\right\}$ *by the queftion.*

Therefore, $x + y + x + z = 12 + 16 = 28$:

That

That is, $2x + y + z = 28$: *But* $y + z = 20$:
Consequently, $2x + 20 = 28$; *or,* $2x = 8$:
Whence, $x = 4$, *and* $y = 12 - x = 12 - 4 = 8$:
And $z = 20 - y = 20 - 8 = 12$.

11. There are four numbers in arithmetical progression ; whereof the product of the two extremes is 112, and the product of the two means 130 ; also, the sum of the first and second terms is 17 : What are those numbers ?

Put x *for the least term, and* y *the common difference ; then will* x, $x + y$, $x + 2y$, $x + 3y$ *be the four numbers required :*

$$\left. \begin{array}{r} And \ x \times \overline{x + 3y} = 112 \\ Also, \ \overline{x + y} \times \overline{x + 2y} = 130 \\ And \ x + \overline{x + y} = 17 \end{array} \right\} \ by \ the \ question.$$

That is, $x^2 + 3xy = 112$:
And $x^2 + 3xy + 2y^2 = 130$:

Whence, $\overline{x^2 + 3xy + 2y^2} - \overline{x^2 + 3xy} = 130 - 112 = 18$:

That is, $2y^2 = 18$; *or,* $y^2 = \frac{18}{2} = 9$, *and* $y = \sqrt{9} = 3$:

But, $x + \overline{x + y} = 17$; *that is,* $2x + 3 = 17$:
Or, $2x = 17 - 3 = 14$:
Consequently, $x = \frac{14}{2} = 7$, *the first term of the progression* ; *and therefore,* $x + y = 10$, *the second term* ; *and* $x + 2y = 13$, *the third term* ; *also,* $x + 3y = 16$, *the fourth term.*

So that 7, 10, 13, *and* 16, *are the numbers required.*

12. There are three numbers in arithmetical progression ; the product of the two extremes, is 128, and the product of the least extreme with the mean, is 96: What are those numbers ?

Call

Call the numbers required, v, y and z ; v and z be-ing the extremes, whereof v is the least.

Then will $vz = 128$
$vy = 96$ } *by the question.*

And $v + z = 2y$ *by the nature of the proportion :*

$z = \dfrac{128}{v}$ *from the first equation :*

And $z = 2y - v$ *from the third :*

Consequently, $\dfrac{128}{v} = 2y - v$ *by equality :*

That is, $128 = 2yv - v^2$: *But* $2yv = 96 \times 2 = 192$:

Therefore, $128 = 192 - v^2$ *by substitution :*

And $v^2 = 192 - 128 = 64$; *or,* $v = \sqrt{64} = 8$:

Also, $z = \dfrac{128}{v} = \dfrac{128}{8} = 16$; *and* $v + z = 2y$;

or, $y = \dfrac{v + z}{2} = \dfrac{8 + 16}{2} = 12$; *and therefore the num-bers sought are* 8, 12, 16.

13. To find a fraction, such that the square of the numerator, added to the denominator, shall make 30 ; and if 2 be added to the denominator, the value of the fraction will be equal to the reciprocal of the numerator.

Put $\dfrac{v}{y}$ *for the fraction sought.*

Then will $v^2 + y = 30$
And $\dfrac{v}{y + 2} = \dfrac{1}{v}$ } *by the question.*

First, $v^2 = 30 - y$:

And $v^2 = y + 2 \times 1 = y + 2$:

Consequently, $y + 2 = 30 - y$; *that is,* $2y = 28$:

Or,

Or, $y = \frac{28}{2} = 14$; *and* $v^2 = 30 - y = 30 - 14 = 16$; *or*, $v = \sqrt{16} = 4.$

So that the fraction sought, is $\frac{4}{14}$; *for*, $\frac{4}{14+2} = \frac{4}{16} = \frac{1}{4} = \frac{1}{v}$; *Therefore*, &c.

14. To find a number consisting of two places, such that the sum of its digits shall be 5, and if 9 be subtracted from it, the digits will be inverted.

Let v *and* y *represent the two digits,* v *that which stands in the tenth's place.*

Then by the nature of notation, we have $10v + y =$ *the number sought.*

$$\left.\begin{array}{c} \textit{Therefore, } v + y = 5 \\ \textit{And } 10v + y - 9 = 10y + v \end{array}\right\} \textit{by the question.}$$

Whence, $9v = 9y + 9$; *or*, $v = \frac{9y + 9}{9} = (by\ division)\ y + 1$:

Also, $v = 5 - y$; *and therefore*, $y + 1 = 5 - y$; *or*, $2y = 4$:

And $y = \frac{4}{2} = 2$; *and* $v = y + 1 = 2 + 1 = 3$: *So that* 32 *is the number required.*

15. A certain company at an inn; when they came to pay their reckoning, found that if there had been two persons less in company, they would have paid a dollar a man more; but if there had been three persons more in company, they would each of them paid a dollar less: What was their reckoning, and the number of persons to pay it?

Put $v =$ *the number of persons, and* y *the number of dollars each paid; then will* $vy =$ *the whole reconing.*

Whence,

Whence, $\dfrac{vy}{v-2} = y+1$
And $\dfrac{vy}{v+3} = y-1$ $\Bigg\}$ *by the question.*

That is, $vy = vy + v - 2y - 2$ *from the first equation:*

Or, $2y + 2 = v$:

And $vy = vy - v + 3y - 3$ *from the second equation:*

Or, $v = 3y - 3$:

Consequently, $3y - 3 = 2y + 2$; *or,* $3y - 2y = 2 + 3$:

Whence, $y = 5$, *the number of dollars each paid:*

And $v = 2y + 2 = 12$, *the number of persons:* *Consequently,* $vy = 60$ *dollars, the whole reckoning.*

16. To find three numbers v, y and w, the product of each with the sum of the other two being given, viz. $v \times \overline{y + w} = 930$; $y \times \overline{v + w} = 1300$, and $w \times \overline{v + y} = 1480$:

Or, $\begin{cases} vy + vw = 930 = a \\ vy + wy = 1300 = b \\ vw + wy = 1480 = c \end{cases}$

Then, $vy + vw + vw + wy = a + c$. But $vy + wy = b$:

And therefore, $2vw = a + c - b$; or, $vw = \dfrac{a+c-b}{2}$

Also, $vy + vw + vy + wy = a + b$: But $vw + wy = c$:

Wherefore, $2vy = a + b - c$; or, $vy = \dfrac{a+b-c}{2}$:

Again, $vy + wy + vw + wy = b + c$: But $vy + vw = a$:

And therefore, we have $2wy = b + c - a$; or, $wy = \dfrac{b+c-a}{2}$

$$\frac{b + c - a}{2}.$$

But $y = \frac{a + b - c}{2v}$; and by writing this value for y

in the equation, $wy = \frac{b + c - a}{2}$, we have $wy =$

$\frac{aw + bw - cw}{2v} = \frac{b + c - a}{2}$; whence by reduction,

$v = \frac{aw + bw - cw}{b + c - a}$: Also, $vw = \frac{a + c - b}{2}$; or, v

$= \frac{a + c - b}{2w}$; and therefore by equality, $\frac{aw + bw - cw}{b + c - a}$

$= \frac{a + c - b}{2w}$:

Or, $2aw^2 + 2bw^2 - 2cw^2 = 2ab + c^2 - b^2 - a^2$:

Whence, $w^2 = \frac{2ab + c^2 - b^2 - a^2}{2a + 2b - 2c}$:

Or, $w = \sqrt{\frac{2ab + c^2 - b^2 - a^2}{2a + 2b - 2c}}$: Which expression

turned into numbers, and the root extracted, w will

be found $= 37$; whence the other numbers are readily

found; for $v = \frac{a + c - b}{2w} = 15$, and $y = \frac{a + b - c}{2v}$

$= 25$:

17. Two women went to market with 42 eggs,
for which they received equal sums of money; af-
terwards says one to the other, if I had sold as many
eggs as you, I should have received 350 cents; says
the other, if I had sold no more than you, I should
have received but 14 cents. Query, the number of
eggs each sold, and the particular prices sold at; al-
so the number of cents each received.

Let

Let $v =$ number of eggs fold by one, and y the number fold by the other; alfo, $u =$ price which v eggs were fold at per egg, and w the price that y eggs were fold per egg.

Then will
$$\left\{ \begin{array}{l} v+y = 42 \\ vu = yw \\ vw = 350 \\ yu = 14 \end{array} \right\}$$
by the queftion.

From the third equation we have, $v = \dfrac{350}{w}$: From the fourth equation, $u = \dfrac{14}{y}$; and therefore, $vu = \dfrac{350}{w} \times \dfrac{14}{y} = \dfrac{4900}{yw}$: But $vu = yw$ from the fecond equation; wherefore, $\dfrac{4900}{yw} = yw$; or $4900 = y^2 w^2$, and $yw = \sqrt{4900} = 70$; whence, $y = \dfrac{70}{w}$: But $y = \dfrac{14}{u}$ from the fourth equation; confequently, $\dfrac{70}{w} = \dfrac{14}{u}$; or, $70u = 14w$; or, $5u = w$: And by writing $5u$ for w in the fecond equation, we have $vu = 5uy$, or dividing both fides by u, we fhall have $v = 5y$: But $v = 42 - y$ from the firft equation; therefore, $5y = 42 - y$; or, $6y = 42$; whence, $y = \dfrac{42}{6} = 7$, $v = 5y = 35$, $u = \dfrac{14}{y} = 2$, and $w = 5u = 10$.

18. Given the fum (s) and product (p) of two quantities, to find the fum of their fquares, cubes, biquadrates, &c.

Let v and w reprefent the two quantities:

Then will
$$\left\{ \begin{array}{l} v + w = s \\ vw = p \end{array} \right\}$$
by the queftion.

And

And $\overline{x + y}|^2 = x^2 + 2xy + y^2 = s^2$ *by involution*:

Or, $x^2 + 2xy + y^2 - 2xy = s^2 - 2p$ *by subtract.*

That is, $x^2 + y^2 = s^2 - 2p =$ *sum of the squares.*

Again, $\overline{x^2 + y^2} \times \overline{x + y} = \overline{s^2 - 2p} \times s$:

That is, $x^3 + xy \times \overline{x + y} + y^3 = s^3 - 2sp$:

Or, $x^3 + sp + y^3 = s^3 - 2sp$ *by writing sp for its equal,* $xy \times \overline{x + y}$; *whence,* $x^3 + y^3 = s^3 - 2sp - sp = s^3 - 3sp =$ *sum of their cubes.*

Also, $\overline{x^3 + y^3} \times x + y = \overline{s^3 - 3sp} \times s$:

That is, $x^4 + xy \times \overline{x^2 + y^2} + y^4 = s^4 - 3s^2p$; or, *(by writing for* $xy \times \overline{x^2 + y^2}$ *its equal,* $s^2p - 2p^2$) $x^4 + \overline{s^2p - 2p^2} + y^4 = s^4 - 3s^2p$; *whence,* $x^4 + y^4 = s^4 - 4s^2p + 2p^2 =$ *sum of their fourth powers.*

And $\overline{x^4 + y^4} \times x + y = \overline{s^4 - 4s^2p + 2p^2} \times s$:

That is, $x^5 + xy \times \overline{x^3 + y^3} + y^5 = s^5 - 4s^3p + 2p^2s$; *and therefore, (by writing for* $xy \times \overline{x^3 + y^3}$ *its equal* $s^3p - 3sp^2$) *we have,* $x^5 + \overline{s^3p - 3sp^2} + y^5 = s^5 - 4s^3p + 2sp^2$; *and by transposition, we get* $x^5 + y^5 = s^5 - 5s^3 + 5sp^2$ *for the sum of their fifth powers; and so on for the rest.*

CHAP. XVI.

Of QUADRATIC EQUATIONS.

A QUADRATIC EQUATION, is an equation of two dimensions involving only one unknown quantity; and is either simple or adfected.

X x

A.

A SIMPLE quadratic, is an equation which involves only the square of the unknown quantity. Thus, $v^2 = a^2$ is a simple quadratic equation.

BUT when you have an equation which involves the square of the unknown quantity, together with its product with some known co-efficient, you have what is called an adfected quadratic equation. Thus, $v^2 + av = bc$, is an adfected quadratic equation.

ALL adfected quadratic equations, fall under the three following forms:

$$\text{viz.} \begin{cases} v^2 + av = bc \\ v^2 - av = bc \\ v^2 - av = -bc \end{cases}$$

THE solution of adfected quadratic equations, or finding the value of the unknown quantity in those equations, is performed by the following

R U L E.

1. TRANSPOSE all the terms that involve the unknown quantity to one side of the equation, and all the terms that are known to the other side.

2. IF the square of the unknown quantity is multiplied with any co-efficient, you must cast off that co-efficient, by dividing all the terms in the equation by it, that the co-efficient of the highest dimension of the unknown quantity may be unity.

3. ADD the square of half the co-efficient prefixed to the unknown quantity, to both sides of the equation; and that side which involves the unknown quantity will then become a complete square.

4. EXTRACT the root from both sides of the equation, which will consist of the unknown quantity connected with half the aforesaid co-efficient; and therefore by transposing this half, the value of the unknown quantity will be determined.   SOL.

SOLUTION of the THREE FORMS Of QUADRATICS ILLUSTRATED.

Let it be required to determine the value of v, in the form $v^2 + av = bc$.

First, $v^2 + av + \dfrac{a^2}{4} = bc + \dfrac{a^2}{4}$ *by adding the sqr. of* $\dfrac{a}{2}$ *to both sides of the equation :* Then $v + \dfrac{a}{2} = \sqrt{bc + \dfrac{a^2}{4}}$ *by extracting the root of both sides ; or,* $v = \sqrt{bc + \dfrac{a^2}{4}} - \dfrac{a}{2}$ *by transposition.* But the square root of any positive quantity, may be either positive, or negative ; that is, the square root of $+ n^2$ may be either $+ n$ or $- n$; for $+ n \times + n$; or, $- n \times - n$, are respectively equal to $+ n^2$. It follows therefore, that all quadratic equations admit of two solutions, that is, the unknown quantity has two values in the given equation. Thus, in the foregoing example, where $v^2 + av + \dfrac{a^2}{4} = bc + \dfrac{a^2}{4}$, we may infer, that $v + \dfrac{a}{2} = \sqrt{bc + \dfrac{a^2}{4}}$ or, $- \sqrt{bc + \dfrac{a^2}{4}}$; for, $+ \sqrt{bc + \dfrac{a^2}{4}} \times + \sqrt{bc + \dfrac{a^2}{4}}$ or, $- \sqrt{bc + \dfrac{a^2}{4}} \times - \sqrt{bc + \dfrac{a^2}{4}}$ are each equal to $bc + \dfrac{a^2}{4}$; and therefore the two values of v, are $v = \sqrt{bc + \dfrac{a^2}{4}} - \dfrac{a}{2}$, and $v = - \sqrt{bc + \dfrac{a^2}{4}} - \dfrac{a}{2}$: which ambiguity

ambiguity is expressed by writing the uncertain sign $\pm$ *before* $\sqrt{bc + \frac{a^2}{4}}$: *Thus,* $v + \frac{a}{2} = \pm \sqrt{bc + \frac{a^2}{4}}$, *or* $v = \pm \sqrt{bc + \frac{a^2}{4}} - \frac{a}{2}$.

In the first expression for the value of v, viz. $\sqrt{bc + \frac{a^2}{4}} - \frac{a}{2}$, the only negative quantity is $\frac{a}{2} = \sqrt{\frac{a^2}{4}}$ which is evidently less than $\sqrt{bc + \frac{a^2}{4}}$; and consequently, the value of v is positive: But in the second expression, viz. $v = -\sqrt{bc + \frac{a^2}{4}} - \frac{a}{2}$, having $\sqrt{bc + \frac{a^2}{4}}$, and $\frac{a}{2}$ both negative; it follows, that the value of v must also be negative.

Again, if $z^2 - az = bc$:

Then will $z^2 - az + \frac{a^2}{4} = bc + \frac{a^2}{4}$ by adding the square of $\frac{a}{2}$ to both sides, and $z - \frac{a}{2} = \pm \sqrt{bc + \frac{a^2}{4}}$ by extracting the root; and therefore, $z = \sqrt{bc + \frac{a^2}{4}} + \frac{a}{2}$ for the positive value of z, and $z = -\sqrt{bc + \frac{a^2}{4}} + \frac{a}{2}$ the negative one; for since $bc + \frac{a^2}{4}$ is greater than $\frac{a^2}{4}$; consequently, $\sqrt{bc + \frac{a^2}{4}}$ is greater than $\sqrt{\frac{a^2}{4}}$;

and

and therefore, $z = -\sqrt{bc + \frac{a^2}{4}} + \frac{a}{2}$ is always a negative quantity.

And in like manner, the value of z determined in the third form, viz. $z^2 - az = -bc$, is $z = \pm\sqrt{\frac{a^2}{4} - bc} + \frac{a}{2}$, where both the values of z will be positive, if $\frac{a^2}{4}$ is greater than bc; for then $z = \sqrt{\frac{a^2}{4} - bc} + \frac{a}{2}$ is evidently a positive quantity; and in the second value of z, viz. $z = -\sqrt{\frac{a^2}{4} - bc} + \frac{a}{2}$, it is plain, that $\frac{a^2}{4}$ is greater than $\frac{a^2}{4} - bc$, since $\frac{a^2}{4}$ is greater than bc; and therefore, the $\sqrt{\frac{a^2}{4}}$ is greater than $\sqrt{\frac{a^2}{4} - bc}$; consequently, $z = -\sqrt{\frac{a^2}{4} - bc} + \sqrt{\frac{a^2}{4}} \left(= \frac{a}{2}\right)$ is a positive quantity: But when bc is greater than $\frac{a^2}{4}$ then $\frac{a^2}{4} - bc$ is a negative quantity; and since the square of any quantity (whether positive or negative) is always positive; it follows, that $\sqrt{\frac{a^2}{4} - bc}$ is impossible, or imaginary; and consequently, $z = \pm\sqrt{\frac{a^2}{4} - bc} + \frac{a}{2}$ is imaginary. Therefore, in the third

form,

form, *when bc is greater than* $\frac{a^2}{4}$ *the solution of the equation will be impossible.*

EXAMPLES

Of determining the value of the unknown quantity in quadratic equations.

Given, $x^2 + 4x = 32$, to find the value of x.

First, $x^2 + 4x + 4 = 32 + 4$, *by adding the square of half the co-efficient to both sides :*

Then, $\sqrt{x^2 + 4x + 4} = \pm \sqrt{36}$:

That is, $x + 2 = \pm 6$; *or,* $x = \pm 6 - 2 = 4$, *or* -8 : *Either of which substituted for x, will produce the given equation.*

Given, $3x^2 - 9x = -6$, to find x.

First, $x^2 - 3x = -2$ *by dividing the whole by 3 :*

Then, $x^2 - 3x + \frac{9}{4} = \frac{9}{4} - 2$ *by completing the square :*

And therefore, $x - \frac{3}{2} = \pm \sqrt{\frac{9}{4} - 2}$ *by extracting the root :*

Or, $x = \frac{3}{2} \pm \sqrt{\frac{9}{4} - 2} = \frac{3}{2} + \frac{1}{2} = \frac{4}{2} = 2$.

Given, $av^2 - bv - c = d$, to find v.

First, $av^2 - bv = d - c$ *by transposition :*

And $v^2 - \frac{b}{a}v = \frac{d-c}{a}$ *by division :*

Therefore,

Therefore, $v^2 - \dfrac{b}{a}v + \dfrac{b^2}{4a^2} = \dfrac{d-c}{a} + \dfrac{b^2}{4a^2}$ *by completing the square :*

Whence, $v - \dfrac{b}{2a} = \pm\sqrt{\dfrac{d-c}{a} + \dfrac{b^2}{4a^2}}$ *by evolution :*

Or, $v = \dfrac{b}{2a} \pm \sqrt{\dfrac{d-c}{a} + \dfrac{b^2}{4a^2}}$ *by transposition.*

ALL equations, wherein there are two terms which involve the unknown quantity, whose index in one term, is just double its index in the other, are reduced to equations of lower dimensions, in the same manner as quadratics.

THUS, $v^6 + bv^3 = d$; and $v^n + bv^{\frac{n}{2}} = c$, are reduced by completing the square, and extracting the root, as in quadratics ; and the value of the unknown quantity determined by extracting the root of the resulting equation ; as in the following

EXAMPLES.

Given, $v^4 - 2v^2 = 224$, to find the value of v.

First, $v^4 - 2v^2 + 1 = 224 + 1 = 225$ *by completing the square :*

And $v^2 - 1 = \sqrt{225}$ *by evolution :*

Or, $v^2 = \sqrt{225} + 1$ *by transposition :*

Whence, $v = \overline{\sqrt{225} + 1}\,^{\frac{1}{2}} = 4$.

Given, $bv^n + cv^{\frac{n}{2}} - d = e$, to find v.

First, $bv^n + cv^{\frac{n}{2}} = e + d$ *by transposition.*

Then, $v^n + \dfrac{c}{b}v^{\frac{n}{2}} = \dfrac{e+d}{b}$ *by division :*

And

And $v^n + \frac{c}{b} v^{\frac{n}{2}} + \frac{c^2}{4b^2} = \frac{e+d}{b} + \frac{c^2}{4b^2}$ by completing the square:

Therefore, $v^{\frac{n}{2}} + \frac{c}{2b} = \pm \sqrt{\frac{e+d}{b} + \frac{c^2}{4b^2}}$ by evolution.

Whence, $v = \pm \left| \sqrt{\frac{e+d}{b} + \frac{c^2}{4b^2}} - \frac{c}{2b} \right|^{\frac{n}{2}}$.

C H A P. XVII.

The SOLUTION of a Variety of QUESTIONS, Producing QUADRATIC EQUATIONS.

1. WHAT two numbers are thofe, whofe fum is 20, and their product 96?

Call one of the numbers w; then will $20 - w$ be the other:

And $w \times \overline{20 - w} = 96$ by the queftion:

That is, $20w - w^2 = 96$:

Or, $w^2 - 20w = -96$ by tranfpofition:

And $w^2 - 20w + 100 = 100 - 96$ by completing the fquare:

Therefore, $w - 10 = \pm \sqrt{100 - 96} = \pm \sqrt{4} = \pm 2$ by evolution:

Or, $w = \pm 2 + 10 = 12$ *or* 8, *and* $20 - w = 20 - 12 = 8$ *the other number.*

2. What two numbers are thofe, whofe fum is 36, and the fum of their fquares 720?

Put w for the greater number:

Then will $36 - w =$ the other:

And

And $w^2 + \overline{36 - w}|^2 = 720$ *by the question:*

That is, $w^2 + 1296 - 72w + w^2 = 720$:

Or, $2w^2 - 72w = -576$ *by transposition* :

And $w^2 - 36w = -288$ *by division* :

Wherefore, $w^2 - 36w + 324 = 324 - 288 = 36$
by completing the square :

Consequently, $w - 18 = \pm \sqrt{36} = 6$ *by evolution:*

Or, $w = 6 + 18 = 24$, *and* $36 - w = 36 - 24$
$= 12$:

3. What number being divided by the product of its two digits, the quotient will be 2 ; and if 27 be added to it, the digits will be inverted ?

Put w *and* y *for the two digits* :

Then will $10w + y$ *be the number sought, by the nature of notation* :

$$\left. \begin{array}{l} And \quad \dfrac{10w + y}{wy} = 2 \\ And \quad 10w + y + 27 = 10y + w \end{array} \right\} \text{ by the question:}$$

Or, $9w = 9y - 27$ *by transposition* :

And $w = \dfrac{9y - 27}{9} = y - 3$:

But $10w + y = 2wy$; *whence, (by writing for* w
its equal $y - 3$, *in the equation* $10w + y = 2wy$) *we
get* $10y - 30 + y = 2y^2 - 6y$:

Or, $17y - 2y^2 = 30$; *or,* $2y^2 - 17y = -30$ *by
transposition* :

Whence, $y^2 - 8\frac{1}{2}y = -15$ *by division* :

And $y^2 - 8\frac{1}{2}y + \dfrac{289}{16} = \dfrac{289}{16} - 15 = \dfrac{49}{16}$ *by com-
pleting the square* :

Or, $y - \dfrac{17}{4} = \pm \sqrt{\dfrac{49}{16}} = \dfrac{7}{4}$ *by evolution:*

Y y   *Consequently,*

Consequently, $y = \frac{17}{4} + \frac{7}{4} = \frac{24}{4} = 6$, *and* $w = y - 3$ $= 3$:

Therefore 36 *is the number required.*

4. To find three numbers in geometrical proportion continued, whose sum is 78; and if the sum of the extremes be multiplied with the mean, the product will be 1080.

Put $v =$ *least extreme, and* z *the greater; also,* $y =$ *mean:*

Then will $v + y + z = 78$
And $\overline{v + z} \times y = 1080$ } *by the question.*

That is, $vy + zy = 1080$; *and* $vy + y^2 + zy = 78y$ *by multiplying the first equation with* y;

Whence, $y^2 = ($ *by writing for* $vy + zy$ *its equal* 1080) $78y - 1080$,

Or, $y^2 - 78y = - 1080$:

And $y^2 - 78y + 1521 = 1521 - 1080 = 441$ *by completing the square:*

And therefore, $y - 39 = \pm \sqrt{441} = \pm 21$ *by evolution:*

Or, $y = 39 \pm 21 = ($ *because* $39 + 21 = 60$, *is greater than the sum of the extremes, which is absurd)* $39 - 21 = 18$:

But, $vz = y^2 = 324$ *by the nature of the proportion:*

Consequently, $v = \frac{324}{z}$, *which wrote for* v *in the equation* $vy + zy = 1080$, *gives* $\frac{324y}{z} + zy = 1080$:

That is, $5932 + 18z^2 = 1080z$:

Or, $18z^2 - 1080z = - 5932$ *by transposition:*

And $z^2 - 60z = - 324$ *by division:*

Therefore, $z^2 - 60z + 900 = 900 - 324 = 576$ *by completing the square:*   *Whence,*

Whence, $z - 30 = \pm \sqrt{576} = \pm 24$ *by evolution:*
Or, $z = 30 + 24 = 54$, *and* $v = 78 - z - y = 78 - 54 - 18 = 6$. *Therefore,* 6, 18, *and* 54, *are the numbers required.*

5. There are three numbers in geometrical progreffion, whofe fum is 117, and the fum of their fquares 7371 : What are thofe numbers ?

Call the numbers x, y *and* v :

Then will $x + y + v = 117$
And $x^2 + y^2 + v^2 = 7371$ } *by the queftion.*

Alfo, $xv = y^2$ *by the nature of the proportion :*
And $x + v = 117 - y$ *by the firft equation :*
Whence, $x^2 + 2xv + v^2 = 13689 - 234y + y^2$ *by involution :*

But, $2xv = 2y^2$, *which fubftituted for* $2xv$ *in the laft equation, gives* $x^2 + 2y + v^2 = 13689 - 234y + y^2$:

Or, $x^2 + v^2 = 13689 - 234y - y^2$:
But, $x^2 + v^2 = 7371 - y^2$ *by the fecond equation :*
Confequently, $7371 - y^2 = 13689 - 234y - y^2$:
Or, $234y = 13689 - 7371 = 6310$:
Whence, $y = \dfrac{6310}{234} = 27$, *and* $xv = y^2 = 729$:

Or, $x = \dfrac{729}{v}$, *which fubftituted in the equation* $x + y + v = 117$, *gives* $\dfrac{729}{v} + 27 + v = 117$; *or,* $\dfrac{729}{v} + v = 117 - 27 = 90$:

Whence, $729 + v^2 = 90v$ *by multiplication :*
Or, $v^2 - 90v = -729$ *by tranfpofition :*
And therefore, $v^2 - 90v + 2025 = 2025 - 729 = 1296$ *by completing the fquare :*

Confequently,

Confequently, $v - 45 = \pm \sqrt{1296} = 36$ *by evolution :*

Or, $v = 45 + 36 = 81$, *and* $x = \dfrac{729}{v} = 9$.

And the numbers required, are 9, 27, 81.

MISCELLANEOUS QUESTIONS,
with their SOLUTIONS.

1. Suppofe two cities, A and B, whofe diftance from each other is 216 miles ; and that two couriers fet out at the fame time, one from A, and the other from B ; the firft travels 10 miles a day, and the other 4 miles lefs than the number of days in which they will meet. Query the number of days before they meet ?

Put $x =$ *number of days required :*

Then will $10x + x - 4 \times x = 216$ *by the queftion :*

That is, $10x + x^2 - 4x = 216$; *or,* $x^2 + 6x = 216$:

And $x^2 + 6x + 9 = 216 + 9 = 225$:

Whence, $x + 3 = \pm \sqrt{225} = 15$; *or,* $x = 15 - 3 = 12$, *the number of days required.*

2. A traveller fets out from the city A, and travels at the rate of 9 miles an hour ; and another at the fame time fets out from the fame city, and follows him, travelling the firft hour 4 miles ; the fecond 5 ; the third 6, and fo on, in arithmetical progreffion : In what time will he overtake the firft ?

Put $x =$ *number of hours in which the firft will be overtaken :*

Then will $9x =$ *the diftance he travels :*

And $x - 1 \times 1 + 4 + 4 = x + 7$:

And

And $\overline{x + 7} \times \frac{1}{2}x = \frac{x^2 + 7x}{2} = $ *distance the other travels before he overtakes the first, by the nature of the proportion*: Consequently, $\frac{x^2 + 7x}{2} = 9x$ *by the question*:

Or, $x^2 + 7x = 18x$: *Whence,* $x + 7 = 18$; or, $x = 11$ *hours, the time required.*

3. There are four numbers in geometrical progression, the sum of the extremes is 84, and the sum of the means 36: What are those numbers?

Put v and y for the means:

Then will $\frac{v^2}{y}$ *and* $\frac{y^2}{v}$ *be the extremes by the nature of the proportion*:

Therefore, $\left.\begin{array}{l} v + y = 36 = a \\ \dfrac{v^2}{y} + \dfrac{y^2}{v} = 84 = b \end{array}\right\}$ *by the question*:

Or, $v^3 + y^3 = vy \times b = $ *(by writing p for vy)* pb.

But, $v^3 + y^3 = $ *(by problem 18 of the last chap.)* $a^3 - 3ap$:

Consequently, $pb = a^3 - 3ap$; or, $p = \dfrac{a^3}{b + 3a} = c$ *by substitution*:

Therefore, $v^3 + y^3 = bc$; or, $v^3 = bc - y^3$:

But, $v = a - y$; *therefore,* $v^3 = a^3 - 3a^2y + 3ay^2 - y^3$:

Consequently, $a^3 - 3a^2y + 3ay^2 - y^3 = bc - y^3$; or, $a^3 - 3a^2y + 3ay^2 = bc$:

Or, $3ay^2 - 3a^2y = bc - a^3$:

And therefore, $y^2 - ay = \dfrac{bc - a^3}{3a}$:

And

$$And \quad y^2 - ay + \frac{a^2}{4} = \frac{bc - a^3}{3a} + \frac{a^2}{4} = \frac{4bc - a^3}{12a}:$$

$$Whence, \quad y - \frac{a}{2} = \pm \left| \frac{4bc - a^3}{12a} \right|^{\frac{1}{2}} = 9; \quad or, \quad y = 9 +$$

$$\frac{a}{2} = 27, \quad and \quad v = 36 - y = 9; \quad therefore, \quad \frac{v^2}{y} = 3,$$

$$and \quad \frac{y^2}{v} = 81,$$

Consequently, 3, 9, 27 and 81, are the numbers required.

4. Suppose two cities, A and B, whose distance from each other is 152 miles; and that two men set out at the same time from those cities to meet each other; the one which goes from A, travels the first day 1 mile, the second day 2, the third day 3, and so on; and the one which sets out from B, goes the first day 4 miles, the second day 7, and the third 10, and so on. Query the number of days before they meet, and the number of miles that each travels?

Put y = number of days before they meet:

Then will $\frac{y^2 + y}{2} + \frac{3y^2 + 5y}{2} = 152$ by the question:

That is, $\frac{4y^2 + 6y}{2} = 152$; or, $4y^2 + 6y = 304$:

Whence, $y^2 + \frac{3}{2}y = 76$:

And $y^2 + \frac{3}{2}y + \frac{9}{16} = 76 + \frac{9}{16} = \frac{1225}{16}$:

Or, $y + \frac{3}{4} = \pm \sqrt{\frac{1225}{16}} = \frac{35}{4}$; and $y = \frac{35}{4} - \frac{3}{4} = 8$.

Consequently,

Confequently, $\dfrac{y^2 + y}{2} = 36$, *the number of miles travelled by the one which fat out from A, and* $\dfrac{3y^2 + 5y}{2} = 116$, *the diftance travelled by the other.*

C H A P.   XVIII.

Of the G E N E S I S, or F O R M A T I O N of E-QUATIONS in GENERAL.

ALL equations of fuperior order, are confider-ed, as produced by the multiplication of equa-tions of inferior orders, that involve the fame un-known quantity.

Thus, a quadratic equation may be confidered as generated by the multiplication of two fimple equa-tions ; a cubic equation by the multiplication of three fimple equations, or one quadratic and one fimple equation ; and a biquadratic equation by the multiplication of four fimple equations, or two quad-ratic equations, or one cubic and one fimple equa-tion.

Suppofe w to be the unknown quantity, and a, b, c, d, &c. its feveral values in any fimple equation :

That is, $w = a$, $w = b$, $w = c$, $w = d$, &c. Then by tranfpofition, $w - a = 0$, $w - b = 0$, $w - c = 0$, $w - d = 0$, &c. And the product of two of thefe equations as $\overline{w - a} \times \overline{w - b} = 0$, gives a quadratic equation, or one of two dimenfions.

The product of any three ; as $\overline{w - a} \times \overline{w - b} \times \overline{w - c} = 0$, produces a cubic equation, or one of three dimenfions.   The

The product of any four of them; as $\overline{w-a} \times \overline{w-b} \times \overline{w-c} \times \overline{w-d} = 0$, produces a biquadratic equation, or one of four dimensions.

Hence it appears, that in every equation, the higest dimension of the unknown quantity, is equal to the number of simple equations that generate that equation; and therefore it follows, that every equation has as many roots, or values of the unknown quantity, as there are units in the highest dimension of that unknown quantity. For suppose an equation $= \overline{w-a} \times \overline{w-b} \times \overline{w-c} = 0$; and that for w you substitute any of its values (a, b or c) in the given equation, then all the terms of an equation will vanish; for if $w = a, w = b$, and $w = c$, then $\overline{w-a} \times \overline{w-b} \times \overline{w-c} = 0$, because each of the factors are equal to nothing. And after the same manner, it appears, that there are three suppositions that give $\overline{w-a} \times \overline{w-b} \times \overline{w-c} = 0$: But since there are no other quantities besides these a, b, c, which substituted for w in the equation $\overline{w-a} \times \overline{w-b} \times \overline{w-c} = 0$, will make all the terms vanish; it follows, that the equation $\overline{w-a} \times \overline{w-b} \times \overline{w-c} = 0$, can have no more than these three roots, or admit of more than three solutions. For if you substitute for w in the proposed equation, any other quantity e, which is neither equal to a, b, nor c; then neither $e-a$, $e-b$, $e-c$, is equal to nothing; and consequently their product $\overline{e-a} \times \overline{e-b} \times \overline{e-c}$, cannot be equal to nothing, but must be some real product: So that no other quantity, besides one of those before-mentioned, will give a true value of w in the proposed equation. And therefore, no equation can have more roots than it contains dimensions of the unknown quantity.　　　To

To be more plain: Suppofe that $x^4 - 10x^3 + 35x^2 - 50x + 24 = 0$, is the equation to be refolved; and that you find it to be the fame as the product of $\overline{x - 1} \times \overline{x - 2} \times \overline{x - 3} \times \overline{x - 4}$: Then you will infer, that the four roots or values of x, are 1, 2, 3, and 4; for any of thefe numbers fubftituted for x, will make that product, and confequently, $x^4 - 10x^3 + 35x^2 - 50x + 24$ equal to nothing, according to the propofed equation.

THE roots of equations are either pofitive or negative, according as the roots or values of the unknown quantity in the fimple equations which produce them, are pofitive or negative. Thus, if $v = -a$, $v = -b$, $v = -c$, $v = -d$; then will $v + a = 0$, $v + b = 0$, $v + c = 0$, and $v + d = 0$; and confequently, $\overline{v + a} \times \overline{v + b} \times \overline{v + c} \times \overline{v + d} = 0$, will be an equation whofe roots $-a$, $-b$, $-c$, $-d$, are all negative. And after the fame manner, if $v = a$, $v = -b$, $v = c$, the equation $\overline{v - a} \times \overline{v + b} \times \overline{v - c}$, will have its roots $+a$, $-b$, $+c$.

BUT to difcover when the roots of an equation are pofitive, and when negative, and how many there are of each kind, it will be neceffary to confider the figns and co-efficients of equations, generated from the multiplication of thofe fimple equations that produce them; which will be beft underftood by confidering the following table, where the fimple equations $v - a$, $v - b$, $v - c$, &c. are multiplied continually with one another, and produce fucceffively the higher equations.

$v - a$

$$
\begin{array}{l}
\quad v - a \\
\times\ v - b \\
\hline
= v^2 - av \\
\qquad\ \ - bv
\end{array} \Bigg\} + ab = 0,\ \textit{a quadratic}
$$

$$\times\ \overline{v - c}$$

$$
\begin{array}{l}
= v^3 - a \\
\qquad\ - b \\
\qquad\ - c
\end{array}\Bigg\} \times v^2
\begin{array}{l}
+ ab \\
+ ac \\
+ bc
\end{array}\Bigg\} \times v - abc = 0,\ \textit{a cubic}
$$
[equation

$$\times\ \overline{v - d}$$

$$
\begin{array}{l}
= v^4 - a \\
\qquad\ - b \\
\qquad\ - c \\
\qquad\ - d
\end{array}\Bigg\} \times v^3
\begin{array}{l}
+ ab \\
+ ac \\
+ ad \\
+ bc \\
+ bd \\
+ cd
\end{array}\Bigg\} \times v^2
\begin{array}{l}
- abc \\
- abd \\
- acd \\
- bcd
\end{array}\Bigg\} \times v + abcd = 0,\ \textit{a biquad.}
$$

&c.

FROM the infpection of thefe equations it appears that the co-efficient of the firft term is unity or 1.

THE co-efficient of the fecond term, is the fum of all the roots (a, b, c, d) with contrary figns.

THE co-efficient of the third term, is the fum of all the products of thofe roots that can poffibly be made by multiplying any two of them together.

THE co-efficient of the fourth term, is the fum of all the products of the roots that can be made by
combining

combining them, three and three : And so on for any other co-efficient. The last term is always the product of all the roots, having their signs changed.

NOTWITHSTANDING those simple equations made use of in the foregoing table, in forming the higher equations, are such as have positive roots ; yet the same reasoning holds, whether the roots are positive or negative. Whence, if $v^4 - pv^3 + qv^2 - rv + s = 0$, represents a biquadratic equation ; then will p be the sum of all the roots, q the sum of all the products made by multiplying any two of them together, r the sum of all the products made by multiplying any three of them together, and s the product of all four.

IT likewise appears from inspection, that the signs of the terms in any equation in the foregoing table, are alternately $+$ and $-$: The first term is always some pure power of v, and is positive : The second term is some power of v, multiplied with the quantities, $-a$, $-b$, $-c$, &c. and since these quantities are all negative, it follows, that the second term must also be negative. The third term hath for its co-efficient the product of any two of these quantities, $(-a, -b, -c, \&c.)$ and since $- \times -$ gives $+$; it follows, that the third term must be positive. For the same reason, the co-efficient of the fourth term, which is formed of the products of any three of these negative quantities, must be negative also, and the co-efficient of the fifth term positive. But in this case, $v = a$, $v = b$, $v = c$, $v = d$, &c. that is, the roots are all positive : Consequently, when the roots of an equation are all positive, the signs of the terms are $+$ and $-$ alternately. But, when the roots are all negative ; that is, $v = -a$, $v = -b$, $v = -c$, $v = -d$, &c. then $\overline{v+a} \times \overline{v+b} \times \overline{v+c} \times \overline{v+d} = 0$, will express the equation produced

duced, whofe terms are evidently all pofitive. And therefore when the roots of an equation are all nega-tive, there will be no change in the figns of the terms. Confequently, there will be as many pofitive roots in an equation, as there are changes in the figns of the terms of that equation, and the reft of the roots will be negative.

HENCE it follows, that the roots of a quadratic e-quation may be both negative, or both pofitive, or one negative and the other pofitive. Thus, in the equation $v^2 \left. \begin{array}{c} - a \\ - b \end{array} \right\} \times v + ab = \overline{(v - a} \times \overline{v - b)}$ o, there are two changes of the figns, viz. the firft term is pofitive, the fecond negative, and the third pofitive ; confequently, the roots are both pofitive.

BUT in the equation $v^2 \left. \begin{array}{c} + a \\ + b \end{array} \right\} \times v + ab = \overline{(v + a}$ $\times \overline{v + b)}$ o, there are no change in the figns, and therefore both the roots are negative.

AND in like manner, in the equation $v^2 \left. \begin{array}{c} + a \\ - b \end{array} \right\} \times$ $v - ab = \overline{(v + a} \times \overline{v - b)}$ o, one of the roots will be pofitive, and the other negative ; for fince the firft term is pofitive, and the laft negative, it is plain, there can be but one change in the figns, whe-ther the fecond term is pofitive or negative.

HENCE alfo it appears, how that a cubic equation may have all its roots pofitive, or all negative, or two pofitive and one negative ; or two negative and one pofitive. For fuppofe the cubic equation is $v^3 \left. \begin{array}{c} - a \\ - b \\ - c \end{array} \right\} \times v^2 \left. \begin{array}{c} + ab \\ + ac \\ + bc \end{array} \right\} \times v - abc = \overline{(v - a} \times$

$v - b$

$\overline{v - b} \times \overline{v - c})$ o, wherein there are three changes in the figns; and confequently all three of the roots pofitive.

AGAIN, fuppofe the cubic equation is of this form,

$$\left.\begin{matrix} v^3 - a \\ - b \\ + c \end{matrix}\right\} \times v^2 \mp \left.\begin{matrix} ab \\ - ac \\ - bc \end{matrix}\right\} \times v + abc = \overline{(v - a} \times$$

$\overline{v - b} \times \overline{v - c})$ o, where there are two changes in the figns; for if $a + b$ is greater than c, then the fecond co-efficient $\overline{-a - b + c}$ muft be negative; if $a + b$ is lefs than c, then the third term will be negative; for its co-efficient $\overline{ab - ac - bc}$ ($= ab - c \times \overline{a + b}$) is, in this cafe negative, becaufe the product $a \times b$ is always lefs than the fquare $\overline{a + b} \times \overline{a + b}$, and confequently, much lefs than $c \times \overline{a + b}$; and fince there cannot be three changes in the figns, the firft and laft terms having the fame fign; it follows, that two of the roots of the propofed equation are pofitive, and the other negative.

IN like manner, the equation $v^3 + \overline{a + b - c}v^2 + \overline{ab - ac - bc}v - abc = $o, will have two of its roots negative, and the other pofitive; for if $a + b$ is lefs than c, the fecond and third terms muft be negative, by what was proved in the laft example; and if the fecond term is pofitive, that is, $a + b$ is greater than c, it is plain there can be but one change in the figns, and confequently but one pofitive root, the other two being negative.

AND by parity of reafon, the pofitive and negative roots of the other equations may be difcovered;

this

this method being general, and extends to all kinds of equations whatever.

CHAP. XIX.

CONCERNING the TRANSFORMATION of EQUATIONS, and EXTERMINATING their INTERMEDIATE TERMS.

ANY equation may be transformed into another, whose roots shall be greater, or less than the roots of the proposed equation by any given difference (e) by the following

RULE.

ASSUME a new unknown quantity (y) and connect it with the given difference (e), with the sign + or —, according as the roots of the proposed equation are to be increased, or diminished; and make this aggregate equal to the unknown quantity (x) in the proposed equation; then instead of the unknown quantity (x) and its powers in the proposed equation, substitute this aggregate, ($y \pm e$) and its powers; and there will arise a new equation, whose roots will be greater or less than the roots of the proposed equation, as required.

EXAMPLES.

1. Let $x^3 - px^2 + qx - r = 0$, be an equation to be transformed into another whose roots shall be less than the roots of the proposed equation, by the difference e. *Assume*

Affume $x = y + e$:

Then will
$$
\begin{aligned}
x^3 &= y^3 + 3y^2 e + 3ye^2 + e^3 \\
-px^2 &= \quad\quad -py^2 - 2ye - pe^2 \\
+qx &= \quad\quad\quad\quad\quad qy + qe \\
-r &= \quad\quad\quad\quad\quad\quad -r
\end{aligned}
\Bigg\} = 0, \textit{ is the e-}\textit{quation requir.}
$$

2. Let $x^2 - 11x + 30 = 0$, be transformed into an equation that shall have its roots less than the roots of the propofed equation by the difference 4.

Affume $x = y + 4$:

Then,
$$
\begin{aligned}
x^2 &= y^2 + 8y + 16: \\
-11x &= \quad\quad -11y - 44 \\
+30 &= \quad\quad\quad\quad\quad +30
\end{aligned}
$$

$y^2 - 3y + 2 = 0$, *is the equation required.*

In the firft example of the foregoing transformations, the co-efficient of the fecond term in the tranf-formed equation, is $3e - p$; and if you fuppofe $e = \frac{1}{3}p$, and therefore, $3e - p = 0$; then the fecond term of the transformed equation will vanifh. Let the propofed equation be of n dimenfions, and the co-ef-ficient of the fecond term $-p$; and fuppofe $x = y + \frac{p}{n}$; then if this value be fubftituted for x in the propofed equation, there will arife a new equa-tion that fhall want the fecond term. For if $p = $ fum of all the roots of the propofed equation, and $x = y + \frac{p}{n}$; it follows, that each value of y in the new equation, will be lefs than the value of x in the pro-pofed equation, by $\frac{p}{n}$; and fince the number of roots is n, it follows, that the fum of the values of y, will

be

be lefs than p, the fum of the values of x, by $n \times \frac{p}{n}$ $= p$; that is, the fum of the values of y, is $+ p - p$ $= 0$; and fince the co-efficient of the fecond term in the equation of y, is the fum of the values of y, viz. $+ p - p$, which is equal to nothing; it follows, that in the equation of y, arifing from the fuppofition of $x = y + \frac{p}{n}$, the fecond term muft vanifh: And therefore the fecond term of any equation may be exterminated by the following

R U L E.

DIVIDE the co-efficient of the fecond term of the propofed equation by the index of the higheft power of the unknown quantity; and affume a new unknown quantity (y) and annex to it the faid quotient with its fign changed; then put this aggregate equal to the unknown quantity (x) in the propofed equation, and inftead of x and its powers, write this aggregate and its powers, and the equation that arifes fhall want the fecond term.

EXAMPLES.

Let the equation $x^2 - 8x + 12 = 0$, be propofed to have its fecond term exterminated.

Firft, $- 8 \div 2 = - 4$:

Therefore, $x = y + 4$, *per rule*:

Then, $x^2 = y^2 + 8y + 16$
$- 8x = \quad - 8y - 32$
$+ 12 = \quad\quad\quad + 12$

$$\overline{\quad y^2 \quad * \quad - 4 = 0}$$

HENCE,

HENCE it appears, that a quadratic equation may be resolved without completing the square, by exterminating the second term; for since $y^2 - 4 = 0$; or, $y^2 = 4$, and $y = \sqrt{4}$, we shall have $x = y + 4 = \sqrt{4} + 4 = 6$.

LET the second term of the equation $x^3 - 9x^2 + 26x - 34 = 0$, be exterminated.

First, $x = y + (\frac{9}{3}) 3$:

$$Then, \quad x^3 = y^3 + 9y^2 + 27y + 27$$
$$- 9x^2 = \quad\quad - 9y^2 - 54y - 81$$
$$+ 26x = \quad\quad\quad\quad\quad + 26y + 78$$
$$- 34 = \quad\quad\quad\quad\quad\quad\quad\quad - 34$$
$$\overline{\quad y^3 \quad\quad * \quad\quad - y - 10 = 0.}$$

WHEN the second term in any equation is wanting, it is plain, that the equation hath both positive and negative roots; and since the co-efficient of the second term in any equation, is the difference between the sum of the positive, and sum of the negative roots; it follows therefore, that when the positive and negative roots are made equal to each other, that difference vanishes. Consequently, when an equation has the second term wanting, the sum of the positive roots is equal to the sum of the negative ones.

HENCE, by the foregoing transformation of equations and the exterminating their second terms, the positive and negative roots are reduced to an equality, and the solution of the equation thereby rendered more easy.

IF the equation $v^3 - pv^2 + qv - r = 0$, be transformed into another, by assuming $v = y + e$, the co-efficient of the third term of the transformed equation will be $3e^2 - 2pe + q$; now if we suppose this

A a a   co-efficient

co-efficient equal to nothing, and resolve the quad-
ratic $3e^2 - 2p + q = 0$ we shall have $e = \frac{p \pm \sqrt{p^2 - 3q}}{3}$
which substituted for e in the equation $v = y + e$,
the third term of the transformed equation will van-
ish: Also, if the proposed equation be of n dimen-
sions, the value of e, by which the third term is to
be exterminated, is found by resolving the quadra-
tic equation $e^2 + \frac{2p}{n} \times e + \frac{2q}{n \times n - 1} = 0$, that is,
by finding the value of e in the co-efficient of the
third term of the transformed equation, when that
co-efficient is equal to nothing. And in like man-
ner, the fourth term of any equation may be exter-
minated, by solving a cubic equation, which is the
co-efficient of the fourth term of a transformed equa-
tion: And after the same manner, the other terms
may be taken away.

THERE are other transformations which are of use
in the resolution of equations; of which the most
useful, and the only one that we shall consider, is,
when the highest term of the unknown quantity is
multiplied with some given quantity, to transform
the equation into another that shall have the co-effi-
cient of the highest term unity.

LET the proposed equation be $av^3 - pv^2 + qv - r = 0$; and suppose $av = y$, then $v = y \div a$, and
this value substituted for v in the proposed equation,
there will arise $\frac{ay^3}{a^3} - \frac{py^2}{a^2} + \frac{qy}{a} - r = 0$, or $\frac{y^3}{a^2}$
$- \frac{py^2}{a^2} + \frac{qy}{a} - r = 0$, and by multiplying the
whole by a^2, we shall have $y^3 - py^2 + qay - ra^2 = 0$; which gives the following

RULE.

RULE.

CHANGE the unknown quantity (v) in the propos-
ed equation, into another (y), prefix no co-efficient
to the first term, pass the second, multiply the third
term with the co-efficient of the highest term of the
unknown quantity in the proposed equation, and the
fourth term by the square of that co-efficient, the
fifth by the cube; and so on, and the highest term
of the unknown quantity in the resulting equation
shall have its co-efficient unity, as required.

EXAMPLES.

Let the equation $2v^3 + 6v - 36 = 0$, be changed
into another that will have unity for the co-efficient
of the highest term of the unknown quantity.
*Thus, $y^2 + 6y - 36 \times 2 = 0$; or, $y^2 + 6y - 72$
$= 0$, is the equation required.*

The finding the roots of the proposed equation,
and all others of the like kind, will be very easy when
the roots of the transformed equation are found; since
$v =$ (in this case) $\frac{1}{2}y$.

Transform the equation $5v^3 - 10v^2 + 16v - 93$
$= 0$, into another that the highest term of the un-
known quantity may have an unit for its co-efficient.
*Thus, $y^3 - 10y + 80y - 2325 = 0$, is the equa-
tion required.*

CHAP.

CHAP. XX.

Of the RESOLUTION of EQUATIONS by DIVISORS.

IF the laft term of an equation is the product of all its roots; it follows, that the roots of an equation when commenfurable, will be found among the divifors of the laft term; which gives the following

RULE.

TRANSPOSE all the terms to one fide of the equation. Find all the divifors of the laft term, and fubftitute them fucceffively for the unknown quantity in the propofed equation; and that divifor, which fubftituted as aforefaid, gives the refult $= 0$, is one of the roots of the equation. But if none of the divifors fucceed, the roots of the equation are for the moft part, either irrational or impoffible.

Note. *If the laft term of the propofed equation is large, and confequently its divifors numerous; they may be diminifhed, by transforming the equation into another, by the rules of the laft chapter.*

EXAMPLES.

Find the roots of the equation $x^3 - 4x^2 + 10x - 12 = 0$.

Here the divifors of the laft term, are $1, 2, 3, 4, 6, 12, -1, -2, -3, -4, -6, -12,$ *which fubftituted fucceffively for x,*

Gives,

$$
\left\{
\begin{array}{l}
1 - 4 + 10 - 12 = -5 \\
8 - 16 + 20 - 12 = 0 \\
27 - 36 + 30 - 12 = 9 \\
64 - 64 + 40 - 12 = 28 \\
216 - 144 + 60 - 12 = 120
\end{array}
\right.
$$
Gives,
$$\&c.$$

WE omit trying the negative divisors, since there are three changes in the signs of the proposed equation, and therefore none of its roots can be negative: And since none of the divisors succeed, except 2; it follows, that 2 is the only rational root of the equation, the other two being either irrational, or impossible.

Let it be required to find the roots of the equation $x^3 + 2x^2 - 40x + 64 = 0$.

Here the divisors of the last term, are 1, 2, 4, 8, 16, 32, which substituted successively for x in the proposed equation,

Gives,
$$
\left\{
\begin{array}{l}
1 + 2 - 40 + 64 = 27 \\
8 + 8 - 80 + 64 = 0 \\
64 + 32 - 160 + 64 = 0
\end{array}
\right.
$$

WHERE the only divisors that succeed, are 2, and 4; and since there are but two changes in the signs of the proposed equation, there must be one negative root: We are therefore to substitute the divisors negatively taken, in order to discover the other value of x; and on trial, we find that -8 succeeds. Therefore the three roots of the proposed equation, are $+2$ $+4$ -8.

BUT when one of the roots of an equation is found, the rest of the roots may be found with less trouble, by dividing the proposed equation by the simple equation, deduced from the root already found, and finding

finding the roots of the quotient, which will be an equation a degree lower than the proposed one.

THUS, in the last example the root $+ 2$ first found, gives $x = 2$; or, $x - 2 = 0$, by which dividing the proposed equation : Thus,

$$x - 2) \, x^3 + 2x^2 - 40x + 64 (x^2 + 4x - 32.$$
$$\underline{x^3 - 2x^2}$$

$$\begin{array}{r} 4x^2 - 40x \\ 4x^2 - 8x \\ \hline \end{array}$$

$$\begin{array}{r} - 32x + 64 \\ - 32x + 64 \\ \hline * \qquad * \end{array}$$

The quotient will be a quadratic equation $x^2 + 4x - 32 = 0$; which is the product of the other two simple equations, from which the proposed cubic was generated ; and whose two roots are consequently, two of the roots of that cubic. But the two roots of the quadratic, are $+ 4$ and $- 8$. Therefore, the three roots of the cubic equation, are 2, 4, $- 8$, the same as before.

THE finding all the divisors of the last term of an equation, especially if that term be large, is much facilitated by the following

RULE.

1. DIVIDE the last term by its least divisor that exceeds unity, and the quotient by its least divisor ; proceeding in this manner, till you have a quotient that is not farther divisible by any number greater than an unit : And this quotient together with those divisors, are the first divisors of the last term.

2. FIND all the products of thofe divifors which arife by combining them two and two, and all the products which arife by combining them three and three, and fo on, until the continued product of the firft divifors, is equal to the quantity to be divided ; and you will have the divifors required.

EXAMPLES.

Thus, fuppofe the laft term of an equation to be 60 : Then $\overline{60 \div 2 = 30}$, $\overline{30 \div 2 = 15}$, $\overline{15 \div 3 = 5}$; therefore, $\overline{2 \times 2}$, $\overline{2 \times 3}$, $\overline{2 \times 5}$, and $\overline{3 \times 5}$, are the combinations of the twos; and $\overline{2 \times 2 \times 3}$, $\overline{2 \times 2 \times 5}$, $\overline{2 \times 3 \times 5}$, the combinations of the threes ; alfo, $\overline{2 \times 2 \times 3 \times 5}$, is the combination of the fours = their continued product, equal to the quantity to be divided. Therefore all the divifors of 60, are 2, 3, 5, 4, 6, 10, 15, 12, 20, 30, 60.

And in like manner, the divifors of $10ab$, are 2, 5, a, b, 10, $2a$, $2b$, $5a$, $5b$, ab, $10a$, $5ab$, $2ab$ and $10ab$.

BUT there is another method for the reduction of equations by divifors, which is lefs prolix, by reducing the divifors to more narrow limits, by the following

RULE.

1. INSTEAD of the unknown quantity in the propofed equation, fubftitute fucceffively the terms of the progreffion, 1, 0, — 1, &c. and find all the divifors of the fums that refult by fuch fubftitution.

2. TAKE out all the arithmetical progreffions that can be found among thofe divifors, whofe terms correfpond with the order of the terms, 1, 0, — 1, &c.

&c. and common difference unity; and the values of x will be found among the divisors which arise from the substitution of $x = 0$, that belong to those progreffions.

> Note. *When the arithmetical progreffion is increasing according to the order of the terms 1, 0, — 1, the value of x will be affirmative; but when the arithmetical progreffion is decreasing, the value of x will be negative.*

EXAMPLES.

Let $x^3 - x^2 - 10x + 6 = 0$, be the proposed equation; and by fubftituting fucceffively for x, the terms 1, 0, — 1, the work will ftand as follows.

Suppofitions.	Refults.		Divifors.	Ar.P.
$x = 1$		$\begin{cases} -4 \\ +6 \\ +14 \end{cases}$	1,2,4	4
$x = 0$ $\Big\}$ $x^3 - x^2 - 10x + 6 =$			1,2,3,6	3
$x = -1$			1,2,7,14	2

HERE the progreffion is decreasing, and 3, that term which ftands againft the fuppofition of $x = 0$; therefore, — 3, fubftituted for x in the proposed equation, gives, $-27 - 9 + 30 + 6 = 0$; where all the terms vanifhing, it follows, that -3 is one of the roots of the proposed equation; and $2 + \sqrt{2}$, and $2 - \sqrt{2}$, the other two roots, found by dividing the proposed equation by $x + 3$, and refolving the quadratic quotient.

Suppofe it be required to find the roots of the equation $v^4 + 3v^3 - 19v^2 - 27v + 90 = 0$.

Then by fubftituting as before, the work will ftand as follows.

Suppositions.	Results.	Divisors.	Arith. Progres.
$v = 1$	48	1, 2, 3, 4, 6, &c.	1, 3, 2, 4, 6
$v = 0$	90	1, 2, 3, 5, 6, &c.	2, 2, 3, 3, 5
$v = -1$	96	1, 2, 3, 4, 6, &c.	3, 1, 4, 2, 4

HERE are five arithmetical progressions; and sub-stituting 2, 3, — 3, — 5, respectively for v in the proposed equation, the whole vanishes; the other progression being in this case useless, since the number of roots are but four. Consequently, 2, 3, — 3, — 5, are the four roots required.

THERE are many other methods beside those which we have here given for the resolution of equations; which the confined limits of our plan obliges us to omit, and proceed to discover the roots of equations by the method of approximation.

CHAP. XXI.

The FINDING the ROOTS of NUMERAL EQUATIONS in GENERAL, by the METHOD of APPROXIMATION.

ALTHOUGH there are other methods for the resolution of equations, than those given in the last chapter, yet the most of them are either very prolex, or confined to particular cases; but the following method of approximation is general, and extends to numeral equations of all kinds whatever, and though not accurately true, gives the value of the root to any assigned degree of exactness you please, by the following

B b b   *RULE.*

R U L E.

1. FIND by trial, a number nearly equal to the root required, and call it r; and put x for the difference between the real root and that already found, then will $r \pm x = v$.

2. INSTEAD of v and its powers in the proposed equation, substitute $r \pm x$ and its powers; and there will arise a new equation involving x and known quantities.

3. THEN by rejecting all the terms of this new equation that involve the powers of x; and assuming the rest equal to nothing, the value of x will be determined by means of a simple equation.

4. ADD the value of x thus found to r, and you will have a nearer value of the root required; which if not sufficiently exact, repeat the operation, by substituting this value for r in the formula exhibiting the value of x, and it will give a correction of the root; which if not yet exact enough, proceed to a third correction; and so on, to any assigned degree of exactness.

EXAMPLES.

Given, $v^2 + 6v - 31 = 0$, to find v by approximation.

The root found by trial is nearly equal to 3:
Therefore, $r = 3$, and $r + x = v$:
Then, $v^2 = r^2 + 2rx + x^2$
$\quad + 6v = 6r + 6x$
$\quad - 31 = - 31$

And, $r^2 + 2rx + 6x + 6r - 31 = 0$:

Whence,

Whence, $x = \dfrac{31 - r^2 - 6r}{2r + 6} = ($ by writing 3 for r

its equal$)\dfrac{31 - 9 - 18}{6 + 6} = \dfrac{4}{12} = .3$; and $v = 3.3$

And if 3.3 be substituted for r in the equation, $x = \dfrac{31 - r^2 - 6r}{2r + 6}$, we shall have $x = \dfrac{31 - 10.89 - 19.8}{6.6 + 6}$

$= \dfrac{.31}{12.6} = .0246$, or rather $x = .0245$, and $v = r + x = 3.3245$:

Again, if this value be substituted for r, we shall have $x = .000005$, and $v = r + x = 3.324505$, for a nearer value of v; and so on, to any assigned degree of exactness.

Given, $v^3 + 2v - 73 = 0$, to find v by approximation.

The root found by trial, is nearly equal 4:

Therefore, $r = 4$, and $r + z = v$:

Then, $v^3 = r^3 + 3r^2z + 3rz^2 + z^3$.
$+ 2v = 2r + 2z$
$-73 = -73$

Whence, $r^3 + 3r^2z + 2r + 2z - 73 = 0$; or,

$z = \dfrac{73 - r^3 - 2r}{3r^2 + 2} = ($ by writing 4 for $z)\dfrac{73 - 64 - 8}{48 + 2}$

$= \dfrac{1}{50} = .02$; and therefore, $v = r + z = 4.02$;

and writing this value for r, in the equation $z = \dfrac{73 - r^3 - 2r}{3r^2 + 2}$;

We shall have $z = \dfrac{73 - 64.964808 - 8.04}{48.4812 + 2}$

$=$

$$= \frac{-.004808}{50.4812} = -.000095; \text{ and } v = r + z = $$
4.019905 *nearly.*

And after this manner of reasoning, we may obtain theorems for approximating to the roots of pure powers.

Thus, if A be a given quantity whose n root is required, r the nearest less root in the integers, and v the difference between r and the root required : Then will $r^n + nr^{n-1}v + n \times \frac{n-1}{2} r^{n-2} v^2 + n \times \frac{n-1}{2} \times \frac{n-2}{3} r^{n-3} v^3$, *&c.* $= A$; *and assuming* $v = \frac{A - r^n}{nr^{n-1}}$; *or, more nearly, taking the three first terms,*

$$v = \frac{A - r^n}{nr^{n-1} + n \times \frac{n-1}{2} r^{n-2} v^2} = (\text{by writing for } v$$

its $= \frac{A - r^n}{nr^{n-1}}) \frac{A - r^n}{nr^{n-1} + n \times \frac{n-1}{2} r^{n-2} \times \frac{A - r^n}{nr^{n-1}}}$

$$= \frac{A - r^n}{nr^{n-1} + \frac{n^2 - nr^{n-2}}{2nr^{n-1}} \times A - r^n} =$$

$$\frac{A - r^n}{nr^{n-1} + \frac{n-1}{2r} \times A - r^n}; \text{ and by writing}$$

$a,$

a, for $A - r^n$, we have $v = \dfrac{a}{nr^{n-1} + \dfrac{n-1}{2r} \times a}$

$= (by\ reduction)\ \dfrac{ra}{nr^n + \dfrac{n-1}{2}a}$, the theorem for

approximating to the value of v, which added to r, will give a correction of the root; which if not sufficiently near the truth, the operation must be repeated, by substituting the new r in the equation exhibiting the value of v.

Thus, for example, suppose the cube root of 3 is required.

Here $r = 1$, *the nearest less root in the integers, and* $r + v = $ *root required.*

Therefore, $v = \dfrac{ra}{nr^n + \dfrac{n-1}{2}a} = \dfrac{2}{2+3} = \dfrac{2}{5} = .4,\ and$

$r + v = 1 + .4 = 1.4$, *which substituted for* r, *and the operation repeated,* v *will be found* $= .0397$; *therefore,* $r + v = 1.4 + .0397 = 1.4397 = $ *cube root of 3, very near.*

C H A P. XXII.

CONCERNING UNLIMITED PROBLEMS.

HAVING gone through, and explained the methods used in arguing limited problems, or such as admit of but one solution; it remains therefore, that we shew the learner how to reason about

those

thofe problems which are unlimited, or admit of various anfwers.

It was obferved in Chap. xv, of this Book, that when the equations expreffing the conditions of the queftion, are lefs in number than the quantities fought, the queftion is unlimited, or capable of innumerable anfwers; yet all the poffible anfwers in whole numbers, are for the moft part limited to a determinate number.

As queftions of this nature admit of fome variations as to their general folution; we fhall therefore confider them in the following problems.

PROBLEM I.

To find the values of v and y in whole numbers, in the equation av ± by ± c = 0; where a, b and c, are given quantities.

RULE.

1. REDUCE the given equation to its leaft terms, by dividing it by its greateft common divifor.

2. FIND the value of v from the given equation; and reduce the refulting expreffion, by expunging all whole numbers from it, until c be lefs than a, and the co-efficient of y becomes unity.

3. ASSUME this laft refult equal to fome known whole number, and the expreffion reduced, will give the value of y in known terms; from which the value of v may be determined in the given equation.

Note. *If after the given equation is divided by its greateft common divifor, the co-efficients of the unknown quantities, are commenfurable to each other, the queftion is impoffible.*

EXAM.

EXAMPLES.

Given, $10v - 8y - 36 = 0$, to find v and y in whole numbers.

First, $5v - 4y - 18 = 0$, *by dividing the whole by* 2; *or,* $5v - 4y = 18$.

Put WN for any whole number:

Then $v = \frac{18 + 4y}{5} = WN$ *by the question:*

But, $\frac{18 + 4y}{5} = 3 + \frac{3 + 4y}{5}$; *therefore,* $\frac{3 + 4y}{5}$ $= WN$, *per axiom* 9. *Also,* $\frac{5y}{5} = WN$: *Consequently,* $\frac{5y}{5} - \frac{3 + 4y}{5} = \frac{y - 3}{5} = WN$, *per axiom* 9; *and therefore,* $\frac{y - 3}{5} = n$; *and for the least value of* y, *assume* $n = 0$, *and we shall have* $y - 3 = 5n = 0$; *or,* $y = 3$, *and* $v = \frac{18 + 4y}{5} = 6$.

Given, $26v + 18y = 140$, to find v and y in whole numbers.

First, $13v + 9y = 70$ *by dividing the whole by* 2:

Then $v = \frac{70 - 9y}{13} = WN$: *But,* $\frac{70 - 9y}{13} = 5$ $+ \frac{5 - 9y}{13}$;

Therefore, $\frac{5 - 9y}{13} = WN$, *per axiom* 9; *also,* $\frac{13y}{13} = WN$; *consequently,* $\frac{5 - 9y}{13} + \frac{13y}{13} = \frac{5 + 4y}{13}$ $=$

$= W N$, per ax. 8; and $\frac{5 + 4y}{13} \times 3 = \frac{15 + 12y}{13}$

$= W N$, per ax. 7. But, $\frac{15 + 12y}{13} = 1 + \frac{2 + 12y}{13}$;

therefore, $\frac{2 + 12y}{13} = W N$, per ax. 9. Also, $\frac{13y}{13} =$

$W N$; whence, $\frac{13y}{13} - \frac{2 + 12y}{13} = \frac{y - 2}{13} = W N$,

per ax. 9: And $\frac{y - 2}{13} = n$; or, $y = 13n + 2$; and

assuming $n = 0$, we have $y = 2$, and $x = \frac{70 - 9y}{13} =$

4.

I owe my friend a moidore, have nothing about me but crowns, and he has nothing but guineas: How muſt we exchange theſe pieces of money, ſo that I may acquit myſelf of the debt? A moidore being valued at 27 ſhillings, a crown at 5 ſhillings, and a guinea at 21 ſhillings.

Put $x =$ number of crowns, and y the number of guineas:

Then $5x - 21y = 27$ by the queſtion:

Or, $x = \frac{27 + 21y}{5} = W N$. But, $\frac{27 + 21y}{5} =$

$5 + 4y + \frac{2 + y}{5}$; conſequently, $\frac{2 + y}{5} = W N$, and

$\frac{2 + y}{5} = n$; or, $2 + y = 5n$; and aſſuming $n = 1$, we

have $y = 3$, the number of guineas, and $x = \frac{27 + 21y}{5}$

$= 18$, the number of crowns. Therefore, I muſt give my friend 18 crowns, and he muſt give me three guineas.

Given,

Given, $4x + 17y = 2900$, to find all the possible values of x and y in whole numbers.

First, $y = \dfrac{2900 - 4x}{17} = WN$; but $\dfrac{2900 - 4x}{17}$

$= 170 + \dfrac{10 - 4x}{17}$; *therefore*, $\dfrac{10 - 4x}{17} = WN$, *per*

ax. 8. *And* $\dfrac{10 - 4x}{17} \times 4 = \dfrac{40 - 16x}{17} = WN$, *per*

ax. 9. *Also*, $\dfrac{17x}{17} = WN$: *Consequently*, $\dfrac{40 - 16x}{17} +$

$\dfrac{17x}{17} = \dfrac{40 + x}{17} = WN$, *per* ax. 7. *But*, $\dfrac{40 + x}{17} = 2$

$+ \dfrac{6 + x}{17}$; *therefore*, $\dfrac{6 + x}{17} = WN$, *per* ax. 8. *And*

assuming this last equation $= n$, *we get* $x = 17n - 6$,
where, if n be taken $= 1$, *we shall have* $x = 17 - 6 =$

11 *for the least value of* x, *and* $y = \dfrac{2900 - 4x}{17} =$

168 *for the greatest value of* y : *And since* $\overline{6 + x} \div 17$
$= n$, *is a whole number* ; *it is plain, that* $n + 1$ *is the*
first augment of $\overline{6 + x} \div 17$ *in whole numbers* ; *and*
therefore, $x = 17n + 17 - 6$, *the second value of* x ;

which substituted for x in the equation $y = \dfrac{2900 - 4x}{17}$

will give the second value of y : *Or, by adding* 17 *suc-*
cessively to the values of x, *and subtracting* 4 *from those*
of y, *we shall have all the possible values of* x *and* y *in*
whole numbers, as follows : viz. $x = 11, 28, 45, \&c.$
to 708 ; *and* $y = 168, 164, 160, \&c.$ *to* 4.

PROBLEM II·

To find the least whole number x, *that being divided*
by the given numbers, $a, b, c, d, \&c.$ shall leave given
remainders, $g, k, l, m, n, \&c.$

C c c

RULE.

R U L E.

1. SUBTRACT each of the remainders from *x*, and divide the several results by their respective divisors, *a, b, c, d,* &c. and the resulting quotients will equal whole numbers.

2. ASSUME the first equation equal *b,* and find the value of *x* in terms of *b.*

3. SUBSTITUTE the value of *x* in terms of *b*, in the second equation; and proceed with the result as in the last problem, by expunging all whole numbers, until the co-efficient of *b* becomes unity, &c.

4. PUT this expression equal *p,* and find the value of *x* in terms of *p*, by means of the equation of *b.*

5. SUBSTITUTE the value of *x* in terms of *p*, in the third equation, with which proceed as before, and so on, through all the given equations; assuming the final result equal to some known whole number, and finding the values of the several substituted letters, *b, p,* &c. from which the value of *x* may be determined in known terms.

EXAMPLES.

To find the least whole number, that being divided by 7 shall leave 6 remainder; but being divided by 6 shall leave 4 remainder.

Put v = number sought.

Then, $\dfrac{v-6}{7} = W N,$ *and* $\dfrac{v-4}{6} = W N.$

Assume $\dfrac{v-6}{7} = b,$ *and we shall have* $v = 7b + 6,$ *which substituted for v in the second equation, gives*
$$7b + 2 \div 6$$

$$\frac{7b+2}{6} = W N: \text{ But, } \frac{6b}{6} = W N: \text{ Confequently,}$$

$$\frac{7b+2}{6} - \frac{6b}{6} = \frac{b+2}{6} = W N, \text{ and affuming } \frac{b+2}{6}$$

$= n$, we fhall have $b = 6n - 2$; where if n be taken $= 1$, we fhall have $b = 4$, and $v = 7b + 6 = 34$, the number required.

To find the leaft whole number, that being divided by 18, fhall leave 14 remainder; but being divided by 28, fhall leave 20 remainder.

Put $v = $ number fought.

Then, $\frac{v-14}{18} = W N:$ And, $\frac{v-20}{28} = W N.$

Affume, $\frac{v-14}{18} = b$; and we have $v = 18b+14$, which fubftituted for v in the fecond equation, gives $\frac{18b-6}{28} = W N$; or, $\frac{9b-3}{14} = W N$ by dividing all the terms by 2; and $\frac{9b-3}{14} \times 3 = \frac{27b-9}{14} = W N.$

Alfo, $\frac{14b}{14} \times 2 = \frac{28b}{14} = W N.$ Confequently, $\frac{28b}{14} - \frac{27b-9}{14} = \frac{b+9}{14} = W N:$ and affuming $\frac{b+9}{14} = n$, we have $b = 14n - 9$; and putting $n = 1$, we have $b = 14n - 9 = 5$, and $v = 18b + 14 = 104$, the number required.

Diophantine Problems.

DIOPHANTINE Problems, fo called, from *Diophantus* their inventor, are fuch as relate to the finding of fquare and cube numbers, &c.

THESE

THESE problems are so exceedingly curious, that nothing less than the most refined Algebra, applied with the utmost skill and judgment, could ever surmount the difficulties which necessarily attend their solution. The peculiar artifice made use of in forming such positions as the nature of the problems require, shews the great use of Algebra, or the analytic art, in discovering those things that otherwise, would be without the reach of human understanding.

ALTHO no general rule can be given for the solution of these problems ; yet the following direction will be very serviceable on many occasions.

DIRECTION.

ASSUME one or more letters, for the root of the required square, cube, &c. such that when involved to the height of the proposed power, either the given number, or the highest term of the unknown quantity may vanish. Then if the unknown quantity in the resulting equation, be of simple dimension, find its value by reducing the equation. But if the unknown quantity be still a square, cube, or other power ; assume other letter or letters, with which proceed as before, until the highest term of the unknown quantity become of simple dimension in the equation.

EXAMPLES.

To find a square number x^2, such that $x^2 + 1$ shall be a square number.

Assume $x - 2$ *for the root of* $x^2 + 1$:

Then will $\overline{x-2}\,^2 = x^2 + 1$; *that is,* $x^2 - 4x + 4 = x^2 + 1$; *or,* $4x = 4 - 1 = 3$; *whence,* $x = \frac{3}{4}$, *and* $x^2 = \frac{9}{16}$, *and* $x^2 + 1 = \frac{9}{16} + 1 = \frac{25}{16}$: *Therefore,*

$\frac{9}{16}$, is the number required. But if we had assumed $\frac{r^4 - 2r^2 + 1}{4r^4}$ for x^2, we should have had $\frac{r^4 - 2r^2 + 1}{4r^2}$

$+ 1 = \frac{r^4 + 2r^2 + 1}{4r^2}$, which is evidently a square number; where r may be taken for any number.

To find two numbers, such that their product and quotient may be both square and cube numbers.

Assume v^9 and v^3 for the required numbers:

Then $v^9 \times v^3 = v^{12}$, and $v^9 \div v^3 = v^6$, are evidently square and cube numbers; where v may be any number taken at pleasure.

To find four square numbers in arithmetical progression.

For the sum of the two extremes, assume $2n^2$; then will the sum of the two means be also $2n^2$ by the nature of the proportion:

For the roots of the two means, assume $n + 3z$, and $n - 4z$:

Then will $\overline{n + 3z}|^2 + \overline{n - 4z}|^2 = 2n^2$:

That is, $n^2 + 6nz + 9z^2 + n^2 - 8nz + 16z^2 = 2n^2$:

Or, $25z^2 - 2nz + 2n^2 = 2n^2$:

Or, $25z^2 = 2nz$; and by dividing by z, we have $25z = 2n$:

Whence, $z = 2n \div 25$; and puting $n = 1$, we have $z = \frac{2}{25}$:

Therefore, $\overline{n + 3z}|^2$, and $\overline{n - 4z}|^2 = \frac{961}{625}$, and $\frac{289}{625}$, are the two means:

And for the roots of the two extremes, assume $n - 2z$, and $n + z$:

Then will $\overline{n - 2z}|^2 + \overline{n + z}|^2 = 2n^2$:

Or, $n^2 - 4nz + 4z + n^2 + 2nz + z^2 = 2n^2$:

And by reduction, $z = 2n \div 5 = \frac{2}{5}$:

Whence,

Whence, $\overline{n-2z}|^2$, *and* $\overline{n+z}|^2 = \frac{1}{25}$, *and* $\frac{49}{25}$, *the two extremes. So that the four square numbers in arithmetical progression, are* $\frac{1}{25}$, $\frac{289}{625}$, $\frac{961}{625}$, $\frac{49}{25}$.

To find a number, such that being multiplied with one tenth part of itself, and the product increased by 36, shall produce a square number.

Put v *for the number sought; then* $v^2 \div 10 + 36$, *is to be a square number :*

Assume the root of this square $= v - 6$, *then will* $\overline{v-6}| = \overline{v^2 \div 10 + 36}$; *that is,* $v^2 - 12v + 36 = v^2 \div 10 + 36$:

Or, $10v^2 - 120v = v^2$; *whence, by reduction* $v = \frac{120}{9}$, *the number required.*

To divide a given number 29, consisting of two known square numbers 4 and 25, into two other square numbers.

For the root of the first square, assume $rv - 2$; *and for the root of the second* $nv - 5$:

Then will $\overline{rv-2}|^2 + \overline{nv-5}|^2 = 29$:

That is, $r^2v^2 - 4rv + 4 + n^2v^2 - 10nv + 25 = 29$:

Or, $\overline{r^2 + n^2}v^2 - \overline{4r - 10n}v + 29 = 29$; *or,* $\overline{r^2 + n^2}v^2 = 4r + 10nv$; *and by dividing by* v, *we have* $\overline{r^2 + n^2}v = 4r + 10n$:

Or, $v = \dfrac{4r + 10n}{r^2 + n^2}$; *and therefore,* $rv - 2 = \dfrac{4r^2 + 10nr}{r^2 + n^2} - 2 = \dfrac{2r^2 - 2n^2 + 10nr}{r^2 + n^2}$; *and* $nv - 5 = \dfrac{4rn + 10n^2}{r^2 + n^2} - 5 = \dfrac{4rn - 5r^2 + 5n^2}{r^2 + n^2}$; *and assuming* $r = 1$, *and* $n = 2$, *we shall have* $\dfrac{2r^2 - 2n^2 + 10nr}{r^2 + n^2} = \dfrac{14}{5}$

$\frac{14}{5}$ *for the root of the first square, and* $\frac{4rn - 5r^2 + 5n^2}{r^2 + n^2}$ $= \frac{25}{5}$ *for the root of the second.*

To find three square numbers in arithmetical progression.

Assume n^2 *for the mean; then will* $2n^2 =$ *the sum of the extremes by the nature of the proportion.*

For the root of the greater extreme, assume $n + 2v$, *and for the root of the less* $n - 3v$:

Then will $\overline{n - 3v}\vert^2 + \overline{n + 2v}\vert^2 = 2n^2$:

That is, $n^2 - 6nv + 9v^2 + n^2 + 4vn + 4v^2 = 2n^2$:

Or, $13v^2 - 2nv + 2n^2 = 2n^2$:

Or, $13v^2 = 2nv$; *and by dividing by* v, *we have* $13v = 2n$; *whence,* $v = 2n \div 13$; *where* n *may be any number at pleasure:*

And by assuming $n = 1$, *we shall have* $v = 2 \div 13$:

Whence, $\overline{1 - \frac{6}{13}}\vert^2 = \frac{49}{169}$ *for the least extreme:*

And $\overline{1 + \frac{4}{13}}\vert^2 = \frac{289}{169}$ *for the greater: Wherefore,*

the numbers required, are $\frac{49}{169}$, 1, *and* $\frac{289}{169}$.

THE END OF VOLUME I.

TABLE OF POWERS.

	2	3	4	5	6	7	8	9
1st. Power	2	3	4	5	6	7	8	9
2d. dit.	4	9	16	25	36	49	64	81
3d. dit.	8	27	64	125	216	343	512	729
4th. dit.	16	81	256	625	1296	2401	4096	6561
5th. dit.	32	243	1024	3125	7776	16807	32768	59049
6th. dit.	64	729	4096	15625	46656	117649	262144	531441
7th. dit.	128	2187	16384	78125	279936	823543	2097152	4782969
8th. dit.	256	6561	65536	390625	1679616	5764801	16777216	43046721
9th. dit.	512	19683	262144	1953125	10077696	40353607	134217728	387420489
10th. dit.	1024	59049	1048576	9765625	60466176	282475249	1073741824	3486784401
11th. dit.	2048	177147	4194304	48828125	362797056	1977326743	8589934592	31381059609
12th. dit.	4096	531441	16777216	244140625	2176782336	13841287201	68719476736	282429536481
13th. dit.	8192	1594323	67108864	1220703125	13060694016	96889010407	549755813888	2541865828329
14th. dit.	16384	4782969	268435456	6103515625	78364164096	678223072849	4398046511104	22876792454961

Explanation of the above Table.

WHEN the root of any number in this table, is required, look for the power at the left hand, then casting your eye along that line towards the right hand, till you observe the number, and the figure standing at the top, is the root required.