THE TEACHING OF ARITHMETIC

SMITH

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PREFACE

This work has been prepared with a view to the needs of Teachers' Reading Circles and of those who are giving instruction or supervising the work in arithmetic in the elementary schools. The effort has been made to free it from the more difficult technicalities of mathematics as far as possible, and from that phraseology of the world of pedagogy that renders educational problems unnecessarily difficult. It is the author's desire to place before the large body of teachers of arithmetic, rather than the few who are interested in the technicalities of experiment, a brief summary of the development of the science and of the reasons for teaching it; a statement of the subject matter that may properly be selected for school purposes, and the arrangement of this matter in a course of study; a consideration of a few of the technical features of arithmetic; and a discussion of the work of the several school years. He feels it quite unnecessary, however, to repeat at length the simple explanations and devices that are found in any good textbook, believing that the teacher will usually find it advantageous in such matters to follow the book that she is using.

The problem of teaching arithmetic is not a particularly complex one. The world demands little of the school, hardly more than the four operations with integers, very simple fractions, and decimal fractions to two places; a small number of tables of denominate numbers; a few cases in mensuration; and the ability to find a given per cent of a number. In America the schools are commonly given eight years in which to accomplish this work, and if they fail it is not for lack of time, but rather because they do not use this time wisely. If they introduce unnecessary features, such as obsolete tables or too elaborate explanations; if they require business arithmetic that is beyond the understanding of the children; and if they use hard methods of solving problems instead of simple ones, they cannot expect to bring the pupils to the standard that the world requires. This seems to be a part of the difficulty, but not all of it, and it is to assist teachers to find the difficulty, so far as it can be found at present, that this book is written.

If the author succeeds in making arithmetic more interesting to teacher and pupil, in making it touch more clearly the vital questions of daily life, in pointing out a line of work that is free from dangerous eccentricities, and in presenting a broad view of the subject as opposed to one in which small things obstruct the vision, the purposes that he has had in mind will be realized. It is with this hope that he submits this little work to his colleagues in the teaching profession.

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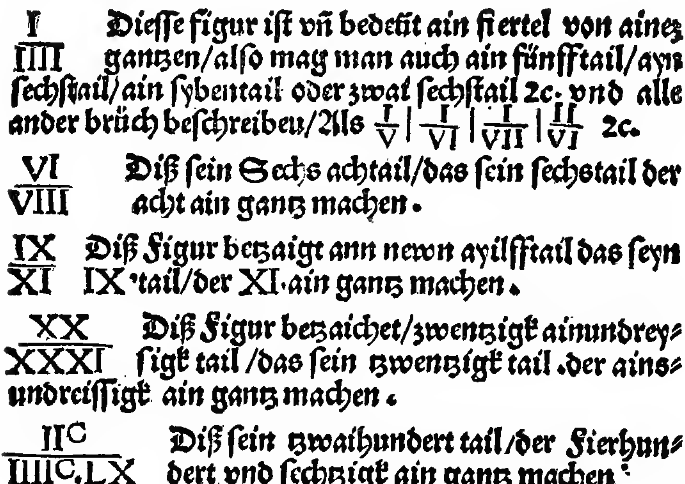
CHAPTER I

THE HISTORY OF ARITHMETIC

Of all the sciences, of all the subjects generally taught in the common schools, arithmetic is by far the oldest. Long before man had found for himself an alphabet, long before he first made rude ideographs upon wood or stone, he counted, he kept his tallies upon notched sticks, and he computed in some simple way by his fingers or by pebbles on the ground. He did not always count by tens, as in our decimal system; indeed, this was a rather late device, and one suggested by his digits. At first he was quite content to count to two, and generations later to three, and then to four. Then he repeated his threes and had what we call a scale of three, and then, as time went on, he used a scale of four, and then a scale of five. In measurements he often used the scale of twelve, because he found that twelve is divisible by more factors than ten, and particularly by two and three and four; but by the time he was ready to write his numbers the convenience of finger reckoning had become so generally recognized that ten was practically the universal radix. Nevertheless, there remain in our language and customs numerous relics of the duodecimal (scale of twelve) idea, such as the number of inches in a foot, of ounces in a troy pound, and of pence in a shilling, all influenced by the Roman inclination to make much use of twelve in practical computation.

The writing of numbers has undergone even more change than the number names. Not only was there usually a notation for each written language in ancient times, but some languages had several sets of numerals, as is seen in the three standard systems of Egypt, the two of Greece, and the somewhat varied forms in use in Rome. The Roman supremacy gave the numerals of these people great influence in Europe, and they were practically in universal use in the West until the close of the Middle Ages. The Romans themselves had no definite standard for their numerals. Whereas we write IV for four, IX for nine, and XL for forty, they usually wrote IIII, VIIII, and XXXX. Occasionally they wrote IIXX for eighteen, and they would have written the year 1914 as MDCCCCXIIII rather than MCMXIV. Soon after books began to be printed a slight effort was made to use Roman numerals in fractions, as in the following facsimile from a German work of 1514:


$\frac{\mathrm{I}}{\mathrm{IIII}}$ Diesse figur ist vñ bedeüt ain fiertel von ainez gantzen/also mag man auch ain fünfftail/ayn sechstail/ain sybentail oder zwai sechstail ꝛc. vnd alle ander brüch beschreiben/Als $\frac{\mathrm{I}}{\mathrm{V}}$ | $\frac{\mathrm{I}}{\mathrm{VI}}$ | $\frac{\mathrm{I}}{\mathrm{VII}}$ | $\frac{\mathrm{II}}{\mathrm{VI}}$ ꝛc.

$\frac{\mathrm{VI}}{\mathrm{VIII}}$ Diß sein Sechs achtail/das sein sechstail der acht ain gantz machen.

$\frac{\mathrm{IX}}{\mathrm{XI}}$ Diß Figur bezaigt ann newn ayilfftail das seyn IX tail/der XI ain gantz machen.

$\frac{\mathrm{XX}}{\mathrm{XXXI}}$ Diß Figur bezaichet/zwentzigk ainundreyssigk tail/das sein zwentzigk tail der ainsundreissigk ain gantz machen.

$\frac{\mathrm{II^C}}{\mathrm{IIII^C.LX}}$ Diß sein zwaihundert tail/der Fierhundert vnd sechtzigk ain gantz machen.

FROM KÖBEL'S ARITHMETIC OF 1514

The Romans, however, never wrote fractions like this, and much that has been quite recently taught in the schools about these numerals is entirely modern.

It may sometimes be wondered how the world was able to handle numbers in the days of Roman numerals. How, for example, did people add CLXVI and DXXIX, or multiply one of these numbers by another? The answer is that they used small disks on a ruled board, or pebbles that were moved on wires or in grooves. The Latin name for "pebble" was *calculus*, and from this we get our word "calculate." The following illustration shows the appearance of the medieval calculating board:

FROM THE TITLE-PAGE OF KÖBEL'S ARITHMETIC, 1514

Meantime there had arisen in the East, probably in India, although very likely subjected to influence from without, our present system of notation, and little by little this permeated the West. When it arose it was without a zero, and hence without such place value as we use to-day; but probably about the seventh century the zero appeared, and the completed system found its way northward into Persia and Arabia, and thence in due time it was transmitted to the West.

Some of the earliest Hindu forms are shown in the following table:[1]

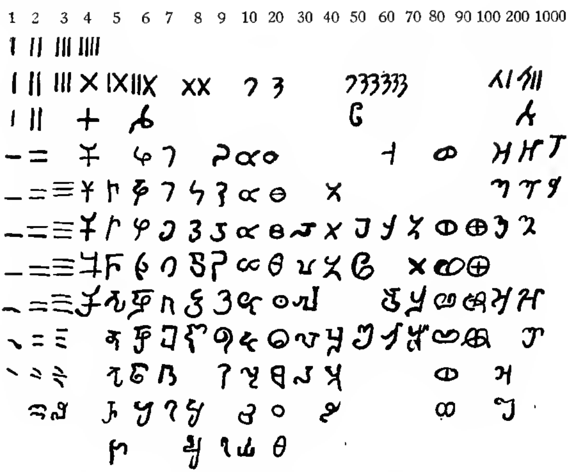
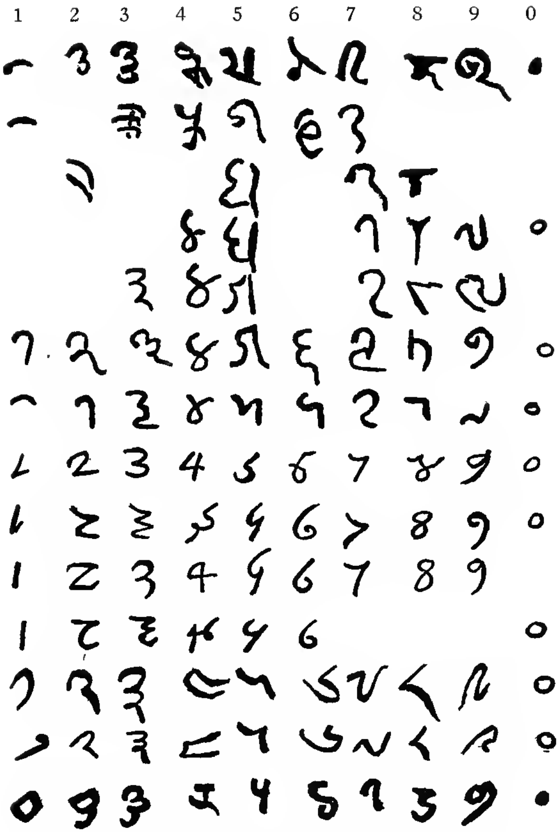


TABLE SHOWING THE PROGRESS OF NUMBER FORMS IN INDIA

After the zero was invented, the numerals appear in various manuscripts with forms like those on page 5.

All of this goes to show how long it took the world to settle upon the forms that we teach our children.

[1] This and similar tables are from Smith and Karpinski's "The Hindu-Arabic Numerals," Ginn and Company, Boston, 1911, to which reference is made for complete descriptions.



Numerals used with Place Value

At first the subject of arithmetic was purely practical, the counting of arrows or sheep or men. For a long time this was all that number meant to the world, until the mystic age developed and philosophy began. Then numbers were differentiated, and odd and even were distinguished, and "There 's luck in odd numbers" became a tenet of faith, while the even numbers became designated as earthly and feminine. The story is long and interesting, that of the development of this mysticism, with its special prominence of three and


Incommincia vna practica molto bona et vtile: a ciascaduno chi vuole vxare larte dela merchadantia. chiamata vulgarmente larte de labbacho.

Regato piu e piu volte da alchuni zouani a mi molto dilectissimi: li quali pretendeuano a douer volere fare la merchadantia: che per loro amore me piacesse affadigarme vno puocho: de dargli in scritto qualche fundamento cerca larte de arismetrica: chiamata vulgarmente labbacho. Unde io constretto per amor di loro: et etiadio ad vtilita di tuti chi pretendano a quella: segondo la picola intelligentia del inzegno mio: ho deliberato se non in tuto: in parte tamen satisfare a loro. acio che loro virtuosi desiderii vtile frutto receuere posseano. In nome di dio adoncha: toglio per principio mio el ditto de algorismo cosi dicendo.

Tute quelle cose: che da la prima origine hanno habuto productione: per raxone de numero sono sta formade. E cosi come sono: hano da fir cognoscude. Pero ne la cognitione de tute le cose: questa practica e necessaria. E per intrar nel proposito mio: primo sapi lectore: che quanto fa al proposito nostro: Numero e vna moltitudine congregata ouero insembrada da molte vnitade. et al meno da do vnitade. come e. 2. el quale e lo primo e menore numero: che se truoua. La vnitade e quella cosa: da la quale ogni cosa si dita vna. Ergo dario sapi: che se truoua numeri de tre maniere. El primo se chiama numero simplice. lal tro numero articulo. El terzo se chiama numero

First Page of the Treviso Arithmetic, 1478

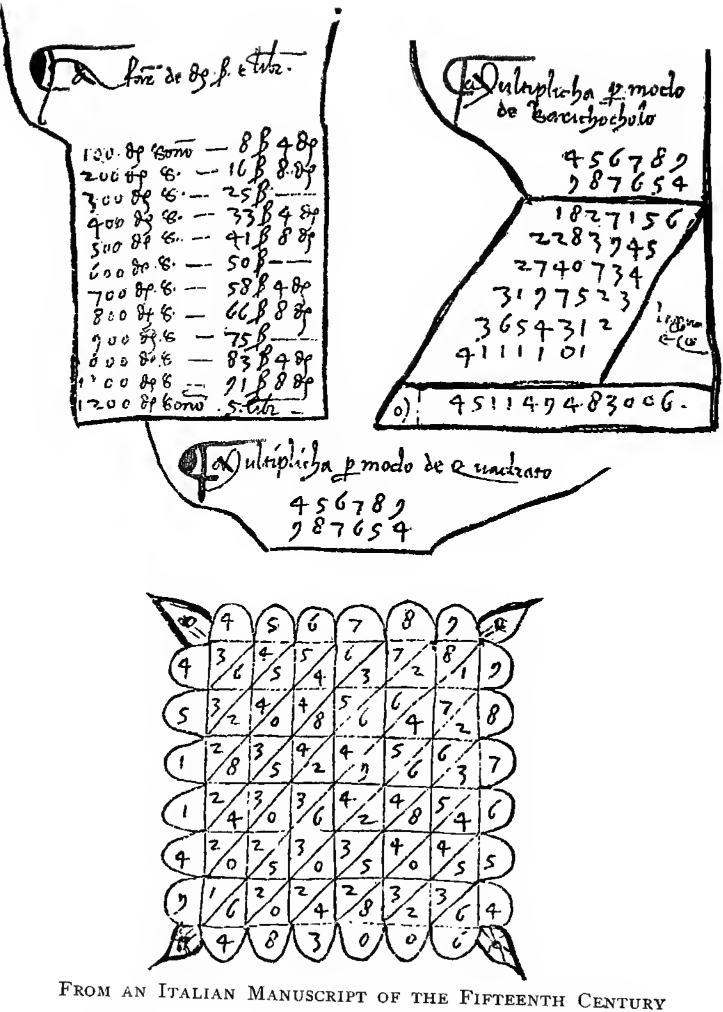
seven. Furthermore, the movement led to a study of the properties of numbers, roots, and series. Number games became connected with the abacus, and so the practical and the mysterious were more or less blended even at times when they are generally regarded as widely separated.

The first printed arithmetic appeared at Treviso, a city situated north of Venice, in 1478. It may add a little to the interest of the subject to see the facsimile of the first page of this very rare book as shown on page 6. The first three lines read as follows: "Here beginneth a practica, very good and useful to anyone who wishes to use the mercantile art commonly called the art of the abacus." "Practica" and "art of the abacus" were then common names for business arithmetic.

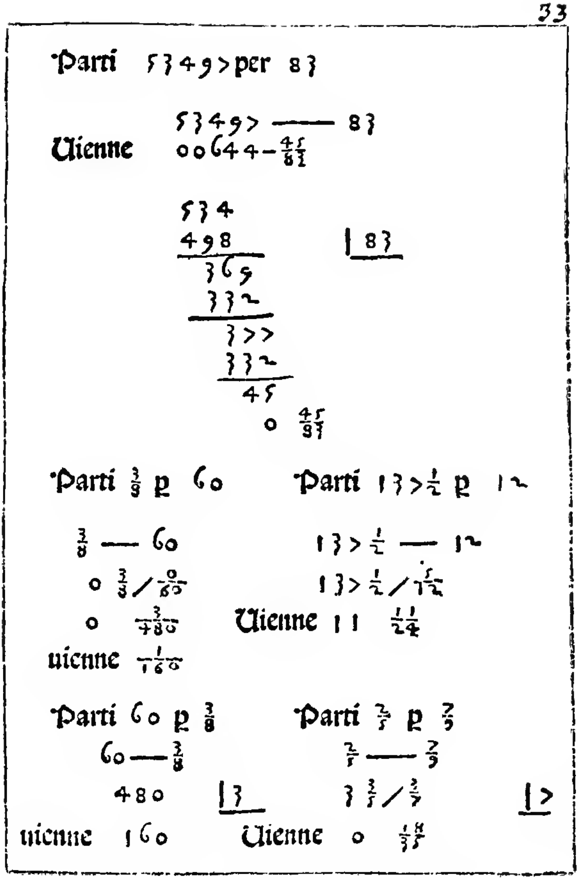
The methods of multiplying have changed greatly in modern times. To illustrate this fact, two cases of multiplication, each showing the product of 456,789 by 987,654, taken from an Italian manuscript of the fifteenth century, are shown on page 8.

Long division has changed very greatly, and it will probably be of interest to the reader to see a facsimile of the first printed page in which our modern form occurs. It is from Calandri's arithmetic of 1491, printed at Florence, and is shown on page 9.

The growth of topics of arithmetic is also an interesting subject for investigation. We say that there are four fundamental operations, although once there was only one, and at another time the world recognized as many as nine. We operate chiefly with decimal fractions, as in working with dollars and cents, although these fractions are scarcely three hundred years old. We are impatient that a child stumbles over common fractions, and yet, so difficult did the world find the subject that for thousands of years only the unit fraction was used. We wonder how the long division form of greatest common divisor ever had place in arithmetic, and yet it was a practical necessity in business until about 1600 A.D. We feel that "partnership involving time" could never have been practical, and yet until a couple of centuries ago it was



From an Italian Manuscript of the Fifteenth Century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From Calandri's Arithmetic, 1491

decidedly so. And thus it is with many topics of arithmetic, — they have changed from century to century, and even in our own time from year to year. It is well for a teacher to know a little of this history of the subject taught, although space does not allow for any serious consideration of the topic in this work. In the bibliography some reference will be found to sources easily available, and the teacher who wishes to see arithmetic in progress, as opposed to arithmetic stagnant and filled with the obsolete, should become acquainted with one or more of these works upon the subject. The history of arithmetic is the best single stimulus to good method in teaching the subject.

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CHAPTER II

THE REASONS FOR TEACHING ARITHMETIC

The ancients had less difficulty than we have in assigning a reason for teaching arithmetic, because they generally differentiated clearly between two phases of the subject. The Greeks, for example, called numerical calculation by the name *logistic*, and this subject was taught solely for practical purposes to those who were going into trade. A man might have been a very good philosopher or statesman or warrior without ever having learned to divide one long number by another. Such a piece of knowledge would probably have been looked upon as a bit of technical training, like our use of the slide rule or the arithmometer, two of the several modern machines by the use of which we can add, subtract, multiply, divide, raise to powers, and extract roots of numbers. On the other hand, the Greeks called their science of numbers *arithmetic*, a subject that had nothing whatever to do with addition, subtraction, multiplication, or division, and that excluded all applications to trade and industry. This subject was taught to the philosopher, and to the man of "liberal education," as we still call him. It considered questions like the factorability of numbers, powers and roots, and series — topics having little if any practical application in the common walks of life. Therefore when a Greek was asked why he taught logistic, his answer was definite: It is to make a business man able to compute sufficiently well for his trade. If asked why he taught arithmetic, as the term was then used, his answer was still fairly clear, although open to debate: I teach it because it makes a man's mind more philosophic.

In the present day we have a somewhat more difficult task when we attempt to answer this question. Arithmetic with us includes the ancient logistic, and we teach the subject to all classes of people: to one who will become a day laborer, belonging to a class that never in the history of the world studied such a subject until very recently; to the tradesman, who never uses or cares to use the chapter on prime numbers; to the statesman, who will probably have little opportunity to employ logarithms in any work that may come to him; to the clergyman, to whom the metric system will soon be merely a name; and to the housewife, the farmer, and all those who travel the multifarious walks of our complex human life. For us to tell why we teach the American arithmetic to all these people is by no means so easy as it was for the Greek to answer his simple question.

In general, however, we may say that as we have combined the ancient logistic (calculation) and arithmetic (theory) in one subject, so we have combined the Greek purposes, and teach this branch because it is useful in a business way to every one, and also because it gives a kind of training that other subjects do not give.

As to the first reason there can be no question. When the great mass of men were slaves the business phase was not so important; but now that every man is to a great extent his own master, receiving money and spending it, some knowledge of calculation is necessary for every American citizen. To elaborate upon this point is superfluous. There is, however, one principle that should guide us in the consideration of this phase of the question: *Whatever pretends to be practical in arithmetic should really be so.* We have no right to inject a mass of problems on antiquated investments, on obsolete forms of partnership, on forgotten methods of mercantile business, or on measures no longer in common use, and make the claim that these problems are practical. If we wish them for some other purpose, well and good; but as practical problems they have no right to appear. To set up a false custom of the business world is as bad as to teach any other untruth; it places arithmetic in particular, and education in general, in a false light before pupils and parents, and is unjustified by any reason that we can adduce. An obsolete business problem has just one reason for being, and this reason is that it has historical interest. We can secure the mental discipline as well by other means, and we have no right to handicap a child's mind with things that he will be forced to forget the minute he enters practical life.

Axiomatic as this statement may seem, however, it is by no means easy to limit the subject matter in arithmetic to the art of calculation and to problems that are representative only of the real business life of to-day. The custom of the school, the familiarity of the teacher with the traditional problem rather than with the demands of modern business, the genuine interest that children find in puzzle problems, the influence of the official curriculum — all militate against as rapid reform as might be desired. This condition of affairs is by no means limited to our own country; it is characteristic of the school everywhere and always. How it affects arithmetic in its details will be seen in the subsequent chapters.

There remains the side of mental discipline which I have elsewhere called, for want of another term and following various other writers, the culture side. What mental training does a child get from arithmetic that he does not get from biology, or Latin, or music? This is a question so difficult to answer that no one has yet satisfied the world in his reply, and no one is likely to do so. There have been elaborate articles written to show that the proper study of arithmetic has an ethical value, though exactly what there is in the subject to make us treat our neighbor better it is a little difficult to say. Others have said that arithmetic, through its very rhythm, has an æsthetic value, as is doubtless true; but that this is generally realized, or that it serves to make us more appreciative of the beautiful, is hardly to be argued with any seriousness. Still others have felt that by coming in contact with exact and provable truth an individual sets for himself a higher standard in all other lines of work, and this again is probably the case, although the measure of its influence has never been satisfactorily ascertained. And to these reasons may be added many more, such as the training of a deductive science, although elementary arithmetic is to a large extent inductive; the training in concentration, although the untangling of a Latin construction requires quite as close attention; the exaltation of mind that comes from the study of numbers that may increase or decrease indefinitely, and others of like nature. And out of it all, what shall we say? That arithmetic can offer no mental discipline that other subjects do not give? No one really feels this, in spite of the fact that the exact nature of this discipline is hard to formulate. Every one is conscious that he got something out of the study, aside from training in calculation and business applications, that has made him stronger, and the few really scientific investigations that have been made as to the effect of mathematical study bear out this intuitive feeling.

Nevertheless, if we ask a class of recently trained teachers if there is any value in the study of arithmetic except the utilitarian one, an animated and general response greets us in the familiar words, "The doctrine of formal discipline has been exploded." These words have been repeated so often that they have lost whatever of meaning they may once have had, and they usually signify but little more than a Latin phrase would to a child. If one asks, What is the mental discipline that has been exploded? there comes no general reply, although a few may answer that "power acquired in one line cannot be transferred." And if we ask what this means, some of the more thoughtful of our hearers may say that the memorizing of rules in arithmetic does not help us to memorize the rules of grammar — if we do either of these things to-day.

Statements like these are interesting, but they mean little. The fact is, some one has set up a man of straw and, with the help of several thousand followers, is slaying him. No one claims any such mental discipline for arithmetic, nor does any one claim that rapidity in acquiring or reciting the multiplication table is going perceptibly to increase a child's rapidity in learning or reciting the names of the trunk lines that radiate from Chicago. No one believes that accuracy in multiplication is a power that will be so transferred as to make a child perceptibly more accurate in map drawing. It would be absurd to assert that a child who comes to like arithmetic would thereby like his mother the better, or like to go to bed or to get up any more ardently. All such "transfers of power" are of course ridiculous, and no one makes or ever has made any such claims for mental discipline on the part of arithmetic. And if any one is tempted to believe that this form of mental discipline has been exploded, the only true reply is that no one ever asserted it.

It has, however, been asserted that arithmetic strengthens the reasoning powers, and there seems to be no doubt that this assertion has been made in times past with unwarranted emphasis. Every one knows of cases where a person reasons well in arithmetic and not in other subjects. This is probably due to the fact that he has a greater interest in mathematics than in other branches of study, and hence gives less attention to the latter. At any rate, it is probable that the claim has been exaggerated, although the efforts of the psychologists to weigh the value of arithmetic, or of any other subject of study, have not progressed far enough to produce results that can be described as other than crude. It is impossible to weigh a child's mind before a certain subject has been studied, and again after it has been completed, and say that the subject has added or subtracted so much of mental power. We must, after all, fall back on our own experience, with such little help as experimental psychology can render.

Not to dismiss the mental-discipline side too summarily, however, and at the same time seeking to avoid the endless verbiage that usually characterizes the discussion, it is well to set forth more clearly some of the objects to be sought on the culture side of arithmetic. In the first place, we seek an absolute accuracy of operation that differs from the kind of accuracy we seek in science or linguistics or music. The fact that we have, in thousands of problems, sought a result so exact as to stand every test leads us to set a higher standard of accuracy in all lines than we could have set without it. This justifies the introduction of any part of theoretical arithmetic for which the pupil is mentally ready. It is one reason why cube root was formerly studied, when pupils were more mature than now, and in the same way it has justified progressions and a more elaborate treatment of primes than any business need would warrant. Here, then, is a reason for teaching arithmetic that is above and beyond the merely practical of the present moment.

A similar and related reason appears in the fact that mathematics in general, and arithmetic in particular, requires a helpful form of analysis that does not stand out so clearly in other studies. "I can prove this if I can prove that; I can prove that if I can prove a third thing; but I can prove that third thing; hence I see my way to proving the first." This is the analytic form that has come down to us from Plato. It appears more evidently in geometry, but is essentially the reasoning of arithmetic as well. "I can find the cost of $2\frac{1}{2}$ yd. if I can find the cost of 1 yd.; but I know the cost of $6\frac{1}{4}$ yd., so I can find the cost of 1 yd.; hence I can solve my problem," is the unworded line of the child's analysis. Such a training, unconsciously received and often unconsciously given, is valuable in every problem we meet, leading us to exclude the nonessential and hold with tenacity to a definite line of argument.

Furthermore, in mathematics in general, and in arithmetic in particular, we set down our results with a clearness and terseness of expression that is not found in other subjects. It is difficult to weigh the influence of a habit thus acquired. It may not show itself to-day or to-morrow, but when the need for it arises, possibly long after arithmetic as a study has been laid aside, it may again appear. The assertion that a statement is "mathematically exact" is not without meaning, and the habit acquired under some helpful, sympathetic, inspiring teacher, of setting forth the work in arithmetic neatly, clearly, and with no superfluous labor, is one of those mental acquisitions that may easily "carry over" into the ordinary work of practical life.

These three phases of the culture side of arithmetic, the side of mental discipline, will then suffice for our present purpose, which is to show that such a side exists: (1) the contact with absolute truth; (2) the acquisition of helpful forms of analytic reasoning; (3) the acquisition of certain habits that "carry over" into related fields of work. With respect to all three it may be said that our psychologists are more sympathetic to-day than they were a few years ago. That these exhaust the list of values of arithmetic from the culture standpoint must not, however, be inferred.

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CHAPTER III

WHAT ARITHMETIC SHOULD INCLUDE

If we look upon arithmetic only with a view to its utilitarian value, to its power to fit us for the computations that the average person needs to perform or know about in daily life, the range of subject matter is not great. Addition, particularly of money, but not involving very large or numerous amounts, is probably the most important topic. Very likely the making of change, one of the forms of subtraction, is next in frequency of use. Perhaps, for it is difficult to say with certainty, the simple fractions $\frac{1}{2}$ and $\frac{1}{4}$ are next in line of relative importance, including $\frac{1}{2}$ of a sum of money, $\frac{1}{4}$ of a length or weight, and so on. Then may come easy multiplications, to find the cost of 5 pounds of sugar, or of 16 yards of cloth, given the price per pound or yard. A few of the most commonly used measures and their relations must then be known, as that 12 inches equal 1 foot and 16 ounces equal 1 pound. Given this equipment, the average run of humanity would be able to get along fairly well. But beyond this there lies a second field of work that every one may need, that a large minority will need, and that we should all at least know something about. This field includes all four fundamental operations with integers, with simple common fractions (say with denominators of one or two figures), with decimal fractions at least to hundredths, and with compound numbers of at least two denominations; the common business cases of percentage, and their applications; the common problems of business, all of which are applications of the operations above mentioned; and a little knowledge of ratio and proportion, chiefly for understanding the meaning of these terms. From the standpoint of business needs this equipment would answer the purposes of nearly every one. Whatever of applied arithmetic lies beyond this is a part of the technical training of a very small minority.

When we consider this list of topics we are struck by its simplicity and brevity. There is little here to try unduly the understanding of a learner. When we think that this is all that the world usually demands of the school, and that we are allowed eight years in which to impart this knowledge, we are led to ask ourselves why the world is not satisfied with the results. Is the difficulty with ourselves in that we include a lot of matter of relatively little value, but which consumes the time without any just return? Or do we fail to insist on the fundamentals while we are teaching the more advanced topics that find place in our schools? When we consider some of these traditional topics we find that many of them have already been eliminated in the past few years. For example, apothecary's measures form part of the technical training of the drug clerk and the physician; the average citizen has long since forgotten them, and happily so. Compound proportion is never used practically, and any mathematician, if called upon to solve its problems, would employ another and a better method. Duodecimals, while interesting historically and philosophically, from the practical standpoint are used by so few as to place them also in the technical training of the very small minority, and so they have long since been dropped out. Subjects like discount and interest are, of course, included under the common applications of percentage. Similarly, with stocks and bonds, for although such securities are purchased by relatively few people, their nature and uses should be understood by all, particularly as we seem to have entered upon an era of extensive coöperation upon a stock basis. The general nature of applications, however, will be discussed later.

We are therefore led to the conclusion that, in spite of the fact that we have weeded out much that is obsolete in the past generation, there is more work of this kind to be done if we wish to have time for the drill upon the essentials of arithmetic. At any rate we are forced to admit that teachers must place less emphasis upon certain topics than they have done, and more emphasis upon others. For example, the inverse cases of percentage contain problems like this: How long will it take $500 to gain $100 at 5% simple interest? Such a problem is not very real, nor are other inverse cases of this kind much better. They have their place because they lead a pupil a little beyond the line of common application, and this is always a desirable thing to do, but the relative emphasis that should be placed upon them is slight.

If we taught arithmetic only from the standpoint of mental discipline, we might use all the material here mentioned, and any other topics that require the securing of accurate results by clear reasoning processes. Obsolete measures, obsolete methods, progressions, cube root and even higher roots, compound proportion — all such topics might have place if we were seeking only the discipline of arithmetic. When, however, we consider that we are seeking to unite these two considerations, and are attempting to make the subject both practical and disciplinary, then we are met by the necessity for mutual concessions. The practical side must concede to the disciplinary by having its processes understood when they are presented, even though the child is not called upon to remember the reasoning. The disciplinary side must concede to the practical by selecting its topics in such way as to give no false notions of business, and as to encourage the pupils to take an interest in the quantitative side of the world about them. On the one hand, we must not teach business arithmetic by mere arbitrary rules that are not understood, since this would be to eliminate its disciplinary nature; on the other hand we must not introduce a style of time draft that is now obsolete in America, or artificial examples in compound proportion, because these inculcate wrong ideas of the business world about us, nor extensive work in equation of payments, because this is part of the technical training of such a very small minority that we can use our time to better advantage.

A word should also be said as to the tendency of some teachers to feel that worthy results are attained when children have been drilled to unnecessary facility in one line or another. It is not difficult to train children to add two columns of figures at a time, or to multiply by cross multiplication, or to distinguish between seven and eight circles at a glance, or to display various other forms of arithmetical ability analogous to the acrobatical feats sometimes allowed in a gymnasium. Such activities have some value as games, and they attract attention on the part of visitors, but it is a question whether the pupil derives any real benefit from most of this kind of work. It is because of this feeling that such features are not found in our best textbooks, the space being devoted to those things that the business man finds useful in everyday life.

Thus it happens that the modern American arithmetic is a fair compromise between the practical without theory and the theoretical without practice, the two distinct phases of the old Greek number work. To keep this balance true is one of the missions of teachers to-day. The tendency is to obtain the mental discipline of arithmetic from problems that are practical, and that this tendency is a healthy one there seems to be no room for reasonable doubt. The nature of these problems is discussed in the next chapter.

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CHAPTER IV

THE NATURE OF THE PROBLEMS

In no way has arithmetic changed so much of late years as in the nature of the problems and the arrangement of the material. The former has come about from two causes — (1) the needs of society, and (2) the study of child psychology. The latter, the arrangement of the material, has been determined almost entirely by psychological considerations. In this chapter it is proposed to speak of the former, the nature of the problems.

It should definitely be stated, however, that this emphasis laid upon applied problems should not be construed to mean that the abstract number work is not quite as important as the concrete. To be sure, we have some advocates of only the concrete, of what the Germans call the "clothed problem," to the entire exclusion of the abstract drill work of the Pestalozzi school. Such extremists are, however, not numerous, and they have but a small following. Every one who looks into the subject is aware that on the score of interest a child prefers to work with abstract numbers, while as to the final results upon his education we seem to neglect altogether too much the ability to get exact results quickly and with a certainty as to their exactness. This phase of the work will be mentioned in a later chapter, and for the time being we may well consider the applied problem.

Within the last few years the question of the practical uses of arithmetic has been a vital one in educational circles, especially in Austria, Germany, and America, resulting in considerable literature upon the subject. These needs, while generally similar in various countries, differ more or less in details. Thus a country whose business is chiefly farming will need to emphasize agricultural problems; one that derives its wealth from its metals or its coal will emphasize mining; a manufacturing nation will find certain kinds of problems relating to the factory peculiarly suited to its needs, while one that derives its wealth chiefly from shipping will require those relating to foreign commerce. The mathematical foundation is the same in all cases, but the material content of the problem will vary.

In America we are unusually cosmopolitan in our needs in this respect, ranking high in all these particulars save only (at present) in ocean traffic. We are therefore fortunate in having at our disposal unlimited problem material that relates to our wide range of national resources and industries. The advantage of using this material instead of the obsolete inherited problems that came down to us from Italy, through England, ought to be so evident as to require no argument. There will always be some who cry out against what they call encyclopedic information in an arithmetic, but surely if a problem is to contain any facts at all, it is better that these facts be American and of the twentieth century than Italian and of the fifteenth. Can there be any doubt that an American boy or girl will get more breadth of view, more interest, and possibly more directly useful information from a problem about the mixing of plant foods for a farm than from one about the mixing of teas that are never mixed in the way the textbook says? If a pupil is to study about goods being transported, is it not better for him to take a practical case relating to our railroads than the old-time one of peddlers carrying their packs? It is well, however, to avoid the unfortunate tendency manifested by some recent writers to introduce problem material that no child is ready to understand, and with which teachers themselves should not be expected to be familiar. Technical information of trades, scientific nomenclature that belongs to the college or to the later years in the high school, problems of the civil engineer or the food chemist — these have no more place in the arithmetic class than has the apothecary's table or the subject of equation of payments. Common information, the subjects of interest to the general public, and those matters that are topics of conversation in the usual walks of life are the bases upon which we may reasonably build our problems. Undertaken in this spirit, we need not fear if critics accuse us of making our schools encyclopedic. Every usable school arithmetic has always been an encyclopedia; what we have to determine is whether it shall now be an encyclopedia of vital, modern facts, or one of obsolete, dull, useless information. The needs of society demand the former; *vis inertiae* holds to the latter. The earnest teacher, awake to the needs of the business community in which a school is located, can hardly fail to introduce genuine problems with local color to enliven the work in arithmetic. No textbook can fit the needs of every locality, and original problems are easily found by the children themselves if the opportunity is given. The awakening of interest in such work, however, will not come from the style of textbook of a generation ago; it will only come in connection with the study of a book that is itself filled with this spirit. Fortunately most of our modern writers are working earnestly to meet the needs of to-day, and our American arithmetics may well lay claim to being among the most progressive that are appearing in this generation.

But what as to the effect of the study of child psychology? Here, too, there has been made very great progress in recent years. Although a problem may represent all that business needs suggest, it still may not be suited to a particular school year. In other words, we have to consider from grade to grade the interests and powers of the child. We would not think of giving to a child in the first grade the problem, If A has 2 shares of railroad stock and B has 3 shares, how many have they together? For while the child can add 2 and 3, he has no knowledge of stocks nor any interest in them. Change the subject to marbles or apples or tops, and it is suited to his mind, but not otherwise. Thus it has come about that teachers are trying to decide what are the larger interests, actual or potential, of the children in the various school years, to the end that the problems of arithmetic may be the better apperceived. Only a beginning has been made, but the future will see the work extended. We know that pride in our national resources renders interesting a style of problem in the fifth school year that would be of no value in the second, even with smaller numbers. On the other hand, we are equally aware that certain problems involving children's games that are part of the genuine applied mathematics of the third year would have no interest whatever in the eighth. And so in general, teachers are seriously attempting at the present time to coördinate the interests of children, the needs of society, and the mathematical powers in each of the grades from the first year through the elementary school.

Teachers should be warned, however, that there are types of applied problems that are open to considerable objection. It is very easy to say, in the seventh and eighth grades, "We are now ready to correlate our arithmetic with our daily life," and to get any number of examples involving price lists of foods, the advancing cost of living, the rate of wages, and so on. Such examples are valuable as giving useful information and as showing the real applications of numbers. But when they are given in Grade VIII, and involve only the arithmetical operations of Grade III, with no difficulties to overcome in the way of reasoning, they entirely fail to serve their purpose as examples in *arithmetic*. We may call them sociological, or vocational, or home informational, but they are not good arithmetical problems because they are so poorly graded as to difficulty that they seem to the pupil unworthy his serious attention. In the study of interest, problems that show the danger of dealing with the "loan shark" of the poorer quarters of our cities are valuable for a city child; they may easily be graded properly and they have high sociological significance; but in the upper grades, save for review drill work, problems involving nothing more than the adding of grocery lists are poor substitutes for those of mathematical difficulty commensurate with the pupil's powers.

It should also be said that problems on a subject like bookbinding, or cabinet work, or carpet weaving, are quite as unreal as those on apothecary's measures, unless the child is, at the precise time when they are given, engaged in these particular industries. The place for such problems, in any considerable number, is in the classes in bookbinding, woodwork, and textiles, where these exist, but not in the class in arithmetic. Problems from the general fields of industry, informational and well graded, are always to be commended; but those from narrow fields that are known to but few children or parents should have little recognition in the schoolroom.

Another mistake that is often made, in the endeavor to make problems real, is that of being ultratechnical. A business man recently said to the writer that we ought to put into some of our arithmetics a certain feature of foreign exchange that is used by large banking houses in New York. But upon the suggestion that not one teacher in ten thousand would or should understand it, that not one pupil in a million would ever have occasion to use it, and that only bankers and mercantile houses engaged in large transactions in foreign exchange paid any attention to it, the value of the scheme diminished to zero. So it is with many industrial problems; they are as technical as those involving Troy weight, and quite as uninteresting and useless. There is a fairly good industrial field in which the teacher and textbook writer may work, but each must be on his guard lest the problems proposed be barren of good results.

Still another danger deserves mention. The excellent work done in certain industrial schools tempts the visitor to try experiments in the general public schools that are certain to yield results that are unsatisfactory. Take, for example, such a school as that so successfully conducted by one of the great electric manufacturing companies. Here is a group of boys who are interested in a special trade. Everything that relates to that trade appeals to their imagination and interests. The diameters of wires for certain currents, the density of copper, the weight of a casting — these and similar topics have an immediate interest for the entire group. Arithmetic does not need to search for applications under such conditions; they are all about and arise without effort. For the teacher of such a class, himself a mechanic, the teaching of arithmetic is a comparatively simple matter, just as it was in the old apprentice schools of which these schools are a survival. It is feasible and natural to teach the subject under circumstances like these, just as it has always been taught in schools of this character, by means of concrete problems, possibly followed by a little practice with abstract numbers. But to transfer this to the public school, with forty pupils in a class, girls as well as boys, representing widely different surroundings and interests, and taught by one who will remain, on the average, only four or five years in the profession — to make this transfer is by no means simple; indeed, it is entirely impracticable. We have much to learn from the spirit of such schools; they inspire us by their interest, but their problems are as ill adapted to the general public school as are many of those in common use to the special schools mentioned. In so far as they teach the public school to introduce daily at least one local problem of general interest and related to the regular work, and encourage the teacher to provoke a spirit of independence and inquiry in connection with all problems, they are helpful to the cause of education. A writer on the work of these special schools has recently suggested as a good general problem the following: If a 4-pound roast cost 25¢ a pound, and the bone was afterwards found to weigh 12 oz., what did the actual meat cost, and what was the percentage of waste? That is a good problem. A problem of this general type may profitably be introduced in each lesson, made up by the pupils or by the teacher. There is always an interest in getting outside the book and in making an attempt to touch home life.

A few words should also be said with respect to the "narrative problem," in which the cost of the waterworks of a village, for example, is figured out. This is sometimes seen in arithmetics in the form of a series of connected problems, the solution of the second depending upon that of the first, and so on. It is not difficult to adduce a number of arguments for such problems, but examples of this nature have not succeeded for textbook purposes. If a pupil or a teacher makes an error in solving the first problem, the solution of every subsequent problem depending upon it is thereby vitiated, and the result is so discouraging as to arouse an unnecessary antagonism to arithmetic. The narrative problem appears to much better advantage in the lecture notes of a professor of education than it does in the classroom of the practical teacher. It is all right as an application of arithmetic, provided it is carefully presented as an outside, local problem, the solution being carried on under the immediate guidance of the teacher. But as a textbook device it is not to be commended. Say what we will about the "made-up" problem, nothing has yet been invented to take the place of a carefully worded statement of some single question that is typical of the actual conditions that the pupil will meet in life and that is suited to his understanding.

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CHAPTER V

THE TEXTBOOK

The first textbooks in practical arithmetic were intended for apprentices in trade. They contained little or no abstract work, the subject being taught by means of such concrete problems as were found in the ordinary commercial life with which the boy was beginning to be familiar. Girls rarely studied arithmetic and, indeed, most women were illiterate. As in the industrial schools mentioned in Chapter IV, so in these apprentice schools, the problem of teaching the necessary arithmetic was a simple one. Great industries were unknown, world commerce had not yet been born, banks were practically nonexistent save in large cities, agriculture was carried on only in a small way, and machinery was still very primitive. The wide range of interest that the world now has, the diversity of problems that result therefrom, the varied demands of all classes to-day as contrasted with the few demands of the trading class then, and the fact that the teacher has at present to keep up the interest of the pupils in many lines instead of a single one make the problem of teaching very much more complex than it was when the Treviso arithmetic mentioned on page 7 appeared.

Moreover, the world had plenty of time in those days. The nervous energy of this generation and the demand for speed that characterizes America to-day were then unknown. The boy might take his time to add or to divide, and the speed tests of our public schools had never been thought of. Hence there was little need for the drill on abstract operations which the world finds essential in the general type of school to-day, the type that seeks to train the class as a whole so that its members may enter indifferently upon a mercantile career, banking, agriculture, manufacturing, railroading, or the professions. Hence when we hear some one say that a textbook should consist merely of applied problems, let us remember that this idea is a very old one, and that it has some merit in schools of the primitive type. In schools of such a type we might go still further and say that, after the child has passed through the primary grades, we need not classify the problems on a mathematical basis, putting all percentage by itself, all proportion by itself, and so on, but may classify by applications. That is, we might put all problems on pulleys together, and all on agriculture by themselves, and so on; or we might take the particular handwork with which we are concerned for the time being and handle the problems as they naturally arise. This has been done with homogeneous groups of pupils, whose interests are alike or who can themselves bring to the class real problems from the workshop; but for the general public school, in the experience of the world, such a plan has nothing to commend it. World experience has shown the necessity of beginning with the concrete, as is always done in primary classes; but it has also shown the necessity for both abstract computation and concrete applications, and for a range of applications not limited to any single narrow field, but related to all the great fields of human activity.

As already stated, the two most noteworthy changes in arithmetic in recent years have related to the nature of the problems and the arrangement of material. The latter has been the result of a more or less serious study of child psychology, namely, of the powers of the individual in the various school years.

Formerly, say a century or more ago, it was the custom to study arithmetic from a single book, after the boy (for the girl, as already stated, seldom understood mathematics of any kind) could read and write. The child was mature enough to understand the subject after a fashion, and he "went through" the book. But as arithmetic began to work its way down to the earliest grades it was found impracticable to follow this plan, the subject as so taught being too difficult for young minds. The ordinary textbook was therefore preceded by a primer on arithmetic, and thus a two-book series was formed. From this beginning numerous experiments have proceeded, seeking to carry the improvement still farther. We have had three-book series, eight-book series, lesson leaflets, two-book courses arranged by grades, and so on. We have had spirals of various degrees of turning, books arranged to follow certain narrow lines of manual training, and so on — all serious efforts for betterment, but many of them too hastily considered to have any material influence.

In the midst of it all, what have the practical teachers of the country done? In every city, in several states, and by numerous associations, courses of study have been arranged setting forth the material that experience has shown can be used to advantage in the several school years, and selections have been made from the current arithmetics to supply what was needed. In other words, the practical teachers of America arranged their own books to a great extent, basing their selection of material upon an empirical psychology, and giving to the child what his interests and capabilities suggested.

It is out of such a movement, spontaneous but thoroughly sound, that the later American textbooks have arisen, and the care and earnestness given to their preparation by various authors and publishers should have the commendation of all.

These books are and probably will continue to be of two distinct types, each with certain peculiar merits, and each capable of producing good work. First there is the topical arithmetic, that is, a book arranged by topics, a subject like percentage being taken once for all, the pupil studying it until it is thoroughly mastered. Such a book has two great merits: it tends to keep the child upon each subject long enough to give him a feeling of mastery that he would not have if he studied some of the scrappy books constructed on the extreme spiral system; and it allows a teacher who wishes to adopt a moderate spiral to do this in a manner that will meet the local conditions. By this latter we mean that some classes need more work in a subject like simple interest, when it is first taken up, than others do. A teacher may therefore select as much or as little as is necessary when a topic is taken up for the first or the second time, and this is perhaps more easily done from a topical than from another form of book. A teacher or a school may also easily adjust books of this type to meet the demands of new or peculiar courses of study. It is always legitimate to omit topics, or to make such moderate rearrangement as may be demanded by the course of study of any particular locality, and changes of this kind should freely be made. On the other hand, it is unfortunate that so many different courses should be adopted from year to year, only to be discarded as soon as there is a change in administration. While it is only by experiment that we make progress, no one can carefully examine the dozens of new courses that are issued every year without seeing that they often show evidence of undue haste and of the desire for mere change. To meet this restlessness the topical arithmetic is probably the better type, for it is more readily adjustable to all kinds of courses, and it has proved its usability with all sorts of teachers and pupils.

The second general type of textbook is one that attempts to fit the course of study in a large number of schools. In general these books agree in a number of particulars: (1) certain subjects, such as the most commonly used operations, appear several times in the course; (2) others, such as the business application of simple interest, appear perhaps two or three times, with gradually increasing difficulty; (3) still others, like board measure, may be sufficiently treated once for all; (4) the closing year of arithmetic should be devoted to a study of such higher problems of business as can be understood by the children. These are some of the articles of agreement, and they go to show the common-sense principles on which our experimental courses of study and textbooks in arithmetic are based. Courses and books conservatively arranged in this way are well adapted to city schools that are thoroughly supervised, but they have not been so successful where this careful supervision is wanting. The topical plan is always safe; the moderate spiral is equally safe, provided we have good supervision or a body of trained teachers who sympathize with the plan.

As to which of these two types is the better it is impossible to give a general decision. It depends largely upon the school and the teacher. With the one book the teacher arranges the matter to suit the needs of the class; in the other it is already arranged to suit the average pupil. Neither will fit each individual case, and the great thing is not so much the arrangement of the matter as the modern spirit displayed in the omission of the obsolete and the introduction of new, vital, interesting, intelligible problems of to-day. On this point great emphasis should be laid. A teacher should never make the mistake of feeling that everything in a textbook should be taken by the class. The best textbooks must give enough work for all needs, but this means that they must give too much for most needs. The teacher must judge when a class has reached the proper stage for taking up a new topic, and when that stage is reached the old topic should be dropped, whether all the drill work and applications have been taken or not. What is needed in one part of a city may not be necessary in another part, and an amount of drill work that may be commended in one community might well be condemned in another. One of the distinguishing features of the work in the best European schools is the freedom with which the teacher omits matter from the textbook and supplies problems of local significance.

It should also be stated in this connection that a similar movement is taking place with reference to algebra and geometry, and one that is destined to reach still further up the grades and into college mathematics. Algebra is old, but algebra textbooks are relatively modern. They were at first based upon the arithmetics that preceded them, and as far as possible they followed the arrangement of matter there found. The ordinary school algebra is therefore merely an old-style arithmetic with letters used instead of numerals, and with a considerable number of its problems taken from the arithmetical collections of earlier days. The arrangement of matter is confessedly not scientific, and we are even now seeing a serious effort made to improve the nature of the problems. The next move of importance may be the rearrangement of the material and the enriching of the applications of algebra and, probably to a less degree, of geometry.

Bibliography. Consult the latest textbooks, comparing the two types and noting the distinctive advantages of each. Consult also those extreme forms in which the spiral arrangement is carried too far. See also Suzzallo, Teaching of Primary Arithmetic, Boston, 1911; Lehmer, On the Use and Abuse of Textbooks, *School Science and Mathematics*, Vol. VIII, p. 13.

CHAPTER VI

METHOD

Of all the terms used in educational circles "method" is perhaps the most loosely defined. Efforts have been made to limit its meaning, to divide its responsibilities with such terms as "mode" and "manner," but it still stands, and is likely to stand, as a convenient name for all sorts of ideas and theories and devices. Nevertheless, it has been most often used in arithmetic in speaking of the general plan of some individual for introducing the subject, as when we speak of the Pestalozzi or Tillich or Grube method, although it is also applied to such arrangements of material as are indicated by the expressions "topical method" and "spiral method," and to such an emphasis of some particular feature as has given name to the ratio method. It is not the intention to attempt any definition of the term that shall include all of these ramifications, but to take it as it stands, to characterize in simple language some of these "methods," and then to speak briefly of the subject as a whole.

Pestalozzi's method was really a creation of his followers. What this great master did for arithmetic was to introduce it much earlier into the school course, to use objects more systematically to make the number relations clear, to abandon arbitrary rules, to drill incessantly on abstract oral work, and to emphasize the unit by considering a number like 6 as "6 times 1." For the time in which he lived (about 1800) all this was a healthy protest against the stagnant education that he found. To-day it is only an incidental lesson to the teacher, although Pestalozzi's spirit and several of his ideas may well command the admiration and respect of all who study the results of his great work.

The method of Tillich, who followed Pestalozzi by a few years, consisted largely of making a systematic use of sticks cut in various lengths, say, from 1 inch to 10 inches. It is evident that such a collection allowed for emphasizing the notion of tens, for treating fractions as ratios, and for visualizing in a very good way the simpler number relations. On the other hand, it is also evident that the use of only a single kind of material is based upon a much narrower idea than that of Pestalozzi, who purposely made use of as wide a range of material as he could.

The Grube method, which created such a stir in America a generation ago, was not very original with Grube (1842). Essentially it was an adaptation of the concentric-circle plan that had already been used, a kind of spiral arrangement of matter to meet the growing powers of the child. It contained some absurdities, such as the exhausting of the work on one number before proceeding to the next, as of studying 25 in all its relations before learning 26; but on the whole it made somewhat for progress by assisting to develop a sane form of the spiral idea as adapted to the first three grades.

It is not worth while to speak of other individual methods, since they have little of value to the practical teacher, and the student of the history of education can easily have access to them. Enough has been said to show that one of the easiest things in the teaching of arithmetic is the creation of "method" — and one of the most useless. We may start off upon the idea that all number is measure, and hence that arithmetic must consist of measuring everything in sight — and we have a "measuring method." It will be a narrow idea — we shall neglect much that is important; but if we put energy back of it, we shall attract attention and will very likely turn out better computers than a poor teacher who is wise enough to have no method, in this narrow sense of the term. Again, we may say that every number is a fraction, the numerator being an integral multiple of the denominator in the case of whole numbers. From this assumption we may proceed to teach arithmetic only as the science of fractions. It will be hard work, but, given enough energy and patience and skill, the children will survive it and will learn more of arithmetic than may be the case with listless teaching on a better plan. We might also start with the idea that every lesson should be a unit, and that in it should come every process of arithmetic, so far as this is possible, and we could stir up a good deal of interest in our "unit method." Or, again, we could begin with the idea that all action demands reaction, and that every lesson containing addition should also contain subtraction; that $6 + 4 = 10$ should be followed by $10 - 6 = 4$ and $10 - 4 = 6$; and that $2 \times 5 = 10$ should be followed by $10 \div 2 = 5$ and $10 \div 5 = 2$. By sufficient ingenuity a very taking scheme could be evolved, and the "inverse method" would begin to make a brief stir in the world. This in fact has been the genesis, rise, and decline of methods: given a strong but narrow-minded personality, with some little idea such as those above mentioned; this idea is exploited as a panacea; it creates some little stir in circles more or less local; it is tried in a greater or less number of schools; the author and his pupils die, and in due time the method is remembered, if at all, only by some inscription in those pedagogical graveyards known as histories of education.

The object in writing thus is manifest. For the teacher with but little experience there is a valuable lesson, namely, that there is no method that will lead to easy victory in the teaching of arithmetic. There are a few great principles that may well be taken to heart, but any single narrow plan and any single line of material must be looked upon with suspicion. Certain of the general principles of Pestalozzi are eternal, but the reckoning-chest of Tillich is practically forgotten.

And here it is proper to say a word as to what schools of observation and practice should stand for in these matters of method and purpose. It would be a very easy thing to concentrate on some single point, some device of teaching, some particular line of problems, and to carry the work to an extreme that would attract attention and produce results that would be remarked upon. This is the temptation of those who direct such schools. But is it a wise policy? These schools are established to train teachers to well-balanced leadership, not to be extremists to the neglect of essential features of education. The graduates of such schools should know the best that there is in every theory of education, but they should also avoid the worst. The prime desideratum in arithmetic is the ability to work accurately, with reasonable rapidity and with interest, and to know how to apply numbers to the ordinary affairs of life. To secure accuracy alone, to secure speed alone, to have arithmetic mere play without accuracy and speed, or to know how to apply numbers to life in a slovenly way — these are extremes that should be avoided at any cost, including the tempting cost of sensationalism. It is the mission of the training school or college to make the earnest, well-balanced teacher, first of all. With this duty goes the laudable one of reasonable experiment, of trying out suggestions from whatever source; but normal schools and teachers colleges must at all hazards guard against the mistake of having it appear that an experiment is an accomplished result, or of sacrificing our children in unnecessary quasi-clinical work that is doomed to failure. In this connection one of the resolutions adopted by the National Educational Association in 1908 may be read with profit as voicing the sentiment of that saner element in education that is, after all, the strength of our profession.

"We recommend the subordination of highly diversified and overburdened courses of study in the grades to a thorough drill in essential subjects; and the sacrifice of quantity to an improvement in the quality of instruction. *The complaints of business men that pupils from the schools are inaccurate in results and careless of details is a criticism that should be removed.* The principles of sound and accurate training are as fixed as natural laws and should be insistently followed. *Ill-considered experiments and indiscriminate methodizing should be abandoned, and attention devoted to the persevering and continuous drill necessary for accurate and efficient training;* and we hold that no course of study in any public school should be so advanced or so rigid as to prevent instruction to any student, who may need it, in the essential and practical parts of the common English branches."

Such advice will be scoffed at by many reformers of twenty-five or thirty, but these very ones will be its stanch advocates at forty. It is the advice of experience, the protest of those who have seen the futility of spasms in education.

One thing that must be said in favor of the multitude of methods, most of them bad, that are announced from time to time, is that they are confined to the first year or so of the primary grades. The harm is therefore limited in extent. This is the case with the much-exploited Montessori method at present, quite the most skillfully advertised one that has appeared in this generation, but one the permanent effect of which on education seems properly to be doubted by scientific observers.

In general it may be said that there has for a century been a tendency away from what is called the direct method of imparting number facts, and toward the rational method. This means that instead of stating to a class that $4 + 5 = 9$, and drilling upon this and similar relations, the schools have generally tended to have the pupils discover the fact and then memorize it. The experience of a century shows that this tendency is a healthy one. A child likes to be a discoverer, to find out for himself how to add and multiply, always under the skillful guidance of the teacher, and to see how to solve a problem before he knows of any dominating rule. It is only when the teacher decides that the child is never to be told anything, never to be helped over difficulties, and never to be shown a short path before habit has determined upon a long one, that the danger of the rational method appears.

The following extracts are taken from the monograph on "Mathematics in the Elementary Schools of the United States," referred to in the bibliography at the close of this chapter. They were written by Professor W. W. Hart, of the University of Wisconsin, and teachers will find it helpful to read the entire report. Most of the points are referred to elsewhere in this work, but it will assist the reader to have them presented again in the succinct form in which they appear in the report itself.

"Certain influences affecting the work of all the subjects of the elementary school have produced, and are producing, changes in the method of instruction in arithmetic. On the part of the schools themselves there has been a growing regard for the ideals of psychology and pedagogy, and an increasing effort to put these ideals into practice as these have become more clearly defined. The ideal of appealing to the interests of the pupils, the maxim of allowing the pupils to learn through their own activity, the falsity of the doctrine of formal discipline as formerly understood, the clearer understanding of the processes of perception and of apperception with the resulting theory of developmental or inductive teaching — these and other ideals have led to changes in the methods of instruction from within the schools. These changes have only started, however, and the present must be regarded as a transition period. On the other hand, there have been other changes brought . . . under pressure from without the school. There has been a growing independence on the part of the public in the matter of educational opinion; the result has been less acceptance as gospel of the statements of the schoolmen, more questioning of the worth of each and all of the subjects taught, and more insistent demand that the schools endeavor to provide instruction which will harmonize with and prepare for the conditions of modern life. There is on all hands growing sympathy for children, as a result of which their burdens are being made lighter, and, in particular, their period of school life is being looked upon not so much as a period of discipline in preparation for a further life of unpleasant tasks, but rather as a portion of their life which they can and should be helped to enjoy. In response to this influence, which has met with ready reception within the schools, there has come a tendency to simplify the work as much as is possible.

"Instruction in arithmetic in the past has been dominated largely by a logical ideal and a disciplinary aim; these influences can be observed in the subject matter, in the organization of it in the textbooks, and in the means employed in presenting it to the pupils. The criterion observed in the selection of material for the course has been 'to what extent is this topic necessary to round out the development of arithmetic as a mathematical topic'; topics were selected frequently because they had certain mathematical interest rather than because they met some actual need of the pupils in either their present or later life. Long lists of abstract problems of unnecessary difficulty and complexity were introduced, regardless of the fact that they were much more involved than any met in actual practice by most people. The concrete problems usually followed the abstract, were equally difficult, and seldom were designed to appeal to the interest of the pupils; up to within recent years the problems often involved matters which are entirely obsolete. This material was organized topically; in the first chapter was given a treatment of notation and numeration which was designed to meet the complete future needs of the pupils, containing such numbers as billions and beyond. Then followed chapters which contained the treatment of the four fundamental operations for integers, and so on through the course. Logically this arrangement was a natural one, but psychologically it was open to criticism. In the presentation of the material the book was again permeated by the logical ideal. Processes were either given by a rule followed by an illustrative example and exercises for the pupils, or some abstract explanation of the rule was attempted. The former method was the more common. This was a period of direct instruction and of drill on the part of the teacher, with little effort to provide any basis of experience for the pupils upon which they might build their arithmetic.

"These ideals of instruction have by no means entirely passed, but in most of the textbooks in use now there is evidence of response to the influences mentioned in a former paragraph. This is a time of transition from the extreme of these views of the past to the more rational methods of the present. In the meantime the schools have passed through a period in which the quest after better methods has led some to adopt certain well-known one-idea methods. As examples of these may be mentioned the Grube method, with its attempted complete study of one number before passing on to the next; and the ratio method, in which the complete development of arithmetic was made to yield to a special mathematical form.

"Whereas in the past instruction in mathematics in the elementary grades has been dominated by logical ideals and a disciplinary aim, at the present the leading purpose is to make the instruction as useful in content and as pedagogical in form as the conditions of school work permit. In regard to its content, remark has been made in a previous paragraph of the tendency to simplify the course through the omission of unduly complicated problems, and of topics which have mathematical rather than general interest; with these omissions has come a greater insistence upon the essentials of arithmetic and additional problems whose special function it is to give the pupils an insight into certain phases of the life of their own country. The mathematician may fear that this process of elimination will result in a lower standard of mathematics; the fear is groundless, however, for the effort to direct attention to a more complete mastery of the fundamentals cannot fail to raise the standard of mathematical equipment of the pupils. . . .

"The material used in the schoolroom as objective aids is limited and highly artificial, consisting of tiles, pegs, splints, toothpicks, squares of cardboard, etc. This material has the special advantage of being adapted to the conditions of schoolroom work; it is inexpensive, compact, easily handled, and not too attractive. Besides this material which can be handled, there is a growing use of geometrical figures, especially of rectangles; these can be quickly drawn, are easily changed and adapted to the conditions of particular problems, are divisible — in fact, lend themselves readily to the needs of the situation. The use of rectangles in teaching fractions is especially to be mentioned. In the selection of objects two qualities which are particularly desirable, in fact necessary in some cases, are not always realized, and when realized are not always readily obtainable; these are the elements of naturalness and variety. The element of naturalness is probably sufficiently appreciated; in the early grades the pupils themselves, the seats in the room, the material which is distributed in connection with the work of all the classes, the selection and grouping of the pupils for their games — these are some of the more natural sources of material for concrete expression of the number facts studied. Similar material for use in the upper grades is not always so readily obtained; of course, in connection with denominate numbers the actual weights and measures are brought into the schoolrooms in the progressive schools. As far as possible, the problem material in all of the grades is selected from the experience of the pupils, and the actual conditions of some of the topics are reproduced in the schoolroom. For example, in teaching stocks and bonds the class may resolve itself into a broker's office, one member of the class acting as the broker and the rest of the class acting as buyers or sellers of stocks and bonds; again, the class in a lower grade may resolve itself into a store, with similar distribution of responsibilities among the members of the class; when discussing commercial paper and business forms, samples are brought into the classroom so that the pupils may get the feeling that there is a real connection between the work they are doing in the schoolroom and that of adults who are engaged in the work of the world. An effort is made to permit the pupils to handle the material themselves.

"The other quality mentioned as desirable, namely variety, is not appreciated as much as it should be. The advantage in having a variety of objects as a means of arousing interest is usually clear, but the necessity for variety in presenting particular cases of a general idea which is being developed inductively is not clearly understood. For example, in presenting the idea of tens, bundles of splints containing 10 splints are often used; the pupil soon learns to associate the word "ten" with one of these bundles without really understanding that the bundle contains 10 splints. This difficulty is obviated somewhat by using a variety of means for representing 10. Other illustrations might be given to make the point that in objective teaching care must be taken to prevent too immediate association of the idea being taught with any particular manner of representing that idea in a concrete way."

To this may be added a further quotation from the same report but from a different subcommittee. The chairman of this subcommittee was Miss Theda Gildemeister, of the State Normal School at Winona, Minnesota, to whom the preparation of this material is due.

"The present chaotic state of our methods in mathematics seems due to a number of causes, some of which are (1) the various views of what number is; (2) difference of opinion as to what shall be selected from the whole field to be taught in Grades I–VI of the elementary school; (3) the bondage we are in to past ideals; (4) the inertia of the school itself or the slowness with which a great institution like the school changes; (5) recent marked progress in the industrial world, demanding different life preparation of elementary-school graduates; (6) social progress; and (7), though by no means least, the great demand for teachers — a demand so urgent that we press into service vast numbers of immature girls who, though earnest and zealous, yet lack that higher and broader professional training spoken of by Münsterberg, which makes a teacher see the aims of education and know well the means available for meeting these aims.

"The many methods which these causes produce may, after all, be classed into two main divisions, — mechanical and thought, — though it is true that most methods are a decided mixture of the two.

"By mechanical methods we mean such as emphasize the symbol, the form, the expression, to the subordination or even exclusion of the thought, the content, the meaning. In direct opposition to this, thought methods, while not ignoring expression, lay greatest stress upon meaning.

"Though the various views of what number is, give a very different content to the number symbols in common use, — that is, 2 may mean how many, how much, location in a series, the ratio of one magnitude to another, etc., according to the individual's view of the essentials of the number idea, — yet thought methods ever give this content first place, whereas mechanical methods center about the symbol. It is to be noted, however, that the human mind tends to make automatic, mechanical, or formal whatever it can. That symbols, then, incline to supersede meaning is due to very natural causes: (1) all must admit (*a*) that conventionalized symbols are needed to insure race progress, (*b*) that the higher the degree of conventionality the further is the symbol removed from its meaning, and, consequently, (*c*) that with this greater conventionality there is increased danger that the child may never get the meaning, that the symbols will grow to have undue importance to him so that he may, indeed, become a mere juggler with figures; and (2) if it is admitted that even any part of the science of number deals not with things but with relations of things, it must readily be seen what difficulty children have in giving symbols any real content. Therefore has the teaching of the past, in a more or less conscious effort to meet children's needs, vacillated from too abstract teaching to teaching so concrete as to hide relations; or, in other words, from no attempt to make children understand,—from avoiding content and giving rules and form to be mechanically learned, — to such profuse and such long-continued use of objects in trying to give a content to number that thinking has been retarded.

"But that, in spite of our keeping away from either extreme, there is still an inevitable tendency among pupils to become formal even in what seems real thought work, was proved satisfactorily to a group of observers who gave Dr. C. W. Stone's tests to one class of pupils, and a month later gave the same children the thought-test problems which they had not done, at the blackboard where work could be watched and where there was a chance to question the pupils. We were convinced that many of what seemed like thought processes really went off mechanically when the cue was at hand. Additional problems were given to the children, among them, this: A man hitches up a horse to a buggy and drives three hours at the rate of six miles an hour. How much farther would he have gone if he had hitched up two horses? All but one child doubled the distance. This child thought 'no farther.' Questioning this one child brought out from the others that the two horses would help one another so that some additional distance would be covered, but certainly not double the distance. Another problem tried with many more individuals was the following (changed, in different cases, to sheep or dog on two or on four legs): If a duck weighs three pounds when it is standing on one leg, what does it weigh when standing on two? It is the exception to receive any other answer than twice the original number of pounds.

"Though this remarked tendency to cast even the thinking into habitual forms must be watched throughout the early years of school life, it yet must be agreed that life demands that we learn many processes automatically. The error of the past was that these processes were the beginning and the end of the teaching. A more thorough study of method in all fields shows us that men have ever passed through three stages in the development of any field: (1) simple or natural art, or mere manner of doing the deed; (2) conscious attention to the method, running into science; and (3) art again, finished art this time.

"Methods which are predominantly thought methods plan (1) that pupils begin with content (having first felt some sensible reason for approaching the subject); (2) that they then pass to a use of symbols, to be handled automatically when expediency demands it, employing a particular form of expression only because that form best expresses the thought held; (3) that they be encouraged in flexibility of expression as well as of thinking, the former, however, always being controlled by the latter; and (4) that they be given many opportunities to exercise choice and judgment in applying the knowledge gained to life situations.

"Games, plays, and construction work are some of the means employed by many grade teachers to give content to, or else a rational motive for, many phases of number work, for example, number combinations, United States money, and fraction study.

"During the current year the writer has witnessed the teaching of some arithmetic lessons dealing with (1) finding areas, perimeters, etc.; (2) subtraction; (3) areas of triangles; (4) fractions; (5) commercial discount; (6) interest; and (7) insurance. The methods employed were mainly thought methods, for each time such an appeal was made to pupils' acquaintance with life situations as to make the plan in common use a most sensible one to the child, a plan he might even think out by himself, and never have given to him as a rule to be learned. The symbols or forms of expression grew from the children's hunting the best method of expressing their thoughts, and though they were plainly guided into adopting conventional modes of expression, the pressure was ever from the thought side.

"Modern psychology, which emphasizes the formative as well as the revealing function of expression, enables us to select our methods of procedure as well as our mathematical curriculum with much more reason than did our predecessors in education. Though, in the main, content should precede form, there will ever be found situations which demand form before content, or if not before content, at least without content at the time of use. When thought methods are in the main employed in the presentation of such work, the teacher is first sure that she has the content herself. She then presents the process in a way to make pupils recognize its rationality, even though they cannot and perhaps should not be asked to explain; drill follows, and yet demands on thinking are made throughout; applications continue the thought work, but the wise teacher patiently waits for maturity to bring full interpretation. (Long multiplication might be cited as an illustration of such work.)

"Finally, methods of teaching arithmetic, besides depending upon the subject matter chosen by makers of the curriculum, vary according to the ideals of the teacher, the school, or the community — whether the end of it all is technical skill for utility or economic purposes, whether it is formal discipline or culture, whether it is to interpret the quantitative side of life experiences, or whether it is a union of all these aims. Furthermore, the still more specific method selected for each recitation in mathematics is determined by the narrow result desired or the means at hand for realizing this end, and hence often is or may be (1) objective, concrete, and rationally motivated, if a new topic is to be developed; (2) abstract and mechanical, if drill or mere repetition is the object; and (3) objective again, and full of original thought work, when application is the motive.

"In conclusion, then, it appears that (1) the aims of teaching mathematics, the selection of subject matter, and the methods employed are constantly acting and reacting upon one another, so that a study of all is necessary to an understanding of one; (2) that topics in arithmetic for Grades I–VI are being rationally motivated, and processes are being steadily rationalized to the child; and (3) that thought methods are gaining ground."

Bibliography. Smith, The Teaching of Elementary Mathematics, pp. 71–97, a rather extensive discussion; Seeley, Grube's Method, New York, 1888; Soldan, Grube's Method, Chicago, 1878; C. A. McMurry, Special Method in Arithmetic, New York, 1905; McLellan and Dewey, The Psychology of Number, New York, 1895; Young, Teaching of Mathematics, pp. 53–150; Hornbrook, Laboratory Method of Teaching Mathematics, New York, 1895; Perry, The Teaching of Mathematics, London, 1902; Suzzallo, Teaching of Primary Arithmetic; Ballard, The Teaching of Mathematics in London Public Elementary Schools, London, 1911; Stephens, The Teaching of Mathematics to Young Children, London, 1911; Brown, An Investigation of the Value of Drill Work, *Journal of Educational Psychology*, Vol. II; Browne, Psychology of the Elementary Arithmetical Processes, *American Journal of Psychology*, Vol. XVII, p. 1; Hedgepeth, Spelling and Arithmetic in 1846 and To-day, *School Review*, Vol. XIV, p. 352; Mathematics in the Elementary Schools of the United States, *Bulletin No. 13* (1911), of the United States Bureau of Education, p. 36, to be obtained gratis by applying to the Bureau.

CHAPTER VII

MENTAL OR ORAL ARITHMETIC

The objection to the expression "mental arithmetic" is fully a generation old. It is argued that written arithmetic is quite as mental as any other kind, and that the opposite to written is oral. As to this there can be no argument, but the word "mental" has so long been used to apply to that phase of arithmetic that is not dependent upon written help that, like a person's proper name, it need not be held strictly to account for what it literally signifies. The expression "mental arithmetic" is therefore employed as well as "oral arithmetic" in this article simply because it is historical and of well-understood significance. Of course it is possible to differentiate still further, and to say that "mental arithmetic" is the silent method, while "oral arithmetic" is that in which the child talks his thoughts, explaining aloud as he proceeds with his work. This differentiation is unnecessary, however, for our purposes, although it is quite logical and leads to some interesting discussion.

What, now, are the relative claims of written and mental arithmetic? Historically, the mental long preceded the written, but only in very simple problems, chiefly involving counting and easy addition. As soon as the writing of numbers was introduced, written arithmetic or else the arithmetic of some form of the abacus became practically universal. In Japan to-day a native shopkeeper will multiply 2 by 6 upon the soroban (abacus), and not only were such mechanical aids retained in western Europe until the sixteenth century, but they are still extensively used in Russia. About the beginning of the last century, however, mental arithmetic underwent a great revival, largely through the influence of Pestalozzi in Europe and Warren Colburn in this country, in each case as a protest against the intellectual sluggishness, lack of reasoning, and slowness of operation of the old written arithmetic. For a long time the mental form was emphasized, in America doubtless unduly so, and was naturally followed by such a reaction that it lost practically all of its standing. The question for teachers to-day is this, What are the fair claims of these two phases of the subject upon the time and energy of pupil and teacher?

There are two points of view in the matter, the practical and the educational or psychological, and fortunately they seem to lead to the same conclusion. Practically, a person of fair intelligence should not need a pencil and paper to find the cost of 6 articles at 2 cents each, or of $5\frac{3}{4}$ yards at 16 cents a yard. The ordinary purchase of household supplies requires a practical ability in the mental arithmetic of daily life, and this ability comes to the mind only through repeated exercise. As will be seen later, it is a fair inference from statistical investigations that a person may be rapid and accurate in written work but slow and uncertain in oral solutions. Therefore it will not do, from the practical standpoint, to drill children only in written arithmetic if we expect them to be reasonably ready in purely mental work On psychological grounds, too, the neglect of mental arithmetic is unwise. It is a familiar law that the memory more firmly retains a fact that is known in several ways (a convenient phrase, if not scientific) than a fact that is known in only one way. A man who knows a foreign word only through the eye may forget it rather easily, but if his tongue has been taught to pronounce it, even though he be deaf, he can the more readily recall it.

If in addition to this his ear has often heard it, he is even more strongly fortified; and if he has also often written it, by pen or by typewriter, there is this further chain that holds it to the memory. In other words, the greater number of stimuli that we can bring to bear, the more certain the reaction. Now arithmetic furnishes merely a special case of this general law. If a child could simply see $9 \times 8 = 72$ often enough, he would come to be able to write it in due time, even if he did not know the meaning. If in addition to this he knows the meaning of these symbols and recalls having taken 9 bundles of 8 sticks each and finding that he had 72 sticks, the impression on the brain is still more lasting. If, furthermore, he has been trained to say "nine times eight are seventy-two" repeatedly, the impression is still stronger, and if he has repeatedly heard this expression (and here is one of the advantages of class recitation), he has then a still further mental grip upon the fact. In other words, mental arithmetic in the form of rapid oral work, with both individual and class recitation, is a valuable aid, psychologically, to the retention of number facts.

There is, however, a danger to be recognized. It is asserted that a child tires more quickly of abstract work than of genuine concrete problems — problems, that is, that are not too manifestly "made up," but that represent some of his actual quantitative experiences. Whether he really tires more quickly of the abstract than of the concrete is by no means certain; for he seems to have more interest in the former than in the latter, probably from the added difficulty that the concrete problem presents in requiring him to know what operations he must perform. At any rate he tires of both, as he does of any other intellectual exercise. It therefore follows that if five minutes of mental work produce a certain efficiency, thirty minutes will by no means produce six times that efficiency.

If, now, this mental work is valuable, how much time and energy should be allotted to it? Possibly we shall have a statistical reply to this question sometime, although it will be a sorry day for good teaching if we should ever accept such a reply as final, any more than we should accept the crude statistics of the health department as determining the prescription our physician gives us for indigestion. The statistics may help us, but they can never control us. However, in absence of even their assistance, what shall we give as an empirical answer? It seems to be the experience of teachers generally that a little mental work, rapid, spirited, perhaps with some healthy, generous rivalry to add spice to the exercise, should form part of every recitation throughout the course in arithmetic. There will often be exceptions, but in general it is a pretty good rule to devote from three to five minutes daily, and sometimes much more time, to this kind of work. In this way a child never gets out of practice, save during the summer holidays, and the practical and psychological benefits can hardly be overestimated.

What should be the nature of this mental work? On the applied side there is no better test for the teacher's ability to adapt herself to her environment, educationally, than this, for the answer varies with the school year, the locality, the related subjects in the course, and with many other factors. In general, however, it may be said that mental arithmetic offers the best means for correlating the subject with the pupil's other work, both within and without the school. To limit it to this field, however, would be an evident mistake, the work with abstract number demanding the major part of the time assigned to this feature. To acquire perfect mechanical reaction to a given stimulus much exercise is required, and for a child to think 72 when stimulated by the ideas 9×8 and 8×9 demands repeated practice, not merely in relatively few applications, but in a multitude of questions involving abstract numbers. Nor is this practice any more irksome than is the solution of applied problems, as any teacher knows. It was almost exclusively by this abstract work that Pestalozzi developed calculators of such ability with concrete problems as astonished those who visited his school, although, if we may place confidence in the results of Dr. Stone's recent investigations, ability in either of these lines does not necessarily imply ability in the other. There are those who feel that abstract drill work (to use the not entirely happy phrase of the schools) has not produced satisfactory results, and should therefore be abandoned in favor of the concrete problems alone. This is somewhat like saying that a man gets indigestion when he eats meat (or vegetables) and that therefore he should eat only vegetables (or meat). But to consider the suggestion seriously, no scientific tests that seem fairly made give the slightest warrant for any such view, while numerous investigations, carefully conducted, show exactly the reverse.

In conclusion, we have two lines of work in mental arithmetic: (1) the concrete, in which the teacher has an excellent opportunity for correlation, for local color, and for stimulating the interest in the uses of arithmetic; (2) the abstract, in which the textbook may be trusted to furnish a considerable part of the material. Each must be cultivated, and ability in one does not necessarily mean a corresponding standard of ability in the other, although a failure in the abstract line must lead to a failure in the concrete. One leads to the acquisition of number facts, the other to the ability to make rational use of these facts in applied problems.

As a practical question for the teacher, how is the material for this oral work to be found? The answer is evident; it must be found exactly as we find material in geography, in history, and in written arithmetic — from a textbook. No teacher can make up on the spur of the moment all of the oral examples necessary, and arrange them properly, and cover all of the important phases of drill work. Either, then, a book must be used that supplies both the oral and the written work, or else two books must be used, one for the oral and one for the written. In either case there should be a good supply of oral problems, to be supplemented by the teacher with such local problems and such correlation with other work as may be advisable.

Teachers sometimes feel that the oral problem should be merely a dictated one; that is, that the pupil should not see the figures on a page before him. This is a serious error. In primary classes, where the numbers are small, the child may be expected to give the result when he hears the words "six times seven." But when the numbers become larger it is reasonable that he should have the assistance of the eye, as he will have in half the cases in practical life. In business computations he will frequently see a price mark and will need mentally to multiply the number it represents by some other number; he will frequently write down his figures and will need quickly to state the result; and, in general, after he leaves the rudiments of the work it is better to place a book in his hands rather than ask him to depend for the problem upon his hearing alone. Otherwise oral arithmetic can hardly be carried on systematically and successfully in the upper grades of the elementary school.

Bibliography. Smith, The Teaching of Elementary Mathematics, p. 117; Smith, Handbook to Arithmetics, Boston, 1904, p. 6; Young, Teaching of Mathematics, p. 230; C. W. Stone, Arithmetical Abilities; Wentworth-Smith, Oral Arithmetic, Boston, 1909; Brown, An Investigation of the Value of Drill Work, *Journal of Educational Psychology*, Vol. II.

CHAPTER VIII

WRITTEN ARITHMETIC

What has been said of mental arithmetic naturally leads us to question the nature of the written work. What shall this be? If the difference in longitude between two ships is $33° 45'$, how shall a pupil find the difference in time? Putting aside the question of the value of such a problem, here are a few possibilities:

(1)

$$15 \overline{) 33° 45'} \quad 2 \text{ hr. } 15 \text{ min.}$$

30

$3° 45' = 225'$

15

75

75

(2)

$33° 45'$

60

1980

45

$15) 2025 (135$ min. $=$ 2 hr. 15 min.

15

52

45

75

75

(3)

$$15° \overline{) 33° 45'} \quad 2 \text{ hr. } 15 \text{ min.}$$

30

$3° 45' = 225'$

15

75

75

(4)

$33° 45'$

$15) 33\frac{3}{4}° \; (2\frac{1}{4}$ hr.

30

$3\frac{3}{4} = \frac{15}{4}$

$\frac{15}{4}$

0

(5)

$33° 45' = 33\frac{3}{4}°$.

$33\frac{3}{4} \times \frac{1}{15}$ hr. $= 2\frac{1}{4}$ hr.

Numerous other forms could be suggested, but these will suffice for our purposes. Which of these should be preferred? In general, should we recommend the form that gives us the result most quickly, or some other that may show clearer reasoning? In Nos. 1 and 4 the forms indicate that we divide degrees by an abstract number and get hours instead of degrees; in No. 3 we seem to divide degrees by degrees and get hours instead of an abstract number; in No. 2 we seem to multiply degrees and get an abstract number, and to divide one abstract number by another and get concrete time in the quotient; in No. 5 we omit part of the work of reduction, but otherwise the solution is a truthful one, with none of the errors of reasoning of the rest. This problem has been selected as the first for consideration because it opens at once such a wide range of possibilities of form, but essentially the same question repeatedly occurs from the very first grade through the pupil's school life. Let us consider a simpler question: If 1 yd. of cloth costs 15¢, what will 6 yd. cost? Here we have these possibilities, among others:

(1)	(2)	(3)
15	15	15¢
6	6	6
90	90¢	90¢

(4) $6 \times 15 = 90$.

(5) $6 \times 15 = 90¢$.

(6) $6 \times 15¢ = 90¢$.

(7) $15 \times 6 = 90¢$.

Out of all these, which shall a child use in writing a solution?

In each of these problems the fundamental question is the same: Shall written work be considered from the standpoint of the answer only, as a business man would be inclined to do, or from the standpoint of the logic of the school, the often nonpractical school?

The answer to such a question ought not to be dogmatic to the extent of saying that any one form is always the best, although it may say that those forms that are untrue in statement are always bad. That is to say, sometimes it is better to write the following:

```
15)33      45            15
   2 15)225               6
          15             --
2 hr. 15 min.            90
```

At other times the step form, with the denomination accurately set forth, is better.

We need to distinguish between two lines of work, equally important; the one relates to accuracy and speed in operation, the getting of an answer as a business man would, with no circumlocution and no superfluous symbols or operations; this is the mechanical part of the problem and there must be abundant exercise on this side. Then there is the equally important side of the reasoning, explaining why the mechanical work is performed as it is, why we multiply instead of divide, and how we know that the result is hours instead of degrees, or cents instead of yards of cloth. Here the step form of analysis may be depended upon to show the pupil's line of reasoning. These two lines of written work are therefore legitimate.

What, then, is illegitimate in written work, and what are the dangers to be guarded against in the form we finally adopt? As to the first, it may be laid down as axiomatic that a form that states or seems to state a falsehood is illegitimate. That is, $30° \div 15 = 2$ hr. is a false statement; it is not even excusable on the score of brevity, since $30 \times \frac{1}{15}$ hr. $= 2$ hr. is as brief, is true, and is as easily explained as any form. So $6 \times 15 = 90¢$ is a false statement and should not be tolerated, although $6 \times 15 = 90$ and $6 \times 15¢ = 90¢$ are both legitimate.

As to the dangers against which to guard, the following advice may be given: (1) To require that every applied problem should be solved in steps is to encourage arithmetical dawdling; the pupils should continually be exercised in rapid solution, the aim being to obtain the correct answer speedily, as a business man would get it. A pupil who lets his mind continually dwell upon dollar signs and well-written steps cannot help dropping away from strict attention to rapidity and accuracy of calculation. The labeling of every number in a solution is unbusinesslike and is a notion of the school that is properly being abandoned. (2) To split hairs on questions of such forms as $9 \times 15¢$ or $15¢ \times 9$ is to get away from the essential point; we must recognize the fact that there is good authority for each, although the former, writing the symbols in the order they are read, is coming into general use in America. The great question is to see, in these analyses, that the thought is clear and that a pupil is not thinking in a hazy way of "15 cents times 9." (3) To require no analyses of the applied problems is an extreme that is about as bad as to require them for all, and perhaps worse. It is quite sure to result in looseness of reasoning that makes correct results a mere matter of luck. (4) To require some particular form of analysis, only to meet the idiosyncrasy of the teacher, is also a danger against which we need to be on our guard. For example, always to require a solution stated in one step, if possible, is a hobby that some teachers like to ride, because it seems to demand continued thought, although it is entirely foreign to the plan that a common-sense business man would adopt, and is not the form of reasoning that we commonly use in mathematics. So to require that a child should always take some unitary form of analysis, finding in every case what one thing costs, may be the means of checking the originality and dampening the ardor of some very promising pupil.

In general, therefore, the teacher should see to it that there is a reasonable amount of rapid, accurate solution, the answer being the paramount object and with no labeling of the numbers that occur in the work. He should also see that there is a reasonable amount of written analysis, accurately stated, preferably in the convenient and terse form of steps, but not limited in any narrow way that would destroy originality or make a solution unnecessarily long.

A word should be said as to another matter in written arithmetic. There are those who advocate forms like 2 ft. × 3 ft. = 6 sq. ft. instead of 2 × 3 sq. ft. = 6 sq. ft. What shall we say of this? It will not do merely to reply that the first is wrong and the second right, for we shall be asked why the first is any more wrong than the second. We may say that the second is justified by the definition of multiplication, while the first is not, since the multiplier is concrete. But then we shall be met with the statement, Then let us change the definition. It is a fact that we have changed the idea of multiplication from time to time, and that it is in the power of the world to change it again if it cares to do so. If we say that multiplication is the process of taking a number as many *times* as there are units in another (a hopeless array of words for a child), we cannot multiply a number by $2\frac{3}{4}$ because "$2\frac{3}{4}$ times" means nothing. Try to pick up a pencil $2\frac{3}{4}$ times. We have simply extended either our understanding of "multiplication" or our understanding of "times" so as to allow for this necessary case. And if we have allowed $2\frac{3}{4} \times \$8$ to enter, why not extend the definition to allow 2 ft. × 3 ft. to be included? The physicist wishes it. He says 2 ft. × 7 lb. = 14 foot pounds, and sometimes 2 hr. × 35 mi. = 70 mi. per hour.

The only reasonable answer is this: It would be entirely possible to extend the definition and idea of multiplication so as to make 2 ft. × 3 ft. = 6 sq. ft. allowable, but in the elementary school we do not need to do so. It is just as easy to write 2 × 3 sq. ft. = 6 sq. ft. and it bothers the child less to do so. So we might extend the idea to include the physicist's 2 ft. × 7 lb. = 14 foot pounds, but the child in the grades does not need this and would not comprehend it. He needs it in physics, and when he arrives there the definition and idea will be extended to meet this need. Hence, in general, it may be said that forms like those suggested are not needed in the elementary school, and that they would only add to the pupil's difficulties. When he needs them, in business or in physics, he will have no difficulty in comprehending their significance. There are hundreds of technicalities of business mathematics that are never touched upon, and never should be touched upon, in school, and we are not justified in adding to the pupil's burden unless it is necessary to do so.

In the marking of papers it should be borne in mind that there is only one test for a question involving a single operation. Either the answer is right or it is wrong. If the problems require some interpretation, a teacher may properly mark both for operations and for method; that is, a pupil may perform his operations correctly, but may have misinterpreted the meaning of the problem. In that case some credit may properly be given for the correct operation. In general, however, papers in arithmetic should be marked, as they are in business, largely by the accuracy of the result. In any single operation the work is *right* or it is *wrong*. A business man will not excuse a bookkeeper who writes $9250.75 instead of $90,250.75. Only a zero is missing, but it means a difference of over $80,000. If the result is wrong, the paper is wrong. The converse of this statement is not true, for the result may be right and yet the paper may justly be criticized for its slovenly appearance and the inaccuracy of the forms used. Where a time limit has been set, and a class has been given twenty minutes to solve as many problems as possible, teachers must use their judgment as to marking pupils who are naturally slow. If their work is accurate, and they have done a reasonable number of examples, they are entitled to credit and should receive commendation.

In connection with the question of written work it should also be mentioned that accuracy does not ordinarily consist in carrying the results to as many decimal places as possible. If a problem requires an answer in dollars and cents, it is wrong to carry the result to tenths of a mill. In other words, common sense should be shown in the use of decimal or common fractions when they appear in a result.

Bibliography. Smith, The Teaching of Elementary Mathematics, pp. 121–129; Handbook to Arithmetics, p. 8; Practical Arithmetic, Boston, 1906, pp. 115, 159; Topical Practical Arithmetic, 1912; Wentworth-Smith, Complete Arithmetic, and Arithmetic, Book II, Boston, 1909 and 1911, p. 191, and other standard textbooks.

CHAPTER IX

CHILDREN'S ANALYSES

The question of mental and written arithmetic leads naturally to that of the analyses to be expected on the part of children. What is their object? What should be their nature? How extensively should they be required?

As to the first, the only defensible object would seem to be that through these analyses a child makes it clear that he understands a particular problem or operation. That he acquires a habit of formal statement that is helpful in other lines of work, or that his memory is strengthened by learning set forms of analysis, has been too often questioned to require consideration. To the extent that this analysis is really an explanation of his process there is an unquestionable advantage, since it enables a teacher to commend or improve the pupil's work. But how often is this the case? Indeed, how often should it be expected to be the case? Is it not the general experience that pupils too often memorize their analyses, and that teachers commend glib repetitions of their own words or those of the textbook, the matter being so imperfectly comprehended by the child that he is able to bear no questioning?

To take a concrete case, we occasionally hear some teacher say that not a child in the class can explain why, in dividing by a fraction, he inverts the divisor and multiplies. But why should he explain it? And if he does, will he do any more than repeat in a perfunctory way the analysis he learned from the book or the teacher? It took the world thousands and thousands of years to learn this process. Not until a thousand years after Euclid made his great geometry was this method of division used, and nearly another thousand years elapsed before it appeared in a printed book. This means that maturity of mind was required to develop such a process, and still greater maturity was needed to embody it in a textbook.

But does this mean that no explanations are to be given or required? By no means. A child should know this process of dividing, and he should learn it by a teacher's questioning; he should thereby know that it is reasonable, and he should feel that for the time he understands why he proceeds in this manner. For that occasion he may be questioned as to all this, but that he should long remember the "why" of it all, or that he should be able, at any time that some teacher or supervisor thinks fit, to give a lucid explanation of such a mature process is as unnatural as it is unscientific. How many readers of this page can tell offhand why it is that a number is always divisible by 3 if the sum of its digits is divisible by 3? And why should they tell it? They may once have known it, but to keep the reasoning in mind for ready repetition would be nonsensical.

So it is with the fundamental operations in general. There is no good reason why a child should remember for any considerable time an explanation for multiplying one integer by another; it is sufficient that he learned the operation as a rational one, and that he can perform it quickly and accurately as we can or as any business man does. If he does give an explanation, it will usually be found to be merely a parrotlike repetition of the teacher's or the textbook's words, without any apparent mental content.

In the matter of the applied problems the case is different. So long as a pupil does not blindly recite formal analyses, there may be a good deal of value in his explanations. If allowed to state his reasons in his own language, with limitations as to tolerable English, he may acquire a habit of succinct and logical statement that will help him in many other lines of expression. This affords, moreover, a very good opportunity for the teacher's commendation and advice — criticism in the best sense of the term, the word too often being employed to signify mere faultfinding.

It has well been said that the problem of teaching children to reason in arithmetic is twofold: (1) it is a matter of the ability to use language; (2) it is a matter of good thinking. The former has been confused with the latter by most teachers, it being felt that if the child repeated the book language of reasoning, he was satisfying the demand for honest thinking. Genuine training in reasoning is not this, however; it is a carefully thought-out process, beginning with problems involving only a single step, and leading gradually to those involving two or more steps.

In all this work it should be borne in mind that there are three things that are properly demanded at one time or another, but not necessarily for each problem that is solved. These three are (1) to work rapidly and accurately, that is, to take the shortest road to the answer, and to be certain that the answer is correct; (2) to put neatly on paper not merely the operation but a brief explanation; (3) to give a brief analysis or oral explanation.

For example, if 5 yd. of cloth cost \$2.10, how much will 12 yd. cost?

(1) *The number work:*

$$\frac{\overset{\$0.42}{12 \times \$2.10}}{5}$$

$$\begin{array}{r} \$0.42 \\ 12 \\ \hline 84 \\ 42 \\ \hline \$5.04 \end{array}$$

(2) *The written explanation:*

5 yd. cost \$2.10. 1 yd. costs $\frac{1}{5}$ of \$2.10.
12 yd. cost $12 \times \frac{1}{5}$ of \$2.10, or \$5.04.

(3) *The oral analysis:*

Since 5 yd. cost \$2.10, 1 yd. will cost $\frac{1}{5}$ of \$2.10, and 12 yd. will cost $12 \times \frac{1}{5}$ of \$2.10, or \$5.04.

Teachers will not need to call for all this work with every example. Sometimes it will be necessary to emphasize (2) and sometimes (3), but the important thing is that (1) should be quickly and neatly and, above all, accurately done. One of the best ways to secure this accuracy and to avoid absurd answers is to estimate the result in advance. The pupil should write down this estimate and compare it with the answer, and if there is a great difference, look over his work again.

For example, if 5 yd. cost \$2.10, we know 12 yd. will cost nearly $2\frac{1}{2}$ times as much, or somewhere near \$5. When we solve, if we find such a result as \$50.40, we see at once that there is a mistake, probably in the position of the decimal point. The correct result is \$5.04.

As an example of written work, involving both the computation and the analysis, the following may be considered:

A merchant bought 800 yd. of linen lawn at $67\frac{1}{2}$¢ a yard, and sold 725 yd. at 80¢ a yard, and the rest at a bargain sale at 65¢ a yard. Find his profit.

0.67\frac{1}{2}$	\$0.65	\$725
800	75	0.80
400	325	\$580.00
536	455	48.75
\$540	\$48.75	\$628.75
		540.
		\$88.75

The written analysis is as follows:

1. $800 \times \$0.67\frac{1}{2} = \$540$, the cost.
2. 800 yd. − 725 yd. = 75 yd., sold at bargain sale.
3. $75 \times \$0.65 = \$48.75$, received at bargain sale.
4. $725 \times \$0.80 = \$580$, received at regular sale.
5. $\$580 + \$48.75 = \$628.75$, total receipts.
6. $\$628.75 - \$540 = \$88.75$, profit.

Of course this solution could easily be shortened, but for beginners it is as well not to attempt too much brevity. The multiplication of \$725 by 0.80, instead of \$0.80 by 725, depends upon the principle that $2 \times \$3 = 3 \times \$2$, and should always be encouraged in similar cases in the actual work of a problem, although in the written analysis (see step 4 above) the form demanded by the reasoning should be given.

In conclusion it may be said that set forms of analysis are more harmful than helpful to children, but that explanations in their own language may be the means of acquiring valuable habits and of offering to the teacher the opportunity for helpful suggestions. To acquire this power the child must be initiated gradually, first in simple examples in one-step reasoning, and second in two-step reasoning. To demand a written analysis for every example would be absurd; to require it for even half of them would probably be a mistake; but to neglect it entirely would be quite as serious an error as to go to the other extreme.

Bibliography. Young, The Teaching of Mathematics, p. 205; Smith, Handbook to Arithmetics, p. 9; Suzzallo, Teaching of Primary Arithmetic.

CHAPTER X

IMPROVEMENTS IN THE TECHNIQUE OF ARITHMETIC

Nothing new goes into arithmetic without a protest, and an equally strong protest is made whenever anything goes out. Nevertheless, there has been an evolution here as everywhere else, and this evolution has made for the betterment of the subject. To take a concrete illustration, the first printed arithmetics had no symbols of operation. What we would write as "4 × 5" was then written "4 times 5," with the natural variation of the word "times" according to the language employed. It was half a century later, and after the symbols + and – were invented, that = was suggested, and some eighty years later still that × was used, and a long time after that before ÷ appeared for division. It was several generations after these were first used that they came into our school arithmetics for the purposes that we use them to-day, and then it was always under strong protest from those who wish to "let well enough alone." It was argued that "4 × 5" was more abstract than "4 times 5," that it was hard because of the symbolism, and that it took arithmetic from the written language and the customs of the common people for whom it was of greatest use. Invented for algebra, the conservatives said that all the symbols ought to remain there and not seek to enter the field of arithmetic. This struggle of symbolism seems strange to us to-day, when a child in the first grade learns at least half a dozen signs of operation and relation, and few would be found to advocate going back to the old custom. We are, however, face to face with similar questions, and many of the very people who would argue to keep $\div$ (a symbol practically unknown save in greater England and in America, strange as this may seem) are the ones who protest most vehemently against letting x stand for a phrase that is too long to write conveniently. But the question is the same. If we use $\div$ as a shorthand way of writing "divided by," why should we not use x as a shorthand way of writing "the number of dollars that the horse cost," or "the amount due the first man," or any other phrase representing a quantity to be sought, an "unknown quantity"? Here, then, is one of the improvements suggested by algebra to assist us in reasoning out the solution of an arithmetical problem. The idea that "$1.10x = \$3300$, therefore $x = \$3000$," is algebra instead of arithmetic is no more true than that "4×5" is algebra, while "4 times 5" is arithmetic. The symbol $\times$ was confined to algebra and excluded from arithmetic for a century or so, and x was so used for over two centuries, but the employment of each to assist in arithmetic does not make "a solution by algebra." This illustration is brought forward as one of the most prominent at the present time. It is impressed upon the writer by numerous letters asking for "a solution by arithmetic instead of algebra" for some little problem that is made clearer by the use of a single symbol in place of a phrase like "the number of bushels," or "the cost of the farm." Teachers should realize that they hereby show an ignorance that is hardly pardonable at the present time, and that such improvements in symbolism are a part of the natural development of the subject.

$$\begin{array}{r} 4 \\ \underline{5} \\ 9 \end{array}$$

Of course there is the danger of overdoing all this. This has often been seen, and is apparent to-day. For example, it is better to train a child's eye to see that 4 and 5 are 9, by putting the symbols in the form here given, the form in which he will usually meet them in computation, than to train it with the symbolism $4 + 5 = 9$, which he rarely sees in practical work. On the other hand, to neglect the latter entirely is to unfit him for reading any save the simplest mathematics, and for expressing his solutions in the condensed step form necessary to allow the eye quickly to grasp the reasoning. So with a symbol like x; we may use it not only where there is no advantage, but where there is a positive disadvantage. It is the part of the textbook and teacher to suggest to the pupil the problems in which it should be employed, and to furnish a reasonable amount of exercise in the subject.

To consider a still more definite illustration, take this problem: If some goods are sold at a profit of $12\frac{1}{2}\%$ for \$1012.50, what did they cost? Here we have several possible lines of attack, among which are the following:

1. If there was a gain of $12\frac{1}{2}\%$, they must have been sold for $112\frac{1}{2}\%$ of their cost. (But where did that 100 come from?) And since $\$1012\frac{1}{2}$ is $112\frac{1}{2}\%$ of the cost, 1% of the cost is $\frac{1}{112\frac{1}{2}}$ (but how are we to read such a fraction?) of $\$1012\frac{1}{2}$, and 100% (but why do we wish 100%?) is 100 times this. Such an explanation, while it could be learned and recited, would be subject to questions like those inclosed in the parentheses, and really would have very little reasoning in it.

2. Let 100% stand for the cost. (But why 100% instead of 200% or 50% or some other number?) Then $100\% + 12\frac{1}{2}\%$ will stand for the selling price. If $112\frac{1}{2}\% = \$1012\frac{1}{2}$ (but an abstract number, $1.12\frac{1}{2}$, cannot equal a concrete number, $\$1012\frac{1}{2}$), then $1\% = \$1012\frac{1}{2} \div 112\frac{1}{2}$ (why? why not multiply? why not divide by $112\frac{1}{2}\%$ instead of $112\frac{1}{2}$?), and 100% (why not find 500%?) $= 100$ times as much, or \$900. This analysis is quite as superficial and unsatisfactory as the first.

3. Let $1 =$ the cost. (But why not let 2 equal the cost, and why take 1 instead of 100%?) The rest of this solution

might follow the lines of solution 1 or solution 2, and in either case it would be equally unsatisfactory. Like 2, it is a relic of the old and long-since-forgotten method of False Position, the discarding of which caused so many conservative teachers to feel that arithmetic was losing its mental discipline. So it would be possible to give a number of other plans of attack, some better than the above, and some much worse. Consider, however, a single other plan.

4. Let $c =$ the cost, a very natural thing to do since it is the initial letter. Then the selling price is $c + 0.12\frac{1}{2}c$.

But $c + 0.12\frac{1}{2}c = 1.12\frac{1}{2}c$, and therefore $1.12\frac{1}{2}c = \$1012.50$. Dividing equals by $1.12\frac{1}{2}$, $c = \$900$.

Now we are not troubled here about any 100%, or 1%, or letting 1 equal something that it cannot equal. As soon as we think of $12\frac{1}{2}\%$ as the same as $0.12\frac{1}{2}$, and know that $c + 0.12\frac{1}{2}c = 1.12\frac{1}{2}c$, just as $c + \frac{1}{2}c = 1\frac{1}{2}c$, the case is exceedingly simple.

Here, then, is one of the places in which the technique of arithmetic has been greatly improved by the introduction of an easy symbol from algebra, as it was years ago improved by the introduction of the symbols of operation. No teacher who has seriously tried to use this symbol has ever willingly abandoned it, but we are still in the period of experiment, so that textbooks can hardly be expected as yet to insist upon these forms. They are for teachers to bring into their teaching, and their sympathetic interest in these ideas is much to be desired.

What has been said for the symbol x might be said for other symbols if we needed them. It is of no particular consequence that we use $4^{\frac{1}{2}}$ instead of $\sqrt{4}$ in arithmetic, because we do not make much use of square root in the school; but if we did use the subject, the more modern symbol would deserve a place for its influence in later work, and the same may be said of the negative number, the parentheses, and other signs of algebra.

Outside of symbolism, however, there are certain improvements in technique that demand our consideration. One relates to subtraction, and in order to have it satisfactorily understood, it becomes necessary first to speak of it historically. There have been several successful plans for teaching subtraction; each has endured for a long period, and some of the plans are in use to-day. Of the most prominent the following may be mentioned:

1. The complementary method. Instead of taking 8 from 13 we may add 10 − 8 to 13 and then drop 10. This depends on the relation $13 - 8 = 13 + (10 - 8) - 10$, and since 10 − 8 is called the complement of 8 (to the number 10), this is known as the complementary method. It is very old, appearing in a famous Hindu work of the twelfth century, in the first printed arithmetic (1478), and in numerous other textbooks. In a case like 452 − 348 the operation would be as follows: $2 + 2 = 4$; since 10 must be dropped, we may add 1 to the 4 instead; then $5 - 5 = 0$, and $4 - 3 = 1$. That is, the complementary idea need be used only when the minuend is less than the subtrahend. This example is actually taken from the first printed arithmetic. The plan is the same as the one used in Pike's famous American arithmetic a century ago, and some teachers still employ it. It is essentially the one used when we employ cologarithms in trigonometry.

$$\begin{array}{r} 13 \\ 8 \\ \hline 5 \end{array} \qquad \begin{array}{r} 13 \\ 2 \\ \hline 5 \end{array} \qquad\qquad \begin{array}{r} 452 \\ 348 \\ \hline 104 \end{array}$$

2. The borrowing and repaying plan, a name that we may use for want of a better one. This may be illustrated by the annexed example, taken from the first great business arithmetic ever printed (Borghi's, of 1484). The operation is as follows: 8 from 14, 6; 8 from 15, 7; 10 from 13, 3; 3 from 6, 3. This was the plan advocated most often in the early printed arithmetics, and the expression "to borrow" was a common one. The plan was

$$\begin{array}{r} 6354 \\ 2978 \\ \hline 3376 \end{array}$$

known to the early Hindu arithmeticians and was also used in Constantinople before the invention of printing.

3. Simple borrowing — to continue to use this old English expression for want of a better one. In this method the computer says, "7 from 12, 5; 2 from 3, 1." This is probably the most common plan in use to-day, and it has much to commend it. It has a long history, appearing in the oldest known manuscript on arithmetic in English, in Spain in the thirteenth century, in Italy in the Middle Ages, and in India still earlier. It has not, however, been as popular as the second plan, although more generally used in our country to-day because it is slightly easier to explain.

$$\begin{array}{r} 42 \\ 27 \\ \hline 15 \end{array}$$

4. The left-to-right method, a plan that has had a long history in which many prominent advocates have appeared. It is more adapted to the needs of a professional computer, however, than to those of the average citizen, and may therefore be dismissed with this mere mention.

5. The addition, "making change," or "Austrian" method, a vague way of naming several related plans. The efforts made to adopt it in the Austrian schools, and the consequent notice taken of it in Germany, have been the cause of its most inappropriate geographical name. As a definite method of subtraction it is not as old as the others, although it appears in the sixteenth century in Italy and has had occasional prominent advocates since. It consists in finding what number must be added to the subtrahend to make the minuend. Thus in thinking of 17 – 8 we think: "8 and 9 are 17," writing down the 9. If the numbers are longer, we may proceed in either of two ways, as in the annexed example. Here we may say: "8 and 5 are 13; 4 and 7 are 11; 1 and 2 are 3." Or we may say: "8 and 5 are 13; 5 and 7 are 12; 2 and 2 are 4."

$$\begin{array}{r} 17 \\ 8 \\ \hline 9 \end{array} \qquad \begin{array}{r} 423 \\ 148 \\ \hline 275 \end{array}$$

Of these two, one is as easily explained as the other. The first might naturally be approached thus:

$$\begin{array}{r}
423 = 400 + 20 + 3 = 300 + 110 + 13 \\
148 = 100 + 40 + 8 = 100 + 40 + 8 \\
\hline
200 + 70 + 5
\end{array}$$

The arrangement for the second would be as follows:

$$\begin{array}{r}
423 = 400 + 20 + 3 \\
148 = 100 + 40 + 8 \\
\hline
\end{array}$$

Since the difference will be the same if we add the same number to both minuend and subtrahend, we will add 10 to each, and then 100 to each, giving the following:

$$\begin{array}{r}
400 + 120 + 13 \\
200 + 50 + 8 \\
\hline
200 + 70 + 5
\end{array}$$

Since the explanations are about equal in difficulty, we may consider the sole question of rapidity in practical use, both as to the method in general and as to these two sub-methods in particular.

Is this general plan the best one? On the side of advantages we have: (1) It is the common method of making change. If I owe $7.65 and pay $10, the merchant finds the change and I verify his work by saying, "65¢ and 5¢ are 70¢ and 30¢ more makes $1, and $2 more makes $10." That is, we find the difference by adding. This is familiar in all business and in the school, and will remain so. It is therefore a natural plan to use in all subtraction. (2) It avoids the necessity for learning a separate subtraction table. Everything is referred at once to the addition table, a table that unfortunately is not at present known any too well. There is therefore an economy of time and an increased efficiency in the very important subject of addition. (3) The facts of addition being used so much more often than those of subtraction, there is naturally an increase in speed and certainty when we employ the addition instead of the subtraction table.

On the other side, it is not desirable to change the customs of the people unless there is a decided gain by so doing. Parents who have been brought up on one plan, and who help their children more or less in their lessons, do not easily adapt themselves to a new method, and the result is a confusion in the child's mind that is most unfortunate. The fair question therefore is, Is it worth while to use the better method?

If we had a standard method known by all, this argument would have much weight, but we have at least three rather common ones, with several variations, already in use in this country. There is therefore bound to be more or less confusion. The only thing for the school to do, then, is to teach the method that will prove in the long run to be the most rapid and accurate, and this seems a priori to be the "Austrian," although a scientific investigation of the matter, on sufficient data, is desirable. And of the two or three submethods, the second one described on page 77 seems without doubt the better.

The question is, it should be repeated, not one of explanation, since any one of the methods is easily explained, and since the explanation is of very slight importance compared with the speed and accuracy of the operation. The question is purely one of practical utility, in which the teacher should divorce himself from all prejudice in favor of the method that was taught to him and with which he is most familiar — a somewhat difficult thing to do in a discussion of this kind. It should also be remarked that it is of doubtful policy to attempt to change a method that a child already knows and handles easily, inasmuch as the difference in value is not great enough to warrant this course. For beginners, however, the plan suggested is probably the best.

These two matters of improvement in technique have been mentioned, not as so important in themselves, but as types of various changes that are worthy of sympathetic consideration. The placing of the quotient over the dividend in long division, instead of at the right as was formerly done, so as to make more clear the position of the decimal point; the multiplying of both dividend and divisor, in the division of decimals, by such a power of 10 as shall make the divisor an integer, thus avoiding the old difficulty of determining the number of integral places in the quotient; the giving of the full form of an operation before the abridged one, in explaining the process; the writing of ratios in the familiar form of fractions in the first steps in proportion; the putting of the unknown quantity first instead of last in writing a proportion and the using of x to represent this quantity — these are some of the other improvements in technique that are getting into our books in these days and that should command our interest. To make these more clear an example of each is given. Thus in long division it is better to use the first of these forms than the second, since it makes clear the position of the decimal point.

```
      14.24
72)1025.28
   72
   ---
   305
   288
   ---
    172
    144
    ---
     288
     288
     ---
```

```
72)1025.28(14.24
   72
   ---
   305
   288
   ---
    172
    144
    ---
     288
     288
     ---
```

Similarly, the first of these forms is better than the second, the problem being to divide 102.528 by 0.72:

```
       142.4
   ________
72)10252.8                 0.72)102.528(142.4
   72                           72
   ---                          ---
   305                          305
   288                          288
   ----                         ----
    172                          172
    144                          144
    ----                         ----
     288                          288
     288                          288
     ---                          ---
```

In the initial stages of explanation the first of the following forms is better than the second:

```
      1424                                1424
   _______                             _______
72)102528                           72)102528
    72000 = 1000 × 72                   72
   ------                               ---
    30528                               305
    28800 =  400 × 72                   288
    -----                               ----
     1728                                172
     1440 =   20 × 72                    144
     ----                                ----
      288                                 288
      288 =    4 × 72                     288
      ---                                 ---
```

In introducing the idea of proportion it is better to begin with known symbols, so as not to confuse the pupil too much. Thus the first of these forms is better than the second in the early stages:

$$\frac{x}{7} = \frac{39}{91} \qquad\qquad 91 : 39 :: 7 : (?)$$

Indeed, it is always better to use the sign of equality (=) than the old sign of proportion (: :), and the latter is now happily passing into oblivion in this country.

There are several other important questions for the future in relation to the technique of computation, but these can be mentioned only briefly. The first relates to limits of accuracy in results. This does not mean that the work may be inaccurate, but that if we know the circumference of a circle only to two decimal places, we cannot from that find the diameter to three or more decimal places. We express this by saying that the result cannot be more accurate than the data. Suppose, for example, that we have made several measurements of the circumference of a steel shaft, and their average is 7.57 in.; it is evidently useless, in dividing this by 3.1416 for the purpose of finding the diameter, to carry the result beyond two decimal places. If we wish to divide so as to avoid unnecessary work, we may proceed by what is known as contracted division, as is here shown.

```
        2.41
3.1416)7.57
       6 28
       ----
       1 29
       1 25
       ----
          4
          3
```

The question naturally arises as to how much of this kind of work the schools are called upon to teach. It is certainly not a thing that many people will need to know, and therefore it is properly omitted from most textbooks to-day. In some localities, however, it might very properly be taught, and when our classes in physics require the mathematics that they might properly demand, and that they do demand in every other important country, it may become necessary to teach contracted multiplication and division in the schools.

Another topic is logarithms. In all engineering computations this labor-saving device is used, and the subject is easily taught. What shall we do with it in arithmetic? So far as grade work is concerned there is nothing to be done at present, because the demand is not sufficiently great. But we can hardly say what the future may bring forth, and the use of logarithms may become much more common than we think if the teachers of physics and the advocates of more mathematics in manual training stir up the question of greater facility in practical calculation. In the same way we may yet see the slide rule (a simple pocket instrument for computing) find a place in business or technical courses in the grades of the elementary school, and we are certain to see an extensive use of tables of roots, powers, reciprocals, and the like. At present our duty as teachers is to acquire a general knowledge of these various matters of technique, and to await the time when some or all may demand more serious attention on the part of the school.

CHAPTER XI

CERTAIN GREAT PRINCIPLES OF TEACHING ARITHMETIC

Before considering the curriculum in arithmetic it is well to devote a little attention to certain great principles that teachers have as a whole agreed upon, in theory if not in practice. Some of these have already been discussed in this article; others will strike the reader as rather trite, which simply means that they are generally accepted; and others will not appeal to all. They will, however, be found to be suggestive of the thought of leaders at the present time, and they may, though stated dogmatically, form the basis for profitable discussions by teachers.

1. Arithmetic is taught chiefly for its usefulness in daily life, but also because of the training that it gives the mind in reasoning, in habits of application, and in exactness of statement. Modern investigations in psychology do not disprove any of these claims.

2. Most of the mental discipline of arithmetic can be secured from those portions that may be called practical, and therefore the practical side of arithmetic may safely be emphasized.

3. But in emphasizing this practical side we need to offer a large amount of abstract work as well as concrete problems, skill in either not necessarily signifying skill in the other.

4. In the concrete problems, whatever pretends to be genuine, representing practical questions of American life, should be so, all obsolete business problems being replaced

by modern questions. In particular, the daily industries of our people should be drawn upon to make arithmetic interesting, informational, and practical.

5. It is better to adopt a good textbook and follow its sequence than to establish some eccentric sequence that is purely local and that may or may not succeed. The world's experience is worth more than that of any small group of individuals. If, however, a moderate spiral arrangement is desired, a teacher should feel free to make such omissions or changes as will adapt the textbook in use to the local conditions.

6. No extreme of "method" should be adopted by any teacher or school, but the best of every "method" should be known as far as possible to all. To measure everything in sight, to base all arithmetic on sticks, to go to extremes on number charts, to put all the time on mental arithmetic, to have all written work placed in steps, or to get into any narrow rut whatever, is to fail of the best teaching and to narrow the horizon of the children in our care. A good, usable textbook, broad in its purpose, modern in its problems, and psychologically arranged, is one of the best balance wheels for us all, and we should depart from its sequence and methods only for reasons that have been very carefully considered, while supplementing its good features by all the problems with local color that we can find time to use.

7. Mental (oral) arithmetic should play a part in every school year, so that children shall have not only an eye training for numerical relations, but also an ear training and a tongue training. A textbook may be expected to furnish a considerable amount of the abstract work in this line, but many of the concrete problems may well be suggested by the teacher or pupil and be correlated with local conditions and with the other work of the class.

8. Children's analyses, instead of being memorized, should be genuine statements of the reasons that prompt them to their solutions. In the problems the analysis should proceed gradually from one-step to two-step cases. In the operations it is not to be expected that children will long remember the reasons involved; they should understand each process when it is first presented, and this is effected through a development by a series of simple questions; but they should not be expected to give a very elaborate explanation of a topic like long division after the process is once understood.

9. Written arithmetic may at one time emphasize the rapid securing of results, and at another the analysis of the problem. Both are important, but in general the accurate result, rapidly secured, is the great desideratum. To say that a child ought not to work merely for the answer is a well-sounding epigram, but if it is interpreted to mean that he may work in a slovenly way, dawdling over his problem and getting an answer that is absurd, no amount of neatly written step work can atone for his mental laziness.

10. Arithmetic should be as attractive as any other subject in school. To this end a teacher should know something of its interesting story, should be familiar with its best applications to local and national life, should know how to treat the oral exercises in sprightly fashion, and should have a fair stock of number recreations. This does not mean that we should teach only what the child likes, but that the child should be led to like that which we know should be taught.

11. The improvements in the technique of arithmetic, including the use of x and the abridged and improved forms of operations, should be understood and appreciated by teachers, to the end that the subject may not stagnate in our schools.

12. It is a matter of relatively little importance that we present fractions, let us say, by sticks or paper folding or

clay cubes or blocks; but it is a matter of great importance that we present the subject in some concrete fashion, so that the child shall not proceed by arbitrary rules, but shall make up his own directions, and that he shall be so guided that these directions are the best that can be evolved at his age. What has been rather pedantically called "heuristic teaching," in its original form as old as Socrates at least, should always be in the teacher's mind — to lead the child unconsciously to feel that he is the discoverer, but to see to it that he is allowed to discover and to fix in mind only what the world has found to be the best. To carry out this policy, in arithmetic or any other subject, is one of the essentials of good teaching.

13. It is a general principle that the child should feel the need for numbers when he begins to study them, and that he should see the bearing of every topic upon life when he first considers it. This ideal is not always easy to realize, but we are approaching it in our education of children, and the tendency is a healthy one.

These are a few of the larger principles that should guide the teacher of arithmetic. The list might be extended, but these suffice to show the spirit in which the subject should be approached. Minor principles will appear as we consider the work of the various grades or school years.

Bibliography. Smith, The Teaching of Elementary Mathematics; Young, The Teaching of Mathematics; De Morgan, On the Study and Difficulties of Mathematics, Chicago, 1898; Clifford, Common Sense of the Exact Sciences, 3d edition, New York, 1892.

CHAPTER XII

SUBJECTS FOR EXPERIMENT

It is well, before leaving this general discussion, to consider a few subjects for legitimate experiment in the teaching of arithmetic that might occupy the attention of schools of observation or practice in connection with institutions for the training of teachers. As every one knows, a great deal of the so-called experimental work of the school is ill considered, arising from a nervous desire for change or for attention. Like promiscuous vivisection, such work is pernicious. But we advance only by "trial and error," and our schools for the training of teachers may safely indulge in experimentation that will be beneficial to the profession.

1. It is desirable to know just how far recreations in number can be used to advantage in teaching arithmetic. Of course we have such games as bean bag, ringtoss, and sometimes dominoes and number games with cards, used in the schoolroom. This, however, is a mere beginning. There are many more games that are usable for children. For example, more people in the history of this world have learned elementary number through dice than in the public school, and this is only one of several widely used number games. It would be very easy to go to a ridiculous and even dangerous extreme in this matter, and a teacher who begins to work upon it will naturally tend to do this, and will need a counterbalance upon his endeavors. Nevertheless, the work has never yet been done scientifically and it ought to be undertaken. This is, however, only a small part of the problem. There is the

whole field of mathematical recreations, touched upon briefly in this work, and this must sometime be examined scientifically. We have a large but undigested literature upon the subject, and no one has ever yet thoroughly studied it from the standpoint of the definite needs of the grades. The result of such a study ought to add greatly to the interest in arithmetic, without leading to any absurd and impractical extremes. Work there must always be in arithmetic, and it ought to be good hard work, but there is no reason why we should not let pupils see the amusements as well as the other interesting phases of the subject. So important does this matter seem that it has been thought best to devote Chapter XIV to its further consideration.

2. It is desirable definitely to map out the chief interests of children from grade to grade, with a view to ascertaining the best field for applied problems from year to year. We know these interests in a general way, and we know the child's mind well enough to judge of his arithmetical powers from grade to grade. But we do not yet know these interests in the exact way that we should know them. For example, when is the game element strongest? When does the interest in the heroic become most pronounced, and is the period the same for boys as for girls so that we may use this information in problem work in a mixed school? When does the interest become manifest in the food supply of our country? When in the clothing supply? When in transportation? When in the mines? When in manufacturing? When in commercial life? We know all this in a general way, but only so; we do not know it exactly, nor have we any definite body of facts concerning it that have been secured as the result of any scientific investigation. When we do know this the rational applications of arithmetic from year to year will be so much better understood that the subject will have an interest that

is now but feebly developed in our schools, and a definite value that we at present appreciate only in part.

3. We also need to know, statistically if possible, the result upon a group of children of emphasizing the abstract problem; upon another group, of emphasizing the concrete; and upon a third, of leaving the two in about the balance that experience has dictated. We have had some scientific investigation in this line, but it has been very slight and is therefore not conclusive. To emphasize the concrete would be to diminish the number of problems and the amount of computation; it would give less exercise in number relations, and it seems to give less satisfactory results. On the other hand, it may create interest in the work, thereby increasing the strength of the impression of the number relations, so that because the child multiplies only one twenty-fifth as many times it will not follow that he knows his subject only one twenty-fifth as well. At present the whole subject is in the domain of doctrinaire argument; what is needed is a scientific investigation of the problem by some school or some person who is unbiased on this matter. The trouble is that even such an investigation will tend to be so limited in its data that claims will be made that are not at all warranted. This is almost invariably the case in experiments of this kind.

4. We are at present entirely unsettled upon the question of time to be assigned to arithmetic. Scientific investigations have been made, but they are incomplete. It is certainly strange that arithmetic requires so much more time here than in other countries. It would seem that excellence in arithmetic work is much less a function of the time assigned to it than has formerly been supposed. Such an investigation would probably require a number of years for its satisfactory completion, but it might be undertaken in any large school system in a single year with helpful if not convincing results.

5. We are quite uncertain as to the relative amount of time to be devoted to oral and written arithmetic in our schools. The wave of oral work, beginning with Pestalozzi and culminating in this country with Warren Colburn, as mentioned in Chapter VII, gradually subsided some years ago. What should we be doing in the matter to-day? It is very easy to talk dogmatically about it, but we need, if it is possible, a scientific investigation as to the practical results of more oral arithmetic, and of less. Would it be well to have the work much more oral than at present, or would we gain by confining our energies more to written work? Is there any scale by which we can definitely measure this matter, and, if so, what is it, and what are the results of the measurement?

6. Just how far are we justified in departing from the old plan of making the operations the basis for a course in arithmetic, and of substituting for them the applications? Shall we ever be justified in giving up multiplication as a topic and in substituting a chapter on housebuilding that shall bring in multiplication as needed? Of course in the latter part of arithmetic we put the application to the front, as in the early part we put the operation there; but just where should we draw the line, and how far should this latter plan encroach upon the former?

I am indebted to my colleague, Professor Henry Suzzallo, for a large number of subjects for experiment in the teaching of arithmetic, and from these I have made certain selections which form most of the remaining part of this chapter. In the first place it is evident that it is not sufficient that a new way or an old way of teaching has succeeded; for example, in the addition of fractions. The test of the worth of a given method is not alone that it gets a thing done efficiently; it must get it done as economically as possible. The method of most worth is the one that obtains the efficient result with the least possible expenditure of energy. The comparative worth of two methods must eventually be decided by experiment, the investigation being conducted upon children of about the same grade, age, and previous training, taught by teachers of fairly even strength of personality, so that approximately the only difference in the conditions of the trial is the difference in the methods involved in the test. Finally, the repetition of the experiment in a considerable number of schools or systems is necessary in order to reduce the danger always present in assuming that a special group of children and teachers is typical of school conditions in general. The following sample experiment will make the method of investigation clear.

In the teaching of the addition combinations it is the requirement of certain courses of study that combinations and their reverses be taught in association with each other. The teaching of "3 and 2 are 5" is followed at once by the teaching of "2 and 3 are 5." In other courses of study or texts there may be quite an interval between the learning of these two combinations. In fact, the pupil may learn each separately without being conscious of any greater intimacy between these two combinations than between any other two; for example, "6 and 3 are 9" and "4 and 2 are 6." Those who advocate the first method imply, if they do not expressly state, that to teach a combination and its reverse simultaneously is more efficient than to teach them separately and unrelated. The problem for the investigator is to determine whether or not this is true.

To illustrate the method: In any large city school there may be two, three, four, or more classes of one grade in which combinations in addition are taught for the first time. In the case where there are four classes, two could be set off against the remaining two, care being taken to equalize the number and quality of grades, the teachers' personalities, and other factors, in so far as they may be equalized within such a limited range. All four classes could then be given the same list of combinations to be learned, and a common method of general procedure could be laid down by the experimenting principal, the only general difference in procedure being that one group of classes would always learn the reverses immediately after the original combination to which it is related, while the other group would learn them in an order that would separate the original combination and its reverse. Thus:

Group I

2	2	3	2	4	3	2	5	3	4	3	5	4	4	5
+2	3	2	4	2	3	5	2	4	3	5	3	4	5	4
4	5	5	6	6	6	7	7	7	7	8	8	8	9	9

Group II

2	2	2	2	3	3	3	3	4	4	4	4	5	5	5
+2	3	4	5	2	3	4	5	2	3	4	5	2	3	4
4	5	6	7	5	6	7	8	6	7	8	9	7	8	9

An equal amount of time having been spent in all the classes up to a point where they have approximately covered the complete list of combinations, a test could be given to determine how far each individual had mastered these number facts. After a lapse of a week or ten days another examination would show how stable the mastery had been in each case. Any difference between the group taught by one method and that taught by the other method as clearly shown by the statistical results would then tend to indicate the relative efficiency of the two methods.

This being the method of experimentation, a few of the general questions may now be considered. First, does the effective use of objective work demand a large amount of this work with relatively young students and a smaller amount with relatively old students? Or should the distribution of objective work be determined by the fact of ignorance or immaturity in any special phase of arithmetic regardless of the age of the student? To what extent does general arithmetical maturity require less objective work in the first formal study of fractions than in the first formal study of addition?

Do all of the fundamental combinations in any given field (addition, multiplication, etc.) require objective development? How many combinations need to be developed objectively before the child will clearly know that succeeding combinations given by the teacher authoritatively stand for real relations? Furthermore, to what extent may the constant handling of objects, pictures, and diagrams, and concrete imaging in general, interfere with rapid abstract manipulation of numbers and number combinations?

To what extent are the difficulties of children with arithmetic problems due to a failure to understand underlying concrete situations because they do not understand the language by which it is intended to convey them? What is the relative difficulty in understanding the significance of a situation when the presentation is (1) objective, (2) oral, and (3) written? Would it be well to postpone the written presentation of problems until a specific number of school years of language training have been given?

How far is it necessary to develop a special terminology for school use in the subject of arithmetic, the terms being little used in ordinary social relations? For example, consider the case of the words "multiplicand" and "dividend," the latter having a radically different meaning in business life. In the case of signs consider such semi-algebraic symbols as ÷, and even +. If special signs are used in examples, to stand for the process of calculation demanded by the situation (which might have been expressed in concrete problem form), how wide and varied should the language of problems describing situations be? If + may be used in an example, while the same relation is expressed in a problem by such words and phrases as "added," "and," "together," "how many," "altogether," how wide should this vocabulary be? To what extent does a lack of practice in dealing with problems presented through varied language explain the failure of children in problem tests given by an outsider, another teacher, the principal, or the superintendent, when the children had always "done perfectly" the problems that their own teacher gave them?

Which types of problems are of most concrete interest to the child: (1) those drawn from his own spontaneous play and work life, (2) those drawn from the facts of actual social life about him, (3) those taken from the textbook, or (4) those in which the puzzle element is prominent? Are all four essential in achieving the aims of arithmetic teaching? If so, is there any special law of advantageous usage for each of the four types? Should problems from the pupil's own life be used to show the necessity for the study of calculation, and those from social life be used for further, later, and final application of the formal processes of calculation that have been mastered? Is a problem concrete to the child merely because it is a concrete reality existing in the world? May not an imagined problem vividly within the grasp of the pupil's own imagination be more concrete as regards his interest in its solution than one which actually exists in the real world? In particular, may not a good puzzle problem, however old, prove to be more interesting than any "narrative problem" that we can devise?

Problems may be done by the pupil (1) silently, (2) orally, (3) in written form on paper or blackboard, or (4) with a mixture of any two or all three of the preceding methods. Which of these methods represent final forms in which efficiency is demanded in ordinary life? Which forms merely represent transitional means used by the teacher to keep track of the workings of the child's mind? What is the proper order and emphasis of these forms in the mastery of a single new line of work, say in a problem where the child needs first to divide and then to multiply?

To what extent do precise oral forms assist in the correct analyses of problems, for example, "If two apples cost 6¢," etc.? To what extent do precise oral forms assist in the memorization of combinations or manipulations; for example, "three 4's are 12," "put down the 3 and 'carry' the 2"? On the other hand, to what extent do precise written arrangements of analyses assist the child in carrying out a strictly logical mode of thinking? Take the following case as an example: "If 3 pencils cost 15¢, 1 pencil costs $\frac{1}{3}$ of 15¢," etc.

Which is the best way to teach young children to count serially from 1 to 100? To have them count by ones from the beginning, extending the series as fast as it can be memorized, without any effort to show them that it repeats with a certain regularity after twenty is passed? Or to have them memorize the names in their order from one to thirty (by which time the regularity is established as a basis) and then have them learn to count by tens, later using the counting by ones and the counting by tens as a double basis for learning to count serially from thirty to one hundred?

Assuming that oral counting leads mainly to the association of a name (27) with a given position in a series of names (between 26 and 28), how far is it advisable for a number to be associated with a given idea of mass or grouping, as when the device of two bundles of ten sticks each, together with seven individual sticks, is used to explain 27? Does the effort toward the association of concrete images and numbers ultimately interfere with the rapid manipulation of figures in complex calculations? How far does the material in objective work need to be varied with first-grade children (sticks, lentils, boys, etc.) so that the idea associated with a number shall be abstract rather than the image of any particular concrete thing or group of concrete things?

How far is group counting (counting by 2's, 3's, etc.) really counting, that is, proceeding from one number to another by an act of absolute memory (saying 3, 6, 9, etc., exactly as one says 1, 2, 3, etc.)? How far is it really a process of consecutive adding (3 and 3 are 6, 6 and 3 are 9, etc.)? If it is a mixture of both, where does one process end and the other begin? If group counting is really adding, should it not always be classified with the work of addition, and placed so as to assist it, rather than be taught independently as alleged counting? How far is group counting as real counting desirable? How far may it be used as a special form of addition? In the latter case should it precede or follow combination work in addition? That is, should 6, 9, 12, etc., precede or follow $6 + 3 = 9$, $9 + 3 = 12$, etc.? If counting forward is an aid to addition, how far can counting backward be an aid to subtraction? How far is real counting backward (by sheer act of consecutive memory) of importance?

How far shall the three processes of (1) oral counting, (2) reading of numbers, and (3) writing of numbers be parallel in the first year of formal arithmetic teaching? Should counting precede reading, and reading precede the writing of numbers? How far are they dependent upon each other? In relation to accomplishment in any one of these processes, when should the teaching of the other begin?

In teaching children to read and write numbers, how far is it useful and how far is it confusing to have them know the place names (unit of units, tens of units, hundreds of units, etc.)? Should such a classification be given to the child finally, or not at all? Is the so-called method of "group reading" superior to the "place" method? In the "group" method a child reads and writes all his numbers as he would numbers of three figures or less, naming them from the commas which mark off the groups of three, as in 34,026, "34" = "thirty-four," "," = "thousand," "026" = "twenty-six." What are the special errors which are peculiar to the "place" method? What are the special errors peculiar to the "group" method? Is the whole discussion nothing more than a splitting of hairs?

It is generally said that there are forty-five fundamental combinations which are the basis of all work in addition. What are the fundamental facts that are required as basic and which, once learned, may be applied in new forms and situations over and over again? There are ten numbers, from 0 up to 9. Each of these may be combined with itself and the nine others, thus making 100 combinations, from $0+0=0$ up to $9+9=18$. The 19 zero combinations are usually left out, leaving 81 combinations. Of the 81 remaining, 36 are reverses ($2+7=9$ is a reverse of $7+2=9$). Omitting these, there are 45 combinations left as fundamental. Is this procedure correct?

How far does the learning of $7+2=9$ also guarantee the acquiring of its reverse, $2+7=9$? Will the second be known without further drill? With how many less repetitions will it be learned because the other combination has been mastered? Will the two combinations mentioned be learned with fewer repetitions when they are constantly presented together, instead of being learned as separate individual combinations the relation of which is not specially kept in mind?

Is there a justification for saying that the zero combinations ($0+3=3$) may be omitted as not being basic? They are necessary for their later use in column addition, $0+4=4$ occurring in $10+4=14$. Is it true that all zeros in column addition are ignored?

Some courses of study require that a combination, such as $5+7=12$, be applied, as soon as it is learned, to the higher decades ($15+7=22$, $25+7=32$, etc.). How much superior in column addition is a class thus trained to one not so trained? Is it necessary to apply all combinations in this way? May it not be that the general idea of application is soon acquired with the first few combinations and that special drill is not required thereafter? Are there certain combinations where special drill must always be insured ($5+6=11$, $15+6=21$) because the rhythm is interfered with? Or may a strictly written presentation do away with the necessity of special drill even here?

2	4	24
3	1	31
2	2	22
7	7	77

Is there any increase of efficiency in drilling on combinations in columns as soon as possible? As soon as the combinations that add up 7 are learned, is there a special advantage in immediately giving the child several columns of numbers to be added in which the sum is 7, as here shown?

23
47
36
16
9
106

In column addition, where carrying is involved, some rationalize the process, and others teach it mechanically as a mere bit of habit formation. In the case here given, some would add each column separately, taking a second total of the partial sums. Others would merely "put down the 6 and add 1 to the next column," writing down only the complete sum. Which will result in accurate and rapid column addition in the shorter time?

The same general questions arise in subtraction. Do children make fewer errors and manifest less confusion where they are formally taught to handle the difficulties prior to being confronted with them in such cases as the following?

(a)	(b)	(c)	(d)
867	867	867	870
− 467	− 400	− 32	− 650
400	467	835	220

Consider the following cases of borrowing from the top:

(e)	(f)	(g)	(h)
128	602	612	612
− 76	− 237	− 318	− 308
52	365	294	304

Consider the following cases in which we might advantageously add to the lower number:

(i)	(j)	(k)	(l)
834	834	804	814
− 406	− 496	− 496	− 406
428	338	308	408

In subtraction, what preparation is needed to insure a sufficient command of the zero combinations to enable the child to perform column subtraction? Is there some general mode of handling zero that will not require separate drill in connection with each number with which it may be combined? How does the above apply to the combinations with ones? Where 1 is involved, is it merely counting downwards or backwards? Or is the subtraction of 1 exactly like the subtraction of 3 or 4 or any other number?

Are the first series of multiplication combinations best presented by the use of objects grouped and counted, or by the use of column addition? Or should these two methods be used as supplementary to each other? Are the combinations with zeros ($6 \times 0 = 0$) and the combinations with ones ($6 \times 1 = 6$) best taught in the tables, or later, in connection with their actual use in column multiplication? As with the addition combinations, is there a gain in teaching the inverses in connection with the combinations to which they are related, as in presenting $6 \times 3 = 18$ immediately after $3 \times 6 = 18$? Since partial products represent only stages in calculation, should their placing be explained, or should it be taught as a mechanical process through habit formation? Is it economical to allow zeros to be recorded which later will be abandoned, as in multiplying by 206?

Is there any need for tables of division combinations? May not the multiplication tables be used for division, precisely as the addition combinations are used for subtraction? For example, from $3 \times 2 = 6$, or "three 2's are 6," may we not step to the case of $2\underline{)6}$? "How many 2's are 6?" "Three 2's are 6."

A large number of questions have here been asked. Are they worth the asking? Are they worth the answering? Certain it is that we are not sure of the answers at the present time. Their mere statement, with the knowledge that it is very easy to ask dozens of similar questions, shows that there are many details still to be settled in the very beginning of the work in arithmetic. Many of the questions of the professional educator are like asking, "Is it better, at dinner, to take a mouthful of meat and then a mouthful of potato, or to begin with the potato?" But most of the topics here suggested are worthy of our attention, and little by little we shall come to find that candid and continued experiment bring answers to some of them that will probably be helpful.

Bibliography. Upon the experimental work consult Professor Suzzallo's Teaching of Primary Arithmetic, Boston, 1911, where the subject is more fully treated than is possible in a chapter like this.

CHAPTER XIII

INTEREST AND EFFORT

There has of late years been a tendency throughout the country to make arithmetic, as well as other subjects, more interesting to children. What the real motive was it is hard to say, since it was probably somewhat subconscious. Such statistical information as we have shows that arithmetic has always been looked upon by children as one of the most interesting subjects of the course, so that the reason was not that it was relatively a dull study. Possibly the desire was that the work of the teacher should become easier through increased interest on the part of the pupils. But whatever the reason, it cannot be questioned that, other things being always kept equal, there is a great gain in increasing the interest in any kind of work.

There is, however, a general danger accompanying this effort to increase interest. If this increase means that the subject is to become anæmic, if its mastery is not to require the same serious effort as heretofore, then it loses a considerable part of the value that has generally been assigned to it. Moreover, through this same cause it loses a considerable part of the very interest that was expected to be fostered. Boys and girls do not like to wrestle with infants or with infantile subjects, and unless a study is suitably graded as to difficulty it will appeal in vain to the interest, the vigorous attack, and the responsive mental effort of the pupils.

Our lesson, therefore, is that we should do all in our power to make arithmetic interesting and even attractive to the children, but that we must not hope to attain this result by offering a sickly substitute for the vigorous subject that has come down to us. Unfortunately we have not been free from this fault of making our arithmetic, and particularly our primary arithmetic, anæmic. Foreign critics frequently comment upon this failing, and claim with good reason that much of our work in the early grades lacks vitality. Certain it is that in spite of many points of superiority in the American school we do not at the end of eight years bring our children as far as European experience would justify us in expecting. Foreigners accuse us of superficiality, and when we compare our results with theirs, we are forced to admit that there is some ground for the accusation. Certain of our state courses of study are very weak in substance, not to speak of the arrangement of matter, and apparently they are so constructed because of the idea that children are interested only in that which requires no effort. Such an idea is not borne out by any scientific study of childhood that has as yet appeared.

How can the interest in the applications of arithmetic be aroused and maintained? The reply has already been made. They must be real if they pretend to be so, they must relate whenever possible to the child's daily environment, and they must reveal the life of America to-day in such a way as to be broadly informational as well as mathematical. This can be accomplished with no less demand for mental power than was required by the obsolete problems of our old-style books. There are, however, various other channels through which we may pass to reach the required end. For example, there are the number games for children in the primary grades, games that have an interest that pleasantly conceals the mental effort required, as tennis does the muscular effort, but that accomplish the result efficiently. This subject is considered later in this work.

Then there are the outside problems, problems brought into class by the pupil or the teacher. Those with some local color are the best, and they rarely fail to arouse a genuine interest in the work. A recent writer has suggested the following problems as being practical, designed to provoke inquiry, and interesting if properly presented:

1. "How many bolts each 4 in. long can be made from three brass rods $\frac{1}{2}$ in. in diameter and 8 ft. long?" This is a real problem, but it is evidently not a textbook question. A textbook with such a problem would be criticized severely as containing an unfair puzzle, namely, the question of what to do with the $\frac{1}{2}$ in. For outside work, however, the problem is a good one, for the teacher can easily guard against the difficulty if necessary.

2. "A casting weighing 4000 lb. is to be raised by means of a crowbar. The distance from the point of the bar under the casting to the pry block is 4 in.; the distance from the pry block to the point where the bar would be grasped is 4 ft. How many men will be required to raise the casting?" Here, again, no textbook writer could use such a problem, the weight of the men not being given; but the suggestion is made, "Use your judgment. Would you send one man or ten to do the job?" For outside work such a problem is commendable, but as a textbook problem it would be worthless.

3. "You deposit $1 per month in a savings bank, paying 4% per year, interest being credited to your account every six months. How does your account stand at the end of 17 months?" As a casual problem suggested by the teacher and leading to class discussion this is good. As a textbook problem, in which all necessary data should appear, it would be a failure. Indeed, as it stands, it cannot have a unique solution, for it depends upon the month and time in the

month when the first deposit is made, the months in which the interest periods terminate, and the custom of the bank as to the balance on which interest is reckoned — sometimes the smallest balance in any quarter.

It would seem better to avoid any uncertain element that might strike the pupil as unfair in such problems, and offer only examples that give clearly the data needed for the solution. Such problems abound in any locality, and they give a reality to the subject that can hardly be expected to come from the miscellaneous exercises in any textbook, however good. If our teachers would learn how to use the textbook as teachers in other countries do, it would be a great gain. They should freely omit problems that are too difficult, freely add to the explanations, freely lead the pupils to derive their own explanations, and freely supplement the problems by others of local interest. A textbook, in its problems, as well as in its explanations, should be a guide, not a slave driver.

The interest in such topics is measurably greater than that in the old subject of equation of payments, or most problems in compound proportion, or examples in the Vermont Rule of Partial Payments (a subject that, however, naturally has its place in the curriculum of that state). On the other hand, the effort may be just as great as we wish to make it. It is only a matter of complicating the problem sufficiently, and using numbers and combinations of proper difficulty, to make a modern problem about the coal industry of Pennsylvania, or the silver output of Colorado, as hard as any example in the arithmetics of fifty years ago.

We are therefore led to the following conclusions: (1) it is possible to bring our arithmetic work to a higher plane of interest, through fostering the game element as well as through the applications; (2) it is possible, with this, to keep the plane of effort as high as we wish; (3) with the increased interest must necessarily come an increase of power that is vital to the improvement of our education.

The game element of mathematics is one of its most valuable educational assets, and its importance is such as to demand for it a special chapter. The immediate interest of the pupil in mathematics is largely because mathematics seems to him to have many of the qualities of a game, and hence it is proper that Chapter XIV should be devoted to this phase of the work.

Bibliography. In the matter of modern problems consult the author's arithmetics or those of other recent writers. All textbook writers seek for good problems of the present day and no one author has a monopoly of them.

CHAPTER XIV

NUMBER GAMES FOR CHILDREN

While the general experience of the world is opposed to the omission of arithmetic, as a definite subject, from the work of the first two years in school, it is also opposed to the policy of making it too formal. It was never Pestalozzi's intention to make it mere drill work, uninteresting to children and unrelated to their interests, although it was he who had most to do with placing it definitely in the curriculum of the first school year. He concealed the drill under the guise of play, but play with a definite purpose.

Every primary teacher recognizes the same principle, and the large majority recognize the necessity of keeping the play element within safe bounds. A teacher who would make the arithmetic hours of the first two years nothing but childish games would have very poor results; and one who would never make any use of number games, but would merely drill day after day on the abstract facts of arithmetic, would have a rather stupid class to pass on to the third grade.

There are many number games adapted to the early years of school, and although some of these are familiar to all teachers, it is felt that a brief list may serve some useful purpose. It must be understood, however, that those here giver are merely types, that many others are equally good, and that they should always be used with common sense and with the definite purpose of making the acquisition of number relations more interesting and the uses of number seem more real to the child. A few miscellaneous games will be given first.

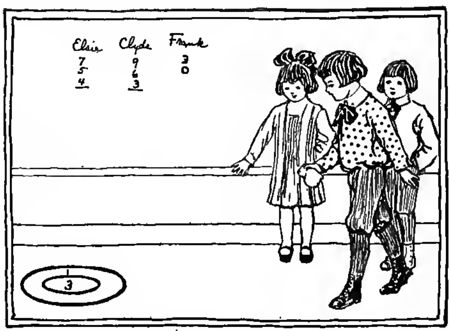
Bird catcher. Arrange the children in a circle, assigning a number to each. Let one child sit or stand at the center and ask for results within the numbers assigned. For example, "How many roses are 3 roses and 6 roses?" The child having the number 9 holds up his hand and announces the number. He has caught the bird.

Sum guessing. One of the pupils says, "I am thinking of two numbers. Their sum is 9. What are they?" The class guesses until the right numbers are given, after which some one else acts as leader. Instead of addition the other operations may be used.

Hide and seek. The teacher places on the blackboard statements like the following, with one number hidden:

$$3 + \square = 7, \qquad 5 + \square = 9, \qquad \square + 4 = 8.$$

In making the full statement the pupils get a good deal of training in repetition of number forms.

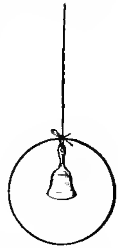


Bean bag. This game can be played in school, during a rainy recess period if desired, or at home. The score requires addition.

It may also be amplified by letting each bag thrown into the circle count 2, each child having six bags. We may also let each bag thrown in count 3, and each one outside count 1 off.

Ringtoss. Rings of different colors and sizes are thrown at a standard, the smaller ones that encircle it counting more than the larger ones.

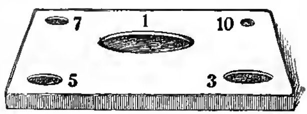
The hoop game. The children may throw bean bags through a hanging hoop in which is a bell. Every bag that goes through without ringing the bell counts 10. If the bell rings, the throw counts only 2. For example, in ten throws each, the score of two pupils may be



0, 0, 2, 10, 0, 0, 2, 2, 0, 10;
10, 0, 0, 0, 0, 2, 0, 0, 10, 2.

What was the score of each? Which won the game?

A marble game. The children may cut holes in the bottom of a cardboard box, like this. They may number the largest 1, the next 3, the next 5, the next 7, and the smallest 10. They may then put the box on the floor and drop marbles from the height of a table.



With 10 marbles each, the score of two pupils might be

1, 1, 3, 1, 0, 10, 7, 5, 0, 1;
0, 5, 0, 1, 1, 1, 1, 7, 7, 10.

What was the score of each?

Odd or even. This is an ancient game, and is played by two persons. Each takes a given number of marbles or peas, say 10. One child places his hands behind his back, arranges the objects to suit himself, and then stretches out his closed hand and says, "Odd or even?" The other guesses, and if he is correct he receives a marble; if incorrect he pays one, the other saying, "Give me one to make it odd," or "even," as the case may be. There is a similar game, played in some parts of the country, called "Hull Gull."

Morra. This is an old Roman game, and it is played with much interest by Italian children in this country. It may be played by two individuals or by a group. In the latter case, at a signal from the teacher, all the children extend their hands, with one or more fingers raised. Each guesses the number of raised fingers, and a count is then made to see which is nearest, a score being kept.

Thumbs up. In this game one pupil acts as the leader. Each player has a number and sits with one thumb up. The leader says, "Simon says 15," at which the thumbs of 3 and 5 (the factors of 15) must at once be turned down. If Simon says 12, then 2, 3, 4, and 6 must be turned down.

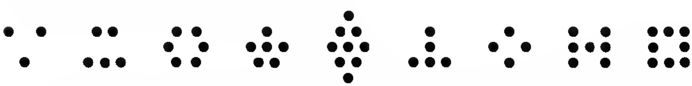
Tag. Each child is given a number, and the leader names any number below 25 (or some other prescribed limit). All the children who have factors of this number must then change seats. If there is only one distinct factor, as in the case of 25, the pupil rises and bows. Children failing to rise are tagged.

Among the various devices for fixing the idea of number may be mentioned the following:

Number touch. Ask a child to close his eyes. Then touch his hand a certain number of times and have him state the number. Children may try this with one another.

Number sound. Ask the pupils to close their eyes and tell the number of taps that one of them makes with a pencil, the number of times the bell is struck, or some other definite number of sounds.

Groups of things. Let the children copy the following and write the figures below them:



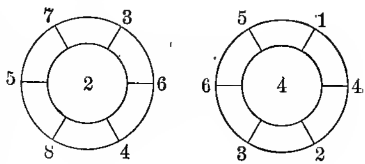
In the same way let them make other groups of dots to represent the various numbers below ten. There is little advantage in carrying the work beyond ten.

Number splints. There is an advantage in using splints to fix the idea of the decimal arrangement of numbers. After

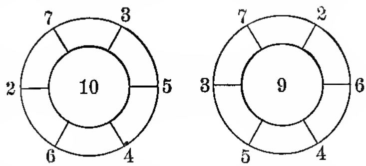
the idea has been clearly fixed in the child's mind, however, the device should be abandoned.

Learning figures. Figures may be cut from sandpaper, so as easily to be felt by the fingers. The children may then close their eyes and tell the figure by the sense of touch. This is one of several devices given in the Montessori method that is now attracting considerable attention.

There are many devices for number drill, but it is possible at this time to mention only a few of the common ones. Others are easily found, and the use of different ones from time to time is desirable.

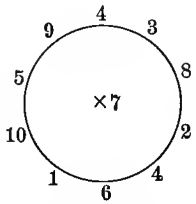


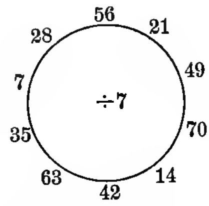
Circle addition. Circles of this kind can be drawn upon the blackboard, the children giving the sums of the interior number and the exterior one to which the teacher points. The class may be separated into "sides," and the captain of each side may do the pointing, a score being kept, or the children sitting when they fail to reply quickly. The numbers may frequently be changed.



Circle subtraction. In the same way, circles can be arranged for subtraction, the numbers being changed as necessary.

Circle multiplication and division. The operations of multiplication and division are carried on in the same way as addition and subtraction, the numbers being selected and arranged according to the progress of the class. There are many other simple arrangements of this general nature that teachers easily invent for themselves, the numbers being arranged about a square, or in rectangles, or in some other simple fashion.





Climbing the ladder. The teacher draws a ladder on the blackboard, placing a number combination on each step. The pupil begins at the bottom and climbs as far as he is able. When he makes a mistake, or does not answer with reasonable promptness, he falls off and another takes his place. The successive combinations, when the table of 3's is being learned, might be

$$2+3,\ 3+2,\ 4+3,\ 3+5,\ 3+4,\ 5+3,\ 3+6.$$

Number tugs. The class may be arranged in two equal groups for a tug of war, each group having a captain. The captains may give combinations to the other side, such as 2×3, 3×1, 3×3, and so on, the pupils being seated as they fail.

Buzz. The teacher gives a number, say 3, to the class, and then asks the children to count in turn, each child saying "Buzz" if he has any number of 3's. Thus, instead of saying "Six," or "Nine," the child would say "Buzz." The game is of value when learning to count by 3's as the basis for the multiplication table.

Rapid drill. It should not be forgotten that all rapid number drill is in itself a game of great interest to children if conducted in a spirited manner.

Among the out-of-door games the following may be mentioned:

Hopscotch. This familiar game is interesting because of its action, and valuable because of the counting involved.

Peggy, trapball, or *tipcat.* The peggy is a piece of wood about four inches long and pointed at each end. One end is struck with a bat as it lies on the ground. This causes it to fly in the air, when the player hits it again. He then estimates the distance it goes in bat lengths, scoring one if he guesses within one length. The game is valuable in training the children to estimate distances.

Marbles. The various games of marbles are too well known to need description.

Shuffleboard. This is a well-known ship game. Ten wooden disks are prepared and, with a stick broadened at one end, are shoved along the floor or sidewalk into a space that has been chalked about twenty feet away. As played on ships the numbers are usually arranged in a "magic square," each row, column, and diagonal having 15 for its sum.

8	1	6
3	5	7
4	9	2

The toy stores supply various number games for home use. Of these a few may be mentioned.

Dominoes. There are several games played with dominoes, all of them useful in fixing the number combinations in mind.

Lotto. There are several games sold under this name. Sometimes numbers are printed on sections of a picture, these sections being fitted in by matching the numbers. The game is valuable when the numbers are being learned.

Card games with numbers. Card games in addition, subtraction, multiplication, division, and fractions are manufactured. These have no relation to card games to which there is any objection on the ground of undesirable associations.

Tenpins. Various forms of this game can be purchased at toy stores.

Pachesi (*Parchesi*). This is one of several games in which counting plays an important part. It involves the ancient game of dice, as is also the case with backgammon.

Teetotum. The teetotum is a kind of top and is spun with the fingers. Sometimes it is six-sided, the sides being marked as they are on dice. Sometimes it is plain, the game being to see on what number, marked on the table, it finally rests.

Top game. There are several games with tops. One of those sold at toy stores has numbered pins that the top knocks down as it is spun. Another has numbered bells that it strikes. There are many games of this kind involving scores. Sometimes marbles are rolled into numbered pockets arranged about a table, or into numbered holes; sometimes they are shot at toy men that are marked with different numbers.

Number blocks. Cubes on which are the letters of the alphabet and the numerals from 1 to 10 are familiar to all. Even plain blocks that the child learns to count as he builds his playhouses are of more value than would at first seem to be the case.

Other games. In nearly every home there are found games involving counting and the keeping of scores. These are all helpful in developing the idea of number.

Besides the games above mentioned there are many interesting tricks and number relations that add interest to the work in arithmetic. Many of these are discussed in the *Teachers College Record* for November, 1912, and a few will now be given. No teacher is expected to remember them all, but one or two may be given to a class now and then to show the curious phases of simple arithmetic.

For example, let some child think of a number consisting of two digits, and then reverse the order of digits, obtaining a new number. Let him then subtract the smaller from the larger and tell you one of the digits in the difference. You will then be able to tell the other by subtracting the digit given from 9. Suppose the number thought of is 37. Reverse the order of the digits and the new number is 73. Subtract 37 from 73 and the result is 36. If, then, 3 is given, subtract 3 from 9 to find the other digit, 6.

The same thing may be done with a number consisting of three digits. The digit in tens' place in the answer will always be 9 and the sum of the other two will be 9. If, then, either the first or the last digit is known, the other two can be found.

The following operations give interesting results:

Let each pupil (1) write any number consisting of three digits; (2) write the number obtained by reversing the order of the digits in the first number; (3) subtract the smaller from the larger; (4) write the difference obtained and write the new number formed by reversing the order of the digits in this difference; (5) add this new number to the difference found in (3). The result of the last operation is always 1089 and all the pupils will have the same answer notwithstanding the fact that each chose his own number. For example, suppose one of the numbers chosen is 643. The successive steps give

$$643 - 346 = 297; \; 297 + 792 = 1089.$$

Another case is somewhat similar. Let each pupil write any number of three digits such that the digits decrease by one from left to right — for example, 432; then reverse the order of the digits and subtract. The answer will always be 198. For example,

$$765 - 567 = 198.$$

The field of multiplication is particularly rich in curious cases and short methods. The test by casting out nines, for example, is interesting, and is so valuable in multiplication that it ought to be used systematically.

There are several combinations which produce results composed of identical digits. These usually appeal to young students as quite remarkable. For example,

$$\begin{array}{r} 12,345,679 \\ 54 \\ \hline 666,666,666 \end{array}$$

The explanation is simple, the number 12,345,679 being the repetend of the fraction $\frac{1}{81}$ and $9 \times \frac{1}{81} = \frac{1}{9}$ whose repetend is 111,111, . . ., while 54 is 6×9.

The pupils may suggest the number which they wish to appear in the answer. To obtain the desired result we need only write as the multiplier 9 times the number chosen. The pupils will need no suggestion to test the accuracy of the result and they will be drilled in multiplication without realizing it.

The same result may be obtained by using as the multiplicand 1,122,334,455,667,789, the repetend of $\frac{1}{891}$. Multiplying this number by 99, we obtain $99 \times \frac{1}{891} = \frac{1}{9}$, whose repetend is 111,111, etc. Therefore to obtain a result composed of identical digits, multiply the above number by 99 times the number which is to appear in the answer. For example,

$$\begin{array}{r} 1,122,334,455,667,789 \\ 297 \\ \hline 333,333,333,333,333,333 \end{array} \qquad \begin{array}{r} 99 \\ \times\ 3 \\ \hline 297 \end{array}$$

We can produce the same product by repeating each digit in the multiplicand three or four times, writing 8 twice or three times respectively, and 9 once. In this case we would multiply by 999 or 9999 respectively, or by one of their first nine multiples.

The multiplier in each of these cases can be written down without stopping to calculate, by applying the familiar rule for multiplying by a number composed entirely of 9's. The first figure in the multiplier will be one less than the figure to appear in the answer. This will be followed by as many 9's less one as there are 1's in the multiplicand. The last digit will be the complement of the first digit.

This case may be made to appear more remarkable by writing in it the form of several products whose sum can be found without any calculation. For example,

$$\begin{array}{r} 1{,}122{,}334{,}455{,}667{,}789 \\ 132 \\ \hline \end{array} \qquad \begin{array}{r} 1{,}122{,}334{,}455{,}667{,}789 \\ 22 \\ \hline \end{array}$$

$$\begin{array}{r} 1{,}122{,}334{,}455{,}667{,}789 \\ 242 \\ \hline \end{array}$$

Since $132 + 22 + 242 = 396$, or 4×99, we have 444,444,444,444,444,444.

The number 37 used as a multiplicand also gives interesting results, as is seen from the following:

$$\begin{array}{rcl} 3 \times 37 = 111 & \quad & 1 + 1 + 1 = 3 \\ 6 \times 37 = 222 & & 2 + 2 + 2 = 6 \\ 9 \times 37 = 333 & & 3 + 3 + 3 = 9 \\ 12 \times 37 = 444 & & \text{etc.} \\ \cdot \quad \cdot \quad \cdot & & \\ 27 \times 37 = 999 & & \end{array}$$

The multiplier is always a factor of three and this factor determines the digit which is to appear in the product. For example,

$$15 \times 37 = 3 \times 5 \times 37 = 555.$$

From the above we also note that the sum of the digits in the product is equal to the number by which 37 was multiplied, as in the case of $18 \times 37 = 666$, where $6 + 6 + 6 = 18$.

The numbers 15,873 and 7 form another combination of this type, as appears from the following:

$$7 \times 15{,}873 = 111{,}111$$
$$7 \times 31{,}746 = 222{,}222$$
$$7 \times 47{,}619 = 333{,}333$$
etc.

The explanation is simple, for

$$7 \times 15{,}873 = 111{,}111,$$

whence $7 \times 31{,}746 = 2 \times 7 \times 15{,}873 = 222{,}222,$

and $7 \times 142{,}857 = 9 \times 7 \times 15{,}873 = 999{,}999.$

If we wish 8 to appear in the answer, the multiplicand will be 126,984, or $8 \times 15{,}873$.

Lucas, in his "L'Arithmétique Amusante," gives the following without explanation:

$$91 \times 1221 = 111{,}111.$$
$$900{,}991 \times 123{,}321 = 111{,}111{,}111{,}111.$$

A little juggling with these figures will reveal that this is simply a modification of the preceding case, for we see at once that $91 \times 1221 = 7 \times 13 \times \frac{15{,}873}{13}$.

Another interesting set of combinations consists of those which produce identical groups of digits. The numbers 143 and 7 belong to this set of combinations. The products obtained by multiplying 143 by any one of the 999 first multiples of 7 consist of two identical sets of numbers each equal to the number by which 7 has been multiplied. For example,

$$2464 \times 143 = 7 \times 352 \times 143 = 352{,}352,$$

the reason for which appears from the fact that

$$7 \times 352 \times 143 = 352 \times 1001 = 352 \times 1000 + 352.$$

The repetend of $\frac{1}{81}$ (12,345,679) multiplied by 3 or any multiple of 3 which is not a multiple of 9 also gives a repeating product. For example,

$$\begin{array}{r} 12{,}345{,}679 \\ 3 \\ \hline 37{,}037{,}037 \end{array} \qquad \begin{array}{r} 12{,}345{,}679 \\ 6 \\ \hline 74{,}074{,}074 \end{array}$$

$$\begin{array}{r} 12{,}345{,}679 \\ 66 \\ \hline 814{,}814{,}814 \end{array} \qquad \begin{array}{r} 12{,}345{,}679 \\ 15 \\ \hline 185{,}185{,}185 \end{array}$$

The same repeating products occur when we use 1122334455667789; 111222333444555666777889; etc., as multiplicands and 33, 333, etc., as multipliers. We also obtain repeating products by using these multiplicands and the multipliers 3, 6, 9, etc., as in the previous case. For example,

$$\begin{array}{r} 1122334455667789 \\ 33 \\ \hline 37037037037037037 \end{array}$$

$$\begin{array}{r} 1122334455667789 \\ 3 \\ \hline 3367003367003367 \end{array}$$

$$\begin{array}{r} 111222333444555666777889 \\ 3 \\ \hline 333667000333667000333667 \end{array}$$

$$\begin{array}{r} 111222333444555666777889 \\ 333 \\ \hline 37037037037037037037037037037 \end{array}$$

Quite remarkable results in lightning multiplication can be produced by using numbers consisting entirely of 9's. For example, to multiply 99,999 by 99,999 (to multiply by itself a number composed entirely of 9's), write the digit 8, to the left of it write as many 9's as there are digits, less one, in either factor, and to the right of it the same number of 0's, and for the last digit of the resulting number write 1. This rule applied to the number given above (99,999) gives the product 9,999,800,001.

A more general form of this problem is that of multiplying a number composed entirely of 9's by a number composed of like digits other than 9, for example, 666 × 999. In this case we can apply the following rule: First multiply one of the digits of the multiplicand by one of the digits in the multiplier. The digit in units' place of this product will be the digit in units' place of the answer. To write the answer, write the digit in tens' place of the first product; to the left of this write the digit of the multiplier as many times as there are digits, less one, in each factor, and to the right of it write the digit which represents the difference between 9 and the digit which appears in the other factor, this being then repeated the same number of times as the digit to the left. At the extreme right, write the units' digit of the first product obtained above. For example, in the case of 666 × 999, the first product will be 6 × 9 = 54. We therefore write 5 (tens' digit), and to the left of it we write 6, the digit of the multiplier, as many times, less one, as there are digits in each factor, that is, twice; and to the right the digit 3, the difference between 9 and 6, the same number of times. We then have 66,533. To complete this number, we place at the extreme right the digit 4, units' digit of the first product (54), and the result becomes 665,334. As a further example,

$$3{,}333{,}333 \times 9{,}999{,}999 = 33{,}333{,}326{,}666{,}667.$$

The first case given above (99,999 × 99,999) is a particular case of the second, the difference between the digits being zero.

A very interesting trick in lightning multiplication consists in finding the sum of two products without performing any actual multiplication. Let some one write at random a number consisting of any number of digits. Using this number as the multiplicand in each case, write two multipliers. The first multiplier may be suggested by the class; the second will be derived from the first by taking the complements of its digits with respect to 9. For example, given

3456	3456
286	9713

Here we can tell without multiplying that the sum of the two products is 34,556,544. The trick is in noticing that the answer consists of the multiplicand minus 1 (in this case 3455) followed by the respective complements of these digits from left to right, 6544. The problem appears quite miraculous at first sight. A little reflection will show that this trick is merely an application of the rule previously used for multiplying by a number composed entirely of 9's — in this case 9999. There should be as many digits in the multiplier as in the multiplicand. If the first multiplier suggested by the class has less digits than the multiplicand, consider the missing digits zero, of which the complements will therefore be 9. Since the same number of digits appears in the multiplicand and multiplier, no 9's will appear in the product.

This problem may be made to appear even more remarkable by using three or more products. For example,

9827	9827	9827
2310	4235	3454

Here we can tell, without multiplying, that the sum of the three products is 98,260,173.

Similarly, in the case of

3567	3567	3567	3567
2412	4342	2015	1230

we can see that the sum of the four products is 35,666,433, because the sum of the multipliers is 9999, as in the examples on page 121.

There are several curious tables which never fail to interest the pupils. The following, involving multiplication by 8, may profitably be introduced at some time when this multiplication table is being reviewed in the upper primary classes:

$$1 \times 8 + 1 = 9$$
$$12 \times 8 + 2 = 98$$
$$123 \times 8 + 3 = 987$$
$$1234 \times 8 + 4 = 9876$$
$$12345 \times 8 + 5 = 98765$$
$$123456 \times 8 + 6 = 987654$$
$$1234567 \times 8 + 7 = 9876543$$
$$12345678 \times 8 + 8 = 98765432$$
$$123456789 \times 8 + 9 = 987654321$$

Multiplication by 9 produces the following tables, characterized by their simplicity and symmetry.

$$1 \times 9 + 2 = 11$$
$$12 \times 9 + 3 = 111$$
$$123 \times 9 + 4 = 1111$$
$$1234 \times 9 + 5 = 11111$$
$$12345 \times 9 + 6 = 111111$$
$$123456 \times 9 + 7 = 1111111$$
$$1234567 \times 9 + 8 = 11111111$$
$$12345678 \times 9 + 9 = 111111111$$
$$123456789 \times 9 + 10 = 1111111111$$

In the preceding table the digits appear in increasing order; in the following they appear in decreasing order:

$$9 \times 9 + 7 = 88$$
$$98 \times 9 + 6 = 888$$
$$987 \times 9 + 5 = 8888$$
$$9876 \times 9 + 4 = 88888$$
$$98765 \times 9 + 3 = 888888$$
$$987654 \times 9 + 2 = 8888888$$
$$9876543 \times 9 + 1 = 88888888$$
$$98765432 \times 9 + 0 = 888888888$$

A very curious table results from the multiplication of numbers composed entirely of 1's. The following is a mechanical rule for the formation of such products. To multiply a number composed entirely of 1's by itself, write the number which represents the sum of the digits in one factor (which, in order that the rule shall hold, must be less than 10), and symmetrically to the left and right of it write the digits less than that one, in natural decreasing order. For example, to multiply 11111 by 11111, write 5, the number of digits in either factor, and symmetrically to the right and left of it write the natural decreasing order of digits, that is, 4, 3, 2, and 1, which gives the product 123454321.

In the above manner form the following table:

$$1 \times 1 = 1$$
$$11 \times 11 = 121$$
$$111 \times 111 = 12321$$
$$1111 \times 1111 = 1234321$$
$$11111 \times 11111 = 123454321$$
$$111111 \times 111111 = 12345654321$$
$$1111111 \times 1111111 = 1234567654321$$
$$11111111 \times 11111111 = 123456787654321$$
$$111111111 \times 111111111 = 12345678987654321$$

The following tables are also interesting:

$$
\begin{aligned}
7 \times 7 &= 49 \\
67 \times 67 &= 4489 \\
667 \times 667 &= 444889 \\
6667 \times 6667 &= 44448889 \\
66667 \times 66667 &= 4444488889 \\
666667 \times 666667 &= 444444888889
\end{aligned}
$$

and so on, as far as we wish to go;

$$
\begin{aligned}
4 \times 4 &= 16 \\
34 \times 34 &= 1156 \\
334 \times 334 &= 111556 \\
3334 \times 3334 &= 11115556 \\
33334 \times 33334 &= 1111155556 \\
333334 \times 333334 &= 111111555556
\end{aligned}
$$

and so on, indefinitely.

The rule for multiplying by a number composed entirely of 9's gives the following:

$$
\begin{aligned}
9 \times 9 &= 81 \\
99 \times 99 &= 9801 \\
999 \times 999 &= 998001 \\
9999 \times 9999 &= 99980001 \\
99999 \times 99999 &= 9999800001
\end{aligned}
$$

Many tables can be formed by applying the rule for multiplying a number composed entirely of 9's by a number composed of identical digits. The following is an example of a table formed in this way:

$$
\begin{aligned}
7 \times 9 &= 63 \\
77 \times 99 &= 7623 \\
777 \times 999 &= 776223 \\
7777 \times 9999 &= 77762223 \\
77777 \times 99999 &= 7777622223
\end{aligned}
$$

And finally, the following case is a simple and interesting one:

$$
\begin{aligned}
1 \times 8 + 1 &= 9 \\
12 \times 8 + 2 &= 98 \\
\end{aligned}
$$

$$
\begin{aligned}
1 \times 8 + 1 &= 9 \\
11 \times 8 + 11 &= 99 \\
111 \times 8 + 111 &= 999 \\
1111 \times 8 + 1111 &= 9999 \\
11111 \times 8 + 11111 &= 99999 \\
111111 \times 8 + 111111 &= 999999 \\
1111111 \times 8 + 1111111 &= 9999999 \\
11111111 \times 8 + 11111111 &= 99999999 \\
111111111 \times 8 + 111111111 &= 999999999
\end{aligned}
$$

Such are a few of the many curiosities of numbers that can be drawn upon occasionally to add a little more interest to the work of the classroom. Although teachers will not usually have access to all of the works that treat of this phase of number teaching, it is thought best to add a somewhat extended bibliography so that those who may care to make a study of the subject can know something of the extensive literature that is available. It need hardly be said that teachers should make only occasional use of the number tricks here given, but the introduction of a little work of this kind from time to time breaks the monotony of the book work and adds interest to the subject.

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CHAPTER XV

WORK OF THE FIRST SCHOOL YEAR

The first question that naturally arises in connection with the arithmetic of the first grade is as to whether or not the subject has any place there at all. For several years past there has been in this country a propaganda in favor of excluding it as a topic from the first grade and even from the second. Like all such efforts, the history of which is not generally known, the very novelty of the suggestion, to many teachers, is sufficient to create a following. It is well to consider briefly the reasons for and against such a suggestion, and to try to weigh these reasons fairly before attempting any decision.

In favor of having no arithmetic as such in the first grade it is argued that the spirit of the kindergarten should extend farther, perhaps even through all of the primary grades; that number work should come in wherever there is need for it, all learning being made attractive and natural, and education appearing to the child as a unit instead of being made up of scattered fragments. Such a theory has much to commend it, not only in the primary school but everywhere else. Opposed to it is the rather widespread idea that most kindergarten work is superficial in aim and unfortunate in result; that children who have had this training are wanting in even the little seriousness of purpose that they should have, that they have no power of application, that they have been "coddled" mentally into a state that requires constant amusement as the condition for doing anything. The dispassionate onlooker in this old controversy probably feels that there is truth in both lines of argument, and that mutual good has been the result. Ancient education was a dreary thing, and to the spirit of the kindergarten, although not to extreme Fröbelism, we are indebted for the brighter spirit of the modern school. On the other hand, to make children self-reliant, independent in thinking, conscious of working for something definite, demands more seriousness of purpose than seems to pervade the ordinary kindergarten.

Now as to arithmetic in the first grade,— shall we leave it to the ordinary teacher to bring in incidentally such number work as she wishes, or shall we lay down a definite amount of work to be accomplished and assign a certain amount of time to it? And in answering these questions are we bearing in mind the average primary teacher throughout the whole country? Are we also bearing in mind that arithmetic was not taught to children just entering school until about a century ago, and that it was largely due to Pestalozzi's influence that the subject was ever placed in the first grade? When, therefore, we advocate having no arithmetic in the first grade, we are going back a hundred years or so, which may be all right, but which is not a new proposition by any means.

Having thus laid a foundation for an answer to the question, it is proper to proceed dogmatically, leaving the final reply to the reader. Not to put arithmetic as a topic in the first grade is to make sure that it will not be seriously or systematically taught there in nine tenths of the schools of the country. The average teacher, not in the cities merely but throughout the country generally, will simply touch upon it in the most perfunctory way. Whatever of scientific statistics we have show that this is true, and that children so taught are not, when they enter the intermediate grades, as well prepared in arithmetic as those who have studied the subject as a topic from the first grade on.

Furthermore, while it is true that the essential part of arithmetic can be taught in about three years, it cannot, for psychological reasons, be as well retained if taught for only a short period. The individual needs prolonged experience with number facts to impress them thoroughly on the mind. We can, for example, teach the metric system in an hour to any one of fair intelligence, but for him to retain it requires long experience in its use.

But more important than all else is the consideration of the child's tastes and needs. Has he such a taste for number as shows him mentally capable of studying the subject at the age of six, and are his needs such as to make it advisable for him to do so? There can be no doubt as to the answer. He takes as much delight in counting and in other simple number work in the first grade as in anything else that the school brings to him, and he makes quite as much use of it in his games, his "playing store," his simple purchases, his reading, and his understanding of the conversation of the home and the playground, as he does of anything else he learns. If we could be certain that in the incidental teaching that is so often advocated he would have these tastes and needs fully satisfied, then arithmetic as a topic might be omitted from the first or any other grade; but since we are pretty sure that this will not be accomplished in the average school, then it is our duty to advocate a definite allotment of time and of work to the subject in every grade from the first through the sixth, and probably through the seventh or eighth.

This being so, what should this allotment of work be? Of course there is no general answer for the whole country. In some schools there are many foreign-born pupils who are unable to speak English when they enter, and therefore the first year's work must be devoted largely to acquiring the language. In other schools the children come from homes where they have already been taught by governesses and are considerably in advance of the average. In general, however, the course here laid down may be considered a fair average for the ordinary American school. In some states, notably New York, the work in the first two school years is unusually strong, resembling that found in the best European courses. In others it is the extreme of what is often called "soft pedagogy," so arranged because of the desire to relieve the child of as much of the new burden of the school as possible. Occasionally the statement is publicly made by school authorities that they propose to find out what is traditional in the course of study and change it because of this characteristic — a rule that would quickly play havoc with our social life if it were made general.

The New York State course, recently adopted, while undoubtedly open to adverse criticism in certain details, has at any rate the merit of thoroughness, and hence a brief summary is given of the work recommended for the first year.

"In the first half year count the numbers to 100. Read the numbers to 100. Write the numbers to 100.

"Memorize the 20 of the 45 combinations in addition, the sum of which does not exceed 9.

"Give plenty of oral drill together with seat work and blackboard work like the following:

2	5	3	4	2	7	2	1	2	3	2	3	7	5	1	2	6
4	2	2	4	4	1	6	8	5	2	3	4	2	3	7	3	2

and have the pupils get correct results by copying, where necessary, the results from the combinations placed upon the blackboard. This work, together with oral drill and tests, will in a short time fix these combinations in mind without the use of objects and the consequent formation of the pernicious habit of counting the fingers in adding.

"From the first, drill in these combinations should be given in such manner as to prepare for subtraction as well as addition. For example, the following development may be given:

Teacher. Five and four?

Pupil. Nine.

Teacher. Five and what are nine?

Pupil. Five and four are nine.

Teacher. Four and what are nine?

Pupil. Four and five are nine.

"When the combinations are learned in this manner, the work in subtraction, if taught by the addition method (sometimes called the Austrian method), is learned at the same time as addition. For example, in the case here shown, we may think as follows: '6 and 3 are 9; 1 and 2 are 3; 3 and 3 are 6; 2 and 5 are 7; 4 and 4 are 8.'

8	7	6	3	9
−4	2	3	1	6

"At the close of the first half year the pupils should be able to count, read, and write numbers to 100. They should know the 20 combinations, the sum of which does not exceed 9, and their use in addition.

"Objects should be used only for consecutive counting and for developing the idea of number in the abstract. Children should never be taught to count two groups of objects to find the sum. This gives the idea that addition is counting, which is a serious hindrance to accuracy and rapidity in work. Nothing has done more injury in number work than the too long-continued and injudicious use of objects in its teaching.

"In the second half year there should be continued drill in addition and subtraction with the 20 combinations learned in the first half.

"Count to 100 by twos, by fives, by tens.

"Drill in adding columns of figures on the blackboard; also arrange cards for the purpose of learning the combinations, the sum not to exceed nine.

"Memorize the remaining 25 combinations in addition.

"Give oral drill and plenty of seat and blackboard work, with examples like the following:

6	8	4	5	3	2	4	8	6	9	3	4	5
4	3	8	6	9	7	8	1	2	4	5	8	8

"Children are here taught to carry in addition.

"As to method, teach the very best model; insist upon its exact imitation and give much repetition and drill. Make no attempt to explain the process. This is the time to teach the art of computation, not the science of numbers."

Lest it should be felt that this is an extreme of formal arithmetic, and that modern ideas make for the mere incidental treatment of number in the first grade, it is well that we look beyond the narrow confines of the school of education and see what the rest of the world is doing. The New York course covers about what is found in Switzerland, France, Italy, Sweden, Belgium, Finland, Germany, and Japan. Counting to 100 and the operations in the number space from 1 to 20 is the general requirement in these countries, and doubtless in others, the reports of which are not at hand as this chapter is being written. World experience, therefore, does not show that the child is being unduly burdened by having some definite number work to do in the first school year.

The leading mathematical feature of the first school year, wherever serious work is undertaken, is the introduction to the addition table, this being at the same time the simplest and the most important table in arithmetic. It is not advisable to use the conventional form of textbook in this year, on account of the children's inability to read the usual kind of print; but a properly designed number primer, prepared with care as to type and arrangement, will be found to systematize the work and to make for the accomplishment of something definite. Such a book, if placed in the hands of the pupils, should not be a compilation of devices for the teacher, but should contain the actual number work that a child needs to see on the printed page.

With respect to number space it has been found best, from the standpoint of mental ability and because of the needs of the children, to set a different limit to the numbers used in counting and those used in the operations. Children like and need to count numbers that are larger than those used in operations. For reading and writing numbers, therefore, they may profitably go as far as 100, meeting these numbers in the paging of books, the numbering of houses, the playing of games, and the counting of various objects. For the operations, however, it is sufficient if they go only as far as 12. Indeed, 10 would make a good limit were it not for the fact that in measuring we so often use 12 inches.

The addition tables should be learned at least as far as 10 or 12. Some prefer to go as far as $9 + 4 = 13$, but it is immaterial so long as the children know the table through 9's before the textbook is used — ordinarily at the middle or the end of Grade II. Appropriate combinations for the first year may therefore be taken as follows:

$$\begin{array}{ccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 \\ \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} & \underline{1} \\ 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \qquad \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} & \underline{2} \\ 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

$$\begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \underline{3} & \underline{3} & \underline{3} & \underline{3} & \underline{3} & \underline{3} & \underline{3} \\ 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \qquad \begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & 6 \\ \underline{4} & \underline{4} & \underline{4} & \underline{4} & \underline{4} & \underline{4} \\ 5 & 6 & 7 & 8 & 9 & 10 \end{array}$$

1	2	3	4	5
5	5	5	5	5
6	7	8	9	10

1	2	3	4
6	6	6	6
7	8	9	10

1	2	3
7	7	7
8	9	10

1	2
8	8
9	10

1
9
10

This arrangement makes the sum the basis for selection. Many prefer, however, to proceed to master the table of 1's, 2's, 3's, and 4's, as mentioned above, thus giving the following combinations:

1	2	3	4	5	6	7	8	9	10
1	1	1	1	1	1	1	1	1	1
2	3	4	5	6	7	8	9	10	11

1	2	3	4	5	6	7	8	9	10
2	2	2	2	2	2	2	2	2	2
3	4	5	6	7	8	9	10	11	12

1	2	3	4	5	6	7	8	9	10
3	3	3	3	3	3	3	3	3	3
4	5	6	7	8	9	10	11	12	13

1	2	3	4	5	6	7	8	9	10
4	4	4	4	4	4	4	4	4	4
5	6	7	8	9	10	11	12	13	14

It is not a matter of great importance which of these two arrangements is adopted in any given school system, at least so far as we are able to judge from any scientific investigations thus far made. The great thing is that the complete table shall be known to $10 + 10$ by the end of the second year.

In each of these cases the combinations of 0 with the other numbers are omitted. This is generally done, but it is possible that something would be gained by teaching $0 + 1$, $0 + 2$, and so on, as part of the table. The experience of the world has not definitely settled this question, and it does not seem to be a matter of great importance.

Every fact learned in addition should, judging from general experience, carry with it the inverse subtraction case. That is, the question "$3 + 2$ equals what number?" should carry with it the questions "$3 +$ what number equals 5?" and "$2 +$ what number equals 5?" or, if preferred, "$5 - 3$ equals what number?" and "$5 - 2$ equals what number?"

Little attention should be given to multiplication in the first grade. The idea that $2 + 2 + 2$ may be spoken of as 3 times 2, and the incidental use of the word "times" in other simple number relations is desirable.

Since multiplication is not taken as a topic, its inverse (division) has no place, save as it appears in the fractions mentioned below.

Children so often hear about the fractions $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{3}$, that these ideas and forms may profitably be introduced at this time, although $\frac{1}{3}$ may be postponed to the next grade. The statement that half the class may go to the blackboard, the idea of $\frac{1}{4}$ of a dollar and that of $\frac{1}{3}$ of a yard, are all common in the first year. In the introduction of these ideas and symbols it is well to avoid extremes that will militate against the child's future progress, such as the extreme of the ratio method, for example. We should remember that a fraction, say $\frac{1}{2}$, is commonly used in three distinct ways, and that it is our duty to see that, little by little, all these become familiar to the child. These ways are as follows: (1) $\frac{1}{2}$ of a single object, the most natural idea of all, the breaking of an object into 2 equal parts; (2) $\frac{1}{2}$ as large, as where a 6-inch stick is $\frac{1}{2}$ as long as a foot rule — not half of it, but half as long as it is; this is essentially the ratio notion, and it is necessary to the child's stock of knowledge, but it is not necessary to make it hard by talking about ratios at this time; (3) $\frac{1}{2}$ of a group of objects, as in the case of $\frac{1}{2}$ of ten children.

Children in this grade should learn the use of actual measures. They should know that 12 in. = 1 ft., 3 ft. = 1 yd., and should employ this knowledge in making measurements. They should know the cent, 5-cent piece, dime, and the dollar as 10 dimes (or even 100 cents), and should use toy money in playing store. They should know the pint and quart, and use these in measuring water or some other convenient substance. Other terms such as pound, week, minute, mile, and gallon may be used incidentally, but they need not be learned in tables at present.

It is important to use objects freely wherever they assist in understanding number relations, but it is equally important to abandon them as soon as they have served their purpose. The continued use of any particular set of objects (blocks, disks, measures, picture cards, etc.) is tiresome and narrowing. Pestalozzi was wiser than many of his successors when he used anything that came to hand to illustrate most of his number work. To continue to use objects after they have ceased to be necessary is like always encouraging a child to ride in a baby carriage.

$$\begin{array}{r} 6 \\ +3 \\ \hline 9 \end{array} \qquad \begin{array}{r} 9 \\ -6 \\ \hline 3 \end{array} \qquad \begin{array}{r} 9 \\ -3 \\ \hline 6 \end{array}$$

It cannot be too strongly impressed upon teachers that the symbols that children should visualize are those that they will need in practical calculation. Thus it is much better to drill upon the annexed forms than upon $6 + 3 = 9$, $9 - 6 = 3$, $9 - 3 = 6$, since the latter are never used in calculation. For ease in printing and writing, symbols like $6 + 3 = 9$ have their important place, but the eye should become accustomed to the perpendicular arrangement so as to catch number combinations, as it must do when we come to actual addition.

While it is proper to begin by reading $6 + 2$ "six *and* two" and $8 - 6$ "eight *less* six," the words "plus" and "minus" should soon enter into the vocabulary of the child as part of the technical language of the subject. It is proper to call a cat a "pussy" for a while, and a horse a "pony," but the time soon comes for "cat" and "horse"—and so for the technical expressions in arithmetic.

In this grade problems of play, of the simplest home purchases, and of interesting measures should dominate. In general, for all grades, the oral problems should have a local color, relating to real things that the children know about. The building of a house near the school, the repairing of a street, the cost of school supplies — these and hundreds of similar ideas may properly suggest problems adaptable to every school year. It is the business of the textbook in the grades where it is used to furnish a large amount of suggestive written work, but it can never furnish all the oral work needed nor can it meet all local conditions.

As a specimen of the early work in this grade the following oral exercise is submitted:

1. How many inches wide is the window pane? How many inches high?

2. How many feet long is your desk, and how many inches over?

3. How many feet and inches from the floor to the bottom of the blackboard?

4. Stepping as you usually do in walking, find how many paces in the length of the room.

5. How many paces wide do you think the room is? Pace the width and see if you are right.

6. How tall do you think you are? Measure. How many feet, and how many inches over?

7. How many inches from the lower left-hand corner of this page to the upper right-hand corner?

8. How wide do you think the door is? Measure. How many feet, and how many inches over?

Such problems suggest measurements of genuine interest to the pupil, relating as they do to his immediate surroundings. They allow for the actual handling of the measures and the forming of reasonably accurate judgments concerning distances.

It is a serious error to neglect abstract drill work in arithmetic, even in the first year. So far as scientific investigations have shown, pupils who have been trained chiefly in concrete problems to the exclusion of the abstract are not so well prepared as those in whose training these two phases of arithmetic are fairly balanced. Abstract work is quite as interesting as concrete; it is a game, and all the joy of the game element in education may be made to surround it. At the same time it is the most practical part of arithmetic, since most of the numerical problems we meet in life are simplicity itself so far as the reasoning goes; they offer difficulties only in the mechanical calculations involved, and constantly suggest to us our slowness and inaccuracy in the abstract work of adding, multiplying, and the like. In the first grade this work is largely but not wholly oral.

It is expected that children in this grade will become familiar with the names of the common solids and polygons needed in their work. For example, the square, rectangle, triangle, oblong, cube, sphere, cylinder, pyramid, prism, and similar forms should be handled and their names should be known. Paper cutting and folding is very helpful in the study of plane figures and in the work with fractions, although, like any other device, it may be carried to an extreme that is to be avoided as a waste of time.

Even in the first grade, and still more in the succeeding years, a time limit should be set on all number work. The children should see how many questions they can answer individually, or as a class, or as half of the class, in a minute or in some other period of time. Unless this is done, or some similar plan is adopted, the tendency to dawdle over the work will begin to crystallize into a habit, and computation will take much more time than necessary. It is also to be observed that, always within reasonable limits, rapid calculation contains less errors than very slow work. The reason is apparent; we concentrate our attention more completely, and other thoughts do not take our minds from the numerical work.

Bibliography. Smith, Handbook to Arithmetics, p. 19; McMurry, C. A., Special Method in Arithmetic. On paper folding consult Sundara Row's work, Geometric Paper Folding (Open Court Publishing Company), illustrated by photographs taken by the author of the present work a few years ago. This work is suggestive, although not adapted to grade work. Consult also Wentworth-Smith, Work and Play with Numbers, Boston, 1912. On all of this work in the grades consult Mathematics in the Elementary Schools of the United States, *Bulletin No. 13* (1911), of the United States Bureau of Education.

CHAPTER XVI

WORK OF THE SECOND SCHOOL YEAR

Whether or not arithmetic has a definite time allotment in the first grade, it usually has one in the second, although some teachers oppose it even there. The argument already advanced holds the more strongly for this grade, especially as, in many schools, the child is quite prepared to use a textbook of the ordinary kind by the middle of the year.

Since the New York State course has been already mentioned as typical of the curricula of the country in which the mathematical element is prominent, the work of the second grade as there laid down will be of interest.

"In the first half year there should be continued drill in the use of the forty-five combinations that enter into addition and subtraction. There should also be drill on series work in addition as here shown. This should be followed by drill in counting by 2's, 3's, 4's, 5's. For example, count to 50 by 2's, beginning with 0, beginning with 1; count to 50 by 3's, beginning with 0, beginning with 1, beginning with 2; count to 50 by 4's, beginning with 0, beginning with 1, beginning with 2, beginning with 3, etc.

2	12	22	32	42	52
5	5	5	5	5	5

"Drill in subtraction by the adding or Austrian method. For example, to take 786 from 1235 proceed as follows:

1235	6 and 9 are 15.
– 786	9 and 4 are 13.
449	8 and 4 are 12.

"The entire time of this half year should be given to the use of and drill upon the facts of number learned as here specified. By the close of the period simple numbers will be added and subtracted with accuracy and facility and much progress will be made in the addition of columns of figures.

"In the second half year there should be notation and numeration of numbers through the first three periods.

"Continued drill in addition and subtraction, especially in the addition of columns.

"Memorize the forty-five combinations in multiplication.

"Teach these combinations so that preparation is given for division at the same time that multiplication is being taught; that is, have the pupil answer the questions, 'How many 6's in 24?' and 'How many 4's in 24?' as well as to state that 4 times 6 are 24.

"Give much oral drill, seat work, and board work in multiplication of the following character:

326453	86245	20	30	40	50	60
2	3	2	3	4	5	6

"Teach the process of carrying in multiplication.

"Let the method be: (1) a good model; (2) imitation of the model; (3) repetition, drill; (4) no explanation."

While this is rather more than is usually attempted in this country, it is about the same as the work of the second school year in Japan, Finland, Holland, Italy, and Sweden, running if anything a little below that of most of these countries. In general the French and German schools are a little ahead of this work from the mathematical standpoint, although the real advance occurs after the primary grades.

In contrast to this we may consider one of the recent city courses of our country. This was prepared for the schools of Indianapolis and represents one of the best attempts of its kind. Less mathematics is given than world experience would suggest, but on the other hand America has a problem that other countries do not have, namely, the teaching of English to the children of immigrants. This necessarily lowers our standard of scholarship in the primary grades.

An outline of the course is as follows:

"All the work of this grade should be objective.

"In the first half year, counting by 10's, 5's, and 1's to 100; relative values of numbers within 100.

"Numbers through 20: counting by 2's, 4's, and 5's; counting backward by 1's. Also $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{3}$ of numbers from 1 to 20 inclusive, which give an integer as result.

"Measures used: inch and foot.

"In the second half year addition and subtraction of numbers through 14; multiplication and division through 20; writing numbers through 100; $\frac{1}{2}$, $\frac{1}{4}$, and $\frac{1}{3}$ of numbers from 1 to 20, giving an integer as result. Addition of single columns, using the combinations learned.

"Measures used: inch, foot, square inch, square foot, pint, quart.

"Review very thoroughly the work of the first half year. Teach addition and subtraction of numbers through 14; multiplication and division through 20. Teach signs for all processes, using the terms 'and,' 'less,' 'times,' 'contains,' and 'equals.'

"Simple concrete problems should be given, using the combinations learned. The order of presentation should be: (1) objects, in learning combinations; (2) concrete examples (oral) with objects present; (3) representation of combinations by figures; (4) recalling combinations without objects — thorough memorizing of results; (5) concrete problems without the use of objects. These problems should be very simple and based upon the child's experience. Very little written work should be required, and the result should be stated in a sentence without any attempt to show the process. Readiness in finding the result of the combination of any two numbers is to be considered of first importance in this grade, because this knowledge must be at command in all succeeding stages of advancement. The addition of short columns, using only the combinations learned, gives additional practice; it is an excellent method for review."

In general it may be said that in schools of average advancement, where the question of language is not as serious as in some cities in the East, children in this grade may be expected to complete the addition tables and to learn the multiplication tables to 10×5.

Children will now take an interest in counting to 1000, first by units to 10, then by 10's to 100, then completely to 100, then by 100's to 1000, and finally completely to 1000. The operations may also be anywhere within this space, although, of course, most of the results will involve only small numbers. In the Roman notation the limit may be set at XII, this sufficing for the reading of time and for the chapter numbers of books.

Without going to an extreme in counting by various numbers where no definite purpose is served, there is a field in which counting is very advantageous. To count by 2's from 2 to 10 and from 1 to 11 has the pleasure of any rhythmic sequence and at the same time gives the addition table of 2's; and the counting by 2's from 2 to 20 gives the corresponding multiplication table. Similarly, counting by 3's from 3 to 30 gives the multiplication table of 3's, while the further counting from 1 and 2 to 13 and 14 gives the different addition combinations. The exercise is interesting to children, and the knowledge secured in this way is more than one would at first think.

The tables should be completed during this year, including the sums of any two one-figure numbers. There are only 45 possible combinations of numbers below 10, namely, $1+1$, $1+2$, and so on to $1+9$; $2+2$, $2+4$, and so on to $2+9$; $3+3$, $3+4$, and so on to $3+9$; and similarly for the others to $9+9$, besides the zero combinations referred to earlier in this work. It is better, however, to continue the sums to include 10 — a simple matter but one that is often helpful. The addition of numbers of two and even of three figures each may be taken during this year.

Subtraction may be carried far enough to include numbers of three figures each. The method to be employed has already been discussed in Chapter X. In both addition and subtraction there should be an effort to cultivate the habit of rapidity, although never to the exclusion of accuracy. The time limit on work, mentioned on page 139, should be employed in all written work. In general in both addition and subtraction the full form should be employed until it is thoroughly understood. For example, in adding 247, 376, and 85, a problem that must have been preceded by many simpler ones, it is well to use the first of the following forms until the reasons are understood, and then to adopt the second, dismissing thereafter the question of reason and dwelling upon speed and accuracy of operation.

$$\begin{array}{r} 247 \\ 376 \\ 85 \\ \hline 18 \\ 190 \\ 500 \\ \hline \end{array} \qquad\qquad \begin{array}{r} 247 \\ 376 \\ 85 \\ \hline 708 \end{array}$$

It should be remembered that it is mechanical efficiency rather than reasoning that we seek in work of this nature.

Likewise, if the addition or Austrian method is taken for subtraction, it is better to begin a problem like $852 - 476$ in the full form, as follows:

$$852 = 800 + 50 + 2$$
$$476 = 400 + 70 + 6$$

The difference between these numbers is the same if we add 10 to each, and also 100 to each, and we add them as follows, so that we can easily subtract the numbers in each order:

$$\begin{array}{r} 800 + 150 + 12 \\ 500 + 80 + 6 \\ \hline 300 + 70 + 6 \end{array} = 376$$

After this is understood we may proceed to the ordinary arrangement.

The multiplication tables may be learned this year as far as 10×5. Some schools, as we have already seen, go even as far as 10×10, and others find it better to postpone all of this work until the third grade. Products should be learned both ways, that is, 5×6 and 6×5. There is a great advantage in reciting all tables aloud, and even in chorus, since this leads to a tongue and ear memory that powerfully aids the eye memory when the pupil needs to recall a number fact. Counting enables the tables to be developed in a rhythmic fashion that is pleasing to the ear, and shows multiplication by integers to be merely an abridged addition, that is, that $3 + 3 + 3 + 3$ is more briefly stated as 4×3. The one danger to be avoided in basing the tables upon counting is that the child may tend to count up every time he wishes a certain product. But we must recognize that counting by 2's is only an interesting way of discovering the relation that $5 \times 2 = 10$, and that the relation must then be memorized and made the object of sufficient drill to fix it distinctly in mind.

The multiplication table should carry with it the division table. This need not be developed as a separate feature, but may be treated as the inverse of the multiplication table, exactly as subtraction is the inverse of addition. The fact that $4 \times 6 = 24$ should bring out the second direct fact that $6 \times 4 = 24$, and the two inverses, $24 \div 6 = 4$, and $24 \div 4 = 6$. These inverses may be introduced in a way that is analogous to that followed in subtraction. That is to say, after learning that $4 + 5 = 9$ we ask, "What number added to 5 equals 9?" "What number added to 4 makes 9?" Similarly, after $4 \times 5 = 20$ we ask, "What number multiplied by 4 equals 20?" "What number multiplied by 5 equals 20?" The answers may then be expressed as $20 \div 4 = 5$, $20 \div 5 = 4$.

In division in this grade we also have an illustration of the fact that the full form may profitably precede the short one. A child more easily grasps the idea of $36 \div 3$ if he sees the first of these forms before he comes to use the second:

$$3\overline{)30 + 6} \qquad\qquad 3\overline{)36}$$
$$\quad 10 + 2 \qquad\qquad\quad 12$$

In the same way, when he comes to divide 36 by 2, it is better to begin with the first of the following forms:

$$2\overline{)20 + 16} \qquad\qquad 2\overline{)36}$$
$$\quad 10 + \ \ 8 \qquad\qquad\quad 18$$

The longer form need be used only occasionally, and by the teacher. The pupil should see from it the reason for proceeding as he does, but thereafter the reason should give way to the mechanics of the operation.

Teachers will find it better to write the quotient below the dividend in short division, even though it is preferably written above in the long process. There is no advantage in trying to change the habit of the world on such a small matter, particularly as there are mathematical advantages in running the division downward instead of upward.

Children know the meaning of $\frac{1}{2}$, $\frac{1}{4}$, and often of $\frac{1}{3}$, on entering this grade. If $\frac{1}{3}$ is not known it should be introduced, and $\frac{1}{8}$, $\frac{1}{6}$, $\frac{1}{5}$ may also be added to the list at this time, although many successful teachers prefer to postpone them to Grade III. The use of objective work is imperative, and it is better to take various simple materials than to confine oneself to elaborate fraction disks or other similar devices. Every school has cubes to work with, and the use of cubes, paper folding, paper cutting, and the common measures is recommended as quite sufficient.

The denominations already learned in Grade I should be frequently used, and to them should be added the relation between the ounce and pound; the pint, quart, and gallon; the quart, peck, and bushel; the reading of time by the clock, and the current dates. The idea of square measure (in square inches) is usually introduced. All of this work should be done with the measures actually in hand so far as this is possible. A table of denominate numbers means very little unless accompanied by the real measures. This will be felt by any American grade teacher who teaches the metric system without the measures, and who tries to think of her weight in kilos, her height in centimeters, and the distance to her home in kilometers.

It has already been said that symbols like $+$, $-$, $\times$, and $\div$ were invented for algebra and have only recently found place as symbols of operation in arithmetic.[1] The desire to employ them has led many teachers to use long chains of operations that are never seen in practical life, and which, while serving

[1] It is true that + and − were first used in Widman's arithmetic of 1489, but not as symbols of operation. See Smith's "Rara Arithmetica," Boston, 1909.

some purpose in oral work, are vicious as written exercises. For example, $2 + 4 \div 2 + 5 \times 6 \div 3 + 3$ is a kind of work that should never appear in the grades. Arithmetically it is easy enough, and the answer is 17, but there is no use in puzzling a child to remember which signs have the preference in such a chain. This is a small technicality of algebra, of which the importance is much overrated even there, and it has no place in the elementary school. With respect to the symbols $2 \times \$3$ and $\$3 \times 2$ there is, however, a reasonable question, since there is good authority for each. Modern usage favors the former because we more naturally say "2 times 3 dollars" than "3 dollars multiplied by 2," and it is better to read from left to right, as in an ordinary sentence. It should be repeated, however, that the forms which the child needs to visualize are not these, but the one he will meet in actual computation, as here shown. The question is, therefore, one of no practical importance.

$$\begin{array}{r}\$3\\ 2\\ \hline \$6\end{array}$$

It is here repeated, as essential to a discussion of the work of the second grade, that objects are necessary in developing certain number relations, but that they should be discarded as soon as the result is attained. Number facts must be memorized by every one, and objects may become harmful if used too often.

The question of the nature of the problems begins to assume considerable importance in this grade, and it has been already discussed in Chapter IV. It may be said in general, however, that several of our recent American arithmetics are making a serious effort to improve the applications of the subject, adapting them to the mental powers and to the environment of the pupils instead of offering obsolete material of no practical value and of little interest.

It is proper at this time to call the attention of teachers to the matter of reviews — not those that naturally occur from time to time during the year, but those that should deeply concern every school at the close of one year and at the opening of the next. Any one who has ever had much to do with the supervision of the work in arithmetic is struck by the general complaint that children are never prepared to enter any particular grade. Every teacher seems to feel that the preceding teacher has imposed a poorly equipped lot of children upon her own grade and that her problem is therefore hopeless. Now if this were only an occasional complaint, the supervisor might well be worried, but he soon recognizes it as part of the tradition of the school, and pays little attention to it accordingly. What does it mean, however, and how should we remedy the evil if evil there be?

If any teacher will herself learn, let us say, the logarithms of the first fifty integers, between September and February, how many will she know in June? And if she knows them all in June, how many will she remember at the end of the summer vacation? And how will she feel, say about the middle of September, if some one asks her to give the logarithm of 37 to six decimal places, telling her, if she fails, that she must have been pretty poorly taught the year before? Now this is a fair illustration of the mental position of a child with respect to the tables learned in Grade II when he enters Grade III. Psychologically it would be strange if he could rapidly and accurately give every product demanded; his brain cells have clogged up or got disarranged or gone through some similar transformation during his nine or ten weeks of careless play. What, therefore, is the teacher's duty? There are two things to do. First, at the close of each school year, in June, there should be a thorough and systematic review of those number facts and operations that are the fundamental features of the year's work. The teacher ought to be convinced that each child leaves the grade with such a mental equipment as shall leave no chance of fair criticism. Her responsibility then ceases. Second, and even more important, at the opening of each school year, in September, there should again be a thorough and systematic review, by the teacher in the next grade, of these same features. But this review should be conducted in the most sympathetic spirit. The teacher should be surprised if the children have not forgotten much rather than if they have failed to remember the facts perfectly. She should think of her own fifty logarithms, for example, and the review should be patiently and helpfully extended until the children's arithmetical brain cells resume their former state. After this has been done in the spirit mentioned, and after the teacher has gone into the next higher grade for a day to see how her own pupils of the preceding year are standing the test, then she may be justified in complaining, but not before.

It need hardly be mentioned that there are few more severe tests of the ingenuity and patience of a teacher than are found in these reviews. The "edge of interest" is already worn off in any review, and it requires all the tact a teacher possesses to maintain the enthusiasm of the pupils in such exercises. The result, however, is well worth the effort, and the school system that carries out the plan will have less of complaint and more of sympathetic coöperation than would at first be thought possible.

CHAPTER XVII

WORK OF THE THIRD SCHOOL YEAR

Since the formal textbook is usually placed in the hands of children at the opening of the third school year, it becomes particularly important to have a systematic review of the work of Grades I and II at the beginning of this year. The textbooks usually provide for this, and by their help these important things are accomplished: (1) The children's memories are refreshed as to the essential features of the preceding year's work, namely, the addition table, and the multiplication table as far as the course of study may require. (2) Children are "rounded up," brought to a certain somewhat uniform standard, so that all can begin the serious use of the textbook with approximately the same equipment. (3) The superior capacity or the defect of the individual has an opportunity to show itself early, allowing for such advancement or special attention as the case demands. In other words, the "lockstep" can be broken without the usual delay. As to further argument for this autumnal review the reader may refer back to Chapter XVI.

In this year rapid written work is an important feature. The oral has predominated until now, but in Grade III the operations involve larger numbers than before, and the child begins to acquire the habit of writing his computations. Multiplication usually extends to two-figure multipliers and long division is begun. The most useful tables of denominate numbers are completed.

It is usually considered sufficient if the child understands numbers to 10,000 in this grade, although he may be allowed to count by 10,000's to 100,000 or even farther. Indeed, as soon as he understands numbers to 1000 he rather enjoys showing his prowess by counting by 1000's and by writing large numbers. Counting always extends far beyond the needs of computation — a law that is true to-day and has been true in the historical development of all peoples. In the writing of Roman numerals there is no particular object in going beyond C in this year. It must be borne in mind that we use the Roman forms chiefly in chapter or section numbers, and less often in reading dates, so that all writing of very large numbers by this system is an obsolete practice and a waste of time. Indeed, it is not strictly a Roman system any more, so much have we changed the numerals from their early forms.

In this grade the counting of Grade II may be continued, including the 6's, 7's, 8's, 9's, and 10's, as a basis for the multiplication tables and as a review of the addition combinations. There is no need for counting beyond certain definite limits, however. Thus in counting by 2's, beginning with 0, we have 0, 2, 4, 6, 8, 10, 12, 14, 16, 18, 20. This suffices for the multiplication table of 2's, and even the last half of this is merely a repetition of the first half with 10 added.

It becomes necessary in this grade to write dollars and cents, and hence forms like $10.75, $25.10, and $32.02 are given. It is not necessary nor even desirable that the children should know any of the theory of decimal fractions at this time. The decimal point should be looked upon by them simply as separating dollars and dimes, and it will give no trouble unless the teacher confuses the class through the ever-present danger of overexplaining.

It is usual in Grade III to introduce incidentally such simple geometric forms as the triangle, rectangle, cylinder, and sphere. Formal definitions are, however, undesirable.

The chief thing is that the child should use the names correctly. Some little paper folding may well be introduced as a basis for simple square measure.

The idea of area (square inches or square feet) may enter into the work of this grade, although some successful teachers prefer to introduce it in Grade IV, along with cubic measure, finishing this work in Grade V. If introduced here, it is of course treated objectively, usually with paper folding, drawing, or inch squares of cardboard. There is hardly any trouble with this work unless the teacher enlarges upon its difficulties. If there is accuracy of language, spoken and written, from the beginning, this will continue; but if the teacher allows expressions like "3 inches times 3 inches equals 9 square inches," instead of "3 times 3 square inches equals 9 square inches," there will be produced loose habits of thought and expression that will lead to great trouble.

2	3	4	5
2	3	4	5
2	3	4	5
2	3	4	5
8	12	16	20

It is still necessary in this grade to make a good deal of use of objective work in treating fractions, and to make the work largely oral during the first half year. There is also an advantage in using columns of figures like those here shown. Here it is very easy to see that $\frac{1}{2}$ of 8 is two 2's, or 4; that $\frac{1}{4}$ of 12 is 3; that $\frac{3}{4}$ of 16 is three 4's, or 12; and that $\frac{1}{2}$ of 20 is the same as $\frac{2}{4}$ of 20, or two 5's, or 10. From the second arrangement it is easy to see that 2 is $\frac{1}{2}$ of 4, $\frac{1}{3}$ of 6, $\frac{1}{4}$ of 8, and $\frac{1}{5}$ of 10; that 4 is $\frac{2}{3}$ of 6, $\frac{1}{2}$ of 8; that 6 is $\frac{3}{4}$ of 8 and $\frac{3}{5}$ of 10, and so on. Devices of this kind add both to the interest in and to a clear comprehension of the subject, and when not carried to an extreme are valuable.

			2
		2	2
	2	2	2
2	2	2	2
2	2	2	2
4	6	8	10

The 45 combinations of one-figure numbers should be reviewed, and in the first half year oral work of the types of $20 + 30$, $25 + 30$, should be given, to be followed in the second half year by cases like $25 + 32$ and $225 + 32$, where no "carrying" is involved. Written work with four-figure numbers including dollars and cents should be given, but long columns of figures should be avoided at present.

As already stated, there is an advantage in introducing any difficulty in operation by using the complete form. While, for example, the annexed problem in addition is not designed as an introduction to the addition of three-figure numbers, it illustrates what is meant by the complete form. The teacher need have no fear that children cannot easily be brought to use the abridged form; "the line of least resistance" will bring that about, while on the score of a clear understanding of the operation this complete form is superior to the other. It should also be mentioned that the pupil should at this early stage be taught to recognize his own liability to error and to do what every computer has to do, namely, to add each column twice, in opposite directions, to be sure of his result — to "check" it, as we say.

$$\begin{array}{r} 427 \\ 326 \\ 452 \\ \underline{49} \\ 24 \\ 130 \\ \underline{1100} \\ 1254 \end{array}$$

Subtraction has been sufficiently treated in Chapter X. The extent of the subject is suggested by the work in addition, and of the various methods the addition, or Austrian, seems at present to be gaining in favor, although it is by no means the most common.

Multiplication, with division, constitutes the special work of the year, addition and subtraction offering no essentially new difficulties. In the first half year it is customary to complete the tables through 10×10, or to review them carefully if they were completed in Grade II. The products must be thoroughly memorized, not merely in tabular form, but when called for in any order. The plan of carrying the tables to 12 × 12, while necessary in England on account of the monetary system used there, has generally been discarded in America, it being felt that the time required for this extra work could be better employed. In the first half year multiplication may be carried so far as to include three-figure multiplicands and one-figure multipliers, and the work may at first be arranged in the complete and later in the common abridged form as here shown. Since all such work is done in the classroom where the teacher can supervise it, there should be a time limit placed upon it, to the end that habits of rapidity as well as habits of accuracy be acquired. In this year the work may safely be extended to the two-figure multipliers, 11 and 12, in which the complete form should again precede the common abridgment, as here shown. There is also introduced in this year such multiplications as that of $2.75 by 7, thus preparing the way for decimal fractions. The latter are not, however, treated in this grade, and the work should not be made difficult by any unnecessary theorizing upon this subject.

$$\begin{array}{r}298\\ \underline{3}\\ 24 = 3 \times 8\\ 270 = 3 \times 90\\ \underline{600} = 3 \times \underline{200}\\ 894 = 3 \times 298\end{array} \qquad \begin{array}{r}298\\ \underline{3}\\ 894\end{array}$$

$$\begin{array}{l}326\\ \underline{12} = 10 + 2\\ 652 = 2 \times 326\\ \underline{3260} = 10 \times 326\\ 3912 = 12 \times 326\end{array} \qquad \begin{array}{r}326\\ \underline{12}\\ 652\\ \underline{326}\\ 3912\end{array}$$

In this year oral division by one-figure divisors is introduced for such simple cases as 484 ÷ 2, 484 ÷ 4, 481 ÷ 2, etc. Short division of numbers like 522 ÷ 6 may properly be introduced by some such form as the annexed, although many teachers find better results by not separating the dividend in this manner. In introducing the idea of remainder there

$$\begin{array}{l}6\overline{)522}\\ 6\overline{)480 + 42}\\ \overline{80 + 7 = 87}\end{array}$$

is some special advantage in doing so, however. Thus the division of 438 by 5 may be first represented in this manner:

$$\begin{array}{l} 5\overline{)40 \text{ tens} + 35 \text{ units} + 3} \\ \quad 8 \text{ tens} + \ \ 7 \text{ units}, \quad 3 \text{ remainder} \end{array}$$

Such separations of the dividend are made for the purpose of having the process seen in its simplest form, and teachers should write problems of this kind on the board often enough to make sure that the process is understood. The children should not be required to use this form, but should get to the practical work of division as soon as possible.

As a preparation for the idea of remainder there should be plenty of rapid oral drill of the following nature:

1. What is the quotient and the remainder of 25¢ ÷ 6¢?

Divide, stating the quotient and remainder:

2.	3.	4.	5.	6.	7.
$4\overline{)23}$	$8\overline{)33}$	$7\overline{)45}$	$5\overline{)52}$	$9\overline{)85}$	$7\overline{)71}$

In certain courses of study long division is taught in the third school year. In case this is done, the suggestions relating to this subject as set forth in Chapter XVIII should be considered. It is impossible to lay down a definite rule that shall guide all schools in deciding this important question. Long division, at least with divisors of two figures, is successfully taught in the third year in many schools in Europe and America. In many other schools, especially in those where the language difficulty is prominent, it may better be relegated to Grade IV. Since it can be taught in Grade III under favorable conditions, and since it is possible to make it interesting to the pupils at that time, it is probable that the tendency will be to introduce it in this grade, at least with simple divisors.

The pupil is now able to use halves, thirds, and fourths, or, if not, this work should be introduced at this time. During this year the work may safely be extended to eighths, relatively little attention being given to fifths, and still less to sevenths, these being fractions for which the child has little use at this time. Oral addition and subtraction of fractions with a common denominator should be freely given. The reduction of halves to fourths, sixths, and eighths, and of thirds to sixths, is introduced by means of objects, the objects being discarded as soon as they have served their purpose. Fractional parts of numbers of three figures or less are safely introduced here, these being selected so as to be multiples of the denominator. The idea of a fraction in the quotient, instead of a remainder in division, also has place in the work of this year, together with a brief introduction to mixed numbers.

Here, as in other grades, it is necessary to review and make frequent use of the tables of denominate numbers already learned. The table of square units is usually introduced and extended to the square yard. The gill is added to the table of liquid measure, and the table of time is completed. Modern teaching finds it advisable to introduce the units of measure only as rapidly as the child develops the need for them and can therefore understand them. In all cases it is desirable to have the measures where they can be seen or in some other way appreciated. For example, when the acre is introduced, somewhat later, a piece of land near the school, approximately an acre in size, should be shown to the class. In the same spirit the pupils should be shown a ton of hay or a ton of coal, a cord of wood where this is possible, a rod, a gill measure, and so on. It is very important that the great basal units used by our people should be visualized by the children, so that bushel, mile, ton, etc. shall not be mere words.

On page 158 there are suggested three practical sets of problems, adapted to this grade, each telling a story that may suggest other topics for original work by the class.

ORAL EXERCISE

1. The coffee for our breakfast cost 6¢, the potatoes 4¢, the meat 32¢, and the bread 4¢. How much did the bread and meat cost? How much did all the food cost?

2. The oatmeal for a breakfast cost 8¢, the milk 4¢, the fruit 10¢, the rolls and butter 5¢, and the eggs 8¢. How much did this food cost?

3. For a dinner the meat cost 30¢, the vegetables 20¢, the dessert 20¢, the coffee 15¢, and the other food 15¢. Find the total cost.

4. The meals for a small family cost $1.70 on one day and $2.20 on another day. How much did they cost for these two days?

ORAL AND WRITTEN EXERCISE

1. A train has 32 freight cars and 11 coal cars. How many cars are there in all?

2. A train has 40 men passengers and 12 women passengers. How many passengers are there in all?

3. A train of 50 cars has 12 switched off. How many cars are left?

4. From a trolley car having 33 passengers 16 passengers went out. How many passengers were left in the car?

5. A train of 4 cars had 46 passengers in the first car, 39 in the second, 48 in the third, and 25 in the fourth. How many were there in all?

6. Of four freight trains the first had 29 cars, the second 49 cars, the third 37 cars, and the fourth 58 cars. How many cars were there in all?

WRITTEN EXERCISE

The toad is one of man's best friends. One toad will keep a garden of 800 sq. ft. free from harmful insects.

1. At this rate, how many toads would protect from insects a garden 80 ft. wide and 100 ft. long?

2. The eggs of four toads were counted and found to be 7547, 9536, 7927, and 11,540 respectively. How many were there in all?

3. If one out of 50 hatched, how many hatched? (Divide all by 50.) If 715 of these were destroyed by other animals, how many survived?

4. If each of these survivors destroys insects that would cause $10 worth of damage, how much are they all worth to a village?

CHAPTER XVIII

WORK OF THE FOURTH SCHOOL YEAR

In this the last year of the primary grades it is well to feel sure that the essentials of arithmetic have all been touched upon. It is therefore desirable to review the four fundamental operations, extending the multiplication and division work to include three-figure multipliers and divisors. The common business fractions should also be included, with simple operations as far as multiplication, and possibly a brief introduction to division. If a child does not know the four operations with integers at the close of this year, he will have trouble with his arithmetic always thereafter; and if he does not know how to handle fractions, at least to eighths, with ease, excluding the case of division, he will be much hampered in his subsequent work. This is a period of habit formation in the child's life, and advantage must be taken of this fact to form habits in numerical calculation that will remain with him through life.

In this year the numbers may extend to 1,000,000. The operations, however, should be confined to the smaller numbers that naturally enter in such business matters as can be appreciated by children of this age.

The prime object of the counting exercises, the developing of the tables of addition and multiplication, has now been accomplished, except when it is desired to carry the multiplication table to 12×12. In that case the counting may now be continued by 11's to 132 and by 12's to 144. Otherwise the only use of this work is for the purpose of review.

There should be much rapid oral work of the following kind:

$$\begin{array}{r} 7 \\ +4 \\ \hline \end{array} \quad \begin{array}{r} 17 \\ +4 \\ \hline \end{array} \quad \begin{array}{r} 37 \\ +4 \\ \hline \end{array} \quad \begin{array}{r} 37 \\ +14 \\ \hline \end{array} \quad \begin{array}{r} 47 \\ +24 \\ \hline \end{array}$$

$$\begin{array}{r} 11 \\ -7 \\ \hline \end{array} \quad \begin{array}{r} 21 \\ -7 \\ \hline \end{array} \quad \begin{array}{r} 41 \\ -17 \\ \hline \end{array} \quad \begin{array}{r} 51 \\ -27 \\ \hline \end{array} \quad \begin{array}{r} 71 \\ -47 \\ \hline \end{array}$$

The written work should be undertaken with the aim of (1) accuracy, secured by always checking the result; (2) rapidity, secured by setting a time limit upon all work. Children should by no means neglect this matter of checks, since it is used in all the business world. Much of the complaint of business men, that boys from the schools are always inaccurate in arithmetic, would be obviated if pupils were required to check their additions by adding in the opposite direction, and their other results in some appropriate manner.

The essentially new feature in most schools, in this year, is long division, although this is taken up in the third grade in some states and cities, with results that seem in every way satisfactory. It is always best to begin with division by 11, and then to pass to division by 21, then by 31, and so on, the units' figure being 1. Then take 12, 22, 32, and so on. It is then found by many teachers to be advantageous to take 19 (which is nearly 20), 29 (which is nearly 30), and so on, although most teachers prefer to take 53, 63, 73, and so on, followed by numbers like 74, 84, and 75, 85, before numbers ending in 8 or 9. Oral work of the following kind will be found useful:

1. Divide 62 by 31; 62 ft. by 31 ft.; $62 by $31.

State the quotients rapidly:

2. 80 ÷ 20.	4. 80 ÷ 40.	6. 120 ÷ 60.
3. 84 ÷ 21.	5. 82 ÷ 41.	7. 122 ÷ 61.

State only the first figure in each quotient:

8. $735 \div 21$.	10. $1353 \div 41$.	12. $2684 \div 61$.
9. $961 \div 31$.	11. $1683 \div 51$.	13. $6237 \div 81$.

In all this work teachers should freely supplement the textbook if they find that they can make the work clearer and easier for the pupil.

The work in common fractions should be confined to fractions that are needed in ordinary business, and chiefly to those from $\frac{1}{2}$ to $\frac{7}{8}$. Of course there is no objection to an occasional example with denominators of two or three figures, but the day of fractions like $\frac{243}{1284}$ is past, decimal fractions having taken the place of all such forms. Children in this grade should also know the aliquot parts of \$1, as that $\$\frac{1}{2} = 50$ cents, and $\$\frac{1}{4} = 25$ cents. The operations may extend as far as easy multiplications of an integer and a fraction, two fractions, or an integer and a mixed number. Unusual forms of operation, not practical in business, should not be given, and the teacher should resist all temptation to depart from this principle on any ground of mental discipline.

A brief introduction to decimal fractions, based on the work already given in United States money, may be allowed in this grade, although the serious treatment of decimals belongs later in the course.

The tables needed in business life are completed in this grade by adding that of cubic measure and completing linear and square measure. In the work of adding and subtracting compound numbers children should feel that there is no principle involved that is not found in integers. For example, consider these three cases:

37	3 ft. 7 in.	3 lb. 7 oz.
25	2 ft. 5 in.	2 lb. 5 oz.
62	6 ft.	5 lb. 12 oz.

In the first case on the preceding page, because $7 + 5 = 12$, which is 1 ten and 2 units, the 1 ten is added to the 10's. In the second, because 7 in. + 5 in. = 12 in., or 1 ft., the 1 ft. is added to the feet. In the third, because 7 oz. + 5 oz. = 12 oz., which does not equal a pound, the 12 oz. is written under ounces. In every case the principle is the same, to add to the next order any units of that order that are found. In general we use compound numbers of only two denominations, and it is on such numbers that we should lay the emphasis. Numbers of four or five denominations are now obsolete, and there is not enough disciplinary value in the subject to warrant using them instead of the numbers of actual business.

As already mentioned, there should be an effort to have children visualize the standard measures of our country, such as the acre, mile, ton, and bushel.

Teachers should be careful at this time that slovenly methods of statement do not become habits. Such forms as the following, for example, are inexcusable:

$$60 \text{ in.} \div 12 = 5 \text{ ft.}$$
$$60 \div 12 = 5 \text{ ft.}$$
$$60 \text{ in.} \div 12 \text{ in.} = 5 \text{ ft.}$$

If we wish to reduce 60 in. to feet, we have three correct forms, any one of which is easily explained:

$$60 \times \tfrac{1}{12} \text{ ft.} = 5 \text{ ft.}$$
$$60 \text{ in.} \div 12 \text{ in.} = 5, \text{ the } \textit{number} \text{ of feet.}$$
$$60 \div 12 = 5, \text{ the } \textit{number} \text{ of feet.}$$

If slovenly forms are allowed here, they must be expected in all subsequent grades, and they must be expected to lead to slovenly thought in the treatment of all kinds of problems.

At the close of the year there should be a review of all the essential features of the work in the primary grades.

This requires skill on the part of the teacher lest it become so stupid and wearisome as to lose its chief value. Original local problems to test the children in the four fundamental operations with integers and (as far as they have gone) with fractions will usually render the work interesting and will hold the attention.

Here as elsewhere the problems should touch the children's interests and be adapted to their mental abilities. The following may be taken as types:

ORAL EXERCISE

1. Tell the cost of some kind of cloth. How much will 2 yd. cost? How much will $10\frac{1}{2}$ yd. cost?

2. Tell the cost of a pair of shoes. How much will two pairs cost?

3. If a man earns $3 for 10 hours' work, how many hours must he work to earn enough to buy his daughter a pair of shoes at $1.50?

4. How many hours must he work to earn enough to buy a $6 suit of clothes for his son?

ORAL AND WRITTEN EXERCISE

1. Fanny and Emily are buying cake and some rolls. They buy $\frac{1}{2}$ lb. of the cake at 40 cents a pound. What does it cost? What do they pay for 6 rolls at 10¢ a dozen?

2. They buy a pie for 15¢ and 2 loaves of bread at 5¢ each. What do these cost?

3. They buy 10¢ worth of doughnuts and 16¢ worth of brown bread. How much do these cost?

4. If the baker pays one of his men $2.25 a day, how much does he pay him in 6 days?

5. If he pays another man $3.15 a day, how much does he pay him in 6 days?

6. If the baker sold 36,000 cakes last year, how many cakes did he sell on an average each month? What is meant by *average?*

7. If he sells 108 pies in 6 days, at 15¢ each, how much does he receive on an average each day from the sale of pies?

WRITTEN EXERCISE

1. Sarah's mother bought $4\frac{1}{4}$ yd. of cloth for a cloak, at $1.20 a yard. What did she pay for it?

2. She also bought $3\frac{1}{2}$ yd. of lining at 50¢ a yard, and $4\frac{1}{4}$ yd. of braid at 20¢ a yard. How much did these cost?

3. She also bought 6 pearl buttons at $1.50 a dozen, and 2 spools of silk at 8¢ a spool. How much did these cost?

4. The dressmaker charged $5 for making the cloak. What did materials and making cost?

5. John's mother bought $2\frac{1}{2}$ yd. of goods for a coat, at $1.20 a yard, and $2\frac{1}{4}$ yd. of lining at 48¢ a yard. How much did these cost?

6. She also bought a dozen buttons at 25¢ a dozen, and 2 spools of silk at 8¢ a spool, and paid $3 for making. How much did the coat cost?

CHAPTER XIX

WORK OF THE FIFTH SCHOOL YEAR

There should in this year be a thorough review of the fundamental operations with integers. This should be followed by the same operations with the common fractions and denominate numbers of business. Percentage may be begun, although it is better to postpone any serious consideration of the subject until the following year.

A new textbook is usually begun in this grade, and this, if properly arranged, offers plenty of material for the review above mentioned, with numbers that are appropriately larger. Teachers should undertake this review in the spirit and for the reason suggested in Chapter XVI.

The textbook may be topical in its arrangement, that is, each general topic like denominate numbers or percentage may be treated once for all; or it may be on the plan of recurring topics, each important subject being met two or three times. As has already been said, each of these types has its advantages. If the school chooses a spiral arithmetic, it can probably be followed rather closely. If, on the other hand, it adopts one arranged by topics, there are two courses open: (1) the teacher may select from the various chapters such material as fits the course of study in use in the particular locality, a task of no great difficulty; or (2) the book may be followed closely, the pupil's work becoming purely topical. We are apt to condemn the latter plan because it is old, but perhaps on this very account it should be commended. The world has used it, and used it successfully, and it has the merit that it brings a feeling of mastery, a sense of thoroughness, and a development of habit that is often lacking with more recent textbooks. In general the question as to which type of book is the better, and as to which plan of using the topical book is to be preferred, may be said to depend upon the school. In a school system with a reasonably permanent staff of teachers, with adequate supervision, and with teachers' meetings that allow classes to keep in touch with one another, the book with recurring topics, or at least the course arranged on this plan, is probably the better. It seems to be somewhat more psychological and it allows for a better grading of material. On the other hand, where teachers change frequently, as in rural schools, it is safer to use the topical book and to follow it rather closely.

In many courses of study the review of the fundamental operations is omitted in the fifth grade. Textbooks must, however, supply it for the reason that, in spite of official courses of study, teachers find that some review of these operations is necessary at this time, at least for many pupils.

The number space is now unlimited, but names beyond billion are of no particular importance. Large numbers should always represent genuine American conditions. It is better to perform several operations on the ordinary numbers of daily life than to perform one on an absurdly long number; but, on the other hand, reasonable operations on large numbers that represent real business cases are to be commended. In general, the numbers used by the average citizen are the ones to drill children upon.

Children should be taught to read columns as nearly as possible as they read a word. When we see the word "book" we do not think "b," "o," "o," "k"—we think "book" without any spelling; so when we see the column of figures on page 167 we should not think "6 and 3 are 9, 9 and 3 are 12, 12 and 5 are 17," nor even "6, 9, 12, 17," if we can do better than this. Probably we cannot train our eyes to see 17 at a glance, as we see "book," but it is well to encourage children to look at this as 9 + 8, thinking of the 6 and 3 as 9, and of the 3 and 5 as 8. In such work we should always check our result by adding in the reverse order. If teachers do not think this necessary, let them add twenty sets of ten five-figure numbers each and see how many mistakes they themselves will make.

$$\begin{array}{r} 5 \\ 3 \\ 3 \\ 6 \\ \hline \end{array}$$

Subtraction has been sufficiently discussed on page 145. The important matter is never the explanation, and particularly true does this statement seem after the technique has been learned. The operation, accurately and rapidly performed, is the desideratum, the check being of great importance in securing the essential accuracy.

It is now advisable to let the children know some good, practical check on their work in multiplication, such as computers actually use. Of these checks, the simplest is that of "casting out 9's." The full theory of this check is too long to be given here, but a brief summary of the process is possible.

The remainder after dividing any integer by 9 is called the *excess of 9's* in the integer. Thus the excess of 9's in 11 is 2, because 11 ÷ 9 = 1, with 2 remainder. The excess of 9's in 48 is 3, the excess in 85 is 4, and the excess in 108 is 0.

Instead of dividing by 9 to find the excess there is a much shorter method. Consider, for example, the number 2348.

We may separate this into 2000 + 300 + 40 + 8.

$$\begin{array}{rl} 2000 = & 2 \times 999 + 2 \\ 300 = & 3 \times \ \ 99 + 3 \\ 40 = & 4 \times \ \ \ \ 9 + 4 \\ 8 = & \qquad\quad\ \ 8 \\ \hline 2348 = & \text{a multiple of } 9, + 2 + 3 + 4 + 8 \end{array}$$

That is, 2348 contains 9 exactly except for the sum of its digits, 2, 3, 4, 8. Therefore, *the excess of 9's in any number is equal to the excess in the sum of the digits.*

In finding the remainder after dividing the sum of the digits by 9, that is, in casting out the 9's, we may of course omit the 9's, or any two or three digits which we see at a glance will make 9. Thus in casting out the 9's from 1,926,754, we see at once that $1 + 2 + 6 = 9$ and $5 + 4 = 9$, and the single 7 will be the remainder. So in 254,786 we reject 5, 4, and 2, 7, and add $8 + 6$; from the sum we reject 9 and have 5 left.

The excess of 9's in a product equals the excess in the product of the excesses in the factors.

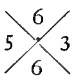
Check the product of 61×47 by casting out 9's. After the 9's are cast out:

from 47 the remainder is 2 } Multiply
from 61 the remainder is 7 }
from 2867 the remainder is 5; from 14 the remainder is 5.

The product of the two *numbers* has an excess 5, and the product of the two *excesses* has 5 remaining after the 9's are cast out. Therefore the work may be assumed to be correct. The check does not, however, disclose errors that are independent of the sum of the digits.

A convenient arrangement of work is as follows:

```
  257
   84
 ----
 1028
2056
-----
21588
```



In the two angles at the sides of the cross write 5 and 3, the excesses in the factors. At the top write the excess in the product of these excesses, 6 being the excess in 3×5, or 15. At the bottom write the excess in the product, 6 being the excess in 21,588. The top and bottom numbers in the cross should then agree.

Of course the child is not supposed to study any of this theory, but he will be interested in the method of checking the work, and if encouraged to use it, he will find it valuable all through life. The theory, as given above, is altogether too difficult for the pupil; but if the teacher once understands it, she will find that the mechanical application of the check is very easy, and that pupils in the fifth or sixth grade will have no difficulty in using it.

The children are now old enough to understand the two forms of division illustrated by the following:

$$\$125 \div \$5 = 25,$$
$$\$125 \div 25 = \$5.$$

There are no generally accepted names to distinguish these, "measuring" and "partition" not meaning much to children. It is sufficient to recognize that there are these two forms, and to see to it that we avoid such inaccuracies as $125 ÷ $25 = 5 cows.

If the theory of the check of 9's in division is not attempted, the practice is simple and is usually interesting to children. It depends merely on the fact that division is the inverse of multiplication, and it may be introduced at any time in the upper grades of the elementary school that a teacher may feel disposed to present it. It involves merely the fact that *the excess of 9's in a dividend equals the excess in the product of the excesses in the divisor and quotient, plus the excess in the remainder.* For, if 1,348,708 ÷ 498 is 2708 with remainder 124, then 1,348,708 = 2708 × 498 + 124. Now 8, the excess in 2708, times 3, the excess in 498, is 24; and 6, the excess in 24, plus 7, the excess in 124, is 13, in which the excess is 4. Since the excess in 1,348,708 is 4, the work may be assumed to be correct.

The subject of factors and multiples formerly played a very important part in arithmetic when large fractions had to be reduced to lower terms. With the introduction of the decimal fraction about the year 1600, however, it lost much of its former importance and need play but a small part in the arithmetic of to-day.

Some objective work will still be necessary in treating common fractions, but it should be dispensed with as soon as possible and the material should not be of one kind alone. In the operations children should not be required to give very elaborate explanations, although they should see clearly the reasons at the time they learn the processes. This has been discussed already in Chapter IX and may therefore be dismissed at this time. The work in common fractions usually constitutes the major part of this year's curriculum. Many teachers argue that the decimal fraction should precede the common fraction, and in this grade it is not a serious matter as to which comes first, since the pupil is already familiar with the initial steps in each. In the primary grades the most important of the practical uses of the common fraction have been considered, together with the leading operations on numbers written as dollars and cents, the latter being essentially decimal fractions. At that time the common fraction was properly taken first as being the less abstract and the more frequently used by children. In the fifth grade, therefore, the child has a background of fraction concepts, and it is not of great consequence whether he reviews his common fractions before he studies decimals, or vice versa. Teachers may therefore safely follow either plan, as the course of study of the state or city may direct. In many cases they will find that both common and decimal fractions are taken in the fifth grade, constituting the main body of the work.

Just as we have seen in the past generation a great change in the teaching of common fractions, notably in the omission of those having large denominators and therefore not used in business, so we are seeing a corresponding change in decimal fractions. The world ordinarily uses decimal fractions with two decimal places; it is rather the exception when three decimal places are used, and the average citizen seldom sees four places, although in science and in large financial transactions such numbers may be employed. Hence the emphasis should be laid upon numbers with two decimal places, enough work being given with more extended decimals to show the processes.

The treatment of decimal fractions is naturally based upon that of common fractions, and this being the method followed in the leading textbooks it demands no attention in this work.

Such minor matters as the attempt to impress upon children a distinction between decimals and decimal fractions, a distinction that the world does not recognize and never will, need not concern us. Indeed, we may even say the same of the convention that it is best to pronounce the word *and* at the decimal point only. Thus 100.023 is read "one hundred and twenty-three thousandths," while 0.123 is read "one hundred twenty-three thousandths." While this is a desirable convention, it is not one of the great things in the treatment of decimal fractions. We must always bear in mind that these great things are the four fundamental operations — not the theory, not the reading of numbers, not even the relation to common fractions, important as this is in the first stages; the large question is, Can the pupil add, subtract, multiply, and divide in the domain of decimal fractions as easily as in the domain of integers?

In order that we may know whether or not we are up to the world's standard, it is well that we again compare our work with that of other countries. In the fifth school year England studies common and decimal fractions, the metric system, and simple bills and invoices, but in London the chief attention seems to be given to common fractions. In France the course includes proportion and simple interest. In Germany they not only study fractions as we do, but they do considerable work with straight lines, angles, and triangles, thus beginning a simple form of geometry. In Bavaria, to speak of one of the most progressive of the German states, work is done in changing from decimal to common fractions, and conversely; in Württemberg this is supplemented by work in proportion and in profit and loss; and in Baden it is supplemented by the study of the properties of triangles, parallelograms, polygons, and circles, and a beginning is made in the use of the ruler and compasses. In

Holland the work includes simple problems in percentage and proportion. In Italy it extends to simple interest, discount, brokerage, and geometric constructions. In Russia it is somewhat more extended than in America, particularly with reference to decimal fractions. In Hungary, a country that stands among the foremost in educational work, there is included not merely the study of decimal and common fractions but also a considerable amount of work in metrical geometry, extending to the mensuration of the circle. Roumania includes in the fifth grade decimal fractions, proportion, simple interest, tests of divisibility, and mensuration as far as the volume of the pryamid and cone.

We therefore see that in the United States we begin, in the fifth grade, to fall appreciably behind most of the other leading countries in the work in arithmetic. There are a few good reasons for this, and many poor ones. Our school year is not as long as that in northern Europe, nor have we as many hours a day in school. Our teachers are not, on the average, so well prepared as those of Europe, nor do they form so permanent a body. We have the language difficulty to face, having to teach English to the children of a million immigrants a year. Our professional educators have not the scholarship that is found in the holders of similar positions in Germany, for example, and their educational ideals are not so high. We tend to destroy courses of study, accepted traditions, methods, and ideals, largely for the nervous pleasure of change, whereas the European teacher tends to conserve what has been proved by experience to be of worth. But after all excuses have been made, we have to face the fact that we are behind the rest of the world at this point, and that we get still further behind before our high-school period is reached. The lamentable thing is that so many of our leaders wish to make matters worse instead of better.

CHAPTER XX

WORK OF THE SIXTH SCHOOL YEAR

The leading features of this year are usually the completion of the work in such denominate numbers as are essential, the study of the most practical measures needed in daily life, and an introduction to percentage. Many children leave school at the end of this year, and with this equipment they are prepared for any of the ordinary work in arithmetic.

The operations with denominate numbers should be a part of the work of the year, only practical cases being taken. To divide a compound number of four denominations by another one of three, for example, is a waste of time. We no longer see in real life such measurements as 4 yd. 1 ft. 2 in., or 4 T. 3 cwt. 95 lb. 4 oz., and it is the retention of such work in some schools that brings the charge that mathematics is of no value and that arithmetic is unnecessarily hard and stupid. The sooner we can eliminate material of this kind, and confine our attention to the real uses of denominate numbers, the better it will be for the children and for the good standing of arithmetic.

Inasmuch as the children now begin to consider problems of more than two steps, it becomes necessary to devote more attention to the methods of solving examples. The step form of analysis, therefore, has a legitimate place in this year's work. If teachers hope for exactness of thought, they must insist upon accuracy of statement in these written exercises.

In this connection it should be stated to a pupil that he will not only be called upon to work rapidly and accurately, but that he may be asked to write an explanation or to give a brief oral analysis. A model for this kind of work is given on page 69 and need not be repeated here. The important thing, of course, is the accuracy of the number work itself, for here is where the child will be judged in business life.

Therefore what has already been said concerning checks on operations should be kept in mind in relation to this work. All computers check their work at every important step, and it is only by acquiring this habit that we can be reasonably certain that our calculations are free from error. This law should be repeated from time to time until the pupil comes to follow it automatically:

Always estimate the answer in advance, write down the estimate, and compare it with the result. If there is a great difference, the work is probably wrong.

A pupil may be excused if he does not understand the wording or the meaning of a problem, but for inaccurate numerical work he must be judged as he will be judged in business — his work is right or else it is wrong. Accuracy is far more important than speed.

It is probable that many teachers will hesitate to introduce the check of casting out nines, already given on page 167. This is because there is a feeling that the theory is too abstract for the children, as it is. But if once teachers can be brought to realize that the theory is not intended for the pupils, that only the practice is suitable for the work in the grades, that this practice is not only easy but interesting, and that the check has a value that can hardly be overestimated, we shall see the casting out of nines once more take the place that it formerly occupied in elementary arithmetic. Such a simple device, eliminating as it does almost all the errors in multiplication and division, may well occupy the attention of teachers in all the upper grades of the elementary school.

The pupil has now advanced to such a stage that the shorter methods of handling the common operations can be understood and used. The school does not exist for the purpose of making "lightning calculators," but there are certain short methods that are of real value to every one. The following are among the simple methods of multiplying and dividing that may profitably be used from this time on:

(1) *To multiply by 10, move the decimal point one place to the right, annexing zero if necessary.*

Thus $10 \times 46.75 = 467.5$, and $10 \times 75\frac{1}{2} = 10 \times 75.5 = 755$.

(2) *To multiply by 100, move the decimal point two places to the right, annexing zeros if necessary.*

Thus $100 \times 0.275 = 27.5$, $100 \times 6.5 = 650$, and $100 \times 72 = 7200$.

(3) *To multiply by 1000, move the decimal point three places to the right, annexing zeros if necessary.*

Thus $1000 \times 0.02365 = 23.65$, and $1000 \times 147 = 147{,}000$.

(4) *To multiply by 50, multiply by 100 and divide by 2.*

For $50 = \frac{100}{2}$; therefore $50 \times 84 = \frac{100}{2} \times 84 = \frac{8400}{2} = 4200$.

(5) *To multiply by 25, multiply by 100 and divide by 4.*

For $25 = \frac{100}{4}$; therefore $25 \times 84 = \frac{100}{4} \times 84 = \frac{8400}{4} = 2100$.

(6) *To multiply by* $12\frac{1}{2}$, *multiply by 100 and divide by 8.*

For $12\frac{1}{2} = \frac{100}{8}$; therefore $12\frac{1}{2} \times 64 = \frac{100}{8} \times 64 = 800$.

(7) *To multiply by 125, multiply by 1000 and divide by 8.*

For $125 = \frac{1000}{8}$; therefore $125 \times 72 = \frac{1000}{8} \times 72 = 9000$.

(8) *To multiply by* $33\frac{1}{3}$, *multiply by 100 and divide by 3.*

For $33\frac{1}{3} = \frac{100}{3}$; therefore $33\frac{1}{3} \times 69 = \frac{6900}{3} = 2300$.

(9) *To multiply by* $16\frac{2}{3}$, *multiply by 100 and divide by 6.*

For $16\frac{2}{3} = \frac{100}{6}$; therefore $16\frac{2}{3} \times 84 = \frac{8400}{6} = 1400$.

(10) *To multiply by 5, multiply by 10 and divide by 2.*

For $5 = \frac{10}{2}$; therefore $5 \times 0.42 = \frac{4.2}{2} = 2.1$.

(11) *To multiply by 75, multiply by 100 and take $\frac{3}{4}$ of the product.*

For $75 = \frac{3}{4}$ of 100; therefore $75 \times \$64.40 = \frac{3}{4}$ of \$6440 = \$4830.

(12) *To multiply by $37\frac{1}{2}$, multiply by 100 and take $\frac{3}{8}$ of the product.*

For $37\frac{1}{2} = \frac{3}{8}$ of 100; therefore $37\frac{1}{2} \times \$72 = \frac{3}{8}$ of \$7200 = \$2700.

(13) *To multiply by $62\frac{1}{2}$, multiply by 100 and take $\frac{5}{8}$ of the product.*

For $62\frac{1}{2} = \frac{5}{8}$ of 100; therefore $62\frac{1}{2} \times 640 = \frac{5}{8}$ of 64,000 = 40,000.

(14) *To multiply by $87\frac{1}{2}$, multiply by 100 and take $\frac{7}{8}$ of the product.*

For $87\frac{1}{2} = \frac{7}{8}$ of 100; therefore $87\frac{1}{2} \times 96 = \frac{7}{8}$ of 9600 = 8400.

(15) *To multiply by $66\frac{2}{3}$, multiply by 100 and take $\frac{2}{3}$ of the product.*

For $66\frac{2}{3} = \frac{2}{3}$ of 100; therefore $66\frac{2}{3} \times 144 = \frac{2}{3}$ of 14,400 = 9600.

(16) *To multiply by 0.1, move the decimal point one place to the left; by 0.01, move the decimal point two places to the left; by 0.001, move the decimal point three places to the left.*

(17) *To multiply by 0.5, take half of the multiplicand.*

(18) *To multiply by 0.25, take a fourth of the multiplicand.*

(19) *To multiply by $0.12\frac{1}{2}$, take an eighth of the multiplicand.*

(20) *To multiply by $0.33\frac{1}{3}$, take a third of the multiplicand.*

(21) *To multiply by $0.16\frac{2}{3}$, take a sixth of the multiplicand.*

(22) *To multiply by 0.75, take three fourths of the multiplicand.*

(23) *To divide by 10, move the decimal point one place to the left; by 100, two places; by 1000, three places; and so on.*

Thus $4936 \div 10 = 493.6$; $\$4876 \div 100 = \$48.76$; $29.5 \div 1000 = 0.0295$.

(24) *To divide by 5, multiply by 2 and divide by 10.*

For to divide by 5 is to multiply by $\frac{1}{5}$, or $\frac{2}{10}$. To divide 32,305 by 5 we may multiply by 2, obtaining 64,610, and cut off the zero, obtaining 6461, a process somewhat easier than actual division.

(25) *To divide by 25, multiply by 4 and divide by 100.*

For $\frac{1}{25} = \frac{4}{100}$. It is much easier to multiply by 4 and move the decimal point two places than to divide by 25.

(26) *To divide by 125, multiply by 8 and divide by 1000.*
(27) *To divide by $33\frac{1}{3}$, multiply by 3 and divide by 100.*

Other similar rules are easily found and easily explained, and their application is not only valuable but interesting, particularly in oral exercises.

In this grade there is usually an introduction to percentage. The pupil should see that "per cent" is only another name for hundredths, and that we may read 0.06 either "6 hundredths" or "6 per cent." In the same way we may think of 6% as either "6 per cent" or "6 hundredths," although it is read "6 per cent." The expression 800% means $\frac{800}{100}$ and equals the whole number 8; 225% equals the mixed number 2.25, or $2\frac{1}{4}$; $\frac{1}{2}\%$ means $\frac{1}{2}$ of $\frac{1}{100}$, or $\frac{1}{200}$, and is read either "$\frac{1}{2}$ per cent" or, quite commonly, "$\frac{1}{2}$ of 1%."

Since 6% means $\frac{6}{100}$, which equals 0.06, or $\frac{3}{50}$, it should be made clear that we may express per cent as a decimal fraction or as a common fraction. It is often convenient to use one form, and often another. Thus if we are multiplying by 13.5%, it is more convenient to think of the multiplier as 0.135; but if we are multiplying by $33\frac{1}{3}\%$, it is better to think of it as $\frac{1}{3}$ instead of $0.33\frac{1}{3}$.

Following up the short methods of operation, the pupil should be told that certain per cents are used so frequently that their equivalent common fractions should be remembered. These are as follows:

$50\% = \frac{1}{2}$,	$37\frac{1}{2}\% = \frac{3}{8}$,	$16\frac{2}{3}\% = \frac{1}{6}$,	$20\% = \frac{1}{5}$,
$25\% = \frac{1}{4}$,	$62\frac{1}{2}\% = \frac{5}{8}$,	$33\frac{1}{3}\% = \frac{1}{3}$,	$40\% = \frac{2}{5}$,
$12\frac{1}{2}\% = \frac{1}{8}$,	$87\frac{1}{2}\% = \frac{7}{8}$,	$66\frac{2}{3}\% = \frac{2}{3}$,	$60\% = \frac{3}{5}$,
$6\frac{1}{4}\% = \frac{1}{16}$,	$3\frac{1}{8}\% = \frac{1}{32}$,	$83\frac{1}{3}\% = \frac{5}{6}$,	$80\% = \frac{4}{5}$.

These are sometimes called *aliquot per cents.* To take $87\frac{1}{2}\%$ of 648 is, therefore, the same as to take $\frac{7}{8}$ of 648.

In the days of solving everything by rule instead of trying to use one's common sense, it was necessary to have such names as "base," "rate," "percentage," "amount," and "difference." It is convenient still to employ "base" and "rate," and the use of "percentage" may also be justified, although it is not of any particular importance. The rest are of little value, and a teacher is quite justified in omitting them, since the business world does not use them in this sense.

Discount being the first and most important of the applications of percentage, considerable attention should be devoted to it. The case of several discounts may, however, be postponed until the following year.

Profit and loss, so closely connected with the business world with which the child is now coming into closer contact, may claim to rank second in importance among the applications of percentage. The principles involved are very simple, and require little explanation. The examples should follow as closely as is possible in the school the common business customs of the mercantile world.

The topic of commission ranks possibly third in importance among the applications of percentage. A considerable field of applications exists, particularly in relation to the sending of farm produce to the cities. The problems can therefore be made to seem real to the children, whether they live in the country or see farm products for sale in the city.

The subject of simple interest is now taken up. Only real cases should, however, be considered. For example, in this school year, at least, there is little advantage in trying to find the principal, given the rate, time, and interest. It is better to spend time in writing promissory notes and in computing the interest than to devote it to questions that seldom arise in business life. If we wish more complicated problems, they are easily secured from genuine mercantile sources.

In each succeeding year the problems now come to relate more and more to the industries of the people, and the range of applications becomes very great. The farm child learns not only of his own surroundings but of the industries of the city, while to the city child the story of the soil and its products opens up a new world. The following farm problems may be taken as types of outside work brought into the class by the pupils or teacher, and suited to this grade:

1. A farmer puts 5 acres into celery, setting out 20,000 plants to the acre. The yield being 1500 doz. heads to the acre, what is the ratio of the number of plants that matured to the number of those that failed to mature?

2. He pays $95 an acre for seeds, fertilizers, labor, and other expenses, and sells the crop at 15¢ a dozen heads. What is his profit on the five acres?

3. Another farmer tries setting out 30,000 plants to the acre, but only 80% mature, and these are so small that he has to put 16 in a bunch to sell for a dozen, and then gets only 14¢ a bunch. His expenses are $100 an acre. At this rate, what is his profit on 5 acres?

4. A farmer has a 30-acre meadow yielding $1\frac{1}{2}$ tons of hay to the acre. If by spending $300 a year for fertilizers he can bring the yield to 4 tons to the acre, how much more will he make a year, hay being worth $12 a ton?

5. A farmer reads that a good mixture of seed for his meadow is, by weight, as follows: timothy 40%, redtop 40%, red clover making up the rest. At 40 lb. of seed to the acre, how many pounds of each should he sow?

6. The following is, by weight, a good mixture of seed for a pasture: Kentucky blue grass 25%, white clover $12\frac{1}{2}$%, perennial rye $28\frac{1}{8}$%, red fescue $9\frac{3}{8}$%, redtop 25%. At 32 lb. to the acre, how many pounds of each are used?

7. A cow weighing 1000 lb. consumes the equivalent of $3\frac{1}{4}$ tons (2000 lb. to the ton) of dry fodder a year; a 100-pound sheep, 770 lb.; every ton of live pork, 12 tons; and every ton of live horseflesh, 8.4 tons. Each class of animals consumes what per cent of its own weight of dry fodder a year?

The following are types of problems relating to home purchases, and others of similar nature may properly be brought to class by the pupils.

1. If 16 lb. of a certain kind of biscuit can be bought for $1.44, how much will 10 lb. cost at the same rate?

2. A woman can buy pastry flour in 10-pound bags at 44¢, or in $24\frac{1}{2}$-pound bags at 98¢. How much does she save a pound by buying the flour in the larger bags?

3. A dealer sells Ceylon tea at 68¢ a pound and 5-pound caddies at $3.14. If two families want $2\frac{1}{2}$ lb. each, how much will each save if they buy a caddy and divide it?

4. A dealer sells coffee in half-pound bags at 17¢, and in 5-pound cans at $1.64. If a woman wishes 5 lb., how much does she save by purchasing it by the can?

5. A woman can buy Dutch cocoa in $\frac{1}{4}$-pound boxes at 22¢, or in 5-pound cans at $2.78. If she wishes 5 lb., how much does she save by purchasing it by the can?

6. A woman can buy maple sirup at 44¢ a quart, or in gallon cans at $1.38. If she wishes 4 qt., how much does she save by purchasing it by the can?

7. A dealer sells large glass jars of comb honey at $5.06 a dozen or 44¢ each. How much more will he receive by selling a gross by the jar than by the dozen?

8. A woman buys a pound each of mustard seed, allspice, and cayenne pepper, paying 60¢ for the three. The mustard seed costs 12¢ a pound, and the allspice costs the same as the pepper. How much does the allspice cost?

9. A woman bought a can each of cinnamon, nutmeg, and pepper, paying 39¢ for the three. The pepper cost 9¢, and the nutmeg cost twice as much as the cinnamon. How much did the cinnamon cost?

10. I bought 3 gal. of best maple sirup at $\$1.12\frac{1}{2}$ a gallon, $2\frac{1}{2}$ gal. New Orleans molasses at 62¢ a gallon, and gave the grocer a 5-dollar bill. How much change did I get?

11. In buying 48 cans of tomatoes, how much is gained by buying two cases of 2 doz. each, at $2.89 a case, over buying at the rate of 3 cans for 40¢?

The older problems of arithmetic were all made for boys, girls rarely studying the subject in early times. It is well for the pupils to make the effort to find problems that appeal to the girl's interests as well as to the boy's. The following may be taken as types:

1. If a dressmaker has $94\frac{1}{2}$ yd. of linen for 9 coat suits, how many yards does this allow to a suit?

2. If it takes 4 yd. 6 in. of material for a skirt, how many yards will it take for 4 such skirts?

3. If it takes 4 yd. of taffeta at $1.10 a yard for a waist, and $3\frac{1}{2}$ yd. of trimming at 60¢ a yard, and the dressmaker charges $1.25 for making, how much will two such waists cost?

4. If it takes $2\frac{3}{4}$ yd. of lawn at 28¢ a yard for a shirt waist, $5\frac{1}{2}$ yd. of trimming at 22¢ a yard, 8 buttons at 60¢ a dozen, and 10¢ worth of thread, and if the labor costs $1.15, how much does the waist cost?

5. How much material is taken up (duplicated) by 32 quarter-inch tucks? If the material is worth $1.60 a yard, how much must be allowed for the extra cost of material in a piece having these tucks? How much must be allowed for the extra cost in three such pieces?

6. How wide must a ruffle be cut to be 6 in. deep when completed, if there is a hem $\frac{3}{4}$ in. wide at the bottom, and 8 one-eighth-inch tucks are placed above the hem?

7. A lady buys a piece of cloth containing 48 yd. at 22¢ a yard. She uses 33 yd. for various purposes, and the rest for making 4 skirts. What did the cloth cost per skirt?

8. If you use 81 in. of ribbon on a dress, how many yards and inches do you use? Express the result also as yards and a fraction of a yard.

9. A dressmaker bought 16 yd. of velvet at $3 a yard, and sold 9 yd. at a profit of 50¢ a yard, and the rest at a profit of half as much. What was the total profit?

10. A dressmaker bought a 50-yd. piece of silk waist lining at 75¢ a yard. She sold 12 yd. at $1 a yard and 10 yd. at 95¢ a yard. The remainder, being kept in stock over the season, had to be sold for 65¢ a yard. What was her per cent of gain or loss?

It is of great advantage to introduce occasionally problems without numbers, so that the pupil can tell how he would go to work to handle a case that might arise outside the textbook. The following are types:

1. If you know the cost of a certain number of things and wish to know the cost of one, how do you proceed?

2. If you know the cost of one thing and wish to find the cost of a certain number of things, how do you proceed?

3. If you know the cost of a fraction of anything and wish to know the cost of the whole, how do you proceed?

4. If you know the cost of a certain number of things and wish to know the cost of a certain other number of the same things, how do you proceed?

5. If you know the cost of a gallon of anything and wish to know the cost of a pint, how do you proceed?

6. If you know the price of cloth per yard, how do you find the cost of a certain number of inches?

7. If you know the price of cloth per yard, how do you find the cost of a certain number of yards plus a certain number of inches?

8. If you know the circumference of a wheel, how do you find how many times the wheel turns in going a mile?

9. If you know how many times a wheel turns in going a mile, how do you find the circumference?

10. If you know the freight rate per hundred pounds, how do you find the freight charge on a ton?

11. If you know the freight rate per hundred pounds, how do you find the freight charge on a certain number of tons plus a certain number of pounds?

12. If you have the product of several numbers to be divided by the product of several others, how may you perform the operations with least labor?

CHAPTER XXI

WORK OF THE SEVENTH SCHOOL YEAR

Most schools devote this year to a thorough study of percentage and to ratio and proportion and the roots. The advanced applications of percentage, those that involve business customs of a higher class, are commonly postponed to Grade VIII when the pupil has a greater interest in commercial affairs. In some sections of the country, however, the subjects of ratio and proportion, and powers and roots, are put in Grade VIII, and the business applications are taken in Grade VII. A tendency is beginning to be manifest to follow the European plan of introducing concrete geometry and vocational algebra of some kind in Grades VII and VIII. How this will develop in this country it is as yet impossible to say.

The subject of percentage, so vital in business life to-day, should be touched upon twice in the elementary school. If the work is sufficiently progressive, the pupils will not find that "the edge of interest" is worn off. In this year there should be a good deal of oral work in the common per cents of business, pupils coming to feel that pencil and paper are unnecessary in finding $12\frac{1}{2}\%$, 25%, $33\frac{1}{3}\%$, 50%, $66\frac{2}{3}\%$, and 75% of ordinary numbers. As to the use of terms like "base," "rate," "percentage," "amount," and "difference," there is, as already stated, but little that can be said in their favor. They were invented in the rule stage of arithmetic, and have served their purpose. Of course, we need "rate," it being a stock term of the business world. "Percentage" is, however, rather confusing than otherwise, (1) because it is understood by the pupils as the name of the subject as a whole, and (2) because the business world does not use it quite as the school does. "Base" means so many things in mathematics that its use is equally confusing, while in the case of "amount" and "difference" this disadvantage is still more noticeable. On the whole, therefore, it is as well not to use these terms, although they are found in most of our leading books to-day because of the demands of teachers.

It should also be remarked that if the use of x is allowed, there is no excuse for the old formulas of percentage. They are nothing but condensed rules; if they are not explained, they defeat part of the purpose of studying arithmetic; if they are explained, they are much harder than the equation form with the single letter x. If the method of the equation is given in the textbook, it should be used in class; if it is not given, it may be optional with the teacher to use it or not. In general, the teacher will find it helpful in Grades VII and VIII.

It is well to bear constantly in mind, in the midst of the large number of possible cases of percentage, that the two important things in the subject are these: (1) to find some per cent of a given number; and (2) to find what per cent one number is of another. All the rest is relatively unimportant, and on these two the emphasis should accordingly be laid.

Simple interest is the leading application of percentage in this year, and the attention of pupils should be concentrated chiefly on the single problem of finding interest in the practical cases of daily life. To find the time, given the principal, rate, and interest, is of very slight importance, and the same may be said of other similar cases. The great point is to find the interest.

Ratio and proportion should be confined largely to the treatment of practical questions, and there are only a few in which this subject can be used to advantage. These are chiefly related to similar figures, although some other questions, like those mentioned below, enter. Compound proportion has no reason to claim a place in our schools to-day. If explained, the process is a very hard one; if not, it is a useless one, since we now have better methods of solving problems. A proportion is merely one method of writing a simple equation, and with the use of the letter x allowed, the equation form is likely to replace that of proportion. When this is not the case, ordinary analysis is likely to be substituted for it. For example, consider this problem: If a shrub 4 ft. high casts a shadow 6 ft. long at a time that a tree casts one 54 ft. long, how high is the tree?

Here we may write a proportion in the form 6 ft. : 4 ft. = 54 ft. : (?), not attempting to explain it, but applying only an arbitrary rule. This is the old plan. Or we may put the work into equation form,

$$\frac{x}{54} = \frac{4}{6},$$

and deduce the rule for dividing the product of the means by the given extreme. Or we may take the same equation and get our result easily by multiplying these equals by 54, giving

$$x = 36.$$

Or we may say: If a 6-foot shadow is cast by a 4-foot object, a 1-foot shadow would be cast by a $\frac{4}{6}$-foot object, and a 54-foot shadow would be cast by a $54 \times \frac{4}{6}$-foot object, or a 36-foot object.

Of these plans the first is the most difficult to explain; the rest are about equally easy, and the third is the shortest.

The following problems show some of the modern applications of ratio and proportion:

1. If a shipment of 5100 lb. of cattle, live weight, sold for $225.42, what would 3500 lb. sell for at the same rate?

2. If a Louisiana farmer paid $75 for 3 T. 1500 lb. of cottonseed meal for fodder, how much would he have to pay for 5000 lb.?

3. If in 225 lb. of milk there are 8.1 lb. of butter fat, how many pounds of milk will be required to produce 27 lb. of butter fat?

4. A farmer had 26 acres planted to potatoes. The crop from 7 acres amounted to 1260 bu. At the same rate, how many bushels did he receive from the whole field?

5. To irrigate a farm at the rate of $\frac{1}{5}$ in. in depth every day requires the flow of 210 gal. an hour through a certain ditch. What flow would be necessary to irrigate it at the rate of 0.3 in. a day?

6. Two boys weighing respectively 100 lb. and 80 lb. sit 9 ft. apart on the ends of a plank. Not counting the weight of the plank, how far from the heavier boy must the fulcrum be placed so that they will just balance?

7. How much pressure will you have to exert on the handles of a pair of shears just 3 in. from the fulcrum (screw or bolt), in order to exert a pressure of 5 lb. at a point 5 in. from the fulcrum?

In each of these examples, however, the ordinary unitary analysis is more satisfactory than proportion, and the latter, as a method of solving such problems, is rapidly going out of use.

Of the ancient Greek theory of numbers the chapter on powers and roots is about all that remains in our textbooks on arithmetic. Cube root is now seldom taught, and square root is taught less fully than was formerly the case. Since the subject is treated sufficiently in the textbooks, it is not necessary to dwell at length upon it.

For purposes of mensuration square root is necessary. Cube root may well be delayed until the pupil studies algebra, because it has so few practical applications. Even square root is more valuable as a bit of logic than as a practical subject, since those who use it most employ tables. The explanation, therefore, is even more important than the technique of the work, and children of this age can easily comprehend it, either by the use of the diagram or by the formula, the latter being quite easily understood by this time. We shall probably see an increasing use of tables of square and cube roots for the solution of practical problems.

Mensuration is now completed so far as the needs of the average person are concerned. The teacher should use simple models that can be made in the schoolroom, as suggested in the best arithmetics. It is not expected that strict geometric demonstrations can be given, but it is entirely possible to avoid arbitrary rules by giving enough objective work to make the matter clear. It is not advisable to introduce work that is not used in ordinary life, such as finding the volume of a frustum of a cone, there being a sufficient amount of more important work to occupy the time and attention of pupils.

The metric system might be taught earlier than the seventh or eighth school year, and there would be some advantage in this. But when we consider that it is not yet used practically by many Americans, it seems as well to postpone it until this time. There are three chief reasons for teaching it now: (1) general information requires us to know a system that is used by a large part of the civilized world, excluding the English-speaking portion; (2) it is used in all scientific laboratories in America; (3) our people should be sympathetic with a system that is likely to replace our own before long in all matters relating to our growing foreign trade; if we sell machines abroad, the measurements must be metric in most cases, and to foster this trade many of our skilled workmen will eventually need to use these instead of the awkward ones with which we are familiar.

At the same time we must not go to an absurd extreme, but must remember that our common system is the one that the people use and that the children must know before all others. In teaching the metric system the results will be poor unless the children use the actual measures and come to visualize the basal units as they should in their own system.

It should be repeated that foreign countries usually begin work in algebra and geometry in this grade, carrying along with it a review of arithmetic. It is not the custom to cover as many business topics of arithmetic as we do, and the study of arithmetic as a large subject closes commonly with the sixth year. As stated above, it is probable that we shall tend more and more to adopt a similar plan.

CHAPTER XXII

WORK OF THE EIGHTH SCHOOL YEAR

The work this year is in the line of business applications, including advanced mensuration. The boy and girl should now begin to feel that the world of business and of life is opening before them. It should therefore be the duty of the school, even more than in the preceding grades, to apply arithmetic to the genuine problems of life, particularly with reference to the common occupations of the people.

In banking, for example, we should not seek to train accountants or bookkeepers or cashiers, but we should seek to give a fair idea of the duties of these men in the ordinary savings bank and bank of deposit. A girl, for example, needs to know how to deposit money in a bank and how to draw checks as well as a boy, and such operations should become as real as the school can make them. School banks, with deposit slips, checks, bank books, cashier, paying teller, and receiving teller, should assist in this work.

The subject of partial payments has not the practical value that it had when banks were not so numerous as now, and when their machinery was not perfected. The old-style problem in partial payments should therefore give place to the more practical cases found in our best modern books.

Partnership is another subject that has entirely changed within a short time. The stock company (corporation) has largely supplanted it, save in its simplest form. The work of the schools should therefore be confined to this common form, the obsolete cases being supplanted by work on corporations.

It is not worth while to teach an elaborate form of book-keeping to the average citizen. On the other hand, it is necessary that every one should know how to keep simple accounts, and this work should be taken up in this year. It should relate to the income and expenditures of daily life, in the home, on the farm, or in the shop, rather than to the technical needs of the merchant, the latter being part of the special training of the individual who enters that line of trade.

With respect to exchange there has been a great change within a few years. The form of time draft given in most of the old-style arithmetics has given place either to sight drafts or to another kind of time draft. Teachers should therefore be particular to use only those types that the ordinary citizen meets to-day, about which girls and boys alike should be informed. In connection with this work a short talk upon the clearing house, upon which any bank will gladly inform the teacher, will add new interest.

The subject of taxes, like others of practical life, should be treated from the standpoint of local conditions as far as possible. It should include the question of tariff, and a few brief talks on civics should make the whole question a real one for the pupils.

Insurance has become so technical that all that the schools can hope to do is to give a general conception of the work of the various kinds of companies, confining the problems to the simplest practical cases that the people need to know about. We should not attempt to enter upon the technicalities of agency work, nor do more than explain briefly some of the common types of policy.

The corporation has, for good or evil, replaced the individual in large business ventures. Our schools must therefore adjust their work to this change. Pupils should know what a corporation is, its chief officials, how it is legally organized, what stocks and bonds are, how dividends are declared and paid, and the legitimate work of stock exchanges. On the other hand, the schools cannot be expected to teach the technicalities of the stockbroker's office, nor to supply information beyond that needed by the general citizen. The newspaper stock reports furnish an excellent basis for the practical problems that the case demands.

The problems of this grade should appeal to the business needs that are soon to come to the children, and the following are suggested as types:

1. A boy who has been working this year at \$25 a month is offered either an increase of 20% for next year or a salary of \$7 a week. Which will bring the larger income, and how much more per year? (Use 52 wk.)

2. A girl who has been working in a factory at \$21.67 a month is offered an increase of 10% where she is or a salary of \$5.60 per week elsewhere. Which will bring the larger income, and how much more per year? (Use 52 wk.)

3. A boy went to work at 90¢ a day. The second year his wages were increased 20%, the third year they were 42¢ a day more than the second, and the fourth they were increased $33\frac{1}{3}$%. At 300 working days to the year, what was his total income for each year?

4. A girl entering a trade school finds that graduates from the dressmaking department receive on an average \$4.60 a week the first year; those from the millinery department, 5% less; those from the embroidery department, 5% more than the dressmakers; and those from the operating department, $66\frac{2}{3}$% as much as the last two together. Find the average wages of each, and tell in which department the girl would receive the highest wage. (Use 52 wk.)

5. A girl leaving the public school finds she can enter a city shop at a salary of \$3 a week the first year, with $16\frac{2}{3}$% more the second year, and a $14\frac{2}{7}$% increase the third year. Instead of this she enters a trade school for a year, tuition free. She then receives a salary of \$5 a week the first year and 20% more the second year. Counting 50 working weeks a year, how much more does she receive in three years by the plan she followed after leaving the public school than she would have received without the trade-school training?

It is well known that in Europe the specialization of schools is carried much farther than has even been thought of here, or than seems possible or desirable in the future. To speak of the arithmetic of these various forms of schools — for foresters, builders, watchmakers, barbers, and so on — would therefore be unprofitable in a work of this general nature. It will not, however, be out of place to give an outline of the work in arithmetic in the eighth school year in a girls' school in Munich, because this shows the tendency at present, in one of the most progressive cities of Europe, to have arithmetic touch the interests and needs of the people. The work is as follows:

1. Simple domestic bookkeeping.

2. Calculation of the prices of foods, bought in large or in small quantities, together with the question of discounts.

3. Cost of meals for the home.

4. Daily, monthly, and yearly supplies for the kitchen, together with the keeping of kitchen accounts.

5. Simple measurements as needed in the household.

6. Food values of different foodstuffs as necessary for a complete meal, with doubtless the application of ratio and percentage.

7. Cost of furnishing a kitchen.

8. Measurements of material, and the cost of buying, renovating, and washing clothing made of various goods, as of woolen or linen.

9. Relative cost of different systems of heating.

10. Relative cost of different systems of lighting.

11. Maintenance of the house, including questions of rent, water, taxes, insurance, and interest on a mortgage.

12. Elementary commercial arithmetic, including such general topics as percentage and its application in discount.

Such a course is highly to be commended. It meets the needs of girls as we are not meeting them in our American schools. Indeed, it becomes a serious question if, in a subject like mathematics, we are not bound to have separate classes for girls and boys after the seventh grade and through our high school. It should be clearly understood, however, that a course of this kind is not primarily a course in mathematics, but one in civics and economics. The mathematics involved is nothing more than primary work. It is doubtful, therefore, whether this should consume the time that may properly be allotted to mathematics.

It may be well also to consider the work done in "Obertertia" in a Prussian Gymnasium, corresponding in years to our eighth grade. This correspondence is not exact in some respects, because the Prussian school year is somewhat longer than ours, but allowing three years before entering the lowest class ("Sexta"), this becomes the eighth school year. In this class three hours a week are allowed to mathematics, the arrangement allowing two to geometry and one to algebra in one week, and two to algebra and one to geometry the next week, and so on. The algebra includes equations of the first degree with two unknown quantities, and the geometry finishes the treatment of the circle, finding the value of π. The arithmetic work, as we call it, is practically completed at the end of the sixth school year (in "Quarta").

This gives an idea of what could be done in America if we should care to set about it. As it is, we may fairly ask whether we should not be justified in materially reducing the arithmetic now given in Grade VII and substituting for it a considerable part of the work of Grade VIII, thus gaining time for an elementary course in algebra in that year.

If algebra be introduced in Grade VIII, what is the purpose and what should be its nature? Aside from the general information thus given, and from the discipline that comes from this or any other subject, there is a need that a few years ago was hardly felt in this country. Boys are apt to leave school after the work of Grade VIII is finished. They go into the shops, into trade, into various occupations. A few years ago algebra was of no practical value to them, but to-day the formula and the graph of a function are common features in our trade journals. Here then is a suggestion to our schools. Why should not elementary algebra be introduced by a study of formulas, so that the simple algebraic expressions of our trade journals or our artisans' manuals can be read easily? Why should we not introduce graphs of functions very early, not in complicated forms, but as used in the journals, the manuals, and the workshop to-day?

As to the geometry, the work in mensuration in arithmetic probably suffices for the present, although it is possible that we may come to adopt the German plan of introducing the scientific treatment of the subject into the elementary grades in the future. In the Horace Mann School, connected with Teachers College, the mathematics in Grades VII and VIII gives only a brief review of arithmetic, devoting the rest of the time to practical mathematics, including a good deal of work out of doors. The result is a great interest in the subject and a desire to continue in serious work in algebra and geometry.

Such are some of the problems in the teaching of arithmetic to-day. Many are solved and many still await solution and are occupying the attention of a large number of teachers. It is with the hope of suggesting some of the larger problems that this book is written, rather than with the desire to treat the minor details that are sufficiently discussed in any good textbook. If the work shall lead to sane experiment, to a conservative view of the reforms to be accomplished, and to a sympathy with the effort to improve the problems of arithmetic, it will have served its purpose.

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ANNOUNCEMENTS

HISTORY OF MODERN ELEMENTARY EDUCATION

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Dean of the College of Education, The University of Chicago

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