each summation being performed with respect to an auxiliary integer m, from $m = 0$ to $m = 1$.

6. Accordingly, *without* using *imaginaries*, it is easy to prove that this expression (5) satisfies all the recent conditions (3), and is therefore a correct expression for the partial sum

$$s^{(p)}_{n,r};$$

while a similar proof of the recent equation 0 = &c.

7. But to form *practically*, with the easiest possible *arithmetic*, a *Table of Values* of s, for any given *period*, p, we are led by No. 3 to construct a *Scheme*, such as the following:—

TABLE OF VALUES OF $s^{(5)}_{n,r}$.

	$r = 5$	4	3	2	1	0	Verification.
$n = 0$	$s = 1$	0	0	0	0	1	$\Sigma s = 1$
1	1	0	0	0	1	1	2
2	1	0	0	1	2	1	4
3	1	0	1	3	3	1	8
4	1	1	4	6	4	1	16
5	2	5	10	10	5	2	32
6	7	15	20	15	7	7	64

The PRESIDENT read the following paper by the late Sir WILLIAM R. HAMILTON:—

ON A NEW SYSTEM OF TWO GENERAL EQUATIONS OF CURVATURE,

Including as easy consequences a new form of the Joint Differential Equation of the Two Lines of Curvature, with a new Proof of their General Rectangularity; and also a new Quadratic for the Joint Determination of the Two Radii of Curvature: all deduced by Gauss's Second Method, for discussing generally the Properties of a Surface; and the latter being verified by a Comparison of Expressions, for what is called by him the Measure of Curvature.

1. NOTWITHSTANDING the great beauty and importance of the investigations of the illustrious GAUSS, contained in his *Disquisitiones Generales circà Superficies Curvas*, a Memoir which was communicated to the *Royal Society of Göttingen* in October, 1827, and was printed in Tom. vi. of

the *Commentationes Recentiores*, but of which a Latin reprint has been since very judiciously given, near the beginning of the Second Part (Deuxième Partie, Paris, 1850) of LIOUVILLE's *Edition* of* MONGE, it still appears that there is room for some not useless Additions to the Theory of *Lines* and *Radii* of *Curvature*, for *any given Curved Surface*, when treated by what Gauss calls the *Second Method* of discussing the *General Properties of Surfaces*. In fact, the *Method* here alluded to, and which consists chiefly in treating the *three* co-ordinates of the *surface* as being so many *functions* of *two* independent variables, does not seem to have been used *at all* by Gauss, for the determination of the *Directions of the Lines of Curvature;* and as regards the *Radii of Curvature* of the *Normal Sections* which *touch* those *Lines* of Curvature, he appears to have employed the *Method, only for the Product*, and *not also* for the *Sum*, of the *Reciprocals*, of those *Two Radii*.

2. As regards the *notations*, let x, y, z be the rectangular co-ordinates of a point P upon a surface (S), considered as *three* functions of *two* independent variables, t and u; and let the 15 partial derivatives, or 15 partial differential coefficients, of x, y, z taken with respect to t and u, be given by the nine differential expressions.

$$\text{(a)} \ldots \begin{cases} dx = x'dt + x_,du; \ dx' = x''dt + x_,'du; \ dx_, = x_,'dt + x_{,,}du; \\ dy = y'dt + y_,du; \ dy' = y''dt + y_,'du; \ dy_, = y_,'dt + y_{,,}du; \\ dz = z'dt + z_,du; \ dz' = z''dt + z_,'du; \ dz_, = z_,'dt + z_{,,}du. \end{cases}$$

3. Writing also, for abridgment,

$$\text{(b)} \ldots e = x'^2 + y'^2 + z'^2; \quad e' = x'x_, + y'y_, + z'z_,; \quad e'' = x_,^2 + y_,^2 + z_,^2$$

we shall have (c) . . $ee'' - e'^2 = K^2$, if (d) . . $K^2 = L^2 + M^2 + N^2$,

and (e) . . $L = y'z_, - z'y_,; \ M = z'x_, - x'z_,; \ N = x'y_, - y'x_,;$

so that (f) . . $Lx' + My' + Nz' = 0, \ Lx_, + My_, + Nz_, = 0.$

Hence $K^{-1}L$, $K^{-1}M$, $K^{-1}N$ are the *direction-cosines* of the *normal* to the surface (S) at P; and if x, y, z be the co-ordinates of any *other* point Q of the same normal, we shall have the equations,

$$\text{(g)} \ldots K(X - x) = LR; \quad K(Y - y) = MR; \quad K(Z - z) = NR;$$

with (h) . . $R^2 = (X - x)^2 + (Y - y)^2 + (Z - z)^2;$

where R denotes the normal line PQ, considered as changing sign in passing through zero.

4. The following, however, is for some purposes a more convenient *form* (comp. (f)) of the *Equations of the Normal;*

$$\text{(i)} \ldots (X - x)x' + (Y - y)y' + (Z - z)z' = 0;$$
$$\text{(j)} \ldots (X - x)x_, + (Y - y)y_, + (Z - z)z_, = 0.$$

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* The foregoing dates, or references, are taken from a note to page 505 of that Edition.

Differentiating these, as if X, Y, Z were constant, that is, treating the point Q as an intersection of two consecutive normals, we obtain these two other equations,

$$(\mathrm{k}) \,.\,.\, \begin{cases} (X - x)dx' + (Y - y)dy' + (Z - x)dz' = x'dx + y'dy + z'dz; \\ (X - x)dx_{,} + (Y - y)dy_{,} + (Z - x)dz_{,} = x_{,}dx + y_{,}dy + z_{,}dz. \end{cases}$$

If, then, we write, for abridgment,

$$(\mathrm{l}) \,.\,.\, \begin{cases} v = du : dt; & E = Lx'' + My'' + Nz''; \\ E' = Lx_{,}' + My_{,}' + Nz_{,}'; & E'' = Lx_{,,} + My_{,,} + Nz_{,,}; \end{cases}$$

we shall have, by (a) (b))g), the two important formulæ:

$$(\mathrm{m}) \,.\,.\, R\,(E + E'v) = K(e + e'v); \quad R\,(E' + E''v) = K\,(e' + e''v);$$

which we propose to call the two general *Equations of Curvature.*

5. In fact, by elimination of R, these equations (m) conduct to a *quadratic in* v, of which the roots may be denoted by v_1 and v_2, which first presents itself under the form,

$$(\mathrm{n}) \,.\,.\, (e + e'v)\,(E' + E''v) = (e' + e''v)\,(E + E'v),$$

but may easily be thus transformed,

$$(\mathrm{o}) \,.\,.\, \begin{cases} Av^2 - Bv + C = 0, \text{ or } Adu^2 - Bdtdu + Cdt^2 = 0, \\ \text{with } A = e'E'' - e''E', \ B = e''E - eE'', \ C = eE'' - e'E; \end{cases}$$

so that we have the following *general relation,*

$$(\mathrm{p}) \,.\,.\, eA + e'B + e''C = 0,$$

(of which we shall shortly see the geometrical signification), between the *coefficients*, A, B, C, of the *joint differential equation* of the system of the two *Lines of Curvatare* on the surface.

6. The root v_1 of the quadratic (o) determines the *direction* of what may be called the *First Line of Curvature,* through the point P of that surface; and the *First Radius of Curvature,* for the same point P, or the radius R_1 of curvature of the *normal section* of the surface which *touches* that *first line,* may be obtained from *either* of the two equations (m), as the value of R which corresponds in that equation to the value v_1 of v. And in like manner, the *Second Radius of Curvature* of the same surface at the same point has the value R_2, which answers to the value v_2 of v, in each of the same two *Equations of Curvature* (m). We see, then, that this *name* for those two equations is justified by observing that when the two independent variables t and u are given or known; and therefore also the seven functions of them, above denoted by e, e', e'', E, E', E'', and K. The equations (m) are satisfied by *two* (but *only two*) *systems of values*, v_1, R_1, and v_2, R_2, of (I.) the *differential quotient* v, or $\frac{du}{dt}$, which determines the *direction* of a *line of curvature* on the surface; and (II.) the symbol R, which determines (comp. No. 4) at once the *length* and the *direction*, of the *radius of curvature* corresponding to that *line.*

7. Instead of eliminating R between the two equations (m), we may *begin* by eliminating v; a process which gives the following quadratic in R^{-1} (the curvature):—

(q) . . $(eR^{-1} - eK^{-1})(e''R^{-1} - e''K^{-1}) = (e'R^{-1} - eK^{-1})^2$;

or (r) . . $R^{-2} - FR^{-1} + G$; where (because $ee'' - e'^2 = K^2$),

(s) . . $F = R_1^{-1} + R_2^{-1} = (eE'' - 2e'E' + e''E)K^{-3}$, and

(t) . . $G = R_1^{-1}R_2^{-1} = (EE'' - E'^2)K^{-4}$.

We ought, therefore, as a *First General Verification*, to find that this last expression, which may be also thus written,

$$(u) \;.\,. \qquad G = R_1^{-1}R_2^{-1} = \frac{EE'' - E'E'}{(L^2 + M^2 + N^2)^2},$$

agrees with that reprinted in page 521 of Liouville's Monge, for what Gauss calls the *Measure of Curvature* (k) of a *Surface;* namely,

$$(v) \;.\,. \qquad k = \frac{DD'' - D'D'}{(AA + BB + CC)^2};$$

which accordingly it evidently does, because our symbols $L\,M\,N\,A\,B\,C$ represent the combinations which he denotes by ABCD D′D″.

8. As a *Second General Verification*, we may observe that if I be the *inclination* of any *linear element*, $du = vdt$, to the *element* $du = 0$, at the point P, then

$$(w) \;.\,. \; \tan I \qquad \frac{Kv}{e + e'v};$$

and therefore, that if H be the *angle* at which the *second crosses the first*, of *any two lines* represented *jointly* by such an equation as

(x) . . $Av^2 - Bv + C = 0$, with v_1 and v_2 for roots, then

$$(y) \;.\,. \; \tan H = \tan(I_2 - I_1) = \frac{K(B^2 - 4AC)^{\frac{1}{2}}}{eA + e'B + e''C};$$

so that the *Condition of Rectangularity* ($\cos H = 0$), for any *two* such lines, may be thus written:

$$(z) \;.\,. \qquad eA + e'B + e''C = 0.$$

But this *condition* (z) had already occurred in No. 5, as an equation (p) which is satisfied generally by the *Lines of Curvature;* we see therefore anew, by this analysis, that those *lines* on *any surface* are in general (as is indeed well known) *orthogonal* to each other.

9. Finally, as a *Third General Verification*, we may assume x and y *themselves* (instead of t and u), as the two independent variables of the problem, and then, if we use *Monge's Notation* of p, q, r, s, t, we shall easily recover all his leading results respecting *Curvatures of Surfaces*, but by transformations on which we cannot here delay.

The following donations were presented:—From

1. E. W. Doyle, Esq., a perfect heptagonal stone quern, and three flint arrow heads.
2. Stanhope Kenny, Esq., a mass of bog butter, found enveloped in the skin of some animal.
3. H. W. Westropp, Esq., his Treatise "On the Fanaux de Cimetières in France, and the Round Towers in Ireland."
4. John T. Gilbert, Esq., Librarian to the Academy, his work entitled "History of the Viceroys of Ireland," Vol. I.

The thanks of the Academy were voted to the donors.

MONDAY, NOVEMBER 13, 1865.

JOHN FRANCIS WALLER, LL. D., Vice-President, in the Chair.

The Secretary reported that a Collection of about 100 Books and 50 Manuscript Volumes had been deposited in the Library of the Academy, agreeably to the will of the late W. Smith O'Brien, Esq.

The Secretary presented the following donations:—From

1. George V. Du Noyer, Esq., a stone Sundial, found in the Churchyard of Kilbeg, near Kells.
2. John Evans, Esq., of Nash Mills, three nuclei of worked Flint, from Pressigny-le-grand.
3. G. H. Kinahan, Esq., a highly finished stone Celt, found a little north of Oughterard, county of Galway.
4. Robert Day, Jun., Esq., a stone Celt, and two flint Lance Heads, found at Toome Bar, county of Antrim.
5. The Rev. John Keleher, P. P., a small brass Box, for holding standard Weights, found near Kinsale.
6. Richard Palmer Williams, Esq., a collection of 327 Autographs, from the addresses of franked Letters, delivered in Dublin.
7. The Secretary also presented some rudely carved circular pieces of Coal found in graves, at Portpatrick, in Galloway.

The thanks of the Academy were voted to the several donors.

The Librarian brought up the Resolution of the Council of Monday, the 6th November—"That the Council do recommend to the Academy to authorize the opening of a Subscription List for the purchase of the MS. Collections of the late John Windele, of Cork;" whereupon it was resolved,

That this recommendation be adopted by the Academy.