NPS-53Sr77021

# NAVAL POSTGRADUATE SCHOOL

## Monterey, California

THE CALCULATION OF $e^{At}$ WITH SOME APPLICATIONS

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February 1977

Final Report for Period
September 1976 - December 1976

Approved for public release; distribution unlimited



repared for:
hief of Naval Research, Arlington, VA 22217

NAVAL POSTGRADUATE SCHOOL
Monterey, California

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The work reported herein was supported in part by the Foundation Research Program of the Naval Postgraduate School with funds provided by the Chief of Naval Research.



This report was prepared by:

| REPORT DOCUMENTATION PAGE | READ INSTRUCTIONS BEFORE COMPLETING FORM |
|---|---|
| 1. REPORT NUMBER NPS- 53Sr77021 / 2. GOVT ACCESSION NO. | 3. RECIPIENT'S CATALOG NUMBER |
| 4. TITLE (and Subtitle) The Calculation of $e^{At}$ With Some Applications | 5. TYPE OF REPORT & PERIOD COVERED Final Report 1 Sep 76 - 31 Dec 76 |
| | 6. PERFORMING ORG. REPORT NUMBER |
| 7. AUTHOR(s) Elmo J. Stewart | 8. CONTRACT OR GRANT NUMBER(s) |
| 9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School Monterey, CA 93940 Code 53Sr | 10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 61152N, RR 000-01-01 N0001477WR70044 |
| 11. CONTROLLING OFFICE NAME AND ADDRESS Office of Naval Research Arlington, VA 22217 | 12. REPORT DATE February 1977 |
| | 13. NUMBER OF PAGES 20 |
| 14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) | 15. SECURITY CLASS. (of this report) Unclassified |
| | 15a. DECLASSIFICATION/DOWNGRADING SCHEDULE N/A |

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

Matrix, characteristic function, eigenvalues, initial value problem, pairwise orthogonal, idempotent, nilpotent.

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

Given A an n x n matrix with real or complex elements, then $e^{At}$ is calculated as the unique solution of an initial value problem. In the process of obtaining this solution n-unknown matrices become involved and must be computed. Characterizing properties of these matrices to be computed become known: such properties as pairwise-orthogonal, idempotent and nilpotent. Finally some applications of the above calculations are given in the field of solutions to systems of differential equations.

THE CALCULATION OF $e^{At}$ WITH SOME APPLICATIONS

## 1. Introduction

Throughout this paper A will be an n x n matrix with real or complex elements, having $f(\lambda)$ as its characteristic function and eigenvalues $\alpha_1, \alpha_2, \ldots \alpha_n$ (not necessarily distinct). In this paper we calculate the exponential matrix $e^{At}$, specify some properties of certain matrices that must be determined in order to describe $e^{At}$ and finally indicate some applications of this calculation.

## 2. An Initial Value Problem and $e^{At}$

In [3] $e^{At}$ is obtained as the unique solution to the following initial value problem. With $f(\lambda)$ as specified above and $D \equiv \frac{d}{dt}$ we wish to obtain the solution to:

$f(D)G(t) = 0$, $G(t)$ an n x n matrix with elements functions of t and such that $G(o) = I$, $G'(o) = A$, $\ldots$ $G(o)^{(n-1)} = A^{n-1}$.

If $e^{At}$ is defined by the equation

$$e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!},$$

then

$$e^{At}\Big|_{t=0} = I, \quad D(e^{At})\Big|_{t=0} = A, \quad \ldots \quad D^{n-1}(e^{At})\Big|_{t=0} = A^{n-1},$$

and $f(D)e^{At} = f(A)e^{At} = 0$ by the Cayley-Hamilton Theorem. Therefore, $e^{At}$ is the unique solution to this initial value problem. Suppose $\alpha_1, \alpha_2, \ldots \alpha_s$ are the distinct eigenvalues of A with multiplicities $\mu_1, \mu_2, \ldots \mu_s$, we may write

$$(1) \qquad e^{At} = \sum_{k=1}^{s} (C_{k1} + C_{k2}t + \ldots + C_{k\mu_k} t^{\mu_k - 1}) e^{\alpha_k t},$$

where the $C_{kj}$ are n x n matrices. From the initial conditions we have:

$$I = \sum_{k=1}^{s} C_{k1}$$

$$A = \sum_{k=1}^{s} (\alpha_k C_{k1} + C_{k2})$$

(2)

$$\vdots \qquad \vdots$$

$$A^{n-1} = \sum_{k=1}^{s} \left(\alpha_k^{n-1} C_{k1} + (n-1)\alpha_k^{n-2} C_{k2} + \ldots + \frac{(n-1)!\alpha_k^{n-\mu_k}}{(n-\mu_k)!(\mu_k-1)!} C_{k\mu_k}\right),$$

from which the n $C_{kj}$ can be determined. We note from the system (2) that a solution for a $C_{kj}$ will be a linear combination of the left side of (2) and therefore, any $C_{kj}$ will be a polynomial in A of degree at most (n-1). Being polynomials in A, the $C_{kj}$ commute.

3. Properties of the $C_{k1}$ when all roots of $f(\lambda)$ are distinct.

We will not use the system (2) to completely solve for the n $C_{kj}$, rather will we obtain another representation for $e^{At}$ satisfying the initial value problem and then equate coefficients of like terms "$t^{\ell} e^{\alpha_k t}$". Before proceeding to this representation let us consider the simple case in which all roots of $f(\lambda)$ are distinct. This case will provide insights on how to handle the more general case of multiple roots.

If all roots of $f(\lambda)$ are distinct, then (1) becomes:

(1)' $$e^{At} = \sum_{k=1}^{n} C_{k1} e^{\alpha_k t} .$$

The inverse of (1)' is

(1)" $$e^{-At} = \sum_{k=1}^{n} C_{k1} e^{-\alpha_k t} .$$

Multiplying (1)' and (1)" and using the commutative property for the $C_{k1}$ we obtain

$$(3) \qquad I - \sum_{k=1}^{n} C_{k1}^2 = \sum_{\substack{k=1 \\ k \neq j}}^{n} \sum_{j=1}^{n} C_{k1} C_{j1} e^{(\alpha_k - \alpha_j)t} .$$

Equation (3) is true for all t, however, the left side of (3) is independent of t which suggests that $C_{k1} C_{j1} = 0$ for $k \neq j$, i.e., the $C_{k1}$ are pairwise orthogonal. (This will be shown below.)

Applying the initial conditions to (1)' we obtain:

$$(2)' \qquad \begin{aligned} I &= C_{11} + C_{21} + \dots + C_{n1} \\ A &= \alpha_1 C_{11} + \alpha_2 C_{21} + \dots + \alpha_n C_{n1} \\ &\vdots \\ A^{n-1} &= \alpha_1^{n-1} C_{11} + \alpha_2^{n-1} C_{21} + \dots + \alpha_n^{n-1} C_{n1} . \end{aligned}$$

Since the $\alpha_k$ are distinct, the coefficient matrix for the system (2)' is a Vandermonde matrix and is non-singular. Therefore, the system (2)' has unique solutions for the $C_{k1}$. If we solve the system (2)' for $C_{k1}$, then the coefficient of $A^{n-1}$ in this solution is:

$$(4) \qquad (-1)^{n+k} \begin{vmatrix} 1 & \dots & 1 & 1 & \dots & 1 \\ \alpha_1 & \dots & \alpha_{k-1} & \alpha_{k+1} & \dots & \alpha_n \\ \vdots & & \vdots & \vdots & & \vdots \\ \alpha_1^{n-2} & \dots & \alpha_{k-1}^{n-2} & \alpha_{k+1}^{n-2} & \dots & \alpha_n^{n-2} \end{vmatrix} \Bigg/ \begin{vmatrix} 1 & 1 & \dots & 1 \\ \alpha_1 & \alpha_2 & \dots & \alpha_n \\ \vdots & \vdots & & \vdots \\ \alpha_1^{n-1} & \alpha_2^{n-1} & \dots & \alpha_n^{n-1} \end{vmatrix}$$

Both numerator and denominator of this coefficient are non-zero Vandermonde determinants and, therefore, $C_{k1}$ ($k = 1, 2, \dots n$) is a polynomial in A of degree precisely n-1. Moreover, the coefficient of $A^{n-1}$ in this solution for $C_{k1}$ is

$$1/\prod_{\substack{j=1\\ j\neq k}}^{n} (\alpha_k - \alpha_j) ,$$

which agrees with the expansion of (4).

Next, suppose we wish to eliminate $C_{k1}$ from the second through the $n^{th}$ equation in (2)', which yields n-1 equations in the n-1 unknown matrices $C_{11}, C_{21}, \ldots C_{k-1,1}, C_{k+1,1}, \ldots C_{n1}$. We do this in order: subtract $\alpha_k$ times the first equation from the second, $\alpha_k^2$ times the first from the third, to finally $\alpha_k^{n-1}$ times the first from the $n^{th}$. Deleting the first of this new system of equations we obtain n-1 equations with the unknown $C_{k1}$ missing. The left members of this new system will be $A - \alpha_k I$, $A^2 - \alpha_k^2 I$, ..., $A^{n-1} - \alpha_k^{n-1} I$, each of which has a factor $A - \alpha_k I$, as does any linear combination of these left members. Therefore, each of the solutions for $C_{11}, C_{21}, \ldots C_{k-1,1}\ C_{k+1,1} \ldots C_{n1}$ will have a factor $A - \alpha_k I$. From this we conclude further that:

(5) $$C_{j1} = a_j \prod_{\substack{\ell=1\\ \ell\neq j}}^{n} (A - \alpha_\ell I) , \qquad j = 1, 2, \ldots n$$

for some scalar $a_j$. We note from (5) that $C_{j1}$ is precisely a polynomial of degree (n-1) in A with leading coefficient $a_j$ which must be

(6) $$a_j = 1/\prod_{\substack{\ell=1\\ \ell\neq j}}^{n} (\alpha_j - \alpha_\ell) \qquad \text{(from (4))} .$$

As an example using (5) and (6), let A be any 3 x 3 matrix with distinct eigenvalues $\alpha_1$, $\alpha_2$, $\alpha_3$, then;

$$e^{At} = \frac{(A - \alpha_2 I)(A - \alpha_3 I)e^{\alpha_1 t}}{(\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)} + \frac{(A - \alpha_1 I)(A - \alpha_3 I)e^{\alpha_2 t}}{(\alpha_2 - \alpha_1)(\alpha_2 - \alpha_3)} + \frac{(A - \alpha_1 I)(A - \alpha_2 I)e^{\alpha_3 t}}{(\alpha_3 - \alpha_1)(\alpha_3 - \alpha_2)} .$$

### 3.1 Orthogonality and Idempotency of the $C_{k1}$.

A further conclusion to be made at this point occurs when, using (5), we multiply $C_{j1}$ and $C_{i1}$ $(i \neq j)$; this yields

$$C_{j1}C_{i1} = a_j a_i \prod_{\substack{\ell=1 \\ \ell \neq j}}^{n} (A - \alpha_\ell I) \prod_{\substack{\ell=1 \\ \ell \neq i}}^{n} (A - \alpha_\ell I)$$

$$= a_j a_i \; f(A) \prod_{\substack{\ell=1 \\ \ell \neq i,\ \ell \neq j}}^{n} (A - \alpha_\ell I) = 0$$

by the Cayley-Hamilton Theorem. Therefore, as conjectured earlier, the $C_{k1}$ are pairwise orthogonal when the eigenvalues of A are distinct.

Using the first of the initial conditions in (2)' and this orthogonality we have $C_{j1} = C_{j1} \sum_{k=1}^{n} C_{k1} = C_{j1}^2$, $j = 1, 2, \ldots n$, i.e., each $C_{j1}$ is idempotent.

We summarize section (3) in the Theorem I. Given A and $f(\lambda)$ with distinct eigenvalues $\alpha_1, \alpha_2, \ldots \alpha_n$ we have

a) $e^{At} = \sum_{k=1}^{n} C_{k1} e^{\alpha_k t}$

b) $C_{k1} = \prod_{\substack{\ell=1 \\ \ell \neq k}}^{n} \frac{(A - \alpha_\ell I)}{(\alpha_k - \alpha_\ell)} \qquad k = 1, 2, \ldots n$

c) $C_{i1}C_{j1} = \begin{cases} C_{i1} & \text{if } i = j \text{ (Idempotent)} \\ 0 & \text{if } i \neq j \text{ (Orthogonal)} \end{cases}$

d) All $C_{k1}$ are polynomials of degree $n - 1$ in A and they commute.

### 4. Properties of the $C_{kj}$ when $f(\lambda)$ has multiple roots.

## 4.1 Minimal Polynomial for a given matrix.

First consider the example:

$$A = \begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \text{for which } f(\lambda) = (\lambda - 1)^4.$$

According to (1)

$$e^{At} = (C_{11} + C_{12}t + C_{13}t^2 + C_{14}t^3)e^t.$$

Using the initial conditions we obtain:

$$C_{11} = I,\ C_{12} = (A - I),\ C_{13} = \frac{1}{2!}(A - I)^2,\ C_{14} = \frac{1}{3!}(A - I)^3 .$$

However:

$$C_{13} = (A - I)^2 = \begin{pmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}^2 = 0 ,$$

and, therefore, $C_{13} = C_{14} = 0$. For the given matrix $(\lambda - 1)^4 = 0$ is the characteristic equation and $(A - I)^4 = 0$. But it is also true for this matrix that $(A - I)^3 = (A - I)^2 = 0$ and $A - I \neq 0$. In this case A not only satisfies its characteristic equation, it also satisfies the equations $(\lambda - 1)^3 = 0$ and $(\lambda - 1)^2 = 0$. For a general square matrix A, the lowest degree monic (leading coefficient equal to 1) polynomial that A satisfies is called the minimal polynomial for A. In the example above $(\lambda -1)^2$ is the minimal polynomial for the given A and for this matrix $(D - 1)^2e^{At} = 0$. Our solution for $e^{At}$ should then have been written

$$e^{At} = (C_{11} + C_{12}t)e^t.$$

In general, if $\psi(\lambda)$ is the minimal polynomial for a given matrix A and the degree of $\psi(\lambda)$ is $m(\leq n)$, then $e^{At}$ satisfies the initial value problem:

$$\psi(D)e^{At} = 0$$

and

$$D^k(e^{At})_{t=0} = A^k \qquad k = 0, 1, 2, \dots m - 1 .$$

4.2 <u>A Redefining of the $C_{kj}$</u>

Let the distinct eigenvalues of A be $\alpha_1, \alpha_2, \dots \alpha_s$ with $\psi(\lambda) = \prod_{k=1}^{s} (\lambda - \alpha_k)^{\mu_k}$ the minimal polynomial for A (written in factored form), $\mu_k \geq 1$, and $\sum_{k=1}^{s} \mu_k = m(\leq n)$ the degree of $\psi(\lambda)$. Following the theory presented in [2] we define $X_k(\lambda) = \psi(\lambda)/(\lambda - \alpha_k)^{\mu_k}$; then $X_k(\lambda)$ and $(\lambda - \alpha_k)^{\mu_k}$ are relatively prime. Therefore, there exist polynomials $p_k(\lambda)$, $q_k(\lambda)$ (degree of $p_k(\lambda) < \mu_k$) such that:

(7) $$p_k(\lambda)X_k(\lambda) + q_k(\lambda)(\lambda - \alpha_k)^{\mu_k} \equiv 1, \quad k = 1, 2, \dots s.$$

Then define:

(8) $$E_k(\lambda) = p_k(\lambda)X_k(\lambda) \quad \text{and}$$
$$E_k(A) = E_k = p_k(A)X_k(A) \qquad k = 1, 2, \dots s.$$

We note from (8) that for $k \neq \ell$ $E_k(\lambda)E_\ell(\lambda)$ is a polynomial multiple of $\psi(\lambda)$ and, therefore, $E_k \cdot E_\ell = 0$ for $k \neq \ell$ (i.e. the $E_k$ are pairwise orthogonal).

From (7) we form the product:

$$\prod_{k=1}^{s} q_k(\lambda)(\lambda - \alpha_k)^{\mu_k} = \psi(\lambda) \prod_{k=1}^{s} q_k(\lambda) = \prod_{k=1}^{s} (1 - E_k(\lambda)).$$

Replacing $\lambda$ by A in this last equation we obtain

$$\psi(A) \prod_{k=1}^{s} q_k(A) = 0 = \prod_{k=1}^{s} (I - E_k) = I - \sum_{k=1}^{s} E_k ,$$

which follows from the definition of $\psi(\lambda)$ and the orthogonality of the $E_k$. From this we have

(9) $$I = \sum_{k=1}^{s} E_k.$$

Multiplying through (9) by $E_\ell$, using the orthogonality, we have

$$E_\ell = E_\ell^2 \qquad \ell = 1, 2, \ldots s$$

i.e., the $E_\ell$ are idempotent.

Next, define:

$$(10) \qquad \begin{aligned} N_k(\lambda) &= (\lambda - \alpha_k)E(\lambda) \qquad \text{and} \\ N_k(A) &= N_k = (A - \alpha_k I)E_k \qquad (\text{for } \mu_k > 1). \end{aligned}$$

If $\mu_k = 1$, then $\alpha_k$ is a simple root of $\psi(\lambda)$ and for such roots $N_k(A) = \psi(A) = 0$. We note from (10) that $N_k^{\mu_k} = (A - \alpha_k I)^{\mu_k} E_k = \psi(A) = 0$ and $N_k^{\mu_k - 1} \neq 0$. The $N_k$ are said to be <u>nilpotent of index</u> $\mu_k$.

Additional conclusions from definitions (8) and (10) are:

(a) All $E_k$, $N_k$ are polynomials in A and, therefore, commute.

(b) $E_k N_k = N_k$, $E_k N_j = N_k N_j = 0$ $(k \neq j)$.

(c) We have the identity:

$$(11) \qquad A = \sum_{k=1}^{s} E_k(\alpha_k I + N_k)$$

which can be seen as follows:

$$\sum_{k=1}^{s} E_k(\alpha_k I + N_k) = \sum_{k=1}^{s} (\alpha_k E_k + AE_k - \alpha_k E_k)$$

$$= \sum_{k=1}^{s} AE_k = A \sum_{k=1}^{s} E_k = AI = A.$$

If we replace A in $e^{At}$ by the identity (11), we have:

$$(12) \qquad \begin{aligned} e^{At} &= e^{\left(\sum_{k=1}^{s} E_k(\alpha_k I + N_k)\right)t} = e^{\sum_{k=1}^{s} E_k \alpha_k t} \cdot e^{\sum_{k=1}^{s} N_k t} \\ &= \sum_{k=1}^{s} e^{\alpha_k t} \left\{E_k + N_k t + \frac{(N_k t)^2}{2!} + \ldots + \frac{(N_k t)^{\mu_k - 1}}{(\mu_k - 1)!}\right\}, \end{aligned}$$

and this must be identical with (1) i.e.

$$= \sum_{k=1}^{s} e^{\alpha_k t} \{C_{k1} + C_{k2} t + \ldots + C_{k\mu_k} t^{\mu_k - 1}\}$$

Rewriting (12) using the definition (10) we have:

$$(13) \qquad e^{At} = \sum_{k=1}^{s} E_k e^{\mu_k t} \{I + (A - \alpha_k I)t + \dots + \frac{(A - \alpha_k I)^{\mu_k - 1} t^{\mu_k - 1}}{(\mu_k - 1)!} \} .$$

In (13) we observe that we have only the $E_k$ ($k = 1, 2, \dots s$) to determine. These can be calculated from the definition (8) or from (2) calculating only the $C_{k1} = E_k (k = 1, 2, \dots s)$.

4.3 Summary of section 4 and examples.

We summarize this section in:

Theorem 2. Given A with minimal polynomial $\psi(\lambda) = \prod_{k=1}^{s} (\lambda - \alpha_k)^{\mu_k}$, $m \leq n$, we have:

$$e^{At} = \sum_{k=1}^{s} E_k e^{\alpha_k t} \{I + (A - \alpha_k I)t + \dots + \frac{(A - \alpha_k I)^{\mu_k - 1} t^{\mu_k - 1}}{(\mu_k - 1)!} \} ,$$

in which the $E_k$ satisfy the following:

(a) $I = \sum_{k=1}^{s} E_k$

(b) $E_k E_j = \begin{cases} 0 \text{ if } i \neq j \\ E_k \text{ if } k = j \end{cases}$

(c) $(A - \alpha_k I)E_k = N_k$ are nilpotent of index $\mu_k$

(d) all $E_k$ and $N_k$ are polynomials in A.

Up to this point we have determined $e^{At}$ for any A(3 x 3) with distinct eigenvalues. Now let us complete this calculation for any 3 x 3 matrix A. To this end we have the following cases and calculations:

(i) All eigenvalues of A are equal to $\alpha$, and $\psi(\lambda) = f(\lambda) = (\lambda - \alpha)^3$.

In this case:

$$e^{At} = e^{t}(E_1 + N_1 t + \frac{N_1 t^2}{2!}) \text{ in which } E_1 = I \text{ and } N_1 = (A - \alpha I).$$

(ii) Again all eigenvalues equal $\alpha$, but $\psi(\lambda) = (\lambda - \alpha)^2$. In this case $E_1 = I$ and $e^{At} = e^{\alpha t}(I + (A - \alpha I)t)$.

(iii) $\psi(\lambda) = (\lambda - \alpha)$. In this case $E_1 = I$ and $e^{At} = e^{\alpha t}I$ a scalar matrix.

(iv) $\psi(\lambda) = f(\lambda) = (\lambda - \alpha_1)^2(\lambda - \alpha_2)$, $\alpha_1 \neq \alpha_2$.

In this case:

$$e^{At} = e^{\alpha_1 t}(E_1 + N_1 t) + e^{\alpha_2 t}E_2 .$$

We can solve for $E_1$ and $E_2$ ($N_1 = (A - \alpha_1 I)E_1$) using the initial conditions (2) or definitions (7) and (8). In view of (7) we have (replacing $\lambda$ by A):

$$E_1 + q_1(A)(A - \alpha_1 I)^2 = I.$$

However, we know that $E_1 + E_2 = I$ and, therefore, $E_2 = q_1(A)(A - \alpha_1 I)^2$ which means we obtain both $E_1$ and $E_2$ simultaneously by using (7) and (8). Accordingly using (7) and (8): $(a\lambda + b)(\lambda - \alpha_2) + c(\lambda - \alpha_1)^2 \equiv 1$ which must hold for all $\lambda$ and, therefore, we have:

$$a = \frac{-1}{(\alpha_2 - \alpha_1)^2}, \quad b = \frac{-(\alpha_2 - 2\alpha_1)}{(\alpha_2 - \alpha_1)^2}, \quad c = \frac{1}{(\alpha_2 - \alpha_1)^2} .$$

Therefore:

$$E_1 = \frac{-1}{(\alpha_2 - \alpha_1)^2}(A - \alpha_2 I)(A + (\alpha_2 - 2\alpha_1)I)$$

$$E_2 = \frac{1}{(\alpha_2 - \alpha_1)^2}(A - \alpha_1 I)^2 \quad \text{and}$$

$$N_1 = \frac{-(A - \alpha_1 I)}{(\alpha_2 - \alpha_1)^2}(A - \alpha_2 I)(A - 2\alpha_1 I + \alpha_2 I) .$$

Accordingly:

$$e^{At} = e^{\alpha_1 t}(E_1 + N_1 t) + e^{\alpha_2 t}E_2.$$

(v) The last case is that in which $f(\lambda) = (\lambda - \alpha_1)^2(\lambda - \alpha_2)$ as in (iv) but $\psi(\lambda) = (\lambda - \alpha_1)(\lambda - \alpha_2)$.

In this case $\alpha_1$ and $\alpha_2$ are simple roots of $\psi(\lambda)$ and therefore,

$$e^{At} = \frac{e^{\alpha_1 t}}{\alpha_1 - \alpha_2}(A - \alpha_2 I) + \frac{e^{\alpha_2 t}}{\alpha_2 - \alpha_1}(A - \alpha_1 I)$$

by virtue of the results in section 3.

In view of the example in section 3 and the 5 cases above we have obtained $e^{At}$ for any 3 x 3 matrix.

It is of interest to note that for any $A(n \times n)$ in which either (a) A has n equal eigenvalues or the opposite extreme (b) A has n distinct eigenvalues we have that $e^{At}$ can be written immediately as:

(a) $$e^{At} = e^{\alpha t} \sum_{k=0}^{n-1} \frac{(A - \alpha I)^k t^k}{k!} .$$

If in this case $\psi(\lambda) = (\lambda - \alpha)^m$, $m < n$, then the summation would extend only to $m - 1$ since $(A - \alpha I)^m = 0$.

(b) $$e^{At} = \sum_{k=1}^{n} e^{\alpha_k t} \prod_{\substack{j=1 \\ j \neq k}}^{n} \frac{(A - \alpha_j I)}{(\alpha_k - \alpha_j)} .$$

## 5. Other representations for $e^{At}$.

In [4] and [6] $e^{At}$ is obtained by use of the Lagrange-Sylvester interpolation polynomial. By using the eigenvalues of A as the interpolation points we obtain the form of $e^{At}$ in equation (12). In [5] and [7] representations of $e^{At}$ are obtained in one case in powers of A and in others in powers of $A - \alpha_i I$ ($\alpha_i$ - eigenvalues of A). In any case, if all these representations were given in powers of the same $(A - \beta_i I)$ then they would, of course, all be the same.

Another representation for $e^{At}$, which has applications to solutions of first order linear systems of simultaneous differential equations with constant coefficients, is obtained as follows. Suppose we have given:

(14) $\chi'(t) = A\chi(t)$

A an n x n matrix with constant elements, $\chi(t)$ an n x 1 vector function of t. Suppose we have found a fundamental set of solutions for (14) namely $\chi_1(t), \chi_2(t), \ldots \chi_n(t)$. Then define:

(15) $X(t) = (\chi_1(t), \chi_2(t), \ldots \chi_n(t))$ ,

which is an n x n matrix whose columns are the elements of the fundamental set. X(0) is nonsingular and we define

(16) $G(t) = X(t)(X(0))^{-1}$;

then

$$G(t) = e^{At}.$$

This equation follows from differentiating (16), which yields

$$G'(t) = (\chi_1'(t), \chi_2'(t), \ldots \chi_n'(t))(X(0))^{-1} .$$

By (14) G'(t) can be written:

$$G'(t) = (A\chi_1(t), A\chi_2(t), \ldots A\chi_n(t))(X(0))^{-1}$$
$$= AX(t)(X(0))^{-1} = AG(t) .$$

From this last equation it follows that:

$$G^{(k)}(t) = A^k G(t) \qquad k=0, 1, \ldots,$$

which in turn yields:

$$G^{(k)}(0) = A^k \qquad k=0, 1, \ldots .$$

Moreover,

$$f(D)G(t) = f(A)G(t) = 0$$

by the Cayley-Hamilton theorem.

Therefore, G(t) satisfies the initial value problem which is also satisfied by $e^{At}$. By uniqueness of such solutions we conclude:

$$G(t) = X(t)(X(0))^{-1} = e^{At}.$$

## 6. Some applications of $e^{At}$.

From the last remarks in section 5, and since $(X(0))^{-1}$ is nonsingular, $X(t)(X(0))^{-1} = e^{At}$ has columns that are linear combinations of the columns of $X(t)$ and form another fundamental set for the differential equation (14):

$$\chi'(t) = A\chi(t).$$

Let us use this fact to solve the system

$$\chi'(t) = \begin{pmatrix} 4 & -5 & 3 \\ 2 & -3 & 2 \\ -1 & 1 & 0 \end{pmatrix} \chi(t),$$

The given matrix has characteristic polynomial $f(\lambda) = \psi(\lambda) = (\lambda - 1)^2(\lambda + 1)$
Using case (iv) in section 4 with $\alpha_1 = 1$ and $\alpha_2 = -1$ we have:

$$e^{At} = e^t \left\{ \begin{pmatrix} 2 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ -1 & 1 & -1 \end{pmatrix} t \right\} + e^{-t} \begin{pmatrix} -1 & 2 & -1 \\ -1 & 2 & -1 \\ 0 & 0 & 0 \end{pmatrix}.$$

$$= \begin{pmatrix} e^t(2+t) - e^{-t} & -e^t(2+t) + 2e^{-t} & e^t(1+t) - e^{-t} \\ e^t - e^{-t} & -e^t + 2e^{-t} & e^t - e^{-t} \\ -te^t & te^t & e^t(1-t) \end{pmatrix}$$

The columns of the latter matrix constitute a fundamental set for the differential equation and its general solution is:

$$\chi(t) = e^{At}\chi(0) ,$$

where $\chi(0)$ forms the initial conditions for $\chi(t)$ (given at $t = 0$).

As a matter of fact, having obtained $e^{At}$ for all 3 x 3 matrices A we have therefore obtained fundamental sets for all systems of differential equations $\chi'(t) = A\chi(t)$ with A a 3 x 3 matrix of constants and $\chi(t)$ a 3 x 1 vector function of t.

Knowing $e^{At}$ and $e^{-At}$ we define

$$\cosh At = 1/2(e^{At} + e^{-At}) \quad \text{and}$$

$$\sinh At = 1/2(e^{At} - e^{-At}).$$

Equally well we know $e^{iAt}$ and $e^{-iAt}$ ($i = \sqrt{-1}$) and define:

$$\cos At = 1/2(e^{iAt} + e^{-iAt})$$

$$\sin At = \frac{1}{2i}(e^{iAt} - e^{-iAt}).$$

As an example, we indicate the expansion of cosh At:

$$\cosh At = \sum_{k=1}^{s} \cosh \alpha_k t(E_k + \frac{N_k^2 t^2}{2!} + \ldots) + \sum_{k=1}^{s} \sinh \alpha_k t(N_k t + \frac{N_k^3 t^3}{3!}$$

with each of these two sums terminating with the term containing either $t^{\mu_k - 2}$ or $t^{\mu_k - 1}$, depending upon whether $\mu_k$ is even or odd.

In [1] T. M. Apostol considers the system of differential equations

$$Y''(t) = AY(t)$$

and writes the solution in terms of two matrix functions

$$C(t) = \sum_{k=0}^{\infty} \frac{t^{2k} A^k}{(2k)!}, \quad S(t) = \sum_{k=0}^{\infty} \frac{t^{2k+1} A^k}{(2k+1)!}.$$

$C(t)$ is precisely $\cosh \sqrt{A}\, t$ and $S(t)$ is $(\sqrt{A})^{-1} \sinh \sqrt{A}\, t$ provided that $\sqrt{A}$ is defined and nonsingular. Clearly one would define $\sqrt{A}$ to be that matrix B such that $B^2 = A$. It turns out that B is not unique (as one would suspect), in fact, there may be as many as $2^n$ matrices B such that $B^2 = A$. However, these B- matrices may be calculated as follows. If A is similar to a diagonal matrix so that

$$A = T^{-1} \text{Diag } \{\alpha_1, \alpha_2, \ldots \alpha_n\} T,$$

then

$$\sqrt{A} = T^{-1} \text{Diag } \{\pm \alpha_1^{1/2}, \pm \alpha_2^{1/2}, \ldots \pm \alpha_n^{1/2}\} T.$$

If A is not similar to a diagonal matrix and is nonsingular, then $\sqrt{A}$ can be obtained as follows: From (11)

$$A = \sum_{k=1}^{s} E_k(\alpha_k I + N_k) = \sum_{k=1}^{s} E_k \alpha_k (I + \frac{N_k}{\alpha_k}) \ ;$$

then

$$\sqrt{A} = \sum_{k=1}^{s} \pm \alpha_k^{1/2} E_k (I + \frac{N_k}{\alpha_k})^{1/2}$$

If $(I + \frac{N_k}{\alpha_k})^{1/2}$ is expanded by the binomial theorem then this expansion would terminate with the term $N_k^{\mu_k - 1}$ since $N_k$ is nilpotent of index $\mu_k$.

Knowing how to compute $\sqrt{A}$ in some cases we then have for these cases the solutions to

(a) $Y''(t) + AY(t) = 0$

given by

$$Y(t) = (\cos \sqrt{A}\, t)Y_1 + (\sin \sqrt{A}\, t)Y_2;$$

or

(b) $Y''(t) - AY(t) = 0$

given by

$$Y(t) = (\cosh \sqrt{A}\, t)Y_1 + (\sinh \sqrt{A}\, t)Y_2.$$

In (a) and (b)

$Y(0) = Y_1$ and $Y'(0) = \sqrt{A}\, Y_2$.

As a last application we will consider: let A be a 3 x 3 matrix with characteristic function $f(\lambda) = \psi(\lambda) = (\lambda - \alpha_1)^2(\lambda - \alpha_2)$ (case (iv) in section 4), and suppose we are given the nonhomogeneous system

(17) $x'(t) = Ax(t) + a_1 e^{\alpha_1 t}$ ,

in which $x(t)$ is a 3 x 1 vector function of t to be determined and $a_1$ is a 3 x 1 constant vector. Multiplying through (17) by $e^{-At}$ yields

$$e^{-At}x'(t) - Ae^{-At}x(t) = D(e^{-At}x(t)) = e^{-At}a_1e^{\alpha_1 t}.$$

Integrating this last equation we obtain

$$(18) \qquad x(t) = e^{At}\int^t e^{-A\tau}a_1e^{\alpha_1\tau}d\tau + e^{At}a_2 ,$$

in which $a_2$ is a 3 x 1 vector whose elements are arbitrary constants. From case (iv)

$$(19) \qquad e^{At} = e^{\alpha_1 t}(E_1 + N_1t) + e^{\alpha_2 t}E_2.$$

Substituting (19) into (18), using the orthogonality, idempotentcy and nilpotentcy of $E_1$, $E_2$ and $N_1$ and integrating we obtain

$$x(t) = e^{\alpha_1 t}\{(E_1 + \frac{N_1t}{2})t + \frac{E_2}{\alpha_1 - \alpha_2}\}a_1 + \{e^{\alpha_1 t}(E_1 + N_1t) + e^{\alpha_2 t}E_2\}a_2$$

as the complete solution to (17).

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