A TREATISE ON

SPHERICAL ASTRONOMY

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A TREATISE ON

SPHERICAL ASTRONOMY

BY

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PREFACE

BY Spherical Astronomy I mean that part of Mathematical Astronomy which lies between the vast domain of Dynamical Astronomy on the one hand and the multitudinous details of Practical Astronomy on the other.

I have aimed at providing for the student a book on Spherical Astronomy which is generally within the limits thus indicated, but I have not hesitated to transgress those limits now and then when there seemed to be good reason for doing so. For example I have just crossed the border of Dynamical Astronomy in Chapter VII., and in two concluding chapters I have so far entered on Practical Astronomy as to give some account of the fundamental geometrical principles of astronomical instruments.

It has been assumed that the reader of this book is already acquainted with the main facts of Descriptive Astronomy. The reader is also expected to be familiar with the ordinary processes of Plane and Spherical Trigonometry and he should have at least an elementary knowledge of Analytic Geometry and Conic Sections as well as of the Differential and Integral Calculus. It need hardly be added that the student of any branch of Mathematical Astronomy should also know the principles of Statics and Dynamics.

As a guide to the student who is making his first acquaintance with Spherical Astronomy, I have affixed an asterisk to the titles of those articles which he may omit on a first reading; the articles so indicated being rather more advanced than the articles which precede or follow.

Such articles as relate to the more important subjects are generally illustrated by exercises. In making a selection from the large amount of available material I have endeavoured to choose exercises which not only bear directly on the text, but also have some special astronomical or mathematical interest. It will be seen that the Tripos examinations at Cambridge and many College examinations at Cambridge and elsewhere have provided a large proportion of the exercises. I have also obtained exercises from many other sources which are duly indicated.

The work on the subject to which I have most frequently turned while preparing this volume is Brünnow's *Spherical Astronomy*, a most excellent book which is available in English and French translations as well as in its original German. Among recent authors I have consulted Valentiner's extensive *Handwörterbuch der Astronomie* which no student of astronomy can afford to overlook, and I have learned much from the admirable writings of Professor Newcomb.

I have to acknowledge with many thanks the assistance which friends have kindly rendered to me. Mr Arthur Berry has furnished me with many solutions of exercises, more especially of Tripos questions. Dr J. L. E. Dreyer has read over the chapter on Aberration and made useful suggestions. Mr W. E. Hartley has helped in the correction of the proofs as well as in the revision of parts of the manuscript. Mr A. R. Hinks has given me help in the correction of the proofs and I am also indebted to him for assistance in the chapter on the Solar Parallax. Dr A. A. Rambaut has devoted much time to the reading of proofs and has assisted in many other ways. Mr F. J. M. Stratton has revised some of the pages, especially those on the rotation of the moon. Dr E. T. Whittaker has given me useful suggestions especially in the chapter on Refraction, and he has also helped in reading proofs, and my son, Mr R. S. Ball, has drawn many of the diagrams. Lastly, I must acknowledge my obligation to the Syndics of the University Press, who have met all my wishes in the kindest manner.

The list of parallaxes of stars (p. 328) is based on more extensive lists given by Newcomb in *The Stars* and Kapteyn in the Groningen publications No. 8. The results stated for α Centauri, Sirius and α Gruis have been obtained by Sir D. Gill; those for Procyon, Altair, Aldebaran, Capella, Vega, Arcturus, by Dr Elkin; that for Cordoba Zone 5^h 243, by Dr De Sitter; that for 1830 Groombridge by Professor Kapteyn; that for 21185 Lalande by Mr H. N. Russell; that for Polaris by Pritchard; and that for 61 Cygni is a mean result.

I ought to add that when I use the word *ephemeris* I refer, so far as works in the English language are concerned, either to the *British Nautical Almanac* or to the *American Ephemeris*.

ROBERT S. BALL.

OBSERVATORY,
CAMBRIDGE,
18*th October*, 1908.

CONTENTS

CHAPTER I.

FUNDAMENTAL FORMULAE.

1. Spherical Trigonometry.

Let a, b, c, A, B, C be as usual the sides and angles of a spherical triangle. It is proved in works on spherical trigonometry that

$$\cos c = \cos a \cos b + \sin a \sin b \cos C \qquad\text{.........(1),}$$

$$\sin c \cos A = \cos a \sin b - \sin a \cos b \cos C \qquad\text{.........(2),}$$

$$\sin c \sin A = \sin a \sin C \qquad\text{..............................(3).}$$

Formula (2) may be conveniently obtained from (1) as follows.

Produce AC (Fig. 1) to H so that

$$CH = 90^\circ - b,$$

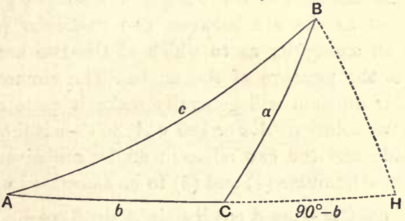


Fig. 1.

then from the triangle BAH, we have by (1)

$$\cos BH = \sin c \cos A,$$

and from triangle BCH

$$\cos BH = \cos a \sin b - \sin a \cos b \cos C.$$

Equating these values of $\cos BH$ we have formula (2).

The various formulae of the type (2) can thus be written down as occasion may require with but little tax on the memory.

The equations (1), (2), (3) are the simplest which can be employed when two sides a and b and the included angle C are given and it is required to find the parts A and c of the spherical triangle. It may at first be a matter of surprise that *three* equations should be required for the determination of only *two* quantities. But a definite solution cannot be obtained if the equations for finding A and c be fewer than three.

Suppose, for example, that only the pair of equations (1) and (2) had been given and that values for A and c had been found which satisfied those equations. It is plain that the same equations would be equally well satisfied by three other sets of values, namely

$$180° + A,\ 360° - c;\quad 360° - A,\ c;\quad 180° - A,\ 360° - c.$$

If, however, we require that the values to be adopted shall also satisfy the equation (3) then the last two pairs of values would be excluded. We thus see that when (1), (2) and (3) are all satisfied by A, c the only other solution is $180° + A$, $360° - c$.

As to this remaining ambiguity it must be remembered that the length of the great circle joining two points A and B on a sphere is generally ambiguous. It may be either AB or $360° - AB$. In like manner if the angle between two great circles is even defined as the arc between two particular poles there will still be an ambiguity as to which of the two arcs between these poles is the measure of the angle. The circumstances of each particular problem will generally make it quite clear as to which of the two solutions A, c or $180° + A$, $360° - c$ is that required.

If one side and the two adjacent angles are given then we require two new formulae (4) and (5) to be associated with (3)

$$\cos C = -\cos A \cos B + \sin A \sin B \cos c \quad \text{.........(4)},$$

$$\sin C \cos a = \cos A \sin B + \sin A \cos B \cos c \quad \text{.........(5)},$$

$$\sin C \sin a = \sin A \sin c \quad \text{.................................(3)}.$$

The formulae (4) and (5) are obtained from (1) and (2) respectively by the general principle of the Polar triangle, viz., that any formula true for all spherical triangles remains true if, instead of a, b, c, A, B, C, we write

$$180° - A,\ 180° - B,\ 180° - C,\ 180° - a,\ 180° - b,\ 180° - c.$$

If we are given two sides and the angle between or two angles and the side between the triangle may also be solved by formulae easily deduced from (2) and (3) and of the type

$$\cot a \sin b = \cot A \sin C + \cos b \cos C \ldots\ldots\ldots\ldots (6).$$

If a, b and C are given this will determine $\cot A$, and thus A is known for there will always be one value of A between 0° and 180° which will correspond to any value of $\cot A$ from $+\infty$ to $-\infty$. Of course $180° + A$ is also a solution.

In like manner if A, C, b were given, this formula would determine $\cot a$.

It may be noted that formula (6) shows the connection between four *consecutive* parts of the triangle as written round a circle (Fig. 2). As we may commence with any one of the elements there are six formulae of this type.

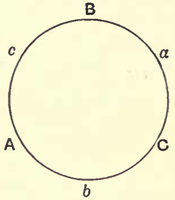


Fig. 2.

The following rule has been given* for remembering the formulae of the type (6).

"Of the angles and sides entering into any one of these formulae, one of the angles is contained by the two sides and may be called the *inner* angle, and one of the sides lies between the two angles and may be called the *inner* side. The formula may then be stated thus:—

(cosine of inner side) (cosine of inner angle)

= (sine of inner side) (cotangent of other side)

− (sine of inner angle) (cotangent of other angle)."

For example, in writing down the formula involving the four parts a, b, C, B we have C as the inner angle and a as the inner side, whence we obtain (6)

$$\cos a \cos C = \sin a \cot b - \sin C \cot B.$$

If two sides a, c and the angle A opposite to a are given, then from (3) we obtain $\sin C$. If $\sin C > 1$ the problem is impossible.

* In Leathem's edition of Todhunter's *Spherical Trigonometry* (1903), p. 27.

If $\sin C < 1$ there is still nothing to show which of two supplementary values is to be given to C, and unless some additional information is obtainable, showing whether C is acute or obtuse, the problem is ambiguous.

If two angles and a side opposite to one of them are given, then, from formula (3) the side opposite the other angle will be determined, subject as before to an ambiguity between the arc and its supplement.

When the ambiguity in either case is removed the problem is reduced to that in which two sides and the angles opposite to both are known. From equations (1) and (2) the following formula is easily deduced

$$\tan b = \frac{\tan a \cos C + \tan c \cos A}{1 - \tan a \cos C \tan c \cos A},$$

and (2) will show whether b or $180° + b$ is to be used. The calculation may be simplified by taking

$$\tan \theta = \tan a \cos C; \ \tan \phi = \tan c \cos A,$$

whence we find $b = \theta + \phi$.

By means of the polar triangle we obtain

$$-\tan B = \frac{\tan A \cos c + \tan C \cos a}{1 - \tan A \cos c \tan C \cos a},$$

from which B may be determined for (5) removes the ambiguity between B and $180° + B$. Also if we take

$$\tan \theta' = \tan A \cos c \ \text{ and } \ \tan \phi' = \tan C \cos a,$$

we find

$$B = 180° - \theta' - \phi'.$$

When the three sides are given, a spherical triangle may be solved as follows. Let $2s = a + b + c$, then

$$\tan \tfrac{1}{2} A = \sqrt{\frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}} \quad \text{................(7)},$$

by which A is found, and by similar formulae we obtain B and C.

If the three angles A, B, C were given, then making

$$2S = A + B + C,$$

we have

$$\tan \tfrac{1}{2} a = \sqrt{-\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}} \quad \text{..........(8)},$$

by which a is found, and similarly for b and c.

In the important case when the triangle is right-angled we make $C = 90°$, and from formulae like (1), (2), (3), we can show that

$$\sin c \cos A = \cos a \sin b \quad \text{......(9)},$$

$$\cos c = \cos a \cos b \quad \text{......(10)},$$

$$\sin c \sin A = \sin a \quad \text{......(11)},$$

$$\cos A = \tan b \cot c \quad \text{......(12)},$$

$$\tan A = \tan a \operatorname{cosec} b \quad \text{......(13)},$$

$$\cos a = \cos A \operatorname{cosec} B \quad \text{......(14)},$$

$$\sec c = \tan A \tan B \quad \text{......(15)}.$$

These formulae may be easily written down by the help of Napier's rules, for the enunciation of which the quantities a, b, $(90° - A)$, $(90° - c)$, $(90° - B)$, often called the "circular parts," are to be arranged inside a circle as shown in Fig. 3.

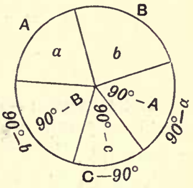


Fig. 3.

Any one of the circular parts being chosen as a "middle," the two on each side are termed "adjacents," and the two others are "opposites."

Formulae (10) to (15) are written down from Napier's rules which are as follows:

sine of middle = product of tangents of adjacents,

sine of middle = product of cosines of opposites.

Ten formulae can thus be obtained, for each of the five parts may be taken as a middle.

It is easy to show that whenever two sides and an angle, or two angles and a side of *any* spherical triangle are given, the triangle can be solved by Napier's rules if divided into two right-angled triangles by a perpendicular from one angle on the opposite side (see Ex. 2 on p. 8).

The formulae for a *quadrantal* triangle ($c = 90°$) can be written down also from the same diagram (Fig. 3). Napier's rules applied to the circular parts on the *outside* of the circumference give the ten formulae for the *quadrantal* triangle. As examples we thus find $\sin A = \sin a \sin C$ and $\cos b = -\tan A \cot C$ where A and $90° - b$ are respectively the middle parts.

The relation here implied between the right-angled triangle and the quadrantal triangle is shown by Fig. 4. If $AB = 90°$, and BC is produced to C' so that $BC' = 90°$, then $\angle C' = 90°$. Napier's rules, applied to the right-angled triangle $AC'C$, give the formulae belonging to the quadrantal triangle ABC.

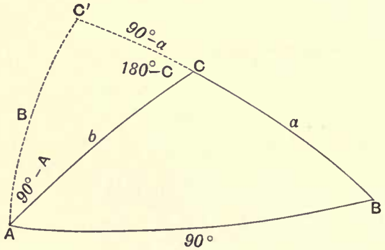


FIG. 4.

LOGARITHMS:—The usual notation employed in writing logarithms of the trigonometrical functions may be illustrated by an example.

The natural cosine of 25° is 0·9063078 and

$$\log\cos 25° = \log 9{\cdot}063078 - \log 10 = -0{\cdot}042724.$$

To obviate the inconvenience of negative logarithms this is sometimes written $\bar{1}{\cdot}957276$ which stands for

$$-1 + 0{\cdot}957276.$$

We shall however generally follow the more usual practice of the tables and add 10 to the logarithm of every trigonometrical function. When this change is made we use L instead of l in writing the word log. Thus in the preceding case the Log would be written 9·957276 and more generally

$$\text{Log}\cos\theta = \log\cos\theta + 10.$$

If it is necessary to state that the trigonometrical function of which the logarithm is used is a negative number it is usual to write (n) after the logarithm.

For example, if cos 155° occurred as a factor in an expression we should write 9·957276 (n) as its Logarithm, where the figures denote Log cos 25°.

It frequently happens that after an angle θ has been determined in the first part of a computation we have to employ certain trigonometrical functions of θ in the second part of the same computation. In this second part we have often a choice as to whether we shall employ one formula depending on $\operatorname{Log}\sin\theta$ or another depending on $\operatorname{Log}\cos\theta$. It is generally immaterial which formula the calculator employs, but if θ be nearly zero or nearly 90° one of the formulae will be uncertain and the other should be used. It is therefore proper to consider the principles on which the choice should be exercised in so far as any general principles can be laid down.

We may assume that, proper care having been taken, the work is free from numerical error so far as the necessary limitations of the tables will permit. But these very limitations imply that the value of θ we have obtained is only an approximate value. The calculator may, generally, protect the latter part of the work from becoming appreciably wrong notwithstanding that it is based on a quantity which is somewhat erroneous. The practical rule to follow is a very simple one. The two quantities $\operatorname{Log}\sin\theta$ and $\operatorname{Log}\cos\theta$ are not generally equal and the formula containing the greater should be used in the remainder of the calculation. This follows from the consideration that if θ be $>(<)\ 45°$ a small error in θ will have less effect on $\sin\theta\,(\cos\theta)$ than on $\cos\theta\,(\sin\theta)$.

Ex. 1. Show how the side a may be determined by the formula

$$\cot a \sin b = \cot A \sin C + \cos b \cos C$$

if we are given

$$A = 117°\,11'\,6''; \quad C = 154°\,13'\,54''; \quad b = 108°\,30'\,30''.$$

$\operatorname{Log}\cot A$	9·7106244 (n),	$\operatorname{Log}\cos C$	9·9545123 (n),
$\operatorname{Log}\sin C$	9·6382230,	$\operatorname{Log}\cos b$	9·5016652 (n),
$\operatorname{Log}\cot A\sin C$	9·3488474 (n),	$\operatorname{Log}\cos C\cos b$	9·4561775;

(Nat.) $\cot A \sin C$	$-0·2232789$,
,, $\cos C \cos b$	$+0·2858759$,
,, $\cot a \sin b$	$+0·0625970$;
$\operatorname{Log}\cot a\sin b$	8·7965535,
$\operatorname{Log}\sin b$	9·9769354,
$\operatorname{Log}\cot a$	8·8196181. $a = 86°\,13'\,24''$.

Ex. 2. Being given $b=57° 42' 39''$; $c=19° 18' 2''$; $A=120° 12' 36''$, find a and B by the method of right-angled triangles.

Draw CP $(=p)$ perp. to AB; then

Log sin b	9·9270432	
sin A	9·9366077	
sin p	9·8636509	$p=46° 55' 58''$
tan b	10·1993454	
cos $(180-A)$	9·7017154	
tan m	9·9010608	$m=38° 31' 45''$
cos p	9·8343291	
cos $(c+m)$	9·7262684	
cos a	9·5605975	$a=68° 40' 48''$
tan p	10·0293218	
cosec $(c+m)$	10·0723887	
tan B	10·1017105	$B=51° 38' 55''$

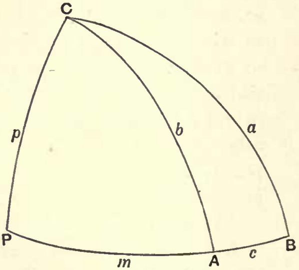


FIG. 5.

2. Delambre's and Napier's analogies.

The following equations are of great utility in spherical astronomy:

$$\sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A-B) = \cos \tfrac{1}{2}C \sin \tfrac{1}{2}(a-b) \quad\ldots\ldots\ldots(16),$$

$$\sin \tfrac{1}{2}c \cos \tfrac{1}{2}(A-B) = \sin \tfrac{1}{2}C \sin \tfrac{1}{2}(a+b) \quad\ldots\ldots\ldots(17),$$

$$\cos \tfrac{1}{2}c \sin \tfrac{1}{2}(A+B) = \cos \tfrac{1}{2}C \cos \tfrac{1}{2}(a-b) \quad\ldots\ldots\ldots(18),$$

$$\cos \tfrac{1}{2}c \cos \tfrac{1}{2}(A+B) = \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(a+b) \quad\ldots\ldots\ldots(19).$$

These equations are often described as Gauss' analogies, but their discovery is really due to Delambre*.

As Delambre's analogies are more convenient for logarithmic calculation than (1), (2), (3) and (4), (5), (6), they are often preferred for the solution of spherical triangles when a, b and C are given or when A, B and c are given.

It is frequently troublesome to remember these formulae without such assistance as is given by Rambaut's rule†.

We write the two rows of quantities

$$\tfrac{1}{2}(A+B),\quad \tfrac{1}{2}(A-B),\quad \tfrac{1}{2}C',$$
$$\tfrac{1}{2}(a+b),\quad \tfrac{1}{2}(a-b),\quad \tfrac{1}{2}c,$$

* For this statement as well as for the proofs of these formulae, reference may be made to Mr Leathem's edition of Todhunter's *Spherical Trigonometry* (1903), p. 36.

† See Dr A. A. Rambaut, *Astronomische Nachrichten*, No. 4135.

where $C' = 180° - C$. Then Rambaut's rule is as follows:

Sum (difference) in one row is always to be associated with cosine (sine) in the other row.

For example to obtain the Delambre analogy which contains $\sin \frac{1}{2}(A - B)$ we conclude from Rambaut's rule:

(1) that $\frac{1}{2}c$ must enter with a *sine* because A and B enter as a *difference*;

(2) that a and b must enter as a *difference* because $\frac{1}{2}(A - B)$ enters with a *sine*;

(3) that $\frac{1}{2}(a - b)$ must enter with a *sine* because A and B enter as a *difference*;

(4) that $\frac{1}{2}C'$ must enter with a *sine* because a and b enter as a *difference*.

Hence the analogy may be written down

$$\begin{aligned} \sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A - B) &= \sin \tfrac{1}{2}C' \sin \tfrac{1}{2}(a - b) \\ &= \cos \tfrac{1}{2}C \ \sin \tfrac{1}{2}(a - b). \end{aligned}$$

As an example of the use of Delambre's analogies we may employ the spherical triangle in which

$$\begin{aligned} a &= 62° \ 48' \ 54'', & A &= 93° \ 46' \ 36'', \\ b &= 57 \ \ 42 \ \ 39, & B &= 71 \ \ 29 \ \ 30, \\ c &= 25 \ \ 46 \ \ 6, & C &= 29 \ \ 11 \ \ 13. \end{aligned}$$

We shall suppose that a, b, C are given and find A, B and c.

The numerical values here set down are the Logs of the corresponding trigonometrical functions:

$$\tfrac{1}{2}C = 14° \ 35' \ 36''{\cdot}5$$

$$\tfrac{1}{2}(a + b) = 60° \ 15' \ 46''{\cdot}5; \ \ \tfrac{1}{2}(a - b) = 2° \ 33' \ 7''{\cdot}5$$

$$\begin{aligned} \sin \tfrac{1}{2}(a - b) \quad & 8{\cdot}6486286 \\ \cos \tfrac{1}{2}C \quad & 9{\cdot}9857578 \\ & 8{\cdot}6343864 = \sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A - B) \\ \sin \tfrac{1}{2}(a + b) \quad & 9{\cdot}9386752 \\ \sin \tfrac{1}{2}C \quad & 9{\cdot}4013301 \\ & 9{\cdot}3400053 = \sin \tfrac{1}{2}c \cos \tfrac{1}{2}(A - B) \\ \cos \tfrac{1}{2}(a - b) \quad & 9{\cdot}9995690 \\ \cos \tfrac{1}{2}C \quad & 9{\cdot}9857578 \\ & 9{\cdot}9853268 = \cos \tfrac{1}{2}c \sin \tfrac{1}{2}(A + B) \end{aligned}$$

$\cos\frac{1}{2}(a+b)$	9·6954999		
$\sin\frac{1}{2}C$	9·4013301		
	$9·0968300 = \cos\frac{1}{2}c\cos\frac{1}{2}(A+B)$		
$\cos\frac{1}{2}c\sin\frac{1}{2}(A+B)$	9·9853268	$\frac{1}{2}(A+B)$	82° 38′ 3″
$\cos\frac{1}{2}c\cos\frac{1}{2}(A+B)$	9·0968300	$\frac{1}{2}(A-B)$	11 8 33
$\tan\frac{1}{2}(A+B)$	0·8884968	$A=$	93 46 36
$\sin\frac{1}{2}c\sin\frac{1}{2}(A-B)$	8·6343864	$\frac{1}{2}(A+B)$	82° 38′ 3″
$\sin\frac{1}{2}c\cos\frac{1}{2}(A-B)$	9·3400053	$\frac{1}{2}(A-B)$	11 8 33
$\tan\frac{1}{2}(A-B)$	9·2943811	$B=$	71 29 30
*$\sin\frac{1}{2}c\cos\frac{1}{2}(A-B)$	9·3400053		
$\cos\frac{1}{2}(A-B)$	9·9917352		
$\sin\frac{1}{2}c$	9·3482701		
†$\cos\frac{1}{2}c\sin\frac{1}{2}(A+B)$	9·9853268		
$\sin\frac{1}{2}(A+B)$	9·9964012		
$\cos\frac{1}{2}c$	9·9889256		
$\sin\frac{1}{2}c$	9·3482701		
$\cos\frac{1}{2}c$	9·9889256		
$\tan\frac{1}{2}c$	9·3593445	$\frac{1}{2}c$	12° 53′ 3″

Hence

$$A = 93°\ 46'\ 36'';\quad B = 71°\ 29'\ 30'';\quad c = 25°\ 46'\ 6''.$$

From Delambre's analogies we easily obtain the following four formulae known as Napier's analogies:

$$\tan\tfrac{1}{2}(a+b) = \frac{\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c \quad \text{.........(20)},$$

$$\tan\tfrac{1}{2}(a-b) = \frac{\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}(A+B)}\tan\tfrac{1}{2}c \quad \text{.........(21)},$$

$$\tan\tfrac{1}{2}(A+B) = \frac{\cos\frac{1}{2}(a-b)}{\cos\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C \quad \text{.........(22)},$$

$$\tan\tfrac{1}{2}(A-B) = \frac{\sin\frac{1}{2}(a-b)}{\sin\frac{1}{2}(a+b)}\cot\tfrac{1}{2}C \quad \text{.........(23)}.$$

As an example of the solution of a triangle by Napier's analogies we may take

$$A = 23°\ 27';\quad B = 7°\ 15';\quad c = 74°\ 29';$$

* We use this rather than $\sin\frac{1}{2}c\sin\frac{1}{2}(A-B)$, because $\cos\frac{1}{2}(A-B)$ is $>\sin\frac{1}{2}(A-B)$ as already explained on p. 7.

† We use this rather than $\cos\frac{1}{2}c\cos\frac{1}{2}(A+B)$, because $\sin\frac{1}{2}(A+B)$ is $>\cos\frac{1}{2}(A+B)$.

and use four-figure logarithms which are quite accurate enough for many purposes.

$\cos\frac{1}{2}(A-B)=9{\cdot}9956$	$\sin\frac{1}{2}(A-B)=9{\cdot}1489$
$\sec\frac{1}{2}(A+B)=0{\cdot}0158$	$\operatorname{cosec}\frac{1}{2}(A+B)=0{\cdot}5772$
$\tan\frac{1}{2}c=9{\cdot}8809$	$\tan\frac{1}{2}c=9{\cdot}8809$
$\tan\frac{1}{2}(a+b)=9{\cdot}8923$	$\tan\frac{1}{2}(a-b)=9{\cdot}6070$
$\frac{1}{2}(a+b)=37°\ 58'$	$\frac{1}{2}(a-b)=22°\ 2'$

$$a=60°\ 0';\quad b=15°\ 56'.$$

As $\frac{1}{2}(a-b)$ and $\frac{1}{2}(a+b)$ are both $<45°$ the proper formula for finding C is (22) which may be written

$$\tan\tfrac{1}{2}C=\cos\tfrac{1}{2}(a-b)\sec\tfrac{1}{2}(a+b)\cot\tfrac{1}{2}(A+B)$$

$$\cos\tfrac{1}{2}(a-b)=9{\cdot}9671$$
$$\sec\tfrac{1}{2}(a+b)=0{\cdot}1033$$
$$\cot\tfrac{1}{2}(A+B)=0{\cdot}5614$$
$$\tan\tfrac{1}{2}C=0{\cdot}6318$$
$$C=153°\ 44'.$$

3. Accuracy attainable in Logarithmic Calculation.

When the logarithm of a trigonometrical function is given it is *generally* possible to find the angle with sufficient accuracy. But we often meet with cases in which this statement ceases to be quite true.

For example, suppose we are retaining only five figures in our logarithms and that we want to find θ from the statement that

$$\text{Log}\sin\theta=9{\cdot}99998.$$

This tells us nothing more than that θ must lie somewhere between 89° 23′ 7″ and 89° 31′ 25″. Nor will the retention of so many as seven places of decimals always prevent ambiguity. We note, for example, that every angle from 89° 56′ 19″ to 89° 57′ 8″ has as its Log sin the same tabular value, viz. 9·9999998.

We thus see that angles near 90° are not well determined from the Log sin, and in like manner angles near zero are not well determined by the Log cos. But all angles can be accurately found from the Log tan as will now be proved.

If θ receive a small increment h'' or in circular measure $h\sin 1''$ and the increment in $\text{Log}_{10}\tan\theta$ be x units in the 7th place of decimals, we have to find the equation between h and x.

Changing the common logs into Napierian logs by the modulus 0·4343, we have

$$x/10000000 = 0{\cdot}4343 \log_e \tan(\theta + h \sin 1'') - 0{\cdot}4343 \log_e \tan\theta$$
$$= 0{\cdot}4343 \log_e (1 + h \sin 1'' \cot\theta)$$
$$- 0{\cdot}4343 \log_e (1 - h \sin 1'' \tan\theta),$$

whence by expanding the logarithms

$$x = 4343000 \sin 1''(\tan\theta + \cot\theta)\, h \text{ very nearly},$$

which may be written

$$h = x \sin 2\theta/42{\cdot}1.$$

The greatest value of h is $x/42{\cdot}1$, and hence the concluded value of θ when Log tan θ is given could never be 1″ wrong unless Log tan θ was itself wrong to the extent of 0·0000042.

Ex. 1. Show that when 5-figure logarithms are used and the computation is exact to within two units of the last decimal the error of an angle determined from its tangent cannot exceed 5 seconds.

Ex. 2. Investigate the change in the value of an angle produced by the alteration of one unit in the last decimal of its Log sin, and show that under all circumstances it is more accurate to determine the angle by its tangent than by its sine.

Ex. 3. Prove that if θ is a small angle its value in seconds is given approximately by the expression $\operatorname{cosec} 1'' \sin\theta (\sec\theta)^{\frac{1}{3}}$, and show that even if θ be as much as 10° this expression will not be erroneous by so much as 1″.

Ex. 4. If θ is a small angle expressed in seconds, show that

$$\text{Log} \sin\theta = \log\theta + S,$$

where $$S = \tfrac{1}{3}(20 + \text{Log}\cos\theta) - 5{\cdot}3144251,$$

and as an example show that when $\theta = 2074''{\cdot}20$

$$\text{Log} \sin\theta = 8{\cdot}0024182.$$

The quantity S is given in Bruhn's tables. (Tauchnitz, Leipzig, 1870.)

Ex. 5. Determine the angle θ of which the Log sin is 8·0123456.

Tables which, like those of Bagay (Paris, 1829), give the functions for each second show that the required angle differs from 0° 35′ 22″ by not more than a small fraction of a second. To determine that fraction we compute $S = \frac{1}{3}(20 + \text{Log}\cos\theta) - 5{\cdot}3144251$ which becomes 4·6855672 by substituting 0° 35′ 22″ for θ in Log cos θ. Then from the equation

$\log\theta = \text{Log}\sin\theta - S$ we obtain $\theta = 2122''{\cdot}16$.

4. Differential formulae in a Spherical Triangle.

Six angles a, b, c, A, B, C will not in general be the sides and angles of a spherical triangle, for they must fulfil three conditions if they are to possess this property. This is obvious from the consideration that if these six quantities were indeed the parts of a triangle, then any three of them being given the other three could be determined.

Let us however assume that these six quantities are indeed the parts of a spherical triangle, and let them all receive small changes Δa, Δb, Δc, ΔA, ΔB, ΔC respectively. The quantities as thus altered $a+\Delta a$, &c. will in general no longer be the parts of a spherical triangle. If they are to be such parts they must satisfy three conditions, which it is now proposed to determine.

Differentiate the fundamental formula (1)

$$\cos a = \cos b \cos c + \sin b \sin c \cos A,$$

and we have

$$\begin{aligned} -\sin a \Delta a = &-\sin b \cos c \Delta b - \cos b \sin c \Delta c \\ &+ \cos b \sin c \cos A \Delta b + \sin b \cos c \cos A \Delta c \\ &- \sin b \sin c \sin A \Delta A. \end{aligned}$$

But from the formula (2) in § 1

$$\sin a \cos B = \cos b \sin c - \sin b \cos c \cos A,$$

$$\sin a \cos C = \sin b \cos c - \cos b \sin c \cos A;$$

whence by substitution, and writing the similar formulae

$$\left.\begin{aligned} \Delta a &= \cos C \Delta b + \cos B \Delta c + H \sin b \sin c \Delta A \\ \Delta b &= \cos A \Delta c + \cos C \Delta a + H \sin c \sin a \Delta B \\ \Delta c &= \cos B \Delta a + \cos A \Delta b + H \sin a \sin b \Delta C \end{aligned}\right\} \dots\dots\dots\text{(i)},$$

where $H = \sin A/\sin a = \sin B/\sin b = \sin C/\sin c.$

Proceeding in like manner from formulae (4), (5) we obtain the equivalent equations

$$\left.\begin{aligned} \Delta A &= -\cos c \Delta B - \cos b \Delta C + H^{-1} \sin B \sin C \Delta a \\ \Delta B &= -\cos a \Delta C - \cos c \Delta A + H^{-1} \sin C \sin A \Delta b \\ \Delta C &= -\cos b \Delta A - \cos a \Delta B + H^{-1} \sin A \sin B \Delta c \end{aligned}\right\} \dots\dots\text{(ii)}.$$

We have thus proved that if a, b, c, A, B, C are the parts of a spherical triangle, either set of equations (i) or (ii) expresses the three necessary and sufficient conditions that

$$a+\Delta a, \quad b+\Delta b, \quad c+\Delta c, \quad A+\Delta A, \quad B+\Delta B, \quad C+\Delta C$$

shall also be the parts of a spherical triangle.

If three of the differentials be zero, then the other three will also in general vanish. This is evident from the equations as it is also from the consideration that if three of the parts of a spherical triangle remain unaltered, then generally the other parts must also remain unaltered.

As an illustration of an exception to this statement let $C=90°$, and $\Delta b=0$, $\Delta c=0$, $\Delta B=0$. The second equation of (i) will in *this case* not require that $\Delta a=0$.

Ex. 1. Under what conditions can a spherical triangle undergo a small change such that $\Delta a=0$, $\Delta b=0$, $\Delta A=0$, $\Delta B=0$ while both Δc and ΔC are not zero?

From (ii) we see that $a=90°$, $b=90°$, whence $A=90°$, $B=90°$

Ex. 2. If a spherical triangle receive a small change which does not alter the sum of its three angles, show that the alterations in the lengths of the sides must satisfy the condition

$$\Delta a \sin(S-A)+\Delta b \sin(S-B)+\Delta c \sin(S-C)=0,$$

where $$S=\tfrac{1}{2}(A+B+C)$$

*5. The Art of Interpolation.

In the calculations of astronomy use is made not only of logarithmic tables but also of many other tables such, for example, as those which are found in every ephemeris. The art of interpolation is concerned with the general principles on which such tables are to be utilised.

Let y be a quantity, the magnitude of which depends upon the magnitude of another quantity x. We then say that y is a function of x and we express the relation thus

$$y=f(x) \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)},$$

where $f(x)$ denotes any function of x. This general form would include as particular cases such equations as

$$y=\log x \quad \text{or} \quad y=\text{Log}\tan x.$$

Suppose that the value zero is assigned to x, then the corresponding value y_0 of y is given by the relation $y_0=f(0)$. Let us next substitute successively h, $2h$, $3h$, ... for x in (i), and let the corresponding values of y be respectively y_1, y_2, y_3, &c. Then the essential feature of a table is that in one column we place the values of x, viz. 0, h, $2h$, $3h$, &c., and in another column beside it the corresponding values of y, viz. y_0, y_1, y_2, y_3, &c.

The value of x, often called the *argument*, advances by equal steps h, and each corresponding value of y, often called the *function*, is calculated with as much accuracy as is demanded by the purpose to which the table is to be applied.

Table for
$y = f(x)$.

x	y
0	y_0
h	y_1
$2h$	y_2
$3h$	y_3
⋮	⋮

The object of such a table is either to show the value of the function corresponding to a given value of the argument, or to show the value of the argument corresponding to a given value of the function.

We often require to know the numerical value of a function corresponding to an argument which is not explicitly shown in the table, but which lies between two consecutive tabular values of the argument. The converse problem also frequently arises, of finding the value of the argument corresponding to a value which lies between two consecutive tabular values of the function. It might at first be imagined that in either of these cases we should have to resort again to the original equation (i). This however is not necessary. The character of the functional relation has been so far imparted to the table that, when either of the quantities x or y is given, the other is ascertained by the art of interpolation now to be explained.

The nature of this art is most clearly illustrated by geometry. We can construct graphically the curve $y = f(x)$ in the usual manner. From the origin O we mark off along the axis of x a series of points A_1, A_2, A_3 at distances h, $2h$, $3h$ from O. We compute the corresponding values $y_1, y_2, y_3, \ldots$ of y from the formula $y = f(x)$. Then we erect ordinates A_1P_1, A_2P_2, &c., at A_1, A_2, &c., (Fig. 6) equal to the corresponding values y_1, y_2, &c. The points P_1, P_2, P_3, &c., will generally be found so placed that a curve can be drawn to pass smoothly through them. If the points A_1, A_2, &c., are sufficiently close together, *i.e. if h be small enough*, the trend of the curve will be so clearly indicated that there will be little ambiguity, and the curve $y = f(x)$ passing through P_1, P_2, P_3 will not, in general, appreciably differ within these limits from the curve just drawn through the same points. The true curve will of course depend upon the character of the function which y is of x. As however, in the art of interpolation, we are concerned with only a small part of the curve, it will be unnecessary to consider the particular characteristics of the special curve involved.

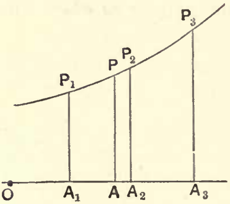


FIG. 6.

We need not therefore make use, for our present purpose, of the true curve $y = f(x)$ but of any osculating curve. We employ at first the osculating circle which, so far as the needs of interpolation are concerned, is sufficiently accurate. It is generally possible to draw this circle, whose arc coincides so nearly with that of the given curve at a given point that for a small distance the departure of the circle from the curve is insensible. We may, therefore, whatever be the true curve, regard that small part which concerns us as a circular arc. Accordingly, we describe a circle through P_1, P_2 and P_3, and we assume that for any point P between P_1 and P_3 the ordinate to the circle is the value of y for the corresponding x. Thus if AP be the ordinate then AP is the value of the function when $x = OA$. We shall make use of the circle to determine an expression for AP which shall involve only its abscissa and the coordinates of P_1, P_2, P_3. This may not, of course, be the value of y as obtained from the formula $y = f(x)$, but it will not differ appreciably therefrom.

Let $TMM'N'N$ be a circle and TLL' the tangent to it at T. Let LN and $L'N'$ be two lines which are both perpendicular to the axis of X. We have, by the property of the circle (Fig. 7),

$$LM \,.\, LN = LT^2,$$
$$L'M' \,.\, L'N' = L'T^2;$$

whence

$$\frac{LM}{L'M'} = \frac{LT^2}{L'T^2} \cdot \frac{L'N'}{LN}$$

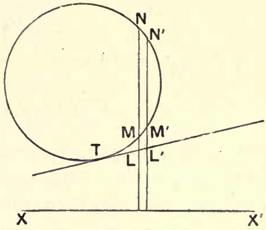


FIG. 7.

Let us now suppose that LN and $L'N'$ approach indefinitely close to T, then $L'N'/LN = 1$ and we have

$$LM : L'M' :: LT^2 : L'T^2.$$

Remembering that the arc of the curve is indistinguishable from that of its osculating circle in the vicinity of the point of contact, we obtain the principle on which interpolation is based and which may be thus expressed.

If a tangent TL be drawn touching a curve at T, and LM be an ordinate contiguous to T, then the intercept LM on that ordinate between the tangent and the curve is proportional to the square of TL.

In Fig. 8, O is the origin, y is the ordinate of P, and y_0 that of T, then, as we have shown, PB varies as BT^2, and therefore, as CT^2, also CB varies as CT; hence, if x be the abscissa of P

$$y - y_0 = lx + mx^2,$$

where l and m are constants for points in the neighbourhood of T. This is of course the equation of a parabola.

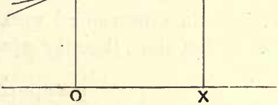


FIG. 8.

With a change in the constants to l' and m' we may write the equation as follows

$$y = y_0 + l'x + m'x(x - h);$$

we find l' and m' from the consideration that (h, y_1); $(2h, y_2)$ are to be points on the curve. The first gives

$$l' = \frac{y_1 - y_0}{h},$$

while $x = 2h$, $y = y_2$ gives

$$y_2 = y_0 + 2(y_1 - y_0) + 2h^2m',$$

or

$$m' = \frac{y_2 - 2y_1 + y_0}{2h^2},$$

and the equation becomes

$$y = y_0 + \frac{x}{h}(y_1 - y_0) + \frac{x(x-h)}{2h^2}(y_2 - 2y_1 + y_0) \quad \text{.........(i).}$$

Let y_0, y_1, y_2 be three consecutive values of the function y, where h is the difference of the arguments between the second and the first value and also between the third and the second. Then for any argument which is greater by x than the first argument but less than the third argument the above formula gives the required function.

The constants of this formula are very easily obtained from the table by the method of differences:

	1st Diff.	2nd Diff.
y_0		
	$y_1 - y_0$	
y_1		$y_2 - 2y_1 + y_0$
	$y_2 - y_1$	
y_2		

The first column contains three consecutive values of y. The second column shows the differences between each value and the preceding one. The third gives the differences between consecutive terms in the second. The third and higher differences are to be similarly formed if required.

If for brevity we write $y_1 - y_0 \equiv \Delta$ and $y_2 - 2y_1 + y_0 \equiv \Delta'$, and if we replace x by t as the time is generally the independent variable in astronomical work, and if we make the difference h the unit of time, then the equation becomes

$$y = y_0 + t\Delta + \frac{t(t-1)}{2}\Delta'.$$

The rate at which y changes with respect to t obtained by differentiating the last equation with respect to t, is

$$\frac{dy}{dt} = \Delta - \tfrac{1}{2}\Delta' + t\Delta',$$

from which it appears that the rate of increase will itself increase uniformly.

In two time-units the function increases from y_0 to y_2, hence its *average* rate of increase per time-unit is $\frac{1}{2}(y_2 - y_0)$, and as the rate increases uniformly it will attain its average value when half the time has elapsed, *i.e.* when the function has the value y_1. Hence we deduce the following result:

The rate at which the function is changing per unit of time at any epoch t is half the difference between the values of the function at one unit of time after t and at one unit of time before t.

Provision is often made in the Ephemeris for a more rapid process of interpolation by giving an additional column indicating the rate of variation of the function at the corresponding moment. We shall illustrate this by finding the South Declination of the Moon at $15 + t$ hours after Greenwich mean noon on Sept. 6, 1905.

The Ephemeris gives the South Declination of the Moon at 15^h G.M.T. to be $18° \, 38' \, 1''\cdot 2$ and the variation in 10 minutes as $23''\cdot 55$, the moon moving south. At 16^h on the same day, the next line of the table shows the variation in 10^m to be $22''\cdot 41$, and as the *rate* of variation may be regarded as declining uniformly, the variation per ten minutes at $(15 + \frac{1}{2}t)$ hours after noon is

$$23''\cdot 55 - 0''\cdot 57t.$$

This may be assumed to be the average rate of variation for the whole interval between 15^h and $15^h + t$, and since t is expressed in hours the total variation in that interval is found by multiplying the average rate by $6t$. We thus find for the South Declination of the Moon at $15^h + t$ on Sept. 6, 1905,

$$18° \, 38' \, 1''\cdot 2 + 141''\cdot 3t - 3''\cdot 42t^2.$$

Formulae of interpolation are also used for the inverse problem of finding the time at which a certain function reaches a specified value. Suppose, for example, that it is required to know the time on September 6th, 1905, when the Moon's South Declination is $18° \, 40'$. We have from the equation just found

$$18° \, 40' = 18° \, 38' \, 1''\cdot 2 + 141''\cdot 3t - 3''\cdot 42t^2.$$

This is a quadratic for t, and by neglecting the last term the root we seek is found to be approximately 0·86. Substituting this value in t^2 in the original equation it becomes

$$118\cdot 8 = 141\cdot 3t - 2\cdot 53,$$

whence $t = 0{\cdot}859$ and the required time is $15^h\,51^m{\cdot}5$. The other root of the quadratic is irrelevant.

It is easy to generalize the fundamental formula of interpolation given above.

Let us assume

$$y = A_0 + A_1 t + A_2 t(t-1) + A_3 t(t-1)(t-2) + A_4 t(t-1)(t-2)(t-3),$$

where A_0, A_1, A_2, A_3, A_4 are undetermined coefficients, to be so adjusted that when t becomes in succession 0, 1, 2, 3, 4, then y assumes the values y_0, y_1, y_2, y_3, y_4 respectively.

Hence by substitution we have

$$\begin{aligned} y_0 &= A_0, \\ y_1 &= A_0 + A_1, \\ y_2 &= A_0 + 2A_1 + 2A_2, \\ y_3 &= A_0 + 3A_1 + 6A_2 + 6A_3, \\ y_4 &= A_0 + 4A_1 + 12A_2 + 24A_3 + 24A_4; \end{aligned}$$

from which,

$$\begin{aligned} A_0 &= y_0, \\ A_1 &= y_1 - y_0, \\ A_2 &= \tfrac{1}{2}(y_2 - 2y_1 + y_0), \\ A_3 &= \tfrac{1}{6}(y_3 - 3y_2 + 3y_1 - y_0), \\ A_4 &= \tfrac{1}{24}(y_4 - 4y_3 + 6y_2 - 4y_1 + y_0). \end{aligned}$$

By this means we obtain the general formula of interpolation

$$y = y_0 + t\Delta_1 + \frac{t(t-1)}{1\,.\,2}\Delta_2 + \frac{t(t-1)(t-2)}{1\,.\,2\,.\,3}\Delta_3 + \frac{t(t-1)(t-2)(t-3)}{1\,.\,2\,.\,3\,.\,4}\Delta_4 \ldots\ldots \text{(ii)},$$

where Δ_1, Δ_2, Δ_3, Δ_4 are the successive differences.

Generally the last term may be neglected as

$$y_4 - 4y_3 + 6y_2 - 4y_1 + y_0$$

will usually be very small. If we make it equal to zero we have

$$y_2 = \tfrac{2}{3}(y_1 + y_3) - \tfrac{1}{6}(y_0 + y_4).$$

The quantities given in the tables are of course generally erroneous to an extent which may amount to almost half a digit in the last place. If y_1 and y_3 were each too large by half a digit in the last place, and y_0 and y_4 were each too small by the same amount, then even under these most unfavourable circumstances the combination can give an error of barely a single digit in the last place in y_2.

We have also from the same equation

$$y_4 = 4y_3 - 6y_2 + 4y_1 - y_0,$$

from which it might appear that from knowing y_0, y_1, y_2, y_3 we could compute y_4. But this "extrapolation" would be unsafe, for if y_1 and y_3 were each half a digit too large, and y_2 and y_0 each half a digit too small, as might conceivably happen, the total error in y_4 would amount to 7 or 8 digits of the last place.

The following method of interpolation due to Bessel should also be noted.

Let t be the argument measured from a point midway between two tabular arguments, and set down that part of the table as far as two tabular arguments on each side of the origin.

	1st Diff.	2nd Diff.	3rd Diff.
y_1			
	$y_2 - y_1$		
y_2		$y_3 - 2y_2 + y_1$	
	$y_3 - y_2$		$y_4 - 3y_3 + 3y_2 - y_1$
y_3		$y_4 - 2y_3 + y_2$	
	$y_4 - y_3$		
y_4			

We shall make

$$a = \tfrac{1}{2}(y_3 + y_2); \qquad b = y_3 - y_2,$$

$$c = \tfrac{1}{2}(y_4 - y_3 - y_2 + y_1); \quad d = y_4 - 3y_3 + 3y_2 - y_1,$$

where a, b, c, d are either quantities on the horizontal line through the origin or the arithmetic means of two adjacent quantities on opposite sides of the line.

We may write down

$$y = -\tfrac{1}{6} y_1 (t + \tfrac{1}{2})(t - \tfrac{1}{2})(t - \tfrac{3}{2}) + \tfrac{1}{2} y_2 (t + \tfrac{3}{2})(t - \tfrac{1}{2})(t - \tfrac{3}{2})$$
$$- \tfrac{1}{2} y_3 (t + \tfrac{3}{2})(t + \tfrac{1}{2})(t - \tfrac{3}{2}) + \tfrac{1}{6} y_4 (t + \tfrac{3}{2})(t + \tfrac{1}{2})(t - \tfrac{1}{2}),$$

for obviously by substituting for t respectively $-\tfrac{3}{2}$, $-\tfrac{1}{2}$, $\tfrac{1}{2}$, $\tfrac{3}{2}$, we obtain y_1, y_2, y_3, y_4, and we assume that for neighbouring values of t the same expression will give the corresponding values of y.

Expanding we obtain

$$\begin{aligned} 48y = &- \ \ y_1(8t^3 - 12t^2 - \ \ 2t + 3) \\ &+ 3y_2(8t^3 - \ \ 4t^2 - 18t + 9) \\ &- 3y_3(8t^3 + \ \ 4t^2 - 18t - 9) \\ &+ \ \ y_4(8t^3 + 12t^2 - \ \ 2t - 3), \end{aligned}$$

and consequently,

$$\begin{aligned} 48y = &(8t^3 - 2t)(d + 3b) + (12t^2 - 3)(2c + 2a) \\ &+ (54t - 24t^3)b + (54 - 24t^2)a, \end{aligned}$$

or $$y = a + bt + c\frac{(2t-1)(2t+1)}{8} + d\frac{t(2t-1)(2t+1)}{24}.$$

If $t = 0$ we have $y = a - \frac{1}{8}c$, by which we see that the value of the function for an argument halfway between two consecutive arguments is equal to the mean of the two adjacent values less one-eighth of the mean of the two second differences on the same horizontal lines as these values.

As an illustration of this method we may take the following problem. The Moon's mean longitude at Greenwich mean noon being given as under for 1st, 2nd, 3rd and 4th March, 1899, it is required to find its mean longitude at midnight on March 2nd.

1899	Moon's mean longitude at noon	1st Diff.	2nd Diff.
March 1st	205° 38′ 38″·1		
		+12° 58′ 6″·9	
„ 2nd	218 36 45 ·0		+13′ 11″·2
		+13 11 18 ·1	
„ 3rd	231 48 3 ·1		+14 40 ·6
		+13 25 58 ·7	
„ 4th	245 14 1 ·8		

The required result is

$$\tfrac{1}{2}(218^\circ\ 36'\ 45''\cdot 0 + 231^\circ\ 48'\ 3''\cdot 1) - \tfrac{1}{16}(13'\ 11''\cdot 2 + 14'\ 40''\cdot 6)$$

$$= 225^\circ\ 10'\ 39''\cdot 5.$$

*EXERCISES ON CHAP. I.

Ex. 1. Show that in any formula relating to a spherical triangle a, b, c, A, B, C may be changed respectively into a, $180°-b$, $180°-c$, A, $180°-B$, $180°-C$ and hence deduce the second of Napier's analogies (21) from the first (20).

Ex. 2. Explain in what sense Delambre's formula (16)

$$\sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A-B) = +\cos \tfrac{1}{2}C \sin \tfrac{1}{2}(a-b)$$

may be also written in the form

$$\sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A-B) = -\cos \tfrac{1}{2}C \sin \tfrac{1}{2}(a-b)$$

and show that there is a similar ambiguity of sign in the remaining three formulae.

Ex. 3. Show that

$$\cot a\,da + \cot B\,dB = \cot b\,db + \cot A\,dA,$$
$$\sin a\,dB = \sin C\,db - \sin B \cos a\,dc - \sin b \cos C\,dA.$$

Ex. 4. If when t assumes the values t_0, t_1, t_2 the corresponding values of y are y_0, y_1, y_2 respectively, show that a formula of interpolation based on these data is given by the equation

$$y = y_0 \frac{(t-t_1)(t-t_2)}{(t_0-t_1)(t_0-t_2)} + y_1 \frac{(t-t_2)(t-t_0)}{(t_1-t_2)(t_1-t_0)} + y_2 \frac{(t-t_0)(t-t_1)}{(t_2-t_0)(t_2-t_1)}.$$

It is sufficient to observe that this is the simplest expression of y in terms of t which obviously gives for y the values y_0, y_1, y_2 when t_0, t_1, t_2 are substituted for t.

Ex. 5. Show that if $t_1 - t_0 = t_2 - t_1 = h$ the formula of interpolation in the last example will reduce to the fundamental formula

$$y = y_0 + \frac{t}{h}(y_1 - y_0) + \frac{t(t-h)}{2h^2}(y_2 - 2y_1 + y_0)$$

if the time be measured from the Epoch t_0.

Ex. 6. Extracting the following from the Ephemeris:

Greenwich mean noon.

		N. Decl. of Sun		
1905. April 7th		6°	40′	49″·9
„ 8th		7	3	22 ·4
„ 9th		7	25	47 ·7

Show that the Sun's declination at 6 p.m. Greenwich mean time on Apr. 7 is 6° 46′ 28″·7.

Ex. 7. The Moon's semi-diameter is as follows:

Greenwich mean noon.

		Semi-diameter of Moon
1909. Sept. 3	... ...	16′ 29″·44
„ 4	... ...	16 18 ·61
„ 5	... ...	16 5 ·97
„ 6	... ...	15 52 ·69

Show that the Moon's semi-diameter at midnight on Sept. 4 is 16′ 12″·44.

Ex. 8. From the following data find the mean time on Aug. 11th, 1909, when Venus and Jupiter have the same R.A.

1909 Mean noon	R.A. of Venus	R.A. of Jupiter
Aug. 11	$11^h\ 10^m\ 40^s{\cdot}24$	$11^h\ 13^m\ 35^s{\cdot}58$
„ 12	11 15 7 ·24	11 14 20 ·36
„ 13	11 19 33 ·61	11 15 5 ·31

If t be the fractional part of a day after noon on Aug. 11 the formulae of Interpolation give the equation

$$11^h\ 10^m\ 40^s{\cdot}24 + t\,267^s{\cdot}00 - 0^s{\cdot}31\, t(t-1)$$
$$= 11^h\ 13^m\ 35^s{\cdot}58 + t\ \ 44^s{\cdot}78 + 0^s{\cdot}08\, t(t-1).$$

It is plain that t must be about $\frac{4}{5}$ Hence the last terms on each side of the equation may be replaced by $+0{\cdot}05$ and $-0{\cdot}01$. Solving the simple equation we have $t = {\cdot}78877$ whence the required answer is $18^h\ 55^m{\cdot}8$.

Ex. 9. We extract from the Ephemeris as follows:

		Right Ascension of Moon
1905. Dec. 21	0 hrs. G.M.T.	$13^h\ 39^m\ 55^s{\cdot}69$
	12 hrs.	14 7 32 ·04
„ 22	0	14 35 38 ·14
	12	15 4 16 ·31

Show from Bessel's formula, neglecting d as it is very small, that the R.A. of the Moon at $(18+12x)$ hours on Dec. 21, 1905, was

$$14^h\ 21^m\ 35^s{\cdot}09 + (28^m\ 6^s{\cdot}10)\,x + 3^s{\cdot}86\,(2x-1)(2x+1).$$

CHAPTER II.

THE USE OF SPHERICAL COORDINATES.

6. Graduated great circles on the sphere.

The circumference of a great circle is supposed to be divided into 360 equal parts by dividing marks. Starting from one of these marks, which is taken as zero, the succeeding marks in regular order will be termed 1°, 2°, 3° and so on up to 359°, after which the next mark is zero so that this point may be indifferently termed 0° or 360°. Thus we obtain what is known as *a graduated great circle,* and it may have subordinate marks by which each interval of 1° is further divided as may be required.

In starting from zero the numbers may increase in either direction, so that there are two perfectly distinct methods of graduating the same circle from the same zero mark.

A man walking on the outside of the sphere along a graduated great circle in the direction in which the numbers increase, *i.e.* from 0° to 1° *not* from 0° to 359°, will have on his left hand that pole of the great circle which may be distinguished by the word *nole*†, and on his right that pole of the great circle which may be distinguished by the word *antinole.*

† The ancient word *nole* being obsolete in its original sense of *head* or *neck* seems available for the purpose now proposed. There being a choice of various spellings that one is preferred which most immediately suggests *north pole.*

Thus when the terrestrial equator is considered as a graduated great circle for longitudes eastward from Greenwich or Paris, the north pole of the earth is the nole of that circle so graduated and its antinole is the south pole of the earth. If on the other hand the equator be graduated so as to show longitudes increasing as the observer moves westward, then the nole of the circle so graduated is the south pole of the earth, and the north pole of the earth is the antinole.

When a point on a sphere is indicated as the nole of a graduated great circle, then, not only is the position of that great circle determined, but also the direction of graduation round it.

If the given point on the sphere had been indicated as the antinole of the graduated great circle, then the direction of graduation would be reversed, for by definition the antinole is on the right hand of a man walking along the great circle in the direction of increasing graduation.

To indicate the direction 0° to 1° on a graduated circle it is sufficient to attach an arrow-head to the circle as shown in Fig. 9 and Fig. 10, and it will be convenient to speak of the direction of increasing graduation as the *positive* direction, and the direction of diminishing graduation as the *negative* direction.

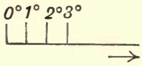


Fig. 9.

7. Coordinates of a point on a sphere.

Any great circle of the sphere graduated from 0° at an origin O being chosen for reference, we can express the position of any point on the sphere by the help of two coordinates α and δ with respect to that graduated great circle.

When specific values are given to α and δ, the corresponding point S on the sphere is obtained in the following way. We measure from O along the great circle in the direction of increasing graduation to a point P so that $OP = \alpha$. At P a great circle is drawn perpendicular to OP, and on this an arc is to be set off equal to δ. If δ is positive, then the

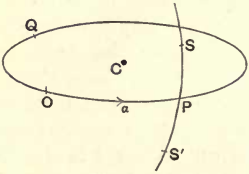


Fig. 10.

required point S is to be taken in the hemisphere which contains the nole. But if δ is negative, then the required point S' is in the hemisphere which contains the antinole. Thus when α, δ are given, the place of a point on the sphere is definitely indicated. It is often convenient to speak of the hemisphere which contains the nole as the positive hemisphere and that which contains the antinole as the negative hemisphere.

Negative values of α need not be considered, for though a point Q might be indicated as $-90°$ if $OCQ = -90°$, yet it would generally be more conveniently indicated by $+270°$, the measurement being made in the positive direction. We hence establish the convention that all values of α are to lie between $0°$ and $+360°$.

It is convenient to restrict the values of δ between $-90°$ and $+90°$, for this dispels some ambiguity while still preserving perfect generality. Two coordinates will indeed always determine one point, but without this limitation of δ it will not follow that one point will have only a single possible pair of coordinates. For example $\alpha = 30°$, $\delta = +20°$ will indicate a point not different from $\alpha = 210°$, $\delta = +160°$. If however we establish the convention that δ shall never lie outside the limits $-90°$ and $+90°$ we are able to affirm that not only does one pair of coordinates determine one point, but that one point, in general, has but one pair of coordinates. The only exceptions then remaining will be the nole and antinole of the fundamental circle. In the former $\delta = +90°$, and in the latter $\delta = -90°$, but in each α is indeterminate.

Ex. 1. Abandoning the restrictions that $0 \ngtr \alpha \ngtr 360°$ and $-90° \ngtr \delta \ngtr 90°$, show that the point $\alpha = 40°$, $\delta = 30°$ would have been equally represented by any of the following pairs of values for α, δ respectively:

220°, 150°; −320°, 30°; −140°, 150°; 400°, 30°; −680°, 30°;
580°, 150°; 40°, 390°

We can always apply $\pm 360°$ to either or both of the coordinates without thereby altering the position of the point to which these coordinates refer.

Ex. 2. Show that the following pairs of coordinates

$$\begin{array}{rr} \alpha\,; & \delta \\ 360° + \alpha\,; & \delta \\ 180° + \alpha\,; & 180° - \delta \\ 180° + \alpha\,; & -180° - \delta \end{array}$$

all indicate the same point, and thus verify that for every point on the sphere a pair of coordinates can be found such that $0 \ngtr \alpha \ngtr 360°$ and $-90° \ngtr \delta \ngtr 90°$.

8. Expression of the cosine of the arc between two points in terms of their coordinates.

Let AA' be the great circle of reference and P its nole, and let S and S' be the two points. As $AS=\delta$ we must have $SP=90°-\delta$, and in like manner $S'P=90°-\delta'$. We have also $AA'=\alpha'-\alpha$, and as PA and PA' are each $90°$,

$$\angle SPS'=\alpha'-\alpha.$$

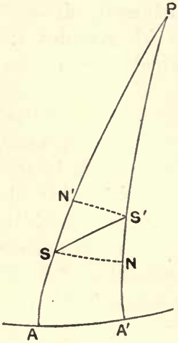


FIG. 11.

Applying fundamental formula (1) to the triangle SPS' we have if $SS'=\theta$

$$\cos\theta=\sin\delta\sin\delta' + \cos\delta\cos\delta'\cos(\alpha'-\alpha) \quad\text{......(i).}$$

When the points S, S' are close together on the sphere a more convenient formula for the determination of their distance is found as follows.

We have

$$\begin{aligned}\cos\theta &= \sin\delta\sin\delta'+\cos\delta\cos\delta'\cos(\alpha-\alpha')\\ &= \sin\delta\sin\delta'\{\cos^2\tfrac{1}{2}(\alpha-\alpha')+\sin^2\tfrac{1}{2}(\alpha-\alpha')\}\\ &\qquad +\cos\delta\cos\delta'\{\cos^2\tfrac{1}{2}(\alpha-\alpha')-\sin^2\tfrac{1}{2}(\alpha-\alpha')\}\\ &= \cos(\delta-\delta')\cos^2\tfrac{1}{2}(\alpha-\alpha')-\cos(\delta+\delta')\sin^2\tfrac{1}{2}(\alpha-\alpha').\end{aligned}$$

Subtracting this from

$$1=\cos^2\tfrac{1}{2}(\alpha-\alpha')+\sin^2\tfrac{1}{2}(\alpha-\alpha'),$$

we have

$$\sin^2\tfrac{1}{2}\theta=\cos^2\tfrac{1}{2}(\alpha-\alpha')\sin^2\tfrac{1}{2}(\delta-\delta')+\sin^2\tfrac{1}{2}(\alpha-\alpha')\cos^2\tfrac{1}{2}(\delta+\delta').$$

This is of course generally true, and when θ is very small it gives the approximate solution

$$\theta^2=(\delta-\delta')^2+(\alpha-\alpha')^2\cos^2\tfrac{1}{2}(\delta+\delta').$$

We can prove this formula geometrically as follows (Fig. 11). Let SN and $S'N'$ be perpendicular to $S'P$ and SP respectively. As $SN'S'$ is a very small triangle

$$SN'^2+N'S'^2=SS'^2,$$

whence approximately

$$(\delta-\delta')^2+(\alpha-\alpha')^2\cos^2\delta'=SS'^2.$$

In like manner from triangle SNS'

$$(\delta-\delta')^2+(\alpha-\alpha')^2\cos^2\delta=SS'^2.$$

These approximate values of SS'^2 only differ in that one contains $\cos\delta$ and the other $\cos\delta'$. Generally one is too large and the other too small and for an approximation we may write instead of $\cos\delta$ or $\cos\delta'$ their mean found as follows:

$$\tfrac{1}{2}(\cos\delta+\cos\delta')=\cos\tfrac{1}{2}(\delta+\delta')\cos\tfrac{1}{2}(\delta-\delta')=\cos\tfrac{1}{2}(\delta+\delta'),$$

which gives by substitution the desired result.

Rectangular Coordinates:—We can determine from (i) the rectangular coordinates of the point α, δ on a sphere of radius r, with reference to axes defined as follows:

$+x$ is from the centre of the sphere to the point $\alpha'=0$, $\delta'=0$.
$+y$ .... $\alpha'=90°$, $\delta'=0$.
$+z$ .... $\delta'=90°$.

We thus see by substitution in (i) that the cosines of the arcs from P to the extremities of the three positive axes are respectively

$$\cos\alpha\cos\delta,\quad \sin\alpha\cos\delta,\quad \sin\delta,$$

and hence the rectangular coordinates are

$$x=r\cos\alpha\cos\delta;\quad y=r\sin\alpha\cos\delta;\quad z=r\sin\delta.$$

Ex. 1. Find the distance θ between S and S' when it is given that

$$\delta=12°\ 24'\ 45'';\quad \delta'=24°\ 15'\ 40'';\quad \alpha'-\alpha=42°\ 38'\ 41''.$$

We calculate the distance θ directly from the formula (i)

			$\cos\delta'$	9·959844
$\sin\delta'$	9·613731		$\cos\delta$	9·989728
$\sin\delta$	9·332334		$\cos(\alpha'-\alpha)$	9·866623
	8·946065			9·816195
	1st term	0·088321		
	2nd „	0·654930		
	$\cos\theta$	0·743251		

$$\theta=41°\ 59'\ 27''.$$

Ex. 2. If $\delta=27°\ 11'\ 6''$, $\delta'=32°\ 17'\ 21''$ and $\alpha'-\alpha=29°\ 11'\ 13''$, show that $\theta=25°\ 46'\ 6''$.

Ex. 3. The coordinates of two stars are α_1, δ_1 and α_2, δ_2 respectively. Show from (i) that the coordinates α, δ of the poles of the great circle joining them are given by the equations

$$-\tan\delta=\cot\delta_1\cos(\alpha-\alpha_1)=\cot\delta_2\cos(\alpha-\alpha_2),$$

and obtain the same equations geometrically.

Ex. 4. Explain how the solution of the last question applies to both poles, and show how to distinguish the nole from the antinole if the positive direction be from the first star to the second.

Ex. 5. Show that if L be the length of the arc of a great circle on the earth (supposed a sphere of radius R) extending from lat. λ_1, long. l_1 to lat. λ_2, long. l_2, then

$$L = R\cos^{-1}(\sin\lambda_1 \sin\lambda_2 \sec^2\phi),$$

where $$\tan^2\phi = \cot\lambda_1 \cot\lambda_2 \cos(l_1 - l_2);$$

and that the highest latitude reached by the great circle will be

$$\cos^{-1}\left(\cos\lambda_1 \cos\lambda_2 \sin(l_1 \sim l_2) \operatorname{cosec}\frac{L}{R}\right).$$

Let S_1S_2 be the two points (Fig. 12), OP_1P_2 the equator, N the north pole; then

$$\cos S_1S_2 = \sin\lambda_1 \sin\lambda_2 + \cos\lambda_1 \cos\lambda_2 \cos(l_1 - l_2).$$

To prove the second part the highest latitude on S_1S_2, produced if necessary, equals $\angle S_1OP_1$.

From ΔNOS_1 we have

$$\cos S_1OP_1 = \sin NOS_1 = \sin NS_1 \sin NS_1O$$
$$= \sin NS_1 \sin NS_2 \sin S_1NS_2 \operatorname{cosec} S_1S_2.$$

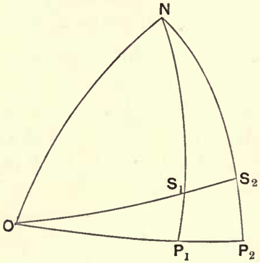


FIG. 12.

Ex. 6. Verify that in the expression of the distance between a point a, δ and another point a_0, δ_0 there is no change if a and δ be altered into $180° + a$ and $180° - \delta$ respectively, and explain why this is necessary.

9. Interpretation of an equation in spherical coordinates.

When α and δ are given, then as we have shown a point of which these quantities are the coordinates is definitely determined on the sphere. If we know nothing with regard to α and δ, except that they satisfy one equation into which they enter in conjunction with other quantities which are known, we have not sufficient data to determine the two unknowns.

Any value of α substituted in the equation will give an equation in δ for which, in general, one or more roots can be found. Repeating the process with different values of α we can obtain an indefinitely numerous series of pairs of coordinates α, δ, each of which corresponds to a point on the sphere. If several of these points be constructed, they will indicate a curve traced on the spherical surface. The original equation may be described as the equation of that curve just in the same way as an equation in x and y indicates a plane curve in analytic geometry.

We shall first show that if the coordinates of a point α, δ satisfy the equation

$$A \sin \delta + B \sin \alpha \cos \delta + C \cos \alpha \cos \delta = 0,$$

where A, B, C are constants, the locus of the point will be a great circle of which the poles will have coordinates α', δ' and $180° + \alpha'$, $-\delta'$, where

$$\tan \alpha' = B/C; \quad \sin \delta' = A/\sqrt{A^2 + B^2 + C^2}.$$

We can make A positive, because if necessary the signs of all the terms can be changed. Assume three new quantities H, α', δ' such that $A = H \sin \delta'$, $B = H \sin \alpha' \cos \delta'$, $C = H \cos \alpha' \cos \delta'$, then by squaring and adding $H = \pm \sqrt{A^2 + B^2 + C^2}$. Taking the upper sign we obtain from the first equation $\sin \delta' =$ a positive quantity, $\not> 1$, hence δ' is positive and as $\delta' \not> 90°$ there is no confusion between δ' and $180° - \delta'$. The second and third equations give $\cos \alpha'$ and $\sin \alpha'$, and thus α' is found without ambiguity, and we have obtained one solution α', δ'. If however we had taken the negative value of H, then instead of δ' we should have had $-\delta'$ from the first equation, and the two last can only be satisfied by putting $\alpha' + 180°$ instead of α'. Thus there are two solutions, α', δ' and $180° + \alpha'$, $-\delta'$. And these are two antipodal points. The original equation then reduces to

$$H \{\sin \delta' \sin \delta + \cos \delta \cos \delta' \cos (\alpha' - \alpha)\} = 0,$$

whence α, δ must be $90°$ from the fixed point α', δ' and therefore its locus is a great circle.

Ex. 1. Show that if the following equation is satisfied:—

$$A \sin \delta + B \sin \alpha \cos \delta + C \cos \alpha \cos \delta = D,$$

the locus of the point α, δ will be in general a small circle of which the radius is

$$\cos^{-1} \{D/(A^2 + B^2 + C^2)^{\frac{1}{2}}\},$$

and that if $D^2 = A^2 + B^2 + C^2$ the equation represents no more than a point.

Ex. 2. If α, δ are the current coordinates of a point on a sphere and a, b are constants, show that the equation

$$\tan \delta = \tan b \sin (a - \alpha)$$

represents a great circle which has the point $\alpha = a + 270°$ and $\delta = 90° - b$ as a pole.

10. The inclination of two graduated great circles is the arc $\not> 180^\circ$ joining their noles.

The inclination of two *ungraduated* great circles is in general unavoidably ambiguous, for it may be either of two supplemental angles, and it is only when the two circles cross at right angles that this ambiguity disappears.

But the inclination of two *graduated* great circles need not be ambiguous because we can always distinguish that one of the two supplemental angles which is to be deemed the inclination of the two circles. The *inclination* is defined to be the angle $\not> 180^\circ$

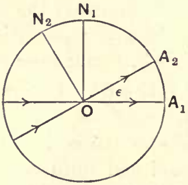


FIG. 13.

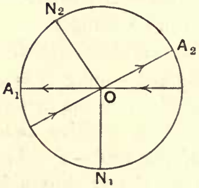


FIG. 14.

between those parts of the circles in which the arrow-heads are both diverging from an intersection or converging towards an intersection.

In Fig. 13 the two segments of the circles diverging from O are OA_1 and OA_2, and consequently the angle is to be $A_1OA_2 = \epsilon$. If however we simply change the direction of the arrow-head on OA_1 without any other alteration in the figure, we have the condition shown in Fig. 14, where the diverging segments OA_1 and OA_2 now contain the angle $A_1OA_2 = 180^\circ - \epsilon$, which is accordingly to be taken as the inclination of the two graduated circles in this case.

If $A_1A_2N_1N_2$ (Fig. 13) is the great circle perpendicular both to OA_1 and OA_2, then since $OA_1 = 90^\circ$ and $OA_2 = 90^\circ$, we have $A_1A_2 = \epsilon$. If N_1 and N_2 be the noles of OA_1 and OA_2 respectively, we have $A_1N_1 = 90^\circ$ and $A_2N_2 = 90^\circ$, and hence

$$N_1N_2 = A_1A_2 = \epsilon.$$

In like manner in Fig. 14 the nole N_1 of OA_1 is now the antinole of the former case. Since $A_2ON_2 = 90^\circ$, we have

$A_1ON_2 = 90° - \epsilon$, and as $A_1ON_1 = 90°$ we have $N_1ON_2 = 180° - \epsilon$, which as already explained is the inclination of the two graduated circles in this case. Then we obtain the important result that *the inclination between two graduated great circles is always measured by the arc between their noles.*

No doubt a question may arise as to the arc N_1N_2 (Fig. 13). Is it the lesser of the two arcs which we should naturally take, or is it the arc reckoned the other way round the circle from N_1 by A_2 and A_1? There are thus two arcs together making 360°, of which either may in one sense be regarded as the inclination. We can however remove any ambiguity thus arising by the convention that the inclination of two graduated great circles is never to exceed 180°.

Ex. 1. If BC, CA, AB be the positive directions on three graduated great circles which form the triangle ABC and if A', B', C' be their respective noles, show that

(1) If $B'C'$, $C'A'$, $A'B'$ be the positive directions on the sides of the polar triangle $A'B'C'$ the noles of those sides are A, B, C respectively.

(2) The sides and angles of $A'B'C'$ are respectively supplementary to the angles and sides of ABC.

Ex. 2. If a_1, δ_1 and a_2, δ_2 be the noles of two graduated circles show that if ϵ is the inclination of the two circles

$$\cos\epsilon = \sin\delta_1 \sin\delta_2 + \cos\delta_1 \cos\delta_2 \cos(a_1 - a_2),$$

and that if a, δ are the coordinates of the intersection of the two circles

$$\sin\delta = \pm\frac{\cos\delta_1 \cos\delta_2 \sin(a_2 - a_1)}{\sin\epsilon},$$

$$\cos\delta\cos a = \pm\frac{\cos\delta_1 \sin\delta_2 \sin a_1 - \sin\delta_1 \cos\delta_2 \sin a_2}{\sin\epsilon},$$

$$\cos\delta\sin a = \pm\frac{\sin\delta_1 \cos\delta_2 \cos a_2 - \sin\delta_2 \cos\delta_1 \cos a_1}{\sin\epsilon},$$

where the upper and lower signs refer to the two intersections.

11. On the intersections of two graduated great circles.

Let C and C' (Fig. 15) be two graduated great circles which intersect in the two diametrically opposite points V and V'. Let N be the nole of C and N' the nole of C'.

A point moving along C' in the positive direction crosses at V into the *positive* hemisphere bounded by C. Thus V is described as the *ascending node* of C' with respect to C.

A point moving along C' in the positive direction crosses at V' into the *negative* hemisphere bounded by C. Thus V' is described as the *descending node* of C' with respect to C.

If O be the origin on C from which coordinates are measured and $OP = \alpha$, $PN' = \delta$, then α and δ are the coordinates of N' the nole of C' with respect to C.

As the angle between two graduated great circles is the arc between their noles (§ 10) we see that $90° - \delta$ is the inclination between C and C'.

We have
$$OV = OP + PV = \alpha + 90°,$$
$$OV' = OV + 180° = \alpha + 270°,$$

and thus we obtain the following general statement:

If α, δ be the coordinates of the nole of one graduated great circle C' with respect to another C, then the inclination of the two circles is $90° - \delta$, the ascending node of C' on C has coordinates $90° + \alpha$, 0, and the descending node of C' on C has coordinates $270° + \alpha$, 0.

If, as is often convenient, we take Ω, 0 as the coordinates of the

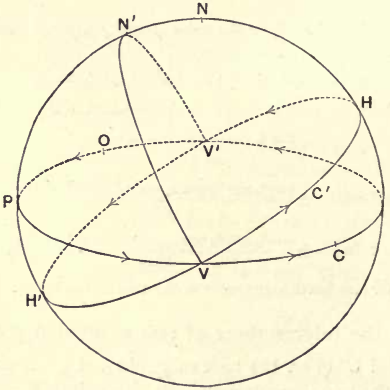


FIG. 15.

ascending node of C' on C and ϵ as the inclination of the two circles, we have $(\Omega + 270°)$, $(90° - \epsilon)$ as the coordinates of the nole of C', the circle of reference being C.

In general to fix the position and direction of graduation of one great circle with respect to another, we must know three parameters of the second circle with regard to the first. We may, for instance, be given the two coordinates of its nole. For this fixes the nole, and then not only is the great circle determined of which that nole is the pole but also the direction in which the graduation advances on the second circle is known. If we had merely been given the coordinates of a pole of the great circle, then no doubt the place of the great circle would be defined, but so long as it is unknown whether the given pole is the nole or the antinole the direction of graduation will remain unspecified. The third parameter is required to fix the *origin* of the graduation on the second circle.

Or we may be given Ω the *ascending* node of the second circle on the first and also ϵ the inclination. Starting from the origin we set off Ω in the positive direction and thus find the ascending node. The second circle is then entering the positive hemisphere of the first. If we make the *two diverging* arcs from the node contain the angle ϵ there is no ambiguity as to the exact place of the circle required.

Ex. 1. Show that the ascending node of C' with regard to C is the descending node of C with regard to C'.

Ex. 2. Show by a figure the difference between two graduated great circles which, having equal inclinations to the great circle of reference, have respectively θ and $180°+\theta$ as the distances of their ascending nodes from the origin.

Ex. 3. If Ω be the longitude of the ascending node of a graduated great circle L and ϵ its inclination to a fundamental circle, and if Ω', ϵ' be the corresponding quantities with regard to another great circle L', determine the coordinates of the ascending node V of L' upon L.

Let N, N' (Fig. 16) be the nodes on the fundamental circle ONN', then V is the ascending node of L' upon L; let x be the distance NV. We have to find x in terms of ϵ, ϵ' and $\Omega'-\Omega$.

From formula (6) in § 1 we obtain

$$\cot x \sin(\Omega'-\Omega) - \cos(\Omega'-\Omega)\cos\epsilon = -\sin\epsilon\cot\epsilon',$$

whence $$\cot x = \frac{\cos(\Omega'-\Omega)\cos\epsilon - \sin\epsilon\cot\epsilon'}{\sin(\Omega'-\Omega)}.$$

To find which value of x is to be taken observe that as

$$\sin x : \sin(\Omega'-\Omega) :: \sin\epsilon' : \sin V$$

and V and ϵ' are both $\not> 180°$, $\sin x$ must have the same sign as $\sin(\Omega'-\Omega)$, which shows whether x or $x+180°$ is the angle required.

When x is known we determine α, δ the coordinates of V with respect to the fundamental circle from the equations

$$\sin\delta = \sin x \sin\epsilon,$$
$$\cos\delta\cos(\alpha - \Omega) = \cos x,$$
$$\cos\delta\sin(\alpha - \Omega) = \sin x\cos\epsilon.$$

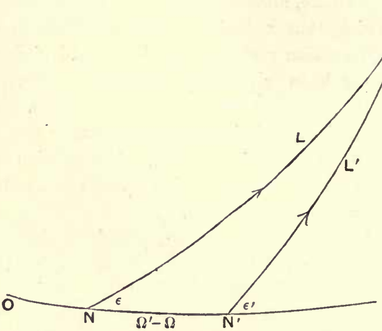


Fig. 16.

Ex. 4. With the data of the last example find the inclination ρ between the two great circles specified by Ω, ϵ and Ω', ϵ' respectively.

We have found that the coordinates of the noles are $\Omega + 270^\circ$, $90^\circ - \epsilon$ and $\Omega' + 270^\circ$, $90^\circ - \epsilon'$, and hence by § 10 Ex. 2 we have

$$\cos\rho = \cos\epsilon\cos\epsilon' + \sin\epsilon\sin\epsilon'\cos(\Omega - \Omega').$$

Ex. 5. If x be the length of the common perpendicular to the two great circles defined by Ω, ϵ and Ω', ϵ' show that

$$\cos x = \cos\epsilon\cos\epsilon' + \sin\epsilon\sin\epsilon'\cos(\Omega - \Omega').$$

12. Transformation of coordinates.

Being given the coordinates of a point with regard to one graduated great circle it is often necessary to determine the coordinates of the same point with regard to a different graduated great circle.

Let α, δ be the original coordinates of a point P and let α', δ' be the coordinates of the same point P in the new system. In like manner let α_0, δ_0 and α_0', δ_0' be the original and transformed coordinates of some other point P_0. Since the transformation cannot affect the distance PP_0 we must have that distance the same whichever be the coordinates in which it is expressed, and consequently (§ 8)

$$\sin\delta' \sin\delta_0' + \cos\delta' \cos\delta_0' \cos(\alpha' - \alpha_0')$$
$$= \sin\delta \sin\delta_0 + \cos\delta \cos\delta_0 \cos(\alpha - \alpha_0) \ldots\ldots\text{(i)}.$$

All the formulae connected with the transformation are virtually contained in this equation.

If we know the coordinates of any point P_0 in both systems, *i.e.* α_0, δ_0, α_0', δ_0' and substitute these values in (i) we obtain an equation connecting in general α, δ and α', δ'. In like manner if the coordinates of a second point are known in both systems we obtain another equation in α, δ and α', δ'. Thus we have two equations for the determination of α', δ' in terms of α, δ.

But two equations are not sufficient for finding α', δ' uniquely in terms of α, δ. The distances PP_0, PP_1 do not fix P without ambiguity. There are obviously two positions which P might occupy. Their distances from a third point P_2 will not however be equal unless indeed P_2 happens to lie on the great circle through P_0P_1. Excluding this case we may say that a point is determinate if its distance from three given points is known. Hence we have to obtain a third equation between α, δ and α', δ' by taking some third point of which the coordinates α_0, δ_0 and α_0', δ_0' in both systems are known and which does not lie on the great circle passing through the two points previously selected.

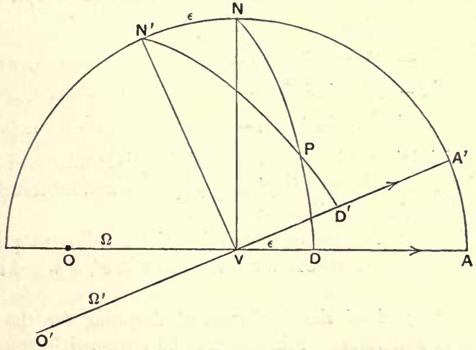


FIG. 17.

Let OA (Fig. 17) be the original great circle graduated in the direction of the arrow from the origin O and having its nole at N.

Let $O'A'$ be the graduated great circle with nole at N' and origin O' to which the coordinates are to be transformed. Let Ω, Ω' be the distances from O and O' respectively of V the ascending node of the second circle on the first. Let ϵ be the inclination of the two graduated circles. Then Ω, Ω', ϵ are the three parameters which completely define in every way the second graduated great circle with reference to the first (§ 11).

We have now to select three points, not on the same great circle, and such that their coordinates in both systems can be directly perceived.

The points we shall choose are respectively V, A and N. It is obvious from the figure that as $VA = VA' = 90°$ the coordinates of these points in the two systems are as follows:

For V $\alpha_0 = \Omega$; $\delta_0 = 0$. and $\alpha_0' = \Omega'$; $\delta_0' = 0$.

„ A $\alpha_0 = 90° + \Omega$; $\delta_0 = 0$. „ $\alpha_0' = 90° + \Omega'$; $\delta_0' = -\epsilon$.

„ N $\alpha_0 = 0$; $\delta_0 = 90°$. „ $\alpha_0' = 90° + \Omega'$; $\delta_0' = 90° - \epsilon$.

Substituting these coordinates successively in the equation (i) we have the general formulae of transformation

$$\left.\begin{aligned} \cos\delta\cos(\alpha-\Omega) &= \cos\delta'\cos(\alpha'-\Omega') \quad\text{.........(ii)},\\ \cos\delta\sin(\alpha-\Omega) &= -\sin\delta'\sin\epsilon + \cos\delta'\cos\epsilon\sin(\alpha'-\Omega') \ldots\text{(iii)},\\ \sin\delta &= \sin\delta'\cos\epsilon + \cos\delta'\sin\epsilon\sin(\alpha'-\Omega') \ldots\text{(iv)}. \end{aligned}\right\}$$

From these we derive

$$\left.\begin{aligned} \cos\delta'\cos(\alpha'-\Omega') &= \cos\delta\cos(\alpha-\Omega) \quad\text{.........(ii)},\\ \cos\delta'\sin(\alpha'-\Omega') &= \sin\delta\sin\epsilon + \cos\delta\cos\epsilon\sin(\alpha-\Omega) \ldots\text{(v)},\\ \sin\delta' &= \sin\delta\cos\epsilon - \cos\delta\sin\epsilon\sin(\alpha-\Omega) \ldots\text{(vi)}, \end{aligned}\right\}$$

for, by multiplying (iii) by $\cos\epsilon$ and adding (iv) multiplied by $\sin\epsilon$ we obtain (v), and by multiplying (iv) by $\cos\epsilon$ and subtracting (iii) multiplied by $\sin\epsilon$ we obtain (vi).

The first set of equations determine the coordinates α, δ when α', δ' are known and the second set determine α', δ' when α, δ are known.

Another proof of the fundamental formulae for the transformation of spherical coordinates may be obtained in the following way.

Since V is the pole of NN' (Fig. 17) we have $\angle VNN' = 90°$ also $\angle VND = \alpha - \Omega$, whence $\angle N'NP = 90° + \alpha - \Omega$. We also see that $\alpha' - \Omega' = \angle VN'D'$ whence $\angle NN'P = 90° - \alpha' + \Omega'$. The figure also shows that $NP = 90° - \delta$, $N'P = 90° - \delta'$ and $NN' = \epsilon$. In the triangle $NN'P$ we thus have expressions for the three sides and two angles, and hence from the fundamental formulae (3), (2), (1) of § 1 we deduce (ii), (iii), (iv) on the last page.

The necessity already pointed out for having three equations in the formulae of transformation may be illustrated from the group (ii), (v) and (vi).

Suppose that we sought α' and δ' from equations (ii) and (v); we have at once

$$\tan(\alpha' - \Omega') = \{\sin\delta \sin\epsilon + \cos\delta \cos\epsilon \sin(\alpha - \Omega)\} \sec\delta \sec(\alpha - \Omega).$$

As all the quantities on the right-hand side are known, $\tan(\alpha' - \Omega')$ is known. Let θ be the angle $\not> 180°$ which has this value for its tangent, then $(\alpha' - \Omega')$ must be either θ or $\theta + 180°$: we can decide which value is to be taken for $\alpha' - \Omega'$ by equation (ii). For as δ and δ' are always between the limits $-90°$ and $+90°$, $\cos\delta$ and $\cos\delta'$ are both necessarily positive. The sign of $\cos(\alpha' - \Omega')$ must therefore be the same as the sign of $\cos(\alpha - \Omega)$. It is thus ascertained whether $\alpha' - \Omega'$ is to be θ or $180° + \theta$, for only one of these angles will have a cosine agreeing in sign with $\cos(\alpha - \Omega)$.

Thus the two equations (ii) and (v) determine $(\alpha' - \Omega')$ without ambiguity and therefore α' is known. We then find $\cos\delta'$ from (ii). At this point the insufficiency of *two* equations becomes apparent, for though the magnitude of δ' is known its sign is indeterminate. Hence the necessity for a third equation like (vi) which gives the value of $\sin\delta'$ and hence the sign of δ'.

The problem of finding α', δ' from (ii), (v) and (vi) might also be solved thus.

Equation (vi) determines $\sin\delta'$ and thus shows that δ' must be one or other of two supplemental angles. It is however understood that $-90° \not> \delta' \not> 90°$ and we choose for δ' that one of the supplemental angles which fulfils this condition. Thus δ' is known and hence $\cos\delta'$. Equation (ii) will then give $\cos(\alpha' - \Omega')$ and (v) will give $\sin(\alpha' - \Omega')$, hence $\alpha' - \Omega'$ is determined without ambiguity as both its sine and cosine are known.

Ex. 1. If $\alpha' = 90° + \Omega'$, $\delta' = 0$, show that $\alpha = 90° + \Omega$, $\delta = \epsilon$, and find the point indicated on the sphere.

Ex. 2. Show that the coordinates of the nole of $O'A'$ in the first and second systems respectively are

$$a = 270° + \Omega, \ \delta = 90° - \epsilon; \quad a' \text{ indeterminate}, \ \delta' = 90°;$$

and verify that these quantities satisfy the equations (ii), (iii), (iv).

Ex. 3. As a verification of the equations (ii), (iii), (iv), show that the sum of the squares of the right-hand members is unity.

Ex. 4. Show that the equations (v), (vi) might have been written down at once from (iii), (iv).

For V is the *descending* node of OA with respect to $O'A'$. This implies that a and δ may be interchanged with a' and δ' if at the same time Ω and Ω' be each increased by 180°.

Ex. 5. If the planes of two graduated great circles are coincident show the connection of the coordinates a, δ on one graduated great circle and a', δ' on the other of the same point on the sphere.

In the general formulae (ii), (v), (vi) we make $\epsilon = 0$ if the two circles are graduated in the same direction, and $\epsilon = 180°$ if they are graduated in opposite directions. In the first case

$$\cos\delta' \cos(a' - \Omega') = \cos\delta \cos(a - \Omega)$$
$$\cos\delta' \sin(a' - \Omega') = \cos\delta \sin(a - \Omega)$$
$$\sin\delta' = \sin\delta,$$

whence $\delta' = \delta$ and $a' = a + \Omega' - \Omega$.

In the second case

$$\cos\delta' \cos(a' - \Omega') = \cos\delta \cos(a - \Omega)$$
$$\cos\delta' \sin(a' - \Omega') = -\cos\delta \sin(a - \Omega)$$
$$\sin\delta' = -\sin\delta,$$
$$\delta' = -\delta, \quad a' = \Omega + \Omega' - a.$$

The coordinate δ here changes sign because the reversal of the direction of graduation interchanges the positive and negative hemispheres.

Ex. 6. Let S be a fundamental graduated great circle and let β, λ be the coordinates of any point P with respect to S. Let S' be another graduated great circle and let β_0, λ_0 be the coordinates of its nole with respect to S. Let Ω_0 denote the degrees, minutes and seconds marked on S' at its ascending node on S. Let β', λ' be the coordinates of P with regard to S'. Show that for the determination of β', λ' in terms of β, λ

$$\begin{cases} \cos\beta' \cos(\lambda' - \Omega_0) = \cos\beta \sin(\lambda - \lambda_0) \\ \cos\beta' \sin(\lambda' - \Omega_0) = \sin\beta \cos\beta_0 - \cos\beta \sin\beta_0 \cos(\lambda - \lambda_0) \\ \sin\beta' = \sin\beta \sin\beta_0 + \cos\beta \cos\beta_0 \cos(\lambda - \lambda_0), \end{cases}$$

and that for the determination of β, λ in terms of β', λ'

$$\begin{cases} \cos\beta \sin(\lambda - \lambda_0) = \cos\beta' \cos(\lambda' - \Omega_0) \\ \cos\beta \cos(\lambda - \lambda_0) = \sin\beta' \cos\beta_0 - \cos\beta' \sin\beta_0 \sin(\lambda' - \Omega_0) \\ \sin\beta = \sin\beta' \sin\beta_0 + \cos\beta' \cos\beta_0 \sin(\lambda' - \Omega_0). \end{cases}$$

Ex. 7. Let a_1, δ_1 and a_2, δ_2 be the coordinates of two stars in the first system and a_1', δ_1' and a_2', δ_2' the corresponding coordinates in the second system. As the distance of the two stars must be the same in both systems we have

$$\sin\delta_1 \sin\delta_2 + \cos\delta_1 \cos\delta_2 \cos(a_1 - a_2) = \sin\delta_1' \sin\delta_2' + \cos\delta_1' \cos\delta_2' \cos(a_1' - a_2');$$

verify this from the equations (ii), (iii), (iv).

Ex. 8. Explain the changes in the coordinates on the celestial sphere according as the sphere is supposed to be viewed from the interior or the exterior and show that the formulae remain unaltered.

Fig. 17 is supposed to be drawn as usual from the appearance of the sphere as seen from the *outside*.

But if we wish Fig. 17 to represent a portion of the sphere as seen from the inside then V is the *descending* node. Instead of Ω and δ we should write $180° + \Omega$ and $-\delta$ and similarly $180° + \Omega'$ and $-\delta'$ for Ω', δ'. These changes make no alteration in the formulae (ii), (v), (vi).

Ex. 9. If a, δ and a', δ' are the coordinates of two points show that the nodes of the great circle joining them are distant from the origin by quantities L and $L + 180°$ where

$$L = \tfrac{1}{2}(a + a') - \tan^{-1}\left(\frac{\sin(\delta' + \delta)}{\sin(\delta' - \delta)} \tan\tfrac{1}{2}(a' - a)\right).$$

13. Adaptation to Logarithms.

If, in calculating the transformed coordinates α', δ', the equations (ii), (v), (vi) (§ 12) be used as they stand, the two terms in (vi) should be evaluated logarithmically and then δ' is taken from a table of natural sines. The equation (ii) determines $\cos(\alpha' - \Omega')$ and (v) is used only to determine the sign of $\alpha' - \Omega'$; for this we need calculate only the logarithms of the two terms on the right-hand side even when they are of opposite signs.

It is, however, often thought convenient to effect a transformation of the formulae (ii), (v), (vi) (§ 12) by the introduction of auxiliary quantities which will make them more immediately adapted for logarithmic calculation. This may be best effected as follows.

Let m be a positive quantity and M an angle between 0° and 360° such that

$$\sin\delta = m\cos M; \quad \cos\delta \sin(\alpha - \Omega) = m \sin M.$$

Hence $\tan M = \cot\delta \sin(\alpha - \Omega)$. If M_0 is the smallest angle which satisfies this, M is either M_0 or $M_0 + 180°$. As m is positive we must choose that value for M which gives $\cos M$ with the same sign as $\sin \delta$. Thus $\log m$ and M become known. By substitution of these auxiliary quantities in (ii), (v), (vi) (§ 12) these equations become

$$\left.\begin{aligned} \cos\delta'\cos(\alpha'-\Omega') &= \cos\delta\cos(\alpha-\Omega) \\ \cos\delta'\sin(\alpha'-\Omega') &= m\sin(M+\epsilon) \\ \sin\delta' &= m\cos(M+\epsilon) \end{aligned}\right\}\quad\text{............(i).}$$

From the last of these formulae δ' is obtained both as to magnitude ($\not> 90°$) and as to sign. This value substituted in the two other formulae determines both $\cos(\alpha'-\Omega')$ and $\sin(\alpha'-\Omega')$. The first gives the magnitude of $\alpha'-\Omega'$ and the second gives its sign.

Ex. 1. The coordinates of a point are $\alpha=75°$, $\delta=15°$. Show by the formulae (ii), (v), (vi), that when transformed to a circle of reference defined by the quantities $\Omega=215°$, $\epsilon=23° \, 30'$, $\Omega'=115°$ the coordinates become $\alpha'=327° \, 13'$, $\delta'=29° \, 0'$.

Ex. 2. If the problem of Ex 1 be solved with the help of the auxiliary quantities M and m show that $M=292° \, 38'$ and $\text{Log}\, m=9{\cdot}8278$.

Ex. 3. If VP (Fig. 17) when produced meets NN' in K show that $m=\cos PK$ and $M=NK$, and obtain the formulae (i) from the right-angled triangle $N'PK$.

CHAPTER III.

THE FIGURE OF THE EARTH AND MAP MAKING.

14. Introductory.

That the earth is globular in form would be suggested by the analogous forms of the sun and moon, and it is demonstrated by familiar facts as set forth in books on geography.

Accurate measurements of the figure of the earth are of fundamental importance in Astronomy and this chapter will be devoted to the elementary parts of this subject as well as to explaining how curved surfaces, such as that of the earth, can be depicted on flat surfaces, as in the art of map making.

It is necessary to explain that by the expression "figure of the earth" we do not mean its irregular surface diversified by continent and ocean as we actually see it, but a surface, part of which is indicated by the ocean at rest, and which in other parts may be defined as coincident with the level to which water would rise at the place if freely communicating with the sea by means of canals which we may imagine traversing the continents from ocean to ocean.

15. Latitude.

If the earth be regarded as a sphere, then the latitude of any station on the earth's surface is the inclination to the plane of the terrestrial equator of the terrestrial radius to that station. But the true figure of the earth is not spherical. It rather approximates to the spheroid of revolution obtained by the rotation of an ellipse about its minor axis. The lengths of the semi-axes of this ellipse as given by Colonel Clarke† are

$$a = 20926202 \text{ feet } [7\cdot3206904],$$
$$= \text{(approximately) } 3963\cdot3 \text{ miles } [3\cdot59806],$$
$$= 6378\cdot2 \text{ kilometres } [3\cdot80470],$$
$$b = 20854895 \text{ feet } [7\cdot3192080],$$
$$= \text{(approximately) } 3949\cdot8 \text{ miles } [3\cdot59657],$$
$$= 6356\cdot5 \text{ kilometres } [3\cdot80322].$$

The figures in square brackets denote the logarithms of the numbers to which they are attached.

If the normal PN to the earth's surface (Fig. 18) meet the plane of the equator in N and CNA be the semi-axis major, then $\angle PNA = \phi$ is the *geographical* latitude of P and $\angle PCA = \phi'$ is its *geocentric* latitude.

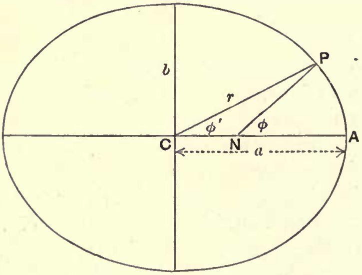


Fig. 18.

If the equation of the ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, and x' and y' be the coordinates of a point P of which λ is the excentric angle, then we easily see that

$$\tan \phi = a \tan \lambda / b, \quad \tan \phi' = b \tan \lambda / a,$$

† *Geodesy*, Clarendon Press, 1880, p. 319.

and ϕ' and ϕ are connected by the relation $\tan\phi' = b^2 \tan\phi/a^2$, by which the geocentric latitude is obtained when the true or geographical latitude is known or *vice versa.*

We obtain r, the geocentric distance of P, as follows:

$$r^2 = x'^2 + y'^2 = a^2 \cos^2\lambda + b^2 \sin^2\lambda = \frac{a^4 \cos^2\phi + b^4 \sin^2\phi}{a^2 \cos^2\phi + b^2 \sin^2\phi}$$

$$= a^2 \frac{\cos^2\phi + (1 - e^2)^2 \sin^2\phi}{1 - e^2 \sin^2\phi} = a^2 (1 - e^2 \sin^2\phi),$$

if powers of e above the second may be neglected.

Under the same conditions

$$\tan(\phi - \phi') = \frac{\tan\phi - \tan\phi'}{1 + \tan\phi \tan\phi'} = \frac{(a^2 - b^2)\tan\phi}{a^2 + b^2 \tan^2\phi} = e^2 \sin\phi \cos\phi,$$

and consequently we obtain the following result.

If the earth be regarded as produced by the revolution of an ellipse of eccentricity e about its minor axis, and if the equatorial radius of the earth be taken as unity, then, a point having the geographical latitude ϕ on the earth's surface will have for its approximate geocentric latitude and radius vector

$$\phi' = \phi - \tfrac{1}{2}[e^2 \operatorname{cosec} 1'' \sin 2\phi]'',$$

$$r = 1 - \tfrac{1}{4} e^2 + \tfrac{1}{4} e^2 \cos 2\phi.$$

Using Clarke's values for a and b we easily find

$$e^2 = (a^2 - b^2)/a^2 = 1/147,$$

and we obtain

$$\phi' = \phi - 702'' \sin 2\phi = \phi - [2{\cdot}846] \sin 2\phi,$$

$$r = {\cdot}9983 + [7{\cdot}2306] \cos 2\phi.$$

Thus $702'' \sin 2\phi$ is the amount to be subtracted from the geographical latitude to obtain the geocentric latitude.

If we desire a higher degree of approximation we may proceed as follows:

$$\tan(\phi - \phi') = \frac{(a^2 - b^2)\tan\phi}{a^2 + b^2 \tan^2\phi} = \frac{(a^2 - b^2)\sin 2\phi}{(a^2 + b^2) + (a^2 - b^2)\cos 2\phi},$$

from which we easily obtain the approximate formula

$$\phi - \phi' = \frac{a^2 - b^2}{a^2 + b^2} \operatorname{cosec} 1'' \sin 2\phi - \tfrac{1}{2} \left(\frac{a^2 - b^2}{a^2 + b^2}\right)^2 \operatorname{cosec} 1'' \sin 4\phi.$$

For the accurate calculation of ϕ' and r the following is the method most generally used.

Taking a as unity we have

$$r\cos\phi' = x' = \cos\lambda = \cos\phi/\sqrt{1-e^2\sin^2\phi},$$

$$r\sin\phi' = y' = b\sin\lambda = (1-e^2)\sin\phi/\sqrt{1-e^2\sin^2\phi}.$$

If therefore we make

$$X = \frac{(1-e^2)}{\sqrt{1-e^2\sin^2\phi}}, \quad Y = \frac{1}{\sqrt{1-e^2\sin^2\phi}},$$

we obtain $\quad r\sin\phi' = X\sin\phi, \quad r\cos\phi' = Y\cos\phi.$

The quantities $\log X$ and $\log Y$ are given in the Ephemeris for each degree of ϕ. As $\sin^2\phi$ is multiplied by e^2 in X and Y, a small error in ϕ will make no appreciable effect on X and Y. Thus $\log X$ and $\log Y$ may be obtained by inspection of the table without troublesome interpolation. Then the accurate values of $\log\sin\phi$ and $\log\cos\phi$ being added to $\log X$ and $\log Y$ respectively, we obtain $\log r\sin\phi'$ and $\log r\cos\phi'$ and thence r and ϕ'†. We may note that $\log X$ and $\log Y$ have a constant difference.

As an illustration of the application of this method we may take the following case.

The geographical latitude of Cambridge being 52° 12′ 52″, show that the reduction to be applied to obtain the geocentric latitude is −11′ 22″, and find the distance of Cambridge from the earth's centre when the earth's equatorial radius is taken as unity.

$\text{Log } X =$	9·9979599	$\log Y =$	0·0009247
$\text{Log}\sin\phi$	9·8977972	$\text{Log}\cos\phi$	9·7872534
$\text{Log } r\sin\phi'$	9·8957571	$\text{Log } r\cos\phi'$	9·7881781
$\text{Log } r\cos\phi'$	9·7881781		
$\tan\phi'$	0·1075790		
$\text{Log } r\sin\phi'$	9·8957571	ϕ	52° 12′ 52″
$\text{Log}\sin\phi'$	9·8966801	ϕ'	52° 1′ 30″
$\text{Log } r$	9·9990770	$\phi' =$	ϕ − 11′ 22″

$$r = \cdot 99788.$$

$\text{Log } r$ could of course also have been found from $r\cos\phi'$, but $r\sin\phi' > r\cos\phi'$, and we adhere to the rule of using the larger of the two quantities, see p. 7.

† E. J. Stone gives a table in *Monthly Notices, R.A.S.* vol. XLIII. p. 102 for aid in computing the reduction of the latitude and log r.

Ex. 1. Using Clarke's elements for the figure of the earth, show that

$$\tan\phi' = [9{\cdot}9970352]\tan\phi,$$

the figures enclosed in brackets representing a Logarithm, and show that as the geographical latitude of Greenwich is 51° 28′ 38″ its geocentric latitude is 51° 17′ 11″.

Ex. 2. If powers of e higher than the second are neglected, show that

$$X = 1 - \tfrac{3}{4}e^2 - \tfrac{1}{4}e^2\cos 2\phi,$$

$$Y = 1 + \tfrac{1}{4}e^2 - \tfrac{1}{4}e^2\cos 2\phi.$$

Ex. 3. Show that the tables for Log X and Log Y so far as five places are concerned may be computed from

$$\text{Log } X = 9{\cdot}99778 - {\cdot}00074\cos 2\phi,$$

$$\log Y = 0{\cdot}00074 - {\cdot}00074\cos 2\phi.$$

*16. Radius of curvature along the meridian.

The curvature of the earth along a meridian at any point is the curvature of the circle which osculates the ellipse at that point. If $a\cos\theta$, $b\sin\theta$ be the coordinates of a point on the ellipse $x^2/a^2 + y^2/b^2 = 1$, then the equation of the normal at that point is

$$ax\sin\theta - by\cos\theta = (a^2 - b^2)\sin\theta\cos\theta \ldots\ldots\ldots\ldots\text{(i)},$$

and for the latitude ϕ, or the angle which the normal makes with the major axis

$$\tan\phi = a\tan\theta/b.$$

The centre of curvature is the intersection of two consecutive normals. Differentiating (i) with regard to θ we see that the coordinates of the centre of curvature must satisfy the equation

$$ax\cos\theta + by\sin\theta = (a^2 - b^2)\cos 2\theta \ldots\ldots\ldots\ldots\text{(ii)}.$$

Solving for x and y from (i) and (ii) we have for the coordinates of the centre of curvature

$$x = (a^2 - b^2)\cos^3\theta/a, \quad y = (b^2 - a^2)\sin^3\theta/b,$$

and for the radius of curvature we then find

$$\rho = (a^2\sin^2\theta + b^2\cos^2\theta)^{\frac{3}{2}}/ab,$$

or in terms of the latitude ϕ,

$$\rho = a^2b^2(b^2\sin^2\phi + a^2\cos^2\phi)^{-\frac{3}{2}}.$$

Hence we see that if s be the distance between two points on the same meridian whose geographical latitudes expressed in radians are ϕ and ϕ_1 respectively, we have

$$s = \int_{\phi}^{\phi_1} \frac{a^2 b^2}{(b^2 \sin^2 \phi + a^2 \cos^2 \phi)^{\frac{3}{2}}} d\phi.$$

It easily follows that if powers of the eccentricity above the second are neglected, we have, as an approximate value of the arc between the latitudes ϕ and ϕ_1,

$$s = (a - \tfrac{1}{2}c)(\phi_1 - \phi) - \tfrac{3}{2}c \sin(\phi_1 - \phi)\cos(\phi_1 + \phi),$$

where $c = a - b$. The quantity c/a is often called the *ellipticity*.

We also obtain the approximate expression

$$a - \tfrac{1}{2}c - \tfrac{3}{2}c \cos 2\phi$$

for the radius of curvature of the meridian at the latitude ϕ, and the approximate length of the quadrant of the meridian is $\pi(a + b)/4$.

Ex. 1. If the lengths of a degree of the meridian measured at latitudes 60° and 45° be s_1 and s_2 respectively, prove that the ellipticity of the earth regarded as a spheroid of revolution is $\frac{4}{3}(1 - s_2/s_1)$. [Math. Trip. I. 1892.]

The radius of curvature of the meridian at lat. ϕ is $a - \frac{1}{2}c - \frac{3}{2}c \cos 2\phi$.
Hence the length of 1° at lat. ϕ is

$$(a - \tfrac{1}{2}c - \tfrac{3}{2}c \cos 2\phi)\, 2\pi/360.$$

$$s_1 = (a + \tfrac{1}{4}c)\, 2\pi/360,$$

$$s_2 = (a - \tfrac{1}{2}c)\, 2\pi/360.$$

Hence $$s_2/s_1 = 1 - 3c/4a.$$

Ex. 2. If the powers of e up to the fourth are to be retained, show that for the radius of curvature ρ of the meridian at a point of geographical latitude ϕ we have the expression

$$\rho = a\,(1 - \tfrac{1}{4}e^2 - \tfrac{3}{64}e^4 - (\tfrac{3}{4}e^2 + \tfrac{3}{16}e^4)\cos 2\phi + \tfrac{15}{64}e^4 \cos 4\phi).$$

Ex. 3. Adopting Clarke's constants as the semi-axes of the earth regarded as a spheroid of revolution, show that the number of metres in a quadrant of the meridian from the pole to the equator is 10000186. (log metre in feet = 0·5159889.)

Ex. 4. In Clarke's *Geodesy*, p. 112, we read "It is customary in geodetical calculations to convert a distance measured along a meridian when that distance does not exceed a degree or so into difference of latitude by dividing the length by the radius of curvature corresponding to the middle point or rather to the mean of the terminal latitudes."

Show that if $\phi+\frac{1}{2}a$ and $\phi-\frac{1}{2}a$ be the extreme latitudes the error of this assumption will be about $\frac{1}{4}(a-b)\sin^3 a\cos 2\phi$.

For as shown above the arc $s=(a-\frac{1}{2}c)a-\frac{3}{2}c\sin a\cos 2\phi$, while the assumed arc is $(a-\frac{1}{2}c)a-\frac{3}{2}ca\cos 2\phi$. The difference is

$$\tfrac{3}{2}c\cos 2\phi\,(a-\sin a)=\tfrac{1}{4}c\sin^3 a\cos 2\phi$$

as a is small. The expression for this difference in inches is approximately

$$214000\sin^3 a\cos 2\phi,$$

which, if $\phi=60°$ and $a=1°$, would be about half an inch.

Ex. 5. Measuring along the meridian from the latitude ϕ until the latitude $\phi+1'$ is reached the number of feet to be traversed will be

$$6077-31\cos 2\phi.$$

Ex. 6. If x be the radius of the parallel of latitude ϕ, and y be the height of the parallel above the equator, both expressed in miles, show that with Clarke's data

$$x=3966{\cdot}7\cos\phi-3{\cdot}4\cos 3\phi,$$

$$y=3946{\cdot}4\sin\phi-3{\cdot}4\sin 3\phi,$$

and that if ρ be the radius of curvature of the meridian at the latitude ϕ

$$\rho=3956{\cdot}6-20{\cdot}2\cos 2\phi.$$

Ex. 7. Show from Clarke's data that at latitude ϕ the length in feet of a degree on the meridian is expressed by

$$364609-1867\cos 2\phi+4\cos 4\phi,$$

where ϕ is the latitude of the middle of the arc. Show also that the length of a degree of longitude is

$$365543\cos\phi-312\cos 3\phi.$$

17. The theory of map making.

By the word *map* is here meant a plane representation of points or figures on a sphere. We have first to consider the methods by which we are to assign to each point on the sphere its corresponding point on the map. We must obtain either a geometrical construction by which each point on the map is connected with the point on the sphere which it represents, or two formulae from which, when the spherical coordinates of a point on the sphere are given, the rectangular coordinates of the corresponding point on the map are determined. Both of these methods are used. We shall commence with the latter.

Let β, λ be respectively the latitude and longitude of a point on the sphere referred to a fundamental great circle. Let x, y be the coordinates of the corresponding point in a plane referred to a pair of rectangular axes. If β and λ are given the problem requires that we must have some means of finding x and y. There is also the converse problem to be considered. If x and y are given we must have some means of finding β and λ. These considerations imply the existence of relations such as

$$x = f_1(\beta, \lambda); \quad y = f_2(\beta, \lambda)$$

where f_1 and f_2 are known functions. This is perhaps the most general conception of the art of map making.

Considerable limitations must however be imposed on the forms of the functions f_1 and f_2 when we bear in mind the practical purposes for which maps are constructed. For a useful map of, let us say, Great Britain, the shapes of the counties on the map must be as far as possible the shapes of the same counties on the spherical surface of the earth. We also expect that the distances of the several towns as shown on the map shall be, at least approximately, proportional to the true distances measured in arc along the earth's surface. It is admitted that the conditions here indicated can under no circumstances be exactly complied with. It is not possible that any plane map could be devised which should represent in their true proportions the distances between *every pair* of points on the sphere. It is however possible in various ways to arrange a correspondence such that every spherical figure, of which each dimension is small in comparison with the diameter of the sphere, shall be represented on the map by a figure essentially similar.

If a spherical triangle is to be represented in a map by a plane triangle, it is obvious that their corresponding angles cannot be equal; indeed as the sum of the three angles of a spherical triangle exceeds 180°, its angles cannot be those of any plane triangle. If however the spherical triangle be small in comparison with the entire surface of the sphere, the spherical excess $(A + B + C - 180°)$ is small, and if it may be neglected we can then in various ways obtain functions f_1 and f_2 such that every small spherical triangle on the sphere shall be similar to the triangle which represents it in the plane.

A map which possesses the property thus indicated is said to be a *conformal representation* of the spherical surface. Let A', B', C' on the map be the representations of three points A, B, C on the sphere and let us assume that these points are adjacent. If the map is conformal, then

$$AB/A'B' = BC/B'C' = CA/C'A',$$

and, unless these relations are generally true for adjacent points A, B, C on the sphere, the map is not conformal.

***18. Conditions that a map shall be conformal.**

The general conditions that a map shall be conformal are thus found.

Let A, B, C be three adjacent points on the sphere, the coordinates of A being β, λ, of B being $\beta + h$, $\lambda + k$, and of C $\beta + h'$, $\lambda + k'$, where h, k, h', k' are small quantities. Then from § 8, we have

$$AB^2 = a^2(h^2 + k^2\cos^2\beta);\quad BC^2 = a^2((h-h')^2 + (k-k')^2\cos^2\beta);$$
$$CA^2 = a^2(h'^2 + k'^2\cos^2\beta),$$

where a is the radius of the sphere.

If A', B', C' be the correspondents of A, B, C and if x, y be the coordinates of A', then we have for coordinates of B'

$$x + \frac{\partial x}{\partial\beta}h + \frac{\partial x}{\partial\lambda}k,\quad y + \frac{\partial y}{\partial\beta}h + \frac{\partial y}{\partial\lambda}k,$$

and for the coordinates of C'

$$x + \frac{\partial x}{\partial\beta}h' + \frac{\partial x}{\partial\lambda}k',\quad y + \frac{\partial y}{\partial\beta}h' + \frac{\partial y}{\partial\lambda}k'.$$

If the triangles ABC and $A'B'C'$ are similar and H^2 is a common factor not depending upon h, k, h', k',

$$\left.\begin{aligned}
&\left(\frac{\partial x}{\partial\beta}h + \frac{\partial x}{\partial\lambda}k\right)^2 + \left(\frac{\partial y}{\partial\beta}h + \frac{\partial y}{\partial\lambda}k\right)^2 = H^2a^2(h^2 + k^2\cos^2\beta)\\
&\left(\frac{\partial x}{\partial\beta}h' + \frac{\partial x}{\partial\lambda}k'\right)^2 + \left(\frac{\partial y}{\partial\beta}h' + \frac{\partial y}{\partial\lambda}k'\right)^2 = H^2a^2(h'^2 + k'^2\cos^2\beta)\\
&\left(\frac{\partial x}{\partial\beta}(h-h') + \frac{\partial x}{\partial\lambda}(k-k')\right)^2 + \left(\frac{\partial y}{\partial\beta}(h-h') + \frac{\partial y}{\partial\lambda}(k-k')\right)^2\\
&\qquad\qquad = H^2a^2\{(h-h')^2 + (k-k')^2\cos^2\beta\}
\end{aligned}\right\}\ \ldots\text{(i)}.$$

These equations will be satisfied for all values of h, k, h', k' if the following equations are satisfied:

$$\frac{\partial x}{\partial\beta}\cdot\frac{\partial x}{\partial\lambda} + \frac{\partial y}{\partial\beta}\cdot\frac{\partial y}{\partial\lambda} = 0\ \ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

$$\left(\frac{\partial x}{\partial\lambda}\right)^2 + \left(\frac{\partial y}{\partial\lambda}\right)^2 = \cos^2\beta\left\{\left(\frac{\partial x}{\partial\beta}\right)^2 + \left(\frac{\partial y}{\partial\beta}\right)^2\right\}\ \ldots\ldots\text{(iii)}.$$

These are the conditions with which x and y must comply when expressed in terms of β and λ if the representation is to be conformal.

When one conformal map has been drawn it is easy to obtain various others as follows.

Let w denote the complex variable $x + iy$ where i is as usual a square root of -1. If we form any function of w, e.g. w^2 or $\sin w$ or $\log \tan w$, &c. or more generally $f(w)$, we obtain another complex variable which may be represented thus

$$f(x + iy) = u + iv,$$

and also

$$f(x - iy) = u - iv.$$

Differentiating both these equations with regard to β and λ

$$f'(x+iy)\left(\frac{\partial x}{\partial \beta} + i\frac{\partial y}{\partial \beta}\right) = \frac{\partial u}{\partial \beta} + i\frac{\partial v}{\partial \beta},$$

$$f'(x+iy)\left(\frac{\partial x}{\partial \lambda} + i\frac{\partial y}{\partial \lambda}\right) = \frac{\partial u}{\partial \lambda} + i\frac{\partial v}{\partial \lambda},$$

$$f'(x-iy)\left(\frac{\partial x}{\partial \beta} - i\frac{\partial y}{\partial \beta}\right) = \frac{\partial u}{\partial \beta} - i\frac{\partial v}{\partial \beta},$$

$$f'(x-iy)\left(\frac{\partial x}{\partial \lambda} - i\frac{\partial y}{\partial \lambda}\right) = \frac{\partial u}{\partial \lambda} - i\frac{\partial v}{\partial \lambda}.$$

Multiplying the first and last and adding the product of the second and third, we have

$$f'(x+iy)f'(x-iy)\left(\frac{\partial x}{\partial \beta}\cdot\frac{\partial x}{\partial \lambda} + \frac{\partial y}{\partial \beta}\cdot\frac{\partial y}{\partial \lambda}\right) = \frac{\partial u}{\partial \beta}\cdot\frac{\partial u}{\partial \lambda} + \frac{\partial v}{\partial \beta}\cdot\frac{\partial v}{\partial \lambda},$$

and as the left-hand side is zero from (ii) because (x, y) is a conformal representation, so must also the right-hand side be zero.

Multiplying the second equation and the last

$$f'(x+iy)f'(x-iy)\left(\left(\frac{\partial x}{\partial \lambda}\right)^2 + \left(\frac{\partial y}{\partial \lambda}\right)^2\right) = \left(\frac{\partial u}{\partial \lambda}\right)^2 + \left(\frac{\partial v}{\partial \lambda}\right)^2,$$

and from the first and the third

$$f'(x+iy)f'(x-iy)\left(\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2\right) = \left(\frac{\partial u}{\partial \beta}\right)^2 + \left(\frac{\partial v}{\partial \beta}\right)^2.$$

We thus see from (iii) that

$$\left(\frac{\partial u}{\partial \lambda}\right)^2 + \left(\frac{\partial v}{\partial \lambda}\right)^2 = \cos^2\beta\left\{\left(\frac{\partial u}{\partial \beta}\right)^2 + \left(\frac{\partial v}{\partial \beta}\right)^2\right\}.$$

Thus we prove the following important theorem.

If x, y be any functions of β, λ which give a conformal representation of the surface of a sphere on a plane, then the coordinates u, v defined by any equation of the form

$$f(x \pm iy) = u \pm iv$$

will also be in conformal correspondence with β, λ.

Ex. If a conformal representation of the points on a sphere is to have for x and y formulae of the type $x = U \cos \lambda$, $y = U \sin \lambda$, where U is a function of β, show from the general conditions for conformal representation that

$$U = h \tan\left(\frac{\pi}{4} \pm \frac{\beta}{2}\right).$$

Substituting for x and y we see that (ii) is identically satisfied and (iii) becomes

$$U^2 = \cos^2 \beta \left(\frac{\partial U}{\partial \beta}\right)^2.$$

*19. The scale in a conformal representation.

The geometrical signification of H (§ 18) should be noted. It is termed the *scale* of the projection under consideration, for it is plain from the first of the equations (i) in which H is introduced that this is the factor to be applied to a small arc on the sphere to give the length of the corresponding arc on the projection.

To obtain the expression for H we may (as the projection is conformal) compare *any* small arc on the sphere in the vicinity of the point with its correspondent. We shall take as the simplest a small arc of length h between the points β, λ and $\beta + h$, λ. Then from (i) we obtain

$$h^2 \left(\frac{\partial x}{\partial \beta}\right)^2 + h^2 \left(\frac{\partial y}{\partial \beta}\right)^2 = H^2 a^2 h^2,$$

whence we obtain the following theorem.

If x, y be the plane rectangular coordinates of a point representing in any conformal map the point with coordinates β, λ on the sphere of radius a, then the scale or factor to be applied to each short arc on the sphere to show the length of the corresponding short line in the projection is

$$\frac{1}{a}\left\{\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2\right\}^{\frac{1}{2}}.$$

20. Mercator's projection.

We have now to consider that representation of the sphere known as "Mercator's projection" which is so useful in navigation. The essential features of this projection are:

(1) That the abscissa of a point on the map is directly proportional to the longitude of the corresponding point on the sphere.

(2) That the ordinate of a point on the map is a function of the latitude (but not of the longitude) of the corresponding point on the sphere.

(3) That the representation is conformal.

To express the first condition we make $x = h'\lambda$. To express the second condition we make $y = f(\beta)$, and to comply with the third we have to determine the form of f so that the representation shall be conformal. The projection would *not* be conformal if we simply made y proportional to β.

The fundamental conditions (ii) and (iii) § 18 must be satisfied. We have

$$\frac{\partial x}{\partial \beta} = 0, \quad \frac{\partial x}{\partial \lambda} = h', \quad \frac{\partial y}{\partial \beta} = f'(\beta), \quad \frac{\partial y}{\partial \lambda} = 0.$$

With these substitutions (ii) vanishes identically and (iii) becomes

$$h'^2 = \cos^2 \beta \, (f'\beta)^2,$$

and we have

$$\frac{df(\beta)}{d\beta} = \pm h' \sec \beta.$$

If we desire that the positive direction of y shall correspond to northwards on the sphere we take the upper sign and

$$f(\beta) = h' \log_e \tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right) + \text{constant}.$$

The constant may be made equal to zero, for then the ordinates on the map are zero for points on the equator. Thus we learn the fundamental theorem on which Mercator's projection depends, and which is thus enunciated.

If λ, β be the longitude and latitude of a point on the sphere, then a map constructed with rectangular coordinates

$$x = h'\lambda, \quad y = h' \log_e \tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right),$$

will be conformal with the sphere.

As λ is here expressed in radians and the logarithm employed is Napierian it is convenient to transform the equations so that λ shall be expressed as usual in degrees of longitude, and that the logarithms shall be changed to common logarithms with the help of the modulus 0·4343. With these changes

$$x = \frac{2\pi h'}{360}\lambda, \quad y = \frac{h'}{0{\cdot}4343}\log_{10}\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right).$$

Introducing instead of h' a new constant h such that $360h = 2\pi h'$ we have

$$x = h\lambda, \quad y = 132h\log_{10}\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right) \quad \text{............(i)},$$

where λ is in degrees and ordinary logarithms are employed.

Ex. 1. Show that the scale in the Mercator projection

$$x=h'\lambda, \quad y=h'\log_e\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)$$

is expressed by $h'\sec\beta/a$.

Ex. 2. If in a Mercator chart of the Atlantic ocean the parallel for north latitude 70° is 185 mm. from the equator, what must be the distance of the parallel of 20°, and the length of 50° on the equator?

We have $$185=132h\log_{10}\tan\left(\frac{\pi}{4}+35^\circ\right),$$

whence h is found to be 1·86 and the equation for the chart is

$$y=245\text{ mm. }\log_{10}\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right),$$

which when $\beta=20^\circ$ gives $y=38$ mm.

As $x=1{\cdot}86\lambda$ we have $1{\cdot}86\times50=93$ mm. for the answer to the second part.

Ex. 3. What difference would be produced in a Mercator chart if instead of taking

$$y=h'\log\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right)$$

we had taken $$y=h'\log\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)?$$

Ex. 4. If s be a small terrestrial arc at latitude β and if s' be its Mercator projection, show that s/s' is the ratio of the length of the terrestrial circle of latitude through β to the length of the equator on the projection.

Ex. 5. In the Mercator projection show that the length of the nautical mile (1′ in latitude) varies as the secant of the latitude.

Ex. 6. In the practical use of Mercator's charts in coasting navigation the mariner, desiring to find by how many nautical miles (*i.e.* minutes of arc) two points A and B are separated, places the points of his dividers on the two points corresponding to A and B on the chart and then by applying the dividers to the graduation for latitude on the margin of *that same chart at about the latitude of A and B* ascertains what he desires. Give the justification for this procedure.

The chart being conformal and representing but a small part of the sphere may be used as if every distance on the chart including the scale of latitudes were strictly proportional to the corresponding distances on the sphere. But the minutes of latitude on charts representing various parts of the earth will generally differ in length even though those charts are all part of the same Mercator projection. Hence the mariner should take his distance scale from the chart he is considering and from about the same latitude as the points whose distance he is measuring.

Ex. 7. Show that on a Mercator map the length of a degree of latitude about Cambridge (Lat. 52° 12′ 52″) is 2·06 times the length of a degree of longitude on the equator.

From the formulae (i) we see that if h be one degree of longitude on the equator the distance between the parallels of latitude β_1 and β_2 on the Mercator map is

$$132h\left(\log_{10}\tan\left(\frac{\pi}{4}+\frac{\beta_1}{2}\right)-\log_{10}\tan\left(\frac{\pi}{4}+\frac{\beta_2}{2}\right)\right).$$

Substituting for β_1 the value 52° 42′ 52″ and for β_2 51° 42′ 52″ the expression becomes $2{\cdot}06h$.

Ex. 8. Prove that the equation of the trace on a Mercator's chart of a great circle will always be of the form

$$2\sin\left(\frac{x}{a}+c\right)=k\left(e^{\frac{y}{a}}-e^{-\frac{y}{a}}\right),$$

where $2\pi a$ is the length on the map of the equatorial circumference, and c, k are constants defining the great circle.

Ex. 9. If β is small enough for $\tan^5\frac{1}{2}\beta$ to be neglected, show that the difference of the distances of a place, whose latitude is β, from the equator on Mercator's chart, and on a chart obtained by projecting from the centre of the earth on the enveloping cylinder touching the earth along the equator is

$$\tfrac{2}{3}\tan^3\tfrac{1}{2}\beta\times\text{the earth's diameter.}$$

The projection on the cylinder of a point on the surface of the sphere gives

$$x=2\pi a\lambda/360,\qquad y=a\tan\beta,$$

where β and λ are the latitude and longitude of the point and a the radius of the sphere.

For the Mercator projection

$$x=2\pi a\lambda/360,\qquad y=a\log\tan(45+\tfrac{1}{2}\beta).$$

The difference of the distances from the equator in the two cases is

$$\begin{aligned}&a\left(\tan\beta-\log\frac{1+\tan\frac{1}{2}\beta}{1-\tan\frac{1}{2}\beta}\right)\\&=a\,(2\tan\tfrac{1}{2}\beta+2\tan^3\tfrac{1}{2}\beta\ldots\ldots-2\tan\tfrac{1}{2}\beta-\tfrac{2}{3}\tan^3\tfrac{1}{2}\beta\ldots\ldots)\\&=\tfrac{2}{3}\tan^3\tfrac{1}{2}\beta\times 2a.\end{aligned}$$

***21. The loxodrome.**

If we assume the earth to be a sphere, then the course taken by a ship which steers constantly on the same course, *i.e.* always making the same angle with the meridian, is called the *loxodrome* or sometimes a *Rhumb-line.*

If λ be the longitude and β the latitude and if θ be the angle at which the curve cuts the successive meridians, then the differential equation of the loxodrome is $\tan\theta = \cos\beta \, d\lambda/d\beta$, whence (by integration)

$$\lambda = \tan\theta \log_e \tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right) + \text{const.}$$

If we substitute this value of λ in the Mercator projection,

$$x = h'\lambda, \quad y = h' \log_e \tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right),$$

we have $x = y \tan\theta + \text{const.}$, showing that the Mercator projection of a loxodrome is a straight line cutting the projections of the meridians at the same angle as that at which the loxodrome is inclined to the meridians on the sphere.

The property just mentioned is of the utmost importance in navigation, for when the mariner joins two points on the Mercator chart by a straight line, the constant angle at which this line cuts the projected meridians indicates the course that is to be steered from one place to the other.

Ex. 1. If r be the radius of a sphere, if θ be the constant angle at which the meridians intersect a loxodrome, if $+z$ be the axis from the centre to the north pole, and if the axes $+x$, $+y$ be the radii to the points on the equator of longitudes 0° and 90° respectively, then the equations of the loxodrome are

$$r \tan\theta \,.\, dz + y\,dx - x\,dy = 0,$$
$$x^2 + y^2 + z^2 = r^2.$$

Ex. 2. If r be the radius of a sphere, if θ be the constant angle at which the meridians intersect a loxodrome, and if s be the length of the arc of the loxodrome whose terminal points are in latitudes β_1, β_2, then

$$r(\beta_1 - \beta_2) = s \cos\theta.$$

Ex. 3. If the earth be regarded as a spheroid produced by the rotation of an ellipse of small eccentricity e about its minor axis, show that the equation connecting λ the longitude and β the latitude of a point on the loxodrome intersecting meridians at the constant angle θ is

$$\lambda = \tan\theta \left(\log \tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right) - e^2 \sin\beta\right) + \text{const.}$$

If ρ be the radius of curvature of the point in the ellipse, and ρ' the intercept on the normal between the curve and the minor axis,

$$\rho' = \frac{a}{\sqrt{1 - e^2 \sin^2 \beta}}, \quad \rho = \frac{\rho'^3}{a^2}(1 - e^2).$$

The differential equation of the loxodrome is

$$\frac{d\lambda}{d\beta} = \frac{\rho \tan \theta}{\rho' \cos \beta} = \frac{\tan \theta}{\cos \beta} - e^2 \tan \theta \cos \beta \text{ very nearly.}$$

Ex. 4. Show that on a Mercator chart on which the unit of length is taken to be 1′ of equatorial longitude the ordinate to the parallel of latitude β will be

$$7916 \log_{10} \tan \left(\frac{\pi}{4} + \frac{\beta}{2} \right) - 3438 e^2 \sin \beta,$$

where e is the eccentricity of the ellipse given by a meridional section of the earth.

From Ex. 3 we see that the point x, y in the projection corresponding to λ, β is given by the equations

$$x = h\lambda,$$

$$y = h \left(\log_e \tan \left(\frac{\pi}{4} + \frac{\beta}{2} \right) - e^2 \sin \beta \right).$$

As λ is in circular measure x is given in minutes by making $h = 3438$ and $3438/0{\cdot}4343 = 7916$.

*22. Stereographic Projection.

One of the most important methods of representing the points on a sphere by a conformal projection is that known as the *stereographic*, which is thus described.

A point O on the sphere having been chosen as the origin of projection, the plane of projection is the plane of the great circle of which O is the pole, or any parallel plane. If P be any other point on the sphere and OP cuts the plane of projection in P', then P' is said to be the stereographic projection of P.

Draw the plane $OP'PC$ where C is the centre of the sphere. The tangent plane at P will cut the plane of projection in a line through M perpendicular to the plane of the paper. Let M_1 be any point on that line. To show that the projection gives a conformal representation we shall consider the inclination of any arc through P to the meridian FPO and the corresponding angle in the projection.

From the properties of the circle $MP = MP'$ and therefore $M_1P = M_1P'$. Hence the triangles M_1PM and $M_1P'M$ are equal and thus $\angle M_1PM = \angle M_1P'M$. But $\angle M_1PM$ is the angle of intersection of two circles on the sphere and $\angle M_1P'M$ is the angle of intersection of their projections.

Perhaps the simplest proof that stereographic projection is conformal is this. The ratio of a line-element at P' to the corresponding line-element at P is OP'/OP, as is at once seen by similar triangles: and the fact that this ratio does not depend on the direction of the line-element shows that the representation is conformal. We also see that the scale is OP'/OP.

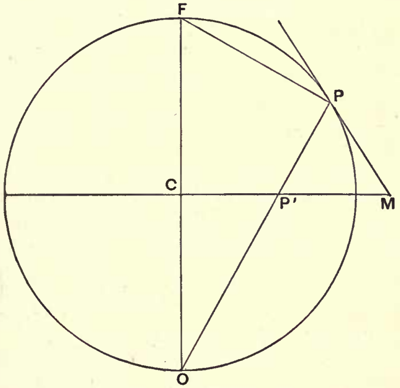


FIG. 19.

It is instructive to show how the stereographic projection can be deduced from Mercator's projection by the principle of § 18, that if $u + iv = f(x + iy)$ then the coordinates u, v give a representation conformal to that given by x, y.

In Mercator's projection

$$x = h\lambda, \quad y = h \log \tan\left(\frac{\pi}{4} + \frac{\beta}{2}\right),$$

therefore

$$\frac{i(x+iy)}{h} = \log \tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right) + i\lambda,$$

and hence

$$ae^{\frac{i(x+iy)}{h}} = a \tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right)\cos\lambda + ia\tan\left(\frac{\pi}{4} - \frac{\beta}{2}\right)\sin\lambda.$$

The left-hand member is a function of $x+iy$ and consequently by § 18

$$u = a\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\cos\lambda \text{ and } v = a\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\sin\lambda$$

are also the coordinates of a conformal representation, and it is easy to show that they correspond to a stereographic projection. For if the plane of the equator be taken as the plane of projection, then the angle FOP in Fig. 19 is $\left(\frac{\pi}{4}-\frac{\beta}{2}\right)$, and

$$CP' = CO\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right).$$

If λ is the longitude of P the projections of CP' in the direction of, and at right angles to, the zero of longitude are

$$CO\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\cos\lambda \text{ and } CO\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\sin\lambda \text{ respectively.}$$

From the formulae of § 19 we can determine the scale at the point β, λ on the sphere in the stereographic projection when, *the apex being at the antinole* of the fundamental circle, the projection is defined by the equations

$$x = a\cos\lambda\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right), \quad y = a\sin\lambda\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right),$$

in which a is the radius of the sphere. We have

$$\frac{\partial x}{\partial \beta} = \frac{a\cos\lambda}{-1-\sin\beta}, \quad \frac{\partial y}{\partial \beta} = \frac{a\sin\lambda}{-1-\sin\beta},$$

and hence

$$\frac{1}{a}\left\{\left(\frac{\partial x}{\partial \beta}\right)^2 + \left(\frac{\partial y}{\partial \beta}\right)^2\right\}^{\frac{1}{2}} = \frac{1}{1+\sin\beta}.$$

Ex. 1. Determine the value of the scale at the point β, λ on the sphere in the stereographic projection, when, the apex being at the point $\lambda = 180°$, $\beta = 0$ on the fundamental circle, the projection is defined by the equations

$$x = \frac{a\cos\beta\sin\lambda}{1+\cos\beta\cos\lambda}, \quad y = \frac{a\sin\beta}{1+\cos\beta\cos\lambda}.$$

Ex. 2. Show that in the stereographic projection of the earth any point and its antipodes will have as their correspondents two points collinear with the centre of the map and such that their distances from the map's centre have a constant product.

Ex. 3. Let x, y be the point in the stereographic projection corresponding to the point of latitude β and longitude λ on the sphere. Let $x+\Delta x$, $y+\Delta y$

be the point corresponding to $\beta+\Delta\beta$, $\lambda+\Delta\lambda$. Show that when Δx, Δy, $\Delta\beta$, $\Delta\lambda$ are small,

$$\Delta x = -y\,\Delta\lambda - x\sec\beta\,\Delta\beta,$$
$$\Delta y = \quad x\,\Delta\lambda - y\sec\beta\,\Delta\beta.$$

Ex. 4. A map of the world is to be constructed in three parts, two circumpolar, on the stereographic projection, and one equatorial, on Mercator's projection. The circumpolar maps are to be such that the scale in latitude a is the same as that of the other map at the equator, and the scale at the bounding latitude ϕ is to be the same for all the maps. Prove that

$$2\tan\phi\,(1+\sin a)=\sin a\,(2+\sin a),$$

and that the scale in latitude ϕ is that at the equator multiplied by

$$1+\frac{\sin^2 a}{2\,(1+\sin a)}.$$

From the scales already shown in § 20, Ex. 1 and § 22, for the Mercator and stereographic projections respectively

$$h/(1+\sin a)=h'/a,\quad h/(1+\sin\phi)=h'\sec\phi/a,$$

whence eliminating h'/ah,

$$\tan\phi+\sqrt{1+\tan^2\phi}=1+\sin a.$$

Solving for $\tan\phi$ the result given is obtained.

The ratio of the scale in lat. ϕ to that at the equator is $\sec\phi$, and solving

$$\sec\phi+\sqrt{\sec^2\phi-1}=1+\sin a,$$

we obtain $\sec\phi$ as required.

*23. The stereographic projection of any circle on the sphere is also a circle.

Let C be the centre of the circle on the sphere and draw the plane through the origin of projection, the centre of the sphere and C.

Let PQ be the intersection of this plane with the plane of the circle. The cone whose apex is O and which passes through all points on the circumference of the circle must have OC as its axis, for since CP is equal to CQ, $\angle COP=\angle COQ$. This must be true for every plane through OC, but this could only be the case if OC were the axis of the cone.

Every cone has two planes of circular section which make equal angles with the axis and whose intersection is perpendicular to the axis. The tangent planes to the sphere at C and O make equal angles with CO and their intersection is perpendicular to CO. But the tangent plane at C is parallel to one circular section

PQ and therefore the tangent plane at *O* must be parallel to the other. Thus the fundamental property of stereographic projection is proved.

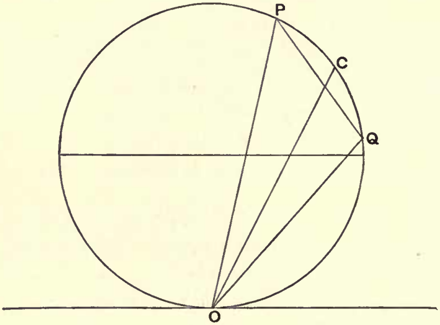


FIG. 20.

As a cone has only two systems of circular sections there can be no other planes except those parallel to the tangent at *O* which give the characteristic feature of stereographic projection.

The same theorem may also be proved as follows.

The generators of a cone touching the sphere in the given circle are each perpendicular to the tangents to the circle drawn at the point of contact. Small portions of the generators at the point of contact may be considered as lying on the sphere. In the projection this cone becomes a pencil of straight lines passing through a point, and as angles are preserved, the projection of the circle must be a curve cutting all these lines at right angles, *i.e.* another circle.

Ex. 1. Show that in the stereographic projection the centre of a circle on the sphere is projected into the centre of its corresponding circle if the diameters of the original circle are small enough to be considered as right lines.

For by the preservation of angles a right-angled triangle inscribed in the original circle becomes a right-angled triangle in the projection, and therefore every diameter of the original circle is projected into a diameter of the corresponding circle.

Ex. 2. Show that in the stereographic projection of the sphere from any point on the surface a system of meridians projects into a system of coaxial circles.

Ex. 3. Show that by the stereographic projection a system of concentric small circles on the sphere project into a system of circles whose centres are collinear and each of which cuts orthogonally the same system of coaxial circles.

For all the great circles through the centre C of the concentric circles invert into a set of coaxial circles, and as angles are preserved in inversion, the inverses of the concentric circles must intersect these coaxial circles orthogonally, and their centres must lie on the line which is the inverse of the great circle OC where O is the centre of projection.

*24. General formulae for stereographic projection.

Let $270°$, β_0 be the coordinates of the origin O of the stereographic projection and let λ, β be the coordinates of any other point P, both referred to the same graduated great circle S.

Let S' be the graduated great circle of which O is the nole.

Let the straight line OP intersect the plane of S' in P'. Thus the stereographic projection of P is P' and we assume the coordinates of P' in the plane S' to be X, Y. The axis $+X$ is from the centre of the sphere to the ascending node of S' on S. The axis $+Y$ passes through $90°$ on S', it being assumed that this node is the origin of graduation on S' as well as S.

We have to find expressions for X and Y in terms of β, λ.

We assume three rectangular axes from the centre of the sphere as follows:

axis $+x$ to the point $\beta=0$, $\lambda=0$,
" $+y$ " " $\beta=0$, $\lambda=90°$,
" $+z$ " " $\beta=90°$, λ is indeterminate.

With reference to these axes the coordinates of O, P', P are as follows:

	x	y	z
O	0	$-a\cos\beta_0$	$a\sin\beta_0$
P'	X	$Y\sin\beta_0$	$Y\cos\beta_0$
P	$a\cos\beta\cos\lambda$	$a\cos\beta\sin\lambda$	$a\sin\beta$.

We express that O, P' and P are collinear and obtain

$$\frac{a\cos\beta\cos\lambda - X}{\cos\beta\cos\lambda} = \frac{a\cos\beta\sin\lambda - Y\sin\beta_0}{\cos\beta\sin\lambda + \cos\beta_0} = \frac{a\sin\beta - Y\cos\beta_0}{\sin\beta - \sin\beta_0}.$$

Solving for X and Y we have

$$X = a\frac{\cos\beta\cos\lambda}{1-\sin\beta\sin\beta_0+\cos\beta\cos\beta_0\sin\lambda} \quad\text{.........(i)},$$

$$Y = a\frac{\sin\beta\cos\beta_0+\cos\beta\sin\beta_0\sin\lambda}{1-\sin\beta\sin\beta_0+\cos\beta\cos\beta_0\sin\lambda} \quad\text{.........(ii)}.$$

If O be the nole of S, then $\beta_0 = 90°$, and we have

$$X = a\frac{\cos\beta\cos\lambda}{1-\sin\beta},\quad Y = a\frac{\cos\beta\sin\lambda}{1-\sin\beta}.$$

If O be the antinole of S, $\beta_0 = -90°$ and

$$X = a\frac{\cos\beta\cos\lambda}{1+\sin\beta},\quad Y = -a\frac{\cos\beta\sin\lambda}{1+\sin\beta}.$$

If O lie on S, $\beta_0 = 0$ and

$$X = a\frac{\cos\beta\cos\lambda}{1+\cos\beta\sin\lambda},$$

$$Y = a\frac{\sin\beta}{1+\cos\beta\sin\lambda}.$$

We have assumed in these formulae that the zero of graduation on S coincides with the ascending node of S' on S. If the zero of graduation had been elsewhere let us suppose that the longitude of the ascending node is Ω. Then in the formulae (i) and (ii) we must put $\lambda - \Omega$ instead of λ and thus obtain

$$X = a\frac{\cos\beta\cos(\lambda-\Omega)}{1-\sin\beta\sin\beta_0+\cos\beta\cos\beta_0\sin(\lambda-\Omega)} \quad \text{...(iii)},$$

$$Y = a\frac{\sin\beta\cos\beta_0+\cos\beta\sin\beta_0\sin(\lambda-\Omega)}{1-\sin\beta\sin\beta_0+\cos\beta\cos\beta_0\sin(\lambda-\Omega)} \quad \text{...(iv)}.$$

By the formulae (i) and (ii) or (iii) and (iv) we can compute X and Y for any given values of λ and β, and thus construct by rectangular coordinates the stereographic chart of any figure on the sphere.

Ex. 1. Show that when the stereographic projection is from the nole of the fundamental circle, and when the axis $+X$ is from the centre to the point $\lambda=0$, $\beta=0$, and the axis $+Y$ is from the centre to the point $\lambda=90°$, $\beta=0$, then the relations between X, Y and λ, β are

$$X = a\cos\lambda\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right),\quad Y = a\sin\lambda\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right).$$

Ex. 2. Show that when the stereographic projection is from the antinole of the fundamental circle, and when the axis $+X$ is from the centre to the point $\lambda=0$, $\beta=0$, and axis $+Y$ is from the centre to the point $\lambda=90°$, $\beta=0$, then the relations between X, Y and λ, β are

$$X = a\cos\lambda\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right),\quad Y = -a\sin\lambda\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right).$$

Ex. 3. If the origin of projection be at Greenwich and the earth be assumed to be spherical, show how the formulae (iii) and (iv) will enable a stereographic map of Australia to be drawn.

Substitute for β_0 the latitude of Greenwich and assuming that the longitudes λ are measured from Greenwich make $\Omega = 90°$. Then if λ, β be the longitude and latitude of any point on the coast of Australia, the corresponding plane rectangular coordinates X and Y will be determined from (iii) and (iv), when a convenient value for the desired size of the map has been assigned to the constant a.

Ex. 4. If λ, β be regarded as variable coordinates but subject to the relation

$$A \cos\lambda \cos\beta + B \sin\lambda \cos\beta + C \sin\beta = 0,$$

where A, B, C are constants, show from (iii) and (iv) that all the points indicated by X, Y will lie on the circumference of the same circle.

25. On the construction of a map in which each area on the sphere is represented by an equal area on the map.

If x, y; x', y'; x'', y'' be three points on the chart, then the area they contain is

$$\tfrac{1}{2}\{x(y''-y') + x'(y-y'') + x''(y'-y)\} \quad \text{............(i).}$$

We take as the three corresponding points on the sphere β, λ; $\beta + k, \lambda$ and $\beta, \lambda + h$, where λ and h are small quantities. The area formed by these points on the sphere is $\tfrac{1}{2}a^2hk\cos\beta$.

We then have for the coordinates x', y'

$$x + \frac{\partial x}{\partial \beta}k, \quad y + \frac{\partial y}{\partial \beta}k,$$

and for x'', y''

$$x + \frac{\partial x}{\partial \lambda}h, \quad y + \frac{\partial y}{\partial \lambda}h.$$

Thus by substitution in (i) we have for the area in the plane

$$\frac{1}{2}\left\{x\left(h\frac{\partial y}{\partial \lambda} - k\frac{\partial y}{\partial \beta}\right) - \left(x + \frac{\partial x}{\partial \beta}k\right)h\frac{\partial y}{\partial \lambda} + \left(x + \frac{\partial x}{\partial \lambda}h\right)k\frac{\partial y}{\partial \beta}\right\}$$

$$= \tfrac{1}{2}hk\left(\frac{\partial x}{\partial \lambda}\cdot\frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \beta}\cdot\frac{\partial y}{\partial \lambda}\right).$$

Equating these two expressions for the area and noting that all surfaces can be built from such elementary areas, we have the following theorem.

If a plane projection of a sphere be such that x and y the coordinates corresponding to the point λ, β on the sphere fulfil the condition

$$\frac{\partial x}{\partial \lambda}\cdot\frac{\partial y}{\partial \beta} - \frac{\partial x}{\partial \beta}\cdot\frac{\partial y}{\partial \lambda} = a^2\cos\beta \quad \text{...............(ii),}$$

then any area on the sphere projects into an equal area.

MISCELLANEOUS EXERCISES ON CHAP. III.

Ex. 1. If the points on a sphere be projected from the centre of the sphere on a plane (Gnomonic Projection), examine by the principles of § 18 whether this projection is conformal.

Ex. 2. If (ϕ, l) be the latitude and longitude of any point lying on a great circle of a sphere, then

$$\tan\phi = A\cos l + B\sin l,$$

where A and B are constants. If then we put

$$(1)\quad x = \cot\phi\cos l, \qquad y = \cot\phi\sin l,$$

$$\text{or } (2)\quad X = \tan\phi\sec l, \qquad Y = \tan l,$$

we get a linear relation between x and y (or X and Y). Plotting x and y (or X and Y) as Cartesian coordinates, all great circles would be therefore straight lines.

Show how both of these charts may be obtained by a perspective projection of the sphere on a plane.

Ex. 3. A circle on the earth's surface has an angular radius ρ, and its centre A is in latitude β_0; show that in a stereographic projection from the north pole on the plane of the equator this circle is represented by a circle (radius ρ'), the distance of its centre from the point which represents A being

$$\rho'\tan\frac{\rho}{2}\tan\left(\frac{\pi}{4}+\frac{\beta_0}{2}\right).$$

Ex. 4. In Gauss' projection of the sphere the meridians are represented as straight lines passing through a point O, and the angle between any two such lines is $h\lambda$, where λ is the difference of longitude between the two corresponding meridians. The parallels of latitude are circular arcs with their centres at O. If the projection is to be conformal, show that the radius of the arc corresponding to colatitude u must be $k(\tan\frac{1}{2}u)^h$ where k is a constant.

We must have $x = U\cos(h\lambda)$, $y = U\sin(h\lambda)$ where U is a function of the latitude. By substitution in equation (iii), § 18 we have

$$h^2U^2 = \cos^2\beta\left(\frac{\partial U}{\partial\beta}\right).$$

Ex. 5. If

$$x = h\left(\frac{\pi}{2}-\lambda\right), \quad y = h\log\tan\left(\frac{\pi}{4}+\frac{\beta}{2}\right),$$

prove that

$$\tan\frac{x+iy}{2h} = u+iv,$$

where

$$u = \cos\beta\cos\lambda/(1+\cos\beta\sin\lambda), \quad v = \sin\beta/(1+\cos\beta\sin\lambda),$$

and hence show that u, v are coordinates giving a conformal representation.

Ex. 6. If the point β, λ on the sphere be represented on a plane by the point whose coordinates are

$$x=\frac{\cos\beta\cos\lambda}{1+\cos\beta\sin\lambda},\quad y=\frac{\sin\beta}{1+\cos\beta\sin\lambda},$$

show that a circle on the sphere with radius ρ and centre β_0, λ_0 will be represented by a circle on the plane having for radius $\sin\rho/(\cos\rho+\cos\beta_0\sin\lambda_0)$, and for the coordinates of its centre $\cos\beta_0\cos\lambda_0/(\cos\rho+\cos\beta_0\sin\lambda_0)$ and $\sin\beta_0/(\cos\rho+\cos\beta_0\sin\lambda_0)$.

Eliminate β and λ with help of the equation

$$\cos\rho=\sin\beta\sin\beta_0+\cos\beta\cos\beta_0\cos(\lambda-\lambda_0).$$

Ex. 7. A map of the northern hemisphere is constructed in such a way that parallels of latitude become concentric circles and meridians radii of these circles, and that equal areas on the earth become equal areas on the map. Find the equation of the curve which a loxodrome becomes on the map, and trace it.

From the conditions of the problem we have

$$x=\rho\cos\lambda,\quad y=\rho\sin\lambda,$$

where ρ is a function of β.

As areas are to be preserved we substitute these values in the condition given in § 25 and find

$$\rho\frac{\partial\rho}{\partial\beta}=-h\cos\beta,$$

where h is some constant connected with the ratio of the areas on the sphere and in the projection.

Integrating and determining the arbitrary constant by the condition that $\rho=0$ if $\beta=90°$,

$$\rho^2=2h(1-\sin\beta),$$

and

$$\rho=2\sqrt{h}\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right).$$

The projection of the loxodrome cutting the meridians at angle ϵ (§ 21) is the result of eliminating β and λ between

$$\lambda=\tan\epsilon\log_e\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right),$$

$$\tan\lambda=y/x,$$

$$\sqrt{x^2+y^2}=2\sqrt{h}\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right),$$

and the result in polar coordinates is

$$r^2(1+e^{2\theta\cot\epsilon})=4h^2.$$

Ex. 8. Show that the greatest distance that could be saved in a single voyage by sailing along a great circle instead of a parallel of latitude is

$$a\left[2\sin^{-1}\frac{2}{\pi}+\sqrt{\pi^2-4}-\pi\right],$$

where a is the earth's radius.

It is obvious that in the case supposed the ports of arrival and departure should have a difference of longitude of 180° so that the great circle joining them should pass through the pole. If ϕ be the latitude the difference of the two voyages is $a(\pi \cos\phi - \pi + 2\phi)$ and for this to be a maximum $\sin\phi = 2/\pi$.

Ex. 9. Show that in sailing from one meridian to a place in the same latitude on another meridian, the distance saved by sailing along a great circle instead of sailing due E. and W. is a maximum for latitude

$$\cos^{-1}(\sqrt{\lambda^2 - \sin^2\lambda}/\lambda \sin\lambda),$$

where λ is the difference of longitude of the two meridians.

Ex. 10. Describe the shortest course of a steamer which is to go from one point to another without going beyond a certain latitude, supposing the great circle course to cross that latitude.

Ex. 11. Cape Clear is in latitude 51° 26′ N., long. 9° 29′ W., and Cape Race is in lat. 46° 40′ N., long. 53° 8′ W.; verify that the great circle course between them would require a vessel to sail in a course from Cape Clear about $17\frac{1}{2}$° further north than the straight course on a Mercator's chart, and that the former course is the shorter by about 28 miles. [Math. Trip. I, 1887.]

Ex. 12. If a be the radius of the sphere, m the distance of the plane of the stereographic projection from the origin, P and P' a pair of corresponding points, r the distance of P' from the diameter through the origin, show that a small arc ρ on the sphere near P will project into a small arc near P' and of length $\rho(m^2 + r^2)/2ma$.

CHAPTER IV.

THE CELESTIAL SPHERE.

26. The celestial sphere.

Let A, B, C (Fig. 21) be three stars and O the position of the observer.

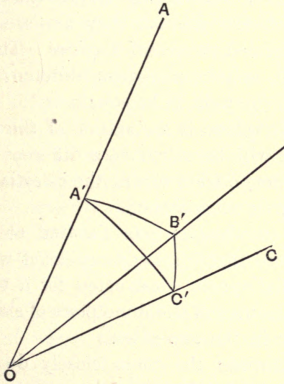


FIG. 21.

With centre O and radius any length OA' a sphere is described cutting OA, OB, OC in A', B', C' respectively, thus giving the spherical triangle $A'B'C'$.

The angle AOB is the angle which the stars A and B subtend at the observer. This is conveniently measured by $A'B'$ the side of the spherical triangle. In like manner, $B'C'$ and $C'A'$ measure the angles BOC and COA respectively.

The *apparent* distance of two stars is measured by the angle they subtend at the eye. For example, the apparent distance of A and C is measured by $\angle AOC$, *i.e.* by $A'C'$. The *apparent* distance of two stars from each other, which is of course only an angle, affords no clue to the *real* distance between them which is, of course, a linear magnitude. To determine the real distance we should also know the linear distance of each of the stars from the observer. The stars in the Pleiades *appear* to be much closer together than the stars in Ursa Major, but it does not necessarily follow that the Pleiades is the lesser group of the two.

Astronomical measurements of the relative positions of celestial bodies generally determine only apparent distances, and these, as we have seen, may be taken as arcs on the sphere described round O. Thus the geometry of astronomical measurements of position is the geometry of the sphere.

The sphere we have been considering shows the *apparent* distances of the celestial bodies just as they are seen on the heavens. This sphere is termed the *celestial sphere.* The length of its radius is immaterial, and in comparing different celestial spheres we shall assume all the radii to be equal.

The centre of a celestial sphere is the station of the observer, and for each station there will be of course a different celestial sphere. We have to consider to what extent the celestial spheres at different stations differ from one another.

Suppose, for instance, an observer was situated at the star Arcturus, the celestial sphere that he would construct would not be the same as the celestial sphere constructed for a terrestrial observer. The apparent distances of the same pairs of stars would be generally quite different in the two cases.

The nearer the two stations the more closely do the two celestial spheres resemble each other which have those stations as their centres. So far as the *fixed stars,* usually so called, are concerned it is correct to say that the celestial spheres constructed for all points on the earth's surface are practically identical. This is because the distances of the fixed stars from the earth are so great that the diameter of the earth is quite inappreciable by comparison. As an illustration we may state that the alteration in the apparent distance of two stars, by a shift of the observer's place from any station on the earth to its antipodes, could in no case exceed the 16,000th part of a second of arc so far as we at present know Stellar distances. An angle would have to be about a thousand times larger than this before it could be appreciable by our measuring instruments.

By the annual motion of the earth round the sun the station of a terrestrial observer is carried round a nearly circular path of mean radius 92,900,000 miles. A terrestrial observer is therefore shifted in the space of six months through a distance about double this amount. But even under these circumstances the great majority of apparent star places are without appreciable alteration and in no case, so far as we know, does the greatest alteration from this cause exceed 1″·5. (See Chap. XV.)

What has been said so far relates only to the fixed stars. We shall see in Chap. XII. that the apparent places on the celestial sphere of the sun and the planets, to some extent, and that of the moon to a large extent, are affected by the position on the earth's surface occupied by the observer.

We are not now considering the individual motions certain of the heavenly bodies possess; these of course affect their positions on the celestial sphere of every observatory.

If we have marked on the celestial spheres only those celestial bodies, such as most of the fixed stars, which are so far off that the apparent distances by which they are separated from each other are sensibly the same from all parts of the solar system, we may make the following statements with regard to the celestial spheres, it being supposed that the radii of all the spheres are equal.

For every station in the solar system there will be a celestial sphere of which that station is the centre.

Every celestial sphere is the same as every other celestial sphere not only as to radius but also as to the stars marked on it.

At any given moment the celestial spheres are all similarly placed, *i.e.* any radius of one sphere to a particular star is parallel to the corresponding radius of any other sphere. It is often convenient to treat of the celestial sphere as if its centre were coincident with the centre of the earth.

Ex. 1. Show that any point at a finite distance may be regarded as the centre of the celestial sphere of which the radius is indefinitely great.

Let O be the centre of the celestial sphere and let X be any point at a finite distance from O and S be a point on the celestial sphere,

$$XS^2 = OS^2 - 2OS \cdot OX \cos XOS + OX^2$$
$$= OS^2\left(1 - 2\frac{OX}{OS}\cos XOS + \frac{OX^2}{OS^2}\right)$$

As OX is finite, we see that as OS approaches infinity OX/OS approaches zero, whence in the limit $XS/OS = 1$. But as OS is constant for all points S on the sphere so must XS be constant, whence X may be regarded as the centre without appreciable error.

Ex. 2. Show that the directions of XS and OS tend in the limit to become identical.

27. The celestial horizon.

Let P be the station of an observer on the earth's surface and let us suppose his celestial sphere to be drawn, the radius of which is incomparably greater than the radius of the earth. A tangent plane drawn to the earth at P will cut this celestial sphere in a great circle, which is known as the *celestial horizon* of P.

The plane of the horizon at any place is also the plane of the surface of a liquid at rest in an open vessel at that place. This plane is normal to the direction of terrestrial gravitation, and consequently the direction of a plumb-line at any place P on the earth's surface is perpendicular to the plane of the horizon at P. The points on the celestial sphere to which the plumb-line points, when continued in both directions, are of the utmost importance in spherical astronomy. The point Z thus indicated overhead is called the *zenith* of P. The other point N in which the direction of the plumb-line supposed continued beneath our feet cuts the celestial sphere is called the *nadir*.

28. The diurnal motion.

The *daily* rotation of the earth on its axis in the approximate period of 23 hrs. 56 m. 4 secs., which is usually called the *sidereal day* (see § 33), causes the celestial sphere to have an apparent rotation in the opposite direction, *i.e.* from east to west, which is known as the *diurnal motion*.

The most direct method of demonstrating that the earth rotates upon its axis is afforded by Foucault's beautiful pendulum experiment. If we assume the earth to be a perfect sphere with centre O the principles of Foucault's pendulum are as follows.

Let ϕ be the north latitude of the observer at P and ω the angular velocity of the earth round its axis. We may suppose ω to be resolved into components $\omega \sin \phi$ round OP and $\omega \cos \phi$ round OQ, where Q is the point with south latitude $90° - \phi$ and on the meridian of P. So far as P and places in its neighbourhood are concerned this latter rotation has only the effect of a translation, so that for our present purpose this component may be neglected. The other component produces a rotation of the plane of the horizon at P round OP with an angular velocity $\omega \sin \phi$. If therefore a vertical plane at P did not partake in the rotation about OP, the angle made with it by any vertical plane which did partake of the rotation about OP would increase with the velocity $\omega \sin \phi$. Foucault's pendulum provides the means of verifying this experimentally. Without entering into practical details the essential feature of the experiment is as follows.

A heavy weight is suspended by a long wire from a fixed point. The weight being drawn aside is carefully released and oscillates slowly to and fro. The plane in which the pendulum oscillates does not partake in the rotation about OP. As however the observer is unconscious of the terrestrial rotation about OP, the plane of oscillation *appears* to revolve with reference to the terrestrial objects around. The direction of this motion and measurements of its magnitude demonstrate the diurnal rotation of the earth. The experiment would be best seen if it could be performed at one of the poles. At a station on the equator the plane of oscillation would have no apparent motion.

All points on the celestial sphere, except two, participate in the diurnal motion; these are of course the North and South Poles of the celestial sphere. The line joining these points passes through the centre of the earth and is the axis about which the earth rotates. It is always to be remembered that the dimensions of the earth are inappreciable in comparison with the celestial sphere, so that for present purposes we regard the earth as no more than a point at the centre of the celestial sphere. The special convenience of this stipulation is that we may not only consider the axis of the celestial sphere as passing through the centre of the earth, but we may always consider also that it passes through the station of any observer wherever he may be situated on the earth's surface. The pole which lies in that part of the celestial sphere within view of the dwellers in northern latitudes, is known as the North Pole. Fortunately for northern astronomers, the locality of the North Pole is conveniently indicated by the contiguous bright Pole star. The similar point in the southern skies and known as the South Pole is not so conveniently indicated, as there is no bright star in its vicinity.

The plane of the earth's equator will, of course, be unaffected by the diurnal rotation. Its intersection with the celestial sphere forms the great circle known as the celestial *equator*, and the poles of this great circle are the north and south poles of the heavens. Any plane parallel to the equator and at a finite distance cuts the celestial sphere in the celestial equator which is the vanishing line of all such planes. Any diameter of the earth (or indeed any straight line rigidly connected with the earth and prolonged indefinitely both ways) will intersect the celestial sphere in two points which as the earth rotates will describe what are called *parallel-circles*. They are in general small circles of the celestial sphere which, when the line producing them is parallel to the earth's axis, merge into the north and south poles respectively; and when the line is perpendicular to the earth's axis coalesce to form the equator.

The celestial horizon divides the celestial sphere into the visible hemisphere and the invisible hemisphere. In the act of passing from below the horizon to above, the star is said to be *rising*, when passing from above to below it is said to be *setting*. If the observer were at the north pole of the earth the celestial north pole would then be in his zenith and his horizon would be the celestial equator. In this case the diurnal motion would make the stars appear to move parallel to the horizon and the phenomena of rising and setting would be unknown; of one half of the celestial sphere no part would ever come above the observer's horizon and no part of the other half would ever set. If the observer were on the terrestrial equator the north and south poles would be on his horizon and the hemispheres into which the horizon divides the celestial sphere would be continually changing. The stars rise perpendicularly to the horizon and each star in the heavens will be above the observer's horizon for one half the sidereal day and below it for the other half†. Thus a contrast between the circumstances of an observer at the pole and an observer at the equator would be that, from the former station, no part of the celestial sphere visible to him at any moment can ever become invisible by diurnal rotation, while to an observer at the equator every portion of the celestial sphere becomes at times invisible.

At a terrestrial station which is neither one of the poles nor on the equator part of the celestial sphere is always above the horizon, part of it is always below the horizon, and the remainder is sometimes above and sometimes below the horizon. Each star in consequence of the diurnal motion revolves in a small circle of the celestial sphere of which one of the celestial poles is the centre. If this circle should lie entirely above the horizon, then the star never sets and is therefore always visible (apart from such interferences as clouds or daylight, which are not at present considered). If the circle should lie entirely below the horizon, then the star never rises and must be permanently invisible from the station in question. If however the circle cuts the horizon, then the star will sometimes be above the horizon and sometimes below.

29. The meridian and the prime vertical.

The great circle which passes through the two celestial poles and the zenith and the nadir of the observer is called the *meridian* of the place where the observer is stationed. The celestial meridian is also the intersection of the plane of the terrestrial meridian of the observer with the celestial sphere. Thus the celestial meridian is the great circle which, starting at right angles to the horizon from the north point N (Fig. 22), meets the horizon again perpendicularly at the south point S and then continues its course below the horizon back to N.

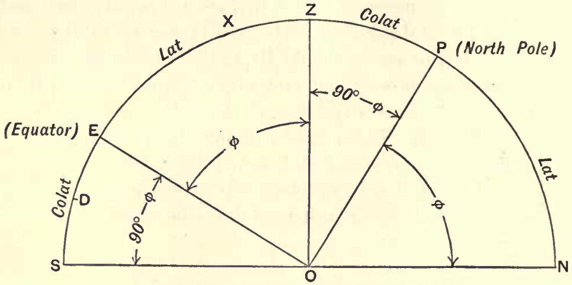
Each star must pass twice across the meridian in the diurnal revolution of the celestial sphere, and on each occasion the star is said to *transit*. The meridian is divided by the north and south poles into two semicircles of which one contains the zenith and the other the nadir. A star in the transit across the first semicircle is said to be at its *upper culmination*, while

† Refraction is not here taken into account.

in transit across the other half of the meridian the star is said to be at its *lower culmination.*

Among the great circles of the celestial sphere the meridian is the most important because it passes through the two most remarkable points of the sphere, namely, the pole P and the zenith Z (Fig. 22). There are also three other points to be specially noted. They are the north point N and the south point S in which the meridian intersects the horizon, and E in which it intersects the celestial equator.

The latitude ϕ is the angle between the direction of a plumb-line and the plane of the equator. Hence (Fig. 22) the latitude



Meridian in Northern Hemisphere at North latitude ϕ

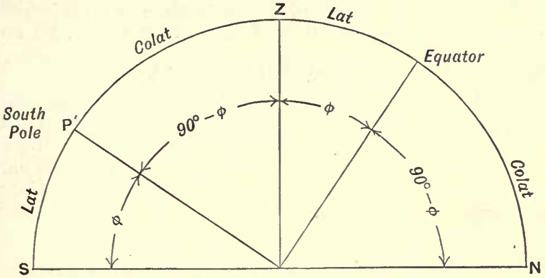
FIG. 22.

of the observer is the angle ZOE, *i.e.* that between the zenith and the equator. Since POE and ZON are both right angles we must have NOP equal to ϕ, and the angle NOP being the elevation of the pole above the horizon is, as we shall see in § 30, called its *altitude.* Thus we obtain the fundamental proposition that *the altitude of the pole is the latitude of the observer.*

The arc $ZP = 90° - \phi$ from the zenith to the elevated pole is generally called the *colatitude.*

It is obvious that a star X does not set unless its distance PX from the elevated pole exceeds the latitude of the observer. A star which does not set is called a *circumpolar star* and its distance EX from the equator towards the north pole, that is to say, its *north declination* (§ 31) must not be less than $90° - \phi$. A star does not rise if its south declination is more than $90° - \phi$.

The corresponding diagram showing the meridian in the southern hemisphere is given in Fig. 23. It may be remarked that *southern* latitudes are often expressed by attaching a minus sign to the numerical value of the latitude. Thus this figure shows the meridian of latitude $-\phi$.



Meridian in Southern Hemisphere at South Latitude φ

FIG. 23.

The great circle through the zenith and at right angles to the meridian is called the *prime vertical.* It passes through the east and west points of the horizon.

Ex. 1. Show that with reference to a station in latitude ϕ the greatest and least zenith distances of a star of declination δ are respectively $180° - \{0° \sim (\phi + \delta)\}$ and $\phi \sim \delta$.

Ex. 2. If the zenith distance of a star is to remain always the same, show that either the observer's latitude is 90° or the star's declination is 90°.

Ex. 3. Show that if a star is always above the horizon, $\{0° \sim (\phi + \delta)\} > 90°$, if it is never above the horizon $\phi \sim \delta > 90°$, and that if it rises and sets $\{0° \sim (\phi + \delta)\} < 90°$ and $\phi \sim \delta < 90°$.

Ex. 4. If the latitude of the observer be known, show how the declination of a star can be obtained from observations of its zenith distance at the moment of transit.

Ex. 5. The latitude of Greenwich being 51° 28′ 38″·1, show that for the meridian of Greenwich (Fig. 22)

$$SE = ZP = 38° \ 31' \ 21''\cdot 9 \quad \text{and} \quad EZ = PN = 51° \ 28' \ 38''\cdot 1.$$

Ex. 6. Show that 51° 29′ is the lowest latitude at which all stars having a north declination exceeding 38° 31′ are circumpolar. Show that all stars having a south declination exceeding 38° 31′ must be there invisible.

Ex. 7. On Nov. 13th the sun is 108° from the north pole, show that in any north latitude exceeding 72° the sun does not rise above the horizon.

Ex. 8. The observatory at Stockholm is in latitude 59° 20′ 33″·0 N., and that at the Cape of Good Hope in latitude 33° 56′ 3″·5 S. The declination of Sirius is −16° 35′ 22″·0. Find the altitudes of Sirius when in culmination in Stockholm and at the Cape of Good Hope respectively.

The distance along the meridian from the north pole to the south point of the horizon is $180°-\phi$, where ϕ is the north latitude (Fig. 22). The distance from the pole to a star of declination δ is $90°-\delta$ (the proper sign being given to δ), whence the distance from the south point of the horizon to the star is

$$180°-\phi-(90°-\delta)=90°-\phi+\delta.$$

Hence in the case of Stockholm (the declination of Sirius being negative), the altitude of Sirius is $90°-(59°\ 20'\ 33''{\cdot}0)-(16°\ 35'\ 22''{\cdot}0)=14°\ 4'\ 5''{\cdot}0$.

At a southern latitude (Fig. 23) the arc from the south pole to the north point is $180°-\phi$, and to a north declination δ is $90°+\delta$. Hence the altitude at culmination is

$$180°-\phi-(90°+\delta)=90°-\phi-\delta,$$

and for Sirius at the Cape

$$90°-(33°\ 56'\ 3''{\cdot}5)+16°\ 35'\ 22''{\cdot}0=72°\ 39'\ 18''{\cdot}5.$$

Ex. 9. If z_1, z_2 be the zenith distances of a circumpolar star at upper and at lower culmination respectively, and both culminations are to the north of the zenith, show that the north latitude of the observer is $90°-\frac{1}{2}(z_1+z_2)$.

30. Altitude and azimuth.

Perhaps the most obvious system of celestial coordinates is that in which the horizon is used as the fundamental circle. We shall suppose that the star is above the horizon, and that a great circle is drawn from the zenith through the star and thence to the horizon, which it cuts at right angles. Such a circle is called a *vertical* circle. The arc of this circle between the horizon and the star is called the *altitude* of the star and is one of the coordinates defining the place of the star. The second coordinate is the *azimuth*, which is reckoned along the horizon in various ways. It seems desirable to adopt a uniform practice in this matter. We shall therefore always measure the azimuth of a celestial object from the north point round by east and south to the foot of the vertical circle through the star†. Thus the azimuth may have any value from 0° to 360°, and the pole of the horizon so graduated is the nadir,—not the zenith. When the azimuth and altitude of a star are known, its position is determined.

† This mode of reckoning azimuths has ancient authority. I have seen it on a compass card of date 1640 kindly shown to me by Professor Silvanus Thompson.

For example, if the azimuth of a star be 310° and its altitude is 15°, then the star can be found as follows. We start from the north point of the horizon and proceed round to the east at the azimuth 90° and thence to the south and the west at azimuths of 180° and 270° respectively, and then 40° further in the same direction indicates the azimuth 310°. The vertical circle thus reached might, no doubt, be described as having an azimuth of −50°, that is, as lying 50° to the westward of the north point. But it is convenient to avoid negative values in this coordinate as can always be done by adding 360°. The point in which the vertical circle meets the horizon being thus defined by its azimuth, a point is then to be taken on the vertical circle at the proper altitude, in this case 15° above the horizon, and we obtain the required position of the star.

Instead of the altitude of a star it is often convenient to use the complement of the altitude which is known as the zenith distance. Thus in the case in question, when the altitude is 15° the zenith distance is 75°.

For approximate measurements of azimuth the magnetic compass is used. The needle points to the magnetic north, which deviates from the true north by what is called the *magnetic declination*. This varies both for different times and for different places. For the British Islands in A.D. 1908 the needle points on an average 18° west of the true north. Thus the azimuth of the magnetic north, *i.e.* the azimuth measured from the true north round by east, south and west, is about 342° for the British Isles in 1908†.

† The following information has been kindly communicated by the National Physical Laboratory:

Mean magnetic declinations for 1906:

Kew	16° 28′·5 W.
Stonyhurst	17° 48′·3 W.
Valencia	21° 6′·3 W.

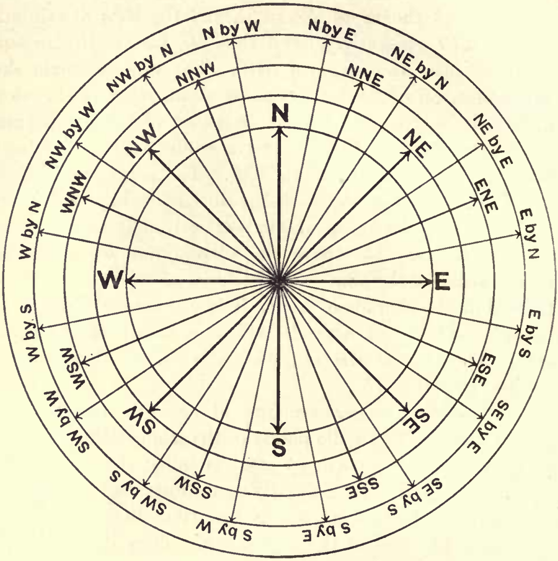
The magnetic declination is decreasing, and for the annual amount of change the mean values at Kew have been as follows for the series of years indicated:

1870 to 1880 ...	−8′·1	1890 to 1900 ...	−5′·8
1880 to 1890 ...	−6′·8	1900 to 1906 ...	−4′·0

Valencia observations commenced in 1901: for the five years 1901 to 1906 the mean values of the annual changes in Declination were:

Stonyhurst ...	−4′·3	Falmouth ...	−4′·0
Kew	−4′·1	Valencia	−4′·3

In the mariner's compass the indications are shown on a card by the division of the circumference into 32 equal *points* of $11\frac{1}{4}°$



each. The four chief points on the card are marked N. (at the magnetic north), E., S., W. at intervals of 90°. Each of these intervals is bisected by points marked NE., SE., SW., NW. respectively. Thus the circumference is divided into eight equal parts of four points each. Each of these parts is again bisected: the bisection of N. and NE. is marked NNE., that of NE. and E. is ENE., and so on. In this way half the points receive their designations. The remaining sixteen points are derived from the first eight, viz. N., E., S., W.; NE., SE., SW., NW. by simply adding the word "by" and appending one of the letters N., E., S., W. For example, "W. by N." means one point from west towards north; "W. by S." in like manner means one point from west towards the south, and "SE. by E." means one point from SE. towards E.

Ex. 1. Find the azimuth measured from the magnetic north of the point NE. by N.

NE. is four points from magnetic north, and "NE. by N." means one point back again towards N. Hence the answer is three points or $33\frac{3}{4}°$.

Ex. 2. Show in like manner that the azimuth from the magnetic north of WNW. is 292°·5.

Ex. 3. If the azimuth of a point as shown by the compass is 73°, find the true azimuth when the magnetic declination is 18°·5 W.

Ex. 4. Find the true azimuth of the magnetic bearing SE. by S. if the magnetic declination is 17° W.

MISCELLANEOUS EXERCISES ON CHAP. IV.

Ex. 1. If r_1, r_2 be the real distances of two stars from the observer, and if θ be the apparent distance between the stars on the celestial sphere, show that the square of the true distance of the stars from each other is

$$r_1^2 - 2r_1r_2\cos\theta + r_2^2.$$

Ex. 2. Show that the prime vertical, the horizon, and the equator intersect in the same two points.

Ex. 3. If a, b be the equatorial and polar radii of the earth, assumed a spheroid, show that the greatest angular difference possible at any point on the earth's surface between the plumb-line and the radius to the earth's centre is

$$\tan^{-1}\frac{a^2-b^2}{2ab}.$$

Ex. 4. If the declination δ of a star exceeds the latitude ϕ, show that the azimuth of the star must oscillate between

$$\sin^{-1}(\cos\delta\sec\phi)$$

on one side of the meridian and the same angle on the other.

Ex. 5. Show that the cosine of the angle which the path of a star as it sets makes with the horizon is equal to the sine of the latitude multiplied by the secant of the declination.

Ex. 6. Two places are of the same latitude and the distance of the pole from the great circle through them is equal to the sun's declination. Prove that at these places the length of the night is equal to their difference of longitude.

CHAPTER V.

RIGHT ASCENSION AND DECLINATION; CELESTIAL LATITUDE AND LONGITUDE.

31. Right ascension and declination. Though the altitude and azimuth are, in one sense, the simplest coordinates of a star, certain other systems are generally more convenient. The altitude and azimuth of a star are continually changing with the time on account of the diurnal motion, and even at the same moment the altitude and azimuth of a star are different for two different observatories. It is often preferable to employ coordinates which remain unaltered by the diurnal motion and are the same whatever may be the latitude and longitude of the observer's station. We can obtain coordinates possessing the required qualities by referring the star to a great circle fixed on the celestial sphere.

The celestial equator as already pointed out (§ 28) remains unaltered in position notwithstanding the diurnal rotation. The equator also possesses such a natural relation to the diurnal motion that it is specially suited to serve as the fundamental circle, and the coordinates most generally useful in spherical astronomy are accordingly referred to the celestial equator. When referred to the equator, the coordinates of a point on the celestial sphere do not change by the diurnal motion, nor do they change when the station of the observer is changed unless the object be near enough to the earth for what is known as *parallax* to be appreciable. This will be discussed in Chap. XII., and need not be further considered here.

To construct the coordinates of a star with respect to the celestial equator we proceed as follows. A great circle *NP*, Fig. 24, is drawn from the north celestial pole *N* through a star *S* and meets the celestial equator at *P*. The arc *PS* intercepted on this circle between the equator and the star is the *declination* of the star. The arc ♈*P* measured from a certain point ♈ on the equator and in the direction which has *N* for its nole is the *right ascension* of the star.

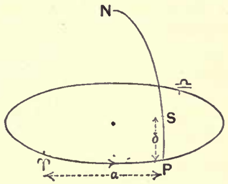


FIG. 24.

The Right Ascension, or "R.A." as it is often written for brevity, is generally expressed by the letter α, and measured from 0° to 360°. The Declination or "Decl." as it is often written, is generally expressed by δ, and a negative sign is attached to δ when *S* is south of the equator. *SN* or $90° - \delta$ is the "North Polar Distance" and is sometimes used instead of δ as the second coordinate of the star.

32. The first point of Aries or ♈. In a subsequent chapter we shall consider the sun's apparent annual movement with respect to the fixed stars. We may, however, here anticipate so far as to say that the sun describes a complete circuit with reference to the stars once in a year in the direction of the diurnal rotation of the earth, *i.e.* from West through South to East. In this movement the centre of the sun appears to follow very closely a great circle on the celestial sphere. This great circle is known as the *ecliptic*, and was so called by the ancients because when Eclipses take place the moon is crossing this circle.

Observing the direction in which the sun moves round the ecliptic, we can distinguish between the two *nodes* or intersections of the ecliptic and the equator. These nodes are to be designated as follows. That at which the sun crosses from S. to N. of the equator is called the *first point of Aries* and is represented by the symbol ♈. The sun passes through ♈ at the moment known as the *vernal equinox*. This occurs each year about March 21. For example, in the year 1909 the vernal equinox is on March 21 at $6^h\ 13^m$, Greenwich Mean Time.

The other node, or that at which the sun crosses from N. to S. of the equator, is called the *first point of Libra* and is represented by the symbol ♎. The sun passes through ♎ at the moment of the *autumnal equinox* (1909, Sept. 23, $4^h\ 45^m$, G.M.T.).

By universal agreement the origin on the equator from which right ascensions are to be measured is the first point of Aries, or ♈. The positive direction along the equator is such that the right ascension of the sun, constantly altering by the sun's motion, is always *increasing*. Thus since the path of the sun among the stars is from W. through S. to E., ♈ is the ascending, ♎ the descending node of the ecliptic on the equator.

As the "first point of Aries" occupies a place of such exceptional importance in astronomy, it may be proper to observe that the word "Aries" has in this expression but little more than historical significance. The node through which the sun passed at the vernal equinox was no doubt at one time in the constellation Aries, but it is not so at present. We shall see in the chapter on Precession (VIII.) that while the plane of the ecliptic shifts only slightly in space, the plane of the equator rotates so that while it makes a nearly constant angle with the ecliptic, its intersection with the ecliptic moves along that circle in the negative direction at the rate of about 50″ annually, so that from this cause alone, in the greater part of the heavens, the R.A. of a celestial body is always increasing.

The present position of ♈ may be approximately indicated as follows. When the great square of Pegasus is towards the south imagine the left vertical side produced downwards to a distance equal to its own length; from the point thus found draw a line to the right, parallel to the lower horizontal side of the square and one-fourth of its length. This terminates at about the present position of "the first point of Aries."

In Fig. 25 ♈HH' is the equator, and ♈KK' is the ecliptic, P and P' are respectively the nole and antinole of the equator, and Q and Q' are the nole and antinole of the ecliptic. The arrow-head on ♈K shows the direction of the apparent motion of the sun relative to stars on the celestial sphere, the arrow-head on ♈H shows the direction in which the right ascensions are measured.

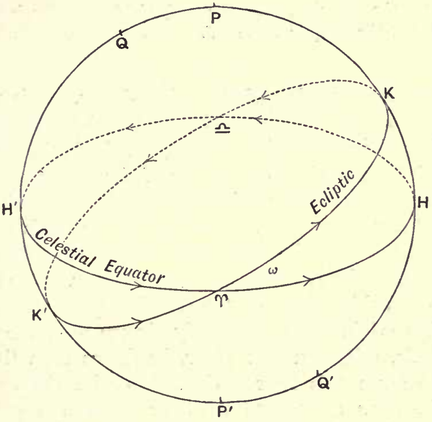


FIG. 25.

The great circle $HKH'K'$ is known as the *solstitial colure* and K, K' are the points at which the sun is found at the *summer* and *winter solstices* respectively. The great circle through P♈♎ is called the *equinoctial colure.*

The inclination between the ecliptic and the equator is generally known as the *obliquity of the ecliptic.* The mean value of the obliquity of the ecliptic as given in the Ephemeris for 1909 is 23° 27′ 4″·04. It is subject to small temporary fluctuation by *nutation* (see Chap. VIII.), and it has also a slow continuous decline at the rate of 46″·84 per century.

Ex. 1. If α be the right ascension and δ the declination of a point on the celestial sphere, show that the values of α, δ for certain points (Fig. 25) on the sphere are as follows, ω being the obliquity of the ecliptic:

	α	δ		α	δ
H	90°	0	P	?	90°
H'	270°	0	P'	?	$-90°$
K	90°	ω	Q	270°	$90° - \omega$
K'	270°	$-\omega$	Q'	90°	$\omega - 90°$
♈	0	0	♎	180°	0

Ex. 2. Show that the right ascension α and the declination δ of the sun will always be connected by the equation

$$\tan \delta = \tan \omega \sin \alpha.$$

Ex. 3. On the 9th May, 1910, the sun's right ascension is 45° 30′, and the obliquity of the ecliptic is 23° 27′. Show that the declination of the sun is +17° 11′·5.

33. The hour angle and the sidereal day. It is sometimes convenient to take as the origin from which coordinates are measured on the equator that point, above the horizon, where the equator is intersected by the meridian of the observer. Owing to the diurnal motion which carries the meridian round the celestial sphere in the course of a sidereal day, this origin is not a fixed point on the celestial sphere, but moves steadily round the equator so as to complete its revolution in a sidereal day. One of the coordinates of an object fixed on the celestial sphere measured from this moving origin must necessarily change with the time. If a great circle, called an *hour circle*, be drawn from the pole to a star, the angle this hour circle makes with the meridian is termed the *hour angle*, and the hour angle of a star and its declination or its polar distance form a system of coordinates which are often convenient.

The declination of an object does not vary in consequence of the diurnal rotation. The hour angle of a star is incessantly altering. Since the star appears to move from upper culmination towards the west we shall measure hour angle from the meridian to the westward. Thus the hour angle is zero when the object is at upper culmination, and gradually increases to 180° as the object passes to lower culmination. From thence the hour angle is continually increasing until it reaches 360° at the next upper culmination. To the west of the meridian an hour angle is therefore between 0° and 180°. On the east of the meridian the hour angle is between 180° and 360°. With this understanding hour angles are always *increasing*, and, since 360° can always be added to or subtracted from any angle when used in a trigonometrical function, we may say that all hour angles are between −180° and +180° and that hour angles west are positive, and hour angles east negative.

The hour angle (unlike the declination in this respect also) changes with the station of the observer. For example when a star is passing the meridian at Greenwich its hour angle is there zero. But at the same moment the star will have passed the meridian of easterly stations and will therefore to such stations show hour angles west. At a place where the longitude is 2 hours east of Greenwich, the star would appear to have an hour angle two hours west at the same moment as an observer at Greenwich has the star on his meridian. More generally we may say that at two places with east longitudes l and l' respectively, the hour angles west of the same object would be simultaneously θ and $\theta + l' - l$.

A *sidereal day* is the interval between two consecutive transits of the first point of Aries across any selected meridian. If we remember that the stars are practically fixed on the celestial sphere, and if we overlook for the present certain small irregularities, we may also say that the time interval between two consecutive transits of the same star across the meridian is a sidereal day. It is also accurate enough for all practical purposes to define the sidereal day as the period of rotation of the earth on its axis (see § 28). Expressed in mean solar time the sidereal day is $23^h\ 56^m\ 4^s{\cdot}0906$.

Like the solar day the sidereal day is divided into 24 equal periods, which are called sidereal hours. A sidereal hour is divided into sixty minutes, and each minute is subdivided into sixty seconds.

In one hour of sidereal time after the meridian passage of a star its hour angle measured in degrees would be 15°, this being the 24th part of the circumference. It is usual to express the hour angle in sidereal time rather than in degrees. If, for example, the star be 3 hours (sidereal) past the meridian, and the secondary from the pole to the equator which passes through the star have an intercept of 35° between the star and the equator, we could express the position of the star at that particular place and at that particular moment by saying that it had a west hour angle of three hours and a north declination of 35°.

The hour angle west of the first point of Aries when turned into time at the rate of 15° per hour is the *sideral time.* When the first point of Aries is on the meridian at upper culmination the sidereal time is $0^h\ 0^m\ 0^s$. When the first point of Arics has passed the meridian so that its hour angle is

15°, then the sidereal time is 1 hr., and when it has passed the meridian so long that the hour angle is 285°, then the sidereal time is 19 hrs.

Let α be the right ascension expressed in time of a star S, and let h be the hour angle west and ϑ the sidereal time.

Let NZ be the meridian (Fig. 26), $N\Upsilon$ the equinoctial colure, then the sidereal time ϑ, as already defined, is measured by $\angle \Upsilon NZ$.

The right ascension of S is ΥNS, and there can be no ambiguity about the sign, for N is the nole of the equator and the R.A. is measured in the positive direction from the equinoctial colure; also ZNS is the hour angle of S; and hence

$$h = \vartheta - \alpha.$$

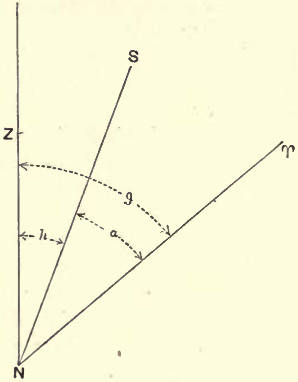


FIG. 26.

Thus we obtain an important relation connecting the hour angle and the right ascension of a body with the sidereal time.

Ex. 1. Show that the sidereal time can be determined by measuring the hour angle of a star whose right ascension is known.

Ex. 2. If the hour angle east of a star be 98° 11′ 15″ and its R.A. be $21^h\ 9^m\ 23^s$, show that the sidereal time is $14^h\ 36^m\ 38^s$.

The hour angle west is $360° - (98°\ 11'\ 15'') = 261°\ 48'\ 45''$ which turned into time at 15° per hour is $17^h\ 27^m\ 15^s$, whence

$$\vartheta = \alpha + h = 38^h\ 36^m\ 38^s = 14^h\ 36^m\ 38^s,$$

as 24^h may always be rejected.

Ex. 3. If θ be an hour angle measured in degrees, show that the angle expressed in circular measure is $2\pi\theta/360°$.

Ex. 4. If t be the number of hours in an hour angle, show that the circular measure of that angle is $\pi t/12$.

Ex. 5. At any place of north latitude ϕ the interval between one of the transits of a star across a vertical circle of azimuth A and one of its transits across the other vertical circle, which makes the same angle with the meridian, is the same for all stars, and equal to $\cot^{-1}(\sin\phi \tan A)/\pi$ of a sidereal day.

Let N (Fig. 27) be the celestial pole, Z the zenith, ZP, ZQ the two vertical circles, NP, NQ great circles perpendicular thereto, P_1, P_2 the points at which any given star crosses ZP and Q_1, Q_2 the points where it crosses ZQ. Then from symmetry

$$\angle P_1NP = \angle PNP_2 = \angle Q_1NQ = \angle QNQ_2,$$

and therefore $$\angle P_1NQ_1 = \angle P_2NQ_2 = \angle PNQ,$$

and is therefore independent of the star chosen.

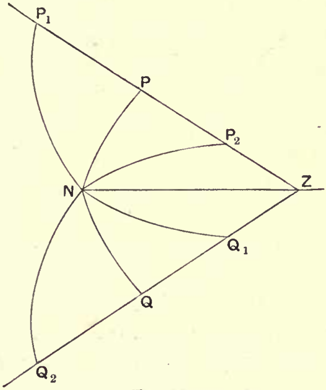


FIG. 27.

Further $\cot PNZ = \sin\phi \tan A$ and the required interval is

$$\cot^{-1}(\sin\phi\tan A)/\pi$$

of the sidereal day.

Ex. 6. If at a place in latitude ϕ, a pair of stars whose coordinates are respectively a, δ and a', δ' ever come on the same vertical, show that

$$\cos\phi > \cos\delta\cos\delta'\sin(a-a')\operatorname{cosec}\theta,$$

where θ is the arc joining the stars.

Let S, S' (Fig. 28) be the two stars. Then the triangle SNS' rotates about N. Let fall $NT(=p)$ perpendicular on SS'. Then no point on the great circle SS' can be at less distance than p from N. But if S, S' are to be on the same vertical, then this arc must pass through the zenith. Therefore $90° - \phi > p$ or $\cos\phi > \sin p$. But $\cos\delta\sin NSS' = \sin p$ and $\sin NSS'\sin\theta = \cos\delta'\sin(a-a')$, whence $\sin p = \cos\delta\cos\delta'\sin(a-a')\operatorname{cosec}\theta$.

Ex. 7. Show that $2^h\ 23^m\ 24^s{\cdot}92$ of mean solar time is equivalent to $2^h\ 23^m\ 48^s{\cdot}48$ of sidereal time and that 15^h of sidereal time will be turned into mean solar time by subtracting $2^m\ 27^s{\cdot}44$.

Ex. 8. Show that 1465 sidereal days are very nearly the same as 1461 mean solar days.

34. Determination of zenith distance and azimuth from hour angle and declination. The required formulae may be written down from the general equations of transformation of

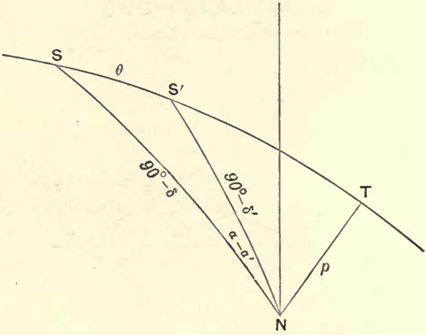


FIG. 28.

coordinates. By our convention for the measurement of azimuth from the north point a is taken in such a direction (§ 30) that the nadir is the nole of the horizon when regarded as a great circle graduated for azimuth. The north pole is of course the nole of the equator when graduated for right ascension. From the definition of a nole (§ 6) it follows that if N and N' are the noles of two graduated great circles L and L', then the nole of NN' ($\ngtr 180°$) is the ascending node of L' on L and the nole of $N'N$ ($\ngtr 180°$) is the ascending node of L on L'. Thus the ascending node of the horizon on the equator is at the point due west, and consequently Ω', *i.e.* the azimuth of the ascending node, is 270° when measured from the origin at the north point. The sidereal time ϑ is the hour angle by which ♈ is to the west of the meridian. Hence remembering the direction in which right ascensions are measured we must have Ω, *i.e.* the right ascension of the ascending node of the horizon on the equator, equal to $270° + \vartheta$. The angle between the equator and horizon is $90° + \phi$, for this is the angle between their two noles (§ 10). Finally as the zenith is the antinole of the horizon, δ' is negative and equal to $z - 90°$. Making the requisite substitutions in the formulae (ii), (iii), (iv), (v), (vi) of § 12, we have the desired equations

$$\left.\begin{aligned} -\sin a \sin z &= \cos\delta \sin(\vartheta - \alpha) \\ \cos a \sin z &= \cos\phi \sin\delta - \sin\phi \cos\delta \cos(\vartheta - \alpha) \\ \cos z &= \sin\phi \sin\delta + \cos\phi \cos\delta \cos(\vartheta - \alpha) \end{aligned}\right\} \quad \ldots\text{(i)},$$

and the equivalent group

$$\left.\begin{aligned} -\sin(\vartheta - \alpha) \cos\delta &= \sin a \sin z \\ \cos(\vartheta - \alpha) \cos\delta &= \cos\phi \cos z - \sin\phi \cos a \sin z \\ \sin\delta &= \sin\phi \cos z + \cos\phi \cos a \sin z \end{aligned}\right\} \quad \ldots\text{(ii)}.$$

By the equations (i) we can calculate the zenith distance and azimuth when the declination and the hour angle $(\vartheta - \alpha)$ are known, and conversely by (ii) we can find the declination and the hour angle when the zenith distance and the azimuth are known.

For a determination of the zenith distance when the hour angle and the declination are known the following process is very convenient. The angle subtended at the star by the arc joining the zenith and the pole is called the *parallactic angle*. This we shall denote by η and for its determination we have from the fundamental formulae, (1), (2), (3) § 1, the following equations in which h is written instead of $(\vartheta - \alpha)$ for the hour angle:

$$\left.\begin{aligned} \cos z &= \sin\phi \sin\delta + \cos\phi \cos\delta \cos h \\ \sin\eta \sin z &= \cos\phi \sin h \\ \cos\eta \sin z &= \sin\phi \cos\delta - \cos\phi \sin\delta \cos h \end{aligned}\right\} \quad \ldots\ldots\text{(iii)}.$$

When h and δ are known, the parallactic angle η and the zenith distance z can both be found from these equations. As $\sin z$ and $\cos\phi$ are both always positive, it follows from the second equation that η and h have the same sign. They are both positive to the west of the meridian and negative to the east.

It is often desirable to make these calculations by the help of subsidiary quantities. We introduce two new angles m and n by the conditions

$$\left.\begin{aligned} \cos n &= \cos\phi \sin h \\ \sin n \cos m &= \sin\phi \\ \sin n \sin m &= \cos\phi \cos h \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

If n_0, m_0 be a pair of values of n and m which satisfy these equations they will be equally satisfied by $360° - n_0$ and $180° + m_0$.

It is a matter of indifference whether in the subsequent work we use n_0, m_0 or $360° - n_0$, $180° + m_0$. Taking one of these two pairs as n, m, we have by substitution in (iii)

$$\left.\begin{aligned} \cos z &= \sin n \sin(\delta + m) \\ \sin \eta \sin z &= \cos n \\ \cos \eta \sin z &= \sin n \cos(\delta + m) \end{aligned}\right\} \text{(v)}.$$

These equations may also be written thus

$$\left.\begin{aligned} \tan \eta &= \cot n \sec(\delta + m) \\ \tan z &= \sec \eta \cot(\delta + m) \end{aligned}\right\} \ldots\text{(vi)}.$$

From the first of these η is found and then the second gives z. Of course z could also be found from the first of (v), but it is always preferable to find an angle from its tangent rather than its cosine (§ 3).

The formulæ (iv) and (v) may be obtained at once geometrically. For if ZL be perpendicular to NP in Fig. 29 we have $NL = m$ and $ZL = 90° - n$.

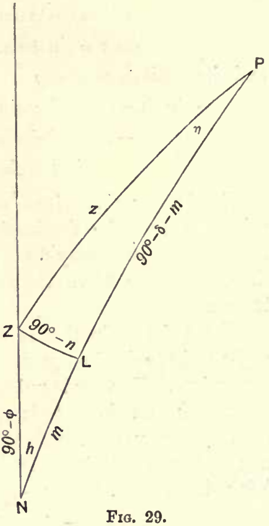


FIG. 29.

It is plain from equations (iv) that as n and m depend only on the latitude and the hour angle they are the same for stars of all declinations. It is therefore convenient to calculate once for all for a given observatory, or rather for a given latitude, a table by which for each particular hour angle at any station on that latitude the values of m and $\operatorname{Log}\cot n$ can be immediately obtained.

Ex. 1. Verify that the equations

$$\tan \eta = \cot n \sec(\delta + m) \text{ and } \tan z = \sec \eta \cot(\delta + m),$$

undergo no change when m and n are changed respectively into $180° + m$ and $360° - n$.

Ex. 2. Determine the zenith distance and parallactic angle of the star 61 Cygni when it is 3^{hrs} E. of the meridian, its declination being $+38° 9'$, and the latitude of the observer being $53° 23'$.

From equations (iv) we find $m = 27° 43'$ and $\operatorname{Log}\cot n = 9{\cdot}6676$ (n). Hence $\delta + m = 65° 52'$ and (vi) $\eta = -48° 41'$, $z = 34° 10'$.

35. Applications of the differential Formulae.

It is convenient to bring together the six differential formulae obtained by applying the fundamental formulae, § 4, to the triangle (Fig. 29) of which the vertices are the Pole N, the Star P, and the zenith Z. The arc NP is the polar distance, $90° - \delta$, the colatitude is NZ or $90° - \phi$; PZ is the zenith distance z, and, of course, the altitude is $90° - z$. The parallactic angle, η, is at P. This angle is positive because it is on the west side of the meridian. The hour angle h is equal to $\vartheta - \alpha$, where ϑ is the sidereal time of observation and α is the right ascension of the star. The azimuth a is measured from the north round by east so that PZN is $360° - a$.

The six differential formulae of § 4, of which only three are independent, may be written

$$\Delta\delta + \cos\eta\Delta z - \cos h\Delta\phi - \sin h\cos\phi\Delta a = 0 \ldots\ldots\ldots (1),$$

$$\Delta z + \cos a\Delta\phi + \cos\eta\Delta\delta + \cos\phi\sin a\Delta h = 0 \ldots\ldots\ldots (2)$$

$$\Delta\phi + \cos a\Delta z - \cos h\Delta\delta + \cos\delta\sin h\Delta\eta = 0 \ldots\ldots\ldots (3),$$

$$\Delta a - \cos z\Delta\eta - \sin\phi\Delta h - \sin h\cos\phi\Delta\delta = 0 \ldots\ldots\ldots (4),$$

$$\Delta h + \sin\delta\Delta\eta - \sin\phi\Delta a - \sin\eta\cos\delta\Delta z = 0 \ldots\ldots\ldots (5),$$

$$\Delta\eta - \cos z\Delta a + \sin\delta\Delta h - \sin a\sin z\Delta\phi = 0 \ldots\ldots\ldots (6).$$

There can be fifteen combinations of four out of the six elements which enter into a triangle. Each set of four are connected by an equation (§ 1). In most cases where variations of the elements are required, two of the elements remain constant, and we seek the relative variations of two other elements. We therefore select that one of the fifteen equations which contains just those two elements that are to be constant and those two whose relative variations are required. The differentiation of that equation with respect to the two variables gives the required relation.

As an example we may take a case which frequently occurs in the determination of latitude by the observation of the zenith distance of a star. Suppose that we know the hour angle and the declination of a star with accuracy, but that there is an error Δz in the assumed zenith distance. We require to see what error will arise in the calculated latitude because the erroneous zenith distance is used in association with the correct hour angle and declination. Here the four quantities concerned are h, δ, z, ϕ, and the formula is therefore

$$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos h.$$

Differentiating and supposing h and δ constant,

$$-\sin z \Delta z = (\cos \phi \sin \delta - \sin \phi \cos \delta \cos h) \Delta\phi,$$

and substituting $\sin z \cos a$ for the coefficient of $\Delta\phi$ we obtain

$$\Delta\phi = -\sec a \Delta z.$$

Of course this might have been obtained directly from formula (2) as just given, by making $\Delta\delta = 0$, $\Delta h = 0$.

As another illustration and one involving the parallactic angle, we shall determine when the parallactic angle of a given star becomes a maximum in the course of the diurnal rotation. The conditions are that while ϕ and δ are both constant, h, z and a shall vary in such a way that there shall be no change in η, *i.e.* $\Delta\eta$ must vanish. The formula involving ϕ, δ, η, h is

$$\tan \phi \cos \delta = \cot \eta \sin h + \sin \delta \cos h.$$

Differentiating we have

$$(\cot \eta \cos h - \sin \delta \sin h) \Delta h = 0,$$

and as the coefficient of Δh must vanish, $\cot \eta = \sin \delta \tan h$, from which we find $\cos a = 0$, and the star must be on the prime vertical.

In this we have another illustration of those exceptional cases in which though three of the variations are zero the formulae do not require that the other three variations shall also be zero (§ 4).

The differential formulae are specially instructive in pointing out how observations should be arranged so that though a small error is made in the course of the observation, the existence of this error shall be as little injurious as possible to the result that is sought.

Suppose, for instance, the mariner is seeking the hour angle of the sun in order to correct his chronometer. What he measures is the altitude of the sun. But from refraction and other causes which no skill can entirely obviate there will be a small error in the altitude and consequently in the zenith distance. The observer measures the zenith distance as z, and concludes that the hour angle is h. But the true zenith distance is $z+\Delta z$, *i.e.* Δz is the quantity which must be added to the observed zenith distance to give the true zenith distance. The true hour angle is therefore not h but some slightly different quantity, $h+\Delta h$, where Δh is the correction to be applied to h, so that Δh is the quantity now sought.

The formula containing only the parts z, ϕ, δ, h is

$$\cos z = \sin\phi \sin\delta + \cos\phi\cos\delta\cos h.$$

Differentiating this and regarding ϕ and δ as constant,

$$-\sin z\Delta z = -\cos\phi\cos\delta\sin h\Delta h,$$

and substituting $$-\sin a \sin z = \sin h \cos\delta,$$

$$-\Delta z = \sin a \cos\phi\Delta h,$$

whence $$\Delta h = -\sec\phi \operatorname{cosec} a\Delta z.$$

The following is a geometrical proof of this formula:

If the sun moves from P to P' (Fig. 30) about the pole N, PP' being a very small arc, its zenith distance changes from ZP

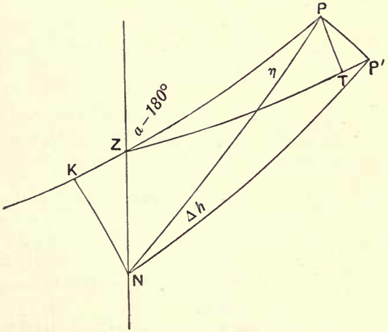


Fig. 30.

to ZP'. If PT be perpendicular to ZP', $\Delta z = TP'$. As $\angle NPP'$ and $\angle ZPT$ are both 90°, $\angle TPP' = \eta$, and $\Delta h \sin NP = PP'$ $= \Delta z \operatorname{cosec}\eta$, whence if NK is perpendicular to ZP we shall have $\Delta z = \Delta h \sin NK$, by which we learn that the rate of change of the zenith distance of the sun with respect to the time is proportional to the sine of the perpendicular from the pole on the vertical circle through the sun. We have also

$$\sin NK = \sin ZN \sin (a - 180^\circ) = -\cos \phi \sin a,$$

whence as before $\Delta h = -\sec \phi \operatorname{cosec} a \Delta z.$

The observation should be so timed that cosec a shall be as small as possible, for then the error Δz will have the smallest possible effect on the determination of the hour angle. It follows that a should be near 90° or 270°. Hence the practical rule so well known to the mariner that for the determination of the time the altitude of the sun should be observed when the sun is on or near the prime vertical.

If the sun does not come to the prime vertical, the smallest value of $\Delta h/\Delta z$ is $\sec \delta$.

Ex. 1. By solving the formulae (1), (2), (3) for $\Delta\delta$, Δz and $\Delta\phi$, show how the formulae (4), (5), (6) can be deduced.

Ex. 2. Show geometrically that if the assumed declination of the sun be erroneous to the extent $\Delta\delta$, the error thence produced on a determination of the hour angle from an observation of the sun's zenith distance will be

$$\cot \eta \sec \delta \,.\, \Delta\delta.$$

Ex. 3. Under what circumstances is the change of zenith distance of a star by the diurnal motion proportional throughout the day to its change of hour angle?

We have from (2) $\Delta z/\Delta h = -\sin a \cos \phi$, and this must be constant, whence a must be constant and the observer must be on the equator and the star must be an equatorial star.

Ex. 4. If the hour angle is being determined from an observed zenith distance of a celestial object of known declination, show geometrically that a small error $\Delta\phi$ in the assumed latitude ϕ will produce an error $-\cot a \sec \phi \Delta\phi$ in the hour angle, where a is the azimuth.

Show also that this error will generally be of little consequence provided the object be near the prime vertical.

The triangle PSZ is formed from the polar distance PS $(=90^\circ - \delta)$, the zenith distance ZS $(=z)$ and the colatitude PZ $(=90^\circ - \phi)$. Fig. 31. The parallactic angle η is negative because it is to the east of the meridian (§ 34).

The triangle $PS'Z'$ is formed from the polar distance PS $(=PS')$, the zenith distance ZS $(=Z'S')$ and the colatitude PZ' $(=90^\circ - \phi - \Delta\phi)$.

Draw $Z'M$ and $S'L$ perpendicular to SZ, then as SZ and $S'Z'$ are very close together $S'Z' = LM$, but as $S'Z' = SZ$ we must have $SL = ZM$.

$\angle SZZ'$ is the azimuth a, so that $SL = ZM = -\cos a \Delta\phi$.

$\angle PSZ = -\eta$ and $SS' = SL \sec (90^\circ + \eta) = +\cos a \operatorname{cosec} \eta \Delta\phi$.

But $\Delta h = SS' \operatorname{cosec} PS$, whence

$$\Delta h = \cos a \operatorname{cosec} PS \operatorname{cosec} \eta \Delta\phi = -\cot a \sec \phi \Delta\phi.$$

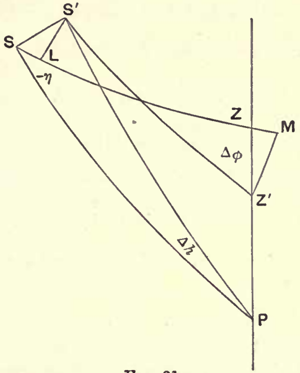


FIG. 31.

Ex. 5. A star of declination δ is observed to have zenith distances z_1 z_2 at instants separated by an interval 2τ; show that the colatitude c can be determined from the equation

$$\sin \tfrac{1}{2} c = \sin \tfrac{1}{2}(z+x) \sin \tfrac{1}{2}\theta \operatorname{cosec} \epsilon,$$

where x, d, z, θ, ϵ are auxiliary angles given by

(i) $\tan x = \cot \delta \cos \tau$,

(ii) $\sin d = \cos \delta \sin \tau$,

(iii) $\cos z = \cos \tfrac{1}{2}(z_1 + z_2) \cos \tfrac{1}{2}(z_1 - z_2) \sec d$,

(iv) $\sin \theta = \sin \tfrac{1}{2}(z_1 + z_2) \sin \tfrac{1}{2}(z_1 - z_2) \operatorname{cosec} z \operatorname{cosec} d$,

(v) $\tan \epsilon = \sin \tfrac{1}{2}(z+x) \operatorname{cosec} \tfrac{1}{2}(z-x) \tan \tfrac{1}{2}\theta$. [Math. Trip.]

Ex. 6. If Δ be the N.P.D. and z the zenith distance of Polaris observed below the pole at an hour angle h from the meridian, show that the colatitude c may be determined from the equations

$$\sin y = \sin \Delta \sin h, \quad \tan x = \tan \Delta \cos h,$$

$$\tan^2 \tfrac{1}{2}(c+x) = \tan \tfrac{1}{2}(z+y) \tan \tfrac{1}{2}(z-y).$$

What is the geometrical significance of the auxiliaries x and y?

[Math. Trip.]

Ex. 7 If δ be a star's declination and A its maximum azimuth, show that in t seconds of time from the moment when the azimuth is A the azimuth has changed by

$$\tfrac{1}{2} 15^2 t^2 \sin 1'' \sin^2 \delta \tan A \text{ seconds of arc.}$$

If there be a maximum value of the azimuth, the star culminates between pole and zenith, and for the maximum azimuth, the zenith distance is tangent to the small circle described by the star in its apparent diurnal motion.

Differentiating $-\cot A = \cos\phi \tan\delta \operatorname{cosec} h - \sin\phi \cot h$,

we obtain $$\operatorname{cosec}^2 A \frac{dA}{dh} = \cos\phi \tan\delta \operatorname{cosec} h \cot h - \sin\phi \operatorname{cosec}^2 h$$
$$= -\cot A \cot h - \sin\phi.$$

Differentiating again and making $\frac{dA}{dh} = 0$, we have

$$\operatorname{cosec}^2 A \frac{d^2 A}{dh^2} = \cot A \operatorname{cosec}^2 h,$$

and $$\frac{d^2 A}{dh^2} = \tan A \sin^2 \delta.$$

Therefore if x be the change of azimuth in t seconds from the moment of maximum azimuth

$$A \sin 1'' = \tfrac{1}{2}\, 15^2 t^2 \sin^2 1'' \sin^2 \delta \tan A. \qquad \text{[Math. Trip.]}$$

*36. On the time of culmination of a celestial body.

At the moment of upper culmination (§ 29) the right ascension of the body is the sidereal time. The problem of finding the time of upper culmination reduces therefore to the discovery of the right ascension of the body at the moment when it crosses the meridian.

THE TIME OF A STAR'S UPPER CULMINATION.

In the case of a star, the computation is a very simple one; for as the apparent right ascension alters very slowly we can always find it by inspection from the tables, and so have at once the sidereal time of upper culmination.

For instance, suppose we seek the time of culmination of Arcturus at Greenwich on 1906 Feb. 12, which for this particular purpose is conveniently reckoned from apparent noon on Feb. 12 to apparent noon on Feb. 13. We find in the ephemeris for 1906 that the R.A. at upper culmination on Feb. 10 is $14^h\, 11^m\, 22^s{\cdot}42$. It increases $0^s{\cdot}29$ in 10 days; and therefore at culmination on Feb. 12 the R.A. is $14^h\, 11^m\, 22^s{\cdot}48$. On that day the sidereal time at mean noon for Greenwich is $21^h\, 26^m\, 29^s{\cdot}91$ (§ 69).

We thus see that Arcturus will reach the meridian at $24^h + (14^h\, 11^m\, 22^s{\cdot}48) - (21^h\, 26^m\, 29^s{\cdot}91) = 16^h\, 44^m\, 52^s{\cdot}57$ of sidereal time after mean noon on Feb. 12. We transform this into mean time by the tables given in the nautical almanac.

16^h	$15^h\ 57^m\ 22^s{\cdot}73$
44^m	$43\ \ 52\ {\cdot}79$
52^s	$51\ {\cdot}86$
${\cdot}57$	${\cdot}57$
	$16\ \ 42\ \ \ 7\ {\cdot}95$

The culmination of Arcturus therefore takes place at $16^h\,42^m\,7^s{\cdot}95$ on Feb. 12.

In the case of a moving body such as a planet or the moon, whose right ascension changes rapidly from hour to hour, we proceed as follows.

Let the right ascension of the body be α_1, α_2, α_3, at three consecutive epochs t_1, t_2, t_3, for which the tables give the calculated values, and such that culmination occurs between t_1 and t_3. Then taking either of the equal intervals $t_2 - t_1$ or $t_3 - t_2$ as the unit of time, and supposing culmination occurs t units after t_1 we have by interpolation for the R.A. at culmination

$$\alpha_1 + t(\alpha_2 - \alpha_1) + \tfrac{1}{2}t(t-1)(\alpha_1 - 2\alpha_2 + \alpha_3).$$

This will be the sidereal time of the body's culmination. Let θ_1 be the sidereal time at the epoch t_1, and let H be the value of the unit in sidereal time. Then at the moment of culmination the sidereal time is

$$\theta_1 + Ht.$$

But this must be equal to the expression already written; whence

$$\theta_1 + Ht = \alpha_1 + t(\alpha_2 - \alpha_1) + \tfrac{1}{2}t(t-1)(\alpha_1 - 2\alpha_2 + \alpha_3).$$

From this equation t is to be determined. The equation is a quadratic; but obviously the significant root for our purpose is indicated by the fact that $\tfrac{1}{2}t(t-1)(\alpha_1 - 2\alpha_2 + \alpha_3)$ is a small quantity. To solve the equation we therefore deduce an approximate value t' for t by solving

$$\theta_1 + Ht' = \alpha_1 + t'(\alpha_2 - \alpha_1);$$

and then we introduce this value t' into the small term and solve the following simple equation for t

$$\theta_1 + Ht = \alpha_1 + t(\alpha_2 - \alpha_1) + \tfrac{1}{2}t'(t'-1)(\alpha_1 - 2\alpha_2 + \alpha_3).$$

THE TIME OF A PLANET'S UPPER CULMINATION.

To illustrate the process we shall compute the time of culmination of Jupiter at Greenwich on Sept. 25, 1906.

From the nautical almanac, p. 247, we have:

Mean noon	R.A. of Jupiter	1st diff.	2nd diff.
1906. Sept. 25	$6^h\,39^m\,53^s{\cdot}59$		
		$+26^s{\cdot}90$	
26	40 20 ·49		$-0^s{\cdot}68$
		+26 ·22	
27	40 46 ·71		

Hence the R.A. of Jupiter t days after noon on Sept. 25 is

$$6^h\,39^m\,53^s{\cdot}59 + 26^s{\cdot}90t - 0^s{\cdot}34t\,(t-1).$$

At the moment of culmination this equals the sidereal time, which is

$$12^h\,13^m\,34^s{\cdot}62 + t\,[24^h\,3^m\,56^s{\cdot}55],$$

whence the equation for t is

$$30^h\,39^m\,53^s{\cdot}59 + 26^s{\cdot}90t - 0^s{\cdot}34t\,(t-1)$$
$$= 12^h\,13^m\,34^s{\cdot}62 + t\,[24^h\,3^m\,56^s{\cdot}55].$$

Neglecting the last term on the left-hand side, and omitting all seconds in the first solution, we have

$$18^h\,26^m = t\,(24^h\,3^m),$$

whence $t = 0{\cdot}77$.

Introducing this approximate value of t into $-0^s{\cdot}34t\,(t-1)$, it reduces to $+0^s{\cdot}06$.

The equation therefore becomes

$$30^h\,39^m\,53^s{\cdot}65 + 26^s{\cdot}90t = 12^h\,13^m\,34^s{\cdot}62 + t\,(24^h\,3^m\,56^s{\cdot}55),$$

whence $$t = \frac{18^h\,26^m\,19^s{\cdot}03}{24^h\;\;3^m\,29^s{\cdot}65} = 0{\cdot}766416.$$

Jupiter's culmination will therefore be 0·766416 of a mean solar day after noon, *i.e.* at $18^h\,23^m\,38^s{\cdot}34$ G.M.T. (see *N. A.*, 1906, p. 272).

THE TIME OF THE MOON'S UPPER CULMINATION.

In the case of the moon the motion is so rapid that the places from hour to hour, as given in the ephemeris, are required. For the sake of illustration we shall compute the time at which the moon culminates at Greenwich on 1906 Oct. 29.

The sidereal time at mean noon on that day is $14^h\,27^m\,37^s{\cdot}42$ (*N. A.*, 1906, p. 165). The moon's R.A. at noon (*N. A.*, p. 175) is $0^h\,23^m\,23^s{\cdot}62$. If there were no motion this would mean that the moon must culminate about ten o'clock in the evening. At 10 o'clock the R.A. of the moon is about $0^h\,43^m$, and this shows that the interval between noon and the moon's culmination is about $10^h\,16^m$ of sidereal time, or about $10^h\,14^m$ of mean solar time. We are therefore certain to include the time of culmination by taking from the ephemeris the following:

			Moon's R.A.	1st diff.	2nd diff.
1906.	Oct. 29.	10 h.	$0^h\,42^m\,52^s{\cdot}03$		
				$+1^m\,56^s{\cdot}40$	
		11 h.	0 44 48 ·43		$-0^s{\cdot}06$
				+1 56 ·34	
		12 h.	0 46 44 ·77		

It follows that the R.A. of the moon at $(10 + t)$ hours after noon is

$$0^h\,42^m\,52^s{\cdot}03 + 116^s{\cdot}40t - 0^s{\cdot}03t\,(t-1).$$

Since t is about $\frac{1}{4}$, the last term amounts to about one two-hundredth of a second, and may be neglected. We thus have the following equation for finding t:

$$0^h\,42^m\,52^s{\cdot}03 + 116^s{\cdot}40t$$
$$= \text{sidereal time at } 10^h \text{ mean time} + t\,[1^h\,0^m\,9^s{\cdot}86],$$

the coefficient of t on the right-hand side being the sidereal value of one mean hour. The sidereal time at mean noon on the day in question is $14^h\,27^m\,37^s{\cdot}42$, if we add to this $10^h\,1^m\,38^s{\cdot}56$ which is the sidereal equivalent of 10^h of mean time we see that the sidereal time at 10^h G.M.T. is $0^h\,29^m\,15^s{\cdot}98$; the equation is therefore

$$0^h\,42^m\,52^s{\cdot}03 - 0^h\,29^m\,15^s{\cdot}98 = t\,(1^h\,0^m\,9^s{\cdot}86 - 1^m\,56^s{\cdot}40),$$

$$t = \frac{13^m\,36^s{\cdot}05}{58^m\,13^s{\cdot}46} = 0{\cdot}233594.$$

This is the fraction of one mean hour after 10 P.M., at which the culmination takes place; that is, at $10^h\,14^m\,0^s{\cdot}94$ (*N. A.*, 1906, p. 167).

THE TIME OF CULMINATION AT LONGITUDE λ.

Suppose it be required to find the time of upper culmination of a heavenly body at a place P in longitude λ to the west of Greenwich.

The R.A. of the body at the moment of culmination will of course be equal to the sidereal time at the place. Let θ be the local mean time; then the mean time at Greenwich at the same instant is $\theta + \lambda$, and the R.A. of the body can be expressed by interpolation as a function of $\theta + \lambda$.

We have therefore only to find the sidereal time at P corresponding to the mean time θ. The ephemeris gives the sidereal time at mean noon at Greenwich, which must be increased by $\lambda/24^h \times$ (the difference in sidereal time between the mean solar and sidereal day) to give the sidereal time at mean noon at P. To this we must add θ, increased in the ratio of the duration of the mean day to the duration of the sidereal day. The resulting sidereal time is to be equated to the Right Ascension, and θ is determined.

For example, let it be proposed to find the time when the moon culminates at the Lick Observatory, Mount Hamilton, California, on Dec. 25, 1906. The longitude is here $8^h\,6^m\,34^s{\cdot}89$; and if θ is the local mean time of culmination, the Greenwich mean time is $8^h\,6^m\,34^s{\cdot}89 + \theta$.

The ephemeris shows that on Dec. 25 the sidereal time at Greenwich mean noon is $18^h\,12^m\,21^s{\cdot}13$; and the moon's R.A. varies from $2^h\,19^m\,29^s{\cdot}84$ at 0^h to $3^h\,3^m\,40^s{\cdot}32$ at 23^h and it is also seen that the culmination at Greenwich takes place *about* $8^h\,22^m$ G.M.T. In the following 8^h the moon's R.A. increases about 15^m; hence culmination will take place at Lick at about $8^h\,37^m$ local mean time, or about $16^h\,43^m$ G.M.T. The portion of the tables to be employed in the accurate calculation is therefore as follows:

	G.M.T.	Moon's R.A.	1st diff.	2nd diff.
1906. Dec. 25.	16 h.	$2^h\,50^m\,9^s{\cdot}73$		
			$+1^m\,55^s{\cdot}55$	
	17	52 5·28		$+0^s{\cdot}08$
			+1 55·63	
	18	54 0·91		

Let $8^h\,6^m\,34^s{\cdot}89 + \theta = 16^h + t$, where t is a fraction of an hour.

Then $$\theta = 7^h\,53^m\,25^s{\cdot}11 + t.$$

The sidereal time at Lick corresponding to the local mean time θ is found as follows.

The sidereal time at Greenwich

mean noon $= 18^h\,12^m\,21^s{\cdot}13$

The Longitude of Lick

$\times (3^m\,56^s{\cdot}56)/24^h =$ 1 19·93

$(7^h\,53^m\,25^s{\cdot}11 + t)$ expressed in

sidereal time $=$ 7 54 42·88 $+ (1^h\,0^m\,9^s{\cdot}86)\,t$

Adding these three lines we obtain the sidereal time of the moon's upper culmination at Lick $= 2^h\,8^m\,23^s{\cdot}94 + (1^h\,0^m\,9^s{\cdot}86)\,t$.

The R.A. of the moon at G.M.T. $(16 + t)^h$ is

$$2^h\,50^m\,9^s{\cdot}73 + t\,(115^s{\cdot}55) + 0^s{\cdot}04t\,(t-1).$$

As t is about $0^h{\cdot}7$ the third term in this expression is $-\,0^s{\cdot}01$, and to find t we have

$$2^h\,8^m\,23^s{\cdot}94 + t\,(1^h\,0^m\,9^s{\cdot}86)$$
$$= 2^h\,50^m\,9^s{\cdot}72 + t\,(115^s{\cdot}55),$$

or $$t = \frac{41^m\, 45^s{\cdot}78}{58^m\, 14^s{\cdot}31} = {\cdot}717103 \text{ hours}$$
$$= 43^m\, 1^s{\cdot}57.$$

Hence the culmination of the moon at Lick took place at $16^h\, 43^m\, 1^s{\cdot}57$ Greenwich mean time, or at $8^h\, 36^m\, 26^s{\cdot}68$ local mean time.

37. Rising and setting of a celestial body.

The time of rising or setting of a celestial body is much affected by refraction. Postponing the consideration of the effect of refraction to a later chapter (VI.) we here give the formulae for finding when a celestial body, atmospheric influences apart, is on the horizon, *i.e.* 90° from the zenith.

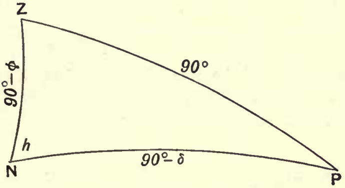


FIG. 32.

In Fig. 32 the points N and Z are the north pole and the zenith respectively. P is a star at the moment of rising or setting when $ZP = 90°$. We have therefore
$$0 = \sin\phi \sin\delta + \cos\phi \cos\delta \cos h,$$
whence $$\cos h = -\tan\phi \tan\delta.$$

Provided the star be one which rises and sets at the latitude of the observer there are two solutions, h ($< 180°$) corresponding to setting, and $360° - h$ corresponding to rising.

Ex. 1. Unless $\tan\phi \tan\delta < 1$ (sign not regarded) show that an object of declination δ neither rises nor sets in a place of latitude ϕ.

Ex. 2. If the N. decl. of a star is 40°, show that the number of hours in the sidereal day during which it will be below the horizon of a place which has latitude 30° is 8·136.

Ex. 3. The declination of Arcturus in 1909 is 19° 39′ N. and the latitude of Cambridge is 52° 13′, find the hour angle through which the star moves between the time at which it rises and that at which it culminates.

Ex. 4. Let S be the sidereal time of rising of a star whose coordinates are a, δ and S' be the time of setting. Show that $S=a+\omega'$, $S'=a+\omega''$, where ω' and ω'' are the two roots of the equation

$$\cos\omega = -\tan\phi\tan\delta.$$

Ex. 5. Under what conditions would the azimuth of a star remain constant from rising to transit?

If the star is to have a constant azimuth it must move along a great circle passing through the zenith. Hence the star must be on the celestial equator and the pole on the observer's horizon, *i.e.* the observer on the terrestrial equator.

Ex. 6. If ϕ be the latitude, δ the declination of a celestial body and h its hour angle when rising or setting, show that when refraction is not considered

$$2\cos^2\tfrac{1}{2}h = \sec\phi\sec\delta\cos(\phi+\delta).$$

Ex. 7. Show that in latitude 45° the interval between the time at which any star passes due East and the time of its setting is constant.

[Math. Trip.]

Let E be the position of the star when due East, Z the zenith and P the pole (Fig. 33). Then $\angle EZP=90°$, $ZP=45°$, and ZP is produced to J so that $PJ=45°$, and $ZH=90°$ is inflected from Z on EPH. Since

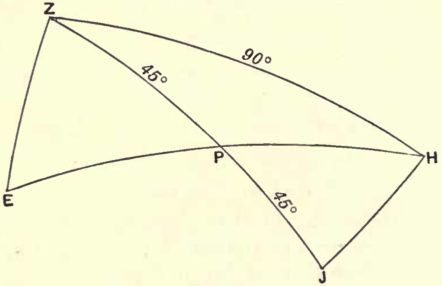


FIG. 33.

$ZJ=ZH=90°$, we have $\angle ZJH=90°$, and therefore in the triangles ZPE and JPH, we have $ZP=PJ$ and $EZP=HJP=90°$. Hence the triangles are equal and $EP=HP$, and as H is 90° from the zenith it is the position of the star at setting, so that half a sidereal day elapses while the star moves from E to H.

Ex. 8. Two stars whose declinations are δ_1, δ_2 are observed to be in the East at the same time and also to set at the same time; show that the latitude of the place of observation is 45°, and that if t be the number of hours between the times at which they rise,

$$2\cos\frac{15^\circ.t}{4}=\sqrt{(1+\tan\delta_1)(1+\tan\delta_2)}+\sqrt{(1-\tan\delta_1)(1-\tan\delta_2)}.$$

The length of the perpendicular from the pole on the great circle joining the two stars being unaffected by the diurnal motion, we see that the pole is equidistant from the prime vertical and the horizon, *i.e.* the latitude is 45°.

Further the time during which a star is above the horizon is twice the hour angle at setting, or

$$2\cos^{-1}(-\tan\phi\tan\delta).$$

Since the stars set together, and ϕ is 45°, the interval between their risings is

$$2\cos^{-1}(-\tan\delta_1)-2\cos^{-1}(-\tan\delta_2),$$

whence the required result.

Ex. 9. If two stars whose coordinates are respectively a, δ and a', δ' rise at the same moment at a station of latitude ϕ, show that

$$\sin^2(a-a')\cot^2\phi=\tan^2\delta+\tan^2\delta'-2\tan\delta\tan\delta'\cos(a-a').$$

Ex. 10. If A be the area of the celestial sphere, show that to an observer in latitude ϕ the stars in a portion $A\sin^2\frac{1}{2}\phi$ will never be above his horizon, the stars in another portion $A\sin^2\frac{1}{2}\phi$ will always be above his horizon, the stars in a portion $A\cos\phi$ will daily rise and set, and a portion $A\cos^2\frac{1}{2}\phi$ will include all the stars with which he can become acquainted.

If a be the radius of a sphere the area cut off by a small circle of radius ϕ is $2\pi a^2(1-\cos\phi)$. Small circles of radius ϕ about the north and south poles respectively show the portions of the sphere always above and always below the horizon.

Ex. 11. If at a place whose north latitude is ϕ, two stars whose N.P.D. are respectively Δ and Δ', rise together, and the former comes to the meridian when the latter sets, prove that

$$\frac{\tan\phi}{\tan\Delta}=1-\frac{2\tan^2\phi}{\tan^2\Delta'}.$$ [Math. Trip.]

It is plain that if h be the hour angle of the second star at rising, that of the first must be $2h$, whence we have

$$0=\cos\Delta\sin\phi+\sin\Delta\cos\phi\cos 2h,$$

$$0=\cos\Delta'\sin\phi+\sin\Delta'\cos\phi\cos h,$$

and the elimination of h gives the desired result.

Ex. 12. If at any instant the plane of vibration of a Foucault's pendulum pass through a star near the horizon, prove that the plane will continue to pass through the star so long as it is near the horizon. [Math. Trip.]

The plane of a Foucault's pendulum appears to rotate round the vertical with an angular velocity found by multiplying the angular velocity of the celestial sphere by the sine of the latitude. In the small time dt the star S (Fig. 34) moves over $SS' = \cos\delta\, dt$. If $S'T$ be perpendicular to ZS we have

$$S'T = S'S \sin S'ST = \cos\delta \cos\eta\, dt = \sin\phi\, dt.$$

Hence $$S'ZS = \sin\phi\, dt.$$

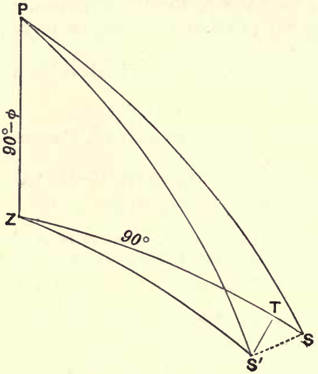


FIG. 34.

38. Celestial latitude and longitude. For certain classes of investigation we have to employ yet another system of coordinates on the celestial sphere. Just as the equator has furnished the means of defining the right ascension and the declination of a star, so the ecliptic is made the basis of a system of coordinates known as celestial longitude and latitude. We employ in this new system the same origin as before. The first point of Aries ♈ is the origin from which longitude is to be measured and the direction of the measurement is to be that of the apparent annual movement of the sun along the ecliptic as indicated by the arrow-head on Fig. 35.

A great circle is drawn from the pole K of the ecliptic through the star S, and the intercept TS on this great circle between the star and the ecliptic is that coordinate which is called the *latitude* of the star. The latitude is positive or negative according as the star lies in the hemisphere which contains the pole or the antipole of the ecliptic. The arc on the ecliptic from the origin ♈ to T, the foot of the perpendicular, is called the *longitude*, which is the second coordinate. This is measured round the circle from 0° to 360°, so that if the right ascension of an object on the ecliptic is increased its longitude is also increased.

The reader will of course observe that the meanings of the words latitude and longitude as here explained in their astronomical significance are quite different from the meanings of the same words in their more familiar use with regard to terrestrial matters. It is usual to employ the letter λ to express astronomical longitude and β to express astronomical latitude, thus $♈T = \lambda$ and $TS = \beta$.

The arc of the solstitial colure LH intercepted between the equator and the ecliptic is equal to the obliquity of the ecliptic.

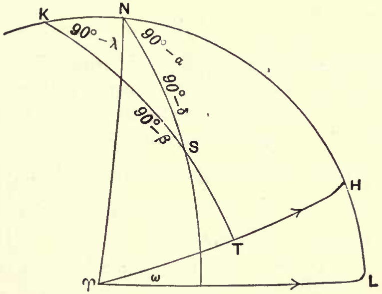


FIG. 35.

If α, δ be the R.A. and decl. of S, then the formulae for transformation are obtained either from the general formulae of § 12 or directly from the triangle SKN (Fig. 35), and for the determination of the latitude and longitude we have the equations

$$\left.\begin{aligned} \sin\beta &= \cos\omega\sin\delta - \sin\omega\cos\delta\sin\alpha \\ \cos\beta\sin\lambda &= \sin\omega\sin\delta + \cos\omega\cos\delta\sin\alpha \\ \cos\beta\cos\lambda &= \cos\delta\cos\alpha \end{aligned}\right\} \ldots\ldots\ldots(1),$$

by which we can determine β and λ when α and δ are known. It is generally easy to see from the nature of the problem whether the longitude is greater or less than 180°. When this is known one of the last two equations may be dispensed with.

We can make these equations more convenient for logarithmic work by the introduction of an auxiliary quantity $M = \angle S\Upsilon L$, so that $\tan M = \operatorname{cosec} \alpha \tan \delta$ (§ 13), and we have

$$\sin \beta = \sin \delta \sin (M - \omega) \operatorname{cosec} M,$$
$$\cos \beta \sin \lambda = \sin \delta \cos (M - \omega) \operatorname{cosec} M,$$
$$\cos \beta \cos \lambda = \cos \delta \cos \alpha.$$

The form of the equations shows that a change of 180° in the adopted value of M does not affect the result.

If we represent by $90° - E$ the angle subtended at S by KN we have from Delambre's formulae

$$\cos \tfrac{1}{2} (E + \lambda) \cos (45° - \tfrac{1}{2} \beta) = \cos \{45° - \tfrac{1}{2} (\delta + \omega)\} \cos (45° + \tfrac{1}{2} \alpha),$$
$$\sin \tfrac{1}{2} (E + \lambda) \cos (45° - \tfrac{1}{2} \beta) = \cos \{45° - \tfrac{1}{2} (\delta - \omega)\} \sin (45° + \tfrac{1}{2} \alpha),$$
$$\sin \tfrac{1}{2} (E - \lambda) \sin (45° - \tfrac{1}{2} \beta) = \sin \{45° - \tfrac{1}{2} (\delta + \omega)\} \cos (45° + \tfrac{1}{2} \alpha),$$
$$\cos \tfrac{1}{2} (E - \lambda) \sin (45° - \tfrac{1}{2} \beta) = \sin \{45° - \tfrac{1}{2} (\delta - \omega)\} \sin (45° + \tfrac{1}{2} \alpha),$$

by which λ and β as well as E can be determined.

If it be required to solve the converse problem, namely, to determine the R.A. and decl. when the latitude and longitude are given, we have by transformation of (1)

$$\left.\begin{aligned} \sin \delta &= \cos \omega \sin \beta + \sin \omega \cos \beta \sin \lambda \\ \cos \delta \sin \alpha &= -\sin \omega \sin \beta + \cos \omega \cos \beta \sin \lambda \\ \cos \delta \cos \alpha &= \cos \beta \cos \lambda \end{aligned}\right\} \quad \ldots\ldots(2).$$

Ex. 1. Show that the right ascension and declination of the pole of the ecliptic are respectively 270°, $90° - \omega$ and that the right ascension and declination of the antipole are 90°, $\omega - 90°$.

Ex. 2. If α, δ are the R. A. and decl. of the point of the ecliptic whose longitude is λ, show that

$$\cos \lambda = \cos \alpha \cos \delta,$$
$$\sin \lambda \sin \omega = \sin \delta,$$
$$\sin \lambda \cos \omega = \sin \alpha \cos \delta.$$

Ex. 3. If α_1, δ_1 and α_2, δ_2 be the R. A. and decl. of two stars which have the same longitude, prove that

$$\sin (\alpha_2 - \alpha_1) = \tan \omega (\tan \delta_1 \cos \alpha_2 - \tan \delta_2 \cos \alpha_1).$$

Ex. 4. The right ascension of α Orionis is 5 h. 49 m., its declination is +7° 23′, and the obliquity of the ecliptic is 23° 27′. Show that the longitude and latitude of the star are respectively 87° 10′, −16° 2′.

Ex. 5. If $\alpha = 6° 33' 29''$, $\delta = -16° 22' 35''$, $\omega = 23° 27' 32''$, show that

$$\lambda = 359° 17' 44'', \quad \beta = -17° 35' 37''.$$

MISCELLANEOUS EXERCISES ON CHAP. V.

Ex. 1. Show that to an observer at the north pole of the earth the altitude of a star would be its declination and would be unaltered by the diurnal motion. Show that in the same case the azimuth of a star (measured from any fixed meridian) would differ from its right ascension only by an arc which would be the same at a given instant for all stars.

Ex. 2. A star of right ascension α and declination δ has a small latitude β. Prove that the longitude of the sun when its R.A. is α, differs from the longitude of the star by $\beta \sin \delta \cot \alpha$ approximately.

Ex. 3. Show that for a place within the arctic or antarctic circle the points of intersection of the ecliptic with the horizon travel completely round the horizon, during a sidereal day, but that for any other place they oscillate about the East and West points. [Math. Trip.]

Ex. 4. The East point is denoted by E, the pole by P, and the places of two stars by A, B. PA meets EB in A', and PB meets EA in B'. The declinations of A, B, A', B' are δ_1, δ_2, δ_1', δ_2' respectively, show that

$$\tan \delta_1' \tan \delta_2' = \tan \delta_1 \tan \delta_2.$$

Let EA and EB intersect the meridian at distances λ, μ respectively from the pole. Then since E is the pole of the meridian, $\tan \lambda / \tan \mu = \tan \delta_1' / \tan \delta_1$ and $\tan \mu / \tan \lambda = \tan \delta_2' / \tan \delta_2$.

Ex. 5. If z be the zenith distance of a star as seen from a station P, then at the same moment at a station P' which is at a small distance s from P the zenith distance will be very nearly z' where

$$z' = z - s \cos \theta + \tfrac{1}{2} s^2 \sin 1'' \cot z \sin^2 \theta,$$

θ being the difference between the azimuths of the star and P' as seen from P.

We assume that $z' - z$ and s are both expressed in arc, their measures in radians are therefore $(z' - z) \sin 1''$ and $s \sin 1''$. We have

$$\begin{aligned} \cos z' &= \cos z \cos s + \sin z \sin s \cos \theta \\ &= \cos z (1 - \tfrac{1}{2} s^2 \sin^2 1'') + s \sin 1'' \sin z \cos \theta. \end{aligned}$$

But we also have

$$\begin{aligned} \cos z' &= \cos (z' - z) \cos z - \sin (z' - z) \sin z \\ &= \cos z \{1 - \tfrac{1}{2} (z' - z)^2 \sin^2 1''\} - (z' - z) \sin 1'' \sin z, \end{aligned}$$

and equating these two values of $\cos z'$ gives

$$z' - z = -s \cos \theta + \tfrac{1}{2} s^2 \sin 1'' \cot z - \tfrac{1}{2} (z' - z)^2 \sin 1'' \cot z;$$

as a first approximation $z' - z = -s \cos \theta$, and with this substitution in the last term the desired result is obtained.

Ex. 6. Let a', δ', η' be respectively the azimuth, the declination and the parallactic angle of a point on the horizon. Show that for a given latitude ϕ these quantities can be calculated for any hour angle by the formulae

$$\tan\delta' = -\cot\phi\cos h,\quad \sin a' = -\sin h\cos\delta',\quad \cos a' = +\sec\phi\sin\delta',$$
$$\tan a' = +\sin\phi\tan h,\quad \cos\eta' = +\sin\phi\sec\delta',\quad \sin\eta' = +\sin h\cos\phi.$$

Ex. 7. If δ, h be respectively the declination and the hour angle of a star, obtain the following formulae by which its azimuth a and zenith distance z can be easily determined when for that latitude the values of a', δ', η' (as defined in the last example), corresponding to h are known.

$$\cos z = \sin(\delta' - \delta)\cos\eta',$$
$$\sin(a' - a)\sin z = \sin(\delta' - \delta)\sin\eta',$$
$$\cos(a' - a)\sin z = \cos(\delta' - \delta).$$

Ex. 8. Show that with the quantities used in the last example we have for determining η the star's parallactic angle:

$$\sin\eta = \sin(a - a')\operatorname{cosec}(\delta' - \delta),$$
$$\cos\eta = \cot z\cot(\delta' - \delta).$$

Ex. 9. As an illustration of the formulae of Exs. 6 and 7, calculate the zenith distance and azimuth of Arcturus at the West hour angle $2^h\, 35^m$, being given that the declination is $+19°\, 44'$ and the latitude $52°\, 13'$.

Ex. 10. Show that the latitude ϕ can be determined by an observation of the altitude a of the pole star which at the time of observation has an hour angle h and a polar distance p and that the formula is approximately

$$\phi = a - p\cos h + \tfrac{1}{2}\sin 1''\, p^2\sin^2 h\tan a.$$

Ex. 11. Show that the hour angle h and the zenith distance z when a star is due East or due West may be found from the equations

$$\sin\delta = \sin\phi\cos z;\quad \sin h\cos\delta = \mp\sin z;\quad \cos h\cos\delta = \cos\phi\cos z;$$

by using the upper sign in the former case and the lower sign in the latter.

Ex. 12. Find the first and second differential coefficients of the zenith distance z of a star with respect to the hour angle.

We can investigate this either from the fundamental formulae or geometrically as follows, Fig. 36.

N is the north pole, Z the zenith, P the star. In the time dh the star has moved to H, where PH is perpendicular to NP and NH. If PQ be perpendicular to ZH, then

$$dz = HQ = HP\sin\eta = \cos\delta\sin\eta\, dh = -\cos\phi\sin a\, dh.$$

We have also

$$da = PQ\operatorname{cosec} z = PH\cos\eta\operatorname{cosec} z = \cos\delta\cos\eta\operatorname{cosec} z\, dh,$$

whence
$$\frac{da}{dh} = \cos\delta\cos\eta\operatorname{cosec} z.$$

To find the second differential coefficient we differentiate dz/dh as above found with respect to h and assuming that a and h are both expressed in radians,

$$\frac{d^2z}{dh^2} = -\cos\phi\cos a\frac{da}{dh},$$
$$= -\cos\phi\cos a\cos\delta\cos\eta\,\mathrm{cosec}\,z.$$

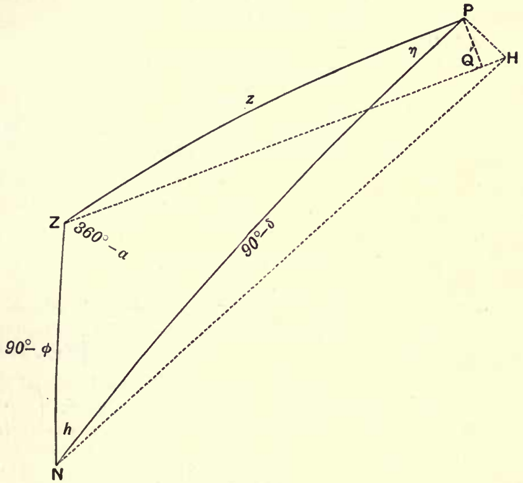


Fig. 36.

Ex. 13. If the declination of a star exceed the latitude, show that the most rapid rate of change in zenith distance by diurnal motion is equal to the cosine of the declination. If the declination be less than the latitude, show that the most rapid rate of change in zenith distance is equal to the cosine of the latitude.

Ex. 14. If z_0 be the zenith distance of an object at the hour angle h_0 and if z be the zenith distance of the same object at the hour angle h which is very near to h_0, show from Ex. 12 that

$$z - z_0 = -15(h-h_0)\cos\phi\sin a - \tfrac{1}{2}\,225\sin 1''\,(h-h_0)^2\cos\phi\cos a\cos\delta\cos\eta\,\mathrm{cosec}\,z,$$

where the zenith distances are expressed in arc and the hour angles in time.

Ex. 15. A series of measurements of zenith distances $z_1 \ldots z_n$ of the same star are made at closely following hour angles $h_1 \ldots h_n$. Let z', h_0 be the arithmetic means of the zenith distances and hour angles. Show that z_0, the value of z corresponding to h_0, is obtained by applying to z' the correction

$$+\frac{1}{2n}\,225\sin 1''\cos\phi\cos a\cos\delta\cos\eta\,\mathrm{cosec}\,z\sum_1^n(h_r-h_0)^2.$$

Making $A=-\cos\phi\sin a$, $B=-\frac{1}{2}225\sin 1''\cos\phi\cos a\cos\delta\cos\eta\operatorname{cosec} z$, we have from the last question

$$z_1=z_0+A\,(h_1-h_0)+B\,(h_1-h_0)^2,$$
$$z_2=z_0+A\,(h_2-h_0)+B\,(h_2-h_0)^2,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$z_n=z_0+A\,(h_n-h_0)+B\,(h_n-h_0)^2,$$

adding and dividing by n

$$z'=z_0+\frac{1}{n}B\sum_1^n(h_r-h_0)^2,$$

which proves the theorem.

This formula is useful when it is desired to obtain the best result from a series of zenith distances taken in rapid succession.

Ex. 16. Show that if the hour angle of a star of declination δ be h when it has the azimuth a and h' when it has the azimuth $180°+a$ the latitude ϕ can be found from the equation

$$\tan\phi=\tan\delta\,\frac{\cos\frac{1}{2}(h'+h)}{\cos\frac{1}{2}(h'-h)}.$$

Ex. 17. In north latitude 45° the greatest azimuth attained by one of the circumpolar stars is 45° from the north point of the horizon. Prove that the star's polar distance is 30°. [Math. Trip.]

Ex. 18. Show how to find the latitude if the local sidereal time be observed at which two known stars have the same azimuth.

The hour angles h, h' at which the two stars have the azimuth a are known and (p. 3)

$$\cot a\sin h=-\cos\phi\tan\delta+\sin\phi\cos h,$$
$$\cot a\sin h'=-\cos\phi\tan\delta'+\sin\phi\cos h',$$

whence eliminating a

$$\tan\phi=\frac{\tan\delta\sin h'-\tan\delta'\sin h}{\sin(h'-h)}.$$

Ex. 19. Two altitudes of the sun, β and $\beta+\Delta\beta$, are simultaneously observed at two neighbouring places on the same meridian at a time when the declination of the sun is δ. Prove that, if ϕ be the latitude of one of the places, the difference of their latitudes is approximately

$$\Delta\beta\cos\beta\cos\phi/\{\sin\delta-\sin\beta\sin\phi\}.$$

[Coll. Exam.]

Ex. 20. Show that if a is the sun's altitude in the prime vertical, L its longitude, and ω the obliquity of the ecliptic, the latitude of the place is

$$\sin^{-1}(\sin\omega\sin L/\sin a).$$

Ex. 21. Show how ϕ the latitude may be accurately found from an observed zenith distance of a body of known declination δ when near the meridian, assuming as an approximate value $\phi_0=z+\delta$.

From the fundamental formula

$$\cos z = \sin\phi \sin\delta + \cos\phi \cos\delta \cos h$$

$$= \cos(\phi - \delta) - 2\sin^2 \tfrac{1}{2} h \cos\phi \cos\delta$$

we obtain

$$\sin \tfrac{1}{2}(z + \delta - \phi) \sin \tfrac{1}{2}(z - \delta + \phi) = \cos\delta \cos\phi \sin^2 \tfrac{1}{2} h,$$

in which the hour angle h is known from the local sidereal time and the right ascension of the body. If we make $x = z + \delta - \phi$, we obtain

$$\sin \tfrac{1}{2}x = \frac{\cos\delta \cos\phi}{\sin(\phi - \delta + \tfrac{1}{2}x)} \sin^2 \tfrac{1}{2} h.$$

But as the star is near the meridian x is small, whence we obtain very nearly (§ 3, Ex. 3)

$$x = \frac{2\cos\delta\cos\phi}{\sin(\phi - \delta)} \sin^2 \tfrac{1}{2} h \frac{\sin(\phi - \delta)}{\sin(\phi - \delta + \tfrac{1}{2}x)} (\sec \tfrac{1}{2}x)^{\frac{1}{3}},$$

or making $$\zeta = \frac{2\cos\delta\cos\phi_0}{\sin(\phi_0 - \delta)} \sin^2 \tfrac{1}{2} h,$$

and observing that ζ is very nearly x, we have

$$x = \zeta \frac{\sin(\phi_0 - \delta)}{\sin(\phi_0 - \delta + \tfrac{1}{2}\zeta)} (\sec \tfrac{1}{2}\zeta)^{\frac{1}{3}},$$

whence $\phi = z + \delta - x$ becomes known.

Ex. 22. If R be the sun's radius vector, $\odot$, β its true longitude and latitude, ω the obliquity of the ecliptic, X the coordinate measured along the line from the earth's centre to the true vernal equinox, Y the coordinate measured along the line in the plane of the equator perpendicular to X and towards the first point of Cancer, *i.e.* to a point whose R. A. is 6^h, and Z the coordinate perpendicular to the equator and towards the north pole, show that (p. 618, *N.A.* 1906),

$$X = R\cos\odot,$$

$$Y = R\sin\odot\cos\omega - 19{\cdot}3\, R\beta,$$

$$Z = R\sin\odot\sin\omega + 44{\cdot}5\, R\beta,$$

where the sun's mean distance is the unit of length and the numerical coefficients are in units of the seventh place of decimals.

From the general formulae of transformation we have

$$\sin\delta = \sin\beta\cos\omega + \cos\beta\sin\omega\sin\odot,$$

$$\cos\delta\cos a = \cos\beta\cos\odot,$$

$$\cos\delta\sin a = -\sin\beta\sin\omega + \cos\beta\cos\omega\sin\odot,$$

whence

$$X = R\cos\beta\cos\odot,$$

$$Y = -R\sin\beta\sin\omega + R\cos\beta\cos\omega\sin\odot,$$

$$Z = R\sin\beta\cos\omega + R\cos\beta\sin\omega\sin\odot.$$

In the case of the sun β is extremely small and making $\sin\beta=\beta\sin 1''$, $\sin\omega=\cdot 3980$ and $\cos\omega=\cdot 9174$ we obtain the desired result. Tables of X, Y, and Z for each day throughout the year are given in the ephemeris.

Ex. 23. Assuming the Milky Way to be a great circle of stars, cutting the equator in R.A. $18^h\ 30^m$, and making an angle 65°, measured northwards, with the equator, determine the R.A. and decl. of its pole.

Ex. 24. A planet's heliocentric orbit is inclined at a small angle i to the ecliptic; show that if its declination is a maximum, either the motion in latitude vanishes, or the longitude is approximately $90°+i\cot\omega\sin a$ where a is the longitude of the ascending node.

As the declination is a maximum the planet P must be 90° from the intersection N of its orbit with the equator. The projection of NP on the ecliptic will also be nearly 90°. Let NT be the perpendicular from N on ♈☊ the ecliptic, where ♈ is the vernal equinox and ☊ the ascending node.

In the small triangle NT♈ we have $\tan NT=\sin$ ♈$T\tan\omega$, and in the triangle NT☊ we have $\tan NT=\sin(a-$♈$T)\tan i$.

Hence $\sin$ ♈$T=\tan i\sin(a-$♈$T)\cot\omega$,

and approximately ♈$T=i\cot\omega\sin a$,

whence the planet's longitude is in general $90°+i\cot\omega\sin a$.

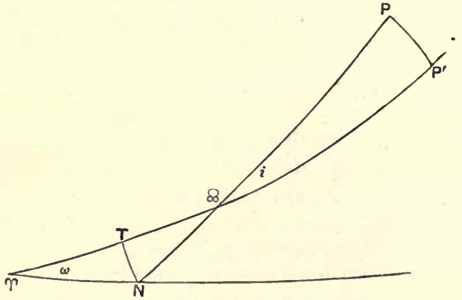


FIG. 37.

Ex. 25. Show that the true distance between Regulus and the moon at 4 P.M. Greenwich mean time on Jan. 6, 1909 is 41° 59′ 31″, being given

	Right Ascension			Declination			
Moon	7^h	12^m	$56^s\cdot 9$	N	24°	15′	40″
Star	10	3	31·6	N	12	24	45

Ex. 26. Prove that for a star which rises to the north of east, the rate at which the azimuth changes is the same when it rises as when it is due east, and is a minimum when the azimuth is $\sin^{-1}\left(\tan\lambda \,.\, \sin\dfrac{a}{2\sqrt{\cos a}}\right)$ north of east, where λ is the latitude and a the altitude of the star when due east.

[Math. Trip. 1902.]

Ex. 27. Show that observations of the altitudes of two known stars at a known Greenwich time are sufficient to determine the latitude and longitude of the observer. Show how from these observations the position of the observer may be found graphically on a terrestrial globe.

If the stars chosen for observation are on opposite sides of the meridian, show that the errors in latitude and longitude due to the small error ϵ in the observed altitude of each star are respectively

$$\epsilon \sec(a_1 + a_2) \cos(a_1 - a_2) \quad \text{and} \quad \epsilon \sec\phi \sec(a_1 + a_2) \sin(a_1 - a_2),$$

where ϕ is the estimated latitude and $2a_1$, $2a_2$ are the azimuths of the stars.

[Oxford Second Public Examination, 1902.]

CHAPTER VI.

ATMOSPHERIC REFRACTION.

39. The laws of optical refraction.

If a ray of light AO (Fig. 38) moving through one transparent homogeneous medium HH enters at O a different transparent homogeneous medium KK the ray generally undergoes a sudden change of direction and traverses the new medium in the direction OO'. This change is known as *refraction*. The ray AO is called the incident ray and the ray OO' the refracted ray, and both the incident ray and the refracted ray lie in the same plane through the normal at O to the surface separating the media.

Let MON be the normal at O to the surface separating the two media, then $\angle NOA = \psi$ is known as the angle of incidence and $\angle MOO' = \phi$ as the angle of refraction, and the fundamental law of refraction is expressed by the formula

$$\sin\psi = \mu \sin\phi,$$

where μ is a certain constant depending on the character of the two media. If by a change in the direction of the incident ray the angle ψ alters, then ϕ must alter correspondingly in such a way that the ratio of the sines of the two angles shall remain the same. We call μ the *index of refraction* from the first medium into the second.

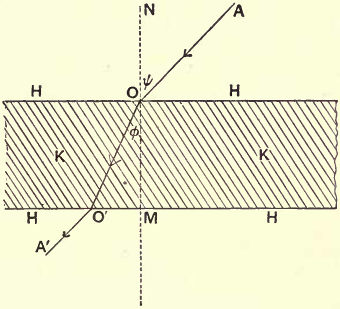


Fig. 38.

It should in strictness be noted that while μ varies, as stated, with the nature of the two media, it also varies with the character of the light. For example, μ would be different for a ray of blue light from what it is for a ray of red light, the media being the same in both cases. We have to consider however only atmospheric refraction and in this case the dispersion, as this phenomenon is called, is not great enough to make it necessary to attend to it for purposes of practical astronomy. We therefore take a mean value of μ which will be sufficiently accurate even though the rays of light with which we have to deal are of a composite nature. The refractive index of the atmosphere at the earth's surface at the temperature 0° C. and pressure 760 mm. is taken to be 1·000294 (Everett, *Units and Physical Constants*, p. 75).

If the direction of the ray were reversed, *i.e.* if a ray went from O' through the medium KK to O and thence emerged into the medium HH the ray would traverse HH precisely along the path OA. This is only a particular case of the general property that the curved or broken line which a ray follows in the course of a series of refractions through any media and at any incidences would also be followed if the direction of propagation of the light were reversed. Hence we see that if the lower surface of the medium KK is parallel to the upper surface, the ray on its emergence at O' into a second layer of the medium HH will pursue a direction $O'A'$, which is parallel to the incident direction AO. Thus we learn that a ray of light on passing through a parallel-sided homogeneous plate is not changed in direction though it will no doubt be shifted laterally. As we are now only concerned with the directions of rays the lateral shift need not be attended to.

Let μ_1 be the refractive index from a medium H_0 into H_1 (Fig. 39). Let μ_2 be the refractive index from H_0 into H_2, it is required to find the refractive index from the medium H_1 into H_2.

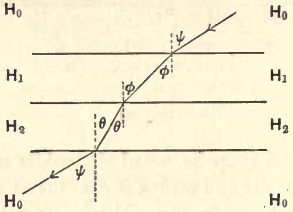


FIG. 39.

A ray from H_0 through parallel plates of H_1 and H_2 emerges in H_0 parallel to its original direction; and if ψ, ϕ, θ be the successive angles of incidence, then from the first incidence and the last emergence we have the equations

$$\sin\psi = \mu_1 \sin\phi \quad \text{and} \quad \sin\psi = \mu_2 \sin\theta,$$

whence
$$\mu_1 \sin\phi = \mu_2 \sin\theta.$$

We thus obtain the following result.

If μ_1 be the index of refraction from a standard medium into another medium H_1, μ_2 the index of refraction from the standard medium into another medium H_2, and if ϕ be the angle of incidence of a ray passing direct from H_1 to H_2, and θ the angle of refraction, then $\mu_1 \sin\phi = \mu_2 \sin\theta$, and the index of refraction for a ray passing directly from H_1 to H_2 is μ_2/μ_1.

40. Astronomical refraction.

The rays of light from a celestial body on passing from outer space through the earth's atmosphere undergo what is known as *astronomical refraction.* In the upper regions of the atmosphere the density of the air is so small that but little is there contributed to the total refraction. The refraction with which astronomers have to deal takes place mainly within a very few miles of the earth's surface. In consequence of refraction a ray of light from a star does not pass through the atmosphere in a straight line. It follows a curve, so that when the observer receives the rays the star appears to him to be in a direction which is not its true direction.

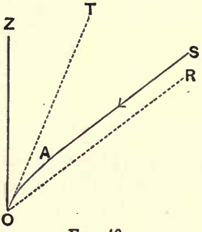


FIG. 40.

A ray of light coming towards us from a distant star in the direction *SA* (Fig. 40) pursues a straight path until it enters the effective atmosphere at *A*, and from thence the path is no longer straight. From *A* to the observer at *O* the ray is passing through atmospheric layers of which the density is continually increasing, so that the ray curves more and more till it reaches *O*. To the observer the rays appear to come from *T*, where *OT* is the tangent to the curve at *O*. If through *O* a line *OR* be drawn parallel to *AS* this line will show the direction in which the star would appear if there had been no refracting disturbance. Thus the effect of refraction is to move the apparent place of the star through the angle *TOR* up towards *Z*, the zenith of the observer. Refraction is greatest at the horizon where objects are apparently elevated by this cause through about 35′.

The observed coordinates of a heavenly body must, in general, receive corrections which will show what the coordinates would have been had there been no refraction. The investigation of the effects of refraction is therefore an important part of practical astronomy.

An approximate table is here given showing the amount by which refraction diminishes the apparent zenith distances of stars. The barometer is supposed to stand at 30 in. and the thermometer at 50° F. See Newcomb's *Spherical Astronomy*, p. 433.

Apparent Zenith Distance	Refraction	Apparent Zenith Distance	Refraction	Apparent Zenith Distance	Refraction
0°	0″	35°	41″	70°	2′ 39″
5°	5″	40°	49″	75°	3′ 34″
10°	10″	45°	58″	80°	5′ 19″
15°	16″	50°	1′ 9″	85°	9′ 51″
20°	21″	55°	1′ 23″	87°	14′ 23″
25°	27″	60°	1′ 41″	88°	18′ 16″
30°	34″	65°	2′ 4″	88° 40′	22′ 23″

For example, at an apparent zenith distance of 50° we learn that the refraction is 1′ 9″ and that consequently the true zenith distance is 50° 1′ 9″. It will be noted that for any zenith distance < 45° the refraction is not so much as 1′, and that for zenith distances up to 20° the refraction is practically 1″ per 1°.

41. General theory of atmospheric refraction.

We shall suppose that the earth is spherical and that the atmosphere is composed of a succession of thin layers bounded by spheres concentric with the earth. The refractive index of the air throughout each layer is to be constant, but it may vary from one layer to another.

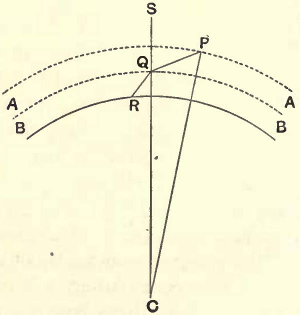


FIG. 41.

Consider two such layers A and B (Fig. 41). The refractive index of the outer layer A is μ_1 relative to free aether and that of B is μ_2. A ray passing through A in the direction PQ is bent into the direction QR as it passes into B.

Let C be the earth's centre,

$$\psi = \angle SQP,\ \phi_1 = \angle QPC,\ \phi_2 = \angle RQC,\ CP = r_1,\ CQ = r_2.$$

Then from the principles of refraction (§ 39) because CQ is normal to the surface of separation we have

$$\mu_1 \sin \psi = \mu_2 \sin \phi_2.$$

But from the triangle PCQ

$$\sin \psi : \sin \phi_1 :: r_1 : r_2,$$

whence eliminating ψ we obtain

$$r_1 \mu_1 \sin \phi_1 = r_2 \mu_2 \sin \phi_2.$$

The same would of course be true for any two consecutive layers, and thus we obtain the following general theorem.

Let the atmosphere be regarded as constituted of a number of thin spherical homogeneous layers, concentric with the earth and varying in density from one layer to another. As a ray of light traverses successive layers, the product of the sine of the angle of refraction by the radius of the layer and by its refractive index is constant.

We may express this theorem in the following formula:

$$r\mu \sin \phi = a\mu_0 \sin z \quad \text{(i)},$$

where z is the *apparent* zenith distance, a the radius of the earth and μ_0 the index of refraction of the lowest layer.

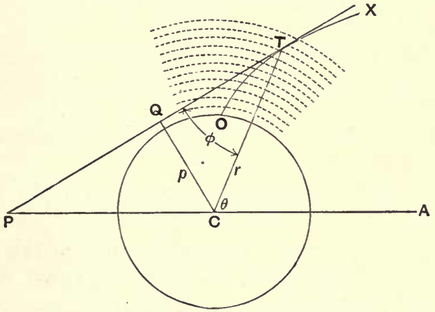


FIG. 42.

If we suppose the layers to be indefinitely thin, then the path of the ray instead of being a broken line would be a curve. Let XTO (Fig. 42) be the curve as it passes through the successive layers and reaches the earth at O. Draw the tangent TQP to the curve at T, where the ray enters a layer whose refractive index is μ, and radius r. The tangent coincides with a small part of the ray and consequently $\angle CTQ = \phi$, the angle of refraction. When the ray first enters the atmospheric strata the tangent to the curve must coincide with the true direction of the star. On the other hand the tangent to the curve at O indicates the direction in which the ray enters the eye of the observer. The angle between these two tangents shows the total change in the direction of the ray. This is the quantity which we seek to determine, for this is what we commonly call the refraction.

If ρ be the refraction then $d\rho$ is the angle between two consecutive tangents $= d\theta - d\phi$ if $\theta = \angle ACT$ and $\phi = \angle CTP$. From geometry we see that $d\theta = -\tan\phi\, dr/r$, whence

$$d\rho = -\tan\phi\, dr/r - d\phi.$$

We can now transform this equation by (i), which may be written

$$\log r + \log \mu + \log \sin\phi = \text{const.},$$

differentiating we have

$$dr/r + d\mu/\mu + \cot\phi\, d\phi = 0 \quad \text{.................(ii)},$$

whence

$$d\rho = \tan\phi\, d\mu/\mu \quad \text{....................(iii)}.$$

Eliminating $\tan\phi$ by the help of (i) we find

$$d\rho = \frac{1}{\mu}\frac{a\mu_0 \sin z}{(r^2\mu^2 - a^2\mu_0^2 \sin^2 z)^{\frac{1}{2}}}\, d\mu.$$

Thus we obtain the differential equation for the refraction.

***42. Integration of the differential equation for the refraction.**

To determine the refraction accurately this equation would have to be integrated between the limits of $\mu = \mu_0$ and $\mu = 1$ the value of μ at the upper layer of atmosphere. It is at this point that the difficulty in the theory of refraction makes itself felt †.

† The general discussion of the integration of this equation is too difficult for insertion here. Reference may be made to Professor Newcomb's *Compendium of Spherical Astronomy* and to Professor Campbell's *Practical Astronomy*. An account of Bessel's elaborate investigation will be found in Brünnow's *Spherical Astronomy*. I am indebted to Prof. E. T. Whittaker for calling my attention to the elegant approximate method here given.

The expression to be integrated contains two variables r and μ which must be related. If the law of this relation were known then we could express r in terms of μ so that the problem would be the integration of a certain function of μ. But we have not precise information as to the law according to which the index of refraction varies with the elevation above the earth's surface. It is however most interesting to find that it is possible to obtain an approximate solution of the problem quite sufficient for most purposes without any knowledge of the law according to which the density of the atmosphere diminishes with the elevation above the earth's surface.

We shall assume $r/a = 1 + s$ where s is a small quantity because the altitude of even the highest part of the atmosphere is small in comparison with the earth's radius. We shall substitute this value for r/a in the expression of $d\rho$ and disregard all powers of s above the first. We thus have

$$\rho = \int_1^{\mu_0} \frac{\mu_0 \sin z\, d\mu}{\mu\,(\mu^2 - \mu_0^2 \sin^2 z + 2s\mu^2)^{\frac{1}{2}}}$$

$$= \int_1^{\mu_0} \frac{\mu_0 \sin z\, d\mu}{\mu\,(\mu^2 - \mu_0^2 \sin^2 z)^{\frac{1}{2}}} \left(1 + \frac{2s\mu^2}{\mu^2 - \mu_0^2 \sin^2 z}\right)^{-\frac{1}{2}}$$

$$= \int_1^{\mu_0} \frac{\mu_0 \sin z\, d\mu}{\mu\,(\mu^2 - \mu_0^2 \sin^2 z)^{\frac{1}{2}}} - \int_1^{\mu_0} \frac{s\mu\mu_0 \sin z\, d\mu}{(\mu^2 - \mu_0^2 \sin^2 z)^{\frac{3}{2}}}.$$

The refraction is thus expressed by two integrals of which the first and most important part expresses what the refraction would be if $s = 0$, *i.e.* if the earth's surface was a plane. This is of course a well-known elementary integral and its value is

$$\sin^{-1}(\mu_0 \sin z) - z.$$

If we denote by x the small quantity $(\mu_0 - 1)$ the integral may be written

$$\sin^{-1}\{(1 + x)\sin z\} - z$$

and this when developed in powers of x by Maclaurin's theorem will be convenient for calculation. If we neglect all powers of x above the second we see that $(\mu_0 - 1)\tan z + \frac{1}{2}(\mu_0 - 1)^2 \tan^3 z$ is the approximate value of the first integral.

In evaluating the second integral we are to notice that s enters as a factor into the integrand and therefore we shall make no appreciable error by putting $\mu = \mu_0 = 1$, for quantities of the order $s(\mu_0 - 1)$ are so small that they may be neglected. Thus the second integral assumes the simple form

$$-\frac{\sin z}{\cos^3 z}\int_1^{\mu_0} s\,d\mu.$$

Let m be the density of that atmospheric shell which has μ as its refractive index, then by Gladstone and Dale's law μ and m are connected by an equation of the form

$$\mu - 1 = cm,$$

where c is a constant quantity, so

$$d\mu = c\,.\,dm.$$

If m_0 be the density of the air at the surface of the earth, then the integral becomes

$$-c\frac{\sin z}{\cos^3 z}\int_0^{m_0} s\,.\,dm.$$

Integrating by parts this becomes

$$-c\frac{\sin z}{\cos^3 z}\int_0^{s'} m\,.\,ds,$$

for the terms independent of the integral vanish at both limits; we also make $s = s'$ when $m = 0$ and $s = 0$ when $m = m_0$. The integral in this expression has a remarkable significance, for it is obvious that it expresses the total mass of air lying vertically over a unit area on the earth's surface and is therefore proportional to the pressure of the atmosphere, *i.e.* to the height of the barometer. Thus the actual law by which the density of the atmosphere may vary with the altitude is not now required in the problem.

The theoretical expression of the refraction has therefore assumed a remarkably simple form. It is the difference between two integrals whereof the first has been found and the second must be proportional to $\tan z + \tan^3 z$. From this we learn that the total refraction must be of the form $A \tan z + B \tan^3 z$ where z is the apparent zenith distance and A, B are certain constants. The values of these constants are to be determined by observation as is shown in § 46.

We can also assume various hypotheses as to the relation between r and μ and compare the results so calculated with those of actual observation. It is noteworthy that several different relations between r and μ each give a theory of refraction the results of which are in fair accordance with observation.

43. Cassini's formula for atmospheric refraction.

By the hypothesis of Cassini, who assumed that the atmosphere is homogeneous, we can obtain an expression for the refraction practically identical with that just found. Of course this hypothesis is untrue, but it should be observed that if the surface of the earth were a plane instead of being a curved surface the successive atmospheric layers would be parallel-sided, and therefore the refractive index of the lowest layer alone would determine the total refraction (§ 39). It is therefore only the curvature of the earth which prevents the formula derived from Cassini's theory from being strictly true.

There are excellent grounds for believing that at an altitude of twenty miles the atmospheric density would be less than a thirtieth part of its amount at the earth's surface. We may therefore conclude that almost all the refraction is produced within twenty miles of the earth's surface.

Let O (Fig. 43) be the place of the observer and OH a ray reaching O in a horizontal direction: such a ray has of course experienced more refraction than any other ray.

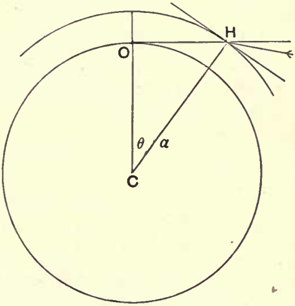


Fig. 43.

Let a be the radius of the earth, and $a + l$ the radius of the shell of atmosphere which the ray first strikes at H. If θ be the angle between the tangents to the shells at O and H, and if HO be taken to be a straight line we have

$$\sin^2\theta = 1 - a^2/(a+l)^2$$
$$= 2l/a = 40/4000 = 1/100,$$

whence θ is about 6°.

Hence the effective layers of air of various densities through which the rays have to pass are so nearly parallel that none of them would have to be altered through an angle exceeding

6° to make them strictly parallel. We might therefore anticipate that no wide departure from truth will arise by assuming the atmosphere to be in horizontal layers, in which case the non-homogeneity produces no effect on the total refraction.

The formula connecting the refraction with the zenith distance in the case of a supposed homogeneous atmosphere has been thus obtained by Cassini.

We shall assume that the atmosphere is condensed into the space between the two spherical shells of radii CS and CV respectively. The atmosphere is considered of uniform density and of refractive index μ.

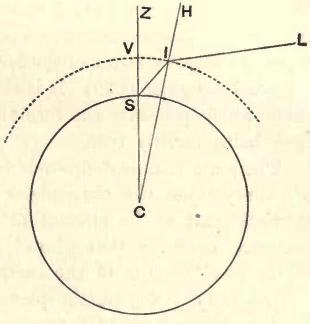


FIG. 44.

The ray LI impinges at I on the atmospheric surface to which CIH is normal and reaches the observer on the earth's surface at S, so that $\angle LIH = \psi$ is the angle of incidence and $\angle SIC = \phi$ is the angle of refraction.

The ray reaches the observer in the direction IS, so that $\angle ISV = z$ is the apparent zenith distance of the object. If a denotes as before the radius of the earth, and l the thickness of the atmosphere SV, we have from the triangle SCI

$$(1 + l/a) \sin \phi = \sin z,$$

and also

$$\sin \psi = \mu \sin \phi.$$

Hence

$$\sin \psi = \mu (1 - l/a) \sin z,$$

very nearly, since l/a is a small quantity estimated at less than 1/800.

If ρ be the whole refraction, *i.e.* the angle through which the incident ray is bent from its original direction, we have $\psi = \phi + \rho$, and assuming ρ to be expressed in seconds of arc

$$\rho \sin 1'' = (\sin \psi - \sin \phi) \sec \phi.$$

Substituting for $\sin \psi$, $\sin \phi$, $\cos \phi$ respectively the expressions

$$\mu (1 - l/a) \sin z, \quad (1 - l/a) \sin z, \quad \sqrt{1 - (1 - l/a)^2 \sin^2 z},$$

we obtain

$$\rho = (\mu - 1) \operatorname{cosec} 1'' \frac{(1 - l/a) \sin z}{\{1 - (1 - l/a)^2 \sin^2 z\}^{\frac{1}{2}}}$$

$$= (\mu - 1) \operatorname{cosec} 1'' \{\tan z - (\tan z + \tan^3 z)\, l/a\}$$

$$= A \tan z + B \tan^3 z \quad \text{.................................(i)},$$

where
$$A = (\mu - 1)(1 - l/a) \operatorname{cosec} 1'',$$
$$B = -(\mu - 1)\, l/a \operatorname{cosec} 1''.$$

For the practical application of the formula which has thus been derived by the different processes of this and the preceding article, we must obtain numerical values for A and B. This has to be done from actual observation of the refraction in at least two particular instances (see § 46), and we shall assume it has been thus found that at temperature 50° F. and pressure 30 in. the refractions at the apparent zenith distances 54° and 74° are 80″·06 and 200″·46 respectively.

The formula (i) will thus give for the determination of A and B the two equations

$$80''{\cdot}06 = A (\tan 54°) + B (\tan 54°)^3,$$
$$200''{\cdot}46 = A (\tan 74°) + B (\tan 74°)^3.$$

Solving these equations we obtain the following general expression for the refraction at mean pressure 30 in. and temperature 50° F.,

$$\rho = 58''{\cdot}294 \tan z - 0''{\cdot}06682 \tan^3 z \text{ ....(ii)}.$$

Thus B/A is only 1/873 so that unless $\tan^2 z$ becomes very great, *i.e.* unless the object is near the horizon, we may neglect the second term.

If the zenith distance does not exceed 70°, the refraction may be computed with sufficient accuracy for many purposes where no extreme temperatures are involved by the simple expression

$$k \tan z,$$

and where we are using only the first term, that is neglecting the term containing $\tan^3 z$, it is slightly more accurate to take $k = 58''{\cdot}2$ rather than 58″·294. The quantity k is called the coefficient of refraction.

Ex. 1. What ought to be the thickness of a homogeneous atmosphere which would give an expression for refraction in accordance with observation?

$$-B/A = l/a,$$

whence $$l/a = 0{\cdot}0668/58{\cdot}3 = 1/873,$$
so that taking $a = 3957$ miles, we find $l = 4{\cdot}5$ miles.

Ex. 2. Show that the refractive index of the atmosphere would be 1·000283 at pressure 30 in. and temp. 50° F. according to Cassini's theory of refraction.

Ex. 3. Show from formula (ii) that 1′ 48″·3 is the refraction at the apparent zenith distance 61° 48′ ($p = 30$ in., temp. = 50° F.).

Ex. 4. Show that if quantities less than the fifth part of a second be disregarded the second term in the expression for the refraction may be omitted whenever the zenith distance does not exceed 55°.

Ex. 5. If we express the refraction as $k' \tan z'$ where z' is the *true* zenith distance instead of in the usual form $k \tan z$ where z is the *apparent* zenith distance, show that if k and k' are both expressed in seconds of arc

$$k' = k\,(1 - k \sec^2 z \sin 1'').$$

44. Other formulae for atmospheric refraction.

It is obvious that the density of the air constituting the atmosphere diminishes as the distance from the earth increases. The index of atmospheric refraction will in like manner diminish from 1·000294 its value at the earth's surface to the value 1 at the upper limits of the refracting atmosphere.

We take a as in § 41 to be the radius of the lowest atmospheric layer for which $\mu = \mu_0$, and r' the radius of the layer when μ has declined to unity. Simpson assumed that $r\mu^{n+1} = r'$, where n is a quantity at present unknown. The assumed equation gives $r = r'$ when $\mu = 1$ as already arranged. As r increases μ is to diminish, and this will be the case provided $(n + 1)$ be positive.

We have seen (§ 41) that $\mu r \sin\phi = \text{const}$. Equating the expressions of this product for the upper and lower limits of the atmosphere

$$\mu_0 a \sin z = r' \sin z',$$

where z' is the angle of incidence at the uppermost layer and z at the lowest. Substituting for r' we have

$$\mu_0 a \sin z = a\mu_0^{n+1} \sin z',$$

whence $$\sin z = \mu_0^{n} \sin z',$$

or $$z' = \sin^{-1} \frac{\sin z}{\mu_0^{n}}.$$

Taking the logarithmic differential of $r\mu^{n+1} = r'$, we have

$$(n+1)\frac{r}{\mu} + \frac{dr}{d\mu} = 0,$$

whence from (ii) (§ 41)

$$\frac{n}{\mu} = \cot\phi \frac{d\phi}{d\mu},$$

and from (iii) (§ 41)

$$\frac{d\rho}{d\phi} = \frac{1}{n}.$$

To find the refraction we have only to integrate this expression between the values of ϕ at the atmospheric boundaries. The angle of refraction is z at the earth's surface, and

$$\sin^{-1}\frac{\sin z}{\mu_0{}^n}$$

at the upper boundary of the atmosphere, whence we have Simpson's formula for the refraction

$$\rho = \frac{1}{n}\left\{z - \sin^{-1}\left(\frac{\sin z}{\mu_0{}^n}\right)\right\}.$$

Ex. 1. Show that if $\mu_0{}^n = 1 + \omega$, where ω is a small quantity of which powers above the second may be neglected, we can obtain from Simpson's formula the following approximate expression for the refraction

$$\rho = \left(\frac{\omega}{n} - \frac{\omega^2}{n}\right)\tan z - \frac{\omega^2}{2n}\tan^3 z.$$

Ex. 2. Assuming that observation has shown the law of refraction to be (§ 42)

$$\rho = 58''\cdot294 \tan z - 0''\cdot06682 \tan^3 z,$$

show that Simpson's formula would give for μ_0 the index of refraction of the air at the earth's surface the value 1·00028, and also that $n = 8$ and

$$\mu^9 = r'/r.$$

Ex. 3. Show that if Simpson's formula were correct the height of the atmosphere so far as it is effective for refraction would be about ten miles.

A convenient formula due to Bradley may be deduced from the expression just obtained:

$$\rho = \frac{1}{n}\left(z - \sin^{-1}\frac{\sin z}{\mu_0{}^n}\right)$$

which may be written

$$\sin(z - n\rho) = \sin z/\mu_0{}^n,$$

whence

$$\frac{\sin z - \sin(z - n\rho)}{\sin z + \sin(z - n\rho)} = \frac{\mu_0{}^n - 1}{\mu_0{}^n + 1},$$

or, as the refraction is small,

$$\rho = \frac{2}{n} \frac{\mu_0^n - 1}{\mu_0^n + 1} \tan (z - \tfrac{1}{2} n\rho).$$

If we introduce the values of μ_0 and n used in Ex. 2, p. 129, we should find as the approximate formula

$$\rho = 59'' \tan (z - 4\rho).$$

We can correct this formula so as to make it exact for two known refractions at standard temperature and pressure, if for example we take

$$z = 50^\circ,\ \rho = 69''\cdot 36 \text{ and } z = 75^\circ,\ \rho = 214''\cdot 10$$

(see Greenwich Tables), we get the final form

$$\rho = 58''\cdot 361 \tan (z - 4\cdot 09\rho).$$

By this formula all refractions up to the zenith distance of 80° can be determined approximately.

Bradley's formulae is suited for observations near the horizon, because $\tan (z - 4\cdot 09\rho)$ does not become indefinitely large as z approaches 90°.

Ex. 1. Show that the formula for refraction given by Bradley and Cassini, viz.

$$\rho = 58''\cdot 361 \tan (z - 4\cdot 09\rho)$$

and

$$\rho = 58''\cdot 294 \tan z - 0''\cdot 06682 \tan^3 z,$$

are practically equivalent until the zenith distance becomes very large.

Ex. 2. On the supposition that the $(n+1)$th power of the index of refraction of the atmosphere varies inversely as the distance from the centre of the earth, prove Bradley's approximate formula for astronomical refraction $\rho = a \tan (z - \frac{1}{2} n\rho)$. Oxford Senior Scholarship, 1903.

Ex. 3. If in the atmosphere the index of refraction vary inversely as the square of the distance from the earth's centre, being μ_0 at the earth's surface and unity at the limit of the atmosphere, show that the corresponding correction for refraction is given by

$$\sin (z + \tfrac{1}{2}\rho) = \sqrt{\mu_0} \sin z.$$ Mathematical Tripos, 1906.

45. Effect of atmospheric pressure and temperature on refraction.

In the formula (ii) for the refraction already obtained (§ 43) we assumed that the barometer stood at 30 inches and the external air at the temperature 50° F. We have now to find the formula to be used when pressure and temperature have any other known values.

We assume that the refraction is proportional to the density of the air at the earth's surface, so that if ρ be the refraction for pressure p and temperature t and ρ_0 the refraction at the standard pressure 30 inches and temperature 50°, we obtain from the properties of gases

$$\frac{\rho}{\rho_0}=\frac{p}{30}\,\frac{460+50}{460+t}=\frac{17p}{460+t}.$$

Introducing the value of ρ_0 already found (§ 42) we obtain the approximate formula for atmospheric refraction at pressure p and temperature t for the apparent zenith distance z.

$$\rho=\frac{17p}{460+t}(58''\cdot 294\tan z-0''\cdot 06682\tan^3 z).$$

In the appendix to the *Greenwich Observations for* 1898, Mr P. H. Cowell has arranged tables of refraction which are used in Greenwich observatory. These tables contain the mean refractions for the pressure 30 inches and temperature 50° F. for every minute of zenith distance from 0° to 88° 40′. The corrections which must be applied for changes in temperature and pressure are given in additional tables.

46. On the determination of atmospheric refraction from observation.

We describe three of the methods by which the coefficients A and B in the expression for the refraction, $A\tan z+B\tan^3 z$ can be determined by observation of meridian zenith distances. The first and second methods can be carried out at a single observatory provided its latitude is neither very great nor very small. The third method requires the cooperation of two observatories, one in the northern and one in the southern hemisphere†.

First Method. A star is selected such that it will be above the horizon both at upper and at lower culmination. If z, z' be the apparent zenith distances at lower and upper culmination respectively and positive to the north of the zenith, then the true zenith distances will be $z+A\tan z+B\tan^3 z$ and $z'+A\tan z'+B\tan^3 z'$. The mean of these two zenith distances

† Of the remaining methods of observing refractions we may mention that of Loewy described by Sir David Gill in the *Monthly Notices of the Royal Astronomical Society*, Vol. XLVI. p. 325.

is, of course, the distance from the zenith to the north pole, *i.e.* the colatitude. Hence we obtain the equation

$$\tfrac{1}{2}\{z + z' + A(\tan z + \tan z') + B(\tan^3 z + \tan^3 z')\} = 90^\circ - \phi.$$

Substituting the observed values of z and z' we obtain a linear equation between the three quantities A, B and ϕ.

Other stars are also observed in the same way and each star gives an equation in the same three unknowns. Three of such equations will suffice to determine A, B, ϕ. The result will, however, be much more accurate if we observe many stars and then treat the resulting equations by the method of least squares to be subsequently described.

As a simple illustration we shall take a case in which the latitude is known and in which, as neither of the zenith distances is excessive, we may assume that the refraction is expressed by the single term $k \tan z$.

At Dunsink in N. latitude 53° 23′ 13″ the star α Cephei is observed to have the apparent zenith distance 8° 48′ 37″ at upper culmination. At lower culmination 12 hours later its apparent zenith distance is 64° 22′ 47″.

The true zenith distances will be

$$8^\circ\, 48'\, 37'' + k \tan(\;8^\circ\, 48'\, 37''),$$
$$64^\circ\, 22'\, 47'' + k \tan(64^\circ\, 22'\, 47'').$$

The sum of these must be double the colatitude (36° 36′ 47″), whence

$$73^\circ\, 11'\, 24'' + k(0{\cdot}155 + 2{\cdot}085) = 73^\circ\, 13'\, 34'',$$

from which $$k = 58''{\cdot}0.$$

Second Method. The constants of refraction can also be determined by observation of the solstitial zenith distances of the sun.

Let z_1, z_2 be the apparent meridional zenith distances of the sun at the solstices. Let ρ_1 and ρ_2 be the corresponding refractions. Then the true zenith distances are $z_1 + \rho_1$ and $z_2 + \rho_2$. Assuming that the sun's latitude may be neglected, or in other words, that the sun's centre is actually in the ecliptic as is always very nearly true, we obtain for the mean of these zenith distances the arc from the zenith to the equator, *i.e.* the latitude. Hence we have

$$2\phi = z_1 + z_2 + \rho_1 + \rho_2.$$

If the latitude be known and if we assume

$$\rho_1 = k \tan z, \text{ and } \rho_2 = k \tan z_2,$$

we obtain an equation for k.

Third Method. In this we require observations of the zenith distances SZ_1 and SZ_2 of the same star S, both from a northern observatory at N. latitude ϕ_1, and a southern observatory at S. latitude ϕ_2 (Fig. 45).

If P and P' be the north and south celestial poles we have

$$SZ_1 = SP - Z_1P = \phi_1 - \delta,$$
$$SZ_2 = SP' - Z_2P' = \phi_2 + \delta.$$

If z_1 and z_2 be the observed zenith distances, and if we assume the refractions to be $k \tan z_1$ and $k \tan z_2$ respectively, then

$$SZ_1 = z_1 + k \tan z_1,$$
$$SZ_2 = z_2 + k \tan z_2,$$

whence

$$z_1 + k \tan z_1 + z_2 + k \tan z_2 = \phi_1 + \phi_2,$$

from which k can be found.

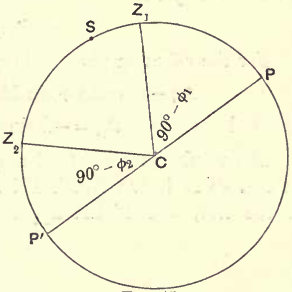


FIG. 45.

We shall take as an example β Andromedæ, which was observed to culminate at Greenwich at an apparent south zenith distance 16° 20′ 3″, the latitude of Greenwich being 51° 28′ 38″ N. The culmination of the star was also observed at the Cape of Good Hope Observatory in 33° 56′ 4″ south latitude, and the apparent north zenith distance was 69° 1′ 50″.

We thus have the equation

$$16° \, 20' \, 3'' + k \tan (16° \, 20') + 69° \, 1' \, 50'' + k \tan (69° \, 2') = 85° \, 24' \, 42'',$$

whence $$k = 58''{\cdot}3.$$

47. Effect of refraction on hour angle and declination.

We may make use of the differential formulae of § 35 to determine the effect of refraction on the hour angle and declination of a star. The effect of refraction is to throw the star upwards towards the zenith. If the observed zenith distance be z the true zenith distance is $z+\Delta z$ where $\Delta z = k \tan z$. We assume that the latitude is known so that $\Delta\phi = 0$, and as the azimuth does not alter by refraction $\Delta a = 0$.

To find the effect on declination we write the formula connecting Δa, $\Delta\phi$, Δz, $\Delta\delta$, § 35 (1),

$$\Delta\delta + \cos\eta\Delta z - \cos h\Delta\phi - \sin h\cos\phi\Delta a = 0,$$

which with the substitution $\Delta a = 0$, $\Delta\phi = 0$, $\Delta z = k\tan z$ gives $\Delta\delta = -k\tan z\cos\eta$, *i.e.* if δ is the observed declination then $\delta - k\tan z\cos\eta$ is the true declination.

To find the effect on hour angle we have (§ 35 (2))

$$\Delta z + \cos a\Delta\phi + \cos\eta\Delta\delta + \cos\phi\sin a\Delta h = 0,$$

from which by the same substitutions

$$\Delta h = k\sin\eta\tan z\sec\delta.$$

For the effect on parallactic angle we use (§ 35 (6))

$$\Delta\eta - \cos z\Delta a + \sin\delta\Delta h - \sin a\sin z\Delta\phi = 0,$$

and find $$\Delta\eta = -k\sin\eta\tan\delta\tan z.$$

The results just obtained may be otherwise proved as follows. In Fig. 46 N is the North Pole, Z the zenith, P the true place of the star, and P' the apparent place of the star as raised

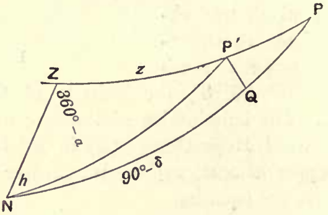


FIG. 46.

towards the zenith by refraction, and $PP' = k\tan ZP' = k\tan z$. $P'Q$ is perpendicular to PN and provided the $\angle PNP'$ is small, as will be the case unless P is near the pole, the change in polar distance is

$$PQ = PP'\cos\eta = k\tan z\cos\eta.$$

The observed declination is $90° - NQ$, but the real declination is $90° - NP$. Hence the observed declination is too large, and consequently the correction $\Delta\delta$ to be applied to the observed declination to obtain the true declination is given by

$$\Delta\delta = -k \tan z \cos \eta.$$

We have also

$$\Delta h = P'NQ = k \tan z \sin \eta \operatorname{cosec} P'N = k \tan z \sin \eta \sec \delta.$$

As $\sin \eta \cos \delta$ is unaltered by refraction we must have

$$\cos \eta \cos \delta \Delta\eta = \sin \eta \sin \delta \Delta\delta,$$

whence by substituting for $\Delta\delta$ we find

$$\Delta\eta = -k \sin \eta \tan \delta \tan z.$$

48. Effect of refraction on the apparent distance between two neighbouring celestial points.

We shall first show that if the refraction be taken as $k \tan z$, then the correction to be added to the apparent distance D in seconds of arc between two neighbouring stars is in seconds of arc

$$kD(1 + \cos^2\theta \tan^2 z) \sin 1'',$$

where z is the zenith distance of the principal star and θ is the angle between the arc joining the two stars and the arc from the principal star to the zenith.

Let Z be the zenith, $ZA = x$, $ZB = y$, $AB = D$, $\angle AZB = a$, $ZAB = \theta$. The effect of refraction is to move the arc AB up to $A'B'$ where

$$AA' = k \tan x$$

and $$BB' = k \tan y.$$

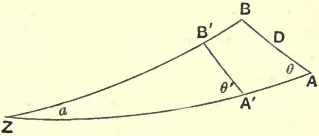


FIG. 47.

Then

$$\cos D = \cos x \cos y + \sin x \sin y \cos a.$$

Differentiating with a as constant, and making

$$\Delta x = -k \tan x$$

we find $$\Delta y = -k \tan y,$$

$$-\sin D \,.\, \Delta D = k \sin x \cos y \tan x + k \cos x \sin y \tan y$$
$$- k \cos a \cos x \sin y \tan x - k \cos a \sin x \cos y \tan y$$
$$= k \sin^2 (x - y) \sec x \sec y + 4k \sin^2 \tfrac{1}{2} a \sin x \sin y.$$

As both these terms are small we may put $x = y = z$, the zenith distance of either star, in the expressions $\sec x \sec y$ and $\sin x \sin y$. Also since $a \,.\, D$ are small we may put

$$\sin D = D \text{ and } \sin^2 (x - y) = D^2 \cos^2 \theta$$

also $$4 \sin^2 \tfrac{1}{2}a = a^2 = D^2 \sin^2 \theta \operatorname{cosec}^2 z$$

and we thus obtain for the decrease in D due to refraction

$$kD(1 + \cos^2 \theta \tan^2 z)$$

or if k, D, ΔD are expressed in seconds of arc

$$kD(1 + \cos^2 \theta \tan^2 z) \sin 1''$$

gives the seconds by which D has been lessened by refraction; this is consequently the correction to the measured distance between two neighbouring stars to clear from the effect of refraction.

We have next to show that θ, the angle which the line joining the two stars makes with the vertical, is increased by refraction to the extent $k \sin \theta \cos \theta \tan^2 z$.

Taking the logarithmic differential of the equation

$$D \sin \theta = \sin a \sin y$$

we have $$\Delta D/D + \cot \theta \Delta \theta = \cot y \Delta y,$$

which becomes by substitution

$$-k(1 + \cos^2 \theta \tan^2 z) + \cot \theta \Delta \theta = -k,$$

whence $$\Delta \theta = k \sin \theta \cos \theta \tan^2 z,$$

and this is the quantity which must be subtracted from the apparent angle $B'A'Z$ to get the true angle BAZ.

The deformation of the circular disc of the sun or moon by refraction is obtained as follows:

Let S (Fig. 48) be the sun's centre, a its radius, P a point on its limb, and Z the zenith, and let $ZS = z$. Let k be the coefficient of refraction which displaces P to P', and let PQ and $P'Q'$ be perpendiculars on ZS. From what we have just seen PQ is displaced by refraction to $P'Q'$. If we take S as origin, SZ as axis of x, and x and y the coordinates of P', then

$$y = P'Q' = (1 - k) PQ = a(1 - k) \sin \theta.$$

Also $$x = SQ' = a \cos \theta + QQ'$$
$$= a \cos \theta + k \tan(z - a \cos \theta)$$
$$= a \cos \theta + k(\tan z - a \cos \theta \sec^2 z),$$

and by eliminating θ we have for the equation of the refracted figure of the sun

$$\frac{(x - k \tan z)^2}{(a - ak \sec^2 z)^2} + \frac{y^2}{a^2(1 - k)^2} = 1.$$

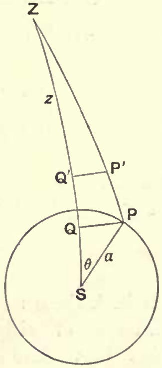


FIG. 48.

The major axis is $a(1-k)$ and the minor $a(1-k\sec^2 z)$, and their ratio is $1-k\tan^2 z$. Of course k is here in radians.

We may note that any short horizontal arc is diminished by refraction in the ratio $1-k:1$, and any small vertical arc at a considerable zenith distance is diminished in the ratio

$$1-k\sec^2 z:1.$$

Ex. 1. If D be the difference of declination between two adjacent stars, and if z be the zenith distance and η the parallactic angle of one of these stars, then the effect of refraction is to diminish the difference of declination by

$$kD(1+\tan^2 z\cos^2\eta)\sin 1'',$$

it being assumed that the refraction is proportional to the tangent of the zenith distance and that k is its coefficient.

D is then the projection of the arc joining the two stars on the hour circle through one of them, and the hour circle makes the angle η with the zenith distance.

Ex. 2. A telescope at an observatory in N. lat. 53° 23′ 13″ is directed to a point on the parallel of 38° 9′ N. decl., and is fixed at an hour angle of 7^{hrs}. Two stars trail successively through the field, and their apparent difference of declination is 68″·02; show that to correct for the effect of refraction this difference should be increased by 0″·09.

(One of the stars is 61 Cygni and the other is one of the comparison stars used at Dunsink in determining the parallax of 61 Cygni by the method of differences of declination.)

Ex. 3. In their unrefracted positions a number of stars lie on a small curve of which the polar equation is $\rho=f(\theta)$, where ρ is the great circle distance from a point O taken as origin to a point P on the curve, and where θ is the angle between OP and OZ where Z is the observer's zenith. Show that on taking account of refraction the polar equation of the curve will be found by the elimination of ρ and θ from the equations

$$\rho=f(\theta),$$
$$\rho'=\rho-k\rho(1+\tan^2 z\cos^2\theta),$$
$$\theta'=\theta+k\sin\theta\cos\theta\tan^2 z,$$

in which ρ' is the radius vector joining the points O' and P', which are the refracted positions of O and P respectively, and θ' is the angle which $O'P'$ makes with $O'Z$.

Ex. 4. It is proposed to determine the angular diameter of the sun. The arithmetic mean of two measured diameters at right angles to one another is D; the coefficient of refraction is k, here expressed in radians; the zenith distance of the sun's centre is z. Show that the true diameter is $D(1+k+\frac{1}{2}k\tan^2 z)$, whatever may have been the position angles in which the two diameters at right angles to one another were measured. (Based on a result in the introduction to the Greenwich Observations.)

The distance from the centre of the ellipse to the point θ is

$a(1-k-k\cos^2\theta\tan^2 z)$. Hence the arithmetic mean of the radii measured at right angles, *i.e.* at θ and $\theta+90°$, is $a(1-k-\frac{1}{2}k\tan^2 z)=\frac{1}{2}D$, whence $2a=D(1+k+\frac{1}{2}k\tan^2 z)$.

***49. Effect of refraction on the measurement of the position angle of a double star.**

Let A, B be respectively the principal star and the secondary star of the pair which form the double star and let P be the north pole.

Imagine a circle with centre A on the celestial sphere and graduated so that the observer is the nole and that $AP\,(<180°)$ cuts the circle at 0°. The point in which AB meets the graduated circle is said to be the *position angle of the star B with respect to A* The mode in which the position angle is measured may be further illustrated as follows. Suppose the double star is on or near the meridian and at its upper culmination, and the secondary star is due east of the principal star. Then the position angle is about 90°. If, however, the secondary star had been due west when the principal star was on the meridian, its position angle would be about 270°, for in each case the direction of measurement from the arc drawn to the pole is the same. Astronomers generally know this as the N.F.S.P. direction, for the measurement proceeds from the *north* point towards the part of the sky which is *following* from the diurnal movement round by the *south* and then back to the *north* by the *preceding* part of the sky.

If P be the pole, Z the zenith, and A the principal star of the double AB (Fig. 49), then the position angle as we have just defined it is $\angle PAB$. The refraction changes the position angle into $PA'B'$. Thus the refraction changes the position angle in two ways, first by altering the parallactic angle $PAZ=\eta$ and secondly by altering BAZ. Both these angles are altered by refraction, and the correction to apply to an observed position angle in the case represented in the figure must be negative. We denote the true position angle by p.

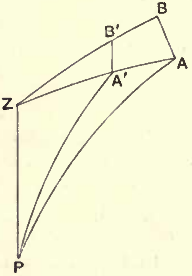


FIG. 49.

We have $\angle BAZ=p-\eta$, and hence (§ 48)

$$\angle B'A'Z=p-\eta+k\sin(p-\eta)\cos(p-\eta)\tan^2 z,$$
$$\angle PA'Z=\eta+k\tan z\tan\delta\sin\eta.$$

If therefore p_r be the position angle as affected by refraction,

$$p_r = p + k \tan z \tan \delta \sin \eta + k \sin (p - \eta) \cos (p - \eta) \tan^2 z.$$

If p_r' and p' be the corresponding quantities with respect to another star with reference to the same primary,

$$p_r' = p' + k \tan z \tan \delta \sin \eta + k \sin (p' - \eta) \cos (p' - \eta) \tan^2 z.$$

Subtracting we easily find

$$p' - p = p_r' - p_r - k \tan^2 z \sin (p' - p) \cos (2\eta - p - p').$$

The true position angle p' of the direction in which A moves by the diurnal motion is 270°. If therefore p_r' be the observed position angle for the movement of A when carried by the diurnal motion,

$$p = p_r + 270° - p_r' + k \tan^2 z \cos p \sin (2\eta - p).$$

Summary. From the last article and the present we obtain the following result for the correction of the observed distance and position angle of a double star for refraction †.

Let D be the distance of the two stars expressed in seconds of arc, z the zenith distance, p the position angle, η the parallactic angle, and k the coefficient of refraction in seconds of arc, then the correction to be added to the *apparent distance* to obtain the true distance is

$$kD \{1 + \tan^2 z \cos^2 (p - \eta)\} \sin 1'',$$

and the correction to be added to the *measured position angle* to obtain the true position angle is

$$k \tan^2 z \cos p \sin (2\eta - p).$$

Ex. If the declination of α Lyræ is 38° 40′ and the position angle of an adjacent star is 150° 58′·0, find the correction for refraction to be applied to the position angle when the hour angle is 7 hours west, the latitude is 53° 23′ 13″, and the coefficient of refraction is 58″·2.

It is first necessary to compute the zenith distance 67° 36′ and the parallactic angle 38° 32′, whence the formula gives 4′·6 as the correction to be added to the observed position angle to clear it from the effect of refraction.

† For tables to facilitate the application of these corrections see *Monthly Notices of the Royal Astronomical Society*, Vol. XLI. p. 445.

MISCELLANEOUS QUESTIONS ON REFRACTION.

Ex. 1. Show that refraction reduces the sine of the zenith distance of an object in the ratio of $(1-k):1$ where k is the coefficient of refraction.

Ex. 2. The north declination of a Aquilæ is 8° 37′ 39″ Show that its *apparent* zenith distance at culmination at Greenwich (lat. 51° 28′ 38″ N.) is 42° 50′ 5″ and at Cape of Good Hope (lat. 33° 56′ 4″ S.) is 42° 32′ 50″.

Ex. 3. If the horizontal refraction be 35′, show that the formula for the hour angle h of the sun's centre at rising or setting when its declination is δ is

$$\cos^2 \tfrac{1}{2}h = \sec\phi \sec\delta \cos(45^\circ + 17'{\cdot}5 - \tfrac{1}{2}\phi - \tfrac{1}{2}\delta)\sin(45^\circ - 17'{\cdot}5 - \tfrac{1}{2}\phi - \tfrac{1}{2}\delta).$$

Ex. 4. Assuming that the moon is depressed at rising by parallax through 59′ and elevated by refraction through 35′, show that if h be the hour angle and δ the declination we have at Greenwich

$$\cos^2 \tfrac{1}{2}h = [\cdot 2056] \sec\delta \cos(19^\circ\, 3'{\cdot}7 - \tfrac{1}{2}\delta)\sin(19^\circ\, 27'{\cdot}7 - \tfrac{1}{2}\delta).$$

Ex. 5. At sunrise at Greenwich (lat. 51° 28′ 38″·1) on Feb. 8th, 1894, the sun's declination is 14° 39′ S. Find its apparent hour angle assuming that the horizontal refraction is 35′.

Ex. 6. The apparent path of a star not far from the pole, projected on the plane of the horizon, is an ellipse of excentricity $\cos\phi$, where ϕ is the latitude. Show that if the zenith distance of the star is not very great, the same will be the case for the apparent path as altered by refraction.

[Coll. Exam.]

Ex. 7. The north declination of a Cygni being 44° 57′ 17″ (1909), show that its apparent zenith distances at upper and lower culmination at the latitude 53° 23′ 13″ are respectively 8° 25′ 49″ and 81° 33′ 18″, assuming that the refraction may be taken as

$$58''{\cdot}294 \tan z - 0''{\cdot}06682 \tan^3 z,$$

where z = the *apparent* zenith distance.

Ex. 8. Prove that if at a certain instant the declination of a star is unaffected by refraction the star culminates between the pole and the zenith, and the azimuth of the star is a maximum at the instant considered.

[Math. Trip. I.]

A great circle drawn from the zenith to touch the small circle described by the star round the pole will give the point in which the zenith distance of the star is at right angles to its polar distance. It is obvious that the star can never have an azimuth greater than when situated at the point of contact.

Ex. 9. Prove that, within the limits of zenith distance in which the refraction may be taken as k tan (zen. dist.) the apparent place of a star describes each sidereal day, a conic section, which is an ellipse or hyperbola according as $\sin^2\phi \gtrless \cos^2\delta$, where δ is the declination of the star and ϕ the latitude of the place.

The great circle from the star's true place to the pole being the axis of x, we have as the plane coordinates of the refracted place

$$x = k\tan z\cos\eta, \quad y = k\tan z\sin\eta,$$

where z and η are respectively the zenith distance and the parallactic angle. From the spherical triangle

$$\sin z\sin\eta = \cos\phi\sin t, \quad \sin z\cos\eta = \cos\delta\sin\phi - \sin\delta\cos\phi\cos t,$$
$$\cos z = \sin\delta\sin\phi + \cos\delta\cos\phi\cos t,$$

$$x = \frac{k\cos\delta\sin\phi - k\sin\delta\cos\phi\cos t}{\sin\delta\sin\phi + \cos\delta\cos\phi\cos t}, \quad y = \frac{k\cos\phi\sin t}{\sin\delta\sin\phi + \cos\delta\cos\phi\cos t},$$

from which we have

$$\sin t = \tan\phi\,\frac{y}{x\cos\delta + k\sin\delta},$$

$$\cos t = \tan\phi\,\frac{k\cos\delta - x\sin\delta}{x\cos\delta + k\sin\delta},$$

whence eliminating t

$$y^2 + (k\cos\delta - x\sin\delta)^2 = \cot^2\phi\,(x\cos\delta + k\sin\delta)^2,$$

which may be written

$$x^2(\sin^2\phi - \cos^2\delta) + y^2\sin^2\phi - xk\sin 2\delta + k^2(\sin^2\phi - \sin^2\delta) = 0,$$

and this is an ellipse or a hyperbola according as $\sin^2\phi - \cos^2\delta$ is positive or negative.

Ex. 10. Assuming that refraction is small and proportional to the tangent of the zenith distance, show that if the same star is observed simultaneously from different stations on the same meridian its apparent places lie on an arc of a great circle.

[Coll. Exam.]

This follows at once from the following geometrical theorem which is easily proved from the rules for quadrantal triangles, p. 5. If AO be a quadrant and a variable great circle through O cut two fixed great circles through A in P, Q respectively, then $\tan OP/\tan OQ$ is constant.

Ex. 11. If δ be the declination of a star, show that, if the horizontal refraction be r'', the time of a star's rising at a place in latitude ϕ is changed approximately by a number of seconds equal to

$$\frac{r}{15\sqrt{\cos^2\delta - \sin^2\phi}}.$$

With the usual notation

$$\cos z = \sin\phi\sin\delta + \cos\phi\cos\delta\cos t.$$

Differentiating $\Delta z = \cos\phi\cos\delta\sin t\,\Delta t$,

but as the star is on the horizon $\sin z = 1$ and

$$0 = \cos z = \sin\phi\sin\delta + \cos\phi\cos\delta\cos t,$$

$$\cos\phi\cos\delta\sin t = (\cos^2\phi\cos^2\delta - \cos^2\phi\cos^2\delta\cos^2 t)^{\frac{1}{2}}$$

$$= (\cos^2\phi\cos^2\delta - \sin^2\phi\sin^2\delta)^{\frac{1}{2}}$$

$$= (\cos^2\delta - \sin^2\phi)^{\frac{1}{2}};$$

whence $\Delta z = (\cos^2\delta - \sin^2\phi)^{\frac{1}{2}}\Delta t.$

If $\Delta z =$ be expressed in seconds of arc and Δt be n seconds of time we put $\Delta z = r''$, $\Delta t = 15n''$, whence we find for n the required result.

Ex. 12. Assuming that the alteration in the zenith distance z of a star owing to refraction is $k\tan z$, where k is small, show that in latitude ϕ the change produced in the hour angle of a circumpolar star is greatest when the angle PSN is a right angle, where P is the pole, S the star, and N the north point of the horizon; and that its maximum value is

$$k\cos\phi\sec\delta\sqrt{\sec z_1\sec z_2},$$

where z_1 and z_2 are the greatest and least zenith distances of the star.

[Coll. Exam.]

The change in the hour angle h by refraction is $k\sec\delta\cos\phi\sin h\sec z$, and if $\sin h\sec z$ is a maximum the point S' found by producing SP through P till $SS' = 90°$ is $90°$ from N.

Ex. 13. Assuming that the refraction of any object S is equal to $k\tan ZS$, prove that the resolved parts of the refraction in R.A. and N.P.D. expressed respectively in seconds of time and seconds of arc are very nearly

$$\frac{k}{15}\frac{\tan ZL}{\sin\Delta\cos(\Delta - PL)} \text{ and } k\tan(\Delta - PL),$$

where Δ is the north polar distance of the object, P the pole, and ZL an arc of a great circle drawn from Z perpendicular to PS.

[Math. Trip. II.]

*Ex. 14. Let ϕ be the latitude of the observer, δ the declination of a star and h its west hour angle, and let the coefficient of refraction be $58''\cdot 4$ (its value for the photographic rays). Show that refraction diminishes the apparent rate of change of hour angle by

$$24^{s}\cdot 5\sin m\cos m(\tan\delta + \cot\phi\sec h)\operatorname{cosec}^2(\delta + m) \text{ per day},$$

where $\tan m = \cot\phi\cos h$.

Show also that the rate of change of refraction in declination is

$$+15''\cdot 3 \cot\phi \sin h \operatorname{cosec}^2(\delta+m)\cos^2 m \text{ per hour.}$$

[Mr A. R. Hinks, *Monthly Notices R.A.S.*, vol. LX. p. 544.]

Refraction raises the star towards the zenith Z from its true place S to an apparent place S'. Let the true hour angle be h and the apparent hour angle h'. Draw the arc $ZL=90^\circ-n$ perpendicular to PS (Fig. 50)

$$(h-h')\cos\delta = k\tan z\sin ZSL$$
$$= k\cos n\sec z$$
$$= \frac{k\cos\phi\sin h}{\sin\phi\sin\delta+\cos\phi\cos\delta\cos h}.$$

Differentiating with respect to t

$$\left(\frac{dh}{dt}-\frac{dh'}{dt}\right)\cos\delta$$
$$= k\cos\phi\,\frac{\cos h(\sin\phi\sin\delta+\cos\phi\cos\delta\cos h)+\cos\phi\cos\delta\sin^2 h}{(\sin\phi\sin\delta+\cos\phi\cos\delta\cos h)^2}\,\frac{dh}{dt}$$
$$= k\cos\phi\,\frac{\cos\phi\cos\delta+\sin\phi\sin\delta\cos h}{\cos^2 z}\,\frac{dh}{dt}$$
$$= k\cos\phi\,\frac{\sin\phi\cos\delta\cos h(\tan\delta+\cot\phi\sec h)}{\sin^2(\delta+m)\sin^2 n}\,\frac{dh}{dt}$$
$$= k\,\frac{\sin\phi\cos\phi\cos\delta\cos h\cos^2 m(\tan\delta+\cot\phi\sec h)}{\sin^2(\delta+m)\sin^2\phi}\,\frac{dh}{dt}$$
$$= k\cos\delta\sin m\cos m\,\frac{\tan\delta+\cot\phi\sec h}{\sin^2(\delta+m)}\,\frac{dh}{dt}.$$

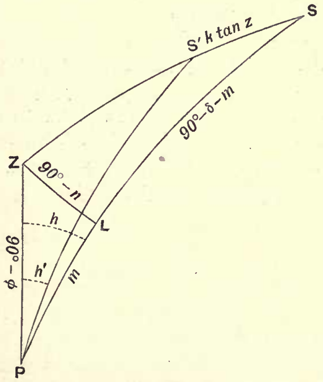


FIG. 50.

The number of seconds in a sidereal day is 86400. Let $86400+r$ be the number of seconds which would be required for a complete revolution if the apparent hour angle of the star continued to increase for the whole day at the same rate as at the moment under consideration. Hence we have

$$\frac{dh}{dt}=\frac{2\pi}{86400}, \quad \frac{dh'}{dt}=\frac{2\pi}{86400+r},$$

and thus

$$\left(\frac{2\pi}{86400}-\frac{2\pi}{86400+r}\right)=k \sin m \cos m \frac{\tan\delta+\cot\phi\sec h}{\sin^2(\delta+m)}\frac{2\pi}{86400}.$$

As r is very small we have by making $k=58{\cdot}4/206265$,

$$r=24^s{\cdot}5 \sin m \cos m (\tan\delta+\cot\phi\sec h)\operatorname{cosec}^2(\delta+m),$$

in which $\tan m=\cot\phi\cos h.$

The case of an equatorial star is instructive,

$$\delta=0 \text{ and } r=24^s{\cdot}5 \cot m \cot\phi \sec h$$
$$=24^s{\cdot}5 \sec^2 h.$$

Thus even in the neighbourhood of the meridian on either side an equatorial star is so affected by refraction that it will only keep time with a sidereal clock when that clock is losing at the rate of $24^s{\cdot}5$ secs. daily.

If x be the refraction in declination expressed in seconds then

$$x=k\tan z\cos ZSL$$
$$=k\tan(90^\circ-\delta-m)=k\cot(\delta+m).$$

Hence differentiating and regarding Δx, Δm and Δh as all expressed in seconds of arc

$$\Delta x=-k\operatorname{cosec}^2(\delta+m)\,\Delta m\sin 1'',$$

but $\tan m=\cot\phi\cos h,$

$$\sec^2 m\,\Delta m=-\cot\phi\sin h\,\Delta h,$$

whence $\sec^2 m\,\Delta x=k\operatorname{cosec}^2(\delta+m)\cot\phi\sin h\,\Delta h\sin 1''.$

If N is the hourly rate expressed in seconds of arc at which the declination is changing then $\Delta x/\Delta h=N/15\times 60\times 60$. With these substitutions we find on introducing the values of k and $\sin 1''$ the desired result namely

$$N=15''{\cdot}3\cot\phi\sin h\operatorname{cosec}^2(\delta+m)\cos^2 m.$$

These results are of practical importance in the art of celestial photography.

CHAPTER VII.

KEPLER'S AND NEWTON'S LAWS AND THEIR APPLICATION.

50. The laws of Kepler and Newton.

The laws according to which the planets move round the sun and which always bear the name of their discoverer Kepler, are as follows.

(1) *The orbit of a planet round the sun is an ellipse; in one focus of which the centre of the sun is situated.*

Let S in Fig. 51 be the centre of the sun. Then the orbit $ABPQ$ of any planet is an ellipse of which S is a focus. The velocity of the planet is not constant and the law according to which the speed varies is given by Kepler's second law.

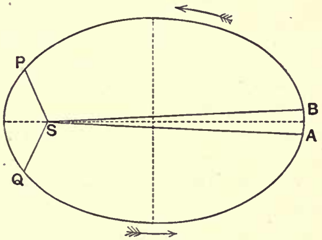


FIG. 51.

(2) *The radius vector, drawn from the centre of the sun to the planet, sweeps over equal areas in equal times.*

For example take any two points AB on the ellipse and also two other points PQ, then if the area ASB = area PSQ the time taken in describing AB will equal the time taken in describing PQ. From this it follows that, with the points as represented in the figure, the velocity of the planet is greater while describing PQ than while describing AB.

In the first two laws of Kepler we are concerned with the motion of a single planet only. In Kepler's third law we obtain a remarkable relation between the movements of two different planets. We define the *mean distance* of a planet to be the semi-axis major of its orbit, and the *periodic time* to be the period in which a planet completes an entire circuit of its orbit. Kepler's third law is then stated as follows.

(3) *The squares of the periodic times of two planets have the same ratio as the cubes of their mean distances from the sun.*

Example:—The periodic times of the earth and Venus are 365·3 and 224·7 days respectively, and the ratio of the squares of these periodic times is $(365{\cdot}3)^2/(224{\cdot}7)^2 = 2{\cdot}643$. The corresponding mean distances are 1 and ·7233 and as $1/({\cdot}7233)^3 = 2{\cdot}643$ we have a verification of Kepler's third law for these two planets.

The three laws of Kepler given above were deduced by him entirely from observations of the movements of the planets and without any reference to the nature of the forces which control these movements. For more than three quarters of a century they remained isolated facts without explanation until Newton showed them to be consequences of the law of universal gravitation which appears to govern the movement of every particle of matter in the universe.

The three *axioms* or *laws of motion*, on which the science of dynamics is built, and which are generally known as Newton's Laws†, may be stated as follows:

LAW I. *Every body continues in its state of rest, or of uniform motion in a straight line, except in so far as it may be compelled to change that state by impressed forces.*

LAW II. *Change of motion is proportional to the impressed force and takes place in the direction of the straight line in which the force acts.*

LAW III. *To every action there is always an equal and contrary reaction; or the mutual actions of any two bodies are always equal and oppositely directed.*

By *change of motion* Newton denoted what is often called the rate of change of momentum, or the product of the mass of the

† The reader who desires a fuller development of Newton's Laws and their applications may refer to Routh's *Dynamics of a particle*, 1898, where the laws as here expressed are given on p. 18.

moving body by the rate of change of its velocity, which may be otherwise expressed as the product of the mass by the acceleration. *Law II* enables us to say that, in the case of a planet for example, the change of motion is proportional to the impressed force and takes place in the direction of the straight line in which the force acts.

Kepler's first and second laws enabled Newton to prove that each planet moves under the control of a force always directed towards the sun and varying inversely as the square of the distance from the sun. Kepler's *third* law enabled Newton to compare the acceleration of one planet with that of another, and from this he was led to the doctrine of universal gravitation with which his name is identified and which states that *every particle of matter attracts every other particle with a force varying as the product of their masses and inversely as the square of the distance between them.*

We shall first prove that if the radius vector drawn to a moving particle from a fixed point sweeps over equal areas in equal times then the force on the particle must always be directed towards the fixed point.

If r is the radius vector, SP, (Fig. 52) and θ the angle it

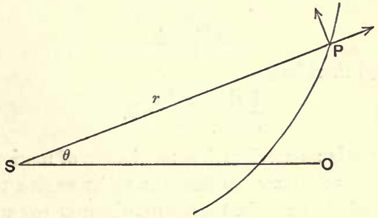


FIG. 52.

makes with any fixed direction, SO, then the velocities along and at right angles to SP are respectively,

$$\frac{dr}{dt} \text{ and } \frac{rd\theta}{dt}.$$

After the short time Δt the velocity along and at right angles to the consecutive radius vector will be

$$\frac{dr}{dt}+\Delta t\frac{d^2r}{dt^2}, \quad \frac{rd\theta}{dt}+\Delta t\frac{d}{dt}\left(r\frac{d\theta}{dt}\right);$$

resolving these velocities along the original radius vector with which the consecutive radius vector makes the angle $\Delta t \, . \, d\theta/dt$, we obtain

$$\frac{dr}{dt} + \Delta t \frac{d^2 r}{dt^2} - \Delta t \frac{r \, d\theta}{dt} \frac{d\theta}{dt},$$

whence if $-F$ be the acceleration towards S we have

$$-F = \frac{d^2 r}{dt^2} - r \left(\frac{d\theta}{dt}\right)^2.$$

In like manner resolving these velocities perpendicular to the original radius vector we find for the resolved part in this direction

$$r \frac{d\theta}{dt} + \Delta t \frac{d}{dt}\left(r \frac{d\theta}{dt}\right) + \Delta t \frac{dr}{dt} \frac{d\theta}{dt},$$

whence for the acceleration perpendicular to the original radius vector we have the expression

$$\frac{1}{r} \frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right).$$

Twice the area swept over by the radius vector in the time dt is $r^2 d\theta$, and if these two quantities are in a constant ratio, as Kepler's second law informs us is the case in the motion of a planet, we have

$$r^2 \frac{d\theta}{dt} = h$$

a constant, and therefore

$$\frac{1}{r} \frac{d}{dt}\left(r^2 \frac{d\theta}{dt}\right) = 0.$$

Hence there is no acceleration, no change of motion, and therefore, by Newton's 2nd law, no force at right angles to the radius vector. The whole force is, therefore, directed towards S. Thus Kepler's second law proves that the planets move under the action of a force directed continually towards the centre of the sun.

The next step is to show that if a body moves in a conic section under a force directed to one of the foci and if the body moves in such a way that the radius vector drawn to it from that focus describes areas proportional to the time, then the force must vary inversely as the square of the focal radius vector.

The equation of a conic referred to the focus is

$$r = p/(1 + e \cos \theta),$$

where p is the semi-latus rectum, e the eccentricity, and θ the angle which the radius vector (r) makes with the line joining the focus to the nearer apse (§ 52).

We have thus the following three equations

$$r = p/(1 + e\cos\theta) \qquad \text{(i)},$$

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = -F \qquad \text{(ii)}$$

and

$$r^2\frac{d\theta}{dt} = h \qquad \text{(iii)},$$

from which to determine F, *i.e.* the acceleration towards the sun.

Differentiating (i) we find

$$\frac{dr}{dt} = \frac{pe\sin\theta}{(1+e\cos\theta)^2}\frac{d\theta}{dt} = \frac{e\sin\theta}{p}r^2\frac{d\theta}{dt} = \frac{he}{p}\sin\theta$$

and

$$\frac{d^2r}{dt^2} = \frac{he}{p}\cos\theta\frac{d\theta}{dt} = \frac{h^2e\cos\theta}{pr^2}.$$

Also

$$r\left(\frac{d\theta}{dt}\right)^2 = \frac{h^2}{r^3};$$

and therefore

$$\frac{d^2r}{dt^2} - r\left(\frac{d\theta}{dt}\right)^2 = \frac{h^2}{r^2}\left\{\frac{e\cos\theta}{p} - \frac{1}{r}\right\}$$

$$= \frac{h^2}{r^2}\left\{\frac{e\cos\theta}{p} - \frac{1+e\cos\theta}{p}\right\} = -\frac{h^2}{pr^2}.$$

Thus we see that the acceleration, and therefore the force, at every point of the orbit varies inversely as the square of the distance from the focus. This is of course true whatever be the value of e and consequently we see that this result holds whether the orbit be an ellipse, an hyperbola or a parabola.

If we denote the acceleration by μ/r^2, where μ is the acceleration at unit distance due to the sun's attraction, we have from the formulæ just given

$$h^2 = \mu p \qquad \text{(iv)}.$$

We are now in a position to prove from Kepler's third law that the constant, μ, is the same for all the planets. For h is twice the area described in the unit of time, and therefore by Kepler's second law if the periodic time be P we must have

$$h = 2\pi ab/P.$$

But $p = b^2/a$. Hence by means of (iv) we find

$$4\pi^2a^3/P^2 = \mu.$$

But according to Kepler's third law a^3/P^2 is the same for all the planets and hence we find that μ is a constant throughout the solar system.

If a perpendicular be drawn from the centre of a planet to the ecliptic, then the angle through which a line from the sun's centre through ♈ would have to be turned in the positive direction in the plane of the ecliptic to meet this perpendicular is termed the *heliocentric longitude* of the planet. The geocentric longitude of the sun increased by 180° is the heliocentric longitude of the earth.

By the *synodic period* of two planets is meant the average interval between two successive occasions on which the planets are in conjunction, *i.e.* have the same heliocentric longitude. If they move uniformly in circular orbits in the same plane and in periods P, p respectively, and if L, l be the heliocentric longitudes of the planets at the time t, then

$$L = 2\pi t/P + L'$$
$$l = 2\pi t/p + l',$$

where L', l' are the longitudes at the time $t = 0$.

Let x be the synodic period and t_0 the time when $L - l = 0$, then $t_0 + x$ is the time when the planets have next the same longitude, and (if $p > P$) $L - l$ is then 2π. We thus have the equations

$$0 = 2\pi t_0/P - 2\pi t_0/p + L' - l'$$
$$2\pi = 2\pi(t_0 + x)/P - 2\pi(t_0 + x)/p + L' - l',$$

whence by subtracting

$$x = Pp/(p - P).$$

If one of the planets is the earth, the year the unit of time, the earth's mean distance the unit of length, and a the mean distance of the other planet from the sun, then from Kepler's third law, we have for an outer planet

$$x = a^{\frac{3}{2}}/(a^{\frac{3}{2}} - 1),$$

and for an inner planet

$$x = a^{\frac{3}{2}}/(1 - a^{\frac{3}{2}}).$$

Ex. 1. Assuming the mean distance of the earth from the sun to be 92·9 (the unit being 1,000,000 miles) and the eccentricity of the earth's orbit to be ·0168, find the side of a square equal to the area swept over daily by the radius vector.

N.B. The year may always be taken to be 365·25 mean solar days unless otherwise stated.

Ex. 2. If v_1, v_2 be the velocities of a planet at perihelion and aphelion respectively and if e be the eccentricity of its orbit, show that

$$(1-e)\,v_1=(1+e)\,v_2.$$

Ex. 3. Show that the velocity of a planet at any moment may be resolved into a component h/p perpendicular to the radius vector and a component eh/p perpendicular to the major axis of the orbit.

Ex. 4. Show from Kepler's 2nd and 3rd laws that two planets in the system describe areas in a given time which are in the ratio of the square roots of their latera recta.

Ex. 5. The mean distance of Jupiter from the sun is 5·203 when the unit of length is the mean distance of the earth from the sun. The periodic time of Jupiter is 11·862 years and of Mercury 0·2408 years: show that the mean distance of Mercury from the sun is 0·387.

Ex. 6. The eccentricity of the orbit of Mars is 0·0933 and its mean distance from the sun is 1·5237 times that of the earth from the sun. Assuming that the earth's distance from the sun is 92,900,000 miles and that the eccentricity of its orbit may be neglected, determine the greatest and least possible distances of Mars from the earth.

Ex. 7. If the periodic time of a planet be P and the length of its semi-axis major be a, show that a small change Δa in the semi-axis major will produce a change $3P\Delta a/2a$ in the periodic time.

Ex. 8. Show that in the motion of a planet in an elliptical orbit about the sun according to the law of nature the angular velocity round the unoccupied focus varies as the square of the sine of the angle between the radius vector and the tangent.

Let ds be an elementary arc of the ellipse at a distance r from the sun and r' from the unoccupied focus. Let p, p' be the perpendiculars from the foci on the tangent at ds. Let θ be the angle which either focal radius makes with the tangent.

From Kepler's second law it follows immediately that p is inversely proportional to the linear velocity of the planet, and hence the time of describing $ds \propto pds$. The angle described about the unoccupied focus is $ds \sin\theta/r'$ and hence the angular velocity round the unoccupied focus

$$\propto ds \sin\theta/r'pds=\sin\theta/r'p=\sin^2\theta/p'p.$$

But from the property of the ellipse $p'p$ is const. and thus the theorem is proved.

Ex. 9. If in an elliptical orbit of a planet about the sun in one focus the square of the eccentricity may be neglected, show that the angular velocity of the planet is uniform about the other focus.

*Ex. 10. Prove by means of the annexed table, extracted from the

Nautical Almanac for 1890, that the eccentricity of the Earth's orbit is ·0168 approximately.

	Sun's longitude
Jan. 1	281° 5′ 30″·6
„ 2	282 6 39 ·7
July 1	99 32 19 ·1
„ 2	100 29 29 ·6

[Coll. Exam.]

*Ex. 11. If the orbit of a minor planet be assumed to be a circle in the ecliptic, prove that two observations of the difference of longitude of the planet and the sun, with a knowledge of the elapsed time are sufficient to determine the radius. Show also that three such observations will determine the orbit if it be assumed to be parabolic. [Math. Trip. I.]

A single observation of the difference of longitude shows that the planet must lie on a known straight line, *i.e.* the line through the earth's centre to the point of the ecliptic at the observed distance from the sun. When two such lines are known a circle with centre at the sun will cut each of these lines in two points. If a point of intersection on one line and a point of intersection on the other subtend the angle at the sun's centre which for that radius gives the observed time interval the problem is solved. Trial will thus determine the radius. An equation could also be found for the radius but this again could only be solved by trial.

*Ex. 12. Prove that in a synodic period an inferior planet crosses the meridian the same number of times as the sun, but that a superior planet crosses it once oftener. [Math. Trip. I. 1902.]

*Ex. 13. The fourth satellite of Jupiter has an orbital period of

$$16^d\ 18^h\ 5^m\ 6^s{\cdot}9 = 16^d{\cdot}753552,$$

while the fifth satellite has a period of $0^d\ 11^h\ 57^m\ 27^s{\cdot}6 = 0^d{\cdot}498236$. Find from Kepler's third law the ratio of the mean distances of these two satellites from the primary.

*Ex. 14. Assuming that Deimos and Phobos, the satellites of Mars, revolve in circular orbits and that at the opposition of Sept. 23rd, 1909, the observed greatest distance of Deimos from the centre of Mars is 1′ 23″·1, show from Kepler's third law that the greatest apparent distance of Phobos is 33″·2, it being given that the periodic time of Phobos is $7^h\ 39^m\ 13^s{\cdot}85$ and of Deimos $30^h\ 17^m\ 54^s{\cdot}86$.

51. Apparent motion of the Sun.

The revolution of the earth about the sun causes changes both in the apparent place and the apparent size of the sun as seen from the earth. We have now to show that the phenomena with which we are concerned in this chapter would be exactly reproduced if the earth were indeed at rest and if the sun revolved round the the earth in an orbit governed by Kepler's laws, and identical in shape and size with the earth's orbit round the sun.

Let S, Fig. 53, be the sun, and E_1 and E_2 two positions of the earth. From E_1 the sun is seen in the direction E_1S and at the distance E_1S.

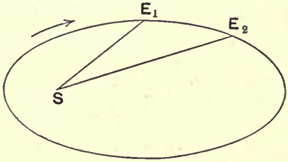


FIG. 53.

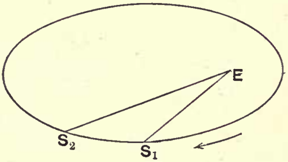


FIG. 54.

Draw ES_1 from E in Fig. 54 parallel and equal to E_1S. In like manner let ES_2 be equal and parallel to E_2S. If this be repeated for other pairs of points E_3, S_3, &c., the ellipse traced out by S_1, S_2, &c. will be exactly the same shape and size as that traced by E_1, E_2, &c. The latter is the true path of the earth round the sun, the former is the path which the sun appears to describe round the earth. At every moment the apparent direction of the sun and the distance of the sun are the same, whether we regard the earth as going round the fixed sun as in Fig. 53, or the sun going round the fixed earth as in Fig. 54.

If a be the radius of the sun and r the distance of the sun's centre from the earth, which we shall here regard as a point, then the angular value of the apparent semi-diameter A of the sun as seen from the earth is $\sin^{-1} a/r$. As this angle is small we may with sufficient approximation take $A = a/r \sin 1''$ as its value in seconds of arc. Thus we see that r varies inversely as A, so that if A be determined by observation at two different dates during the year, the relative distances of the sun at those two dates are immediately obtained.

Ex. On Jan. 3rd, 1909, the sun, being then at its least distance from the earth, has the angular semidiameter 16′ 17″·58. On July 4th, 1909, the sun, being then at its greatest distance, has the angular semidiameter 15′ 45″·37. Show from these data that the eccentricity of the Earth's orbit is ·0167.

52. Calculation of elliptic motion.

Let F be the earth's centre and OPO' the ellipse, with F as focus, in which the sun appears to make its annual revolution. OO' is the major axis of the ellipse and C its centre. The circle OQO' has its centre at C and its radius $CO = \frac{1}{2}OO' = a$. The line QPH is perpendicular to OO'; and $FP = r$. Let $\angle OFP = v$, and $\angle OCQ = u$. Thus v, r are the polar coordinates of P with respect to the origin F and axis FO. The angles v and u are called respectively the *true anomaly* and the *eccentric anomaly*.

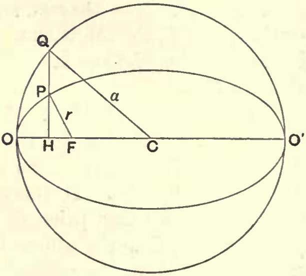


FIG. 55.

The points O and O' being the extremities of the major axis of the ellipse are termed the *apses* of the orbit. That *apse* O which is nearest the earth is termed the *perigee*. The other apse O' is called the *apogee*. The time is to be measured from that moment known as the *epoch*, at which the sun passes through the perigee O. If we had been considering the true motion of the earth round the sun, then the points O and O' would have been termed the *perihelion* and the *aphelion* respectively. We should also note that $CF = eCO = ea$.

We have now to show how the polar coordinates of the sun are to be found when the time is given. It is not indeed possible to obtain finite values for r and v in terms of t. We can, however, with the help of the eccentric anomaly u, obtain expressions in series which enable the values of r and v to be calculated to any desired approximation.

From Kepler's second law we see that if t be the time in which the sun moves from O to P, and if T be the periodic time of the orbit,

$$t : T :: \text{area } OFP : \text{area of ellipse}.$$

Introducing n to signify the *mean motion*, *i.e.* the circular measure of the average value of the angle swept over by the radius vector in the unit of time, we have $n = 2\pi/T$, and as the area of the ellipse is πab we have

$$nt = 2 \text{ area } OFP/ab.$$

The angle nt is of much importance; it is called the *mean anomaly*, and is usually denoted by m.

From the properties of the ellipse $PH/QH = b/a$, whence

$$\text{area } OHP = b \,.\, OHQ/a = b\,(OCQ - HCQ)/a = \tfrac{1}{2}\,ab\,(u - \sin u \cos u).$$

Also

$$\text{area } FHP = b \,.\, QH \,.\, FH/2a = \tfrac{1}{2}\,ab\,(\sin u \cos u - e \sin u),$$

whence $\qquad OFP = OHP + FHP = \tfrac{1}{2}\,ab\,(u - e\sin u),$

and finally

$$m = u - e \sin u \quad \text{.......................(i).}$$

Thus m is expressed in terms of u, and we express v in terms of u as follows:

From the ellipse we see at once

$$r \cos v = a \cos u - ae,$$

$$r \sin v = b \sin u,$$

whence, squaring and adding, we obtain

$$r = a\,(1 - e \cos u) \quad \text{....................(ii).}$$

$$\begin{aligned} 2r \sin^2 \tfrac{1}{2}v = r\,(1 - \cos v) &= a\,(1 - e \cos u - \cos u + e) \\ &= a\,(1 + e)\,(1 - \cos u), \\ 2r \cos^2 \tfrac{1}{2}v = r\,(1 + \cos v) &= a\,(1 - e \cos u + \cos u - e) \\ &= a\,(1 - e)\,(1 + \cos u), \end{aligned}$$

and finally

$$\tan \tfrac{1}{2}v = \sqrt{\frac{1+e}{1-e}} \tan \tfrac{1}{2}u \quad \text{.................(iii).}$$

*[APPLICATION OF LAGRANGE'S THEOREM. If we could eliminate u from (i) and (iii) we should have the relation between m and v, but owing to the transcendental nature of the equations such an elimination in finite terms is impossible. With the help of Lagrange's theorem we may, however, express v in terms of m by a series ascending in powers of e which for given values of m and e will enable us to compute v with any degree of accuracy required.

Lagrange's theorem may be thus stated :—If we are given

$$z = x + y\phi(z) \quad \text{.......................}(a),$$

in which x and y are independent variables, and if $F(z)$ be any function of z, then

$$F(z) = F(x) + y\phi(x)F'(x) + \frac{y^2}{1 . 2}\frac{d}{dx}[\{\phi(x)\}^2 F'(x)] + \dots$$

$$+ \frac{y^n}{1 . 2 \dots n}\frac{d^{n-1}}{dx^{n-1}}[\{\phi(x)\}^n F'(x)] + \text{etc.} \dots \text{(A)},$$

in which $F'(x)$ as usual denotes $\frac{d}{dx}\{F(x)\}$.

To apply this to the case before us we see that, if we write u for z, m for x, e for y, and if we make $\phi(u) = \sin u$, equation (a) is identical with equation (1). If further we write (iii) in the form $v = F(u)$ then we have from equation (A)

$$v = F(u) = F(m) + e \sin m\, F'(m) + \frac{e^2}{\lfloor 2}\frac{d}{dm}\{\sin^2 m F'(m)\}$$

$$+ \frac{e^3}{\lfloor 3}\frac{d^2}{dm^2}\{\sin^3 m\, F'(m)\} + \text{etc.} \dots\dots \text{(B)}.$$

But from equation (iii) we find by a well-known trigonometrical expansion which is proved on p. 160,

$$v = F(u) = u + 2\{c \sin u + \tfrac{1}{2}c^2 \sin 2u + \tfrac{1}{3}c^3 \sin 3u + \text{etc.} \dots\},$$

where $c = \{1 - \sqrt{1 - e^2}\}/e$. Hence

$$F(m) = m + 2\{c \sin m + \tfrac{1}{2}c^2 \sin 2m + \tfrac{1}{3}c^3 \sin 3m + \text{etc.} \dots\},$$

and therefore

$$F'(m) = 1 + 2\{c \cos m + c^2 \cos 2m + c^3 \cos 3m + \text{etc.} \dots\}.$$

Hence all terms on the right-hand side of equation (B) may be evaluated, and thus v may be obtained with any required degree of accuracy. See formula (vii), p. 161.]

KEPLER'S PROBLEM. To effect the solution of equation (i), *i.e.* to determine u when m is known is often called Kepler's problem.

Suppose u_0 is an approximate value of u, which has been arrived at by estimation or otherwise, and let

$$u_0 - e \sin u_0 = m_0.$$

If the true value of u be $u_0 + \Delta u_0$, then by substitution in (i) we have approximately

$$\Delta u_0 = \frac{m - m_0}{1 - e \cos u_0} \dots\dots\dots\dots\dots\dots\dots \text{(iv)}.$$

Cagnoli has shown that this method of approximation is improved if instead of the formula (iv) we use

$$\Delta u_0 = \frac{m - m_0}{1 - e\cos\{u_0 + \frac{1}{2}(m - m_0)\}}.$$

As pointed out by Adams †, both these methods are virtually given by Newton.

Many processes have been employed for the solution of Kepler's problem by the assistance of *graphical methods.* I shall here give one of these graphical solutions, for which I am indebted to Dr Rambaut ‡.

Draw three concentric circles (Fig. 56) with radii respectively

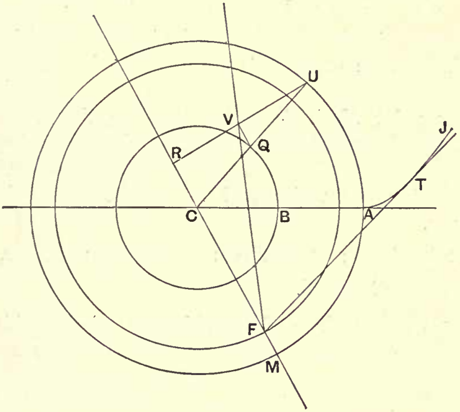


Fig. 56.

$CB = b$, $CF = ae$ and $CM = a$. These circles are referred to as the minor circle, the focal circle, and the major circle respectively. Draw the involute ATJ to the major circle starting from any point A. Let CA be the direction from which the mean anomaly $m = \angle ACM$ is measured. The values of r, u, v corresponding to m can now be found.

Let CM cut the focal circle in F. The normal to the involute at T is a tangent to the major circle: let U be its point of contact: then CU which crosses the minor circle at Q is parallel

† *Collected Works*, Vol. I. p. 291.

‡ Compare *Monthly Notices R.A.S.*, Vol. LXVI. p. 519.

to FT. From the essential property of an involute it follows that the arc AU is equal to UT, but

$$UT = CF \sin FCU = ae \sin FCU,$$

whence $$ae \sin FCU = a(\angle FCU - \angle ACF)$$

which, if we make $\angle FCU = u$, becomes simply

$$m = u - e \sin u.$$

Drawing perpendiculars UR from U on CF and QV from Q on UR we have

$$FV \cos \angle MFV = CU \cos u - CF = a \cos u - ae,$$
$$FV \sin \angle MFV = CQ \sin u \qquad = b \sin u.$$

Thus when the three circles and the involute ATJ have been drawn the solution of Kepler's problem may be summarized as follows:

Take a point M on the major circle so that $\angle ACM = m$. From F the intersection of CM with the focal circle, draw the tangent FT to the involute, and through C draw CQU parallel to FT, cutting the major and minor circles in U and Q respectively.

Then $\angle ACM = m$; $\angle MFV = v$; $\angle MCU = u$; $FV = r$,

and the problem is solved.

*[Tables such as those of Bauschinger† greatly facilitate the solution of the problem of finding u when m and e are given. We illustrate their use in the following question.

Being given the following assumed elements for the orbit of Halley's Comet,

Eccentricity, $e = 0{\cdot}961733$,

Time of Perihelion Passage = 1910, May 24,

Period, $P = 76{\cdot}085$ years,

find the eccentric, and true anomalies of the comet on 1900, May 24.

We have the mean motion equal to $360°/P$, and since the time to perihelion is 10 years we have

$$m = \frac{10 \times 360 \times 60 \times 60''}{76{\cdot}085} = 170335''{\cdot}8 = 47°\ 18'\ 55''{\cdot}8.$$

† *Astronomical Tables* by Bauschinger, published by Engelmann, Leipzig.

Entering Bauschinger's Tables of double entry with the arguments $m = 47°{\cdot}3$ and $e = 0{\cdot}96$ we find the approximate value of the eccentric anomaly

$$u_0 = 101°{\cdot}3.$$

Then from formula (iv) we calculate Δu_0 as follows

$$\begin{aligned}
\text{Log}\sin u_0 &= 9{\cdot}9914984\\
\text{Log}\, e &= 9{\cdot}9830547\\
\log\text{cosec}\, 1'' &= 5{\cdot}3144251\\
\log e\sin u_0 &= 5{\cdot}2889782\\
e\sin u_0 &= 194526''{\cdot}2\\
&= 54°\ 2'\ 6''{\cdot}2\\
u_0 &= 101\ \ 18\ \ 0\ {\cdot}0\\
m_0 &= 47\ \ 15\ 53\ {\cdot}8\\
m &= 47\ \ 18\ 55\ {\cdot}8\\
\therefore\ m - m_0 &= 182''{\cdot}0
\end{aligned}$$

$$\begin{aligned}
\text{Log}\cos u_0 &= 9{\cdot}29214\, n\\
\text{Log}\, e &= 9{\cdot}98305\\
\text{Log}\, e\cos u_0 &= 9{\cdot}27519\, n\\
\therefore\ 1 - e\cos u_0 &= 1{\cdot}1884\\
\log(m - m_0) &= 2{\cdot}26007\\
\log(1 - e\cos u_0) &= 0{\cdot}07496\\
\log\Delta u_0 &= 2{\cdot}18511\\
\Delta u_0 &= 153''{\cdot}15\\
&= 0°\ \ 2'\ \ 33\ {\cdot}15\\
u_0 &= 101\ \ 18\ \ \ 0\ {\cdot}00\\
\therefore\ u_1 &= 101\ \ 20\ \ 33\ {\cdot}15
\end{aligned}$$

This must be very nearly the true value of u. To verify it we proceed to a second approximation:

$$\begin{aligned}
\text{Log}\sin u_1 &= 9{\cdot}9914338\\
\text{Log}\, e &= 9{\cdot}9830547\\
\log\text{cosec}\, 1'' &= 5{\cdot}3144251\\
\log e\sin u_1 &= 5{\cdot}2889136\\
e\sin u_1 &= 194497''{\cdot}31\\
&= 54°\ \ 1'\ 37''{\cdot}31\\
u_1 &= 101\ \ 20\ \ 33\ {\cdot}15\\
m_1 &= 47\ \ 18\ \ 55\ {\cdot}84\\
m &= 47\ \ 18\ \ 55\ {\cdot}80\\
m - m_1 &= -0{\cdot}04
\end{aligned}$$

This small difference is quite negligible, but if it were to be attended to we remark that $1 - e\cos u_1$ will not differ sensibly from $1 - e\cos u_0$ already calculated, and we have

$$\Delta u_1 = \frac{m - m_1}{1 - e\cos u_1} = \frac{m - m_1}{1 - e\cos u_0} = \frac{-0''{\cdot}04}{1{\cdot}2} = -0''{\cdot}03,$$

and thus finally $\qquad u = 101°\,20'\,33''{\cdot}12.$

Having found the eccentric anomaly $u = 101°\,20'\,33''{\cdot}12$ we substitute this in equation (iii) to find v. For this purpose it is convenient to write equation (iii) in the form

$$\tan\tfrac{1}{2}v = \tan(\tfrac{1}{4}\pi + \tfrac{1}{2}\phi)\tan\tfrac{1}{2}u,$$

where $\sin\phi = e$.

Although Bauschinger's Tables are useful as enabling us at once to obtain a good approximation to the required value they are not indispensable. Any of the graphical methods would readily determine u to within three or four degrees of the true value. We may then obtain a value as accurate as that of the tables by the help of four place logarithms. If, for example, we have found $u_0 = 105°$ by a graphical process the next step may be conducted as follows:

$$\begin{aligned}
\text{Log} \sin u_0 &= 9{\cdot}9849 \\
\log e \operatorname{cosec} 1'' &= \underline{5{\cdot}2975} \\
\log e \sin u_0 &= 5{\cdot}2824 \\
e \sin u_0 &= 191600'' \\
&= 53°\ 13'{\cdot}3 \\
u_0 &= 105 \quad 0{\cdot}0 \\
m_0 &= \underline{51 \quad 46{\cdot}7} \\
m &= 47 \quad 18{\cdot}9 \\
m - m_0 &= -4 \quad 27{\cdot}8 \\
&= -4°{\cdot}46
\end{aligned}$$

$$\begin{aligned}
\text{Log} \cos u_0 &= 9{\cdot}4130\, n \\
\text{Log}\, e &= \underline{9{\cdot}9831} \\
\text{Log}\, e \cos u_0 &= 9{\cdot}3961\, n \\
1 - e \cos u_0 &= 1{\cdot}249 \\
\log (m - m_0) &= 0{\cdot}6493\, n \\
\log (1 - e \cos u_0) &= \underline{0{\cdot}0966} \\
\log \Delta u_0 &= 0{\cdot}5527\, n \\
\Delta u_0 &= -3°{\cdot}6 \\
u_0 &= \underline{105{\cdot}0} \\
u_1 &= 101{\cdot}4
\end{aligned}$$

The problems which arise in the majority of cases are those in which the eccentricity is very small; for example in the motion of the earth about the sun the eccentricity is no more than 1/59·7. For such cases it is best to obtain an approximate expression for the sun's true anomaly v in terms of m in the form of a series which need not for most purposes be carried beyond e^3.]

Writing $\sin \phi$ instead of e we have from § 52 (iii)

$$\tan \tfrac{1}{2} v = \tan \tfrac{1}{2} u \,(1 + \tan \tfrac{1}{2}\phi)/(1 - \tan \tfrac{1}{2}\phi),$$

whence if ϵ be the base of Napierian logarithms,

$$(\epsilon^{iv/2} - \epsilon^{-iv/2})/(\epsilon^{iv/2} + \epsilon^{-iv/2})$$
$$= (1 + \tan \tfrac{1}{2}\phi)\,(\epsilon^{iu/2} - \epsilon^{-iu/2})/(1 - \tan \tfrac{1}{2}\phi)\,(\epsilon^{iu/2} + \epsilon^{-iu/2}),$$

or
$$\epsilon^{iv} = \epsilon^{iu}\,(1 - \epsilon^{-iu} \tan \tfrac{1}{2}\phi)/(1 - \epsilon^{iu} \tan \tfrac{1}{2}\phi),$$

and by taking logarithms of both sides

$$v = u + 2\,(\tan \tfrac{1}{2}\phi \sin u + \tfrac{1}{2} \tan^2 \tfrac{1}{2}\phi \sin 2u + \ldots).$$

To express the formula in terms of the eccentricity e, we have

$$\tan \tfrac{1}{2}\phi = (1 - \sqrt{1 - e^2})/e = \tfrac{1}{2} e + \tfrac{1}{8} e^3 + \ldots$$

and by substitution

$$v = u + (e + \tfrac{1}{4} e^3) \sin u + \tfrac{1}{4} e^2 \sin 2u + \tfrac{1}{12} e^3 \sin 3u \ldots\ldots \text{(v)}.$$

It remains to eliminate u between this formula and

$$m = u - e \sin u.$$

As a first approximation

$$u = m + e \sin m.$$

If terms beyond e^2 are neglected,

$$u = m + e \sin(m + e \sin m)$$
$$= m + e \sin m + \tfrac{1}{2}e^2 \sin 2m,$$

and from this we have

$$\sin u = (1 - \tfrac{1}{8}e^2) \sin m + \tfrac{1}{2}e \sin 2m + \tfrac{3}{8}e^2 \sin 3m.$$

By substitution of this in

$$u = m + e \sin u,$$

we find

$$u = m + (e - \tfrac{1}{8}e^3) \sin m + \tfrac{1}{2}e^2 \sin 2m + \tfrac{3}{8}e^3 \sin 3m \quad \text{...(vi)}.$$

We have also to the first power of e,

$$\sin 2u = \sin 2m + e(\sin 3m - \sin m).$$

Introducing these values into (iv) we obtain

$$v = m + (2e - \tfrac{1}{4}e^3) \sin m + \tfrac{5}{4}e^2 \sin 2m + \tfrac{13}{12}e^3 \sin 3m \text{...(vii)}.$$

This is a fundamental equation in astronomy. It gives the true anomaly of a planet in terms of its mean anomaly. It has been here computed to the third power of the eccentricity, but for our present purposes the third power is generally too small to require attention and consequently

$$v = m + 2e \sin m + \tfrac{5}{4}e^2 \sin 2m$$

will be here regarded as a sufficiently accurate formula.

The difference between the true anomaly and the mean anomaly, or $v - m$, is called the *equation of the centre*, and is represented by

$$2e \sin m + \tfrac{5}{4}e^2 \sin 2m.$$

Expression of the mean anomaly in terms of the true. The elementary area swept over by the radius vector when the planet's true anomaly increases by dv is $\tfrac{1}{2}r^2 dv$. If dt be the time required to describe this area, and if T be the periodic time of the planet then from Kepler's second law

$$\tfrac{1}{2}r^2 dv : \pi ab :: dt : T.$$

If dm be the increase in the mean anomaly in the time dt then

$$dm : 2\pi :: dt : T,$$

whence

$$\frac{dm}{dv} = \frac{r^2}{ab} \quad \text{.........................(viii).}$$

This equation may be written thus

$$\frac{dm}{dv} = \frac{(1-e^2)^{\frac{3}{2}}}{(1+e\cos v)^2},$$

whence

$$m = (1-e^2)^{\frac{3}{2}} \int_0^v (1 - 2e\cos v + 3e^2\cos^2 v - 4e^3\cos^3 v)\, dv,$$

and by integration

$$m = v - 2e\sin v + \tfrac{3}{4}e^2 \sin 2v - \tfrac{1}{3}e^3\sin 3v \quad \text{.........(ix),}$$

where powers of e above the third are neglected.

*[*General Expansion.* This series can be obtained as follows. We have from (viii)

$$\frac{dm}{dv} = \cos^2\phi \frac{d}{d\phi}\left(\frac{\sin\phi}{1+\sin\phi\cos v}\right) \text{ where } e = \sin\phi.$$

If we make $x = \epsilon^{iv}$, it is easy to verify that

$$\frac{\sin\phi}{1+\sin\phi\cos v} = \tan\phi\left\{\frac{1}{1+\tan\frac{1}{2}\phi \,.\, x} - \frac{\tan\frac{1}{2}\phi\,.\,x^{-1}}{1+\tan\frac{1}{2}\phi\,.\,x^{-1}}\right\}$$
$$= \tan\phi\,\{1 + 2\Sigma(-1)^k \cos kv \,.\, \tan^k \tfrac{1}{2}\phi\},$$

and hence

$$\frac{dm}{dv} = \cos^2\phi \frac{d}{d\phi}[\tan\phi\,\{1 + 2\Sigma(-1)^k\cos kv\,.\,\tan^k\tfrac{1}{2}\phi\}]$$
$$= 1 + 2\Sigma(-1)^k\cos kv\,.\,\tan^k\tfrac{1}{2}\phi$$
$$+ 2\sin\phi\cos\phi\Sigma(-1)^k\cos kv\,.\,\frac{d}{d\phi}(\tan^k\tfrac{1}{2}\phi)$$
$$= 1 + 2\Sigma(-1)^k\cos kv\,.\,\tan^k\tfrac{1}{2}\phi$$
$$+ 2\sin\phi\cos\phi\Sigma(-1)^k\cos kv\,.\,\tan^k\tfrac{1}{2}\phi\,.\,k\frac{1+\tan^2\frac{1}{2}\phi}{2\tan\frac{1}{2}\phi}$$
$$= 1 + 2\Sigma(-1)^k\cos kv\,.\,\tan^k\tfrac{1}{2}\phi\,(1 + k\cos\phi).$$

Integrating we find

$$m = v + 2\Sigma(-1)^k\frac{\tan^k\frac{1}{2}\phi}{k}(1 + k\cos\phi)\sin kv,$$

the constant of integration being zero since m and v vanish together. The first four terms of this series are

$$m = v - 2\tan\tfrac{1}{2}\phi\,(1+\cos\phi)\sin v$$
$$+ \tan^2\tfrac{1}{2}\phi\,(1+2\cos\phi)\sin 2v$$
$$- \tfrac{2}{3}\tan^3\tfrac{1}{2}\phi\,(1+3\cos\phi)\sin 3v.$$

If powers of e above the third may be neglected we have

$$\phi = e + \tfrac{1}{6}e^3,\ \cos\phi = 1 - \tfrac{1}{2}e^2 \text{ and } \tan\tfrac{1}{2}\phi = \tfrac{1}{2}e + \tfrac{1}{8}e^3$$

and therefore we obtain as before

$$m = v - 2e\sin v + \tfrac{3}{4}e^2\sin 2v - \tfrac{1}{3}e^3\sin 3v.]$$

Ex. 1. Being given that

$$m = v - 2e\sin v + \tfrac{3}{4}e^2\sin 2v - \tfrac{1}{3}e^3\sin 3v,$$

where e is a small quantity of which all powers above the third are neglected, show by reversal of the series that

$$v = m + (2e - \tfrac{1}{4}e^3)\sin m + \tfrac{5}{4}e^2\sin 2m + \tfrac{13}{12}e^3\sin 3m.$$

Ex. 2. Show that the angle between the direction of a planet's motion and the planet's radius vector has as its tangent $\sqrt{1-e^2}/e\sin u$.

Ex. 3. If the eccentricity $\sin\phi$ be very nearly unity, show that the mean anomaly m can be expressed in terms of the true anomaly v by the following formula in which $x = \tan\frac{1}{2}v$,

$$m = \frac{2\cos^3\phi}{(1+\sin\phi)^2}\left(x + \frac{x^3}{3} - 2\,\frac{1-\sin\phi}{1+\sin\phi}\left(\frac{x^3}{3} + \frac{x^5}{5}\right)\right).$$

Ex. 4. Prove the following graphical method given by J. C. Adams† of solving for u from the equation $m = u - e\sin u$.

Draw the curve of sines $y = \sin x$. From the origin O measure $OM = m$ along the axis of x. Through M draw a line inclined to the axis of x at the angle $\cot^{-1}e$, and let P be the point in which it cuts the curve, then u is the abscissa of P.

Ex. 5. Prove Leverrier's rule for the solution of $m = u - e\sin u$ if powers of e above the third may be neglected,

$$u = m + \frac{e\sin m}{1 - e\cos m} - \tfrac{1}{2}\left(\frac{e\sin m}{1 - e\cos m}\right)^3.$$

Ex. 6. If θ' be the longitude of a planet measured from an apse round the empty focus, prove that

$$\theta' = nt + \tfrac{1}{4}e^2\sin 2nt,$$

if powers of e above the second be neglected.

† See also Routh, *Dynamics of a particle*, p. 225, and *Monthly Notices R.A.S.*, Vol. L. p. 301.

*Ex. 7. If $C(\theta)$ is an abbreviation for $\frac{1}{2}(\theta+\cos\theta)$ show that the equation $m=u-e\sin u$ may be written $m=C(\phi+u)-C(\phi-u)$ if $e=\sin\phi$, and show how a table of the values of $C(\theta)$ will facilitate the solution of Kepler's problem.

See the paper by Mr Aldis in *Monthly Notices R.A.S.* Vol. LXII. p. 633, where the table is given with illustrations of its use.

53. Formulæ of elliptic motion expressed by quadratures.

We take ϖ to be the longitude of the perihelion of the planet measured from a fixed direction in the plane of the orbit, θ the longitude of the planet and $v=(\theta-\varpi)$ the true anomaly. The quantity b^2/a is represented by p. The periodic time is P.

From the properties of the ellipse we have for the radius vector r,

$$r=\frac{p}{1+e\cos(\theta-\varpi)} \quad \text{.....................(i).}$$

For any body moving round the sun, we have (§ 50)

$$r^2 d\theta/dt=\sqrt{\mu p} \quad \text{........................(ii).}$$

Solving (ii) for dt, substituting for r from (i) and integrating, we have

$$t=\frac{p^{\frac{3}{2}}}{\sqrt{\mu}}\int_{\varpi}^{\theta}\frac{d\theta}{\{1+e\cos(\theta-\varpi)\}^2} \quad \text{...........(iii),}$$

where t is the time in which the planet moves from perihelion to the true anomaly $v=(\theta-\varpi)$ in an orbit of which e is the eccentricity and p the semi-latus rectum. The equation may also be written in the homogeneous form

$$\frac{t}{P_0}=\frac{p^{\frac{3}{2}}}{2\pi a_0^{\frac{3}{2}}}\int_0^v\frac{1}{(1+e\cos v)^2}\,dv,$$

where a_0, P_0 are the mean distance and the periodic time respectively of the earth.

Differentiating (i) with regard to t we obtain

$$\frac{dr}{dt}=\frac{pe\sin(\theta-\varpi)}{\{1+e\cos(\theta-\varpi)\}^2}\frac{d\theta}{dt}=e\sin(\theta-\varpi)\frac{r^2}{p}\frac{d\theta}{dt}=e\sqrt{\mu}\sin(\theta-\varpi)/\sqrt{p},$$

also

$$r\frac{d\theta}{dt}=\sqrt{\mu}\,\{1+e\cos(\theta-\varpi)\}/\sqrt{p},$$

and for the square of the velocity of the planet

$$\left(\frac{dr}{dt}\right)^2+r^2\left(\frac{d\theta}{dt}\right)^2=\mu\{1+2e\cos(\theta-\varpi)+e^2\}/p=2\mu/r-\mu/a,$$

which may be expressed more conveniently for calculation in the homogeneous form

$$\frac{4\pi^2 a_0^3}{P_0^2}\left(\frac{2}{r}-\frac{1}{a}\right).$$

In the case of a parabolic orbit such as that in which the great majority of comets revolve, $e=1$ and $a=\infty$, so that the formulæ (i) and (iii) become

$$\left.\begin{aligned} r &= \tfrac{1}{2}p\sec^2\tfrac{1}{2}(\theta-\varpi) \\ t &= \frac{P_0 p^{\frac{3}{2}}}{4\pi a_0^{\frac{3}{2}}}\left\{\tan\tfrac{1}{2}(\theta-\varpi)+\tfrac{1}{3}\tan^3\tfrac{1}{2}(\theta-\varpi)\right\}\end{aligned}\right\}\ldots\text{(iv)}.$$

The result at which we have arrived may be thus stated. Let P_0, a_0 be respectively the periodic time and the mean distance of any planet, for example, the earth. If the semi-latus rectum of the parabolic orbit of a comet be p, then the time in which the comet passes from perihelion to the true anomaly v is

$$P_0 p^{\frac{3}{2}}(\tan\tfrac{1}{2}v+\tfrac{1}{3}\tan^3\tfrac{1}{2}v)/4\pi a_0^{\frac{3}{2}}.$$

Euler's Theorem. A remarkable property of parabolic motion is expressed in *Euler's theorem*, which is thus enunciated.

If r and r' be the radii vectores from the sun to two points C and C' in the parabolic orbit of a comet, and if k be the distance CC', the time required by the comet to move from C to C' is

$$\frac{P_0}{12\pi}\left\{\left(\frac{r+r'+k}{a_0}\right)^{\frac{3}{2}}-\left(\frac{r+r'-k}{a_0}\right)^{\frac{3}{2}}\right\}$$

where P_0 is the length of the sidereal year and a_0 the earth's mean distance.

For brevity we make

$$q=P_0p^{\frac{3}{2}}/4\pi a_0^{\frac{3}{2}};\ x=\tan\tfrac{1}{2}v;\ x'=\tan\tfrac{1}{2}v';\ s=(r+r'+k)/2;$$

then

$$\begin{aligned} t'-t &= q\left\{x'-x+\tfrac{1}{3}(x'^3-x^3)\right\} \\ &= \tfrac{1}{3}q(x'-x)(1+x^2+1+x'^2+1+xx'). \end{aligned}$$

But from the properties of the parabola

$$1+x^2=2r/p;\quad 1+x'^2=2r'/p;$$

$$1+xx'=\sec\tfrac{1}{2}v\sec\tfrac{1}{2}v'\cos\tfrac{1}{2}(v'-v)=\frac{2\sqrt{rr'}}{p}\sqrt{\frac{s(s-k)}{rr'}},$$

whence

$$\begin{aligned} t'-t &= 2q(x'-x)(r+r'+\sqrt{s\,.\,s-k})/3p \\ &= 2q(x'-x)\{s+s-k+\sqrt{s\,.\,s-k}\}/3p. \end{aligned}$$

But we have also

$$(x'-x)^2 = 1 + x^2 + 1 + x'^2 - 2(1 + xx')$$
$$= 2\{s + s - k - 2\sqrt{s(s-k)}\}/p$$
$$= 2\{\sqrt{s} - \sqrt{s-k}\}^2/p,$$

whence
$$t' - t = 2q\sqrt{2}\{s^{\frac{3}{2}} - (s-k)^{\frac{3}{2}}\}/3p^{\frac{3}{2}}$$
$$= q\{(r + r' + k)^{\frac{3}{2}} - (r + r' - k)^{\frac{3}{2}}\}/3p^{\frac{3}{2}},$$

and by restoring the value of q the desired result is obtained.

*LAMBERT'S THEOREM. An important extension of Euler's theorem for motion in a parabola to the more general case of motion in an ellipse is given by *Lambert*, and may be stated as follows.

If t is the time occupied by the planet in moving from the position indicated by the radius vector r to the position indicated by the radius vector r', and if k is the chord between the two positions, then

$$2\pi t/P = (\eta - \sin\eta) - (\eta' - \sin\eta'),$$

where

$$\sin\tfrac{1}{2}\eta = \tfrac{1}{2}\sqrt{\frac{r + r' + k}{a}}; \quad \sin\tfrac{1}{2}\eta' = \tfrac{1}{2}\sqrt{\frac{r + r' - k}{a}};$$

and P is the periodic time of the planet.

We have†

$$r = a(1 - e\cos u), \quad r' = a(1 - e\cos u'),$$
$$k^2 = a^2(\cos u - \cos u')^2 + a^2(1 - e^2)(\sin u - \sin u')^2,$$
$$2\pi t/P = u - u' - e(\sin u - \sin u'),$$
$$= u - u' - 2e\sin\tfrac{1}{2}(u - u')\cos\tfrac{1}{2}(u + u'),$$

whence

$$(r + r')/2a = 1 - e\cos\tfrac{1}{2}(u + u')\cos\tfrac{1}{2}(u - u'),$$
$$k^2/4a^2 = \sin^2\tfrac{1}{2}(u - u')\{1 - e^2\cos^2\tfrac{1}{2}(u + u')\},$$
$$2\pi t/P = u - u' - 2e\cos\tfrac{1}{2}(u + u')\sin\tfrac{1}{2}(u - u').$$

We thus see that if a and therefore P are known then $(r + r')$, k, and t are functions of the two quantities $u - u'$ and $e\cos\frac{1}{2}(u + u')$.

† The proof here given is due to Adams, *Collected Papers*, Vol. I. p. 411. See also Routh, *Dynamics of a particle*, p. 228.

Let us now make

$$u - u' = 2\alpha \text{ and } e\cos\tfrac{1}{2}(u + u') = \cos\beta,$$

then $\quad (r + r')/2a = 1 - \cos\alpha\cos\beta;\ \ k/2a = \sin\alpha\sin\beta,$

therefore $\quad (r + r' + k)/2a = 1 - \cos(\beta + \alpha),$

$$(r + r' - k)/2a = 1 - \cos(\beta - \alpha),$$

also $\quad 2\pi t/P = 2\alpha - 2\sin\alpha\cos\beta$

$$= \{\beta + \alpha - \sin(\beta + \alpha)\} - \{\beta - \alpha - \sin(\beta - \alpha)\},$$

whence making $\beta + \alpha = \eta$ and $\beta - \alpha = \eta'$ we have Lambert's theorem as enunciated.

Ex. 1. Show that m the mean anomaly in an elliptic orbit of which the mean distance is a may be variously expressed as follows

$$m = 2\pi a_0^{\frac{3}{2}} t/a^{\frac{3}{2}} P_0 = 2\pi a_0^{\frac{3}{2}} (1 - e^2)^{\frac{3}{2}} t/p^{\frac{3}{2}} P_0 = \sqrt{\mu} t/a^{\frac{3}{2}}.$$

where a_0, P_0 are respectively the mean distance of the earth from the sun and the length of the sidereal year.

*Ex. 2. If m be the mean anomaly, v the true anomaly, and e the eccentricity, show that

$$\tfrac{1}{2}m = (1 - e)\left(\frac{1 - e}{1 + e}\right)^{\frac{1}{2}} \tan\tfrac{1}{2}v - \frac{1 - 3e}{3}\left(\frac{1 - e}{1 + e}\right)^{\frac{3}{2}} \tan^3\tfrac{1}{2}v$$
$$+ \frac{1 - 5e}{5}\left(\frac{1 - e}{1 + e}\right)^{\frac{5}{2}} \tan^5\tfrac{1}{2}v - \&c. \ldots\ldots,$$

and transform this equation into

$$\frac{\sqrt{\mu}t}{2p^{\frac{3}{2}}} = \frac{1}{(1 + e)^2}\tan\tfrac{1}{2}v - \frac{1 - 3e}{3(1 + e)^3}\tan^3\tfrac{1}{2}v$$
$$+ \frac{1 - 5e}{5}\,\frac{1 - e}{(1 + e)^4}\tan^5\tfrac{1}{2}v - \&c. \ldots\ldots.$$

[Edinburgh Degree Examination, 1907.]

We have $\quad m = u - e\sin u$

$$= 2\left\{\tan^{-1}\left(\sqrt{\frac{1 - e}{1 + e}}\tan\tfrac{1}{2}v\right) - e\,\frac{\sqrt{\dfrac{1 - e}{1 + e}}\tan\tfrac{1}{2}v}{1 + \dfrac{1 - e}{1 + e}\tan^2\tfrac{1}{2}v}\right\}.$$

For $\sqrt{\dfrac{1 - e}{1 + e}}\tan\tfrac{1}{2}v$ write λ and we have

$$\tfrac{1}{2}m = \tan^{-1}\lambda - e\frac{\lambda}{1 + \lambda^2}$$
$$= \lambda - \tfrac{1}{3}\lambda^3 + \tfrac{1}{5}\lambda^5 - \&c. \ldots\ldots - e\lambda\{1 - \lambda^2 + \lambda^4 - \lambda^6 + \&c. \ldots\ldots\}$$
$$= (1 - e)\lambda - \frac{1 - 3e}{3}\lambda^3 + \frac{1 - 5e}{5}\lambda^5 - \&c. \ldots\ldots,$$
$$= (1 - e)\left(\frac{1 - e}{1 + e}\right)^{\frac{1}{2}}\tan\tfrac{1}{2}v - \frac{1 - 3e}{3}\left(\frac{1 - e}{1 + e}\right)^{\frac{3}{2}}\tan^3\tfrac{1}{2}v$$
$$+ \frac{1 - 5e}{5}\left(\frac{1 - e}{1 + e}\right)^{\frac{5}{2}}\tan^5\tfrac{1}{2}v - \&c. \ldots\ldots$$

But we have

$$n=\frac{\sqrt{\mu}}{a^{\frac{3}{2}}}=(1-e^2)^{\frac{3}{2}}\frac{\sqrt{\mu}}{p^{\frac{3}{2}}}, \quad \therefore \quad m=(1-e^2)^{\frac{3}{2}}\frac{\sqrt{\mu}}{p^{\frac{3}{2}}}.t.$$

Hence

$$\frac{\sqrt{\mu}t}{2p^{\frac{3}{2}}}=\frac{1}{(1+e)^2}\tan\tfrac{1}{2}v-\frac{1-3e}{3(1+e)^3}\tan^3\tfrac{1}{2}v+\frac{1-5e}{5}\frac{1-e}{(1+e)^4}\tan^5\tfrac{1}{2}v-\&c.\ldots\ldots$$

This equation corresponds for the ellipse or hyperbola to equation (iv) of § 53 for the parabola. If we put $e=1$ in this expression we get simply

$$\sqrt{\mu}t=\tfrac{1}{2}p^{\frac{3}{2}}\{\tan\tfrac{1}{2}v+\tfrac{1}{3}\tan^3\tfrac{1}{2}v\},$$

since all terms after the second have $(1-e)$ as a factor.

Ex. 3. Show that the time spent by a comet within the earth's orbit is $\sqrt{2}(1-m)^{\frac{1}{2}}(1+2m)/3\pi$ parts of a year where m is the perihelion distance of the comet the unit being the earth's heliocentric distance regarded as constant. The orbit of the comet is presumed to be parabolic, and in the plane of the ecliptic.

As $p=2m$ we find for the time from perihelion to a true anomaly v

$$m^{\frac{3}{2}}(\tan\tfrac{1}{2}v+\tfrac{1}{3}\tan^3\tfrac{1}{2}v)/\pi\sqrt{2},$$

we have also

$$r=m\sec^2\tfrac{1}{2}v,$$

so that $\cos^2\frac{1}{2}v=m$ will determine the true anomaly of the point where the comet crosses the earth's orbit. Hence substituting for $\tan\frac{1}{2}v$ we have

$$\frac{m^{\frac{3}{2}}}{\pi\sqrt{2}}\left\{\sqrt{\frac{1-m}{m}}+\tfrac{1}{3}\left(\frac{1-m}{m}\right)^{\frac{3}{2}}\right\},$$

as the time from the earth's orbit to perihelion, and double this period gives the answer to the question.

This expression has its greatest value $2/3\pi$ when $m=1/2$.

*Ex. 4. Two planets are moving in coplanar orbits. Show that when these planets are nearest to each other their longitudes θ and θ' must satisfy the two following equations:

$$T+p^{\frac{3}{2}}\int_{\varpi}^{\theta}\frac{d\theta}{(1+e\cos(\theta-\varpi'))^2}-T'-p'^{\frac{3}{2}}\int_{\varpi'}^{\theta'}\frac{d\theta'}{(1+e'\cos(\theta'-\varpi'))^2}=0,$$

and

$$\frac{e}{\sqrt{p}}\sin(\theta-\varpi)\{r-r'\cos(\theta-\theta')\}+\frac{e'}{\sqrt{p'}}\sin(\theta'-\varpi')\{r'-r\cos(\theta-\theta')\}$$
$$+\sin(\theta-\theta')\left(\sqrt{p}\frac{r'}{r}-\sqrt{p'}\frac{r}{r'}\right)=0,$$

in which T and T' are the epochs at which the planets pass through perihelion.

The first equation merely expresses that the planets have the longitudes θ and θ' at the same moment.

To find the second equation we note that $r^2-2rr'\cos(\theta-\theta')+r'^2$ is to be a minimum whence

$$r\frac{dr}{dt}-r'\frac{dr}{dt}\cos(\theta-\theta')-r\frac{dr'}{dt}\cos(\theta-\theta')+r'\frac{dr'}{dt}$$
$$+rr'\sin(\theta-\theta')\left(\frac{d\theta}{dt}-\frac{d\theta'}{dt}\right)=0.$$

Substituting $\frac{dr}{dt}=\frac{e}{\sqrt{p}}\sin(\theta-\varpi)$, $\frac{d\theta}{dt}=\frac{\sqrt{p}}{r^2}$ we obtain the second equation.

If e and e' are both small, θ and θ' are nearly equal and the second equation may be written

$$(a-a')\{e\sqrt{a'}\sin(\theta-\varpi)-e'\sqrt{a}\sin(\theta'-\varpi')\}+(a'^{\frac{3}{2}}-a^{\frac{3}{2}})\sin(\theta-\theta')=0.$$

*Ex. 5. Show that the distance of the earth from a planet, whose orbit is in the ecliptic, will not in general be a minimum when the planet is in opposition unless the earth is at one or other of two points in her orbit, but that if the perihelia of the two orbits have the same heliocentric longitude and the latera recta are in the duplicate ratio of the eccentricities, the distance will be a minimum at every opposition.

[Math. Trip. I. 1900.]

This may be deduced from the last question or obtained otherwise as follows.

Let P, Q Fig. 57 be the simultaneous positions of the two planets at the time t and P', Q' their positions at the time $t+dt$. If then PQ is a minimum or maximum we must have $PQ=P'Q'$ whence

$$PP'\cos P'PN=QQ'\cos Q'QN.$$

Let $\angle AFP=\theta$, $\angle AFQ=\theta'$, $\angle ANP=\omega$, $\angle FPP'=\phi$, $FQQ'=\phi'$, $FP=r$, $FQ=r'$,

then

$$PP'\cos\phi=-dr,\quad PP'\sin\phi=rd\theta,$$

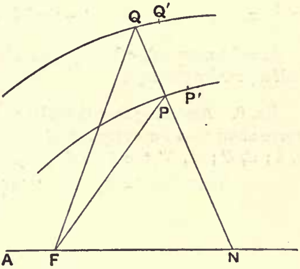


FIG. 57.

whence

$$PP'\cos P'PN=PP'\cos(\phi-\theta+\omega)=-dr\cos(\theta-\omega)+rd\theta\sin(\theta-\omega)$$
$$=\{-e\sin(\theta-\varpi)\cos(\theta-\omega)/\sqrt{p}+\sqrt{p}\sin(\theta-\omega)/r\}\sqrt{\mu}\,dt$$
$$=\{e\sin(\varpi-\omega)/\sqrt{p}+\sin(\theta-\omega)/\sqrt{p}\}\sqrt{\mu}\,dt.$$

If therefore $PQ=P'Q'$ we must have

$$e\sin(\varpi-\omega)/\sqrt{p}+\sin(\theta-\omega)/\sqrt{p}=e'\sin(\varpi'-\omega)/\sqrt{p'}+\sin(\theta'-\omega)/\sqrt{p'}.$$

If $\theta=\theta'=\omega$ the planet is in opposition, and we have

$$e\sin(\varpi-\theta)/\sqrt{p}=e'\sin(\varpi'-\theta)/\sqrt{p'}.$$

Hence there are two values for θ differing by 180°. This is the first part of the question. Also if $\varpi=\varpi'$ and $e/\sqrt{p}=e'/\sqrt{p'}$, the condition is satisfied at every opposition.

Ex. 6. Show how Euler's theorem for the time of describing the arc of a parabola can be deduced from Lambert's theorem.

In this case both β and α will become indefinitely small.

Ex. 7. The sun passed through the first point of Aries on March 20, 1898, at $2^h\ 5^m$, and through the first point of Libra on Sept. 22 at $12^h\ 35^m$; show that the interval is consistent with the facts that the eccentricity of the earth's orbit is about 1/60, and that the apse line is nearly at right angles to the line of equinoxes. [Coll. Exam.]

If the sun is at an Equinoctial point and if e^2 is negligible it can easily be shown that

$$\sin(\varpi+u)=e\sin\varpi,$$

whence we have the two values of u, namely $e\sin\varpi-\varpi$ and $\pi-\varpi-e\sin\varpi$.

If t_1 and t_2 be the times of passage through ♈ and ♎ respectively, T the time of passing through perihelion and P the length of the year,

$$2\pi\frac{t_1-T}{P}=e\sin\varpi-\varpi-e\sin(e\sin\varpi-\varpi),$$

$$2\pi\frac{t_2-T}{P}=\pi-\varpi-e\sin\varpi-e\sin(\varpi+e\sin\varpi),$$

whence

$$t_2-t_1=\tfrac{1}{2}P-\frac{2}{\pi}Pe\sin\varpi.$$

If ϖ be near 90°, e be $\frac{1}{60}$, and $P=365\frac{1}{4}$, we see that t_2-t_1 differs from half a year by 3·8 days.

Ex. 8. Assuming that the orbit of the earth relative to the sun is a plane curve show that for every set of three observations of the solar coordinates α, δ; α', δ'; α'', δ'' the following equation is satisfied

$$\tan\delta\sin(\alpha'-\alpha'')+\tan\delta'\sin(\alpha''-\alpha)+\tan\delta''\sin(\alpha-\alpha')=0.$$

[Coll. Exam.]

CHAPTER VIII.

PRECESSION AND NUTATION.

54. How luni-solar precession is observed. The important phenomenon which we know as the precession of the equinoxes is most easily made apparent when the observed right ascension and declination of any fixed star at one epoch are compared with the observed right ascension and declination of the same star at a later epoch sufficiently distant from the earlier one. For example the coordinates of Polaris (the Pole Star) were determined as follows:

$$\text{Polaris 1st Jan. 1850}\left\{\begin{array}{llll}\text{R.A.} & 1^h & 5^m & 23^s \\ \delta & +88^\circ & 30' & 49''.\end{array}\right.$$

These are now to be compared with the coordinates of the same star as determined 50 years later:

$$\text{Polaris 1st Jan. 1900}\left\{\begin{array}{llll}\text{R.A.} & 1^h & 23^m & 0^s \\ \delta & 88^\circ & 46' & 53''.\end{array}\right.$$

The differences between these two sets of coordinates amounting to more than a quarter of an hour in right ascension and more than a quarter of a degree in declination must receive close attention.

At first it might appear that the change in the apparent position of Polaris must be attributed to actual movements of that star. But we can show that the phenomena cannot be thus explained. Changes in the coordinates of a point may be caused by changes in the axes with regard to which those coordinates are measured as well as by absolute changes in the position of the point itself. We have to show that the changes in the place of Polaris are only apparent. They are to be attributed to changes not in the place of the star but in the place of the great circle with reference to which the star's place is determined. These changes are due to the phenomena known as *Precession* and *Nutation*.

Consider first the declination of Polaris which in the course of half a century is shown by observation to have increased no less than 16′ 4″, or at the average rate of 19″ annually. This means that the distance between the Pole and Polaris has been diminishing 19″ annually. It follows that either the Pole or Polaris, or both, must be in movement.

But no appreciable portion of the change in the polar distance of Polaris can be attributed to the proper motion (§ 60) of that star. Measurements of the distance of Polaris from other neighbouring stars show no variation comparable with that in the distance from Polaris to the Pole. Any true proper motion which Polaris may possess is far too small to account for the changes observed in its declination. It is also to be noticed that while, in the course of fifty years, other stars generally exhibit large changes in their polar distances they do not show considerable changes in their distances from each other. We are thus led to the conclusion that the changes in the distance between Polaris and the Pole are not to be attributed to the movement of Polaris itself, but to a movement of the Celestial Pole, and we have now to study the character of this movement.

If the Pole shifts its position continually on the celestial sphere the celestial equator must also be in constant motion, because under all circumstances every point on the equator must be 90° from the Pole. But though the equator moves yet it always preserves the same average inclination to the ecliptic. The angle only fluctuates some seconds to one side or the other of its mean value. The declination of the sun at midsummer is the obliquity of the ecliptic, and this was practically the same in 1850 as in 1900 (see p. 187). Hence we see that the equator must move so that it cuts the ecliptic, regarded as fixed, at a nearly constant angle, while the equinoctial points move along the ecliptic in the opposite direction to the earth's motion. The pole of the ecliptic may be regarded as fixed on the celestial sphere, and the motion with which we are at present concerned causes the pole of the equator to describe a small circle around the pole of the ecliptic. This is the movement which is known as the *luni-solar precession of the equinoxes.* It manifests itself most simply by a continuous increase in the longitude of a star, while the star's latitude remains unaltered. In general luni-solar precession produces change both in the declination and the right ascension of a celestial body.

55. Physical explanation of luni-solar precession and nutation.

The direction of the axis about which the earth performs its diurnal rotation undergoes very slow changes, and these changes produce the phenomena of precession and nutation. The disturbance of the earth's axis from the constant direction it would otherwise retain is due to the fact that the resultant attraction of an external body (moon or sun) on a spheroidal body like the earth is not a single force through the centre of gravity of the earth.

If the earth were a truly spherical rigid body, and if the density along the surface of each internal concentric spherical shell was constant, then the attraction of any external body, such as the moon or the sun, would be equivalent to a force acting at the centre of the sphere. A force whose line of action passes through the centre of gravity of the body on which it acts would be without effect on the rotation of the body about its centre of gravity. But under the conditions existing in the solar system the attraction of neither sun nor moon passes in general through the earth's centre of gravity Hence arise those disturbances of the earth's rotation which we are now to consider.

Although for reasons indicated later, the moon is more effective than the sun in producing precession, we consider first the effect of the sun because its motion relative to the earth is simpler than that of the moon.

If we assume that the earth is a solid of revolution and symmetrical about the equator, then *NS*, Fig. 57, being its axis and C its centre, P any external particle, the plane NSP divides the earth symmetrically, and therefore the resultant attraction of P on the earth lies in the plane NSP: further if P be in the plane of the equator, the resultant attraction will also be in that plane. Hence if P lies in the equatorial plane AB the resultant attraction will be along CP.

If P were in the axis NS (a case we need not consider) it is clear that the resultant attraction would be along CP, but for any other position of P, such as that indicated in Fig. 57, it can be shown that the resultant attraction does not pass through C but along a line such as HP in the plane NSP.

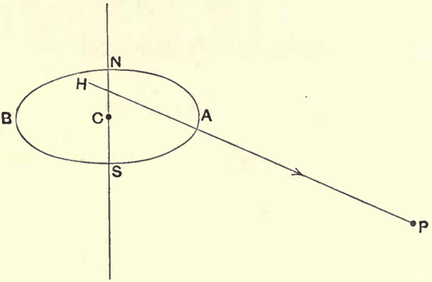


FIG. 57.

At first sight it might seem as if this force would tend towards turning NS into a direction perpendicular to HP, in other words as the sun is the attracting body, the immediate effect would seem to be to force the earth's equator towards the ecliptic. But the fact that the earth is in rapid rotation produces the apparently paradoxical effect that the axis NS at each instant moves in a direction not in the plane NSP but at right angles thereto.

This is a phenomenon well illustrated by the common pegtop, though in this case we are dealing not with the rotation about the centre of gravity of a body free in space, but with rotation about a fixed point—which is mathematically a very similar problem. While the pegtop is in rapid rotation about its axis of symmetry that axis is itself slowly describing a cone round the vertical. Thus the axis of the pegtop is at each instant moving in a direction at right angles to that in which the force of gravity would appear to urge it and which it is only prevented from following by the fact that the top has a rotation about its axis very much more rapid than the conical motion of the axis itself.

The diurnal rotation of the earth appears very rapid when compared with the conical motion of the earth's axis inasmuch as the period of the latter is about 24,500 years. Applying the analogy of the conical rotation of the axis of the pegtop to the case of the rotation of the earth as disturbed by the sun we should expect to find that the terrestrial axis NS would slowly describe a right circular cone about the normal to the plane of the ecliptic.

The precessional action of the moon is more important than that of the sun, for though the total attraction of the moon on the earth is very much less than that of the sun, yet as the precessional effect depends upon the *difference between the attractions exercised by the disturbing body on different parts of the earth* the greater proximity of the moon raises its precessional effect to double that of the sun.

The plane of the moon's orbit is very near the ecliptic, being inclined thereto only at the small angle of 5°, and the moon's orbit while preserving this inclination is in continuous motion, so that each of its nodes accomplishes a complete circuit of the ecliptic in about 19 years, a very small quantity in comparison with the precessional period of 26,000 years. As the moon is always near the ecliptic and is as much below the ecliptic as above, and as the average position of its orbit coincides with the ecliptic, it follows that the principal part of the moon's precessional action is of the same general tendency as that of the sun. The sun's action and this part of the moon's action together constitute what is called *luni-solar precession* by which ♈ moves on the ecliptic in the opposite direction to increasing longitudes at the rate of $50\frac{1}{4}''$ annually. About two-thirds of this quantity is due to the action of the moon and the remainder to that of the sun. The obliquity of the ecliptic ω remains unaltered by luni-solar precession.

But the moon has also an important influence from the circumstance that its movement, though near the ecliptic, is not exactly in that plane. The precessional action of the moon tends to make the axis of the earth describe a cone round the pole of the lunar orbit, which is itself describing a circle of radius 5° about the pole of the ecliptic. The influence of this on the plane of the equator is twofold. It gives to ♈ a small periodic movement of oscillation to and fro on the ecliptic about its mean place as determined by luni-solar precession. It also gives to ω a small oscillation to and fro about its mean value. These phenomena are known as *nutation*, and their discovery was one of Bradley's great achievements. The sun has also some effect in producing nutation, but it is very small compared with that of the moon.

*56. **Planetary precession.** The luni-solar precession and nutation relate, as we have seen, to change in the relative position of the equator and the ecliptic due to the motion of the former. We have now to learn that the ecliptic is itself not quite a fixed plane, and its changes have to be taken into account, though these

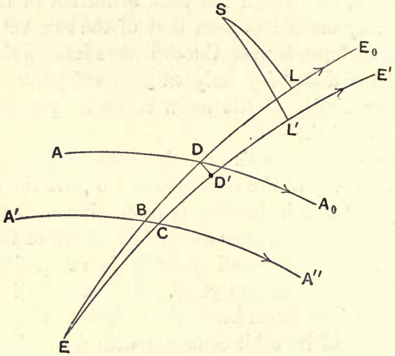


FIG. 58 (after Brunnow).

changes are so small that they may for many purposes be regarded as non-existent and the ecliptic be treated as absolutely fixed.

The movements of the ecliptic are due to the attractions of the other planets on the earth. The irregularity thus caused in the positions of the equinoctial points is accordingly known as *Planetary precession*†.

We must take some standard position of the ecliptic to which its position at other dates shall be referred and we use for this purpose the great circle with which the ecliptic coincided at the beginning of the year 1850, EE_0, Fig. 58. Let EE' be

† The reader who desires to learn more about Planetary Precession than is contained in the slight sketch here given is referred to Newcomb's *Ccmpendium of Spherical Astronomy*, from which source the numerical values here used have been obtained.

the position of the ecliptic in the year $1850 + t$. Let AA_0 be the equator at the commencement of 1850 and let the equator have moved by luni-solar precession to $A'A''$ at the time $1850 + t$. Let SL and SL' be perpendiculars from a star S on EE_0 and EE'. Let DD' be drawn perpendicular to EE' from the intersection of EE_0 and AA_0. Then we have the following statements.

BD is the *Luni-solar Precession* in t years,

$\angle D'CA''$ is the obliquity of the *true* ecliptic in $1850 + t$,

$\angle DBA''$ is the obliquity of the *fixed* ecliptic in $1850 + t$.

BC being the distance along the equator through which the node has been shifted by the motion of the ecliptic in t years is known as the *Planetary Precession* and its magnitude has been found to be $0''\cdot13t$.

CD' is the *General Precession* in longitude. It is the displacement of the intersection of the equator with the apparent ecliptic on the latter, and its annual increase at the date $1850 + t$ is

$$50''\cdot2453 + 0''\cdot0002225t.$$

This quantity is known as the *constant of precession.* It changes with extreme slowness, thus in 1900 its value is $50''\cdot2564$ and in 1950 it is $50''\cdot2675$. It will be accurate enough for our purposes to take the present constant of precession as $50''\cdot26$.

At the date $1850 + t$ the angle between the equator and the ecliptic of the same date (neglecting periodic terms) is

$$23° \, 27' \, 32''\cdot0 - 0''\cdot47t,$$

and the second term is called the *secular* change in the obliquity. Observing the directions of the arrow heads we see that E is the *descending* node of the true ecliptic on the fixed ecliptic and consequently $180° - EC$ is the longitude of the *ascending* node of the true ecliptic on the fixed ecliptic.

The longitude of the star S which was DL in 1850 becomes BL in $1850 + t$ by the *luni-solar* precession. The latitude of S, or SL, is unaltered by the luni-solar precession.

If planetary precession as well as luni-solar precession be considered then the longitude of S, which was DL at the date 1850, becomes CL' at the date $1850 + t$, and in like manner the latitude changes from SL to SL'.

57. General formulae for precession and nutation in right ascension and declination.

We shall generally assume that the plane of the ecliptic is unchanged, and that the position of the equator with respect to the ecliptic is subjected to slow changes due to precession and nutation in the only ways in which a great circle of the sphere can change, *i.e.* the line of nodes changes and the obliquity of the ecliptic also changes. Any alterations in the great circles with respect to which the coordinates of a star are measured necessarily involve changes in those coordinates even though, as we shall at present suppose, there is actually no change in the place of the star on the celestial sphere.

Consider two positions of the equator: the first cutting the ecliptic at an equinoctial point ♈, with obliquity ω, the second cutting the ecliptic at an equinoctial point ♈′, which has moved along the ecliptic through an arc k, in the direction of diminishing longitude, while the obliquity has changed from ω to ω' (Fig. 59).

Let α and δ be the R.A. and declination of a star S referred to the first equator and equinox (System I.); and let α' and δ' be the coordinates of the same star referred to the second equator and equinox (System II.).

Let α_1, δ_1 and α_1', δ_1' be the corresponding coordinates of a second star S' referred to the two systems.

Then since the length of the arc SS' is the same whichever system of coordinates be used, we have the fundamental equation as used in § 12,

$$\sin\delta\sin\delta_1 + \cos\delta\cos\delta_1\cos(\alpha-\alpha_1)$$
$$= \sin\delta'\sin\delta_1' + \cos\delta'\cos\delta_1'\cos(\alpha'-\alpha_1').$$

We shall now introduce into this equation three cases in which α_1, δ_1 and α_1', δ_1' are at once evident and thus obtain the three equations of transformation.

If the second star S' is at ♈ its coordinates in System I. are

$$\alpha_1 = 0, \quad \delta_1 = 0.$$

The coordinates of the same star in System II. are given by the equations

$$\sin\delta_1' = \sin k \sin\omega',$$
$$\cos\delta_1'\sin\alpha_1' = \sin k\cos\omega',$$
$$\cos\delta_1'\cos\alpha_1' = \cos k.$$

And making these substitutions in the fundamental equation we have

$$\cos\delta\cos\alpha = \sin k \sin\omega' \sin\delta' + \cos k \cos\delta' \cos\alpha' + \sin k \cos\omega' \cos\delta' \sin\alpha' \ldots\ldots(\text{i}).$$

In the same way by taking S' at ♈′ we find

$$\cos\delta'\cos\alpha' = -\sin k \sin\omega \sin\delta + \cos k \cos\delta \cos\alpha - \sin k \cos\omega \cos\delta \sin\alpha \ldots\ldots(\text{ii}).$$

Finally, suppose the second star S' is at the pole of the ecliptic.

Its coordinates in System I. are

$$\alpha_1 = 270°, \quad \delta_1 = 90° - \omega,$$

and in System II.

$$\alpha_1' = 270°, \quad \delta_1' = 90° - \omega'.$$

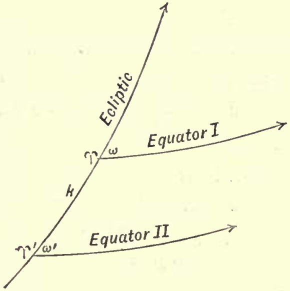


FIG. 59.

Making these substitutions in the fundamental equation we have

$$\sin\delta\cos\omega - \cos\delta\sin\omega\sin\alpha = \sin\delta'\cos\omega' - \cos\delta'\sin\omega'\sin\alpha' \ldots\ldots(\text{iii}),$$

and we thus obtain the three general equations connecting α, δ with α', δ' and the necessary constants k, ω, ω'.

We may note that (iii) is symmetrical in accented and unaccented letters and it is easily seen how (ii) might have been obtained from (i) by the interchange of accented and unaccented letters and by changing the sign of k.

If the known quantities are α', δ', then from (i), (ii), (iii) we can express $\sin\delta$ and $\cos\delta\sin\alpha$ each in terms of α', δ' and we thus group the three equations (iv), (i), (v) from which α, δ can be found without ambiguity:

$$\begin{aligned}\sin\delta = \sin\delta'\,(\cos k\sin\omega\sin\omega' + \cos\omega\cos\omega')\\ -\cos\delta'\cos\alpha'\sin\omega\sin k\\ +\cos\delta'\sin\alpha'\,(\cos k\sin\omega\cos\omega' - \cos\omega\sin\omega')\ldots\quad\text{(iv)},\end{aligned}$$

$$\begin{aligned}\cos\delta\cos\alpha = \sin\delta'\sin k\sin\omega'\\ +\cos\delta'\cos\alpha'\cos k\\ +\cos\delta'\sin\alpha'\sin k\cos\omega' \quad\ldots\ldots\ldots\ldots\text{(i)}\end{aligned}$$

$$\begin{aligned}\cos\delta\sin\alpha = \sin\delta'\,(\cos k\cos\omega\sin\omega' - \sin\omega\cos\omega')\\ -\cos\delta'\cos\alpha'\cos\omega\sin k\\ +\cos\delta'\sin\alpha'\,(\cos k\cos\omega\cos\omega' + \sin\omega\sin\omega')\ldots\ldots\text{(v)}.\end{aligned}$$

If it be desired to determine α', δ' when α and δ are given, we find in the same way

$$\begin{aligned}\sin\delta' = \sin\delta\,(\cos k\sin\omega\sin\omega' + \cos\omega\cos\omega')\\ +\cos\delta\cos\alpha\sin\omega'\sin k\\ +\cos\delta\sin\alpha\,(\cos k\sin\omega'\cos\omega - \cos\omega'\sin\omega)\quad\ldots\text{(vi)},\end{aligned}$$

$$\begin{aligned}\cos\delta'\cos\alpha' = -\sin\delta\sin k\sin\omega\\ +\cos\delta\cos\alpha\cos k\\ -\cos\delta\sin\alpha\sin k\cos\omega\ \ldots\ldots\ldots\ldots\text{(ii)}\end{aligned}$$

$$\begin{aligned}\cos\delta'\sin\alpha' = \sin\delta\,(\cos k\cos\omega'\sin\omega - \sin\omega'\cos\omega)\\ +\cos\delta\cos\alpha\cos\omega'\sin k\\ +\cos\delta\sin\alpha\,(\cos k\cos\omega\cos\omega' + \sin\omega\sin\omega')\ \ldots\text{(vii)}.\end{aligned}$$

For the calculation of precession we may usually regard k as so small that powers above the first may be neglected and we also take $\omega = \omega'$, so that the formulæ (vi), (ii), (vii) become

$$\sin\delta' = \sin\delta + k\sin\omega\cos\delta\cos\alpha,$$

$$\cos\delta'\cos\alpha' = \cos\delta\cos\alpha - k\sin\omega\sin\delta - k\cos\omega\cos\delta\sin\alpha,$$

$$\cos\delta'\sin\alpha' = \cos\delta\sin\alpha + k\cos\omega\cos\delta\cos\alpha.$$

From which we easily obtain the approximate results

$$\alpha' - \alpha = k\cos\omega + k\sin\omega\tan\delta\sin\alpha\ldots\ldots\ldots\text{(viii)},$$

$$\delta' - \delta = k\sin\omega\cos\alpha\ \ldots\ldots\ldots\ldots\text{(ix)}$$

These are the fundamental formulæ for precession.

Ex. 1. Show that, if a, δ, be the R.A. and decl. of a star, its annual increase of right ascension in consequence of precession when expressed in seconds of arc will be very nearly $46''+20''\tan\delta\sin a$ and in declination $20''\cos a$.

Ex. 2. k is the angular velocity of the pole of the equator round the pole of the ecliptic, L is the longitude of the instantaneous axis of rotation of the ecliptic, and η its angular velocity. Show that these changes in the planes of reference produce annual rates of change

$$m+n\sin a\tan\delta \text{ and } n\cos a$$

in a, δ, the R.A. and decl. of a star, where

$$m=k\cos\omega-\eta\sin L\operatorname{cosec}\omega$$

and
$$n=k\sin\omega,$$

ω being the inclination of the equator to the ecliptic.

[Sheepshanks Exhibition, 1903.]

Ex. 3. Prove that the points on the celestial sphere whose declinations undergo the greatest change in a given period, owing to the precession of the equinoxes, lie on two arcs of a great circle; and that the points whose declinations are, at the end of the period, unchanged lie on another great circle.

[Math. Trip. I. 1901.]

Let P, P' be the poles of the equator at the beginning and end of the period. Then it is obvious geometrically that the greatest possible change of declination by precession in this period is equal to the arc PP', and that the stars which undergo this greatest change lie on the great circle through PP', outside the limits of the arc PP' and its antipodal arc. The stars whose declinations are, at the end of the period, unchanged lie on the great circle bisecting the arc PP' at right angles.

Ex. 4. Show that if a star lie on the solstitial colure it has no precession in declination, and that all stars on the equinoctial colure have the same precession in right ascension and also in declination.

Ex. 5. Prove that if S be a star without precession in R.A., and P, K the poles of the equator and the ecliptic respectively, then SP and SK will be at right angles.

[Math. Trip.]

Ex. 6. Show that all stars whose R.A. is not at the moment being altered by precession lie on an elliptic cone passing through the poles of the equator and the ecliptic.

[Coll. Exam.]

The condition is, see (viii),

$$\cos\omega+\sin\omega\tan\delta\sin a=0.$$

If we make
$$x=r\cos a\cos\delta,$$
$$y=r\sin a\cos\delta,$$
$$z=r\sin\delta,$$

and eliminate r, a, δ, we have as the equation of the cone

$$yz\sin\omega+(x^2+y^2)\cos\omega=0.$$

Ex. 7. Show that for all stars for which the rate of variation in the declination due to the motion of the node of the equator along the ecliptic has its greatest value A, the rate of variation in right ascension due to the same cause is $A \cot \omega$, where ω is the angle between the ecliptic and the equator.

Ex. 8. Show from the formulæ (i), (ii), (iii) that we have the following expressions for the differential coefficients of a', δ' with respect to ω' and k

$$\frac{\partial a'}{\partial \omega'} = -\tan \delta' \cos a'; \quad \frac{\partial \delta'}{\partial \omega'} = \sin a';$$

$$\frac{\partial a'}{\partial k} = \cos \omega' + \sin \omega' \tan \delta' \sin a'; \quad \frac{\partial \delta'}{\partial k} = \sin \omega' \cos a'.$$

Differentiating (vi) with regard to ω' and regarding a, δ, k, ω as constants we have from (vii)

$$\cos \delta' \frac{\partial \delta'}{\partial \omega'} = \cos \delta' \sin a'$$

whence excluding the case of $\delta' = 90°$

$$\frac{\partial \delta'}{\partial \omega'} = \sin a'.$$

Differentiating (ii) with regard to ω'

$$\cos \delta' \sin a' \frac{\partial a'}{\partial \omega'} + \sin \delta' \cos a' \frac{\partial \delta'}{\partial \omega'} = 0$$

whence after substituting for $\partial \delta' / \partial \omega'$ we have

$$\frac{\partial a'}{\partial \omega'} = -\tan \delta' \cos a'.$$

Differentiating (vi) with regard to k we have

$$\cos \delta' \frac{\partial \delta'}{\partial k} = \sin \omega' (-\sin \delta \sin k \sin \omega + \cos \delta \cos a \cos k - \cos \delta \sin a \sin k \cos \omega)$$
$$= \sin \omega' \cos \delta' \cos a'$$

whence
$$\frac{\partial \delta'}{\partial k} = \sin \omega' \cos a'.$$

Finally differentiating (iii) with regard to k and substituting the value just found for $\partial \delta' / \partial k$

$$0 = \cos \delta' \cos \omega' \sin \omega' \cos a' + \sin \delta' \sin \omega' \sin a' \sin \omega' \cos a'$$
$$- \cos \delta' \sin \omega' \cos a' \frac{\partial a'}{\partial k}$$

whence
$$\frac{\partial a'}{\partial k} = \cos \omega' + \sin \omega' \tan \delta' \sin a'.$$

Ex. 9. Show that notwithstanding the precessional movement the celestial equator always touches two fixed small circles.

Ex. 10. If there be a change in the obliquity $\Delta\omega$ without any change in ♈ prove that

$$\cos a \cos \delta = \cos a_0 \cos \delta_0,$$
$$\sin a \cos \delta = \sin a_0 \cos \delta_0 \cos \Delta\omega - \sin \delta_0 \sin \Delta\omega,$$
$$\sin \delta = \sin a_0 \cos \delta_0 \sin \Delta\omega + \sin \delta_0 \cos \Delta\omega,$$

where a, δ, and a_0, δ_0 are the right ascension and declination of a star as affected and unaffected respectively by the alteration.

Ex. 11. Let a, δ be the right ascension and declination of a star at a given epoch, k the constant of precession, and ω the obliquity of the ecliptic.

If A denote the expression $\sin\omega\sin\delta+\cos\omega\cos\delta\sin a$ and B denote $\cos a\cos\delta$, then after t years the values of these expressions for the same star will be

$$A\cos kt+B\sin kt, \text{ and } B\cos kt-A\sin kt.$$

Also if the declination change to δ' in this period,

$$\sin\delta-\sin\delta'=\sin\omega\ \{A(1-\cos kt)-B\sin kt\}.$$

[Coll. Exam. 1901.]

We note that

$$(\sin\omega\sin\delta+\cos\omega\cos\delta\sin a)^2+(\cos a\cos\delta)^2$$

is an invariant so far as precession is concerned and it is easy to see that this expression is always the square of the cosine of the latitude. The expression

$$\sin\delta\cos\omega-\cos\delta\sin a\sin\omega$$

being the sine of the latitude is of course also an invariant, and from this circumstance Formula (iii) might have been at once written down.

Ex. 12. Show that, owing to precession, the R.A. of a star at a greater distance than $23\frac{1}{2}°$ from the pole of the ecliptic will undergo all possible changes, but that the R.A. of a star at a less distance than $23\frac{1}{2}°$ will always be greater than 12 hours.

If $x=\tan\frac{1}{2}k$, then from (ii) and (vii) we obtain

$$\begin{aligned}&x^2(2\sin\delta\sin\omega\cos\omega+\cos\delta\sin a\cos 2\omega-\tan a'\cos\delta\cos a)\\&-2x(\cos\delta\cos a\cos\omega+\tan a'\sin\delta\sin\omega+\tan a'\cos\delta\sin a\cos\omega)\\&+\tan a'\cos\delta\cos a-\cos\delta\sin a=0,\end{aligned}$$

the condition that this quadratic shall have real roots is easily seen to be

$$\tan^2 a'\cos^2\beta+\cos^2\omega-\sin^2\beta>0,$$

where β is the latitude of the star. If $\beta<(90°-\omega)$ a real value of k can be found for every value of a'.

We have also (Ex. 11)

$$\sin\delta'\cos\omega-\cos\delta\sin a'\sin\omega=\sin\beta.$$

If $\beta>(90°-\omega)$ we must have $\sin a'$ always negative.

*Ex. 13. Let x, y, z be the coordinates of a star referred to rectangular axes, the axis of x through the vernal equinox, the axis of y at right angles to it in the plane of the equator, the axis of z the polar axis of the earth. Assume that the ecliptic is fixed, and that precession may be represented as a revolution of the pole of the equator round the pole of the ecliptic at an angular rate q. After an interval of t years let the coordinates of the star, referred to the new positions of the axes, be ξ, η, ζ.

Show that the relations between the two sets of coordinates are

$$\begin{aligned}&\xi=x\cos qt-y\cos\omega\sin qt-z\sin\omega\sin qt,\\&\eta=x\cos\omega\sin qt+y(\cos^2\omega\cos qt+\sin^2\omega)+z\cos\omega\sin\omega(\cos qt-1),\\&\zeta=x\sin\omega\sin qt+y\cos\omega\sin\omega(\cos qt-1)+z(\sin^2\omega\cos qt+\cos^2\omega),\end{aligned}$$

where ω is the obliquity of the ecliptic.

[Prof. H. H. Turner, *Monthly Notices, R.A.S.* LX. 207.]

We have

$$x = \cos\delta\cos\alpha, \qquad \xi = \cos\delta'\cos\alpha'$$
$$y = \cos\delta\sin\alpha, \qquad \eta = \cos\delta'\sin\alpha',$$
$$z = \sin\delta, \qquad \zeta = \sin\delta'.$$

Hence putting $k = qt$ and letting $\omega' = \omega$ the results follow at once from (ii), (vi), (vii).

*Ex. 14. Supposing the pole of an orbit progresses with uniform velocity in a small circle, find on what great circles the motion of the nodes is (1) uniform, (2) continuous but variable, (3) oscillatory; and show that in the last case the progressive motion of the node takes longer than the regressive.

Let ω' be the radius $\not> 90°$ of the circle described by the moving pole P about the fixed point P_0. Then the great circle C of which P is pole intersects C_0 of which P_0 is pole at the constant angle ω'. The node moves uniformly along C_0 and there is no other great circle except C_0 on which the node moves uniformly. Draw two small circles C_1 and C_2 parallel to C_0 and on opposite sides of it at the constant distance ω' from C_0. Then as no point on C can be at a distance

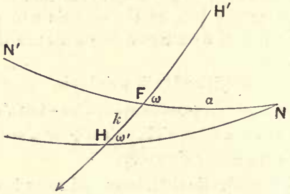


FIG. 60.

from C_0 greater than ω' we see that all the points on C must be confined to the zone Z between C_1 and C_2. Hence all possible nodes of C with any other circle O are limited to this zone Z.

The circle C is intersected by its consecutive position at its points of contact with C_1 and C_2. Hence if the node in which C intersects any other circle O be stationary that node must lie on either C_1 or C_2.

If the node in which a fixed circle O is intersected by C is to advance continuously it must not become stationary at any point, and consequently O must have no real intersections with C_1 and C_2; it must therefore be confined within the zone Z.

If O be not confined within Z then the nodes can only oscillate, for as we have seen that the nodes lie within Z it follows that they can never enter the portions of O exterior to Z, and consequently each node must oscillate in one of the two arcs intercepted on O by Z.

Let T_1 and T_2 be the points of contact of C with C_1 and C_2 and let O_1 and O_2 be the points in which an arc of O terminates in C_1 and C_2. Let the arc from O_1 to O_2 make an acute angle with that direction in which the nodes of C on C_0 are moving. When T_1 becomes coincident with O_1 then direct motion of the node will be commencing on O_1O_2. But this will not be completed till T_2 becomes coincident with O_2, and for this C will have to be turned more than half-way round, *i.e.* the direct oscillation takes more than half the whole period of C. But after T_2 has passed O_2 then the retrograde motion commences, and it will be finished when T_1 again reaches O_1 and therefore requires less than half the complete revolution.

We can also investigate the question thus. Let $H'FH$ (Fig. 60) be the circle C_0, HN be C and FN be O. Then from the triangle FHN we have by § 1 (6)

$$\cos a \sin k + \sin a \cos k \cos \omega - \sin a \cot \omega' \sin \omega = 0 \text{ ....(i).}$$

To find the corresponding changes of a and k we differentiate, treating ω and ω' as constant, and obtain

$$\frac{\Delta a}{\Delta k} = \frac{\cos a \cos k - \sin a \sin k \cos \omega}{\sin a \sin k - \cos a \cos k \cos \omega + \cos a \cot \omega' \sin \omega}.$$

If N is a stationary node then $\cos a \cos k - \sin a \sin k \cos \omega = 0$ or $HN = 90°$, which means that ω' is the perpendicular from N on FH, this being of course the same condition as that N shall lie on C_1. We hence find that $\cos k = \tan \omega' \cot \omega$, and thus we see that H moves over an arc $2k$, while the node retrogrades from the stationary node N' on C_2 to N. As $\tan \omega' \cot \omega$ is positive in the case represented we have $k < 90°$ and $2k$ is less than half the circumference, so that the regression of the nodes in the oscillatory movement takes less time than the progression.

*Ex. 15. On account of precession the interval between two passages of a given meridian through the same star differs from a mean sidereal day. If the colatitude of the star be less than that of the pole, show that this difference will vanish when the difference of longitudes of the pole and star is

$$\cos^{-1} \frac{\tan\text{ (colat. of star)}}{\tan\text{ (colat. of pole)}}.$$

*Ex. 16. If p_0 be the position angle of the smaller component of a double star at the epoch T_0, show that if the effect of precession only be considered, the position angle p at any other epoch T will be given by the equation

$$p = p_0 + 0''{\cdot}3342\,(T - T_0) \sin a \sec \delta,$$

where a, δ are the R.A. and decl. of the principal star of the pair and where T and T_0 are expressed in years.

58. Movement of the first point of Aries on the ecliptic.

In consequence of precession and nutation the intersection of the equator and ecliptic, which we call the first point of Aries (♈), is in motion on the ecliptic supposed fixed. Its position is therefore a function of the time, and if Ω be the distance of ♈ measured from some fixed point O on the ecliptic we may write

$$\Omega = a + bt + P.$$

In this equation t is the time measured from some fixed epoch and a and b are constants, while P consists of periodic terms only. These terms contain t in the expressions of angles which enter P solely by their sines and cosines. There is thus a fundamental difference between the quantities bt and P; the former is capable of indefinite increase in proportion to the time, and b is in fact the constant of precession. The value of P, on the other hand, is restricted between limits—it can never become greater than some quantity $+P_0$ nor less than $-P_0$ where P_0 is a finite quantity. The quantity P is the nutation by which Ω fluctuates about the uniformly moving position it would have if the nutation were absent.

Let N be a point moving uniformly on the ecliptic so that its distance from O at any time t is represented by $a+bt$. ♈ will be sometimes in advance of N and sometimes behind N, but the distance ♈N can never exceed P_0. The movement of ♈ will be the same on the average as that of N, and consequently N may be regarded as the *mean vernal equinoctial point* which moves uniformly along the ecliptic and in the immediate neighbourhood of which the first point of Aries is always to be found.

As the longitude of a star is measured from ♈ along the ecliptic it is clear that the longitude must be generally increasing by the motion of ♈ even though the star itself be devoid of proper motion. Introducing the numerical values† of the principal terms we have the following expression for the true longitude λ of a star on the ecliptic

$$\lambda = \lambda_0 + 50''\cdot 26t - 17''\cdot 235 \sin ☊ - 1''\cdot 27 \sin 2L,$$

where

λ_0 is the longitude of the star at the beginning of the year with reference to N;

t is the fraction of the year which has elapsed at the time under consideration;

☊ is the geocentric longitude of the moon's ascending node;

L is the sun's mean longitude which for our present purpose may with sufficient accuracy be regarded as the sun's true geocentric longitude.

† The values of the coefficients in this expression were adopted by a conference of astronomers which met in Paris in May 1896 and are those now used in the Nautical Almanac.

The second term in the expression of λ is due to precession. It corresponds to an annual increase of $50''\cdot26$ in the longitude of the star. As this term contains t as a factor, it is capable of indefinite increase and may become by far the most important of the three variable terms.

The third term involves ☊, the longitude of the moon's ascending node on the ecliptic. This term may make the longitude of the first point of Aries vary from $+17''\cdot235$ to $-17''\cdot235$ on either side of its mean value. As the moon's nodes revolve round the ecliptic in about $18\frac{1}{2}$ years, nutation causes ♈ to be in advance of its mean place for about 9 years and then to be behind its mean place for about 9 years. The last term is the nutation in longitude due to the sun, it is expressed in terms of L the mean longitude of the sun, and has a period of about six months.

Besides its effect on longitude, nutation has also a periodic effect on the obliquity of the ecliptic so that to find the true obliquity at any given time the mean obliquity for the beginning of the year must be increased by $9''\cdot21 \cos ☊ + 0''\cdot55 \cos 2L$. We should here remember that there is another minute variation in the obliquity of the ecliptic namely that due to the planetary precession (§ 56). The whole amount of the variation so caused is a diminution at the rate of $0''\cdot468$ per annum.

The joint effect of the nutation (omitting the small terms) and the planetary precession gives for the date T the following value for the obliquity of the ecliptic†:

$$23^\circ\ 27'\ 3''\cdot58 - 0''\cdot468\,(T - 1910) + 9''\cdot21 \cos ☊ + 0''\cdot55 \cos 2L.$$

The last two terms represent the nutation with sufficient accuracy for almost every purpose. The complete expression is given in the ephemeris. (See Ex. 5.)

Ex. 1. Newcomb's value of the constant of Precession as used in *N.A.* (see p. v) is

$$50''\cdot 2453 + 0''\cdot0002225t,$$

where t is the interval in years from 1850·0.

Show that this gives $50''\cdot2584$ for the constant of precession in 1909.

† The following values of the mean obliquity for the eight equidistant epochs from 1750 to 2100 are given by Newcomb, *Spherical Astronomy*, p. 238:

1750	23° 28′ 18″·51	1950	23° 26′ 44″·84
1800	23 27 55 ·10	2000	23 26 21 ·41
1850	23 27 31 ·68	2050	23 25 57 ·99
1900	23 27 8 ·26	2100	23 25 34 ·56

Ex. 2. If the origin of longitudes is the position of the mean equinoctial point at 1908·0, find the longitude of the first point of Aries and the obliquity of the ecliptic on June 29, 1908, being given that ☊ $=94^\circ\cdot 9$, $L=97^\circ$, and that $t=0\cdot 493$.

Precession in longitude for the interval is $24''\cdot 8$ and the nutation terms are $-17''\cdot 1$ and $+0''\cdot 3$ respectively, so that the answer is $8''\cdot 0$. In like manner the obliquity is shown to be $23^\circ\ 27'\ 2''\cdot 96$.

Ex. 3. Show that on 7th November, 1909, the precession in longitude from the beginning of the year is $42''\cdot 7$ and the nutation is $-17''\cdot 3$, being given that $L=226^\circ\cdot 1$ and ☊ $=68^\circ\cdot 7$.

Ex. 4. If $23^\circ\ 27'\ 4''\cdot 04$ be the mean value of the obliquity of the ecliptic in 1909·0, show that the apparent value on June 10th, 1909, when ☊ $=76^\circ\cdot 6$ and $L=78^\circ\cdot 2$, will be $23^\circ\ 27'\ 5''\cdot 48$.

*Ex. 5. If the nutation of the obliquity of the ecliptic $\Delta\omega$ is computed from the more complete expression (*N.A.* 1910, p. v)

$$\Delta\omega = +9''\cdot 210 \cos ☊ - 0''\cdot 090 \cos 2☊ + 0''\cdot 551 \cos 2L - 0''\cdot 009 \cos (L - 78^\circ\cdot 6) + 0''\cdot 022 \cos (3L + 78^\circ\cdot 6),$$

in which ☊ is the longitude of the moon's ascending node and L is the mean longitude of the sun, show that the nutation of the obliquity on 1st May, 1909, is $+1''\cdot 97$, being given ☊ $=78^\circ\cdot 7$ and $L=38^\circ\cdot 8$.

*Ex. 6. If the nutation of longitude ΔL is computed from the more complete expression (*N.A.* 1910, p. v)

$$\Delta L = -17''\cdot 235 \sin ☊ + 0''\cdot 209 \sin 2☊ - 1''\cdot 270 \sin 2L + 0''\cdot 107 \sin (L + 74^\circ\cdot 3) - 0''\cdot 05 \sin (3L + 78^\circ\cdot 6),$$

show that its value on 27 December, 1909, is $-15''\cdot 37$, being given ☊ $=66^\circ\cdot 00$ and $L=275^\circ\cdot 33$.

*59. The independent day numbers.

Even if a star is devoid of any proper motion (§ 60), as we shall at present suppose to be the case, its coordinates must be continually altering by precession and nutation. We may now assume that the ecliptic is a fixed circle and the mean equinoctial point is defined as moving *uniformly* on the ecliptic so that its average distance from the first point of Aries is zero. The mean equator at the date T intersects the ecliptic at the mean equinoctial point and as above explained is inclined to the ecliptic at the angle

$$23^\circ\ 27'\ 3''\cdot 58 - 0''\cdot 468\,(T - 1910).$$

By the mean right ascension and declination of a star we are to understand the R.A. and declination of that star as referred to the mean equator at the commencement of the year. The problem now before us is to determine α', δ' the apparent coordinates of a star on any particular day when we are given its mean coordinates α and δ for the year in which that day is contained.

The general formulæ (vi), (ii), (vii) of § 57 will provide us with the required equations and for our present purpose we may regard both k and $\omega' - \omega$ as small quantities whose squares or product may be neglected. Under these circumstances the equations reduce to

$$\sin \delta' = \sin \delta + \sin k \sin \omega \cos \delta \cos \alpha + \sin (\omega' - \omega) \cos \delta \sin \alpha,$$

$$\cos \delta' \cos \alpha' = \cos \delta \cos \alpha - \sin k \sin \omega \sin \delta - \sin k \cos \omega \cos \delta \sin \alpha,$$

$$\cos \delta' \sin \alpha' = \cos \delta \sin \alpha + \sin k \cos \omega \cos \delta \cos \alpha - \sin (\omega' - \omega) \sin \delta.$$

From which we obtain

$$\cos \delta \sin (\alpha' - \alpha) = \sin k (\cos \omega \cos \delta + \sin \omega \sin \delta \sin \alpha) - \sin (\omega' - \omega) \sin \delta \cos \alpha,$$

$$2 \sin \tfrac{1}{2} (\delta' - \delta) = \sin k \sin \omega \cos \alpha + \sin (\omega' - \omega) \sin \alpha.$$

We thus have approximately, if $\alpha' - \alpha$ be expressed in seconds of time and $\delta' - \delta$, k, $\omega' - \omega$ in seconds of arc,

$$\left.\begin{aligned} \alpha' - \alpha &= \tfrac{1}{15} k \cos \omega + \tfrac{1}{15} \{k \sin \omega \sin \alpha - (\omega' - \omega) \cos \alpha\} \tan \delta \\ \delta' - \delta &= k \sin \omega \cos \alpha + (\omega' - \omega) \sin \alpha \end{aligned}\right\} \ldots \text{(i)}.$$

We now assume three new quantities f, g, G determined by the equations

$$f = \tfrac{1}{15} k \cos \omega ; \quad g \cos G = k \sin \omega ; \quad g \sin G = -(\omega' - \omega) \ldots \text{(ii)},$$

and the equations (i) become

$$\left.\begin{aligned} \alpha' - \alpha &= f + \tfrac{1}{15} g \sin (G + \alpha) \tan \delta \\ \delta' - \delta &= g \cos (G + \alpha) \end{aligned}\right\} \ldots\ldots\ldots\ldots \text{(iii)}.$$

It will be observed that f, g, G are independent of the coordinates of the star, they only vary with the day of the year and they are called the *independent day numbers*.

To facilitate the computation of the effects of precession and nutation upon the coordinates of a star, tables are provided in the ephemeris in which the values of the independent day numbers will be found for each day of the year. The accurate formulæ are given each year in the ephemeris (see for example *N.A.* 1910, p. 233) by which the computation of the day numbers f, g, G as well as other day numbers to which we have not as yet referred is to be effected. So far as we are at present concerned the following approximate equations will suffice

$$\left.\begin{aligned} f &= \tfrac{1}{15}\cos\omega\,(50''\cdot26t - 17''\cdot2\sin\Omega - 1''\cdot3\sin 2L) \\ &= 3^{s}\cdot073\,(t - 0\cdot342\sin\Omega - 0\cdot025\sin 2L) \\ g\cos G &= \sin\omega\,(50''\cdot26t - 17''\cdot2\sin\Omega - 1''\cdot3\sin 2L) \\ &= 20''\cdot05\,(t - 0\cdot342\sin\Omega - 0\cdot025\sin 2L) \\ g\sin G &= -9''\cdot2\cos\Omega - 0''\cdot6\cos 2L \end{aligned}\right\}\ldots\text{(iv)}.$$

In these equations L and Ω are (as on p. 186) the sun's mean longitude and the longitude of the moon's ascending node on the ecliptic.

The time t is the fractional part of the year which has elapsed since the commencement of the year†.

We can obtain the annual precession in R.A. and declination directly from formulæ (iii) by writing instead of f, g, G the values of those quantities that would be derived from formulæ (iv) if we omitted the terms due to nutation. We thus substitute in (iii) $3^{s}\cdot073t$ for f, $20''\cdot05t$ for g and zero for G, and find for the star α, δ that as in Ex. 1, § 57

$$\left.\begin{aligned} &\text{one year's precession in R.A. changes } \alpha \text{ into} \\ &\qquad\qquad \alpha + 3^{s}\cdot073 + 1^{s}\cdot336\sin\alpha\tan\delta \\ &\text{,,}\qquad\text{,,}\qquad\text{,, Decl. changes } \delta \text{ into} \\ &\qquad\qquad \delta + 20''\cdot05\cos\alpha \end{aligned}\right\}\ldots\text{(v)}.$$

We are now able to solve the general problem of precession and nutation which may be stated as follows.

Being given α_0, δ_0 the mean R.A. and decl. of a star at the beginning of the year T_0 it is required to find α_1, δ_1 the apparent R.A. and decl. of the same star for a certain day in the year T_1 so far as precession and nutation are concerned.

We have first to find the coordinates of the star referred to the mean equator on Jan. 1st in the year $T_1 (> T_0)$. These are obtained by adding to the given mean R.A. and Decl. the following precessions

† It should be noted that if the strictest accuracy is required the beginning of the year is taken to be the moment when the sun's mean longitude is exactly 280°. In the year 1910 this is at $5^{h}\ 40^{m}$ A.M. on Jan. 1st which may also be expressed as Jan. $0^{d}\cdot735$. For the Greenwich mean time of the beginning of each adopted tropical year between the dates 1900 and 2000, see Appendix to Newcomb's *Spherical Astronomy*, p. 403, where other useful tables will also be found.

$$\text{Precession in R.A.}\ (3^s{\cdot}073 + 1^s{\cdot}336 \sin \alpha_0 \tan \delta_0)(T_1 - T_0),$$
$$\text{,, ,, Decl.}\ (20''{\cdot}05 \cos \alpha_0)(T_1 - T_0).$$

Having thus obtained the mean place for Jan. 1st in the year T_1 we obtain from the ephemeris for that year the values f_1, g_1, G_1 for the particular day for which the coordinates α_1, δ_1 are required and apply formulæ (iii), which give

$$\left.\begin{aligned} \alpha_1 &= \alpha_0 + (3^s{\cdot}073 + 1^s{\cdot}336 \sin \alpha_0 \tan \delta_0)(T_1 - T_0) \\ &\qquad + f_1 + \tfrac{1}{15} g_1 \sin (G_1 + \alpha_0) \tan \delta_0 \\ \delta_1 &= \delta_0 + 20''{\cdot}05 \cos \alpha_0 (T_1 - T_0) + g_1 \cos (G_1 + \alpha_0) \end{aligned}\right\} \ldots\text{(vii)}.$$

As an example of the application of these formulæ we shall calculate the apparent R.A. and Decl. of β Geminorum at Greenwich mean midnight on 1910, Nov. 7th, so far as precession and nutation are concerned †.

In the Greenwich second Ten-Year Catalogue of 6892 stars we find for the mean place of β Geminorum for 1890

$$\alpha = 7^h\ 38^m\ 35^s{\cdot}06, \quad \delta = 28^\circ\ 17'\ 28''{\cdot}4.$$

Substituting these values in $3^s{\cdot}073 + 1^s{\cdot}336 \sin \alpha \tan \delta$ we see that the annual precession is $3^s{\cdot}727$ so that as $T_1 - T_0$ is in this case 20 years the precession in R.A. from the mean place for 1890 to the mean place for 1910 is $1^m\ 14^s{\cdot}54$. In like manner the annual precession in declination is $20''{\cdot}05 \cos \alpha = -8''{\cdot}36$ so that in 20 years it amounts to $-(2'\ 47''{\cdot}2)$. Thus we see that the mean place of β Geminorum for 1910 is

$$\alpha = 7^h\ 39^m\ 49^s{\cdot}60, \quad \delta = 28^\circ\ 14'\ 41''{\cdot}2.$$

We have now to apply the corrections for giving the apparent place on 1910, Nov. 7th. From the *N.A.* p. 250 we obtain for that day

$$f = 1{\cdot}75, \quad \log g = 1{\cdot}1099, \quad G = 332^\circ\ 10'.$$

The equivalent of α in arc is $114^\circ\ 57'\ 24''$ so that $G + \alpha = 87^\circ\ 7'$, whence $\frac{1}{15} g \sin (G + \alpha) \tan \delta = 0^s{\cdot}46$ and thus the correction to α is $1^s{\cdot}75 + 0^s{\cdot}46 = 2^s{\cdot}21$. The correction to δ is $g \cos (G + \alpha) = 0''{\cdot}7$ so that we have finally for the desired apparent place on 1910, Nov. 7th

$$\alpha' = 7^h\ 39^m\ 51^s{\cdot}81, \quad \delta' = 28^\circ\ 14'\ 41''{\cdot}9.$$

† See *N.A.* 1910, p. 583, where the further corrections for aberration and proper motion are also attended to. See also Chap. XI, § 91.

If at the time $t=0$, α_0 be the right ascension with reference to the mean equinox of a star on the equator then the true right ascension of that star at the time t (when t is expressed in years) will be so far as the motion of ♈ is concerned

$$\alpha = \alpha_0 + 3^s\cdot073t - 1^s\cdot06 \sin ☊ - 0^s\cdot08 \sin 2L.$$

In this formula $3^s\cdot073$ is the annual change in R.A. due to precession and the first two terms form the *mean right ascension at the time t.* The last two terms are due to nutation. We thus see that the variations of the right ascension of an equatorial star from its mean value are comprised between the limits $+1^s\cdot14$ and $-1^s\cdot14$. So far as concerns the principal term of the nutation a complete cycle of the possible changes is run through in $18\frac{1}{2}$ years, this being, as already mentioned, the period in which ☊ increases through an angle of 360°.

Let $\Delta ☊$ and ΔL be the daily changes in the longitude of the moon's node and in the sun's mean longitude respectively, then the daily change in ♈ due to nutation is

$$-1^s\cdot06 \cos ☊ . \Delta ☊ - 0^s\cdot16 \cos 2L . \Delta L.$$

The values of $\Delta ☊$ and ΔL expressed in radians are approximately $-0\cdot000927$ and $0\cdot0172$, and consequently the diurnal change in ♈ is very nearly

$$0^s\cdot001 \cos ☊ - 0^s\cdot003 \cos 2L.$$

This expression must lie between the limits $-0^s\cdot004$ and $+0^s\cdot004$ and consequently the difference between any sidereal day and the mean sidereal day cannot exceed $0^s\cdot004$ (excess or defect).

We have already (§ 33) defined the sidereal day as the interval between two successive transits of ♈, and now it appears that owing to the fact that the movement of ♈ is not absolutely uniform all sidereal days would not be strictly equal. It might therefore be thought that we should distinguish the average sidereal day from the apparent sidereal day included between two transits of ♈, and therefore slightly variable. We are reminded of the distinction between the apparent solar day and the mean solar day to be subsequently considered, but there is no real analogy. The difference between two apparent solar days in the same year may be several thousand times as much as the greatest difference between two sidereal days (see p. 215).

If we had an ideally perfect clock which would keep time without any correction whatever for $18\frac{1}{2}$ years, so that throughout that period the hands showed $0^h\ 0^m\ 0^s$ at the completion of each average sidereal day, then ♈ would culminate daily for $18\frac{1}{2}$ years at various clock times which would lie between $23^h\ 59^m\ 58^s{\cdot}86$ and $0^h\ 0^m\ 1^s{\cdot}14$. But as even the best clocks require frequent correction by comparison with observation, the errors arising between one correction and the next, and attributable to the irregularities in ♈ are neglected because they are inappreciable in comparison with the other sources of error. Thus we define the sidereal day as commencing with the culmination of the *true* first point of Aries.

To illustrate the actual extent of the influence of the movement of ♈ on the measurement of sidereal time we take the case of 1909, June 10 and 20. On the first date the ephemeris gives for the nutation $-1^s{\cdot}05$ and on the second $-1^s{\cdot}02$. Assuming all other sources of error absent this would be equivalent to a daily clock rate averaging ·003 secs. So small a quantity would be masked by the much larger changes in the rate of the clock arising from ordinary mechanical or climatic causes. Nor would the error arising from the irregularity of ♈ accumulate, for on Oct. 18 the nutation is again $-1^s{\cdot}05$, so that from June 10 to Oct. 18 the average apparent change of rate of the clock from this cause would be zero.

Thus the frequent determination of the error of the clock will obviate not only the small irregularities unavoidable in a piece of mechanism however carefully made, but will at the same time allow us to assume that the sidereal time as shown by the clock after the correction has been applied is with all needful accuracy the hour angle of the first point of Aries.

It will be useful to investigate the effects of precession and nutation on the place of a star in another manner as follows.

As the longitude of a star is measured from the first point of Aries the precessional movement of the equator will alter the longitude of a star while its latitude remains unaltered. Thus if λ be the longitude of a star at any time, and if the first point of Aries move so that the longitude of the star becomes $\lambda + \Delta\lambda$, while at the same time the obliquity ω becomes $\omega + \Delta\omega$, we have the two following systems of equations. Of these (i), (ii), (iii) give the values of α and δ at the first Epoch, and then (iv), (v), (vi) give

$\Delta\alpha$ and $\Delta\delta$ by which the coordinates are changed by precession in the interval

$$\cos\delta\sin\alpha = \sin\lambda\cos\beta\cos\omega - \sin\beta\sin\omega \quad\text{.........(i)},$$

$$\cos\delta\cos\alpha = \cos\lambda\cos\beta \quad\text{..............................(ii)},$$

$$\sin\delta = \sin\lambda\cos\beta\sin\omega + \sin\beta\cos\omega \quad\text{......(iii)},$$

and

$$\cos(\delta+\Delta\delta)\sin(\alpha+\Delta\alpha) = \sin(\lambda+\Delta\lambda)\cos\beta\cos(\omega+\Delta\omega) - \sin\beta\sin(\omega+\Delta\omega) \quad\text{...(iv)},$$

$$\cos(\delta+\Delta\delta)\cos(\alpha+\Delta\alpha) = \cos(\lambda+\Delta\lambda)\cos\beta \quad\text{..................(v)},$$

$$\sin(\delta+\Delta\delta) = \sin(\lambda+\Delta\lambda)\cos\beta\sin(\omega+\Delta\omega) + \sin\beta\cos(\omega+\Delta\omega) \quad\text{...(vi)}.$$

These equations determine $\Delta\alpha$ and $\Delta\delta$ when $\Delta\lambda$ and $\Delta\omega$ are given, and the solution is effected without ambiguity in the most general case. But in the case of most general use in astronomy the four quantities $\Delta\lambda$, $\Delta\omega$, $\Delta\alpha$, $\Delta\delta$ are all small quantities, and we proceed directly as follows.

Differentiating (iii) and dividing by $\cos\delta$ (for we need not consider the case of $\delta = 90°$) we have after a slight reduction

$$\Delta\delta = \cos\alpha\sin\omega\,\Delta\lambda + \sin\alpha\,\Delta\omega;$$

also differentiating (i) and dividing by $\cos\delta$

$$\cos\alpha\,\Delta\alpha - \tan\delta\sin\alpha\,\Delta\delta = \cos\alpha\cos\omega\,\Delta\lambda - \tan\delta\,\Delta\omega,$$

we thus obtain the following results by which the effects of precession and nutation on right ascensions and declinations can be calculated with sufficient precision for most purposes.

If the position of the first point of Aries be displaced along the ecliptic so that all longitudes are increased by the small quantity $\Delta\lambda$, and if the obliquity be increased by a small angle $\Delta\omega$, then the corresponding changes, $\Delta\alpha$ and $\Delta\delta$, in the right ascension and declination of a star are given by

$$\Delta\alpha = (\cos\omega + \sin\alpha\tan\delta\sin\omega)\,\Delta\lambda - \tan\delta\cos\alpha\,\Delta\omega,$$

$$\Delta\delta = \cos\alpha\sin\omega\;\Delta\lambda + \sin\alpha\,\Delta\omega.$$

Ex. 1. Show that on any given day the stars whose declination are increased by precession are divided from those whose declination is diminished by precession by a great circle the stars on which have on that day no precession in declination.

For if $\cos(G+a)=0$ then all stars whose R.A. is $90° - G$ or $270° - G$ are unchanged as to declination by precession.

Ex. 2. Show how the independent day numbers will enable the apparent obliquity of the ecliptic to be readily computed.

From (ii) § 59 we have $g \sin G = -(\omega' - \omega)$ and therefore $\omega' = \omega - g \sin G$.

For example on March 2nd, 1910, we find, *N.A.* p. 245, that $\log g = 0{\cdot}7232$ and $G = 243^\circ\ 49'$, hence $g \sin G = -4''{\cdot}74$, and consequently as the mean obliquity 1910·0, *N.A.* p. 1, is $23^\circ\ 27'\ 3''{\cdot}58$ the obliquity when corrected for nutation is $23^\circ\ 27'\ 8''{\cdot}32$. As however the mean obliquity steadily diminishes by $0''{\cdot}468$ annually we must further apply a reduction of $0''{\cdot}08$ so that the apparent obliquity on March 2nd is $23^\circ\ 27'\ 8''{\cdot}24$ as in *N.A.* p. 217.

Ex. 3. Show how to compute from the independent day numbers the position of ♈ the apparent equinoctial point on the ecliptic with respect to the mean equinoctial point $♈_0$ at the beginning of the year.

$♈♈_0$ is the quantity k which we see from § 59 (ii) is $(225f^2 + g^2 \cos^2 G)^{\frac{1}{2}}$. For example 1910, Dec. 25th (midnight) we have $f = 2{\cdot}274$, $\log g = 1{\cdot}2018$, $G = 338\ 47^\circ$ (*N.A.* p. 251), whence $k = 37''{\cdot}20$.

Ex. 4. Show that $-6^s{\cdot}30$ is the annual precession in right ascension of ϵ Ursae minoris, being given that $\alpha = 16^h\ 56^m\ 12^s$; $\delta = 82^\circ\ 12'$ (1900).

Ex. 5. Explain how to find out, by the aid of a celestial globe, what constellations visible in the latitude of Cambridge 2000 years ago are no longer visible there; and indicate in what part of the heavens they lie.

[Math. Trip. I.]

60. Proper motions of stars. Besides those changes in right ascension and declination which arise from changes in the great circles to which the coordinates of the star are referred there are in the case of many stars real changes of place arising from the actual movements of the stars themselves. Such changes are called *proper motions*. The star in the northern hemisphere with the largest known motion of this kind is a small star of 6·5 magnitude in the constellation *Canes Venatici*. It bears the number 1830 in Groombridge's catalogue and its coordinates for 1900 are

$$\alpha = 11^h\ 47^m{\cdot}2, \quad \delta = +38^\circ\ 26'.$$

This star moves over an arc of $7''$ annually, and as its distance is also known it can be shown that its velocity must be not less than 150 miles per second. This motion is exceeded by that of a small star (magnitude 8·5) in the southern hemisphere, which was discovered by Kapteyn and Innes to have a proper motion of $8''{\cdot}7$ annually: its coordinates for 1900·0 are

$$\alpha = 5^h\ 7^m{\cdot}7, \quad \delta = -45^\circ\ 3'.$$

Among the bright stars the largest proper motion is that of α Centauri $\{\alpha = 14^h\ 32^m{\cdot}8,\ \delta = -60^\circ\ 25'\ (1900)\}$ which amounts to $3''{\cdot}7$ annually and it is so directed as to produce an annual change $-0^s{\cdot}49$ in R.A. and $+0''{\cdot}7$ in Declination. Arcturus $\{\alpha = 14^h\ 11^m{\cdot}1,\ \delta = +19^\circ\ 42'\ (1900)\}$ has a proper motion of $2''{\cdot}3$ per annum, corresponding to a speed of 257 miles per second, and the annual effect on the R.A. is $-0^s{\cdot}08$ and in decl. $-2''{\cdot}0$. In giving the apparent places of the stars throughout the year in the ephemeris the proper motion, if appreciable, is of course taken into account.

The proper motions just referred to are those which affect the coordinates of a star on the celestial sphere. If a star is moving in the line of sight the spherical coordinates are not changed by that motion and the existence of such a motion can be deduced only from spectroscopic observations. Thus Groombridge 1830, is found to be approaching our system at the rate of 59 miles per second. We have already seen that the tangential speed of this star is 150 miles per second so that its total velocity in space relation to the sun appears to be about 160 miles per second.

61. Variations in terrestrial latitudes. The axis about which the earth rotates was found by Küstner to be affected by a small motion with respect to the earth. The effect of such a change in the earth's axis is to alter the positions of the terrestrial poles and consequently of the terrestrial equator, and hence the latitude of any point on the earth's surface undergoes changes not due to actual motion in the point, but to changes in the base from which the latitudes are measured. The first systematic investigation of this subject was made in 1891 by Chandler, who showed that the observed changes in latitude could apparently be explained by the supposition that the pole of the earth described a circle with a radius of thirty feet in a period of about fourteen months. Later investigations by Chandler himself and others, while substantiating the general fact that the pole is in movement, have shown that the character of that movement is not quite so simple as had been at first supposed. We reproduce here the diagram (Fig. 61) given by Professor Albrecht in *Astronomische Nachrichten*, Nr. 4187, as a provisional result of the work of the International Geodetic Association. Reference may also be made to an account given by Mr Sidney D. Townley in the Publications of the Astronomical Society of the Pacific, Vol. XIX. p. 152.

In this diagram the origin at the centre of the figure is the mean position of the north pole in the earth, and the points marked on the curves indicate the actual positions of the pole at the corresponding dates. Thus for example the curve nearest the centre shows the movement of the pole from 1899·9 to 1901·0, from whence it can be traced forward in its various convolutions up to 1907·0. It will be seen that the positions of the pole are included within a square, each side of which subtends about 0″·50 at the earth's centre. The movements of the pole during the six years are thus comprised within a square of which the sides are not more than 50 feet. The individual positions are no doubt subject to considerable uncertainty.

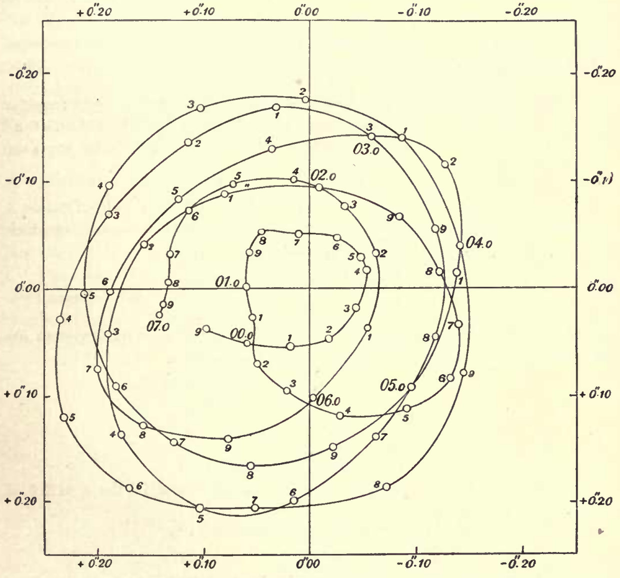


FIG. 61.

EXERCISES ON CHAPTER VIII.

Ex. 1. Assuming that the constant of precession is $50''\cdot2453 + 0''\cdot0002225t$ where t is the interval in years from 1850·0, find the number of years that must elapse before ♈ makes a complete circuit of the ecliptic.

Integrating we find the movement of ♈ in t years and if x be the number sought we have

$$50\cdot2453x + 0\cdot00011125x^2 = 1296000.$$

Of the two roots of this quadratic one is negative and irrelevant, the other root is 24468 or in round numbers 24,500.

*Ex. 2. Show that the points on the celestial sphere where the correction to R.A. for precession and nutation is zero on any given day lie on the cone

$$f(x^2+y^2) + \tfrac{1}{15}gz(x\sin G + y\cos G) = 0,$$

where the origin is at the sun's centre and the axes $+X$, $+Y$, $+Z$ pass respectively through the points whose R.A. and δ are (0°, 0°); (90°, 0°), (0°, 90°) and where f, g, G are the independent day numbers for the day in question. If the nutation be omitted deduce Ex. 6, § 57.

Ex. 3. Neglecting nutation show that the interval between two successive returns of the star (a, δ) to the meridian will exceed by $0^s\cdot00366 \sin a \tan\delta$ the sidereal day as defined by successive transits of ♈, it being supposed that the star has no proper motion.

Ex. 4. The right ascension of a star on the ecliptic is a, its declination δ, its longitude l. The precessions in right ascension, declination, and longitude are respectively a', δ', l'. Prove the relations

$$\delta' \cot\delta = l' \cot l = a' \cot a \cos^2\delta.$$

[Coll. Exam.]

Ex. 5. The stars on the celestial sphere regarded as a rigid system are supposed to be subjected to three rotations as follows.

(1) Through a small angle η round ♈ as pole,

(2) " " " " ξ " B " "

(3) " " " " ζ " P " "

where P is the north pole and B the point $a = 90°$, $\delta = 0$.

Show that if Δa, $\Delta\delta$ are the changes thus produced in the a and δ of a star, then

$$\Delta a = -\eta\cos a \tan\delta - \xi\sin a\tan\delta + \zeta,$$

$$\Delta\delta = \eta\sin a - \xi\cos a.$$

This is proved most easily by infinitesimal geometry each of the three rotations being considered separately.

*Ex. 6. Show that the apparent place of the equator as affected by precession and nutation at any date T during the year can be obtained by applying to the position of the equator at the commencement of the year the three following rotations.

(1) Through the small angle $g \sin G$ round ♈ as nole,
(2) " " " " $g \cos G$ " B " "
(3) " " " " $-15f$ " P " "

where P is the north pole and B the point $\alpha = 90°$, $\delta = 0$.

The point where ♈ was situated at the commencement of the year has as its coordinates with respect to the equator of the date T, $\alpha' = 15f$; $\delta' = g \cos G$ where f is expressed in seconds of time. In like manner the point B at the beginning of the year has as its coordinates with respect to the equator at time T, $\alpha' = 90° + 15f$, $\delta' = -g \sin G$. It is obvious from geometry that rotations $g \sin G$, $g \cos G$, $-15f$ round noles ♈, B, P will convey the two points in question from the equator at the beginning of the year to the equator at the date T.

Ex. 7. Show geometrically that the effect of precession and nutation upon the R.A. and decl. of the stars during an interval t is equivalent to that produced by rotating the celestial sphere (*i.e.* the sphere containing the stars but not the circles of reference), about a diameter passing through the point whose longitude is zero and latitude is

$$\tan^{-1}\left(\frac{pt + \Delta L}{\Delta\omega}\right),$$

the angle of rotation being

$$\{(pt + \Delta L)^2 + (\Delta\omega)^2\}^{\frac{1}{2}},$$

and its direction retrograde, where p is the constant of precession, and ΔL, $\Delta\omega$ are the nutations in longitude and obliquity respectively.

The effect of precession and nutation in longitude could be produced by rotating the celestial sphere round P the pole of the ecliptic through the angle $pt + \Delta L = VPV_1$ (Fig. 62). Thus any point R on PV is conveyed to R_1 on PV_1. The direction of this rotation is determined by the necessity that

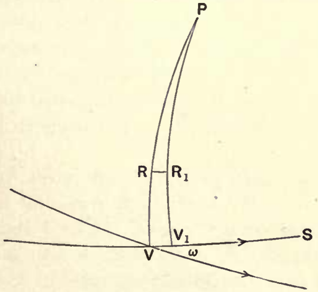


Fig. 62.

it shall increase the longitude of each point. To effect the change arising from the increase of ω by the nutation $\Delta\omega$ the celestial sphere is to be rotated round V. The movement of the equator through the angle $\Delta\omega$ increases the angle between the ecliptic VS and the equator while the ecliptic remains fixed. The effect will be the same as if all points had an anti-clockwise rotation $\Delta\omega$ round V. Each point on PV_1 will be moved to the left and there will be some point R_1 which will be moved back to its original place R. Thus so far as this point is concerned the two rotations neutralize. The two rotations round V and P will therefore compound into one rotation about R.

If θ be the latitude of R, then $VR=\theta$ and

$$RR_1=(pt+\Delta L)\cos\theta=\Delta\omega\sin\theta,$$

whence $\tan\theta=(pt+\Delta L)/\Delta\omega$, and as the component rotations are at right angles the resultant is the square root of the sum of their squares, *i.e.*

$$\sqrt{(pt+\Delta L)^2+(\Delta\omega)^2}.$$

Ex. 8. Show that on a given day the greatest displacement of apparent position which a star can have by precession and nutation is

$$\sqrt{(pt+\Delta L)^2+(\Delta\omega)^2},$$

that all stars which have this displacement must lie on a great circle, whose equation is

$$\cos a\cos\delta\,\Delta\omega+(\sin\delta\cos\omega-\sin a\cos\delta\sin\omega)(pt+\Delta L)=0,$$

and finally that the displaced position lies also on the same great circle.

CHAPTER IX.

SIDEREAL TIME AND MEAN TIME.

62. Sidereal time.

We have already seen (Ex. 1, p. 198) that in the course of about 24,500 years ♈ accomplishes a complete revolution of the heavens, and in such a direction that in this period the stars have made one complete apparent revolution less than ♈. The duration of the earth's rotation bears to the sidereal day (§ 33) the ratio of 24,500 years + 1 day to 24,500 years. Thus the period of rotation of the earth exceeds by about one hundredth of a second the sidereal day as actually used in the observatory. It has been pointed out (§ 59) that the variations in the length of the sidereal day, due to the irregularities in the motion of ♈, are too small to be perceptible.

The sidereal clock, by which we mean a clock regulated to keep sidereal time, carries a dial divided into 24 equal spaces by figures marked 0 to 23. When ♈ is on the meridian of the observer, then if the sidereal clock has no error it will show $0^h\ 0^m\ 0^s$, and if in addition the rate of the clock is correct it will again show $0^h\ 0^m\ 0^s$ when ♈ returns to the meridian.

The special convenience of sidereal time in the observatory is due to the fact that, subject to certain small corrections, the same star crosses the meridian each day at the same sidereal time †.

Ex. 1. If the proper motion by which a star shifts its place on the celestial sphere amounts to p seconds of arc annually, show that so far as this is concerned the interval between two successive transits of this star could never differ from a sidereal day by more than $\cdot 00018 p \sec \delta$ seconds, where δ is the declination of the star.

Ex. 2. If the distance of the first point of Aries measured from a fixed equatorial star is

$$p + qt + A \cos mt + B \sin mt,$$

where p, q, A, m, B are constants and where t is the time expressed in years, show that the interval between two successive upper transits of the first point of Aries will have as its extreme limits

$$24^{\text{h}} + m\sqrt{A^2+B^2}/366\cdot 24 \text{ and } 24^{\text{h}} - m\sqrt{A^2+B^2}/366\cdot 24.$$

Let t' be a moment of culmination of ♈, then the next culmination will take place approximately at $t' + \frac{1}{366\cdot 24}$. The distance of ♈ from its original position will have changed by the amount

$$p + q\left(t' + \frac{1}{366\cdot 24}\right) + A \cos mt' - mA\,\frac{1}{366\cdot 24}\sin mt'$$
$$+ B \sin mt' + mB\,\frac{1}{366\cdot 24}\cos mt'$$
$$-(p + qt' + A \cos mt' + B \sin mt').$$

Of this the periodic part is $\frac{m}{366\cdot 24}(B \cos mt' - A \sin mt')$, and there is no value for t' which can make this numerically greater than $\frac{m}{366\cdot 24}\sqrt{A^2+B^2}$.

63. The setting of the astronomical clock.

The practical method for determining the correction of the astronomical clock, is, in its simplest form, as follows.

The ephemeris shows for every tenth day the apparent right ascensions of some hundreds of fundamental stars, so distributed over the heavens that at every place and at every hour one or more of these stars is approaching the meridian. The *correction* of the clock is obtained by subtracting the clock time of transit over the meridian as found by observation from the Right Ascension of the star as deduced by interpolation from the ephemeris. Thus

† In the ephemeris tables will be found for transforming intervals of sidereal time into the corresponding intervals of mean time and *vice versa*.

the correction is positive when the clock is slow, for an *addition* has then to be made to the clock time to obtain the true time. When the clock is fast the correction is negative.

Suppose, for example, that an observation of the transit of the star β Eridani is made on 1910, Feb. 10, and that when all due corrections have been applied we have:

Clock time of transit of β Eridani	$5^h\ 3^m\ 42^s{\cdot}6$
Ephemeris gives for the apparent R.A. of β Eridani	5 3 25·6
Correction of clock	-0 17·0

If therefore the correction $-17^s{\cdot}0$ is applied to any reading of the clock the correct corresponding time is obtained. Thus at the moment when the first point of Aries is on the meridian this clock, which ought to show $0^h\ 0^m\ 0^s$, does in fact show $0^h\ 0^m\ 17^s{\cdot}0$. To obtain greater accuracy the mean of the corrections derived from a number of fundamental stars should be used.

The *rate* of the clock is found by comparing the corrections to the clock found at suitable intervals. Thus suppose

on June 14, at 20^h S.T. the correction is $+18^s{\cdot}64$,

on June 15, at 21 „ „ „ $+20{\cdot}80$.

The rate per day at which the clock has been losing during the interval is therefore $\frac{24}{25}\times 2^s{\cdot}16 = 2^s{\cdot}07$.

When the rate of the clock is known the difference of Right Ascensions of two stars is determined by observing the difference of their times of transit and then applying the correction for the rate of the clock in the interval.

Thus if the Right Ascension of even one celestial body is known we can determine, subject to certain qualifications, the Right Ascensions of other celestial bodies. We have therefore to show how a single fundamental R.A. is to be ascertained, and as the position of ♈ is determined by the sun's motion, it is obvious that the sun must be the body to be observed for this purpose.

If ω be the obliquity of the ecliptic and α, δ the R.A. and Declination of the sun, the centre of which is supposed to be in the ecliptic, then

$$\sin\alpha = \tan\delta\cot\omega \quad\text{.......................(i).}$$

We shall assume that ω is known (§ 64) and that δ has been observed. Then α can be calculated from this equation. If τ be the time of transit as shown by the astronomical clock, then $\alpha - \tau$, the error of the clock, is known.

As an example of this process we may take the following.

Suppose that on the meridian of Greenwich the clock time of the transit of the sun on March 28, 1909, is $0^h\ 26^m\ 49^s\cdot2$ and the observed declination of the sun's centre is $2°\ 51'\ 1''\cdot3$ N. The obliquity of the ecliptic is known to be $23°\ 27'\ 6''\cdot1$, and we seek the correction to the clock. We find the R.A. of the sun at transit by the formula (i), and the calculation is as follows:

$$\begin{array}{ll} \text{Log tan } 2°\ 51'\ 1''\cdot3 & 8\cdot6971357 \\ \text{log cot } 23\ \ 27\ \ 6\ \cdot1 & \underline{0\cdot3627002} \\ \text{Log sin } 0^h\ 26^m\ 21^s\cdot7 = & 9\cdot0598359. \end{array}$$

The correction of the clock is therefore

$$0^h\ 26^m\ 21^s\cdot7 - (0^h\ 26^m\ 49^s\cdot2) = -\ 27^s\cdot5.$$

By the application of this correction to any clock time and allowing for the rate of the clock (assumed constant) the true corresponding sidereal time is found.

In the following method of finding the right ascension of a star we shall suppose the effects of precession and nutation to have been already allowed for.

Let α be the unknown R.A. of a star, and on a certain day at a certain place let t_1 be the interval in sidereal time by which the transit of the sun precedes the transit of the star. The R.A. of the sun is therefore $\alpha - t_1$, and if δ be its declination and ω the obliquity of the ecliptic

$$\sin(\alpha - t_1) = \tan\delta_1 \cot\omega \text{ ....(ii).}$$

On another occasion in the course of the year let the transit of the sun with declination δ_2 precede that of the same star by the time t_2 and we have

$$\sin(\alpha - t_2) = \tan\delta_2 \cot\omega \text{ ....(iii),}$$

subtracting these equations and then adding them we easily deduce

$$\cot\{\alpha - \tfrac{1}{2}(t_1 + t_2)\} = \cot\tfrac{1}{2}(t_2 - t_1)\sin(\delta_1 - \delta_2)\,\text{cosec}\,(\delta_1 + \delta_2)\text{...(iv).}$$

Hence from observing δ_1 and δ_2 and the time intervals t_1 and t_2 we have the means of finding α without a previous knowledge of ω.

Ex. 1. If E_0 be the correction to the astronomical clock at a clock time T_0 and if r be the gaining rate of the clock, expressed in seconds per day, show that the correction to be applied to any clock time T to obtain the true time is $E_0-(T-T_0)r/24$, where T and T_0 are expressed in hours.

Ex. 2. On a little shelf attached to the middle of the pendulum of a mean time clock a number of small equal masses are carried, each just so heavy that an addition of one to their number causes an increase in the rate of the clock of one second daily. It is arranged that any small number of these masses may be placed on the shelf or removed from the shelf while the clock is going without disturbing the clock's motion.

If the correction of the clock at noon yesterday was E_1 seconds and at noon to-day is E_2, show that the number of the masses to be added to the shelf at noon to-day to make the clock right at noon to-morrow is $2E_2-E_1$.

Ex. 3. On March 25, 1909, the sun crosses the meridian $5^h\,34^m\,47^s$ before α Orionis, and on September 17 the sun crosses the meridian $5^h\,47^m\,28^s$ after α Orionis, the corresponding declinations of the sun being $+1°\,40'\,27''$ and $+2°\,24'\,37''$.

Show that the R.A. of α Orionis is approximately $5^h\,50^m\,14^s$.

[Math. Trip. I. 1901.]

64. The obliquity of the ecliptic.

The obliquity of the ecliptic (see p. 187) is found by measurement of the declination of the sun about the time of a solstice. If this measurement could be made exactly at the time of the solstice then the obliquity would be equal to this measured declination. But an observation will not generally be feasible actually at the moment of the solstice, so the problem we have to consider is how the obliquity is obtained from an observed declination of the sun *about* the time of the solstice and at a known Right Ascension.

We have as in last section

$$\tan\omega=\tan\delta\operatorname{cosec}\alpha \qquad \text{(i).}$$

It would at first seem that there could be no more simple formula than this for the determination of ω when δ and α are given. We have however to show that a more practical formula for the actual calculations can be obtained, even though its form is more complicated and even though it is only an approximate formula, while the formula (i) just written is exact.

We have from (i), for the summer solstice,

$$\tan(\omega-\delta)=\frac{\tan\delta(1-\sin\alpha)}{\sin\alpha+\tan^2\delta}$$

$$=\sin\delta\cos\delta(1-\sin\alpha), \text{ since } \sin\alpha \text{ is nearly } 1,$$

whence $$\omega-\delta=\sin 2\delta\sin^2(45°-\tfrac{1}{2}\alpha)\operatorname{cosec}1'' \qquad \text{(ii).}$$

This is the proper formula to be used in this calculation, because in (ii) we are computing not ω but only $\omega - \delta$; and as ω is very nearly equal to δ we have only to find the small quantity $\omega - \delta$. This will be illustrated by taking a particular case.

On 22 June, 1909, at apparent noon at Greenwich, the sun's apparent Declination is 23° 27′ 4″·3. Its Right Ascension is $6^h\ 1^m\ 43^s{\cdot}29$ (= 90° 25′ 49″·35). We now calculate $(\omega - \delta)$ from formula (ii), using only three decimal places in the logarithms:

$$\begin{aligned} &\text{Log}\sin 2\delta &&= 9{\cdot}863 \\ &\text{Log}\sin(45° - \tfrac{1}{2}\alpha) &&= 7{\cdot}574n \\ &\quad ,, \qquad ,, &&= 7{\cdot}574n \\ &\log\operatorname{cosec} 1'' &&= \underline{5{\cdot}314} \\ &\log(\omega - \delta) &&= 0{\cdot}325 \qquad \omega - \delta = +2''{\cdot}1 \\ & && \qquad\qquad\qquad \omega = 23°\ 27'\ 6''{\cdot}4. \end{aligned}$$

There would be no advantage in using more than three figures in the logarithms, for neglect of the remaining figures could by no possibility make a difference of 0″·1 in ω. It also appears that δ need be taken to only the nearest minute when $\log\sin 2\delta$ is being written down.

If we attempted to find ω by using logarithms of three figures in formula (i) we have

$$\begin{aligned} \text{Log}\tan\delta &= 9{\cdot}637 \\ \text{Log}\sin\alpha &= \underline{0{\cdot}000} \\ \text{Log}\tan\omega &= 9{\cdot}637 \end{aligned}$$

from which it would seem that ω may be any angle between 23° 24′ 48″ and 23° 27′ 42″. Thus we see that while formula (ii) determines ω correctly to 0″·1, formula (i) gives a value of ω which may be wrong by nearly 3′, although the same number of decimal places has been used in the logarithms in each case. A few further trials will show that the 3-figure logs applied to an approximate formula (ii) actually give a more correct result than 4, 5 or even 6-figure logs applied to an exact formula (i); and this is true notwithstanding that formula (ii) has been deduced from formula (i).

Of course (i) must give the accurate result if a sufficient number of decimal places in the logarithms be employed. For example, using 7 figures,

$$\begin{aligned} \text{Log}\tan\delta &= 9{\cdot}6372895 \\ \text{Log}\sin\alpha &= \underline{9{\cdot}9999878} \\ \text{Log}\tan\omega &= 9{\cdot}6373017 \end{aligned}$$

and we obtain the correct result $\omega = 23^\circ\ 27'\ 6''\cdot 4$. This however cannot be obtained without interpolation even if we employ Bagay's tables which give the logarithms of the trigonometrical functions for each second of arc.

The point here illustrated is important not only in connection with the determination of the obliquity of the ecliptic but in other astronomical problems in which an unknown quantity is sought and in which a choice of the most suitable formula for the calculation has to be made.

In general we should select a formula which, as in (ii), gives an expression not exactly for the unknown itself but rather for the difference between the unknown and a known approximate value. When such a formula is obtained, troublesome interpolation in the calculation can generally be dispensed with, and a small number of decimal places suffices in the logarithms.

Ex. 1. Show that near the time of the winter solstice the obliquity of the ecliptic ω is given by the formula $\omega = \delta + \operatorname{cosec} 1'' \sin 2\delta \sin^2(45^\circ + \frac{1}{2}a)$, a being the right ascension and δ the southern declination of the sun and apply this formula to show that when $\delta = 23^\circ\ 26'\ 58''\cdot 2$ S. and $a = 17^h\ 57^m\ 47^s\cdot 98$ (22 Dec. 1907) the obliquity of the ecliptic is $23^\circ\ 27'\ 1''\cdot 9$.

Ex. 2. Prove from the following observation and data that on Jan. 1, 1893, the mean obliquity of the ecliptic was $23^\circ\ 27'\ 11''\cdot 36$:—

Observed :—

⊙'s apparent declination June 19 at apparent noon $23^\circ\ 26'\ 42''\cdot 90$ N.

Extracted from Nautical Almanac :—

⊙'s apparent R.A. June 19 at apparent noon $5^h\ 52^m\ 52^s\cdot 11$.

⊙'s apparent latitude June 19 is $0''\cdot 45$ N.

Nutation in obliquity June 19 $+7''\cdot 73$.

Secular change in obliquity $-0''\cdot 476$ annually.

[Math. Trip. II.]

In consequence of the planetary perturbations the earth swerves to a small extent now to one side of the ecliptic and now to the other, thus the centre of the sun has apparently a very small latitude β which though generally neglected is taken account of in this question. The value of ω on June 19 is easily seen to be

$$\omega = \delta - \beta \sin\omega \operatorname{cosec}\delta + \sin 2\delta \sin^2(45^\circ - \tfrac{1}{2}a) \operatorname{cosec} 1''.$$

Introducing the given values we have

$$\omega = 23^\circ\ 26'\ 42''\cdot 90 + 35''\cdot 95 = 23^\circ\ 27'\ 18''\cdot 85.$$

Applying the nutation and the secular change of the obliquity for half a year we must correct this by $-7''\cdot 73$ and $+0''\cdot 24$ and we thus have for the mean obliquity at the beginning of the year $23^\circ\ 27'\ 11''\cdot 36$ as proposed.

Ex. 3. If the observed right ascension of the sun be $90° - u$, and its declination δ, find the following formula for the determination of the obliquity of the ecliptic from observations taken near the solstice

$$\omega - \delta = \frac{\tan^2 \frac{1}{2}u}{\sin 1''} \sin 2\omega - \frac{\tan^4 \frac{1}{2}u}{\sin 2''} \sin 4\omega + \ldots$$

where ω is the required obliquity, and $\omega - \delta$ is measured in seconds.

Discuss carefully the following questions arising from the formula:— (1) A knowledge of the position of the first point of Aries is required in order to determine u. (2) A correction is required for δ owing to the small latitude of the sun. (3) The sought quantity ω appears on the right-hand side.

[Coll. Exam.]

65. Estimation of the accuracy obtainable in the determination of Right Ascensions.

It is useful to examine the degree of accuracy attainable in the determination of the origin from which Right Ascensions are measured.

Let us first suppose that there was an error $\Delta\omega$ in the value of the obliquity of the ecliptic assumed in calculating the R.A. of the sun from observed values of the Declination. Differentiating the equation $\sin \alpha = \tan \delta \cot \omega$ and regarding δ as constant,

$$\cos \alpha \Delta\alpha = -\tan \delta \operatorname{cosec}^2 \omega \Delta\omega,$$

or

$$\Delta\alpha = -2 \tan \alpha \operatorname{cosec} 2\omega \Delta\omega.$$

Substituting in this for ω its approximate value 23° 27′, we have

$$\Delta\alpha = -2{\cdot}74 \tan \alpha \Delta\omega.$$

We thus see the advantage obtained by making these observations as nearly as possible at the equinox. The greater is α the greater becomes $\Delta\alpha$ for a given value of $\Delta\omega$. As therefore we want $\Delta\omega$ to produce the smallest possible effect on α, we should have α as small as possible.

Suppose the adopted value of the obliquity were erroneous to the extent of one second of arc then $\Delta\omega = 1''$, and if this produce an error of x seconds of time in α we have $\Delta\alpha = 15x$, whence

$$x = -0^s{\cdot}183 \tan \alpha.$$

If then the R.A. is to be correct to within $0^s{\cdot}1$ we must have $\tan \alpha \not> {\cdot}548$ or $\alpha \not> 1^h\ 54^m$. This is the R.A. of the sun on the 20th April. Hence we see that for about a month on either side of the equinox this method may be relied on to give the position of ♈ accurately to the tenth part of a second, provided the assumed value of the obliquity of the ecliptic is known to within a second of arc. It is, of course, in this assumed that there is no error in the observed value of the declination.

We must next consider what would have been the effect of an error in the observed Declination on the concluded value of the R.A. of the sun.

Differentiating $\sin\alpha = \tan\delta\cot\omega$ with respect to α and δ, and regarding ω as constant, we obtain

$$\Delta\alpha = \sec\alpha\sec^2\delta\cot\omega\Delta\delta.$$

which may also be written in the form

$$\Delta\alpha = \sec\alpha\,(1 + \sin^2\alpha\tan^2\omega)\cot\omega\,\Delta\delta.$$

As δ is the measured quantity we must take care to arrange our observations so that any error $\Delta\delta$ (and, of course, such errors are unavoidable) shall not unduly affect α. The factor $\cot\omega$ is constant, and as the declination of the sun never exceeds ω there will be no great variations in $\sec^2\delta$. As however $\sec\alpha$ may have any value from 1 to ∞ it is plain that we must have $\sec\alpha$ at its lowest value to keep $\Delta\alpha$ as small as possible, *i.e.* α should be nearly zero or 180°, and hence the sun should be near ♈ or ♎, and consequently the observations must be made near either the vernal or the autumnal equinox.

Substituting its numerical value for ω we easily find that the values of $\Delta\alpha$, corresponding to different Right Ascensions of the sun, are as follows:

R.A. of sun.	$\Delta\alpha$.
0^h	$2{\cdot}3\ \Delta\delta$
2^h	$2{\cdot}8\ \Delta\delta$
4^h	$5{\cdot}3\ \Delta\delta$

and at the solstice the coefficient of $\Delta\delta$ would be infinite.

Thus we see how important it is to minimize errors by making the observations near one of the equinoxes.

If the right ascension be required within the tenth part of a second $\Delta\alpha = 0^s{\cdot}1 = 1''{\cdot}5$, and consequently an error of a tenth of a second of time may arise from an error of $0''{\cdot}65$ in the determination of the sun's declination even close to the equinox.

66. The sidereal year and the tropical year.

In consequence of the revolution of the earth the sun appears to a terrestrial observer to make a complete circuit of the heavens once a year. It is important to distinguish the different meanings which may be assigned to the word *year*.

The *sidereal* year is the time interval in which the sun's centre performs a complete revolution with reference to the stars, or more precisely with reference to any one star situated in the ecliptic and devoid of proper motion. The sidereal year is also the periodic time in which the earth completes one sidereal revolution round the sun, when the earth is regarded as a planet belonging to the solar system. At the epoch 1900 the duration of the sidereal year is 365·2564 mean solar days.

The *tropical* year is the average interval between two successive returns of the sun to the first point of Aries. This point moves on the ecliptic by precession and advances to meet the sun at the rate (1900) of $50''\cdot2564$ annually (Newcomb). The tropical year is therefore less than the sidereal year in the ratio of $(360° - 50''\cdot2564)/360°$, and is found to be 365·2422 mean solar days. We have already mentioned (see note on p. 190) that in astronomical reckoning the commencement of the tropical year is the moment when the sun's mean longitude is exactly 280°, which in the year 1910 corresponds to Jan. $0^d\cdot735$.

It is the tropical year and not the sidereal year which is adopted as the basis in determining the *civil* year. According to the Julian Calendar the tropical year was assumed to be 365·25 days and it was arranged that of every four consecutive civil years three should have 365 days each and the fourth year, *i.e.* that divisible by 4, should be *leap* year and have a 29th of February added to make the number of days 366. This arrangement made the average civil year about 11 minutes longer than the tropical year.

The Gregorian correction to the Julian Calendar was introduced to bring about a closer correspondence between the *average* civil year and the tropical year. By this correction three of the leap years given by the Julian rule in every four centuries were suppressed. If the number expressing a year terminated in two cyphers, then such a year being divisible by 4 would be of course a leap year according to the Julian rule. But according to the Calendar with the Gregorian correction such a year is not to be a leap year unless the number formed by its two first digits is to be divisible by 4. Thus 1800, 1900, 2100, 2200, 2300 though Julian leap years are not Gregorian leap years but 2000 and 2400 are leap years in both systems. It is the Julian Calendar with the Gregorian correction that we now employ.

The present Calendar thus contains 97 leap years in every four centuries, and consequently the number of days in the four centuries is $365 \times 400 + 97 = 146097$ so that the average length of the civil year according to our present system is 365·2425 days. This agrees with the tropical year to within 0·0003 of a day. This approximation is so close that an error of a day in the reckoning would not be reached for some thousands of years.

Ex. 1. Show that at any observatory the number of upper culminations of the first point of Aries in the course of a tropical year (*i.e.* the number of sidereal days between two consecutive passages of the sun through ♈) exceeds by unity the number of upper culminations of the sun at the same observatory and in the same year.

At the first transit of ♈ after the year has commenced the sun must culminate somewhat later than ♈. At the second, third, and subsequent culminations of ♈ the sun will be ever more and more behind until when the year draws near its completion the sun will have fallen behind by nearly the whole circumference. The nth culmination of the sun will then be speedily followed by the $(n+1)$th culmination of ♈. If the sun overtakes ♈ when ♈ is not in culmination the year is complete but the number of culminations of the sun is one short. If the sun overtakes ♈ at the moment of its culmination then at the expiring moment of the year one more culmination is added to both sun and ♈, thus leaving the sun still one short.

Ex. 2. In a certain country the rule for leap year is :—If there are any cyphers at the end of the number for the year, strike off as many pairs of cyphers as possible ; then if the remaining number is divisible by 4, it is leap year. In another country the rule is :—Divide the number of the year by 33, then if there is a remainder and if that remainder is divisible by 4, it is leap year. Prove that the reckoning will never differ by more than a day in the two countries.

[Math. Trip.]

In 33 consecutive periods each of 400 years there must be one period but only one which commences with a year of which the number is divisible by 33. That year will not be a leap year, and the total number of leap years in the second country in that period of 400 years will be 96 and it thus falls 1 day in arrear. In each of the other 32 periods the number of leap years is 97. Hence the total number in $33 \times 400 = 13200$ years is $33 \times 97 - 1$.

In the first country there will be generally 97 leap years per 400 years, but in a period of 13,200 years there will be according to the condition no leap year in the year 10,000 and the count will fall one day in arrear. Hence the total number of leap years in each 13,200 years is $33 \times 97 - 1 = 3200$. Thus we see that each cycle of 13,200 years contains exactly 3200 leap years in either country.

67. The geometrical principle of a mean motion.

A point P is moving in the circumference of a circle (Fig. 63), and in such a way that at the time t the angle $OCP = \theta$, measured from a fixed radius CO, is defined by the equation

$$\left.\begin{aligned}\theta = a + \frac{2\pi t}{T_0} + A_1 \sin\frac{2\pi t}{T_0} + B_1 \cos\frac{2\pi t}{T_0} \\ + A_2 \sin\frac{4\pi t}{T_0} + B_2 \cos\frac{4\pi t}{T_0} \\ + A_3 \sin\frac{6\pi t}{T_0} + B_3 \cos\frac{6\pi t}{T_0} \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\end{aligned}\right\} \quad \ldots\ldots\ldots\ldots \text{(i)}$$

where a, T_0, A_1, B_1, A_2, B_2, A_3, B_3 ... are constants. The very general expression of θ in terms of t here given will include as a particular case the formula for the longitude of the sun in terms of the time.

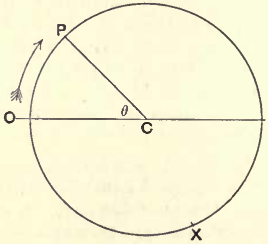


FIG. 63.

If $t + T_0$ be written in the expression for θ instead of t, then θ becomes $\theta + 2\pi$, *i.e.* P returns to the point from which it started. Thus T_0 is the periodic time of the motion of P.

Differentiating θ with respect to t we have

$$\left.\begin{aligned}\frac{d\theta}{dt} = \frac{2\pi}{T_0} + \frac{2\pi A_1}{T_0}\cos\frac{2\pi t}{T_0} - \frac{2\pi B_1}{T_0}\sin\frac{2\pi t}{T_0} \\ + \frac{4\pi A_2}{T_0}\cos\frac{4\pi t}{T_0} - \frac{4\pi B_2}{T_0}\sin\frac{4\pi t}{T_0} \ldots\ldots\ldots\ldots\end{aligned}\right\} \ldots\text{(ii)},$$

an expression which is not altered by changing t into $t + T_0$. Thus we see that if v be the velocity with which P passes through any point X in one circuit, then v will also be the velocity with which P passes through the same point X on every circuit.

It thus appears that every circuit exactly reproduces every other circuit. Not only is the time taken for the circuit the same but the actual velocity at every point is the same in every circuit.

That part of the expression of $d\theta/dt$, namely $2\pi/T_0$, which is obtained by omitting the trigonometrical functions, is termed the *mean angular velocity*. Let P_0 be a point moving uniformly round the circle with the angular velocity $2\pi/T_0$, and so that at every moment CP_0 makes with the fixed radius CO the angle $a + 2\pi t/T_0$, then we have the following properties of the *mean position* P_0 and the true position P:

(1) The periodic times of P and P_0 are identical.

(2) The distance between P and P_0 can never exceed a certain finite limit.

(3) The average difference between P and P_0 in the course of a complete circuit is zero.

(1) is evident because each of these periodic times is T_0.

(2) follows because the distance from P to P_0 can never exceed the sum $\pm A_1 \pm A_2 \pm A_3 \pm B_1 \pm B_2 \pm B_3$ &c. where each sign is so taken that the corresponding term is positive.

(3) For whenever n is an integer we have

$$\int_0^{T_0} \sin \frac{2n\pi t}{T_0}\, dt = 0 \quad \text{and} \quad \int_0^{T_0} \cos \frac{2n\pi t}{T_0}\, dt = 0.$$

Hence it follows that if we subtract the angle representing the position of P_0 from the corresponding θ the average value of the difference is zero, for that difference consists of periodic terms only, and the average value of each one is zero.

It follows that in moving uniformly round the circle P_0 is sometimes in advance of P and sometimes behind P, and that on the average P_0 will be just as much before P as it is behind. Thus the movement of P_0 is rightly described as the *mean motion* of P. We may regard the true motion of P as an oscillatory movement about the mean place P_0.

Ex. 1. If $\theta_0, \theta_1, \ldots, \theta_5$ be the values of θ at the times 0, $T_0/6$, $2T_0/6$, $3T_0/6$, $4T_0/6$, $5T_0/6$ respectively, show that for a the part of θ independent of t we have

$$a=\tfrac{1}{6}(\theta_0+\theta_1+\theta_2+\theta_3+\theta_4+\theta_5)-150°,$$

if A_4, B_4 and all higher terms be omitted.

By successive substitutions in the general formula (i)

$$\begin{aligned}
\theta_0&=a & &+B_1 & &+B_2 & &+B_3,\\
\theta_1&=a+60°+A_1\sqrt{3}/2 & &+B_1/2+A_2\sqrt{3}/2 & &-B_2/2 & &-B_3,\\
\theta_2&=a+120+A_1\sqrt{3}/2 & &-B_1/2-A_2\sqrt{3}/2 & &-B_2/2 & &+B_3,\\
\theta_3&=a+180 & &-B_1 & &+B_2 & &-B_3,\\
\theta_4&=a+240-A_1\sqrt{3}/2 & &-B_1/2+A_2\sqrt{3}/2 & &-B_2/2 & &+B_3,\\
\theta_5&=a+300-A_1\sqrt{3}/2 & &+B_1/2-A_2\sqrt{3}/2 & &-B_2/2 & &-B_3,
\end{aligned}$$

whence by addition

$$a=\tfrac{1}{6}(\theta_0+\theta_1+\theta_2+\theta_3+\theta_4+\theta_5)-150°.$$

If therefore we know the values of θ at the six epochs which divide a whole period of revolution T_0 into six equal parts we can determine a and thence $a+2\pi t/T_0$ or the mean position P_0 at any time t is known.

Ex. 2. Show how the general formula (i) is simplified if the motion is symmetrical about the axis CO.

In this case $d\theta/dt$ will be the same for t and T_0-t, if t be measured from the time of passage through O, whence by substitution in (ii)

$$\frac{2\pi B_1}{T_0}\sin\frac{2\pi t}{T_0}+\frac{4\pi B_2}{T_0}\sin\frac{4\pi t}{T_0}+\frac{6\pi B_3}{T_0}\sin\frac{6\pi t}{T_0}=0,$$

as this must be true for all values of t, $B_1=B_2=B_3=0$, and thus the formula (i) reduces to

$$\theta=2\pi t/T_0+A_1\sin 2\pi t/T_0+A_2\sin 4\pi t/T_0+A_3\sin 6\pi t/T_0.$$

Ex. 3. Assuming that the movement is symmetrical and that the axis of symmetry is the axis from which θ is measured and that A_3 and higher coefficients may be regarded as zero, show that if $A_1<2A_2$ there will be three real points in which the mean position of P coincides with its true position.

Ex. 4. From the Nautical Almanac for 1909 we obtain the following values for the apparent longitude of the sun at mean noon:—

1909 Mean noon	Apparent longitude of ☉
Jan. 1st	280° 28′ 1″·1
Ap. 2nd	12 6 30 ·9
July 2nd	99 55 40 ·4
Oct. 1st	187 39 27 ·2

Show that $280°{\cdot}499+360°\,t/T_0$ is the mean longitude of the sun where t is the number of mean solar days elapsed since mean noon on Jan. 1st, 1909, and where T_0 is the length of the tropical year expressed in mean solar days.

In applying formula (i) we discard A_3, B_3 and higher terms. In the periodic terms we may, with sufficient accuracy, make t successively 0, $\frac{1}{4}T_0$, $\frac{1}{2}T_0$, $\frac{3}{4}T_0$, and we must obviously increase each of the apparent longitudes on the last three dates by 360°. Thus we have from (i)

$$\begin{aligned} 280^\circ\ 28'\ \ 1''\cdot 1 &= a & &+B_1 \quad +B_2,\\ 372\ \ \ 6\ 30\ \cdot 9 &= a + 360^\circ \times\ 91/T_0 & &+A_1 \quad -B_2,\\ 459\ \ 55\ 40\ \cdot 4 &= a + 360\ \times 182/T_0 & &-B_1 \quad +B_2,\\ 547\ \ 39\ 27\ \cdot 2 &= a + 360\ \times 273/T_0 & &-A_1 \quad -B_2, \end{aligned}$$

whence by addition and making $T_0 = 365\cdot 2422$,

$$1660^\circ\ 9'\ 39''\cdot 6 = 4a + 538^\circ\ 9'\ 48''\cdot 6,$$

and finally $$a = 280^\circ\cdot 499.$$

The daily increase of mean longitude of the sun is 0°·98565 and the mean longitude is zero 80·656 days after the commencement of the year, *i.e.* March 22nd.

When several small terms which are here omitted have been attended to the sun's mean longitude is

$$280^\circ\cdot 49942 + 360^\circ\, t/T_0.$$

Ex. 5. Show from the last example that on Nov. 7th, 1906, the mean longitude of the sun is 226°·05.

Ex. 6. Being given that the sun's mean longitude is 9°·20768 on April 1st, 1909, and that the daily increase is 0°·98565, show that the sun's mean longitude on Jan. $0^d\cdot 493$ was 280°.

68. Mean time.

Though it is essential for the special work of the observatory to employ sidereal time, yet it is obvious that the astronomical clock would not serve the ordinary purposes of civil life. For this latter object a day of which the length is regulated by the sun and not by the stars is required. We therefore use what is known as the *mean solar day* for our ordinary time measurement.

Since the movement of the sun in Right Ascension is not uniform, the interval between two successive returns of the sun's centre to the meridian is not constant. As an illustration we here give the sidereal length of the solar day at four equidistant dates throughout the year 1909:

1909.	Sidereal interval.
Apparent noon Jan. 1st to apparent noon Jan. 2nd	$24^h\ 4^m\ 24^s\cdot 9$
„ „ April 2nd „ „ „ April 3rd	24 3 38 ·5
„ „ July 3rd „ „ „ July 4th	24 4 7 ·5
„ „ Oct. 2nd „ „ „ Oct. 3rd	24 3 37 ·6

The first line of this table states that if the time at which the sun's centre crosses the meridian of the observer be taken by an astronomical clock on Jan. 1st, 1909, and the observation be repeated on the following day, the astronomical clock, if due allowance be made for its rate, will show that an interval of $24^h\ 4^m\ 24^s{\cdot}9$ of sidereal time has elapsed between the two transits.

We observe that the apparent solar day, commencing at apparent noon on Jan. 1st, is 47·3 sidereal seconds longer than that commencing at apparent noon on Oct. 2nd. It thus appears that the length of the apparent solar day is not constant throughout the year, and its variations certainly exceed three-quarters of a minute. On account of these irregularities the true solar day is not a suitable unit for ordinary time measurement. We adopt as the unit a mean solar day, the length of which is the average duration of the apparent solar days in a large number of years. The average duration of the four days in the list just given for 1909 is $24^h\ 3^m\ 57^s{\cdot}1$, and this is an approximate value of the mean solar day. When the mean of an exceedingly large number of consecutive apparent solar days has been taken it is found that the equivalent in sidereal time to one mean solar day is $24^h\ 3^m\ 56^s{\cdot}555$.

To avoid circumlocution astronomers have found it convenient to imagine a fictitious body (or point rather) which at every moment is on the apparent equator, and has an apparent right ascension equal to the mean longitude of the sun. This imaginary body is called the *mean sun*. It will be shown in § 74 that the apparent right ascension of the sun is equal to the sum of the mean longitude of the sun and periodic terms. Thus the apparent R.A. of the sun and the apparent R.A. of the mean sun differ by periodic terms only. Hence in a long interval of time the average difference in apparent R.A. between the true sun and the mean sun will tend to zero. If we could overlook the movement of the equator by precession and nutation then the mean sun might be described as a body moving uniformly in the equator so that at every moment its R.A. is equal to the sun's mean longitude.

When the mean sun is in the meridian a clock properly regulated to show local mean time will read $0^h\ 0^m\ 0^s$. Thus the time shown by the mean time clock indicates at any moment the hour angle of the mean sun from the meridian. For civil purposes the day commences at midnight, and the hours are counted from 1^h to 12^h (noon), and then again from 1^h to 12^h (midnight), the former hours being distinguished by the letters A.M. and the latter by P.M. In astronomical reckoning the day extends from noon to noon: noon is called 0^h, and the following hours are numbered consecutively, up to 23^h. Thus 12.30 P.M. civil reckoning is called $0^h\ 30^m$ in astronomical reckoning.

Ex. 1. Find the length of the mean solar day in sidereal time from the following data :—

On 4th July, 1836, the apparent R.A. of the centre of the sun was found by observation at transit at Greenwich to be $6^h\ 54^m\ 7^s{\cdot}03$.

Similarly on 4th July, 1890, the apparent R.A. of the centre of the sun was found to be $6^h\ 53^m\ 54^s{\cdot}61$.

We have first to determine the sidereal interval between $6^h\ 54^m\ 7^s{\cdot}03$ sidereal time on 4th July, 1836, and $6^h\ 53^m\ 54^s{\cdot}61$ sidereal time on 4th July, 1890.

This is an interval of 54 years and therefore there will be 54 more transits of the first point of Aries than of the sun (§ 66, Ex. 1). Of the latter there are 19723, so that there are 19777 of ♈, and consequently the total interval expressed in sidereal time is

$$19777^d\ 6^h\ 53^m\ 54^s{\cdot}61 - (6^h\ 54^m\ 7^s{\cdot}03).$$

Dividing this by 19723 we find the sidereal value of the mean solar day to be $24^h\ 3^m\ 56^s{\cdot}555$.

Ex. 2. The length of the mean solar day in sidereal time is determined as in the last example by comparing the right ascensions of the sun at two epochs of which one is 30 years later than the other. Show that errors as great as 5^s in both of the right ascensions cannot affect the value found for the mean solar day by more than the thousandth of a second.

Ex. 3. An approximate rule for converting mean solar time into sidereal may be stated thus :—For every $1^h\ 1^m$ add 10^s; for every remaining $1^m\ 1^s$ add $\frac{1}{6}^s$; for every remaining 4^s add $0^s{\cdot}01$. What is the error in finding by this rule the length of a mean solar day?

[Sheepshanks Exhibition.]

Ex. 4. If in the expression of the duration of the tropical year in days, hours, minutes and seconds of mean solar time the number of days is increased by unity while the hours, minutes, and seconds are left unaltered, the result expresses the duration of the tropical year in days, hours, minutes and seconds of sidereal time.

69. The sidereal time at mean noon.

The right ascension of the mean sun or more precisely the distance at a given moment of the mean sun from ♈ is as already explained (§ 68) the mean longitude of the sun and we have for this the expression (Ex. 4, § 67)

$$280^\circ{\cdot}49942 + 360^\circ\, t/T_0,$$

where T_0 is the length of the tropical year and where t/T_0 is the fractional part of the tropical year which has elapsed since noon on Jan. 1st, 1909.

Transforming this expression into time at the rate of 15° to an hour we find that at t mean solar days, after Greenwich mean noon on 1909, January 1, the right ascension of the mean sun is

$$18^h\, 41^m\, 58^s{\cdot}84 + 236^s{\cdot}5554t.$$

We may explain as follows the nature of the observations by which a the first term of this expression has been obtained. Divide the year T_0 into a sufficient number of equal parts. At each point of division we shall suppose the R.A. of the sun to be observed with results $\alpha_1, \alpha_2, \alpha_3 \ldots$ respectively. We shall assume that the average of these right ascensions is the average of the right ascensions of the mean sun at the same epochs. This assumption is justified because each of the chief periodic terms runs through a complete cycle of changes in an interval which is contained an exact number of times in a year. If therefore we take a number of instants dividing the year into equal parts the average value of any one of these terms for those instants will be zero (provided the number of instants taken be sufficiently large, § 67). Hence the average right ascension of the true and the mean sun at these instants will be equal. We may illustrate the process by taking 6 such epochs for which the sun's mean longitudes are respectively

$$a,\ (a+4^h),\ (a+8^h),\ \text{etc.}$$

where a is the unknown to be found. If we determine the right ascensions ($\alpha_1, \alpha_2 \ldots \alpha_6$) of the true sun at these epochs we have

$$a + (a+4^h) + (a+8^h) + (a+12^h) + (a+16^h) + (a+20^h)$$
$$= \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6.$$

To obtain a we take as a particular case the values of $\alpha_1 \ldots \alpha_6$ as shown in the following table:

		Equidistant dates G.M.T.		R.A. of sun	
1909.	Jan.	1^d 0^h	... ...	18^h 45^m 33^s	
	March	2 21	... ...	22 54 12	
	May	2 18	... ...	2 38 41	$+24^h$
	July	2 15	... ...	6 45 47	$+24^h$
	Sep.	1 12	... ...	10 41 54	$+24^h$
	Nov.	1 9	... ...	14 25 40	$+24^h$

and by substitution in the formula we have

$$a = 18^h\ 41^m\ 58^s.$$

Making the slight alteration of less than 1^s in a, which is necessitated when due attention is paid to many small details which it would not have been possible for us to consider here, we have the required value $18^h\ 41^m\ 58^s{\cdot}84$.

When the mean sun comes to the meridian its right ascension is of course the sidereal time at the moment. Thus we obtain the important element in practical astronomy known as *the sidereal time of mean noon.* This is an indispensable quantity in transforming sidereal time to mean time and *vice versâ.* The sidereal time at mean noon is given in the ephemeris for each day.

Ex. 1. Show that the sidereal time at Greenwich at mean noon on 1909, March 27, is $0^h\ 17^m\ 6^s$.

At mean noon on March 27 the interval from Jan. 1 is 85 days. Putting this value for t into the expression

$$18^h\ 41^m\ 58^s{\cdot}84 + 236^s{\cdot}5554t,$$

the required result is obtained.

Ex. 2. Find at what date in 1909 the mean sun passes through the first point of Aries.

Ex. 3. Given that the mean times of transit at Greenwich of the first point of Aries on 1896, March 21, are $0^h\ 1^m\ 59^s{\cdot}70$ and $23^h\ 58^m\ 3^s{\cdot}79$, determine the moment at which the Greenwich mean and sidereal times are the same.

[Math. Trip.]

Ex. 4. Show that the right ascension of the mean sun at t mean solar days after mean noon on January 1st, 1900, is $18^h\ 42^m\ 43^s{\cdot}51 + 236^s{\cdot}5554t$ and the mean longitude of the sun is

$$280^\circ{\cdot}681 + 0^\circ{\cdot}98565t.$$

70. Determination of mean time from sidereal time.

The determination of the mean solar time at any station must, of course, depend directly or indirectly on observations of the sun. The mariner usually obtains the time from observations of the sun made with his sextant in the morning or the evening. This is an example of the direct method. But the astronomer, who has the use of fixed instruments of much greater power and precision than the sextant, generally deduces the mean time by calculation from the sidereal time, which, as already explained (§ 63), he obtains from observation of certain so-called "clock stars." The places of the clock stars he finds from the ephemeris. Those places depend on the position of ♈, which is determined from solar observations. Hence in the clock star method of finding the mean time the observations of the sun are only indirectly involved.

The ephemeris gives the true sidereal time at which the clock star culminates and the observer notes the time shown by his sidereal clock. The difference is the correction to his clock, and thus the sidereal time is known. The ephemeris also gives the sidereal time at Greenwich mean noon, so that a comparison of the mean time clock with the sidereal clock if made at noon will show the error of the former. The comparison of the mean clock and the sidereal clock cannot however generally be made at noon, nor will the longitude of the observer generally be zero. We therefore proceed as follows:

Let S be the local sidereal time,

„ T be the simultaneous local mean time,

„ l be the longitude of the observer west of Greenwich,

„ n be the number of mean solar days in a tropical year $= 365{\cdot}2422$,

„ M be the sidereal time at preceding mean noon at Greenwich.

There are $n+1$ sidereal days in n mean solar days, hence an interval of solar time is transformed into the equivalent sidereal time by the factor $(n+1)/n$, and an interval of sidereal time into the equivalent solar time by the factor $n/(n+1)$. In the case proposed the longitude is l, and we notice that this implies both of the following statements:

(1) The first point of Aries will move, in l hours of *sidereal* time, from the meridian of Greenwich to the meridian of the observer.

(2) The mean sun will move in l hours of *mean* time from the meridian of Greenwich to the meridian of the observer.

As S and T are the local sidereal and mean times at the instant considered, it follows that $S+l$ and $T+l$ are the corresponding Greenwich sidereal and mean times.

The interval $T+l$ of mean time is reduced to sidereal time by the factor $(n+1)/n$. By subtracting this from $S+l$ we must obtain the sidereal time at mean noon at Greenwich the same day, hence

$$M = S + l - (n+1)(T+l)/n,$$

which may be written in the equivalent forms often found convenient for use with the tables in the Ephemeris

$$T + l = (S + l - M)\,n/(n+1),$$
$$S + l = M + (T + l)\,(n+1)/n.$$

Probably the most practical method for determining mean time from sidereal time is as follows:

If we put $T=0$ in any one of the three equivalent equations just written and make M_l the local sidereal time of local mean noon we shall have

$$M = M_l + l - l\,(n+1)/n,$$

or

$$M_l = M + l/n.$$

This l/n is a *constant quantity for the particular meridian*. It is added to the sidereal time at mean noon at Greenwich to obtain the local sidereal time at local mean noon.

Then we have simply

$$T = (S - M_l)\,n/(n+1),$$

which may be very simply computed by the tables for converting intervals of sidereal time into the corresponding intervals of mean time.

Ex. 1. If M be the sidereal time at mean noon at Greenwich, show that M_l, the sidereal time at mean noon on the same day, at a station of which l is the longitude west from Greenwich, expressed in hours, is given by

$$M_l = M + 9^s{\cdot}8565 \times l.$$

Ex. 2. Show that if an interval of time is expressed by t when reckoned in mean time and by t' when reckoned in sidereal time, then

$$t' = t + 9^s{\cdot}8565t,$$
$$t = t' - 9^s{\cdot}8296t',$$

where t and t' are expressed in hours and fractional parts of an hour in the last term of each expression.

Ex. 3. On 1909, Feb. 18th, the sidereal time at Greenwich mean noon is $21^h\ 51^m\ 13^s{\cdot}55$. Show that the transit of the first point of Aries takes place at $2^h\ 8^m\ 25^s{\cdot}35$ mean time.

Ex. 4. Show that the Greenwich mean time of sidereal noon at Greenwich is $(24^h - M)\,n/(n+1)$, where M is the sidereal time at mean noon and n the number of mean solar days in the tropical year.

Show also that the local mean time of sidereal noon at west longitude l is obtained by subtracting $l/(n+1)$ from the Greenwich mean time of sidereal noon at Greenwich.

N.B. By sidereal noon is meant the moment of culmination of ♈.

Ex. 5. Show that $21^h\ 2^m\ 39^s$ is the sidereal time at Madras (longitude $5^h\ 21^m\ 0^s$ E.) at 1 P.M. Greenwich mean time on 1908, Nov. 1, being given that the sidereal time at Greenwich at mean noon is $14^h\ 41^m\ 29^s$.

Ex. 6. Columbia College, New York, is in longitude $4^h\ 55^m\ 54^s$ West of Greenwich. The sidereal time at mean noon at Greenwich on 1908, Dec. 12, is $17^h\ 23^m\ 8^s$. Show that on the same day when the sidereal time at Columbia College is $20^h\ 8^m\ 4^s$ the local mean time is $2^h\ 43^m\ 41^s$.

Ex. 7. The *sidereal* time in which the sun's semidiameter passes the meridian on 1908, July 1, being $1^m\ 8^s{\cdot}73$, show that the corresponding *mean* time is found by subtracting $0^s{\cdot}19$ from the sidereal time.

71. The terrestrial date line.

The notion of the *terrestrial date* line may be conveniently introduced by a particular illustration as follows:

Suppose the epoch to be 10 A.M. at Greenwich on Wednesday, June 14th, 1905. We have to consider for the same epoch what is the hour and more especially the *name* of the day on every other meridian east or west.

On the meridian $9^h\ 59^m$ west of Greenwich the time at the stated epoch is just after midnight, *i.e.* Wednesday has commenced. But on the meridian $10^h\ 1^m$ W. the time is $11^h\ 59^m$ P.M., and therefore on this meridian it is still Tuesday, June 13th. If we imagine each meridian all round the globe to be labelled with the day of the week (or month) pertaining to it at the epoch 10 A.M. June 14th, 1905, at Greenwich, there will be an abrupt change in the names on the labels when we come to the meridian 10^h W. from Greenwich.

But it is easily seen that another breach of continuity must present itself at some other meridian. For imagine we could move with the quickness of thought westward from the meridian 10^h W. all round the globe, we should begin by crossing meridians labelled Tuesday, but when the journey was near completion and we were approaching the meridian of 10^h W. from the east, we should find ourselves crossing meridians labelled Wednesday. There must therefore have been some other transition from a meridian labelled with one day to that labelled with another day.

This second breach of continuity in the labels on the meridians cannot have arisen as the first did by the occurrence of midnight. The change at a midnight point would be in the wrong direction, and indeed 10^h W. is the only meridian on the whole globe then at midnight. Every parallel of latitude must therefore possess a second point at which there is a breach of continuity in the dates pertaining to different places along that parallel. Any point on the parallel might be assigned for this purpose: we therefore choose it arbitrarily to suit general convenience. The convention followed is that the point shall be as near as is convenient to the meridian 12^h from Greenwich if it cannot be actually taken on that meridian. The actual "date line" as it is called is drawn from pole to pole. In so far as the meridian of 12^h passes through the open ocean, as it does during the greater part of its course, the date line coincides with that meridian. At other places the date line may swerve a little to one side or the other of the meridian of 12^h, so as not to pass, for example, across inhabited land in Alaska or to divide the Aleutian Islands in a way which would be inconvenient to their inhabitants.

In the case proposed the day is Wednesday, June 14th, at all west longitudes up to 10^h W. For two hours more of west longitude, *i.e.* from 10^h W. to 12^h W., or more accurately from 10^h W. up to the point where the date line crosses, the day is Tuesday, June 13th. But as the parallel crosses the date line the date suddenly alters. From being about 10 P.M. on Tuesday, June 13th, on one side of that line the date becomes about 10 P.M. on Wednesday, June 14th, on the other side of the line. Thus Wednesday, June 14th, prevails over all E. longitudes from 0^h to about 12^h. It thus appears that at the moment considered about 22 hours of longitude have the date Wednesday, June 14th, and 2 hours have Tuesday, June 13th.

As another example let the hour be 6 P.M. at Greenwich on

Sunday. Then at $5^h\ 59^m$ E. long. the time is $11^h\ 59^m$ P.M. on Sunday. At $6^h\ 1^m$ E. long. the time is $0^h\ 1^m$ A.M. on Monday. Thus as we move eastwards from 6^h E. long. to 12^h E. long., or more accurately to the date line on this parallel the day is Monday, but at the date line (where the actual time is about 6 A.M.) the date suddenly changes to Sunday at the same hour, and Sunday prevails at all west longitudes from the date line to Greenwich.

EXERCISES ON CHAPTER IX.

Ex. 1. If λ be the longitude of the sun, a its R.A. and ω the obliquity of the ecliptic, show that the greatest value of $\lambda - a$ occurs when $\tan \lambda = \sqrt{\sec \omega}$ and $\tan a = \sqrt{\cos \omega}$.

Ex. 2. On the 22nd of September the sun's declination at transit was observed to be $17' \ 2''{\cdot}80$ N., and on the 23rd it was observed to be $6' \ 21''{\cdot}56$ S.; also the sidereal interval of the two transits was $24^h\ 3^m\ 35^s{\cdot}50$. What was the sun's R.A. at the second observation?

Where would the chief errors be likely to occur in determining the first point of Aries by the method of this example?

[Coll. Exam.]

Ex. 3. The R.A. of Polaris is $1^h\ 21^m\ 18^s$; the sidereal times of mean noon at Greenwich on April 11 and 12 are respectively $1^h\ 19^m\ 50^s{\cdot}60$ and $1^h\ 23^m\ 47^s{\cdot}15$. Find the mean times of the three transits of Polaris at Greenwich on April 11.

[Coll. Exam.]

Ex. 4. Given from the *Nautical Almanac*

Sidereal time of mean noon March 21, 1898 ...... $23^h\ 56^m\ 5^s{\cdot}87$

Sidereal time of mean noon March 22, 1898 ...... 0 0 2·42,

find approximately the mean time at which the mean sun passed the vernal equinox.

[Coll. Exam.]

Ex. 5. On Feb. 7 a star, the R.A. of which is $5^h\ 9^m\ 43^s{\cdot}9$, is in transit at Sydney (longitude $151° \ 12' \ 23''$ E.), when the time by the observer's watch which should keep local time is $8^h\ 0^m\ 3^s$. Given that the mean sun's R.A. at mean noon at Greenwich on Feb. 7 is $21^h\ 8^m\ 36^s{\cdot}1$, and that 1^h of sidereal time is equivalent to $59^m\ 50^s{\cdot}2$ of mean time, find to the nearest second how much the watch is fast or slow.

[Math. Trip.]

Ex. 6. Show that a single altitude of a known star is sufficient to determine the latitude if the local sidereal time be known, and to determine the local sidereal time if the latitude be known.

If the observed altitude have an error x minutes of arc, then the deduced sidereal time will have an error $\frac{1}{15}x \sec\lambda \operatorname{cosec} a$ minutes of time; where λ is the latitude of the place and a the azimuth of the star at the instant of observation. [Math. Trip.]

Ex. 7. If a star of declination δ has a zenith distance z when observed near the meridian at an hour angle t, show that unless $\phi-\delta$ is very small the latitude ϕ may be determined accurately by the equation

$$\phi = z + \delta \quad \frac{2\cos\delta\cos\phi}{\sin(\phi-\delta)}\sin^2\tfrac{1}{2}t,$$

in the last term of which an approximate value of ϕ may be used.

*Ex. 8. If the sidereal clock times when the sun arrives at equal altitudes on each side of the meridian are u' and u, and if the change of declination δ of the sun in the interval be $d\delta$, and the right ascension of the sun at culmination is a, show that the correction to be applied to clock time to obtain the true sidereal time is

$$a - \tfrac{1}{2}(u+u') - \tfrac{1}{2}\left(\frac{\tan\delta}{\tan\frac{1}{2}(u'-u)} - \frac{\tan\phi}{\sin\frac{1}{2}(u'-u)}\right)d\delta,$$

and explain why no account need be taken of the sun's movement in right ascension between the two observations.

*Ex. 9. Show that, if $l-\delta$ is the zenith distance of the sun observed near the meridian when it is in declination δ, and h is its hour angle measured in seconds of time, the latitude of the place is approximately

$$l - \frac{\cos l\cos\delta\sin 1''}{2\sin(l-\delta)}(15h)^2.$$

Show also that, if the observation is made from a ship in motion in a direction making an angle θ with the meridian, the greatest altitude occurs when the sun is approximately h seconds of time from the instantaneous meridian, where

$$h = \frac{\sin(\delta-\phi)}{\cos\phi\cos\delta}\left(\frac{v\cos\theta}{\rho\sin 1''} - m\right)\frac{1}{15^2.60^2.\sin 1''},$$

v is the space described by the ship per hour, ρ the radius of the earth, ϕ the latitude of the place, δ the declination of the sun, and m the change of declination per hour measured in seconds of arc. [Math. Trip.]

CHAPTER X.

THE SUN'S APPARENT ANNUAL MOTION

72. The reduction to the equator.

As the sun performs its annual circuit of the ecliptic its true longitude ☉ measured of course from ♈ and in the direction of the sun's motion is continuously, though not uniformly, increasing. In like manner the sun's right ascension α is continuously, though not uniformly, increasing. The difference between the right ascension and the longitude, that is to say the quantity $(\alpha - \odot)$ which must be added to the sun's longitude to give the sun's right ascension, is called the *reduction to the equator*. We are now to consider the variations in the reduction to the equator in the course of the year. The centre of the sun is presumed to be in the ecliptic, as we need not here take account of its small latitude which is $< 1''$.

Let α, δ be the right ascension and declination of a point on the ecliptic. The longitude of the point is ☉ and if ω be the obliquity of the ecliptic we have

$$\tan \alpha = \cos \omega \tan \odot \quad \text{.....................(1)},$$

and we can transform this equation into

$$\sin (\alpha - \odot) = - \tan^2 \tfrac{1}{2} \omega \sin (\alpha + \odot) \quad \text{...........(2)}.$$

Thus we see that $\alpha - \odot$ must lie between the limits $- \sin^{-1} \tan^2 \frac{1}{2} \omega$

and $+\sin^{-1}\tan^2\frac{1}{2}\omega$ or if we take for ω its mean value for 1910, viz. 23° 27′ 4″, we may say that the reduction to the equator varies between $-\theta$ and $+\theta$ where $\theta = 2° \; 28' \; 8''$. The variations of the reduction as ☉ increases from 0° to 360° may be indicated as follows:

As α and ☉ start together from ♈ where they are both zero ☉ is at first the greater so that the reduction is at first negative and attains a minimum $-\theta$ when ☉ is $45° + \frac{1}{2}\theta$. Then the right ascension begins to gain on the longitude so that ☉ and α reach 90° together and the reduction is zero. In the second quadrant α gradually increases its lead over ☉ until α becomes $135° + \frac{1}{2}\theta$ when ☉ is $135° - \frac{1}{2}\theta$ and the reduction has its maximum value of $+\theta$. In the third quadrant there is another minimum $-\theta$ when $\odot = 225° + \frac{1}{2}\theta$ and in the fourth quadrant there is another maximum of $+\theta$ when $\odot = 315° - \frac{1}{2}\theta$. Finally the values of ☉ and α coincide at 360° when the circuit is complete.

For the calculation of the reduction we use the formula (3) which is easily derived from (1)

$$\tan(\alpha - \odot) = -\tan^2\tfrac{1}{2}\omega \sin 2\odot / (1 + \tan^2\tfrac{1}{2}\omega \cos 2\odot) \ldots(3)$$

by which the reduction is obtained at once for any given longitude. It is also convenient to obtain an expression for $(\alpha - \odot)$ in a series ascending by powers of the small quantity $\tan^2\frac{1}{2}\omega$. This is most readily deduced from equation (1) by a well known expansion. (See Todhunter's *Plane Trigonometry*, p. 238.)

$$\alpha - \odot = -\tan^2\tfrac{1}{2}\omega \sin 2\odot + \tfrac{1}{2}\tan^4\tfrac{1}{2}\omega \sin 4\odot$$
$$-\tfrac{1}{3}\tan^6\tfrac{1}{2}\omega \sin 6\odot + \ldots \quad \ldots\ldots(4).$$

The terms in this formula are expressed in radians but the reduction to the equator is more conveniently expressed by putting for each radian its equivalent of $86400/2\pi = 13751$ seconds of time. If we multiply the expression for $(\alpha - \odot)$ given in (4) by 13751 and if we further reduce it by substituting for ω its mean value already given we have

$$\alpha - \odot = -592^s{\cdot}38 \sin 2\odot + 12^s{\cdot}76 \sin 4\odot - 0^s{\cdot}36 \sin 6\odot \ldots(5).$$

The coefficients of the terms in series (5) decrease so rapidly that there is no need to take account of more than the three terms there written and even the last of these may be generally omitted.

If we had assumed that no more than two terms of (4) would be required, then those terms could have been obtained otherwise from (3) for we have by Gregory's series

$$\alpha - \odot = \tan(\alpha - \odot) - \tfrac{1}{3}\tan^3(\alpha - \odot) + \ldots$$
$$= -\sin 2\odot \tan^2\tfrac{1}{2}\omega\,(1 + \cos 2\odot \tan^2\tfrac{1}{2}\omega)^{-1} + \tfrac{1}{3}\sin^3 2\odot \tan^6\tfrac{1}{2}\omega$$

which gives the desired expression when quantities smaller than $\tan^6\frac{1}{2}\omega$ are neglected.

Ex. 1. Prove the following graphic method of obtaining the reduction to the equator for any given longitude $\odot$.

Describe a circle with centre C (Fig. 64) and radius $CA = \tan^2\frac{1}{2}\omega$, and take a fixed point O so that $CO = 1$. Find the point P on the circle such that $\angle OCP = 2\odot$ and let P' be the point on the circle diametrically opposite to P. Then $\angle P'OC$ is the reduction with its sign changed.

Let A and B be the points in which CO cuts the circle. Join AP' and

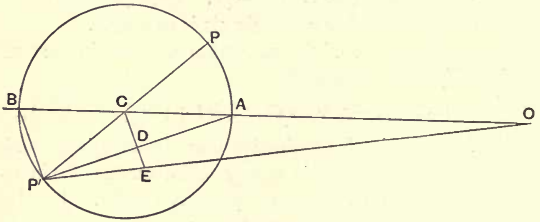


FIG. 64.

BP'. Draw CD perpendicular to AP' and produce it to meet OP' in E. Then the anharmonic ratio of the pencil $P'(OACB)$ is

$$OA/OB = (1 - \tan^2\tfrac{1}{2}\omega)/(1 + \tan^2\tfrac{1}{2}\omega) = \cos\omega.$$

But since AP' is perpendicular to BP' and to CD we have the same anharmonic ratio also equal to

$$ED/DC = \tan EP'D/\tan DP'C.$$

Hence $\tan EP'D = \cos\omega \tan DP'C = \cos\omega \tan\odot$.

Therefore $EP'D = \alpha$, and since $CAP' = CP'A = \odot$ we have $P'OC = \odot - \alpha$.

Ex. 2. Prove the following construction. Take any line AB and cut off a part AC such that $AC = AB\cos\omega$. At A erect AL perpendicular to AB. Draw the line CP to meet AL in P so that $\angle ACP = \odot$. Join BP. Then $\angle ABP = \alpha$, and $\angle BPC$ is the reduction to the equator.

Ex. 3. Show from Ex. 1 that the greatest value of the reduction is $\sin^{-1}(\tan^2\frac{1}{2}\omega)$ and that in this case AP in Ex. 2 is a tangent to the circle circumscribing CBP and that α and $\odot$ are complementary.

Ex. 4. Show that, if the sun be supposed to move uniformly in the ecliptic and another body to move at the same uniform rate in the equator, the difference of their right ascensions will vanish four times in the year only if the interval between their passages through the first point of Aries be less than $\sin^{-1}(\tan^2\frac{1}{2}\omega)/2\pi$ of a year. [Coll. Exam.]

Let t be the fraction of a year that has elapsed between the passage of the sun through ♈ and the passage of the body moving in the equator through ♈. Then, when their right ascensions are both a,

$$\tan(2\pi t + a)\cos\omega = \tan a.$$

We thus have for a the equation

$$\tan^2 a \tan 2\pi t - (1-\cos\omega)\tan a + \tan 2\pi t \cos\omega = 0.$$

The roots of this will be real if

$$2\pi t < \sin^{-1}(\tan^2 \tfrac{1}{2}\omega).$$

There will thus be two real values of $\tan a$ and four of a.

Ex. 5. Assuming the sun's apparent orbit to be circular, show that the ratio of the sidereal times occupied by the passage of the sun's diameter across the meridian at an equinox and at a solstice is approximately $(\cos\omega - \cdot 0027\sin^2\omega)$, where ω is the obliquity of the ecliptic.

[Math. Trip.]

If R be the sun's radius and δ its declination, then at the moment when the preceding limb is on the meridian the hour angle of its centre is $-R\sec\delta$. If t_1 denote the sidereal time and a_1 the sun's R.A. at this moment we have

$$t_1 - a_1 = -R\sec\delta.$$

Similarly, when the following limb is on the meridian, we have

$$t_2 - a_2 = +R\sec\delta.$$

And therefore $$(t_2 - t_1) - (a_2 - a_1) = 2R\sec\delta.$$

Differentiating the equation $\tan a = \cos\omega \tan \odot$, and remembering that $\cos\odot = \cos a \cos\delta$, we find

$$\frac{da}{dt} = \cos\omega \sec^2\delta \frac{d\odot}{dt}.$$

But since t increases 360° in one day, whereas $\odot$ increases by the same amount in about 365 days, we have $d\odot/dt = 1/365 = \cdot 0027$. Hence

$$da/dt = \cdot 0027 \cos\omega \sec^2\delta.$$

And $$a_2 - a_1 = (t_2 - t_1)\, da/dt,$$

therefore $$(t_2 - t_1)\{1 - \cdot 0027\cos\omega\sec^2\delta\} = 2R\sec\delta,$$

or $$t_2 - t_1 = 2R/\{\cos\delta - \cdot 0027\cos\omega\sec\delta\}.$$

At the equinoxes $\delta = 0$ and at the solstices $\delta = \pm\omega$, hence the sidereal time occupied by the passage of the sun's diameter across the meridian at the equinox is to that at the solstice in the ratio

$$(\cos\omega - \cdot 0027)/(1 - \cdot 0027\cos\omega) = \cos\omega - \cdot 0027\sin^2\omega.$$

73. The equation of the centre.

Let ♈A be the equator, and ♈B, Fig. 65, be the ecliptic, where S is the position of the sun and P the perigee of the sun's apparent orbit, i.e. the point occupied by the sun when nearest to

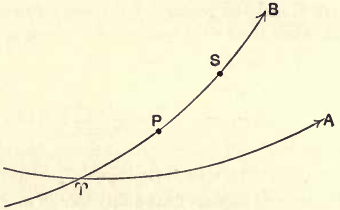


FIG. 65.

the earth

♈$P \equiv \varpi$, the longitude of the perigee,

$PS \equiv v$, the sun's true anomaly,

♈$S \equiv \odot \equiv \varpi + v$, the sun's true longitude.

Let n be the average value for the whole year of the apparent angle daily swept over by the radius vector drawn from the earth's centre to the sun's centre. The sun's mean longitude is expressed by $L \equiv nt + \epsilon$, where t is the time in days and ϵ is the value of the mean longitude at the epoch from which the time is measured. The sun's mean anomaly is $L - \varpi$ and the corresponding true anomaly is $\odot - \varpi$.

In § 52 we have determined the relation between the true anomaly and the mean anomaly in an elliptic orbit, and substituting in the formula there given $(\odot - \varpi)$ for v and $L - \varpi$ for m, we obtain

$$\odot = L + (2e - \tfrac{1}{4}e^3)\sin(L - \varpi) + \tfrac{5}{4}e^2 \sin(2L - 2\varpi) + \tfrac{13}{12}e^3 \sin(3L - 3\varpi) \quad \text{.........(1)},$$

where e is the eccentricity of the earth's orbit.

The terms involving e^3 are too small to require attention for most purposes. We shall neglect them as before and write simply

$$\odot = L + 2e\sin(L - \varpi) + \tfrac{5}{4}e^2 \sin(2L - 2\varpi) \text{.........(2)}.$$

We have thus obtained the expression for the true longitude of the sun in terms of its mean longitude.

Reversing this series we obtain by omitting powers of e above the second

$$L = \odot - 2e \sin(\odot - \varpi) + \tfrac{3}{4}e^2 \sin(2\odot - 2\varpi) \quad \text{......(3)},$$

which expresses the sun's mean longitude in terms of its true longitude.

It is now necessary to know the numerical values of e and ϖ, and we have to show how continued observations of the sun's right ascension enable us to determine these quantities. This is done by formula (3) which we shall transform by making $x = e \cos \varpi$, $y = e \sin \varpi$, $L = nt + \epsilon$, and in the first instance neglecting e^2 as very small, we have

$$nt + \epsilon = \odot - 2x \sin \odot + 2y \cos \odot \quad \text{...........(4)},$$

as the approximate formula.

In this equation there are four unknowns, n, ϵ, x, y, and to determine them we must suppose that a series of determinations of the longitude $\odot_1$, $\odot_2$, $\odot_3$, $\odot_4$...... have been calculated from observations of the right ascension made at certain times t_1, t_2, t_3, t_4....... Each of these quantities represents the number of mean solar days since a moment taken as the epoch. Each value of $\odot$ and its corresponding date t when substituted in (4) will give a linear equation connecting n, ϵ, x, y. Four such equations would therefore make these quantities determinate, though for increased accuracy the result should be based on very many observations extended over many years. Thus x and y and therefore e and ϖ become known approximately. At the same time n and ϵ become known and the expression for the mean longitude L is determined. We now substitute the approximate values for e and ϖ in the term involving e^2 in equation (3), for as this term is so small there will be no appreciable error in its value even though e and ϖ may not be quite correct. Thus a more accurate linear equation between n, ϵ, x, y is obtained and each observation will supply one such equation. In this way e and ϖ may be obtained with all desirable precision.

The length of the tropical year is $360/n$ days, and of course if we had been at liberty to assume as we have so often done already that the tropical year is 365·2422 days we should not have described n as an unknown. But it is necessary to point out that it is by such an investigation as that now given that this value of the tropical year has been itself determined so that having found n from the system of equations we obtain $360/n$. The quantity ϵ is the sun's mean longitude at the epoch. We thus obtain the formula for L which has been already determined in a more elementary manner in § 67, Ex. 4.

It remains to give equations (2) and (3) their numerical forms by introducing the actual values of e and ϖ for the earth's orbit. These are for the year 1900

$$e = 0{\cdot}01675, \qquad \varpi = 281° \; 13',$$

and though on account of the perturbations caused by the other planets these quantities are not strictly constant, their changes from year to year are far too minute to be of any consequence for our present purpose. Substituting these values and using the value 3438′ for one radian we obtain

$$\odot = L + 115'{\cdot}2 \sin (L - 281°{\cdot}2) + 1'{\cdot}2 \sin (2L - 202°{\cdot}4) \ldots (5),$$

$$L = \odot - 115'{\cdot}2 \sin (\odot - 281°{\cdot}2) + 0'{\cdot}7 \sin (2\odot - 202°{\cdot}4) \ldots (6).$$

We may thus make the approximate statements:

The true longitude $\odot$ of the sun at any epoch is obtained by adding to the mean longitude L of the sun at the same epoch the quantity which has been defined in § 52 as the *equation of the centre*, and for which we have now found the expression

$$115' \sin (L - 281°).$$

The mean longitude L of the sun at any epoch is obtained by adding to the true longitude $\odot$ of the sun at the same epoch the quantity

$$-115' \sin (\odot - 281°).$$

Ex. 1. Show that the equation of the centre is never zero unless the sun is at one of the apsides.

*Ex. 2. Show that if attention is paid to the seconds of arc, the formula (5) is to be written

$$\odot = L + 1344'' \sin L + 6778'' \cos L - 67'' \sin 2L + 28'' \cos 2L.$$

74. The equation of time.

We can now express α, the right ascension of the sun, in terms of its mean longitude L, for if $\odot$ be the true longitude then from §§ 72, 73,

$$\alpha = \odot - \tan^2 \tfrac{1}{2}\omega \sin 2\odot + \tfrac{1}{2} \tan^4 \tfrac{1}{2}\omega \,.\, \sin 4\odot,$$

$$\odot = L + 2e \sin (L - \varpi) + \tfrac{5}{4} e^2 \sin (2L - 2\varpi).$$

The expression of α in terms of L involves a number of terms with small coefficients. As the formulae need not be encumbered with terms which are too small to produce an appreciable effect we shall not retain any power or product of powers of e and $\tan^2 \frac{1}{2}\omega$ which is less than 1/10,000. This condition excludes all except $\tan^2 \frac{1}{2}\omega = 1/23{\cdot}21$, $e = 1/59{\cdot}70$, $\tan^4 \frac{1}{2}\omega = 1/538{\cdot}7$, $e \tan^2 \frac{1}{2}\omega = 1/1385$ and $e^2 = 1/3564$.

Eliminating $\odot$ we obtain

$$\alpha = L + 2e \sin(L - \varpi) + \tfrac{5}{4}e^2 \sin 2(L - \varpi)$$
$$- \tan^2 \tfrac{1}{2}\omega \{\sin 2L + 4e \sin(L - \varpi) . \cos 2L\} + \tfrac{1}{2}\tan^4 \tfrac{1}{2}\omega . \sin 4L,$$

which may be written

$$\alpha = L + 2e \sin(L - \varpi) - \tan^2 \tfrac{1}{2}\omega . \sin 2L + 2e \tan^2 \tfrac{1}{2}\omega \sin(L + \varpi)$$
$$+ \tfrac{5}{4}e^2 \sin 2(L - \varpi) - 2e \tan^2 \tfrac{1}{2}\omega \sin(3L - \varpi) + \tfrac{1}{2}\tan^4 \tfrac{1}{2}\omega . \sin 4L.$$

As the quantities e^2, $e \tan^2 \frac{1}{2}\omega$ and $\tan^4 \frac{1}{2}\omega$ are very small, the first two terms of the expression $(\alpha - L)$ are by far the most important, and the others may for our present purposes be neglected, so that we have

$$\alpha = L + E,$$

where $$E = 2e \sin(L - \varpi) - \tan^2 \tfrac{1}{2}\omega \sin 2L.$$

The quantity E is called the *equation of time*. It is to be added to the mean longitude of the sun to give the sun's right ascension.

E is here expressed in radians. We transform it into time at the rate of 2π radians to 24 hours, and consequently for the equation of time in hours, we have

$$12 \{2e \sin(L - \varpi) - \tan^2 \tfrac{1}{2}\omega \sin 2L\}/\pi,$$

or in seconds of time

$$13751 \{2e \sin(L - \varpi) - \tan^2 \tfrac{1}{2}\omega \sin 2L\}.$$

If in this we make $e = {\cdot}001675$, $\varpi = 281^\circ{\cdot}2$, we obtain the approximate result

$$\alpha = L + 90^s \sin L + 452^s \cos L - 592^s \sin 2L,$$

which is equivalent to the statement that

$$E = 90^s \sin L + 452^s \cos L - 592^s \sin 2L \quad \text{.........(i)}$$

is the equation of time when small terms are omitted.

At any sidereal time ϑ the hour angle of the true sun is $\vartheta - \alpha$, or

$$\text{apparent solar time} = \vartheta - \alpha.$$

At the same instant the hour angle of the mean sun is equal to $\vartheta - L$, or

$$\text{mean solar time} = \vartheta - L = (\vartheta - \alpha) + (\alpha - L).$$

The equation of time is therefore defined to be *the correction to be added algebraically to the apparent solar time to find the mean solar time.*

Ex. 1. Determine approximately the equation of time at mean noon on Dec. 27, 1910, being given that the sun's mean longitude is then 275°.

By substitution in (i) we find $E = +53^s$. If all the terms now neglected had been taken account of we should have obtained $52^s \cdot 81$ as given in the ephemeris. The R.A. of the true sun a is thus 53^s greater than L, which is the R.A. of the mean sun. At apparent noon the mean sun is already past the meridian by 53^s, so that to obtain the mean time we have to *add* 53^s to the apparent time.

Ex. 2. Show that the equation of time is about $7\frac{1}{2}^m$ at the vernal equinox and about $-7\frac{1}{2}^m$ at the autumnal.

Ex. 3. Find the sun's true R.A. at apparent noon on Nov. 1st, 1902, given that the equation of time at mean noon that day is $-16^m\ 18^s$, and that the sidereal time of mean noon on June 14 is $5^h\ 27^m\ 23^s$. (Take a tropical year to be $365\frac{1}{4}$ days.)

[Oxford Second Public Examination, 1902.]

Ex. 4. Show that at the summer solstice, the equation of time has an hourly increase of about 0·53 seconds, it being assumed that the daily motion of the mean sun in arc is 59′ 8″·32.

We have (1) $90^s \sin L + 452^s \cos L - 592^s \sin 2L$ for the equation of time. For a small change ΔL in L this increases by

$$(90^s \cos L - 452 \sin L - 1184 \cos 2L)\, \Delta L.$$

In one hour $\Delta L = 147''\cdot 85$, or, in radians, ·000717. Hence the hourly change in the equation of time is

$$\Delta E = 0^s \cdot 064 \cos L - 0^s \cdot 324 \sin L - 0^s \cdot 849 \cos 2L.$$

In the particular case supposed $L = 90°$, and $\Delta E \doteq 0^s \cdot 525$.

Ex. 5. Show that the greatest value of the equation of time arising from the eccentricity is $24e/\pi$ hours.

75. Formulae connected with the equation of time.

It is convenient to bring together various formulae connected with the equation of time. The observer is supposed to be in longitude l west of Greenwich, and at a certain moment he observes the apparent solar time. The following is the notation employed:

α	is the sun's R.A. at the moment of observation,				
A	„	apparent time	„	„	„
T	„	local mean time	„	„	„
ϑ	„	local sidereal time	„	„	„
E	„	equation of time	„	„	„
E_0	„	equation of time at the preceding G.M.N.,			
E_1	„	„ „ „ following „			
M_0	„	Greenwich sidereal time at preceding G.M.N.,			
M_1	„	„ „ „ following „			

We have from the definition of E (see p. 234)

$$T = A + E \quad \text{.........................(i)},$$

At the moment of observation the G.M.T. is $A + E + l$, and assuming that E changes uniformly, we have

$$E = E_0 + (A + E + l)(E_1 - E_0)/24^h \quad \text{...........(ii)}.$$

We have also

$$\alpha = \vartheta - A \quad \text{.........................(iii)}.$$

When the Greenwich mean time is $T + l$ the Greenwich sidereal time is $\vartheta + l$. The sidereal interval since the preceding Greenwich mean noon is therefore $\vartheta + l - M_0$, and this is converted into mean time by applying the factor $24^h/(24^h + M_1 - M_0)$. We thus have the equation

$$T + l = 24^h(\vartheta + l - M_0)/(24^h + M_1 - M_0),$$

from which we obtain

$$T = \vartheta - M_0 - (M_1 - M_0)(\vartheta - M_0 + l)/(24^h + M_1 - M_0) \text{...(iv)}.$$

From the equations (i), (ii), (iii), (iv) involving the six quantities α, A, T, ϑ, E, l we can determine any four when the two others are given. It is understood that E_0, E_1, M_0, M_1 are constants for the day obtained from the ephemeris.

Ex. 1. Show that at any place and at any moment the sidereal time ϑ, the mean time T, the right ascension of the sun a and the equation of time E are connected by the relation

$$\vartheta - a + E - T = 0.$$

Ex. 2. If t be the Greenwich mean time, M_0, M_1, E_0, E_1 the sidereal times and the equations of time at the preceding and the succeeding Greenwich mean noons, and if A be the observed apparent solar time, find the longitude, and show that the local sidereal time is

$$M_0 + E_0 + t(M_1 + E_1 - M_0 - E_0)/24^h + A.$$

[Coll. Exam.]

Ex. 3. If at Greenwich a, a' are the hour angles (in degrees) of the sun at t and t' hours mean time, show that the equations of time at the preceding and following mean noons expressed in fractions of an hour are respectively

$$\frac{a't-at'}{15(t'-t)}, \qquad 24+\frac{a'(24-t)-a(24-t')}{15(t'-t)}.$$

[Math. Trip.]

Ex. 4. Show from (ii) that

$$E=24^h E_0/(24^h+E_0-E_1)+(A+l)(E_1-E_0)/(24^h+E_0-E_1),$$

and prove from the formulae given on the last page that the corresponding mean time at Greenwich is

$$24^h(A+E_0+l)/(24^h+E_0-E_1).$$

Ex. 5. Find the sidereal time at New York, in longitude 73° 58′ 24″·6 West at apparent noon on October 1, given that the numerical values of the equation of time at Greenwich mean noon on October 1 and October 2 are $10^m\ 23^s{\cdot}28$ and $13^m\ 42^s{\cdot}28$ respectively, and that the sidereal times at Greenwich mean noon on those days are $12^h\ 40^m\ 57^s{\cdot}62$ and $12^h\ 44^m\ 54^s{\cdot}17$ respectively.

[Math. Trip.]

Ex. 6. On April 15th and 16th, 1895, at Greenwich mean noon, the equation of time is given as $1^s{\cdot}57$ and $13^s{\cdot}09$, to be subtracted from and added to mean time respectively. Find the apparent hour angle of the sun at a place 4° E. of Greenwich at $11^h\ 58^m$ local mean time on April 16.

[Coll. Exam.]

76. Graphical representation of the equation of time.

It appears from § 74 that the equation of time, E, when expressed in hours of mean solar time is with sufficient approximation

$$E \equiv 12\,\{2e \sin(L-\varpi) - \tan^2\tfrac{1}{2}\omega \sin 2L\}/\pi.$$

Making in this expression the following approximate substitutions,

$$\tan^2\tfrac{1}{2}\omega = 1/23{\cdot}2, \quad e = 1/59{\cdot}7, \quad \varpi = 360° - 79°,$$

we obtain after reduction (or directly from (i), p. 233)

$$E \equiv 0^h{\cdot}128 \sin(L+79°) - 0^h{\cdot}165 \sin 2L$$
$$\equiv 7^m{\cdot}68 \sin(L+79°) - 9^m{\cdot}90 \sin 2L.$$

We then plot (Fig. 66) the two curves of which the equations are

$$y = 7^m{\cdot}68 \sin(L+79°) \quad\text{.....................(i)},$$

and

$$y = 9\ \ {\cdot}90 \sin 2L \quad\text{.........................(ii)},$$

where the mean longitude L is taken as abscissa, the ordinates being laid off positively and negatively as shown in the left-hand margin. The curves are plotted for every longitude from 0° to 360°, and are thus available from the vernal equinox of one year to that of the next. The diagram can be used without appreciable alteration for a great number of successive years.

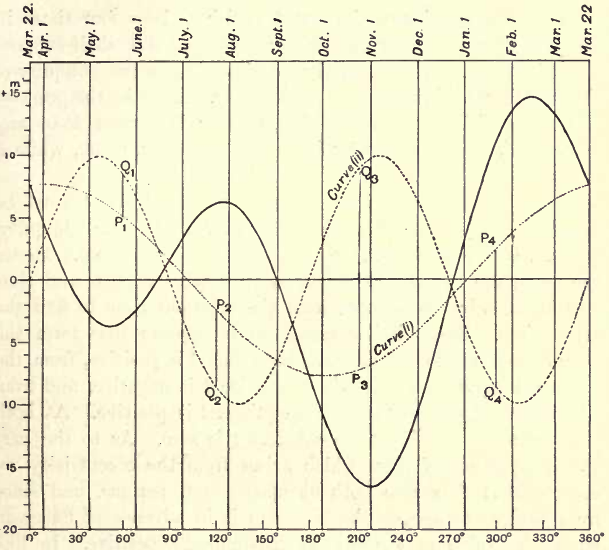


Fig. 66.

The use of the curves depends upon the fact that the equation of time is equal to the ordinate of curve (i) *minus* the ordinate of curve (ii), it being understood that in accordance with the usual convention, ordinates on the upper side of the horizontal axis are positive and those below are negative.

Thus on May 22nd the equation of time is Q_1P_1 and is negative. On July 22nd it is Q_2P_2 and positive. On Oct. 22nd it is Q_3P_3 and negative, and on Jan. 22nd it is Q_4P_4 and positive.

In this way, taking as ordinate the difference of the ordinates of curves (i) and (ii) with its proper sign, we obtain the continuous curve in Fig. 66, the ordinates of which represent the equation of time for every day in the year.

There are four places at which the curves (i) and (ii) intersect and at which consequently the equation of time is zero. Thus we learn that the equation of time vanishes four times a year and the continuous curve cuts the horizontal axis at four points which show the corresponding dates.

That the equation of time must vanish at least four times in the year may be shown otherwise as follows. We shall suppose that t is the part of the equation of time due to the obliquity of the ecliptic, and t' that due to eccentricity. Let k be the greatest value of t without regard to sign, then k is greater than any value of t'. The value of k is, as we have seen, $9^{m}\cdot90$, while t' never exceeds $7^{m}\cdot68$.

From the vernal equinox to the summer solstice t must be negative, because so far as the inequality arises from obliquity, the R.A. of the mean sun exceeds that of the true sun. Hence the mean sun crosses the meridian after the true sun, and thus a subtraction has to be made from the apparent time to find the mean time. From similar reasoning it appears that from the summer solstice to the autumnal equinox t is positive, from the autumnal equinox to the winter solstice t is negative, and from the winter solstice to the vernal equinox t is positive. At both the equinoxes and both the solstices t is zero. As to the part of the equation of time which arises from the eccentricity we observe that t' is zero both at apogee and perigee, and since from perigee to apogee the true sun is in advance of its mean place, the value of t' must be continuously positive. In like manner t' must be negative all the way from apogee to perigee.

Let P, A (Fig. 67) be the perigee and apogee respectively, S, W the positions of the sun at the summer and winter solstices, and ♈, ♎ the equinoctial points.

Let M be the point occupied by the sun at the moment when t, which is zero at ♈ and at S, has its greatest negative value. Then remembering that E, the equation of time, is $t + t'$, we see that from P to ♈ the value of E must be continuously positive, for t and t' are both positive.

At M we have $E = t' - k$, and as t' can never equal k, we must have E negative at M. Since E is positive at ♈, negative at M, and again positive at S, there must be some point between ♈ and M, and also another between M and S, where $E = 0$. Thus the equation of time must vanish at least twice between the vernal equinox and the summer solstice.

From S to A both t and t' are positive, and therefore E is continuously positive from the summer solstice to the apogee. But from A to ♎ t' is negative, and as t is zero at ♎ and t' is still negative, E must be negative at ♎ and positive at A. It follows that E must be zero somewhere between A and ♎, and thus the equation of time must vanish at least once again between apogee and the autumnal equinox.

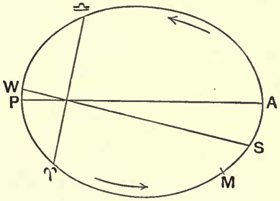


FIG. 67.

From ♎ to W both t and t' are continuously negative, and thus the equation of time cannot vanish between these points of the orbit. At P we find E has become positive again, and therefore it must vanish at least once between W and P.

We thus learn that the equation of time vanishes at least twice between the vernal equinox and the summer solstice, at least once between the apogee and the autumnal equinox, and at least once between the winter solstice and the perigee.

Ex. 1. If we take x as the tangent of the sun's mean longitude L, show that the days on which the equation of time vanishes can be found graphically as the intersections of the curve $y=x(1+x^2)^{-\frac{1}{2}}$ with a straight line; and if the equation of this line be

$$y=0{\cdot}07x+0{\cdot}38,$$

estimate roughly the dates in question.

[Coll. Exam.]

In the equation $\tan^2\frac{1}{2}\omega\sin 2L=2e\sin(L-\varpi)$ we make $\tan L=x$ and the equation in x thus obtained is obviously the result of eliminating y between the two equations

$$y=ex\cos\varpi\cot^2\tfrac{1}{2}\omega-e\sin\varpi\cot^2\tfrac{1}{2}\omega,$$

$$y=x(1+x^2)^{-\frac{1}{2}}.$$

Ex. 2. It has been shown that the equation of time expressed in seconds is

$$90\sin L+452\cos L-592\sin 2L,$$

where L is the mean longitude of the sun. Prove from this expression that the equation of time vanishes at least four times annually.

If we substitute successively 0°, 45° for L in the expression, the sign changes from + to −, hence there must be some value for L between 0° and 45° which is a root of the equation

$$90 \sin L + 452 \cos L - 592 \sin 2L = 0.$$

We have further changes of sign for values of L between 45° and 90°, between 90° and 180° and between 180° and 360°. Hence there must be four real roots, and when some other small terms are taken account of it is found that the roots are approximately

$$23°, \quad 83°, \quad 159°, \quad 272°,$$

and the corresponding dates on which the equation of time vanishes are

15th Ap., 14th June, 31st Aug., 24th Dec.

Ex. 3. Show that the sum of the solar longitudes on the four occasions on which the equation of time vanishes must be 540° if the square of e and the 4th power of $\tan \frac{1}{2}\omega$ are neglected.

We have $$2e \sin (L - \varpi) = \tan^2 \tfrac{1}{2}\omega \sin 2L,$$

and making

$$x = \tan L, \quad m = e \cos \varpi \cot^2 \tfrac{1}{2}\omega, \quad n = e \sin \varpi \cot^2 \tfrac{1}{2}\omega,$$

we find $$m^2x^4 - 2mnx^3 + (m^2 + n^2 - 1)\, x^2 - 2mnx + n^2 = 0.$$

The coefficients of x^3 and x in the equation are equal, and if we express this fact in terms of the longitudes L_1, L_2, L_3, L_4 corresponding to the four roots, we have the condition

$$\tan (L_1 + L_2 + L_3 + L_4) = 0.$$

This equation shows that $L_1 + L_2 + L_3 + L_4 = k \,.\, 180°$, where k is an integer. But we have seen (Ex. 2) that $L_3 > 90°$ and $L_4 > 270°$, hence $k > 2$. Also $L_1 + L_2 < 180°$, $L_3 < 180°$ and $L_4 < 360°$, hence $k < 4$. Thus the only admissible value of k is 3, and accordingly

$$L_1 + L_2 + L_3 + L_4 = 540°.$$

It is also easy to show that

$$\sin L_1 + \sin L_2 + \sin L_3 + \sin L_4 + \sin L_1 \sin L_2 \sin L_3 + \sin L_1 \sin L_3 \sin L_4 + \sin L_1 \sin L_2 \sin L_4 + \sin L_1 \sin L_2 \sin L_3 = 0.$$

Ex. 4. If the eccentricity of the earth's orbit be 1/60, the cosine of the obliquity 11/12, and the line of equinoxes be taken as perpendicular to the major axis of the orbit, prove that the longitudes of the sun when the equation of time due to both causes conjointly is numerically a maximum are angles whose sines are approximately 0·617 and −0·809.

[Math. Trip. I.]

The equation of time being

$$2e \sin (L - \varpi) - \tan^2 \tfrac{1}{2}\omega \sin 2L,$$

becomes a maximum for $\varpi = 90°$ when

$$e \sin L - \tan^2 \tfrac{1}{2}\omega \cos 2L = 0.$$

Introducing the given constants the equation is $\frac{1}{60} \sin L - \frac{1}{23} \cos 2L = 0$, whence we obtain a quadratic for $\sin L$ whose roots are the given numbers.

77. General investigation of stationary equation of time.

We shall now determine when the equation of time is a maximum or a minimum independently of any assumption with regard to the obliquity of the ecliptic or the eccentricity of the earth's orbit. We shall however suppose that the movement of the earth about the sun takes place in a fixed ellipse and that the movement of the equator is neglected. We obtain the necessary equations from § 52, and they are as follows:

$$\tan v = \sqrt{1-e^2} \sin u/(\cos u - e); \quad m = u - e \sin u;$$
$$\tan \alpha = \cos \omega \tan \odot; \qquad \odot = v + \varpi;$$

where v, m, u are the true, mean, and eccentric anomalies, $\odot$ the sun's true longitude, e the eccentricity of the orbit, and ϖ the longitude of perihelion.

Differentiating these equations with regard to the time t, we obtain

$$\frac{dv}{dt} = \frac{\sqrt{1-e^2}}{1 - e\cos u} \cdot \frac{du}{dt} \quad \text{.............(i)},$$

$$(1 - e\cos u)\frac{du}{dt} = \frac{dm}{dt} \quad \text{..........................(ii)},$$

$$(\cos^2\odot + \cos^2\omega \sin^2\odot)\frac{d\alpha}{dt} = \cos\omega \frac{dv}{dt} \quad \text{....................(iii)}.$$

The equation of time is obtained by subtracting the mean longitude of the sun $(m + \varpi)$ from its right ascension α, and when the equation of time is stationary its differential coefficient with regard to the time is zero whence

$$\frac{d\alpha}{dt} = \frac{dm}{dt},$$

or by elimination of the differential coefficients

$$(1 - e\cos u)^2 (\cos^2\odot + \cos^2\omega \sin^2\odot) = \sqrt{1-e^2}\cos\omega.$$

From the geometrical properties of an ellipse we have

$$(1 - e^2) = (1 - e\cos u)\{1 + e\cos(\odot - \varpi)\},$$

whence

$$(1 - e^2)^{\frac{3}{2}} (\cos^2\odot + \cos^2\omega \sin^2\odot) = \cos\omega \{1 + e\cos(\odot - \varpi)\}^2.$$

This formula involves no limitation in the magnitude of e and is a general equation for the determination of $\odot$ when the equation of time is stationary.

Ex. 1. Show that the stationary values of the equation of time occur when the projection of the sun's radius vector on the plane of the equator is $(1-e^2)^{\frac{1}{4}}(\cos\omega)^{\frac{1}{2}}$ times the mean distance, a, where e is the eccentricity of the orbit, ω the obliquity of the ecliptic.

Let ρ be the projection, then if δ be the sun's declination,

$$\rho = a(1-e^2)\cos\delta/\{1+e\cos(\odot-\varpi)\},$$

but

$$\cos\delta = (\cos^2\odot + \cos^2\omega\sin^2\odot)^{\frac{1}{2}},$$

and from what has just been proved,

$$\frac{(\cos^2\odot+\cos^2\omega\sin^2\odot)^{\frac{1}{2}}}{1+e\cos(\odot-\varpi)} = \frac{(\cos\omega)^{\frac{1}{2}}}{(1-e^2)^{\frac{3}{4}}},$$

whence

$$\rho = a(1-e^2)^{\frac{1}{4}}(\cos\omega)^{\frac{1}{2}}.$$

Ex. 2. In general, supposing the sun's path relative to the earth to be an exact ellipse with the earth in the focus, and a second ellipse to be constructed by projecting the former on the plane of the equator, then the projections of the sun's position when the equation of time is greatest are the intersections of the second ellipse with a circle whose centre is at the earth, and whose area is equal to the area of this ellipse.

[Math. Trip. 1905.]

Ex. 3. In the general case show that whatever be the eccentricity the equation of the centre is a maximum, when the radius vector is a geometric mean between the major and minor axes.

78. The cause of the seasons.

The apparent annual path of the sun in the heavens is divided into four quadrants by the equinoctial and solstitial points. The corresponding intervals of time are called the seasons, *Spring*, *Summer*, *Autumn*, and *Winter*. *Spring* commences when the sun enters the *sign* of Aries, that is to say when its longitude is zero. When the sun reaches the solstitial point (longitude = 90°) *Summer* begins. *Autumn* commences when the sun enters Libra (longitude = 180°) and *Winter*, commencing when the sun's longitude is 270° continues until the vernal equinox is regained.

The changes in the meteorological conditions of the earth's atmosphere, which constitute the phenomenon known as the variation of the seasons, are determined chiefly by the changes in the amount of heat received from the sun as the year advances.

The amount of heat received from the sun at any place on the surface of the earth depends upon the number of hours during which the sun is above the horizon and its zenith distance at noon. At a place situated in latitude ϕ the interval from sunrise to sunset is equal to $24h/\pi$, where h is the angle, expressed in radians, given by the equation

$$\cos h = -\tan\phi \tan\delta \quad \text{.....................(i)},$$

and the zenith distance at noon is $\phi \sim \delta$, δ being the declination of the sun.

As the sun moves along the ecliptic from the first point of Aries its declination is positive (see Fig. 68) and increases to a maximum at the summer solstice, when the sun is at the first point of Cancer marked by the symbol ♋, the declination being

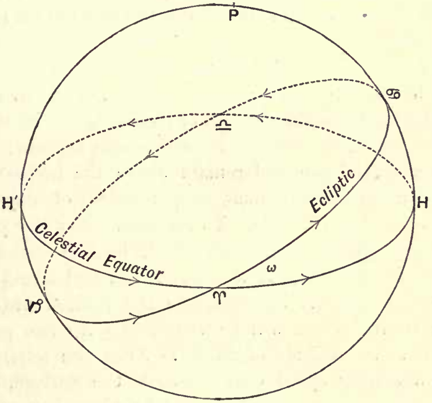


FIG. 68.

then equal to the obliquity of the ecliptic, viz. 23° 27′. From this point the solar declination diminishes until it vanishes at the autumnal equinox ♎, from which the declination becomes negative diminishing until a minimum (−23° 27′) is reached at the winter solstice in Capricornus, marked by the symbol ♑, after which it begins once more to increase and vanishes again at the following equinox.

In considering the seasonal changes it is convenient to divide the earth into five zones which are bounded by circles parallel to the equator in latitudes ± 23° 27′ and ± 66° 33′. The zone included between the parallels of 23° 27′ north and south is called the *Torrid Zone*, and its northern and southern bounding circles are termed the Tropics of Cancer and Capricorn respectively. The parallels of latitude 66° 33′ north and south are called the *Arctic* and *Antarctic* Circles respectively. The zone included between the Arctic Circle and the Tropic of Cancer is known as the North Temperate Zone, while that bounded by the Tropic of Capricorn and the Antarctic Circle is the South Temperate Zone. Lastly, the regions round the North and South Poles bounded by the Arctic and Antarctic Circles respectively, are known as the North and South *Frigid* Zones.

At the time of the summer solstice $\delta = +23° 27'$ and we have then for any point on the Arctic Circle $\tan\phi \tan\delta = 1$. Under these circumstances the hour angle of the sun at rising or setting is 180°. That is to say the diurnal course of the sun is then a circle parallel to the equator, touching the horizon at the north point, so that at midnight one-half of its disc would be visible (we are not here taking the effect of refraction into account). Within the frigid zone the sun will remain above the horizon without setting for a continually increasing number of days, as the observer approaches the pole. To an observer at the pole itself the sun would appear to move round the horizon at the equinox, after which it will describe a spiral round and round the sky, gradually increasing its height above the horizon until at the solstice its diurnal track will be very nearly a circle parallel to the horizon at an altitude of 23° 27′. After the solstice it will return in a similar spiral curve towards the horizon, which it reaches at the autumnal equinox. In the winter half of the year the sun will be continuously below the horizon.

The phenomena in the south temperate and south frigid zones will be similar to those in the corresponding northern zones, but they will occur at opposite epochs of the year. Thus the spring of the southern hemisphere coincides in point of time with autumn in the northern hemisphere, the summer of the North with the winter of the South, and *vice versa*.

In the torrid zone the conditions are as follows. On the equator, since $\phi = 0$, we have from (i) $\cos h = 0$ whatever may be be the value of δ. Hence $h = \frac{1}{2}\pi$, or the length of the day is 12 hours all the year round. The meridian zenith distance of the sun will however vary from day to day. At the vernal equinox the sun's meridional zenith distance will be nearly equal to zero (it would be *exactly* zero if the sun crossed the meridian of the place at the moment when it was passing through the first point of Aries). As spring advances the meridian zenith distance will gradually increase until the solstice, when the sun culminates about 23° 27′ north of the zenith. At the autumnal equinox the sun again passes nearly through the zenith at noon, and at the winter solstice it culminates 23° 27′ south of the zenith. At places situated between the equator and either tropic the amount of heat received from the sun will, so far as it is affected by the sun's zenith distance at noon, reach a maximum twice a year when the sun's declination is equal to the latitude of the place.

Though the four parts into which the great circle of the ecliptic is divided by the equinoxes and solstices are equal in length, the times occupied by the sun in passing over them are not equal.

To find the lengths of the seasons we employ equation (3) of § 73, connecting the mean and true longitudes of the sun, namely

$$L = \odot - 2e \sin (\odot - \varpi) + \tfrac{3}{4} e^2 \sin 2 (\odot - \varpi).$$

For our present purpose we may neglect the third term in this expression, and write simply

$$L = \odot - 2e \sin (\odot - \varpi).$$

When the sun is in ♈ we have $\odot = 0$, and putting L_0 to represent the mean longitude at that moment, we have

$$L_0 = 2e \sin \varpi.$$

In like manner, putting L_1, L_2, L_3, and L_4 respectively to denote the mean longitudes at the summer solstice, autumnal equinox, winter solstice, and vernal equinox next succeeding, we find

$$L_1 = \tfrac{1}{2}\pi - 2e \cos \varpi,$$

$$L_2 = \pi - 2e \sin \varpi,$$

$$L_3 = \tfrac{3}{2}\pi + 2e \cos \varpi,$$

$$L_4 = 2\pi + 2e \sin \varpi.$$

The lengths of the seasons are found by multiplying the difference between each consecutive pair of the five mean longitudes by the factor $365{\cdot}24/2\pi$. Writing K for this factor we have for the northern hemisphere

No. of days in

$$\left.\begin{aligned}\text{Spring} &= K(L_1 - L_0) = 91{\cdot}310 - 2eK(\sin\varpi + \cos\varpi)\\ \text{Summer} &= K(L_2 - L_1) = 91{\cdot}310 - 2eK(\sin\varpi - \cos\varpi)\\ \text{Autumn} &= K(L_3 - L_2) = 91{\cdot}310 + 2eK(\sin\varpi + \cos\varpi)\\ \text{Winter} &= K(L_4 - L_3) = 91{\cdot}310 + 2eK(\sin\varpi - \cos\varpi)\end{aligned}\right\} \quad \ldots\text{(i)}.$$

Taking the values of e and ϖ as given in § 73 we obtain

$$2eK \sin\varpi = -1{\cdot}910 \text{ days},$$

and

$$2eK \cos\varpi = +0{\cdot}379 \quad \text{,,}$$

from which we deduce the lengths of the four seasons as follows:—

	Days	Hours
Spring contains	92	20·2
Summer ,, ...	93	14·4
Autumn ,, ...	89	18·7
Winter ,, ...	89	0·5.

Thus we see that the spring and summer seasons together last for 186 days 10·6 hrs., whereas the autumn and winter together contain only 178 days 19·2 hrs. The reverse of this is the case in the southern hemisphere, the summer half of the southern year lasting for 178 days 19·2 hrs., whereas the southern winter lasts for 186 days 10·6 hrs.

Ex. 1. Assuming that ϖ increases uniformly show that in the course of time the lengths of the four seasons will have as their extreme limits

$$91{\cdot}310 \pm \sqrt{2} \times 365{\cdot}24 \times e/\pi.$$

Ex. 2. If P be the number of days in the year and if summer is longer than spring by Q days and longer than autumn by R days, find the eccentricity of the orbit and the longitude of perigee.

EXERCISES ON CHAPTER X.

Ex. 1. On the assumption that the earth's orbit is a nearly circular ellipse and that the apsidal and solstitial lines have the same longitude, prove that the eccentricity is approximately equal to

$$\frac{E_1 - E_2}{E_1 + E_2} \tan^2 \tfrac{1}{2}\omega,$$

where E_1, E_2 are the hourly variations in the equation of time at perigee and apogee, and ω is the obliquity of the ecliptic.

[Math. Trip. I. 1900.]

Ex. 2. A clock at Cambridge keeps Greenwich mean time. Find what time it indicated when the sun's preceding limb arrived on the meridian on Jan. 6, 1875, having given

Longitude of Cambridge	$22^s{\cdot}75$ E.,
Time of ☉'s semi-diameter passing meridian ...	1^m 10 ·62,
Equation of time	6 2 ·88.

Ex. 3. Show that the columns in the *Nautical Almanac* which give the 'Variation of the sun's right ascension in one hour' and the 'Time of the semi-diameter passing the meridian' increase and diminish together, the former quantity being practically proportional to the square of the latter.

[Math. Trip. I.]

Ex. 4. If the eccentricity of the earth's orbit be e and if the line of equinoxes be perpendicular to the axis major of the orbit, show that the number of days' difference in the time taken by the earth in moving from ♈ to ♎ and from ♎ to ♈ is $465e$ very nearly.

Ex. 5. Show that the greatest equation of the centre is $2e+11e^3/48$ and that when this is the case

$$v=\tfrac{1}{2}\pi+\tfrac{3}{4}e+\tfrac{21}{128}e^3,\quad m=\tfrac{1}{2}\pi-\tfrac{5}{4}e-\tfrac{25}{384}e^3,\quad u=\tfrac{1}{2}\pi-\tfrac{1}{4}e-\tfrac{37}{384}e^3.$$

CHAPTER XI.

THE ABERRATION OF LIGHT.

79. Introductory.

We have already learned that in consequence of atmospheric refraction there is generally a difference between the true place of a celestial body and the place which that body seems to occupy. We have here to consider another derangement of the place of a celestial body which is due to the fact that the velocity of light, though no doubt extremely great, is still not incomparably greater than the velocity with which the observer is himself moving. Any apparent change in the place of a celestial body arising from this cause is known as *aberration*. The true coordinates of a celestial body cannot therefore be ascertained until certain corrections for aberration have been applied to the apparent coordinates as indicated by direct observation†. The nature of these corrections is now to be investigated.

† That this must be the case was perceived by Roemer, when he discovered the gradual propagation of light in 1675. This appears in a letter he wrote to Huygens (*Oeuvres complètes de C. Huygens*, T. VIII. p. 53). Though a periodic change in the place of the Pole Star, really due to aberration, was announced in 1680 by Picard, the credit of discovering the general phenomenon of aberration is due to Bradley (1728), who also gave the correct explanation of it.

80. Relative velocity.

Let AB (Fig. 69) represent both in magnitude and direction the velocity of a body P. Let CB, in like manner represent the velocity of a body Q. On account of his own movement an

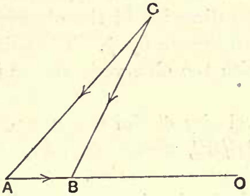


Fig. 69.

observer on P will attribute to Q a movement different from that which Q actually possesses. We have therefore to consider the movement of Q *relative* to P.

Two points moving with equal velocities and in parallel directions have no relative motion, for their distance does not change, nor does the direction of the line joining them. It follows that any equal and parallel velocities may be compounded with the original velocities of two particles without affecting their relative motion.

Observing the directions of the arrows in Fig. 71, it is plain from the triangle of velocities that the velocity CB may be resolved into the two velocities CA and AB. But P has the velocity AB. If we take away the velocity AB from both P and Q we do not alter their relative motion, but this operation would leave P at rest, and show that CA is the relative velocity of Q. We therefore learn that the velocity of Q relative to P is to be obtained by compounding the true velocity of Q with a velocity equal and opposite to that of P.

81. Application to aberration.

From what we have just seen it follows that to an observer who is himself in motion, the apparent direction of a star will be obtained by compounding the velocity of the rays of light from the star with a velocity equal and opposite to that of the observer.

Thus, although the real direction of the star C is BC (Fig. 70), yet its apparent direction will be AC, if the observer moves uniformly along AB, with a velocity which bears to the velocity of light the ratio AB/BC. The angle ACB is called the *aberration* and is denoted by ϵ, while AC the apparent direction of the star makes with AB, the direction of the observer's motion, an angle CAO which we shall denote by ψ. The point O on the celestial sphere towards which the observer's motion is directed is termed the *apex*.

Let v be the velocity of the observer and μ the velocity of light, then $v/\mu = AB/BC$, whence $\sin\epsilon = v\mu^{-1}\sin\psi$, which is the fundamental equation for aberration.

The angle ϵ is the inclination between the actual direction of the telescope when pointed by the moving observer to view the star, and the true direction in which the telescope would have to be pointed if the observer had been at rest. As ϵ is always small we may use its circular measure instead of its sine. We have taken ψ to be the angle between the *apparent* place of the star and the apex. As, however, $\sin\psi$ is multiplied in the equation by v/μ, which is a small quantity, we may often without sensible error in the value of ϵ use, instead of ψ, the angle between the *true* place of the star and the apex.

Ex. 1. Show that the aberration of a star S, resolved in any direction SS' (Fig. 70), is $\kappa\cos AS'$, where A is the apex, $SS' = 90°$, and $\kappa = v/\mu$.

$$\cos AS' = \sin AS \cos\theta,$$

$$\kappa \cos AS' = \kappa \sin AS \cos\theta.$$

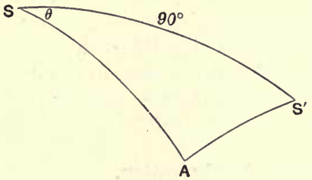


FIG. 70.

But $\kappa\sin AS$ is the aberration, and multiplied by $\cos\theta$ it gives the component in the direction SS'.

Ex. 2. Let S_1, S_2 be the true places of two stars, O be the point bisecting S_1S_2, and A the apex of the earth's way. Show that aberration diminishes S_1S_2 by $2\kappa\sin\frac{1}{2}S_1S_2\cos OA$.

On the great circle S_1S_2 produced, take points $S_1'S_2'O'$, so that

$$S_1S_1' = S_2S_2' = OO' = 90°.$$

Then it follows from Ex. 1 that the change in S_1S_2 owing to aberration is

$$\kappa(\cos AS_1' - \cos AS_2') = 2\kappa \sin O'S_1' \sin O'A \cos OO'A = 2\kappa \sin \tfrac{1}{2} S_1S_2 \cos OA.$$

Ex. 3. Show that stars on the circumference of a great circle will be apparently conveyed by aberration to the circumference of an adjacent small circle, and that the planes of the two circles are parallel.

82. Effects of aberration on the coordinates of a celestial body.

Let μ be the velocity of light, η, ζ the true coordinates of the star on the celestial sphere; we are to seek the apparent coordinates of the star as affected by aberration. Let η_0, ζ_0 be the coordinates of the apex towards which the observer is moving with a velocity v. Let t, t' be the moments at which a ray of light coming from a star, supposed at rest, passes through first the object-glass and secondly the eye-piece of a telescope supposed to be in motion parallel to itself.

Let x, y, z be the rectangular coördinates of the centre of the eye-piece at the time t, referred to axes $+X$, $+Y$, $+Z$ through the earth's centre and towards the points whose spherical coordinates are (0°, 0°), (90°, 0°), (0°, 90°) respectively. At the time t' the coordinates of the eye-piece will therefore be

$$x + v(t' - t)\cos\zeta_0 \cos\eta_0,\quad y + v(t' - t)\cos\zeta_0 \sin\eta_0,\quad z + v(t' - t)\sin\zeta_0.$$

Let l be the length of the telescope, i.e. the length of the line from the centre of the eye-piece to the centre of the object-glass, (η', ζ') the coordinates of the point on the celestial sphere to which this line is directed or the apparent direction of the star. Therefore at the time t

$$x + l\cos\zeta' \cos\eta',\quad y + l\cos\zeta' \sin\eta',\quad z + l\sin\zeta'$$

are the coordinates of the centre of the object-glass.

In the time $(t' - t)$ the ray of light has moved over the distance from the object-glass at the time t to the eye-piece at the time t'; this length is $\mu(t' - t)$, and its components parallel to the axes are

$$\mu(t' - t)\cos\eta \cos\zeta,\quad \mu(t' - t)\sin\eta \cos\zeta,\quad \eta(t' - t)\sin\zeta.$$

But these quantities when added to the corresponding coordinates of the centre of the eye-piece must give the coordinates of the centre of the object-glass. Hence if we write $\mu' = l/(t' - t)$ we obtain the equations

$$+\mu\cos\zeta\cos\eta = +\mu'\cos\zeta'\cos\eta' - v\cos\zeta_0\cos\eta_0 \;\ldots(1),$$
$$+\mu\cos\zeta\sin\eta = +\mu'\cos\zeta'\sin\eta' - v\cos\zeta_0\sin\eta_0 \;\ldots(2),$$
$$+\mu\sin\zeta \quad = +\mu'\sin\zeta' \quad - v\sin\zeta_0 \quad \ldots(3).$$

Multiply (2) by $\cos\eta'$ and subtract it from (1) multiplied by $\sin\eta'$, and we have

$$\mu\cos\zeta\sin(\eta' - \eta) = -v\cos\zeta_0\sin(\eta' - \eta_0) \;\ldots\ldots(4).$$

Multiply (2) by $\sin\frac{1}{2}(\eta + \eta')$ and add it to (1) multiplied by $\cos\frac{1}{2}(\eta + \eta')$ and then divide by $\cos\frac{1}{2}(\eta' - \eta)$, and we have

$$\mu\cos\zeta = \mu'\cos\zeta' - v\cos\zeta_0\cos\{\eta_0 - \tfrac{1}{2}(\eta + \eta')\}\sec\tfrac{1}{2}(\eta' - \eta)\ldots(5).$$

Again, multiplying (3) by $\cos\zeta'$ and subtracting it from (5) multiplied by $\sin\zeta'$, we have

$$\mu\sin(\zeta' - \zeta) = v\sin\zeta_0\cos\zeta'$$
$$- v\cos\zeta_0\sin\zeta'\cos\{\eta_0 - \tfrac{1}{2}(\eta + \eta')\}\sec\tfrac{1}{2}(\eta' - \eta)\ldots(6).$$

Equations (4) and (6) can be much simplified by taking advantage of the fact that v/μ is a very small quantity. This shows that $\eta' - \eta$ is small, and consequently we may replace η' by η in the right-hand member of (4), and thus obtain the effect of aberration on the coordinate η in the form

$$\eta' - \eta = -v\mu^{-1}\cos\zeta_0\sec\zeta\sin(\eta - \eta_0) \;\ldots\ldots\ldots(7).$$

We thus find $\eta' - \eta$, and thence η' with sufficient accuracy for most purposes. If a further approximation be required, as might, for example, be the case if ζ were nearly 90°, we can introduce the approximate value of η' found from this equation into the right-hand side of (4), and thus obtain again $\sin(\eta' - \eta)$.

In like manner, we can find $\zeta' - \zeta$ from (6). The first approximation, quite sufficient in most cases, is obtained by replacing ζ' and η' by ζ and η in the right-hand side. We thus obtain

$$\zeta' - \zeta = v\mu^{-1}\{\sin\zeta_0\cos\zeta - \cos\zeta_0\sin\zeta\cos(\eta_0 - \eta)\}\ldots(8).$$

If further approximation is required, the approximate values of ζ' and η', obtained in (7) and (8), may be introduced into the right-hand side of (6). If θ be the angle whose cosine is $\sin\zeta_0\sin\zeta + \cos\zeta_0\cos\zeta\cos(\eta_0 - \eta)$, then $v\mu^{-1}\sin\theta$ is the distance that aberration has apparently moved the star.

The formulae (7) and (8) are fundamental results for aberration whether the observer's motion be the annual movement of the earth round the sun or be of any other kind.

83. The different kinds of aberration.

The expressions we have given in § 82 show in what way the aberration depends on η_0, ζ_0, the coordinates of the apex. If η_0 and ζ_0 change, then in general, the effect of aberration on the apparent place of the star will also change. If η_0 and ζ_0 change periodically, then the effect of aberration on the coordinates of the apparent place will also be periodic. If however η_0 and ζ_0 do not change, then the effects of such an aberration would be constant for each star. Aberration of this type would no doubt displace a star from the position in which it would be seen if there were no such aberration, but it would always displace the same star in precisely the same way. This being so, observation could not disclose the amount or even the existence of the aberration, for what the coordinates of the star would be if unaffected by aberration are unknown.

Aberration of the class here referred to, does undoubtedly exist. It must arise from the motion of the solar system as a whole. So far as our present knowledge is concerned, the position of the apex of this motion is presumably constant, nor have we any grounds for supposing that the velocity is otherwise than uniform, so far at least as the few centuries during which accurate observation has been possible are concerned. The amount of aberration of each star from this cause is therefore constant, and its effect is not by us distinguishable in the coordinates of the star's place. Nor can we calculate the amount of this aberration, because we do not know the velocity of the solar system, nor the position of the apex with sufficient accuracy. All we can affirm is that the right ascension and declination of each star as they appear to us, are different by unknown amounts from what they would be if this aberration were absent.

The aberrations which are of practical importance in Astronomy are those in which the motion of the observer is such that the apex has a periodic motion on the celestial sphere. There is thus a periodical alteration in the apparent position of any star, which is of the greatest significance and interest. The annual motion of the earth in its orbit is one of these periodic movements, and it produces what is known as annual aberration. Another aberrational effect arises from the rotation of the earth on its axis, and this causes what is known as diurnal aberration; of these the first is by far the more important, and whenever the word "aberration" is used without the prefix "diurnal," it is always the annual aberration which is to be understood.

84. Aberration in right ascension and declination.

We shall now apply the general formulae (7) and (8) of § 82 to obtain the expressions for the aberration of a star in those particular coordinates which we term right ascension and declination. If the point (0°, 0°) be ♈, if (90°, 0°) be that point on the celestial equator of which the R.A. is 90°, and if (0°, 90°) be the north celestial pole, then η is the right ascension α, and ζ is the declination δ, and we have

$$\alpha' - \alpha = v\mu^{-1} \cos \delta_0 \sec \delta \sin (\alpha_0 - \alpha) \quad\text{.....................(1)},$$

$$\delta' - \delta = v\mu^{-1} \{\sin \delta_0 \cos \delta - \cos \delta_0 \sin \delta \cos (\alpha_0 - \alpha)\} \quad\text{.....(2)},$$

from which we obtain the effect of aberration on the R.A. and declination respectively.

We assume for the present that the orbit of the earth is a circle. In other words, we take the velocity of the earth as constant and equal to the mean velocity in the actual elliptic orbit. The ratio v/μ is called the constant of aberration and is denoted by κ as before. Let $\odot$ be the longitude of the sun; then, since the earth is moving in the direction of the tangent to the orbit and longitudes increase in the direction of the sun's apparent motion, it follows that $\odot - 90°$ is the longitude of the apex, while its latitude is zero. To illustrate this, suppose the time to be noon at the summer solstice. As the apparent annual motion carries the sun among the stars from west to east, the true motion of the earth to which this apparent solar motion is due must be from east to west. At noon in the summer solstice ♈ is in the westerly point of the horizon. This is the apex, and its longitude is zero, while the longitude of the sun is 90°.

Let ♈ (Fig. 71) be the first point of Aries, A the apex and S the sun. Then ♈$S = \odot$ and ♈$A = \odot - 90°$. Let fall a perpendicular

AP on the equator ΥP. Then $AP = \delta_0$ and $\Upsilon P = \alpha_0$. From the right-angled triangle ΥAP, we have

$$\sin\delta_0 = -\cos\odot\sin\omega,$$
$$\cos\delta_0\cos\alpha_0 = \sin\odot,$$
$$\cos\delta_0\sin\alpha_0 = -\cos\odot\cos\omega.$$

Making these substitutions in (1), we see that if α and δ be the true R.A. and decl. of a star, κ the constant of aberration

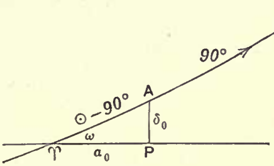


Fig. 71.

and $\odot$ the longitude of the sun, then the apparent R.A. and declination as affected by aberration are respectively

$$\alpha - \kappa\sec\delta(\sin\alpha\sin\odot + \cos\alpha\cos\omega\cos\odot)\ldots\ldots\ldots\ldots(3),$$
$$\delta - \kappa(\cos\delta\sin\omega\cos\odot + \sin\delta\cos\alpha\sin\odot - \sin\delta\sin\alpha\cos\omega\cos\odot)\ldots(4).$$

Ex. 1. If the aberration of a star in R.A. is stationary, prove that the R.A. of the star equals that of the sun; and that if a_0 be the R.A. of the apex,

$$\tan a\tan a_0 + \cos^2\omega = 0,$$

where ω is the obliquity of the ecliptic.

If the aberration in R.A. is stationary then by equating to zero the differential coefficient of (3) we have

$$\sin a\cos\odot = \cos a\cos\omega\sin\odot,$$

whence $\tan a = \tan\odot\cos\omega$, which shows that a must be the R.A. of the sun no less than of the star. In general the R.A. and declination of the apex, i.e. a_0 and δ_0, are respectively

$$-\tan^{-1}(\cot\odot\cos\omega),\quad -\sin^{-1}(\cos\odot\cos\omega),$$

and when the aberration in R.A. is stationary $\tan a\tan a_0$ becomes

$$-\tan\odot\cos\omega\cot\odot\cos\omega = -\cos^2\omega.$$

We have also in the same case

$$\sin\delta_0 = -\cos a\cos\omega\sin\omega/\sqrt{(\sin^2 a + \cos^2 a\cos^2\omega)},$$
$$\cos\delta_0\cos a_0 = \sin a/\sqrt{(\sin^2 a + \cos^2 a\cos^2\omega)},$$
$$\cos\delta_0\sin a_0 = -\cos a\cos^2\omega/\sqrt{(\sin^2 a + \cos^2 a\cos^2\omega)},$$

whence we easily verify that

$$\sin\delta_0\cos\delta_0\tan(a_0 - a)\cos a_0\tan\omega = 1.$$

Ex. 2. If a, δ be the R.A. and declination of a star and if $\odot$ be the longitude of the sun and ω the obliquity of the ecliptic, show that when the star's aberration in declination has its greatest value,

$$\tan \odot (\sin \omega \cos \delta - \cos \omega \sin \delta \sin a) = \sin \delta \cos a.$$

By § 81, Ex. 1, the aberration in declination of S (Fig. 71) is $\kappa \cos AS'$, if A be the apex and $SS' (=90°)$ passes through the pole P. If AS' is a minimum it must be the perpendicular from S' to the ecliptic at A and must therefore contain K, the pole of the ecliptic, whence the result follows from the triangle $S'KP$.

Ex. 3. Prove that when the aberration in declination attains its greatest numerical value for the year the arcs on the celestial sphere joining the star to the sun and to the pole of the equator are at right angles.

Ex. 4. Prove that for a given position of the sun the aberration in right ascension of a star on the equator will be least when

$$\tan a = \tan \odot \sec \omega,$$

a being the right ascension of the star, $\odot$ the sun's longitude, and ω the obliquity of the ecliptic.

Ex. 5. Prove that all stars, whose aberration in R.A. is a maximum at the same time that the aberration in declination vanishes, lie either on a cone of the second order whose circular sections are parallel to the ecliptic and equator, or on the solstitial colure.

[Math. Trip.]

As the aberration in declination is zero, we have

$$\tan \delta \cos (a_0 - a) = \tan \delta_0 = \tan \omega \sin a_0,$$

whence

$$\tan a_0 = \tan \delta \cos a / (\tan \omega - \tan \delta \sin a),$$

$$\tan \delta_0 = \tan \omega \tan \delta \cos a / (\tan^2 \omega + \tan^2 \delta - 2 \tan \omega \tan \delta \sin a)^{\frac{1}{2}}.$$

But as the aberration in R.A. is a maximum, we have (Ex. 1)

$$\sin \delta_0 \cos \delta_0 \tan (a_0 - a) \cos a_0 \tan \omega = 1,$$

and eliminating a_0, δ_0 and reducing, we obtain

$$(\tan^2 \omega - 2 \tan \omega \tan \delta \sin a + \tan^2 \delta)(1 + \tan \omega \tan \delta \sin a) = 0.$$

It is impossible for the first factor to vanish unless

$$\sin a = \pm 1, \text{ and } \tan \delta = \sin a \tan \omega.$$

Thus there are two points on the solstitial colure which satisfy the condition.

If we transform the second factor by making

$$x = r \cos \delta \cos a; \quad y = r \cos \delta \sin a; \quad z = r \sin \delta;$$

we obtain $x^2 + y^2 + yz \tan \omega = 0$, which may be written thus:—

$$x^2 + y^2 + z^2 - z(z - y \tan \omega) = 0,$$

which is the equation of a cone whose circular sections are parallel to

$$z = 0 \text{ and } z - y \tan \omega = 0,$$

i.e. the equator and the ecliptic.

Ex. 6. Show that the effect of annual aberration on *any coordinate* of a star may be expressed in the form

$$a \cos(\odot + A),$$

where $\odot$ is the longitude of the sun and a, A constants depending upon the position of the star.

85. Aberration in longitude and latitude.

To apply the formula of § 82 to this case we must make $\eta_0 = \odot - 90°$ and $\zeta_0 = 0$. Then λ and β, the longitude and latitude, replace η and ζ respectively, and thus we find that aberration increases the longitude of the star by

$$-\kappa \sec\beta \cos(\odot - \lambda),$$

and increases the latitude of the star by

$$-\kappa \sin\beta \sin(\odot - \lambda).$$

It will be remembered that in these expressions it is assumed that the earth's orbit is circular.

Ex. 1. The angular distance between two stars which have the same latitude β is θ, and the mean of their longitudes is ϕ; show that the increment of θ due to aberration is

$$2\kappa \tan\tfrac{1}{2}\theta \sin(\phi - \odot)(\cos^2\beta - \sin^2\tfrac{1}{2}\theta)^{\frac{1}{2}},$$

where $\odot$ is the sun's longitude.

[Math. Trip. I.]

If β' be the latitude of the point bisecting the arc between the two stars, then the increment of θ (§ 81, Ex. 2) is $2\kappa \sin\frac{1}{2}\theta \cos\beta' \sin(\phi - \odot)$, and eliminate β by help of $\sin\beta = \sin\beta' \cos\frac{1}{2}\theta$.

Ex. 2. Show that the distance between two stars at β, λ and β_0, λ_0 respectively is not altered by aberration if the sun's longitude $\odot$ satisfies the equation

$$\cos\beta \sin(\odot - \lambda) + \cos\beta_0 \sin(\odot - \lambda_0) = 0.$$

Ex. 3. The apparent longitude and latitude, λ' and β', of a star being given, show that, when the earth's orbit is taken as circular, the terms in the aberrations in latitude and longitude depending on the square of κ are

$$\tfrac{1}{4}\kappa^2 \sin 1'' \tan\beta' \cos 2(\odot - \lambda'),$$

and

$$\tfrac{1}{2}\kappa^2 \sin 1'' \tan^2\beta' \sin 2(\odot - \lambda');$$

where $\odot$ is the true longitude of the sun. To what stars would these corrections be applied?

[Math. Trip. I.]

Ex. 4. If (x, y, z) be the direction cosines of a star referred to rectangular axes, (l, m, n) direction cosines of the point towards which the earth is travelling, show that the direction cosines of the star's apparent place as affected by aberration are $x + \kappa(l - x\cos\theta)$, and two similar expressions when κ is the quantity v/μ (p. 250) and $\cos\theta = lx + my + nz$.

If on a great circle three points R_1, R_2, R_3 be set off at distances ρ_1, ρ_2, ρ_3 respectively from an origin on the circle, and if x_1, y_1, z_1; x_2, y_2, z_2; x_3, y_3, z_3 be the direction cosines of R_1, R_2, R_3 respectively from the centre of the sphere, then

$$x_1 \sin(\rho_2-\rho_3)+x_2 \sin(\rho_3-\rho_1)+x_3 \sin(\rho_1-\rho_2)=0,$$
$$y_1 \sin(\rho_2-\rho_3)+y_2 \sin(\rho_3-\rho_1)+y_3 \sin(\rho_1-\rho_2)=0,$$
$$z_1 \sin(\rho_2-\rho_3)+z_2 \sin(\rho_3-\rho_1)+z_3 \sin(\rho_1-\rho_2)=0.$$

To apply this to the present case, we make

$$\rho_2-\rho_3=\kappa \sin\theta;\quad \rho_3-\rho_1=-\theta;\quad \rho_1-\rho_2=\theta-\kappa\sin\theta.$$

86. The geometry of annual aberration.

We now investigate the shape of the small closed curve which the star appears to describe on the celestial sphere in consequence of annual aberration.

Let ST be the perpendicular from S, the true place of the star, to the ecliptic AT, where A is the apex (Fig. 72), and

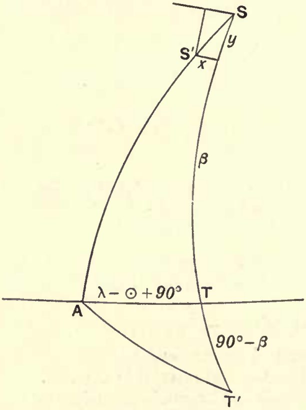


FIG. 72.

produce T to T' so that $ST' = 90°$.

Let S' be the point to which the star is displaced by aberration, then since SS' is small we may regard the locus of S' as a plane curve. If x, y be the rectangular coordinates of S' as indicated in the figure, we have (§ 81)

$$y = \kappa \cos AT' = \kappa \sin(\odot - \lambda)\sin\beta,$$

and $$x = \kappa \sin SA \sin AST = \kappa \cos(\odot - \lambda),$$

whence $$x^2 + y^2 \operatorname{cosec}^2\beta = \kappa^2.$$

We thus obtain the following results with regard to the effect of annual aberration on the apparent position of a star.

In consequence of annual aberration the apparent place of each star describes an ellipse, known as the ellipse of aberration, in the course of a year and the centre of the ellipse is the true place of the star.

The axis minor of the ellipse is perpendicular to the ecliptic.

The semi-axis major of the ellipse is the constant of aberration and is therefore the same for all stars.

For a star on the ecliptic the ellipse becomes a straight line. For a star at the pole of the ecliptic the ellipse becomes a circle, and in general the semi-axis minor of the ellipse is the product of the sine of the star's latitude and the constant of aberration.

Ex. 1. Assuming that the sun's motion is uniform, show that at four consecutive epochs at intervals of three months, the apparent place of the star will occupy successively the four extremities of a pair of conjugate diameters of the ellipse of aberration.

Ex. 2. Let λ be the longitude of a star and β its latitude, show geometrically that the effect of aberration will be to displace the star by a distance which is the square root of

$$\tfrac{1}{2}\kappa^2\{1+\sin^2\beta+\cos^2\beta\cos 2(\odot-\lambda)\}.$$

Ex. 3. Show that the ellipse of aberration is the orthogonal projection of a circle in the plane of the ecliptic on the tangent plane touching the celestial sphere in the true place of the star.

Ex. 4. Show that the effect of annual aberration upon the apparent places of the fixed stars would be produced if each star actually revolved in a small circular orbit parallel to the plane of the ecliptic and if the earth were at rest.

***87. Effect of the elliptic motion of the earth on aberration.**

We have now to consider the influence of the eccentricity of the earth's orbit on the annual aberration.

Let $\odot$ be as usual the sun's geocentric longitude, then $180° + \odot$ is the earth's heliocentric longitude, ϖ the longitude of perihelion and θ the true anomaly, so that $\odot + 180° = \varpi + \theta$. The earth's radius vector is r and if v, δ_0, α_0 have the significations already given to them (§ 84), we must have

$$\left.\begin{aligned} v\cos\delta_0\cos\alpha_0 &= -\frac{d}{dt}(r\cos\odot),\\ v\cos\delta_0\sin\alpha_0 &= -\frac{d}{dt}(r\sin\odot\cos\omega),\\ v\sin\delta_0 &= -\frac{d}{dt}(r\sin\odot\sin\omega).\end{aligned}\right\}\quad\text{.........(i)}$$

The first of these equations is obtained by identifying two expressions for the velocity of the earth parallel to the line ♎ ♈. The third equation is obtained from identifying expressions for the velocity of the earth parallel to the earth's polar axis, and the second equation is obtained in like manner from the axis perpendicular to those already mentioned.

To make use of equations (i) we must obtain from the elliptic motion the values of dr/dt and of $-rd\odot/dt = rd\theta/dt$. Kepler's second law shows that $rd\theta/dt \propto 1/r$ (§ 50), and hence from the polar equation of the ellipse, viz. $r = a(1-e^2)/(1+e\cos\theta)$, we obtain

$$rd\theta/dt = C(1+e\cos\theta),$$

where C is a constant; by substituting this in the logarithmic differential of the polar equation of the ellipse, we find

$$dr/dt = Ce\sin\theta.$$

Expanding (i) and making these substitutions

$$\begin{aligned} v\cos\delta_0\cos\alpha_0 &= C\{-e\sin\theta\cos\odot + \sin\odot(1+e\cos\theta)\},\\ v\cos\delta_0\sin\alpha_0 &= C\cos\omega\{-e\sin\theta\sin\odot - \cos\odot(1+e\cos\theta)\},\\ v\sin\delta_0 &= C\sin\omega\{-e\sin\theta\sin\odot - \cos\odot(1+e\cos\theta)\},\end{aligned}$$

whence, remembering that $180° + \odot = \theta + \varpi$, we obtain

$$\left.\begin{aligned} v\cos\delta_0\cos\alpha_0 &= C(\sin\odot - e\sin\varpi),\\ v\cos\delta_0\sin\alpha_0 &= C\cos\omega(-\cos\odot + e\cos\varpi),\\ v\sin\delta_0 &= C\sin\omega(-\cos\odot + e\cos\varpi).\end{aligned}\right\} \quad \text{......(ii)}$$

Substituting in equations (i) and (ii) (§ 84), and making $C/\mu = \kappa$, which is called the *constant of aberration*,

$$\begin{aligned} \alpha' - \alpha = {} & \kappa\sec\delta(-\sin\alpha\sin\odot - \cos\alpha\cos\odot\cos\omega)\\ & + \kappa e\sec\delta(\sin\alpha\sin\varpi + \cos\alpha\cos\varpi\cos\omega),\\ \delta' - \delta = {} & \kappa(\cos\omega\sin\alpha\sin\delta\cos\odot - \sin\omega\cos\delta\cos\odot\\ & \qquad - \cos\alpha\sin\delta\sin\odot)\\ & + \kappa e(+\sin\omega\cos\varpi\cos\delta + \sin\varpi\cos\alpha\sin\delta\\ & \qquad - \cos\omega\cos\varpi\sin\alpha\sin\delta).\end{aligned}$$

As e is only about 1/60 it is plain that the eccentricity of the earth's orbit has but a very small effect on the aberration. The peculiar character of that effect is however worthy of notice. The terms in $\alpha' - \alpha$ and $\delta' - \delta$ which contain e do not contain $\odot$. Consequently these terms do not change during the course of the year, and indeed it is only after the lapse of many centuries that any change in such terms would be large enough to be appreciable. The effect therefore of these terms is to produce changes in the R.A. and δ of each star which are quite different in character from the annual effect which is the chief result of aberration. We might allow for these by means of the values just found, but since they are constant for many centuries it is more convenient to include this part of the aberration in the adopted R.A. and declination. The catalogued mean coordinates of stars are therefore to a very minute extent distorted in consequence of the eccentricity of the earth's orbit.

Ex. 1. The apparent positions of a star when the earth is in perihelion and aphelion are P and Q respectively; show that the true position of the star is at a point R in PQ, such that $PR : RQ :: 1+e : 1-e$, where e is the eccentricity of the earth's orbit. Prove that PQ is conjugate to the diameter of the ellipse formed by drawing a great circle through its centre and the apses of the earth's orbit.

[Math. Trip. I.]

Ex. 2. Prove that in the case of a star the result of replacing the usual assumption of the earth's orbit being a circle with the mean radius by that of the elliptic form is equivalent to (1) modifying the constant in the displacement towards the point ninety degrees behind the sun in the ecliptic, and (2) regarding as included in the star's *mean* position a constant aberrational displacement to a point on the ecliptic in the direction at right angles to the apse line of the orbit.

[Math. Trip. I.]

Ex. 3. Show that the constant of aberration C/μ (see p. 260) is $2\pi a/\mu T(1-e^2)^{\frac{1}{2}} \sin 1''$, where a, T, e are the mean distance, the periodic time, and the eccentricity of the earth's orbit, and μ the velocity of light.

Ex. 4. By a well known theorem in elliptic motion the velocity of the observer P relative to the sun S is compounded of a velocity C perpendicular to SP, and a velocity eC perpendicular to the axis major when C is the constant on p. 260. Deduce from this the equations (ii).

88. Determination of the constant of aberration.

The investigation of the constant of aberration is now most frequently based on the observation of zenith distances of stars specially selected to meet the requirements of the problem. These observations are preferably made with an instrument known as a zenith telescope. We shall take a simple case in which only two stars are employed.

Let S_1 and S_2 be two stars which culminate as nearly as possible at the zenith, one a little north and the other a little south of the zenith. The stars are to be chosen so that their right ascensions differ by about 12 hours. The first observations of zenith distance of both stars are to be made on a day when S_1 has its upper culmination at 6 A.M., and S_2 will on the same day have its upper culmination at 6 P.M. These are to be combined with observations made six months later, when S_1 culminates at 6 P.M. and S_2 at 6 A.M. These conditions can hardly be exactly realized, but they indicate the most perfect scheme for an accurate result when only two stars are used. The reasons for these requirements will presently be made clear.

Let α_1, δ_1 be the mean values of the right ascension and declination of S_1 for the beginning of the year taken from some standard catalogue. Even the most excellent determinations of star places must be presumed to be in some degree erroneous. No doubt the errors of the coordinates are very small, and for most purposes they may be quite overlooked. But such minute errors as are unavoidable in the declinations adopted for the stars would be quite large enough to vitiate a determination of the coefficient of aberration which depended on the declination. In the present method the observations are so combined that the declinations disappear from the result and consequently their errors are void of effect.

We shall assume for the moment that a value of the constant of aberration is approximately known. We may, for example, take the constant to be $20''\cdot5 + \kappa_1$, where κ_1 is some very small fraction of a second. The determination of κ_1 is then the object of the investigation. By this device we secure the convenience that the quantity sought is very small in comparison with the total amount of aberration, and consequently in computing the coefficients by which κ_1 is to be multiplied we are permitted to use approximate methods that would not be valid if these coefficients were to be multiplied by any quantity other than a very small one.

The first operation is to deduce the apparent places of S_1 and S_2 for the days of observation. We must by the known processes compute the precession and nutation. We must further calculate the aberration, using the approximate value $20''\cdot5$ for the coefficient. The correction thus obtained for the declination of S_1

on the first day of observation we denote by p_1. It is a complete correction except in so far as we have used an incorrect value of the constant of aberration. We must therefore increase p_1 by $A_1\kappa_1$, where A_1 is the coefficient of $v\mu^{-1}$ as given in equation (1), § 84. Thus we see that the apparent declination of S_1 on the first day of observation is $\delta_1 + p_1 + A_1\kappa_1$. We assume, however, as above explained, that there may be an unknown error in δ_1.

Let z_1 be the observed zenith distance, which we shall suppose cleared from refraction (Chap. VI.). Then since the latitude ϕ is the sum of the zenith distance (in this case supposed to be south) and the declination, we have

$$\phi = z_1 + \delta_1 + p_1 + A_1\kappa_1 \quad \text{..................(1)}.$$

On the same day, about 12 hours later, we observe the second star, and as in that time the latitude will not have changed appreciably, we have also,

$$\phi = z_2 + \delta_2 + p_2 + A_2\kappa_1 \quad \text{..................(2)},$$

where by the changes in the suffices we indicate that this formula relates to the second star. Six months later, the observations are to be repeated on the same stars, and we must then suppose the latitude has changed to ϕ', which generally differs from ϕ on account of certain minute periodic alterations (§ 61). The zenith distances are different at the second epoch of observation, and so are also p_1, p_2, A_1, A_2, but δ_1 and δ_2 being the mean values of the declinations at the beginning of the year are the same at both epochs. Using accented letters to distinguish the quantities relating to the second epoch, we thus have

$$\phi' = z_1' + \delta_1 + p_1' + A_1'\kappa_1 \quad \text{..................(3)},$$

$$\phi' = z_2' + \delta_2 + p_2' + A_2'\kappa_1 \quad \text{..................(4)},$$

from (1), (2), (3), (4) we easily obtain the following equation for κ_1

$$z_1 - z_2 - z_1' + z_2' + p_1 - p_2 - p_1' + p_2' + (A_1 - A_2 - A_1' + A_2')\,\kappa_1 = 0,$$

and therefore the aberration is $20''\cdot5 + \kappa_1$, where

$$\kappa_1 = -\frac{z_1 - z_2 - z_1' + z_2' + p_1 - p_2 - p_1' + p_2'}{A_1 - A_2 - A_1' + A_2'}.$$

The numerator and denominator are both known quantities, and hence κ_1 is found.

If in the computation of p_1, p_2, p_1', p_2' no allowance had been made for aberration the formula just given would have afforded an approximate value of the aberration, and the approximate value we have used, $20''\cdot5$, may be regarded as having been thus obtained.

We are to notice that δ_1 and δ_2 have both disappeared. If therefore these quantities had been affected by small errors as, of course, will generally be the case, those errors will not have imparted any inaccuracy to κ_1, except in so far as A_1, p_1, &c. depend on the adopted values of δ. As ϕ and ϕ' have also both disappeared, any uncertainty as to the latitude at either the first epoch or the last will also have very little influence.

It is by the observed quantities z_1, z_2, z_1', z_2' that errors of observation are introduced into the expression for κ_1. How far these errors will influence the value of κ_1 depends upon the denominator $A_1 - A_2 - A_1' + A_2'$. The larger this denominator the larger will be the quantity by which the errors will be divided, and consequently the smaller will be the influence of the errors of observation on the result. The observations are therefore to be arranged so as to make this denominator as great as circumstances will permit. To determine the most suitable arrangement we may use approximate values for A_1, A_2, A_1', A_2', though of course the true values must be used in the actual determination of κ_1.

As the stars culminate near the zenith we may for our present object suppose that their declinations are equal to the latitude ϕ, and thus we have (§ 84)

$$A_1 = \sin\delta_0\cos\phi - \cos\delta_0\sin\phi\cos(\alpha_1 - \alpha_0),$$

$$A_2 = \sin\delta_0\cos\phi - \cos\delta_0\sin\phi\cos(\alpha_2 - \alpha_0);$$

$$A_1 - A_2 = \cos\delta_0\sin\phi\,\{\cos(\alpha_2 - \alpha_0) - \cos(\alpha_1 - \alpha_0)\}$$

$$= 2\cos\delta_0\sin\phi\sin\tfrac{1}{2}(\alpha_1 - \alpha_2)\sin\tfrac{1}{2}(\alpha_1 + \alpha_2 - 2\alpha_0).$$

In like manner

$$A_1' - A_2' = 2\cos\delta_0'\sin\phi'\sin\tfrac{1}{2}(\alpha_1 - \alpha_2)\sin\tfrac{1}{2}(\alpha_1 + \alpha_2 - 2\alpha_0'),$$

where α_0', δ_0' is the position of the apex at the time of the second observation.

As the apex is on the ecliptic, $\cos\delta_0$ and $\cos\delta_0'$ have as extreme limits 1·00 and 0·92. We shall therefore take with sufficient accuracy $\cos\delta_0 = \cos\delta_0' = 0{\cdot}96$. It is perhaps hardly necessary to add that we may for this purpose take $\phi = \phi'$, and thus we have

$$(A_1 - A_2) - (A_1' - A_2')$$
$$= 4\cos\delta_0 \sin\phi \sin\tfrac{1}{2}(\alpha_1 - \alpha_2)\sin\tfrac{1}{2}(\alpha_0' - \alpha_0)\cos\tfrac{1}{2}(\alpha_1 + \alpha_2 - \alpha_0 - \alpha_0').$$

To make this numerically as large as possible, we first put

$$\sin\tfrac{1}{2}(\alpha_1 - \alpha_2) = \pm 1,$$

whence $\alpha_1 - \alpha_2 = 180°$, or the two stars should differ by 12 hours in right ascension. Similarly to make the factor $\sin\frac{1}{2}(\alpha_0' - \alpha_0)$ as large as possible the sun must, between the two sets of observations have moved 180° in right ascension and therefore 180° in longitude. This requires that the interval between the two sets of observations should be six months. The factor $\cos\frac{1}{2}(\alpha_1 + \alpha_2 - \alpha_0 - \alpha_0')$ will have unity as its greatest value, in which case $\sin(\alpha_1 + \alpha_2 - \alpha_0 - \alpha_0')$ will be zero or

$$\sin\{(\alpha_1 - \alpha_2) + (\alpha_0 - \alpha_0') + 2(\alpha_2 - \alpha_0)\} = 0.$$

Expanding this and noting the conditions already obtained we see that $\sin 2(\alpha_2 - \alpha_0) = 0$. This condition will be satisfied if $\alpha_2 = \alpha_0$, which requires that the two stars shall lie on the hour circle through the two antipodal positions of the apex. It follows that the conditions will be most favourable when one of the stars culminates about 6 A.M. and the other about 6 P.M. respectively.

In the application of this as well as in other methods of determining this important constant there are many difficulties and the results hitherto obtained are therefore somewhat less accordant than the present state of astronomical work of precision would lead us to desire, so that the exact value cannot be given within a few hundredths of a second of arc, but it must be very close to 20″·47, which is the final result of the best determinations made up to the present.

*89. Diurnal aberration.

We have now to consider the particular kind of aberration produced by that movement of the observer which arises from the diurnal rotation of the earth. This aberration is described as *diurnal*, to distinguish it from the far more important phenomenon of *annual* aberration with which we have been hitherto occupied.

At the latitude ϕ the velocity of the observer arising from the earth's rotation is $463 \cos\phi$ metres per second, and as the velocity of light is about 300,000 kilometres per second we see that the coefficient of diurnal aberration is

$$463 \operatorname{cosec} 1'' \cos\phi / 300000000 = 0''{\cdot}32 \cos\phi.$$

This coefficient is so small that diurnal aberration may always be neglected except when great refinement is required.

The diurnal rotation carries the observer towards the east point of the horizon. Hence $\delta_0 = 0$, and $\alpha_0 - \alpha = 90° + h$, where h is the west hour angle of the star. Making these substitutions in § 84, we find that the R.A. and declination of the star when affected by diurnal aberration become

$$\alpha + 0''{\cdot}32 \cos\phi \cos h \sec\delta,$$

$$\delta + 0''{\cdot}32 \cos\phi \sin h \sin\delta.$$

When a star is on the meridian, $h = 0$ and the effect of diurnal aberration in declination vanishes, while the transit is delayed by the amount $0^s{\cdot}021 \cos\phi \sec\delta$. For lower meridian transits $h = 180°$, and the transit is accelerated by $0^s{\cdot}021 \cos\phi \sec\delta$.

To find the effect of diurnal aberration on the zenith distance of a star which is not on the meridian, we differentiate the equation

$$\cos z = \sin\phi \sin\delta + \cos\phi \cos\delta \cos h,$$

and substitute for dh and $d\delta$ the values $-0''{\cdot}32 \cos\phi \cos h \sec\delta$ and $+0''{\cdot}32 \cos\phi \sin h \sin\delta$ respectively, and obtain

$$dz = -0''{\cdot}32 \cos\phi \cos\delta \sin h \cot z.$$

Ex. 1. Show that to an observer in latitude ϕ, a star of declination δ will, owing to diurnal aberration, appear to move in an ellipse whose semi-axes are $m \cos\phi$ and $m \cos\phi \sin\delta$, where m is the ratio of the circumference of the earth to the distance described by light in a day, and the angles are in circular measure.

[Coll. Exam.]

Ex. 2. Show that the effect of diurnal aberration on the observed zenith distance z of a star may be allowed for by subtracting $t \cos z$ seconds from the time of observation, where t is the time in seconds that light would take to travel a distance equal to the earth's radius.

[Math. Trip. I.]

*90. Planetary aberration.

Up to the present we have assumed that the star whose aberration was under consideration was itself at rest. But if the star be in motion, it is obvious that the formulae already given must receive some modification. The general principle on which planetary aberration depends may be best illustrated by assuming for the moment the corpuscular theory of light.

Let x_0, y_0, z_0 be the coordinates of a planet at the time t_0, the velocity of which has as its components $\dot{x}_0$, $\dot{y}_0$, $\dot{z}_0$. Let x, y, z be the coordinates of the earth at the time t_0 and $\dot{x}$, $\dot{y}$, $\dot{z}$ its component velocities. We shall suppose that these components remain unaltered during the time the light travels from the planet to the earth, in other words we overlook for this brief period the curvatures of the orbits and the changes of velocity of both bodies. Let $\dot{X}$, $\dot{Y}$, $\dot{Z}$ be the component velocities of a ray of light which at the time t_0 left the point x_0, y_0, z_0, regarded as a fixed point.

As the ray of light regarded as a projectile from the planet will start with a velocity which has components $\dot{X}+\dot{x}_0$, $\dot{Y}+\dot{y}_0$. $\dot{Z}+\dot{z}_0$ it will in the time τ have reached the position with coordinates

$$x_0+(\dot{X}+\dot{x}_0)\,\tau,\quad y_0+(\dot{Y}+\dot{y}_0)\,\tau,\quad z_0+(\dot{Z}+\dot{z}_0)\,\tau,$$

and if this fall on the earth we must have

$$x_0+(\dot{X}+\dot{x}_0)\,\tau=x+\dot{x}\tau,$$
$$y_0+(\dot{Y}+\dot{y}^0)\,\tau=y+\dot{y}\tau,$$
$$z_0+(\dot{Z}+\dot{z}_0)\,\tau=z+\dot{z}\tau.$$

These equations may be written in the form

$$x_0+\dot{X}\tau=x+(\dot{x}-\dot{x}_0)\,\tau,$$
$$y_0+\dot{Y}\tau=y+(\dot{y}-\dot{y}_0)\,\tau,$$
$$z_0+\dot{Z}\tau=z+(\dot{z}-\dot{z}_0)\,\tau.$$

This proves that planetary aberration may be calculated by compounding with the actual velocity of the earth a velocity equal and opposite to that of the planets, and then regarding the planet as at rest.

The formulae here arrived at by the corpuscular theory of light have been shown to be equally true when the undulatory theory of light is adopted.

It will be sufficient to take the case of the earth and a planet, which are assumed to move uniformly in circular orbits in the same plane.

Let S (Fig. 73) be the sun, T the earth moving in the direction TT', perpendicular to ST, with a velocity v. The planet V is moving in the direction of the tangent VV' and with a velocity $v\sqrt{r}/\sqrt{r'}$, where r and r' denote ST and SV. The elongation of the planet from the sun, as seen from the earth, is $\angle STV$, and that of the earth from the sun as seen from the planet is $\angle SVT$. We shall denote these elongations by E and P.

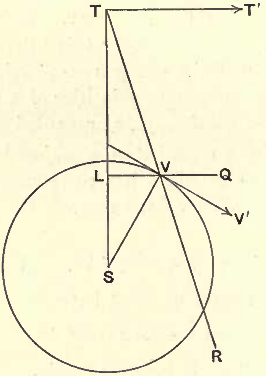


FIG. 73.

If we write for brevity $v\sqrt{r}/\sqrt{r'} = v'$ then the components of v' parallel to TT' and TS are $-v'\cos(E+P)$ and $+v'\sin(E+P)$ respectively. To obtain the planetary aberration we are to compound these velocities when their signs have been changed with the velocity of the earth, which then has components $\{v + v'\cos(E+P)\}$ from T towards T' and $-v'\sin(E+P)$ from T towards S. If TT', TS be the positive directions of axes x and y then by making $\zeta = 0$, $\zeta' = 0$, $\zeta_0 = 0$, $\eta = 90° - E$, $\eta' = 90° - E'$, $v\cos\eta_0 = v + v'\cos(E+P)$, $v\sin\eta_0 = -v'\sin(E+P)$ we have from equations (1) and (2) in § 82

$$\mu\sin E = \mu'\sin E' - v - v'\cos(E+P),$$
$$\mu\cos E = \mu'\cos E' + v'\sin(E+P),$$

whence by eliminating μ' and remembering that $E' - E$ is very small

$$\mu\sin(E' - E) = v\cos E + v'\cos P.$$

Let v_0 be a planet of the system at the distance r_0, then we have

$$v = v_0 \sqrt{r_0}/\sqrt{r}, \quad v' = v_0 \sqrt{r_0}/\sqrt{r'}.$$

With this substitution we have for the planetary aberration $(E' - E)$

$$E' - E = \frac{v_0}{\mu} \sqrt{r_0} (\cos E/\sqrt{r} + \cos P/\sqrt{r'}).$$

When the correction for planetary aberration has been applied, we obtain the place of the planet as it was when the ray of light left it. But this will not be the actual place of the planet at the time the observation is made. It will be the place of the planet at an epoch earlier by $498^s \cdot 5 \times \Delta$, where Δ is the distance of the planet from the earth expressed in terms of the sun's mean distance, because $498^s \cdot 5$ is the time occupied by light in moving through a distance equal to the mean distance of the sun.

Ex. 1. The orbits of two planets are circular and in the same plane; prove that when there is no aberration in the position of either of them as seen from the other, the distance from the sun of the line joining them is $ab(a^2 + ab + b^2)^{-\frac{1}{2}}$, where a and b are the radii of their orbits.

[Math. Trip.]

Ex. 2. Two planets move in coplanar circular orbits of radii R, r; show that when the difference of their longitudes is θ, the aberration is proportional to

$$\frac{(\sqrt{R} + \sqrt{r})\{(R - \sqrt{Rr} + r)\cos\theta - \sqrt{Rr}\}}{\sqrt{Rr}\,(R^2 - 2Rr\cos\theta + r^2)}.$$

Ex. 3. If two planets move in circles round the sun, show that the aberration of one as seen from the other will be less in conjunction than in opposition in the ratio

$$\frac{\sqrt{R} - \sqrt{r}}{\sqrt{R} + \sqrt{r}},$$

R and r being the radii of the two orbits.

[Math. Trip.]

***91. Formulae of reduction from mean to apparent places of stars.**

By the *mean place* of a star we are to understand the position of the star as it would appear if it could be viewed by an observer who was in the centre of the sun and at rest. The *apparent place* of the star is the position which it seems to occupy to a terrestrial observer, and it differs from the mean place both by refraction which we have already considered in Chap. VI., and which need not be now referred to, and by aberration which we are now to consider. When the mean place of a star is expressed by its R.A. and decl., the equator and equinox adopted are those of the beginning of the year, or more strictly, of the moment when the sun's mean longitude is exactly 280°, as explained in § 59.

We have already explained in § 59 the compendious methods by which we can calculate the changes in the coordinates of a star caused by precession and nutation. We are now to explain the more complete process in which the effects of aberration as well as those of precession, nutation and proper motion on the coordinates of any particular star can be readily computed, so that the apparent place can be obtained when the mean place is known. The necessary formulae are given in the ephemeris for each year, see, for example, p. 233 in the *Nautical Almanac* for 1910. We shall here set down the formulae for what are known as Bessel's day numbers A, B, C, D, the expressions for which, retaining only the terms of chief importance, are as follows:

$$\left.\begin{aligned} A &= -20''\cdot47 \cos\omega \cos \odot, \\ B &= -20''\cdot47 \sin \odot, \\ C &= t - 0\cdot342 \sin \Omega - 0\cdot025 \sin 2L, \\ D &= -9''\cdot210 \cos \Omega - 0''\cdot551 \cos 2L, \end{aligned}\right\} \quad \ldots\ldots\ldots\ldots(\text{i})$$

where ω is the obliquity of the ecliptic, $\odot$ the sun's true longitude, L the sun's mean longitude, Ω the longitude of the moon's ascending node, each for the time t, which may be expressed with sufficient accuracy for our present purpose as the fraction of the year which has elapsed since noon of the preceding January 1st. The quantities A, B, C, D do not involve the star coordinates, they are common to all stars and depend only on the time. The logarithms of A, B, C, D are given daily throughout the year in the ephemeris, and in forming them all the terms, including those which on account of their smallness we have here omitted, are duly attended to. To apply the day numbers to finding the corrections for any particular star we have to calculate certain other quantities a, b, c, d, a', b', c', d' which depend on the place of the star, but do not depend on the time. They are as follows:

$$\left.\begin{array}{ll} a = \frac{1}{15}\cos\alpha\sec\delta, & a' = \tan\omega\cos\delta - \sin\alpha\sin\delta, \\ b = \frac{1}{15}\sin\alpha\sec\delta, & b' = \cos\alpha\sin\delta, \\ c = 3^s{\cdot}073 + 1^s{\cdot}336\sin\alpha\tan\delta, & c' = 20''{\cdot}046\cos\alpha, \\ d = \frac{1}{15}\cos\alpha\tan\delta, & d' = -\sin\alpha, \end{array}\right\} \text{(ii)}$$

where α, δ are the mean right ascension and declination for the beginning of the year.

We also take account of the proper motion of the star if it be sufficiently large to be appreciable by assuming

Δc = the annual proper motion in right ascension,
$\Delta c'$ = the annual proper motion in declination.

Then for the time represented by t, we have

$$\left.\begin{array}{l} \text{Apparent R.A. in time} = a + Aa + Bb + Cc + Dd + t\Delta c, \\ \text{Apparent declination} = \delta + Aa' + Bb' + Cc' + Dd' + t\Delta c'. \end{array}\right\} \ldots\text{(iii)}$$

The convenience of these formulae will be readily perceived, for the quantities $\log a$, $\log a'$, &c., $\log b$, $\log b'$, &c. can be calcucated once for all for any given star, and then for any particular day on which the reductions are required, $\log A$, $\log B$, &c. can be taken from the ephemeris.

The proof of equations (iii) follows from formulae which have already been given. The aberration in right ascension which has been found in § 84, is precisely what we have here represented by $Aa + Bb$, and in like manner the effect of precession and nutation in R.A. is that which we have here expressed as $Cc + Dd$. The second formula of (iii) can be explained in like manner.

For some purposes formulae (iii) may with much advantage be superseded by others. The transformation is effected by introducing the independent day numbers f, $\log g$, G, $\log h$, H, $\log i$, of which we have already discussed the use in § 59 so far as f, g, G are concerned. For convenience we here collect all the formulae showing how the independent day numbers are connected with Bessel's day numbers by the equations

$$\left.\begin{array}{ll} 3^s{\cdot}073C = f, & B = h\cos H, \\ 20^s{\cdot}046C = g\cos G, & A = h\sin H, \\ D = g\sin G, & A\tan\omega = i. \end{array}\right\} \ldots\ldots\text{(iv)}.$$

With these substitutions we have

$$\text{Apparent R.A. in time} = \alpha + f + t\Delta c + \tfrac{1}{15} g \sin (G + \alpha) \tan \delta + \tfrac{1}{15} h \sin (H + \alpha) \sec \delta \ldots (\text{v}),$$

$$\text{Apparent declination} = \delta + i \cos \delta + t\Delta c' + g \cos (G + \alpha) + h \cos (H + \alpha) \sin \delta \ldots (\text{vi}).$$

The independent day numbers i, h, H which are connected with the aberration can be directly calculated from the following formulae:

$$h \cos H = -20''\cdot47 \sin \odot ; \quad h \sin H = -20''\cdot47 \cos \omega \cos \odot ;$$
$$i = -20''\cdot47 \sin \omega \cos \odot ;$$

in which we may without restriction of generality always take h to be a positive quantity. It is easily seen that $(180° - H)$ and $\tan^{-1}(i/h)$ are respectively the R.A. and the decl. of the apex.

Ex. 1. Show that the displacement of a star by aberration when expressed in seconds of arc is the square root of the quantity

$$\{i \cos \delta + h \cos (H + a) \sin \delta\}^2 + \{h \sin (H + a)\}^2.$$

Ex. 2. If the mean R.A. of Capella on 1910 Jan. 1st be $5^h\, 10^m\, 2^s\cdot31$ and its mean declination be $+45°\, 54'\, 26''\cdot5$, show that for its apparent place on 1910 Nov. 27, the R.A. should be increased by $4^s\cdot68$ and the declination by $7''\cdot7$, it being given that on Nov. 27, $f = 1\cdot95$, $\log g = 1\cdot145$, $G = 335°\, 32'$, $\log h = 1\cdot305$, $H = 23°\, 16'$, $\log i = +0\cdot539$, and that the annual proper motion is $+0^s\cdot009$ in R.A. and $-0''\cdot4$ in declination.

*Ex. 3. If D be the apparent distance on a certain day between a star a, δ, and an adjacent star at the apparent position angle p, and if f, g, G, h, H, i be the corresponding independent day numbers for correcting the apparent places of stars for aberration, precession and nutation, show that the distance between the mean places of the two stars on the preceding Jan. 1st was

$$D + D\{i \sin \delta - h \cos (H + a) \cos \delta\} \sin 1'',$$

and the position angle was

$$p - g \sin (G + a) \sec \delta - h \sin (H + a) \tan \delta.$$

To find the position angle at a date n years earlier than the Jan. 1st immediately preceding the observation, show that a further correction of $-20''\cdot046 \sin a \sec \delta$ must be applied to p to allow for the precessional motion of the pole.

Let a, δ be the *apparent* right ascension and declination of the principal star of the pair.

Let $a + \phi$, $\delta + \psi$ be the corresponding mean coordinates when reduced to the beginning of the year.

Let α', δ' be the apparent R.A. and declination of the adjacent star, and when the corrections are applied to α' and δ' to bring them to the commencement of the year they become by Taylor's theorem

$$\alpha' + \phi + \frac{\partial \phi}{\partial \alpha}(\alpha' - \alpha) + \frac{\partial \phi}{\partial \delta}(\delta' - \delta) \quad \text{.......................(1)},$$

$$\delta' + \psi + \frac{\partial \psi}{\partial \alpha}(\alpha' - \alpha) + \frac{\partial \psi}{\partial \delta}(\delta' - \delta) \quad \text{.......................(2)}.$$

Let $D + dD$ and $p + dp$ be the corresponding distance and position of the two stars when in their mean places for the commencement of the year. Then we have approximately

$$D \cos p = \delta' - \delta,$$

$$D \sin p = (\alpha' - \alpha) \cos \delta,$$

and by differentiating and substituting

$$\cos p\, dD - D \sin p\, dp = \frac{\partial \psi}{\partial \alpha}(\alpha' - \alpha) + \frac{\partial \psi}{\partial \delta}(\delta' - \delta) \quad \text{..................................(3)}$$

$$\sin p\, dD + D \cos p\, dp = -\psi(\alpha' - \alpha)\sin\delta + (\alpha' - \alpha)\cos\delta\frac{\partial \phi}{\partial \alpha} + (\delta' - \delta)\cos\delta\frac{\partial \phi}{\partial \delta} \text{...(4)},$$

but these may be written

$$\cos p\, dD - D \sin p\, dp = \quad D \sin p \sec \delta \frac{\partial \psi}{\partial \alpha} + D \cos p \frac{\partial \psi}{\partial \delta} \quad \text{.......................(5)},$$

$$\sin p\, dD + D \cos p\, dp = -D \sin p \tan \delta \,.\, \psi + D \sin p \frac{\partial \phi}{\partial \alpha} + D \cos p \cos \delta \frac{\partial \phi}{\partial \delta} \text{ ...(6)}.$$

Introducing for ϕ and ψ their values

$$\phi = -f - g \sin(G + \alpha) \tan \delta - h \sin(H + \alpha) \sec \delta,$$

$$\psi = -i \cos \delta - g \cos(G + \alpha) - h \cos(H + \alpha) \sin \delta,$$

performing the differentiations indicated and then solving for dD and dp the result assumes the form

$$dD = \quad D\{i \sin \delta - h \cos(H + \alpha) \cos \delta\} \sin 1'' \quad \text{...............(7)},$$

$$dp = -g \sin(G + \alpha) \sec \delta - h \sin(H + \alpha) \tan \delta \text{ ....(8)}.$$

The quantities g, G are absent from dD, for it is obvious that changes in the equator cannot produce any effect on the distance between two stars. Though these formulae are concerned with such small quantities they become of importance in the investigation of the annual parallax of stars.

For the last part of the question we have to reduce formula (8), so that it shall contain the effect only of precession for n whole years, aberration and nutation being both made zero. This is done by making

$$h = 0, \quad G = 0, \quad g = 20''{\cdot}046n,$$

whence the correction to reduce to the mean pole n years earlier is

$$-20''{\cdot}046n \sin \alpha \sec \delta.$$

*Ex. 4. If by a small change in the equator the coordinates α, δ of each star become $a+\phi$, $\delta+\psi$ without any change in the position of the star, show that ϕ and ψ being functions of the coordinates

$$\frac{\partial\psi}{\partial\delta}=0,$$

$$\frac{\partial\phi}{\partial a}-\psi\tan\ \delta=0,$$

$$\sec\delta\frac{\partial\psi}{\partial a}+\cos\delta\frac{\partial\phi}{\partial\delta}=0.$$

From Ex. 3 we see that that the change dD in the distance D of two adjacent stars is given by

$$dD=D\cos^2 p\frac{\partial\psi}{\partial\delta}+D\sin^2 p\left(\frac{\partial\phi}{\partial a}-\psi\tan\delta\right)$$
$$+D\sin p\cos p\left(\sec\delta\frac{\partial\psi}{\partial a}+\cos\delta\frac{\partial\phi}{\partial\delta}\right),$$

and as dD must be zero whatever be the value of p the required result is at once obtained.

*Ex. 5. Show that if a number of stars lie on a circle of which the arcual radius is very small the effect of aberration on these stars is to convey them to the circumference of an adjacent circle (Brünnow).

This follows from the absence of p from equations (7), (8), Ex. 3.

*Ex. 6. Let A and B be two stars which appear to be conveyed by aberration to A' and B' towards an apex C. Show that the aberration changes the angle at A into $A-\kappa\tan\frac{1}{2}c\sin p$ where c is the arc AB and p is the perpendicular from C on AB.

Let the two stars at B and A be at distances a, b respectively from C. Then in the spherical triangle we have

$$\cos b\cos C=\sin b\cot a-\sin C\cot A,$$

differentiating and making

$$\Delta a=-\kappa\sin a,\quad \Delta b=-\kappa\sin b,\quad \Delta C=0,$$

we obtain

$$\kappa\sin b\operatorname{cosec} a\,(\sin a\sin b\cos C+\cos a\cos b-1)=\sin C\operatorname{cosec}^2 A\,\Delta A,$$

whence

$$\Delta A=-\kappa\tan\tfrac{1}{2}c\sin A\sin b,$$
$$=-\kappa\tan\tfrac{1}{2}c\sin p.$$

If the stars are adjacent c is small, so that when multiplied by κ the product is very small, and ΔA becomes inappreciable.

*Ex. 7. From the star defined by ($a=59°\ 53'$; $\delta=37°\ 45'$) the distance $237''\cdot3$ and position angle $207°\ 14'$ of an adjacent star were measured on 6th Jan. 1880. Show that the corrections to be applied to the distance and position angle to reduce them to the date 1879·0 are $-0''\cdot015$ and $-0'\cdot66$ respectively.

From *N. A.* 1880, p. 303, we have for 1880 Jan. 6,

$$\log g = +0{\cdot}8734,\ G = 343^\circ\ 40',\ \log h = +1{\cdot}3079,\ H = 345^\circ\ 8',\ \text{Log}\, i = -0{\cdot}3541.$$

Aberration in distance, Log 1st term $= -7{\cdot}202$, Log 2nd term $= -8{\cdot}116$, Log nutation in position $-9{\cdot}036$, Log aberration in position $-9{\cdot}268$. Correction for one year's precession $-0''{\cdot}366$.

*Ex. 8. Show that the effect of aberration on the distance D between two stars must always be $< D/10000$.

EXERCISES ON CHAPTER XI.

Ex. 1. If the earth be supposed to move round the sun in a circle with velocity v, and the velocity of an observer on the earth's surface due to its rotation be nv, then x the aberration of any star σ is accurately given by the formula

$$\tan x = \frac{k\,(\sin^2 O\sigma + 2n \sin O\sigma \sin \sigma E \cos O\sigma E + n^2 \sin^2 \sigma E)^{\frac{1}{2}}}{1 + k \cos O\sigma + nk \cos \sigma E},$$

where O is a point on the ecliptic 90° behind the sun, E a point on the equator differing in R.A. from the sun by the complement of the hour angle, and k the ratio of the velocity of the earth's centre to the velocity of light.

[Math. Trip.]

If in a spherical triangle an arc CO of length s be drawn from the vertex C dividing the base into two segments $BO = l$ and $AO = m$, then

$$\sin^2 s \sin^2 (l+m) = \sin^2 b \sin^2 l + 2 \sin a \sin b \sin l \sin m \cos C + \sin^2 m \sin^2 a.$$

If the star be at C and if B be the apex of the rotational motion and A that of the orbital, and if x be the resultant aberration, then $\mu \sin x = \rho \sin (s - x)$, where ρ is the resultant velocity of the observer, and μ the velocity of light, and we have

$$\rho \operatorname{cosec} (l+m) = v \operatorname{cosec} l - nv \operatorname{cosec} m,$$

from which

$$\tan x = \frac{k \sin s \sin (l+m)}{\sin l + k \cos s \sin (l+m)},$$

but

$$\cos s \cos m = \cos b - \sin s \sin m \cos O,$$

$$\cos s \cos l = \cos a + \sin s \sin l \cos O,$$

whence

$$\cos s (\cos m + n \cos l) = \cos b + n \cos a,$$

which with the formula above proves the theorem.

Ex. 2. Show that the locus of all stars whose zenith distance at a given place and a given instant are unaltered by aberration is an elliptic cone, one of whose circular sections is horizontal, and the other is perpendicular to the ecliptic.

[Coll. Exam.]

In this case the angle subtended by the zenith and the apex at the star must be 90°, from which the desired result is easily obtained.

Ex. 3. Prove that at every place there is always at a given instant one position for a star for which the aberration is entirely counteracted by the refraction. Show also that at midnight on the shortest day the zenith distance of this position is given by an equation of the form

$$\sin^2 z + \lambda \sin z = 1,$$

if the correction for refraction be assumed proportional to the tangent of the zenith distance, and the earth's orbit be assumed to be circular.

[Math. Trip. I. 1900.]

*Ex. 4. If by any small change in the equator the coordinates a, δ of each point on the celestial sphere become $a+\phi$, $\delta+\psi$, show that we must have

$$\phi = C + A \sin(a+B) \tan\delta,$$

$$\psi = A \cos(a+B),$$

where A, B, C are constants independent of the coordinates, and verify that this transformation leaves the distance between every pair of stars unaltered.

CHAPTER XII.

THE GEOCENTRIC PARALLAX OF THE MOON.

92. Introductory.

By the word *Parallax* we mean the angle OSO' (Fig. 74) between the direction $O'S$ in which a point S is seen by the observer at O' and the direction in which the same point S would be seen if the observer occupied a standard position O. If S be the sun or the moon or a planet or a comet or, in fact, any body

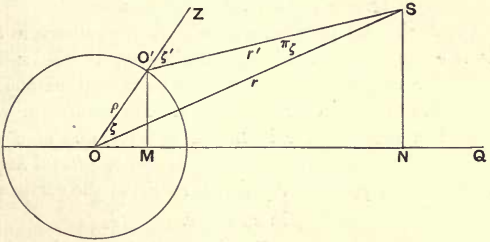


FIG. 74.

belonging to the solar system, then the standard position O is always taken to be the centre of the earth, and the parallax is said to be *geocentric*. If S be a star, then O is taken to be the centre of the sun and the parallax is generally described as *annual*.

The geocentric parallax of the sun is the angle OSO' where S is the centre of the sun, ρ is the earth's radius OO', and ζ', ζ represent the angles $SO'Z$ and SOZ respectively. Then the effect of parallax may be said to throw the apparent place of the object away from the direction OO' by the angle $\zeta' - \zeta$ which we shall represent by the symbol π_ζ. Of course, if the earth were regarded as a sphere, then ζ' and ζ would be the apparent and real zenith distances and the influence of parallax would merely depress the apparent place of the object further from the zenith. As the earth is not spherical, the effect of parallax is to depress the body not exactly from the zenith but from the point in which the earth's radius when continued will meet the celestial sphere. The arc between this point and the true zenith is of course the quantity already considered in § 15.

From the triangle OSO' we have

$$\sin \pi_\zeta = \rho \sin \zeta'/r \text{ ....(i).}$$

We now introduce the angle π_ϕ defined by the equation

$$\sin \pi_\phi = \rho/r \text{ ....(ii),}$$

and hence from (i),

$$\sin \pi_\zeta = \sin \pi_\phi \sin \zeta'.$$

Thus we see that π_ϕ is the greatest value of π_ζ and this will be attained when ζ' is 90° which, if refraction were not considered, would mean that the centre of the sun was on the horizon. We accordingly term π_ϕ the *horizontal parallax.*

As the horizontal parallax depends on ρ as shown in (ii) and as ρ is not the same for all latitudes owing to the spheroidal form of the earth, it follows that the horizontal parallax must vary with the latitude of the observer. Its maximum value is attained when the observer is on the equator, and as ϕ is then zero we express by π_0 what is known as the *equatorial horizontal parallax,* so that if ρ_0 is the equatorial radius of the earth we have

$$\sin \pi_0 = \rho_0/r.$$

If the sun be at its mean distance so that r equals a the semi-axis major of the sun's apparent orbit, then the quantity π_a is defined to be the *mean equatorial horizontal parallax of the sun,* and is given by the equation

$$\sin \pi_a = \rho_0/a.$$

We shall always take $\pi_a = 8''\cdot 80$.

The symbols already given apply to the geocentric parallax of the *sun*. By the addition of a dash to π we denote the corresponding quantities for the moon, thus

$\pi_{☾}'$ is the *geocentric parallax of the moon, i.e.* the angle which the centre of the earth and the position of the observer subtend at the centre of the moon.

π_ϕ' is the angle whose sine is the ratio of the distances of the observer and the moon's centre from the centre of the earth. This is the *horizontal parallax of the moon at latitude* ϕ.

π_0' is the value of π_ϕ' when the observer is on the equator, this is the *equatorial horizontal parallax of the moon.*

π_a' is the value of π_0' when the moon is at its mean distance. This is the *mean equatorial horizontal parallax of the moon.* We shall take $\pi_a' = 3422''$.

The moon is here regarded as a sphere and the semi-vertical angle of the cone which this sphere subtends at the earth's centre, *i.e.* the apparent semi-diameter of the moon, varies from 16′ 47″ to 14′ 43″, and has a mean value of 15′ 34″.

From the formula (ii) we obtain

$$r = \rho \operatorname{cosec} \pi_\phi'.$$

The radius of the earth is a known quantity, and if π_ϕ' is also known, then in this equation the right-hand side is known and therefore r is known. Thus we obtain the important result that the distance of a celestial body can be determined when its horizontal parallax is known. It is in fact only by determining the parallax of a celestial body by observation that we can ascertain its distance, and as the determination of these distances is of the utmost importance in Astronomy it is obvious that the subject of parallax merits careful attention.

The *geocentric* parallax of a star properly so called is far too minute to be sensible. In the case of even the nearest star (α Centauri) the horizontal parallax would be only 0″·00003, and no parallax could be detected by our measurements which was not more than a thousand times greater than this quantity. It is therefore impossible to determine the distance of a star by its geocentric parallax. For such an investigation we have to resort to *annual parallax,* and the consideration of this is deferred to Chap. XV. Our present problem is that of geocentric parallax and especially in its application to the moon, the mean equatorial horizontal parallax of which is 57′ 2″. In Chapters XIII. and XIV. we shall consider the geocentric parallax of the sun and other bodies in the solar system.

Ex. 1. Show that

$$\tan \pi_\zeta' = \sin \pi_\phi' \sin \zeta / (1 - \sin \pi_\phi' \cos \zeta).$$

Ex. 2. Show that parallax increases the apparent semi-diameter of the moon in the ratio $\sin \zeta' : \sin(\zeta' - \pi_\zeta')$, where ζ' is the apparent zenith distance and the earth is assumed to be spherical.

Ex. 3. Show that if the horizontal parallax π_0' be a quantity whose square may be neglected the apparent daily path of a celestial body as seen from the earth's surface (supposed spherical), is a small circle of radius $90° - \delta + \pi_0' \sin \phi \cos \delta$ described about a point depressed $\pi_0' \cos \phi \sin \delta$ below the pole. [Coll. Exam.]

From § 35 (1) we obtain if z be the zenith distance

$$\Delta\delta + \cos \eta \, \Delta z - \cos h \, \Delta\phi - \sin h \cos \phi \, \Delta a = 0.$$

In the present case $\Delta z = \pi_0' \sin z$, $\Delta a = 0$, and if $\Delta\phi = -\pi_0' \cos \phi \sin \delta$, we have

$$\Delta\delta = -\pi_0' (\cos \eta \sin z + \cos \phi \sin \delta \cos h) = -\pi_0' \sin \phi \cos \delta.$$

93. The fundamental equations of geocentric parallax.

To obtain the desired equations it is necessary to express the coordinates of the points on the celestial sphere to which the lines OO', OS, $O'S$ (Fig. 74) are severally directed. It is convenient to take for this purpose the celestial equator as the fundamental circle and ♈ as the origin. Thus the coordinates we employ are right ascensions and declinations.

In the general investigation to which we are now proceeding the earth is to be regarded as a spheroid and ρ is the distance from the observer to the earth's centre. The inclination of the line OO' to the equator is the geocentric latitude ϕ' of the observer, and therefore ϕ' is the declination of the point on the celestial sphere to which OO' is directed. The right ascension of the same point is the right ascension of the point where the observer's meridian intersects the celestial equator, but this is the sidereal time ϑ which is of course the W. hour angle of ♈.

The directions of OS, $O'S$ will be defined respectively by the coordinates (α, δ), (α', δ').

If the parallax, *i.e.* $\angle OSO'$, be inappreciable then OS and $O'S$ are sensibly parallel and α', δ' are indistinguishable from α, δ. If the parallax be appreciable then the point α, δ called the *true place* is not the same as α', δ' called the *apparent place.* We have first to obtain equations from which α', δ' can be obtained in terms of α, δ or *vice versa.*

Draw through O (Fig. 74) a line OQ (not necessarily in the plane $OO'S$) to the point (λ, μ) and let $O'M$ and SN be perpendiculars on OQ, so that OM, ON are the projections of OO', OS on OQ: then the projection of $O'S$ is $MN = ON - OM$, and therefore (§ 8) we have the general formula

$$\begin{aligned} r'\{\sin\delta' \sin\mu + \cos\delta' \cos\mu \cos(\alpha' - \lambda)\} &= \\ r\{\sin\delta \sin\mu + \cos\delta \cos\mu \cos(\alpha - \lambda)\} & \\ - \rho\{\sin\phi' \sin\mu + \cos\phi' \cos\mu \cos(\vartheta - \lambda)\} & \quad \text{.........(1).} \end{aligned}$$

This equation must be true whatever be the line OQ. If therefore we take in succession the three cases where λ, μ are respectively (0, 0); (90°, 0); (0°, 90°); we obtain the three fundamental equations for parallax in the form

$$\left.\begin{aligned} r'\cos\delta' \cos\alpha' &= r\cos\delta\cos\alpha - \rho\cos\phi'\cos\vartheta \text{......(2),} \\ r'\cos\delta' \sin\alpha' &= r\cos\delta\sin\alpha - \rho\cos\phi'\sin\vartheta \text{......(3),} \\ r'\sin\delta' &= r\sin\delta \qquad\quad - \rho\sin\phi' \text{............(4).} \end{aligned}\right\}$$

These equations might also have been obtained by equating to zero the coefficients of $\sin\mu$, $\cos\mu\cos\lambda$, $\cos\mu\sin\lambda$ in the equation (1), for these coefficients must vanish because the equations have to be true for all values of λ, μ.

The formulae just obtained will perhaps appear to the beginner to express all that could be necessary for the determination of the effect of parallax on the coordinates of a celestial body. If α, δ, r are given we have here three equations for α', δ', r' or if α', δ', r' are given we have here three equations for α, δ, r. But in their present form the equations are not nearly so easy to employ or so accurate in their results as are certain other equations which we shall deduce from them. It certainly might seem at first sight that if two sets of equations are mathematically equivalent the calculations from one set of equations should be equally accurate with those made from the other set. But as we had occasion to mention already in another problem (§ 64) this is not necessarily the case. It must be remembered that logarithms of the trigonometrical functions like other logarithms are only approximate. Thus every formula into which logarithms are introduced becomes, to some extent, erroneous from this cause. Equations which are mathematically correct in their symbolic form generally part with mathematical accuracy when numerical logarithms are introduced and the extent of the inaccuracy varies according to circumstances. It is the art of the astronomical computer to select from among the different possible transformations of a given set of equations that particular set which when solved shall afford results as little influenced as possible by the inevitable logarithmic inaccuracies. Thus it happens that though (2), (3), (4) are theoretically sufficient for the determination of α', δ', r' yet we shall obtain greater accuracy and have much less trouble with the logarithms if we employ in our calculations certain other equations such as (7) and (15) in which the unknowns are $(\alpha' - \alpha)$ and $(\delta' - \delta)$ instead of merely α' and δ'. It will be found that the work can be done more accurately by using only five-figure logarithms in (7) and (15) than by using seven-figure logarithms in (2), (3), (4).

The rationale of the matter may be thus illustrated. Let A and B be two points one kilometre apart, and let it be desired to set off on the line AB a point O which shall be one metre from A and therefore 999 metres from B. If our instruments of measurement were mathematically perfect we could set off O with equal precision by measuring either from A or from B. But our instruments are not perfect, and this being so, it is no longer a matter of indifference whether the measurements be from A or from B. For suppose that our measuring instruments habitually gave a result which was one millionth part in excess of the truth. Then in setting off BO there would be an error of almost a millimetre. But in setting off AO the error would only be the thousandth part of a millimetre. Hence we should make our measurements from A and not from B. Substituting α for AB, $\alpha - \alpha'$ for AO, and α' for BO we see how much more satisfactory it will be to proceed by calculating $\alpha - \alpha'$ rather than by making the more precarious calculation of deducing α' from (2), (3), (4). We must therefore obtain from (2), (3), (4) formulae giving $\alpha' - \alpha$, and $\delta' - \delta$ and employ these formulae rather than the original formulae in the subsequent calculations.

The equation for $(\alpha' - \alpha)$ will be obtained by multiplying (3) by $\cos\alpha$ and subtracting from it (2) multiplied by $\sin\alpha$, thus giving

$$r'\cos\delta'\sin(\alpha'-\alpha) = -\rho\cos\phi'\sin(\vartheta-\alpha)\ldots\ldots\ldots(5),$$

in which $\vartheta-\alpha$ is the moon's hour angle West.

This equation might indeed have been obtained directly from (1) which has to be true for all values of λ, μ. If we make $\lambda=\alpha+90°$, $\mu=0$, then (1) becomes (5).

Multiplying (2) by $\cos\alpha$ and adding to it (3) multiplied by $\sin\alpha$, we find

$$r'\cos\delta'\cos(\alpha'-\alpha) = r\cos\delta - \rho\cos\phi'\cos(\vartheta-\alpha)\ \ldots(6),$$

and this might have been obtained at once from (1) by making $\lambda=\alpha$, $\mu=0$.

Dividing (5) by (6) we obtain the fundamental equation for parallax in right ascension in the form

$$\tan(\alpha'-\alpha)$$
$$= -\sin\pi_\phi'\cos\phi'\sin(\vartheta-\alpha)/\{\cos\delta-\sin\pi_\phi'\cos\phi'\cos(\vartheta-\alpha)\}\ldots(7),$$

where we have replaced ρ/r by $\sin\pi_\phi'$.

All the quantities on the right-hand side being known, then $\tan(\alpha'-\alpha)$ is determined. We shall assume that in all the cases to which this equation is to be applied $\sin\pi_\phi'$ is a small quantity. Hence the numerator of the expression for $\tan(\alpha'-\alpha)$ is a small quantity. If δ be small, *i.e.* if the body be near the equator, as is of course the case with the sun and the moon and the principal planets, which are the only bodies that concern us at present, then the denominator is nearly unity, so that $\tan(\alpha'-\alpha)$ must be itself small and so is also $(\alpha'-\alpha)$. But it should be remarked that if the body had a very high declination so that $\cos\delta$ was very small, then the denominator of $\tan(\alpha'-\alpha)$ would be very small when $\sin\pi_\phi'$ was small, so that $\alpha'-\alpha$ need not be a small quantity. A comet which passed close to the pole would be a case in point, and we would then have to distinguish between the two roots of equation (7), viz. $(\alpha'-\alpha)$ and $180°+(\alpha'-\alpha)$ by observing that (5) had to be satisfied.

We have next to find $(\delta'-\delta)$, that is the correction to be applied to the true declination to give the apparent declination. This is not quite so simple a matter as the parallax in right ascension. We multiply (2) by $\cos\frac{1}{2}(\alpha'+\alpha)$ and (3) by $\sin\frac{1}{2}(\alpha'+\alpha)$ and add so that after division by $\cos\frac{1}{2}(\alpha'-\alpha)$ we obtain

$$r'\cos\delta' = r\cos\delta - \rho\cos\phi'\sec\tfrac{1}{2}(\alpha'-\alpha)\cos\{\vartheta-\tfrac{1}{2}(\alpha'+\alpha)\}\ldots(8).$$

This equation might also have been obtained directly from (1) by introducing the values $\lambda = \frac{1}{2}(\alpha + \alpha')$, $\mu = 0$.

We shall now employ two auxiliary quantities β and γ defined by the following equations

$$\beta \sin \gamma = \sin \phi' ; \quad \beta \cos \gamma = \cos \phi' \sec \tfrac{1}{2}(\alpha' - \alpha) \cos \{\vartheta - \tfrac{1}{2}(\alpha' + \alpha)\}.$$

Dividing one of these equations by the other we have $\tan \gamma$ and we may choose between γ and $180° + \gamma$ by deciding that β shall be positive. Thus β and γ are both definitely known from the equations

$$\tan \gamma = \tan \phi' \cos \tfrac{1}{2}(\alpha' - \alpha) \sec \{\vartheta - \tfrac{1}{2}(\alpha' + \alpha)\} \quad \text{.........(9)},$$

$$\beta = \sin \phi' \operatorname{cosec} \gamma \quad \text{.................................(10)}.$$

With these substitutions equations (4) and (8) assume the form

$$r' \sin \delta' = r \sin \delta - \rho\beta \sin \gamma \quad \text{..............(11)},$$

$$r' \cos \delta' = r \cos \delta - \rho\beta \cos \gamma \quad \text{..............(12)},$$

multiplying (11) by $\sin \delta$ and (12) by $\cos \delta$ and adding, we have

$$r' \cos (\delta' - \delta) = r - \rho\beta \cos (\delta - \gamma) \quad \text{...........(13)},$$

multiplying (11) by $\cos \delta$ and subtracting from it (12) when multiplied by $\sin \delta$, we obtain

$$r' \sin (\delta' - \delta) = \rho\beta \sin (\delta - \gamma) \quad \text{..............(14)},$$

whence dividing (14) by (13) and writing $\sin \pi_\phi'$ for ρ/r

$$\tan (\delta' - \delta) = \beta \sin \pi_\phi' \sin (\delta - \gamma)/\{1 - \beta \sin \pi_\phi' \cos (\delta - \gamma)\} \quad \text{...(15)}.$$

From this we obtain $\delta' - \delta$, which, when applied to the true declination, will give the apparent declination as affected by parallax.

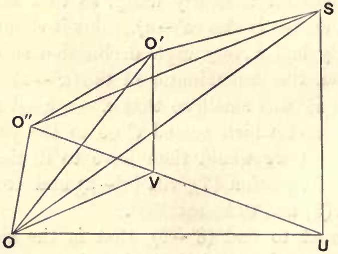


Fig. 75.

Ex. 1. If A' be the place of the moon as disturbed by geocentric parallax, show that the displacement by parallax along any direction $A'O$ will be $\sin \pi_0 \cos ZO$, where Z is the zenith $A'O = 90°$, and the earth is supposed spherical.

*Ex. 2. Show how the equations (11) and (12) by which the parallax in declination is obtained can be deduced directly from geometrical construction and explain the geometrical meaning of the introduced quantities β and γ.

Let SU, $O'V$ (Fig. 75) be perpendiculars on the equatorial plane through O: on UV take O'' so that $UO''=UO$, and join $O''O'$, $O''S$.

The triangle $O''US$ is equal in all respects to OUS. Therefore

$$O''SO'=(\delta-\delta').$$

Let $O''O'=\rho\beta$ and $\angle UO''O'=\gamma$. Since $O''V$, OV have the same projection on the bisector of $\angle OUV$, we have

$$\rho\beta\cos\gamma=O''V=OV\cos\{\vartheta-\tfrac{1}{2}(a+a')\}\sec\tfrac{1}{2}(a'-a)$$

$$=\rho\cos\phi'\sec\tfrac{1}{2}(a'-a)\cos\{\vartheta-\tfrac{1}{2}(a+a')\}.$$

Also $\rho\beta\sin\gamma=VO'=\rho\sin\phi'$,

thus determining β, γ. We take γ of the same sign as ϕ' and numerically $<180°$, so that β is positive.

We now have from the triangle $O''O'S$,

$$r'\cos(\delta'-\delta)=r-\rho\beta\cos(\delta-\gamma),$$

$$r'\sin(\delta'-\delta)=\rho\beta\sin(\delta-\gamma),$$

whence as before,

$$\tan(\delta'-\delta)=\rho\beta\sin(\delta-\gamma)/\{r-\rho\beta\cos(\delta-\gamma)\}.$$

Ex. 3. Show that when seen from a latitude ϕ the parallax in declination of a celestial body vanishes when $\tan\phi=\tan\delta\cos h$ in which δ and h are the declination and hour angle. The earth is supposed spherical.

Ex. 4. Show that if h', δ' be the hour angle and the declination of the moon as seen from the earth's surface at a place of geocentric latitude ϕ', and h, δ the hour angle and the declination as seen from the centre, then

$$\sin(h'-h)=a\sec\delta\sin h',$$

$$\tan\delta'\operatorname{cosec}h'=(1-b\operatorname{cosec}\delta)\tan\delta\operatorname{cosec}h,$$

where $a=\sin\pi_\phi'\cos\phi',\quad b=\sin\pi_\phi'\sin\phi'.$

[Coll. Exam.]

Writing $a=\vartheta-h$ and $a'=\vartheta-h'$ we find from (2) and (3)

$$r\cos\delta\sin h=r'\cos\delta'\sin h',$$

and $r\cos\delta\cos h-\rho\cos\phi'=r'\cos\delta'\cos h'$,

which equations are otherwise obvious, since each side of the first merely expresses the distance of the moon from the meridian, while the second may be obtained directly from equation (1), p. 281, by making $\mu=0$ and $\lambda=\vartheta$, for the equation has to be true for all values of μ and λ. These equations combined with (4) give the desired result.

Ex. 5. Show that the angular radii R' and R of a planet, as seen from a place P on the earth and from the centre of the earth respectively, are connected by the relation

$$\sin R'=\frac{\sin(\delta'-\gamma)}{\sin(\delta-\gamma)}\sin R,$$

where γ is an auxiliary angle defined by the equation

$$\cot\gamma=\sec\tfrac{1}{2}(a'-a)\cot\phi'\cos\{\vartheta-\tfrac{1}{2}(a+a')\},$$

ϕ' being the geocentric latitude of the place, ϑ the sidereal time of the observation, α' and δ' the right ascension and declination of the planet as seen from P, and α and δ the right ascension and declination as seen from the centre of the earth.

[Coll. Exam.]

Ex. 6. Show that the parallaxes in right ascension and declination of the moon are respectively

$$\pi_\alpha' = -\tan^{-1}\{a \sin h/(1 - a\cos h)\},$$

$$\pi_\delta' = -\delta - \tan^{-1}\{(\tan\delta - a\tan\phi')\sin\pi_\alpha'/a\sin h\},$$

where π_ϕ' is the horizontal parallax, h the hour angle, ϕ' the geocentric latitude and $a = \sin\pi_\phi' \cos\phi' \sec\delta$.

[Math. Trip. 1907.]

This follows at once from the equations (4), (5), (6).

94. Development in series of the expressions for the parallax.

Assuming that $\sin\pi_\phi'$ is a small quantity and that the object which has this horizontal parallax is sufficiently distant from either of the celestial poles to prevent $\cos\delta$ from being very small we may develop formula (7) § 93 as in (4) on p. 227

$$\alpha' - \alpha = -\frac{\sin\pi_\phi'\cos\phi'\sin(\vartheta - \alpha)}{\cos\delta\sin 1''} - \frac{\sin^2\pi_\phi'\cos^2\phi'\sin 2(\vartheta-\alpha)}{\cos^2\delta\sin 2''} - \frac{\sin^3\pi_\phi'\cos^3\phi'\sin 3(\vartheta-\alpha)}{\cos^3\delta\sin 3''} \quad \ldots(1).$$

In like manner, we have from (15) § 93,

$$\delta' - \delta = \frac{\beta\sin\pi_\phi'\sin(\delta-\gamma)}{\sin 1''} + \frac{\beta^2\sin^2\pi_\phi'\sin 2(\delta-\gamma)}{\sin 2''} + \frac{\beta^3\sin^3\pi_\phi'\sin 3(\delta-\gamma)}{\sin 3''} \quad \ldots(2).$$

We have not written more than three of the terms of each series because all the higher terms are far too small to be appreciable. Formula (1) gives $(\alpha' - \alpha) = \pi_\alpha'$, which is the correction which must be applied to the true right ascension of the moon in order to obtain the apparent right ascension. Formula (2) gives the corresponding correction π_δ' for the declination.

The first term in each of the series is by far the most important, but the second must not be overlooked in the parallax of the moon and even the third is appreciable when the highest accuracy is sought. In the corresponding expressions for the sun or the planets, only the first term is required.

The equation Ex. 1, § 92, viz.

$$\tan \pi_\zeta' = \sin \pi_\phi' \sin \zeta / (1 - \sin \pi_\phi' \cos \zeta),$$

in which $\cos \zeta = \sin \delta \sin \phi' + \cos \delta \cos \phi' \cos (\vartheta - \alpha)$

can also be expressed as a series, and we thus have for π_ζ' the parallactic displacement

$$\pi_\zeta' = \sin \pi_\phi' \sin \zeta \operatorname{cosec} 1'' + \sin^2 \pi_\phi' \sin 2\zeta \operatorname{cosec} 2'' + \sin^3 \pi_\phi' \sin 3\zeta \operatorname{cosec} 3'' \text{.........(3)}.$$

For an approximate calculation of the moon's parallax in declination and right ascension we may regard the earth as a sphere and consider only the two first terms of (1) and (2). If the observer's latitude is ϕ and the moon's hour angle $= \vartheta - \alpha = h$, we have $\beta \sin \gamma = \sin \phi$ and $\beta \cos \gamma = \cos \phi \cos h$ very nearly, whence

$$\pi_\alpha' = \alpha' - \alpha = - \pi_0' \cos \phi \sin h \sec \delta \text{ ....(4)},$$

$$\pi_\delta' = \delta' - \delta = \pi_0' (\cos \phi \cos h \sin \delta - \sin \phi \cos \delta) \text{ ...(5)}.$$

We can obtain an approximate determination of the moon's parallax in hour angle by the use of the following short table, which is easily constructed from formula (4). The table is formed on the supposition that the horizontal parallax is 60′, and that the declination of the moon is zero. Under these conditions the parallax when expressed in minutes of time becomes $4 \cos \phi \sin h$, from which the table is calculated.

The parallax in hour angle for a given hour angle and latitude is shown along the top line, and the use of the table may be

Minutes of parallax in hour angle

Hour angle	0·5	1	1·5	2	2·5	3	3·5	Hour angle
1	61	15			Latitude North or South			11
2	76	60	41	0				10
3	80	69	58	45	28			9
4	82	73	64	55	44	30		8
5	83	75	67	59	50	39	25	7
6	83	76	68	60	51	41	29	6

made clear by an example:—Let us suppose the hour angle west of the moon is 3^{hrs} and the latitude is 58°, then the table shows that the parallax in minutes of hour angle is 1·5, this is accordingly the amount to be subtracted from the apparent R.A. to obtain the true R.A. If, as is generally the case, the moon's declination is not zero, then a fractional addition must be made to the parallax which is thus indicated:

Declination of moon whether + or − ...	10°	15°	20°	25°
Percentage to be added to parallax given by table	2	4	6	10

It will of course not be generally true that the horizontal parallax is 60′, we must therefore add to (or subtract from) the parallax of the table one-sixtieth part for each minute that the parallax is greater (or less) than 60′. These points as well as the necessary interpolations for latitudes other than those which appear in the table are illustrated by the following example:— The latitude of the observer is 42°, the declination of the moon is 10°, its hour angle is 5^{hrs}, its horizontal parallax is 57′. Find from the table the parallax in hour angle.

The table shows that for 5^{hrs} hour angle and 3^m parallax the latitude is 39°, while for $2\frac{1}{2}^m$ parallax the latitude would be 50°. It is therefore plain that for 42° latitude the parallax would be about 172 secs. The correction for declination adds 2 per cent., *i.e.* 3 secs., while $\frac{1}{20}$th or 9 secs. has to be taken off because the parallax is 57′ and therefore 3′ less than the standard taken in the table. Hence we conclude that the parallax in hour angle is $2^m\ 46^{secs}$, and this is the amount by which parallax increases the hour angle if the moon is west of the meridian and decreases it if the moon is in the east, for we must remember that eastern hour angles are negative.

Ex. 1. Show that in formula (3) for the moon's geocentric parallax the second term $\sin^2 \pi_\phi' \sin 2\zeta \operatorname{cosec} 2''$ may reach 33″, but the third term $\sin^3 \pi_\phi' \sin 3\zeta \operatorname{cosec} 3''$ must always be under 0″·5.

N.B. The greatest horizontal parallax of the moon is 61′·5.

Ex. 2. Show that when the hour angle of the moon changes by the small quantity Δh the corresponding change of the parallax in hour angle is approximately $-\pi_0' \cos\phi \cos h \sec\delta \,.\, \Delta h$, and the change in the parallax in declination is $-\pi_0' \cos\phi \sin h \sin\delta$.

Ex. 3. Show that if the geocentric latitude of the observer be 39° 45′ 55″, and if the moon's declination be +26° 23′ 3″·6, its hour angle 32° 39′ 49″·5, and its horizontal parallax 57′ 7″·5 then the parallax in R.A. will be 26′ 46″·5, whereof the 1st term of (1) contributes 1587″·2, the 2nd 19″·1 and the 3rd 0″·2.

*Ex. 4. Show that 16′ 15″·8 is the moon's parallax in declination when observed at the Western Reserve College, Ohio, in geographical latitude 41° 14′ 42″, being given that the moon's declination is + 26° 24′ 31″·5, its hour angle 23° 13′ 12″·0, its horizontal parallax 57′ 7″·7, and its parallax in R.A. 19′ 12″·6. [From Loomis' *Practical Astronomy*, p. 196.]

95. Investigation of the distance of the moon from the earth.

The general expression for the effect of geocentric parallax on the declination of the moon § 94 (2) becomes much simplified in the particular case when the moon is on the meridian. The true and the apparent R.A. of the moon are then coincident, being both equal to the sidereal time. We thus have $\alpha' = \alpha = \vartheta$, and consequently from (9), (10), § 93 we see that $\beta = 1$ and $\gamma = \phi'$. With this substitution we have

$$\delta' - \delta = \frac{\rho \sin(\delta - \phi')}{r \sin 1''} + \frac{\rho^2 \sin 2(\delta - \phi')}{r^2 \sin 2''} + \frac{\rho^3 \sin 3(\delta - \phi')}{r^3 \sin 3''} \quad \ldots(1),$$

and we have now to show how by suitable observations made at two observatories this equation will provide us with a determination of r.

As the moon is passing the meridian of a certain observatory A, it is observed with the transit circle, and as will be explained in a later chapter, its apparent declination δ' is thereby ascertained. When this value is substituted in (1) we obtain a formula which, as ϕ' and ρ are known, may be regarded as an equation between two unknowns δ and r.

Let the observation be also made at some other observatory A_1, and let the quantities δ_1', δ_1, ρ_1, ϕ_1', r have the same significations with regard to A_1 as δ', δ, ρ, ϕ', r have with regard to A. It is desirable that A and A_1 be nearly on the same meridian, so that the interval between the two observations shall be as small as possible, for as the moon is in motion its true declination is generally changing and thus δ_1 and δ are not the same at the two stations.

Even if the meridians of the two observatories did not differ by more than an hour δ_1 and δ might differ by as much as 17′, which would be nearly a third of the whole parallax. In like manner of course r_1 and r will in general differ. The rate per hour at which the moon is changing its declination at each particular date is however known, and the interval between the two transits is known, so that if we make $\delta_1 = \delta + \Delta\delta$ we may consider that $\Delta\delta$ is known. As to ρ_1 and ϕ_1 they are known from the locality of the second observatory as ρ and ϕ are known from that of the first, and if a be the earth's equatorial radius we may make $\rho = a(1-n)$ and $\rho_1 = a(1-n_1)$, where n and n_1 are small known quantities. Finally, we may make $r_1 = r(1+k)$ where k is a small quantity depending on the rate of change of the moon's distance at the particular moment. This, like $\Delta\delta$, may be regarded as a known quantity in the present investigation. With these substitutions the two equations for finding δ and a/r become

$$\delta' - \delta = \frac{a(1-n)\sin(\delta - \phi')}{r \sin 1''} + \frac{a^2(1-n)^2 \sin 2(\delta - \phi')}{r^2 \sin 2''} \quad \ldots(2),$$

$$\delta_1' - \delta - \Delta\delta = \frac{a(1-n_1)\sin(\delta + \Delta\delta - \phi_1')}{r(1+k)\sin 1''} + \frac{a^2(1-n_1)^2 \sin 2(\delta + \Delta\delta - \phi_1')}{r^2(1+k)^2 \sin 2''} \quad \ldots(3),$$

in which we have only written two terms in the right-hand side of each equation, but the third may be added if extreme accuracy be desired.

To solve these equations we first reject the terms containing a^2/r^2 and introduce for δ the value $\frac{1}{2}(\delta' + \delta_1') = \delta_0$ into the terms containing a/r, thus obtaining two simple equations in δ and a/r

$$\delta' - \delta = \frac{a(1-n)\sin(\delta_0 - \phi')}{r \sin 1''} \quad \ldots\ldots\ldots\ldots\ldots\ldots(4),$$

$$\delta_1' - \delta - \Delta\delta = \frac{a(1-n_1)\sin(\delta_0 + \Delta\delta - \phi_1')}{r(1+k)\sin 1''} \quad \ldots\ldots\ldots(5),$$

from which the first approximate values of δ and a/r are determined. Substituting this value of δ in both terms on the right-hand side of (2) and (3) and this value of a/r in the final terms of (2) and (3) we again obtain two simple equations by solving which δ and a/r are given with all desired accuracy. Thus the moon's distance is determined.

It is important to consider how the stations should be chosen so that r may be found with the highest accuracy. To study the conditions conducive to this object we may suppose the earth to be spherical and the two observatories on the same meridian, in which case (4) and (5) become

$$\delta' - \delta = \frac{a \sin(\delta_0 - \phi')}{r \sin 1''} \dots\dots\dots\dots\dots\dots\dots (6),$$

$$\delta_1' - \delta = \frac{a \sin(\delta_0 - \phi_1')}{r \sin 1''} \quad \dots\dots\dots\dots\dots\dots (7).$$

Subtracting we have

$$a/r = \tfrac{1}{2}(\delta' - \delta_1') \sin 1'' \operatorname{cosec} \tfrac{1}{2}(\phi_1' - \phi') \sec\{\delta_0 - \tfrac{1}{2}(\phi_1' + \phi')\} \dots (8).$$

Suppose that owing to errors in making the observations an error of E seconds had crept into the value of $\frac{1}{2}(\delta' - \delta_1')$, then the error thus arising in a/r is

$$\tfrac{1}{2} E \sin 1'' \operatorname{cosec} \tfrac{1}{2}(\phi_1' - \phi') \sec\{\delta_0 - \tfrac{1}{2}(\phi_1' + \phi')\}.$$

The observations should be so arranged that errors like E which are to some extent inevitable shall vitiate the concluded value of a/r as little as possible. The smallest possible error in a/r would be $\frac{1}{2} E \sin 1''$, and this would require that $\phi_1' = 90°$, $\phi' = -90°$, $\delta_0 = 0$, in other words for this extreme case the observatory A should be at the south, and A' at the north, terrestrial pole, and the moon should be in the equator. These conditions are of course impossible, but we learn that one of the two observatories should be in the highest possible northern latitude and the other in the lowest possible southern latitude and that the declination of the moon should be as nearly $\frac{1}{2}(\phi_1' + \phi')$ as possible.

Ex. 1. If s be the semi-vertical angle of the tangential cone to the moon from the earth's centre when the moon's horizontal parallax is p and if s', p' be another similar pair, show that the earth being supposed spherical

$$\sin s : \sin s' :: \sin p : \sin p'.$$

Ex. 2. At noon on Jan. 7th, 1904 it was found that $s' = 16' 20''$ and $p' = 59' 51''$; determine the apparent semidiameter of the moon when the horizontal parallax is 3422″.

Ex. 3. Assuming that the earth's equatorial radius is 3963 miles and that the moon's equatorial horizontal parallax is 57′, show that the distance of the moon from the earth's centre is 239,000 miles.

96. Parallax of the moon in azimuth.

If the earth were a perfect sphere the effect of parallax would be solely manifested in depressing the moon in a vertical circle, so that it would have no effect on the azimuth. But when we take into account the spheroidal shape of the earth the circumstances are somewhat different. The parallactic effect depresses the moon from the point on the celestial sphere indicated by the direction of the earth's radius to the place of observation, and owing to the ellipticity of the earth this point is not generally coincident with the zenith. Hence there is generally a parallactic effect on the azimuth of the moon, though that effect is no doubt a small one. An approximate calculation which is sufficiently accurate for most purposes may be made as follows. Let Z (Fig. 76) be the true zenith and Z' be the point of the celestial sphere to which the radius from the earth's centre to the observer is directed, then $ZZ' = \phi - \phi'$, the difference between the astronomical latitude and the geocentric latitude. Parallax depresses

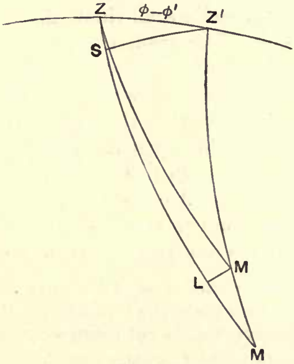


FIG. 76.

the moon from M to M', and if ML and $Z'S$ be perpendicular to ZM' the effect on azimuth is

$$\begin{aligned}\angle MZL &= \sin ML \operatorname{cosec} ZM = \sin MM' \sin LM'M \operatorname{cosec} ZM\\ &= \sin \pi_\phi' \sin Z'M' \sin LM'M \operatorname{cosec} ZM\\ &= \sin \pi_\phi' (\phi - \phi') \sin Z'ZS \operatorname{cosec} ZM\\ &= -\sin \pi_\phi' (\phi - \phi') \sin a \operatorname{cosec} z,\end{aligned}$$

where a and z are respectively the azimuth and the zenith distance of the moon and $\angle Z'ZS = a - 180°$.

We can also investigate this problem as follows. Draw through the position of the observer O' (Fig. 74) three rectangular axes of which the positive directions are towards the north and east points of the horizon and the zenith. Let a', z' be the apparent azimuth and zenith distance of the moon to the observer and a, z the corresponding quantities for parallel axes through the centre of the earth, then the direction cosines with reference to these axes will be

of OS	$\sin z \cos a$,	$\sin z \sin a$,	$\cos z$,
of $O'S$	$\sin z' \cos a'$,	$\sin z' \sin a'$,	$\cos z'$,
and of OO'	$-\sin(\phi - \phi')$,	0,	$\cos(\phi - \phi')$.

By equating the projection of $O'S$ to the difference of the projections of OS and OO' on each of the three axes we obtain as in equations (2), (3), (4) § 93,

$$r' \sin z' \cos a' = r \sin z \cos a + \rho \sin(\phi - \phi') \quad \text{.........(i)},$$

$$r' \sin z' \sin a' = r \sin z \sin a \quad \text{.........................(ii)},$$

$$r' \cos z' = r \cos z - \rho \cos(\phi - \phi') \quad \text{...........(iii)}.$$

From which we easily obtain as in (7) § 93,

$$\tan(a' - a) = -\sin \pi_\phi' \sin(\phi - \phi') \sin a / \{\sin z + \sin \pi_\phi' \cos a \sin(\phi - \phi')\},$$

which gives the parallax in azimuth.

Multiplying (i) by $\cos \frac{1}{2}(a + a')$ and (ii) by $\sin \frac{1}{2}(a + a')$ and adding, we obtain after dividing by $\cos \frac{1}{2}(a' - a)$

$$r' \sin z' = r \sin z + \rho \sin(\phi - \phi') \cos \tfrac{1}{2}(a' + a) \sec \tfrac{1}{2}(a' - a).$$

Following the procedure of § 93, we now introduce two auxiliary quantities β', γ' defined by the equations

$$\beta' \cos \gamma' = \cos(\phi - \phi');$$

$$\beta' \sin \gamma' = -\sin(\phi - \phi') \cos \tfrac{1}{2}(a' + a) \sec \tfrac{1}{2}(a' - a),$$

and obtain

$$r' \sin z' = r \sin z - \rho\beta' \sin \gamma', \quad r' \cos z' = r \cos z - \rho\beta' \cos \gamma',$$

whence, as in (13), § 93,

$$\tan(z' - z) = \beta' \sin \pi_\phi' \sin(z - \gamma') / \{1 - \beta' \sin \pi_\phi' \cos(z - \gamma')\},$$

which gives the parallax in zenith distance.

The results at which we have arrived may of course be expanded in series and we obtain

$$a' - a = -\sin\pi_\phi' \sin(\phi - \phi') \operatorname{cosec} z \sin a$$
$$+ \tfrac{1}{2}\sin^2\pi_\phi' \sin^2(\phi - \phi') \operatorname{cosec}^2 z \sin 2a \ldots,$$
$$z' - z = \beta' \sin\pi_\phi' \sin(z - \gamma') + \tfrac{1}{2}\beta'^2 \sin^2\pi_\phi' \sin 2(z - \gamma') \ldots .$$

Ex. Prove that parallax diminishes the moon's azimuth by

$$\tfrac{1}{2}e^2 \sin 2\phi \sin\pi_\phi' \sin a \operatorname{cosec} z,$$

where e is the eccentricity of the earth regarded as a spheroid, ϕ the latitude, π_ϕ' the moon's horizontal parallax at the place, z the zenith distance, a the azimuth of the moon.

97. Numerical value of the lunar parallax.

The movement of the moon is principally determined by the attraction of the earth. But the disturbing attractions of the sun, and to some extent of the planets, cause the actual motion of the moon to be very much more complex than is the mere elliptic motion already considered in Chap. VII. The theoretical expression for the parallax of the moon has however been calculated by mathematicians from the dynamical theory of the moon's motion while duly taking the disturbances alluded to into account. We cannot here discuss the researches by which the result has been arrived at. It will however be useful to know the theoretical value which has been found for this important quantity so we shall give the essential parts of the expression determined by Adams†. He finds for the number of seconds of arc in the moon's equatorial horizontal parallax

$$\pi_0' = 3422'' + 187'' \cos x + 10'' \cos 2x$$
$$+ 28'' \cos 2t + 34'' \cos(2t - x) + 3'' \cos(2t + x) \ldots (1).$$

In this expression t and x are functions of the time and therefore constitute the variable elements in the expression. It should be added that in the expression as given by Adams there are a very large number of terms besides the six here written. As however these terms have but little effect on the total result, we need not now consider them. Each coefficient in the neglected terms is under two seconds and even in the terms retained we have discarded fractions of a second in the coefficients.

† *Collected Scientific Papers*, vol. I. p. 109.

The first term in (1) is the only term which does not contain a sine or a cosine of the function of the time. We therefore regard 3422″ as the mean value of the moon's equatorial horizontal parallax π_a', because in forming the mean value of the other terms over a sufficiently extended period, the trigonometrical functions appear now with one sign and now with another, so that their effect tends to vanish from the mean.

It is plain that for real values of x and t the expression for the parallax can never become greater than 3684 (which is simply 3422 increased by the sum of the coefficients of the other terms) nor less than 3160.

As so many small terms in π_0' have been omitted, we cannot conclude that its limits are exactly those just written, but we may always assume $61'\cdot 5 > \pi_0' > 53'\cdot 9$.

Ex. 1. Show that the distance of the moon's centre from the earth's centre will always lie between 222,000 miles and 253,000 miles.

*Ex. 2. From the Nautical Almanac for 1896, we extract the following values of the moon's equatorial horizontal parallax,

1896		
Aug. 8th, noon	... ...	59′ 2″·6
„ „ 12^{hrs}	... ...	59 21 ·4
Aug. 9th, noon	... ...	59 37 ·6

show that at t hours after noon on Aug. 8th we have for π_0', the equatorial horizontal parallax, the expression

$$\pi_0' = 59'\ 2''\cdot 6 + 1''\cdot 68t - \cdot 009t^2.$$

EXERCISES ON CHAPTER XII.

Ex. 1. The observed zenith distance of the moon's limb when corrected for refraction is ζ, the equatorial horizontal parallax is π_0' and the geocentric semidiameter D. Prove that, assuming the earth and moon spherical, the geocentric zenith distance of the moon's centre z is given by

$$\sin(\zeta - z) = \sin\pi_0' \sin\zeta \mp \sin D.$$

Ex. 2. If δ and δ' are the true and apparent distances between a planet and the moon, a and a' the true and apparent altitudes (corrected for refraction) of the planet, β and β' of the moon, π_0', σ_0, the equatorial horizontal parallaxes of the moon and planet for the place of observation, then

$$\cos\delta = \frac{\cos a \cos\beta}{\cos a' \cos\beta'} \cos\delta' + \sin a \sin\pi_0' + \sin\beta \sin\sigma_0,$$

very nearly. [Coll. Exam.]

Ex. 3. If m and s are the observed altitudes of the moon and a star, μ and σ the corrections to m and s for parallax and refraction, and Δ the correction to be applied to the observed distance d to get the true distance, prove that

$$\Delta \sin d \cos m \cos s = \mu \cos s (\sin s - \sin m \cos d) - \sigma \cos m (\sin m - \sin s \cos d).$$

[Coll. Exam.]

Ex. 4. Taking the correction for refraction in the form $k \tan z$, show that, when the zenith distance of the moon is $\cos^{-1}(k/\pi_0')$, the horizontal diameter is unaltered, and that, when the zenith distance is $\cos^{-1}(k/\pi_0')^{\frac{1}{3}}$ the vertical diameter is unaltered, by the combined effects of refraction and diurnal parallax; π_0' being the horizontal parallax of the moon.

[Math. Trip.]

Ex. 5. Prove that, if R be the angular geocentric radius of the moon, r_0 its apparent radius when on the meridian of a place in latitude ϕ, r its apparent radius when the geocentric hour angle of its centre is h, then

$$\sin^2 R (\operatorname{cosec}^2 r - \operatorname{cosec}^2 r_0) = 4 \sin\pi_0' \cos\phi \cos\delta \sin^2 \tfrac{1}{2}h,$$

where π_0' is the horizontal parallax of the moon, δ its geocentric declination, and the earth is regarded as spherical.

[Coll. Exam.]

CHAPTER XIII.

THE GEOCENTRIC PARALLAX OF THE SUN.

98. Introductory.

The determination of the distance of the sun from the earth is of unique importance in astronomy. When it has been found then the dimensions of the sun are easily ascertained; so are also the distances of the planets and of their satellites; and the sizes and even the masses of these bodies are also deduced. But the determination of the sun's distance is not important merely because it gives us the measurements in the solar system. We shall find that the distances of the stars can be determined only with reference to the sun's distance, so that the sun's distance forms as it were the base line by which sidereal measurements are conducted. It is not indeed too much to say that almost all lineal measurements relating to celestial bodies, those with respect to the moon being excepted, are based on our knowledge of the distance of the sun. This problem, at once so fundamental and at the same time so difficult of accurate solution, must now engage our attention.

We must first clear the problem before us from a certain ambiguity. We are to seek the distance of the sun from the earth. That distance is, however, constantly changing between certain limits, so we have to consider what is meant by the *mean* distance of the sun, for this is indeed the element which we have to determine by observation. As explained in § 50 the earth moves round the sun in an ellipse and the sun occupies one of the foci. Thus the distance of the sun fluctuates in correspondence with the changes in the radius vector from the focus of an ellipse to a point on its circumference.

The semi-axis major of the elliptic orbit of the earth is represented by a, $\odot$ is the sun's true longitude, ϖ the longitude of the sun at the time when nearest the earth or as it is often called the longitude of the Perigree, then from the well-known polar equation of the ellipse

$$r = \frac{a(1-e^2)}{1+e\cos(\odot - \varpi)} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1).$$

As e is so small (0·0168) we may for our present purpose treat its square and higher powers as negligible, so that the formula may be written

$$r = a\{1 - e\cos(\odot - \varpi)\} \ldots\ldots\ldots\ldots\ldots\ldots (2),$$

which may be still further simplified, because as the true longitude $\odot$ differs from the mean longitude L only by terms involving the eccentricity (§ 73), we may replace $\odot$ by L, because we are omitting terms introducing e^2, and may use the equation

$$r = a\{1 - e\cos(L - \varpi)\} \ldots\ldots\ldots\ldots\ldots\ldots (3).$$

By the *mean distance* of the sun we are to understand the average value of r during the complete revolution. This average value is to be computed as follows: Let $t_0, t_1, t_2, t_3, \ldots t_n$ be a series of epochs so chosen that $t_1 - t_0 = t_2 - t_1 = t_3 - t_2 = \ldots = t_n - t_{n-1}$. We are also to suppose that $t_n - t_0$ is the whole period of revolution. Then the *mean distance* is $(r_1 + r_2 \ldots + r_n)/n$, where $r_1, r_2, r_3 \ldots$ are the distances at the respective epochs $t_1, t_2, t_3, \ldots$, and where n is indefinitely great. We can compute this quantity directly from (2), because L increases proportionally to the time, so that the mean distance is

$$\int_0^{2\pi} r\,dL \div \int_0^{2\pi} dL,$$

which by substitution for r reduces to a. Thus we learn that the semi-axis major of the ellipse is also the mean distance of the planet from the sun. We had already assumed this to be the case in the statement of Kepler's third law § 50.

We shall assume that the eccentricity of the sun's apparent orbit is 0·0168, that the mean value of the sun's horizontal parallax is 8″·80, that its mean semidiameter is 961″, and that 281°·2 is the longitude of the perigee of the sun's apparent orbit (§ 73), and hence we have the following approximate results when the sun's mean distance is taken as unity:

The sun's	distance	is	$\{1 - 0{\cdot}0168 \cos (L + 78^\circ{\cdot}8)\}$,
„ „	hor. par.	„	$8''{\cdot}80 \{1 + 0{\cdot}0168 \cos (L + 78^\circ{\cdot}8)\}$,
„ „	semidiam.	„	$961'' \{1 + 0{\cdot}0168 \cos (L + 78^\circ{\cdot}8)\}$,
„ „	longitude	„	$L + 1^\circ{\cdot}92 \sin (L + 78^\circ{\cdot}8)$,
„ „	R.A.	„	$L + 1^\circ{\cdot}92 \sin (L + 78^\circ{\cdot}8) - 2^\circ{\cdot}47 \sin 2L$.

Solar Parallax in Right Ascension and Declination. Assuming the earth to be a sphere and that α, δ are the true geocentric R.A. and decl. of the sun and α', δ' the coordinates as affected by parallax when the observer's latitude is ϕ and the sun's hour angle is h, then from (4) (5) § 94 we obtain

$$\alpha' - \alpha = -8''{\cdot}80 \cos \phi \sec \delta \cos h,$$

$$\delta' - \delta = -8''{\cdot}80 (\sin \phi \cos \delta - \cos \phi \sin \delta \cos h).$$

The total parallactic displacement is of course $8''{\cdot}80 \sin z$,

where $$\cos z = \sin \phi \sin \delta + \cos \phi \cos \delta \cos h.$$

Ex. 1. Show that the parallax in right ascension of a body with declination δ and hour angle h is the same as for a body of declination $-\delta$ and hour angle h if their horizontal parallaxes be the same.

Ex. 2. Assuming the distance of a body from the earth to be so great that the sine and circular measure of the parallax may be considered equal, show that the locus of all bodies which, at a given instant and place, have their parallaxes in right ascension equal will be a right circular cylinder touching the plane of the meridian along the axis of the earth.

[Godfray's *Astronomy*.]

Ex. 3. Show that when the sun's mean longitude is 51° its semidiameter is 15′ 51″, its horizontal parallax 8″·71 and 1·01 is the ratio of its distance to the mean distance.

Ex. 4. Being given that the sun is at its least distance from the earth on Jan. 1st and that its mean longitude has a daily increase of 0°·9856, show that the sun's distance from the earth has its most rapid rate of decrease on Oct. 2nd.

Ex. 5. The semidiameter of the sun at the earth's mean distance being $16' \, 1''\cdot18$ and the equatorial horizontal parallax of the sun at the earth's mean distance being $8''\cdot80$, find the diameter of the sun in terms of the earth's equatorial radius.

Ex. 6. Given that a kilometre is the arc of a meridian in latitude 45° which subtends an angle of one centesimal minute at the centre of the earth, that the ellipticity of the surface of the earth is $\frac{1}{295}$, and that the sun's mean equatorial horizontal parallax is $8''\cdot76$, prove that the mean distance of the sun is $1\cdot5 \times 10^8$ kilometres.

[Math. Trip.]

Ex. 7. Show that, on the assumption that the sun's horizontal parallax is $8''\cdot80$, the time during which the sun is below the horizon at either pole is longer on account of parallax by $7 \operatorname{cosec} \omega$ minutes, where ω is the obliquity of the ecliptic.

[Math. Tripos, 1903.]

Ex. 8. Prove that the difference due to parallax in the apparent position of the sun as determined by simultaneous observations at two stations is a maximum and equal to $2\pi_0 \sin a$ when the zenith distances are the same, where $2a$ is the angle subtended at the earth's centre by the arc joining the two stations. [Math. Tripos, 1902.]

Ex. 9. Assuming that the sun's semidiameter at mean distance is 961″, show that the number of sidereal seconds which the semidiameter takes to cross the meridian is t, where

$$\frac{961}{r \cos \delta} = 15 \left(1 - \frac{\cos \omega}{r^2 (\tau + 1) \cos^2 \delta}\right) t,$$

δ being the declination, τ the length of a solar year in days, and r the sun's distance, the mean distance being unity.

[Coll. Exam.]

99. The sun's horizontal parallax.

It might at first be supposed that the methods of the last chapter, which are successful in finding the parallax of the moon would also be successful in finding the parallax of the sun, but the cases are not parallel. It is an essential point of difference between them that the brightness of the sun is incomparably greater than the brightness of the moon. Stars can easily be observed while apparently quite close to the moon, yet the light from any star is invisible, even with the most powerful telescope, when that star is close to the sun. Thus the measurements of the differences of declination between the moon and adjoining stars by which the moon's parallax is accurately determined have no counterpart in possible solar observations for finding the sun's parallax in the same manner. As already explained in the last chapter the *horizontal parallax* of the sun is the angle of which the sine is the ratio of the earth's equatorial radius to the distance of the sun.

For this reason we are unable to effect the determination of the solar parallax satisfactorily by observations of the sun in the way in which the parallax of the moon is obtained and therefore we have to resort to other methods. Of such methods there are several, which may be classified as follows:

I. *Direct observational methods.*

a. Parallax of Venus in transit across the sun.

b. Parallax of an exterior planet by the diurnal method.

c. Parallax of an exterior planet by simultaneous observations at distant stations.

II. *Gravitational methods.*

d. Parallax of the sun from the mass of the earth.

e. Parallax of the sun from the parallactic inequality of the moon.

III. *Physical methods.*

f. Parallax of the sun from the constant of aberration and the velocity of light.

g. Parallax of Jupiter from the light equation of his satellites.

The direct observational methods are founded on Kepler's third law, which states that the squares of the periodic times of the planets are proportional to the cubes of their mean distances. The periodic times of the planets are known with the highest accuracy, because an error of any importance in the assumed value of a planet's periodic time would in the course of oft repeated revolutions attain a cumulative magnitude that would inevitably lead to its detection. Since then the periodic times in the solar system are known, the values of the major axes of all their orbits can be computed if that of the earth's orbit be taken as unity. Repeated observations of the right ascensions and declinations of the planets will give further particulars of their orbits. We deduce from such observations the longitudes of their ascending nodes and the inclinations of the planes of their orbits, as well as the eccentricities and the longitudes of the perihelia. Thus taking the sun's mean distance as unity, the other measurements of the planetary system are known. That is we know the form of the system and all that is lacking is what we may term its scale. If therefore we can measure in terms of the earth's radius the mean distance of any one planet we obtain the scale of the whole system. Thus by measuring the parallax of Venus as the transit of Venus enables us to do, we learn the distance of Venus and thence the distance of the sun and the other dimensions in the solar system. This method will be considered in the next chapter. It is of much historical interest, though it may now be considered to be superseded by methods *b* and *c*.

The parallax of a planet exterior to the earth as supposed in methods *b* and *c* can be obtained by observing its displacement among the stars from different observing stations. In the case of *c* the two stations are in different geographical localities as in the method employed in observing the moon (§ 95). In the process marked *b* there is but one geographical station, and the base line is obtained by the diurnal movement which between evening and morning carries the observer some thousands of miles. This method has the great practical advantage that observations can be continued for several months about the time of the planet's opposition when it lies as nearly as possible between the earth and the sun. There can be no doubt that the observation of a minor planet is much the best method of determining the distance of the sun by direct measurement, for as the minor planet is a star-like point its apparent distances from the neighbouring stars admit of being measured with a high degree of accuracy.

The gravitational methods II. and the physical methods III. have many points of great scientific interest. It must, however, always be borne in mind that the problem before us is purely one of *geometrical* measurement and that for this purpose methods involving only geometrical considerations, such as the methods in group I., must be deemed more reliable than the methods of groups II. and III., which depend to some extent on physical assumptions that cannot pretend to geometrical rigour.

It will illustrate the history of the problem of the sun's distance to examine briefly the successive values of the mean equatorial horizontal solar parallax used in the nautical almanac during the 19th century.

From 1801 to 1833 the adopted value was 9″. The origin of this is not clear. It was perhaps chosen merely as a round number, based upon observations of the parallax of Mars on the meridian, and the preliminary results from the transits of Venus of 1761 and 1769.

From 1834 to 1869 the value in use was 8″·5776, derived by Encke from his discussion of the transits of Venus.

From 1870 to 1881 the parallax 8″·95 was used, as found by Leverrier in 1858 from the parallactic inequality of the moon.

From 1882 to 1900 the value was 8″·848, as found by Newcomb in 1867 from a discussion of a number of determinations by different methods (*Washington Observations*, 1865, Appendix II.).

The conference of directors of nautical almanacs which met at Paris in 1896 decided to adopt from 1901 the value 8″·80, derived from Sir David Gill's heliometer observations of minor planets and supported by the results of other methods.

***100. Parallax of an exterior planet by the diurnal method.**

We shall describe briefly the investigation carried out by Sir David Gill at the Island of Ascension of the parallax of the planet Mars during the opposition of that planet in 1877. The opportunity thus taken advantage of was particularly favourable for the determination of the parallax by the diurnal method from a station near the equator, since the parallax of the planet had attained nearly its maximum value.

The programme of work was to measure each evening and morning the distance of Mars from selected comparison stars, whose places were well determined by cooperation of a number of observatories in meridian observations.

Since the effect of parallax is always to displace the planet downwards from the zenith, the displacement with reference to the stars will be in opposite directions evening and morning. Thus the change in the position of the observer by the rotation of the earth in the interval between the time at which the observations are made in the evening and in which they are repeated in the following morning gives the base line required for parallax determination.

For the investigation of the parallax of Mars it would not be always practicable to find suitable stars close enough to the planet to admit of the necessary measurements being made unless an instrument possessing the exceptional range of the *heliometer* were employed.

The principle of this instrument, so important in modern astronomy, may be indicated here. The heliometer is an equatorially mounted telescope constructed to measure directly the distances of neighbouring points on the celestial sphere. The essential feature of the heliometer lies in its bisected object glass. The object glass is cut in two along a diameter, and the two halves are mounted on slides which can be separated by sliding them equal distances in opposite directions along the line of section and perpendicular to the axis of the telescope. The separation of the segments is measured by two scales, placed almost in contact along the inner edges of the two slides.

The principle of the procedure depends on the optical fact that when the image of a star A made by one part of the object glass is coincident with the image of a star B made by the other part, then the angle between the two stars equals the angle subtended at the focus by the distance through which one-half of the object glass has been moved relative to the other. Thus the scale measurement of this distance provides the means of determining the arc between the two stars. In this way angular distances up to 7000″ may be accurately measured by the heliometer, while the ordinary micrometers for measuring the arcs between adjacent stars are hardly available for a twentieth part of such a distance.

The apparent distance between the planet and the star will be in a state of continuous change for various reasons. As to the change caused by refraction, we have already considered it in § 48. We may however remark that for the present purpose where the distances are much greater than those already presumed the more extensive formulae of Seeliger (*Theorie des Heliometers*, p. 96) will be often required in the practical conduct of the work, though it will not be necessary for us to discuss them in this place. There will remain two other causes of change in the apparent distance which must now be attended to. The actual motion of the planet in the interval will of course be the cause of some alteration, and the parallactic displacement experienced by the planet but not by the star and for which we are now searching will have an effect on the distance which we must calculate.

Let α, δ be the geocentric R.A. and decl. of a planet when ♈ is on the meridian, and let $\dot{\alpha}$, $\dot{\delta}$ be the rates of change of these two quantities by the motion of the planet per sidereal day. Then the geocentric R.A. and decl. of the planet at the sidereal time ϑ will be $\alpha+\dot{\alpha}\vartheta$, $\delta+\dot{\delta}\vartheta$. The hour angle of the planet is $\vartheta-\alpha$, and we have already seen (p. 287) that the corrections to be applied to the geocentric coordinates of the planet to obtain the apparent coordinates are respectively $-\sigma_0\cos\phi\sec\delta\sin(\vartheta-\alpha)$ in R.A. and $-\sigma_0\sin\phi\cos\delta+\sigma_0\cos\phi\sin\delta\cos(\vartheta-\alpha)$ in decl. where σ_0 is the horizontal parallax of Mars.

To obtain the apparent coordinates of the planet at the time ϑ we unite the two different corrections, and thus obtain for the apparent right ascension and declination

$$\alpha+\dot{\alpha}\vartheta-\sigma_0\cos\phi\sec\delta\sin(\vartheta-\alpha),$$

and
$$\delta+\dot{\delta}\vartheta-\sigma_0\sin\phi\cos\delta+\sigma_0\cos\phi\sin\delta\cos(\vartheta-\alpha).$$

At the sidereal time ϑ' these coordinates will become

$$\alpha+\dot{\alpha}\vartheta'-\sigma_0\cos\phi\sec\delta\sin(\vartheta'-\alpha),$$
$$\delta+\dot{\delta}\vartheta'-\sigma_0\sin\phi\cos\delta+\sigma_0\cos\phi\sin\delta\cos(\vartheta'-\alpha),$$

and hence in the time interval $\vartheta'-\vartheta$ the apparent coordinates will have undergone changes $\Delta\alpha$ and $\Delta\delta$ where

$$\Delta\alpha=\dot{\alpha}(\vartheta'-\vartheta)-2\sigma_0\cos\phi\sec\delta\sin\tfrac{1}{2}(\vartheta'-\vartheta)\cos\tfrac{1}{2}(\vartheta'+\vartheta-2\alpha),$$
$$\Delta\delta=\dot{\delta}(\vartheta'-\vartheta)-2\sigma_0\cos\phi\sin\delta\sin\tfrac{1}{2}(\vartheta'-\vartheta)\sin\tfrac{1}{2}(\vartheta'+\vartheta-2\alpha).$$

Let θ be the angle between the geocentric position of the point with coordinates α, δ, *i.e.* the centre of the disc of the planet Mars and the star α_0, δ_0. Then

$$\cos\theta=\sin\delta_0\sin\delta+\cos\delta_0\cos\delta\cos(\alpha_0-\alpha)\ \dots\dots (1).$$

The small change $\Delta\theta$ in the value of θ arising from changes $\Delta\alpha$, $\Delta\delta$ in the values of α and δ is determined by differentiation

$$-\sin\theta\Delta\theta=\{\sin\delta_0\cos\delta-\cos\delta_0\sin\delta\cos(\alpha_0-\alpha)\}\Delta\delta$$
$$+\cos\delta_0\cos\delta\sin(\alpha_0-\alpha)\Delta\alpha\dots(2).$$

Substituting in this the values for $\Delta\alpha$ and $\Delta\delta$ we obtain an equation involving $\Delta\theta$, θ, δ_0, δ, α_0, α, $\dot{\alpha}$, $\dot{\delta}$, ϑ', ϑ, ϕ and σ_0. Of these quantities α_0, δ_0, α, δ being the coordinates of the star and of the planet at a certain time are known, $\dot{\alpha}$, $\dot{\delta}$ are known because the movements of the planet with regard to the surrounding stars is carefully determined by repeated independent observations made with this particular object. θ is known because it can be calculated from α, δ, α_0, δ_0 by (1). The quantities ϑ', ϑ are the times of observation and therefore known, and ϕ is the latitude of the observer. Thus equation (2) reduces to a relation between $\Delta\theta$ and σ_0. The heliometer, as we have already stated, provides the means of measuring the distance between the planet and the star. This is repeated when the bodies have reached the suitable position some hours later. The difference between the two distances is $\Delta\theta$ and thus σ_0 becomes known, for we have just shown how it can be expressed in terms of $\Delta\theta$.

In the practical application of this process there are many technical matters to be attended to and for their discussion we must refer to Sir David Gill's work. To obtain increased accuracy many observations obtained during the whole period in which the planet is in or near opposition and thus at its smallest distance from the earth have to be combined.

Such is in outline the principle of the determination of the solar parallax by the diurnal method, for when σ_0 the horizontal parallax of Mars has been ascertained, we can find the parallax of the sun in the manner explained in § 99.

The result of the observations at Ascension Island was to assign a horizontal parallax of $8''\cdot778$ to the sun.

When a numerical value has been deduced as the outcome of an investigation it is customary and extremely useful to supplement the numerical value by expressing also what is known as its *probable error*. Thus in the present case the probable error is stated to be $\pm 0''\cdot012$. The meaning of this is as follows. The exact parallax of the sun is unknown, but what is known, so far as this research is concerned, is that $8''\cdot778$ must be very nearly the exact parallax. It is practically certain that this result cannot be two seconds wrong or one second wrong, and it is highly improbable that it could be half a second wrong; on the other hand it might perhaps be $0''\cdot01$ wrong, and it is probable that it is $0''\cdot002$ wrong and highly probable that it is at least $0''\cdot001$ wrong. There must be some fraction of a second between, shall we say, $0''\cdot001$ and $0''\cdot5$ which would possess the character that the error of the determination was as likely to be below this fraction as above it. In the present case the discussion of the observations showed it to be as probable that the parallax lay between the limits $8''\cdot778 - 0''\cdot012$ and $8''\cdot778 + 0''\cdot012$ as that it did not. In this case $0''\cdot012$ is the probable error, and the smaller the probable error of a determination the narrower are the limits within which the quantity probably lies and the better the quality of the investigation by which it has been found. The statement of the probable error of any determination is the numerical method of indicating the degree of confidence with which the result should be accepted.

There is one cause which may possibly introduce appreciable error into the determination of the solar parallax from observations of Mars. The effect of the dispersion of light in the atmosphere at considerable zenith distances is to give a coloured fringe to the disc of the planet, blue above and red below, which in the case of a reddish planet observed in a blue twilight sky would make the planet appear systematically too low, and apparently increase the parallactic displacement. It is therefore preferable to employ minor planets, whose discs are indistinguishable from stars. This was done in 1888 and 1889 by Sir David Gill at the Cape, working in conjunction with four observers in the northern hemisphere, during oppositions of the minor planets Victoria, Iris, and Sappho. The results are discussed in *Annals* of the Cape Observatory, Vols. VI. and VII. On this occasion it was not found possible to occupy a station near the equator, and the diurnal method was replaced by the method of making more or less simultaneous observations at stations separated by great distances. The principles of the calculation are very similar to those developed above, but the details are more complex and more difficult to illustrate by examples of the actual procedure. We have therefore chosen the earlier work to illustrate the method of determining the solar parallax by the heliometer, though the results of that work have been superseded by the value of the parallax derived from the three minor planets, viz.

$$\pi = 8''\cdot802 \pm 0''\cdot005.$$

This value may perhaps be regarded as the best that can at present be derived from direct observation, but it may be superseded when the photographs and measures of the planet Eros made during the opposition of 1900–1 have been completely discussed. Eros comes nearer to the earth than any other planet, and therefore offers exceptional advantages in this problem.

Ex. Let ϕ_0 be the geocentric latitude of the observatory, ϕ its astronomical latitude, h the hour angle, δ the declination of a planet, and Δ its distance from the earth in terms of the mean radius of the earth's orbit. Let the value of the solar parallax be $8''\cdot80$.

Show that the motion of the planet in R.A. due to change of parallax is at any moment at the rate of

$$+3^s\cdot69 \times 1/\Delta \times \cos\phi \cos\phi_0 \sec\delta \cos h \text{ per day,}$$

and the corresponding rate in declination is

$$-2''\cdot3 \times 1/\Delta \times \cos\phi \cos\phi_0 \sin\delta \sin h \text{ per hour.}$$

[Mr Hinks, *Mon. Not. R.A.S.* Vol. LX. p. 545.]

*101. The solar parallax from the constant of aberration.

When the constant of aberration is known, and the velocity of light in kilometres per second, we can find the mean velocity of the earth, and thence the mean radius of the earth's orbit and the sun's parallax.

It follows from p. 260 and Ex. 3, p. 261 that if

κ is the constant of aberration,

μ the observed velocity of light,

e the eccentricity of the earth's orbit,

n mean motion of earth during sidereal year of 365·256 days in seconds of arc per second of mean time,

ρ the equatorial radius of the earth,

π_a the equatorial horizontal parallax of the sun,

then $\pi_a = \dfrac{\rho n \operatorname{cosec} 1''}{\kappa\mu\sqrt{1-e^2}}$.

Putting $\rho = 6378\cdot249$ kilometres (Clarke),

$$n \doteq \frac{360 \times 60 \times 60}{24 \times 60 \times 60 \times 365\cdot256} = \frac{15}{365\cdot256},$$

$\mu = 299860$ kilometres per second (Newcomb, *Astronomical constants*),

$1 - e^2 = 0\cdot999719$,

we have $\pi_a = 180\ 2/\kappa$,

so that $\pi_a = 8''\cdot803$ if κ be $20''\cdot47$.

The value of this indirect method of finding the solar parallax is impaired by the curious discordance in recent determinations of the constant of aberration κ. All determinations of this constant made before 1892 were necessarily entangled with the then unknown variation of latitude (§ 61). Special precautions have been taken to eliminate this in subsequent work, but the latest results are still somewhat uncertain.

*102. The solar parallax by Jupiter's satellites.

The time at which an eclipse of one of the satellites of Jupiter is observed is later than the time at which it actually takes place owing to the fact that light does not travel instantaneously from the satellite to the earth, and the interval varies with the varying distance of Jupiter from the earth. The law of this variation is known with great accuracy, and there is little uncertainty in the experimental determination of the velocity of light in space. If then it were possible to compute with accuracy the instant at which the eclipse takes place, and to observe with equal accuracy the instant at which it appears to take place as seen from the earth it would be possible to calculate successively the time which the light had taken to travel, the distance of Jupiter from the earth, and the distance of the earth from the sun. Owing to the unsatisfactory state of the theory of the motions of Jupiter's satellites, and the difficulty of observing with sufficient accuracy the instant when the satellite is exactly half immersed in the shadow, the determinations of the distance of the sun hitherto made by this method are of but little weight. Recent measures of the places of the satellites made at the Royal Observatory, Cape of Good Hope, by Sir David Gill, Dr De Sitter, and Mr Bryan Cookson, have however improved the elements of the orbits of the satellites; and Prof. R. A. Sampson has discussed the photometric observations made by Professor E. C. Pickering at Harvard College Observatory which will, it is hoped, reduce the second difficulty.

*103. The solar parallax from the mass of the earth.

There is a well determined relation between the masses of the earth and the sun, the equatorial radius of the earth, the length of the seconds pendulum, and the distance of the sun, which is the basis of the lunar theory. If π_a is the solar parallax, and M the ratio of the mass of the sun to that of the earth,

$$\pi_a^3 M = [8{\cdot}35493]$$

(Newcomb, *Astronomical Constants*, p. 100).

The value of M may be derived from the perturbations in the motions of the other planets, particularly Venus and Mars, produced by the attraction of the earth. As the result of an exhaustive discussion of the whole subject, for which reference may be made to the above work, Professor Newcomb concludes that the value of the solar parallax derived by this method is 8″·76, and that "unknown actions and possible defects of theory aside, this value is less open to doubt from any known cause than any determination that can be made." In view of the divergence of this result from the mean of all other good determinations of the solar parallax, the reservation is important, for there are outstanding discrepancies in the motions of the inner planets which are at present unexplained.

But it may be remarked that in thirty or forty years time this method may perhaps be applicable in a new direction. It has been shown by Mr H. N. Russell of Princeton University, New Jersey, that there is a large periodic inequality in the motion of the planet Eros due to the attraction of the earth, which may in time afford a new and effective method of determining the mass of the earth.

***104. The solar parallax from the parallactic inequality of the moon.**

One of the principal inequalities in the motion of the moon depends upon the fact that the perturbing effect of the sun is greater when the moon is in the half of her orbit nearer to the sun than when she is in the other half. The result of this is an inequality whose coefficient is proportional to the solar parallax. It has been shown by Prof. E. W. Brown that the hitherto accepted theoretical value of this coefficient (Delaunay's) is somewhat erroneous. Prof. Brown finds (*Mon. Not. R.A.S.* Vol. LXIV. p. 535) that if the value of the solar parallax is 8″·790 the expression for the parallactic inequality is

$$-124''{\cdot}92 \sin D,$$

where D is the moon's ecliptic longitude. If the solar parallax has any other value π_a, the coefficient of sin D becomes

$$-124''\cdot 92\pi_a/8''\cdot 790.$$

This is the value derived from the theory of the moon's motion. If we compare it with the value of the coefficient derived from observation of the Moon we have a means of determining the value of π_a. The most recent and accurate determination of the coefficient by observation is that made by Mr Cowell from a discussion of the Greenwich observations of the moon, 1847—1901 (*Mon. Not. R.A.S.* Vol. LXIV. pp. 96, 585), where he gives the value $-124''\cdot 90$. Equating the theoretical expression to the observed, we find $\pi_a = 8''\cdot 789$.

It should, however, be noted that it is scarcely possible to separate completely the observed value of the parallactic inequality from the uncertainty in the semi-diameter of the moon, and this may affect the deduced value of π_a by at least $0''\cdot 01$. For a discussion of this question, reference may be made to a number of papers by Mr Cowell and Prof. Turner in *Mon. Not. R.A.S.* Vols. LXIV. and LXV.

CHAPTER XIV.

ON THE TRANSIT OF A PLANET ACROSS THE SUN.

105. Introductory.

If the orbit of Venus were in the plane of the ecliptic then whenever the geocentric longitude of Venus was the same as that of the sun the planet would appear near the centre of the solar disc. About three hours previously the terrestrial observer would have seen the planet entering on the sun's disc, about three hours later the planet would pass off from the disc and during the six hours of its passage the planet would be said to have been in *transit across the sun's disc.* As the orbit of Venus does not lie in the plane of the ecliptic the phenomena of a transit of Venus are by no means so simple as the hypothetical transit just indicated. The inclination of the orbit of Venus to the ecliptic is $3° 23' 35''$, and it may therefore happen and indeed generally does happen that when Venus and the sun have the same geocentric longitude, the planet passes above the sun or below the sun and so a transit does not take place. It is indeed obvious that a transit cannot occur unless the apparent distance of the planet from the sun's centre is less than the sun's apparent semi-diameter. But owing to the inclination of the orbit of Venus it may happen that even at a conjunction the apparent distance of the planet from the sun's centre may be many times as much as the sun's apparent semi-diameter.

The geometrical relations of the sun, the earth and the planet at the time of a transit can be studied by supposing that the diameters of the earth and the planet are evanescent in comparison with the diameter of the sun so that the earth is represented by its centre E and Venus by its centre V.

If a transit is on the point of commencing or of ending the line EV should be a tangent to the solar globe. It is therefore easy to see that the small angle expressing the heliocentric elongation of Venus from the earth at the moment of commencement or ending of a transit of Venus must be approximately $R(r-b)/br$ when R is the radius of the sun and r, b the respective distances of Venus and the earth from the sun. If we take the mean value of the sun's apparent angular semi-diameter to be $16'$ and for r and b the values 1 and 0·72 respectively, we find that the required elongation is approximately $16' \times \cdot 28/\cdot 72 = 6'\cdot 2$. It thus appears that at a transit the heliocentric elongation of Venus from the earth must not exceed about $6'$. The conditions under which the transit takes place and its variations as seen from different points on the earth's surface are so complicated as to require a general investigation of the problem to which we now proceed and in which we shall regard the sun and the planet as exact spheres

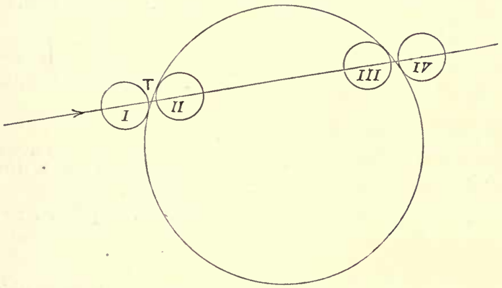


FIG. 77.

When a transit of Venus is about to commence the circular disc of the planet, Fig. 77, comes into apparent contact with the circular disc of the sun. This initial stage of the phenomenon is known as the first external contact and is denoted by I. The planet then appears to enter slowly upon the disc of the sun, and in due course the next stage II known as *first internal contact* is reached. From this point the planet, now seen as a black disc on the brilliant background, advances across the sun's disc and after the lapse of perhaps four hours reaches the third critical stage III at what is known as *second internal contact.* Then the planet begins to pass off the sun's disc, and finally arrives at IV or last external contact, and the phenomenon is at an end. As the external contacts cannot be observed so satisfactorily as the internal contacts the former are comparatively of small importance, and our attention will be devoted to the two internal contacts II and III.

To understand the geometrical problem involved in the transit of Venus we shall imagine a line drawn from the observer to T the point of apparent contact of the globes in stage II. It is evident that this line, though meeting both spheres, does not cut either of them. It must therefore be a common tangent to the two spheres. But such common tangent lines to the two spheres are generators of that common tangent cone which has its vertex exterior to the two spheres and hence we see that at the moments of II and III the observer must be situated at some point on that tangent cone. At the moments of external contact represented in I and IV the observer must be situated at some point on the other common tangent cone, namely that one which has its vertex between the two spheres.

The theory of the transit of Venus must therefore be based on that of the common tangent cones to two spheres which will be discussed in the next article.

Ex. Assuming that the inclination of the orbit of Mercury to the plane of the ecliptic is 7° 0′ 8″ and the longitude of its ascending node is 46° 52′ 19″, show that for a transit of Mercury to take place near that node, when the sun's diameter is 16′ 11″, the heliocentric longitude of the planet must be

$$>44° \, 40' \text{ and } <49° \, 5'.$$

106. Tangent cones to the sun and planet both regarded as spherical.

Let O, C, Fig. 78, be the centres of the sun and the planet of which the radii are R, r_1 and let b denote OC. Then the double tangents PQ and ST meet OC in L and M the vertices of the common tangential cones to the two spheres.

Let x_1, y_1, z_1 be the coordinates of C with respect to three rectangular axes through the origin O. Then

$$x_1 R/(R \mp r_1), \quad y_1 R/(R \mp r_1), \quad z_1 R/(R \mp r_1)$$

with the upper signs are the coordinates of L and with the lower signs are the coordinates of M. If x, y, z are the coordinates of any point F on the cone with its vertex at L then the coordinates of any

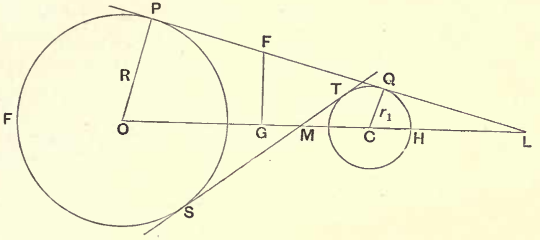


FIG. 78.

other point on the line PL will be obtained by assigning certain values to f and g in the expressions

$$\frac{fx+gx_1R/(R-r_1)}{f+g}\quad \frac{fy+gy_1R/(R-r_1)}{f+g},\quad \frac{fz+gz_1R/(R-r_1)}{f+g}.$$

When these coordinates are substituted in the equation of either of the spheres they will give a quadratic in f/g corresponding to the two points in which FL meets that sphere. This equation for the sphere with centre O becomes

$$\begin{aligned}f^2(x^2+y^2+z^2-R^2)(R-r_1)^2 \\ +2fgR(xx_1+yy_1+zz_1-R^2+Rr_1)(R-r_1) \\ +g^2R^2\{b^2-(R-r_1)^2\}=0.\end{aligned}$$

Expressing the condition that this quadratic shall have equal roots because FL touches the sphere we find for the equation of the common tangent cone with vertex L

$$\begin{aligned}(xx_1+yy_1+zz_1-R^2+Rr_1)^2 \\ =(x^2+y^2+z^2-R^2)(b^2-R^2+2Rr_1-r_1^2)\ \ldots\ldots\ldots(1).\end{aligned}$$

In like manner we obtain for the cone with its vertex at M

$$\begin{aligned}(xx_1+yy_1+zz_1-R^2-Rr_1)^2 \\ =(x^2+y^2+z^2-R^2)(b^2-R^2-2Rr_1-r_1^2)\ \ldots\ldots\ldots(2).\end{aligned}$$

Seen by an observer on cone (1) between Q and L the two circles appear on the same side of their common tangent. Seen by an observer on ST produced beyond T the two circles are on opposite sides of their common tangent.

*Ex. 1. If the equations of the two spheres were given in the form

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2-r_1^2=0,$$
$$(x-x_2)^2+(y-y_2)^2+(z-z_2)^2-r_2^2=0,$$

show that the equations of the common tangential cones would be

$$\{(x-x_1)(x_2-x_1)+(y-y_1)(y_2-y_1)+(z-z_1)(z_2-z_1)-r_1^2\pm r_1 r_2\}^2$$
$$=\{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2-r_1^2\}\{(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2-(r_2\mp r_1)^2\},$$

and explain the geometrical meanings of the different factors in this equation.

*Ex. 2. Show that the equations of cones (1) and (2) may also be expressed in the form

$$(xx_1+yy_1+zz_1-b^2\mp Rr_1+r_1^2)^2$$
$$=\{b^2-(R\mp r_1)^2\}\{(x-x_1)^2+(y-y_1)^2+(z-z_1)^2-r_1^2\}.$$

*Ex. 3. If from the point F (Fig. 80) the line FG be drawn perpendicular to OC, show that

$$OG\,.\,OC-PF\,.\,PQ=R\,(R-r_1),$$

and hence obtain the equation (1).

107. Equation for determining the times of internal contact II and III.

The transit of Mercury or Venus across the sun's disc must take place, as we have seen, when the planet is sufficiently near one of its nodes, the limits being $\sin^{-1}\{\cot i\tan D\}$ on either side of a node, where i is the inclination of the planet's orbit and D the semi-diameter of the sun. We shall suppose the planet to be near its *ascending* node on the ecliptic, and we shall confine our attention to the internal contacts and obtain the equation by which they are determined.

The axes of reference and the symbols to be employed are as follows:

O is the origin of coordinates at the sun's centre.
$+X$ is from O towards ♈.
$+Y$ „ „ the celestial point of long. 90° and lat. zero.
$+Z$ „ „ the nole of the ecliptic.

The coordinates of the observer with respect to these axes are x, y, z and those of the centre of the planet are x_1, y_1, z_1.

O' is the centre of the earth. $O'X'$, $O'Y'$, $O'Z'$ are the axes through O' parallel to OX, OY, OZ, and x', y', z' are the coordinates with respect to $O'X'$, $O'Y'$, $O'Z'$ of the point on the earth's surface occupied by the observer.

λ is the earth's heliocentric longitude.
r, b are the radii vectores from sun to earth and Venus.
ρ is the distance from the centre of the earth to the observer.
ϕ is the geocentric latitude of the observer.
ϑ is the sidereal time on the meridian of the observer.
Ω is the longitude of the planet's ascending node.
ϵ is the inclination of the planet's orbit to the ecliptic.
θ is the angle round O swept over by the planet in its orbit since passing through its ascending node.

Equation (i), § 106, determines the times of interior contact of the planet if for $x, y, z, x', y', z', x_1, y_1, z_1$ we substitute:

$$\left.\begin{aligned} x &= r\cos\lambda + x' \\ y &= r\sin\lambda + y' \\ z &= z' \end{aligned}\right\} \quad \text{(i)},$$

$$\left.\begin{aligned} x' &= \rho\cos\phi\cos\vartheta \\ y' &= \rho\cos\omega\cos\phi\sin\vartheta + \rho\sin\omega\sin\phi \\ z' &= -\rho\sin\omega\cos\phi\sin\vartheta + \rho\cos\omega\sin\phi \end{aligned}\right\} \quad \text{(ii)}.$$

$$\left.\begin{aligned} x_1 &= b\cos\Omega\cos\theta - b\sin\Omega\sin\theta\cos\epsilon \\ y_1 &= b\sin\Omega\cos\theta + b\cos\Omega\sin\theta\cos\epsilon \\ z_1 &= b\sin\theta\sin\epsilon \end{aligned}\right\} \quad \text{(iii)}.$$

These equations are obtained as follows:

The coordinates of O' are $r\cos\lambda$, $r\sin\lambda$, 0, and to find x, y, z we must add severally to the coordinates of O' the corresponding

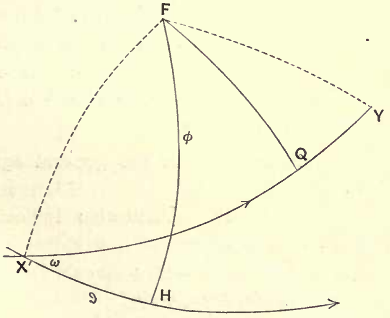


Fig. 79.

coordinates x', y', z' of the observer with respect to the parallel axes through O'.

We may obtain equations (ii) for x', y', z' from Fig. 79 in which F is the position of the observer, FH his meridian and $X'H$ the terrestrial equator. A plane through the earth's centre parallel to the ecliptic meets the earth's surface in $X'Y'$ and $X'Y' = 90°$. As $O'X'$ is parallel to $O♈$ the arc $X'H$ which is increasing by the earth's rotation must be the west hour angle of ♈ for the meridian FH of the observer, that is, the local sidereal time ϑ. If FQ be perpendicular to $X'Y'$ the coordinates of the observer relative to the axes through O' are therefore

$$x' = \rho \cos FX',$$
$$y' = \rho \cos FY' = \rho \sin FX' \cos (FX'H - \omega),$$
$$z' = \rho \sin FQ \;\; = \rho \sin FX' \sin (FX'H - \omega),$$

and thus since

$$\sin FX' \cos FX'H = \cos \phi \sin \vartheta, \quad \sin FX' \sin FX'H = \sin \phi,$$

and $$\cos FX' = \cos \phi \cos \vartheta,$$

we see that x', y', z' have the values shown in (2).

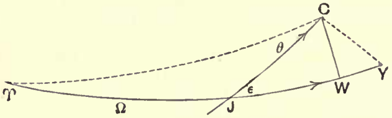


FIG. 80.

In Fig. 80 C is the centre of the planet and J the ascending node of its orbit on the ecliptic $♈Y$. If CW be perpendicular to the ecliptic and $♈Y = 90°$, the coordinates x_1, y_1, z_1 of C are $b \cos C♈$, $b \cos CY$, $b \sin CW$, and from the formulae (§ 1) we obtain the values of x_1, y_1, z_1 given in (iii).

108. Approximate solution of the general equation for internal contact.

We are now to make the substitution indicated in the preceding article, and we have

$$xx_1 + yy_1 + zz_1 = br \{\cos \theta \cos (\lambda - \Omega) + \cos \epsilon \sin \theta \sin (\lambda - \Omega)\}$$
$$+ x'x_1 + y'y_1 + z'z_1,$$
$$x^2 + y^2 + z^2 - R^2 = r^2 + 2rx' \cos \lambda + 2ry' \sin \lambda + \rho^2 - R^2.$$

We next avail ourselves of the fact that r_1^2/b^2, ρ^2/r^2, $\rho R^2/r^3$, and R^4/r^4, being respectively about

$$1/(18000)^2,\ 1/(23000)^2,\ 1/23000 \times 215^2, \text{ and } 1/215^4,$$

are very small quantities (see *Table of elements of the solar system* at the end of the volume) and may be neglected as insensible. Thus we find approximately

$$(x^2 + y^2 + z^2 - R^2)^{\frac{1}{2}} = r - R^2/2r + x' \cos\lambda + y' \sin\lambda,$$
$$\{b^2 - (R - r_1)^2\}^{\frac{1}{2}} = b - R^2/2b + Rr_1/b.$$

Making these substitutions, equation (1), § 106, becomes, after taking the square root of both sides and rejecting the negative sign for the radical because that relates only to the passage of the planet *behind* the sun,

$$\begin{aligned}\cos\theta\cos(\lambda - \Omega) &+ \cos\epsilon\sin\theta\sin(\lambda - \Omega)\\ &= 1 - R^2(r - b)^2/2r^2b^2 + Rr_1(r - b)/rb^2\\ &\quad + (x'b\cos\lambda + y'b\sin\lambda - x'x_1 - y'y_1 - z'z_1)/rb \quad \ldots\text{(i)}.\end{aligned}$$

The time, which is the unknown quantity we are seeking, does not appear explicitly in the equation as at present written. It is, however, implicitly contained in the expressions for λ, θ, x', y', z', x_1, y_1, z_1, so that the equation appears to be of great complexity. But this complexity is unavoidable because the equation as it stands at present has to apply to transits of the planet for all time past and future. When we restrict our view to a single transit the equation admits of reduction to a manageable form giving all that is necessary for that particular transit.

We begin by considering the times at which the transit would commence and end if it could be viewed from the centre of the earth, in which case x', y', z' are all zero and the equation may be written

$$\begin{aligned}\cos\theta\cos(\lambda - \Omega) &+ \cos\epsilon\sin\theta\sin(\lambda - \Omega)\\ &= 1 - R^2(r - b)^2/2r^2b^2 + Rr_1(r - b)/rb^2 \quad \ldots\ldots\ldots\text{(ii)}.\end{aligned}$$

Each side of this equation expresses the cosine of the angle ψ subtended at the centre of the sun by the centres of Venus and the earth. If $\dot\theta$, $\dot\lambda$ be the known rates per hour at which the true anomalies of Venus and the earth are increasing, and if t_0 and t_1 be the Greenwich mean times at which the earth and Venus respectively arrive at the node, we have approximately

$$\theta = \dot\theta(t - t_1), \quad \lambda - \Omega = \dot\lambda(t - t_0).$$

On the occasion of the most recent transit of Venus on December 6th, 1882, we had

$$r = 0{\cdot}9850, \quad b = 0{\cdot}7205, \quad R = 0{\cdot}004663, \quad r_1 = 0{\cdot}00004026,$$

when the mean distance of the earth from the sun is taken as unity. With these figures

$$R^2(r-b)^2/2r^2b^2 = 0{\cdot}000001510,$$
$$Rr_1(r-b)/rb^2 = 0{\cdot}000000097.$$

The equation may therefore be written as follows:

$$\cos\psi \equiv \cos\dot\theta(t-t_1)\cos\dot\lambda(t-t_0) + \cos\epsilon\sin\dot\theta(t-t_1)\sin\dot\lambda(t-t_0)$$
$$= 1 - 0{\cdot}000001413 \quad \text{.....................(iii).}$$

Thus ψ is a small angle of $5'\,47''$, so that as ϵ is $3^\circ\,23'\,31'$ it is easy to see that neither $\dot\theta(t-t_1)$ nor $\dot\lambda(t-t_0)$ can exceed $1^\circ\,40'$. We may therefore express this equation with sufficient accuracy as follows:

$$1 - \tfrac{1}{2}(t-t_1)^2\dot\theta^2 - \tfrac{1}{2}(t-t_0)^2\dot\lambda^2 + \dot\theta\dot\lambda(t-t_1)(t-t_0)\cos\epsilon$$
$$= 1 - 0{\cdot}000001413,$$

which gives a quadratic for t in which all the other quantities are known. When the substitutions for $\dot\theta$, $\dot\lambda$, ϵ, t_0, t_1 are made it is found that the roots of this equation are real, which shows that a transit takes place. If they were imaginary there would not be a transit. If they were equal then Venus would appear just to graze the sun's limb.

Suppose the real roots of the quadratic are t', t'' and that $t'' > t'$. Then t' is the time at which the planet will appear to enter fully on the sun's disc (II) and at t'' the planet will begin to leave the disc (III). The duration of the transit is $t'' - t'$. Thus the problem has been solved for the transit of Venus if it could be viewed from the centre of the earth.

109. On the application of the transit of Venus to the determination of the sun's distance.

This application depends upon observations of the second and third contacts made from different stations, and we have first to obtain the theoretical expressions for such times of contact.

We see from equations (ii) and (iii) (§ 107) that we may write

$$x' = \rho\alpha' \text{ and } x_1 = b\alpha_1,$$
$$y' = \rho\beta' \quad ,, \quad y_1 = b\beta_1,$$
$$z' = \rho\gamma' \quad ,, \quad z_1 = b\gamma_1,$$

where α', β', γ', α_1, β_1 and γ_1 are functions of the several angles ϕ, ϑ, ω, Ω, θ and ϵ and are independent of the linear quantities ρ and b.

Thus the last term in equation (i), § 108, viz.:

$$(x'b\cos\lambda + y'b\sin\lambda - x'x_1 - y'y_1 - z'z_1)/rb,$$

becomes

$$(\alpha'\cos\lambda + \beta'\sin\lambda - \alpha'\alpha_1 - \beta'\beta_1 - \gamma'\gamma_1)\,\rho/r.$$

To obtain the times $t' + \Delta t'$ and $t'' + \Delta t''$ of second and third contact, as seen by the observer whose terrestrial coordinates are $x'y'z'$, we first compute A' which is the value of

$$\alpha'\alpha_1 + \beta'\beta_1 + \gamma'\gamma_1 - \alpha'\cos\lambda - \beta'\sin\lambda,$$

when the values of ϑ, θ and λ corresponding to the time t' have been introduced. In like manner A'' expresses the value of the same function corresponding to the time t''. Thus we have for the second contact

$$\cos\psi - \sin\psi\,.\,\dot{\psi}\Delta t' = 1 - R^2(r-b)^2/2r^2b^2 + Rr_1(r-b)/rb^2 - A'\rho/r,$$

whence we obtain

$$\Delta t' = A'\rho/r\dot{\psi}\sin\psi,$$

and consequently the observer at x', y', z' sees second contact at the time

$$t' + A'\rho/r\dot{\psi}\sin\psi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(i)}.$$

In like manner it is shown that for the same observer the time of third contact will be

$$t'' - A''\rho/r\dot{\psi}\sin\psi \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots\text{(ii)},$$

and accordingly for this observer the duration of the transit from second to third contact will be

$$t'' - t' - (A' + A'')\,\rho/r\dot{\psi}\sin\psi \ldots\ldots\ldots\ldots\ldots\text{(iii)}.$$

If the same transit be also observed from another station and if for this second station B', B'' be the quantities corresponding to A', A'', then the duration of the transit as there seen will be

$$t'' - t' - (B' + B'')\,\rho/r\dot{\psi}\sin\psi \;\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

Hence if D be the difference between the durations of the transit of the planet from second to third contact as seen from the two stations, we have

$$D = (B' + B'' - A' - A'')\,\rho/r\dot{\psi}\sin\psi \ldots\ldots\ldots\ldots\text{(v)}.$$

In this equation A', A'', B', B'' are calculated by the formulae of § 107. The angle ψ is given by the equation (iii), § 108, and $\dot{\psi}$ is obtained by differentiation with regard to the time.

If finally D is determined by observation, then as ρ is known, r is found from equation (v). This is the famous method of determining the sun's distance proposed by Halley. It requires that both second and third contacts should be observed at each of the two stations.

There is also another method of deriving the distance of the sun from observations of the transit of Venus, which bears the name of its originator, De Lisle. This method has the advantage over Halley's that only two successful observations instead of four are needed and consequently the risks of failure by bad weather are correspondingly reduced.

Suppose that the times of observed second contact are obtained at two stations, then the interval will be from (i)

$$(t' + A'\rho/r\dot{\psi}\sin\psi) - (t' + B'\rho/r\dot{\psi}\sin\psi) = (A' - B')\,\rho/r\dot{\psi}\sin\psi.$$

If therefore this interval can be determined we shall have an equation for r.

Of course De Lisle's process can also be applied to a pair of observations of third contact made from two different stations.

The chief drawback to the transit of Venus as a method for the determination of the sun's distance arises from the difficulty of observing exactly the moment of contact between the disc of the planet and the limb of the sun. The movement of the planet is so slow and the limb of the sun is so ill-defined that an uncertainty of several seconds is liable to be found in each observation.

EXERCISES ON CHAPTER XIV.

Ex. 1. Show that A', the quantity used in equation (i), p. 32, is very nearly $\psi \sin z$ where z is the sun's zenith distance and where the observer is supposed to be in the plane which contains the centres of the sun, the earth and the planet.

Ex. 2. Explain why Halley's method of determining the Solar Parallax by the Transit of Venus would not be equally applicable to the Transit of Mercury.

Taking the maximum value $A' = \psi \sin z$ we see that $\Delta t' = \sigma \sin z/\dot{\psi}$ or $\sigma \sin z = \dot{\psi}\Delta t'$. If therefore σ is to be obtained from an observation of $\Delta t'$ it is obvious that the smaller $\dot{\psi}$ the smaller will be the effect of errors in the value of $\Delta t'$ on the concluded value of σ. The quantity $\dot{\psi}$ is inversely proportional to the synodic period of the planet, which in the case of Mercury is 116 days and in that of Venus 584. Hence an error of determination of the moment of contact of Mercury will produce more than five times as much error in the concluded parallax of the sun as in the case of Venus. It is supposed that the zenith distance of the sun is the same in both cases.

Ex. 3. Prove that there will be a transit of Venus, provided that when the planet crosses the ecliptic, the heliocentric angular distance between the earth and Venus does not exceed 41′. The sun's apparent angular semi-diameter is taken as 16′, the distance of Venus from the sun as ·72 times the earth's distance, and the inclination of its orbit to the ecliptic, $\sin^{-1} 1/17$.

[Math. Trip. I. 1902.]

Ex. 4. If in taking the square root of the equation (i), § 108, a negative sign had been given to the radical instead of the upper sign as has been actually used, show that the solution of the equation thence obtained would, as is there stated, refer to those occasions on which the planet passed behind the sun.

Ex. 5. Supposing the planes of the earth's equator and the orbit of Mercury to coincide with the ecliptic, show that to an observer in latitude ϕ, on the same meridian with an observer at the equator who sees Mercury projected on the centre of the sun's disc at midday, the duration of a transit will be $2h$ hours nearly, when

$$r(r-b)\,\omega h + b\rho\cos\phi\sin(\pi h/12) = \sqrt{R^2(r-b)^2 - b^2\rho^2\sin^2\phi},$$

r, b being the radii of the orbits of the earth and Mercury, R, ρ the radii of the sun and the earth, and ω the difference of the apparent horary motions of Mercury and the sun. [Math. Trip.]

If η be the hour angle of the sun when the transit commences its duration will be 2η,

$$x = r\cos\lambda - \rho\cos\phi\cos(\eta+\lambda), \qquad x_1 = b\cos\theta,$$
$$y = r\sin\lambda - \rho\cos\phi\sin(\eta+\lambda), \qquad y_1 = b\sin\theta,$$
$$z = \rho\sin\phi, \qquad z_1 = 0.$$

Expressing the condition that the line from the observer through Mercury (supposed a point) touches the sun,

$$(r^2+\rho^2-2r\rho\cos\phi\cos\eta - R^2)(b^2-R^2)$$
$$= \{br\cos(\theta-\lambda) - b\rho\cos\phi\cos(\eta+\lambda-\theta) - R^2\}^2,$$

which may be transformed into

$$R^2\{r^2+b^2+\rho^2-2r\rho\cos\phi\cos\eta - 2br\cos(\theta-\lambda) + 2b\rho\cos\phi\cos(\eta+\lambda-\theta)\}$$
$$= b^2\{r\sin(\theta-\lambda) - \rho\cos\phi\sin(\theta-\lambda-\eta)\}^2 + b^2\rho^2\sin^2\phi.$$

This is the equation when none of the quantities have been rejected on account of smallness, but if we remember that $\theta-\lambda$, ρ/r, R/r are all small, the equation reduces to

$$b\{r(\theta-\lambda) + \rho\cos\phi\sin\eta\} = \sqrt{R^2(r-b)^2 - b^2\rho^2\sin^2\phi}.$$

We also have $\omega\eta = (\theta-\lambda)\,b/(r-b)$ where $\eta = \pi h/12$, and the desired result is obtained.

Ex. 6. In five synodic periods the motion of the planet from its node is about 2° 22′ less than 13 complete revolutions, and a transit across the sun will take place when the planet's distance from the node at the time of its conjunction with the earth is less than 1° 43′ Deduce a general explanation of the fact that the intervals between successive transits of Venus recur in the order

$$8, \quad 121\tfrac{1}{2}, \quad 8, \quad 105\tfrac{1}{2} \text{ years nearly.}$$

Will this order of recurrence be perpetual?

[Smith's Prize Exam.]

Ex. 7. Supposing that objects can be observed only when their altitude is greater than a, show that the greatest possible interval between accelerated and retarded ingress of Venus in transit obtainable on the earth is about $(11^m\, 35^s) \cos a$, the solar parallax being taken as $8''\cdot 93$, and the periodic times of Venus and the earth to be 224·7 and 365·25 days respectively.

[Math. Trip. I.]

Ex. 8. If the orbits of Venus and the earth be regarded as circular and coplanar with the earth's equator, and if the periodic times of Venus and the earth be respectively 224·7 days and 365·25 days and the solar parallax be $8''\cdot 93$: if also t_1 and t_2 be the moments at which ingress (or egress) of Venus on the sun's disc be observed at two stations on the parallel of ϕ, and if h_1 and h_2 be the west hour angles of the sun at the two stations at the moment of observation respectively, show that the number of minutes in the difference of observed ingress (or egress) is given by the equation

$$t_1 - t_2 = (5^m \cdot 794) \cos \phi \left(\sin \frac{\pi h_2}{12} - \sin \frac{\pi h_1}{12} \right).$$

We have from Ex. 5,

$$b \left\{ r(\theta - \lambda) + \rho \cos \phi \sin \frac{\pi h}{12} \right\} = \sqrt{R^2 (r-b)^2 - b^2 \rho^2 \sin^2 \phi}.$$

Let $$\theta = n_1 t + \epsilon_1, \quad \lambda = n_2 t + \epsilon_2,$$

$$br(n_1 - n_2) t_1 + br(\epsilon_1 - \epsilon_2) + b\rho \cos \phi \sin \frac{\pi h_1}{12} = \sqrt{R^2 (r-b)^2 - b^2 \rho^2 \sin^2 \phi},$$

$$br(n_1 - n_2) t_2 + br(\epsilon_1 - \epsilon_2) + b\rho \cos \phi \sin \frac{\pi h_2}{12} = \sqrt{R^2 (r-b)^2 - b^2 \rho^2 \sin^2 \phi},$$

whence $$r(n_1 - n_2)(t_1 - t_2) = \rho \cos \phi \left(\sin \frac{\pi h_2}{12} - \sin \frac{\pi h_1}{12} \right).$$

But n_1 and n_2 are respectively 0·000019419 and 0·000011947, being the angle in radians described by Venus and the earth respectively in 1^m. Also $\rho / r = 8\cdot 93 \sin 1''$, and substituting these values the desired formula is obtained.

Ex. 9. Taking the orbits of the earth and of Mercury to be circles of radii 1·000 and 0·387 respectively, the parallax of the sun to be 8″·80 and his diameter 32′ 4″·0; find the greatest inclination of Mercury's orbit to the ecliptic which would allow of a transit being visible at *some* place on the earth at every inferior conjunction. [Sheepshanks Exhibition.]

Ex. 10. It is found that at two places on the earth's surface in the plane of the ecliptic and at opposite extremities of a diameter of the earth, the differences of times of ingress and egress of Venus at transit over the sun's disc are $11^m\, 19^s$ and $11^m\, 21^s$ respectively. Being given that the synodic period of Venus is 584 days, calculate the sun's parallax in seconds of arc to two decimal places. [Coll. Exam. 1900.]

Ex. 11. Assuming that the diameters of the earth and Venus are negligible, show that ψ, the heliocentric elongation of Venus from the earth at the moment of the commencement or the end of a transit, is given accurately by the equation

$$b^2 r^2 \cos^2 \psi - 2brR^2 \cos \psi + R^2 (b^2 + r^2) - b^2 r^2 = 0,$$

where R is the sun's radius and b, r the distances of Venus and the earth from the sun's centre.

Ex. 12. Let λ, θ be the heliocentric angular velocities of the earth and Venus, x the heliocentric elongation of the earth from Venus when the planet is crossing the ecliptic, ϵ the inclination of the orbit of Venus, ψ the angle defined in Ex. 11. Show that if a transit of Venus is to take place $x \ngtr \psi (\dot{\theta} - \dot{\lambda})/\dot{\theta} \sin \epsilon$ approximately.

Ex. 13. Show that if a transit of Venus is to take place the heliocentric latitude of Venus must be $\ngtr \psi$ when the planet is in conjunction with the earth in longitude and that neither the heliocentric elongation of the earth nor the planet from the node can exceed ψ cosec ϵ.

CHAPTER XV.

THE ANNUAL PARALLAX OF STARS.

110. Introductory.

In the investigation of the distance of the moon or a planet we have found that a base line of adequate length can be obtained if its terminals are properly chosen terrestrial stations. By measurements from both ends of the base line the required distance can be ascertained.

If the measurement of the distance of a star is to be effected the base line to be employed must be a magnitude of a much higher order than the diameter of the earth (see p. 279). The base line for our sidereal measurements is the diameter of our earth's annual orbit which is $(206265/8{\cdot}80) = 23400$ times the earth's diameter. The terrestrial observer is transferred in six months from one end of a diameter of the earth's orbit to the other. From each end of the diameter he makes observations of the apparent places of a star, and if there is an appreciable difference between those apparent places, the means of determining the distance of that star are provided.

Let ρ be the mean distance of the earth from the sun, and r the distance of the star from the sun, then $\rho/r \sin 1''$ is defined to be the *annual parallax* of the star. It is the number of seconds of arc in the vertical angle of an isosceles triangle of which ρ is the base and r each of the equal sides. We may for brevity denote the annual parallax by the symbol σ.

Let T (Fig. 81) be the earth, O be the sun, and S be the star, and draw TS' parallel to OS. Then TS' is the true direction of the star, *i.e.* the direction in which it would be seen from the sun's centre, and the angle $S'TS = TSO$ is the effect of parallax on the apparent place of the star. As this angle is very small we may replace its sine by the angle itself, and denoting the star's elongation STO from the sun by E we have for the Parallax in seconds of arc

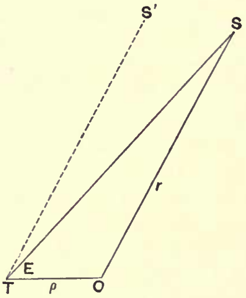


Fig. 81.

$$\angle TSO = \sin E \,.\, \rho/r \sin 1'' = \sigma \sin E.$$

Hence we see that the effect of parallax is to throw the star from its mean place towards the sun through an angle which is proportional to the sine of the angle between the star and the sun. Thus $\sigma \sin E$ or the product of the *annual parallax* σ and the sine of the star's elongation from the sun shows the parallactic displacement.

It should be remarked that so far as the present work is concerned we may, in considering stellar parallax, neglect the ellipticity of the earth's orbit and regard ρ as a constant.

The diameter of the earth's orbit amounting to 186,000,000 miles provides the longest base line available to the terrestrial astronomer for his measurement of star distances. The distances of the great majority of stars are, however, so vast, that the changes in their apparent positions, when viewed first from one end of this base line and then from the other, are hardly appreciable. The largest annual parallax known up to the present is that of α Centauri $0''\cdot75$. It is the first on the following list (*Annuaire publié par le Bureau des Longitudes*).

It will be obvious from this list that the determination of the parallaxes of stars is a work of much delicacy and refinement. The annual parallax of Arcturus having a circular measure 1/8,600,000 shows that the earth's orbit viewed from that star would appear no larger than a circle a foot in radius would appear

PARALLAXES OF STARS.

Name	Magnitude	R.A. 1900·0 h. m. s.	Decl. 1900·0 ° ′	Proper Motion ″	Annual Parallax ″	Millions of times sun's distance	Light years
α Centauri	0·2	14 32 48	−60 25	3·7	0·75 ± 0·01	0·28	4·4
21185 Lalande	7·5	10 57 53	+36 38	7·3	0·48 ± 0·02	0·43	6·8
61 Cygni	4·8	21 2 25	+38 16	5·2	0·37 ± 0·02	0·56	8·8
Sirius	−1·4	6 40 44	−16 35	1·3	0·37 ± 0·01	0·56	8·8
Cordoba Zone 5^h 243	8·5	5 7 42	−44 59	8·7	0·31	0·67	11
Procyon	0·5	7 34 4	+ 5 29	1·3	0·31	0·69	11
Altair	0·9	19 45 54	+ 8 36	0·7	0·28 ± 0·02	0·74	12
Aldebaran	1·1	4 30 11	+16 19	0·2	0·17 ± 0·02	1·4	22
Capella	0·2	5 9 18	+45 54	0·4	0·12 ± 0·02	1·7	27
Vega	0·1	18 33 33	+38 41	0·4	0·12 ± 0·02	1·7	27
1830 Groombridge	6·4	11 47 33	+38 26	7·0	0·10 ± 0·02	2·0	33
Polaris	2·1	1 22 13	+88 46	0·0	0·07 ± 0·02	3·0	47
Arcturus	0·2	14 11 6	+19 42	2·3	0·024	8·7	140
α Gruis	2·2	22 1 56	−47 27		0·015	14	220

when viewed from a distance of 1630 miles. It will readily be admitted that the determination of the distance of an object about 1630 miles away from observations made at the ends of a base line only two feet long would be a very delicate undertaking. It is only by the accumulation of a long series of careful observations in which the errors of observation are gradually eliminated that success can be obtained.

Even the best meridian observations of the places of the stars are of but little service for the determination of stellar parallax. We require for this purpose observations of the class termed *differential*, and what this term signifies will now be explained.

If a star were at an infinitely great distance its parallactic displacement would be zero, and its place would therefore be the same when seen from all points of the earth's orbit. The great majority of the stars have a parallax too small to affect our measurements. In such a case we make no appreciable error by treating it as zero, and the observations now to be considered are those in which the position of a star which is affected by parallax is compared with a star which has no parallax, but is so placed that, as projected on the celestial sphere, the two stars seem closely adjacent. The two stars should appear sufficiently close to be in the same field of view of the telescope. We then make differential measurements of these two stars. In this way certain errors, for example, those arising from the flexure of the instrument and many others, are eliminated, for they affect both stars equally. The influence of refraction can also be allowed for with accuracy, because the irregularities in the refraction may also be presumed to affect both stars equally, and will therefore disappear from a differential measurement. If the second star is also near enough to have an appreciable parallax then the result determined is the difference of the parallaxes of the two stars.

The observations made are usually those of the distance and the position angle (§ 49) between the two stars. These observations, repeated on as many occasions as possible throughout the course of at least one year, afford the data by which, after due reduction, the parallax is to be determined.

Ex. 1. If σ be the annual parallax of a star and n be the number of years in which light from that star would reach the earth, show that $n = 13/4\sigma$.

Ex. 2. Show that the light from 61 Cygni of which the parallax is 0″·37 takes 8·8 years to come from the star to the earth. (See Table, p. 328.)

111. Effect of annual parallax on the apparent right ascension and declination of a fixed star.

If r be the distance of a star from the centre of the sun, α, δ the true R.A. and decl. of the star, *i.e.* those referred to the centre of the sun, r', α', δ' the corresponding coordinates referred to the centre of the earth, $\odot$ the longitude of the sun, ρ its distance from the earth, and ω the obliquity of the ecliptic, then we have the fundamental equations as in § 93

$$r' \cos \delta' \cos \alpha' = r \cos \delta \cos \alpha + \rho \cos \odot \quad \text{(i)},$$

$$r' \cos \delta' \sin \alpha' = r \cos \delta \sin \alpha + \rho \sin \odot \cos \omega \quad \text{(ii)},$$

$$r' \sin \delta' = r \sin \delta + \rho \sin \odot \sin \omega \quad \text{(iii)},$$

whence we obtain

$$\tan \alpha' = \frac{r \cos \delta \sin \alpha + \rho \sin \odot \cos \omega}{r \cos \delta \cos \alpha + \rho \cos \odot},$$

but as $(\alpha' - \alpha)$ is a small quantity expressed in seconds of arc,

$$\tan \alpha' = \tan (\alpha + \alpha' - \alpha) = \tan \alpha + \sec^2 \alpha \, (\alpha' - \alpha),$$

whence, using σ to denote the annual parallax, ρ/r, we obtain the annual parallax in R.A.

$$\alpha' - \alpha = \sigma \sec \delta \, (\cos \alpha \cos \omega \sin \odot - \sin \alpha \cos \odot) \quad \text{...(iv)}.$$

Squaring and adding (i) and (ii) and remembering that ρ is small compared with r,

$$r'^2 \cos^2 \delta' = r^2 \cos^2 \delta + 2r\rho \, (\cos \delta \cos \alpha \cos \odot + \cos \delta \sin \alpha \cos \omega \sin \odot),$$

and taking the square root we find

$$r' \cos \delta' = r \cos \delta + \rho \, (\cos \alpha \cos \odot + \sin \alpha \cos \omega \sin \odot).$$

Dividing this into (iii) we have

$$\tan \delta' = \frac{r \sin \delta + \rho \sin \odot \sin \omega}{r \cos \delta + \rho \, (\cos \alpha \cos \odot + \sin \alpha \cos \omega \sin \odot)}.$$

Substituting for $\tan \delta'$ the expression

$$\tan \delta + \sec^2 \delta \, (\delta' - \delta),$$

we deduce the annual parallax in Decl.

$$\delta' - \delta = \sigma \, [\cos \delta \sin \omega \sin \odot - \sin \delta \cos \alpha \cos \odot - \sin \delta \sin \alpha \cos \omega \sin \odot] \quad \text{......(v)}.$$

Thus the parallactic displacement of a star in R.A. and in decl. is made to depend on the sun's longitude $\odot$ as the single variable element and we can avail ourselves of this fact to obtain more concise expressions by the introduction of six new quantities a, b, a', b', a'', b'' defined by the equations

$$a\cos b = \sin\alpha;\quad a'\sin b' = \sin\omega;\quad a''\sin b'' = a'\sin(b'-\delta);$$
$$a\sin b = \cos\alpha\cos\omega;\quad a'\cos b' = \cos\omega\sin\alpha;\quad a''\cos b'' = \cos\alpha\sin\delta.$$

As these quantities involve only the position of the star and the obliquity of the ecliptic, they are constant throughout the year. Thus the formulae

$$(\alpha-\alpha')\cos\delta = \sigma a\ \ \cos(b\ +\odot),$$
$$\delta-\delta' = \sigma a''\cos(b''+\odot),$$

enable the parallactic effect at different parts of the year to be computed with all needful simplicity by the introduction of the corresponding values of $\odot$.

Let S (Fig. 82) be the true place of the star, S' the place to which the star is apparently carried by parallax, and draw ST perpendicular to $S'P$, where P is the pole. Then $PS = 90° - \delta$, and we have

$$(\alpha-\alpha')\cos\delta = ST;\quad \delta-\delta' = S'T,$$

showing that $\sigma a\cos(b+\odot)$ is the distance by which the parallax displaces the star parallel to the equator. It is plain from

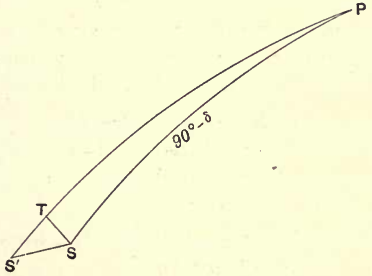


Fig. 82.

this formula that the apparent place of the star as disturbed by parallax describes an ellipse in the course of the year. For taking ST and SP as the axes of x and y respectively we have

$$x = \sigma a\cos(b+\odot);\quad y = -\sigma a''\cos(b''+\odot);$$

and the elimination of $\odot$ gives an elliptic orbit for S'.

Ex. 1. In 1876 the mean place of the great spiral nebula 51*M* was $\alpha = 13^h\ 24^m\ 35^s$, $\delta = +47°\ 50'$. If its annual parallax was σ and if its apparent place as affected by parallax was α', δ', show that if $\odot$ be the sun's longitude

$$(\alpha - \alpha')\cos\delta = \sigma\,[9{\cdot}9678]\cos(\odot + 247°\ 8'),$$

$$\delta - \delta' = \sigma\,[9{\cdot}9348]\cos(\odot + 143°\ 27').$$

Determine the dates on which the parallax in declination is as great as possible and find the maximum parallax in R.A.

N.B. The figures enclosed in brackets are logarithms.

Ex. 2. Prove that (assuming uniform motion of the sun in longitude) the correction to the time of transit of a star of R.A. α, due to annual parallax, has its greatest magnitude $\frac{1}{2\pi}\,.\,365\tfrac{1}{4}\,.\tan^{-1}(\sec\omega\tan\alpha)$ days after a solstice, ω being the obliquity of the ecliptic.

[Coll. Exam.]

If σ be the star's parallax its effect on right ascension is

$$\alpha' - \alpha = \sigma\sec\delta\,(\cos\alpha\sin\odot\cos\omega - \sin\alpha\cos\odot).$$

For this to be a maximum,

$$\tan(\odot - 90°) = \sec\omega\tan\alpha.$$

Hence the sun's longitude exceeds that of the solstice by $\tan^{-1}(\sec\omega\tan\alpha)$ radians. But the sun describes $2\pi/365\tfrac{1}{4}$ radians of longitude per diem, whence the desired result.

Ex. 3. Show that the maximum effects of annual parallax on the right ascension and declination of a star are given by the expressions

$$\sigma\sec\delta\,(1 - \cos^2\alpha\sin^2\omega)^{\frac{1}{2}} \text{ and } \sigma\,(\sin^2\lambda + \sin^2\omega\cos^2\alpha)^{\frac{1}{2}},$$

where σ is the coefficient of annual parallax, ω is the obliquity of the ecliptic, and α, δ, λ are the right ascension, declination, and latitude of the star.

[Math. Trip. I.]

The maximum value of (iv) for any real value of $\odot$ is

$$\sigma\sec\delta\,(\cos^2\alpha\cos^2\omega + \sin^2\alpha)^{\frac{1}{2}} = \sigma\sec\delta\,(1 - \cos^2\alpha\sin^2\omega)^{\frac{1}{2}},$$

and the maximum value of (v) for any real value of $\odot$ is

$$\sigma\,\{(\cos\delta\sin\omega - \sin\delta\cos\omega\sin\alpha)^2 + \sin^2\delta\cos^2\alpha\}^{\frac{1}{2}}$$
$$= \sigma\,\{(\sin\delta\cos\omega - \cos\delta\sin\omega\sin\alpha)^2 + \sin^2\omega\cos^2\alpha\}^{\frac{1}{2}}$$
$$= \sigma\,(\sin^2\lambda + \sin^2\omega\cos^2\alpha)^{\frac{1}{2}}.$$

Ex. 4. Show that the general effect of the annual parallax of a star is to alter its position in the small ellipse which it appears to describe annually owing to aberration, and for a given star state how this alteration varies with the time of year. [Math. Trip. I.]

Let S be the sun, S_0 (Fig. 83) a point 90° behind the sun and on the ecliptic. Let O be a star of coordinates λ, β and parallax σ. The line OX is perpendicular to SS_0. Let OY be perpendicular to OX and we seek the coordinates x, y with respect to those axes of a star at O disturbed both by parallax and aberration. It is assumed that κ is the constant of aberration and that the squares and higher powers of σ/κ may be neglected. Since aberration

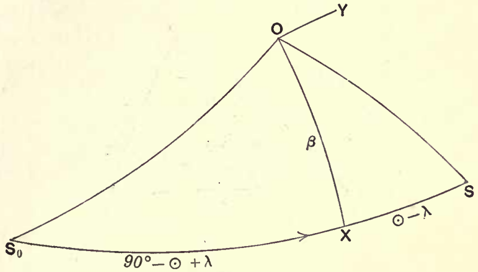


FIG. 83.

moves the star along OS_0 to a distance $\kappa \sin OS_0$ and parallax moves the star towards S through a distance $\sigma \sin OS$ we have

$$x = \kappa \sin\beta \sin(\odot - \lambda) + \sigma \sin\beta \cos(\odot - \lambda),$$
$$y = -\kappa \cos(\odot - \lambda) + \sigma \sin(\odot - \lambda),$$

which may be written

$$x = \kappa \sin\beta \sin(\odot + \sigma/\kappa - \lambda),$$
$$y = -\kappa \cos(\odot + \sigma/\kappa - \lambda),$$

so that $$x^2 \operatorname{cosec}^2\beta + y^2 = \kappa^2.$$

We thus see that taking parallax into account has simply the effect of changing the apparent place of the star on the aberrational ellipse from the point corresponding to $\odot$ to the point corresponding to $\odot + \sigma/\kappa$.

112. Effect of parallax in a star S on the distance and position angle of an adjacent star S'.

In Fig. 84 let O be the sun, ΥO the ecliptic, P the north pole. Let α, δ be the R.A. and decl. of S and α', δ' those of the sun. Let D, p be the distance SS' and position angle PSS' of S' with respect to S.

The effect on S of a parallax σ is to convey the star from its mean position S to an apparent position along the direction SO, so that $ST = \sigma \sin SO$.

The apparent distance of the two stars is TS', and this is sensibly equal to HS' if TH is a perpendicular on SS'. It follows that SH, which we shall represent as D', measures the effect of parallax on the distance D between the stars. As parallax changes the direction $S'S$ into $S'T$, the angle $SS'T$ is, with sufficient approximation, the change in the position angle of S' with regard to S.

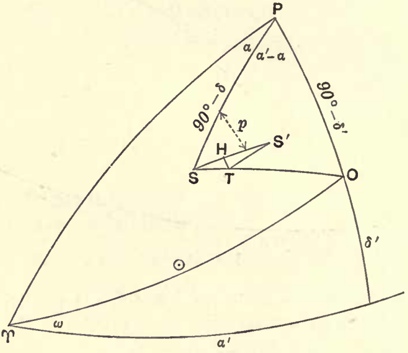


FIG. 84.

The effect of parallax on the apparent distance is calculated as follows:

$$D' = SH = \sin ST \cos TSH = \sigma \sin SO \cos (PSO - p),$$

but

$$\sin SO \sin PSO = \cos \delta' \sin (\alpha' - \alpha),$$

$$\sin SO \cos PSO = \sin \delta' \cos \delta - \cos \delta' \sin \delta \cos (\alpha' - \alpha),$$

and thus we find for D' the parallax in distance

$$D' = \sigma \sin p \cos \delta' \sin (\alpha' - \alpha) + \sigma \cos p \{\sin \delta' \cos \delta - \cos \delta' \sin \delta \cos (\alpha' - \alpha)\}.$$

To express this in terms of the sun's longitude we have

$$\cos \odot = \cos \delta' \cos \alpha',$$

$$\cos \omega \sin \odot = \cos \delta' \sin \alpha',$$

$$\sin \omega \sin \odot = \sin \delta',$$

and hence

$$D' = \sigma \cos \odot (-\cos \alpha \sin \delta \cos p - \sin \alpha \sin p) + \sigma \sin \odot (-\sin \alpha \sin \delta \cos \omega \cos p + \cos \delta \sin \omega \cos p + \cos \alpha \cos \omega \sin p).$$

In like manner we can compute $TS'S$ or p' where p' is the correction to be applied to the observed position angle of S' from S

in order to obtain the position angle as it would be seen from the sun

$$p' = TS'S = \sigma \sin SO \sin (PSO - p) \operatorname{cosec} D$$
$$= \sigma \cos \odot (-\cos p \sin \alpha + \sin p \cos \alpha \sin \delta) \operatorname{cosec} D$$
$$+ \sigma \sin \odot (+\cos \alpha \cos \omega \cos p$$
$$+ \sin \alpha \sin \delta \cos \omega \sin p - \cos \delta \sin \omega \sin p) \operatorname{cosec} D.$$

As these formulae have $\odot$ as the only variable they can be put into a much more convenient form by the introduction of certain auxiliary quantities m, M, m', M', defined by expressions in which an approximate value will suffice for p:

$$m \cos M = -\cos \alpha \sin \delta \cos p - \sin \alpha \sin p,$$
$$m \sin M = -\sin \alpha \sin \delta \cos \omega \cos p$$
$$+ \cos \delta \sin \omega \cos p + \cos \alpha \cos \omega \sin p,$$
$$m' \cos M' = -\cos p \sin \alpha + \sin p \cos \alpha \sin \delta,$$
$$m' \sin M' = + \cos \alpha \cos \omega \cos p$$
$$+ \sin \alpha \sin \delta \cos \omega \sin p - \cos \delta \sin \omega \sin p.$$

By these substitutions we obtain

$$D' = \sigma m \cos (\odot - M),$$
$$p' = \sigma m' \cos (\odot - M') \operatorname{cosec} D,$$

in which D' and p' as well as σ are expressed in seconds of arc.

Ex. 1. Compute the effect of an annual parallax σ on the star No. 182 in Schjellerup's catalogue of red stars ($\alpha = 15^h\ 45^m$; $\delta = +39°\ 57'$) with respect to an adjacent star without parallax at the distance 392″ and position angle 340° 59′.

The parallax in distance expressed in seconds is

$$[9{\cdot}96381]\, \sigma \cos (\odot - 85°\ 52').$$

The parallax in position angle expressed in seconds is

$$[2{\cdot}68936]\, \sigma \cos (\odot + 13°\ 52').$$

Ex. 2. Show that on the 9th Jan. 1877, when $\odot = 289°\ 25'$, the observed distance (see last example) must have the correction $-0{\cdot}843\sigma$, to clear from the effect of parallax, and the observed position angle must have the correction 268σ.

Ex. 3. Let α, δ be the R.A. and decl. of a star which has an annual parallax σ. Let p, D be the observed position angle and distance of an adjacent star without parallax. If ρ, θ, λ, μ are auxiliary quantities defined by the equations

$$\rho \cos \theta = \sin \delta', \qquad \lambda \cos \mu = \cos \delta' \sin (\alpha' - \alpha),$$
$$\rho \sin \theta = \cos \delta' \cos (\alpha' - \alpha), \qquad \lambda \sin \mu = \rho \cos (\theta + \delta),$$

then the correction for parallax to the observed position angle and distance to reduce them to what they would be as seen from the sun are respectively

$$\sigma\lambda \cos(p+\mu) \operatorname{cosec} D \quad \text{and} \quad \sigma\lambda \sin(p+\mu),$$

if we assume the earth's orbit to be a circle.

Ex. 4. If the observations are made when the star is 90° from the sun, show that $\sin(\theta+\delta)=0$ and $\lambda^2=1$, so that the expressions become

Parallax in distance ... ... $xR\sin(p+\mu)$,
" position angle ... ... $xR\cos(p+\mu)\operatorname{cosec} D$,

where μ is determined by

$$\sin\mu=\sin\delta'/\cos\delta;\quad \cos\mu=\cos\delta'\sin(\alpha'-\alpha).$$

Ex. 5. A star S at the position ($\alpha=16°\,33'$, $\delta=+46°\,51'$) with a parallax σ has another star S' (supposed without parallax) at the position angle 91° 32′. The distance of the two stars was measured on 28 Feb. 1877, when the apparent coordinates of the sun were $\alpha'=22°\,46'\,20''$, $\delta'=-7°\,48'\,19''$. Show that $+{\cdot}982\sigma$ is the correction which must be applied to the observed value of the distance to reduce it to what it would have been if the observer had been at the sun instead of on the earth.

113 Parallax in latitude and longitude of a star.

If β, λ, r be the heliocentric latitude, longitude and distance of a star with annual parallax σ, and if β', λ', r' be the geocentric latitude, longitude and distance of the same star when $\odot$ is the longitude of the sun and ρ its distance, then by making $\omega=0$ in (i), (ii), (iii) (§ 111), and writing β, λ, β', λ' for δ, α, δ', α',

$$r'\cos\beta'\cos\lambda' = r\cos\beta\cos\lambda + \rho\cos\odot \quad \text{......... (i)},$$
$$r'\cos\beta'\sin\lambda' = r\cos\beta\sin\lambda + \rho\sin\odot \quad \text{......... (ii)},$$
$$r'\sin\beta' = r\sin\beta \quad \text{.........................(iii)}.$$

From these we obtain

$$r'\cos\beta'\sin(\lambda'-\lambda) = \rho\sin(\odot-\lambda),$$

whence, if squares and higher powers of $\sigma=\rho/r$ may be neglected,

$$\lambda'-\lambda = \sigma\sec\beta\sin(\odot-\lambda).$$

From (i) and (ii) we obtain

$$r'\cos\beta' = r\cos\beta + \rho\cos(\odot-\lambda),$$

whence with (iii) we have

$$\beta'-\beta = -\sigma\sin\beta\cos(\odot-\lambda).$$

We thus learn that if the heliocentric latitude and longitude of a star of parallax σ are β, λ, then the corresponding geocentric quantities are

$$\{\beta-\sigma\sin\beta\cos(\odot-\lambda)\} \quad \text{and} \quad \{\lambda+\sigma\sec\beta\sin(\odot-\lambda)\}.$$

We can investigate in another way the effect of annual parallax on the distance and position angle of a star β, λ with respect to another star β_0, λ_0, of which the parallax is regarded as zero. Consider the spherical triangle A, B, C, of which A is β_0, λ_0; B is β, λ and C is the nole of the ecliptic. Let the points A, C remain fixed while B undergoes a slight displacement, then $\Delta b = 0$, and from formulae (i), p. 13, we have, if $H = \sin A/\sin a$,

$$\Delta a = \cos B \Delta c + H \sin b \sin c \Delta A,$$

$$\Delta c = \cos B \Delta a + H \sin a \sin b \Delta C,$$

whence, by eliminating Δc and solving for ΔA, we have

$$\Delta A = -\sin a \operatorname{cosec} c \cos B \Delta C + \operatorname{cosec} c \sin B \Delta a.$$

If the displacement of the star at B arise from parallax, then as shown above

$\Delta C = \Delta\lambda = \sigma \sec \beta \sin (\odot - \lambda)$ and $\Delta a = -\Delta\beta = \sigma \sin \beta \cos (\odot - \lambda)$ where $\beta = 90° - a$, and we thus obtain

The displacement in position angle or ΔA is

$$\sigma \operatorname{cosec} c \{-\cos B \sin (\odot - \lambda) + \sin B \sin \beta \cos (\odot - \lambda)\},$$

and the displacement in distance or Δc is

$$\sigma \{\sin \beta \cos B \cos (\odot - \lambda) + \sin B \sin (\odot - \lambda)\},$$

where B, c are determined from the triangle

$$BC = 90° - \beta, \quad AC = 90° - \beta_0, \quad \angle ACB = \lambda - \lambda_0.$$

As a verification of these results we note that the square of the total displacement in consequence of parallax divided by σ^2 must be both $(\sin c\Delta A)^2 + (\Delta a)^2$ and $(\Delta\beta)^2 + (\cos \beta \Delta\lambda)^2$, and each of these reduces to $\sin^2 (\odot - \lambda) + \sin^2 \beta \cos^2 (\odot - \lambda)$.

Ex. 1. The latitudes of two stars are β and β_0 and their difference of longitude is l; the parallax of the second is insensible and that of the first is σ. Show that the angle between the extreme positions of the arc joining them is approximately

$$\frac{2\sigma \sqrt{\{\sin^2 (\beta - \beta_0) + \sin 2\beta \sin 2\beta_0 \sin^2 \tfrac{1}{2} l\}}}{\sin^2 (\beta - \beta_0) + \sin 2\beta \sin 2\beta_0 \sin^2 \tfrac{1}{2} l + \cos^2 \beta \cos^2 \beta_0 \sin^2 l}.$$

[Math. Trip. I.]

As the displacement in position angle by parallax is

$$\sigma \operatorname{cosec} c \{-\cos B \sin (\odot - \lambda) + \sin B \sin \beta \cos (\odot - \lambda)\},$$

its extreme values must be

$$\pm 2\sigma \operatorname{cosec} c (\cos^2 B + \sin^2 \beta \sin^2 B)^{\frac{1}{2}},$$

and in the triangle whose sides are $90° - \beta_0$, $90° - \beta$ and included angle l the angle opposite $90° - \beta_0$ is B, and c is the side opposite l, whence B, c can be eliminated and the desired result is obtained.

Ex. 2. Show that the greatest variation in the apparent distance of a star S with parallax σ from a star S' which has no parallax, is

$$2\sigma(\sin^2\beta\cos^2 B + \sin^2 B)^{\frac{1}{2}},$$

when β is the latitude of S' and where B is the angle which S' and either pole of the ecliptic subtends at S.

Ex. 3. Prove that the cosine of the angle between the directions in which a star is displaced on the celestial sphere by annual aberration and by annual parallax is

$$\sin 2(\odot - \lambda)\cos^2\beta\,[4\sin^2\beta + \cos^4\beta\sin^2 2(\odot - \lambda)]^{-\frac{1}{2}},$$

where β is the latitude and λ the longitude of the star, and $\odot$ the longitude of the sun. [Coll. Exam.]

*114. On the determination of the parallax of a star by observation.

We are now to show how by continued observation of the distance and position of a star S which has a parallax σ from a star S' which has no parallax we can determine the value of σ. If we could assume that the question was not complicated by any proper motion in one or both of the stars S and S', and if we could also assume that the errors of our observations were insignificant in comparison with the quantity sought, then the determination of the parallax from observations either of the distance or the position angle would be a simple matter.

Suppose that the distance from S to S' as seen from the sun was D, then the observed distance is $D - D'$ where

$$D' = \sigma m \cos(\odot - M),$$

in which m, M are known because these are determined once for all for the particular pair of stars by the formulae already given (§ 112). Suppose that two observations D_1 and D_2 are made when the longitudes of the sun are $\odot_1$ and $\odot_2$, then we have the equations

$$D = D_1 + \sigma m \cos(\odot_1 - M),$$
$$D = D_2 + \sigma m \cos(\odot_2 - M),$$

whence we have for the parallax

$$\sigma = \frac{D_1 - D_2}{m\{\cos(\odot_2 - M) - \cos(\odot_1 - M)\}}.$$

All the quantities on the right-hand side being known, σ is determined. As errors are unavoidable in making the observations $D_1 - D_2$ will be necessarily erroneous to an extent which is of course unknown, but the effect of which on σ we desire to have as small as possible. If $\Delta\sigma$ be the error in σ caused by an error $\Delta(D_1 - D_2)$, then by differentiation we have

$$\Delta\sigma = \frac{\Delta(D_1 - D_2)}{m\{\cos(\odot_2 - M) - \cos(\odot_1 - M)\}}.$$

To have $\Delta\sigma$ as small as possible we must have $\Delta(D_1 - D_2)$ as small as possible and $\{\cos(\odot_2 - M) - \cos(\odot_1 - M)\}$ as large as possible. The former condition we seek to attain by making our observations with the utmost care. For the latter condition we choose certain particular dates for the observations. If $\odot_2 - M = 0°$ and $\odot_1 - M = 180°$, then the denominator of $\Delta\sigma$ becomes $2m$ and it cannot exceed this amount. Hence by choosing the dates of our observations on the two days six months apart, when $\odot_1 = 180° + M$ and $\odot_2 = M$ respectively, we have the most favourable conditions and obtain

$$\Delta\sigma = \Delta(D_1 - D_2)/2m.$$

It happens that the parallaxes of all stars, so far as at present known, are so small that two observations as here supposed would not suffice for our purpose. Where the parallax is only a few tenths of a second and where the casual errors of observation may also be a few tenths of a second a single pair of observations cannot give a reliable result. Not fewer than 30 or 40 observations fairly distributed over the year would be necessary, and we now discuss the procedure to be adopted, adding however the remark that in the actual conduct of the investigation various minor points not here treated of must be attended to. We shall suppose the search for the parallax to be made by observations of the distance SS', though such an investigation can also be made from observations of the position angle.

Suppose that at n epochs $t_1, t_2, \ldots$ extending over a year or longer, measurements $D_1, D_2, \ldots D_n$ of the apparent distances between S and S' have been obtained. We shall assume that these observations have been already corrected for refraction by the principles explained in § 48 for finding the effect of refraction on the apparent distance of two adjacent stars.

In the first place there will generally be a small proper motion of one or both of the two stars by which their distance is continually changing. As the distance of the stars is many times greater than the relative proper motion during the year over which the observations extend, we shall introduce no appreciable error by regarding the change in distance due to this cause as being simply proportional to the time. It is thus distinguished from the parallactic change which is essentially periodic.

If there is any relative proper motion the apparent path of the star is not of course the ellipse which parallax alone would make it, nor the straight arc which proper motion alone would make it, but rather a sinuous arc which is the resultant of both. It often happens that the change due to proper motion greatly exceeds the displacement due to parallax.

The increase of the distance between the two stars by their relative proper motion we shall represent by yt, where y is an unknown which will be determined in the course of the investigation and where t is the fraction of the year which has elapsed since Jan. 1st preceding.

The true distance between S and S' as it would be seen from the sun on Jan. 1st is unknown, so we shall assume it to be x, and hence the true distance at the time of observation t_1 is

$$x + yt_1.$$

Let $\odot_1$ be the sun's longitude at the time t_1, then the correction for parallax to be applied to the observed distance D_1 is as already shown

$$\sigma m \cos(\odot_1 - M),$$

and consequently the true distance is

$$D_1 + \sigma m \cos(\odot_1 - M).$$

Equating these values of the true distance and forming the similar equations for all the other epochs, we have

$$\left.\begin{array}{l} x + yt_1 - \sigma m \cos(\odot_1 - M) - D_1 = 0 \\ x + yt_2 - \sigma m \cos(\odot_2 - M) - D_2 = 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ x + yt_n - \sigma m \cos(\odot_n - M) - D_n = 0 \end{array}\right\} \ldots\ldots\ldots\ldots\text{(i)}.$$

Thus the observations give n linear equations among three unknowns x, y, σ, so that the same investigation which finds the parallax of S shows us also x the true distance SS' at the beginning of the year and y the annual rate at which that distance increases.

No doubt three of these n equations would suffice for finding x, y, σ if D_1, D_2 and D_3 were absolutely accurate. But owing to the errors in D_1, D_2, D_3 it is found that the values of x, y, σ from any three of these equations will not precisely satisfy the remaining equations. The only possible course is to obtain such values for the three unknowns as shall give the most reasonable representation of the whole system. For this we must adopt the method of least squares, of which we shall now explain the principle.

We shall denote by $\phi(\Delta)\,d\Delta$ the probability that in making a measurement of some unknown quantity an error shall be committed which lies between Δ and $\Delta + d\Delta$. This important function $\phi(\Delta)$ is called the error function and its form is to be determined from the assumption that if $a_1, a_2 \ldots a_n$ are n measurements conducted under uniform conditions of such an unknown quantity as the arcual distance between two stars, then the arithmetic mean $(a_1 + a_2 + \ldots + a_n)/n$ is the most probable value of that quantity.

Let x be the value of the unknown, then the errors are $(a_1 - x)$, $(a_2 - x) \ldots (a_n - x)$ and the probabilities that each of these errors shall be separately committed are respectively $\phi(a_1 - x)$, $\phi(a_2 - x) \ldots \phi(a_n - x)$. It follows from the laws of probabilities that the probability that just these errors shall have been committed is the continued product of all the separate probabilities or

$$\phi(a_1 - x) \,.\, \phi(a_2 - x) \ldots \phi(a_n - x).$$

We may regard this function as the expression of the probability that x is the true value of the unknown. Therefore the value of x which makes this function a maximum is the most probable value of the unknown.

Equating the logarithmic differential of this expression to zero we have

$$(a_1 - x)\frac{\phi'(a_1 - x)}{(a_1 - x)\,\phi(a_1 - x)} + (a_2 - x)\frac{\phi'(a_2 - x)}{(a_2 - x)\,\phi(a_2 - x)} + \ldots$$
$$+ (a_n - x)\frac{\phi'(a_n - x)}{(a_n - x)\,\phi(a_n - x)} = 0.$$

But by our fundamental assumption this equation for x must not differ from

$$(a_1 - x) + (a_2 - x) + \dots + (a_n - x) = 0,$$

whence we obtain

$$\frac{\phi'(a_1 - x)}{(a_1 - x)\,\phi(a_1 - x)} = \frac{\phi'(a_2 - x)}{(a_2 - x)\,\phi(a_2 - x)} = \dots = -2h^2,$$

in which h^2 is a constant. We thus see that the error function $\phi(\Delta)$ must satisfy the condition

$$\frac{\phi'(\Delta)}{\Delta\phi(\Delta)} = -2h^2,$$

and consequently

$$\phi(\Delta) = Ae^{-h^2\Delta^2},$$

where A is a constant introduced by the integration.

As some error (zero of course included) must have been committed the sum of the probabilities for every error from $-\infty$ to $+\infty$ must be unity, whence

$$1 = \int_{-\infty}^{+\infty} \phi\Delta \, . \, d\Delta = A\int_{-\infty}^{+\infty} e^{-h^2\Delta^2} d\Delta = Ah^{-1}\int_{-\infty}^{+\infty} e^{-\Delta^2} d\Delta,$$

and we have to evaluate this definite integral.

Let a surface be formed by the extremities of perpendiculars of length e^{-r^2} erected at every point P in a plane where r is the distance of P from a fixed point O in the plane. Then the volume between the surface and the plane is

$$\int_0^\infty e^{-r^2} \, . \, 2\pi r \, . \, dr = \pi\int_0^\infty e^{-r^2} dr^2 = \pi.$$

But if rectangular axes x and y are drawn through O the volume of the surface can be shown to be equal to

$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} e^{-x^2-y^2} dx\,dy = \int_{-\infty}^{+\infty} e^{-x^2} dx \int_{-\infty}^{+\infty} e^{-y^2} dy = \pi,$$

whence

$$\int_{-\infty}^{+\infty} e^{-x^2} dx = \sqrt{\pi}.$$

Hence we see that $1 = Ah^{-1}\sqrt{\pi}$, and consequently we have for the error function

$$\phi(\Delta) = \frac{h}{\sqrt{\pi}} e^{-h^2\Delta^2}.$$

The Method of Least Squares. Let k be an observed quantity and p, q, r unknowns connected with k by the linear equation

$$k = Ap + Bq + Cr,$$

where A, B, C are known quantities depending on the circumstances of each particular observation. Then for a series of observations $k_1, k_2 \dots k_n$ we have a series of equations which may be written as follows

$$\left.\begin{array}{l} k_1 - A_1 p - B_1 q - C_1 r = \Delta_1 \\ k_2 - A_2 p - B_2 q - C_2 r = \Delta_2 \\ \dots\dots\dots\dots\dots\dots\dots\dots \\ k_n - A_n p - B_n q - C_n r = \Delta_n \end{array}\right\} \dots\dots\dots\dots\dots \text{(ii)}.$$

If our observations were perfect there should be values of p, q, r which would make $\Delta_1 = \Delta_2 = \dots = \Delta_n = 0$, but this is generally not true. The probability that all these errors $\Delta_1, \Delta_2, \dots$ shall have arisen is the product of the probabilities that each one should singly have arisen, *i.e.* the probability of the occurrence of just this system of errors is

$$\left(\frac{h}{\sqrt{\pi}}\right)^n e^{-h^2(\Delta_1^2 + \Delta_2^2 + \dots + \Delta_n^2)}.$$

The most probable values of p, q, r will therefore be those which make this quantity greatest, and therefore $\Delta_1^2 + \Delta_2^2 \dots + \Delta_n^2$ is to be a minimum. Thus we have the method of least squares which consists in determining p, q, r so that

$$(k_1 - A_1 p - B_1 q - C_1 r)^2 + (k_2 - A_2 p - B_2 q - C_2 r)^2 + \dots \\ + (k_n - A_n p - B_n q - C_n r)^2$$

shall be as small as possible.

The reasonableness of the method of least squares in such a problem can be perceived in a very elementary manner, as follows.

To satisfy as nearly as possible the group of equations found by making all the right-hand members of (ii) zero we should have values for p, q, r which shall on the whole make the actual residuals $\Delta_1, \Delta_2, \dots \Delta_n$ as small as possible, and symmetry suggests that some expression like

$$\Delta_1^m + \Delta_2^m + \dots \Delta_n^m$$

should be made a minimum. Plainly m must be an *even* integer for otherwise there would be no assurance that even if the sum was small the individual quantities would all be small. The simplest process would therefore be to make $m = 2$, which is the method of least squares.

Applying this principle in the present case to the determination of the parallax σ of a star from observations of its distance, we have to make the following quantity a minimum.

$$\{x + yt_1 - \sigma m \cos(\odot_1 - M) - D_1\}^2$$
$$+ \{x + yt_2 - \sigma m \cos(\odot_2 - M) - D_2\}^2$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$+ \{x + yt_n - \sigma m \cos(\odot_n - M) - D_n\}^2.$$

We equate to zero the differential coefficients of this expression with regard to x, y, σ taken as independent variables and thus we have as the fundamental equations by which x, y, σ are to be determined

$$nx + y\Sigma t' - \sigma m \Sigma \cos(\odot_1 - M) - \Sigma D_1 = 0,$$
$$x\Sigma t_1 + y\Sigma t_1^2 - \sigma m \Sigma t_1 \cos(\odot_1 - M) - \Sigma t_1 D_1 = 0,$$
$$x\Sigma \cos(\odot_1 - M) + y\Sigma t_1 \cos(\odot_1 - M) - \sigma m \Sigma \cos^2(\odot_1 - M)$$
$$- \Sigma D_1 \cos(\odot_1 - M) = 0,$$

in which the summations represented by Σ extend from 1 to n. Solving these linear equations for x, y, σ we determine not only σ the annual parallax but also x the mean distance of the two stars at the beginning of the year and y the annual rate at which their proper motions affect the distance.

The principle of the method of least squares is of the utmost importance in Astronomy, for there are so many problems in which we have to find the most probable solution of equations whose number exceeds that of the unknowns. (See Appendix to Vol. II. of Chauvenet's *Practical and Spherical Astronomy.*)

EXERCISES ON CHAPTER XV.

Ex. 1. The parallax of 61 Cygni is $0''\cdot37$ and its proper motion perpendicular to the line of sight is $5''\cdot2$ a year; compare its velocity in that direction with that of the earth in its orbit round the sun.

[Math. Trip. I.]

If n be the number of seconds in the annual proper motion of a star at the distance r miles, then in one year the star moves $rn \sin 1''$ miles. If the star's annual parallax is σ seconds, then $\sigma \sin 1'' = a/r$, where a is the sun's mean distance. Hence the star's annual movement is an/σ. The earth's annual movement is $2\pi a$ and hence the ratio of the star's velocity to the earth's velocity is $n/2\pi\sigma$. In the present case this reduces to 2·23.

Ex. 2. Prove that, if the sun have a proper motion in space towards a point in the heavens of right ascension A and of declination D, the rates of variation of the coordinates of a star, right ascension a, declination δ, annual parallax σ, contain terms of the form

$$\dot{a}=\frac{\sigma}{T}\frac{\cos D\sin(a-A)}{\cos\delta},\qquad \dot{\delta}=\frac{\sigma}{T}\frac{\cos D\cos(a-A)\sin(\delta-\phi)}{\cos\phi},$$

$$\dot{r}=-\frac{a}{T}\frac{\cos D\cos(a-A)\cos(\delta-\phi)}{\cos\phi},$$

where $\tan\phi\equiv\tan D\sec(A-a)$, a is the radius of the earth's orbit, T is the time of the sun's traversing the distance a, and $\dot{r}$ is the velocity of separation of the sun and star. [Math. Trip.]

The axes of x, y, z are drawn through an origin at the position of the sun at the time $t=0$ to the points whose R.A. and decl. are respectively (0°, 0°); (90°, 0°); (0°, 90°). The coordinates of the sun at the time t are

$$at\cos A\cos D/T,\quad at\sin A\cos D/T,\quad at\sin D/T.$$

If x', y', z' are the coordinates of the star with respect to the origin and $r\cos a\cos\delta$, $r\sin a\cos\delta$, $r\sin\delta$ the coordinates of the star with respect to parallel axes through the sun,

$$r\cos a\cos\delta=x'-at\cos A\cos D/T \quad\text{.....................(i)},$$

$$r\sin a\cos\delta=y'-at\sin A\cos D/T \quad\text{.....................(ii)},$$

$$r\sin\delta=z'-at\sin D/T \quad\text{.............................(iii)}.$$

Squaring and adding and observing that a is very small with reference to x', y', z',

$$r^2=x'^2+y'^2+z'^2-2a(x'\cos A\cos D+y'\sin A\cos D+z'\sin D)\,t/T,$$

whence by differentiating

$$r\dot{r}=-a(x'\cos A\cos D+y'\sin A\cos D+z'\sin D)/T,$$

or

$$\dot{r}=-a(\cos D\cos\delta\cos(A-a)+\sin D\sin\delta)/T$$

$$=-\frac{a}{T}\frac{\cos D\cos(a-A)\cos(\delta-\phi)}{\cos\phi}\quad\text{...........................(iv)}.$$

Dividing (ii) by (i), and reducing

$$x'^2\tan a=x'y'+aty'\cos A\cos D/T-atx'\sin A\cos D/T,$$

differentiating $\quad r^2\cos^2\delta\,.\,\dot{a}=a\cos D(y'\cos A-x'\sin A)/T$

$$=ar\cos\delta\cos D\sin(a-A)T,$$

whence

$$\dot{a}=\frac{\sigma}{T}\frac{\cos D\sin(a-A)}{\cos\delta}.$$

Finally, differentiating (iii),

$$\dot{r}\sin\delta+r\dot{\delta}\cos\delta=-a\sin D/T,$$

whence substituting for $\dot{r}$ from (iv), we obtain $\dot{\delta}$.

CHAPTER XVI.

ECLIPSES OF THE MOON.

115. An eclipse of the moon.

An eclipse of the moon is caused by the entrance of the moon into the shadow cast by the earth. The geometrical conditions under which this takes place will now be investigated.

Let M (Fig. 85) be the moon just arriving at the point T where it is in contact with PQ, one of the generators of the

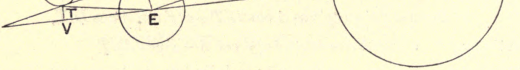


FIG. 85.

external common tangent cone to the earth whose centre is E and the sun whose centre is S. The moon is then on the point of entering the earth's shadow or *umbra* as it is called to distinguish it from the penumbra to be referred to later. A lunar eclipse is accordingly commencing.

We have first to calculate $\angle TEV$ or the angle subtended at the earth's centre by a radius of the circular section of the shadow cone made by a plane passing through T and perpendicular to ES. If EQ' be parallel to PQ we have

$$\angle Q'ES = (QS - PE)/ES = r_{\odot} - \pi_0,$$

where $r_{\odot}$ is the angular semi-diameter of the sun subtended at the earth's centre and π_0 is the horizontal parallax of the sun. The angle PTE may be taken to be π_0' the horizontal parallax of the moon with quite sufficient accuracy for our present purpose and hence we have

$$\angle TEV = \pi_0 + \pi_0' - r_{\odot}.$$

We thus prove the following statement.

The angular semi-diameter subtended at the earth's centre by the section of the earth's shadow at the distance of the moon equals the excess of the sum of the horizontal parallaxes of moon and sun over the angular semi-diameter of the sun.

As an example we may find the angular radius of the shadow on the occasion of the total eclipse of the moon which occurred on Feb. 8th, 1906, when $\pi_0 = 9''$, $\pi_0' = 58' \, 1''$, $r_{\odot} = 16' \, 13''$ and consequently $\angle TEV = 41' \, 57''$.

This is the radius of the shadow if the earth's atmosphere be not taken into account. But it is found that in consequence of the atmosphere the effective shadow has a radius about one-fiftieth part greater than that of the purely geometrical shadow in which the atmosphere is not considered. We must therefore add $50''$ and thus we find $42' \, 47''$ as the effective radius of the shadow.

The horizontal parallax of the moon, which we take from the ephemeris is, of course, the *equatorial* horizontal parallax and as, in the calculation of lunar eclipses, we shall regard the earth as a sphere, it would be rather more correct to employ a horizontal parallax corresponding to some mean latitude such as 45° rather than the parallax of an equatorial station. This reduces π_0' by about one five-hundredth part of its total amount. It is, however, quite futile to attend to such a refinement in the calculation because this correction, if introduced, would be much less than the margin of uncertainty unavoidably accompanying the correction introduced to allow for the influence of the atmosphere.

Let $\dot{\eta}$ be the fraction of an hour per hour at which the moon's right ascension is gaining on that of the centre of the shadow at the moment of opposition, *i.e.* when the two right ascensions coincide. Then at t hours after opposition the difference of the two right ascensions is $\dot{\eta}t$. Let δ and δ' be the declinations of the centre of the moon and the centre of the shadow at opposition and $\dot{\delta}$, $\dot{\delta}'$ the rate per hour at which δ and δ' change; then at the time t the declinations are $\delta + \dot{\delta}t$, $\delta' + \dot{\delta}'t$.

If D be the distance in seconds of arc between the centres of the moon and the shadow at the time t, then as the distance is small we have from § 8

$$D^2 = (\delta + \dot{\delta}t - \delta' - \dot{\delta}'t)^2 + 54000^2\dot{\eta}^2t^2\cos^2\tfrac{1}{2}(\delta + \delta'),$$

for in the last term we may, without appreciable error, take for the two declinations their values at opposition, and 54000 is the number of seconds of arc in one hour of right ascension. If we simplify this equation by making

$$A = (\dot{\delta} - \dot{\delta}')^2 + 54000^2\dot{\eta}^2\cos^2\tfrac{1}{2}(\delta + \delta'),\ B = (\delta - \delta')(\dot{\delta} - \dot{\delta}'),$$
$$C = (\delta - \delta')^2,$$

we have $$D^2 = At^2 + 2Bt + C \ldots\ldots\ldots\ldots\ldots\ldots\ldots (1).$$

This is the fundamental equation by which the various phases of an eclipse of the moon are to be investigated.

When the eclipse is commencing or ending the moon is just in outer contact with the shadow and D_1 the distance of the two centres must be found (Fig. 85) by increasing the apparent radius of the shadow by $r_☽$ the angular semi-diameter of the moon or

$$D_1 = (\pi_0 + \pi_0' - r_\odot)\,51/50 + r_☽ \ldots\ldots\ldots\ldots\ldots (3).$$

When the eclipse has become total the moon must be entirely immersed in the shadow and for the beginning and end of this phase we must have

$$D_2 = (\pi_0 + \pi_0' - r_\odot)\,51/50 - r_☽ \ldots\ldots\ldots\ldots\ldots (4).$$

Introducing first D_1 as the value of D into equation (1) we obtain a quadratic for t which will show if there is to be any eclipse and if so the two roots of t will give the moments at which the partial eclipse commences and ends.

The equation

$$At^2 + 2Bt + C - D_1 = 0$$

will have as its roots

$$-B/A \pm (B^2 - AC + AD_1)^{\frac{1}{2}}/A,$$

and if there is an eclipse these roots must be real. Whence

$$(\pi_0 + \pi_0' - r_\odot)\,51/50 + r_☽ > (AC - B^2)/A \ldots\ldots\ldots (5).$$

As A, B, C are all known quantities we can thus find the necessary and sufficient condition that there shall be at least a partial eclipse.

The duration of the eclipse is the difference of the two roots and is therefore

$$2(B^2 - AC + AD_1)^{\frac{1}{2}}/A.$$

By substituting D_2 for D_1 we find in like manner that if there is to be a *total* eclipse

$$(\pi_0 + \pi_0' - r_\odot)\,51/50 - r_{☽} > (AC - B^2)/A \;\;\ldots\ldots\ldots(6),$$

and that the duration of totality is

$$2(B^2 - AC + AD_2)^{\frac{1}{2}}/A.$$

At the middle of the eclipse the centres of the moon and the shadow are nearest to each other so that $At^2 + 2Bt + C$ is a minimum. This distance of the centres is $(AC - B^2)^{\frac{1}{2}}/A^{\frac{1}{2}}$ and it occurs at the time $t = -B/A$ as measured from the conjunction in right ascension.

The magnitude of a partial eclipse of the moon is usually measured by the fraction eclipsed of that diameter of the moon which points to the centre of the shadow at the moment when the distance between the centres is least. The radius of the umbra being $(\pi_0 + \pi_0' - r_\odot)\,51/50$ and $D = (AC - B^2)^{\frac{1}{2}}/A^{\frac{1}{2}}$ being the shortest distance of the centres the magnitude of the eclipse is easily seen to be

$$\{(\pi_0 + \pi_0' - r_\odot)\,51/50 + r_{☽} - D\}/2r_{☽}.$$

Ex. 1. Show that the distance of the vertex of the umbra from the centre of the earth is $s \operatorname{cosec} 1''/(r_\odot - \pi_0)$, where s is the earth's radius and where $r_\odot$ and π_0 are the apparent radius of the sun and its horizontal parallax both expressed in seconds of arc.

Ex. 2. Show that the parallax of the point where the moon comes in contact with the penumbra does not differ from the parallax of the moon's centre by so much as a third of a second of arc.

Ex. 3. Show that the duration of a lunar eclipse does not necessarily contain the instant of opposition in longitude if the eclipse be partial but must do so if the eclipse be total.

Ex. 4. The sum of the angular radii of the earth's shadow at the distance of the moon, and of the moon at the moment of conjunction in right ascension near a node is r. The square of the angular distance between the centre of the earth's shadow and the centre of the moon at a time t hours after conjunction is $At^2+2Bt+C$; where A, B, C involve the elements of the sun's and moon's positions at the moment of conjunction, and their hourly changes. The hourly change in the parallax of the moon is ϖ and its angular radius is ρ. Prove that there will be an eclipse if

$$\{A(C-r^2)-B\}^2 < \{C(\varpi+\rho)^2-2Br(\varpi+\rho)\}. \quad \text{[Coll. Exam.]}$$

Ex. 5. Assuming the earth, moon, and sun to be spherical, prove that when the moon is partially or totally eclipsed the geocentric angular distance of her centre from the axis of the earth's shadow must be less than

$$\sin^{-1}(\sin\pi_0+\sin D)-\sin^{-1}(\sin d-\sin\pi_0'),$$

where π_0, π_0' are the horizontal parallaxes, and D, d the semi-diameters of the sun and moon respectively. [Math. Trip. I. 1900.]

Ex. 6. Show that the interval between the middle of an eclipse of the moon, and the time of opposition is approximately

$$\frac{m\Delta}{m^2+n^2\cos\delta\cos\delta'} \text{ hours,}$$

where m and n are the differences of hourly motion of the moon and the centre of the earth's shadow in declination and right ascension respectively, Δ is the difference in declination of the moon and the centre of the earth's shadow at the time of opposition, and δ, δ' are the mean declinations of the shadow and the moon during the eclipse. [Coll. Exam.]

116. The penumbra.

Up to the present we have been considering only the case in which the moon enters the *umbra* or shadow of the earth. It remains to consider the conditions under which the moon enters the *penumbra* in which it is partially shaded from the sun or in which an observer on the moon would see a partial eclipse of the sun. As the moon enters the penumbra it must come into contact with the internal common tangent cone to the earth and the sun.

Let M (Fig. 86) be the moon which has just arrived at the point T on the internal common tangent PQ. When the moon

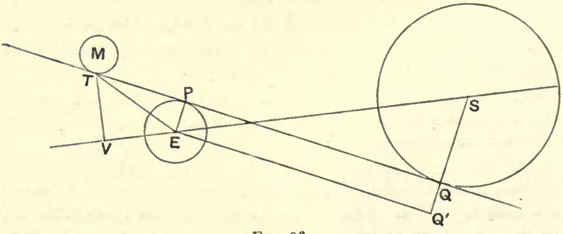


FIG. 86.

passes T it enters the penumbra. The line TV perpendicular to SE is the radius of the section of the penumbral cone at the distance of the moon. We require to find the angle which TV subtends at E.

If EQ' be parallel to PQ we have approximately

$$\angle TEV = \angle ETP + \angle SEQ' = EP/ET + QQ'/ES + SQ/ES$$
$$= \pi_0' + \pi_0 + r_\odot.$$

This proves the following statement:

The angle subtended at the earth's centre by the semi-diameter of the earth's penumbra at the distance of the moon is equal to the horizontal parallax of the moon + the horizontal parallax of the sun + the sun's angular semi-diameter.

We thus see from § 115 that the equation

$$\{(\pi_0 + \pi_0' + r_\odot)\, 51/50 \pm r_☽\}^2 = At^2 + 2Bt + C$$

when solved as a quadratic for t will give the times of first and last external contact of the moon with the penumbra when the upper sign is given to $r_☽$ and first and last internal contact when the lower sign is used.

117. The ecliptic limits.

Let x be the angular distance of the line joining the centres of earth and sun from the moon's node at the time the moon is crossing the ecliptic. Let $\dot\theta$, $\dot\phi$ be the angular velocities expressed in radians per hour of the sun and moon about the earth's centre and in the planes of their respective orbits, and let i be the inclination of the moon's orbit to the ecliptic. Let t be the time in hours from the moment of the passage of the moon's centre through its node. We may regard the triangle formed by the node and the centres of the moon and the shadow as a plane triangle and at the time t the distances of the centre of the shadow and the centre of the moon from the node are respectively $x + \dot\theta t$ and $\dot\phi t$. If then D be the distance between the centre of the moon and the centre of the shadow we have

$$D^2 = (x + \dot\theta t)^2 - 2\dot\phi t\,(x + \dot\theta t)\cos i + \dot\phi^2 t^2.$$

We can give this equation the form

$$D^2 = \frac{x^2 \dot\phi^2 \sin^2 i}{\dot\theta^2 - 2\dot\theta\dot\phi\cos i + \dot\phi^2}$$
$$+ (\dot\theta^2 - 2\dot\theta\dot\phi\cos i + \dot\phi^2)\left\{t + \frac{x(\dot\theta - \dot\phi\cos i)}{\dot\theta^2 - 2\dot\theta\dot\phi\cos i + \dot\phi^2}\right\}^2 \ldots(1).$$

As the second term may be zero but can never become negative the minimum value of D must be

$$x\dot{\phi}\sin i/(\dot{\theta}^2 - 2\dot{\theta}\dot{\phi}\cos i + \dot{\phi}^2)^{\frac{1}{2}},$$

so that if a particular phase of an eclipse is to occur at a given conjunction we must have x the distance of the centre of the shadow when the moon is passing through the node within the limit

$$x < D(\dot{\theta}^2 - 2\dot{\theta}\dot{\phi}\cos i + \dot{\phi}^2)^{\frac{1}{2}}/\dot{\phi}\sin i$$

where D is the distance between the centre of the moon and the centre of the shadow corresponding to the given phase.

To illustrate the numerical computation of the limit of x we shall take mean values as follows

$$\pi_0 = 9'',\ \pi_0' = 3422'',\ r_{\odot} = 961'',\ r_{☽} = 934'',\ \dot{\theta}/\dot{\phi} = 3/40,\ i = 5°9',$$

which gives $(\dot{\theta}^2 - 2\dot{\theta}\dot{\phi}\cos i + \dot{\phi}^2)^{\frac{1}{2}}/\dot{\phi}\sin i = 10{\cdot}3,$

and introducing the factor 51/50 to make the atmospheric correction as already explained, we have for the various values of D corresponding to different phases of the eclipse

$$(\pi_0 + \pi_0' + r_{\odot})\,51/50 + r_{☽} = 90'2,$$
$$(\pi_0 + \pi_0' + r_{\odot})\,51/50 - r_{☽} = 59{\cdot}1,$$
$$(\pi_0 + \pi_0' - r_{\odot})\,51/50 + r_{☽} = 57{\cdot}6,$$
$$(\pi_0 + \pi_0' - r_{\odot})\,50/51 - r_{☽} = 26{\cdot}5.$$

Applying to these quantities the factor 10·3 we learn that if when the moon is at one node the sun is 15°·5, 10°·2, 9°·9 or 4°·6 respectively from the other the moon will partially enter the penumbra, wholly enter the penumbra, partially enter the umbra or wholly enter the umbra.

Of course as these results are obtained only for *mean* values they must be accepted as only average results. The particular values of the several quantities given in the ephemeris should be employed when accuracy is required.

Ex. 1. Show that the maximum duration of totality of a lunar eclipse is

$$\frac{2(\pi_0 + \pi_0' - \odot - ☾)}{m \sec i}\left(1 + \frac{s\cos^2 i}{m}\right) \text{ hours}$$

approximately, if the atmospheric influence be neglected, and where π_0, π_0', $\odot$, ☾, s, m are the horizontal parallaxes, semi-diameters, and hourly motions in longitude of the sun and moon, respectively, and i is the inclination of the moon's orbit to the ecliptic. [Math. Trip. I.]

Ex. 2. Show that an eclipse of the moon will occur, provided that at full moon the sun is within nine days of the moon's node.

[Coll. Exam. 1905.]

Ex. 3. If the distance of the moon from the centre of the earth is taken to be 60 times the earth's radius, the angular diameter of the sun to be half a degree, and the synodic period of the sun and moon to be 30 days, show that the greatest time which can be occupied by the centre of the moon in passing through the umbra of the earth's shadow is about 3 hours.

[Coll. Exam.]

Ex. 4. Determine the greatest latitude that the moon can have at the instant of opposition in longitude that a total lunar eclipse may be possible, having given the moon's parallax 61′ 32″, the moon's semi-diameter 16′ 46″, the sun's parallax 9″, the sun's semi-diameter 15′ 45″, and the inclination of the moon's orbit to the ecliptic 5° 52′. [Coll. Exam.]

Ex. 5. Given that on 1894 September 21 the altitude of the moon was greater than at any other time during the last nineteen years, show that an eclipse of the moon must have taken place on 1895 March 10. About what hour did the moon culminate at London on 1894 September 21?

(The length of the synodic month is $29\frac{1}{2}$ days, the lunar parallax may be taken as 1°, the inclination of the orbits of the sun and moon as 5° and the semi-diameters of each as 30′.) [Coll. Exam.]

118. Point on the moon where the eclipse commences.

It remains to find the point on the moon's limb which first begins to be eclipsed.

Let M, V, Fig. 87, be the centres of the moon and the shadow respectively when the first external contact takes place at T. Let P be the pole, then MP crosses the moon's limb at N, the most northerly point on the moon's disc. We require to find the angle

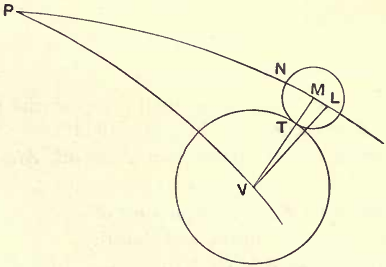


Fig. 87.

NMT, *i.e.* the angle measured anti-clockwise round the moon's limb from the most northerly point to the point of contact. Draw *VL* perpendicular to *PM*. We may with sufficient accuracy regard *VML* as a plane triangle and $ML = PV - PM = \delta - \delta'$ where δ' and δ are the declinations of *V* and *M* respectively, which are known because, as we have already seen, the time of the first contact is known. We have therefore

$$\cos NMT = (\delta' - \delta)/VM,$$

whence $\angle NMT$ is determined. In like manner the point on the moon's limb at which the obscuration finally departs is also determined.

If r be the distance between the centres of the moon and the shadow and if R be the radius of the shadow and $r_☽$ that of the moon, then $R + r_☽ - r$ is the greatest portion in shade of any lunar diameter and the ratio of this to the diameter or $(R + r_☽ - r)/2r_☽$ is said to be the magnitude of the eclipse.

Ex. In a partial lunar eclipse the first contact with the shadow occurs at an angle α from the most northerly point of the moon's limb toward the east, and the last contact at an angle β towards the west. Prove that the proportion of the moon's diameter eclipsed is

$$\tfrac{1}{2}(1 + s/m)\{1 \mp \cos \tfrac{1}{2}(\alpha + \beta)\},$$

where s and m are the semi-diameters of the shadow and moon respectively, the upper sign being taken when the moon's centre passes to the north of that of the shadow and the lower sign when it passes to the south.

Let P be the pole, T_1, T_2 the first and last points of contact of the moon with the shadow of which the centre is V. Then since PT_1 and PT_2 are inclined at only a small angle and T_1T_2 is small we have $\angle T_1VT_2 = \tfrac{1}{2}(\alpha + \beta)$ or $180° - \tfrac{1}{2}(\alpha + \beta)$, whence the shortest distance of the centres of the moon and shadow is $\pm(m + s)\cos\tfrac{1}{2}(\alpha + \beta)$. The greatest part of a diameter in shadow is therefore

$$(m + s)\{1 \mp \cos \tfrac{1}{2}(\alpha + \beta)\},$$

and the ratio of this to $2m$ is the quantity required.

119. Calculation of an eclipse.

To illustrate the formulae we shall compute the total eclipse of the moon which occurred on Feb. 8th, 1906.

The following are the data (see *Nautical Almanac*, 1906, p. 483):

The epoch or Greenwich mean time of conjunction of moon and centre of shadow in R.A. is ... ... $19^h\ 49^m\ 59^s$

Right ascension of moon at epoch $= \alpha$	9^h	28^m	22^s
Moon's declination at epoch $= \delta$ ...	14°	48′	16″
Declination of centre of shadow at epoch $= \delta'$	14	55	24
Hourly motion of moon in R.A. $= \dot{\alpha}$ ...		34	28
Hourly motion of shadow in R.A. $= \dot{\alpha}'$...		2	29
Hourly motion of moon in decl. $= \dot{\delta}$ ...		−7	42
Hourly motion of shadow in decl. $= \dot{\delta}'$...			48
Moon's equatorial horizontal parallax		58	1
Sun's equatorial horizontal parallax ...			9
Moon's angular semi-diameter $= r_☽$ ...		15	47
Sun's angular semi-diameter $= r_\odot$ ...		16	13

Substituting these values in the expressions for A, B, C already given (p. 348) we obtain

$$D^2 = 183000 + 354000t + 3610000t^2,$$

where D is the distance in seconds of arc between the centre of the moon and the centre of the shadow, where t is the time in hours since the epoch, and where not more than three significant figures are retained.

Solving this equation for t we have

$$t = -\cdot 0491 \pm \sqrt{(D/1900)^2 - (\cdot 2198)^2}.$$

If we make $\cos\theta = 418/D$, this becomes

$$t = -\cdot 0491 \pm \cdot 2198 \tan\theta,$$

and we find that the centres of the moon and the shadow are at the distance D at the Greenwich mean times

$$19^h\ 47^m\!\cdot\!1 \pm 13^m\!\cdot\!2 \tan\theta.$$

The shortest distance D is 418″, for θ would otherwise be imaginary, and the corresponding time, *i.e.* the middle of the eclipse, is $19^h\ 47^m\!\cdot\!1$.

To find the first and last contacts with the penumbra we make

$$D = (p_☽ + p_\odot + r_\odot)\ 51/50 + r_☽ = 5499,$$

$$\cos\theta = 418/5499 = \cdot 0760 \text{ and } \tan\theta = 13\cdot 1,$$

and accordingly the required times are

$$19^h\ 47^m \pm 2^h\ 53^m = 16^h\ 54^m \text{ and } 20^h\ 40^m.$$

For the first and last contacts with the shadow

$$D = (p_{☽} + p_{\odot} - r_{\odot})\ 51/50 + r_{☽} = 3510,$$

$$\cos\theta = 418/3514 = \cdot 119, \quad \tan\theta = 8\cdot 35,$$

and accordingly the required times are

$$19^{h}\ 47^{m}\cdot 1 \pm 1^{h}\ 50^{m}\cdot 2 = 17^{h}\ 56^{m}\cdot 9 \text{ and } 21^{h}\ 37^{m}\cdot 3.$$

For the first and last moments of internal contact with the shadow

$$D = (p_{☽} + p_{\odot} - r_{\odot})\ 51/50 - r_{☽} = 1620,$$

$$\cos\theta = 418/1620 = \cdot 258, \quad \tan\theta = 3\cdot 75,$$

and accordingly the required times are

$$19^{h}\ 47^{m}\cdot 1 \pm 49^{m}\cdot 4 = 18^{h}\ 57^{m}\cdot 7 \text{ and } 20^{h}\ 36^{m}\cdot 5.$$

To find the point on the moon's limb at which first contact with the shadow takes place we have to find the declinations of the moon and the shadow at $17^{h}\ 57^{m}\cdot 0$. This is $1^{h}\ 53^{m}$ before the epoch, but the moon is moving southwards in declination at the rate of 7′ 42″ per hour. Hence at the time of first contact the declination of the moon must have been 14′·6 greater than at the epoch, and therefore it was 15° 2′·9. In this time the sun would move 1′·5 north and the shadow therefore 1′·5 south. Hence the declination of the centre of the shadow at the time of first contact must have been = 14° 56′·9. Hence from p. 354 we have

$$\cos NMT = -360/3514 = -0\cdot 102,$$

and the point of first contact is 96° from the north point of the moon towards the east.

To find the terrestrial station from which the eclipse can be best seen we determine the latitude and longitude of the place on the earth which will lie directly between the centres of the earth and moon at the middle of the eclipse.

The middle of the eclipse is at G.M.T. $19^{h}\ 47^{m}\cdot 1$ and therefore $2^{m}\cdot 9$ before the conjunction in R.A. with the centre of the shadow. In $2^{m}\cdot 9$ the moon moved $1^{m}\cdot 7$ in R.A. and − 0′·4 in declination, and therefore the coordinates of the moon at the middle of the eclipse were as follows:

$$\text{R.A.} = 9^{h}\ 28^{m}\cdot 3 - 1^{m}\cdot 7 = 9^{h}\ 26^{m}\cdot 6,$$

and

$$\text{decl.} = 14°\ 48'\cdot 2 + 0'\cdot 4 = 14°\ 48'\cdot 6.$$

The line joining the centre of the earth to the centre of the moon will pierce the earth's surface at the point with geocentric latitude $14^\circ\, 48'\cdot 6$. To find the corresponding true latitude we must add to this the angle of the vertical (§ 15), which is in this case about $5'$. Hence we obtain $14^\circ\, 54'$ as the true latitude of the station from which the eclipse can be best seen.

To find the longitude of the station we learn from the ephemeris that on Feb. 8th the sidereal time at mean noon was $21^h\, 10^m\cdot 7$. The mean time interval of $19^h\, 47^m\cdot 1$ between Greenwich noon and the middle of the eclipse is $19^h\, 50^m\cdot 3$ of sidereal time. Hence the Greenwich sidereal time of the middle of the eclipse is

$$21^h\, 10^m\cdot 7 + 19^h\, 50^m\cdot 3 = 17^h\, 1^m\cdot 0,$$

for of course we can omit 24^h. The right ascension of the moon is to be the sidereal time at the station in question, *i.e.* $9^h\, 26^m\cdot 6$. Hence the west longitude of the station must be

$$17^h\, 1^m\cdot 0 - 9^h\, 26^m\cdot 6 = 7^h\, 34^m\cdot 4,$$

or in arc $= 113^\circ\cdot 6$.

The magnitude of the eclipse is

$$\{(p_{☽} + p_{\odot} - r_{\odot})\, 51/50 + r_{☽} - D\}/2r_{☽},$$

where D in this case is to have its smallest value 418. Substituting for the other quantities as before we obtain 1·64 as the magnitude of the eclipse.

Ex. Show from the following data that the eclipse of the moon on 1898, July 3rd, was only partial:

Moon's latitude at opposition in longitude	...	− 30′	40″
„ hourly motion in latitude	... ...	+ 3	40
„ „ „ longitude	... ...	38	2
Sun's „ „ „	... ...	2	22
Moon's equatorial horizontal parallax	... ...	61	21·4
Sun's „ „ „	... ...		8·7
Moon's true semi-diameter	... ...	16	43
Sun's „ „	... ...	15	44

[Coll. Exam.]

The hourly motion of moon with respect to the axis of the earth's shadow is $35'\,40'' = 2140''$ in longitude and $220''$ in latitude. Hence the least distance of the moon from the axis of the earth's shadow is very nearly

$$(30'\,40'') \times \frac{214}{\sqrt{214^2 + 22^2}} = (30'\,40'') \times 0\cdot 995 = 30'\,31'',$$

which lies between

$$61'\,21''\cdot 4 + 8''\cdot 7 - 16'\,43'' - 15'\,44''$$

and

$$61'\,21''\cdot 4 + 8''\cdot 7 + 16'\,43'' - 15'\,44''.$$

Hence there was an eclipse but only a partial one, p. 352.

CHAPTER XVII.

ECLIPSES OF THE SUN.

120. Introductory.

If the orbit of the moon were in the plane of the ecliptic there would be an eclipse of the sun at every new moon. As however the orbit of the moon is inclined to the ecliptic at an angle of about five degrees, it is plain that at the time of new moon the moon will generally be too much above or below the sun to make an eclipse possible. But when the moon is near a node of its orbit about the time of new moon, then an eclipse of the sun may be expected.

We have already mentioned in § 58 that ☊, the moon's ascending node, moves backwards along the ecliptic under the influence of nutation. In about $18\frac{1}{2}$ years, or more accurately 6798·3 days, ☊ makes a complete circuit of the ecliptic, and on account of this movement the sun, in its apparent motion, passes through the ascending node of the moon's orbit at intervals of 346·62 days. We thus find that 19 complete revolutions of the sun with respect to ☊ are performed in 6585·8 days. The *lunation* or average interval between two successive new moons is 29·5306 days, so that 223 lunations amount to 6585·3 days. The close approximation in the duration of 223 lunations and 19 revolutions of the sun with respect to ☊ is not a little remarkable. They each differ from a period of 18 years and 11 days by no more than half a day. This curious period, known as the Saros, is of much significance in connection with solar eclipses.

Suppose that at a certain epoch the moon is new when the sun is at ☊, and an eclipse of the sun therefore takes place. After the lapse of a Saros the sun has performed just 19 revolutions with regard to ☊ and therefore the sun is again at ☊. But we also find that the moon is again new because an integral number of lunations (223) are contained in the Saros, and consequently the conditions under which an eclipse is produced will have recurred. Of course the same would be true with regard to the moon's descending node.

The Saros is related to lunar eclipses also. We have seen in Chapter XVI. that there is an eclipse of the moon when at the time of full moon the sun is sufficiently near one of the moon's nodes. Thus we perceive that an eclipse of the moon will, after the lapse of a Saros, be generally followed by another eclipse of the moon, so that every eclipse of either kind will generally be followed by another eclipse of the same kind after an interval of *about* 18 years and 11 days.

For instance there were eclipses in 1890 on June 16 (sun), Nov. 25 (moon), and Dec. 11 (sun), and accordingly in 1908 there are eclipses on June 28 (sun), Dec. 7 (moon), and Dec. 22 (sun).

As another numerical fact connected with the motion of the moon it should be noted that 235 lunations make 6939·69 days while 19 years of 365·25 days amount to 6939·75 days. Thus we have the cycle of Meton consisting of 19 years, which is nearly identical with 235 lunations.

Hence we may generally affirm that 19 years after one new moon we shall have another new moon, *e.g.* 1890 July 17 and 1909 July 17.

When an eclipse of the sun is on the point of commencing or ending, the circular disc of the moon as projected on the celestial sphere from the position of the observer is in external contact with the projected disc of the sun. It is evident that at this moment a plane through the position of the observer and the apparent point of contact, but which does not cut either of the discs, must be a common tangent plane to the spherical surfaces of the sun and moon. It is also obvious that the line joining the actual points of contact of this tangent plane with the two spheres must pass through the position of the observer, for if it did not do so the two bodies would not appear to him to be in contact. The geometrical conditions involved in solar eclipses are therefore analogous to those which we have already had to consider in Chapter XIV. when discussing the transit of Venus.

When the partial phase of a solar eclipse is about to commence or about to end the observer must therefore occupy a position on the surface of that common tangent cone to the sun and moon which has its vertex between the two bodies, as in the parallel case of the lunar eclipse, § 115. This cone is known as the *penumbra*, the other common tangent cone to the sun and moon in which the vertex and the sun are on opposite sides of the moon being termed the *umbra*. The observer who sees the beginning or end of a total eclipse, or an annular eclipse, must be situated on the umbra. In the former the moon will completely hide the disc of the sun. In the latter a margin of the brilliant disc of the sun is visible round the dark circular form of the moon.

121. On the angle subtended at the centre of the earth by the centres of the sun and the moon at the commencement of a solar eclipse.

Let the external common tangent TQ (Fig. 88) to the sun S and moon MQ be supposed to advance till it just touches the earth

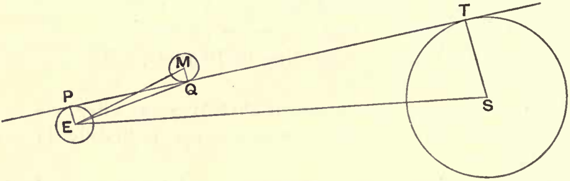


FIG. 88.

E, at P, and let s, l, ρ be the radii of the sun, moon and earth respectively, and $ES = r$, $EM = r'$, $\angle PEM = \theta$ and $\angle MES = x$, then we have from the figure

$$r \cos(\theta + x) + s = \rho \text{ ....(1)},$$

$$r' \cos\theta = \rho + l \text{ ....(2)}.$$

Subtracting (1) divided by r from (2) divided by r' we have

$$2 \sin \tfrac{1}{2} x \sin (\theta + \tfrac{1}{2} x) = \rho/r' + l/r' - \rho/r + s/r,$$

but equation (2) shows that $\cos \theta$ is very small, or that θ is nearly 90°, and hence as x is small we have very nearly

$$x = \pi_0' - \pi_0 + r_{☽} + r_{\odot}.$$

The quantities in this expression are of course variable and for their values from day to day reference must be made to the ephemeris.

122. Elementary theory of solar eclipses.

Let S_1, M_1 (Fig. 89) be the centres of the sun and moon as they would appear if the two bodies could be seen from the centre of the earth about the time of a solar eclipse.

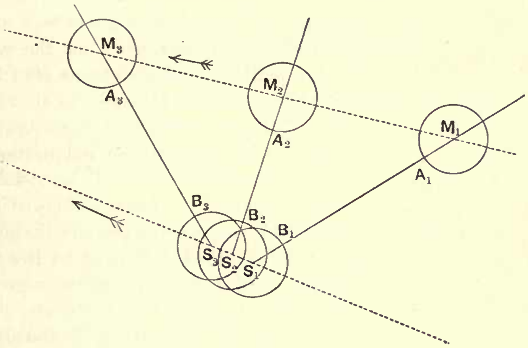


Fig. 89.

Let S_2 and M_2 and S_3 and M_3 represent the centres of the sun and moon as similarly presented at two later stages.

To an observer in the position we have supposed and under the circumstances represented in the figure the moon would evidently pass quite clear of the sun and there would be no eclipse. But the circumstances as witnessed from a point on the earth's surface will generally be different from those here represented. We may suppose that the effect of the sun's parallax (8″·80) is insensible, *i.e.* that the apparent place of the sun on the celestial sphere is, so far as solar eclipses are concerned, practically the same when viewed from any point on the earth's surface as when viewed from the earth's centre. The parallax of the moon (3422″) being nearly 389 times that of the sun causes the apparent place of the moon to be shifted to an extent which may be nearly double the lunar diameter. Thus, although as viewed from the earth's centre the moon may pass clear of the sun, yet as viewed from a point on the earth's surface parallax may interpose the moon, wholly or partially, between the observer and the sun, and thus produce a solar eclipse.

We have already seen (Chapter XII.) that the effect of parallax is to depress the moon from the zenith of the observer towards the horizon, and that the amount of this depression is proportional to the sine of the zenith distance.

We can now consider whether at or about a given conjunction in longitude of sun and moon, *i.e.* at or about a given new moon, there will be an eclipse visible from any place on the earth. If this is to be the case the parallax of the moon as seen from such a place must project the moon towards the sun so that their limbs overlap. Suppose that S_2M_2 is the shortest distance between the centres of sun and moon at the conjunction in question as seen from the earth's centre. Then an eclipse will be visible at any place if, but only if, the parallax of the moon, as viewed from that place, appears to thrust the moon towards the sun through a distance exceeding A_2B_2. It follows that A_2B_2 must be less than the moon's horizontal parallax. If A_2B_2 be equal to or greater than the horizontal parallax, then there will be no eclipse.

The critical point on the earth's surface from which the moon's limb if visible would just appear to graze the sun is determined as follows. The moon is depressed by parallax along the great circle $M_2A_2B_2S_2$, but parallax at any place always depresses the moon from the zenith of that place. It therefore follows that under the circumstances supposed this zenith must also lie on the continuation of this great circle $M_2A_2B_2S_2$. As the lower limb of the moon appears to be on the horizon when its parallax is greatest (we need not here consider any question of atmospheric refraction), it follows that the zenith of the place must be distant from S_2 by $90° +$ the apparent semi-diameter of the sun. Thus the point on the celestial sphere, which is the zenith of the place of observation, is determined and the time is also known because it is that at which the true angular distance of the centres of sun and moon is a minimum. But the declination of the zenith is the latitude of the place and the right ascension of the zenith minus the Greenwich sidereal time is the longitude of the place. In this way we indicate geometrically the latitude and the longitude of the terrestrial station at which the eclipse is just a grazing contact, while at no other station is there any eclipse whatever.

If the eclipse be larger than the limiting case just considered, then the track of the moon $M_1M_2M_3$ must pass nearer to the sun. If A_1B_1 (Fig. 89) be equal to the horizontal parallax of the moon, then just as before a point may be found on the continuation of the great circle S_1M_1 which is the zenith of the terrestrial station, from which the two limbs are just brought into contact by parallax. This is the point on the earth's surface at which the phase of partial eclipse is just commencing as a graze of the limbs of sun and moon, and in like manner a zenith is determined along S_3M_3 where the eclipse ends in a similar manner.

It is easy to see how other eclipse problems can be illustrated in this manner. Suppose, for example, it be required to find the terrestrial station at which the central phase of a total eclipse shall take place when the sun has the greatest possible altitude.

We plot, as before, a series of corresponding geocentric positions S_1, S_2, S_3 of the sun, and M_1, M_2, M_3 of the moon about the time of conjunction where S_2M_2 is the minimum geocentric distance of the two bodies. As the sun and moon are to have the greatest altitude possible at the phase in question it is plain that the moon must have the least possible parallax which will suffice to make its centre appear to coincide with the centre of the sun. This shows that the zenith of the place required must lie on the continuation of the great circle S_2M_2, and the position of the zenith Z is defined by observing that the parallax is exactly S_2M_2, so that

$$\sin ZS_2 = \sin S_2M_2 \div \sin \pi_0',$$

where $\sin \pi_0'$ is the sine of the moon's horizontal parallax. Thus Z is found, and the time being known, the required station on the earth's surface becomes determined.

The line of central eclipse on the earth can also be constructed. For if on the great circle joining S_1, M_1 a pair of corresponding positions of sun and moon, a point Z_1 be chosen so that

$$\sin Z_1S_1 = \sin S_1M_1 \div \sin \pi_0',$$

then Z_1 will be the zenith of the place from which, when the sun and moon are at S_1 and M_1 respectively, the eclipse will appear central. By taking other pairs of corresponding positions any number of points on the central line, and thus the terrestrial line of central eclipse can be constructed.

123. Closest approach of sun and moon near a node.

Let β be the latitude of the moon M at the time of a new moon which is supposed to occur when the moon is in the vicinity of its node N. Let S be the position of the sun at the same moment (Fig. 90).

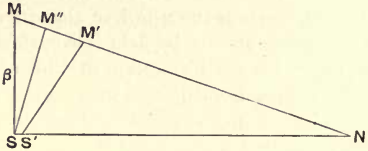


FIG. 90.

Let M', S' be the positions to which the moon and sun have advanced a little later. Let $MM' = x$, then $SS' = mx \cos I$, where m is the ratio of the sun's apparent velocity in longitude to the moon's apparent velocity in longitude, and I is the inclination of the moon's orbit to the ecliptic.

It will be approximately correct to regard the triangle MNS as a plane triangle, and hence if D denote $M'S'$

$$D^2 = (\beta \cot I - mx \cos I)^2 + (\beta \operatorname{cosec} I - x)^2$$
$$- 2 \cos I (\beta \cot I - mx \cos I)(\beta \operatorname{cosec} I - x),$$

which may be written in the form

$$D^2 = (1 - 2m \cos^2 I + m^2 \cos^2 I)\left\{x - \frac{\beta \sin I}{1 - 2m \cos^2 I + m^2 \cos^2 I}\right\}^2$$
$$+ \beta^2 \frac{(1-m)^2 \cos^2 I}{1 - 2m \cos^2 I + m^2 \cos^2 I}.$$

We observe that the first part of the expression for D^2 can never be negative, and therefore to obtain the smallest value of D' we make

$$x = \frac{\beta \sin I}{1 - 2m \cos^2 I + m^2 \cos^2 I},$$

for then the first term of M^2 vanishes. If therefore β_0 represent the smallest distance of sun and moon at that conjunction we have

$$\beta_0 = \frac{\beta(1-m)\cos I}{(1-2m\cos^2 I + m^2\cos^2 I)^{\frac{1}{2}}},$$

and assuming an angle I' defined by the equation

$$\tan I' = \tan I/(1-m)$$

we see that

$$\beta_0 = \beta \cos I'.$$

If we substitute 3/40 for m and 1/11·1 for $\sin I$ we have approximately

$$\beta_0 = \beta \cos I\,(1 - 0\cdot0006),$$

the difference between β_0 and $\beta\cos I$ is thus shown to be so small that it will be quite accurate enough in the calculation of eclipses to assume that the latter, *i.e.* the perpendicular from S on NM, is the shortest distance between the geocentric positions of the sun and moon at the given conjunction.

If an eclipse is to take place then (§ 121) β_0 must not exceed

$$\pi_0' - \pi_0 + r_{☽} + r_{\odot}.$$

Hence
$$\beta < (\pi_0' - \pi_0 + r_{☽} + r_{\odot}) \sec I$$
$$< (\pi_0' - \pi_0 + r_{☽} + r_{\odot})(1 + \tfrac{1}{2}\sin^2 I).$$

The mean value of $\pi_0' - \pi_0 + r_{☽} + r_{\odot}$ is 1° 28′·6 and this mean value may be used in the part of the expression multiplied by $\frac{1}{2}\sin^2 I\,(=1/246)$ which part thus becomes 0′·4. As π_0 may always be taken for this purpose to be 0′·1 we see that when β is the geocentric latitude of the moon at conjunction, then for a solar eclipse to be visible from some part of the earth's surface about the time of this conjunction it is necessary that

$$\beta \not> \pi_0' + r_{☽} + r_{\odot} + 0'{\cdot}3.$$

The greatest values of π_0', $r_{☽}$ and $r_{\odot}$ are respectively 61′·5, 16′·8, 16′·3. The sum of these quantities increased by 0′·3 is 1° 34′·9. If therefore the North or South geocentric latitude of the moon at the time of new moon exceeds 1° 34′·9 there can be no solar eclipse at that conjunction.

The expression *superior ecliptic limit* denotes the greatest possible distance of the sun from a node at the time of new moon if an eclipse is to take place. If x be the distance then

$$\sin x = \tan\beta \cot I \text{.......................(i)},$$

and the greatest value of x will be found when β has its largest value $1^\circ\,34'\cdot9$ and I its least value $4^\circ\,58'\cdot8$. We thus find $18^\circ\cdot5$ as the superior ecliptic limit.

The *inferior ecliptic limit* is found by taking the lowest possible values of π_0', $r_☽$ and $r_\odot$ namely $53'\cdot9$, $14'\cdot7$, and $15'\cdot8$ respectively. If the geocentric latitude of the moon at conjunction be less than $53'\cdot9 + 14'\cdot7 + 15'\cdot8 + 0'\cdot3 = 1^\circ\,24'\cdot7$ then an eclipse of the sun from some terrestrial stations must take place about the time of that conjunction. The maximum value of the inclination of the moon's orbit to the ecliptic is $5^\circ\,18'\cdot6$. If the values $1^\circ\,24'\cdot7$ and $5^\circ\,18'\cdot6$ be substituted for β and I in the formula (i) we find $x = 15^\circ\cdot3$. Thus we see that whenever at the time of new moon the sun's longitude is within $15^\circ\cdot3$ of the node then an eclipse of the sun must take place at that conjunction. The inferior ecliptic limit is therefore $15^\circ\cdot3$.

Finally we see that if $\beta < 1^\circ\,24'\cdot7$ then an eclipse must happen. If $\beta > 1^\circ\,34'\cdot9$ then there cannot be an eclipse. If

$$1^\circ\,24'\cdot7 < \beta < 1^\circ\,34'\cdot9$$

then an eclipse may happen or it may not. To decide the question we must calculate $\pi_0' + r_☽ + r_\odot + 0'\cdot3$ and there will be an eclipse or not according as β is less or greater than the quantity so found.

Ex. 1. If in Fig. 90, SM'' is the perpendicular from S on MN and $S'M'$ the shortest distance between the centres of the sun and moon, show that approximately

$$M''M' = 2m\beta \sin I.$$

Ex. 2. If the inferior ecliptic limits are $\pm\epsilon$ and if the satellite revolves n times as fast as the sun, and its node regredes θ every revolution the satellite makes round its primary, prove that there cannot be fewer consecutive solar eclipses at one node than the integer next less than

$$\frac{2(n-1)\epsilon}{n\theta + 2\pi}.$$

[Math. Trip.]

Let λ be the diurnal movement of the sun in longitude, then $n\lambda$ is that of the moon and $-n\lambda\theta/2\pi$ that of the node. The duration of a lunation is $2\pi/(n-1)\lambda$ and the time taken by the sun to pass from the distance ϵ on one side of the node to a distance ϵ on the other is $2\epsilon/\{\lambda + n\lambda\theta/2\pi\}$, and the number of entire lunations contained in this gives the required answer.

Ex. 3. At a certain conjunction of the sun and moon, the moon just grazes the sun, but there is no sensible partial eclipse at any point of the earth's surface. Prove that

$$(\pi_0' - \pi_0 + r_☽ + r_\odot)^2 = \frac{(\delta_m - \delta_s)^2\,(\dot{a}_m - \dot{a}_s)^2 \cos\delta_m \cos\delta_s}{(\dot{\delta}_m - \dot{\delta}_s)^2 + (\dot{a}_m - \dot{a}_s)^2 \cos\delta_m \cos\delta_s},$$

where $r_{\odot}$ and r_{D} are the angular radii of the sun and moon, π_0 and π_0' their parallaxes, δ_s and δ_m their declinations at the instant of conjunction in R.A., and $\dot{a}_s$, $\dot{a}_m$, $\dot{\delta}_s$, $\dot{\delta}_m$ their hourly motions in R.A. and declination.

[Coll. Exam. 1904.]

Find the minimum distance between the centres which at the time t is approximately the square root of

$$\{\delta_m - \delta_s + t(\dot{\delta}_m - \dot{\delta}_s)\}^2 + t^2(\dot{a}_m - \dot{a}_s)^2 \cos\delta_m \cos\delta_s.$$

Ex. 4. Prove that there are more solar eclipses than lunar eclipses on the average, but that the moon's face is dimmed by the penumbra, though not necessarily eclipsed, rather more frequently than the sun is eclipsed.

[Coll. Exam.]

Ex. 5. Prove that at *a given terrestrial station* lunar eclipses will be more frequent than solar.

Ex. 6. The horizontal parallaxes and semi-diameters of the sun and moon being known, find the maximum inclination of the moon's orbit to the ecliptic which would ensure a solar eclipse every month.

[Coll. Exam.]

124. Calculation of the Besselian elements for a partial eclipse of the sun.

The following method of computing the circumstances of an eclipse of the sun at a given terrestrial station is that now generally employed. It is due to Bessel*.

Through the centre of the earth a line is supposed to be drawn parallel to the line joining the centres of the sun and moon at any moment. We shall regard this as the axis of z and the plane normal thereto through the earth's centre is known as the fundamental plane. The positive side of the plane of z is that on which the sun and moon are situated.

The plane of x is that which contains the axis of the earth and the axis of z. The positive side of x is that which contains the point of the equator which the earth's rotation is carrying from the positive side of z to the negative side. This criterion can never become ambiguous because the plane of z can never coincide with the equator.

The plane of y is perpendicular to the planes of x and z and the positive side of y is that which contains the earth's north pole. This also cannot become ambiguous.

Let a, d be the right ascension and declination of the celestial

* See also Chauvenet's *Practical and Spherical Astronomy*.

point D which is pointed at by the positive direction of the axis of z. Then the R.A. and declination of the points on the celestial sphere pointed to by $+x$, $+y$, $+z$ respectively are $90°+a$, 0; $180°+a$, $90°-d$; a, d. We hence obtain the cosines of the angles between the point α, δ and the three points just given by the formula (i), p. 28, and thus derive the expressions

$$\left.\begin{aligned} x &= \Delta \cos\delta \sin(\alpha - a) \\ y &= \Delta\{\sin\delta \cos d - \cos\delta \sin d \cos(\alpha - a)\} \\ z &= \Delta\{\sin\delta \sin d + \cos\delta \cos d \cos(\alpha - a)\} \end{aligned}\right\} \text{.........(i)},$$

where x, y, z are the coordinates with respect to the fundamental axes of a body in the direction α, δ and at the distance Δ.

Let α_1, δ_1 and α_2, δ_2 be the R.A. and decl. of the centres of the sun and the moon respectively, then as the line joining these points is parallel to z we must have the x coordinates of the sun and moon equal and the y coordinates must also be equal, whence

$$\Delta_1 \cos\delta_1 \sin(\alpha_1 - a) - \Delta_2 \cos\delta_2 \sin(\alpha_2 - a) = 0,$$

$$\Delta_1 \sin\delta_1 \cos d - \Delta_1 \cos\delta_1 \sin d \cos(\alpha_1 - a)$$
$$- \Delta_2 \sin\delta_2 \cos d + \Delta_2 \cos\delta_2 \sin d \cos(\alpha_2 - a) = 0.$$

From the first of these $\tan a$ is found. This gives two values for a of which one exceeds the other by 180°. As, however, the value of a must be very nearly the right ascension of the sun there can be no doubt as to which value of a is to be chosen. This substituted in the second equation gives $\tan d$, and here also there can be no ambiguity as to which of the two values of d differing by 180° should be chosen, for d being a declination must lie between $+90°$ and $-90°$.

As the point D, of which a, d are the coordinates, is so near the centre of the sun and as at the time of an eclipse α_2 and δ_2 are very near to α_1 and δ_1 respectively, the following approximate solution gives a and d with all needful accuracy.

If in the first equation we write the small angles $\alpha_1 - a$ and $\alpha_2 - a$ instead of their sines, and if we make $\cos\delta_1 = \cos\delta_2$ and $\Delta_2 \div \Delta_1 = 1/391$ we have

$$a = \alpha_1 + (\alpha_1 - \alpha_2)/391.$$

In the second equation we make $\cos(\alpha_1 - a)$ and $\cos(\alpha_2 - a)$ each unity and thus substituting the small angles $\delta - d$ and $d - \delta'$ for their sines we have

$$d = \delta_1 + (\delta_1 - \delta_2)/391.$$

The west hour angle of D from Greenwich at the Greenwich sidereal time ϑ is $\vartheta - a$, this is the Besselian element μ, which must first of all be calculated for each separate half hour during the eclipse.

At any particular epoch the values of a, d substituted in (i) give x and y for the values of $\alpha_1, \delta_1, \Delta_1$ and $\alpha_2, \delta_2, \Delta_2$. These quantities are of course variable on account of the movements of the heavenly bodies in question. The eclipse, indeed, depends on their relative changes. It is therefore necessary to compute the values of x and y for several epochs about the time of conjunction of the sun and moon. It is convenient to make a table showing x and y for intervals of 10 minutes during the progress of each solar eclipse, the unit of length in which these coordinates are expressed being the earth's equatorial radius. We designate by x', y' the rates at which x and y change per minute. All these quantities will be found in the ephemeris and if T be the epoch for which x and y are calculated then at the time $T+t$ that is at t minutes after the epoch T, x and y will change into $x + x't$ and $y + y't$ respectively.

Imagine the internal tangent cone or penumbra (Fig. 91) to be drawn to the sun S and the moon M, then we have to find f the semi-angle of the cone at its vertex O and the radius $l = PQ$ of the section of this cone by the fundamental plane.

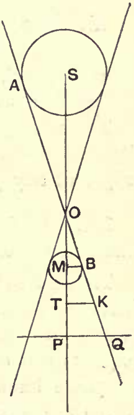


Fig. 91.

Let R be the ratio of the actual distance of the sun from the earth to its mean distance. The distance of the moon is about $R/391$ and therefore at the time of a solar eclipse when the earth, the sun and the moon are in a line

$$MS = PS - PM = R - R/391 = 390R/391.$$

If a be the radius of the sun then $a/401$ is the radius of the moon and as f is small

$$\tan f = \sin f = a\,(1 + 1/401)/MS,$$

whence $R \tan f = a\,391 \times 402/(390 \times 401)$.

From the choice of units it follows that a is the sine of the angle subtended by the semi-diameter of the sun at its mean distance and it is found that for the computation of eclipses this angle should be $15' 59''\cdot 6$. We hence obtain

$$\text{Log}\, R \tan f = 7\cdot 6700.$$

The radius $l = PQ = (PS - OS) \tan f = R \tan f - a$. But we have $(R - MP) \tan f = a + b$, where b is the radius of the moon, whence $l = MP \tan f + b$. If we take as is most convenient the earth's equatorial radius as the unit of distance for the measurement of l then π_0' being the moon's horizontal parallax, and $0\cdot 2725$ the ratio of the moon's radius to the earth's equatorial radius we have

$$l = 0\cdot 2725 + \tan f \operatorname{cosec} \pi_0'.$$

For example. In the annular eclipse of March 5, 1905, we have $\text{Log}\, R = 9\cdot 9966$, whence

$$\text{Log} \tan f = 17\cdot 6700 - 9\cdot 9966 = 7\cdot 6739.$$

The moon's horizontal parallax on this occasion is $54' 9''$, and with the value of f just found we obtain

$$l = \cdot 5728.$$

All these quantities, viz. x, y, x', y', $\text{Log} \tan f$, $\text{Log} \sin d$, $\text{Log} \cos d$, μ, are known as the Besselian elements, and it will be observed that they relate to the whole earth rather than to particular stations thereon.

The next part of the calculation shows how the Besselian elements are to be applied to determine the circumstances of an eclipse at any particular station.

125. Application of the Besselian elements to the calculation of an eclipse for a given station.

The critical phenomena of an eclipse are presented when the observer is on the penumbra or the umbra. In the former case the contact of the limbs of the sun and moon is external and the partial eclipse is just beginning or ending. In the case of a total eclipse the phase known as "totality" is just commencing or ending when the observer is on the umbra. In the case of an annular eclipse the first or second internal contact takes place when the observer occupies this position. We shall now study the case of the commencement or ending of an eclipse.

It has been already stated that μ is the westerly hour angle from Greenwich of the point D and therefore with respect to the observer's station K which has an easterly longitude λ, the west hour angle of D is $\mu + \lambda$. The geocentric zenith of the observer has therefore a right ascension $a + \mu + \lambda$ and a declination ϕ', where ϕ' is the geocentric latitude of K. If therefore ρ be the distance of K from the earth's centre and ξ, η, ζ the coordinates of K with respect to the fundamental axes, we have

$$\left.\begin{aligned} \xi &= \rho \cos \phi' \sin (\mu + \lambda) \\ \eta &= \rho \{\sin \phi' \cos d - \cos \phi' \sin d \cos (\mu + \lambda)\} \\ \zeta &= \rho \{\sin \phi' \sin d + \cos \phi' \cos d \cos (\mu + \lambda)\} \end{aligned}\right\} \quad \ldots\ldots \text{(i)}.$$

The values of ξ and η and also of ξ' and η' are to be calculated for the particular locality and for the same epoch T which was used in calculating x and y. Hence at the time $T + t$ where t is expressed in minutes of mean time and is understood to be a small quantity (as of course it will be if T be properly chosen) the values of ξ and η become $\xi + \xi' t$ and $\eta + \eta' t$ respectively.

We have now to find ξ' and η', that is to say the rate per minute at which ξ and η are changing about the time when the eclipse is visible at the place in question.

ξ and η depend on ρ, ϕ', λ, d, μ. Of these the three first are fixed for a given locality and hence the changes in ξ and η at a given place can only arise through changes of d or μ or both. As to d it is very nearly the declination of the sun and this at most only changes at the rate of a second of arc per minute. The changes of ξ and η which now concern us are due to the changes in μ. This is very nearly the west hour angle of the sun at Greenwich and its variation in one minute of mean time is about one minute of sidereal time $= 15'$, or expressed in radians $1/229{\cdot}2$.

Differentiating the expressions for ξ and η with regard to the time and representing the differential coefficients by ξ' and η', we have

$$\left.\begin{aligned} \xi' &= \rho \cos \phi' \cos (\mu + \lambda)/229{\cdot}2 \\ \eta' &= \xi \sin d/229{\cdot}2 \end{aligned}\right\} \ldots\ldots\ldots\ldots\ldots \text{(ii)}.$$

The distance of the observer from the axis of the penumbra is $l - \zeta \tan f$, which for brevity is represented by L. It is obvious that a small change in ζ, being multiplied as it is by the small quantity $\tan f$, is insensible, and consequently we have as the fundamental equation for the determination of the commencement or ending of the partial eclipse

$$\{(x-\xi)+t(x'-\xi')\}^2+\{(y-\eta)+t(y'-\eta')\}^2=L^2 \ldots\text{(iii)}.$$

The solution of this equation is effected as follows.

We make the substitutions

$$\left.\begin{aligned} m\sin M &= x-\xi; \quad n\sin N = x'-\xi' \\ m\cos M &= y-\eta; \quad n\cos N = y'-\eta' \end{aligned}\right\} \ldots\ldots\ldots\ldots\text{(iv)},$$

in which m, n, M, N are four auxiliary quantities. This gives $\tan M=(x-\xi)\div(y-\eta)$ from which two values of M differing by 180° are determined. We choose that value which shall make $\sin M$ have the same sign as $x-\xi$, then $\cos M$ must have the same sign as $y-\eta$, and m will be the positive square root of

$$(x-\xi)^2+(y-\eta)^2.$$

In like manner N is determined so that n shall be the positive square root of

$$(x'-\xi')^2+(y'-\eta')^2.$$

Substituting in the equation (iv) we have

$$n^2t^2+2mnt\cos(M-N)+m^2=L^2,$$

where a minute of mean time is as already stated the unit of t.

We introduce another angle ψ such that

$$L\sin\psi=m\sin(M-N).$$

As ψ is given only by its sine there is a choice of two supplemental angles for ψ. We choose that one which lies between $+90°$ and $-90°$ so that $\cos\psi$ is positive then

$$\begin{aligned} n^2t^2+2mnt\cos(M-N)&+m^2\cos^2(M-N) \\ &=L^2-m^2+m^2\cos^2(M-N)=L^2\cos^2\psi, \end{aligned}$$

whence $$nt=-m\cos(M-N)\mp L\cos\psi \ldots\ldots\ldots\ldots\text{(v)},$$

since $\cos\psi$ is positive as well as L and n, the upper sign gives t_1 and the lower t_2 and the Greenwich mean times of the commencement and ending of the eclipse are $T+t_1$ and $T+t_2$ respectively. If we denote by T_1 and T_2 the local mean times of the beginning and ending we have

$$T_1=T+t_1+\lambda; \quad T_2=T+t_2+\lambda,$$

where λ is the longitude of the observer.

It remains to determine the points on the limb of the sun at which the eclipse commences and ends.

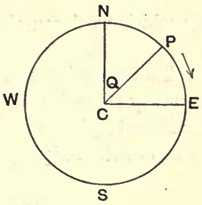


FIG. 92.

In Fig. 92 the fundamental plane is in the plane of the paper. C is the centre of the circle $NESW$ which is the intersection of the penumbra with the fundamental plane.

If NC is parallel to y then the generator of the penumbral cone through N touches the sun's apparent disk in its most northerly point, because the earth's axis lies in the plane normal to x. If CE be parallel to x then the generator through E touches the sun's apparent disk in its most easterly point, and if S and W are points on the circle diametrically opposite to N and E they lie on the generators which touch the apparent disk of the sun in its most southerly and westerly points respectively.

If the point ξ, η, ζ lies on the generator through P then

$$L \sin Q = (x + x't) - (\xi + \xi' t),$$
$$L \cos Q = (y + y't) - (\eta + \eta' t).$$

We therefore substitute in this the values of t corresponding to the beginning and the end of the eclipse and we have

$$\begin{aligned} L \sin Q &= x - \xi + t(x' - \xi') \\ &= m \sin M + \sin N(-m \cos(M - N) \mp L \cos \psi) \\ &= m \cos N \sin(M - N) \mp L \cos \psi \sin N \\ &= L \sin \psi \cos N \mp L \cos \psi \sin N \\ &= \mp L \sin(N \mp \psi). \end{aligned}$$

In like manner

$$L \cos Q = \mp L \cos(N \mp \psi).$$

If Q_1 be the value of Q at the beginning of the eclipse we take the upper signs

$$\sin Q_1 = \sin(N - \psi + 180^\circ),$$
$$\cos Q_1 = \cos(N - \psi + 180^\circ).$$

If Q_2 be the value of Q at the end of the eclipse we use the lower signs and

$$\sin Q_2 = \sin(N + \psi),$$
$$\cos Q_2 = \cos(N + \psi),$$

whence
$$Q_1 = N - \psi + 180°,$$
$$Q_2 = N + \psi,$$

by which we learn the points of the solar disk which the moon touches at the first and last moments of the eclipse.

To determine the circumstances of the eclipse with greater accuracy the calculation should be repeated, using the values found for T_1 and T_2 instead of T according to whether it is the beginning or end of the eclipse that is sought.

EXERCISES ON CHAPTER XVII.

Ex. 1. Prove that at the instant of conjunction in right ascension the ratio of the distance of the sun from the moon to the distance of the sun from the earth is

$$\{\sin \pi_0' - \sin \pi_0 \cos(\delta - \delta')\}/\sin \pi_0',$$

where δ' and δ are the declinations of the sun and the moon, π_0 and π_0' are the horizontal parallaxes of the sun and the moon, and the square of $\sin \pi_0$ is neglected.

Also prove that at the same instant, if $\dot{a}'$ and $\dot{a}$ be the hourly change of the right ascensions of the sun and the moon respectively, and $\dot{A}$ the hourly change in the right ascension of the line from the centre of the earth parallel to the line joining the moon's centre to the sun's centre, then

$$\dot{A} = \dot{a}' - \frac{\sin \pi_0 \cos \delta}{\sin \pi_0' \cos \delta'}(\dot{a} - \dot{a}').$$

[Coll. Exam.]

Ex. 2. The geocentric angular distance between the centres of the sun and moon at the instant of conjunction in right ascension is d, and the declination of the sun is δ'. The rates of separation of the sun and moon in right ascension and declination are $\dot{a}$ and $\dot{\delta}$. Prove that, if the sun be eclipsed, the time from conjunction to the middle of the geocentric eclipse is approximately $\dot{\delta} d/(\dot{\delta}^2 + \dot{a}^2 \cos^2 \delta')$.

Prove that the approximate difference of the right ascension of the point, where the celestial sphere is intersected by the axis of the cone of shadow during an eclipse of the sun, and the geocentric right ascension of the sun is

$$\frac{b \cos \delta \sec \delta' \sin (a - a')}{\sin 1''} + \frac{b^2 \cos^2 \delta \sec^2 \delta' \sin 2(a - a')}{2 \sin 1''};$$

where a, a' are respectively the geocentric right ascensions of the moon and sun, δ and δ' their declinations, b the ratio of the moon's geocentric distance to the sun's geocentric distance. [Math. Trip.]

Ex. 3. Using five-figure logarithms, show that from the earliest beginning, as seen from the earth's surface, of the solar eclipse of Aug. 8, 1896 to the latest ending is an interval of about $4^h\ 49^m$, having given,

Moon's latitude at conjunction in longitude	42′ 11″ N.
Moon's hourly motion in longitude	35′ 55″
Sun's " " "	2′ 24″
Moon's hourly motion in latitude	3′ 18″ S.
Moon's equatorial horizontal parallax	59′ 28″
Sun's " " "	9″
Moon's true semi-diameter	16′ 14″
Sun's " "	15′ 48″

[Math. Trip.]

Ex. 4. Show that, neglecting the sun's parallax, the equations to determine the place where a solar eclipse is central at a given time are

$$\frac{\cos\phi\cos l-\rho\cos\delta}{\cos(\alpha'-\alpha)\cos\delta'}=\frac{\cos\phi\sin l}{\sin(\alpha'-\alpha)\cos\delta'}=\frac{\sin\phi-\rho\sin\delta}{\sin\delta'},$$

where α, δ, α', δ' are the geocentric right ascensions and declinations respectively of the moon and sun, and ρ the ratio of the moon's distance to the earth's radius, ϕ the latitude of the place, and l the hour angle of the moon. [Coll. Exam.]

Ex. 5. If a lunation be 29·5306 days, and the period of the sidereal revolution of the moon's node 6798·3 days, prove that after a period of 14558 days eclipses may be expected to recur in an invariable order.

[Math. Trip. 1881.]

The moon's node having a retrograde motion, the daily approach of the sun to the node in degrees is $\frac{360}{6798 \cdot 3}+\frac{360}{365 \cdot 24}$. Dividing this into 360 we obtain 346·62 as the number of days in the revolution of the sun with respect to the moon's node. Multiplying this by 42 we obtain 14558·0. We also find that 493 lunations make 14558·6 days. We thus see that in about 14558 days there are 493 lunations and that in the same period 42 complete revolutions are made by the sun with regard to the node. Thus after the lapse of this period from a conjunction the sun and moon are again in conjunction, and at the same distances from the node as they were at the beginning.

CHAPTER XVIII.

OCCULTATIONS OF STARS BY THE MOON.

126. The investigation of an occultation.

It occasionally happens that the moon in the course of its movement passes between the observer and a star. This phenomenon is called an *occultation.* As the star may, for this purpose, be regarded as a mathematical point, the extinction of the star by the moon's advancing limb is usually an instantaneous phenomenon, though occasionally, owing doubtless to the marginal irregularities of the moon's limb, the phenomenon is not quite so simple. The reappearance of the star when the moon has just passed across it may also be observed, though in this case the observer should be forewarned as to the precise point on the moon's limb where the star will suddenly emerge.

It is easy to see the astronomical significance of the observation of an occultation. The time of its occurrence depends both on the movement of the moon and the position of the observer. The place of the star being known with all desirable precision, an accurate observation of the moment of the star's disappearance gives a relation between the place of the moon and the position of the observer. The observation may be available for an accurate determination of the place of the moon or it may be used for finding the longitude of the observer if compared with the similar observation at another station of known longitude.

The following method of calculating the time at which the disappearance or reappearance of an occulted star takes place is due to Lagrange and Bessel.

The symbols used are thus defined:

A Apparent R.A. of the star.
D „ decl. „

α Apparent R.A. of moon from earth's centre.
δ „ decl. „ „ „
π_0' Equatorial horizontal parallax of moon.
$r_☽$ Angular semi-diameter of moon from earth's centre.
α' Apparent R.A. of moon from place of observer.
δ' „ decl. „ „ „
$r_☽'$ Semi-diameter of moon from place of observer.

ϑ Sidereal time at place of observer.
ϕ Latitude of observer.
ϕ' Geocentric latitude of observer.
ρ Distance of observer from the earth's centre when the earth's equatorial radius is taken as unity.

Let S, M, P be the star, the moon and the pole respectively (Fig. 93) and let Σ be the angular distance from the star S to the centre of the moon as they appear to the observer at any moment.

Let θ be the spherical angle MSP subtended at S by the pole P and the centre of the moon. It is assumed that θ is measured from SP in the direction according with the usual convention for position angles (p. 138). There will be no confusion between θ and $360° - \theta$ if it be remembered that when $\alpha' > A$ then θ lies between 0° and 180°, and when $\alpha' < A$ then we must take for θ an angle between 180° and 360°.

Thus we may write the following formulae (§ 1):

$$\left.\begin{aligned}\sin\Sigma\sin\theta &= \cos\delta'\sin(A-\alpha')\\ \sin\Sigma\cos\theta &= \sin\delta'\cos D - \cos\delta'\sin D\cos(A-\alpha')\\ \cos\Sigma &= \sin\delta'\sin D + \cos\delta'\cos D\cos(A-\alpha')\end{aligned}\right\}\ldots\text{(i)}.$$

Consider three rectangular axes through the earth's centre such that $+x$ is towards the point on the equator whose R.A. is 90°, $+y$ is toward ♈, $+z$ is towards the north pole.

The sidereal time ϑ is the hour angle of ♈. Hence for the coordinates of the point of observation with respect to these axes

$$x = \rho\cos\phi'\sin\vartheta;\quad y = \rho\cos\phi'\cos\vartheta;\quad z = \rho\sin\phi'.$$

The coordinates of the moon referred to the same axes are

$$x = \cos\delta \sin\alpha \operatorname{cosec} \pi_0'; \quad y = \cos\delta \cos\alpha \operatorname{cosec} \pi_0';$$
$$z = \sin\delta \operatorname{cosec} \pi_0'.$$

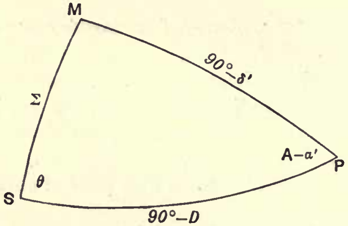


FIG. 93.

If Δ be the ratio which the distance of the moon from the observer bears to its distance from the centre of the earth then $\Delta \operatorname{cosec} \pi_0'$ will be the distance of the moon from the place of observation, and the projections of this distance on the three axes respectively are

$$\Delta \cos\delta' \sin\alpha' \operatorname{cosec} \pi_0', \quad \Delta \cos\delta' \cos\alpha' \operatorname{cosec} \pi_0', \quad \Delta \sin\delta' \operatorname{cosec} \pi_0';$$

whence we obtain

$$\cos\delta \sin\alpha \operatorname{cosec} \pi_0' = \Delta \cos\delta' \sin\alpha' \operatorname{cosec} \pi_0' + \rho \cos\phi' \sin\vartheta,$$
$$\cos\delta \cos\alpha \operatorname{cosec} \pi_0' = \Delta \cos\delta' \cos\alpha' \operatorname{cosec} \pi_0' + \rho \cos\phi' \cos\vartheta,$$
$$\sin\delta \operatorname{cosec} \pi_0' = \Delta \sin\delta' \operatorname{cosec} \pi_0' + \rho \sin\phi',$$

which may be changed into

$$\Delta \cos\delta' \sin\alpha' = \cos\delta \sin\alpha - \rho \cos\phi' \sin\pi_0' \sin\vartheta,$$
$$\Delta \cos\delta' \cos\alpha' = \cos\delta \cos\alpha - \rho \cos\phi' \sin\pi_0' \cos\vartheta,$$
$$\Delta \sin\delta' = \sin\delta - \rho \sin\phi' \sin\pi_0'.$$

Multiplying formulae (i) by Δ and eliminating α' and δ' by the expressions just given we have

$$\left.\begin{aligned}
\Delta \sin\Sigma \sin\theta &= -\cos\delta \sin(\alpha - A) \\
&\qquad + \rho \cos\phi' \sin\pi_0' \sin(\vartheta - A) \\
\Delta \sin\Sigma \cos\theta &= \sin\delta \cos D - \cos\delta \sin D \cos(\alpha - A) \\
&\qquad - \rho \sin\pi_0' \{\sin\phi' \cos D - \cos\phi' \sin D \cos(\vartheta - A)\} \\
\Delta \cos\Sigma &= \sin\delta \sin D + \cos\delta \cos D \cos(\alpha - A) \\
&\qquad - \rho \sin\pi_0' \{\sin\phi' \sin D + \cos\phi' \cos D \cos(\vartheta - A)\}
\end{aligned}\right\} \ldots \text{(ii)}.$$

These formulae enable Σ and θ to be determined and are specially adapted for the study of occultations because at the beginning or the end of an occultation the star is on the moon's limb and we have $\Sigma = r_{☽}'$. As the sine of the moon's semi-diameter varies inversely as its distance we must have $\Delta \sin r_{☽}' = \sin r_{☽}$, and therefore $\Delta \sin \Sigma = \sin r_{☽}$. Introducing this into the equations (ii) we obtain the following remarkable formulae, true at the moments of disappearance or reappearance of an occulted star

$$\left.\begin{aligned} \sin r_{☽} \sin \theta &= -\cos\delta \sin(\alpha - A) \\ &\qquad + \rho \sin \pi_0' \cos\phi' \sin(\vartheta - A) \\ \sin r_{☽} \cos\theta &= \sin\delta\cos D - \cos\delta \sin D \cos(\alpha - A) \\ &\qquad - \rho \sin\pi_0' \{\sin\phi' \cos D - \cos\phi' \sin D \cos(\vartheta - A)\} \end{aligned}\right\} \ldots\text{(iii)}.$$

It is plain that $r_{☽}$ and π_0' are connected by the constant relation

$$\sin\pi_0'/\sin r_{☽} = \text{radius of earth/radius of moon}.$$

The ratio of the moon's radius to that of the earth is termed k and is equal to 0·2725. Thus we have from (iii)

$$\left.\begin{aligned} k \sin\theta &= -\cos\delta \sin(\alpha - A) \operatorname{cosec} \pi_0' + \rho \cos\phi' \sin(\vartheta - A) \\ k\cos\theta &= \{\sin\delta \cos D - \cos\delta \sin D \cos(\alpha - A)\} \operatorname{cosec}\pi_0' \\ &\qquad - \rho\{\sin\phi' \cos D - \cos\phi' \sin D \cos(\vartheta - A)\} \end{aligned}\right\} \ldots\text{(iv)}.$$

Finally squaring and adding we obtain the following fundamental equation, which contains the theory of the time of commencement or ending of an occultation

$$\left.\begin{aligned} k^2 = {} & \{\cos\delta \sin(\alpha - A) \operatorname{cosec}\pi_0' - \rho\cos\phi' \sin(\vartheta - A)\}^2 \\ & + [\{\sin\delta\cos D - \cos\delta \sin D \cos(\alpha - A)\} \operatorname{cosec}\pi_0' \\ & - \rho\{\sin\phi' \cos D - \cos\phi' \sin D \cos(\vartheta - A)\}]^2 \end{aligned}\right\} \ldots\text{(v)}.$$

If the coordinates of the observer be given, then the only unknown in this equation is ϑ the time. The solution of this equation for ϑ will therefore show the moment of the beginning or the end of an occultation.

The equation for ϑ is necessarily a transcendental equation, for it has to represent all possible occultations in infinite time. To apply it to any particular occultation we must use approximate methods.

Let T be an assumed time very near the true time $T + t$ at which a certain occultation takes place, t is thus a small quantity and the terms of the equation can be expanded in a rapidly converging series of powers of t.

We shall make

$$\left.\begin{aligned} \cos\delta\sin(\alpha - A)\operatorname{cosec}\pi_0' &= p + p't \\ \{\sin\delta\cos D - \cos\delta\sin D\cos(\alpha - A)\}\operatorname{cosec}\pi_0' &= q + q't \\ r\cos\phi'\sin(\vartheta - A) &= u + u't \\ r\{\sin\phi'\cos D - \cos\phi'\sin D\cos(\vartheta - A)\} &= v + v't \end{aligned}\right\}\ldots\text{(vi)}.$$

It is supposed that p, q, u, v are calculated for the time T and p', q', u', v' are terms involving t for which we may first assume the approximate value zero.

The equation (v) then becomes

$$k^2 = \{p - u + (p' - u')t\}^2 + \{q - v + (q' - v')t\}^2.$$

A solution of this will give t, which we can then substitute in p', q', u', v' and thus by repeating the solution obtain a more accurate value of t.

For a convenient solution of these equations we make

$$\left.\begin{aligned} p - u = m\sin M;\ \ p' - u' = n\sin N \\ q - v = m\cos M;\ \ q' - v' = n\cos N \end{aligned}\right\}\ldots\ldots\ldots\text{(vii)};$$

where m, n, M, N are four auxiliary quantities

$$\begin{aligned} k^2 &= (m\sin M + n\sin Nt)^2 + (m\cos M + n\cos Nt)^2 \\ &= m^2\sin^2(M - N) + \{m\cos(M - N) + nt\}^2. \end{aligned}$$

We now introduce another auxiliary quantity ψ, such that

$$m\sin(M - N) = k\cos\psi.$$

Then $$k^2\sin^2\psi = \{m\cos(M - N) + nt\}^2,$$

or $$nt = -m\cos(M - N) \mp k\sin\psi.$$

We assume that ψ is less than 180°, and then the upper sign corresponds to the disappearance of the star behind the moon, and the lower to its reappearance.

If $$m\sin(M - N) > k,$$

then ψ is imaginary and there is no occultation. In drawing this conclusion it would be proper to remember that further approximation may be necessary to decide whether this condition is truly satisfied. To examine this we take for t its mean value

$$-m\cos(M - N)/n,$$

and insert this value in p', u', q', v', so that by repeating the calculations we can find whether ψ is a real quantity.

If there is an occultation of a star and t' and t'' the two roots of the equation have been finally ascertained then the time $T+t'$ of the star's disappearance and $T+t''$ of its reappearance are determined.

How to find the points on the moon's limb at which the star disappears and reappears. Substituting from (vi) in formulae (iv), we have

$$k \sin \theta = -p - p't + u + u't,$$
$$k \cos \theta = q + q't - v - v't,$$

which from (vii) may be written

$$\left.\begin{aligned} k \sin \theta &= - m \sin M - nt \sin N \\ k \cos \theta &= + m \cos M + nt \cos N \end{aligned}\right\} \quad \text{......... (viii).}$$

Introducing the value for t into the first of these formulae,

$$k \sin \theta = - m \sin (M - N) \cos N \pm k \sin N \sin \psi.$$

But $$m \sin (M - N) = k \cos \psi,$$

whence, substituting and dividing by k,

$$\sin \theta = - \cos (N \pm \psi).$$

In like manner from the second of the formulae (viii)

$$\cos \theta = - \sin (N \pm \psi),$$

and consequently

$$\tan \theta = \cot (N \pm \psi) = \tan \{90° - (N \pm \psi)\}.$$

Hence $$\theta = w\, 180° + 90° - (N \pm \psi),$$

where w is any integer, and from this

$$\sin \theta = \cos (w\, 180°) \cos (N \pm \psi).$$

But we have already seen

$$\sin \theta = - \cos (N \pm \psi).$$

Hence $$\cos (w\, 180°) = -1 \text{ or } w = 1.$$

and finally $$\theta = 270° - (N \pm \psi).$$

Thus we obtain the position angle of the centre of the moon from the star at the moment of disappearance or reappearance. This shows the points on the limb at which the phenomena take place.

The angle subtended at the moon's centre by the star and the pole at the moments of disappearance and reappearance is very nearly

$$180° - \theta = N \pm \psi - 90°.$$

Thus the necessary formulae for solving the problem of an occultation have been obtained.

For convenient methods of conducting the calculations reference may be made to the ephemeris in which tables are given by which the work is facilitated.

Ex. 1. At midnight on 27 Oct. 1909 the declination of the moon is 4° 36′ 46″·7 and the R.A. and declination of the moon are then increasing in every 10^m by $23^s·0$ and 164″ respectively. Show that a star in conjunction in R.A. with the moon at midnight cannot be occulted at or about the time of this conjunction if the star's declination is less than 3° 10′·4, it being given that the sum of the moon's semi-diameter and horizontal parallax is 78′·0.

The movement of the moon in R.A. in 10^m is 344″. The inclination of the moon's movement to the hour circle is therefore

$$\tan^{-1}(344/164) = 64° \, 30'.$$

Hence the sine of the difference between the moon's declination and the star's at the time of conjunction must not exceed

$$\sin 78'·0 \times \operatorname{cosec}(64° \, 30') = ·0251 = \sin 86'·4.$$

Ex. 2. If the inclination of the moon's orbit to the ecliptic be 5° 20′ 6″, show that the moon will at some time or other occult any star whose latitude north or south is less than 6° 38′ 24″. [Coll. Exam.]

Ex. 3. Show that a star in the ecliptic will be occulted by the moon at some station on the earth between 17 and 22 times at each passage of a node of the moon's orbit through the star. Assume semi-diameter of moon between 16′ 46″ and 14′ 44″, horizontal parallax of moon between 61′ 18″ and 53′ 58″, inclination of orbit to ecliptic between 5° 19′ and 4° 57′. [Math. Trip.]

Ex. 4. On Feb. 29, 1884, the moon occulted Venus. It was stated in the *Times* that Venus would be on the meridian at 2.30 p.m., the moon being then three days old, and that the occultation would last about an hour and a quarter. Find roughly when it commenced, and show that the statement as to the duration of occultation is not inconsistent with the known angular diameter of the moon. [Math. Trip.]

CHAPTER XIX.

PROBLEMS INVOLVING SUN OR MOON.

127. Phenomena of rising and setting.

Let R (Fig. 94) be the sun at sunrise, ♎ the first point of Libra, where ERN is the horizon, R♎ the ecliptic, E♎ the equator. As E is the easterly point, the equator produced for

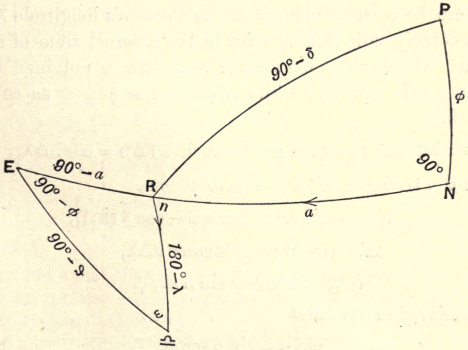


Fig. 94.

90° beyond E in the direction from ♎ to E will reach the meridian, and produced for a further distance equal to ϑ, the sidereal time will reach ♈ the first point of Aries. Hence we have

$$E♎ = 90° - \vartheta,$$

The longitude of the sun is λ and $R♎ = 180° - \lambda$, because λ is measured from ♈ in the direction of the arrow head. a is the azimuth of the rising sun measured from N, the north point of the horizon in the usual direction by east and south. P is the pole and $PN = \phi$, the altitude of the pole above the horizon. The angle at E between the equator and the horizon is equal to $90° - \phi$ as that is the distance from the zenith to the pole, $180° - n$ is the inclination of the ecliptic to the horizon (§ 10), and ω is the obliquity of the ecliptic.

Many questions that can be proposed with regard to the rising and setting of the sun can be solved by the triangle ER♎. If we assume that ϕ and ω are given, and that one of the other elements of the triangle is known, the remaining elements can be determined.

To find the time of sunrise or sunset when the longitude λ is given we deduce from the formula (6), p. 3,

$$\sin\vartheta\cos\omega + \cos\vartheta\cot\lambda + \sin\omega\tan\phi = 0 \;\ldots\ldots\ldots (i).$$

This equation may be put into the form $\sin(\vartheta + A) = \sin B$ whence $\vartheta = B - A$ or $\vartheta = 180° - B - A$ one of which corresponds to the rising and the other to the setting.

If there be a small change $\Delta\lambda$ in the sun's longitude λ there will be a corresponding change $\Delta\vartheta$ in the sidereal time of sunrise or sunset. The relation between $\Delta\lambda$ and $\Delta\vartheta$ is obtained by differentiating this equation while regarding ω and ϕ as constant, we thus find

$$\sin\lambda(\sin\lambda\cos\vartheta\cos\omega - \cos\lambda\sin\vartheta)\,\Delta\vartheta = \cos\vartheta\,\Delta\lambda,$$

but from the triangle ER♎ we have at once

$$\sin a = \sin\lambda\cos\vartheta\cos\omega - \cos\lambda\sin\vartheta,$$

whence $$\Delta\vartheta = \cos\vartheta\,\text{cosec}\,\lambda\,\text{cosec}\,a\,\Delta\lambda,$$

also $$\cos\vartheta\,\text{cosec}\,\lambda = \sin n\sec\phi$$

from the triangle ER♎ and

$$\cos a = \sin\delta\sec\phi$$

from the triangle *PRN*, whence

$$\operatorname{cosec} a = \cos\phi/(\cos^2\delta - \sin^2\phi)^{\frac{1}{2}};$$

hence finally

$$\Delta\vartheta = \sin n\,(\cos^2\delta - \sin^2\phi)^{-\frac{1}{2}}\,\Delta\lambda \quad \text{............(ii)}.$$

This equation gives the daily retardation in the hour of rising of the sun and it will also explain certain phenomena connected with the rising of the moon. We may for the present purpose suppose the orbit of the moon to be coincident with the plane of the ecliptic, the inclination being only about 5°. The pole of the ecliptic describes a small circle round P with a radius of 23°·5, and when it comes nearest to the zenith n has its lowest value. If the moon be in Aries then $\cos\delta = 1$ and the denominator of the expression for $\Delta\vartheta/\Delta\lambda$ is greatest. Thus for a double reason the daily retardation in the hour of rising of the moon is small. If at the same time the sun is in Libra the moon will then be full, and consequently near the autumnal equinox the full moon rises for several consecutive nights nearly at the same time. This is the phenomenon known as the harvest moon.

If it be desired to find a, the azimuth of the point at which a celestial body rises or sets, we have the equations

$$\sin n \sin a = \cos\omega\cos\phi + \sin\omega\sin\phi\sin\vartheta,$$
$$\sin n \cos a = \sin\omega\cos\vartheta,$$

from which $\tan a$ can be found when ϑ is known.

Ex. 1. Prove that at a place on the Arctic circle 18^h is the sidereal time of sunrise for one half the year, and of sunset for the other half; and that the azimuth of the point of the horizon where the sun rises, measured from the nearest point of the meridian, is throughout the year either $90° \sim l$ or $270° \sim l$, where l is the sun's longitude. [Math. Trip.]

Once a day the pole of the ecliptic passes through the zenith. The sun must then be rising or setting according as it is E. or W. of the meridian, but as ♈ is at E. point of horizon the sidereal time is 18^h.

Ex. 2. Prove that at a place on the Arctic circle the daily displacement of the point of sunset is equal to the sun's change in longitude during the same interval. [Math. Trip.]

Ex. 3. Show that near the vernal equinox the sidereal time of sunrise is decreasing at places within the Arctic circle and increasing at other places within the northern hemisphere. [Math. Trip.]

In the general equation (i) we make λ small and the equation becomes

$$\cos\vartheta + \lambda(\sin\vartheta\cos\omega + \sin\omega\tan\phi) = 0.$$

But at sunrise about the time of the vernal equinox we have $\vartheta = 270^\circ - x$, where x is very small, whence

$$x \cos\phi = -\lambda \sin(90^\circ - \omega - \phi).$$

Hence x is positive if $\phi > 90^\circ - \omega$.

Ex. 4. Find the latitude by observing the angular distance of the extreme points of the horizon on which the sun appears at rising in the course of a year: and if a, β are the distances of those points from the point in which the sun rises when its declination is δ, prove that, ω being the obliquity of the ecliptic,

$$\sin\delta = \sin\omega \frac{\sin\frac{1}{2}(a+\beta)}{\sin\frac{1}{2}(a-\beta)}. \qquad \text{[Coll. Exam.]}$$

If a_1 and a_2 be the extreme azimuths at rising measured from the north point then $\tan a_1 = m$, $\tan a_2 = -m$, where

$$m = (\cot^2\omega \cos^2\phi - \sin^2\phi)^{\frac{1}{2}}.$$

If a be the azimuth of the sunrise at decl. δ, then $a = \cos^{-1}(\sin\delta \sec\phi)$,

$$\frac{\sin\frac{1}{2}(a+\beta)}{\sin\frac{1}{2}(a-\beta)} = \frac{\sin a + \sin\beta}{\sin(a-\beta)} = \frac{\sin(a_1 - a) + \sin(a_2 - a)}{\sin(a_1 - a_2)}$$

$$= \frac{m\cos a - \sin a + m\cos a + \sin a}{2m}\sqrt{1+m^2} = \sin\delta \operatorname{cosec}\omega.$$

Ex. 5. Prove that in the latitude of Cambridge, 52° 13′, the minimum retardation of the sidereal time of sunrise from day to day is about 96 sec., having given that the obliquity of the ecliptic is 23° 27′, and the daily increase of the sun's R.A. at the vernal equinox is $3^{\text{m}}\ 38^{\text{s}}$.

[Math. Trip.]

In general

$$\Delta\vartheta = \sin n (\cos^2\delta - \sin^2\phi)^{-\frac{1}{2}} \Delta\lambda.$$

For the minimum retardation $n = 90^\circ - \phi - \omega$, and $\delta = 0$, so that

$$\Delta\vartheta = \cos(\omega + \phi)\sec\phi\,\Delta\lambda.$$

We are given $\cos\omega \,.\, \Delta\lambda = 3^{\text{m}}\ 38^{\text{s}}$, whence $\Delta\vartheta = 96^{\text{secs}}$.

Ex. 6. Show that in latitude 45° the difference between the times from sunrise to apparent noon and from apparent noon to sunset is

$$\frac{D}{365}\tan\delta\sec\delta(\sec 2\delta)^{\frac{1}{2}}\cot(360^\circ\ T/365),$$

where D is the length of the day, δ the sun's declination and T the number of days since the vernal equinox, the earth's orbit being supposed circular.

Ex. 7. Show that the time taken by the sun's disc to rise above the horizon is greatest at the solstices and least at the equinoxes.

[Coll. Exam.]

Let z be the zenith distance, then as usual if h is the hour angle,

$$\cos z = \sin\phi\sin\delta + \cos\phi\cos\delta\cos h.$$

Differentiating

$$\sin z \,.\, \Delta z = \cos\phi\cos\delta\sin h \,.\, \Delta h,$$

and when $z = 90^\circ$,

$$\Delta h = \Delta z \sec\phi\sec\delta\operatorname{cosec} h \quad \text{.........................(i).}$$

If n be the number of seconds of time in dh and D be the sun's diameter in seconds of arc, we obtain by eliminating h from (1),

$$n = \tfrac{1}{15} D (\cos^2 \phi - \sin^2 \delta)^{-\frac{1}{2}},$$

and n is greatest when $\sin \delta$ is greatest.

Ex. 8. When the sun's declination is δ, it rises at a point distant α and β from the extreme points of rising. Show that

$$\tan \tfrac{1}{2}\alpha : \tan \tfrac{1}{2}\beta :: \tan \tfrac{1}{2}(\omega + \delta) : \tan \tfrac{1}{2}(\omega - \delta).$$

[Math. Trip. I. 1900.]

Ex. 9. At a certain place in latitude ϕ the sun is observed on one day to rise h hours before noon, and on the next day to rise m minutes later. The sun's declination on the first day is δ. Prove that the distance in minutes of arc between the two points of the horizon at which it rises is

$$15m \cos^2 \delta \operatorname{cosec} \phi.$$

[Coll. Exam.]

Ex. 10. An observer in latitude 45° climbs an isolated hill whose height is $1/n$ of a nautical mile. Show that he will see a star which rises in the N.E. point approximately $8\sqrt{(6/n\pi)}$ minutes earlier than if he had remained below. [Math. Trip. I.]

We find as in Ex. 7 (i) $\Delta z = \cos \phi \cos \delta \sin h \Delta h$,

but now $\cos \delta \sin h = 1/\sqrt{2}, \quad \cos \phi = 1/\sqrt{2}$,

whence $\Delta h = 2\Delta z$.

A point on the horizon and the centre of the earth subtend at the observer an angle $90° - \Delta z$ where Δz known as the dip of the horizon $= \{2 \text{ height}/\text{radius of earth}\}^{\frac{1}{2}}$, and thus the required result easily follows.

Ex. 11. The setting sun slopes down to the horizon at an angle θ. Prove that in latitude ϕ, at the time of year when the declination of the sun is δ, a mountain whose height is $1/n$ of the earth's radius will catch the sun's rays in the morning $12\sqrt{2} \operatorname{cosec} \theta \sec \delta / \pi \sqrt{n}$ hours before the sun rises on the plain at its base; and estimate to the nearest minute the value of this expression at the summer solstice for a mountain three miles high in latitude 45°. [Math. Trip. I.]

Ex. 12. Show that in a place of latitude ϕ the sunrise at the equinoxes will be visible at the top of a mountain h feet high, about $4\sqrt{h} \sec \phi$ seconds before being seen at the foot of the mountain. [Coll. Exam.]

Ex. 13. At a certain place the moon rises on two consecutive days at the same sidereal time. Show that the place of observation lies within five degrees of the Arctic or Antarctic circles. [Coll. Exam.]

When the plane of the moon's orbit coincides with the horizon, the sidereal time of rising on consecutive days will be the same.

Ex. 14. Show that once a month in places about the latitude of London the moon sets for two or three days with the least retardation in the hour of setting, and find approximately the age of the moon and the time of day when the phenomenon occurs in June.

The moon must be in Libra and in June the sun is in Cancer. The moon will then be nearest its first quarter and will set about midnight.

Ex. 15. Taking the horizontal refraction as 35′ and the sun's semi-diameter as 16′, and defining the beginning and end of daylight as the moments when the sun's upper limb appears on the horizon, show that the increase in the duration of daylight, taking account of the refraction and semi-diameter, varies from $6^m{\cdot}8 \sec\phi$ at the equinoxes to

$$6^m{\cdot}8 \{\sec(\phi+\omega)\sec(\phi-\omega)\}^{\frac{1}{2}} \text{ at the solstices.}$$

[Coll. Exam. 1902.]

By elimination of h the equation (i) of Ex. 7 may be written

$$\Delta h = \{\sec(\phi-\delta)\sec(\phi+\delta)\}^{\frac{1}{2}}\,\Delta z.$$

$\Delta z = 35' + 16'$ or in time $3^m{\cdot}4$, and the total gain of daylight at rising and setting is $2\Delta h$.

Ex. 19. Show that near the equator the phenomenon known as the harvest moon will not be so marked as in the temperate regions, but that it will recur at each equinox. [Math. Trip. 1902.]

At the equator $n = 90° - \omega$ is the least value of n and it is the same at each equinox, and the equation

$$\Delta\vartheta = \cos\omega\,\Delta\lambda$$

gives the least retardation.

Ex. 20. Two stars used to come simultaneously to the horizon of a place in geocentric latitude $\cot^{-1}(\sqrt{3}\,.\sin\omega)$ at 0^h sidereal time. When the precession has reached 60°, the same stars will come simultaneously to the horizon of a place whose latitude is $\omega + \cot^{-1}(2\tan\omega)$ at 6^h sidereal time, where ω is the obliquity of the ecliptic. [Coll. Exam.]

If a, δ be the R.A. and declination of one of the stars, we have

$$\tan\delta = -\cot(\text{lat.})\cos a = -\sqrt{3}\sin\omega\cos a.$$

In the general formulae of § 57 we make $k = 60°$, and $\omega = \omega'$ so that

$$\sin\delta' = -\tfrac{1}{2}\sin\omega\cos\omega\cos\delta\,(\sin a + \sqrt{3}\cos\omega\cos a),$$

$$\cos\delta'\sin a' = \tfrac{1}{2}\cos\delta\,(1+\sin^2\omega)(\sin a + \sqrt{3}\cos\omega\cos a),$$

whence $$\sin\delta'/\cos\delta'\sin a' = -\sin\omega\cos\omega/(1+\sin^2\omega),$$

but for a star a', δ' rising at 6^h at latitude ϕ' we have

$$\sin\delta'\sin\phi' + \cos\delta'\cos\phi'\sin a' = 0,$$

whence $\tan\phi' = (1+\sin^2\omega)/\sin\omega\cos\omega$ and $\phi' = \omega + \cot^{-1}(2\tan\omega)$.

128. To find the mean time of rising or setting of the sun.

From the formula

$$\cos z = \sin\phi\sin\delta + \cos\phi\cos\delta\cos h \quad \text{............(i)}$$

we easily show that $\tan\frac{1}{2}h$ is equal to

$$\pm\{\sin\tfrac{1}{2}(z+\phi+\delta)\sin\tfrac{1}{2}(z-\phi-\delta)\sec\tfrac{1}{2}(z+\phi-\delta)\sec\tfrac{1}{2}(z-\phi+\delta)\}^{\frac{1}{2}}.$$

If ϕ be the latitude of the observer, and δ the declination of any celestial body without appreciable parallax, then h is the hour angle at its rising or setting if we take the horizontal refraction to be 35′ and $z = 90° \ 35'$.

If the object whose rising we are considering had been a *star* then the time it would require to reach the meridian would be h sidereal hours. In the case of the sun the actual movement through the heavens is slower than the star on account of the apparent annual motion of the sun. Of course the amount varies according to circumstances, but on the average the motion of the sun in hour angle relatively to the motion of a star in hour angle is in the proportion of mean solar time to sidereal time. In the case now before us we may always assume with sufficient accuracy that the actual motion of the sun is the same as its mean motion, and consequently the sun reaches the meridian h hours of mean solar time after rising.

At actual noon the apparent time is 12 hours and the mean time is $12^h + \epsilon$, where ϵ is the equation of time. The sunrise took place h mean hours previously, and accordingly the civil time of sunrise is

$$12^h + \epsilon - h.$$

The hour of sunrise being thus known within a minute or two, the declination of the sun at the time can be obtained accurately, and with this corrected declination the computation of h can be repeated and a corrected time of rising is thus ascertained. It is, however, needless to take this trouble since the calculation is affected by the horizontal refraction, the amount of which is quite uncertain. We have taken it as 35′, but it may differ from this by at least 1′.

To obtain the mean time of sunset we observe that at apparent noon the correct mean clock shows a time ϵ, and the ensuing sunset will take place h hours of mean solar time later, whence the time of sunset is

$$\epsilon + h.$$

As an example we shall find the time of sunrise and sunset at Greenwich (Lat. 51° 29′) on 6 June 1908. We have from the ephemeris $\delta = 22° \ 39'$, whence from (i) we compute $h = 122° \ 49'$, or in time $8^h \ 11^m{\cdot}3$. This is the hour angle of the sun when apparently on the horizon. The equation of time is $-1^m{\cdot}6$ and by substitution in $12^h + \epsilon - h$ and $h + \epsilon$, we see that the times of sunrise and sunset are respectively $3^h\ 47^m$ a.m. and $8^h\ 10^m$ p.m.

About the time of the solstice the declination of the sun does not vary more than about 1′ from its mean value in a week. Hence for that week h will be nearly constant and any variations in the mean time of rising and setting can be attributed only to changes in the equation of time. A small effect arising from this is noticeable at the winter solstice. The equation of time is then increasing, so that if ϵ be the equation of time at the solstice and ϵ_1 what it becomes a few days later we have

$$12 + \epsilon_1 - h > 12 + \epsilon - h,$$

and therefore the sun rises later, according to mean time, a few days after the solstice than it did at the solstice. Also if ϵ_2 be the equation of time a few days before the solstice, then

$$\epsilon + h > \epsilon_2 + h,$$

so that a few days before the winter solstice the sun sets earlier than it does at the solstice.

For example on 1908 Dec. 14 the sun sets at Greenwich at $3^h\ 49^m$, and on 1908 Dec. 22 (solstice) it sets at $3^h\ 52^m$. On the other hand the sun rises on the solstice at $8^h\ 6^m$ and a week later it rises at $8^h\ 8^m$.

129. Rising and setting of the moon.

In considering the rising and setting of the moon we have to take the parallax of the moon into account. Parallax tends to depress the moon from the zenith distance through the average distance of 57′. Hence when the moon appears to be on the horizon its true zenith distance measured from the earth's centre is $90° - 57′$. It is, however, apparently raised 35′ by refraction, so that in formula (i) § 128 we are to take $z = 90° - 57′ + 35′ = 89°\ 38′$.

If the place of the moon were known at the time of rising we could find h by the formula already given. The sidereal time at the moment of rising would therefore be $\alpha - h$, where α is the moon's right ascension. The mean time would then have to be calculated from the sidereal time.

But of course this process as here described is impracticable, for the place of the moon cannot be determined until the hour of rising is known. The problem must therefore be solved by approximation.

To illustrate the method we shall compute the time of moonrise at Greenwich on 10th Feb. 1894.

For the first approximation it will suffice to take for the place of the moon $\alpha = 0^h\ 55^m$; $\delta = +6^\circ\ 17'$ as given in the ephemeris at noon on the day in question. Introducing this value of δ into (i) we find $h = 6^h\ 29^m$. At rising, therefore, the hour angle of the moon was *about* $6^h\ 29^m$, and as its R.A. is *about* $0^h\ 55^m$ it follows that the sidereal time at the time of rising was *about* $18^h\ 26^m$. The sidereal time at mean noon on Feb. 10th was $21^h\ 22^m{\cdot}2$. Hence it follows that the rising of the moon must have been *about* 3 hours before noon, *i.e.* about 9 a.m., that is, in astronomical language, 21 hours on Feb. 9th.

We therefore repeat the calculations, taking for the R.A. and decl. of the moon their values, for Feb. 9th at 21 hours, viz. $\alpha = 0^h\ 49^m{\cdot}4$; $\delta = 5^\circ\ 31'$, and we find that the accurate value of the moon's hour angle at rising was $6^h\ 25^m{\cdot}5$. Subtracting this from the R.A. $0^h\ 49^m{\cdot}4$ we see that the sidereal time of rising was $18^h\ 23^m{\cdot}9$. As the sidereal time at mean noon on the 10th is $21^h\ 22^m{\cdot}2$, it follows that the interval between rising and noon is $2^h\ 58^m{\cdot}3$ of sidereal time. Transformed into solar time this becomes $2^h\ 57^m{\cdot}8$, and consequently the moon rises on the morning of 10th Feb. 1894 at $9^h\ 2^m$ a.m.

To find the mean time at which the moon sets on the day in question we can dispense with part of the work when the time of rising has been found. The hour angle of the moon at rising on Feb. 10th 1894 we have seen to be $6^h\ 25^m$, and if we neglected the motion of the moon this would also be the hour angle of setting. The setting would then be $12^h\ 50^m$ after the rising. But as the moon's motion would carry it over about half-an-hour this period would be $13^h\ 20^m$. As the rising took place at $9^h\ 2^m$ a.m. the setting would therefore be between 10 p.m. and 11 p.m. We may therefore assume for the R.A. and the δ of the moon their tabular values for 10.30 p.m., viz.: $\alpha = 1^h\ 15^m{\cdot}8$; $\delta = 8^\circ\ 57'$. The hour angle at setting is then calculated by (i) to be $6^h\ 43^m{\cdot}2$, and the R.A. of the moon being then $1^h\ 15^m{\cdot}8$, the sidereal time at setting is $7^h\ 59^m{\cdot}0$. Increasing this by 24^h and then subtracting the sidereal time at mean noon $21^h\ 22^m{\cdot}2$, we have as the sidereal interval after mean noon at which the moon sets $10^h\ 36^m{\cdot}8$, and consequently the mean time is $10^h\ 35^m$ p.m.

For the practical computations of the hour of moonrise at a particular place as required in an almanac, it is an assistance to form a table of single entry in which for the given latitude the hour angle of the moon, at rising or setting, is shown for each degree of lunar declination.

130. Twilight.

The twilight after sunset and before sunrise has been shown to be an indirect sunlight which we receive by reflection of sunbeams from particles suspended in the atmosphere. When the sun is not more than 18° below the horizon its beams illumine floating particles which are still above the horizon, and each of these becomes a source of light. Thus after sunset there is still some light until the sun is 18° below the horizon, and in like manner the approach of day is announced by the twilight which begins when the sun comes within 18° of the horizon.

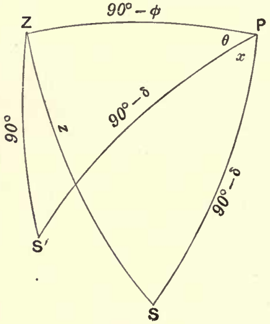


FIG. 95.

To find x, the duration of twilight, we shall investigate in general the time that elapses at a given latitude ϕ between the moment when the sun S (Fig. 95) is at the zenith distance z, and the moment of its arrival on the horizon at S'. Let θ be the hour angle of the sun when on the horizon, then $\theta + x$ is its hour angle when twilight begins, and

$$\cos z = \sin\phi \sin\delta + \cos\phi \cos\delta \cos(\theta + x),$$
$$0 = \sin\phi \sin\delta + \cos\phi \cos\delta \cos\theta,$$

adding and subtracting

$$\cos z - 2\sin\phi \sin\delta = 2\cos\phi \cos\delta \cos(\theta + \tfrac{1}{2}x)\cos\tfrac{1}{2}x,$$
$$-\cos z = 2\cos\phi \cos\delta \sin(\theta + \tfrac{1}{2}x)\sin\tfrac{1}{2}x,$$

multiplying the first by $\sin\frac{1}{2}x$ and the second by $\cos\frac{1}{2}x$, then squaring and adding, we eliminate θ and obtain

$$(\cos z - 2\sin\phi\sin\delta)^2 \sin^2\tfrac{1}{2}x + \cos^2 z \cos^2\tfrac{1}{2}x$$
$$= \cos^2\phi \cos^2\delta \sin^2 x \;\ldots\; \text{(i)}.$$

This equation gives x when δ is known, and for the problem of the duration of twilight we are to make $z = 108°$. Of course δ is known when the time of year is known.

To find the time of year at which the twilight is stationary we express that $dx/d\delta = 0$, and thus obtain

$$\sin^2 \tfrac{1}{2}x = \tfrac{1}{2}\sec^2\phi\,(2 - \sin\phi \operatorname{cosec}\delta \cos z),$$

$$\cos^2 \tfrac{1}{2}x = \tfrac{1}{2}\sin\phi \sec^2\phi \operatorname{cosec}\delta\,(\cos z - 2\sin\phi \sin\delta),$$

which by substitution in (i) gives after a little reduction

$$\cos z = 2\sin\delta \sin\phi/(\sin^2\delta + \sin^2\phi),$$

or

$$\sin\delta/\sin\phi = \tan(45° - \tfrac{1}{2}z),$$

and making $z = 108°$

$$\sin\delta = -\tan 9° \sin\phi.$$

When the latitude is known δ can be calculated from this equation and the time of year is thus found.

Ex. 1. Supposing that twilight begins or ends when the sun is 18° below the horizon, show that, so long as the sun's declination is less than 18°, all places have a day of more than 12 hours, including the twilights.

Ex. 2. Show that at a place in latitude ϕ the shortest duration of twilight when expressed in hours is

$$\tfrac{2}{15}\sin^{-1}(\sin 9° \sec\phi),$$

where $\sin^{-1}(\sin 9° \sec\phi)$ is expressed in degrees.

Ex. 3. Assuming that the sun moves uniformly in the ecliptic in 365 days, show that in latitude ϕ the number of nights in which there is twilight all night is the integer next greater than

$$\tfrac{73}{36}\cos^{-1}\{\cos(\phi + 18°)/\sin\omega\},$$

where 18° is the greatest angular distance below the horizon for twilight to be possible and ω is the obliquity of the ecliptic. [Coll. Exam.]

Ex. 4. If the length of the day be defined to be the period during which the sun is within $90° + a°$ of the zenith, show that at a station on the equator the day is $12 + \tfrac{2}{15}a \sec\delta$ mean solar hours, if δ be the declination of the sun, and that if $\sin a \sin\delta + \sin\phi = 0$ the lengths of two consecutive days will be equal at a station in latitude ϕ.

Ex. 5. Show that no two latitudes have the same length of day except at the equinoxes; but if daylight be considered to begin and terminate when the sun is θ degrees below the horizon, there are two latitudes which have the same duration of daylight so long as the sun's declination is numerically less than θ degrees. [Math. Trip.]

131. The sun-dial.

We may suppose that the place of the sun on the celestial sphere does not change appreciably in 24 hours, and a plane through the earth's axis and the sun will intersect the terrestrial equator in two points which move uniformly round the equator in consequence of the sun's apparent diurnal rotation.

In like manner we see that if a post were fixed perpendicularly into the earth at the north pole so as to be coincident with the earth's axis its shadow would move uniformly round the horizon, so that the position of the sun, and therefore the apparent time would be indicated by the point in which the shadow crossed a uniformly graduated circle with its centre in the axis of the post, and its plane perpendicular to the earth's axis. Thus we have the conception of the sun-dial.

As the dimensions of the earth are so inconsiderable in comparison to the distance of the sun, we may say that, if at any point of the earth's surface a post, or *style* as it is called, be fixed parallel to the earth's axis, the shadow of the style cast by the sun in its daily motion on a plane perpendicular to the style will move round uniformly, and by suitable graduation will show the apparent time. The hour lines on the dial are to be drawn at equal intervals of 15°. The inclination of the style to the horizon equals the latitude, and the inclination of the dial to the horizon is the colatitude. Thus we have what is known as the *equatorial* sun-dial.

While the style is always parallel to the earth's axis the plane of the dial may be arranged in different positions, horizontal, vertical, or otherwise. The graduation of the dial is uniform only in the equatorial sun-dial, and we have now to consider the graduation of the dial when otherwise placed so that the shadow of the style shall indicate the apparent time.

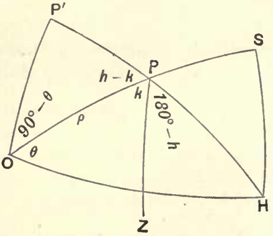


FIG. 96.

Suppose that the pole of the plane of the dial is at the point O of the celestial sphere of which the north polar distance is ρ and west hour angle k.

Let P (Fig. 96) be the north celestial pole, PZ the meridian, PP' the hour circle containing the sun, SH the plane of the dial. The point S is called the *substyle* and $PS = 90° - \rho$ is the *height of the style.* The hour line corresponding to the hour circle PP' is given by H where $OH = 90°$.

To graduate the dial we require to know the arc $SH = \theta$ corresponding to each solar hour angle h. Produce HP to P' so that $HP' = 90°$. P' must be a right angle, because $OH = 90°$ and consequently

$$\tan\theta = \cos\rho\tan(h - k) \qquad \text{(i)}.$$

As ρ and k are known this equation gives the value of $\theta = SH$ for each value of h.

To mark the hour lines on any particular instrument by observation we proceed as follows. It is assumed that there is an ordinary graduation from 0° to 360° on the dial, the centre of graduation being the point in which the style meets the plane of the dial and the origin from which the angles are measured being the line through this point and S the substyle. It is also assumed that ρ is a known angle. When the sun has a known hour angle h_1, let the observed position of the shadow be θ_1, and we have

$$\tan\theta_1 = \cos\rho\tan(h_1 - k) \qquad \text{(ii)}.$$

We thus find k and consequently for each value of h we can compute from (i) the corresponding value of θ. Thus the sun-dial will show at any moment the hour angle of the sun or the apparent time, and by application of the equation of time the mean time is ascertained.

The form of sun-dial most usually seen is the so-called horizontal sun-dial in which, as the dial is to be horizontal, O must coincide with the zenith Z; we thus have $k = 0$, and

$$\rho = PZ = 90° - \phi,$$

where ϕ is the latitude. Thus the equation (i) becomes

$$\tan\theta = \sin\phi\tan h.$$

The last hour lines that need be drawn on the dial are those corresponding to the case where the sun reaches the horizon when its declination is greatest. In this case if h' be the hour angle

$$\cos(180 - h') = \tan\phi\tan(23°\ 28'),$$

when the value of h' thus obtained is substituted for h in (i) we obtain θ.

An extreme type of sun-dial† is that in which the dial is parallel to the meridian, and the style is parallel to the earth's axis but not in the plane of the dial.

Let PZP' (Fig. 97) be the dial parallel to the plane of the meridian, AB is a thin rectangle standing perpendicular to the plane of the paper, and of which the upper edge AB, parallel to the terrestrial axis PP', is the style. The sun in the diurnal motion may be supposed to be carried by a plane rotating uniformly round AB, and hence the shadow $A'B'$ of the edge AB will always be parallel to AB and at, let us say, the distance x. When the sun is in the meridian x is infinite. When the sun's hour angle is 6^h then $x=0$. In general if d be the height of the style above the dial

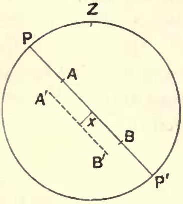


FIG. 97.

$$x = d \cot h,$$

where h is the sun's hour angle. From this equation the value of x for each value of h can be found.

Ex. 1. Show how to construct a sun-dial of which the dial shall be vertical and facing due south, and in which the style is directed to the south pole.

This may be obtained as a particular case of the general theory by making $k=0$, $\rho=\phi$ in equation (i) or directly as follows.

Let S (Fig. 98) be the point on the horizon due south, N the nadir, P' the south pole, $P'H$ the hour circle containing the sun, then from the triangle $NP'H$ we have

$$\tan NH = \cos\phi \tan h.$$

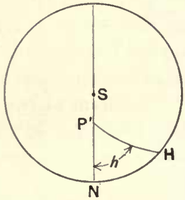


FIG. 98.

Ex. 2. Prove that any sun-dial can be graduated by the following rule: Let T_0 be the time at which the shadow of the style was projected normally on the disc, ω the N.P.D. on the celestial sphere of the normal to the disc, then the mark for time T is inclined to the mark for time T_0 at an angle

$$\tan^{-1}\{\cos\omega \tan(T-T_0)\}.$$

[Coll. Exam.]

† An example of this kind of sun-dial occurs at Wimborne Minster.

Ex. 3. The lengths of the shadows of a vertical rod of unit length are observed when the sun is on the meridian on two days separated by a quarter of a year, and are found to be x, x'. Supposing the sun to move uniformly in the ecliptic, prove that the longitude L of the sun at the date of the earlier observation is given by

$$\sin 2L = \frac{\sin^2 \omega - \sin^2 \beta}{\sin^2 \omega \cos \beta},$$

where
$$\tan \beta = \frac{x \sim x'}{1 + xx'},$$

and ω is the obliquity. [Math. Trip. I. 1900.]

Ex. 4. In a horizontal sun-dial of the usual form, show that the locus traced out by the end of the shadow of the style during one day is approximately a conic section of eccentricity

cos (latitude) cosec (declination of sun).

Ex. 5. If x be the angle between the graduations on a horizontal sun-dial indicating h_1, h_2 hours after noon, then

$$\tan x = \frac{\sin \lambda \sin \left\{(h_2 - h_1)\frac{\pi}{12}\right\}}{\cos \left\{(h_2 - h_1)\frac{\pi}{12}\right\} - \cos^2 \lambda \sin \frac{h_1 \pi}{12} \sin \frac{h_2 \pi}{12}},$$

where λ is the latitude for which the dial is made. [Coll. Exam.]

Ex. 6. Show that at a place outside the Arctic and Antarctic regions the end of the shadow of a vertical post cast by the sun on a horizontal plane approximately describes in the course of the day one branch of an hyperbola, and that as the hyperbola varies from day to day its asymptotes touch a fixed parabola the focus of which is the foot of the post.

[Math. Trip. I. 1904.]

Ex. 7. A sun-dial is constructed of a reflecting cylinder whose cross-section is a cycloid, mounted upon a card so that the generating lines of the cylinder are parallel to the earth's axis and perpendicular to the plane of the card, whilst the axis of the cycloidal cross-section lies in the plane of the meridian. Prove that, if the distance between the cusps of the cycloid on the card be provided with a proper uniform graduation, the cusp of the caustic due to the reflection of the solar rays will always indicate apparent solar time. [Coll. Exam.]

132. Coordinates of points on the sun's surface.

From the observation of sun spots it has been shown that the sun rotates about an axis inclined to the ecliptic at an angle of 82° 45′. The direction of this rotation is the same as that in which the earth and the other planets revolve round the sun. A plane through the sun's centre perpendicular to the axis of rotation intersects the sun's surface in a great circle known as the *solar equator*. Points on the sun's surface are said to have heliographic latitude and longitude with reference to the solar equator. The heliographic latitude of a solar point S is the perpendicular arc from S to the solar equator and the longitude of S is the arc from a standard point O' on the solar equator to the foot of the perpendicular.

In Fig. 99 ON is the section of the surface of the sun by

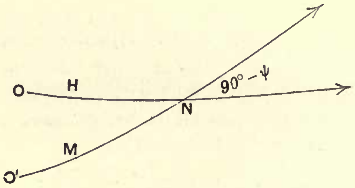


FIG. 99.

the plane of the ecliptic, where O is the point in which the sun's surface is met by the line from the sun's centre to ♈ and the longitudes increase in the direction shown by the arrow-head. $O'N$ is the solar equator and N is the ascending node of the solar equator on the ecliptic. This point remains fixed in the plane of the ecliptic because the sun's equator has no recognisable motion of precession. The longitude H of N measured on the ecliptic from O the equinox of 1909·0 is 74 °29′·4. As the sun is not a solid body and therefore cannot have a permanent "Greenwich," resort is made to a special method for indicating the point O' which is adopted as the origin of *solar* longitudes. The point O' is defined to be the particular point of the solar equator which happened to be passing through N at Greenwich mean noon on 1st Jan. 1854. By the rotation of the sun O' is carried towards N with a uniform motion which would bring it round the circumference in 25·38 days. The solar equator is inclined to the ecliptic at the angle $90° - \psi = 7° \ 15'$.

The coordinates β, λ are the latitude and longitude of a point P on the sun's surface with respect to ON and measured from the origin O. In like manner λ', β' are the heliographic coordinates of P with respect to $O'N$ and measured from the origin O'.

From the general formulae of transformation, § 12, we have

$$\left.\begin{aligned}\sin\beta' &= \sin\beta\sin\psi - \cos\beta\cos\psi\sin(\lambda - H)\\ \cos\beta'\cos(\lambda' - M) &= \cos\beta\cos(\lambda - H)\\ \cos\beta'\sin(\lambda' - M) &= \sin\beta\cos\psi + \cos\beta\sin\psi\sin(\lambda - H)\end{aligned}\right\}\ldots\text{(i)}.$$

To obtain the heliographic latitude and longitude, usually termed D and L, of the apparent centre of the sun's disc we substitute for β, λ in the equations just found the values 0, $180° + \odot$, where $\odot$ is the geocentric longitude of the sun and we have

$$\left.\begin{aligned}\sin D &= \cos\psi\sin(\odot - H)\\ \cos D\cos(L - M) &= -\cos(\odot - H)\\ \cos D\sin(L - M) &= -\sin\psi\sin(\odot - H)\end{aligned}\right\}\ldots\ldots\ldots\text{(ii)},$$

from which D, L the required heliographic coordinates of the centre of the sun's disc can be obtained without ambiguity.

We now seek the expression for $\cos P$ where P is the arc on the sun's limb between the northernmost point of the disc and the projection of the solar axis on the plane of the disc.

The longitude and latitude of S the nole of the sun's equator on the celestial sphere are found by making $\lambda' = 0$, $\beta' = 90°$ in (i) from which we see that $\lambda = 270° + H$, $\beta = \psi$. The solution $\beta = 180° - \psi$ is of course rejected because $\psi = 82° 45'$ and $\beta \not> 90°$. The longitude and latitude of E the nole of the earth's equator on the celestial sphere are given by $\lambda = 90°$, $\beta = 90° - \omega$. The longitude and latitude of T the heliocentric position of the earth are given by $\lambda = 180° + \odot$, $\beta = 0$ where $\odot$ is the sun's geocentric longitude. Then P the angle required is equal to $\angle STE$. To obtain the expression for it we have

$$\cos ST = \cos\psi\sin(\odot - H); \quad \cos ET = -\sin\omega\sin\odot;$$
$$\cos ES = \sin\psi\cos\omega - \cos\psi\sin\omega\cos H,$$

and substituting in

$$\cos P = (\cos ES - \cos ST\,.\cos ET)/\sin ST\,.\sin ET,$$

we have

$$\cos P = \pm\frac{\cos\omega\sin\psi - \sin\omega\cos\psi\cos\odot\cos(\odot - H)}{(\cos^2\omega + \sin^2\omega\cos^2\odot)^{\frac{1}{2}}\{\sin^2\psi + \cos^2\psi\cos^2(\odot - H)\}^{\frac{1}{2}}},$$

and

$$\sin P = -\frac{\sin\omega\sin\psi\cos\odot + \cos\omega\cos\psi\cos(\odot - H)}{(\cos^2\omega + \sin^2\omega\cos^2\odot)^{\frac{1}{2}}\{\sin^2\psi + \cos^2\psi\cos^2(\odot - H)\}^{\frac{1}{2}}}.$$

To show that the negative sign should be attributed to $\sin P$ it suffices to take the case of $\psi = 90°$, $\odot = 180°$. It is then obvious that the position angle P must be $+\omega$, but this would not be the case unless the radical in the expression of $\sin P$ had a negative sign.

As $\sin P$ may be written in the form $f \cos(\odot + h)$ where f is a negative quantity and where h is independent of $\odot$ it is easily shown that P is positive for one half the year (from July 7 to Jan. 5) and negative for the remaining half. The maximum value of P is $+26°\cdot42$ on October 8 and the minimum is $-26°\cdot44$ on April 6.

Ex. 1. It is required to find the value of P on July 15th 1909 from the following data:

$$\omega = 23°\ 27'; \quad \psi = 82°\ 45'; \quad \odot = 112°\ 19'; \quad H = 74°\ 29'.$$

It is easy to see that $\sin \omega \sin \psi \cos \odot = -\cdot14991$;

$$\cos \omega \cos \psi \cos(\odot - H) = \cdot09144; \quad \cos^2 \omega + \sin^2 \omega \cos^2 \odot = \cdot86446;$$

$$\sin^2 \psi + \cos^2 \psi \cos^2(\odot - H) = \cdot99400, \text{ whence } P = 3°\cdot62.$$

We find in the appendix to the nautical almanac the values of P as well as of the other elements D, L.

Ex. 2. The meridian of the sun which passed through the ascending node on the ecliptic of the sun's equator on 1854 Jan. 1, Greenwich mean noon, is the zero meridian for physical observations on the sun and heliographic longitudes are measured from this zero meridian and heliographic latitudes from the solar equator. Assuming that the node remains fixed determine its heliographic longitude at noon on 1909 July 15, if the period of the sun's rotation be 25·38 days.

From mean noon on 1854 Jan. 1 to mean noon on 1909 July 15 is an interval of 20283 days (1900 is *not* a leap year). By dividing 20283 by 25·38 we obtain the number of rotations of the sun 799·17258. The zero meridian has therefore advanced ·17258 of a complete revolution beyond the node, *i.e.* $\cdot17258 \times 360° = 62°\ 7'\cdot7$. Hence the heliographic longitude of the node of the solar equator on the ecliptic is

$$360° - (62°\ 7'\cdot7) = 297°\ 52'\cdot3.$$

The longitude of the ascending node measured on the ecliptic from the first point of Aries (1909·0) is 74° 29′·4.

Ex. 3. It is required to find when the position angle P of the sun's axis is a maximum.

Differentiating the expression for $\sin P$ with regard to $\odot$ and equating the result to zero, we have for the determination of l the equation $A \cdot B = 0$, where

$$A = \sin \omega \cos \psi \cos(\odot - H) \cos \odot - \cos \omega \sin \psi,$$

$$B = \{\sin^2 \psi + \cos^2 \psi \cos^2(\odot - H)\} \sin \omega \cos \omega \sin \odot$$
$$+ (\cos^2 \omega + \sin^2 \omega \cos^2 \odot) \sin(\odot - H) \sin \psi \cos \psi.$$

The equation $A=0$ could give no real values of $\odot$ for as ψ is near 90° the second term is larger than the first could become. We therefore seek the values of $\odot$ from the second factor $B=0$, which may be written

$$\{\tan^2\psi+\cos^2(\odot-H)\}\tan\omega\sin\odot+(1+\tan^2\omega\cos^2\odot)\sin(\odot-H)\tan\psi=0.$$

As a first approximation we may omit $\cos^2(\odot-H)$ and $\tan^2\omega\cos^2\odot$, and we then have $3{\cdot}41\ \sin\odot=\sin(74°\ 29'-\odot)$, whence $\odot=14°\ 42'$. Using this approximate value in the terms omitted before, we obtain

$$2{\cdot}91\sin\odot=\sin(74°\ 29'-\odot),$$

whence $\odot=16°\ 51'$ or $196°\ 51'$. The first is on April 7th and the second on Oct. 10th. Substituting either of these values for $\odot$ in the original expression for $\sin P$ we see that $P=26°{\cdot}5$.

Ex. 4. On what days in the year does P become zero?

If $\sin P=0$, we must have

$$\tan\odot=-\frac{\sin\omega\sin\psi+\cos\omega\cos\psi\cos H}{\cos\omega\cos\psi\sin H},$$

and with the values of the constants ω, ψ, H already given, we have $\odot=104°\ 40'$ and $\odot=284°\ 40'$. By reference to the ephemeris we see that the sun has these longitudes on July 7 and Jan. 5 respectively.

*133. Rotation of the moon.

The character of the rotation of the moon about its centre of gravity is very approximately given by the three following laws, known as the Laws of Cassini†.

1. The moon rotates round its axis in a time which is accurately equal to the time of the moon's revolution round the earth.
2. The inclination of the lunar equator to the ecliptic is permanently 1° 32′ 6″.
3. The ascending node of the moon's equator on the ecliptic coincides with the descending node of the moon's orbit on the ecliptic.

The third law may be expressed by stating that the longitude of the nole of the moon's equator exceeds by 90° the longitude of the ascending node of the moon's orbit. The latitude is of course $90°-1°\ 32'\ 6''=88°\ 27'\ 54''$.

From these rules we can find for each day the three following quantities: i the inclination of the moon's equator to the terrestrial equator, ☊′ the R.A. of the ascending node of the moon's equator

† See Dr J. Franz, *Observations at the Observatory of Kœnigsberg*, Vol. 38. The value of the inclination of the lunar equator used in expressing Law 2 is that given by Hayn in *Astronomische Nachrichten*, No. 4083.

on the terrestrial equator, and Δ the arc of the moon's equator from its ascending node on the earth's equator to its ascending node on the ecliptic. ☊ is as usual the longitude of the ascending node of the moon's orbit on the ecliptic.

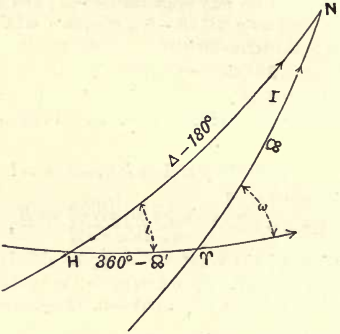


FIG. 100.

♈ (Fig. 100) is the vernal equinox, N is the ascending node of the moon's orbit on the ecliptic ♈N and therefore by Law 3 the descending node of the moon's equator HN, and as Δ is measured from H to the ascending node we have

$$HN = \Delta - 180°.$$

In this spherical triangle ω and I have the values 23° 27′ 4″ and 1° 32′ 6″ respectively. ☊ is a function of the time given in the ephemeris for intervals of ten days throughout the year. For each value of ☊ the quantities i, Δ, ☊′ are computed by the following formulae

$$\begin{cases} \cos i = \cos\omega\cos I + \sin\omega\sin I\cos ☊, \\ \sin i\sin\Delta = -\sin\omega\sin ☊, \\ \sin i\cos\Delta = \cos\omega\sin I - \sin\omega\cos I\cos ☊. \end{cases}$$

$$\begin{cases} \cos i = \cos\omega\cos I + \sin\omega\sin I\cos ☊, \\ \sin i\sin ☊' = -\sin I\sin ☊, \\ \sin i\cos ☊' = \cos I\sin\omega - \sin I\cos\omega\cos ☊. \end{cases}$$

These equations are not independent and indeed the first and the fourth are identical. But the first three enable i and Δ to be found without ambiguity and the last three in like manner give i and ☊′ and the coincidence of the two values of i thus independently found provides a useful check on the accuracy of the work.

As the period of rotation of the moon coincides with that of its revolution round the earth the moon keeps nearly the same face to the earth. Owing however to the inclination of the moon's equator to the ecliptic and to other circumstances connected with its motion a certain margin round the moon's limb occasionally passes out of view and a corresponding margin on the other side comes into view. This phenomenon is known as the *libration* of the moon.

Ex. 1. On Sept. 28th 1908 the longitude of the moon's ascending node is 70° 46′·2, determine the inclination of the moon's equator to the earth's equator, the R.A. of the ascending node of the moon's equator on the terrestrial equator and the arc from the ascending node on the earth's equator to the ascending node on the ecliptic.

We obtain from the above formulae

$$I = 22° \, 59', \quad \Delta = 254° \, 9', \quad ☊' = 356° \, 19'.$$

Ex. 2. Show from the laws of Cassini that the nole on the moon's surface of the moon's equator may be obtained by the following construction.

From the centre of the moon regarded as a sphere draw lines to the noles of the moon's orbit and of the ecliptic and let them meet the moon's surface in A and B respectively. Produce the arc AB beyond B to C so that $BC = 1° \, 32' \, 6''$. Then C is the nole on the moon's surface of the moon's equator.

134. Sumner's method of determining the position of a ship at sea.

If from the centre of the earth a line be drawn towards the centre of the sun the line will cut the earth's surface in what is known as the *subsolar point*. Thus there is, at every moment, a subsolar point somewhere. This is the only spot on the earth at which the sun is at that moment in the zenith. The geocentric latitude of the subsolar point is obviously the declination of the sun. The longitude of the subsolar point measured eastwards† from Greenwich is 24^h − (apparent time at Greenwich).

Let us suppose the earth to be a sphere with centre E

† It is often convenient to adopt as we are here doing the continuous measurement of terrestrial longitudes *eastward* from Greenwich. The north pole is then the nole of the necessary graduation of the equator throughout the circumference.

(Fig. 101) and neglect the sun's parallax. Let ES be the direction of the sun and P the subsolar point. Let O be the position of an observer who sees the sun in the direction OS' parallel to ES and at the altitude $l = \angle HOS'$, then $\angle OEP = 90° - l$, and we see that the altitude of the sun is the complement of the angular distance of the observer from the subsolar point. Whenever an altitude l of the sun is observed the observer knows that he must be situated at that moment on the circumference of a small circle of the earth described around the subsolar point with the radius $90° - l$. If the observer knows the Greenwich time and the solar declination he knows the geographical position of the subsolar point and consequently he can draw on a stereographic chart (§ 23) the circumference of a circle on which his position must lie. Of course the observer will already know his position approximately, and hence he will not need more than a very small arc which is practically a straight line, and is called the Sumner line after the inventor of this method. Thus a single observation of the sun's altitude enables the mariner to rule a short line on his chart which passes through his actual position. To obtain that position he must repeat the observation when the sun is at a different altitude some hours later. He can then draw another Sumner line and the intersection of the two lines will give his actual position.

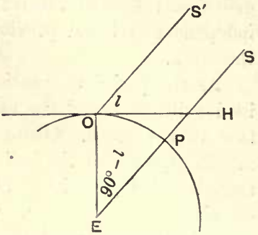


FIG. 101.

In this we have assumed that the position of the observer has not changed between the two observations. If he has been in motion and knows the course he has taken and the number of miles run he proceeds as follows. Take any point A on the first Sumner line and set off on the chart a point B so that AB represents both in magnitude and direction the distance run. Through B draw a parallel to the original Sumner line, then the ship must lie at the time of the second observation somewhere on that parallel. The intersection of this parallel with the second Sumner line gives the position of the ship at the time of the second observation.

We can find as follows the equation of the stereographic projection of the Sumner line in which the latitude and longitude of the subsolar point are δ and θ and where l is the observed altitude of the sun.

Let β, λ be the latitude and longitude of the observer, then

$$\sin l = \sin\beta \sin\delta + \cos\beta \cos\delta \cos(\lambda - \theta).$$

If x, y be the coordinates in the projection of the point corresponding to β, λ, then from the equations on p. 64

$$x = a\cos\beta\cos\lambda/(1-\sin\beta),\quad y = a\cos\beta\sin\lambda/(1-\sin\beta),$$

whence

$$1/(1-\sin\beta)$$
$$= (a\sin\delta - x\cos\theta\cos\delta - y\sin\theta\cos\delta)/(a\sin\delta - a\sin l),$$

and $$x^2 + y^2 = a^2\{2/(1-\sin\beta) - 1\},$$

eliminating $(1-\sin\beta)$ we have the desired equation of the circle

$$(x^2+y^2)(\sin\delta - \sin l) + 2a\cos\delta(x\cos\theta + y\sin\theta)$$
$$- a^2(\sin\delta + \sin l) = 0$$

As a verification we may note that if $l = 90°$ the equation becomes simply

$$\{x - a\cos\delta\cos\theta/(1-\sin\delta)\}^2 + \{y - a\cos\delta\sin\theta/(1-\sin\delta)\}^2 = 0,$$

in which case the circle reduces to the point of the chart which corresponds to the subsolar point.

Ex. 1. Two altitudes A_1, A_2 of the sun are taken, at an interval of time $2h$, and the Sumner lines cut orthogonally. Show that

$$\sin A_1 \sin A_2 = 1 - 2\sin^2 h\cos^2\delta,$$

where δ is the sun's declination. [Coll. Exam. 1903.]

Let S_1, S_2 be the two subsolar points, P the terrestrial north pole and O the station of the observer, then $\angle S_1PS_2 = 2h$, also $S_1OS_2 = 90°$ and OS_1, OS_2 are respectively $90° - A_1$, $90° - A_2$ and $PS_1 = PS_2 = 90° - \delta$.

Ex. 2. When the latitude and longitude are found by simultaneous observations of the altitudes a_1 and a_2 of two known stars, as in Sumner's method, prove that the two possible places of observation will have the same longitude if

$$\sin a_1/\sin a_2 = \sin\delta_1/\sin\delta_2,$$

where δ_1 and δ_2 are the declinations of the two stars.

[Coll. Exam. 1902.]

Ex. 3. At Greenwich sidereal time t the zenith distances of two stars of right ascensions a_1 and a_2 and equal declinations δ are observed to be z_1 and

z_2 respectively. Show that the west longitude of the place of observation exceeds $t-\frac{1}{2}(a_1+a_2)$ by ϕ, where $\cot\phi=\cos\lambda\cot x\pm\sin\lambda\operatorname{cosec} x\tan z$, and z, x and λ are auxiliary angles given by

(i) $\cot\lambda=\cot\delta\cos\frac{1}{2}(a_1-a_2)$,

(ii) $\sin\theta=\cos\delta\sin\frac{1}{2}(a_1-a_2)$,

(iii) $\tan x=\tan\frac{1}{2}(z_1-z_2)\tan\frac{1}{2}(z_1+z_2)\cot\theta$,

(iv) $\cos z=\cos\frac{1}{2}(z_1-z_2)\cos\frac{1}{2}(z_1+z_2)\sec x\sec\theta$.

[Math. Trip.]

λ is the declination of the point midway between the stars, 2θ the distance between the stars, z is the perpendicular from the zenith on the arc joining the stars and x the arithmetic mean of the distances from the foot of this perpendicular to the stars, $-\phi$ is the hour angle of the point midway between the two stars, and this point with the pole and the zenith form a triangle by which from formula (6) on p. 3 the required result is obtained.

Ex. 4. The sun's declination being 15° N. and the chronometer indicating $2^h\ 0^m$ Greenwich mean time, and the sun's observed zenith distance being 45°, prove that the equation of the corresponding Sumner line on the map formed by stereographic projection from the south pole on to a plane parallel to the equator is (in polar coordinates referred to the north pole as pole and the meridian of Greenwich as initial line)

$$r^2=2cr\cos(\theta+30°)+c^2(2\sqrt{3}-3)=0.$$

The equation of time is neglected, and c is a constant depending on the scale of the map. [Coll. Exam.]

CHAPTER XX.

PLANETARY PHENOMENA.

135. Introductory.

As we have already seen (§ 50), each of the planets moves round the sun in accordance with the laws of Kepler. As the earth is one of the planets, and consequently follows Kepler's laws, the observed movement of any other planet is complicated by the motions of the terrestrial observer. Thus the apparent movements of the planets with regard to the fixed stars are generally from west to east, but they are occasionally stationary or move from east to west.

The following nomenclature is used.

The *line of nodes* is the intersection of the plane of a planet's orbit with the ecliptic.

Regarding the ecliptic and the planetary orbit as graduated great circles in the directions of motions of the earth and planet, the inclination of the two circles is the *inclination* of the planetary orbit.

The *ascending node* of the planetary orbit is that in which the direction of movement crosses from the side of the ecliptic which contains the antinole of the ecliptic to that which contains the nole. The other node is known as the descending node.

The places of planets are defined by their *latitudes* and *longitudes*, and these places are termed *heliocentric* if they are as they would be seen by an observer from the sun and *geocentric* as they would be seen by an observer on the earth.

Thus the *heliocentric latitude* of a planet is its angular distance from the ecliptic as seen from the sun. The *heliocentric longitude* is the angle subtended at the sun by the first point of Aries and the foot of the perpendicular from the planet on the ecliptic and measured from ♈ in the positive direction.

In like manner the geocentric latitude and longitude of a celestial body are defined when the observer is presumed to be on the earth or more strictly at the centre of the earth.

To determine completely the orbit of a planet, six quantities are necessary. These are as follows:

(1) The longitude Ω of the ascending node on the ecliptic.

(2) The inclination i of the planetary orbit to the ecliptic.

(3) The longitude ϖ of the perihelion, which is measured from ♈ along the ecliptic in the positive direction to the planet's ascending node and thence in the plane of the planet's orbit in the direction of the planet's motion to the perihelion or point of its orbit in which the planet is nearest the sun.

(4) The semi-axis major a of the ellipse. This quantity is generally known as the mean distance (see p. 298).

(5) The eccentricity e of the ellipse.

(6) The epoch t, or date at which the planet passes the perihelion.

Of the six data the two first give the plane of the orbit, the third gives the position of the axis of the ellipse and the fourth and fifth the form and dimensions of the ellipse. The sixth is necessary to determine the position of the planet in its orbit.

136. Approximate determination of the orbit of a planet from observation.

As the orbits of the important planets are nearly circular we shall for this approximation suppose them to be exactly circular though in different planes, and we have first to show that if this is the case two observations of each planet would suffice to determine its orbit.

An observation of a planet, by which we understand a determination of the position of the planet on the celestial sphere sufficient to enable its latitude and longitude to be found, indicates no more than the position in space of a straight line on which at the moment the planet is somewhere situated. At the time of observation the earth's place is of course known and the observation shows the direction of a line A from the earth in which the planet must lie. An observation at a subsequent date gives, in like manner, a second straight line B in which the planet then lies and the time interval between the two observations is noted.

It has been assumed that the orbit of the planet is a circle and, of course, the centre of that circle, being the centre of the sun, is known. We have therefore to construct a circle with its centre S at a given point and such that its circumference shall intersect two given straight lines A and B. There are of course an infinite number of solutions of this problem, for choose *any* point P on A and draw the sphere with centre S and radius SP. Let Q be either of the points in which the sphere cuts B. Then the plane SPQ cuts the sphere in a circle which has its centre at S and which intersects A and B. It might therefore seem at first as if the problem of finding the circular orbit of a planet from two observations was indeterminate.

But the observation of the time interval required by the planet to move from P to Q removes the indeterminateness. When the plane SPQ is drawn the orbit thus found has a periodic time determined by Kepler's third law from the length of its radius. If the year be the unit of time and the mean distance of the earth the unit of length, and if T be the periodic time expressed in years we have $T^2 = (SP)^3$ or $T = (SP)^{\frac{3}{2}}$. The time interval between P and Q is therefore $(SP)^{\frac{3}{2}} \times \angle PSQ \div 2\pi$. This is to be compared with the time interval observed, and P is to be altered in successive trials until the observed and calculated time intervals coincide. SPQ is then the required orbit.

The following is the analytic method of investigating the same problem.

Let x, y, z be the heliocentric coordinates of the planet and a the radius of its orbit, then the equations of the orbit, where axis of x passes through ♈ and z is normal to ecliptic, are

$$x^2 + y^2 + z^2 = a^2 \quad\text{.........................(i)},$$

$$z = px + qy \quad\text{..................(ii)}.$$

Let β, λ, ρ be respectively the geocentric latitude, longitude and distance of the planet at the first observation, and R, L be respectively the distance of the earth from the sun and the earth's heliocentric longitude, then

$$x = \rho \cos\beta \cos\lambda + R \cos L,$$
$$y = \rho \cos\beta \sin\lambda + R \sin L,$$
$$z = \rho \sin\beta,$$

whence by substitution in (i) and (ii)

$$\rho^2 + 2\rho R \cos\beta \cos(L - \lambda) + R^2 = a^2 \text{............(iii)},$$
$$\rho \sin\beta = p(\rho \cos\beta \cos\lambda + R \cos L)$$
$$+ q(\rho \cos\beta \sin\lambda + R \sin L) \text{.........(iv)}.$$

In like manner from the second observation we obtain two similar equations

$$\rho'^2 + 2\rho' R' \cos\beta' \cos(L' - \lambda') + R'^2 = a^2 \text{.........(v)},$$
$$\rho' \sin\beta' = p(\rho' \cos\beta' \cos\lambda' + R' \cos L')$$
$$+ q(\rho' \cos\beta' \sin\lambda' + R' \sin L') \text{.........(vi)}.$$

If t be the time, then $2\pi t a^{-\frac{3}{2}}$ is the angle through which the planet has moved, as the earth's distance and the year are the units of distance and time respectively, hence

$$a^2 \cos(2\pi t a^{-\frac{3}{2}}) = xx' + yy' + zz'$$
$$= \rho\rho' \cos\beta \cos\beta' \cos(\lambda - \lambda') + RR' \cos(L - L')$$
$$+ \rho R' \cos\beta \cos(\lambda - L') + \rho' R \cos\beta' \cos(\lambda' - L) \text{...(vii)}.$$

There are thus five equations (iii—vii), and they contain five unknowns, viz. ρ, ρ', p, q, a.

We then proceed as follows. Assuming a value for a we obtain two values of ρ from (iii) and two of ρ' from (v) and see if one of the four pairs will satisfy (vii). Further trials must then be made until a value of a is found which gives values of ρ and ρ' that satisfy (vii). Then from (iv) and (vi) p and q are determined linearly. The node and inclination of the planet's orbit can then be found. For if x', y', 0 is the node, then $px' + qy' = 0$ or $\tan\Omega = -p/q$ and Ω or $\Omega + 180°$ is found, and $\sec i = \sqrt{1 + p^2 + q^2}$.

An approximate determination of the orbits of most of the planets can be made in this way because, the eccentricity being generally small, the orbits do not differ much from circles†.

† On the calculation of orbits see Gauss, *Theoria Motus Corporum Coelestium.*

**The argument of the latitude.* The heliocentric coordinates of a planet can be conveniently expressed in terms of the angular distance through which the planet has moved round the sun after passing its ascending node on the ecliptic. This angle is, in each case, to be measured in the direction of the motion. It is designated by u and is termed the *argument of the latitude.*

We now take as axes of $+x$, $+y$, $+z$ respectively lines from the sun's centre to the points whose R.A. and decl. are (0°, 0°), (90°, 0), (0°, 90°). Thus the coordinates of the planet at the distance r and equatorial coordinates α, δ become

$$r\cos\delta\cos\alpha, \quad r\cos\delta\sin\alpha, \quad r\sin\delta,$$

or, expressed in terms of the longitude and latitude λ, β, we easily find from (i), p. 107

$$\begin{aligned} x &= r\cos\beta\cos\lambda, \\ y &= - r\sin\beta\sin\omega + r\cos\beta\cos\omega\sin\lambda, \\ z &= r\sin\beta\cos\omega + r\cos\beta\sin\omega\sin\lambda. \end{aligned}$$

If i be the inclination of the planet's orbit to the ecliptic and Ω the longitude of its ascending node,

$$\sin\beta = \sin u\sin i, \quad \cos\beta\sin(\lambda - \Omega) = \sin u\cos i,$$
$$\cos\beta\cos(\lambda - \Omega) = \cos u,$$

from which we easily obtain

$$\cos\beta\cos\lambda = \cos u\cos\Omega - \sin u\cos i\sin\Omega,$$
$$\cos\beta\sin\lambda = \cos u\sin\Omega + \sin u\cos i\cos\Omega.$$

Eliminating β and λ from the expressions for x, y, z, we have for the coordinates of the point in the orbit corresponding to u

$$x = r\sin a\sin(A + u), \quad y = r\sin b\sin(B + u),$$
$$z = r\sin c\sin(C + u),$$

where a, b, c, A, B, C, are known as the *constants for the equator* and are found as follows:

$$\begin{aligned} \sin a\sin A &= \cos\Omega, \\ \sin a\cos A &= - \cos i\sin\Omega, \\ \sin b\sin B &= \cos\omega\sin\Omega, \\ \sin b\cos B &= \cos i\cos\omega\cos\Omega - \sin i\sin\omega, \\ \sin c\sin C &= \sin\omega\sin\Omega, \\ \sin c\cos C &= \sin i\cos\omega + \cos i\sin\omega\cos\Omega. \end{aligned}$$

It is also easy to prove that

$$\cos a = \sin i \sin \Omega,$$
$$\cos b = -\sin i \cos \omega \cos \Omega - \cos i \sin \omega,$$
$$\cos c = -\sin i \sin \omega \cos \Omega + \cos i \cos \omega,$$
$$\tan i = \operatorname{cosec} a \sin b \sin c \sec A \sin (C - B).$$

We take an example given in Watson's *Theoretical Astronomy*, where

$$\Omega = 206^\circ\ 43'\ 33''\cdot 74,$$
$$i = 4\ \ 36\ \ 50\ \cdot 11,$$
$$\omega = 23\ \ 27\ \ 24\ \cdot 03,$$

and the reader may verify that for the constants of the equator we have

$$A = 296^\circ\ 39'\ \ 5''\cdot 07 \qquad \text{Log} \sin a = 9\cdot 9997156,$$
$$B = 205\ \ 55\ \ 27\ \cdot 14 \qquad \text{Log} \sin b = 9\cdot 9748252,$$
$$C = 212\ \ 32\ \ 17\ \cdot 74 \qquad \text{Log} \sin c = 9\cdot 5221920.$$

137. On the method of determining geocentric coordinates from heliocentric coordinates and vice versa.

Let us take three rectangular axes whereof the origin is at the centre of the sun, the axis of $+x$ is the line to ♈, the axis of $+y$ to the point whose latitude and longitude are 0°, 90° and the axis of $+z$ to the nole of the ecliptic.

Let r be the distance of the planet from the sun's centre and λ, β the heliocentric longitude and latitude of the planet. Then, if x, y, z are the heliocentric coordinates of the planet,

$$x = r \cos \beta \cos \lambda\ ; \quad y = r \cos \beta \sin \lambda\ ; \quad z = r \sin \beta.$$

If R be the distance of the earth and L its longitude, and if X, Y, Z be the coordinates of the earth,

$$X = R \cos L, \quad Y = R \sin L, \quad Z = 0.$$

Let x', y', z' be the coordinates of the planet with regard to a set of parallel axes through the earth's centre, then

$$x = X + x'\ ; \quad y = Y + y'\ ; \quad z = Z + z' \quad\ldots\ldots\ldots\ldots\text{(i)},$$

and if λ', β' be the geocentric longitude and latitude of the planet and ρ its distance from the earth's centre,

$$x' = \rho \cos \beta' \cos \lambda'\ ; \quad y' = \rho \cos \beta' \sin \lambda'\ ; \quad z' = \rho \sin \beta',$$

and thus from (i) we obtain

$$\left.\begin{aligned} r\cos\beta\cos\lambda &= R\cos L + \rho\cos\beta'\cos\lambda' \\ r\cos\beta\sin\lambda &= R\sin L + \rho\cos\beta'\sin\lambda' \\ r\sin\beta &= \rho\sin\beta' \end{aligned}\right\}\quad\ldots\ldots\ldots\text{(ii)}.$$

Multiplying the first of (ii) by $\cos L$ and the second by $\sin L$ and adding

$$r\cos\beta\cos(L-\lambda) = R + \rho\cos\beta'\cos(L-\lambda');$$

multiplying the same two equations by $\sin L$, $\cos L$ respectively and subtracting

$$r\cos\beta\sin(L-\lambda) = \rho\cos\beta'\sin(L-\lambda'),$$

whence

$$\tan(L-\lambda') = \frac{r\cos\beta\sin(L-\lambda)}{r\cos\beta\cos(L-\lambda) - R}.$$

As the time of observation is known L and R are both known, and hence when the heliocentric coordinates λ, β of the planet are known $L-\lambda'$, and thence λ', are determined.

Also squaring and adding (ii) after transferring $R\cos L$ and $R\sin L$ to the other side

$$\rho^2 = r^2 - 2rR\cos\beta\cos(L-\lambda) + R^2,$$

by which ρ is found.

From the two first equations of (ii) we have

$$\rho^2\cos^2\beta' = r^2\cos^2\beta - 2rR\cos\beta\cos(L-\lambda) + R^2,$$

whence from the last of (ii)

$$\tan\beta' = \frac{r\sin\beta}{(r^2\cos^2\beta - 2rR\cos\beta\cos(L-\lambda) + R^2)^{\frac{1}{2}}},$$

whence β' is known.

In like manner we can obtain β and λ when β' and λ' are given.

138. Geocentric motion of a planet.

Let S be the sun, E the earth, P the planet (Fig. 102). It is supposed that the earth and planet revolve in circles in coplanar orbits with radii a, b respectively, that their heliocentric longitudes are L and l, and that the geocentric longitude and distance of the planet are λ and ρ.

We have

$$\left.\begin{aligned}\rho\sin\lambda &= b\sin l - a\sin L\\ \rho\cos\lambda &= b\cos l - a\cos L\end{aligned}\right\}\ldots\text{(i)},$$

whence

$$\tan\lambda = (b\sin l - a\sin L)\,/(b\cos l - a\cos L),$$

by which the geocentric longitude is obtained.

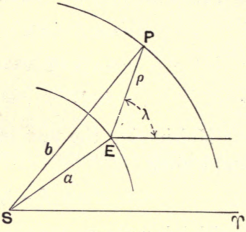


FIG. 102.

By Kepler's laws the mean motion of the planet is proportional to $b^{-\frac{3}{2}}$. We shall choose the units of time and distance such that the mean motion, *i.e.* the heliocentric angular velocity, shall be not only proportional to but actually equal to $b^{-\frac{3}{2}}$, so that

$$\frac{dl}{dt} = b^{-\frac{3}{2}}, \quad \frac{dL}{dt} = a^{-\frac{3}{2}}.$$

To investigate the changes in the angular velocity of P with respect to E we have, by differentiating (i),

$$\left.\begin{aligned}\rho\cos\lambda\frac{d\lambda}{dt} + \sin\lambda\frac{d\rho}{dt} &= \ \ b^{-\frac{1}{2}}\cos l - a^{-\frac{1}{2}}\cos L\\ -\rho\sin\lambda\frac{d\lambda}{dt} + \cos\lambda\frac{d\rho}{dt} &= -b^{-\frac{1}{2}}\sin l + a^{-\frac{1}{2}}\sin L\end{aligned}\right\}\ldots\ldots\text{(ii)},$$

whence from (i)

$$\rho^2\frac{d\lambda}{dt} = a^{\frac{1}{2}} + b^{\frac{1}{2}} - (ab^{-\frac{1}{2}} + ba^{-\frac{1}{2}})\cos(L - l)\ldots\ldots\ldots\text{(iii)}.$$

We have also from the figure

$$\rho^2 = a^2 - 2ab\cos(L - l) + b^2,$$

whence eliminating $\cos(L - l)$

$$\rho^2\frac{d\lambda}{dt} = a^{\frac{1}{2}} + b^{\frac{1}{2}} - \frac{ab^{-\frac{1}{2}} + ba^{-\frac{1}{2}}}{2ab}(a^2 + b^2 - \rho^2),$$

which becomes after a little reduction

$$\frac{d\lambda}{dt} = \tfrac{1}{2}(a^{-\frac{3}{2}} + b^{-\frac{3}{2}})\left\{1 - \frac{(b^2 - a^2)(b^{\frac{3}{2}} - a^{\frac{3}{2}})}{\rho^2(a^{\frac{3}{2}} + b^{\frac{3}{2}})}\right\}\ \ldots\ldots\ \text{(iv)}.$$

Equation (iii) may be written

$$\rho^2\frac{d\lambda}{dt} = (a^{\frac{1}{2}} + b^{\frac{1}{2}})\left\{1 - \frac{(a^{\frac{1}{2}} - b^{\frac{1}{2}})^2 + a^{\frac{1}{2}}b^{\frac{1}{2}}}{a^{\frac{1}{2}}b^{\frac{1}{2}}}\cos(L - l)\right\},$$

which shows that if $d\lambda/dt = 0$

$$\cos(L - l) = a^{\frac{1}{2}} b^{\frac{1}{2}}/(a - a^{\frac{1}{2}} b^{\frac{1}{2}} + b) \text{..............(v).}$$

Thus there will always be a real value of $L - l$ and consequently there must always be points at which one planet has no angular velocity as seen from the other; such points are known as *stationary* points.

Let α be an angle defined by the equation

$$\cos\alpha = a^{\frac{1}{2}} b^{\frac{1}{2}}/(a - a^{\frac{1}{2}} b^{\frac{1}{2}} + b) = (a^{\frac{1}{2}} + b^{\frac{1}{2}})/(ab^{-\frac{1}{2}} + ba^{-\frac{1}{2}}),$$

then (iii) may have the form

$$\rho^2 \frac{d\lambda}{dt} = (ab^{-\frac{1}{2}} + ba^{-\frac{1}{2}})(\cos\alpha - \cos\phi),$$

where for brevity ϕ is written instead of $L - l$.

In the course of the synodic period (§ 150) ϕ assumes all values from 0° to 360° When the planets are in conjunction $\phi = 0$ and $d\lambda/dt$ is negative, so the motion is retrograde. On the other hand when $\phi = 180°$ we have $d\lambda/dt$ positive and the motion is direct. We therefore see that for one value of ϕ between 0° and 180° and for another between 180° and 360° the planet must be stationary to the terrestrial observer. Hence during the synodic revolution the movement is retrograde while ϕ increases through an angle 2α, and is direct while ϕ increases through an angle $360° - 2\alpha$, and if p, P be the periodic times of the planet and the earth, we have

$$\frac{180° - \alpha}{180°} \frac{Pp}{P - p} \text{ and } \frac{\alpha}{180°} \frac{Pp}{P - p}$$

for the periods of the direct and retrograde motions during the synodic period.

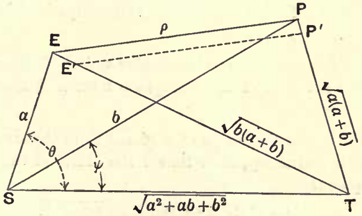


FIG. 103.

Investigation of the stationary points in a planet's orbit supposed to be circular but not in the plane of the ecliptic. If two planets are revolving in circular orbits round S, but not in the same plane, the stationary points E, P (Fig. 103) can be investigated as follows.

Let E' and P' be the positions to which the two planets move in the short time dt, then $E'P'$ must be parallel to EP and consequently EE', PP' are coplanar and intersect at some point T. Hence remembering that the velocity of each planet is inversely proportional to the square root of the radius of its orbit,

$$ET/PT = EE'/PP' = a^{-\frac{1}{2}}/b^{-\frac{1}{2}},$$

and making $ET = xa^{-\frac{1}{2}}$ and $PT = xb^{-\frac{1}{2}}$ we have

$$ST^2 = a^2 + x^2/a = b^2 + x^2/b,$$

because $\angle SET = 90°$ and $\angle SPT = 90°$, whence $x^2 = ab(a+b)$ and

$$PT = \sqrt{a(a+b)}, \quad ET = \sqrt{b(a+b)},$$

$$ST = \sqrt{a^2 + ab + b^2}.$$

If $\theta = \angle EST$ and $\psi = \angle PST$, then

$$a \sec\theta = b \sec\psi = \sqrt{a^2 + ab + b^2}.$$

Let α be the angle between the planes of the orbits. Imagine a sphere with centre S and intersected by SE, SP, ST in points E_1, P_1, T_1 respectively, then $E_1P_1 = \phi$, $E_1T_1 = \theta$, $P_1T_1 = \psi$ and $\angle P_1T_1E_1 = i$ and we have

$$\cos\phi = \cos\theta\cos\psi + \sin\theta\sin\psi\cos i.$$

If we substitute for θ and ψ we obtain

$$\cos\phi = \frac{ab + \sqrt{ab}\,(a+b)\cos i}{a^2 + ab + b^2}.$$

Ex. 1. Show that if the earth were at rest a superior planet could never appear stationary.

Ex. 2. Assuming the orbits to be circular and co-planar, find the distance of a planet if the duration of the retrograde motion is the kth part of the period of the planet.

Ex. 3. Let E be the elongation of a planet from the sun at the moment when the planet is stationary, show that if the orbits of earth and planet be circular and coplanar

$$b/a = \tfrac{1}{2}\tan^2 E + \tfrac{1}{2}\tan E\sqrt{4 + \tan^2 E}.$$

[Maddy's *Astronomy*, p. 273.]

Ex. 4. If θ be the angle subtended at the earth by the sun and a stationary point of a planet's orbit, and ϕ be the greatest elongation of the planet, prove that

$$2 \cot \theta = \sec \tfrac{1}{2} \phi + \operatorname{cosec} \tfrac{1}{2} \phi.$$

[Godfray's *Astronomy*, p. 320.]

Ex. 5. If m, m' be the mean motions in longitude of the earth and a planet in circular and coplanar orbits, and ϕ the difference of their longitudes, show that the planet's geocentric longitude is increasing at the rate

$$(mm')^{\frac{2}{3}} (m^{\frac{1}{3}} + m'^{\frac{1}{3}}) \frac{(mm')^{\frac{1}{3}} - \{m^{\frac{2}{3}} - (mm')^{\frac{1}{3}} + m'^{\frac{2}{3}}\} \cos \phi}{m^{\frac{4}{3}} - 2 (mm')^{\frac{2}{3}} \cos \phi + m'^{\frac{4}{3}}}.$$

[Math. Trip.]

Obtained at once from (iii), p. 414, by making

$$\phi = L - l, \quad m = a^{-\frac{3}{2}}, \quad m' = b^{-\frac{3}{2}}.$$

Ex. 6. Show that the number of times an inferior planet appears to change from direct motion to retrograde in the course of one revolution of the superior planet round the sun is the integral part of

$$(b/a)^{\frac{3}{2}} \text{ or of } (b/a)^{\frac{3}{2}} - 1,$$

where a and b are the radii of the orbits ($b > a$).

[Math. Trip.]

Let ϕ' be the smallest positive value of ϕ which corresponds to a change from progreding to regreding, then similar changes will occur when ϕ is $2n\pi + \phi'$, whatever integer n may be (while the changes from regreding to progreding correspond to $2n\pi - \phi'$). The angular velocity with which the inferior planet gains on the superior is $a^{-\frac{3}{2}} - b^{-\frac{3}{2}}$, while the periodic time of the superior planet is $2\pi b^{\frac{3}{2}}$. Hence the increase of ϕ during a revolution of the superior planet is $2\pi (b^{\frac{3}{2}}/a^{\frac{3}{2}} - 1)$. The number of integral values of n which make $n + \phi'/2\pi$ less than $l + k$, where l is the integral and k the fractional part of $b^{\frac{3}{2}}/a^{\frac{3}{2}} - 1$, must be $l - 1$ or l. Adding the case of $n = 0$ we have the desired result.

Ex. 7. If u and v are the velocities of two planets in circular orbits in the same plane, show that the period of direct motion is to the period of regression as $(180^\circ - \theta) : \theta$ where $\cos \theta = uv/(u^2 - uv + v^2)$.

[Coll. Exam. 1900.]

Ex. 8. Show that, if the earth and a planet be supposed to describe circles in the same plane about the sun, and the difference of the longitudes of the sun and planet be θ, the rate of change of θ is

$$\frac{2\pi}{S} \left(1 \sim \frac{a}{c} \cos \theta\right),$$

where S is a synodic period, a the radius of the earth's orbit, and c the distance of the planet from the earth at the moment.

[Coll. Exam.]

Let ϕ be the difference of heliocentric longitudes of the earth and the planet, then $\dot{\phi}=2\pi/S$.

Differentiating $\rho^2=a^2-2ab\cos\phi+b^2$ we have $\dot{\rho}=2a\pi\sin\theta/S$, and by differentiating $\rho\sin\theta=b\sin\phi$ the desired result is obtained.

Ex. 9. Prove that the time of most rapid approach of an inferior planet to the earth is when its elongation is greatest, and that the velocity of approach is then that under which it would describe its orbit in the synodic period of the earth and the planet. Give the corresponding results for a superior planet. The orbits are to be taken circular and in the same plane.

[Math. Trip. I.]

For from the last $\dot{\rho}=2a\pi\sin\theta/S$.

Ex. 10. If the line joining two planets to one another subtend an angle of 60° at the sun, when the planets appear to one another to be stationary, show that $a^2+b^2=7ab$ where a, b are the distances of the planets from the sun.

[Math. Trip.]

Ex. 11. Assuming that the orbits of Mercury and the earth are circular and in one plane, and that the angle subtended at the earth by the sun and Mercury when at a stationary point, is $\cot^{-1}3$, prove that the distances of the planets from the sun are as 39 to 100 nearly.

[Math. Trip.]

Ex. 12. Prove that when a planet is absolutely stationary as seen from the earth, its direction of motion and that of the earth must intersect on the line of nodes of its orbit with the ecliptic, and that its projection on the plane of the ecliptic is also stationary; the plane of the planet's orbit not coinciding with the ecliptic.

[Math. Trip. 1.]

Relative velocity is along PE and therefore the projection of the relative velocity on the plane of the ecliptic is along the line joining E to the projection of P.

Ex. 13. The orbits of two planets being supposed circular but not in one plane, prove that they will be stationary with regard to one another if their angles of separation from a node of their orbits, measured in the same direction, be respectively

$$\tan^{-1}\{b^{\frac{1}{2}}(a+b)^{\frac{1}{2}}/a\} \text{ and } \tan^{-1}\{a^{\frac{1}{2}}(a+b)^{\frac{1}{2}}/b\},$$

a and b being the radii of the orbits.

[Math. Trip.]

Ex. 14. The synodic period of Jupiter is 399 days, and his distance from the sun is 5·2 times the radius of the earth's orbit; find the sidereal period of Jupiter, and represent in a figure his geocentric motion during his sidereal period.

[Coll. Exam.]

139. The phases and brightness of the moon and the planets.

By the "phase" of a celestial body we are to understand the ratio of that part of the disc of the body which is seen to be illuminated to the whole disc. The phase is measured by the fraction of the diameter perpendicular to the line of cusps which lies in the illuminated portion. The hemisphere ACB (Fig. 104) of the celestial body turned towards the sun is illuminated by sunlight. The hemisphere XAY is that presented towards the earth.

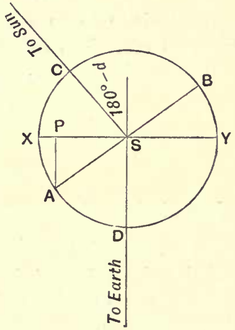


FIG. 104.

Fig. 105 represents the aspect of the celestial body as seen from the earth. The figure $EPHX$ represents the illuminated portion of the disc turned towards the observer. The area of the curve is

$$\tfrac{1}{2}\pi ES.PX = \tfrac{1}{2}\pi ES^2(1 + \cos d),$$

where d is the elongation of the earth from the sun as seen from the planet. Thus the expression $\frac{1}{2}(1 + \cos d)$ measures the "phase" of the celestial body. It is also plain that the quantity of light received varies inversely as the square of x, the distance from the earth L to the planet M

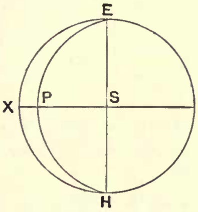


FIG. 105.

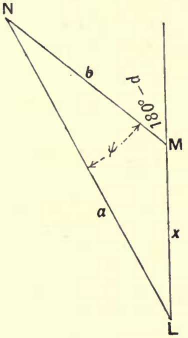


FIG. 106.

(Fig. 106), and hence it appears that the brightness of the planet as seen by the terrestrial observer is proportional to

$$(1 + \cos d)/x^2.$$

If a, b are the distances of the earth and planet from the sun S this may be written

$$(x^2 + 2bx - a^2 + b^2)/2bx^3,$$

and if this is to be a maximum we have by equating the differential coefficient to zero

$$x^2 + 4bx + 3(b^2 - a^2) = 0,$$

or $$x = \sqrt{b^2 + 3a^2} - 2b; \ \cos d = \{\sqrt{3a^2 + b^2} - 4b\}/3b.$$

As a particular case we consider the planet Venus where

$$a = 1, \ b = 0{\cdot}7233.$$

We find that when brightest

$$x = 0{\cdot}430, \ d = 117^\circ\, 55', \ \psi = 22^\circ\, 20',$$

and the elongation of the planet from the sun is 39° 43′. If the maximum brightness of Venus be unity the brightness at greatest elongation is ·727. Greatest brightness takes place 36·2 days after inferior conjunction. In this calculation we have regarded the orbits of the planet and the earth as circular, and the above results are consequently only approximately correct.

It is instructive to plot the brightness as the ordinate of a curve of which the abscissa is the angle subtended at the sun by the earth and planet.

Ex. 1. The difference between the first and second quarters of a lunation is half an hour; compare the distances from the earth of the sun and moon.

[Coll. Exam. 1893.]

The moon moves from 1st quarter to quadrature in $\frac{1}{4}$ hour in which time it describes an angle of about 8′, and the cosec of 8′ is approximately the ratio of the sun's distance to that of the moon.

Ex. 2. If x be the phase of the moon as seen from the earth, and y the phase of the earth as seen from the moon, prove that approximately

$$y = 2 - x + b(2x - x^2)/a,$$

the phase of full moon being denoted by 2, and a, b being respectively the radii of the orbits of the earth and moon.

Prove also that the first quarter of the moon terminates before the last quarter of the earth commences, and the last quarter of the moon begins after the termination of the earth's first quarter.

[Math. Trip.]

Let S, M, T (Fig. 107) be sun, moon and earth, then the diameter of the moon perpendicular to MS and that of the earth perpendicular to TS indicate the illuminated hemispheres. If E and L are the respective elongations, then the phase of the moon is $x=1+\cos L$ and of the earth $y=1+\cos E$, also

$$a \sin (L+E)=b \sin L,$$

and since b/a is a small quantity

$$\cos E=-\cos L+b \sin^2 L/a,$$

whence

$$y=2-x+b(2x-x^2)/a.$$

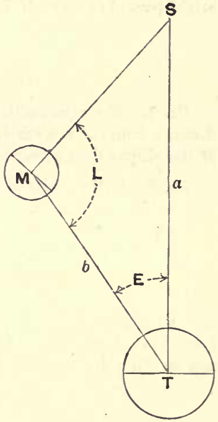


FIG. 107.

Ex. 3. If the phase of full moon be taken as unity, prove that midway between new moon and the first quarter the phase is slightly greater than $\frac{1}{7}$th.

Ex. 4. Show that the phase of a superior planet, as seen from the earth, is least when the earth appears half illuminated to the planet: but that the apparent brightness of a superior planet is greatest at opposition and least at conjunction. [Coll. Exam.]

Ex. 5. If r, R are the radii vectores of Venus and the earth and if Δ is the distance of Venus from the earth, show that the brilliancy of Venus is proportional to

$$(r+\Delta+R)(r+\Delta-R)/r^3\Delta^3.$$

Let θ be the angle which earth and sun subtend at Venus. The proportion of the disc of Venus which we see luminous is $(1+\cos\theta)/2$. The intrinsic brightness of the planet varies inversely as the square of its distance from the sun and its apparent brightness varies inversely as the square of its distance from the earth. Hence the brilliancy varies as $(1+\cos\theta)/r^2\Delta^2$ and substituting for $\cos\theta$ its value $(r^2+\Delta^2-R^2)/2r\Delta$ the desired result is obtained.

Ex. 6. Prove that if B be the brightness of an inferior planet as it would be seen from the sun, its greatest brightness as seen from the earth is

$$\frac{Bb(\sqrt{3a^2+b^2}-b)}{3(\sqrt{3a^2+b^2}-2b)^2},$$

a being the radius of the earth's orbit, and b that of the planet's, assumed circular and in the same plane.

[Math. Trip.]

Ex. 7. One planet whose mean distance from the sun is a appears to have a phase E to another planet whose mean distance from the sun is b, and the latter appears to the former to have a phase V; prove that if the inclination of the orbits to one another and their eccentricities be neglected,

$$b^2 V(1-V)=a^2 E(1-E).$$

Hence from the datum that the distance of Venus (from the sun) is ·7232 times that of the earth from the sun, prove that at least five-sixths of the bright part of the earth's disc is visible from Venus.

[Math. Trip.]

EXERCISES ON CHAPTER XX.

Ex. 1. The equatorial and polar semidiameters of Jupiter at its mean distance from the sun are 18″·71 and 17″·51 (Schur). Find the eccentricity of the ellipse thus presented by the disc of Jupiter.

Ex. 2. If the orbits of the earth and a planet are assumed to be ellipses, but in different planes, show that, if they are stationary as seen from one another, the perpendiculars from them to the line of nodes will be in the subduplicate ratio of the latera recta of the orbits.

[Math. Trip. I.]

It is easily shown from § 138 that if two planets are moving in two different orbits, their velocities may be represented by $s\sqrt{l}/p$ and $s\sqrt{l'}/p'$ where l and l' are the latera recta of their orbits, p and p' the perpendiculars from the sun on their directions of motion, and s is a constant for the solar system.

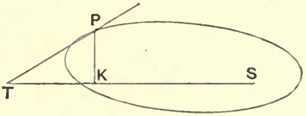


Fig. 108.

The tangents meet at T on the line of nodes and the velocities u, v are proportional to PT and QT, but

$$\frac{u}{PT}=\frac{h}{ST.\sin STP.PT}=\frac{h}{ST.PK},$$

hence $$PK : QL = h : h' = \sqrt{l} : \sqrt{l'}.$$

Ex. 3. The orbits of two planets are ellipses of latera recta $2l$ and $2l'$. If the latera recta lie in the line of nodes, prove that the distances from the sun when the planets are stationary obey the relation

$$e\sqrt{l}/(l-r)=e'\sqrt{l'}/(l'-r'),$$

e and e' being the eccentricities.

[Coll. Exam. 1900.]

Ex. 4. If the orbits of two planets are conics with equal latera recta in the same plane, each planet would appear stationary to an observer on the other when the line joining the planets was parallel to the line joining the sun to the intersection of their directions of motion.

[Math. Trip.]

Let S be the sun, P and E the planets, T the point of intersection of their directions of motion, p and p' the perpendiculars from S on ET and PT respectively. Then $ET : PT :: 1/p : 1/p'$, whence $\Delta STE = \Delta STP$ or ST and EP are parallel.

Ex. 5. If the orbit of an outer planet be an ellipse of eccentricity e and semi-axis a, and if it is in opposition at perihelion, show that its motion will then appear to be direct if $a/b < (1+e)/(1-e)$.

Ex. 6. Two comets move in coaxial parabolas in one plane round a centre of force in the focus. Find the condition that they may appear stationary to one another when one is at the vertex of, and the other at the end of the latus rectum of, their respective paths. [Coll. Exam.]

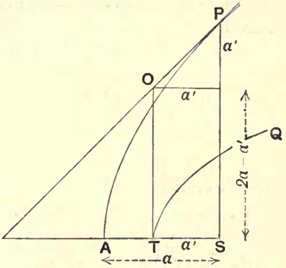


FIG. 109.

Let S be the sun, AP, TQ be the two parabolas, P and T the planets. The (velocities)² at P and T must be in the ratio $PO^2 : TO^2 = 2a'^2 : (2a-a')^2$, but the squares of the velocities are inversely as $2a$ and a', whence

$$4aa' = (2a-a')^2.$$

Ex. 7. A comet is describing a parabola in a plane inclined to the earth's orbit, which is assumed to be circular, and the line of nodes coincides with the axis of the parabola. If T years be the time the comet takes to move from the vertex to the end of the latus rectum, prove that, whatever be the inclination of the orbits, when the comet appears stationary the angular distance of the earth from the line of nodes is given by

$$2\sec\phi - \sin^2\phi = (3\pi T^{\frac{2}{3}}).$$ [Math. Trip. I. 1903.]

Let C be the comet (Fig. 110), E be the earth, S the sun. As before tangents at E and C meet in T and the velocities are proportional to TE and TC. Let V, v be the velocities at C and E, and take $SE=1$ and the latus rectum of the parabola as $4a$.

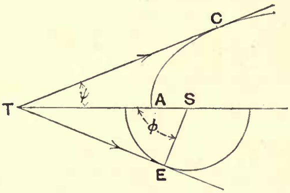


FIG. 110.

Then

$$V : v = TC : TE$$

$$= 2a \operatorname{cosec}^2\psi \cos\psi : \tan\phi.$$

Also $V = \sin\psi\sqrt{2\mu/a}$, $v = \sqrt{\mu}$, $ST = \sec\phi = a \operatorname{cosec}^2\psi$,

whence eliminating ψ and reducing we have

$$2a = 2\sec\phi - \sin^2\phi,$$

and $$T = \frac{2 \text{ area } SAL}{h} = \frac{\frac{8}{3}a^2}{\sqrt{2\mu a}} = \frac{4\sqrt{2}}{3}\frac{a^{\frac{3}{2}}}{\sqrt{\mu}} = \frac{2\sqrt{2}}{3\pi}a^{\frac{3}{2}},$$

because the year is the unit of time.

Ex. 8. A comet moving in a parabolic orbit, and a planet moving in a circular orbit, are in line with the sun when the comet is in perihelion. Determine the ratio of the perihelion distance of the comet to the distance of the planet, in order that each may then appear stationary when seen from the other.

Ex. 9. If a planet is describing a circular orbit whose radius is equal to the semi-latus rectum of the parabolic orbit of a comet whose plane of motion is coincident with that of the planet, show that the planet and comet will be absolutely stationary with regard to each other when the planet is 60° distant from the apse of the comet, provided its angular distance from the apse was approximately −39° when the comet was passing through the apse.

[Math. Trip. I.]

When the comet is 120° from apse and the planet 60°, the directions of their movements are identical and their velocities are equal and equal areas are described by comet and planet in equal times. The area moved over by a comet in moving to an angle of 120° from apse is equal to a sector of the planet's orbit containing an angle 60° + 39°.

Ex. 10. If a, δ be the coordinates of the sun and a', δ' be those of a planet ($a' > a$), and if Q be the position angle measured from the centre of the planet of the point of greatest defect of illumination on the planet, *i.e.* the angle from the northernmost point of the disc of the planet measured round by east to that point on the limb of the planet which is apparently at the greatest distance from the sun; if ρ be the distance on the celestial sphere from the centre of the planet to the centre of the sun, show that for the determination of ρ and Q we have the equations

$$\begin{aligned}\sin\rho\sin Q &= \cos\delta\sin(a'-a),\\ \sin\rho\cos Q &= -\sin\delta\cos\delta'+\cos\delta\sin\delta'\cos(a'-a),\\ \cos\rho &= \sin\delta\sin\delta'+\cos\delta\cos\delta'\cos(a'-a).\end{aligned}$$

These equations appear at once from the spherical triangle formed by the pole and the centres of the sun and the planet.

Ex. 11. At noon on May 30th 1908 the apparent R.A. and decl. of Venus are $7^h\ 19^m\ 4^s$ and N. 24° 59′ 26″ and the Log of the true distance from the earth is 9·65439. The corresponding quantities for the sun are $4^h\ 27^m\ 39^s$ N. 21° 45′ 16″ and ·00606 respectively. Show that 94°·4 and 50° 37′ 45″ are respectively the values of Q and ρ.

Ex. 12. When a planet is gibbous (*i.e.* disc more than half illuminated) show that the corrections to an observation of the R.A. and decl. of the defective limb are respectively $\tau(1-\cos\phi)$ and $a(1-\cos\psi)$ where τ is the sidereal time in which the semidiameter passes the meridian, a is the semidiameter of the planet and ϕ and ψ are determined from the equations

$$\sin\phi = \sin d\sin Q, \quad \sin\psi = \sin d\cos Q,$$

d being the angle between the earth and sun as seen from the planet and Q the position angle as defined in Ex. 10. (See *Nautical Almanac*, 1908, Appendix, p. 31.)

Show also that when the corrections are small they are very nearly $\frac{1}{2}\tau \sin^2 d \sin^2 Q$ and $\frac{1}{2}a \sin^2 d \cos^2 Q$ respectively.

The correction in R.A. is $\tau(1 - CH/a)$ and the correction in decl. is $a - CK$. But from the properties of the ellipse the perpendiculars (Fig. 111) CH and CK are respectively

$$\sqrt{a^2 \cos^2 Q + b^2 \sin^2 Q} \text{ and } \sqrt{a^2 \sin^2 Q + b^2 \cos^2 Q}.$$

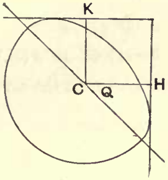


FIG. 111.

Since $b = a \cos d$ these become

$$CH = a\sqrt{1 - \sin^2 d \sin^2 Q} \text{ and } CK = a\sqrt{1 - \sin^2 d \cos^2 Q}.$$

Hence

$$\tau(1 - CH/a) = \tau(1 - \cos\phi) \text{ and } a - CK = a(1 - \cos\psi).$$

When the corrections are small we have

$$\tau(1 - CH/a) = \tfrac{1}{2}\tau \sin^2 d \sin^2 Q \text{ and } a - CK = \tfrac{1}{2}a \sin^2 d \cos^2 Q.$$

Ex. 13. Show that when the planet is horned (*i.e.* less than half illuminated) an observation of the declination of the cusp should receive the correction

$$\text{semi-diameter } (1 \pm \sin Q),$$

the sign being taken so as to make the quantity within the bracket less than unity.

Ex. 14. Show that the correction, to the declination observation of the moon's defective limb necessary to reduce the observation to what would have been observed if the moon were full is

$$\text{moon's semi-diameter} \times \text{versin } \theta,$$

where

$$\sin\theta = -\sin\delta_s \cos\delta_m + \cos\delta_s \sin\delta_m \cos P,$$

δ_m being the moon's declination, δ_s the sun's declination, and P the hour angle of the sun.

[Coll. Exam.]

Ex. 15. The periodic times of Mars and Jupiter are 687 and 4333 days respectively; show that the defect of Mars due to phase, *i.e.* the greatest fraction of a diameter which can be in the dark part, is never more than one-eighth, and that Jupiter is always seen with a nearly full face.

[Coll. Exam.]

If b, a be the relative distances of the earth and the planet from the sun, and θ be the elongation of the earth from the sun as seen from the planet, then the defect is $\frac{1}{2}(1-\cos\theta)$, the greatest value of θ is $\sin^{-1} b/a$, and thus the defect can never exceed $\frac{1}{2}-\frac{1}{2}\sqrt{1-b^2/a^2}$. In the case of Mars

$$b^2/a^2 = \left(\tfrac{365}{687}\right)^{\frac{4}{3}} = \cdot 430.$$

In the case of Jupiter the influence of b^2/a^2 is negligible.

Ex. 16. On May 30th 1908 at Greenwich mean noon the apparent place of the sun is

$$\alpha = 4^{\text{h}}\ 27^{\text{m}}\ 39^{\text{s}}\cdot 20,$$

$$\delta = 21^\circ\ 45'\ 16''\cdot 5\ (\text{N.}),$$

and the log of a the sun's distance from the earth is ·00606. The apparent place of Venus is

$$\alpha' = 7^{\text{h}}\ 19^{\text{m}}\ 4^{\text{s}}\cdot 25,$$

$$\delta' = 24^\circ\ 59'\ 26''\cdot 3\ (\text{N.}),$$

while the log of ρ the distance of Venus from the earth is 9·65439.

Show that Venus appears to have ·270 of its disc illuminated.

We have first to compute E the elongation of Venus from the formula

$$\cos E = \sin\delta\sin\delta' + \cos\delta\cos\delta'\cos(\alpha-\alpha'),$$

and then to find d, the angle which the earth and sun subtend at Venus, from the formulae $b\sin d = a\sin E$ and $b\cos d = \rho - a\cos E$, where b is the distance of Venus from the sun.

The calculation is as follows :—

$\sin\delta$	9·56894	a	·00606	$b\sin d$	9·80783
$\sin\delta'$	9·62579	$\sin E$	9·80177	$\sin d$	9·94822
Log (1)	9·19473	$b\sin d$	9·80783	b	9·85961
$\cos\delta$	9·96791	a	·00606	$b\cos d$	9·52293 (n)
$\cos\delta'$	9·95731	$\cos E$	9·88858	$\cos d$	9·66331 (n)
$\cos(\alpha-\alpha')$	9·86516	$a\cos E$	9·89464	b	9·85962
Log (1)	9·79038				
(1)	·15658	(ρ)	·45122		
(2)	·61713	($a\cos E$)	·78459		
($\cos E$)	·77371	($b\cos d$)	−·33337		
E	39° 18′ 40″				
		$b\sin d$	9·80783		
		$b\cos d$	9·52293 (n)		
		$\tan d$	·28490 (n)		
		d	117° 25′ 30″		

The required fraction

$$k = \tfrac{1}{2}(1+\cos d) = \tfrac{1}{2}(1-\cdot 4606) = \cdot 270.$$

See *N.A.* 1908, p. 30, Appendix.

Ex. 17. A satellite revolves in a circle (radius b) about a primary which revolves about a fixed centre in a circle (radius a), the angular velocity of the satellite being m times that of the primary; show that the satellite as seen from the fixed centre will have a certain part of its path convex, and its motion therein retrograde, if $a>b$ and $<mb$. [Math. Trip.]

The equation of the satellite's path is thus given

$$x=a\cos\theta+b\cos m\theta,$$
$$y=a\sin\theta+b\sin m\theta.$$

The tangent to the path at the point θ has for its equation

$$x(a\cos\theta+bm\cos m\theta)+y(a\sin\theta+bm\sin m\theta)$$
$$=a^2+b^2m+ab(m+1)\cos(m-1)\theta.$$

When the orbit passes from concave direct to convex retrograde with respect to the centre the tangent must pass through the centre. If therefore there are to be such changes it must be possible to obtain a value of θ which will make the right-hand side zero. But this requires

$$ab(m+1)>a^2+b^2m,$$

or

$$0>(a-bm)(a-b).$$

Let $y/x=\tan\phi$, then $d\phi/d\theta$ has the same sign as

$$a^2+mb^2+ab(m+1)\cos(m-1)\theta,$$

and thus the movement will be alternately direct and retrograde between successive stationary points.

Ex. 18. Assuming the orbits of the earth around the sun and of the moon around the earth to be circular, show that the moon's path is everywhere concave towards the sun.

We see from Ex. 17 that it cannot retrograde since $a>b$ and $a>bm$. The condition that the orbit should change from concave to convex without retrograding is that the radius of curvature should pass through the value ∞, or that $d^2y/dx^2=0$. This condition gives us the relation

$$a^2+m^3b^2+ab(m^2+1)\cos(m-1)\theta=0.$$
$$\therefore\ ab(m^2+1)>a^2+m^3b^2,$$

or

$$0>(a-m^2b)(a-mb).$$

Since $a>mb$ we should therefore require $m^2>a/b$. But $m^2<169$, whereas $a/b=387$.

Ex. 19. A small satellite is eclipsed at every opposition: find an expression for the greatest inclination which its orbit can have to the ecliptic. [Math. Trip.]

The expression is $\sin^{-1}(a/r)-\sin^{-1}\{(s-a)/R\}$, where a, s are the diameters of earth and sun and r, R distances of satellite and sun from the earth.

Ex. 20. If the moon be treated as spheroidal, show that the boundary of the illuminated portion seen from the earth is composed of two semi-ellipses, neglecting the parallaxes of the earth and sun as viewed from the satellite. [Coll. Exam.]

Ex. 21. It frequently happens that the time from new to full moon exceeds the time from that full moon to the following new moon by a day or more. Explain the chief cause of this, and show that it is adequate, having given that the maximum and minimum apparent diameters are approximately 33′ and 29′·5. [Math. Trip. I.]

The greatest difference due to the eccentricity will be found when the sun is on the latus rectum of the moon's orbit.

Ex. 22. Given that the periodic time of Venus is nearly two-thirds that of the earth, find roughly what time of year is indicated in these consecutive extracts from an almanac:

First month: Venus is an evening star. Enters Libra.

Next month: Venus is too close to the sun to be easily seen.

[Math. Trip.]

Ex. 23. If the earth and Saturn move in circular orbits of radii 1 and n^2 in the same plane, to which the plane of Saturn's rings is inclined at a finite angle, show that the condition that Saturn's rings may disappear or reappear to an observer on the earth may be written

$$\sin(t+\epsilon)=n^2\sin n^{-3}t,$$

where t is the time and ϵ a constant.

Hence show, by a graphical method of solving the equation or otherwise, that the occasions of disappearance or reappearance occur in groups of 1 or 3, 3 or 5, 5 or 7, &c. as n increases; and find approximately the critical value of n separating the first and second cases.

[Sheepshanks Exhibition.]

Let E and P be the earth and Saturn respectively, then when the rings are presented edgewise to the earth (Fig. 112) PE will be the intersection of the plane of Saturn's ring with the plane of the ecliptic. Draw ALB and CMD parallel to PE. Then the conditions required can only be met while Saturn is moving from A to C or D to B. If $SP=n^2$ and $ES=1$ and if we measure longitudes from EP and the time from the moment when the longitude of Saturn is zero, we have from triangle ESP

$$n^2\sin n^{-3}t=\sin(t+\epsilon).$$

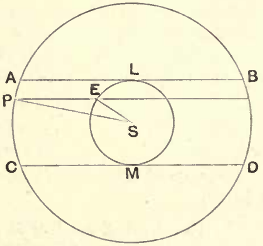


FIG. 112.

The time T Saturn takes in passing from A to C, compared with a year T_0, is given by the equation

$$T/T_0=\frac{n^3}{\pi}\sin^{-1}n^{-2}=\frac{n}{\pi}+\frac{1}{6\pi n^3}=\cdot 986,$$

for in the case of Saturn $n=3\cdot 09$.

Supposing PE to start from AL, it reaches CM in ·986 of a year and E will have overtaken the moving parallel certainly once and possibly three times. If $T/T_0>1\cdot 5$ then E must have overtaken the parallel three times and possibly five times. The value of n is given by

$$1\cdot 5=\frac{n}{\pi}+\frac{1}{6\pi n^3},$$

in which of course $1\cdot 5\pi$ may be substituted for n in the second term.

Ex. 24. It has been supposed that Mercury rotates on his axis in the same period as that of his revolution round the sun in an elliptic orbit ($e=0{\cdot}205$). Describe the phenomena of day and night on his surface, if this suggestion is correct.

[Sheepshanks Exhibition.]

Ex. 25. Let the elevation of the earth above the plane of Jupiter's equator be B and the principal semiaxes of the plane be a, b. Show that the Jovigraphical latitude B'' of the apparent centre of the planet's disc is given by

$$\tan B''=a^2 \tan B/b^2.$$

[Mr A. C. D. Crommelin, *Monthly Notices Roy. Ast. Soc.*, Vol. LXI. p. 116.]

If we represent the points on the surface of Jupiter in the usual way by the eccentric angle, $x=a\cos\phi$, $y=b\sin\phi$, then the line from the centre of Jupiter to the observer will cross the surface of the planet at a point marked by the eccentric angle ϕ when $\tan B=b\tan\phi/a$. The normal to Jupiter at the point ϕ cuts the plane of Jupiter's equator at an angle B'' and

$$\tan B''=a\tan\phi/b.$$

Ex. 26. The mean distance of Venus from the sun is ·72 of that of the earth. Determine the greatest altitude at which Venus, supposed to have a circular orbit in the plane of the ecliptic, can be visible after sunset in a given latitude, and the time of year at which it may occur.

[Math. Trip. I.]

If ω be the obliquity of the ecliptic and ϕ the latitude, the greatest distance of the pole of the ecliptic from the zenith is $90°-\phi+\omega$. Hence the sine of the greatest altitude required is $\cdot 72\cos(\phi-\omega)$ and the time is the vernal equinox.

Ex. 27. If an inferior planet were brightest at the moment of its greatest elongation from the sun, show that $b=a/\sqrt{5}$, where a, b are the respective distances of the earth and planet from the sun. Prove that Mercury is brightest before greatest eastern elongation and after greatest western elongation, while Venus is brightest after greatest eastern elongation and before greatest western elongation.

(The values of b for Mercury and Venus are respectively $0{\cdot}3871a$ and $0{\cdot}7233a$.)

Ex. 28. Assuming that the earth and Venus both move in the ecliptic in circles of radii 10 and 7 respectively, show that at superior conjunction the interval between consecutive transits of Venus across the meridian may exceed one mean day by about $1^{\text{m}}{\cdot}6$, assuming 12/11 for the secant of the obliquity.

[Coll. Exam. 1904.]

Let $b^{-\frac{3}{2}}t$ and $a^{-\frac{3}{2}}t+\epsilon$ be the respective longitudes of Venus and the earth. If a' be the apparent R.A. of Venus as seen from the earth, then

$$\sec\omega\tan a'=\frac{b\sin b^{-\frac{3}{2}}t-a\sin(a^{-\frac{3}{2}}t+\epsilon)}{b\cos b^{-\frac{3}{2}}t-a\cos(a^{-\frac{3}{2}}t+\epsilon)}.$$

Differentiating with respect to t and then making $a^{-\frac{3}{2}}t+\epsilon=180^\circ+b^{-\frac{3}{2}}t$,

$$\sec\omega\,\frac{da'}{dt}=\cos^2 a'\,\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}\,(a+b)\cos^2 b^{-\frac{3}{2}}t}$$

$$=\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}\,(a+b)\,(\cos^2 b^{-\frac{3}{2}}t+\cos^2\omega\sin^2 b^{-\frac{3}{2}}t)},$$

and thus the greatest value of da'/dt is

$$\sec\omega\,\frac{\sqrt{a}+\sqrt{b}}{\sqrt{ab}\,(a+b)}.$$

This is made homogeneous by the factor $2\pi a^{\frac{3}{2}}\div P$ where P is the year.

$$\frac{da'}{dt}=\frac{2\pi\sec\omega}{P}\,\frac{a\,(\sqrt{a}+\sqrt{b})}{\sqrt{b}\,(a+b)}.$$

Thus in one day the change in R. A. may be as much as

$$\tfrac{1}{365}\times 1440^m\sec\omega\,\frac{a\,(\sqrt{a}+\sqrt{b})}{\sqrt{b}\,(a+b)}=3^m\cdot 95\sec\omega\,\frac{a\,(b+\sqrt{ab})}{b\,(a+b)}$$

$$=4\cdot 31\times 1\cdot 29=5^m\cdot 6.$$

Hence the apparent R. A. of Venus in such cases may increase by $5^m\cdot 6$, and, remembering that the mean day is about 4^m longer than the sidereal day, the required result is obtained.

Ex. 29. It has been variously stated that the earth's orbit about the sun is (1) a circle having its centre near the sun, and (2) an ellipse having one focus at the sun's centre. Prove that, if the same observations were used for determining the apses, it would be impossible, by direct observation of the sun, to distinguish between the two orbits, unless the sun's diameter could be measured within about a quarter of a second.

(The greatest and least values of the sun's diameter are about 32′ 36″ and 31′ 32″ respectively.)

[Math. Trip. I.]

In one case the orbit would have the equation

$$r=a\,(1+e\cos\theta-\tfrac{1}{2}e^2\sin^2\theta),$$

and in the other

$$r=a\,(1+e\cos\theta-e^2\sin^2\theta).$$

It would be impossible to discriminate between these equations by observation of the sun's diameter unless quantities such as $\frac{1}{2}e^2\times$semi-diameter can be measured.

CHAPTER XXI.

THE GENERALIZED INSTRUMENT.

140. Fundamental principles of the generalized instrument.

By the expression "generalized instrument" we are not to understand a particular instrument actually employed in the observatory, but rather a geometrical abstraction, the theory of which includes, as special cases, the principles of the fundamental instruments used in practical astronomy.

When we have obtained the equations giving the theory of the generalized instrument it will be found that these same equations comprise as particular cases the formulae necessary for the study of the following instruments among others: the altazimuth, the meridian circle, the prime vertical instrument, the almucantar, and the equatorial. Some of these instruments will be discussed separately in Chap. XXII.

The following diagrammatic representation (Fig. 113) is intended to exhibit the essential parts of the generalized instrument.

The fundamental axis AB, distinguished as axis I, is capable of rotation about bearings which may be represented by the

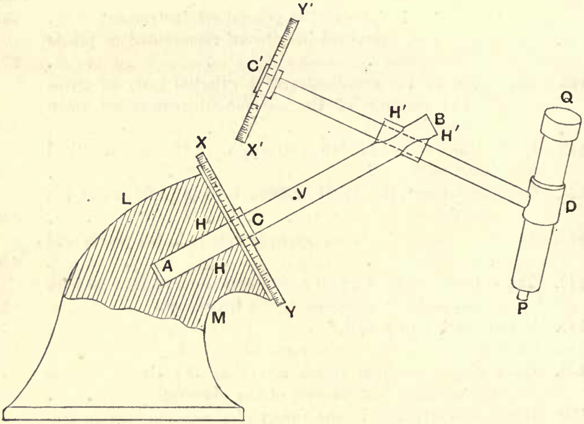


FIG. 113.

cylindrical socket HH in the base LM. It must be understood that axis I may be horizontal or vertical or in any other position, but its direction is fixed relatively to the base LM, and of course it has no freedom for any other motion than rotation.

The bearings at H' are fixed to AB and carry $C'D$, known as axis II, which can rotate freely in its bearings, though longitudinal motion through its bearings is not presumed. As AB is rotated $C'D$ is carried with it and the angle between $C'D$ and AB is fixed.

XY is the diameter of a graduated circle fixed rigidly to LM and of which the plane is perpendicular to AB. The graduation of this circle is to be from 0° to 360°. The nole of this circle is to be on the same side as B, which of course means that to an observer looking from the side which contains B the graduations will appear to increase counter-clockwise.

P is at the eye piece, and Q at the object glass, of a telescope of which the optical axis, *i.e.* the line joining the centre of the object glass to the intersection of two cross lines at the focus, is PQ. This telescope is rigidly fixed to $C'D$ so that there can be no alteration in the angle between PQ and $C'D$.

$X'Y'$ is the diameter of a graduated circle perpendicular to axis II and rigidly attached thereto, so that as axis II is rotated in its bearings this circle also rotates. The nole of this graduated circle is on the same side as D so that the graduation appears counter-clockwise when viewed from D, and this second circle like the first is graduated from 0° to 360°.

A pointer (not shown in the figure) *rigidly attached to axis* I at a point V will indicate a different reading R on the fixed circle XY for every different position of axis I. To obtain the necessary delicacy this pointer will of course take the form of a vernier or a microscope in the actual instrument, but for the geometrical theory we consider the pointer only as a straight line.

A pointer *also rigidly attached to axis* I will be used to read the circle $X'Y'$. As axis II is turned round in its bearings $H'H'$ the position of the circle $X'Y'$ will be indicated by this pointer. This reading we shall call R'.

The use of the generalized instrument is as follows. By suitable rotations about axes I and II the optical axis of the telescope can be directed to any star with certain limitations to be considered subsequently. When the optical axis is in the line required the two pointers give readings R and R'. It is now required from these two quantities to determine the place of the star. We seek therefore to express the coordinates of the star on the celestial sphere in terms of R and R'.

It is to be noted that the generalized instrument or rather its geometrical equivalent which we are now considering is a congeries of straight lines. The axes of AB and $C'D$ (I and II) are straight lines, the axis of the telescope is a straight line. We are also to remember that the graduations can be completely indicated by the radii of the two circles. We further observe that each of these lines has *sense* as well as direction. Thus the sense of the axis AB

is *from* the centre of the circle XY *to* the nole of that circle, the axis $C'D$ is *from* the centre of $X'Y'$ *to* the nole of $X'Y'$. The axis of the telescope is *from* the eye piece *to* the object glass and the radii of the circles are *from* their respective centres *to* the circumference.

The pointers are as has been said rigidly attached to AB. If we think of the pointer as a straight line rigidly attached to AB and perpendicular thereto, this line will be parallel to some radius of the graduated circle. As the axis AB is turned round 360° this parallel radius will also move completely round the circumference. If there be an arrow-head on the pointer to indicate its sense then we can definitely assume the reading to be that indicated by the radius drawn from the centre of the circle parallel to the pointer and in the direction indicated by its arrow-head.

In like manner a pointer for the circle $X'Y'$ must be fixed rigidly to axis I and be perpendicular to axis II. So far as the geometrical theory is concerned we may make the same pointer do for both circles. We have only to imagine the pointer as the common perpendicular to AB and $C'D$ and rigidly fixed to AB. Then this line will be parallel to the planes of both circles and the radii in each circle parallel to this line will give the corresponding readings for each circle.

Let R be the graduation in circle I (*i.e.* XY) indicated by the radius of that circle parallel to the pointer just described and in the sense shown by the arrow-head on the pointer. Let R' be the graduation in circle II (*i.e.* $X'Y'$) indicated by the radius of that circle parallel to the pointer and in the sense shown by the arrow-head on the pointer.

Then whatever pointers be actually used, provided only that they are fixed to axis I, their indications can only be $R+\Delta R$ and $R'+\Delta R'$ respectively, where ΔR and $\Delta R'$ are certain *index errors* which are constant for the instrument. It will duly appear later on how the quantities ΔR and $\Delta R'$ are to be determined. We shall first investigate the relations between R and R' and the coordinates of the body on the celestial sphere.

141. The lines in the generalized instrument represented as points on the sphere.

We shall now study the generalized instrument by the help of points on the celestial sphere corresponding to the lines in the generalized instrument.

Draw from any finite point O lines parallel to the lines of the generalized instrument, as already explained, each in the sense of the arrow-head on the corresponding line. Each radius of the celestial sphere supposed so drawn will terminate in a point on the sphere and the arc between any two such points will be equal to the angle between the two corresponding lines of the instrument.

Draw also from O a line in the direction of the celestial body, supposed to be a star, which is under observation. This line will be coincident with the line drawn through O parallel to the axis of the telescope when the telescope is directed upon the same star. Let this point be S (see Fig. 114); in like manner let B be the point corresponding to axis I, D to axis II, and V to the common pointer for the two circles.

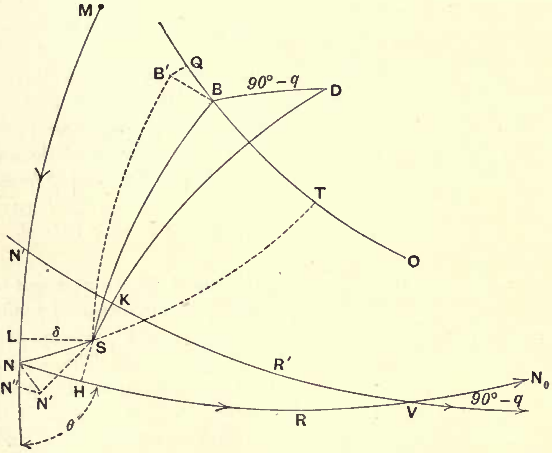


Fig. 114.

Let NV be the polar circle of B so that NV is the great circle representing the plane of circle I. Any two points on NV will therefore be separated by an arc equal to the angle between the two corresponding radii of XY. As B is the nole of the circle NV the graduation increases from N to V (as shown by the arrow-head). We have already settled that the graduation at the point V is to be R. As the circle XY remains in the same position however the instrument be rotated about AB or $C'D$, we may, so far as such movements of the instrument are concerned, regard NV as a fixed circle on the celestial sphere.

142. Expression of the coordinates of a celestial body in terms of the readings of the generalized instrument when directed upon it.

Let MN (Fig. 114) be the equator or the ecliptic or any other fixed great circle which is adopted as the standard of reference for the coordinates of points on the celestial sphere. Let M be the origin from which in the direction indicated by the arrow-head a coordinate α $(=ML)$ is to be determined for the star S. Let δ $(=LS)$ be the other coordinate of S which is to be taken as positive because S lies on the same side of MN as does the nole of MN.

We have now to define the two quantities which will express the position of NV (which is of course the plane of circle I) with respect to the standard circle MN. These quantities are the arc MN $(=\lambda)$ from M to the ascending node N of NV and the angle VNN'' $(=\theta)$ between the two great circles both diverging from N, where θ is an angle between 0° and 180°. The point N may be taken as the zero of graduation on NV and we then have the arc $NV=R$.

It was stated that AB and $C'D$ (Fig. 113) are at a constant angle. It follows that no matter how the instrument be moved we must have the point D, which is the nole of $N'V$, always at the same distance from B, which is the nole of NV. This constant arc between the noles of the circles I and II we shall represent by $90° - q$, so that q is always between $+90°$ and $-90°$. As the angle between two graduated circles is the arc between their noles we see that the angle NVN' (Fig. 114) is also $90° - q$.

We have already agreed that the point V on $N'V$ is to have the reading R' and it remains to choose a situation for the zero of graduation on $N'KV$, that is, on the circle $X'Y'$. In this case we could not make use of N' the intersection of VN' with the standard circle MN, for the node N' is constantly changing in the use of the instrument.

A convenient zero on $X'Y'$ is thus suggested. The plane through $C'D$ and PQ (Fig. 113) will always cut $X'Y'$ in the same diameter no matter how the instrument be used. Let it be agreed that the extremity of that diameter which lies on the same side of $C'D$ as the object glass of the telescope shall mark the zero of graduation on $X'Y'$.

In Fig. 114 we take S as the star on which the telescope is pointed and therefore the arc SD must intersect $N'V$ in K the zero of graduation. We thus have $R' = KV$. We also take $90° + r$ as the inclination of PQ to $C'D$ (Fig. 113), where r lies between $+90°$ and $-90°$. Hence in Fig. 114 we have $DS = 90° + r$ and $KS = r$.

We have now to show how α and δ can be expressed in terms of the observed quantities R, R' and the four constants λ, θ, r and q.

The required equations are obtained from the four right-angled triangles NLS, NHS, VKS, VHS.

From NHS, NLS we have

$$\left.\begin{aligned} \sin NS \sin HNS &= \sin HS \\ \sin NS \cos HNS &= \sin HN \cos HS \\ \cos NS &= \cos HN \cos HS \end{aligned}\right\} \quad \text{............(i).}$$

$$\left.\begin{aligned} \sin NS \sin LNS &= \sin LS \\ \sin NS \cos LNS &= \sin LN \cos LS \\ \cos NS &= \cos LN \cos LS \end{aligned}\right\} \quad \text{............(ii).}$$

$$LNS + HNS = 180° - \theta,$$

$$\sin LNS = \sin\theta \cos HNS + \cos\theta \sin HNS,$$

$$\cos LNS = -\cos\theta \cos HNS + \sin\theta \sin HNS,$$

and substituting these values in (ii) we have by virtue of (i)

$$\left.\begin{aligned} \sin LS &= \cos\theta \sin HS + \sin\theta \cos HS \sin HN \\ \sin LN \cos LS &= \sin\theta \sin HS - \cos\theta \cos HS \sin HN \\ \cos LN \cos LS &= \cos HS \cos HN \end{aligned}\right\} \quad \text{...(iii).}$$

These equations express the sides LS, LN of the quadrilateral $SLNH$, right-angled at L and H, in terms of the other two sides and the exterior angle θ at N.

Similarly from $SKVH$, right-angled at K and H, the exterior angle at V being $90° + q$, we have

$$\left.\begin{aligned} \sin HS &= -\sin q \sin KS + \cos q \sin KV \cos KS \\ \sin HV \cos HS &= \cos q \sin KS + \sin q \sin KV \cos KS \\ \cos HV \cos HS &= \cos KV \cos KS \end{aligned}\right\} \quad \text{...(iv).}$$

As $$HN = R - HV,$$

$$\cos HS \cos HN = \cos R \cos HS \cos HV + \sin R \cos HS \sin HV$$

and

$$\cos HS \sin HN = \sin R \cos HS \cos HV - \cos R \cos HS \sin HV.$$

Substituting these values in (iii) and reducing by means of (iv) we have

$$\begin{aligned}\sin LS = &- \cos\theta \sin q \sin KS + \cos\theta \cos q \sin KV \cos KS\\ &+ \sin\theta \sin R \cos KV \cos KS - \sin\theta \cos R \cos q \sin KS\\ &- \sin\theta \cos R \sin q \sin KV \cos KS,\end{aligned}$$

$$\begin{aligned}\sin LN \cos LS = &- \sin\theta \sin q \sin KS\\ &+ \sin\theta \cos q \cos KS \sin KV\\ &- \cos\theta \sin R \cos KV \cos KS\\ &+ \cos\theta \cos R \cos q \sin KS\\ &+ \cos\theta \cos R \sin q \cos KS \sin KV,\end{aligned}$$

$$\begin{aligned}\cos LN \cos LS = & \cos R \cos KV \cos KS\\ &+ \sin R \cos q \sin KS\\ &+ \sin R \sin q \cos KS \sin KV.\end{aligned}$$

Let M (Fig. 114) be the origin of the spherical coordinates, and let ML $(= \alpha)$ and LS $(= \delta)$ be the coordinates of S. As MN is λ we have

$$LN = \lambda - \alpha \text{ and } KS = r,\ KV = R'.$$

With these changes we obtain the following fundamental formulae for the generalized instrument.

$$\left.\begin{aligned}\sin\delta = &- \cos\theta \sin q \sin r\\ &- \sin\theta \cos q \sin r \cos R\\ &+ \cos\theta \cos q \cos r \sin R'\\ &+ \sin\theta \cos r \sin R \cos R'\\ &- \sin\theta \sin q \cos r \cos R \sin R'\end{aligned}\right\} \ldots (1).$$

$$\left.\begin{aligned}\sin(\lambda - \alpha)\cos\delta = &- \sin\theta \sin q \sin r\\ &+ \cos\theta \cos q \sin r \cos R\\ &+ \sin\theta \cos q \cos r \sin R'\\ &- \cos\theta \cos r \sin R \cos R'\\ &+ \cos\theta \sin q \cos r \cos R \sin R'\end{aligned}\right\} \ldots (2).$$

$$\left.\begin{aligned}\cos(\lambda - \alpha)\cos\delta = &+ \cos q \sin r \sin R\\ &+ \cos r \cos R \cos R'\\ &+ \sin q \cos r \sin R \sin R'\end{aligned}\right\} \ldots (3).$$

The required quantities α and δ can be calculated by these formulae from the observed quantities R and R', it being assumed that the constants of the instrument, viz. θ, λ, q, r, are known.

Ex. 1. The sum of the squares of the three left-hand members of the equations (1), (2), (3) of the generalized instrument is equal to unity. Verify that the same is true of the sum of the squares of the three right-hand members.

Ex. 2. Determine what the equations of the generalized instrument become when axis I is perpendicular to axis II ($q=0$) when there is no error of collimation in the telescope ($r=0$) and when a_0, δ_0, the coordinates of the nole of circle I, are the only instrumental constants in the expressions.

It is obvious that $\lambda=90°+a_0$ and $\theta=90°-\delta_0$, whence eliminating λ and θ and making $q=r=0$ the equations (1), (2), (3) become

$$\sin\delta=\sin\delta_0\sin R'+\cos\delta_0\sin R\cos R',$$
$$\cos(a_0-a)\cos\delta=\cos\delta_0\sin R'-\sin\delta_0\sin R\cos R',$$
$$\sin(a_0-a)\cos\delta=-\cos R\cos R'.$$

Ex. 3. Show that $q+r=0$ is the condition necessary that the telescope of the generalized instrument can be directed by a real setting of R and R' to the nole of circle I and that $q-r=0$ is the similar condition for the antinole.

Ex. 4. If a, δ be the coordinates of the nole of circle II while the instrument is so placed that R is the reading of circle I, show that

$$\sin\delta=\cos\theta\sin q+\sin\theta\cos q\cos R,$$
$$\sin(\lambda-a)\cos\delta=\sin\theta\sin q-\cos\theta\cos q\cos R,$$
$$\cos(\lambda-a)\cos\delta=-\cos q\sin R.$$

If $r=-90°$ in the fundamental equations (1), (2), (3); it is plain that the telescope is invariably directed on the nole of circle II. If r had been made $+90°$ then we should have found the coordinates of the antinole of circle II.

Ex. 5. If ρ be the arc from the nole of circle I to the star to which the telescope is pointed when the reading on circle II is R', show that

$$\cos\rho=-\sin q\sin r+\cos q\cos r\sin R'$$

and explain why R is absent from the expression.

Ex. 6. Find the region on the celestial sphere within which an object can be reached by the generalized instrument.

We see from Ex. 5 that the extreme values of $\cos\rho$ correspond to

$$R'=-90° \text{ and } R'=+90°,$$

whence at the limits

$$\cos\rho=\cos\{(90°+r)+(90°-q)\} \text{ and } \cos\rho=\cos\{(90°+r)-(90°-q)\}.$$

If therefore circles on the celestial sphere be described with radii $(q+r)$ and $180°-(q\sim r)$ respectively, and the nole of circle I as centre, then the zone between these circles will be the region of visibility.

Ex. 7. Let P_1, P_2 be two diametrically opposite points on the celestial sphere and let R' be the reading of circle II when the generalized instrument is directed to P_1. Show that the instrument cannot be directed to P_2 unless

$$\cos^2\tfrac{1}{2}(90-R') \nless \tan q \tan r \text{ (if } \tan q \tan r > 0),$$
$$\sin^2\tfrac{1}{2}(90-R') \nless -\tan q \tan r \text{ (if } \tan q \tan r < 0).$$

Ex. 8. Show how the absence of θ from Equation (3) is to be accounted for geometrically.

143. Inverse form of the fundamental equations of the generalized instrument.

We refer again to Fig. 114 where, in addition to the notation already explained, we now take $NN_0 = 90°$, in which case it is easily seen that the coordinates of N_0 are $\lambda + 90°$, θ. Then remembering that the cosine of the distance between α, δ and α', δ' is

$$\sin\delta \sin\delta' + \cos\delta \cos\delta' \cos(\alpha - \alpha')$$

we have, by substituting the coordinates of S, B, N, N_0, the equations

$$\cos SB = \sin\delta\cos\theta + \cos\delta\sin\theta\sin(\lambda - \alpha),$$
$$\cos SN_0 = \sin\delta\sin\theta - \cos\delta\cos\theta\sin(\lambda - \alpha),$$
$$\cos SN = \cos\delta\cos(\lambda - \alpha).$$

But we can obtain other expressions for $\cos SB$, $\cos SN_0$, $\cos SN$.

In the triangle BDS the angle $BDS = 90° - R'$, for since V is the pole of BD we have $VDB = 90°$, and as D is the pole of KV we have $VDK = R'$. Hence

$$\begin{aligned}\cos SB &= \cos(90° - q)\cos(90° + r) \\ &\qquad + \sin(90° - q)\sin(90° + r)\cos(90° - R') \\ &= -\sin q \sin r + \cos q \cos r \sin R'.\end{aligned}$$

From the triangle SVN we have

$$\begin{aligned}\cos SN &= \cos SV\cos NV + \sin SV \sin NV \cos(90° - q - SVK) \\ &= \cos SV \cos NV + \sin NV \sin q \sin SV \cos SVK \\ &\qquad + \sin NV \cos q \sin SV \sin SVK \\ &= \cos r \cos R \cos R' + \sin q \cos r \sin R \sin R' \\ &\qquad + \cos q \sin r \sin R.\end{aligned}$$

By writing $R - 90^\circ$ for R in this expression we get the value of $\cos SN_0$, viz.

$$\cos SN_0 = \cos r \sin R \cos R' - \sin q \cos r \cos R \sin R' - \cos q \sin r \cos R.$$

Equating the three sets of expressions to those already obtained for the three quantities we have, from the values of $\cos SB$, $\cos SN_0$, $\cos SN$ respectively

$$\left.\begin{aligned}&\cos\theta \sin\delta + \sin\theta\cos\delta\sin(\lambda-\alpha)\\ &\qquad = -\sin q \sin r + \cos q \cos r \sin R'\end{aligned}\right\} \ldots \text{(i)},$$

$$\sin\theta\sin\delta - \cos\theta\cos\delta\sin(\lambda-\alpha)$$
$$= \cos r \sin R \cos R' - \cos q \sin r \cos R - \sin q \cos r \cos R \sin R' \quad \ldots\ldots\ldots\text{(ii)},$$

$$\cos\delta\cos(\lambda-\alpha)$$
$$= \cos r \cos R \cos R' + \cos q \sin r \sin R + \sin q \cos r \sin R \sin R' \quad \ldots\ldots\ldots\text{(iii)}.$$

These equations may be written in the equivalent form

$$\left.\begin{aligned}\cos q \cos r \sin R' = {} & \sin q \sin r\\ & + \cos\theta\sin\delta\\ & + \sin\theta\cos\delta\sin(\lambda-\alpha)\end{aligned}\right\}\ldots\ldots\ldots\ldots\ldots\text{(iv)}.$$

$$\left.\begin{aligned}\cos r \cos R' = {} & \sin\theta\sin\delta\sin R\\ & - \cos\theta\cos\delta\sin(\lambda-\alpha)\sin R\\ & + \cos\delta\cos(\lambda-\alpha)\cos R\end{aligned}\right\}\ldots\ldots\ldots\text{(v)}.$$

$$\left.\begin{aligned}\sin r = {} & -\cos\theta\sin q\sin\delta\\ & - \sin\theta\sin q\cos\delta\sin(\lambda-\alpha)\\ & + \cos q\cos\delta\cos(\lambda-\alpha)\sin R\\ & - \sin\theta\cos q\sin\delta\cos R\\ & + \cos\theta\cos q\cos\delta\sin(\lambda-\alpha)\cos R\end{aligned}\right\}\ldots\text{(vi)}.$$

These can, of course, also be deduced from the three formulae (1), (2), (3) already given in § 142. The present forms are useful as they contain the solution of the inverse problem in the theory of the generalized instrument, viz. given α and δ find R and R' when the quantities θ, λ, q, r are known.

Ex. 1. Let α_1, δ_1 and α_2, δ_2 be the celestial coordinates of two stars and let R_1, R_1' and R_2, R_2' be the corresponding pairs of readings of the generalized instrument. If we write

$$A_1 = \cos q \sin r \sin R_1 + \cos r \cos R_1 \cos R_1' + \sin q \cos r \sin R_1 \sin R_1',$$

$$B_1 = \cos q \sin r \cos R_1 - \cos r \sin R_1 \cos R_1' + \sin q \cos r \cos R_1 \sin R_1',$$

$$C_1 = \cos q \cos r \sin R_1' - \sin q \sin r,$$

and also similar expressions with the suffix 2, prove that

$$\sin \delta_1 \sin \delta_2 + \cos \delta_1 \cos \delta_2 \cos (\alpha_1 - \alpha_2) = A_1 A_2 + B_1 B_2 + C_1 C_2.$$

It is easy to show that A_1, B_1, C_1 are the direction cosines with respect to the rectangular axes ON, ON_0, OB of the line OS, where O is the centre of the celestial sphere and S_1 the star whose coordinates are α_1, δ_1.

Ex. 2. If A_0, B_0, C_0 be the values of A, B, C for a standard point α_0, δ_0, show that the errors $\Delta\alpha$, $\Delta\delta$ in the coordinates of any other point α, δ arising from an error ΔR in the determination of R satisfy the relation

$$\{\cos \delta \sin \delta_0 - \sin \delta \cos \delta_0 \cos (\alpha - \alpha_0)\} \Delta\delta - \cos \delta \cos \delta_0 \sin (\alpha - \alpha_0) \Delta\alpha = (BA_0 - AB_0) \Delta R.$$

For we have

$$\sin \delta \sin \delta_0 + \cos \delta \cos \delta_0 \cos (\alpha - \alpha_0) = AA_0 + BB_0 + CC_0.$$

Differentiating with respect to R, and noting that

$$\frac{\partial A}{\partial R} = B \text{ and } \frac{\partial B}{\partial R} = -A; \quad \frac{\partial C}{\partial R} = 0,$$

we obtain the required result.

Ex. 3. Show that the equations of the generalized instrument may be expressed in the form

$$\cos q \cos r \sin R' = L + \sin q \sin r,$$

$$\cos r \cos R' = M \sin R + N \cos R,$$

$$\sin r = -L \sin q - M \cos q \cos R + N \cos q \sin R,$$

in which

$$L = \cos \theta \sin \delta + \sin \theta \cos \delta \sin (\lambda - \alpha),$$

$$M = \sin \theta \sin \delta - \cos \theta \cos \delta \sin (\lambda - \alpha),$$

$$N = \cos \delta \cos (\lambda - \alpha).$$

Ex. 4. Show that

$$\tan R' = \pm \tfrac{1}{2} \left\{ \left(\frac{\cos (q - r) + L}{\cos (q + r) - L} \right)^{\frac{1}{2}} - \left(\frac{\cos (q + r) - L}{\cos (q - r) + L} \right)^{\frac{1}{2}} \right\},$$

where L has the same meaning as in the last question.

Ex. 5. Show how the quantities q, r can be determined when the readings R_n, R_n' and R_s, R_s' for each of two opposite points have been obtained.

It is easily seen that we have two equations

$$F \tan r \cos q + G \sin q + H = 0,$$

$$F' \tan r \cos q + G' \sin q + H' = 0,$$

where F, G, H, F', G', H' are known quantities and by solution of these equations we obtain both $\tan r \cos q$ and $\sin q$.

There are thus two possible solutions, viz. q_0, r_0 and $180° - q_0$, $180° - r_0$, but as r_0 lies between $+90°$ and $-90°$ we can determine which pair of roots is to be taken.

Ex. 6. Show that for every real point on the celestial sphere except the nole and antinole of circle I the corresponding readings R and R' will be either both real or both imaginary and that in the excepted cases R' is indeterminate.

We have, Ex. 3,

$$M \sin R + N \cos R = \cos r \cos R'$$

$$-M \cos R + N \sin R = \sin r \cos q + \sin q \cos r \sin R'.$$

If $\sin R'$, $\cos R'$ are both real, $\sin R$ and $\cos R$ are both real and vice versa, unless in the case when $M=0$ and $N=0$. See Ex. 8.

Ex. 7. Show that if R' satisfies the equations (iv), (v), (vi), then $180° - R'$ will also satisfy them.

This is obviously true for (iv) and to prove it for (v) and (vi) we square and add the equations of Ex. 6, and replacing M and N by their values and observing that $L^2 + M^2 + N^2 = 1$ we have as the result of the elimination of R

$$1 = L^2 + \cos^2 r \cos^2 R' + (\sin r \cos q + \sin q \cos r \sin R')^2,$$

but this if true for R' is true for $180° - R'$.

Ex. 8. Show that, in general, the telescope could not be directed to the nole of circle I unless the reading on that circle indicated one or other of the imaginary circular points at infinity.

In § 142, Ex. 2 we have mentioned that the coordinates of the nole of circle I are given by $a = \lambda - 90°$ and $\delta = 90° - \theta$, and on substituting these in $M = \sin\theta \sin\delta - \cos\theta \cos\delta \sin(\lambda - a)$ and $N = \cos\delta \cos(\lambda - a)$ we see that $M=0$ and $N=0$. To satisfy under these conditions the equations

$$M \sin R + N \cos R = \cos r \cos R'$$

$$-M \cos R + N \sin R = \sin r \cos q + \sin q \cos r \sin R',$$

we must have $\sin R$ or $\cos R$ infinite. In this case $\tan R = \pm i$ and R must be one of the imaginary circular points at infinity.

144. Contrast between the direct and the inverse problems of the generalized instrument.

We are now to notice a fundamental difference between the direct problem of finding α and δ when R and R' are given and the inverse problem of finding R and R' when α and δ are given.

In the former we introduce the observed values of R and R' into the equations (1), (2), (3), § 142, and remembering that

$$-90° \ngtr \delta \ngtr 90°,$$

we obtain α and δ from the three equations without any ambiguity. This is the direct problem which has therefore always one solution and only one.

But in the inverse problem α and δ are given and we are to seek R and R' from the equations (iv), (v), (vi), § 143. There are two solutions, real, imaginary or coincident, to this inverse problem, so that if the generalized instrument can be pointed on a star in one way it can, in general, be also pointed on the same star in quite a different way. It may not be possible to direct the instrument on the star by any real setting, but if it is there are generally two totally different dispositions of the instrument by which the star can be observed. There are thus two different pairs of values for R and R' which equally correspond to one pair of values for α and δ.

From the equation (iv), § 143, we can determine $\sin R'$, and if this is $\not> 1$ we can satisfy (iv) by either of the two real angles R' and $180° - R'$. Introducing the first of these supplementary values into (v) and taking this in conjunction with (vi) we have two linear equations from which both $\sin R$ and $\cos R$ are determined, and thus R is known without ambiguity as to its quadrant. The value of R so found we shall term R_1.

When $180° - R'$ is substituted in (v) the equation so obtained if taken in conjunction with (vi) will, in like manner, give another value of R which we shall term R_2 (§ 143, Ex. 7). Thus for given values of α, δ we have two solutions, viz, R', R_1 and $180° - R'$, R_2, and we learn that if there is one then there are generally two different positions in which the generalized instrument can be pointed upon a given star. One of these is called the right position and the other the left, and the operation of changing the instrument from one of these positions to the other is called *reversal*.

Ex. 1. Show that the expression

$$-\cos q \sin r \cos R + \cos r \sin R \cos R' - \sin q \cos r \cos R \sin R'$$

does not change if the generalized instrument be reversed and redirected to the same point of the celestial sphere, and explain the geometrical meaning of the fact.

We see from (ii), p. 441, that the given expression is equal to

$$\sin \delta \sin \theta - \sin (\lambda - a) \cos \delta \cos \theta.$$

Ex. 2. Show that the expression

$$\cos(\lambda - a)\sin R - \sin\theta\tan\delta\cos R + \cos\theta\sin(\lambda - a)\cos R$$

has the same value whether the generalized instrument is in the right position or in the left when set on the star a, δ.

Ex. 3. If λ, θ be respectively the longitude and inclination of the ascending node of the circle I of the generalized instrument with respect to the circle of reference, prove that

$$\sin\tfrac{1}{2}(R_1+R_2)[\sin\theta\sin\delta - \cos\theta\cos\delta\sin(\lambda - a)] + \cos\tfrac{1}{2}(R_1+R_2)\cos\delta\cos(\lambda - a) = 0,$$

where R_1 and R_2 are the readings of circle I in the right and left positions when directed on the same point of the celestial sphere, and account geometrically for the absence of q and r from this equation.

Ex. 4. Let R_1, R_2 be the readings of circle I when the instrument is directed on the same star in both right and left positions respectively, it being understood that the coordinates of the star have not changed in the interval. Let R_1', R_2' be the corresponding readings of circle II. Prove the general formula

$$\cos q\sin r\sin\tfrac{1}{2}(R_1 - R_2) + \sin q\cos r\cos\tfrac{1}{2}(R_1' - R_2')\sin\tfrac{1}{2}(R_1 - R_2) - \cos r\sin\tfrac{1}{2}(R_1' - R_2')\cos\tfrac{1}{2}(R_1 - R_2) = 0.$$

145. Determination of the index error of circle II in the generalized instrument.

The first constant of the instrument which must be determined is the so-called *Index error* of the graduation with respect to the pointer or microscope by which the movable circle $X'Y'$ is read. The index error is the constant quantity which should be added to the observed value of R' so as to obtain the value which R' should have if the instrument were geometrically perfect.

Let us suppose this error to be Δ, and we are to understand that the correction to be applied to the observed reading R' is Δ, so that $R' + \Delta$ is to be the geometrical arc KV (Fig. 114), *i.e.* the quantity which we have hitherto taken to be R'.

Let the telescope be directed upon some distant mark and let the reading be R_1', then the corrected reading becomes $R_1' + \Delta$. Next let the instrument be reversed and directed again on the same mark, which is presumed to have remained unaltered. Let the reading of circle II be now R_2', then since the correction applicable to the observed reading is always the same in the same instrument, we see that the corrected reading will be $R_2' + \Delta$.

As already shown in § 144, $\sin R'$ is not changed by reversal, *i.e.* the two values of R' in a perfectly adjusted instrument would be supplemental. Hence

$$R_1' + \Delta + R_2' + \Delta = 180^\circ,$$

or

$$\Delta = 90^\circ - \tfrac{1}{2}(R_1' + R_2') \quad \text{.....................(i).}$$

Thus from a single pair of right and left readings on a distant object we determine Δ.

If the distant mark be a star it is to be noted that the diurnal movement of the heavens will in certain cases make the coordinates of the star different in the second observation from what they were in the first. The following procedure will generally suffice to remove this difficulty.

There are to be two observations of the "star" in the "right" position and one observation R_2' of the same star in the "left" position at a moment which is midway between the two right observations. The mean of the two former is to be taken for R_1'. Thus we can eliminate the effect of the diurnal motion for most practical purposes.

The determination of this particular instrumental constant is so simple that in what follows we shall always presume that the correction has been made so that the R' of our formulae is indeed the arc KV of Fig. 114. The index error of circle I or XY (Fig. 113) cannot be determined until certain other constants connected with the instrument have been investigated.

146. The determination of q and r by observations of stars in both right and left positions of the instrument.

Let R_1 and R_2 be the readings of circle I in the right and left positions of the instrument when directed to the same distant mark, it being understood that if the mark is a star the effect of any apparent movement is to be eliminated in the way already explained. It will be shown that the index error of circle I has no effect on the finding of q and r by the present process and therefore we may regard it as zero, while the index error of circle II we have already corrected. We shall now write the formula for $\sin \delta$ ((1) § 142) for both the right and left positions. We have for the right position

$$\begin{aligned}\sin\delta = &-\cos\theta\sin q\sin r\\ &-\sin\theta\cos q\sin r\cos R_1\\ &+\cos\theta\cos q\cos r\sin R'\\ &+\sin\theta\cos r\sin R_1\cos R'\\ &-\sin\theta\sin q\cos r\cos R_1\sin R',\end{aligned}$$

and for the left position

$$\begin{aligned}\sin\delta = &-\cos\theta\sin q\sin r\\ &-\sin\theta\cos q\sin r\cos R_2\\ &+\cos\theta\cos q\cos r\sin R'\\ &-\sin\theta\cos r\sin R_2\cos R'\\ &-\sin\theta\sin q\cos r\cos R_2\sin R'.\end{aligned}$$

Identifying these two values of $\sin\delta$ we find the terms involving $\cos\theta$ disappear, so, omitting the case of $\sin\theta = 0$, we may divide by $\sin\theta$ and obtain the result

$$A\sin\tfrac{1}{2}(R_1 + R_2) = 0,$$

in which A is an abbreviation for

$$(\cos q\sin r + \sin q\cos r\sin R')\sin\tfrac{1}{2}(R_1 - R_2) + \cos r\cos R'\cos\tfrac{1}{2}(R_1 - R_2).$$

In like manner we obtain for the right position of the instrument (2), § 142,

$$\begin{aligned}\cos(\lambda - \alpha)\cos\delta = &\cos q\sin r\sin R_1\\ &+\cos r\cos R_1\cos R'\\ &+\sin q\cos r\sin R_1\sin R',\end{aligned}$$

and for the left

$$\begin{aligned}\cos(\lambda - \alpha)\cos\delta = &\cos q\sin r\sin R_2\\ &-\cos r\cos R_2\cos R'\\ &+\sin q\cos r\sin R_2\sin R'.\end{aligned}$$

Identifying these expressions we have

$$A\cos\tfrac{1}{2}(R_1 + R_2) = 0.$$

But we have already seen that

$$A\sin\tfrac{1}{2}(R_1 + R_2) = 0,$$

by squaring and adding we see that $A = 0$ or

$$(\cos q\sin r + \sin q\cos r\sin R')\sin\tfrac{1}{2}(R_1 - R_2) + \cos r\cos R'\cos\tfrac{1}{2}(R_1 - R_2) = 0.$$

As R_1 and R_2 enter only in the combination $R_1 - R_2$, the index error of circle I has been eliminated. We thus obtain a formula showing how the two internal constants q and r may be found by observation. As α and δ are absent, the formula does not depend on the star or mark chosen, while λ and θ, which define the aspect of the instrument, also vanish.

If for brevity we write

$$A = \sin \tfrac{1}{2}(R_1 - R_2); \quad B = \sin R' \sin \tfrac{1}{2}(R_1 - R_2);$$
$$C = \cos R' \cos \tfrac{1}{2}(R_1 - R_2);$$

the equation may be expressed

$$A \cos q \sin r + B \sin q \cos r + C \cos r = 0,$$

in which A, B, C involve only quantities known by observation.

The same operation applied to another star or mark will give a similar equation

$$A' \cos q \sin r + B' \sin q \cos r + C' \cos r = 0,$$

whence $$(BA' - AB') \sin q = AC' - CA'.$$

We thus learn $\sin q$ and consequently there appear to be two supplemental values for q of which either will satisfy the required conditions. We have, however, agreed that $90° - q$ is the inclination of circle II to circle I and it is a convention (p. 33) that the angle denoting an inclination shall lie between $0°$ and $180°$, so that q must be between $-90°$ and $+90°$. Accordingly we can distinguish which one of the two supplemental angles must be taken, and thus q is known without ambiguity. We find also

$$(AC' - A'C) \tan r = (B'C - BC') \tan q.$$

From this r is known, for as between r and $r + 180°$ we choose the value which lies, as r must lie, between $-90°$ and $+90°$.

Thus q and r, the two internal constants of the generalized instrument, may be determined.

147. Determination of λ and θ.

The determination of these quantities may be made by means of formula (iv) of § 143, which can be written

$$X \sin \delta + Y \cos \delta \cos \alpha + Z \cos \delta \sin \alpha$$
$$+ \sin q \sin r - \cos q \cos r \sin R' = 0 \ldots\ldots \text{(i)},$$

where $$X = \cos \theta, \quad Y = \sin \theta \sin \lambda, \quad Z = -\sin \theta \cos \lambda.$$

We have just shown how q and r may be determined, so that if R' be observed and corrected for index error (§ 145), and if the star is one whose coordinates are known, then all the coefficients in equation (i) are known. Two other known stars will provide two more such equations and thus we have three equations of the first degree from which X, Y, Z can be determined.

As $\cos\theta$ is thus known and $0° \ngtr \theta \ngtr 180°$, we see how θ is definitely determined, and as Y and Z are known, then $\sin\lambda$ and $\cos\lambda$ are known and therefore λ is also known. Of course as in other parallel cases we require both $\sin\lambda$ and $\cos\lambda$. If only $\sin\lambda$ was given there would be nothing to show whether the required angle was λ or $180° - \lambda$. If only $\cos\lambda$ was given there would be nothing to show whether the required angle was λ or $360° - \lambda$.

Ex. Show that in the three fundamental equations (1), (2), (3) (§ 142) we may substitute for R' the expression $90° + \frac{1}{2}(R_1' - R_2')$, where R_1' and R_2' are the readings of circle II when the telescope of the generalized instrument is set on the star α, δ in the right and left positions respectively.

Show that when this substitution has been made the equations are true whatever be the index error in circle II, though the equations are not true in their original form if there is any index error in circle II.

148. Determination of the index error of circle I.

We have shown how q, r, λ, θ can be determined and also the index error of circle II, so that the angle R' as it will be now employed is a known angle, for it is the reading of circle II to which the known correction for index error has been applied. To complete the theory of the generalized instrument it remains to show how the index error of circle I can be determined by observations of a star α, δ in both the right and left positions of the instrument.

From the equation Ex. 3, § 143, we have

$$\cos r \cos R' = M \sin R + N \cos R \quad\ldots\ldots\ldots\ldots\text{(i)},$$

where
$$M = \sin\theta \sin\delta - \cos\theta \cos\delta \sin(\lambda - \alpha),$$
$$N = \cos\delta \cos(\lambda - \alpha).$$

This formula is only true if the quantity R be $R_1 + y$, where R_1 is the actual observed angle on circle I and y is the index error which has to be added to R_1 to produce the true distance NV (Fig. 116). We thus have

$$\cos r \cos R' = M \sin(R_1 + y) + N \cos(R_1 + y) \quad\ldots\ldots\text{(ii)}.$$

If the instrument be reversed and directed again on the same star α, δ, we know that R' is changed into $180^\circ - R'$, the reading R_1 becomes R_2 and y is unaltered, whence

$$-\cos r \cos R' = M \sin(R_2 + y) + N \cos(R_2 + y) \ldots\ldots \text{(iii)}.$$

In equations (ii) and (iii) both M and N are known, for as the place of the star is known α, δ are known. The observations give R', R_1, R_2. There are therefore two linear equations in $\sin y$ and $\cos y$ in which the coefficients are known. From these $\sin y$ and $\cos y$ are determined, so that y is ascertained without ambiguity.

We have thus shown how all the constants of the generalized instrument may be ascertained.

***149. On a single equation which comprises the theory of the fundamental instruments of the Observatory.**

Let α_1, δ_1 be the coordinates of a star S_1, for which the readings of the generalized instrument are R_1, R_1'. In like manner let a second star S_2, with coordinates α_2, δ_2, have the readings R_2, R_2'. The coordinates may be altitude and azimuth, or right ascension and declination, or latitude and longitude, or any other system.

For the cosine of the angle between the two stars we have the expression

$$\sin \delta_1 \sin \delta_2 + \cos \delta_1 \cos \delta_2 \cos(\alpha_1 - \alpha_2),$$

which may be written

$$\sin \delta_1 \sin \delta_2 + \cos \delta_1 \cos \delta_2 \cos\{(\lambda - \alpha_1) - (\lambda - \alpha_2)\}.$$

From the general formulae of (1), (2), (3), § 142, we can substitute in the expression just written for $\sin \delta_1$, $\sin(\lambda - \alpha_1) \cos \delta_1$, $\cos(\lambda - \alpha_1) \cos \delta_1$ their equivalents in terms of R_1 and R_1', and the constants of the instrument θ, q, r. In like manner we can substitute for $\sin \delta_2$, $\sin(\lambda - \alpha_2) \cos \delta_2$, $\cos(\lambda - \alpha_2) \cos \delta_2$ their equivalents in terms of R_2 and R_2' and θ, q, r, and thus obtain an expression for the cosine of the angle between the two stars in terms of R_1, R_1', R_2, R_2' and the constants of the instrument.

The work may be simplified by observing that θ cannot enter into the result, for it is obvious that the angle between the two stars must be independent of the position of the fundamental circle with regard to which the coordinates are measured. It is therefore permissible, for this particular calculation, to assign to θ any arbitrary value we please without restricting the generality of the result. If we make $\theta = 90°$ the equation becomes as follows:

$$\begin{aligned}&\sin\delta_1\sin\delta_2+\cos\delta_1\cos\delta_2\cos(\alpha_1-\alpha_2)\\&=(-\cos q\sin r\cos R_1+\cos r\sin R_1\cos R_1'-\sin q\cos r\cos R_1\sin R_1')\\&\times(-\cos q\sin r\cos R_2+\cos r\sin R_2\cos R_2'-\sin q\cos r\cos R_2\sin R_2')\\&+(\cos q\sin r\sin R_1+\cos r\cos R_1\cos R_1'+\sin q\cos r\sin R_1\sin R_1')\\&\times(\cos q\sin r\sin R_2+\cos r\cos R_2\cos R_2'+\sin q\cos r\sin R_2\sin R_2')\\&+(-\sin q\sin r+\cos q\cos r\sin R_1')(-\sin q\sin r+\cos q\cos r\sin R_2'),\end{aligned}$$

which gives the following fundamental equation:

$$\begin{aligned}&\sin\delta_1\sin\delta_2+\cos\delta_1\cos\delta_2\cos(\alpha_1-\alpha_2)\\&=+\sin^2 q\sin^2 r\\&\quad+\cos^2 q\sin^2 r\cos(R_1-R_2)\\&\quad+\cos^2 q\cos^2 r\sin R_1'\sin R_2'\\&\quad+\cos^2 r\cos R_1'\cos R_2'\cos(R_1-R_2)\qquad\qquad\text{(i).}\\&\quad+\sin^2 q\cos^2 r\sin R_1'\sin R_2'\cos(R_1-R_2)\\&\quad+\cos^2 r\sin q\sin(R_1-R_2)\sin(R_1'-R_2')\\&\quad+\cos q\sin r\cos r\sin(R_1-R_2)(\cos R_2'-\cos R_1')\\&\quad+\sin q\cos q\sin r\cos r\{\cos(R_1-R_2)-1\}(\sin R_1'+\sin R_2').\end{aligned}$$

It is obvious that a rotation of circle I in its plane must be without effect on the distance S_1S_2. Hence R_1 and R_2 enter into $\cos S_1S_2$ only by their difference $R_1 - R_2$ and consequently the index error of circle I does not enter into the expression. We might indeed have further abbreviated the work by making $R_2 = 0$ before multiplying to form the equation (i) if after the multiplication we replaced R_1 by $(R_1 - R_2)$. We may suppose that the index error of circle II is Δ, in which case R_1' and R_2' should be replaced by $R_1' + \Delta$ and $R_2' + \Delta$. We have already shown how Δ might be found by right and left observations of the same object. It may, however, be determined otherwise, as will presently appear.

By assigning suitable values to q and r, this formula can be made to apply to the following astronomical instruments:—the altazimuth, the meridian circle, the prime vertical instrument, the equatorial, and the almucantar. We shall see later that for the meridian circle q and r should be each as near zero as possible, and for the almucantar q is the latitude and r quite arbitrary. The following general proof will show that the complete theory of each of the instruments named must be included in this one formula.

From any such instrument we demand no more than that the two readings R and R' obtained by directing the instrument to any particular star shall enable us to calculate the coordinates α, δ of that star free from instrumental errors.

Let S_1, S_2, S_3 be three standard stars of which the coordinates are known, and let each of these stars be observed with the generalized instrument with results R_1, R_1'; R_2, R_2'; R_3, R_3' respectively. Substituting for each of the three pairs $(S_1 S_2)$, $(S_2 S_3)$, $(S_3 S_1)$ in the typical formula (i), we obtain three independent equations. From these equations, q, r, and Δ can be found. Nor will there be any indefiniteness in the solution, for in each case we may regard these quantities as approximately known, so that to obtain the accurate values of q, r and Δ we shall have to solve only linear equations. We may thus regard (i) as an equation connecting α_1 δ_1, α_2 δ_2, R_1, R_1', R_2, R_2', and known quantities.

Let S be the star whose coordinates α, δ are sought. We write the equation (i) for the pair $(S S_1)$, and substitute their numerical values for α_1, δ_1, R_1, R_1'. We thus have an equation connecting the coordinates α, δ of *any* star with its corresponding R, R' and known numerical quantities. When we substitute for R and R' the values observed for S, the formula reduces to a numerical relation between the α and δ of the particular star S. From the pair $(S S_2)$ we find in like manner another quite independent numerical equation involving α, δ. As, however, two equations are not generally sufficient to determine α, δ without ambiguity, we obtain a third equation from $(S S_3)$. This equation is not independent of the others, but if we make $x = \sin\delta$, $y = \cos\delta\cos\alpha$, $z = \cos\delta\sin\alpha$, we shall obtain three linear equations in x, y, z by the solution of which α and δ are found without ambiguity.

All the ordinary formulae used in connection with the different instruments named can be deduced as particular cases of the general equation (i).

Ex. Show that if q and r be small quantities such that their second and higher powers may be omitted, formula (i) may be written as follows:

$$\begin{aligned}&\sin\delta_1\sin\delta_2+\cos\delta_1\cos\delta_2\cos(a_1-a_2)\\ &\quad=\cos R_1'\cos R_2'\cos(R_1-R_2)+\sin R_1'\sin R_2'\\ &\qquad+q\sin(R_1-R_2)\sin(R_1'-R_2')+r\sin(R_1-R_2)(\cos R_2'-\cos R_1').\end{aligned}$$

***150. Differential formulae in the theory of the generalized instrument.**

If the angle λ, see § 142, were increased by a small quantity $\Delta\lambda$ while θ, q, r were all left unchanged, then the readings R and R' of the instrument when directed upon a star α, δ would in general be affected by certain changes ΔR and $\Delta R'$. In like manner if θ were changed to $\theta + \Delta\theta$ while λ, q, r were left unaltered, then R and R' would also undergo certain small changes. The dotted lines in Fig. 116 show the modification the figure receives when the changes indicated have been made.

If the changes in λ and θ were made simultaneously, then ΔR and $\Delta R'$ will be each linear functions of $\Delta\lambda$ and $\Delta\theta$. Of course it will not, in general, be the case that either ΔR or $\Delta R'$ is zero. As however $\Delta\lambda$ and $\Delta\theta$ are both arbitrary, there must obviously be some ratio between these quantities which would make the resulting value of $\Delta R'$ zero. We shall now investigate in this case the relations between $\Delta\lambda$, $\Delta\theta$ and ΔR.

For this we have to find the effect upon the coordinates of a small change in θ. As the position of S with respect to the fundamental circle is unchanged and as the figure SKV and the angle $90° - q$ are unaltered by the changes of λ and θ (for $\Delta R' = 0$), we see that the figure $SKVN$ receives a small rotation η about S, bringing N to N''. Let O be the nole of MN. Then as MN is the fundamental circle of reference it will remain unaltered by this rotation and therefore O is unaltered. But the rotation about S will move NV and thus B, the nole of NV, will be conveyed to B', where $SB' = SB$ and $BSB' = \eta$.

The angle θ between NV and NN'' must be equal to BO, the arc between their noles. Hence after the rotation about S we have OB' as the altered value of θ. Let fall $B'Q$ perpendicular on OB, then the difference between OB' and OB is BQ, and hence

$$\Delta\theta = BQ = B'B \cos B'BQ = B'B \sin SBT$$
$$= \eta \sin SB \sin SBT = \eta \sin ST,$$

where ST is the production of NS, for N is a pole of OB, but

$$\eta \sin ST = \eta \cos NS = \eta \cos(\lambda - \alpha) \cos\delta,$$

whence
$$\Delta\theta = \eta \cos(\lambda - \alpha) \cos\delta.$$

We have next to express $\Delta\lambda$ and ΔR by means of η.

If N' be the new position of the point originally at N and N'' the new position of the ascending node of NV on MN, then

$$\angle\, N'N''N = 180^\circ - \theta,$$

but
$$N'N'' = NN' \sin N'NN'' \operatorname{cosec} \theta,$$

or
$$\sin \theta \Delta R = \eta \sin SN \cos SNL,$$

whence
$$\sin \theta \Delta R = \eta \sin (\lambda - \alpha) \cos \delta.$$

If, finally, a perpendicular $N'N'''$ is drawn to MN, then

$$NN''' = NN'' + N''N''',$$

$$\eta \sin NS \sin SNL = \Delta\lambda + \cos \theta \Delta R,$$

or
$$\eta \sin \delta = \Delta\lambda + \cos \theta \Delta R.$$

Thus we obtain the three formulae

$$\left.\begin{aligned} \Delta\theta &= \eta \cos (\lambda - \alpha) \cos \delta \\ \sin \theta \Delta R &= \eta \sin (\lambda - \alpha) \cos \delta \\ \Delta\lambda + \cos \theta \Delta R &= \eta \sin \delta \end{aligned}\right\} \quad \text{............(i).}$$

We shall now investigate the effect upon α and δ of a change of θ into $\theta + \Delta\theta$, it being supposed that λ, q, r, R, R' remain unaltered while this change takes place. The change is of course equivalent to a rotation of the figure $NVKS$ round N through an angle $\Delta\theta$ while this figure remains unaltered in form.

NS is unchanged and S moves perpendicular to NS through the small distance $\sin NS \,.\, \Delta\theta$, it is obvious from the figure that this increase of θ diminishes the declination by

$$\sin NS \sin NSL \Delta\theta = \sin (\lambda - \alpha) \Delta\theta.$$

We thus obtain

$$\Delta\delta = - \sin (\lambda - \alpha)\ \Delta\theta \text{ ....(ii).}$$

We have also

$$\Delta\alpha = - \cos (\lambda - \alpha) \tan \delta \Delta\theta \text{ ....(iii).}$$

***151. Application of the differential formulae.**

The formulae (i), (ii), and (iii), § 150, will enable us now to deduce from the first of the formulae for the generalized instrument (1), § 142, the remaining formulae (2) and (3).

The first formula is

$$\begin{aligned} \sin \delta = &- \cos \theta \sin q \sin r \\ &- \sin \theta \cos q \sin r \cos R \\ &+ \cos \theta \cos q \cos r \sin R' \\ &+ \sin \theta \cos r \sin R \cos R' \\ &- \sin \theta \sin q \cos r \cos R \sin R'. \end{aligned}$$

As this must be universally true it must be true if θ be increased by $\Delta\theta$ while δ receives its corresponding variation. Performing the differentiation, substituting for $\Delta\delta$ from (ii) and dividing by $\Delta\theta$, we have

$$\begin{aligned}\sin(\lambda-\alpha)\cos\delta = &-\sin\theta\sin q\sin r\\ &+\cos\theta\cos q\sin r\cos R\\ &+\sin\theta\cos q\cos r\sin R'\\ &-\cos\theta\cos r\sin R\cos R'\\ &+\cos\theta\sin q\cos r\cos R\sin R'.\end{aligned}$$

Thus we see how the first leads to the second of the fundamental formulae (2), § 142.

Finally let the equation just obtained be submitted to the differentiation as already explained in § 150 with respect to $\Delta\theta$, $\Delta\lambda$, ΔR, all other quantities remaining constant, and we have

$$\begin{aligned}&\cos(\lambda-\alpha)\cos\delta\Delta\lambda\\ &\quad=\sin\delta\Delta\theta-(\cos q\sin r\sin R+\cos r\cos R\cos R'\\ &\qquad+\sin q\cos r\sin R\sin R')\cos\theta\Delta R.\end{aligned}$$

Eliminating $\Delta\theta$, ΔR, $\Delta\lambda$ by equation (i), § 150, we obtain the third of the three fundamental formulae for the generalized instrument, § 142, viz.

$$\begin{aligned}\cos(\lambda-\alpha)\cos\delta = &\ \cos q\sin r\sin R\\ &+\cos r\cos R\cos R'\\ &+\sin q\cos r\sin R\sin R'.\end{aligned}$$

Thus we see how the third of the three fundamental formulae may also be deduced from the first.

152. The generalized transit circle.

An important case of the generalized instrument is that in which axis I is simply the earth itself. If the equator is taken as the fundamental plane MN, Fig. 114, then since this is normal to the earth's axis we must have $\theta = 0$, and with a suitable adjustment of the origin the coordinates α, δ will become the R.A. and decl. The pointer which is carried with the earth in the diurnal rotation will indicate on circle I (in this case the celestial equator) a reading R which can differ only by a constant from the sidereal time ϑ. This constant may be included in λ and thus the fundamental equations (§ 143) become

$$\left.\begin{aligned}\cos q\cos r\sin R' &= \sin q\sin r + \sin\delta\\ \cos r\cos R' &= \cos\delta\cos(\vartheta+\lambda-\alpha)\\ \sin r &= -\sin q\sin\delta + \cos q\cos\delta\sin(\vartheta+\lambda-\alpha)\end{aligned}\right\}\text{(i).}$$

We may distinguish this case of the generalized instrument as the *generalized transit circle.*

To find expressions for the right ascension α_0 and the declination δ_0 of the nole of circle II in the generalized transit circle we observe that if r had been $-90°$ the telescope would be necessarily always pointed to α_0, δ_0, and therefore by substitution of $-90°$ for r in equations (i) they must be satisfied by α_0, δ_0 whence

$$-\sin q + \sin\delta_0 = 0;\quad \cos\delta_0\cos(\vartheta+\lambda-\alpha_0) = 0;$$
$$\sin q\sin\delta_0 - \cos q\cos\delta_0\sin(\vartheta+\lambda-\alpha_0) = 1.$$

From the first we obtain $\delta_0 = q$ for we reject the solution $180° - q$ because $-90° \not> q \not> 90°$. The second equation requires that $\vartheta + \lambda - \alpha_0$ shall be either 90° or 270°, and of these the former is inadmissible for it would not satisfy the third equation. Hence for the coordinates of the nole of circle II we have

$$\alpha_0 = \vartheta + \lambda - 270°;\quad \delta_0 = q,$$

and equations (i) may be written thus:

$$\left.\begin{aligned}\cos\delta_0\cos r\sin R' &= \sin\delta_0\sin r + \sin\delta\\ \cos r\cos R' &= \cos\delta\sin(\alpha_0-\alpha)\\ \sin r &= -\sin\delta_0\sin\delta - \cos\delta_0\cos\delta\cos(\alpha_0-\alpha)\end{aligned}\right\}\text{(ii).}$$

When the generalized transit circle is still further specialized to form the *meridian circle* of our observatories the telescope must be at right angles to axis II, so that $r = 0$ and axis II must lie due east and west. The instrument may be in two positions according as the nole of circle II is in the east point of the horizon or the west point. In the former case $\alpha_0 = \vartheta + 90°$; $\delta_0 = 0$, and equations (ii) become

$$\sin R' = \sin\delta;\quad \cos R' = \cos\delta\cos(\alpha-\vartheta);\quad \cos\delta\sin(\alpha-\vartheta) = 0,$$

from which we have two solutions, viz.

$$\alpha = \vartheta;\quad \delta = R',$$

and also

$$\alpha = \vartheta + 180°;\quad \delta = 180° - R',$$

the first solution corresponding to the upper culmination and the second to the lower.

If the nole of circle II be at the west point $\alpha_0 = \vartheta - 90°$; $\delta_0 = 0$, then equations (ii) become

$$\sin R' = \sin \delta; \quad \cos R' = -\cos \delta \cos (\alpha - \vartheta); \quad \cos \delta \sin (\alpha - \vartheta) = 0,$$

and there are, as before, two solutions,

$$\alpha = \vartheta \qquad ; \quad \delta = 180° - R',$$
$$\alpha = \vartheta + 180°; \quad \delta = R',$$

the first corresponding to the upper culmination and the second to the lower.

In the instrument known as the *prime vertical instrument* axis II is also horizontal but it lies due north and south, we also have $r = 0$ and the nole of circle II coincides with either the north point $\alpha_0 = \vartheta + 180°$; $\delta_0 = 90° - \phi$ or the south point $\alpha_0 = \vartheta$; $\delta_0 = \phi - 90°$, and the last equation of (ii) becomes in both cases

$$\cos (\vartheta - \alpha) \tan \phi = \tan \delta.$$

If the telescope be fastened rigidly to a body floating on mercury then axis II is vertical. In the actual instrument circle II is not graduated, we may however assume graduations of which the nadir is the nole, in which case $\alpha_0 = \vartheta + 180°$; $\delta_0 = -\phi$. The last equation of II becomes

$$\sin r = \sin \phi \sin \delta + \cos \phi \cos \delta \cos (\vartheta - \alpha),$$

where $90° - r$ is the constant angle between the zenith and the point in which the axis of the telescope meets the celestial sphere. This instrument is known as the *almucantar*, being so designated by Chandler its inventor.

We thus see that when the generalized instrument is specialized to become the meridian circle both r and q are zero. When it is specialized to become the prime vertical instrument then r is zero but q is not zero. When it is specialized to become the almucantar neither r nor q is zero. What are known as the *errors* of the instruments will be considered in the next chapter.

CHAPTER XXII.

THE FUNDAMENTAL INSTRUMENTS OF THE OBSERVATORY.

153. The reading of a graduated circle.

We shall first consider the theory of the graduated circle as actually employed in the construction of astronomical instruments.

The circle so used is generally made of gun-metal and round the circumference a thin strip of silver or other suitable metal is inlaid on which the dividing lines or "traits" as they are often called are engraved. The principal traits are numbered from 0° to 359°, thus dividing the circumference into 360 equal parts or degrees. Between each two traits there are subsidiary divisions. In some of the finest instruments such as PISTOR and MARTIN'S meridian circles there are no fewer than 29 subsidiary traits in every degree and thus the circumference is really graduated to spaces of 2′. It is however more usual in ordinary instruments to find traits at only 5′ or 10′ intervals.

The further subdivisions of the circumference between two consecutive traits are obtained, as will be presently explained, by the help of microscopes, so that seconds and even tenths of a second can be taken account of. In small instruments such as sextants the subdivision of the interval between traits is effected by the vernier, a contrivance we may regard as generally known from its employment in the barometer.

If the fixed pointer happened to coincide exactly with one of the traits on the circle, then by the reading of the circle for that particular position we are to understand the number of degrees and minutes by which such trait is designated. It will however most usually happen that the pointer is not in coincidence with one of the traits. To read the circle under these circumstances we require an artifice by which the spaces between the traits can be subdivided. For this reason among others the pointer of the generalized instrument is superseded in the meridian circle by the spider line of the reading microscope.

The microscope is attached to a fixed support and is so directed that its field of view shows a small portion of the divided circle (Fig. 115). The spider line AB is stretched across the focus of the microscope and consequently the images of two consecutive traits T_1 and T_2 and the line AB are both shown distinctly to an observer who looks through the eye-piece of the microscope.

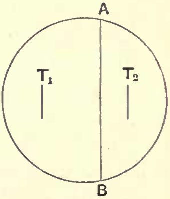


FIG. 115.

The measurement is effected by the line AB, which by means of a carefully wrought screw with a divided head can be moved parallel to itself and perpendicularly to the axis of the microscope. The position of AB is read by a scale which shows the number of complete revolutions of the micrometric screw, and the divided head shows the fractional part of one revolution which is to be added thereto. When the position of the screw is such that its reading is zero the line AB may be regarded as taking the place of the pointer.

We now move AB from the zero position and bring it to coincidence with T_1 ($< T_2$). The reading of the scale and the screw-head will then give the distance from the pointer to T_1, where the unit is the distance AB advances in a single revolution of the screw. The value of this unit in seconds of arc is determined as one of the constants of the instrument by measurement of known angular distances by the micrometer. Thus we find the number of seconds and fractional parts of a second from T_1 to the pointer. The quantity so found added to the degrees and minutes proper to T_1 gives the reading of the circle.

The single line AB may be advantageously replaced by two parallel lines close together. In this case the instrument is set for reading by placing the two lines so that the trait T_1 lies symmetrically between them. It is found that this can be done with greater exactitude than by trying to bring a single line into coincidence with the trait.

Ex. If when AB is placed on T_1 the micrometer screw reads n_1 and when AB is placed on T_2 the micrometer screw reads n_2, find the reading of the circle when the micrometer screw reads n. It is assumed that the interval between two consecutive traits is $2' = 120''$, and that the reading of the screw increases from left to right.

One revolution of the screw corresponds in seconds to $120''/(n_2 - n_1)$ and accordingly the reading is

$$T_1 + 120''(n - n_1)/(n_2 - n_1).$$

154. The error of eccentricity in the graduated circle.

The error of eccentricity arises from the absence of coincidence between the centre O (Fig. 116) round which the circle was rotated when on the dividing engine by which the graduations were engraved, and the centre O' about which the circle rotates when in use as part of a meridian circle or other astronomical instrument.

Let $OA_0 (= a)$ be the radius of the circle and let $OO' (= m)$ meet the circle in A_0. We take two other points A_1 and A_2 on the circle, and we shall suppose that R_0, R_1, R_2 are the graduations corresponding to A_0, A_1, A_2.

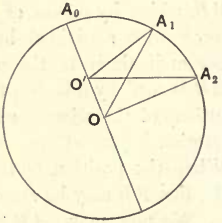


FIG. 116.

Let the instrument be rotated so that a line originally at $O'A_1$ is moved to $O'A_2$. Then $A_1O'A_2$ is the angle through which the instrument has really been turned though the arc indicated by the pointer is

$$A_1A_2 = R_2 - R_1 = \angle A_2OA_1.$$

The error caused by eccentricity is the difference between the angles subtended by A_1A_2 at O and O' respectively. If $\angle A_0O'A_1 = \psi_1$ then

$$\angle O A_1 O' = \psi_1 - \angle A_0OA_1 = \psi_1 - R_1 + R_0,$$

and from the triangle OA_1O' we have

$$a \sin(\psi_1 - R_1 + R_0) = m \sin \psi_1,$$

which, as m/a is very small, may be changed into

$$a\psi_1 = a(R_1 - R_0) + m \sin(R_1 - R_0) \qquad \text{(i).}$$

In like manner if $\angle A_0O'A_2 = \psi_2$ we find

$$a\psi_2 = a(R_2 - R_0) + m \sin(R_2 - R_0) \qquad \text{(ii).}$$

By subtracting (i) from (ii) we find for the error of eccentricity

$$(\angle A_2O'A_1 - \angle A_2OA_1) = (\psi_2 - \psi_1 - R_2 + R_1)$$

the expression

$$\frac{2m}{a} \sin \tfrac{1}{2}(R_2 - R_1) \cos \tfrac{1}{2}(R_2 + R_1 - 2R_0),$$

this is the circular measure of the angle to be added to the observed angle $(R_2 - R_1)$ to find the true angle $A_2O'A_1$. The correction for eccentricity when expressed in seconds of arc can never exceed $2m/a \sin 1''$.

Ex. 1. Show geometrically that the angle through which the circle has been rotated will be the mean of the arcs AA' and BB' if $A'B'$ be the position into which a chord AB through the centre of rotation is carried by the rotation.

Ex. 2. Assuming the eccentricity to be small show that its effect has disappeared from the mean of the measurements made by an even number of microscopes symmetrically placed round the circle.

Ex. 3. If the radius of a graduated great circle be $0^{m}{\cdot}450$ and if the centre from which the graduations have been struck is one-hundredth of a millimetre from the centre about which the circle is rotated, show that a reading by a single microscope of the angle through which the circle has been turned might be in error as much as $9''$.

Ex. 4. When the circle is in any position four microscopes at right angles show readings R_1, R_2, R_3, R_4. The circle is now turned through an angle θ very nearly equal to $180°$ and the microscopes now read R_1', R_2', R_3', R_4'. Show that e, the ratio which the distance between the centres bears to the radius (OO'/OA_0), is given by the equation

$$16e^2 = (R_1 - R_1' - R_3 + R_3')^2 + (R_2 - R_2' - R_4 + R_4')^2.$$

Ex. 5. The following pairs of readings are taken at two of the microscopes of a meridian circle,

22° 10′ 9″,	140° 39′ 44″,	250° 14′ 51″,
82° 14′ 13″,	201° 10′ 6″,	310° 45′ 3″;

prove that, assuming the rim to be circular and correctly graduated, the error of centering indicated by these readings is approximately the fraction ·0048 of the radius of the circle.

[Math. Trip.]

Subtracting equation (i) from (ii) we have

$$a(\psi_2 - \psi_1) = a(R_2 - R_1) + m \sin(R_2 - R_0) - m \sin(R_1 - R_0).$$

If we make $x = m/a \operatorname{cosec} R_0$ and $y = m/a \sec R_0$, this equation may be written

$$\psi_2 - \psi_1 = R_2 - R_1 + x(\cos R_1 - \cos R_2) + y(\sin R_2 - \sin R_1),$$

where $\psi_1 = A_0O'A_1$ and $\psi_2 = A_0O'A_2$ correspond to the first two positions, and R_2 and R_1 are the readings of the first microscope. If ψ_3 be $A_0O'A_3$ for the third position

$$\psi_3 - \psi_1 = R_3 - R_1 + x(\cos R_1 - \cos R_3) + y(\sin R_3 - \sin R_1).$$

If we take dotted letters R_1', R_2', R_3' to be the readings of the second microscope in the three positions of the circle, then we must have

$$R_2 - R_1 + x(\cos R_1 - \cos R_2) + y(\sin R_2 - \sin R_1)$$
$$= R_2' - R_1' + x(\cos R_1' - \cos R_2') + y(\sin R_2' - \sin R_1'),$$
$$R_3 - R_1 + x(\cos R_1 - \cos R_3) + y(\sin R_3 - \sin R_1)$$
$$= R_3' - R_1' + x(\cos R_1' - \cos R_3') + y(\sin R_3' - \sin R_1').$$

Substituting for R_1, R_2, R_3, R_1', R_2', R_3' the given readings we have two equations for x and y and $\sqrt{x^2 + y^2} = m/a$ is the quantity wanted.

155. The errors of division in the graduated circle.

We have hitherto assumed that the engraving of the traits on a graduated circle has succeeded in the object desired, which is of course to make the intervals between every pair of consecutive traits equal. But even the most perfect workmanship falls short of the accuracy demanded when the more refined investigations of astronomy are being conducted. Consecutive traits are not strictly equidistant and we have to consider how the observations may be combined so as to be cleared as far as possible from the effects of "Errors of Division." Such errors are no doubt small. The skilful instrument maker can adjust the actual place of each trait so that it will not be more than a few tenths of a second from the place it ought to occupy, but in the best work such errors must not be overlooked.

The errors may be divided into two classes. First. The systematic errors which rise and fall gradually from trait to trait according to some kind of law. Second. The casual errors which do not seem to follow any law and vary irregularly from trait to trait.

As to the latter there is no certain method of completely eliminating their effect unless by the actual determination of the error of each separate trait all round the circumference, followed by a rigorous application of its error to the reading of every trait involved. As this would require a separate investigation for each one of several thousand traits the task would be a colossal one, and indeed is not generally attempted. The errors of individual traits are tested at different parts of the circle and if they are found to be small it may then be hoped that in the mean of several observations taken with several microscopes, the influence of the casual errors will not appreciably affect the final result.

As regards the systematic errors in the division of the circle the assurance of their disappearance from the final result has a more satisfactory foundation. Errors of this class may arise from the mechanism used in the dividing engines by which the traits are engraved on the circle. The toothed wheels in the dividing engine are not and cannot be absolutely truly shaped and absolutely centred. Such errors in the traits may to a large extent be deemed periodic, so that when the wheels of the engine have performed a certain number of revolutions and a certain advance has been made in the engraving the same errors will be repeated. This is at least one of the chief sources from which systematic errors arise in the places of the traits.

Let R be the reading of a certain trait and let $R + \Delta R$, where ΔR is a small quantity, be the true reading of that point on the circle at which the trait is actually situated. Then ΔR is the error of that trait. We shall assume that ΔR can be represented by an expression of the form

$$\left.\begin{aligned}\Delta R = A_0 + A_1 \cos R + A_2 \cos 2R + \&c.\\ + B_1 \sin R + B_2 \sin 2R + \&c.\end{aligned}\right\} \quad \ldots\ldots\ldots \text{(i)},$$

where $A_0, A_1, A_2 \ldots, B_1, B_2 \ldots$ are constants whose values depend upon the individual peculiarities of the graduation.

It is some justification of this assumption that R and $R + 360°$ of course indicate only the same reading, and therefore when ΔR is expressed in terms of R the expression must be unaltered if R be changed into $R + 360°$. This condition is obviously fulfilled if ΔR has the form (i).

It should be noted that in the necessary limitation of the number of terms of the series we tacitly assume that there is no large break in the continuity of the errors all round the circumference. We assume in fact that changes in the errors at different points round the circumference do not take place very abruptly, and this of course would be the case with careful workmanship. Discontinuity is not presumed and $A_1, A_2 \ldots, B_1, B_2 \ldots$ are each finite and their number is small.

We may illustrate this remark as follows. Suppose the dividing engine started from $0° \, 0'$ and in engraving the traits round the circumference persistently made each trait one hundredth of a second too far from the preceding trait but made no other error. This error would accumulate so that the trait $359° \, 55'$ would have an error of $43''\cdot 19$, while the error of the next trait, *i.e.* $0° \, 0'$, is zero. At this one place the difference of two consecutive errors will be $43''\cdot 19$, while in all other cases the difference is only $0''\cdot 01$. This is in effect a case of discontinuity. Such an arrangement of the errors, or one in which the discontinuity was even still more violent could, as is well known, be represented in a series of the kind we are considering if we were allowed to take a great number of terms in the series for ΔR, but it could only be very ill represented if we were restricted to a few terms. It will be assumed in what follows that a small number of terms in the expression for ΔR does represent the errors of any particular circle.

Suppose that there are n microscopes symmetrically placed round the circle and that the corresponding readings of the circle in a certain position are $R_1, R_2 \ldots R_n$. Let the circle be now rotated through an angle λ and let the readings be $R_1', R_2' \ldots R_n'$. If the instrument were theoretically perfect we should of course have

$$R_1 - R_1' = R_2 - R_2' \ldots = R_n - R_n' = \lambda.$$

Owing, however, to errors of division and other errors, such for instance as the error of eccentricity which we have already considered, these quantities will not be all equal and we take for λ the mean of all the different quantities $(R_1 - R_1')$, $(R_2 - R_2')$, &c. as indicated by each microscope separately.

If θ be approximately the angle between two microscopes then $n\theta = 360°$. The reading R_1 of the first microscope when duly corrected by the addition of ΔR_1 will be

$$R_1 + A_0 + A_1 \cos R_1 + A_2 \cos 2R_1 \ldots$$
$$+ B_1 \sin R_1 + B_2 \sin 2R_1 \ldots.$$

In like manner we have for the corrected reading of the second microscope

$$R_2 + A_0 + A_1 \cos (R_1 + \theta) + A_2 \cos (2R_1 + 2\theta) \dots$$
$$+ B_1 \sin (R_1 + \theta) + B_2 \sin (2R_2 + 2\theta) \dots,$$

and for the nth microscope

$$R_n + A_0 + A_1 \cos (R_1 + (n-1)\,\theta) + A_2 \cos (2R_1 + 2\,(n-1)\,\theta) \dots$$
$$+ B_1 \sin (R_1 + (n-1)\,\theta) + B_2 \sin (2R_1 + 2\,(n-1)\,\theta) \dots.$$

Owing to the symmetrical disposition of the microscopes the sum of the n readings admits of a remarkable reduction. The coefficient of A_k in that sum is

$$\cos kR_1 + \cos (kR_1 + k\theta) + \dots + \cos (kR_1 + (n-1)\,k\theta) \dots \text{(i)},$$

which may of course be written in the form

$$P \cos (kR_1 + \epsilon) \dots\dots\dots\dots \text{(ii)},$$

where P and ϵ are independent of R_1 and are given by the equations

$$P \cos \epsilon = 1 + \cos k\theta + \cos 2k\theta \dots + \cos (n-1)\,k\theta,$$
$$P \sin \epsilon = \sin k\theta + \sin 2k\theta \dots + \sin (n-1)\,k\theta.$$

But (i) is unaltered if R_1 be changed into $R_1 + \theta$, for this merely changes the first term into the second, the second into the third, &c., and as $n\theta = 360°$ the last into the first. It follows that (ii) must be unaltered if R_1 be changed into $R_1 + \theta$, whence

$$P \cos (kR_1 + \epsilon) = P \cos (kR_1 + \epsilon + k\theta).$$

This has to be true for all values of R_1 and it is therefore true when

$$kR_1 + \epsilon = 0,$$

in which case

$$P = P \cos k\theta.$$

This can be satisfied only by making $P = 0$ unless $k\theta$ is an integral multiple of 2π when the series reduces to $n \cos kR_1$. Similar reasoning is easily seen to apply to the coefficient of B_k in the sum of the n corrected readings. Thus all the terms disappear from the mean of the corrected readings of the n microscopes except those in which k is an integral multiple of n, and there remains

$$(R_1 + R_2 \dots + R_n)/n + A_0$$
$$+ A_n \cos nR_1 + A_{2n} \cos 2nR_1 \dots$$
$$+ B_n \sin nR_1 + B_{2n} \sin 2nR_1 \dots.$$

Let us now suppose the circle turned into a different position when the readings are $R_1', R_2' \dots R_n'$. The angle λ through which the circle has been turned is, as already stated, the mean of the differences of the n readings in the two positions. We may, as an illustration, take the most usual case of four microscopes, when we have, if $k \not> 11$,

$$\lambda = \tfrac{1}{4}(R_1 + R_2 + R_3 + R_4) - \tfrac{1}{4}(R_1' + R_2' + R'_3 + R_4')$$
$$+ A_4(\cos 4R_1 - \cos 4R_1') + A_8(\cos 8R_1 - \cos 8R_1')$$
$$+ B_4(\sin 4R_1 - \sin 4R_1') + B_8(\sin 8R_1 - \sin 8R_1').$$

It is noteworthy that A_1, A_2, A_3, B_1, B_2, B_3 have disappeared from this formula. The terms corresponding to these quantities are however by far the most important in the expressions for ΔR. These terms will include most of the effects of systematic errors in the division and, as has been already shown, all the effects of the error in eccentricity. Thus by taking readings at four equidistant microscopes λ is determined free from the principal errors of the graduated circle.

By observing a case in which λ is known we obtain a linear equation in A_4, B_4, A_8, B_8. Other similar observations will provide further equations and from a sufficiently large number the values of A_4, B_4, A_8, B_8 can be determined by the methods of least squares. It may be stated generally that the effect is to show that these four quantities are too small to need attention. Thus for finding the angle through which the circle has been rotated we simply use the formula

$$\lambda = \tfrac{1}{4}(R_1 + R_2 + R_3 + R_4) - \tfrac{1}{4}(R'_1 + R_2' + R_3' + R_4').$$

Finally the reasons for having an even number of reading microscopes placed symmetrically round a graduated circle may be summarised thus:

(1) We eliminate the effects of eccentricity by taking the mean of the readings of two microscopes at the ends of a diameter and *a fortiori* at the ends of several diameters.

(2) We eliminate the greater part of the errors in division by taking the mean of the readings of four microscopes placed at intervals of 90°.

156. The transit instrument and the meridian circle.

It has been shown in § 152 that the theory of the generalized instrument includes among many other special cases the theory of the instrument known as the meridian circle or transit circle by which zenith distances as well as transits can be observed. The importance of the meridian circle is, however, so great, being as it is the fundamental instrument of the astronomical observatory, that it is useful to develop its theory in another and more direct manner.

The general description of the meridian circle may be briefly summarized as follows. A graduated circle is rigidly attached to an axis A through its centre and normal to its plane. A telescope whose optical axis is perpendicular to A, and therefore parallel to the graduated circle, is also rigidly attached to A. Thus when A moves the graduated circle and also the telescope move with it as one piece. The axis A is mounted horizontally and its extremities terminate in pivots which rest in bearings lying due east and due west.

In some instruments of this class arrangements are made by which the instrument, after being raised from its bearings, can be turned round 180° in the horizontal plane and then replaced so that the pivot which was originally towards the east shall be placed towards the west and *vice versa*. In such instruments the nole of the graduated circle may therefore be turned towards the east or towards the west, according to the positions of the pivots.

It will be observed that whether the instrument be in the nole-east position or in the nole-west, the graduated circle and the optical axis of the telescope will both be parallel to the plane of the meridian if the adjustments be perfect.

In the plane of the focus of the object-glass of the telescope there are two† spider lines at right angles to the telescope. One of these is parallel to the axis about which the telescope revolves and is called the *horizontal wire*. The other is perpendicular to the horizontal wire and is called the *meridional wire*. When the image of a star is on the meridional wire that star is in the act of transit. A line from the intersection of these wires to the centre of the object glass is the *optical axis* of the instrument. When the telescope is said to be set upon a star it is to be understood that the image of the star is coincident with

† In the actual meridian circle there are usually several fixed meridional wires and the single horizontal wire is often replaced by two parallel wires placed close together.

the intersection of the two cross wires, this is equivalent to saying that the optical axis of the telescope is directed upon the star.

An observation with the meridian circle has for its object the determination of both the right ascension and the declination of a star or other celestial body. The first is obtained by noting the time by the sidereal clock when the star crosses the meridian. If the clock be correct that time is the right ascension of the star. In so far as the determination of this element is concerned the meridian circle is what is called a *transit instrument* and the graduated circle is not concerned. The declination of the star is obtained from its zenith distance, which is observed by means of the graduated circle at the moment of transit.

The ideal conditions of the meridian circle as here indicated can of course be only approximately realized in the actual instrument. In the first place the axis A will not be quite horizontal, and we shall assume that the point on the celestial sphere indicated by the nole of the graduated circle shall have an easterly azimuth $90° - k$ and a zenith distance $90° + b$, where b and k are both small quantities. The axis of the telescope is of course approximately at right angles to the axis A. We shall suppose it to be directed to a point on the celestial sphere $90° - c$ from the nole of the circle. The small quantities k, b, c are called the errors of *azimuth*, of *level* and of *collimation* respectively. If the instrument and its adjustment were perfect, all these quantities should be zero, but in practice they are not zero and they are not even constant from day to day. They have to be determined whenever the instrument is used, and the methods of doing so will be duly explained. When k, b, c are known we can apply corrections to the observed time of transit and thus learn at what time the transit across the true meridian actually took place.

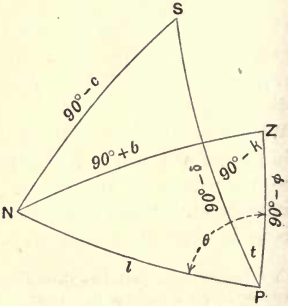


Fig. 117.

In Fig. 117 P is the north celestial pole, Z is the zenith, N is the nole of the graduated circle. According to the definitions of b and k already given, we have

$NZ = 90° + b$, $PZ = 90° - \phi$, and $\angle PZN = 90° - k$, so that when P and Z are known the quantities b and k define the position of the nole.

Let S be the star of declination δ, then $PS = 90° - \delta$, and when S is on the cross wires of the telescope, *i.e.* at the moment of observation, $NS = 90° - c$.

The base NP being fixed and the lengths NS and PS being both given there may be two possible positions of S. These correspond respectively to upper and lower culmination of the star (§ 29). In the present case we shall deal with upper culmination, and, the observer being supposed to be in the northern hemisphere, the point S will be towards Z as in Fig. 117.

If $b = c = k = 0$, then S would lie on the great circle through Z and P, *i.e.* on the meridian. But when b, c, k are different from zero, then $\angle SPZ$ is in general not zero, and consequently at the moment when in the instrument the star appears to be on the meridian it is still at an east-hour angle ZPS. If then the observer notes the time by his clock when the star is on the cross wires in his telescope he must add to the observed time to find the true time of transit the quantity t, which is the angle ZPS turned into time. Thus t is said to be the *correction* of the observed time of transit for the errors of the instrument.

From the triangle PZN and making $PN = l$ we have

$$\cos l = -\sin\phi \sin b + \cos\phi \cos b \sin k,$$

$$\sin l \sin\theta = \cos b \cos k,$$

$$\sin l \cos\theta = -\cos\phi \sin b - \sin\phi \cos b \sin k,$$

and from the triangle PSN

$$\sin c = \cos l \sin\delta + \sin l \cos\delta \cos(\theta - t) \quad \text{.........(i).}$$

Eliminating l and θ we have the fundamental equation for t,

$$\sin c + \sin\phi \sin b \sin\delta - \cos\phi \cos b \sin k \sin\delta$$
$$- \cos b \cos k \cos\delta \sin t + (\cos\phi \sin b + \sin\phi \cos b \sin k) \cos\delta \cos t = 0.$$

To apply this general equation to the meridian circle when used for observing a star at upper culmination we make t, b, k, c so small that their squares or products are neglected, and we may write it as follows:

$$t \cos\delta = c + b \cos(\phi - \delta) + k \sin(\phi - \delta),$$

whence

$$t = b \cos(\phi - \delta) \sec\delta + k \sin(\phi - \delta) \sec\delta + c \sec\delta \quad \text{......(ii).}$$

This is the fundamental formula for the reduction of meridian observations. The quantity t is the correction to be added to the observed sidereal time of the transit to obtain the true sidereal time. This expression for t is generally known as Mayer's formula. It may be transformed in various ways. For example Bessel introduced two new quantities m and n, determined by the equations

$$m = b \cos \phi + k \sin \phi, \quad n = b \sin \phi - k \cos \phi,$$

and thus we have the following convenient formula:

$$t = m + n \tan \delta + c \sec \delta \quad \text{.................(iii)}.$$

The quantities m and n are functions of the errors in level and azimuth and of the latitude, and are independent of the star. We easily see that $m = \theta - 90^\circ$ and $n = l - 90^\circ$. (See Fig. 117.)

We may also notice Hansen's formula, which is obtained by substituting for m its value $b \sec \phi - n \tan \phi$, with which change the last formula becomes

$$t = b \sec \phi + n (\tan \delta - \tan \phi) + c \sec \delta \quad \text{.........(iv)}.$$

By any of these formulae we obtain the correction which must be applied to an observed time of transit in order to obtain the clock time of transit across the true meridian.

157. Determination of the error of collimation.

The quantity c, known as the error of *collimation* in the meridian circle or indeed in any form of transit instrument, can be determined by the aid of what are known as collimating telescopes, of which we shall here describe the use.

In the focus of the telescope of the meridian circle is a frame carrying a line which can be moved from coincidence with the fixed meridional line into any parallel position in the plane perpendicular to the optical axis of the telescope. This movement is effected by a micrometer screw with a graduated head, so that by counting the revolutions and the fractional parts of a revolution the distance through which the movable wire has been displaced from the fixed wire becomes exactly known. We shall first show how by this contrivance we could determine the error of collimation if we could observe two diametrically opposite points on the heavens.

If t, δ be the hour angle and declination of a point on the celestial sphere, then from (i) we have

$$\sin c = \cos l \sin \delta + \sin l \cos \delta \cos (\theta - t).$$

As c, l, θ are fixed quantities connected with the instrument, it is plain that this equation will not, in general, be satisfied for a given pair of values t, δ. This means of course no more than the obvious fact that as the meridian circle has only one degree of freedom, *i.e.* rotation about a single axis, its telescope cannot be directed to any point on the celestial sphere except those which lie on a certain circle C. If, however, we give the instrument a second degree of freedom then it can, within certain limits, which for our present purpose are quite narrow limits, be directed upon any point in the vicinity of the circumference of C.

This second degree of freedom is given by the movable wire just described. By moving this wire to a distance x_1 from the fixed wire and regarding the intersection of the wire in its new position with the horizontal wire as the line of collimation of the telescope, the error of collimation is now $c + x_1$ and the equation (i) becomes therefore

$$\sin (c + x_1) = \cos l \sin \delta + \sin l \cos \delta \cos (\theta - t).$$

The quantity x_1 is determined by simply screwing the movable wire until the axis of the telescope can be directed to the point P, of which the coordinates are t, δ.

Let us now suppose the telescope directed to the celestial point P' with coordinates $(t + 180°)$, $-\delta$, which is 180° from the former point P. Again let the movable wire be set at the distance x_2 so that P' shall lie on the intersection of the movable wire and the fixed horizontal wire, and we have

$$\sin (c + x_2) = - \cos l \sin \delta - \sin l \cos \delta \cos (\theta - t),$$

or

$$\sin (c + x_1) + \sin (c + x_2) = 0,$$

and, as all the quantities c, x_1, x_2 are small, this may be written

$$c + x_1 + c + x_2 = 0,$$

or

$$c = - \tfrac{1}{2} (x_1 + x_2).$$

Thus c is found in terms of the observed quantities x_1 and x_2.

In the application of this process we obtain a pair of diametrically opposite points by what are known as a pair of collimating telescopes. The principle involved, one of much importance in the theory of astronomical instruments, is explained in the annexed diagram. AB is the transit instrument or telescope of the meridian circle, of which the central cube is pierced by a cylindrical hole LM of which the axis is XX' when the telescope is in the vertical position AB. The axis about which the transit instrument itself rotates is perpendicular to the plane of the paper, and the pivot at the westerly extremity is shown in the figure, while one of the positions which the instrument may assume during its rotation is indicated by the dotted lines. The two collimating instruments XY and $X'Y'$ are fixed horizontally north and south of the meridian circle, and cross wires are placed at the foci F and F' of each of these subsidiary instruments as in the focus of the great instrument itself.

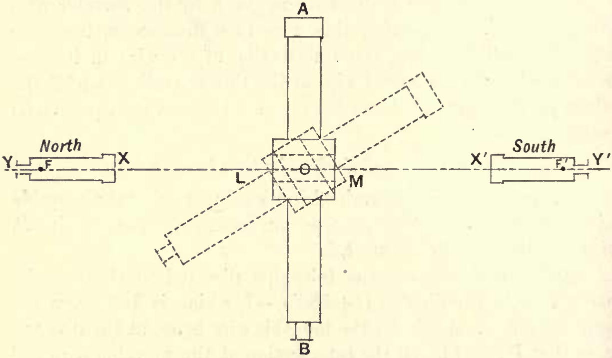


Fig. 118.

If light be admitted to the north collimator at Y, then the rays from the focal cross wires F diverge till they fall on the object-glass X, from which they emerge as a parallel beam, and, after passing through the hole LM (which they can do when the axis of the great telescope is vertical), fall on the object-glass X' of the second collimator. As these rays have been rendered parallel, an image is formed at F' of the cross wires at F. Thus the observer, on looking into the south collimator at Y', sees both the wires F' and the image of those at F simultaneously. By movement of the frame carrying the wires in F' he is able to bring the two intersections into coincidence, and when this adjustment is made the axes of the two collimating telescopes are exactly parallel, and consequently the axes of the two collimators, continued each way to the celestial sphere, indicate two points at a distance of 180°.

To use this apparatus for the determination of the error of collimation the meridian circle is rotated round its own axis till the telescope is directed on the north collimator, when the images of the wires at F will be seen in the same field as the wires in the focus of the telescope. The movable wire is then to be set carefully on the image of the intersection at F and as already explained x_1 is to be read off. Then the meridian circle is turned round 180° to the south collimator and in like manner x_2 is obtained. Thus c which is $-\frac{1}{2}(x_1 + x_2)$ becomes known.

158. Determination of the error of level.

When c the error of collimation has been found by the method just described, we can easily find b the error of level if we have the means of directing the telescope to a point S whose declination and hour angle are known quantities δ_0 and t_0. For if on increasing c (which has been already determined), by the measured quantity c'', we can direct the axis of the telescope on the point S, we have the following equation (§ 156):

$$\begin{aligned}\sin(c + c'') &+ \sin\phi \sin b \sin\delta_0 \\ &- \cos\phi \cos b \sin k \sin\delta_0 - \cos b \cos k \cos\delta_0 \sin t_0 \qquad \text{(i)} \\ &+ (\cos\phi \sin b + \sin\phi \cos b \sin k)\cos\delta_0 \cos t_0 = 0.\end{aligned}$$

The zenith would of course be a very convenient point to take for S, were it not that we have no means of knowing when the telescope is pointed to the zenith. We have, however, an excellent method of knowing when the telescope is pointed to the nadir. If a basin of mercury be placed beneath the centre of the meridian circle so that the telescope can be directed vertically downwards upon it, we can then, by looking though the eyepiece, compare the cross wires in the telescope with their images reflected from the mercury. For it is obvious that a beam of rays diverging from the focus of the telescope will emerge as a parallel beam from the object-glass and being reflected from the surface of the mercury are returned as a parallel beam to the object-glass and are transmitted again through the object-glass in the opposite direction and will therefore form an image of the cross wires at the focus beside the wires themselves. We have then only to shift the movable wire through such a measured distance c'' as shall make the intersection of the cross wires coincide with its reflected image, and then we know that the axis of the telescope must be perpendicular to the surface of mercury and must therefore point to the nadir. The declination of the nadir is $-\phi$, while its hour angle is 180°. With these substitutions the equation (i) reduces to

$$\sin(c + c'') = \sin b,$$

and therefore $b = c + c''$, for as all the quantities are small, the solution $180° - b = c + c''$ must be rejected. Thus b is known, for c the error of collimation is supposed to have been previously found and c'' is, as already stated, the quantity which has just been measured.

159. Determination of the error of azimuth and the error of the clock.

We shall assume that c and b, the errors of collimation and of level, have both been determined by the methods indicated, that T is the sidereal clock time of transit of a star α, δ and that ΔT is the error of the clock and k the error of azimuth. We can apply the corrections for b and c to the time T, and thus we have from Mayer's formula (iii), § 156, applied to two known stars (α_1, δ_1) and (α_2, δ_2) observed at a short interval during which ΔT may be presumed not to vary,

$$\alpha_1 = T_1 + \Delta T + k \sin(\phi - \delta_1) \sec \delta_1,$$
$$\alpha_2 = T_2 + \Delta T + k \sin(\phi - \delta_2) \sec \delta_2.$$

We thus have two equations in two unknowns ΔT and k, and solving these equations we obtain:

$$\Delta T = \{(\alpha_1 - T_1) \cos \delta_1 \sin(\delta_2 - \phi) - (\alpha_2 - T_2) \cos \delta_2 \sin(\delta_1 - \phi)\} \times \sec \phi \operatorname{cosec}(\delta_2 - \delta_1),$$

$$k = \{(\alpha_1 - \alpha_2) - (T_1 - T_2)\} \cos \delta_1 \cos \delta_2 \sec \phi \operatorname{cosec}(\delta_2 - \delta_1).$$

In these values of the desired quantities we are to note that errors of observation affect T_1 and T_2, and it is essential that the observations be so planned that the multipliers of T_1 and T_2 shall be as small as possible, hence $\operatorname{cosec}(\delta_2 - \delta_1)$ must be as small as possible. It is therefore necessary that one of the two declinations shall be near zero and the other near 90°. Hence we learn the important practical rule that for determining the error of the clock and the azimuth of the instrument one of the stars chosen should be near the pole and the other should be near the equator.

It will be observed that while b and c can be found without observation of celestial bodies this is not true with regard to ΔT and k.

160. Determination of the declination of a star by the meridian circle.

It is the function of the meridian circle to enable the observer to determine both the right ascension and the declination of a celestial object at the same transit. We have already shown how the right ascension is found and it remains to show how the declination is measured.

As nearly as possible at the moment of culmination the observer moves the telescope so that the star appears to run along the horizontal wire stretched across the focus of the telescope. The circle is then to be read by the microscopes in the manner already explained (§ 153). It is essential that at least two microscopes at opposite ends of a diameter be employed, but four microscopes symmetrically placed round the circumference are required for the best instruments and sometimes even more than four are used. The mean R of the readings of these microscopes is then adopted as the reading for this particular observation (see § 155).

We have seen how the collimation is determined by reflection from a basin of mercury placed under the meridian circle. We are now to move the instrument about its axis until it is so adjusted that the fixed horizontal wire in the principal focus coincides with its image reflected from the mercury as seen by an observer looking vertically downwards through the eyepiece. By this operation the telescope is directed to the nadir and the four microscopes are to be read and the mean R_0 is to be found. We may then assert that $180° + R_0$ would be the reading of the instrument when directed towards the zenith, and consequently $180° + R_0 - R$ is the apparent distance of the star from the zenith at the moment of culmination. This must then be corrected for refraction (see Chap. VI.), and thus the true zenith z distance is found. Assuming that the latitude ϕ is known we have the declination from the equation $\delta = \phi - z$.

It is often possible to dispense with an observation of the nadir by making use of stars whose declinations are already known. If such a star be observed with the reading R_1 we obtain $180° + R_0 - R_1$ as the expression of its apparent zenith distance, and again correcting for refraction we obtain the true zenith distance. But this is $\phi - \delta_1$, where δ_1 is the star's declination, and thus if r denote the refraction

$$180° + R_0 - R_1 + r = \phi - \delta_1,$$

by which R_0 can be ascertained. Thus we can learn the value of R_0 without directly making an observation of the nadir.

Ex. 1. The observed time of transit of a known star whose declination is 30° is found to be correct, i.e. to agree with the star's right ascension, while the observed times for stars in declination 15° and 60° are found to be $-7^s{\cdot}4$ and $+31^s{\cdot}5$ respectively in error. Prove that the error to be expected for a star in declination 45° is about 11^s.

[Math. Trip. I.]

Using Bessel's formula (iv) § 156, we obtain the following four equations, from which m, n, c can be eliminated, and the resulting equation for X will give the desired result

$$\begin{aligned} m + n\tan 30° + c\sec 30° &= 0, \\ m + n\tan 15 + c\sec 15 &= -7{\cdot}4, \\ m + n\tan 60 + c\sec 60 &= 31{\cdot}5, \\ m + n\tan 45 + c\sec 45 &= X. \end{aligned}$$

Ex. 2. In a transit instrument of 10 feet focal length which is correct, except for collimation error, a star of declination 60° is observed to cross the meridian 2^s too soon. Show that to adjust the instrument the cross wires must be moved a distance $0^{in}{\cdot}0087$. In which direction should the wires be moved?

[Math. Trip. I. 1900.]

The correction for collimation is $c\sec\delta = 2^{secs} = 30''$ whence $c = 15''$. The circular measure of this angle $\times$ 10 feet gives $0^{in}{\cdot}0087$. If we remember that the image in an astronomical telescope is reversed right and left it is obvious that the wires must be moved to the east

Ex. 3. Show that a transit instrument could conceivably be so adjusted that all stars passing to the south of the zenith would appear to be late in crossing the meridian by a constant error k.

[Coll. Exam.]

Ex. 4. If two stars of different declinations δ_1 and δ_2 can be found for which the three errors of adjustment produce no error in the time of transit, show that the correction to be added to the observed time of transit of a star of declination δ is

$$2c \sin \tfrac{1}{2}(\delta - \delta_1) \sin \tfrac{1}{2}(\delta - \delta_2) \sec \delta \sec \tfrac{1}{2}(\delta_1 - \delta_2),$$

where c is the collimation error.

[Math. Trip.]

From Bessel's formula we have

$$m + n \tan \delta_1 + c \sec \delta_1 = 0,$$
$$m + n \tan \delta_2 + c \sec \delta_2 = 0,$$

whence

$$m = -c \cos \tfrac{1}{2}(\delta_2 + \delta_1) \sec \tfrac{1}{2}(\delta_2 - \delta_1), \quad n = -c \sin \tfrac{1}{2}(\delta_1 + \delta_2) \sec \tfrac{1}{2}(\delta_2 - \delta_1),$$

and $m + n \tan \delta + c \sec \delta$ is obtained as desired.

Ex. 5. The level error of a transit instrument is b, its azimuthal error is k, and its collimation error is c. Prove that the error in the time of transit of a star due to these three errors in the instrument is a minimum for a star whose declination is

$$\sin^{-1} \{(k \cos \phi - b \sin \phi)/c\},$$

if it be a real angle; where ϕ is the latitude of the observatory.

[Coll. Exam.]

Ex. 6. A close circumpolar star is observed for error of azimuth, but the assumed level error is in error by a quantity x. Shew that the deviation error will be consequently in error by a quantity $x \tan \phi$, and that the time of transit of all stars must consequently be corrected by an amount $x \sec \phi$, where ϕ is the latitude of the place of observation, and it is assumed that there is no collimation error.

[Math. Trip. 1907.]

An observation of a known star will give the value of

$$b \cos(\phi - \delta) \sec \delta + k \sin(\phi - \delta) \sec \delta,$$

a simultaneous addition of x to b and y to k will leave this unaltered, if

$$x \cos(\phi - \delta) + y \sin(\phi - \delta) = 0,$$

or $y = x \cot(\delta - \phi)$, and as the star is close to the pole we may make

$$\delta = 90^\circ \text{ or } y = x \tan \phi.$$

Whence for any star there must be a correction to the time of transit of

$$x \{\cos(\phi - \delta) + \tan \phi \sin(\phi - \delta)\} \sec \delta = x \sec \phi.$$

Ex. 7. Show how Mayer's formula for the correction of observations with the meridian circle may be obtained directly from the equations of the generalized instrument (§ 142).

Ex. 8. Show that whatever be the magnitudes of the errors of a meridian circle the arithmetic mean of the hour angles t_1 and t_2 of a star at its upper and lower instrumental culminations is independent of the collimation of the instrument.

We may write the equation (ii) (§ 156) in the form

$$A \sin t + B \cos t + C = 0.$$

If t_1 and t_2 be the two different values of t which satisfy this equation

$$A \sin t_1 + B \cos t_1 + C = 0,$$
$$A \sin t_2 + B \cos t_2 + C = 0,$$

whence by subtraction and dividing out by $\sin \frac{1}{2}(t_1 - t_2)$ we have

$$\tan \tfrac{1}{2}(t_1 + t_2) = A/B,$$

while the collimation enters into C only.

Ex. 9. A transit instrument is mounted at a place of known latitude ϕ in a vertical plane which is not the meridian. Find an equation to determine the azimuth A of the instrument in terms of the observed time θ between two successive transits of a circumpolar star of declination δ.

Show that a small error $\Delta\theta$ in the observed difference θ of hour angles will lead to an error in the azimuth of magnitude

$$\tfrac{1}{2} \sin^2 A \tan A \cos^2 \phi \tan^2 \delta \tan \tfrac{1}{2}\theta \sec^2 \tfrac{1}{2}\theta \, \Delta\theta.$$

[Math. Trip. 1905.]

In the general formula § 156 we make $c=0$, $b=0$, $k=A$ and the formula becomes

$$-\cos\phi \sin A \sin\delta - \cos A \cos\delta \sin t + \sin\phi \sin A \cos\delta \cos t = 0,$$

which may be written thus

$$\sin\phi \tan A \cos t - \sin t = \cos\phi \tan A \tan\delta.$$

This must be true if $t-\theta$ be substituted for t and making $t - \frac{1}{2}\theta = P$ and $\frac{1}{2}\theta = Q$, we have the two formulae

$$\sin\phi \tan A \cos(P+Q) - \sin(P+Q) = \cos\phi \tan A \tan\delta \quad \text{............(i)},$$
$$\sin\phi \tan A \cos(P-Q) - \sin(P-Q) = \cos\phi \tan A \tan\delta \quad \text{............(ii)}.$$

Subtracting (i) from (ii) and dividing by $\sin Q$ we obtain

$$\sin\phi \tan A \sin P + \cos P = 0 \quad \text{............................(iii)}.$$

Multiplying (i) by $\sin(P-Q)$ and subtracting (ii) multiplied by $\sin(P+Q)$,

$$\sin\phi \tan A \sin 2Q = 2\cos\phi \tan A \tan\delta \cos P \sin Q,$$

whence dividing by $\sin Q$

$$\sin\phi \cos Q = \cos\phi \tan\delta \cos P,$$

whence from (iii)

$$\cos Q \cot A = -\cos\phi \tan\delta \sin P,$$

and therefore $$(\sin^2\phi + \cot^2 A)\cos^2 Q = \cos^2\phi \tan^2\delta.$$

Restoring to Q its value $\frac{1}{2}\theta$, we easily obtain

$$\cos^2\phi \sin^2 A = \cos^2 \tfrac{1}{2}\theta / (\cos^2 \tfrac{1}{2}\theta + \tan^2\delta),$$

which is the required equation between A and θ.

The second part of the question is obtained by differentiating A with regard to θ.

Ex. 10. Show that the difference of right ascension of the limb and the centre of a planet at transit is $\frac{1}{15}\rho R_0 / r \cos\delta$, where ρ is the semi-diameter of the planet in seconds of arc when the sun is at its mean distance R_0 where r is the planet's true distance and δ is the planet's declination and where the motion of the planet is neglected.

Ex. 11. The transit of Sirius was observed at Greenwich on Feb. 13th, 1851, at each of four wires at the times

$$6^h\,37^m\,43^s{\cdot}7\,;\ 6^h\,37^m\,58^s{\cdot}2\,;\ 6^h\,38^m\,12^s{\cdot}6\,;\ 6^h\,38^m\,26^s{\cdot}9.$$

The sum of the equatorial intervals for the observed wires is $-82^s{\cdot}905$, and the cosine of the star's declination is ·95871. Find the time of the star's transit.

[Coll. Exam.]

Let d_1, d_2, d_3, d_4 be the equatorial intervals of the four wires used expressed in time at the rate of 1 sec. to 15″. Then we have from the several observations the following times of meridian passage:

$$6^h\ 37^m\ 43^s{\cdot}7+d_1\sec\delta,\qquad 6^h\ 38^m\ 12^s{\cdot}6+d_3\sec\delta,$$
$$6\ \ 37\ \ 58\,{\cdot}2+d_2\sec\delta,\qquad 6\ \ 38\ \ 26\,{\cdot}9+d_4\sec\delta.$$

The mean of the four is

$$6^h\ 38^m\ 5^s{\cdot}3+\tfrac{1}{4}(d_1+d_2+d_3+d_4)\sec\delta,$$

but $d_1+d_2+d_3+d_4=-82^s{\cdot}905$ and $\sec\delta=1{\cdot}043$. Hence the reduction to the meridian is $-21^s{\cdot}6$ and the required time of transit is $6^h\ 37^m\ 43^s{\cdot}7$.

Ex. 12. Wires inclined at an angle of 45° to each other are placed in the focus of a transit instrument, pointed so that the declination of the intersection of the wires is 30°. A star whose declination is nearly 30° crosses from one wire to the other in $1^m\ 54^s$. Find the declination of the star.

[Coll. Exam.]

Ex. 13. A star's image S', Fig. 119, is bisected on the horizontal wire TS' of a transit-circle when the star is crossing a vertical wire distant i from the meridian PS: show that the correction to the observed declination to be applied for curvature of the star's path is

$$\tfrac{1}{2}i^2\sin 1'\tan\delta.$$

Let P be the pole, then

$$PS=PS'=90°-\delta,$$

and $S'TQ$ is perpendicular to PS. The required correction is ST.

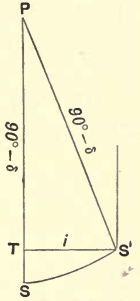


FIG. 119.

Ex. 14. Show that the correction for curvature (see last exercise) may be thus expressed, in seconds of arc

$$[6{\cdot}43569]\times\sin 2NPD\times t^2,$$

where NPD is the north polar distance of the star, and where t is the interval between the time of meridian passage and the time of transit expressed in seconds of time. The number in square brackets is a Logarithm.

[*Greenwich Obs.* 1898, p. xlviii.]

Ex. 15. If the zenith distance z of a star of declination δ be observed when very near the meridian at the hour angle t, and if ϕ be the latitude, show that to obtain the true meridional zenith distance we subtract from z the quantity expressed in seconds,

$$\frac{2\sin^2\frac{1}{2}t\cos\phi\cos\delta}{\sin 1''\sin z} - \frac{2\cot z}{\sin 1''}\left(\frac{\sin^2\frac{1}{2}t\cos\phi\cos\delta}{\sin z}\right)^2.$$

161. The altazimuth and the equatorial telescope.

The *altazimuth* is as its name indicates an instrument for measuring the altitude and azimuth of a celestial body. It is that particular case of the generalized instrument in which axis I is vertical and axis II is horizontal. In its well known form of the theodolite the altazimuth is of great use in surveying. It has also its uses in the astronomical observatory, but a much more important instrument for celestial observation is that known as the Equatorial which may also be regarded as a particular case of the generalized instrument (see § 142). The equatorial is characterized by the circumstances that axis I is parallel to the earth's axis and that axis II is parallel to the plane of the equator, but is not otherwise restricted.

To apply the equations of the generalized instrument to the equatorial we shall take the equator as the fundamental plane and as we shall at first assume that the instrument is in perfect adjustment we make $\theta = 0$, $q = 0$, $r = 0$, and thus (1), (2), (3) § 142 become

$$\sin\delta = \sin R'; \quad \sin(\lambda - \alpha)\cos\delta = -\sin R\cos R';$$
$$\cos(\lambda - \alpha)\cos\delta = \cos R\cos R' \quad\quad (1).$$

When α and δ are given there are in general two solutions to this set of equations. We may have

$$R' = \delta, \quad R = \alpha - \lambda,$$

or we may have

$$R' = 180° - \delta, \quad R = 180° - \lambda + \alpha.$$

This means, as already explained, that there are two ways of setting the instrument on a given point α, δ and that in one of these ways δ is the reading of R' and in the other δ is the supplement of R'.

If $R = 0$ we have $\alpha = \lambda$ from which we learn that the quantity λ is the right ascension of the origin of graduation on circle I. It is convenient to arrange that this point shall be the southerly point of the circle I. In this case $\lambda = \vartheta$ and $\lambda - \alpha$ is the hour angle west of the star α, δ, so that when the equatorial, being in perfect adjustment, is directed on a star, the reading R of circle I gives the hour angle east of that star.

The convenience of that particular mounting of a telescope which makes it an equatorial depends mainly on the fact that when the telescope has been pointed on a star a rotation of the instrument about its axis I, which in this case is generally called its polar axis, will counteract the effect of the diurnal motion. Mechanism known as an equatorial clock is provided by which the instrument is rotated about its polar axis with a velocity equal and opposite to that of the rotation of the earth about its axis. When all is in perfect adjustment and the equatorial clock keeping perfect time, a star appears fixed in the field of view.

We have supposed in the equatorial that axis I points exactly to the pole and that axis II is at right angles to axis I. Of course these conditions are not perfectly realized in the actual instrument and we shall now prove the following theorem† which is of practical importance in the adjustment of the equatorial instrument.

The axis of an equatorial is directed to a point at a small distance l from the true pole and in hour angle h. The telescope is directed so that the image of a star, in declination δ and hour angle h_1 (expressed in seconds of time), coincides with the intersection of two wires respectively parallel, and at right angles to, a great circle passing through the pole of the instrument. If the clockwork gains n seconds per day, then by the time the hour angle of the star has increased to h_2, the displacements of the star's image parallel to the two wires will be

$$-2l \sin \tfrac{1}{2}(h_2 - h_1) \cos \{h - \tfrac{1}{2}(h_1 + h_2)\} \sin \delta + \frac{n(h_2 - h_1) \cos \delta}{24 \times 60 \times 60},$$

and $$2l \sin \tfrac{1}{2}(h_2 - h_1) \sin \{h - \tfrac{1}{2}(h_1 + h_2)\}$$

respectively if the refraction be neglected.

Let A (Fig. 120) be the true pole and AZ the meridian. Let A' be the point towards which the axis of the instrument is directed. Then $AA' = l$ and $\angle A'AZ = h$.

Let C be the position of the star to which the telescope is

† Communicated by Dr Rambaut.

directed; then while the star moves from C to B so that $AB = AC$, the telescope is carried from C to B' so that $A'B' = A'C$. The angle $CAZ = h_1$. Let θ, ϕ, ψ, and ρ denote the angles $A'B'B$, BCB', ACA' and the arc BB' respectively.

We have thus two isosceles triangles ABC and $A'B'C$ whose sides and angles differ by small quantities of the order l.

Since $b = c$, $\angle ABC = \angle ACB$, $b' = c'$, and $\angle A'B'C = \angle A'CB'$, we have

$$\Delta b = \Delta c, \quad \Delta B = \Delta C.$$

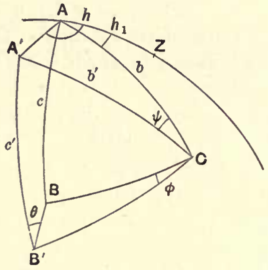


FIG. 120.

By the differential formulae of § 4, we have in general

$$\Delta a = \cos C \,.\, \Delta b + \cos B \,.\, \Delta c + \sin b \sin C \,.\, \Delta A,$$

$$\Delta c = \cos B \,.\, \Delta a + \cos A \,.\, \Delta b + \sin b \sin A \,.\, \Delta C,$$

or in this case $\quad \Delta a = 2 \cos C \,.\, \Delta b + \sin b \sin C \,.\, \Delta A,$

and $\quad \Delta C = 2 (\sin^2 \tfrac{1}{2} A - \cos^2 C) \operatorname{cosec} b \operatorname{cosec} A \,.\, \Delta b$
$\qquad\qquad - \sin C \cos C \operatorname{cosec} A \,.\, \Delta A.$

In the triangle ACA' we have

$$\sin l \sin (h - h_1) = \sin b' \sin \psi,$$

$$\sin l \cos (h - h_1) = \sin b \cos b' - \cos b \sin b' \cos \psi,$$

or, to the first order of small quantities,

$$\psi = \quad l \sin (h - h_1) / \sin b,$$

$$\Delta b = - l \cos (h - h_1).$$

We have also $\quad C - \psi = C' - \phi; \; \phi = \Delta C + \psi.$

In the triangle $BB'C$, since $\angle B' = \angle C' = \angle A'CB'$, we have

$$\sin\rho \sin(C' - \theta) = \sin a \sin\phi,$$

$$\sin\rho \cos(C' - \theta) = \cos a \sin a' - \sin a \cos a' \cos\phi,$$

or approximately,

$$\rho \sin(C - \theta) = \phi \sin a,$$

$$\rho \cos(C - \theta) = \Delta a.$$

Hence
$$\rho \sin\theta = \Delta a \sin C - \phi \sin a \cos C,$$

$$\rho \cos\theta = \Delta a \cos C + \phi \sin a \sin C,$$

or
$$\rho \sin\theta = \sin C \,.\, \Delta a - \sin a \cos C \,.\, \Delta C - \sin a \cos C \,.\, \psi,$$

$$\rho \cos\theta = \cos C \,.\, \Delta a + \sin a \sin C \,.\, \Delta C + \sin a \sin C \,.\, \psi.$$

Substituting in the first of these the values already found for Δa and ΔC, we obtain after a little reduction

$$\begin{aligned}\rho \sin\theta &= 2\left\{\sin C \cos C - \sin^2 \tfrac{1}{2}A \cot C + \cos^2 C \cot C\right\} \Delta b \\ &\qquad + \sin b \,.\, \Delta A - \sin a \sin C \cot C \,.\, \psi \\ &= \left\{2 \cos^2 \tfrac{1}{2}A \,.\, \Delta b - \sin a \sin C \,.\, \psi\right\} \cot C + \sin b \,.\, \Delta A.\end{aligned}$$

But $\cot C = \tan \frac{1}{2}A \cos b$ and $\sin a \sin C = \sin b \sin A$, therefore

$$\begin{aligned}\rho \sin\theta &= 2 \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A \cos b \,.\, \Delta b \\ &\qquad - 2 \sin^2 \tfrac{1}{2}A \cos b \sin b \,.\, \psi + \sin b \,.\, \Delta A.\end{aligned}$$

Substituting here the values found for Δb and ψ, we obtain

$$\rho \sin\theta = -2l \sin \tfrac{1}{2}A \cos(h - h_1 - \tfrac{1}{2}A) \cos b + \sin b \,.\, \Delta A.$$

Similarly

$$\begin{aligned}\rho \cos\theta &= 2 \sin^2 \tfrac{1}{2}A \,.\, \Delta b + \sin b \sin A \,.\, \psi \\ &= -2l \sin^2 \tfrac{1}{2}A \cos(h - h_1) + 2l \sin \tfrac{1}{2}A \cos \tfrac{1}{2}A \sin(h - h_1) \\ &= 2l \sin \tfrac{1}{2}A \sin(h - h_1 - \tfrac{1}{2}A).\end{aligned}$$

But $\rho \sin\theta$ and $\rho \cos\theta$ are the displacements in, and at right angles to the parallel, from which the required result follows.

CONCLUDING EXERCISES.

Ex. 1. An equatorial instrument being supposed in perfect adjustment except that the polar axis, though in the meridian, has an inclination error θ, show that even if the equatorial clock is running perfectly, the apparent place of a circumpolar star instead of being permanently at the centre of the field of view, traces out an ellipse whose principal semi-axes are θ and $\theta \sin\delta$.

[Math. Trip. 1905.]

Let P be the true pole, P' the actual pole of the equatorial (Fig. 121). Then h, δ are the true hour angle and declination of S, and $h+\Delta h$, $\delta+\Delta\delta$ the apparent values.

Let ST be perpendicular to the meridian, then

$$\sin PT \,.\, \tan h = \tan ST.$$

Taking the logarithmic differentials

$$\theta \cot PT + \sec h \operatorname{cosec} h \,.\, \Delta h = 0,$$

but $\cot PT = \tan\delta \sec h,$

whence $\Delta h = -\theta \tan\delta \sin h,$

and $\Delta\delta = -\theta \cos h.$

Thus the star appears displaced by quantities

$$x = -\theta \cos h,$$

$$y = -\theta \tan\delta \sin h \times \cos\delta,$$

whence $\dfrac{x^2}{\theta^2} + \dfrac{y^2}{\theta^2 \sin^2\delta} = 1.$

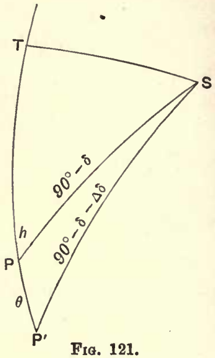


FIG. 121.

Ex. 2. The polar axis of an equatorial telescope is slightly out of adjustment, so that in hour angles h_1 and h_2 its declination circle, whose zero is in adjustment, reads too great by amounts d_1 and d_2. Draw two lines PS, PT, proportional to d_1, d_2, and making an angle $h_2 - h_1$. Describe a circle through PST. Show that the position of the instrumental pole is represented by P', where PP' is a diameter of the circle; and find the errors of adjustment in altitude and azimuth.

If S be any star whose polar distance and hour angle are Δ and h_1 and if P' is the instrumental pole whose position is similarly defined by the quantities λ and h, we have in the triangle SPP'

$$\cos\Delta' = \cos\Delta \cos\lambda + \sin\Delta \sin\lambda \cos(h - h_1).$$

We are also given that $\Delta' = \Delta - d_1$, hence

$$\cos(\Delta - d_1) = \cos\Delta \cos\lambda + \sin\Delta \sin\lambda \cos(h - h_1),$$

or neglecting squares and higher powers of the small quantities λ and d_1 we find

$$d_1 = \lambda \cos(h - h_1).$$

Similarly

$$d_2 = \lambda \cos(h - h_2).$$

Hence the construction follows, and we see that the diameter of the circle is equal to λ.

Ex. 3. The declination axis of an equatorial makes an angle $90° + x$ with the polar axis, and the telescope an angle $90° + y$ with the declination axis. The telescope is then pointed on a star in the meridian and on the equator, and the position wire of the micrometer set so that the star runs along it when the instrument is not being driven by the clock. It is then pointed on a star at declination δ also in or near the meridian. Show that this star will cross the field at an angle $-x(\sec\delta - 1) + y\tan\delta$ to the position wire.

[Sheepshanks Exhibition, 1900.]

Let P be the pole of the heavens, A the point to which the declination axis is directed and S a star in declination δ. Then

$$AP=90^\circ+x, \quad AS=90^\circ+y \text{ and } PS=90^\circ-\delta.$$

If Q denote the angle ASP we have

$$\cos Q=\frac{-\sin x+\sin y\sin\delta}{\cos y\cos\delta},$$

or approximately,

$$90^\circ-Q=-x\sec\delta+y\tan\delta.$$

The star will move across the field in a direction perpendicular to the meridian. Hence $90^\circ-Q$ is the angle its path will appear to make with AS which is fixed in the instrument.

At the equator we have

$$90^\circ-Q_0=-x.$$

Hence $$Q_0-Q=-x(\sec\delta-1)+y\tan\delta.$$

Ex. 4. An equatorial telescope whose axis is adjusted to the apparent pole is pointed to a star very near the meridian; show that, if the telescope is to follow the star accurately, the rate of the clock must be diminished in the ratio of

$$1-\kappa\cot\lambda\tan z : 1,$$

where λ is the latitude of the place of observation.

[Math. Trip.]

Let P be the true pole, Z the zenith and S the position of a star in any hour angle h. Let P' and S' be the positions of the pole and of the star as affected by refraction. Then $PP'=\kappa\cot\lambda$ and $SS'=\kappa\tan z$, where z is the zenith distance of the star.

If h' be the angle $ZP'S'$, *i.e.* the apparent hour angle of S', on applying the differential formulae (i) of § (4) to the triangle ZPS, we find

$$\Delta h=h'-h=\Delta\lambda\sin h\tan\delta+\Delta z\frac{\sin h\cos\lambda}{\sin z\cos\delta},$$

or $$h'-h=\kappa\left\{\cot\lambda\tan\delta-\frac{\cos\lambda}{\cos z\cos\delta}\right\}\sin h.$$

Hence

$$\frac{dh'}{dt}-\frac{dh}{dt}=\kappa\left\{\cot\lambda\tan\delta-\frac{\cos\lambda}{\cos z\cos\delta}\right\}\cos h\frac{dh}{dt}-\kappa\frac{\cos\lambda\sin z}{\cos^2 z\cos\delta}\sin h\frac{dz}{dt}.$$

On the meridian we have $h=0$ and $\delta=\lambda-z$, hence in this case

$$\frac{dh'}{dt}-\frac{dh}{dt}=\kappa\cot\lambda\left\{\tan\delta-\frac{\sin\lambda}{\cos z\cos\delta}\right\}\frac{dh}{dt}$$

$$=\kappa\cot\lambda\frac{\sin(\lambda-z)\cos z-\sin\lambda}{\cos z\cos(\lambda-z)}\frac{dh}{dt}=-\kappa\cot\lambda\tan z\frac{dh}{dt}.$$

Therefore $$\frac{dh'}{dt}=\{1-\kappa\cot\lambda\tan z\}\frac{dh}{dt}.$$

Ex. 5. A telescope is mounted on a stand with free movement in altitude and azimuth. Show that it may be made to move equatorially if a wire be attached to the end of the telescope connecting it with a certain fixed point.

[Sheepshanks Exhibition.]

Ex. 6. A photographic plate attached to an equatorial telescope of focal length F is exposed for one hour on the pole with the driving clock in action. Show that if there be an error of adjustment a in the polar axis, the trails of all star images on the plate will be arcs of equal circles and the lengths of the trails will be $\pi aF/12$.

[Math. Trip.]

Ex. 7. Show how the small errors of adjustment in the axis of an equatorial telescope may be determined by measures of the apparent difference of declination of two pairs of stars whose true differences of declination are given, and point out how the stars of each pair should be situated in hour angle to obtain the most reliable results.

[Dr Rambaut.]

Let AZ be the meridian, A the pole and A' the point towards which the axis is pointing.

Let $AA'=\lambda$ and $\angle A'AZ=h$.

Let B and C be one pair of stars, and let the telescope be pointed at C so that its image falls on the intersection of the cross wires, one of which is supposed to lie in the great circle $A'C$, and the other at right angles to it.

Let the telescope be turned in hour angle so that the N. and S. wire shall pass through the star B. If $A'B$ is produced to B' so that $A'B'=A'C$, then B' will be the position of the cross wires and the distance BB' $(=y)$ is the measured difference of declination.

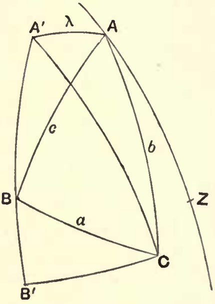


FIG. 122.

We have therefore

$$y=BB'=A'C-A'B,$$

or, if the sides and angles of the $\Delta A'BC$ are denoted by the letters a', b', c', A', B', C' we have

$$y=b'-c' \quad \text{......................................(i).}$$

The sides and angles of the $\Delta A'BC$ differ from those of ABC only by quantities of the order λ, and since BC is common to both we see that $da=0$. Also

$$db=-\lambda\cos A'AC \text{ and } dC=-A'CA=-\lambda\sin A'AC/\sin b.$$

If we denote the angle $A'AZ$ by h, CAZ by h_2, and BAZ by h_1, we have

$$da=0;\ db=-\lambda\cos(h-h_2);\ dC=-\lambda\sin(h-h_2)/\sin b.$$

We have also in general (see § 4)

$$dc = \cos B da + \cos A db + H \sin a \sin b \, dC,$$

substituting the values just found for da, db and dC in this we obtain

$$dc = -\lambda \cos A \cos (h - h_2) - \lambda \sin A \sin (h - h_2).$$

From (i) we have

$$y = b - c + db - dc,$$

therefore $\quad y = b - c + \lambda \sin A \sin (h - h_2) - \lambda (1 - \cos A) \cos (h_1 - h_2)$

$$= b - c + 2\lambda \sin \tfrac{1}{2} A \, . \sin \{h - \tfrac{1}{2}(h_1 + h_2)\}.$$

If δ_1 and δ_2 are the declinations of B and C respectively, then

$$b - c = \delta_1 - \delta_2,$$

and therefore

$$y = \delta_1 - \delta_2 + 2\lambda \sin \tfrac{1}{2}(h_1 - h_2) \sin \{h - \tfrac{1}{2}(h_1 + h_2)\}.$$

If we write $X = \lambda \sin h$ and $Y = \lambda \cos h$, then X is the distance of A' from the meridian, and Y is the projection of λ on the meridian towards the zenith, and we find

$$2 \sin \tfrac{1}{2}(h_1 - h_2) \cos \tfrac{1}{2}(h_1 + h_2) X - 2 \sin \tfrac{1}{2}(h_1 - h_2) \sin \tfrac{1}{2}(h_1 + h_2) Y = y - (\delta_1 - \delta_2) \quad \text{......(ii)},$$

which may be written

$$(\sin h_1 - \sin h_2) X + (\cos h_1 - \cos h_2) Y = y - (\delta_1 - \delta_2) \quad \text{......(iii)}.$$

Similar observations on another pair of stars will furnish a second equation of the same form which with (iii) will enable us to determine X and Y.

To find X most favourably we must make its coefficient in (ii) or (iii) as large as possible. It is clear that this is a maximum when the hour angles are 90° and 270°, *i.e.* the two stars should lie on the six-hour circle. In this case we find

$$2X = y - (\delta_1 - \delta_2).$$

To find the most favourable conditions for determining Y we make the two hour angles 0° and 180°, and we have

$$2Y = y - (\delta_1 - \delta_2).$$

In this case the two stars should be on the meridian.

Ex. 8. An equatorial telescope in north latitude λ is driven by clockwork gearing at true sidereal rate, and has its polar axis at the right elevation but in a vertical plane inclined at a small angle a to the plane of the meridian. The telescope is pointed to a star of south declination δ which is in the middle of the field of view when it is crossing the meridian. Neglecting refraction, prove that it will remain in the field of view as long as it is above the horizon, provided that the angular radius of the field is greater than

$$a \sec \delta \{\cos^2 \lambda - \sin^2 \delta \sin^2 (\lambda + \delta)\}^{\frac{1}{2}}.$$

[Math. Trip. I.]

Ex. 9. If the declination wire in the meridian circle instead of being exactly horizontal make an angle $90° - I$ with the meridian, and if δ' be the observed declination of a star of true declination δ, and near the meridian with an hour angle t; then show that

$$\tan\delta = \tan\delta'\cos t + \sec\delta'\sin t\tan I.$$

Ex. 10. Hence show that if the pole star with true declination δ appear when about an hour on one side of the meridian to have the declination δ' and the hour angle t', and when about an hour on the other side of the meridian to have the declination δ'', and the hour angle t'', the small inclination I can be found from the equation

$$\tan I = \frac{\tan\delta''\cos t'' - \tan\delta'\cos t'}{\sec\delta''\sin t'' + \sec\delta'\sin t'}.$$

Ex. 11. Show that the effect of the error of azimuth on the zenith distance of a star observed with the meridian circle is

$$\tfrac{1}{2}a^2\cos\phi\sin z\sec(\phi - z)\sin 1'',$$

where a is the azimuth error, ϕ the latitude, and z the zenith distance.

[Coll. Exam. 1903.]

Let $z + X$ be the apparent zenith distance as altered by azimuth of the instrument

$$\sin(\phi - z) = \cos(z + X)\sin\phi - \sin(z + X)\cos\phi\cos a,$$

whence $$X\cos(\phi - z) = \tfrac{1}{2}a^2\sin z\cos\phi\sin 1''.$$

Ex. 12. Show that in the reduction of a transit-circle observation of the R.A. of the moon's bright limb, the effect of the motion of the moon in R.A. and of its increase in semi-diameter, upon the reduction from the mean of the wires observed to the centre wire, is expressed by multiplying the ordinary reduction to centre by the factor

$$\frac{3600 + I}{3600} \times \frac{\sin(\text{moon's geocentric Z.D.})}{\sin(\text{moon's apparent Z.D.})} \times \sec(\text{moon's geoc. decl.}),$$

where I is the rate of increase of the moon's R.A., expressed in seconds of time per hour, for the instant of observation.

[Introduction to *Greenwich Obs.*]

Ex. 13. Show that it is possible for a transit instrument to point correctly to the zenith and to the south point of the horizon, but to be incorrect between them. If c'' be the collimation error of such a telescope, the error in the time of transit of a star of zenith distance 45° is $\cdot0276c\,\text{cosec}(45° + \phi)$ seconds of time, where ϕ is the latitude of the place of observation. Should this be added to or subtracted from the observed time?

[Coll. Exam. 1898.]

We have as in § 156

$$\sin c + (\sin\phi\sin b - \cos\phi\cos b\sin k)\sin\delta - \cos b\cos k\sin t\cos\delta$$
$$+ (\cos\phi\sin b + \sin\phi\cos b\sin k)\cos\delta\cos t = 0.$$

For the meridian circle b, c, k and therefore t are small quantities. Hence we may write

$$c+(b\sin\phi-k\cos\phi)\sin\delta+(b\cos\phi+k\sin\phi)\cos\delta-t\cos\delta=0.$$

By the conditions of the problem $t=0$ when $\delta=\phi$ (i.e. for a point at the zenith) whence $c+b=0$, also $t=0$ when $\delta=\phi-90^\circ$ (i.e. for the south point), whence $c+k=0$.

Hence for any other point we have

$$t\cos\delta=c\{1-(\sin\phi-\cos\phi)\sin\delta-(\cos\phi+\sin\phi)\cos\delta\}$$
$$=c\{1-\sqrt{2}\cos(\phi-\delta-45^\circ)\}.$$

If the zenith distance of a star at culmination be 45° then $\delta=\phi-45^\circ$ and therefore $\sin(45^\circ+\phi).\,t=-c(\sqrt{2}-1)$.

If c be expressed in seconds of arc and t in seconds of time,

$$15t=-\cdot 4142c\,\text{cosec}\,(45^\circ+\phi).$$

As t is negative the hour angle of the star is east when on the instrumental meridian, and consequently the correction is

$$+\cdot 0276c\,\text{cosec}\,(45^\circ+\phi).$$

Ex. 14. Show that if θ, q, r are all so small that powers above the first may be neglected the equations of the generalized instrument (1), (2), (3), § 142, assume the form

$$\sin\delta=\sin R'+\theta\sin R\cos R',$$
$$\sin(\lambda-a)\cos\delta=-\sin R\cos R'+\theta\sin R'+\cos R\,(r+q\sin R'),$$
$$\cos(\lambda-a)\cos\delta=\cos R\cos R'+\sin R\,(r+q\sin R').$$

Show that for these equations we have

First Solution.

$$R=a-\lambda+r\sec\delta+q\tan\delta+\theta\cos(a-\lambda)\tan\delta,$$
$$R'=\delta-\theta\sin(a-\lambda).$$

Second Solution.

$$R=180^\circ+a-\lambda-r\sec\delta-q\tan\delta+\theta\cos(a-\lambda)\tan\delta,$$
$$R'=180^\circ-\delta+\theta\sin(a-\lambda),$$

and explain how these formulae are applicable either to the altazimuth or the equatorial as well as to the meridian circle.

EXPLANATION OF THE TABLES I. AND II.

In Table I. I have followed Newcomb and Hill in the "Astronomical Papers for the American Ephemeris." The semi-axes major are the natural numbers corresponding to the logarithmic values given by Newcomb and Hill, and in expressing them in miles, I have, in conformity with the units adopted in this volume, assumed 8″·80 as the solar parallax, and Clarke's value, 3963·3 miles, as the equatorial semi-diameter of the Earth.

In Table II. will be found a consistent set of elements depending upon the angular semi-diameters as at present used in the Nautical Almanac, the masses of Newcomb and Hill, Clarke's semi-diameter of the earth (3963·3 miles), the Solar Parallax 8″·80 and the Earth's mean density 5·56, as found by Cornu and Baille.

TABLE I.

ELEMENTS OF THE SOLAR SYSTEM. Epoch 1900.

Name of Planet	Symbol	Semi-Major Axis of Orbit ⊖ = unity	Semi-Major Axis in millions of miles	Sidereal Period		Mean daily motion	Longitude of Perihelion	Longitude of Ascending Node	Inclination of Orbit	Eccentricity
				Mean Solar days	Julian years					
Mercury	☿	0·3870986	36·0	87·96926	0·24	4° 5′ 32″·4	75° 53′ 59″	47° 8′ 45″	7° 0′ 10″	0·205614
Venus	♀	0·7233315	67·2	224·7008	0·62	1 36 7·7	130 9 50	75 46 47	3 23 37	0·006821
Earth	⊖	1·000000	92·9	365·2564	1·00	59 8·2	101 13 15	0 0 0	0 0 0	0·016751
Mars	♂	1·523688	141·6	686·9797	1·88	31 26·5	334 13 7	48 47 9	1 51 1	0·093309
The Asteroids ..										
Jupiter	♃	5·202803	483·3	4332·588	11·86	4 59·1	12 36 20	99 26 42	1 18 42	0·048254
Saturn...........	♄	9·538844	886·2	10759·20	29·46	2 0·5	90 48 32	112 47 12	2 29 39	0·056061
Uranus	♅	19·19098	1782·8	30586·29	83·74	42·2	169 2 56	73 29 25	0 46 22	0·047044
Neptune.........	♆	30·07067	2793·5	60187·65	164·78	21·5	43 45 20	130 40 44	1 46 45	0·008533

TABLE II.

ELEMENTS OF THE SOLAR SYSTEM.

Name of Planet	Symbol	Axial Rotation	Equatorial Semi-diameter			Mass		Mean Density		Force of Gravity at Surface ⊖ = 1	Ellipticity	Inclination of Equator to Orbit
			Angular †	Miles	⊖ = 1	⊙ = 1	⊖ = 1	⊖ = 1	Water = 1			
Sun	☉	25·38 days	16′ 1″·18	432890	109·2	1/1	329390	0·25	1·40	27·61	?	7° 15′ ‡
Mercury	☿	?	3·34	1504	0·380	1/6000000 ?	0·055 ?	1·00 ?	5·56 ?	0·38 ?	?	?
Venus ...	♀	?	8·40	3783	0·955	1/408000	0·807	0·93	5·14	0·89	?	?
Earth ...	⊖	h. m. s. 23 56 4·09	8·80	3963	1·000	1/329390	1·000	1·00	5·56	1·00	1/293·465	23° 27′ 8″
Mars	♂	24 37 22·74	4·68	2108	0·532	1/3093500	0·106	0·71	3·92	0·38	?	24 52
Jupiter..	♃	9 56 ±	1 37·36	43850	11·06	1/1047·35	314·50	0·25	1·37	2·57	1/17	3 5
Saturn...	♄	10 15 ±	1 24·75	38170	9·63	1/3501·6	94·07	0·12	0·64	1·01	1/9	26 49
Uranus..	♅	?	34·28	15440	3·90	1/22869	14·40	0·24	1·35	0·95	1/95 ?	?
Neptune	♆	?	36·56	16470	4·15	1/19700	16·72	0·23	1·29	0·97	?	?

† The angular semi-diameter given in this column is the angle which the semi-diameter would subtend at a distance equal to the earth's mean distance from the sun.

‡ This is the inclination of the plane of the sun's equator to the plane of the ecliptic.

INDEX AND GLOSSARY

The numbers refer to the pages.

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