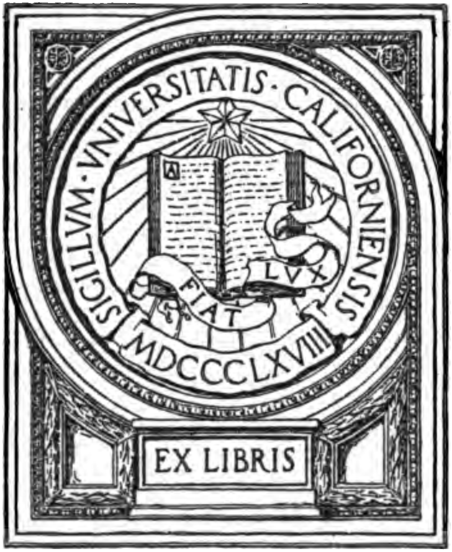
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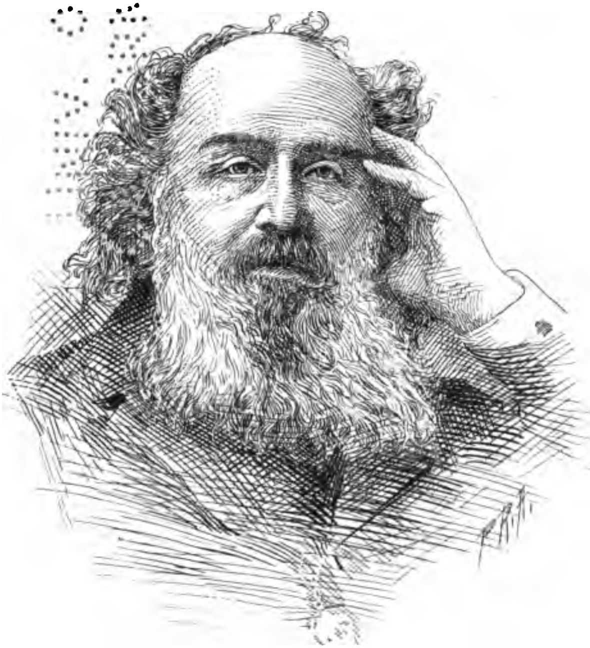

JAMES JOSEPH SYLVESTER (1814–1897)

(*See page 295.*)

ADVANCED ALGEBRA

BY

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COLLINS'S ADV. ALG.

W. P. I

PREFACE

THE College Entrance Board recognizes two standards of requirement in algebra: first, the ordinary one, for colleges of liberal arts; and second, the standard for entrance to technical schools. This book is arranged to follow a First Year Course, and is adapted to meet either of these two requirements, as well as the needs of students desiring to secure a maximum amount of mathematical training in the high school. Part I is a review of the First Year Course; Part II includes the remaining topics belonging to elementary algebra; and Part III, the usual topics of advanced algebra, as, the general theory of equations, determinants, etc. Containing as it does the chapters on advanced algebra, the book is adapted for use in those colleges which seek to lead their students through the gateway of algebra to the study of other college mathematics rather than to train them in the refinements of algebraic logic. For college students, especially for those who do not come fresh from the study of elementary algebra, the review will be found a great convenience.

The review of the First Year Course to radicals is intentionally rapid, being presented as succinctly as clearness and thoroughness permit. The exercises are moderate in number and difficulty. Since radicals and quadratics usually come at the end of the First Year Course, they are seldom as well understood or remembered as the earlier topics, and therefore should be reviewed in more detail. One aim sought in Part I was to *unify* the preceding mathematics, — arithmetic, algebra, and plane geometry, — as effectively as possible. In accordance with a suggestion of the Society for the Promotion of Engineering Education, there will be found on page 89 a *Summary* of the fundamental principles of elementary algebra, for study and reference.

Special attention is directed to the chapters on logarithms, permutations and combinations, determinants, and graphs. In the chapter on graphs a uniform plan and notation are followed throughout, designed to show the connection between the given equations, the tables of values of coördinates derived from them, and the corresponding graphs constructed on the diagrams from the values given in the tables. This method of presenting the matter makes a rather difficult subject much easier to learn.

In the advanced portion, covering the college algebra topics proper, the aim, as in other parts of the study, has been to secure simplicity, clearness, and conciseness, without sacrifice of rigor. The practical character of the exercises will also commend itself to teachers and students.

The author acknowledges with thanks valuable suggestions made on the manuscript or on the proofs by Professor Ernest B. Lytle, of the University of Illinois, Urbana; Professor Paul Prentice Boyd, State University, Lexington, Ky.; Professor Robert E. Moritz, of the University of Washington, Seattle; Professor E. A. Lyman, of State Normal College, Ypsilanti, Mich.; Mrs. Eva S. Maglott, of the Ohio Northern University, Ada; Lewis Omer, of the Northwestern University Academy at Evanston, Ill.; Ralph P. Bliss, Commercial High School, Brooklyn; J. Henry Graham, Central High School, Philadelphia; Arthur F. M. Petersilge, East High School, Cleveland; J. S. Counselman, High School, Birmingham; W. F. Moncrieff, Hume-Fogg High School, Nashville; C. M. Bookman, High School of Commerce, Columbus; Marie Gugle, Central High School, Toledo; T. H. McCormick, High School, Ft. Wayne; and W. E. Beck, High School, Sioux City.

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4. TABLE OF SQUARE ROOTS

The following (read as a logarithmic table by getting the first figure in the left column and the second figure at the top) contains the square roots of numbers from 10 to 99, and from 0 to 10 by tenths. Thus, $\sqrt{32} = 5.657$; $\sqrt{3.2} = 1.789$.

From it can also be obtained the square root of any number consisting of two figures followed or preceded by any number of zeros. Thus, $\sqrt{5900} = 76.81$; $\sqrt{460} = 21.45$; $\sqrt{.084} = .2898$; $\sqrt{.006} = .07746$; etc.

	0 or .0	1 or .1	2 or .2	3 or .3	4 or .4	5 or .5	6 or .6	7 or .7	8 or .8	9 or .9
0	0.000 0.000	1.000 .316	1.414 .447	1.732 .548	2.000 .632	2.236 .707	2.449 .775	2.646 .837	2.828 .894	3.000 .949
1	3.162 1.000	3.317 1.049	3.464 1.095	3.606 1.140	3.742 1.183	3.873 1.225	4.000 1.265	4.123 1.304	4.243 1.342	4.359 1.378
2	4.472 1.414	4.583 1.449	4.690 1.483	4.796 1.517	4.899 1.549	5.000 1.581	5.099 1.612	5.196 1.643	5.292 1.673	5.385 1.703
3	5.477 1.732	5.568 1.761	5.657 1.789	5.745 1.817	5.831 1.844	5.916 1.871	6.000 1.897	6.083 1.924	6.164 1.949	6.245 1.975
4	6.325 2.000	6.403 2.025	6.481 2.049	6.557 2.074	6.633 2.098	6.708 2.121	6.782 2.145	6.856 2.168	6.928 2.191	7.000 2.214
5	7.071 2.236	7.141 2.258	7.211 2.280	7.280 2.302	7.348 2.324	7.416 2.345	7.483 2.366	7.550 2.387	7.616 2.408	7.681 2.429
6	7.746 2.449	7.810 2.470	7.874 2.490	7.937 2.510	8.000 2.530	8.062 2.550	8.124 2.569	8.185 2.588	8.246 2.608	8.307 2.627
7	8.367 2.646	8.426 2.665	8.485 2.683	8.544 2.702	8.602 2.720	8.660 2.739	8.718 2.757	8.775 2.775	8.832 2.793	8.888 2.811
8	8.944 2.828	9.000 2.846	9.055 2.864	9.110 2.881	9.165 2.898	9.220 2.915	9.274 2.933	9.327 2.950	9.381 2.966	9.434 2.983
9	9.487 3.000	9.539 3.017	9.592 3.033	9.644 3.050	9.695 3.066	9.747 3.082	9.798 3.098	9.849 3.114	9.899 3.130	9.950 3.146

5. TABLE OF SYMBOLS

I. **Of Number:** 1, 2, 3, ⋯ 10, 11 ⋯ I, V, X, L, C, D, M; Italic letters, a, b, c, ⋯; a', b'', c''', ⋯ (read 'a prime,' 'b second,' 'c third' ⋯); a_1, a_2, a_3, ⋯ (read 'a sub one,' or 'a one,' ⋯); Greek letters, α, β, γ, δ, ⋯ ω; 0, zero, ∞, infinity, but ∞ does not denote a definite number.

II. **Of Operation:** $+$, $-$, $\pm$ (plus and minus), $^+$ exeess, $^-$ defect, $\times$, $\cdot$, $\div$, :, /, vinculum in a fraction, $\sqrt{\ }$, $\sqrt[n]{\ }$, exponents (integral, fractional, positive, and negative). Absence of any sign between letters, between figures and letters, and between parentheses denotes multiplication, but between *figures* in arithmetic and algebra it denotes addition. The solidus line / is sometimes used to mean 'over.' Thus, a/b denotes $\frac{a}{b}$. *Cancellation* is denoted by one oblique line, as $\frac{\cancel{a+b}}{}$; *crossing out* equal quantities with opposite signs by two oblique lines, as, $\xcancel{3a^2b} - \xcancel{3a^2b}$. Such quantities are said to *destroy* each other.

III. **Of Relation:** $=$, is numerically equal to; $\equiv$, is identically equal to; $\neq$, is not equal to; $>$, is greater than; $<$, is less than; $\ngtr$, is not greater than; $\nless$, is not less than; $\propto$, varies as (little used); $\doteq$, approaches the limit. Formerly $\equiv$ was not in use, and $=$ is still largely employed throughout literal arithmetic in identical equations.

IV. **Of Aggregation:** Parenthesis ( ); brace { }; bracket [ ]; vinculum $\overline{\quad}$.

V. **Of Continuation:** a, b, ⋯, read 'a, b, and so on'; a, b, ⋯, k, read 'a, b, and so on to k.'

VI. **Of Inference:** $\therefore$, hence or therefore; $\because$, since or because.

VII. **Miscellaneous:** $i = \sqrt{-1}$; $f(x)$ for f function of x.

(x, y), the point whose Cartesian coördinates are x and y.

$\log_a n$, the logarithm of n in a system whose base is a.

${}_nP_r$, the number of permutations of n things r together.

${}_nC_r$, the number of combinations of n things r together.

$n!$, *n factorial*, or the product of all the numbers from 1 to n.

$\sum_{n=1}^{n=r} u_n$, summation from $n = 1$ to $n = r$ of u_n.

$.\dot{6}\dot{3}$ for .636363 ⋯, called repeating decimal.

$\frac{\dot{1}}{2} + \frac{\dot{1}}{4}$ for recurring continued fraction (see § 298).

$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}$, etc., determinant notation.

ADVANCED ALGEBRA

PART I. REVIEW OF ELEMENTARY ALGEBRA

CHAPTER I

REVIEW OF LITERAL ARITHMETIC

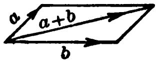
I. THE FOUR FUNDAMENTAL OPERATIONS

1. Algebra and Arithmetic. Both algebra and arithmetic treat of numbers, but algebra uses general characters as well as figures to represent numbers, and recognizes both positive and negative numbers.

Much of algebra has to do with equations. The symbol of a *conditional equation*, "$=$," means that the number on its left is, or becomes, the same as the number on its right when a letter or letters are given certain values and any indicated operations on either or both sides have been performed. The symbol of an *identical equation*, "$\equiv$," denotes that the expression on its left is either the same, or reduces to the same, as the expression on its right. Any letter in an identical equation can have any value. A conditional equation is essentially an interrogative sentence, and an identity a declarative sentence.

The transformations of this chapter yield identical equations or identities. The next chapter deals with conditional equations.

2. Meaning of Addition. The word *addition* means literally "putting together." The sum of two numbers contains as many units as there are in both numbers. The sum of two line segments is found by placing them end to end on the same straight line. The sum of two forces, represented by two adjacent sides of a parallelogram, is a force represented by the diagonal of the parallelogram drawn from the intersection of the two given adjacent sides. Thus,



The sum of the complex numbers

$$a + b\sqrt{-1} \text{ and } c + d\sqrt{-1}$$

is

$$(a + c) + (b + d)\sqrt{-1}.$$

It is evident from these illustrations that the operation of addition, denoted by the sign +, is difficult to define in precise terms.

Addition in algebra is governed by the following laws, which may be considered as constituting a part of the definition of addition:

Laws of Addition. 1. *Any two numbers have a sum that is the result of adding them. This sum is uniquely determined, that is, it has only one value.*

2. *The commutative law, or* $a + b = b + a$.
3. *The associative law, or* $(a + b) + c = a + (b + c)$.
4. *If* $a + x = a + y$, *then* $x = y$.

3. Meaning of Subtraction. Subtraction may be defined as the operation of finding x in the equation $b + x = a$; or, of finding a number which added to one of two given numbers produces the other.

Law of Subtraction. *There is one and only one value of* x *that satisfies the equation* $b + x = a$.

Using "$-$" for the sign of subtraction, we write $x = a - b$.

If $b = a$, $x = a - a$. This value of $a - a$ is denoted by 0.

If $x + a = 0$, then $x = 0 - a$, or $-a$.

In this way $-$ becomes a sign of the *quality* of the quantity to which it is prefixed. Thus, $-$ before a number indicates that it is below 0, and +, that it is above 0. When no sign stands before a quantity, + is understood. When used as a sign of quality, + or $-$ is frequently inclosed with its quantity in a parenthesis.

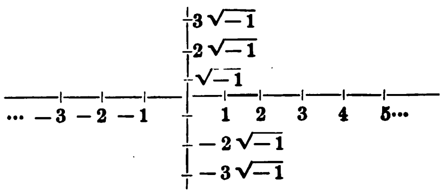
In algebra, subtraction is commonly included with addition.

Thus, $$6+4-3-7+2$$

can be written $$6+4+(-3)+(-7)+2.$$

4. Meaning of Multiplication. Multiplication in its simplest form is an extension of the idea of addition, the numbers added being equal. It is denoted by $\times$, $\cdot$, and by absence of sign between letters, figures and letters, letters and parentheses, and parentheses in juxtaposition.

Not only can we have two ordinary numbers multiplied together giving a real product, as $2 \times -3 = -6$, but we can have two imaginary numbers giving a real product, as $2\sqrt{-1} \times -3\sqrt{-1} = +6$, since $\sqrt{-1} \times \sqrt{-1} = -1$. On a graphical representation these numbers are located as indicated in the figure. Thus we see that the product of two numbers on the vertical line gives a number on the horizontal line.



It is evident from the preceding illustrations that it is difficult to define the term multiplication precisely.

Multiplication in algebra is governed by the following laws, which may be considered as constituting a part of the definition of multiplication:

Laws of Multiplication. 1. *The product of two numbers, a and b, as $a \times b$, or ab, is a number; this number is uniquely determined.*

2. *The commutative law, or $ab = ba$.*

3. *The associative law, or $(ab)c = a(bc)$.*

4. *If $a \times x = a \times y$, then $x = y$, provided $a \neq 0$.*

5. *The distributive law, or $(a+b)c = ac + bc$; or,*

$$(a+b)(c+d) = ac + ad + bc + bd.$$

a. The distributive law gives a general rule in mathematics, which is that every part of the multiplicand must be multiplied by every part of the multiplier and the partial products must be added.

6. *The laws of signs:* $(+a) \times (+b) = +ab$; $(+a) \times (-b) = -ab$; $(-a) \times (+b) = -ab$; $(-a) \times (-b) = +ab$.

Or, like signs in the two factors give a positive product, and unlike signs give a negative product.

b. Various explanations of the laws of signs, based on the distributive law and addition, are given in elementary algebras because they serve to satisfy the mind of the pupil as regards their reasonableness. These laws, although assumptions, have been proved by experience to be adapted to the needs of algebra.

5. Meaning of Division. Division may be defined as finding the value of x in the equation $ax = b$, or as finding one factor when the product and the other factor are given.

Law of Division. 1. *If a and b are two numbers ($a \neq 0$), there is one and but one value of x that satisfies $ax = b$.*

Division by 0 is always excluded, since it has no meaning.

2. The law of signs in division is easily deduced from that for multiplication, since division is the *inverse* process to multiplication. *Like signs in dividend and divisor give a positive quotient, and unlike signs a negative quotient.*

Definitions

6. A **quantity** in algebra is a *number*, which may be represented by a single figure or letter, or by a more or less complicated expression of figures, letters, and signs.

7. An **exponent** is a number written to the right of, and higher than, a quantity. When it is a positive whole number, it shows how many times the quantity is to be taken as a factor.

Thus, $5^3 = 5 \times 5 \times 5$; $x^4 = x \times x \times x \times x$.

8. An **integral power** of a number is the continued *product* arising from using the number one or more times as a factor. When a number is used once as a factor, the other factor is 1. Students often fail to distinguish between a power and the *exponent* of the power. Thus, the power denoted by 2^4 is not 4, but 16.

When two or more equal factors multiplied together produce a number, one of these equal factors is called a **root** of the number.

The sign $\sqrt{}$ denotes that one of the equal factors of the number following the sign is to be taken. Thus, $\sqrt[3]{64} = 4$. The small figure over the radical sign is called the **index** of the root and shows into how many equal factors the number is separated.

9. A **term,** or monomial, is a quantity not separated into parts by a + or a − sign. In a term, there is usually a numerical **coefficient** * and a **literal,** or **letter, part.** Thus, $3\,a^4b$ is a term in which 3 is the numerical coefficient.

A **binomial** is a quantity consisting of two terms, a **trinomial,** of three terms, and a **polynomial,** of two or more terms. By a "**polynomial in** x" is meant a sum of multiples of powers of x; as, $3\,x^3 + 2\,x^2 + ax - 1$.

10. Similar terms are those having the same letters with the same exponents for the corresponding letters. Similar terms may differ in signs and coefficients. Thus, $6\,a^3bc^2$ and $-\,11\,a^3bc^2$ are similar terms.

11. The **numerical value** of a quantity is found by assigning certain values to its letters and simplifying the result.

Numerical values are frequently used to *check* answers. Roots of equations are *verified,* and exercises can often be *proved* correct, as by multiplication in factoring. If verifying or proving is correctly done, the answer must be right; but answers can *check* and yet be wrong. Thus, if a^3 is the true answer, while a^2 is the answer obtained, if $a = 1$ is substituted, the answer checks.

12. Symbols of aggregation are the **parentheses** ( ), the **bracket** [ ], the **brace** { }, and the **vinculum** $\overline{}$. They are used to indicate that the quantity within is to be regarded as a single quantity.

* Formerly, algebraists often thought of letters as denoting magnitudes (such as lines, etc.) not expressed as numbers.

Order of Operations in Algebra

13. Rules of Precedence of Operations. This topic is very important.

1. *In any term (§ 9), symbols of aggregation being absent, raising to powers and extracting roots must be performed before multiplications and divisions.*

Thus, $2 \times 3^2 = 18$; $3x^2 = 3xx$; while $(3x)^2 = 9xx$.

2. *Symbols of aggregation being absent, multiplications and divisions must be performed before additions and subtractions.*

Thus, $3 + 4 \times 5 = 23$; $5 \times 2^2 - 2^3 \div 4 = 18$.

3. *Operations inside symbols of aggregation must be performed before those outside.*

Thus, $8(3 + 4) = 8 \times 7 = 56$. $(2 \times 3)^2 = 6^2 = 36$.

EXAMPLE. If $a = 3$, $b = 2$, $c = 64$,

$$15ab^2 - 3b^3\sqrt[3]{c}$$

becomes $15 \times 3 \times 2^2 - 3 \times 2^3 \times \sqrt[3]{64}$ (By § 11)

or, $15 \times 3 \times 4 - 3 \times 8 \times 4$, or 84. (By § 8)

a. Multiplications and divisions denoted by $\times$, $\cdot$, $\div$ are performed in order from left to right, but when multiplications are denoted by juxtaposition, as in $4c \div 3ab$, the multiplications in dividend and divisor must be performed first. Hence, in translating this notation to that of $\times$, $\cdot$, $\div$, parentheses must be introduced.

Thus, if $a = 12$, $b = 3$, $c = 4$.
$a \div b \times c = 12 \div 3 \times 4 = 16$; while $a \div bc = 12 \div (3 \times 4) = 1$.

14. Importance of the Rules of Precedence of Operations. If strict attention is not paid to the order in which operations are to be performed, great confusion may result.

Thus, $(2+3) \times 4^2 = 80$; while $2 + (3 \times 4)^2 = 146$; and $(2 + 3 \times 4)^2 = 196$; and $2 + 3 \times 4^2 = 50$, by the rules of § 13.

Although other rules than those just described *might* have been adopted by those who introduced the algebraic notation, yet since the rules of § 13 are well adapted to their purpose and are universally used, those who study algebra are required to make their calculations conform to these laws.

15. Exercise in Simplifying Expressions and Finding Values.

1. $3+7-6+4-2.$
2. $3+6\div 2.$
3. $4(6-3)-2.$
4. $8\times 6\div 3-12.$
5. $24\div(2\times 3).$
6. $(5-2)(11-3\times 2).$
7. $2+16\div 2.$
8. $2\cdot 7 \quad 5-3^2\cdot 4.$
9. $2(13+7^2-41)-15.$
10. $\sqrt{[3(19+50-42)]}+26.$

Translating in Algebra

16. Exercise in Translating Algebraic Expressions into Ordinary Language. Changing algebraic language into ordinary words is, in many respects, like translating from one language to another, as from English to German, or from Latin to English. The student will find that quantities which cannot be described in ordinary language without much circumlocution are expressed readily in the algebraic notation. This is why we can easily work with and reason about quantities written in the algebraic notation, while the complexity of the language would make it difficult to deal with the same quantities in ordinary language. The great mathematician Fourier said of the algebraic notation, "There can be no more universal or more simple language, no language more exempt from error and obscurity."

1. Translate $4\,ab^2-3\,a^3\sqrt{c}+9\,c$ into ordinary language, and then find its numerical value when $a=2$, $b=3$, $c=4$.

TRANSLATION. "From 4 times a times b times b take 3 times a times a times a times the square root of c, and add 9 times c to the remainder."

To evaluate $4\,ab^2-3\,a^3\sqrt{c}+9\,c,$

we have $4\times 2\times 3\times 3-3\times 2\times 2\times 2\times\sqrt{4}+9\times 4,$

that is, $72-48+36$, or 60. *Ans.*

2. Translate $3\,x^3+2\,xy^2-y\sqrt{z}$, and evaluate when $x=3$, $y=1$, $z=9$.

3. Translate $3x(y^2-z)$, and evaluate when $x=5$, $y=8$, $z=25$.

SUGGESTION. "3 times x times the quantity which is the difference between y times y and z."

a. It is convenient to use the phrase "the quantity," or "the binomial," etc., to indicate what is included in parentheses.

Translate the following 15 exercises and evaluate them when $x=3$, $y=5$, $z=4$:

4. $3x(y-z)$.	**5.** $4(xy-z)$.	**6.** $3x^2+2y^2$.
7. $9x^2yz-2y^2$.	**8.** $3xy^2-2z\sqrt{z}$.	**9.** $4(x^2+y^2-z^2)$.
10. $3(xy^2z)^2$.	**11.** $4(xy^2-x^2y)$.	**12.** $2(3xy^2-z)^2$.
13. $(2x-y)^2z$.	**14.** $2\sqrt{y^2-x^2}$.	**15.** $(x+2y)(3y-z)$.
16. $4(2xy)^4z$.	**17.** $\sqrt{3\cdot 2y^xz}$.	**18.** $12z^x+x^xy^x$.

19. If c represents the cost of n articles and p the price of one article, translate $c=np$ into words.

20. If a is area of a circle, and r is radius, put $a=\pi r^2$ into words.

21. If a represents the area of a trapezoid, b and b' its bases, and h its altitude, translate $a=\frac{1}{2}(b+b')h$ into words.

22. If s represents the distance through which a body moves, v its uniform velocity or speed, and t the number of seconds it is in motion, translate $s=vt$ into ordinary language.

23. If a and b represent the two legs of a right triangle and h its hypotenuse, translate $a^2+b^2=h^2$ into words.

24. If s is the half sum of the sides a, b, c, of a triangle, and A is its area, translate into words $A=\sqrt{s(s-a)(s-b)(s-c)}$.

Exponents

17. Laws of Exponents.

1. *Multiplication.*

Let a be any quantity and m and n positive integers. Then

$a^m = aaaaa \cdots$ to m factors, $\quad a^n = aaaa \cdots$ to n factors. (§ 7)

Therefore, $a^m \times a^n = aaa \cdots$ to m factors $\times\, aaa \cdots$ to n factors or, $$a^m a^n = aaaaa \cdots \text{ to } m + n \text{ factors.}$$

Therefore, $$\boldsymbol{a^m a^n = a^{m+n}}.$$

Hence, add the exponents of the same quantity in the factors for the exponent of this quantity in the product.

$x^2 \times x^5 = ?$ $\quad y^7 \times y^4 = ?$ $\quad (a+b)^2 \times (a+b) = ?$ $\quad x^{2m} \times x^{n+2} = ?$

2. *Division.*

With the same quantities as those used in multiplication, we have

$$\frac{a^m}{a^n} = \frac{aaa \cdots \text{ to } m \text{ factors}}{aaa \cdots \text{ to } n \text{ factors}} = a^{m-n}. \quad \text{(Canceling common factors.)}$$

Hence, subtract the exponent of a quantity in the divisor from the exponent of the same quantity in the dividend for the exponent of the quantity in the quotient.

$a^5 \div a^2 = ?$ $\quad a^{2m} \div a^m = ?$ $\quad (m+n)^7 \div (m+n)^2 = ?$

3. *Powers.*

$$(a^m)^n = a^m \times a^m \times a^m \times \cdots \text{ to } n \text{ factors} \qquad (\S\ 7)$$
$$= a^{m+m+m+\cdots \text{ to } n \text{ terms}} \qquad \text{(By 1, page 8)}$$

or, $$(a^m)^n = a^{mn}.$$

Hence, multiply the exponent of a quantity by the exponent of the power to which its power is to be raised for the exponent of the quantity in the result.

$(x^3)^2 = ?$ $\quad (b^x)^2 = ?$ $\quad ((a+y)^4)^7 = ?$

4. *Roots.*

$$\sqrt[n]{a^{mn}} = \sqrt[n]{a^m \times a^m \times \cdots \text{ to } n \text{ factors}} = a^m$$ (since by § 8, *one* of n equal factors is the nth root of their product)

or, $$\sqrt[n]{a^{mn}} = a^{\frac{mn}{n}}$$

Hence, divide the exponent of the quantity by the index of the root for the exponent of the quantity in the result.

$\sqrt{x^6} = ?$ $\quad \sqrt[3]{a^9} = ?$ $\quad \sqrt[4]{x^{12n}} = ?$ $\quad \sqrt[5]{(b-c)^{10}} = ?$

$\sqrt[6]{(x^4)^3} = ?$ $\quad \sqrt{a^4 \times a^6} = ?$ $\quad \sqrt[3]{m^{10p} \div m^p} = ?$

Performing Fundamental Operations

18. How the Fundamental Operations are Performed.

1. **Addition of similar monomials** (§ 10) is performed by taking the arithmetical difference between the total of the positive and the total of the negative coefficients, giving it the sign of the numerically greater total, and annexing to it the common literal part.

Thus, $6a^2b - 11a^2b + 5a^2b - 7a^2b + 2a^2b = -5a^2b.$

The result of such addition can be *checked* by adding the terms seriatim; that is, the third term to the sum of the first two, the fourth to the sum of the first three, and so on.

a. Dissimilar monomials are added by simply *indicating the addition.*

Thus, the sum of $3a^2$, $2b$, and $-4c$ is $3a^2 + 2b - 4c$.

b. Two equal and opposite terms *crossed out*, as $\cancel{5a^2b} - \cancel{5a^2b}$, are said to *destroy* each other, giving 0 for the sum. The word *cancel* and the cancellation mark should be reserved for division.

2. **Addition of polynomials** is performed by placing the quantities to be added so that similar terms will fall in the same columns, and adding these columns, using the rule for monomials.

EXAMPLE.

$$\begin{array}{rrrrr} 11a^2 & -12ab & +13b^2 & & \\ -2a^2 & +7ab & -4b^2 & +7bc & \\ 4a^2 & -ab & +6b^2 & -12bc & +c^2 \\ \hline 13a^2 & -6ab & +15b^2 & -5bc & +c^2 \end{array}$$

3. **Subtraction** can be performed either by finding a quantity which when added algebraically to the subtrahend gives the minuend, or by conceiving the sign of each term of the subtrahend to be changed and adding the result to the minuend.

Addition in algebra is most easily understood by thinking of the addition of debts and credits.

Subtraction in algebra is most clearly grasped by thinking of it as finding the *distance* between the minuend and the subtrahend on the algebraic number scale, and marking the result + or − according as the minuend is above or below the subtrahend on the scale.

Thus, on a thermometer scale, the difference between $+12°$ and $-19°$ is the distance between them, or $+31°$; the difference between $-17°$ and $-5°$

is the *distance* between them, or $-12°$, the answer being marked $-$, since the minuend (the quantity named first) is *below* the subtrahend. Evidently, the remainder $-12°$ and the subtrahend $-5°$ *added* give the minuend $-17°$.

EXAMPLES.

$$\begin{array}{r} 6\,a^2 \\ -11\,a^2 \\ \hline 17\,a^2 \end{array} \qquad \begin{array}{r} 3\,x^2 - 2\,xy - 11\,y^2 \\ -x^2 + 5\,xy - 7\,y^2 \\ \hline 4\,x^2 - 7\,xy - 4\,y^2 \end{array}$$

4. **Multiplication of monomials** is performed by multiplying together the coefficients of the factors for the coefficient of the product, and adding the exponents (§ 17, 1) of the same letter or quantity in the factors for its exponent in the product, the letters being arranged alphabetically. If there is an odd number of $-$factors, the product is negative; otherwise it is positive.

Thus, $$-6\,a \times -3\,a^2b^n \times 2\,ab^2c = 36\,a^4b^{n+2}c.$$

5. **Multiplication of polynomials** is performed by use of the distributive law, § 4, 5.

Thus,
$$\begin{array}{l} 3\,a^2 - 2\,ab + b^2 \\ a - 2\,b \\ \hline 3\,a^3 - 2\,a^2b + ab^2 \\ - 6\,a^2b + 4\,ab^2 - 2\,b^3 \\ \hline 3\,a^3 - 8\,a^2b + 5\,ab^2 - 2\,b^3 \end{array}$$

6. **Division of monomials** is performed by dividing the coefficient of the dividend by the coefficient of the divisor for the coefficient of the quotient, and subtracting the exponent of any letter or quantity in the divisor from its exponent in the dividend for its exponent in the quotient. The rule for signs is given in § 5.

Thus, $$30\,a^2b^mc \div (-5\,ab^n) = -6\,ab^{m-n}c.$$

7. **Long division** is performed by first arranging both dividend and divisor according to the ascending (or descending) powers of one leading letter. Then the first term of the dividend is divided by the first term of the divisor, giving the first term of the quotient. Next the whole divisor is multiplied by this quotient term and the product is subtracted from the dividend. Then the first term of the remainder (in arranged form) is divided by the first term of

the divisor for the second term of the quotient, and the whole divisor is multiplied by this quotient term, the product being subtracted from the first remainder. This process is continued until there is no remainder, or until the first term of the last remainder does not contain the first term of the divisor.

The work can be checked by multiplying divisor and quotient together, adding the remainder to the product, and seeing if the sum is the same as the dividend.

Example.

$$\begin{array}{l|l} \multicolumn{1}{c}{\text{Dividend}} & \\ 6x^2 + 5xy - 4y^2 & 3x + 4y \quad \text{Divisor} \\ \underline{6x^2 + 8xy} & \overline{2x - y} \quad \text{Quotient} \\ \quad - 3xy - 4y^2 & \\ \quad \underline{- 3xy - 4y^2} & \end{array}$$

8. **A monomial is raised to a power** by raising its numerical coefficient to the required power for the coefficient of the result, and multiplying the exponent of each letter or quantity by the exponent of the power to which the monomial is to be raised for the exponents of the several factors in the result. (See § 17, 3.)

Thus, $(-3a^nb^2c^3)^4 = 81a^{4n}b^8c^{12}$; $(a^mb^{2n}c)^x = a^{mx}b^{2nx}c^x$.

9. **The root of a monomial is extracted** by extracting the required root of its numerical coefficient and dividing each of the exponents of its literal factors by the index of the required root for the exponents of the several factors in the result. The sign of any even root is + or −, written ±. The sign of any odd root of a quantity is the same as the sign of the quantity itself.

Thus, $\sqrt[3]{-8a^3b^6c^{9n}} = -2ab^2c^{3n}$; $\sqrt{4x^2y^2} = \pm 2xy$.

19. Removal and Insertion of Parentheses.

1. *Symbols of aggregation preceded by + can be removed without changing the signs of the quantities inside them.*

Thus, $(a + 2b) + (c - 3b) - 2b$

becomes $a + 2b + c - 3b - 2b$, or $a - 3b + c$.

2. *Symbols of aggregation preceded by − can be removed by changing the sign of every term within them.*

Thus, $-(a^2 - 2b^2 + c^2) - (-3b^2 + 4c^2)$
becomes $-a^2 + 2b^2 - c^2 + 3b^2 - 4c^2$, or $-a^2 + 5b^2 - 5c^2$.

Notice that the signs before the parentheses call for the changing of the signs within and themselves disappear in this operation.

3. *If symbols of aggregation contain others, all can be removed in one operation by leaving unchanged in sign all those terms preceded and affected by + signs or by an even number of − signs, and by changing the signs of those terms preceded and affected by an odd number of − signs.*

Thus, $3a - \{2a - [3c - (2a - b)]\}$,
becomes $3a - 2a + 3c - 2a + b$, or $-a + b + 3c$.

A good plan to follow is to check, *i.e.* mark, the terms whose signs are to be changed before removing the symbols.

4. *Quantities can be inserted in parentheses preceded by + without changing the signs of their terms, and in parentheses preceded by − by changing the sign of each of their terms.*

Thus, $a + b - 2c - 3d + 4e$
becomes $(a + b) - (2c + 3d - 4e)$.

20. Exercise in Performing the Fundamental Operations.

1. Add $3a$, $5a$, $-7a$, $6a$, $-11a$, $2a$. *Ans.* $-2a$.

2. Add $2a^mb^n$, $-7a^mb^n - 12a^mb^n$, a^mb^n, $-5a^mb^n$.

3. Add $4a^{\frac{1}{2}}b^{\frac{2}{3}}$, $-2\frac{1}{4}a^{\frac{1}{2}}b^{\frac{2}{3}}$, $-11\frac{1}{3}a^{\frac{1}{2}}b^{\frac{2}{3}}$, $6\frac{1}{2}a^{\frac{1}{2}}b^{\frac{2}{3}}$.

4. Add $x^3 - 2x^2 + 5$, $-3x^3 + 7x - 4$, $5x^3 - 2x^2 - 11x$, and $-x^2 - 15x + 12$.

Subtract in the following four problems:

5. $\begin{array}{r} 6a^2 \\ \underline{8a^2} \end{array}$ **6.** $\begin{array}{r} -9m^2 \\ \underline{-3m^2} \end{array}$ **7.** $\begin{array}{r} 2a^2b \\ \underline{-11a^2b} \end{array}$ **8.** $\begin{array}{r} -7bc^3 \\ \underline{-19bc^3} \end{array}$

9. From $11a^3 - 6a^2b - 7ab^2 - b^3$ take $a^3 - 12a^2b + 4ab^2 - 6b^3$.

10. $(2m^3 - m^2 - 4m - 6) - (3m^3 - 5m^2 - 2m + 12) = ?$

11. $(6x^2 - 5ax - 11a^2) + (2x^2 - 2a^2) - (5x^2 - 2ax) = ?$

Perform the multiplications indicated in the following:

12. $6a \times (-2a^2)$. **13.** $5a^2b \times 11a^3b^n$. **14.** $2x^{\frac{1}{3}}y^{\frac{1}{2}} \times -3x^{\frac{5}{6}}y^2$.

15. $(2a^2 + 3ay)(3a^2 - 2ay)$. **16.** $(x^3 - x^2 + 2x)(x^3 - 5x + 4)$.

17. $(3x^{\frac{4}{5}} - 2x + 3x^{\frac{2}{5}} - 1)(5x^{\frac{3}{5}} - x^{\frac{1}{5}} + 2)$.

18. $(m^{2a} - m^an^b + 5n^{2b})(m^{2a} - 2m^an^b - 7n^{2b})$.

Perform the divisions indicated in the following:

19. $15a^3 \div 5a$. **20.** $25a^4 \div 3a^2$. **21.** $a^{\frac{1}{2}} \div a^{\frac{1}{3}}$.

22. $3x^{2m+n} \div x^{m-2n}$. **23.** $12c^{\frac{3}{4}}d^{\frac{1}{5}}e^{\frac{1}{4}} \div -7c^{\frac{2}{3}}d$.

24. $(4a^3 - 8a^2b - 6abc) \div 2a$. **25.** $(x^{\frac{1}{2}} - 4x^{\frac{1}{3}} + 3x^{\frac{1}{5}}) \div x^{\frac{1}{6}}$.

26. $(x^3 - 7x - 6) \div (x - 3)$. **27.** $(x^6 - 2x^3 + 1) \div (x^2 - 2x + 1)$.

28. Divide $a^4 - 8a^3 + 24a^2 - 30a + 12$ by $a^2 - 4a + 4$, and prove the answer correct by multiplying the divisor and quotient together and adding the remainder, thus getting the dividend.

29. $(21a^3 - 4a^2 - 12 - 42a) \div (4a + 2 - 3a^2)$.

30. Divide $1 + 2x$ by $1 - 3x$, getting five terms in the quotient, and prove the answer correct.

31. $(x^{m+n} + x^ny^n + x^my^m + y^{m+n}) \div (x^n + y^m)$.

Find in the quickest way the numerical value in Exs. 32–34.

32. $x^4 + x^3 - 4x^2 + 5x - 3$ divided by $x^2 + 2x - 3$ if $x = 3$.

33. $(x^4 - 6x^3y + 9x^2y^2 - 4y^4) \div (x^2 - 3xy + 2y^2)$ when $x = 3$, $y = -1$.

34. $(x^4 - 6xy - 9x^2 - y^2) \div (x^2 + 3x + y)$ when $x = 4$, $y = 2$.

Remove symbols of aggregation and simplify in the following:

35. $(1 - 2x + 3x^2) + (3 + 2x - x^2)$.

36. $(a + 2b - 3c) - (b + c - 3d) + (2e + d - f) - (f + g - e)$.

37. $\overline{a - b} - c + \overline{b + c - d} - \overline{e - d - f} - \overline{-f + g - e}$.

SUGGESTION. Test by writing parentheses for vinculums and solving.

38. $1 - \{1 - [1 - (1 - x)]\}$.

39. $3c + (2a - [5c - \{3a + \overline{c - 4a}\}])$.

40. $2m - [3m - \{m - (2m - \overline{3m + 4})\} - (5m - 2)]$.

Simplify in the following:

41. $(a + b)(b + c) - (c + d)(d + a) - (a + c)(b - d)$.

42. $[x - (2y + 3z)][x - (2y - 3z)]$.

43. $(x + y)(x^3 - y^3)[x^2 - y(x - y)]$.

44. $x(x + 1)(x + 2)(x + 3) + 1 - (x^2 + 1)(x^2 + 1)$.

Raise the following quantities to the powers indicated or extract the roots:

45. $(3m^3)^2$; $(x^3)^3$; $(a^m)^{2n}$; $(-6a^2b^n)^3$; $(a^2)^m$; $(\frac{1}{2}a^2)^4$.

46. $\sqrt{16a^4}$; $\sqrt{4a^{2m}}$; $\sqrt[3]{-8a^3}$; $\sqrt[3]{27m^9}$; $\sqrt[n]{a^{2n}b^{3n}}$; $\sqrt[3]{\frac{1}{64}a^{3n}}$.

47. Divide $x^5 - 28x^3 + 8x^2 - 35x + 21$ by $x^3 - 5x^2 - 7$ by the method of "detached coefficients."

SOLUTION. We write the coefficients only in regular order, filling in missing powers of x with 0 coefficients.

$$
\begin{array}{l|l}
1 + 0 - 28 + 8 - 35 + 21 & 1 - 5 + 0 - 7 \\
\underline{1 - 5 + 0 - 7} & \overline{1 + 5 - 3} \\
\quad 5 - 28 + 15 - 35 & \text{or, } x^2 + 5x - 3. \quad \textit{Ans.} \\
\quad \underline{5 - 25 + 0 - 35} & \\
\qquad - 3 + 15 + 0 + 21 & \\
\qquad \underline{- 3 + 15 + 0 + 21} &
\end{array}
$$

The student who does not understand this solution should perform the division in the usual way, supplying missing powers of x with 0 coefficients. By comparing his solution with that above he will understand the process.

Perform the divisions in the following by the detached coefficients method:

48. Divide $x^3 + 4x^2 + 7x + 6$ by $x^2 + 2x + 3$.

49. $(a^4 - 8a^2x^2 + 16x^4) \div (a^2 + 4ax + 4x^2)$.

SUGGESTION. Write the dividend $1 + 0 - 8 + 0 + 16$, thinking of it as standing for $a^4 + 0a^3x - 8a^2x^2 + 0ax^3 + 16x^4$.

50. $(15x^4 + 7x + 7x^3 + 15x^2 + 4) \div (3x^2 + 2x + 1)$.

51. $(x^4 + x^2y^2 + y^4) \div (x^2 - xy + y^2)$.

52. $(1 + 5x^3 - 6x^4) \div (1 - x + 3x^2)$.

II. FORMULAS AND THEOREMS

21. Formulas and Theorems. The **formulas** that follow are derived by simple multiplications and divisions. They are examples of identical equations defined in § 1. The student should perform the multiplications and divisions, verifying the truth of the formulas. Thus, in Theorem I he should show that

$$(a+b)(a+b)=a^2+2\,ab+b^2.$$

The **theorems** are simply a translation of the formulas into words.

I. $$(a+b)^2=a^2+2\,ab+b^2.$$

THEOREM. *The square of the sum of two quantities equals the square of the first, plus twice the product of the first by the second, plus the square of the second.*

a. The student should point to the corresponding symbols in the formulas of this article as he says the words of the theorems. Notice that the exponent 2 is read first, then + between a and b; and so on.

EXAMPLES. $(3\,a^2+2\,b^8c)^2=(3\,a^2)^2+2(3\,a^2)(2\,b^8c)+(2\,b^8c)^2$

$=9\,a^4+12\,a^2b^8c+4\,b^6c^2.$ *Ans.*

$(4\,a^8+3\,x^2)^2=\,?$ $(5\,a^3m+6\,my^4)^2=\,?$

II. $$(a-b)^2=a^2-2\,ab+b^2.$$

State the theorem that is derived from this formula.

EXAMPLES. $(2\,a^2-3\,x)^2=\,?$ (Use theorem.) $(m^n-2\,p)^2=\,?$

III. $$(a+b)(a-b)=a^2-b^2.$$

THEOREM. *The product of the sum and difference of two quantities is equal to the difference of their squares.*

EXAMPLES. $(6\,a^2+5\,b^2)(6\,a^2-5\,b^2)=(6\,a^2)^2-(5\,b^2)^2$

$=36\,a^4-25\,b^4.$ *Ans.*

$(5\,x^3+1)(5\,x^3-1)=\,?$ $(3\,x^n-2\,y^p)(3\,x^n+2\,y^p)=\,?$

IV. $$(x+a)(x+b)=x^2+(a+b)\,x+ab.$$

THEOREM. *The product of two binomials having a common term equals the square of the common term, and the algebraic sum of the other terms times the common term, and the algebraic product of the other terms.*

EXAMPLES. $(a+2)(a+5) = a^2 + 7a + 10.$

$$(x^2+7)(x^2-12) = x^4 - 5x^2 - 84.$$

$$(3xy - 9z)(3xy + 12z) = 9x^2y^2 + 3z(3xy) - 108z^2$$
$$= 9x^2y^2 + 9xyz - 108z^2.$$

$$(4x^2 + 7z^2)(4x^2 - 13z^2) = ? \qquad (3x^n + 5)(3x^n - 14) = ?$$

If in the above examples *the right member is given to find the left member*, then two numbers must be found whose algebraic sum is the given middle coefficient and whose algebraic product is the given third coefficient. Hence, we have

CONVERSE THEOREM. *A trinomial that is the product of two binomials has for the first term of each binomial the square root of its square term, and for the two second terms two quantities whose sum is coefficient of first term in the trinomial's second term, and whose algebraic product is its third term.*

EXAMPLES. $m^2 - 11m - 26 = (m - 13)(m + 2).$

$$x^2y^4 - 12xy^2 - 64 = ? \qquad x^{2m} + 9x^m - 112 = ?$$

$$9a^4 - 33a^2 + 28 = ? \qquad 4c^2 + 16cd - 33d^2 = ?$$

V. $$(a-b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

Write the theorem that is derived from this formula. Then repeat it while pointing to the corresponding symbols of the formula.

EXAMPLES. $(3a^2 - 2)^3 = (3a^2)^3 - 3(3a^2)^2 \times 2 + 3(3a^2) \times 2^2 - 2^3$

$$= 27a^6 - 54a^4 + 36a^2 - 8.$$

$$(4m - 5n)^3 = ? \qquad (2a^n - 7b^{2p})^3 = ?$$

CONVERSELY: $27a^6 - 54a^4 + 36a^2 - 8 = (3a^2 - 2)^3.$

$$64m^9 - 240m^6n + 300m^3n^2 - 125n^3 = ?$$

VI. $$a^2 - b^2 = (a+b)(a-b).$$

State the theorem that is derived from this formula. It is the converse of Theorem III.

EXAMPLES. $16a^{2n} - 9 = (4a^n + 3)(4a^n - 3).$

$$81c^6 - 49d^8 = ? \qquad 9a^{2x} - 25b^{2y} = ?$$

$$(a+b)^2 - c^2 = ? \qquad m^2 - (n-p)^2 = ?$$

VII. $$a^3 - b^3 = (a - b)(a^2 + ab + b^2).$$

THEOREM. *The difference of the cubes of two quantities equals the product of the difference of the quantities multiplied by the square of the first, plus the product of the first by the second, plus the square of the second.*

EXAMPLES. $8a^3 - 27b^3 = (2a - 3b)(4a^2 + 6ab + 9b^2)$.

$64x^9y^6 - 1 = ?$ $216p^{3n} - b^3 = ?$

The formula $a^3 - b^3 = (a - b)(a^2 + ab + b^2)$ is included in the following more general formula:

$$a^n - b^n = (a - b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \cdots + b^{n-1}).$$

GENERAL THEOREM. *The difference of any same powers of two quantities is exactly divisible by the difference of the quantities.*

This theorem will be proved in chapter XI.

VIII. $$a^3 + b^3 = (a + b)(a^2 - ab + b^2).$$

State the theorem for this formula.

EXAMPLES. $64m^{3a} + 125n^6 = (4m^a + 5n^2)(16m^{2a} - 20m^an^2 + 25n^4)$.

$27y^6 + 64z^{12} = ?$ $216y^3 + 343z^{3n} = ?$

This theorem is included in the following:

$$a^{2n+1} + b^{2n+1} = (a + b)(a^{2n} - a^{2n-1}b + a^{2n-2}b^2 - \cdots + b^{2n}).$$

Here $2n + 1$ is used to denote that the power must be odd.

State the theorem, commencing "The sum of the same odd powers . . ." as in the general theorem in VII above.

IX. $$(a + b + c + d)^2 = a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd.$$

This formula may be generalized so as to include the sum of *any* number of terms instead of the sum of four terms, as in the formula.

THEOREM. *The square of the sum of any number of quantities equals the sum of their squares increased by twice the product of each term by each term that follows it.*

EXAMPLES. $(2a - 3b + 5c)^2 = (2a)^2 + (-3b)^2 + (5c)^2 + 2(2a)(-3b) + 2(2a)(5c) + 2(-3b)(5c)$

$= 4a^2 + 9b^2 + 25c^2 - 12ab + 20ac - 30bc.$ *Ans.*

$$(m^2 - 2n^2 + 3pq - 4r^3)^2 = (m^2)^2 + (-2n^2)^2 + (3pq)^2 + (-4r^3)^2 + 2(m^2)(-2n^2) + 2(m^2)(3pq) + 2(m^2)(-4r^3) + 2(-2n^2)(3pq) + 2(-2n^2)(-4r^3) + 2(3pq)(-4r^3).$$

$$= m^4 + 4n^4 + 9p^2q^2 + 16r^6 - 4m^2n^2 + 6m^2pq - 8m^2r^3 - 12n^2pq + 16n^2r^3 - 24pqr^3. \quad \textit{Ans.}$$

$(2x - 4y + 3z^2)^2 = ?$ $(a - b^2 + 3c^3 - 4d^4)^2 = ?$

22. Exercise in the Use of the Formulas and Theorems I–V, IX.

1. $(a^2 - 3b)^2$.
2. $(2a + 3b)(2a - 3b)$.
3. $(m^2 + 2n^y)^2$.
4. $(x - 5y)(x + 7y)$.
5. $(2c^n - d^2e)^2$.
6. $(2x - 7y)(2x + 6y)$.
7. $(2x - 4y)^3$.
8. $(3x^n + 5y^n)(3x^n - 11y^n)$.
9. $(a^{2n} - 2b^{3n})^3$.
10. $(4a^m - 6y^n)(4a^m + 6y^n)$.
11. $(\frac{2}{3}a^2x^3 - \frac{5}{6}by)^2$.
12. $(6x^3 - 7y^2)(6x^3 + 11y^2)$.
13. $(\frac{1}{2}a^2 - \frac{2}{5}b^3)^3$.
14. $(12abc^n - 14)(12abc^n - 26)$.
15. $(x^3 - y^3)^3$.
16. $(x^3 - y^3)(x^3 + 21y^3)$.
17. $(100 - 2)^2$.
18. $(80 + 3)(80 - 3)$.
19. 97^2; 103^2.
20. 67×73; 54×46.
21. $(1000 + 5)^2$.
22. $(\frac{1}{2}m^2 + \frac{1}{3}n^3)(\frac{1}{2}m^2 - \frac{1}{3}n^3)$.
23. $(2m^2 - 3p)^3$.
24. 96^2; 101^2; 1002^2.
25. 99^3; 102^3; 1004^3.
26. $(-3a^3 + b^n)(-3a^3 - 6b^n)$.
27. $(a + 2b - 3c)^2$.
28. $(m^2 + 3p - 2q + r)^2$.
29. $(a^2 - 2x^2 - 3y^3)^2$.
30. $(a^m + 2b^n - 4c^p - 5d^q)^2$.
31. $(\frac{1}{3}m^4 - 12pq)^3$.
32. $(\frac{1}{4}b^4 - 15cd)(\frac{1}{4}b^4 + \frac{1}{2}cd)$.
33. $(ap^m - bq^n)^3$.
34. $(ab^3 - 25cd^3)(ab^3 + 11cd^3)$.

III. FACTORING

23. Definition. Classes of Problems in Factoring. Factoring is separating a given quantity into other quantities whose product is the given quantity.

The following is a classification of all the more common exercises in factoring.

I. Monomial Factor in Polynomial Expression

$$2ma + mb - mc = m(2a + b - c).$$

$$2xy - 3by + 2cy + y = (2x - 3b + 2c + 1)y.$$

II. Problems Solved by Theorems VI–VIII, § 21

1. *Difference of two squares.*

Formula: $a^2 - b^2 = (a + b)(a - b).$ (Th. VI.)

EXAMPLES. $9a^2 - 16b^4 = (3a + 4b^2)(3a - 4b^2).$

$25m^4 - 1 = ?$ $\quad 36p^2 - 49q^2 = ?$

$(x + y)^2 - z^2 = ?$ $\quad (a - b)^2 - c^2 = ?$

2. *Difference of two cubes.*

Formula: $a^3 - b^3 = (a - b)(a^2 + ab + b^2).$ (Th. VII.)

EXAMPLES. $27x^6 - 125y^3 = (3x^2 - 5y)(9x^4 + 15x^2y + 25y^2).$

$8m^3 - 1 = ?$ $\quad a^{3p} - b^{3p} = ?$

3. *Difference of the same powers.* (See § 21, VII, Gen. Th.)

EXAMPLES. $x^5 - 32 = (x - 2)(x^4 + 2x^3 + 4x^2 + 8x + 16).$

$x^7 - y^7 = ?$ $\quad 32n^5 - 243 = ?$

4. *Sum of two cubes.*

Formula: $a^3 + b^3 = (a + b)(a^2 - ab + b^2).$ (Th. VIII.)

EXAMPLES. $64x^3 + 343 = (4x + 7)(16x^2 - 28x + 49).$

$8a^6 + 125y^6 = ?$ $\quad r^{3s} + 64 = ?$

5. *Sum of the same odd powers.* (See § 21, VIII, Gen. Th.)

EXAMPLES. $x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).$

$m^7 + 1 = ?$ $\quad 32a^5 + b^5 = ?$

III. Problems Solved by the Trial Method

1. *Trinomials.*

Two forms of the trial method for trinomials are described below.

1st Method. By setting down two binomials as factors chosen so that their product gives correctly the first and last terms of the given (arranged) trinomial, and multiplying them together to see if they give the middle term. If they do not, other trial sets of coefficients are chosen until a set of factors is found whose product is the given trinomial.

EXAMPLES. $6x^2 + 17xy - 14y^2 = (3x - 2y)(2x + 7y)$.

Here $3x$ and $2x$ are first chosen to give $6x^2$, and $-2y$ and $+7y$ to give $-14y^2$. The sum of their "cross products" is $17xy$, as required.

$$8m^2 - 22m + 15 = ? \qquad 15a^2 + 2ac - 24c^2 = ?$$

2d Method. By use of the converse of Theorem IV, § 21.

EXAMPLES. Factor $6x^2 + 17xy - 14y^2$.

$6^2x^2 + 17y(6x) - 6 \times 14y^2$. (Multiplying the given trinomial by 6, the smaller end coefficient, putting 6 with x in parenthesis in the second term.)

$(6x - 4y)(6x + 21y)$. (Choosing two numbers whose product is $-84y^2(-6 \times 14y^2)$ and whose sum is $17y$, as § 21, IV, Converse requires.)

$(3x - 2y)(2x + 7y)$ *Ans.* (Removing factors 2 and 3 inserted when we multiplied by 6 at the start.)

Solve each of the following by both methods.

$$6x^2 + 11xy - 10y^2 = ? \qquad 8x^4 + 10x^2yz - 75y^2z^2 = ?$$

$$6x^{2a} + 38x^ay^b - 28y^{2b} = ? \qquad 12m^6 - 23m^3 - 77 = ?$$

SUGGESTION. $12 \times 77 = (2 \times 2 \times 3)(7 \times 11) = (4 \times 11)(3 \times 7)$.

2. *Polynomials.*

Two trinomial factors are set down whose product includes the three leading terms of the given polynomial. Then multiplication shows whether their product is the given polynomial. If the product is wrong, other sets of coefficients are tried until the correct set is found.

EXAMPLES.

$2x^2 - 10xy + 12y^2 - xz + 7yz - 10z^2 = (x - 3y + 2z)(2x - 4y - 5z)$

$6x^2 - 19xy + 15y^2 + 7xz - 13yz - 20z^2 = ?$

IV. Problems Solved by Extracting the Roots of Certain Terms

The sum of these roots being squared, cubed, etc., produces the given quantity.

EXAMPLES.

1. $x^3 - 3x^2y + 3xy^2 - y^3 = (x - y)^3$. *Ans.* (See § 21, V.)
2. $x^4 - 4x^3y + 6x^2y^2 - 4xy^3 + y^4 = (x - y)^4$. *Ans.*
3. $a^2 + b^2 + c^2 + 2ab - 2ac - 2bc = (a + b - c)^2$. *Ans.* (§ 21, IX.)

V. Problems Solved by Placing Sets of Terms in Parentheses

After removing monomial factors from each of the expressions, the presence of a common polynomial factor becomes apparent.

EXAMPLES. $3a^3 + 3ab - a^2bc - b^2c$

$= 3a(a^2 + b) - bc(a^2 + b)$. (23, I; 19, 4.)

$= (a^2 + b)(3a - bc)$. *Ans.* (Prove by multiplication.)

$3a^2x^3 - 4acx^2 - 6abx + 8bc = ?$ $a^3x - a^2by - ab^2x + b^3y = ?$

VI. Problems Solved by Adding a Quantity

A quantity is added to a given binomial or trinomial and then subtracted from it, thus changing the given quantity into a quadrinomial that is the difference of two squares.

1. $a^4 + 4b^4 = (a^4 + 4a^2b^2 + 4b^4) - 4a^2b^2$

$= [(a^2 + 2b^2) + 2ab][(a^2 + 2b^2) - 2ab]$. *Ans.* (§ 21, VI.)

Prove by multiplication after removing parentheses.

$4m^4 + 81p^4 = ?$ $x^4 + 64y^4z^4 = ?$ $p^8 + 324 = ?$

2. $25x^4 - 41x^2y^2 + 16y^4 = (25x^4 - 40x^2y^2 + 16y^4) - x^2y^2 =$

$(5x^2 + xy - 4y^2)(5x^2 - xy - 4y^2)$. *Ans.* (§ 21, VI.)

$9m^4 + 38m^2n^2 + 49n^4 = ?$ $4p^4 - p^2q^2 + 4q^4 = ?$

VII. Problems Solved by the Remainder Theorem

The remainder theorem states that *if a "polynomial in x" (§ 9) equals 0 when $x = a$, then the polynomial is divisible by $x - a$.* This theorem will be proved in a later chapter. (XIV.)

EXAMPLES. $x^3 - 4x^2 - x + 4$ is exactly divisible by $x - 1$,
because $1^3 - 4 \times 1^2 - 1 + 4 = 0$.

By what is $x^3 + 6x^2 + 3x - 10$ divisible? $x^3 - 2x - 4$?

24. Exercise in Factoring. As a preparation for solving the problems of this article the student should fix in mind the following system for factoring composite quantities.

System for Factoring

(1) *Test first for a possible monomial factor contained in each of the terms of the given quantity, and if one is found, divide the given quantity by it. Then proceed, as below, to separate the polynomial factor into its prime factors.*

(2) *Look next to see whether the quantity to be factored is a binomial, a trinomial, a quadrinomial, or a polynomial of five or more terms.*

(3) *If the quantity is a binomial, it can be factored by one of the theorems of* § 23, II, *generally as it stands, but occasionally after arrangement as in* § 23, VI. *Always try to factor the given quantity or any of its factors by* II, 1 *before using the other theorems of* II.

(4) *If the quantity is a trinomial, it can be factored by either of the trial methods (page 21), except sometimes when it is of the form* $x^4 \pm ax^2y^2 + y^4$, *in which case it may have to be solved by the method of* § 23, VI.

(5) *If the quantity is a quadrinomial, it can usually be factored by the method of* § 23, V, *or* II, 1, *but it may have to be solved by one of the other methods.*

(6) *If the quantity is a polynomial of five or more terms, it may have to be solved by any one or other of the methods explained in* § 23.

Factor the following, and, if there is any doubt of the correctness of the work, test it by multiplying together the factors found, either by ordinary multiplication or by a theorem:

1. $a^4 - 9$.
2. $4a^2y + 4aby - 7bny - 7any$.
3. $6x^2 + 27x + 12$.
4. $8x^3 - 12x^2 + 6x - 1$.
5. $8a^3 - 64a^6x^6y^6$.
6. $(a+b)^2 - (c-d)^2$.
7. $10{,}000x^2 + 80x^5$.
8. $ab - bx^n + x^ny^m - ay^m$.
9. $4x^2 - 4xz - 35z^2$.
10. $9x^2 - 24xy + 16y^2$.

11. $16x^4 + 4x^2y^2 + y^4$.
12. $x^3 + 11x + 12$. (§ 23, VII.)
13. $a^4b^4 + 64c^4$.
14. $21a^2 - a - 10$.
15. $x^3 - 6x + 5$.
16. $8x^6 - 3 - 23x^3$.
17. $x^6 - m^6$.
18. $a^5 - 32b^5$.
19. $a^3 - a^2 - a + 1$.
20. $10x^{2n} - 30x^n - 40$.
21. $a^8 - 81$.
22. $9x^4y^4 - 3x^3y^5 - 6x^2y^6$.
23. $380 - x - x^2$.
24. $4(m-n)^3 - (m-n)$.
25. $x^4 + x^2 + 1$.
26. $(m^4 - m^2 + 5)^2 - 25$.
27. $a^2 - b^2 - c^2 - 2bc$.
28. $4x^2 - 12xy + 9y^2 - 81$.
29. $H^2 - 3H + 2$.
30. $4t^2 + t - 14$.
31. $\pi R^2 - \pi r^2$.
32. $x^2 + (a+b)x + ab$.
33. $x^{16} - y^{16}$.
34. $mnpq + 2 + pq + 2mn$.
35. $8c^2 - 6cd - 5d^2$.
36. $18x^2 + 33axy + 14a^2y^2$.
37. $\frac{1}{2}ab + \frac{1}{2}ac$.
38. $\frac{1}{2}ab + \frac{1}{2}ab' + \frac{1}{2}ab''$.
39. $\frac{4}{3}\pi R^3 - \frac{4}{3}\pi r^3$.
40. $x^9 - 64x^3 - x^6 + 64$.
41. $x^4 + 2x^2y^2 + 9y^4$.
42. $(x-y)^2 - 14(x-y) + 40$.
43. $2\pi r^2 + 2\pi rh$.
44. $a(a-c) - b(b-c)$.
45. $\frac{1}{6}\pi d^3 + \frac{1}{6}\pi D^3$.
46. $(x^2 + y^2 - z^2)^2 - 4x^2y^2$.
47. $\frac{2}{3}\pi rh + \frac{1}{2}\pi r^2$.
48. $x^{2m} - (y-z)^{2m}$.
49. $2ma^{6m} - 2mb^{6m}$.
50. $1 + (b - a^2)x^2 - abx^3$.
51. $a^{5m} - b^{5m}$.
52. $A^2 - B^2 + C^2 - 2AC$.
53. $1 - (x-y)^3$.
54. $x^3 + y^3 + 3xy(x+y)$.
55. $x^4 + 2x^3 - 4x^2 - 2x + 3$. (See § 23, VII.)
56. $m^4n - m^2n^3 - m^3n^2 + mn^4$.
57. $2x^2 + 5xy - 3y^2 - 4ax + 2ay$.
58. $x^2 + 2ax + a^2 - y^2 + 2yz - z^2$.
59. $x^2 + 9y^2 + 25z^2 - 6xy - 10xz + 30yz$.
60. $3a^2 + 6ab - 2ad + 3ac - 4bd - 2cd$.
61. $a^4 - 8a^3m + 24a^2m^2 - 32am^3 + 16m^4$.
62. $a^2b^2 - m^2n^2 - 2a^2b - 2amn$.
63. $a^2c^2 - p^2q^2 - 3a^2c + 3apq$.

IV. FRACTIONS

25. A **fraction** in algebra is an expressed division.

26. Integral Quantities. A polynomial is **integral** if it is the algebraic sum of integral terms, or terms that contain no letter in any denominator. Its numerical coefficients, however, can be fractional.

Thus, $3x^4 - 4ax^3 - \frac{2}{5}bx^2 - \frac{1}{9}x + 7$ is integral.

27. Rational Quantities. By rational quantities are meant ratios or fractions. The quotient of two integral quantities, each expressed without the use of radical or root signs, is called a rational quantity or expression.

$$\text{Thus, } \frac{3x^3 - x + \frac{4}{3}}{7x^2 - \frac{2}{5}} \text{ is rational.}$$

Since the denominator of a rational quantity, by definition, may be 1, integral quantities are included among rational quantities. Such quantities are said to be **rational integral quantities.**

28. Reduction of Fractions. By **reduction** in arithmetic and algebra is meant changing the form of a quantity without altering its value.

29. Fundamental Principle in Fractions. *Both terms of a fraction may be multiplied or divided by the same quantity without altering its value.*

PROOF. Let $\frac{a}{b}$ = any fraction, r = its value, and m = any number ($\neq 0$).

Then,	$a = br$,	(Since the dividend equals the quotient × divisor.)
also,	$ma = mbr$,	(By the multiplication axiom, § 36.)
or,	$ma = r \cdot mb$.	(By the commutative and associative laws of multiplication, § 4.)
Therefore,	$\frac{ma}{mb} = r$,	(By the division axiom, dividing through by mb.)
or,	$\frac{ma}{mb} = \frac{a}{b}$.	(Each equals r. Numbers equal to the same number are equal to each other.)

It should be observed in the preceding demonstration, that since a and b may represent any numbers whatever, whether integral, fractional, ordinary irrational, as $\sqrt{3}$, or other, we *assume* that they have a definite quotient, and assign r to take the place of this quotient. When we multiply through by m, we *assume* that the multiplication axiom holds true for all values of m and for the quantities which are multiplied by it.

30. Signs in Fractions. To avoid confusion it is often desirable to make changes in the signs of fractions or in their numerators or denominators. The following simple rules should be applied:

1. To change the sign of a quantity, whether monomial or polynomial, multiply it by the factor -1.

2. A fraction, as $+\frac{+a}{+b}$, can be regarded as having three signs: that before the fraction, that before the numerator, and that before the denominator. Any two of these can be changed without altering the value of the fraction; but changing one or all three changes the sign of the fraction, thus changing its value.

Thus, by the rule of signs in § 5,

$$\frac{a}{-b} = -\frac{a}{b} = \frac{-a}{b}; \quad \frac{-a}{-b} = \frac{a}{b} = -\frac{a}{-b} = -\frac{-a}{b}.$$

Notice that $-\frac{a}{b}$ is obtained from $\frac{a}{-b}$ by changing the signs of the fraction and of the denominator. Notice also that in $\frac{a}{-b}$ unlike signs in dividend and divisor give a negative quotient, or $-\frac{a}{b}$, and then $-\left(-\frac{a}{b}\right) = \frac{a}{b}$. Similar explanations can be given for the other equalities.

Examples.

$$\frac{y-x}{n-m} = \frac{x-y}{m-n}; \quad -\frac{c-a}{2m-n} = +\frac{a-c}{2m-n}; \quad -\frac{p-2q}{3n-m} = \frac{p-2q}{m-3n}.$$

In the example below, the sign of $x^2 - a^2$ should be changed to agree with $a - x$. The change is effected by multiplying the quantity $x^2 - a^2$ by -1, and, at the same time, changing the sign before the fraction, so as not to change the problem.

Thus $\frac{x}{a-x} + \frac{2ax}{x^2-a^2}$ becomes $\frac{x}{a-x} - \frac{2ax}{a^2-x^2}$.

3. Changing the sign of an even number of *factors* in the numerator or the denominator or both, evidently does not change the value of the fraction. Changing the sign of an odd number of factors changes the sign of the fraction. Thus,

$$\frac{(q-p)(b-a)}{c-d}=\frac{(p-q)(a-b)}{c-d};\quad \frac{y-x}{(3q-2p)(d-c)}=-\frac{x-y}{(2p-3q)(c-d)}.$$

31. Operations in Fractions.

1. Reduction of Fractions to their Lowest Terms.

To reduce fractions to their lowest terms, factor both numerator and denominator into their prime factors, and cancel all the common factors. This operation depends on the fundamental principle, § 29.

EXAMPLES. (1) $\frac{6a^2bc^4}{9a^3c}=\frac{2bc^3}{3a}$.

(2) $\frac{x^2+9x+20}{x^2-3x-28}=\frac{(x+4)(x+5)}{(x+4)(x-7)}=\frac{(x+5)}{(x-7)}$. *Ans.*

(3) $\frac{x^2-y^2}{2x+2y}=?$ (4) $\frac{x^2-2x+1}{3x^2-3x}=?$ (5) $\frac{27a+a^4}{18a-6a^2+2a^3}=?$

2. Addition of Fractions.

To add fractions, or integral quantities and fractions, proceed as in arithmetic, reducing the given quantities to equivalent fractions having a lowest common denominator, adding the resulting numerators, and placing the sum over the lowest common denominator. See that the answer is in its lowest terms.

To reduce fractions or integral quantities to equivalent fractions having a lowest common denominator, write 1 for the denominator of all integral quantities; then divide the lowest common denominator (or quantity of lowest degree in which each of the denominators is exactly contained) in its factored form by each of the given factored denominators in turn, and multiply both terms of the several fractions (§ 29) by the respective quotients.

EXAMPLES. (1) $\frac{2}{3}+\frac{5}{6}+\frac{7}{8}+\frac{15}{16}=?$ (2) $\frac{15}{16}-\frac{7}{12}=?$

(3) $\frac{x}{x+1} - \frac{x^2-3x+2}{x^2-1} + 3 = ?$ Check with $x = 2$.

Solution. The l.c.d. (lowest common denominator) is $(x+1)(x-1)$. The denominator of the integral quantity, 3, is 1 understood.

$$\frac{x(x-1)}{(x+1)(x-1)} - \frac{x^2-3x+2}{(x+1)(x-1)} + \frac{3(x^2-1)}{(x+1)(x-1)}.$$

(Dividing the l. c. d. by the denominator of each fraction, and then multiplying both terms by the quotient, § 29.)

$$\frac{x^2-x-(x^2-3x+2)+3x^2-3}{(x+1)(x-1)}.$$

(Indicating the addition of the numerators and placing the sum over the common denominator.)

$$\frac{3x^2+2x-5}{(x+1)(x-1)} = \frac{(3x+5)(x-1)}{(x+1)(x-1)} = \frac{3x+5}{x+1}. \quad \textit{Ans.}$$

(Simplifying and reducing to lowest terms.)

Check. Let $x = 2$.

Given quantity $= \frac{x}{x+1} - \frac{x^2-3x+2}{x^2-1} + 3 = \frac{2}{3} - \frac{4-6+2}{4-1} + 3 = 3\frac{2}{3}$.

Answer $= \frac{3x+5}{x+1} = \frac{3\times2+5}{2+1} = 3\frac{2}{3}$.

a. In checking, values that make any denominator 0 must be avoided. Thus, $x = 1$ could not have been used in the preceding problem.

Throughout fractions it is desirable that the student make considerable use of checking, since this is the best means available to uncover mistakes in performing operations.

(4) $\frac{a+x}{a-x} + \frac{a-x}{a+x} + \frac{4ax}{a^2-x^2} = ?$ (5) $m - \frac{m^2+n^2}{m-n} + n = ?$ (6) $a^2 - \frac{a^8-b^8}{a} - 1 = ?$

3. Multiplication and Division of Fractions.

To multiply fractions, factor both terms of every fraction into their prime factors, cancel any factors common to numerators and denominators, and multiply together the remaining factors of the numerator for the numerator of the product and the remaining factors of the denominator for the denominator of the product.

Cancellation of factors in numerators and denominators evidently depends on the principle that both terms of a fraction may be divided by the same quantity without altering its value.

In division of fractions invert the divisor and multiply.

EXAMPLES. (1) $\frac{a^2x - x^3}{5a} \times \frac{3a}{2ax - 2x^2} \times 2 = ?$

SOLUTION. $\frac{\cancel{x}(a+x)\cancel{(a-x)}}{5\cancel{a}} \times \frac{3\cancel{a}}{\cancel{2}\cancel{x}\cancel{(a-x)}} \times \cancel{2} = \frac{3(a+x)}{5}$. *Ans.*

(2) $-\frac{x^2 + 5x + 6}{x^2 - 1} \div -\frac{x^2 - 9}{x^2 - 2x - 3}$.

SOLUTION. $-\frac{\cancel{(x+3)}(x+2)}{\cancel{(x+1)}(x-1)} \times -\frac{\cancel{(x-3)}\cancel{(x+1)}}{\cancel{(x-3)}\cancel{(x+3)}} = \frac{x+2}{x-1}$. *Ans.*

(3) $\frac{ab^2}{cd^2} \times \frac{c^2d}{c^2f} \times \frac{ef}{ab} = ?$ (4) $\frac{2ax - x^2}{ax + a^2} \div \frac{4a^2 - x^2}{3x + 3a} = ?$

4. Complex Fractions.

A **complex fraction** is a fraction that has a fraction in one or both of its terms. It is an example of division of fractions, and can be solved as such. However, complex fractions can often be simplified much more quickly by multiplying both terms of the fraction by the lowest common multiple of the denominators in both terms.

EXAMPLES. (1) $\frac{\frac{a}{x} + \frac{b}{y}}{\frac{c+d}{z}} = \frac{ayz + bxz}{cxy + dxy}$. (Multiplying both terms by xyz.)

(2) $\frac{x + \frac{2x}{x-3}}{x - \frac{2x}{x-3}} = ?$ (3) $\frac{\frac{1}{x-y} - \frac{1}{x+y}}{\frac{y}{x-y}} = ?$ (4) $\frac{1}{2x + \frac{1}{3x + \frac{x}{4}}} = ?$

5. Fraction to Mixed Quantity.

A **mixed quantity** consists of an integral part and a fractional part. A fraction can be reduced to a mixed quantity by dividing the numerator by the denominator and adding the remainder divided by the divisor to the quotient.

EXAMPLES. (1) $\frac{a^2 + b}{a} = a + \frac{b}{a}$. *Ans.* (3) $\frac{ab + b^2}{a} = ?$

(2) $\frac{x^2 - 6x + 3}{x - 2} = x - 4 - \frac{5}{x - 2}$. *Ans.* (4) $\frac{x^4 + 1}{x - 1} = ?$

32. Type Forms in Fractions.

1. $\frac{a}{b}=\frac{ma}{mb}$.
2. $\frac{a}{c}+\frac{b}{c}=\frac{a+b}{c}$.
3. $\frac{a}{c}-\frac{b}{c}=\frac{a-b}{c}$.
4. $a\pm\frac{b}{c}=\frac{ac\pm b}{c}$.
5. $\frac{a}{b}\pm\frac{c}{d}=\frac{ad\pm bc}{bd}$.
6. $\frac{a}{b}\times c=\frac{ac}{b}$.
7. $\frac{a}{b}\div c=\frac{a}{bc}$.
8. $\frac{a}{b}\times\frac{c}{d}=\frac{ac}{bd}$.
9. $\frac{a}{b}\div\frac{c}{d}=\frac{ad}{bc}$.
10. $\frac{\frac{a}{b}}{c}=\frac{a}{b}\div\frac{c}{1}=\frac{a}{bc}$.
11. $\frac{a}{\frac{b}{c}}=\frac{a}{1}\div\frac{b}{c}=\frac{ac}{b}$.
12. $\frac{\frac{a}{b}}{\frac{c}{d}}=\frac{a}{b}\div\frac{c}{d}=\frac{a}{b}\times\frac{d}{c}=\frac{ad}{bc}$.
13. Translate the foregoing formulas into principles or rules.

33. Exercise in Simplifying Fractional Expressions.

1. $\frac{a^2-b^2}{(a+b)^2}$. *Ans.* $\frac{a-b}{a+b}$.
2. $\frac{3a^2+3ab}{a^4+ab^3}$.
3. $\frac{x^3-a^3}{x^2-2ax+a^2}$.
4. $\frac{x^2-5x}{x^2-4x-5}$.
5. $\frac{2x^2+17x+21}{3x^2+26x+35}$.
6. $\frac{x+3}{4}-\frac{x-2}{5}+\frac{x-4}{10}-\frac{x+3}{6}$. *Ans.* $\frac{15-x}{60}$.
7. $\frac{2x+5}{x}-\frac{x+3}{2x}-\frac{27}{8x^2}$.
8. $\frac{a^2-b^2}{bc}-\frac{ac-b^2}{ac}+\frac{ab-c^2}{ab}$.
9. $\frac{x-4}{x-2}-\frac{x-7}{x-5}$.
10. $\frac{y}{x(x^2-y^2)}+\frac{x}{y(x^2+y^2)}$.
11. $a+b-\frac{b^3}{a^2-ab+b^2}$.
12. $m-\frac{m^2+n^2}{m-1}+n+1$.
13. $\frac{x^2-5x+6}{x^2-7x+12}$.
14. $\frac{1}{D-2}+\frac{2}{(D-2)^2}$.

15. $\frac{a^2b^2+3\,ab}{4\,a^2-1} \div \frac{ab+3}{2\,a+1}$.

16. $\frac{3-x}{1-3\,x}-\frac{3+x}{1+3\,x}-\frac{1-16\,x}{9\,x^2-1}$.

17. $\left(1-\frac{a^4}{b^4}\right) \div \frac{a^2+b^2}{ab}$.

18. $\frac{2}{x}-\frac{3}{2\,x-1}-\frac{2\,x-3}{4\,x^2-1}$.

19. $(1-G^2) \div \frac{(1-G)^2}{G}$.

20. $\frac{1}{R-1}-\frac{1}{R+2}-\frac{3}{(R+2)^2}$.

21. $\dfrac{\frac{ax-x^2}{(a+x)^2}}{\frac{x^2}{a^2-x^2}}$.

22. $\dfrac{x^2+\frac{1}{x^2}+2}{x+\frac{1}{x}}$.

23. $\dfrac{\frac{a+b}{c+d}+\frac{a-b}{c-d}}{\frac{a+b}{c-d}+\frac{a-b}{c+d}}$.

24. $\dfrac{\frac{a+x}{a-x}+\frac{a-x}{a+x}}{\frac{a+x}{a-x}-\frac{a-x}{a+x}}$.

25. $\frac{x^2+xy}{x^2+y^2} \times \frac{x^3-y^3}{xy(x+y)}$.

26. $\left(\frac{x^2}{a^2}-\frac{x}{a}+1\right)\left(\frac{x^2}{a^2}+\frac{x}{a}+1\right)$.

27. $\left(x^4-\frac{1}{x^4}\right) \div \left(x+\frac{1}{x}\right)$.

28. $\frac{3}{x-3}-\frac{2}{x-4}-\frac{x-6}{(x-2)(x-5)}$.

29. $\frac{R^2-r^2}{t^2} \times \frac{t^2-t}{R^2-2\,Rr+r^2}$.

30. $\frac{a^{x-2}b^{x+1}c^x}{x^{a-b}y^cz^{a-1}} \times \frac{x^{a+b}y^bz^3}{a^2b^{x+1}c^3}$.

31. $\dfrac{1}{1+\frac{q+1}{n-q}}$.

32. $\frac{2\,a}{a^4+a^2+1}-\frac{1}{a^2-a+1}+\frac{1}{a^2+a+1}$.

33. $\left(4\,x^2+14\,x+\frac{98\,x-27}{2\,x-7}\right)\left(\frac{1}{6}-\frac{3\,x+29}{12\,x^2+18\,x+27}\right)$.

34. $\frac{4\,a(a^2-x^2)}{3\,b(c^2-x^2)} \div \left[\frac{a^2-ax}{bc+bx} \times \frac{(a+x)^2}{(c-x)^2}\right]$.

35. $\frac{2\,a}{(x-2\,a)^2}-\frac{x-a}{x^2-5\,ax+6\,a^2}+\frac{2}{x-3\,a}$.

36. $\frac{1}{b}\left(\frac{1}{a-b}-\frac{1}{a+2\,b}\right)-\frac{2}{a^2+ab-2\,b^2}$.

37. $\left[\frac{x^4-a^4}{x^2-2\,ax+a^2} \div \frac{a^2+ax}{x-a}\right] \div \frac{x^5-a^2x^3}{x^3+a^3}$.

CHAPTER II

REVIEW OF EQUATIONS

I. SIMPLE EQUATIONS CONTAINING ONE UNKNOWN QUANTITY

34. An **equation** is a true statement that two quantities have the same numerical value (§ 11).

Equations are of two kinds, **identical** and **conditional** (§ 1).

A **conditional equation** is one in which the two sides have the same numerical value only upon substitution of a certain value, or values, for the unknown number.

We are to study conditional equations in this chapter.

The values just referred to are called the **roots** of the conditional equation. To find these roots is called **solving** the equation. Showing that a value found for a root is a true one is called **verifying** the root. Such a root is said to *satisfy* the equation.

A **literal equation** is one in which one or more of the known numbers are represented by letters.

35. Equations Classified. Equations may be classified according to their degrees.

The **degree of a term** is the sum of the exponents of its literal factors. Thus, the degree of $6\,a^2b^3$ is 5.

The **degree of an equation** containing one unknown quantity is the greatest exponent of this unknown in any term, both sides or members of the equation being integral quantities.

Thus, $3\,x + 4 = 7\,x - 5$ is of the first degree.

$4\,x^2 + 5\,x + 6 = 2\,x^2 - 9$ is of the second degree.

$ax^3 + bx^2 = cx - d$ is of the third degree.

An equation of the first degree is also called a **simple** or **linear equation.** An equation of the second degree is called a **quadratic equation.** Equations of the third, fourth, fifth, etc. degrees are called **cubic, quartic** (or **biquadratic**), **quintic,** etc. equations.

36. The Axioms. An **axiom** is a statement *assumed* to hold true and used as a basic principle in a study.

The solution of equations depends in large measure on the use of the axioms.

1. **Identity Axiom.** *Quantities equal to the same quantity are equal to each other.* (See § 6.)

2. **Addition Axiom.** *Equals added to equals give equals.*

3. **Subtraction Axiom.** *Equals subtracted from equals leave equals.*

4. **Multiplication Axiom.** *Equals multiplied by the same quantity, or by equals, give equals.*

5. **Division Axiom.** *Equals divided by the same quantity (not 0), or equals, give equals.*

6. **Power Axiom.** *Equals raised to the same power give equals.*

7. **Root Axiom.** *Corresponding roots of equals are equal.*

8. **General Axiom.** *If only real positive quantities are considered, the same operation performed on equals gives equals.*

The axioms will be referred to as "Add. Ax.," "Mult. Ax.," etc.

37. Solution of Simple Equations.

A simple equation is solved for the unknown by first "*clearing it of fractions*" (if any appear); next, "*transposing*" all the terms containing the unknown to the left "member" or side, and the other terms to the right side of the equation; then, combining similar terms in each member, and, in literal equations, factoring the left member so that the unknown is one factor; last of all, dividing both members by the coefficient of the unknown.

Clearing of fractions always depends on the multiplication axiom; transposing on the addition or the subtraction axiom; and dividing on the division axiom.

To clear of fractions, each side of the equation is multiplied by the lowest common denominator. Thus, each term on both sides is multiplied by the lowest common multiple of the denominators, and this cancels all the denominators on both sides.

To transpose a term, it is subtracted from both sides of the equation; or, the term with its sign changed is added to both sides of the equation.

1. Solve the equation $\frac{3n-1}{n-3} - \frac{5n+28}{n+2} = -2$, and verify root.

Solution. L. c. d. $= (n-3)(n+2)$.

$$(3n-1)(n+2) - (5n+28)(n-3) = -2(n-3)(n+2) \quad \text{(Mult. Ax.)}$$
$$3n^2 + 5n - 2 - (5n^2 + 13n - 84) = -2(n^2 - n - 6) \quad \text{(Multiplying.)}$$
$$5n - 13n - 2n = 2 - 84 + 12 \quad \text{(Sub. Ax.)}$$
$$-10n = -70 \quad \text{(Adding.)}$$
$$n = 7. \quad \textit{Ans.} \quad \text{(Div. Ax.)}$$

Verification. $\frac{3 \times 7 - 1}{7 - 3} - \frac{5 \times 7 + 28}{7 + 2} = -2$; or $5 - 7 \equiv -2$.

2. $\frac{x-3}{2} - \frac{x-2}{3} = \frac{3x-21}{12}$.

3. $\frac{x}{2} - 2 = \frac{x}{4} + \frac{x}{5} - 1$.

4. $x + \frac{11-x}{3} = \frac{26-x}{2}$.

5. $\frac{x-2}{5} - \frac{x-3}{4} = \frac{x-7}{10}$.

38. Exercise in Equations Containing One Unknown.

Solve and verify the answer in the following:

1. $\frac{x}{3} - \frac{5x+4}{3} = \frac{4x-20}{6}$.

2. $\frac{x-1}{7} = 7 - \frac{4+x}{4} - \frac{23-x}{5}$.

3. $2(16-x) + 3(5x-4) = 12(3+x) - 2(12-x)$.

4. $5(x+1)^2 + 7(x+3)^2 = 12(x+2)^2$.

5. $\frac{x}{x+2} + \frac{4}{x+6} = 1$.

6. $\frac{6x+3}{15} - \frac{3x-1}{5x-25} = \frac{2x-9}{5}$.

7. $\frac{7}{t} = \frac{23-t}{3t} + \frac{t-1}{4t}$.

8. $\frac{p+4}{3p-8} - \frac{p+5}{3p-7} = 0$.

9. $\frac{3n+7}{4n+8} = \frac{6n-2}{8n-5}$.

10. $\frac{4}{x+2} + \frac{7}{x+3} = \frac{37}{x^2+5x+6}$.

11. $\frac{1}{5-\frac{1}{x}} = \frac{2}{7}$. (§ 31, 4)

12. $3 - \frac{1}{3} = \frac{1}{\frac{1}{3}+\frac{1}{t}}$.

13. $\frac{8}{n} + \frac{n-3}{n+3} = \frac{n+1}{n-1}$.

14. $\frac{3}{4} - \frac{\frac{3}{4}x + \frac{3}{4}}{x + \frac{3}{4}} = \frac{\frac{3}{4}}{x + \frac{3}{4}} - \frac{3}{4}$.

15. $3 + \frac{n}{.5} = 10 - \frac{n}{.2}$.

16. $\frac{2x+19}{5x^2-5} - \frac{17}{x^2-1} = \frac{3}{1-x}$.

17. $u = 3u - \frac{1}{2}(4-u) + \frac{1}{3}$.

18. $\frac{1}{4}(4 + \frac{3}{2}x) - \frac{1}{7}(2x - \frac{1}{3}) = \frac{31}{28}$.

19. $\frac{1}{3}(s-3)-\frac{1}{4}(s-8)+\frac{1}{5}(s-5)=0.$

20. $(x-1)^3+x^3+(x+1)^3=3x(x^2-1).$

21. $(x+\frac{5}{2})(x-\frac{3}{2})-(x+5)(x-3)+\frac{3}{4}=0.$

22. $(x+1)^2=\{6-(1-x)\}x-2.$ See § 13, 2.

23. $\dfrac{2z^2-z+3}{3z+2}-\dfrac{2z^2+3z-1}{3z-2}=\dfrac{3-6z-20z^2}{9z^2-4}.$

24. $\dfrac{x^2+3}{2(x^3-8)}+\dfrac{1}{6(x-2)}=\dfrac{2x-1}{3(x^2+2x+4)}.$

25. $ax-b=cx+d.$

SOLUTION. $ax-cx=b+d.$ (Sub. Ax., putting unknowns on left side and knowns on right side.)

$(a-c)x=b+d.$ (Factoring in left member.)

$x=\dfrac{b+d}{a-c}.$ (Div. Ax.)

VERIFICATION. $a\left(\dfrac{b+d}{a-c}\right)-b=c\left(\dfrac{b+d}{a-c}\right)+d.$ (Substituting its value for x in the given equation.)

$\dfrac{ab+ad}{a-c}-b=\dfrac{bc+cd}{a-c}+d.$ (Multiplying.)

$ab+ad-b(a-c)=bc+cd+d(a-c).$ (Mult. Ax.)

$ab+ad-ab+bc=bc+cd+ad-dc.$

$ad+bc\equiv ad+bc.$

In the following, solve in each case for the last letter of the alphabet in the problem:

26. $ax+bx=ac.$ 27. $m^2y+n^2y=p^2.$ 28. $bx+2ax-a=3ax-2c.$

29. $\dfrac{a}{x}-1=\dfrac{b}{x}-9.$

30. $\dfrac{1+x}{1-x}=\dfrac{a}{b}.$

31. $\dfrac{u}{a}+\dfrac{u}{b}=c.$

32. $\dfrac{1+2t}{1-3t}=\dfrac{a}{b}.$

33. $\dfrac{v+a}{v-b}=\dfrac{1}{m}.$

34. $\dfrac{x+a}{b}-\dfrac{b}{a}=\dfrac{x-b}{a}+\dfrac{a}{b}.$

39. Solution of Applied Formulas for any Letter. The following formulas come from arithmetic, geometry, physics (including sound, light, heat, electricity), shop work, trigonometry, engineering, etc. They are all solved by use of the axioms in much the same manner as the equations of § 38.

Solve the following formulas for each letter in them (excepting known numbers like π) not given directly:

1. $s = ba$ (area of a rectangle = base × altitude.)

SOLUTION. $ba = s$. (Reversing members.)

$a = \frac{s}{b}$. *Ans.* (Div. Ax.). $b = \frac{s}{a}$. *Ans.* (Div. Ax.)

2. $c = \frac{b}{h}$.

3. $s = \frac{a}{h}$.

4. $t = \frac{a}{b}$.

5. $d = \frac{w}{v}$.

6. $w = fd$.

7. $c = pn$.

8. $a = \frac{1}{2} bh$.

9. $A = \pi ab$.

10. $ST = st$.

11. $t = \frac{1}{k}$.

12. $F = \frac{Wa}{g}$.

13. $l = \frac{\pi dn}{360}$.

14. $\frac{PV}{T} = \frac{P'V'}{T'}$.

15. $E = \frac{nCN}{10^8}$.

16. $R = 3 + \frac{V}{6}$.

17. $P \times P_a \times S_a = W \times W_a \times s_a$.

18. $b = 1.155(f - 0.2)$

19. $R = 0.3788\,GT$.

20. $\frac{P}{W} = \frac{dr}{2\pi lR}$.

21. $a = b(1 + r)$.

22. $a = p(1 + rt)$.

23. $v = 4N(l + r)$.

24. $Mts = M't's'$.

25. $d = d' + \frac{2}{p}$.

26. $A = \frac{1}{2} a(b + b')$.

27. $H = \frac{PLAN}{33000}$.

28. $C = \frac{SE}{Sb + R}$.

SOLUTION FOR S. $SCb + CR = SE$. (Mult. Ax.)

$SCb - SE = -CR$. (Sub. Ax. Unknowns to left, knowns to right member.)

$S(E - Cb) = CR$. (Changing signs, Mult. Ax., and factoring.)

$S = \frac{CR}{E - Cb}$. *Ans.* (Div. Ax.)

Other answers:

$$E = \frac{SbC + RC}{S};\ R = \frac{SE - SbC}{C};\ b = \frac{SE - CR}{CS}.$$

29. $C = \frac{EP}{b + RP}$. 30. $c = \frac{SEP}{Sb + RP}$. 31. $R = \frac{gs}{g + s}$.

32. $s = \frac{lr - a}{r - 1}$. 33. $S = \frac{W}{W - W'}$. 34. $\frac{P}{W} = \frac{p}{2\pi R}$.

35. $\frac{1}{F} = \frac{1}{D} + \frac{1}{D'}$. 36. $S = \frac{w_1}{w_1 + w_2 - w_3}$. 37. $\frac{2R - h}{h} = \frac{l}{2p}$.

38. $\frac{P + p}{2p} = \frac{P}{P'}$. 39. $l = \frac{\pi D + \pi D'}{2} + 2d$. 40. $a = \frac{2n - 4}{n}$.

41. $\frac{1}{R} = \frac{1}{r} + \frac{1}{r'} + \frac{1}{r''}$. 42. $ms(t - t_m) = m's'(t_m - t')$.

Pupils should examine geometries, books on physics, and shop mathematics for similar formulas, and solve those found for any letter. But if in any formula the unknown appears to the second or higher powers, the method of solution is different from that here explained. It will be given later.

40. Making and Using of Formulas in Arithmetic.

1. If b represents base, r rate, and p percentage, make a formula giving the value of p. Solve this formula for b; for r.

a. The word rate means *ratio*, that is, the ratio of the percentage to the base; or, if r = rate, $r = \frac{p}{b}$. In the solution of problems it is often preferable to let $r\%$, or $\frac{r}{100}$ = the rate, thus making r ordinarily an integral or mixed number, rather than a decimal. Then $\frac{r}{100} = \frac{p}{b}$.

Solve the following numerical problems by substituting in the appropriate formula:

2. A trader having $1960 spent 15 % of it. What sum did he spend?

3. If I sell $\frac{5}{9}$ of a carload of wheat for what $\frac{2}{3}$ of the whole carload cost, what is the gain per cent?

SUGGESTION. Find first what part of cost whole carload would be sold for, and then find gain.

4. $68\frac{3}{4}$ is $9\frac{1}{6}\%$ of what number?

5. If b represents base, r rate, a amount, or sum of base and percentage, and d "difference," or base less percentage, write a formula for value of a; also a formula for value of d. Solve the latter for b.

6. A has $3009, which is 18% more than B has. How much has B? Also A has 15% less than C. How much has C?

7. If p represents principal, r rate, t number of years principal bore simple interest, and a amount, make a formula giving the value of a.

a. This formula can be used to solve *any* problem in simple interest.

8. Find the amount of $520 at 5% for 4 yr. 4 mo. 24 da. by using formula.

9. Find the rate when $1652.64 is amount, principal is $1320, and time is 3 yr. 7 mo. 6 da. Solve the formula for r first.

10. Find the principal when the amount is $279.18, rate is 3%, and time is 1 yr. 1 mo. 6 da. Solve the formula for p first.

11. Find the time when the amount is $1148, the rate is $4\frac{1}{2}\%$, and the principal is $1025. Solve the formula for t first.

12. Potatoes whose value was p dollars were shipped to a commission merchant to sell, who charged $r\%$ for selling or buying. With proceeds from the sale of the potatoes he was instructed to buy salt. What is the value of the salt he should ship back? Suppose $p = 3198$ and $r = 2\frac{1}{2}$.

13. If goods that cost $\$c$ are sold for $\$s$, what is the rate per cent r of gain? Find rate r when $c = 450$, $s = 531$.

14. Goods that cost c¢ were marked to sell at m¢, but $r'\%$ was thrown off from the marked price. What was the actual gain per cent r?

Given $c = 8$, $m = 12$, $r' = 16\frac{2}{3}$, to find r.

15. Three discounts of $r\%$, $s\%$, $t\%$ were taken from a bill of $\$b$. What was the net cost c?

Given $b = 2250$, $r = 20$, $s = 15$, $t = 8$, to find c.

16. Find the amount $\$a$ of a note of $\$p$ which drew simple interest at $r\%$ for n yr. m mo. d da., assuming 12 mo. of 30 da. each to a month. Then find a when $p = 400$, $r = 6$, $n = 3$, $m = 9$, $d = 13$.

17. Find B's tax $\$t$ on $\$p$ worth of property when the total assessed valuation is $\$A$ and the tax to be raised is T. Find t when $p = 4000$, $A = 1{,}946{,}500$, and $T = 33090.50$.

18. Find the insurance premium $\$p$ to be paid on a house valued at $\$h$ when insured for $v\%$ of its valuation at an $r\%$ rate. A carriage factory and stock worth $40,000 were insured for 90% of their value at $3\frac{1}{2}\%$. Find p.

19. What is the bank discount $\$d$ on a non-interest-bearing note for $\$a$ if discounted at $r\%$, n days before it is due? Find discount $\$d$ on a note for $300 discounted at 7% 45 days before it was due.

20. A railroad stock is quoted at $n\%$ above par and its annual dividends are $r\%$; what interest rate R is realized by an investor? Suppose $n = 28$, $r = 8$, to find R.

21. What is the duty on n yd. of cloth worth $\$c$ a yard, taxed at $r\%$ ad valorem, plus a specific duty of s¢ a yard? What would be the duty on 75 yd. of cloth worth $2 a yard, at 55% ad valorem, and 11¢ a yard?

22. The interest on $\$x$ for t yr. and m mo. at a certain rate per cent was $\$i$. What was the rate? Find rate when $t = 2$, $m = 3$, $i = 69.75$, and $x = 620$.

23. The population of a town in 1900 was p and in 1910 it was q. What was the gain per cent r? Make the calculation for the following cities:

	p	q
Cleveland, O.,	381,768	560,663
Atlanta, Ga.,	89,872	154,839
Seattle, Wash.,	80,671	237,194
Los Angeles,	102,479	319,198
New York,	3,437,202	4,766,883

II. SIMULTANEOUS EQUATIONS

(*If desired,* §§ 116–128 *on graphs can be taken up at this time, or after* § 45.)

41. Simultaneous equations are groups of equations that are satisfied by the same set or sets of values of the unknown quantities.

Such equations are solved by **elimination.** One or other of two methods of elimination is commonly used, viz., **substitution,** or **addition and subtraction.**

42. Elimination by Substitution. This is accomplished by finding the value of one unknown from one of the given equations and *substituting* this value for the same unknown in the other equation.

a. For convenience of reference the equations are numbered (1) and (2), and their changed forms, (1_1), (2_1), (1_2), etc.

1. Given $\begin{cases} (1)\ \dfrac{7}{x-3}=\dfrac{8}{y-5}, \\ (2)\ \dfrac{9}{2x-1}=\dfrac{5}{3y+4}, \end{cases}$ to find values of x and y that satisfy both equations.

SOLUTION.

(1_1)	$7y-35=8x-24$	(Mult. Ax.)
(1_2)	$7y=8x+11$	(Sub. Ax.)
(1_3)	$y=\dfrac{8x+11}{7}$	(Div. Ax.)
(2_1)	$27y+36=10x-5$	(Mult. Ax.)
	$27\left(\dfrac{8x+11}{7}\right)+36=10x-5$	(Substituting value of y.)
	$216x+297+252=70x-35$	(§ 31, 3, Mult. Ax.)
	$146x=-584$	(Sub. Ax.)
	$x=-4$. *Ans.*	(Div. Ax.)
(1_3)	$y=\dfrac{8\times -4+11}{7}=-3$. *Ans.*	(Substituting -4 for x.)

VERIFICATION. (1) $\dfrac{7}{-4-3}\equiv\dfrac{8}{-3-5}$; (2) $\dfrac{9}{-8-1}\equiv\dfrac{5}{-9+4}$.

2. $\begin{cases} (1)\ 9x+8y=57, \\ (2)\ 6x+7y=48. \end{cases}$

3. $\begin{cases} (1)\ 4x-3y=26, \\ (2)\ 3x-4y=16. \end{cases}$

4. $\begin{cases}(1)\ \frac{1}{2}x+\frac{1}{3}y=6,\\(2)\ \frac{1}{4}x+\frac{1}{2}y=5.\end{cases}$

5. $\begin{cases}(1)\ x-\dfrac{4y-9}{11}=5,\\(2)\ \dfrac{9}{2}-\dfrac{x+5}{3}=-3y.\end{cases}$

6. $\begin{cases}\dfrac{2}{u+3}=\dfrac{3}{v-2},\\5(u+3)=3(v-2)+2.\end{cases}$

7. $\begin{cases}\dfrac{m+n}{5}-\dfrac{m-n}{2}=3,\\\dfrac{m-n}{2}+\dfrac{m+n}{10}=0.\end{cases}$

8. $\begin{cases}x-2y=a,\\2x+8y=b.\end{cases}$

9. $\begin{cases}ax+by=c,\\mx+by=d.\end{cases}$

10. Solve for a and d as the unknowns in

(1) $p=a+(m-1)d.$ (2) $q=a+(n-1)d.$

43. Elimination by Addition or Subtraction.

1. Given $\begin{cases}(1)\ \dfrac{x}{4}+\dfrac{y}{5}=5,\\(2)\ \dfrac{2x}{3}+y=18,\end{cases}$ to find values of x and y which satisfy both equations.

SOLUTION.

$(1_1)\quad 5x+4y=100$ (Mult. Ax.)
$(2_1)\quad 2x+3y=54$ (Mult. Ax.)
$(1_2)\quad 10x+8y=200$ (Multiplying (1_1) by 2, Mult. Ax.)
$(2_2)\quad 10x+15y=270$ (Multiplying (2_1) by 5, Mult. Ax.)
$7y=70$ (Subtracting (1_2) from (2_2), Sub. Ax.)
$y=10.$ *Ans.* (Div. Ax.)
$(2_1)\quad 2x+3\times 10=54.$ (Substituting its value for y.)
$x=12.$ *Ans.*

VERIFICATION. (1) $\dfrac{12}{4}+\dfrac{10}{5}\equiv 5$; (2) $\dfrac{2\times 12}{3}+10\equiv 18.$

2. **Rule.** *After reducing both equations to the form $ax+by=c$, each equation is multiplied by such a number that the coefficients of x (or y if easier) are made the same in both new equations. Then, one of these equations is subtracted from the other when the equal coefficients have like signs, or added to it when the equal coefficients have unlike signs.*

3. $\begin{cases} (1)\ 2x + 3y = 18, \\ (2)\ 3x - 2y = 1. \end{cases}$

4. $\begin{cases} (1)\ 4x + 8y = 2.4, \\ (2)\ 10.2x - 6y = 3.48. \end{cases}$

5. $\begin{cases} x + \dfrac{y}{3} = \dfrac{14}{3}, \\ \dfrac{4x}{5} - \dfrac{2y}{5} = 4. \end{cases}$

6. $\begin{cases} (1)\ \dfrac{4x + 5y}{40} = x - y, \\ (2)\ \dfrac{2x - y}{3} - 2y = \dfrac{1}{2}. \end{cases}$

7. $\begin{cases} ax + by = p, \\ ax + dy = q. \end{cases}$

8. $\begin{cases} \dfrac{u}{a+b} + \dfrac{v}{a-b} = 2a, \\ u - v = 4ab. \end{cases}$

9. $\begin{cases} a^2u + b^2v = c^2, \\ a^3u + b^3v = c^3. \end{cases}$

10. $\begin{cases} ax + by = a^2 + b^2, \\ bx + ay = 2ab. \end{cases}$

11. $\begin{cases} \dfrac{5}{x} + \dfrac{6}{y} = 3, \\ \dfrac{15}{x} + \dfrac{3}{y} = 4. \end{cases}$

SUGGESTION. Do not clear of fractions in this problem but solve directly for $\frac{1}{x}$ and $\frac{1}{y}$. Then, at the last, find x and y.

12. $\begin{cases} \dfrac{6}{x} - \dfrac{7}{y} = 2, \\ \dfrac{2}{x} + \dfrac{14}{y} = 3. \end{cases}$

13. $\begin{cases} \dfrac{7}{ax} + \dfrac{3}{by} = 2, \\ \dfrac{5}{ax} - \dfrac{2}{by} = 7. \end{cases}$

a. **Elimination by Comparison**, a third method, is less important than the others. It is performed by finding the value of the same unknown from both equations and setting these values equal to each other.

44. Special Methods of Elimination. Instead of eliminating immediately, it is often better to proceed, as in the following solution, to get a third equation with smaller coefficients. Equation (1) is multiplied by 4 and equation (2) by 3. By subtracting (2_1) from (1_1), we get (3), in which the coefficient of x and of y is 1. This new equation can now be combined with one of the old equations, or with another new equation obtained in a similar way. Any pair of such equations are said to constitute a *system*, since they suffice to find the values of the unknowns.

1. Given $\begin{cases} (1)\ 13x - 29y = 97, \\ (2)\ 17x - 39y = 129, \end{cases}$ to find values of x and y.

SOLUTION.

$$\begin{array}{lrl} (1_1) & 52x - 116y = 388 & \text{(Mult. Ax.)} \\ (2_1) & 51x - 117y = 387 & \text{(Mult. Ax.)} \\ (3) & x + \quad y = 1 & \text{(Sub. Ax.)} \\ (3_1) & 13x + 13y = 13 & \text{(Ax. ?)} \\ (1) & 13x - 29y = 97 & \\ & 42y = -84 & \text{(Ax. ?)} \\ & y = -2. \quad \textit{Ans.} & \text{(Ax. ?)} \end{array}$$

(3) $x + (-2) = 1$; $x = 3$. *Ans.*

VERIFICATION. (1) $13 \times 3 - 29(-2) \equiv 97$; (2) $17 \times 3 - 39(-2) \equiv 129$.

(*a*) If the student will solve this problem by the regular process of solution, that is, by multiplying (1) by 17 and (2) by 13, he will find that the labor of solution is two or three times as great as that of the preceding solution. Evidently the plan is to multiply both members of each of the two equations by two small numbers so as to bring the corresponding coefficients *close together* in value instead of to make them equal.

2. (1) $11x - 21y = 26$; (2) $21x - 40y = 50$.

3. (1) $19x + 35y = 127$; (2) $28x + 53y = 190$.

4. (1) $23x + 29y = -1$; (2) $29x + 23y = 53$.

SUGGESTION. Add and divide through by 52, getting (3); then subtract and divide through by 6, getting (4). Use (3) and (4) to finish solution.

45. Systems of Equations. Equivalence. Two systems of equations (§ 44) are **equivalent** when one set can be derived from the other by axiomatic processes, and the values of the unknowns obtained from each set are the same. Thus, we saw in the last article that values of the unknowns derived from using a new third equation together with one of the given equations verified in both of the given equations; also in Ex. 4, that values of the unknowns obtained from the two *new* equations (3) and (4) verified in the original equations.

It may be said here that, in general, throughout simple equations, values of the unknowns found from a set of derived equations will verify in the original ones. Nevertheless the student should make it a rule to test all answers by substituting them in the original equations.

46. Simultaneous Equations Containing Three or more Unknowns.

1. Given $\begin{cases} (1)\ 2x + 3y - z = 21, \\ (2)\ 6x - 7y + 5z = 55, \\ (3)\ 9x + 5y - 2z = 71, \end{cases}$ to find values of x, y, z which will satisfy each of these equations.

SOLUTION. Of the unknowns z is most easily eliminated.

$$\begin{array}{lrl}
(1_1) & 10x + 15y - 5z = 105 & \text{(Mult. Ax.)} \\
(2) & 6x - 7y + 5z = 55 & \\
(4) & 16x + 8y = 160 & \text{(Ax. ?)} \\
(4_1) & 2x + y = 20 & \text{(Ax. ?)} \\
(3) & 9x + 5y - 2z = 71 & \\
(1_2) & 4x + 6y - 2z = 42 & \text{(Ax. ?)} \\
(5) & 5x - y = 29 & \text{(Ax. ?)} \\
(4_1) & 2x + y = 20 & \\
 & 7x = 49 & \text{(Ax. ?)} \\
 & x = 7. & \textit{Ans.}\ \text{(Ax. ?)} \\
(4_1) & 2 \times 7 + y = 20. & \text{(Substituting its value 7 for } x.) \\
 & y = 6. & \textit{Ans.} \\
(1) & 2 \times 7 + 3 \times 6 - z = 21. & \text{(Substituting their values for } x \text{ and } y.) \\
 & z = 11. & \textit{Ans.}
\end{array}$$

VERIFICATION. (2) $6 \times 7 - 7 \times 6 + 5 \times 11 \equiv 55$,
(3) $9 \times 7 + 5 \times 6 - 2 \times 11 \equiv 71$.

2. $\begin{cases} 4x - 3y + z = 9, \\ 9x + y - 5z = 16, \\ x - 4y + 3z = 2. \end{cases}$

3. $\begin{cases} 12l + 5m - 4n = 29, \\ 13l - 2m + 5n = 58, \\ 17l - m - n = 15. \end{cases}$

4. Make a rule for solving sets of three simultaneous equations.

47. Exercise in Solving Simultaneous Equations.

1. $\begin{cases} 5x + 6y = 17, \\ 6x + 5y = 16. \end{cases}$

2. $\begin{cases} 6x + 17y = 35, \\ 14x - 3y = 39. \end{cases}$

3. $\begin{cases} 12x - 6y = 8, \\ 48x - 9y = 92. \end{cases}$

4. $\begin{cases} 17a - 18b = 52, \\ 5a - 12b = 22. \end{cases}$

5. $\begin{cases} \dfrac{x}{2} - \dfrac{y}{5} = 4, \\ \dfrac{x}{7} + \dfrac{y}{15} = 3. \end{cases}$

6. $\begin{cases} \dfrac{3x - 5y}{2} - 2 = \dfrac{2x + y}{7}, \\ 4 - \dfrac{x - 2y}{4} = \dfrac{x}{2} + \dfrac{y}{4}. \end{cases}$

7. $\begin{cases} u + v + 6t = 63, \\ v + t - 6u = -130, \\ 3u + v - t = 76. \end{cases}$

8. $\begin{cases} \frac{1}{6}x - \frac{1}{5}y + \frac{1}{4}z = 3, \\ \frac{1}{5}x - \frac{1}{4}y + \frac{1}{5}z = 1, \\ \frac{1}{4}x - \frac{1}{8}y + \frac{1}{2}z = 5. \end{cases}$

9. $\begin{cases} 2p - 3q = 3, \\ 3q - 4r = 7, \\ 4r - 5p = 2. \end{cases}$

SUGGESTION. Eliminate q, using (1) and (2), getting (4). Then take (3) and (4) together.

10. $\begin{cases} \dfrac{2}{x} + \dfrac{1}{y} - \dfrac{3}{z} = 0, \\ \dfrac{3}{z} - \dfrac{2}{y} - 2 = 0, \\ \dfrac{1}{x} + \dfrac{1}{z} - \dfrac{4}{3} = 0. \end{cases}$

SUGGESTION. Do not clear of the denominators, but solve directly for $\dfrac{1}{x}, \dfrac{1}{y}, \dfrac{1}{z}$, and then get x, y, z.

11. (1) $ax + by = c$; (2) $mx + ny = p$.

SOLUTION

(2_1) $amx + any = ap$	(1_2) $anx + bny = cn$	(Ax. ?)
(1_1) $amx + bmy = cm$	(2_2) $bmx + bny = bp$	(Ax. ?)
$any - bmy = ap - cm$	$anx - bmx = cn - bp$	(Ax. ?)
$y = \dfrac{ap - cm}{an - bm}$. *Ans.*	$x = \dfrac{cn - bp}{an - bm}$. *Ans.*	

VERIFICATION (1) $a \cdot \dfrac{cn - bp}{an - bm} + b \cdot \dfrac{ap - cm}{an - bm} = c,$

or, $acn - abp + abp - bcm \equiv acn - bcm.$ (Mult. Ax.)

12. (1) $ax - by = 0$; (2) $x + y = c$.

13. (1) $x + ay = b$; (2) $ax - by = c$.

14. (1) $3ax - 2by = c$; (2) $a^2x + b^2y = 5bc$.

15. $\begin{cases} x + y = a + b, \\ ax - by = b^2 - a^2. \end{cases}$

16. $\begin{cases} ax + by = m, \\ bx - ay = c. \end{cases}$

17. $\begin{cases} \dfrac{x}{a} - \dfrac{y}{b} = m, \\ \dfrac{x}{c} + \dfrac{y}{d} = n. \end{cases}$

18. $\begin{cases} \dfrac{x+1}{y+1} = \dfrac{a+b+1}{a-b+1}, \\ x - y = 2b. \end{cases}$

19. $\begin{cases} \dfrac{x-y}{2} - \dfrac{y+z}{3} = a - b, \\ \dfrac{y+z}{3} + \dfrac{z-x}{4} = b + c, \\ \dfrac{z-x}{4} - \dfrac{x-y}{2} = c - a. \end{cases}$

20. $\begin{cases} \dfrac{3}{x} - \dfrac{4}{5y} + \dfrac{1}{z} = \dfrac{38}{5}, \\ \dfrac{1}{3x} + \dfrac{1}{2y} + \dfrac{2}{z} = \dfrac{61}{6}. \\ \dfrac{4}{5x} - \dfrac{1}{2y} + \dfrac{4}{z} = \dfrac{161}{10}. \end{cases}$

21. $\begin{cases} 3x + 6y + 2z + u = 2, \\ x - y - 3z - 4u = 3, \\ x + 2y - 2z - 2u = 0, \\ 2x + y - z - 3u = 5. \end{cases}$

SUGGESTION. Eliminate x first. Combine (2) and (3), eliminating x, getting (5). Then combine (1) and (2), eliminating x, getting (6). Then combine (3) and (4), eliminating x, getting (7). Then solve (5), (6), and (7) as in preceding problems.

III. PROBLEMS

48. Problems in Equations. There are two parts to the solution of a **problem** in algebra:

The **statement**, or *construction* of the equation or equations.

The **solution** of the equation or equations.

In the solution of a problem, four steps may be described:

(1) *Reading the problem carefully, getting all its conditions in mind, and letting a letter, or letters, represent its unknown number or numbers.*

(2) *Constructing the functions, or expressions involving the unknown or unknowns, described in the problem.*

(3) *Writing as the members of an equation two different expressions, which are said in the problem to be equal, repeating this procedure if there is given another equation or equations.*

(4) *Solving the equation or equations found, and verifying the answers by seeing if they satisfy the reading of the given problem.*

The choice of the number of unknowns to be used in the solutions is left to the student.

1. A grocer has two kinds of kerosene oil, one worth 12¢, and the other 15¢ a gallon. How many gallons of each shall he take to make a mixture of 55 gal. worth $7.20?

2. The garrison of a certain town consists of 125 men, partly cavalry and partly infantry. The monthly pay of a cavalryman is $20, and that of an infantry man is $15, and the whole garrison receives $2050. What is the number of cavalry and of infantry?

3. If a certain number is multiplied by 5, 24 is subtracted from the product, the remainder is divided by 6, and this quotient is increased by 13, the result is the number itself. What is the number?

4. What is the property of a person whose income is $860, when he has $\frac{5}{8}$ of it invested at 8%, $\frac{1}{4}$ at 6%, and the remainder at 4%?

5. A farm laborer engaged for 48 days at the rate of $2 per day and his board. But for every day he might be idle, he was to pay $1 for his board. At the end of the time he received only $42. How many days did he work?

6. A tree standing vertically on level ground is 60 ft. high. Upon being broken over in a storm, the upper part reached from the top of the trunk to the ground just 30 ft. from the foot of the trunk. What was the length of the part broken off?

7. A boatman who can row 5 mi. an hour in still water rows a certain distance upstream and back in 4 hr. How many miles upstream does he go if the stream itself flows 3 mi. in 2 hr.?

8. Two men 27 mi. apart, setting out at the same time, meet in 9 hr. if they walk in the same direction; but if they walk in opposite directions, that is, toward each other, they meet in 3 hr. Find their rates.

9. Two men, A and B, 330 mi. apart, set out in automobiles, B 45 min. after A, and travel towards each other, A at the rate of 20 mi. an hour and B at the rate of 15 mi. an hour. How far will each have traveled when they meet?

10. The area of the United States in 1850 was 559,452 sq. mi. more than 3 times what it was in 1800. By 1910, exclusive of Alaska and the islands, it had increased 45,535 sq. mi. over what it was in 1850. The total increase from 1800 to 1910 was 2,260,675 sq. mi. What was the area of continental United States at each epoch?

11. The population of continental United States in 1850 was 383,408 less than 6 times what it was in 1790, and in 1910 it was 795,238 less than 4 times what it was in 1850. The increase from 1850 to 1910 was 10,992,404 more than 3 times what it was from 1790 to 1850. Find the population at each date and verify.

12. Of the six great continents Australia has 386,000 sq. mi. less area than Europe. Africa has 14,000 sq. mi. less than three times the area of Europe, and North America has 2,081,000 sq. mi. less than Africa. South America has 828,000 sq. mi. less than twice the area of Europe, and Asia has 769,000 sq. mi. more than both Americas. The total area of all these continents is 52,153,000 sq. mi. Find the area of each continent.

13. If the smaller of two numbers is divided by the greater, the quotient is .21 and the remainder .0057; but if the greater is divided by the smaller, the quotient is 4 and the remainder 1.742 What are the numbers?

14. Fifty laborers were engaged to build a bridge on a railroad. Some of them were to receive \$2.50 per day, and others \$5. There was paid them \$150. No memorandum having been made, it is required to find how many worked at each rate.

15. If a certain number is divided by the sum of its two digits, the quotient is 6 and the remainder 4. If the digits are interchanged and the resulting number is divided by the sum of its digits, the quotient is 4 and the remainder 6. What is the number?

SUGGESTION. Let $10x + y =$ the number.

16. If the sides of a rectangular field were each increased by 2 yd., the area would be increased by 164 sq. yd. If the length were increased by 5 yd. and the breadth diminished by 5 yd., the area would be diminished by 65 sq. yd. What is the area?

17. A cistern can be filled by two pipes in 9 min. and 12 min., and emptied by two others in 15 min. and 18 min. respectively. After the first pipe had been running one minute, the third was opened; at the end of the second minute the second was opened; and at the end of the third minute, the fourth was opened. With all the pipes running, how long would it take to fill the cistern, counting from the time the first pipe was opened?

SUGGESTION. "Work," "cistern," and similar problems are solved by use of *reciprocals*. Thus, if it takes 9 minutes to fill a cistern, $\frac{1}{9}$ of the cisternful runs in in one minute. The cisternful is the unit in terms of which the members of the equation are expressed.

18. If A and B can perform a certain work in m days, A and C in n days, and B and C in p days, in what time can each perform it alone?

SUGGESTION. See Ex. 11, § 43.

19. Railroads join four cities A, B, C, D, thus forming a quadrangle. If I go from A to D through B and C I must pay $6.10 fare. If I go from A to B through D and C, I must pay $5.50. Going from A to C through B I pay the same as from A to C through D. On the other hand from B to D through A costs 40¢ less than from B to D through C. What are the distances AB, BC, CD, DA if the fare is 2¢ a mile?

20. A, B, and C in a hunting excursion killed 96 birds, which they wished to share equally. In order to do this, A, who had most, gave to B and C as many as each already had; next B gave to A and C as many as they each had after the first division; and lastly, C gave to A and B as many as they each had after the second division. It was then found that they all had the same number. How many had each at first?

21. A man distributed a cents among n persons, some receiving b cents each, and others c cents each. How many received b cents each, and how many c cents?

22. A was m times as old as B a years ago, and will be n times as old as B in b years. Find the age of each at present.

23. A and B can do a piece of work in a days, or if A works m days alone, B can finish the work by working n days. In how many days can each do the work?

24. A man invested p dollars, part at r per cent, and the remainder at s per cent. His annual income from both investments was i dollars. What was the amount of each investment?

25. Two trains are scheduled to leave A and B, d miles apart, at the same time and to meet in h hours. If the train that leaves B starts b hours late, and runs at its usual rate, it will meet the other train in k hours. What is the rate of each train?

IV. INDETERMINATE EQUATIONS

49. Indeterminate Equations. Thus far as many equations have always been given as there were unknowns in the equations. When there were two unknowns, two equations were given; when there were three unknowns, three equations were given, and so on.

If the number of given equations is less than the number of unknowns, we have what are called **indeterminate equations.** These equations can have an indefinite number of values of x and y. Thus, we can assign x *any* value and get a corresponding value of y. They are solved sometimes for *integral* values of the unknowns, *but the method of solution is altogether different* from the solution of determinate equations. We will solve a few of these.

Get all the positive integral solutions in the following problems:

1. Separate 71 into two parts, one of which is divisible by 5 and the other by 8 without remainders.

SOLUTION. Let $5x =$ one part, and $8y =$ other part.

Then $5x + 8y = 71$ is the only equation obtainable.

Solving, we get $x = \frac{71 - 8y}{5}$.

Now, by the conditions of the problem, x is a whole number. Then

$$\frac{71-8y}{5}, \text{ or } 14-y+\frac{1-3y}{5} \text{ (§ 31, 5), is a whole number.}$$

But as y must be a whole number, making $14-y$ a whole number, then $\frac{1-3y}{5}$ also must be a whole number. Then $2 \times \frac{1-3y}{5} = \frac{2-6y}{5}$ is a whole number. (We multiply by such a number that when we divide by the denominator, the coefficient of y in the remainder is 1.) Then $-y+\frac{2-y}{5}$ (§ 31, 5) is a whole number. In order that $\frac{2-y}{5}$ shall be zero or a whole number, $y=2$, or 7, or 12, or 17, etc. Substituting these values of y in the original equation, we find for $y=2$, $x=11$; for $y=7$, $x=3$; for $y=12$, $x=-5$. The last value of x is regarded as inadmissible here, as would be other values of x obtained from subsequent values of y. Hence, $5x=55$, $8y=16$, and $5x=15$, $8y=56$ are the values sought.

CHECK. Draw the graph of the equation and find pairs of integral coördinates on it. (See § 121.)

2. Separate 97 into two parts, one of which is divisible by 7 and the other by 9.

3. A woman paid $106 for silk at $2.50 a yard, and velvet at $3.50 a yard. How many yards of each did she buy?

4. A person bought 40 animals, consisting of pigs, geese, and chickens, for $40. The pigs cost $5 apiece, the geese $1, and the chickens 25¢ each. Find the number of each bought.

SUGGESTION. In this problem we get two equations and have three unknowns. After eliminating one of the unknowns between the two given equations, we proceed with the resulting equation as in Ex. 1.

5. A farmer buys oxen, sheep, and hens. The number bought is 100, and the total cost, $100. If the oxen cost $35, the sheep $3, and the hens 25¢ each, how many of each does he buy?

CHAPTER III

INVOLUTION AND EVOLUTION

I. INVOLUTION

50. Involution is the operation of raising quantities to powers.

51. Law for Raising Quantities to Powers. Let it be required to show that $(x^m)^n = x^{mn}$, m and n being integers.

$(x^m)^n = (xxx \cdots \text{to } m \text{ factors}) \times (xxx \cdots \text{to } m \text{ factors}) \times \cdots \text{to } n$ times.

$= xxxx \cdots \text{to } mn$ times.

$= x^{mn}$.

Hence, to raise a quantity with any integral exponent to any integral power, multiply the exponent of the quantity by the exponent of the power for the exponent of the quantity in the result.

52. Raising Monomials to Powers.

Type form: $(x^a y^b)^n = x^{an} y^{bn}$.

1. Square: $3\,m^2$; $-5\,a$; $-6\,a^m$; $\frac{2}{3}\,x^3$; $-3\,a^3b^n$.

2. Cube: $-6\,xy$; $3\,abc$; $-2\,m^2y$; $\frac{2}{5}\,a^2bc$; $-3\,a^mb^n$.

3. Simplify: $(3\,ab)^4$; $(2\times 3^2\times 5)^3$; $(3\times 2\,a)^4$; $(-\frac{2}{5}\,bc^n)^3$; $(2^2\times 5)^4$; $(a^m)^{2p}$; $(a^mb^pc^q)^3$; $(2^2\times 3^3\times 5)^n$; $(4^2\times 5^n\times 7)^a$.

4. Simplify: $3(2\,x)^2$ (see § 13); $5(2\times 3\,a)^4(3\,b)^2$; $\frac{1}{3}(\frac{1}{2}\,a)^2(4\,b)^3$; $6(-3\,b)^2(-2\,c)^3$; $2(3\times 4)^2(2\times 5)^5$; $2(2\,a)^n(2\,b)^m$.

53. Raising Binomials to Powers. Newton's Theorem. By a theorem, or by actual multiplication, show that:

$$(a-b)^2 = a^2 - 2\,ab + b^2.$$
$$(a-b)^3 = a^3 - 3\,a^2b + 3\,ab^2 - b^3.$$
$$(a-b)^4 = a^4 - 4\,a^3b + 6\,a^2b^2 - 4\,ab^3 + b^4.$$
$$(a-b)^5 = a^5 - 5\,a^4b + 10\,a^3b^2 - 10\,a^2b^3 + 5\,ab^4 - b^5.$$

Examining the right members of these equations, we see;

(1) *That the signs are alternately + and −.*

(2) *That the exponent of **a** in the first term in each case is the same as the exponent of (**a** − **b**) in the left member, and that the exponents of **a** decrease by 1 from term to term; also that **b** appears first in each case in the second term, and its exponents increase by 1 from term to term.*

(3) *That the first coefficient in the right member in each case is 1 (understood), and the second coefficient is the same as the exponent of (**a** − **b**) in the left member; also that each succeeding coefficient can be obtained by multiplying the coefficient of the preceding term by the exponent of the leading letter **a**, and dividing the product by the exponent of the other letter increased by 1.*

1. Raise $(a-b)$ to the 8th power.

SOLUTION. $(a-b)^8 = a^8 - 8a^7b + 28a^6b^2 - 56a^5b^3 + 70a^4b^4 - 56a^3b^5 + 28a^2b^6 - 8ab^7 + b^8.$

2. $(a-b)^{10}$. **3.** $(m+n)^6$. **4.** $(H-K)^9$.

5. Raise $2a^2 - 3b$ to the fifth power.

SOLUTION. $(x-y)^5 = x^5 - 5x^4y + 10x^3y^2 - 10x^2y^3 + 5xy^4 - y^5$.

Then, $(2a^2-3b)^5 = (2a^2)^5 - 5(2a^2)^4(3b) + 10(2a^2)^3(3b)^2 - 10(2a^2)^2(3b)^3 + 5(2a^2)(3b)^4 - (3b)^5$

(See § 13, 1.) $= 32a^{10} - 240a^8b + 720a^6b^2 - 1080a^4b^3 + 810a^2b^4 - 243b^5.$

6. $(3m^2-5n)^3$. **7.** $(2m^2n^3-1)^6$. **8.** $(4m^n - \frac{1}{3}p^3)^4$.

9. $(10-1)^4$. **10.** $(14\frac{1}{4})^3 = (14+\frac{1}{4})^3$. **11.** $(12\frac{1}{6})^3$

In cases of raising a sum or difference of two quantities of which one is small as compared with the other, Newton's theorem furnishes a quickly and easily obtained *approximate* result, since after the first or second terms, the numerical values of the succeeding terms diminish rapidly and some or all of the latter terms can be neglected. Thus,

$$(122\tfrac{1}{3})^3 = 122^3 + 3 \times 122^2 \times \frac{1}{3} + 3 \times 122 \times \frac{1}{9} + \frac{1}{27}$$
$$= 122^2(122+1) + 41 \text{ nearly, or } 1{,}830{,}773.$$

12. $(100-\frac{1}{2})^3$. **13.** $(18\frac{1}{6})^3$. **14.** $(2\frac{1}{16})^4$.

15. $(5\frac{19}{20})^5=(6-\frac{1}{20})^5$. **16.** $(1\frac{1}{12})^6$. **17.** 999^4.

54. Raising Polynomials to Powers. This is accomplished by changing the given polynomial into a binomial by the use of parenthesis, and applying Newton's theorem.

1. Cube $2a+3b-c$.

SOLUTION. $(2a+3b-c)^3=[(2a+3b)-c]^3$
$$=(2a+3b)^3-3(2a+3b)^2c+3(2a+3b)c^2-c^3.$$
The solution is continued by expanding each term of this result, then multiplying and combining terms when this can be done.

2. $(m^2-3mn+4n^2)^3$. 3. $(a^m+a^n-2c^{2p})^3$.

4. *Indicate* the expansion of $(2x-3y+4z-5n)^4$ without expanding the several terms.

II. EVOLUTION

55. Evolution is the process of extracting roots, that is, of finding equal factors of quantities.

56. Law for Extracting Roots.

Let it be required to show that $\sqrt[q]{x^{pq}}=x^p$.

We have, $x^{pq}=(x^p)^q$. (By § 51.)

Then, $\sqrt[q]{(x^p)^q}=x^p$. (Since the qth root of the qth power of a quantity equals that quantity, by the definitions of power and root, § 8.)

Hence, to extract a root of a quantity, divide the exponent of the quantity by the index (§ 8) *of the root for the exponent of the quantity in the answer.*

57. Extraction of the Roots of Monomials.

Type form: $\sqrt[n]{a^{mn}b^{np}}=a^mb^p$.

Even roots of positive quantities can have for sign either + or − and are marked ± (see § 18, 9). Even roots of negative quantities cannot be real quantities. Thus, $\sqrt{-4x^2}$ is not $2x$, since $(2x)^2=+4x^2$; neither is it $-2x$, since $(-2x)^2$ also equals $+4x^2$, and not $-4x^2$. Even roots of negative quantities have been called **imaginaries.**

1. $\sqrt{36\,a^2}$. 2. $\sqrt[3]{-64\,a^3b^6}$. 3. $\sqrt[5]{32\,a^{10}}$.

4. $\sqrt[m]{a^{2m}b^{5m}c^{3m}}$. 5. $\sqrt{8 \times 3^3 \times 6}$. 6. $\sqrt[3]{-2^6 \times 5^3 \times 7^3}$.

7. $\sqrt[4]{\dfrac{16\,a^4}{81\,b^8}}$. 8. $\sqrt[10]{\dfrac{x^{10}y^{20}}{z^{30}}}$. 9. $\sqrt{\dfrac{5 \times 3^3 \times 2^8}{2^2 \times 5^{13} \times 3^5}}$.

58. Square Root of Polynomials.

Let it be required to derive a process for the extraction of the square root of a polynomial from a study of the formula

$$a^2 + 2\,ab + b^2 = (a + b)^2.$$

SOLUTION.

$$\begin{array}{r|l} & a^2 + 2ab + b^2 \;\underline{|\, a + b} \\ & a^2 \\ \hline 2a + b & 2ab + b^2 \\ & \underline{2ab + b^2} \end{array}$$

How is a, the first term of the root, found? How is $2\,a$, the trial divisor, found from the first term of the root? How is b, the second term of the root, found? In what two places is b written when found? How is the operation concluded?

1. Extract square root of $4\,x^4 - 12\,x^3 + 5\,x^2 + 6\,x + 1$.

SOLUTION.

$$\begin{array}{r|l} & 4x^4 - 12x^3 + 5x^2 + 6x + 1 \;\underline{|\, 2x^2 - 3x - 1 \text{ Ans.}} \\ & 4x^4 \\ \hline 4x^2 - 3x & -12x^3 + 5x^2 \\ & \underline{-12x^3 + 9x^2} \\ (4x^2 - 6x) - 1 & -4x^2 + 6x + 1 \\ & \underline{-4x^2 + 6x + 1} \end{array}$$

REMARK. After two terms of the root are found, to find the next trial divisor, the two terms are regarded as one quantity or as a monomial and are doubled, giving $4\,x^2 - 6\,x$.

2. $81\,x^4 - 432\,x^3 + 864\,x^2 - 768\,x + 256$.

3. $1 - 2z + 2\,z^2 - z^3 + \dfrac{z^4}{4}$.

4. Extract fourth root, *i.e.* square root twice, of $a^4 + 8\,a^3b + 24\,a^2b^2 + 32\,ab^3 + 16\,b^4$.

5. Make a rule for the extraction of the square root of algebraical quantities, mentioning first the matter of arrangement of terms before beginning.

6. $x^6 + 25x^2 + 10x^4 - 4x^5 - 20x^3 + 16 - 24x.$

7. $\sqrt{x(x+1)(x+2)(x+3)+1} = ?$

Solve the two following mentally by getting the first and last terms, and then the middle one. Test the answer carefully, paying special attention to the sign of the last term.

8. $\sqrt{x^4 - 2x^3 + 3x^2 - 2x + 1} = ?$

9. $\sqrt{9x^4 - 12x^3 + 16x^2 - 8x + 4} = ?$

59. Extraction of the Square Root of Arithmetical Numbers.

1. Extract the square root of 1156.

SOLUTION. Since 1156 lies between 900 and 1600, its square root lies between 30 and 40, *i.e.* between 3 tens and 4 tens.

In general counting from the unit's order, each pair of figures, or "period," in the number gives one figure in the root, except that the left-hand period may have only one figure. Thus, the square root of 1′49 is 12+. The periods are usually marked by little lines above and between the figures.

For guidance in the solution the following formula is used:

$$(t+u)^2 = t^2 + 2tu + u^2 = t^2 + (2t+u)u.$$

Notice in the last expression that $2t$ is the *trial* divisor, and $2t + u$ is the *complete* divisor. The process at the right below is just like that at the left. That at the left was explained in the last article.

$$\begin{array}{r|l} & t^2 + 2tu + u^2 \ \underline{|\ t+u} \\ & t^2 \\ \hline 2t+u & 2tu + u^2 \\ & 2tu + u^2 \\ \hline \end{array}$$

$$\begin{array}{r|l} & \qquad t + u \\ & 11'56 \ \underline{|\ 30 + 4} \\ t^2 = & 9\,00 \\ \hline \text{Trial divisor} = 2t = 60 & 2\,56 \\ u = \ \ 4 & \\ \hline \text{Complete divisor} = 2t + u = 64 & 2\,56 \end{array}$$

2. Extract the square root of 43,347.24 and prove answer.

SOLUTION.

$$\begin{array}{r|l} & 4'33'47'.24 \ \underline{|\ 208.2} \\ & 4 \\ \hline 408 & 33\,47 \\ & 32\,64 \\ \hline 4162 & 83\,24 \\ & 83\,24 \end{array}$$

$(208.2)^2 = 43347.24$ by actual multiplication.

EXPLANATION. In the solution to Ex. 1, the ciphers were retained to make the process clear. In this solution no cipher is written that is not essential to the solution.

When the first figure of the root is doubled for trial divisor it is *understood* to be 40, though no zero is written after 4 at first. Now 40 is not contained in the period 33 brought down, so 0 is written in the root and after 4, and a new period, 47, is brought down. Next, 40 is understood to be 40 tens or 400. This 400 is contained in 3347 eight times. When the decimal period is brought down, the decimal point is inserted in the root.

3. 14,356,521. 4. 33,790,969. 5. 16,803.9369.

Solve the following, getting three decimal places in the root, and prove answers by squaring the root and adding in remainder thus getting given number.

6. 2.5. 7. $\frac{1}{70}$. 8. .008. 9. .1.

a. Every sequence of figures has two square roots due to the position of the decimal point. Thus $\sqrt{25} = 5$; $\sqrt{2.5} = 1.6^-$. This shows importance of pointing off number into periods correctly, commencing at decimal point. Notice $\sqrt{.5}$ must be written $\sqrt{.50}$ to get results correct. It is important to check the square root of decimals by multiplication, especially when first beginning the subject.

10. Make a rule for extracting square root of arithmetical numbers, explaining carefully exceptional cases.

11. 2. 12. .056. 13. $\frac{2}{7}$. 14. .00003.

Solve following mentally, getting one decimal place in root:

15. 2. 16. 3. 17. .5. 18. 209.

*** 60. Extraction of Cube Root of Algebraic Quantities.** To derive a process for the extraction of the cube root of polynomials from the formula:

$$(a+b)^3 = a^3 + 3a^2b + 3ab^2 + b^3 = a^3 + (3a^2 + 3ab + b^2)b.$$

SOLUTION

$$\begin{array}{r|l} & a^3 + 3a^2b + 3ab^2 + b^3 \;\underline{|a+b} \\ & a^3 \\ \hline 3a^2 + 3ab + b^2 & 3a^2b + 3ab^2 + b^3 \\ & \underline{3a^2b + 3ab^2 + b^3} \end{array}$$

How is first term of root found? How is trial divisor $3a^2$ found from first term of root? How is second term of root found? What two terms

*This subject is often omitted. It is not required for college entrance.

are added (annexed) to the trial divisor to make complete divisor? Explain precisely how they are obtained. How is process completed? Is it important that the terms be arranged before beginning?

1. $8x^6 + 66x^4 - 63x^3 + 33x^2 - 9x + 1 - 36x^5.$

SOLUTION

$$\begin{array}{r|l}
 & 8x^6 - 36x^5 + 66x^4 - 63x^3 + 33x^2 - 9x + 1 \,\underline{|\, 2x^2 - 3x + 1} \\
 & 8x^6 \\
\hline
12x^4 - 18x^3 + 9x^2 & -36x^5 + 66x^4 - 63x^3 \\
 & -36x^5 + 54x^4 - 27x^3 \\
\hline
12x^4 - 36x^3 + 27x^2 & 12x^4 - 36x^3 + 33x^2 - 9x + 1 \\
6x^2 - 9x & \\
+1 & \\
\hline
12x^4 - 36x^3 + 33x^2 - 9x + 1 & 12x^4 - 36x^3 + 33x^2 - 9x + 1 \\
\hline
\end{array}$$

EXPLANATION. 1st trial divisor $= 3(2x^2)^2$.

2d trial divisor $= 3(2x^2 - 3x)^2$.

$3(2x^2 - 3x) \times 1 = 6x^2 - 9x.$

$1^2 = 1.$

2. $x^3 - 24x^2 + 192x - 512.$

3. $x^6 - 3x^5y + 6x^4y^2 - 7x^3y^3 + 6x^2y^4 - 3xy^5 + y^6.$

4. $8x^6 + 48cx^5 + 60c^2x^4 - 80c^3x^3 - 90c^4x^2 + 108c^5x - 27c^6.$

5. Make a rule for finding the cube root of algebraical quantities.

6. $39x^4 - 99x^3 - 9x^5 + x^6 + 64 - 144x + 156x^2.$

7. Find the first three terms of the cube root of $1 - x$. Prove by cubing root (§ 54) and adding remainder.

*** 61. Extraction of the Cube Root of Arithmetically Expressed Numbers.**

1. Extract cube root of 405,224.

SOLUTION. Since 405,224 lies between 343,000, which is the cube of 70, and 512,000, which is the cube of 80, the cube root of 405,224 lies between 7 tens and 8 tens. In general, counting from unit's order, each set of three figures or "period" in the number gives one figure in the root, except that the left-hand period may have three or two figures or only one figure. Thus, the cube root of 1'728 is 12.

* This subject is often omitted. It is not required for college entrance.

Solution with Explanation

	$t + u$
	405′224 | 70 + 4
$t^3 =$	343 000
$3t^2 =$ 14700	62 224
$3tu =$ 840	
$u^2 =$ 16	
15556	62 224

Regular form of Solution

	405′224 | 74
	343
$3 \times 70^2 =$ 14700	62 224
$3 \times 70 \times 4 =$ 840	
$4^2 =$ 16	
15556	62 224

2. Extract the cube root of 232435.510 to one decimal place and prove answer.

Solution. 232′435.′510 | 61.4

216	
10800	16 435
180	
1	
10981	10 981
1116300	5454.510
7320	
16	
1123636	4494.544
	959.966

Explanation

$3t^2 = 3 \times 60^2 = 10800,$
$3tu = 3 \times 60 \times 1 = 180,$
$u^2 = 1^2 = 1,$
$3t^2 = 3 \times 610^2 = 1116300,$
$3tu = 3 \times 610 \times 4 = 7320,$
$u^2 = 4^2 = 16.$

Check. $(61.4)^3 + 959.966 = 232435.510.$

The answer to the nearest tenth is 61.5, since the next figure of the root after 4 would be more than 5.

3. 12,167. 4. 12,812.904. 5. 167.284151.

In the following get 2 places in the root and verify:

6. 3. 7. 0.2. 8. $\frac{1}{254}$.

9. Write out a rule for cube root, including peculiarities.

In following get answer to nearest tenth mentally:

10. 10. 11. .37. 12. 150.

62. Symmetry in Algebra. A quantity is **symmetrical** if the letters in it can change places without changing its value.

Thus, $a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4$ is symmetrical, as is the sum of a and b raised to any integral power, since a and b can interchange without altering the value of the expression.

Similarly, $x^3 + y^3$, $x^2 \pm xy + y^2$, $\frac{a^2 + b^2}{2ab}$, etc., are all symmetrical.

CHAPTER IV

FRACTIONAL EXPONENT QUANTITIES AND RADICALS.*

I. FRACTIONAL EXPONENT EXPRESSIONS

63. Formulas Giving the Laws for Integral Exponents.

1. Addition and subtraction formulas:

$ax^m + bx^m = (a+b)x^m$; $ax^m - bx^m = (a-b)x^m$. (§ 18.)

2. Multiplication formula: $x^m \times x^n = x^{m+n}$. (§ 17.)

3. Division formula: $x^m \div x^n = x^{m-n}$; $\frac{x^m}{x^n} = x^{m-n}$. (§ 17.)

4. Power formulas: $(x^m)^n = x^{mn}$; $(xyz)^m = x^m y^m z^m$. (§ 51.)

5. Root formulas: $\sqrt[n]{x^m} = x^{\frac{m}{n}}$; $\sqrt[n]{x^m y^p} = x^{\frac{m}{n}} y^{\frac{p}{n}}$. (§ 56.)

64. Meaning of Fractional Exponents. Integral exponents were defined in § 7. We are now to determine a meaning for fractional exponents. Such definition will naturally have to agree with the definition and formulas for integral exponents, since integral exponents are merely special cases of fractional exponents having the denominator 1.

If we let the laws of § 51 and § 17 hold true for fractional exponents, we have:

$$(x^{\frac{1}{2}})^2 = x^{\frac{2}{2}} = x;\ x^{\frac{1}{2}} \times x^{\frac{1}{2}} = x^{\frac{1}{2}+\frac{1}{2}} = x^1 = x.$$

Thus, $x^{\frac{1}{2}}$ *is the square root of* x, since it is *one* of two equal factors whose product is x. (See § 8.)

Again, $(x^{\frac{1}{3}})^3 = x^{\frac{3}{3}} = x$; $x^{\frac{1}{3}} \times x^{\frac{1}{3}} \times x^{\frac{1}{3}} = x^{\frac{1}{3}+\frac{1}{3}+\frac{1}{3}} = x^1 = x$.

* Throughout this chapter many problems can be solved mentally and should be so solved.

Thus, $x^{\frac{1}{3}}$ *is the cube root of* x, since it is *one* of three equal factors whose product is x.

In general, $(x^{\frac{m}{n}})^n = x^{\frac{mn}{n}} = x^m$;

$$x^{\frac{m}{n}} \times x^{\frac{m}{n}} \times x^{\frac{m}{n}} \times \cdots \text{ to } n \text{ factors} = x^{\frac{m}{n}+\frac{m}{n}+\frac{m}{n}+\text{to } n \text{ terms}} = x^{\frac{mn}{n}} = x^m.$$

Thus, $x^{\frac{m}{n}}$ is the *nth* root of x^m, since it is *one* of n equal factors whose product is x^m.

We are therefore led to attach the following meaning to a fractional exponent:

The numerator of a fractional exponent denotes the power to which the quantity of which it is the exponent is to be raised, and the denominator denotes the root to be taken of this result.

The student will see this more clearly from the following examples; $\sqrt{a^4} = a^{\frac{4}{2}} = a^2$; $\sqrt[3]{x^6} = x^{\frac{6}{3}} = x^2$; $\sqrt[4]{b^{12}} = b^{\frac{12}{4}} = b^3$; then $\sqrt[3]{a^7} = a^{\frac{7}{3}}$, and $a^{\frac{7}{3}}$ means the cube root of the 7th power of a.

65. Exercise dealing with Quantities having Fractional Exponents.

1. Calculate $8^{\frac{2}{3}}$.

SOLUTION. $8^2 = 64$; $\sqrt[3]{64} = 4$. *Ans.* Or, $\sqrt[3]{8} = 2$; $2^2 = 4$. *Ans.*

Evidently it is easier to extract the root *first*.

ANOTHER SOLUTION. $8^{\frac{2}{3}} = (2^3)^{\frac{2}{3}} = 2^2 = 4$. *Ans.*

Here the number is first expressed as the power of a prime.

a. To extract the root of a fraction extract the root of each term.

$$\text{Thus, } \sqrt{\frac{4}{9}} = \frac{2}{3}. \quad \left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}.$$

2. Calculate: $4^{\frac{1}{2}}$; $16^{\frac{1}{4}}$; $27^{\frac{2}{3}}$; $4^{\frac{3}{2}}$; $9^{\frac{3}{2}}$; $(\frac{1}{8})^{\frac{2}{3}}$; $(-\frac{1}{27})^{\frac{2}{3}}$; $(\frac{8}{27})^{\frac{4}{3}}$; $(a^m)^{\frac{1}{m}}$; $(x^{pq})^{\frac{1}{q}}$.

3. Calculate: $81^{\frac{1}{4}}$; $16^{\frac{1}{2}}$; $49^{\frac{3}{2}}$; $(36)^{\frac{5}{2}}$; $(\frac{1}{4})^{\frac{7}{2}}$.

4. $(9\,m^4 - 30\,m^2n + 25n^2)^{\frac{1}{2}} = ?$

5. $(a^3 - 6\,a^2b + 12\,ab^2 - 8\,b^3)^{\frac{1}{3}} = ?$

6. Express with fractional exponents: $\sqrt[5]{x^4}$ (*Ans.* $x^{\frac{4}{5}}$); $\sqrt[3]{a^2}$; $\sqrt[6]{m^{11}}$; $\sqrt[m]{y^n}$; $\sqrt[3]{m^2}$.

7. Express with radical signs: $x^{\frac{4}{5}}$; $a^{\frac{3}{4}}$; $p^{\frac{1}{2}}$; $3\,z^{\frac{1}{3}}$.

8. Calculate $243^{\frac{2}{5}}$; $36^{\frac{3}{2}}$; $9^{\frac{5}{2}}$; $81^{\frac{3}{4}}$.

66. Principles underlying Operations with Quantities containing Fractional Exponents.

We know that $(abc)^2 = a^2b^2c^2$; $(abc)^3 = a^3b^3c^3$; etc. The question arises, is it likewise true that

$$(abc)^{\frac{1}{2}} = a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}};\ (abc)^{\frac{1}{3}} = a^{\frac{1}{3}}b^{\frac{1}{3}}c^{\frac{1}{3}};\ \text{etc.}\,?$$

Or, stated generally, is it true that

$$(1) \qquad (abc\cdots)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots?$$

Now we can test the truth of this equation by raising both sides to the nth power by the power axiom (§ 36). We have

$$[(abc\cdots)^{\frac{1}{n}}]^n = abc\cdots. \quad \text{(By the definitions of root and power.)}$$

and

$$(a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots)^n = (a^{\frac{1}{n}})^n(b^{\frac{1}{n}})^n(c^{\frac{1}{n}})^n\cdots$$
$$= abc\cdots. \quad \text{(By the definitions of root and power.)}$$

Thus, we see that raising the two members of equation (1) to the same power gives equals. This, however, does not *prove* that $(abc\cdots)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots$, since in algebra unequals raised to the same power may give equals. For example, $(-2)^2 = +4$, and $(+2)^2 = +4$; but this does not prove that -2 equals $+2$, for they are *not* equal.

If, however, we limit the roots considered to arithmetical values, that is, to positive numbers, then it is always true that if two numbers raised to the same power give equals, they are equal.

Hence, with the limitation in meaning stated, we have

$$(abc\cdots)^{\frac{1}{n}} = a^{\frac{1}{n}}b^{\frac{1}{n}}c^{\frac{1}{n}}\cdots.$$

Changing this formula into a principle, we obtain:

1. **Fundamental Principle.** *The arithmetical root of a product is equal to the product of the same arithmetical roots of the several factors, and conversely.*

By means of this principle any **factor** within a sign denoting a root may be removed outside the sign provided the desired root of the factor can be extracted.

For example, $12^{\frac{1}{2}} = (4 \times 3)^{\frac{1}{2}} = 4^{\frac{1}{2}} \times 3^{\frac{1}{2}} = 2(3)^{\frac{1}{2}}$.

By means of this principle also like roots of quantities can be multiplied together. Thus, $6^{\frac{1}{2}} \times 5^{\frac{1}{2}} = 30^{\frac{1}{2}}$.

2. *If both terms of a fractional exponent are multiplied by the same number, the value of the quantity is not changed.*

This principle holds, likewise, only for arithmetical roots.

For, while $9^{\frac{1}{1}} = 9$, $9^{\frac{2}{2}}$ may equal either $+9$ or -9, since the square root (denoted by the denominator 2 of the exponent) may be either positive or negative.

3. *A quantity with a fractional exponent may have its value calculated either by raising to the power first and extracting the root afterwards, or vice versa.* See Ex. 1, § 65.

This principle also holds true only for arithmetical roots.

Thus, $25^{\frac{2}{2}} = (625)^{\frac{1}{2}} = \pm 25$, while $(25^{\frac{1}{2}})^2 = (\pm 5)^2 = +25$ only.

***67. Proofs based on principles of § 66 to show that same laws govern the use of fractional exponents as governed integral ones.**

1. *Proof that in multiplication the exponents of like factors are added.*

Let $x^{\frac{m}{n}}$ and $x^{\frac{p}{q}}$ be any two quantities with fractional exponents, x being any quantity, and m, n, p, q being integral numbers.

Then,

$$x^{\frac{m}{n}} \times x^{\frac{p}{q}} = x^{\frac{mq}{nq}} \times x^{\frac{np}{nq}} \qquad (\S\ 66,\ 2.)$$

$$= (x^{mq})^{\frac{1}{nq}} \times (x^{np})^{\frac{1}{nq}} \qquad (\S\ 66,\ 3.)$$

$$= (x^{mq} \times x^{np})^{\frac{1}{nq}} \qquad (\S\ 66,\ 1.)$$

$$= (x^{mq+np})^{\frac{1}{nq}} = x^{\frac{mq+np}{nq}} \qquad (\S\ 17,\ \S\ 66,\ 3.)$$

* This article may be omitted, if desired. A special case was assumed in § 64 to get the meaning of a fractional exponent.

But $\frac{mq+np}{nq}$ is the **sum** of the exponents $\frac{m}{n}$ and $\frac{p}{q}$.

Hence the rule to add the exponents of the same letter in the factors for the exponent of this letter in the product holds for fractional as well as for integral exponents.

2. Derive the corresponding rule for division.

3. *Proof that in raising a quantity to a power the exponent, of the quantity is multiplied by the index of the power for the exponent of the quantity in the result.*

Let $a^{\frac{m}{n}}$ be the given quantity and $\frac{p}{q}$ be the exponent, a being any quantity and m, n, p, q being integral numbers.

Then $(a^{\frac{m}{n}})^{\frac{p}{q}} = ([(a^m)^{\frac{1}{n}}]^{\frac{1}{q}})^p$ (By § 66, 3.)

$= [(a^m)^{\frac{1}{nq}}]^p$ (Since the qth root of the nth root means the nqth root by the definition of root, § 8, the nqth root being one of q equal factors of *one* of the n equal factors of a^m.)

$= [(a^m)^p]^{\frac{1}{nq}}$ (By § 66, 3.)

$= (a^{mp})^{\frac{1}{nq}}$ (Since the pth power of the mth power is the mpth power, § 51.)

$= a^{\frac{mp}{nq}}$. (By § 66, 3.)

But $\frac{mp}{nq}$ is the *product* of the exponent of the quantity and the exponent of the power.

Hence the rule to multiply the exponent of the quantity by the exponent of the power holds for fractional as well as integral exponents.

68. Exercise in performing Operations involving Quantities having Fractional Exponents.

Work mentally the first ten and as many more as possible.

1. Simplify $3\,a^{\frac{2}{3}} \times -5\,a^{\frac{3}{5}}$. *Ans.* $-15\,a^{\frac{19}{15}}$.

2. $9\,a^{\frac{7}{6}}b \times 3\,a^{\frac{5}{6}}b^{\frac{2}{3}}$. 3. $3\,m^{\frac{2}{3}}n^{\frac{4}{5}} \times 6\,m^{\frac{1}{6}}n^{\frac{1}{5}}p$. 4. $a^{\frac{1}{3}}b^{\frac{1}{2}}c^{\frac{5}{6}} \times a^{\frac{1}{2}}bc^{\frac{1}{3}}$.

5. $a^{\frac{m}{n}} \times a^{\frac{p}{q}} \times a^{\frac{r}{s}}$. 6. $12\,x^{\frac{1}{3}}y^{\frac{4}{5}}z^{\frac{1}{2}} \div -2\,x^{\frac{1}{5}}y^{\frac{2}{5}}$. 7. $6\,x^{\frac{a}{b}} \div -2\,x^{\frac{c}{d}}$.

8. $(3\,a^{\frac{1}{2}})^8$. REMARK. Notice that just as $(ab)^2 = a^2b^2$, so here $(3\,a^{\frac{1}{2}})^8 = 3^8 \times (a^{\frac{1}{2}})^8$.

9. $(9\,a^{\frac{5}{6}})^{\frac{1}{2}}$; $(8\,a^{\frac{6}{7}})^{\frac{1}{3}}$. 10. $(a^{\frac{1}{3}})^{\frac{2}{5}}$; $(-4\,a^{\frac{1}{5}})^2$; $(-27\,a^{\frac{2}{3}})^{\frac{1}{3}}$.

11. $(x^{\frac{3}{4}} - x^{\frac{1}{2}} + 2\,x^{\frac{1}{4}} + 3)(x^{\frac{1}{4}} - 2)$.

12. $(x - y) \div (x^{\frac{1}{3}} - y^{\frac{1}{3}})$; $(a^{\frac{2}{3}} + a^{\frac{1}{3}} + 1)(a^{\frac{1}{3}} - 1)$.

13. $(x^{\frac{1}{4}} - 6\,y^{\frac{1}{3}})(x^{\frac{1}{4}} + 5\,y^{\frac{1}{3}})$. (§ 21, IV.)

14. $(x^{\frac{1}{4}} + y^{\frac{1}{4}})^2$; $(a^{\frac{1}{3}} - 5\,b^{\frac{1}{3}})^3$; $(3\,m^{\frac{1}{5}} - 2\,n^{\frac{2}{5}})^4$.

15. From the sum of $5\,ax^{\frac{1}{2}} - (x+y)^{\frac{1}{2}} + (a-b)^{\frac{1}{3}}$ and $-7\,ax^{\frac{1}{2}} + 2(x+y)^{\frac{1}{2}} - 3(a-b)^{\frac{1}{3}}$ take $3\,ax^{\frac{1}{2}} + 4(x+y)^{\frac{1}{2}} - 5(a-b)^{\frac{1}{3}}$

16. Calculate $36^{1\frac{1}{2}} + 4^{3\frac{1}{2}} - 100^{0.5} - 81^{0.75}$.

17. Find the product of $\left(\frac{ay}{x}\right)^{\frac{1}{2}}$, $\left(\frac{bx}{y^2}\right)^{\frac{1}{3}}$, and $\left(\frac{y^2}{a^2b^2}\right)^{\frac{1}{4}}$.

18. Divide $x^{\frac{4}{3}} + x^{\frac{2}{3}}y^{\frac{2}{3}} + y^{\frac{4}{3}}$ by $x^{\frac{2}{3}} + x^{\frac{1}{3}}y^{\frac{1}{3}} + y^{\frac{2}{3}}$.

19. $(a^{\frac{1}{2}} + b^{\frac{1}{2}} + c^{\frac{1}{2}})^2 = ?$ $(a^{\frac{3n}{2}} + b^{\frac{3n}{2}}) \div (a^{\frac{n}{2}} + b^{\frac{n}{2}})$.

20. $\sqrt{(1 + 4\,x^{\frac{1}{3}} - 2\,x^{\frac{2}{3}} - 4\,x + 25\,x^{\frac{4}{3}} - 24\,x^{\frac{5}{3}} + 16\,x^2)} = ?$

21. Factor $a^{\frac{3}{2}} - 1$; $a^2 + ab + b^2$. (§ 23, VI.)

22. $(x^{\frac{6}{5}} - 9\,x + 33\,x^{\frac{4}{5}} - 63\,x^{\frac{3}{5}} + 66\,x^{\frac{2}{5}} - 36\,x^{\frac{1}{5}} + 8)^{\frac{1}{3}} = ?$

23. Expand $(x^{\frac{2}{5}} - 4)(x^{\frac{2}{5}} + 5)$.

24. Expand $(x^{\frac{2}{3}} + 4\,y^{\frac{2}{3}})(x^{\frac{2}{3}} - 4\,y^{\frac{2}{3}})$.

25. Expand $(x^{\frac{n}{3}} - 2\,y^{\frac{2p}{3}})^3$.

26. Expand $(x^{\frac{1}{2}} - 2\,y^{\frac{1}{2}} - 3\,z^{\frac{1}{2}})^2$.

69. Meaning of Zero and Negative Exponents.

1. Zero exponents arise naturally in division when a letter has the same exponent in both divisor and dividend.

Thus, $x^m \div x^m = x^{m-m} = x^0$. (§ 17.)

But $x^m \div x^m = 1$.

Hence, if the rule for subtracting exponents is to continue to hold, we must take $x^0 = 1$. (§ 17, 2.) We have then this important result:

2. **Theorem concerning 0 Exponent.** *Any finite quantity with exponent* 0 *is equal to* 1.

The meaning of this theorem is understood better if we think of x^0 as being associated with other factors.

For example, $\dfrac{6\,x^2yz}{x^2uv} = \dfrac{6\,x^0yz}{uv} = \dfrac{6\,yz}{uv}$.

The exponent 0 shows that x is used *no* times as a factor of the product, or has dropped out and so does not affect the product of the other factors. x^0 cannot have the value 0, for that would make the whole expression 0, and it is *not* 0.

3. The meaning to attach to negative exponents can be obtained from that for zero exponent. We have

$x^m \times x^{-m} = x^{m+(-m)} = x^0 = 1$. (As just shown).

Or $x^m \times x^{-m} = 1$. (Things equal to same thing are equal to each other.)

Then $x^{-m} = \dfrac{1}{x^m}$ (Division Axiom).

Thus, we are led to take a quantity with a negative exponent to mean the *reciprocal* of the same quantity with a positive exponent.

If $b^{-2} = \dfrac{1}{b^2}$, and $c^{-3} = \dfrac{1}{c^3}$, as just laid down,

$$\frac{ab^{-2}}{c^{-3}d} = \frac{a \times \dfrac{1}{b^2}}{\dfrac{1}{c^3} \times d} = \frac{\dfrac{a}{b^2}}{\dfrac{d}{c^3}} = \frac{ac^3}{b^2d}.$$

Here we see b^{-2} in the numerator of the first fraction become b^2 in the denominator of the last, and c^{-3} in the denominator of the first fraction become c^3 in the numerator of the last. In this way we get the following theorem:

4. **Theorem concerning Negative Exponents.** *Any factor can be transferred from the numerator to the denominator or from the denominator to the numerator of a fraction provided the sign of its exponent is changed.*

70. Exercise in Using Zero and Negative Exponents.

Write the values of the following:

1. 10^0. 2. 1000^0. 3. a^0b^0c. 4. $\frac{4x^2y^3}{x^2y}$.

Express the following with positive exponents:

5. $2a^{-5}b$. 6. $m^{-1}n^{-2}$. 7. $5a^{\frac{1}{2}}c^{-\frac{1}{3}}$. 8. $10a^{-\frac{5}{4}}b^{\frac{2}{3}}$.

Write the following fractions without denominators:

9. $\frac{2}{x^{-3}}$. 10. $\frac{1}{a^{-2}b^3}$. 11. $\frac{a^2m^4}{2b^{\frac{1}{3}}}$. 12. $\frac{5}{3^{-1}c^{\frac{2}{3}}d^{-\frac{1}{4}}}$.

(*a*) When no denominator is written, 1, of course, is understood. When all the factors of a numerator are transferred to the denominator (as in Ex. 6), the factor 1 is understood to remain in the numerator.

71. Rules giving the Laws for Exponents whether Integral, Fractional, or Negative.

1. *In addition, add the coefficients of similar quantities. The exponents in the literal part remain unchanged.*

2. *In multiplication, add the exponents of the same quantity in the factors for the exponent of this quantity in the product.*

3. *In division, subtract the exponent of a factor in the divisor (denominator) from the exponent of the same factor in the dividend (numerator) for the exponent of this quantity in the quotient.*

4. *In raising to powers, multiply the exponent of a factor quantity by the index of the power to which it is to be raised for the exponent of this quantity in the answer.*

5. *In extracting roots, divide the exponent of a factor quantity by the index of the root for the exponent of this quantity in the answer.*

6. *Any finite quantity with exponent* 0 *is equal to* 1.

7. *Any factor of the numerator or denominator of a fraction can be transferred to the other term of the division by changing the sign of its exponent.*

72. General Exercise in the Simplification of Quantities involving Fractional and Negative Exponents. Answers are to have only positive exponents. Solve as many as possible mentally.

In solving numerical exercises write quantities as powers of prime factors, and then, after simplifying, change all negative exponents to positive ones.

Thus, $$8^{-\frac{2}{3}} = (2^3)^{-\frac{2}{3}} = 2^{-2} = \frac{1}{2^2} = \frac{1}{4}.$$

1. $16^{\frac{3}{4}}$.
2. $81^{\frac{1}{4}}$.
3. $100^{-\frac{1}{2}}$.
4. $625^{0.75}$.
5. $x^{\frac{2}{3}} \times x^{-\frac{1}{2}}$.
6. $m^{\frac{2}{3}} \times m^{\frac{3}{2}}$.
7. $(a^{\frac{2}{3}})^{\frac{3}{5}}$.
8. $16^{-\frac{1}{4}}$.
9. $a \times a^{-\frac{2}{5}}$.
10. $(a^4 b^{-\frac{2}{3}})^{\frac{3}{2}}$.
11. $100^{-\frac{5}{2}}$.
12. $4\,a^{\frac{1}{3}} \div 2\,a^{-\frac{5}{6}}$.
13. $(\frac{4}{25}\,a^2)^{-\frac{3}{2}}$.
14. $(\frac{27}{125})^{-\frac{2}{3}}$.
15. $a^{\frac{1}{3}} \times a^{-\frac{3}{4}} \times a^{\frac{5}{12}}$.
16. $((a^{-2})^2)^{\frac{3}{4}}$.
17. $(xy)^{x+y} \div x^y y^x$.
18. $(a^{\frac{1}{2}} - a^2)^2$.
19. $(a - b) \div (a^{\frac{1}{2}} - b^{\frac{1}{2}})$.
20. $(x^{\frac{2}{3}} + 2\,x^{-\frac{1}{3}} + 1)^2$.
21. $\left(\frac{a^{-2}b}{a^3b^{-4}}\right)^{-3} \div \left(\frac{ab^{-1}}{a^{-3}b^2}\right)^5$.
22. $\left(\frac{x^{-2}y^2}{x^3y^{-2}}\right)^{-\frac{1}{2}} \times \frac{(y^3x^{-2})^{-1}}{x^3y^{-3}}$.
23. $(a^{\frac{1}{n-1}}a^{\frac{1}{n+1}})^{\frac{1}{n}}$.
24. $8^{-\frac{2}{3}} \div 256^{-\frac{1}{4}}$.
25. $\frac{m^{-1} - n^0}{m^{-2} - n^0m^{-1}} \times m^{-2}$.
26. Square $a^{\frac{1}{2}} + b^{\frac{1}{2}} - c^{\frac{1}{2}}$.
27. $12^0 + 4^{\frac{1}{2}} - 9^{-1} + (-64)^{-\frac{1}{3}} + 27^{\frac{2}{3}}$.
28. $(x^{-\frac{9}{4}} + 2\,x^{-\frac{3}{2}} + 4\,x^{-\frac{3}{4}} + 8)(x^{-\frac{3}{4}} - 2)$.

The table at the top of the next page will be found convenient for reference in simplifying and evaluating radical quantities, and evaluating the roots of quadratic equations. (See also p. ix.)

TABLE OF POWERS AND SQUARE ROOTS

x	x^2	x^3	x^4	x^5	x^6	$x^{\frac{1}{2}}$	x	$x^{\frac{1}{2}}$	x	$x^{\frac{1}{2}}$
1	1	1	1	1	1	1.000	10	3.162	19	4.359
2	4	8	16	32	64	1.414	11	3.317	20	4.472
3	9	27	81	243	729	1.732	12	3.464	21	4.583
4	16	64	256	1024	4096	2.000	13	3.606	22	4.690
5	25	125	625	3125		2.236	14	3.742	23	4.796
6	36	216	1296	7776		2.449	15	3.873	24	4.899
7	49	343	2401			2.646	16	4.000	25	5.000
8	64	512	4096			2.828	17	4.123	26	5.099
9	81	729	6561			3.000	18	4.243	27	5.196

II. RADICAL EXPRESSIONS

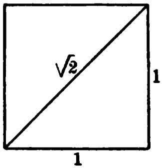
73. Radical Quantities. An indicated root of a quantity is called a **radical quantity**, or a **radical.** Thus, $\sqrt[3]{5}$, $2^{\frac{1}{2}}$ are radical quantities. In $4\sqrt[3]{5a}$, or $4(5a)^{\frac{1}{3}}$, 4 is the **coefficient** of the radical, 3 is the **index** of the root, and $5a$ is the **radicand.**

Radical quantities may be either *rational,* as $\sqrt{25}$, $\sqrt{\frac{4}{9}}$, or *irrational,* as $\sqrt{7}$. A **rational** quantity is the **ratio** of two integral numbers. An **irrational** (or **surd**) quantity cannot be the ratio of two integral numbers.

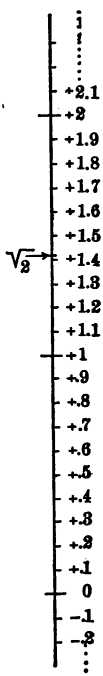
Rational quantities, as $\frac{4}{1}$, $\frac{2}{5}$, $\frac{15}{111}$, give rise, when the numerator is divided by the denominator, to either integers, finite decimals, or repeating decimals, that is, to decimals that *repeat* certain sets of figures. Thus, $\frac{5}{37} = .135135135\cdots$; $\frac{3}{7} = .428571'428571\cdots$. When dividing by 7, after ciphers begin to be annexed to the dividend, the possible remainders are 1, 2, 3, 4, 5, 6. When such remainders are all exhausted, the figures of the quotient must repeat in the same order.

Irrational quantities, on the other hand, never give rise to decimals like the preceding classes, but to decimals which do not end and which do not repeat the figures in regular order. An irrational number and unity have no common divisor, that is, are *incommensurable.*

In particular, it can be shown by the continued division process of finding the greatest common divisor that the diagonal of a square and its side, whose ratio equals $\sqrt{2}:1$, are incommensurable. By the process of § 59, we have $\sqrt{2} = 1.4142^{+}$.



The student can get a fairly good idea of the different kinds of numbers, — positive, negative, integral, rational, and irrational from the following diagram, on which the length $\sqrt{2}$, from 0 to the point indicated by the arrow, is found from the length 1 on this scale by obtaining it as the diagonal of a square whose side is 1. The diagram shows negative numbers as extending *downwards* from 0, and positive numbers *upwards* from 0; it shows also the integral numbers 1 and 2. The decimal division numbers also are examples of rationals, and $\sqrt{2}$ is an example of an irrational number.



To see the difference between rational and irrational numbers, we observe that $\sqrt{2}$ gives rise to a never ending series of decimal figures. As indicated on the diagram, we thus learn, first, that $\sqrt{2}$ lies *between* 1 and 2; next that it lies *between* 1.4 and 1.5. Now, if we subdivided the line from 1.4 to 1.5 into ten equal divisions, $\sqrt{2}$ would lie *between* 1.41 and 1.42; and if we subdivided the line from 1.41 to 1.42 into ten equal divisions, $\sqrt{2}$ would lie *between* 1.414 and 1.415; and if we subdivided the line from 1.414 to 1.415 into ten equal parts, it would lie *between* 1.4142 and 1.4143; and so on indefinitely. Thus, no matter into how many equal decimal divisions the line from 1.4 to 1.5 is divided, $\sqrt{2}$ will always lie *between* two such divisions. It can be shown that the same thing would hold true if the line were divided into any other equal divisions than tenths. The distance from 0 to a point *on* any one of the divisions described would be denoted by a rational quantity. Thus, an irrational number always locates a point different from one located by any rational number.

74. The Duplicate Notation for Radicals. As we have seen, both radical signs and fractional exponents are used to denote roots.

For example:

$$8^{\frac{1}{3}} = \sqrt[3]{8};\ a^{\frac{2}{5}} = \sqrt[5]{a^2};\ 4\,x^{\frac{1}{3}}y^{\frac{2}{3}} = 4\sqrt[3]{xy^2}.$$

The reason for the existence and use of both notations is easily explained. The radical sign $\sqrt{\ }$ came into use a century before Newton and Wallis found out that fractional exponents were naturally adapted to denote roots. The use of the radical sign in the meantime had become too firmly intrenched to admit of its displacement. The radical sign notation, it may be remarked, is often shorter than the other. Thus, $(a+b)^{\frac{1}{2}}$ is more troublesome to write than $\sqrt{a+b}$, because both numerator and denominator of the fractional exponent are not written in the radical sign notation.

The use of this double notation, however, is oftentimes quite confusing to the student, and it makes the subject much more difficult than it would otherwise be.

75. Simplification of Radical Quantities. There are four kinds of **reductions** in radicals which we now proceed to investigate.

A radical is not in its simplest form:

1. *If some required root of its radicand can be extracted.*

Thus, $\quad (36\,x^2)^{\frac{1}{2}} = 6\,x;\ 16^{\frac{1}{4}} = 2;\ (27\,m^3n^6)^{\frac{1}{6}} = (3\,mn^2)^{\frac{1}{2}}.$

a. By § 8 we see that taking the square root of the cube root gives the sixth root; the square root of the square root gives the 4th root; the square root of the fourth root gives the 8th root, etc. Thus,

$$(16\,b^4)^{\frac{1}{8}} = [(16\,b^4)^{\frac{1}{4}}]^{\frac{1}{2}} = (2\,b)^{\frac{1}{2}};\ (m^3n^6)^{\frac{1}{6}} = [(m^3n^6)^{\frac{1}{3}}]^{\frac{1}{2}} = (mn^2)^{\frac{1}{2}}.$$

Type form: $\quad (a^m)^{\frac{1}{mn}} = a^{\frac{1}{n}}.$

2. *If any factor of the radicand can have the desired root taken of it.*

$$(18)^{\frac{1}{2}} = (9 \times 2)^{\frac{1}{2}} = 3(2)^{\frac{1}{2}}.\quad (\S\ 66,\ 1.)$$

$$5(16\,x^5)^{\frac{1}{3}} = 5(8\,x^3 \times 2\,x^2)^{\frac{1}{3}} = 10\,x(2\,x^2)^{\frac{1}{3}}.$$

If the required root of any factor of the radicand can be taken, that factor may be removed from the radicand, and its root used as a factor of the coefficient of the radical.

b. In searching for square factors, test first for 4, then for 9, then for 25, etc.

Thus, $(252)^{\frac{1}{2}} = (4 \times 9 \times 7)^{\frac{1}{2}} = 6(7)^{\frac{1}{2}}$; $450^{\frac{1}{2}} = (9 \times 25 \times 2)^{\frac{1}{2}} = 15(2)^{\frac{1}{2}}$.

Testing for cube factors, look first for 8, then for 27, etc.

Type form: $$(a^n b)^{\frac{1}{n}} = a(b)^{\frac{1}{n}}.$$

3. *If the radicand is fractional.*

To simplify, we multiply both terms of the radicand by a factor such that we can extract the desired root of the resulting denominator.

$$(\tfrac{3}{5})^{\frac{1}{2}} = (\tfrac{3}{5} \times \tfrac{5}{5})^{\frac{1}{2}} \text{ (§ 98) } = (\tfrac{1}{25} \times 15)^{\frac{1}{2}} = \tfrac{1}{5}(15)^{\frac{1}{2}}. \quad \text{(§ 66, 1.)}$$

$$3\left(\frac{5x}{4y}\right)^{\frac{1}{3}} = 3\left(\frac{5x}{4y} \times \frac{2y^2}{2y^2}\right)^{\frac{1}{3}} = 3\left(\frac{1}{2^3y^3} \times 10xy^2\right)^{\frac{1}{3}} = \frac{3}{2y}(10xy^2)^{\frac{1}{3}}.$$

Type form: $$\left(\frac{a}{b}\right)^{\frac{1}{n}} = \frac{1}{b}(ab^{n-1})^{\frac{1}{n}}.$$

To simplify a quantity whose radicand is fractional, multiply both terms of the radicand by such a quantity as will make the denominator a perfect power of which the required root can be taken. Extract this root and write the result as a factor of the denominator of the coefficient. If possible, simplify the numerator as in 2 above.

4. *If there is an irrational quantity in its denominator.*

Simplifying such a quantity is called "rationalizing its denominator."

Thus, $$\frac{1}{6^{\frac{1}{2}}} = \frac{1}{6^{\frac{1}{2}}} \times \frac{6^{\frac{1}{2}}}{6^{\frac{1}{2}}} = \frac{6^{\frac{1}{2}}}{6}. \quad (\text{Since } 6^{\frac{1}{2}} \times 6^{\frac{1}{2}} = 6^{\frac{1}{2}+\frac{1}{2}} = 6.)$$

$$\frac{a}{b^{\frac{2}{3}}} = \frac{a}{b^{\frac{2}{3}}} \times \frac{b^{\frac{1}{3}}}{b^{\frac{1}{3}}} = \frac{ab^{\frac{1}{3}}}{b}. \quad (\text{Since } b^{\frac{2}{3}} \times b^{\frac{1}{3}} = b^{\frac{2}{3}+\frac{1}{3}} = b.)$$

$$\frac{4}{2+3^{\frac{1}{2}}} = \frac{4}{2+3^{\frac{1}{2}}} \times \frac{2-3^{\frac{1}{2}}}{2-3^{\frac{1}{2}}} = \frac{4(2-3^{\frac{1}{2}})}{4-3} = 4(2-3^{\frac{1}{2}}).$$

Verify this last result by calculating $\frac{4}{2+3^{\frac{1}{2}}}$ and the answer $4(2-3^{\frac{1}{2}})$ each to three decimal places. Notice by this simplification a long division is saved.

To rationalize a quantity with a monomial denominator, multiply both terms by a monomial which will make the denominator rational.

To rationalize a quantity with a binomial denominator, multiply both terms by the "conjugate surd" to the denominator; that is, the denominator with the sign between its terms changed.

Notice that multiplying by the conjugate surd always makes the radical signs disappear. Why?

76. Exercise in Simplifying Radical Quantities. All problems given in the radical sign notation should be immediately changed into the fractional exponent notation, then solved, and last of all changed back. *Before changing back into the radical sign notation, separate quantities with fractional exponents into two factors, one with integral exponents, and the other with proper fraction exponents.*

Thus, $5\,a^{\frac{5}{2}}b^{\frac{3}{2}} = 5\,a^2b \times a^{\frac{1}{2}}b^{\frac{1}{2}} = 5\,a^2b\sqrt{ab}.$

Solve as many as possible of the following mentally:

1. $(25\,a^3b)^{\frac{1}{2}}$.
2. $(27\,a^4b^3c^2)^{\frac{1}{2}}$.
3. $(108\,a^3b)^{\frac{1}{3}}$.
4. $(x^2y^4z^6)^{\frac{1}{4}}$.
5. $(1000)^{\frac{1}{6}}$.
6. $(1\frac{1}{16})^{\frac{1}{2}}$.
7. $\sqrt[3]{a^4x^2y^3}$.
8. $7\sqrt{396\,x}$.
9. $5\sqrt{726}$.
10. $\sqrt{\frac{1}{2}}$.
11. $\sqrt[3]{\frac{1}{2}}$.
12. $\sqrt[3]{\frac{5\,a^2}{9b}}$.
13. $\frac{3}{5}\sqrt{1\frac{2}{3}}$.
14. $\frac{a}{b}\sqrt{\frac{c^2}{d}}$.
15. $\frac{3}{45^{\frac{1}{2}}}$.
16. $\left(\frac{a^{-1}b^{-2}}{x}\right)^{\frac{1}{2}}$.
17. $\sqrt{a^2b^2+a^2c^2}$.
18. $\sqrt[3]{a^{7x}b^{3x}}$.
19. $\sqrt[5]{729\,a^6}$.
20. $\sqrt[6]{64\,a^{18}x^7}$.
21. $\sqrt{(a^2-1)(1+a)}$.

22. $\sqrt[3]{\frac{8}{9}}$. 23. $\dfrac{3}{3^{\frac{1}{2}} + 5^{\frac{1}{2}}}$. 24. $\dfrac{1}{\sqrt{5} - \sqrt{2}}$.

25. $\sqrt[4]{\dfrac{9\,a^{-2}}{36\,x^{2}}}$. 26. $\sqrt{\dfrac{a^3}{x^4} - \dfrac{a^5}{x^6}}$. 27. $\sqrt[10]{\dfrac{32}{x^{15}y^{20}}}$.

28. Find the numerical value of $\sqrt{4\,a^2b}$ when $a = 3.51$, $b = 16.81$.

a. Notice the result can be had very much more quickly by first simplifying the radical. Check result by calculating radical in original form, noting the difference in the labor of calculation.

29. Calculate $\sqrt{L^2Fm}$ when $L = 25.7$, $F = 14$, $m = 4\frac{4}{7}$.

30. Calculate area of triangle whose sides are 8, 10, and 14 (§ 16, 24), being given from a table (p. 69) that $\sqrt{6} = 2.449$.

31. Calculate the diagonal of a rectangular box from one lower corner to opposite upper corner if dimensions are 3, 4, 5 ft. and if it is known from a table (p. 69) that $\sqrt{2} = 1.414$.

32. Find u from formula $u = \sqrt{v^2 + 2\,fs}$ when $v = 3$, $f = 31\frac{5}{6}$, $s = 3$.

33. Find one side of a square equal to the sum of two squares whose sides are respectively 13 m. and 9 m., using the table of square roots.

34. Calculate $\sqrt{8^3 \times 6^3 \times 72}$, using table of square roots.

SUGGESTION. First write: $8^3 = (2^3)^3 = 2^9$; $6^3 = (2 \times 3)^3 = 2^3 \times 3^3$; $72 = 2^3 \times 3^2$.

35. Calculate $\sqrt{\dfrac{32^2 \times 4^2 \times 12^5}{8^4 \times 36}}$, using table.

77. Insertion of Coefficients under the Root Sign.

1. $5(3\,a)^{\frac{1}{2}}$.

SOLUTION. $5(3\,a)^{\frac{1}{2}} = 25^{\frac{1}{2}} \times (3\,a)^{\frac{1}{2}} = (75\,a)^{\frac{1}{2}}$. (§ 66, 1.)

2. $9(2\,m)^{\frac{1}{2}}$. 3. $3\,x(5\,x)^{\frac{1}{3}}$. *Ans.* $(135\,x^4)^{\frac{1}{3}}$.

4. $2\,y(7\,y^3)^{\frac{1}{4}}$. 5. $3\,x^2\sqrt{5\,xy}$. 6. $\frac{1}{2}\sqrt[3]{4}$.

7. $\dfrac{ax}{a-x}\sqrt{\dfrac{a^2 - x^2}{a^2x^2}}$. 8. $5\,x(25\,x^2)^{\frac{1}{n}}$. 9. $a^m\sqrt[n]{2\,a^{mn+2}}$.

78. Addition and Subtraction of Radicals. To add radical quantities they are first simplified or reduced to similar radicals.

1. Add $18^{\frac{1}{2}}$, $2\left(\frac{1}{2}\right)^{\frac{1}{2}}$, $5(64)^{\frac{1}{4}}$, and $\frac{3}{8^{\frac{1}{2}}}$.

Solution		Checking
$18^{\frac{1}{2}} = 3(2)^{\frac{1}{2}}$	(§ 75, 2.)	$\sqrt{18} = 4.24+$
$2(\frac{1}{2})^{\frac{1}{2}} = 2^{\frac{1}{2}}$	(§ 75, 3.)	$2\sqrt{.50} = 1.41+$
$5(8)^{\frac{1}{2}} = 10(2)^{\frac{1}{2}}$	(§ 75, 1, 2.)	$5\sqrt{8} = 14.14+$
$\frac{3}{8^{\frac{1}{2}}} = \frac{3}{4}(2)^{\frac{1}{2}}$	(§ 75, 4.)	$\frac{3}{\sqrt{8}} = 1.06-.$
Sum $= 14.75(2)^{\frac{1}{2}} = 20.86-.$		$20.85+$

a. By this radical simplification, which is easily and quickly performed, *four* root extractions and one long division are changed into *one* root extraction and one multiplication.

Change quantities written in the radical sign notation to the other before solving. Check some of the solutions, assigning values to the letters partly to test answers and partly for drill in root extraction. Solve as many as possible mentally.

2. $8(125)^{\frac{1}{2}} + 2(80)^{\frac{1}{2}}$.

3. $\sqrt{48\,ab^2} - b\sqrt{75\,a}$.

4. $2(\frac{1}{4})^{\frac{1}{3}} + 8\sqrt[3]{\frac{1}{32}}$.

5. $4\sqrt{128} - 5\sqrt{162} + 16\sqrt{\frac{1}{2}}$.

6. $3\sqrt{75\,a^4b} - 2\,a^2b\sqrt{\frac{12}{b}}$.

7. $3\sqrt[3]{189} - 3\sqrt[3]{875} - 7\sqrt[3]{56}$.

8. $3\,b^2(a^3c)^{\frac{1}{2}} - 2\,ab^3\left(\frac{ac}{b^2}\right)^{\frac{1}{2}}$.

9. $\frac{2}{\sqrt{2}} + \frac{3}{\sqrt{8}} - \frac{1}{\sqrt{18}}$.

10. $2\sqrt{3} - \sqrt{12} + \sqrt[4]{9}$.

11. $\sqrt[3]{\frac{1}{2}} - 2\sqrt[6]{16} + \sqrt[3]{32}$.

12. $\sqrt{\frac{1}{2}} + \sqrt{\frac{2}{9}} + \sqrt{\frac{1}{8}}$.

13. $2\sqrt{\frac{5}{3}} + \sqrt{60} - \sqrt{15} + \sqrt{\frac{3}{5}}$.

14. $3\sqrt{\frac{5}{2}} + \sqrt{40} + \sqrt{\frac{2}{5}} - \frac{1}{\sqrt{10}}$.

15. $\sqrt[4]{a^2b^8} + \sqrt[3]{a^6b^2} - \sqrt[8]{a^8b^4}$.

16. $\sqrt{2\,ax^2 - 4\,ax + 2\,a} - \sqrt{2\,ax^2 + 4\,ax + 2\,a}$.

17. From $(a-x)(a^2-x^2)^{\frac{1}{2}}$ take $a(a-x)\left(\frac{a+x}{a-x}\right)^{\frac{1}{2}}$.

18. $(54\,a^{m+6}b^3)^{\frac{1}{3}} - (16\,a^{m-3}b^6)^{\frac{1}{3}} + (2\,a^{4m+9})^{\frac{1}{3}} + (2\,c^3a^m)^{\frac{1}{3}}$.

SUGGESTION. $(54\,a^{m+6}b^3)^{\frac{1}{3}} = (27\,a^6b^3 \times 2\,a^m)^{\frac{1}{3}}$; etc.

19. The areas of three lots in the form of squares lying side by side along a road are 6, 24, and 54 sq. rd., respectively. What is their total frontage on the road? Use table of square roots, page 69.

20. The areas of three squares are 32, 8, and 18 acres respectively. Find the number of rods of fence needed to inclose all of them.

21. One square lot has an area of 75 sq. rd. and another an area of 12 sq. rd. How much more fence will one need than the other?

79. Multiplication and Division of Radicals Having a Common Index. Use the fractional exponent notation in solving.

Type form: $a^{\frac{1}{n}} \times b^{\frac{1}{n}} = (ab)^{\frac{1}{n}}$. (See § 66, 1.)

1. Multiply $4\sqrt{15}$ by $3\sqrt{35}$.

SOLUTION. $4(5\times3)^{\frac{1}{2}} \times 3(5\times7)^{\frac{1}{2}} = 12(5^2\times21)^{\frac{1}{2}} = 60(21)^{\frac{1}{2}} = 60\sqrt{21}$. *Ans.*

CHECK. $15.49+ \times 17.75+ = 274.9+ = 275-$.

2. $2(15)^{\frac{1}{2}} \times 3(5)^{\frac{1}{2}}$.

3. $8\sqrt{12} \times 3\sqrt{24}$.

4. $3(8\,a^2)^{\frac{1}{2}} \times (6\,a^3)^{\frac{1}{2}}$.

5. $\sqrt[3]{x+2} \times \sqrt[3]{x-2}$.

6. $4\sqrt{12\,a^3} \div 2\sqrt{3\,a}$.

7. $70\sqrt[3]{9} \div 7\sqrt[3]{18}$.

8. $\sqrt[4]{\frac{b}{a}} \div \sqrt[4]{\frac{a}{b}}$.

9. $\frac{3}{x}\sqrt{\frac{a^2}{x}} \times \frac{4}{3}\sqrt{\frac{x^3}{2\,a^4}}$.

10. $(6^{\frac{1}{2}} + 3^{\frac{1}{2}})(6^{\frac{1}{2}} + 5^{\frac{1}{2}})$.

11. $(9(2)^{\frac{1}{2}} - 7)(9(2)^{\frac{1}{2}} + 7)$.

12. $(7 + 3\sqrt{7})(2\sqrt{7} - 7)$.

13. $(2\sqrt{6} - \sqrt{12} - 3\sqrt{24}) \times 3\sqrt{2}$.

14. $(2\,a + 3\sqrt{x})(3\,a - 2\sqrt{x})$.

15. $(2\sqrt{8} - \sqrt{12} - 5\sqrt{18}) \div 3\sqrt{2}$.

16. $\frac{2}{5}\sqrt{\frac{4}{5}} \times \frac{1}{3}\sqrt{\frac{1}{20}}$.

17. $\frac{3}{8}\sqrt{21} \div \frac{9}{16}\sqrt{\frac{7}{20}}$.

18. $(m + n - \sqrt{mn})(\sqrt{m} + \sqrt{n})$.

19. $(9\sqrt[3]{2} - 6\sqrt[3]{6} - 3\sqrt[3]{8}) \div 3\sqrt[3]{2}$.

20. $(4\sqrt[3]{4} - 3\sqrt[3]{2})(2\sqrt[3]{6} + \sqrt[3]{9})$.

21. $(\sqrt{2} - 3\sqrt{3})^3$.

22. $(5\sqrt{8} + 6\sqrt{12} - 2\sqrt{20})(7\sqrt{2} - 3\sqrt{3} + 4\sqrt{5})$.

Rationalize the denominators of the following, evaluating the first two (see § 75, 4):

23. $\dfrac{2(5)^{\frac{1}{2}}}{5^{\frac{1}{2}}+3^{\frac{1}{2}}}$.

24. $\dfrac{15+14\sqrt{3}}{15-2\sqrt{3}}$.

25. $\dfrac{6-3\sqrt{12}}{2\sqrt{6}+\sqrt{12}}$.

26. $\dfrac{1}{\sqrt{a}+\sqrt{b}}-\dfrac{1}{\sqrt{a}-\sqrt{b}}$.

27. $\dfrac{1}{\sqrt{10}-\sqrt{2}-\sqrt{3}}$.

SUGGESTION. Make a binomial out of denominator and rationalize (§ 75, 4). Then rationalize the new denominator in a second operation.

28. $\dfrac{\sqrt{2}-\sqrt{3}+\sqrt{5}}{\sqrt{2}+\sqrt{3}-\sqrt{5}}$.

29. $\dfrac{\sqrt{x^2+a^2}+\sqrt{x^2-a^2}}{\sqrt{x^2+a^2}-\sqrt{x^2-a^2}}$.

30. If $t=2\pi\sqrt{\dfrac{l}{g}}$, and $t'=2\pi\sqrt{\dfrac{l'}{g}}$ show that $t:t'=\sqrt{l}:\sqrt{l'}$.

In the following exercises the results are to be reduced to their simplest form whenever they can be simplified.

31. Find the surface of a cube whose volume is 60; whose volume is v.

32. Find the volume of a cube whose surface is 20; whose surface is s.

33. Ratio of area of one square to a second is $\frac{1}{3}$. Find ratio of side of first to side of second. (N. B. *The areas of similar figures are proportional to the squares of the corresponding sides.*)

34. One side of an equilateral triangle is 3. Calculate its altitude, and then find ratio of one half of one side to the altitude. Find same ratio when one side is a. Why are ratios the same?

35. Find the ratio of a side to the diagonal of a square. Let side $=a$.

36. One leg of a right triangle is one half the hypotenuse R. Find the other leg, and then the area.

37. The ratio of one leg of a right triangle to another is 2. Find the ratio of each leg to the hypotenuse.

38. The ratio of one leg of a right triangle to the hypotenuse is $\frac{1}{3}$. Find the ratio of the given leg to the other.

39. Find the area of an equilateral triangle one side of which is 6.

40. Find a side and the area of a square whose diagonal is d.

41. If a regular decagon is inscribed in a circle whose radius is R, one side of the decagon is $\frac{R(\sqrt{5}-1)}{2}$. Find the ratio of the radius to one side.

42. One leg of an isosceles triangle is 6, and the base is $\sqrt{5}$. Find the altitude and area.

43. If an equilateral $\triangle$ is inscribed in a circle whose radius is R, the perpendicular distance from the center to one side is $\frac{1}{2}R$. Find ratio of a side of the triangle to the radius.

80. Multiplication and Division of Radicals which do not have a Common Index. To perform the multiplication, such quantities must be reduced to a common index. Change quantities written in radical sign notation to fractional exponent notation.

1. Multiply $18^{\frac{1}{3}}$ by $6^{\frac{1}{2}}$.

SOLUTION.

$$18^{\frac{1}{3}} \times 6^{\frac{1}{2}} = (2 \times 3^2)^{\frac{2}{6}} \times (2 \times 3)^{\frac{3}{6}} = (2^2 \times 3^4)^{\frac{1}{6}}(2^3 \times 3^3)^{\frac{1}{6}}. \quad (\S\ 66,\ 2,\ 3.)$$
$$= (2^2 \times 3^4 \times 2^3 \times 3^3)^{\frac{1}{6}} = (2^5 \times 3^7)^{\frac{1}{6}}. \quad (\S\ 66,\ 1.)$$
$$= (3^6 \times 2^5 \times 3)^{\frac{1}{6}} = 3(2^5 \times 3)^{\frac{1}{6}} = 3(96)^{\frac{1}{6}}. \quad \textit{Ans.} \quad (\S\ 75,\ 2.)$$

2. $\sqrt[3]{6} \times \sqrt{2}$. **3.** $18^{\frac{1}{2}} \times 6^{\frac{1}{3}}$. **4.** $\sqrt[5]{64} \div 2$.

5. $\sqrt[3]{2} \times \sqrt[4]{3}$. **6.** $\sqrt[3]{10} \times \sqrt[4]{2}$. **7.** $\sqrt[m]{x^p} \div \sqrt[n]{x^q}$.

8. $\sqrt[3]{6} \times \sqrt[6]{\frac{1}{3}} \div \sqrt{8}$. **9.** $a\sqrt[m]{x} \times b\sqrt[n]{y}$. **10.** $16^{\frac{1}{3}} \times 2^{\frac{1}{2}} \times 32^{\frac{5}{6}}$.

11. $2\sqrt[3]{6} \div 6\sqrt{2}$. **12.** $\sqrt[4]{\frac{a}{b}} \times \sqrt[3]{\frac{a^2}{b^2}} \div \sqrt{\frac{b}{a}}$. **13.** $\sqrt[3]{4^2} \times \sqrt[9]{8} \times \sqrt[3]{4}$.

14. $(2\sqrt{3} - \sqrt[3]{2})(2 - \sqrt{6})$. **15.** $(\sqrt{2} - \sqrt[3]{2})^3$.

16. Arrange in order of magnitude $\sqrt{\frac{4}{3}}$, $\sqrt[3]{\frac{3}{2}}$, $\sqrt[4]{\frac{7}{4}}$.

81. Powers and Roots of Radicals. Change quantities to fractional exponent notation.

1. $[3(ab)^{\frac{1}{2}}]^3$. *Ans.* $27\,ab(ab)^{\frac{1}{2}}$.

2. $(9^{\frac{1}{3}})^2$. 3. $(3\cdot 3^{\frac{1}{3}})^2$. 4. $(\sqrt{ax^2})^3$.

5. $(2\sqrt{\frac{1}{3}})^5$. 6. $\sqrt{\sqrt{a^3}}$. 7. $\sqrt[5]{\sqrt{32\times 10^5}}$.

8. $(\sqrt[6]{3c^2})^3$. 9. $(\sqrt[m]{a^2b})^{m+1}$. 10. $(\sqrt{3}-\sqrt{2})^4$.

11. Simplify $\left(\frac{\sqrt{5}-3}{2}\right)^2+3\left(\frac{\sqrt{5}-3}{2}\right)+1$.

12. Simplify $2\left(\frac{7-\sqrt{17}}{4}\right)^2-7\left(\frac{7-\sqrt{17}}{4}\right)+4$.

13. Verify that $x=3+\sqrt{2}$ is a root of equation $x^2-6x+7=0$.

14. Verify that $\frac{1}{2}(3\pm 2\sqrt{6})$ are roots of $4x^2-12x=15$.

15. Verify that $\frac{-11\pm\sqrt{21}}{10}$ are roots of $5y^2+11y+5=0$.

82. The Square Root of Binomial Surds.

We have, $(\sqrt{2}\pm\sqrt{3})^2=2\pm 2\sqrt{6}+3=5\pm 2\sqrt{6}$.

To extract the square root of the last expression by the method of § 58, the rational term, 5, must be separated into two parts such that the product of their square roots is $\sqrt{6}$. This can usually be done by inspection when the coefficients are small.

Extract the square root of the following by inspection:

1. $4+2\sqrt{3}$.

SOLUTION. $\sqrt{3+2\sqrt{3}+1}=\sqrt{3}+1$.

2. $6-2\sqrt{5}$. 3. $9-2\sqrt{14}$. 4. $23-8\sqrt{7}$.

5. $10-\sqrt{96}$. 6. $11+\sqrt{72}$. 7. $28-5\sqrt{12}$.

When the coefficients are large, mathematicians have recourse to a formula to solve these problems. Evidently the square root of any binomial surd will have the *form* $\sqrt{x}\pm\sqrt{y}$. It is clear, also, that if the square of $\sqrt{x}+\sqrt{y}$ gives the sum, $a+\sqrt{b}$, the square of $\sqrt{x}-\sqrt{y}$ will give the difference, $a-\sqrt{b}$.

8. Let $a+\sqrt{b}$ or $a-\sqrt{b}$ denote any binomial surd, and $\sqrt{x}+\sqrt{y}$ the square root of the first and $\sqrt{x}-\sqrt{y}$ the square root of the second.

Then, (1) $\sqrt{x}+\sqrt{y}=\sqrt{a+\sqrt{b}}.$

(2) $\sqrt{x}-\sqrt{y}=\sqrt{a-\sqrt{b}}$

whence, $x-y=\sqrt{a^2-b}$ (Mult. Ax., multiplying (1) by (2).)

and $\underline{x+y=a}$ (By squaring eq. (1) and squaring eq. (2), adding, and dividing through by 2.)

$\therefore 2x=a+\sqrt{a^2-b}.$ (Add. Ax.)

$2y=a-\sqrt{a^2-b}.$ (Sub. Ax.)

whence $\sqrt{x}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}.$ (Div. and Sq. Root Axs.)

$\sqrt{y}=\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$ (Div. and Sq. Root Axs.)

But $\sqrt{a\pm\sqrt{b}}=\sqrt{x}\pm\sqrt{y}.$ (By hypothesis.)

$$\therefore \sqrt{a\pm\sqrt{b}}=\sqrt{\frac{a+\sqrt{a^2-b}}{2}}\pm\sqrt{\frac{a-\sqrt{a^2-b}}{2}}.$$

9. Extract the square root of $37-20\sqrt{3}$, using the formula just found.

Solution. $37-20\sqrt{3}=37-\sqrt{1200}.$ (§ 77.)

Then, $a=37,\ b=1200,$

giving $\sqrt{37-\sqrt{1200}}=\sqrt{\frac{37+\sqrt{37^2-1200}}{2}}-\sqrt{\frac{37-\sqrt{37^2-1200}}{2}},$

$=\sqrt{\frac{37+13}{2}}-\sqrt{\frac{37-13}{2}},$

$=5-\sqrt{12}=5-2\sqrt{3}.$

Proof. $(5-2\sqrt{3})^2=25-20\sqrt{3}+4\times 3=37-20\sqrt{3}.$

10. $17+\sqrt{288}.$ 11. $56+24\sqrt{5}.$ 12. $87-12\sqrt{42}.$

13. $47-4\sqrt{99}.$ 14. $56+32\sqrt{3}.$ 15. $1+\frac{2}{3}\sqrt{2}.$

16. $x-2\sqrt{x-1}.$ 17. $4\frac{1}{3}-\frac{4}{3}\sqrt{3}.$ 18. $x+xy-2x\sqrt{y}.$

III. IMAGINARIES

83. Imaginary Quantities. We have seen that such an expression as $\sqrt{-4}$ equals neither $+2$ nor -2, since the square of each of these is $+4$. The same thing is true of the square root of *any* negative number. Because such roots are neither positive nor negative ordinary numbers, they have been called *imaginaries*, though this term is somewhat misleading.

A **pure imaginary** is an even root of a negative number, as $\sqrt{-9}$, $(-6)^{\frac{1}{4}}$. The sum of a real quantity and a pure imaginary, as $2+5\sqrt{-1}$, is called a **complex number**.

The **unit** for ordinary positive numbers is $+1$; that for negative numbers is -1; and that for pure imaginaries is $\sqrt{-1}$, or $(-1)^{\frac{1}{2}}$. The $\sqrt{-1}$ unit is often denoted by the letter i.

1. By the definitions of root and power (§ 8), we have:

$(\sqrt{-1})^2=-1$, (Since $\sqrt{-1}$ means a number which

or, $([-1]^{\frac{1}{2}})^2=-1$. multiplied by itself gives -1.)

$$(\sqrt{-1})^3=(\sqrt{-1})^2(\sqrt{-1})=-1\sqrt{-1},$$

$$(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2=-1\times-1=1.$$

Thus, $(\sqrt{-1})^2=-1$; $(\sqrt{-1})^3=-\sqrt{-1}$; $(\sqrt{-1})^4=+1$,

or, $i^2=-1$; $i^3=-i$; $i^4=+1$.

The student is advised to solve some of the following problems as they stand, and then with i substituted for $\sqrt{-1}$, until he gets accustomed to associating the two notations. (See § 75.)

2. Add $4+3\sqrt{-1}$ and $7+5\sqrt{-1}$. *Ans.* $11+8\sqrt{-1}$.

3. $(3+2\sqrt{-1})+(4-5\sqrt{-1})$.

4. $(-6-11\sqrt{-1})-(5-3\sqrt{-1})$.

5. $\sqrt{-4}+\sqrt{-49}-2\sqrt{-9}-3\sqrt{-25}$. *Ans.* $-12\sqrt{-1}$.

6. $\sqrt{-9a^4}+5\sqrt{-16a^4}$. (§ 191.) 7. $3\sqrt{-25}-\sqrt{-81}$.

8. $3(-20)^{\frac{1}{2}}-(-80)^{\frac{1}{2}}+6(-45)^{\frac{1}{2}}$. *Ans.* $20(-5)^{\frac{1}{2}}$.

9. $6(-16)^{\frac{1}{2}}-[-5-(-36)^{\frac{1}{2}}]$. 10. $(-a^2+2ab-b^2)^{\frac{1}{2}}$.

11. Multiply $6+2\sqrt{-5}$ by $3-4\sqrt{-5}$.

SOLUTION. In all multiplications, divisions, and raising to powers of imaginary and complex numbers the safe plan is to replace $\sqrt{-1}$ by i. The reason is that following the ordinary rule for multiplication *may mislead.*

Thus, $(\sqrt{-1})^2=\sqrt{-1}\times\sqrt{-1}=\sqrt{+1}=-1$ *only.*

For, by definition of power and root $(\sqrt{-1})^2=-1$.

To change $2\sqrt{-5}$ into the i notation we write $2\sqrt{5}\sqrt{-1}=2\sqrt{5}i$. Then, in the answer $18\sqrt{5}i$ is changed back to $18\sqrt{-5}$.

For the value of i^2 see 1 of this article.

$$\begin{array}{l} 6+\ \ 2\sqrt{5}i \\ \underline{3-\ \ 4\sqrt{5}i} \\ 18+\ \ 6\sqrt{5}i \\ \underline{\quad -24\sqrt{5}i-8(\sqrt{5})^2i^2} \\ 18-18\sqrt{5}i-8\times5\times-1=58-18\sqrt{-5}. \end{array}$$ *Ans.*

12. $(5+\sqrt{-3})(7+4\sqrt{-3})$.

13. $(3+\sqrt{-7})(2-\sqrt{-6})$.

14. $(4+(-5)^{\frac{1}{2}})(-6-(-8)^{\frac{1}{2}})$.

15. $(6-2\sqrt{-1})^2$.

16. $\sqrt{-36}\times\sqrt{-25}\times 5$.

17. $3(-6)^{\frac{1}{2}}\times 2(-4)^{\frac{1}{2}}\times(-7)^{\frac{1}{2}}$.

18. $(-3+\sqrt{-4})^3$.

19. $6(-3)^{\frac{1}{2}}\div 2(-5)^{\frac{1}{2}}$. *Ans.* $\frac{3}{5}(15)^{\frac{1}{2}}$.

20. $2\sqrt{-8}\div\sqrt{-2}$.

21. $2(-32)^{\frac{1}{2}}\div 2(-2)^{\frac{1}{2}}$.

22. $\dfrac{1+\sqrt{-1}}{1-\sqrt{-1}}$. (See § 75, 4.)

23. $\dfrac{\sqrt{3}-\sqrt{-5}}{\sqrt{3}-\sqrt{-2}}$.

24. Form the product of $x+a$, $x-a$, $x+a\sqrt{-1}$, and $x-a\sqrt{-1}$.

25. Form the product of $y-3$, $y-7$, $y+2+\sqrt{-7}$, and $y+2-\sqrt{-7}$.

26. Verify that both $x=-4+\sqrt{-2}$ and $x=-4-\sqrt{-2}$ satisfy the equation $x^2+8x+18=0$.

27. Verify that $x=6a\pm 2a\sqrt{-1}$ satisfy $x^2-12ax+40a^2=0$.

84. Theorems concerning Equations containing Irrationals and Imaginaries.

1. *One irrational quantity cannot equal the sum of a rational quantity and another irrational quantity.*

Let $\sqrt{m}$ and $\sqrt{n}$ denote two different irrationals, m, n, and a being rational quantities, and *suppose* $\sqrt{m} = a + \sqrt{n}$.

Then, $\quad m = a^2 + 2a\sqrt{n} + n,$ (Squaring Axiom.)

whence $\quad -2a\sqrt{n} = a^2 + n - m,$ (Sub. Ax.)

or $\quad \sqrt{n} = \dfrac{m - n - a^2}{2a}.$ (Div. Ax.)

Thus, the irrational $\sqrt{n}$ equals the rational quantity $\dfrac{m-n-a^2}{2a}$, which is contrary to § 73. Hence an irrational quantity cannot equal the sum of a rational quantity and another irrational quantity.

2. *In any equation containing rationals and irrationals, the rational part on one side equals the rational part on the other, and the irrational part on one side equals the irrational part on the other.*

GIVEN $a + \sqrt{b} = c + \sqrt{d}$, in which $\sqrt{b}$ and $\sqrt{d}$ are irrationals.

TO PROVE $\quad a = c$, and $\sqrt{b} = \sqrt{d}$.

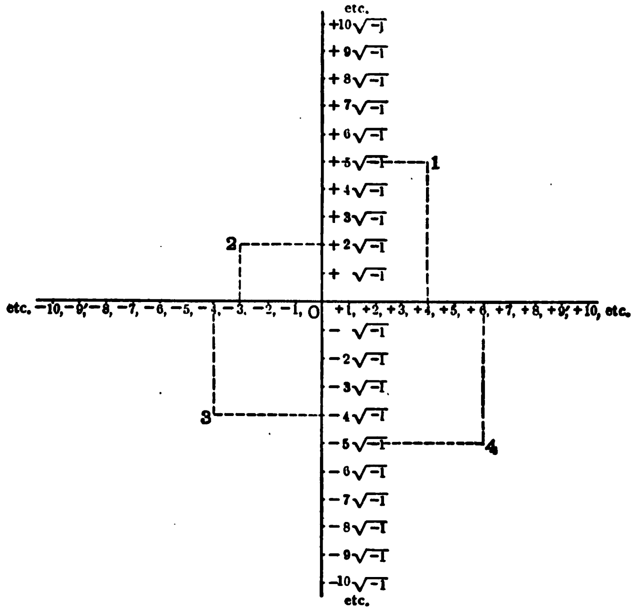
PROOF. $\quad \sqrt{b} = c - a + \sqrt{d}.$ (Sub. Ax.)

But this equation cannot exist, by 1 above, so long as $c - a$ is different from 0. Thus, $c - a$ must equal 0, whence $c = a$, and then $\sqrt{b} = \sqrt{d}$. Q.E.D.

The above theorems hold true for real and imaginary quantities as well as for rationals and irrationals.

85. Geometrical Interpretation of Imaginaries or Orthotomic Numbers. Argand's Diagram. If the word imaginary is used to describe the quantities of the preceding article with the idea that *no* real interpretation of them is possible, it is a misnomer.

It can be explained how -1 is a multiplier or operator which reverses the direction of the multiplicand. Now $\sqrt{-1} \times \sqrt{-1} = -1$. Thus $\sqrt{-1}$ used as a factor *twice* accomplishes as much as -1 used once. Accordingly it appears that $\sqrt{-1}$ can be regarded as a multiplier which turns a multiplicand number through an angle of 90°. Arbitrarily choosing as the positive direction for angles that opposite to the motion of the hands of a clock, we get the accompanying Argand diagram, in which 1, 2, 3, etc., multiplied by $\sqrt{-1}$ give $\sqrt{-1}$, $2\sqrt{-1}$, $3\sqrt{-1}$, etc., respectively; $\sqrt{-1}$, $2\sqrt{-1}$, $3\sqrt{-1}$, etc., multiplied by $\sqrt{-1}$ give -1, -2, -3, etc.; -1, -2, -3, etc., multiplied by $\sqrt{-1}$



give $-\sqrt{-1}$, $-2\sqrt{-1}$, $-3\sqrt{-1}$, etc.; and $-\sqrt{-1}$, $-2\sqrt{-1}$, $-3\sqrt{-1}$, etc., multiplied by $\sqrt{-1}$ give 1, 2, 3, etc., again. The series written vertically containing $\sqrt{-1}$ have been called *orthotomic* numbers instead of imaginaries, because they *cut at right angles* the original algebraic series.

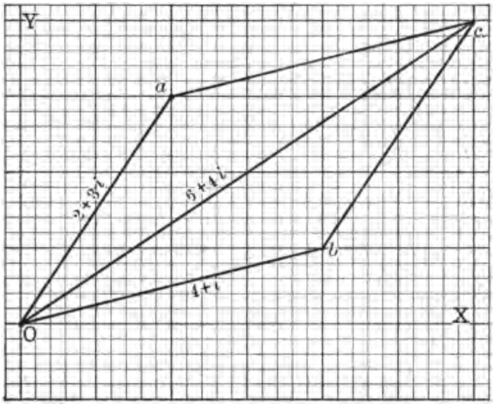
Any ordinary, orthotomic, or complex number (§ 83) represents some point in the plane of the first two. Thus, -3 represents the point three units to the left of the origin; $3\sqrt{-1}$, the point three units directly above the origin; $4+5\sqrt{-1}$ represents the point marked 1 on the diagram, whose abscissa is $+4$ and ordinate $+5$; $-4-4i$, the point marked 3, whose abscissa is -4

and ordinate -4; and so for any number. Evidently the sign conventions are the same as on the graph diagram.

We have seen complex numbers added, subtracted, multiplied, divided, and raised to powers. These operations and their results can all be given a meaning on the diagram.

86. Graphical Representation of Sums on Argand's Diagram.

1. Represent graphically the sum of $2 + 3\sqrt{-1}$ and $4 + \sqrt{-1}$, or $2 + 3i$ and $4 + i$.



SOLUTION. Point a (see diagram) is located by its coördinates (2, 3), and b by its coördinates (4, 1). Joining these points to the origin and completing the parallelogram $Oacb$, we see that c, the vertex opposite O, has for its coördinates (6, 4). Adding $(2 + 3i)$ and $(4 + i)$ as in § 83, we get $6 + 4i$. Thus,

The sum of two complex numbers is a point which is the vertex opposite the origin of the parallelogram formed on the lines from the origin to the points located by the two given complex numbers.

a. Evidently the addition of complex numbers bears a resemblance to the addition of forces or velocities in physics.

Add the following complex numbers graphically, using graph paper, and check the answer by an ordinary solution.

2. $3 + 5i$ and $5 + 2i$. 3. $10 + 2i$ and $1 + 5i$.

4. $6+3i$ and $2-4i$.
5. $-2-3i$ and $4-i$.
6. $-6-3i$ and $5-4i$.
7. $-5-4i$ and $-3-4i$.
8. $3+6i$, $5-7i$, and $-2-5i$.

Show in Ex. 8 that the sum is the same whether the third quantity is added graphically to the sum of the first two, or whether the sum point is constructed directly by adding all the real quantities for the abscissa of the answer and all the imaginary quantities for the ordinate of the answer.

In the following add the short way:

9. $2+11i$, $0+5i$, and $4-7i$.
10. $2+4i-6-11i-5-12i-2-9i$.

87. Graphical Representation of Differences on Argand's Diagram. Subtract graphically in the following problem:

1. $(6+4i)-(2+3i)$.

SOLUTION. Notice that the difference line is one side of a parallelogram of which the minuend is the diagonal and the subtrahend the other side.

Construct first the minuend point $6+4i$ and the line from the origin to it, *Oc*. (See figure for addition, § 86.) Next, construct the subtrahend point, $2+3i$, and the line to it, *Oa*. Join *a* and *c*. Last of all draw from the origin a line *Ob* equal and parallel to *ac*. The point *b* is the remainder sought.

The coördinates of this point *b* can be found very quickly by subtracting the real number in the subtrahend from that in the minuend for the abscissa of the answer, and the coefficient of the imaginary in the subtrahend from that in the minuend for the ordinate of the answer point.

Subtract in the following, locating the minuend, subtrahend, and remainder points, getting the coördinates of the latter by simple subtraction, as just described:

2. $(7+4i)-(4+i)$.
3. $(-6-3i)-(4-5i)$.
4. $(10+i)-(2-6i)$.
5. $(-5+10i)-(-7+3i)$.
6. $(3-6i)-(-2-5i)$.
7. $(3-6i)-(0+7i)$.

88. Absolute Value of a Complex Number. By the **absolute value** or **modulus** of a complex number is meant the positive square root of the sum of the squares of its coefficients. The absolute value of a complex number is evidently *the distance from the origin to the* *point located by it.* Two notations are used to denote absolute value or modulus. One uses "Mod," and the other writes two vertical lines one on each side of the number.

Thus, $\text{Mod}(x + iy) = +\sqrt{x^2 + y^2},$

and $|x + iy| = +\sqrt{x^2 + y^2}.$

89. Conjugate Complex Numbers. Two complex numbers differing only in sign of second term, as $x + iy$ and $x - iy$ are called **conjugates.** Two conjugate numbers have the same **absolute value.**

IV. EQUATIONS CONTAINING RADICALS

90. Solution of Equations involving Radicals. The solution of equations involving radicals differs from that of ordinary equations. The difference appears in the first part of the solution in which the radicals are eliminated.

1. Solve $2\,x^{\frac{5}{2}} = 64$, and verify.

SOLUTION.

$x^{\frac{5}{2}} = 32.$ (Ax. ?)

$x^{\frac{1}{2}} = 2.$ (Extracting the fifth root of equals.)

$x = 4.$ (Squaring equals, Power Axiom.)

VERIFICATION. $2(4)^{\frac{5}{2}} = 2 \times 32 \equiv 64.$

2. $3\,y^{\frac{3}{4}} = 24.$ 3. $2\sqrt{x} = 16.$ 4. $5\,y^{\frac{4}{3}} = 80.$

5. $x^{-\frac{1}{2}} = \frac{1}{5}.$ 6. $2\,z^{\frac{5}{4}} = -486.$ 7. $2\,x^{-\frac{2}{3}} = \frac{1}{18}$

8. Solve $\sqrt{x^2 - 1} - x = -1$, and verify.

SOLUTION. $(x^2 - 1)^{\frac{1}{2}} = x - 1.$ (Sub. Ax. The radical is put *alone* on one side of equation.)

$x^2 - 1 = x^2 - 2\,x + 1.$ (Squaring Ax.)

$2\,x = 2.$ (Ax. ?)

$x = 1.$ (Ax. ?)

VERIFICATION. $\sqrt{1 - 1} - 1 \equiv -1$, or $0 - 1 \equiv -1.$

a. The plan is to get the *radical* containing the unknown quantity by itself on one side of the equation. Then, by squaring (or cubing, etc.) both sides, we make the radical sign disappear. From this point the solution is like those with which we are now familiar.

9. $\sqrt{16+x^2}-x-2=0.$ 10. $(ax+2\,ab)^{\frac{1}{2}}-a=b.$

11. $\sqrt{4\,x^2-6\,x-6}=2\,x+4.$ 12. $1+2\sqrt{x}=7-\sqrt{x}.$

13. $(x^3-9\,x^2)^{\frac{1}{3}}+3=x.$ 14. $2\sqrt{x}-x=x-8\sqrt{x}.$

15. $\sqrt{12\,x-5}-\sqrt{3\,x-1}=\sqrt{27\,x-2}.$

Solution

$$12\,x-5-2\sqrt{36\,x^2-27\,x+5}+3\,x-1=27\,x-2. \quad \text{(Power Ax. § 21, II.)}$$
$$\sqrt{36\,x^2-27\,x+5}=-6\,x-2. \quad \text{(Sub. and Div. Axs.)}$$
$$36\,x^2-27\,x+5=36\,x^2+24\,x+4. \quad \text{(Squaring Ax.)}$$
$$\therefore x=\tfrac{1}{51}.$$

Verification. $9\sqrt{-\frac{1}{17}}-4\sqrt{-\frac{1}{17}}=5\sqrt{-\frac{1}{17}}.$

b. The plan in solving such equations is to place *one* radical quantity (preferably the least simple, when there are two or more) by itself on one side of the equation, and then to square, thus removing the root sign. Then the remaining radical quantity, if any, is placed by itself on one side, and both members are again squared. The most frequent error in such problems consists in failure to apply Theorems I and II in squaring. Notice that $\sqrt{12\,x-5}-\sqrt{3\,x-1}$ is a binomial, and must be squared as such.

16. $\sqrt{4\,x+5}-\sqrt{x}=\sqrt{x+3}.$

17. $\sqrt{x+3}+\sqrt{x+8}-\sqrt{4\,x+21}=0.$

18. $\sqrt{x}+\sqrt{a-\sqrt{ax+x^2}}=\sqrt{a}.$

19. $x+a=\sqrt{a^2+x\sqrt{b^2+x^2}}.$

20. Solve $v=\sqrt{\dfrac{e}{d}}$ for d; also for e.

21. Solve $v=332.4\sqrt{1+0.003665\,t}$ for t.

22. Solve $t:t'=\sqrt{l}:\sqrt{l'}$ for l; also for l'.

23. Solve $s=\sqrt{2-\sqrt{4-q}}$ for q.

24. Solve $t=\sqrt{\dfrac{2\,s-2\,vt}{g}}$ for s; also for v.

V. FUNDAMENTAL PRINCIPLES IN ALGEBRA

91. Fundamental Principles in Algebra. Students are apt to think that the principles of algebra are very numerous, and, as a consequence, hard to remember. In truth, their number is comparatively small, and they are easy to learn. The student should fix them all firmly in mind.

They may be listed as follows:

1. *The laws of precedence of operations* (§ 13).

2. *The rule for addition* based on the bookkeeper's procedure in adding debts and credits separately. The rule for subtraction is based on that for addition (§ 18).

3. *The law of signs in multiplication* (§ 18, 4). The law of signs in division follows from that for multiplication (§ 5).

4. *The associative, commutative, and distributive laws* (§ 2, § 4).

5. *The laws for exponents in simple operations* (§ 17).

6. *The nine theorems of multiplication and division* (§ 21).

7. *The fundamental principle in fractions* (§ 29). All operations in fractions except multiplication depend on this principle.

8. *The solution of equations* depends on the axioms, on simple addition, and, in the case of literal equations, on factoring the terms of the left member. (See § 38, 25.)

9. Power and root operations with polynomials depend on *Newton's theorem* and on the formulas in § 21, I, V.

10. In *radicals* there is only one fundamental principle (§ 66, 1). See also § 66, 2, 3. On § 66, 1 depend simplification of radicals, and multiplication of radicals having a common index. Nearly all the other operations in radicals depend on the latter two.

a. After memorizing the foregoing, the student should test his knowledge by reference to concrete exercises in various places. Thus, in the solution on the opposite page we see in turn: the power axiom used; theorem II, § 21, applied: $((12x-5)^{\frac{1}{2}})^2 = 12x-5$; $\sqrt{12x-5}\times\sqrt{3x-1}=\sqrt{36x^2-27x+5}$; algebraic additions performed; etc. In §§ 18–20, 2–5 above are in evidence throughout, and in § 53, 1, 5, 9 above are used; and so in general.

VI. COMMON ERRORS IN ALGEBRA

92. Such errors are due to the students' breaking of some law or convention. A list of the most common errors with an explanation of *why* they are errors will, it is hoped, help the pupils to guard against them.

I. Errors in Simple Operations

1. Probably the most common error consists in breaking the laws of precedence of operations (§ 13). These laws were made arbitrarily, but are recognized the world over, and the student must accept and memorize them.

2. To say that $a(b + c)$ equals $ab + c$, or that $\frac{ab + c}{a}$ equals $b + c$, is an error, since it breaks the distributive law (§ 4, 5).

3. To say that $a(b \times c)$ equals $ab \times ac$ is an error since, by the associative law, $a(b \times c) = abc$. Similarly, $(ab \times ac) \div a$ is not equal to bc, but to abc.

The errors in 2 and 3 are often due to the pupil's remembering *one* form when he is dealing with the other. A good plan to convince a pupil that he is in the wrong is to substitute figures for the letters. Thus, $(8 \times 6) \div 2$ equals 24, and not 12.

4. In long division and in the extraction of square and cube roots of polynomials, errors are often caused by failure to keep all the quantities (including remainders) arranged according to the descending (or ascending) powers of the leading letter or letters.

II. Errors in Theorems and Factoring

5. The error of saying that $(a + b)^2$ equals $a^2 + b^2$ or that $(a \pm b)^3$ equals $a^3 \pm b^3$ is often made. Compare the true (expanded) values with the false ones here given. The errors may arise from a failure to distinguish between $(a + b)^2$ and $(ab)^2$, and between $(a \pm b)^3$, and $(ab)^3$.

6. The commonest failing in factoring consists in not first taking monomial factors out of given polynomial quantities. As a result, the student finds himself unable to proceed to the correct solution.

7. A common mistake in factoring consists in trying to factor *prime* quantities, such as $a^2 + b^2$, $a^4 + b^4$, $a^2 \pm ab + b^2$, etc. Such quantities should be memorized as primes.

III. Errors in Fractions

8. A common mistake in fractions consists in canceling *terms* of a numerator and a denominator instead of *factors* of each. To do this breaks the fundamental principle, § 29.

Thus, it is not allowable to cancel the $3x$'s in $\frac{a+3x}{b+3x}$ and get $\frac{a}{b}$, since not the whole numerator and denominator but only parts of them have been divided by the same number. The absurdity of such cancellation can best be shown by using figures. Thus, if $\frac{17}{33}$ is written in the form $\frac{16+1}{32+1}$, the careless student may, by absurd cancellation, get $\frac{1}{3}$, $\frac{1}{2}$, or 0 as a result, according to the manner of canceling and the interpretation of the resulting expression. Once the fundamental principle is given up, any fraction can be made to have almost any value.

9. A common error in subtraction of fractions, due sometimes to ignorance and sometimes to forgetfulness, consists in failing to change *every* sign of the subtrahend numerator when subtracting it from the minuend numerator. Thus, $\frac{a}{b} - \frac{c-d}{b}$ does not become $\frac{a-c-d}{b}$ but equals $\frac{a-c+d}{b}$. A corresponding error is very often made in solving equations. The use of parentheses about polynomial subtrahends is advisable as long as there is danger of error.

10. In changing the sign of a polynomial numerator or denominator of a fraction, students often fail to change the sign of every term. The best way to change the sign of a quantity is to multiply it by -1.

11. An error sometimes made in the reduction of an improper fraction to a mixed number consists in failing to write + between a quotient and a remainder over the divisor. In arithmetic the sign + is not needed, since absence of sign in arithmetic means addition. But in algebra absence of sign means *multiplication*.

12. In the multiplication of a fraction by an integral quantity, as $\frac{a}{b} \times c$, students sometimes thoughtlessly multiply *both* terms of $\frac{a}{b}$ by the whole number. By the fundamental principle in fractions this leaves $\frac{a}{b}$ *unaltered* instead of multiplied by c. In fractional operations such mistakes are avoided by regularly writing 1 for the denominator of integral quantities. Thus, $\frac{a}{b} \times \frac{c}{1} = \frac{ac}{b}$.

13. Students who have been studying equations, a subject in which denominators are made to *disappear* by use of the multiplication axiom, often write only the numerator of the answer in addition of fractions.

IV. Errors in Equations

14. A common error in solving fractional equations is the failure to multiply every term on both sides of the equation by the lowest common denominator. This error often occurs when the right member is an integral quantity. Notice that both the multiplication axiom and the distributive law are violated.

15. Failure to verify answers permits of the occurrence of mistakes and of their repetition because they are not discovered.

16. Students sometimes fail to get and keep all the terms containing the unknown on the left side of the equation. Since to solve an equation is to get the value of the unknown expressed in known quantities only, no answer that contains an unknown is a solution of the equation.

V. Errors in Powers, Roots, and Radicals

17. The commonest mistake in extracting the square and the cube root of arithmetical numbers comes from not commencing at the decimal point to point off the given number into periods of two or three figures each, *and in failing to fill out decimal periods at the right.*

Thus, $\sqrt{.5}$ must be set down .′50′00 ; $\sqrt[3]{.3}$ must be set down .′300′000.

18. A common mistake of those who imitate and do not think is to say that $\sqrt{a} + \sqrt{b}$ is the same as $\sqrt{a+b}$, after the analogy of $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$. That $\sqrt{a} + \sqrt{b}$ is not equal to $\sqrt{a+b}$ may be shown by pointing out that $\sqrt{4+9} = \sqrt{13} = 3.6+$, whereas $\sqrt{4} + \sqrt{9} = 5$. This error is often made also in the converse form. Thus, $\sqrt{a^2+b^2}$ is said to be equal to $\sqrt{a^2} + \sqrt{b^2}$, or $a + b$; or $\sqrt{a^2+b}$ is said to be equal to $a\sqrt{b}$.

19. It is easy to fall into error in the multiplication of imaginaries. Thus, $\sqrt{-4} \times \sqrt{-9}$ appears to equal $+\sqrt{+36}$ or 6. But the negative root of $\sqrt{36}$ must be taken, since $2\sqrt{-1} \times 3\sqrt{-1} = -6$, because $(\sqrt{-1})^2 = -1$.

20. In the solution of radical equations, when both members are squared and one is a binomial, the error of II, 5 is often repeated. Thus, it is said that the square of $\sqrt{x+3} + \sqrt{5}$ is equal to $(x+3) + 5$, instead of to $x + 3 + 2\sqrt{5(x+3)} + 5$. See Theorem I, § 21.

VI. Errors in Quadratic Equations

21. The commonest error in quadratics consists in forgetting to write the sign $\pm$ before the right member when the square root of both members is taken. From the theory of radicals negative roots were excluded, and in the solution of concrete problems negative roots often have to be discarded; but in the solution of quadratic equations the student should be particular to write the $\pm$ sign, else he will get only one root.

22. In the solution of a quadratic by completing the square there are six or eight single steps and there is always danger of error in some of them.

CHAPTER V

QUADRATIC EQUATIONS*

93. A **quadratic equation** is one which, when reduced to integral form, contains the second but no higher power of the unknown quantity; or, a quadratic equation is an equation of the second degree. (See § 35.)

94. Classification of Quadratics. Determinate† quadratic equations may be classified into two kinds:

(1) Those which contain but one unknown quantity;

(2) Simultaneous quadratics.

Quadratic equations containing but one unknown are either complete or incomplete. A **complete quadratic** contains *both* the second and first powers of the unknown: as, $ax^2 + bx + c = 0$. An **incomplete** (sometimes called **pure**) **quadratic** contains only the second power of the unknown: as, $ax^2 + c = 0$.

In the earlier sections of this chapter we shall take up the more common types of quadratics, and in the last section the more unusual and difficult ones.

I. INCOMPLETE QUADRATICS

95. Solution of Incomplete Quadratics. Incomplete quadratic equations are solved in much the same way as simple equations.

1. Given $\dfrac{x}{x+1} + \dfrac{x}{x-1} = \dfrac{8}{3}$, to find its root and verify.

* The study of the chapter on logarithms, Chapter VIII, can be taken up here if desired. It follows naturally the chapter on exponents, since logarithms are exponents.

† For indeterminate equations, see § 49; also, under Diophantus, in § 115.

SOLUTION. $3x^2 - 3x + 3x^2 + 3x = 8x^2 - 8.$ (Mult. Ax.)

$6x^2 - 8x^2 = -8.$ (Sub. Ax.)

$-2x^2 = -8.$ (Simplifying left member.)

$x^2 = 4.$ (Ax. ?)

$x = \pm 2.$ (Root Ax.)

VERIFICATION. $\frac{2}{2+1} + \frac{2}{2-1} \equiv \frac{8}{3}.$ $\frac{-2}{-2+1} + \frac{-2}{-2-1} \equiv \frac{8}{3}.$

a. Nothing is gained by writing $\pm x = \pm 2$, since this equation can give only $x = +2$, or $x = -2$.

b. Notice that the degree of an equation given in the fractional form cannot be told until after it is cleared of fractions. Some equations, apparently like the one just solved, give rise, when cleared, to simple equations, others to incomplete quadratics, others still to complete quadratics.

Solve and verify in the following:

2. $\frac{4x+5}{7x-1} = \frac{x+2}{5x-3}.$

3. $\frac{x+2}{x-2} + \frac{x-2}{x+2} = \frac{26}{x^2-4}.$

4. $\frac{5}{4x} - \frac{13}{8x} = -\frac{2}{3}x.$

5. $\frac{1+x}{1-x} = \frac{x+25}{x-25}.$

6. $ax^2 + b = bx^2 + a.$

7. $a^2x^2 - b^2 + c^2x^2 = 0.$

8. $\sqrt{4x^2-1} = 6x - 4\sqrt{4x^2-1}.$

9. Solve $2a^2 + 2b^2 = 4m^2 + c^2$ for m.

Solve as many as possible of the following mentally, Exs. 10–13 for x, and 14–19 for the letter named:

10. $13x^2 - 19 = 7x^2 + 5.$

11. $(3x-7)(3x+7) = 32.$

12. $x^2 - 3 = \frac{4x^2+18}{9}.$

13. $mx^2 = a^2 - nx^2.$

14. $a^2 = b^2 + c^2 - 2cp$ for b.

15. $d = \sqrt{a^2 + b^2 + c^2}$ for c.

16. $A = \pi(R^2 - r^2)$ for r.

17. $s = \frac{1}{2}gt^2$ for t.

18. $V = \pi r^2 a$ for r.

19. $H = E^2t/R$ for E.

96. Problems involving the Solution of Incomplete Quadratics.

1. Find the area correct to one decimal place of a circle whose area is 243 sq. in., from the formula $a = \pi r^2$, using 3.1416 for π.

2. Find correct to two decimal places one side of a square whose diagonal is 72 m.

3. Find the altitude of an equilateral triangle one side of which is 6 ft.

4. Find the radius of a sphere whose surface is 600 sq. in., from the formula $s = 4\pi r^2$.

5. Find to two decimal places the radius of the base of a cylinder whose volume is 231 cu. in. and whose altitude is 6 in., from the formula $v = \pi a r^2$.

6. Find to two decimal places the radius of the base of a cone whose volume is 251 cu. cm. and whose altitude a is 22 cm., from the formula $v = \frac{1}{3}\pi a r^2$.

7. The area of a sector of a circle is 25 sq. ft. and its angle at the center is 20°. Find the radius from the formula $a = \frac{20°}{360°}\pi r^2$.

II. COMPLETE QUADRATICS

97. Solution of Complete Quadratic Equations. There are three distinct methods of solving complete quadratics with which the student is asked to familiarize himself, viz.: (1) **By factoring**; (2) **By completing the square**; (3) **By use of a formula.**

Of these the factoring method is not practical when irrationals appear in the roots; otherwise, it generally gives the quickest and easiest solution. Review § 23, III.

98. Solution of Complete Quadratics by Factoring.

1. Solve and verify in the equation $x^2 - 13x = 48$.

SOLUTION. $x^2 - 13x - 48 = 0$. (Ax.? Right member made 0.)

$(x - 16)(x + 3) = 0$. (Factoring.)

$\therefore x - 16 = 0$, whence $x = 16$ *Ans.*
$x + 3 = 0$, whence $x = -3$ *Ans.* (Setting each factor equal to 0.)

VERIFICATION. $16^2 - 13 \times 16 \equiv 48$; $(-3)^2 - 13(-3) \equiv 48$.

2. $x^2 - 11x = 42$.

3. $x^2 - 12x + 35 = 0$.

4. $x^2 - 15x = 0$.

5. $2x^2 - 11x + 12 = 0$.

6. $6x^2 - x = 15.$ 7. $4x^2 + 7x + 3 = 0.$

8. $\frac{x^2}{10} + 350 - 12x = 0.$ 9. $96x^2 = 4x + 15.$

10. $x^2 - (a + b)x + ab = 0.$ 11. $\sqrt{4v + 17} = 4 - \sqrt{v + 1}.$

12. Separate the number 266 into two factors whose difference is 5.

13. Find three consecutive numbers whose sum is equal to the product of the first two.

14. A man traveled 60 mi., and if he had traveled 3 mi. more per hour, he would have required 1 hr. less to perform the journey. At what rate did he travel?

a. Algebra is a *formal* science, built up without reference to particular problems. When it is used to solve problems which by their nature do not admit of negative answers, such answers are discarded. Thus, -15 in Ex. 14 is ignored. Only in problems which admit of negatives are negative answers retained.

15. The perimeter of a field is 360 rd. and its area is 50 acres; find its length and breadth.

16. A and B start at the same time from the same place. After $2\frac{1}{2}$ hr., A, who covers 1 km. in 3 min. less time than B, has traveled 2.5 km. more than B. How many minutes does each need to travel a kilometer?

99. Construction of Equations, their Roots being given. Converse operation to that of preceding article.

1. Construct the equation whose roots are 4 and -2.

SOLUTION. $[x - 4][x - (-2)] = [x - 4][x + 2] = x^2 - 2x - 8 = 0.$
Hence $x^2 - 2x - 8 = 0$ is the equation sought.

PROOF. $x^2 - 2x - 8 = (x - 4)(x + 2) = 0.$ (Factoring.)
Then, $x - 4 = 0$, whence $x = 4$; and $x + 2 = 0$, whence $x = -2$.

Form the equations whose roots are as follows:

2. $2, -1.$ 3. $3, 7.$ 4. $-2, 6.$ 5. $-3, -8.$ 6. a and b.

7. $\sqrt{2}$ and $-\sqrt{2}.$ 8. 25 and $-70.$ 9. $-20a$ and $30a.$

10. Make a rule to construct an equation, being given its roots.

11. Construct the equation whose roots are $2+\sqrt{3}$ and $2-\sqrt{3}$.

SOLUTION

$$[x-(2+\sqrt{3})][x-(2-\sqrt{3})]=[x-2-\sqrt{3}][x-2+\sqrt{3}]=0,$$

or, $x^2-4x+1=0.$ *Ans.* (See § 79.)

12. Construct the equation whose roots are $-3-\sqrt{7}$ and $-3+\sqrt{7}$.

100. Solution of Quadratic Equations by "Completing the Square." In order to solve an equation by this method, it is *first* reduced to the form $a^2x^2+bx=c$, if it is not already in this form.

The *second* step consists in completing the square in the left member. We regard the two terms of the equation which contain x^2 and x as corresponding to a^2+2ab in $a^2+2ab+b^2$. The problem is to find b^2 from the other two terms and add it to them. The third term is found from formula as follows:

$$(2ab \div 2\sqrt{a^2})^2=b^2, \text{ or } \left[\frac{\text{2d term}}{2\times\text{square root of 1st term}}\right]^2.$$

Solve the following equations by completing the square, and verify:

1. $3x^2-10x+8=0.$

SOLUTION

$3x^2-10x=-8.$ (Sub. Ax. Known term to right member.)

$\therefore 9x^2-30x=-24.$ (Mult. Ax. Coefficient of x^2 made a positive perfect square.)

$\left[\frac{-30x}{2\sqrt{9x^2}}\right]^2=25.$ (The second term is divided by twice the square root of the first term, and the quotient is squared.)

$\therefore 9x^2-30x+25=1.$ (Add. Ax. Result just found, 25, is added to both members; $25-24=1$.)

$\therefore 3x-5=\pm 1.$ (Root Ax. The square roots of equals are equal.)

$\therefore 3x=5+1=6, \text{ or } 5-1=4.$ (Sub. Ax.)

$\therefore x=2, \text{ or } \frac{4}{3}.$ *Ans.* (Div. Ax.)

VERIFICATION. $3\times 2^2-10\times 2+8\equiv 0.$ $3(\frac{4}{3})^2-10\times\frac{4}{3}+8\equiv 0.$

2. $x^2 - 10x - 56 = 0.$

3. $x^2 - 6x = 135.$

4. $x^2 - 28x = 60.$

5. $x^2 - 18x - 63 = 0.$

6. $x^2 - 0.4x = 23.$

7. $x^2 - 0.04x = 1.254.$

8. $x^2 - 11x = -28.$

a. Fractions can always be avoided by first multiplying the equation through by four times the coefficient of x^2, merely indicating the multiplications in the left member.

This is called the *Hindu Rule* for solving quadratics. Notice that the quantity squared and added to complete the square by this rule is always the coefficient of x in the given equation. It is not always necessary to multiply by so large a multiplier, or, often, any multiplier at all, to avoid fractions. Thus, this was not necessary in Exs. 2–7 above.

b. A simple rule which will solve any quadratic directs us first to *divide the equation through by the coefficient of x^2. Then the quantity added to complete the square is always the square of half the coefficient of x.* Explain why.

An objection to this rule is that it often gives rise to large fractions, and gives the student no chance to exercise his judgment, before he begins, as to what is the wisest thing to do, whether to multiply or divide through, and by what quantity, so as to shorten most the subsequent calculations, and avoid error. What is best to do in any given problem is learned from experience.

In the next four problems, following the Hindu Rule is the best that can be done, if fractions are to be avoided. Solve them by the rules in *a* and *b*, and compare the solutions.

9. $2x^2 + 3x = 27.$

10. $3x^2 - 13x = -10.$

11. $5x^2 - 7x = -2.$

12. $2x^2 - 5x = 42.$

13. $7x^2 + 2x = 32.$

14. $8x^2 - 22x = 21.$

15. $84x^2 + 45 = 129x.$

16. $3x + 4 = 39x^{-1}.$

17. $\frac{19}{5}x = \frac{11}{5} - x^2.$

18. $\dfrac{x+22}{3} - \dfrac{4}{x} = \dfrac{9x-6}{2}.$

19. $\dfrac{m+3}{m-2} - \dfrac{24}{m} = -2.$

20. $\dfrac{2t^2-4t-3}{2t^2-2t+3} = \dfrac{t^2-4t+2}{t^2-3t-2}.$

21. $\sqrt{5n+11} = \sqrt{3n+1} + 2.$

22. $\sqrt{2z} + 2\sqrt{2z+5} = 2\sqrt{6z+4}.$

For the verification in the following eight problems see § 81, Ex. 13, and § 83, Ex. 27.

23. $x^2 - 4x + 2 = 0.$

24. $m^2 + 5m + 5 = 0.$

25. $(x-1)(x-2)=1.$

26. $8x^2-20x=21.$

27. $x+\frac{1}{x}=\frac{4}{\sqrt{3}}.$

28. $x^2-6x+10=0.$

29. $\frac{1}{x+\frac{1}{x}}=1.$

30. $7(x+7)+\frac{7(3x+50)}{x}=0.$

31. $x^2-2ax-3a^2=0.$

32. $y^2-4by-5b^2=0.$

33. $ax^2+bx+c=0.$

SOLUTION. $4a^2x^2+4abx=-4ac.$ (Hindu Rule, § 100, *a*.)

$4a^2x^2+4abx+b^2=b^2-4ac.$ (Completing the square.)

$2ax+b=\pm\sqrt{b^2-4ac}$ (Sq. root Axiom.)

$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$ (Sub. and Div. Axs.)

VERIFICATION.

$$a\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)^2+b\left(\frac{-b+\sqrt{b^2-4ac}}{2a}\right)+c=0.$$

$$\frac{+b^2-2b\sqrt{b^2-4ac}+b^2-4ac}{4a}+\frac{-b^2+b\sqrt{b^2-4ac}}{2a}+c=0.$$

$$(2b^2-2b\sqrt{b^2-4ac}-4ac)+(-2b^2+2b\sqrt{b^2-4ac})+4ac\equiv 0.$$

34. $mx^2-2nx+p=0.$

35. $x^2-(a+b)x+ab=0.$

36. $x^2+(n+1)x+n=0.$

37. $y^2-(4a+10)y+40a=0.$

38. $9a^4b^4z^2-6a^3b^3z=b^2.$

39. $(m-n)x^2-nx=m.$

40. $bx^2-ax+a+cx^2=b-cx.$

SUGGESTION. In all such problems the terms containing x^2 must be combined into one term by factoring, and also the terms containing x, and the known terms are to be transposed to the right member, so as to reduce the equation to the form $mx^2+nx=p$.

41. $t^2+3bt=5ct+15bc.$

42. $x^2-3bx=2ax-6ab.$

43. $\frac{1}{x+a}+\frac{1}{x+b}=\frac{1}{a-b}.$

44. $\frac{my}{a-y}+m=\frac{n(y+2m)}{m+n}.$

45. Make a rule for the solution of quadratic equations by completing the square covering all the different steps. Include literal equations.

46. $s+\sqrt{1-s^2}=1.2.$

47. $2s-\sqrt{1-s^2}=.25.$

48. $t+\frac{1}{t}=24.$

49. $2c+\sqrt{1+c^2}=17.$

In Ex. 50 solve for s and in Ex. 51 for c.

50. $2s + \sqrt{s^2 - 1} = m.$

51. $c + \sqrt{1 - c^2} = p.$

52. The diagonals of a rectangle are $2x^2 + x$ and $x^2 + 13x - 35$. Find both values of x and the diagonals.

53. If a train had traveled 5 mi. an hour faster, it would have needed one hour less to cover 150 mi. Find the rate of the train.

54. The perimeters of two similar polygons are expressed by $2x^2 + 3x$ and $3x + 5$ and two corresponding sides by .03 and .05. Find values of x first and then find the perimeters.

55. Two launches race over a course of 12 mi. The first goes $7\frac{1}{2}$ mi. an hour. The other, having a start of 10 min., passes over the first half of the course with a certain speed, but increases its speed over the second half by 2 mi. per hour, winning the race by 1 min. What is the speed of the second launch in the first half of the course?

101. Solution of Quadratic Equations by Means of a Formula. Solve the quadratic $ax^2 + bx + c = 0$, regarding it as a type of all quadratics, getting (§ 100, Ex. 33)

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

To solve any particular quadratic, we reduce it to the form $ax^2 + bx + c = 0$, if not already in this form, and then substitute the values of a, b, and c from the given problem in the formula just found.

The pupil should solve the equation $ax^2 + bx + c = 0$ repeatedly, until he becomes familiar with the process and remembers the answer, and can solve the equation mentally, writing down the answer formula.

1. Solve $3x^2 - 7x = 6$ and verify.

SOLUTION. Writing the given equation underneath the type form we have

$$ax^2 + bx + c = 0,$$
$$3x^2 - 7x - 6 = 0.$$

We see now that in this particular problem $a = 3$, $b = -7$, $c = -6$. Substituting these values for a, b, and c in

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a},$$

we get
$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - (4 \times 3 \times -6)}}{2 \times 3},$$

or
$$x = \frac{7 \pm \sqrt{49 + 72}}{6} = \frac{7 \pm 11}{6} = 3, \text{ or } -\frac{2}{3}.$$

VERIFICATION. $3(3)^2 - 7 \times 3 \equiv 6.$ $3(-\frac{2}{3})^2 - 7 \times (-\frac{2}{3}) \equiv 6.$

2. $9x^2 + 18x + 8 = 0.$

3. $7r^2 + 20r + 12 = 0.$

4. $12x^2 - 11x + 2 = 0.$

5. $6x^2 + 17x = 3.$

6. $x^2 + (n+1)x + n = 0.$

7. $5x^2 + 9x + 12 = 4x^2 + x.$

8. $12 - 51x = 36 + 6x^2.$

9. $10 + 15T + T^2 = 26T.$

10. $(m-n)x^2 - nx = m.$

11. $y^2 - 1 = 4ay - a^2.$

12. $\frac{4x}{x-1} - \frac{x+3}{x} = 4.$

13. $\frac{x}{9(x-1)} = \frac{x-2}{6}.$

14. $\frac{x}{a} + \frac{a}{x} = \frac{2}{a}.$

15. $\sqrt{a+x} - \sqrt{2x} = \frac{2a}{\sqrt{a+x}}.$

16. Find two consecutive numbers the sum of whose reciprocals is $\frac{9}{20}$.

17. A rectangular park 56 rods long and 16 rods wide was surrounded by a street of uniform width containing 4 acres. What was the width of the street?

18. The perimeter of a field is a rods and it contains b acres. What are its dimensions?

19. A merchant sold a coat for $11, and gained a number of per cent equal to the number of dollars the coat cost him. What was his per cent of gain?

20. A boatman rowed 8 miles up a stream and back in 3 hours. If the velocity of the current was 2 miles an hour, what was his rate of rowing in still water?

III. HIGHER EQUATIONS CONTAINING ONE UNKNOWN AND QUADRATICS

102. Higher Equations solved as Quadratics. By *higher* equations are meant equations of a degree higher than the second. (See § 93.)

1. Solve $x^4 - 5x^2 = -4$.

SOLUTION. $4x^4 - 20x^2 = -16$. (Hindu Rule.)

$4x^4 - 20x^2 + 25 = 9$. (Completing Square, Ax. ?)

$2x^2 - 5 = \pm 3$. (Ax. ?)

$x^2 = 4$, or 1. (Ax. ?)

$x = \pm 2$ or ± 1. (Ax. ?)

VERIFICATION. $(\pm 2)^4 - 5(\pm 2)^2 \equiv -4$. $(\pm 1)^4 - 5(\pm 1)^2 \equiv -4$.

2. $x^4 - 13x^2 + 36 = 0$.

3. $x^4 - 18x^2 + 32 = 0$.

4. $z^4 - 4z^2 = 45$.

5. $9u^4 + 5u^2 = 4$.

6. $t^4 - 24t^2 = 81$.

7. $7y^4 + y^2 = 350$.

8. $x^6 - 7x^3 = 8$.

SUGGESTION. Solve first for x^3 by completing the square, and then for x, getting two values for x.

9. $y^6 + 37y^3 = 1728$.

10. $m^6 - 19m^3 = 216$.

11. $x^2 + \frac{1}{x^2} = \frac{17}{4}$.

12. $\frac{x^2 - 1}{9} + \frac{1}{x^2} = 1$.

13. $16\left(x^2 + \frac{1}{x^2}\right) = 257$.

14. $x^2 - 4 + \frac{72}{x^2 - 8} = 21$.

15. $x^2 + \frac{a^2b^2}{x^2} = a^2 + b^2$.

16. $(x^2+2)^2 + 198 = 29(x^2+2)$.

103. Construction of Higher Equations.

In § 98 we saw how to solve quadratics by the factoring method, and in § 99 how to construct an equation when its roots are given. Equations of higher degree are solved and constructed in a similar manner.

1. Construct the equation whose roots are 3, 4, and -5.

SOLUTION. $[x - 3][x - 4][x - (-5)] = x^3 - 2x^2 - 23x + 60 = 0$. *Ans.*

Form the equations whose roots are as follows:

2. $2, 1, -3.$ 3. $1, 2, 4.$ 4. $-5, 4, 0.$ 5. $-6, -7, 1.$

6. $1, 2, 3, 2.$ 7. $1, \frac{1}{2}, \frac{1}{3}, 2.$ 8. $a, b, c.$ 9. $a, -b, c, -d.$

10. Make a rule to construct an equation of any degree when its roots are given.

11. $1, -1, 2, -2, 3.$ 12. $\frac{1}{2}, \frac{1}{3}, -\frac{1}{4}, 0.$

13. $2 \pm \sqrt{3}, 1, -3.$ 14. $1, -3, 3 \pm \sqrt{-1}.$

15. $2 \pm \sqrt{-2}, 1 \pm \sqrt{-1}.$ 16. $\pm \sqrt{3}, 2 \pm \sqrt{-3}, 0.$

104. Number of Roots in an Equation. The preceding article makes it clear that the number of roots in an equation equals the degree of the equation, but some of the roots are very often imaginary.

105. Solution of Equations of Higher Degrees by Factoring. Reverse operation. Before beginning this article the pupil should review §§ 23, 24, and 98.

1. Solve $x^3 = a^3$ and verify.

SOLUTION. $x^3 - a^3 = (x-a)(x^2+ax+a^2) = 0.$ (§ 23.)

$x - a = 0, x = a.$ *Ans.* (Setting each factor equal to 0.)

$x^2 + ax + a^2 = 0,$

$4x^2 + 4ax = -4a^2,$ (Hindu rule.)

$4x^2 + 4ax + a^2 = -3a^2.$ (Completing square, Ax. ?)

$2x + a = \pm a\sqrt{-3}.$ (Ax. ?)

$x = \dfrac{-a \pm a\sqrt{-3}}{2}.$ *Ans.* (Ax. ?)

VERIFICATION. $a^3 = a^3$; $\left(\dfrac{-a \pm a\sqrt{-3}}{2}\right)^3 = a^3.$

The solution shows that there are *three* cube roots of a^3 instead of one.

Find all the roots of the following by factoring method, setting each prime factor equal to zero, and solving prime factors of the second degree equal to zero by the method of § 100:

2. $x^3 = 8.$ 3. $x^3 + 64 = 0.$ 4. $x^4 = a^4.$

5. $x^6 - 1 = 0.$ 6. $216 + 125x^3 = 0.$ 7. $x^6 - 64 = 0.$

8. $6x^3+17x^2+12x=0$.
9. $x^3+x^2+x+1=0$.
10. $x^3+4x^2-4x-16=0$.
11. $x^4-2x^3+32x=16x^2$.
12. $x^4-13x^2+40=0$.
13. $y^3+3y^2+3y+1=0$.
14. $x^4-16=0$.
15. $x^3-19x-30=0$. (§ 23, VII.)
16. $x^3-7x+6=0$.
17. $12x^3-28x^2+17x-3=0$.
18. $2x^4-3x^3-4x^2+3x+2=0$.
19. $4x^3-12x^2+9x-1=0$.
20. $6x^4+25x^3+5x^2-60x-36=0$.
21. $x^4+a^4=0$. SUGGESTION. Factor as in § 23, VI, and treat $\sqrt{2}$ as if it were a rational quantity, solving by § 100.
22. $x^4+64=0$.
23. $4x^4+81=0$.
24. $4y^4+625=0$.

From § 103 it is clear that if we have any means of knowing any of the roots of an equation, we can remove these roots from the equation by dividing its left member (right member having been made 0) by the binomial differences between x and such roots.

25. $x^3+4x^2+6x-11=0$.

SOLUTION. If this equation has a rational root it is either 1 or 11. Substituting $x=1$ in the equation we see it is satisfied. Hence $x-1$ is a factor of the left member by the divisibility theorem, § 23, VII.

Dividing $x^3+4x^2+6x-11$ by $x-1$ we get $x^2+5x+11$.

Thus, $x^2+5x+11=0$ is the equation containing the remaining roots. Solve this equation.

26. $x^4+2x^3-x=30$.
27. $x^4-6x^3+27x=10$.

IV. SIMULTANEOUS QUADRATICS

106. Simultaneous Quadratics. Degree of Simultaneous Equations. The **degree** of an integral equation is the greatest exponent, or sum of exponents, of the unknown or unknowns in any one term.

Notice that $xy=5$ is a quadratic, as well as $x^2+y^2=16$. We may think of x^2, xy, y^2 as denoting *areas*, while x and y terms denote lengths only.

107. Classes of Simultaneous Quadratics. The variety of problems and solutions in simultaneous quadratics is large. We shall next study examples of three or four of the most common cases in simultaneous quadratic equations, and then in the last section of this chapter examples of the more difficult kinds of problems.

108. Simultaneous Equations, one of which is Quadratic and the other Simple, can always be solved by *substitution* method (§ 42).

1. Given $\begin{cases}(1)\ 3x^2+2y^2-y=48,\\(2)\ 2x-y+3=0.\end{cases}$ to find x and y.

SOLUTION. $(2_1)\ 2x=y-3;\ (2_2)\ x=\dfrac{y-3}{2}.$ (Axs. ?)

(1) $$3\left(\frac{y-3}{2}\right)^2+2y^2-y=48. \qquad (§\ 42.)$$

$$\frac{3y^2-18y+27}{4}+2y^2-y=48.$$

$$3y^2-18y+27+8y^2-4y=192. \qquad \text{(Ax. ?)}$$

$$11y^2-22y-165=0. \qquad \text{(Ax. ?)}$$

$$y^2-2y-15=0. \qquad \text{(Ax. ?)}$$

$$(y-5)(y+3)=0. \qquad (§\ 98.)$$

$\therefore y=5.$ *Ans.*, Or $y=-3.$ *Ans.*

$(2_2)\ x=\dfrac{5-3}{2}=1.$ *Ans.* $(2_2)\ x=\dfrac{-3-3}{2}=-3.$ *Ans.*

VERIFICATION. (1) $3\times1^2+2\times5^2-5\equiv48.$

$$3(-3)^2+2(-3)^2-(-3)\equiv48.$$

2. $\begin{cases}x^2+2y^2=34,\\x+y=7.\end{cases}$

3. $\begin{cases}x+4y=23,\\x^2+3xy=54.\end{cases}$

4. $\begin{cases}3x-y=11,\\3x^2-y^2=47.\end{cases}$

5. $\begin{cases}3x(y+1)=12,\\3x=2y.\end{cases}$

6. $\begin{cases}x^2+xy+y^2=343,\\2x-y=21.\end{cases}$

7. $\begin{cases}2m^2-3mn+n^2=14,\\2m-n=7.\end{cases}$

8. $\begin{cases}2u-3v=11,\\\dfrac{4}{u}-\dfrac{3}{v}=-\dfrac{17}{7}.\end{cases}$

9. $\begin{cases}u+\dfrac{1}{3}=\dfrac{2u+v}{3},\\\dfrac{u+v}{u}=\dfrac{4u-v}{2}.\end{cases}$

10. $\begin{cases}ax+by=p,\\cx^2+dy^2=q.\end{cases}$

11. $\begin{cases}s=ut+\frac{1}{2}gt^2,\\v=u+gt.\end{cases}$ u and t unknowns.

109. Simultaneous Quadratic Equations solved by Squaring, Adding or Subtracting, and Extracting the Square Root. — This method is applicable only for certain combinations of the given coefficients.

1. Given $\begin{cases} (1) \quad 9x^2 + 4y^2 = 394, \\ (2) \quad 3x - 2y = 2, \end{cases}$ to find x and y.

SOLUTION

$$\begin{array}{lllll}
(1) & 9x^2 & & +4y^2 & = 394 \\
(2_1) & 9x^2 & -12xy & +4y^2 & = 4 \quad \text{(Ax. ?)} \\
\hline
(3) & & 12xy & & = 390 \quad \text{(Ax. ?)} \\
(1) & 9x^2 & & +4y^2 & = 394 \\
\hline
(4) & 9x^2 & +12xy & +4y^2 & = 784 \quad \text{(Ax. ?)} \\
(4_1) & 3x & & +2y & = \pm 28 \quad \text{(Ax. ?)} \\
(2) & 3x & & -2y & = 2 \\
\hline
 & 6x & & & = 30, \text{ or } -26 \quad \text{(Ax. ?)} \\
 & x & & & = 5, \text{ or } -\tfrac{13}{3}. \ \textit{Ans.} \quad \text{(Ax. ?)} \\
 & & & 4y & = 26, \text{ or } -30. \quad \text{(Ax. ?)} \\
 & & & y & = 6\tfrac{1}{2}, \text{ or } -7\tfrac{1}{2}. \ \textit{Ans.} \quad \text{(Ax. ?)}
\end{array}$$

VERIFICATION. (1) $9(5)^2+4(6.5)^2 \equiv 394$, or (2) $9(-\tfrac{13}{3})^2+4(-\tfrac{15}{2})^2 \equiv 394$.

a. The general plan of this method is to calculate the value of the sum, or difference, or both of two quantities by processes like that just given. Thus, in the preceding problem we *had given* the value of $3x - 2y$, and we proceeded to calculate the value of $3x + 2y$. From these two it was easy to calculate x and y themselves. In Ex. 2, calculate the value of $x - y$.

2. $\begin{cases} x^2 + y^2 = 178, \\ x + y = 16. \end{cases}$

3. $\begin{cases} x^2 + y^2 = 89, \\ xy = 40. \end{cases}$

4. $\begin{cases} xy = 518, \\ x + y = 51. \end{cases}$

5. $\begin{cases} x+y=9, \\ x^2+xy+y^2=61. \end{cases}$

6. $\begin{cases} 2x + y = 7, \\ 4x^2 + y^2 = 25. \end{cases}$

7. $\begin{cases} 2x + 5y = 19, \\ 4x^2+25y^2=241. \end{cases}$

8. $\begin{cases} x^3 + y^3 = 243, \\ x + y = 9. \end{cases}$

SUGGESTION. Begin by dividing Eq. (1) by (2), getting (3). Then use (3) and (2).

9. $\begin{cases} x^3 - y^3 = 152, \\ x - y = 2. \end{cases}$

10. $\begin{cases} x^3 + y^3 = 37, \\ x^2 - xy + y^2 = 37. \end{cases}$

11. $\begin{cases} x^4 + x^2y^2 + y^4 = 21, \\ x^2 + xy + y^2 = 7. \end{cases}$

12. $\begin{cases} x^2 + xy = 77, \\ xy + y^2 = 44. \end{cases}$

13. $\begin{cases} 9x^2 - 13xy + 9y^2 = 101, \\ 3xy = 12. \end{cases}$

14. $\begin{cases} x^2 + xy + y^2 = 7, \\ x^2 - xy + y^2 = 19. \end{cases}$

15. $\begin{cases} u^2 + 2uv + 4v^2 = 21, \\ u - 2v = 9. \end{cases}$

16. $\begin{cases} x^2 - 3xy + y^2 = -1, \\ 3x^2 - xy + 3y^2 = 13. \end{cases}$

17. $\begin{cases} 36x^2 + 64y^2 = 85, \\ 6x + 8y = 11. \end{cases}$

18. $\begin{cases} 3m - 2n = 11, \\ 9m^2 + 4n^2 = 241. \end{cases}$

V. DIFFICULT QUADRATICS

110. Solutions of More Difficult Quadratics. The variety of forms taken by quadratic equations, or equations quadratic in character, especially simultaneous quadratics, is large. To become expert in the solution of such equations requires considerable experience. The practical value of such skill is not very great, since there is little call in the applications of mathematics for the solution of such equations. If the student desires to prepare himself as quickly as possible for solving such problems, his best plan is to try to get a grasp of the various *types* of problems, *methods* of solutions, and *special devices* that are commonly met with in textbooks. For these reasons, in what follows, solutions and suggestions are offered freely, but the number of exercises given is small.

111. Solutions of Quadratics containing one Unknown Quantity. Classes of Problems.

I. *Equations quadratic in character containing fractional exponents.*

1. Solve $x^{\frac{1}{2}} + 4x^{\frac{1}{4}} = 21$, and verify.

SOLUTION. Notice that $x^{\frac{1}{2}} = (x^{\frac{1}{4}})^2$. Thus, this equation is a complete quadratic since it contains a certain unknown in one term, and the *square* of this unknown in another term, besides a term not containing the unknown.

(In this problem the unknown is a simple expression. Later in this section we shall see a function of the unknown in one term and the square of this function in another term giving a quadratic.)

If we put $x^{\frac{1}{4}} = z$, then $x^{\frac{1}{2}} = z^2$. Making these substitutions, the given equation becomes $z^2 + 4z = 21$. Thus, we see that this equation can easily be solved by any of the methods of solving a quadratic.

Solving the equation by factoring (§ 98), we get

$$(x^{\frac{1}{4}} + 7)(x^{\frac{1}{4}} - 3) = 0.$$

$$\therefore x^{\frac{1}{4}} = -7, \text{ and } x^{\frac{1}{4}} = 3. \qquad \text{(Ax.?)}$$

$$\therefore x = 2401, \text{ and } x = 81. \qquad \text{(Ax.?)}$$

VERIFICATION. $(2401)^{\frac{1}{2}} + 4(2401)^{\frac{1}{4}} \neq 21$; $(81)^{\frac{1}{2}} + 4(81)^{\frac{1}{4}} \equiv 21$.

Thus, $x = 2401$ is *not* a root of this equation if positive square roots are taken, while 81 is a root.

2. $x + 4x^{\frac{1}{2}} = 5$.
3. $x^{\frac{1}{2}} - 5x^{\frac{1}{4}} = -6$.
4. $x^3 - 24x^{\frac{3}{2}} = 81$.
5. $7\sqrt[3]{z^2} + \sqrt[3]{z} = 350$.
6. $12x^{-\frac{3}{4}} - x^{-\frac{3}{8}} = \frac{1}{16}$.
7. $ax^{2n} + bx^n = c$.
8. $x^{-\frac{1}{2}} - 6x^{\frac{1}{2}} = 1$.
9. $x^{\frac{3}{2}} - 4x = 5x^{\frac{1}{2}}$. SUGGESTION. Divide through by $x^{\frac{1}{2}}$.

II. *Equations solved first for functions of the unknown.*

1. Solve $x^2 = 9 + \sqrt{x^2 - 3}$, and verify.

SOLUTION. In such problems the radical quantity is taken as the unknown at first. By adding to both members just the right quantity the equation can be put in the quadratic form. Thus, we can write here

$$x^2 - 3 - \sqrt{x^2 - 3} = 6. \quad \text{(Sub. Ax.)}$$

To see more clearly that this is quadratic in character, put $\sqrt{x^2 - 3} \equiv z$. Then $x^2 - 3 = z^2$, and the equation becomes $z^2 - z = 6$.

Then $(z - 3)(z + 2) = 0$. (Factoring.)

Whence $z = 3$, and $z = -2$.

Now, we have $\sqrt{x^2 - 3} = 3$, or -2,

whence $x^2 = 12$, or 7,

and $x = 2\sqrt{3}$, or $\sqrt{7}$. *Ans.*

VERIFICATION. $(2\sqrt{3})^2 \equiv 9 + \sqrt{(2\sqrt{3})^2 - 3}$; $(\sqrt{7})^2 \neq 9 + \sqrt{(\sqrt{7})^2 - 3}$.

Thus, we see the root $\sqrt{7}$ does *not* verify in the original equation with the sign of the radical taken positive.

2. $x^2 - 6 = 2\sqrt{x^2 + 9}$.
3. $x^2 = \sqrt{x^2 - 7} + 13$.
4. $x^2 - 7x + \sqrt{x^2 - 7x + 18} = 24$.
5. $x^2 - x - \sqrt{x^2 - x + 4} = 8$.
6. $2\sqrt{x^2 - 3x + 11} = x^2 - 3x + 8$.
7. $\sqrt{x^2 - 8x + 31} + (x - 4)^2 = 5$.

8. $x^4 + x^3 - 4x^2 + x + 1 = 0.$

SOLUTION. $x^2 + x - 4 + \frac{1}{x} + \frac{1}{x^2} = 0,$ (Div. Ax.)

or, $\left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 4 = 0,$ (Arranging.)

Then, $\left(x^2 + 2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) - 6 = 0,$ (Adding 2 in one term and subtracting it in another to make first quantity a perfect square).

Now put $z \equiv x + \frac{1}{x}$. Then, we have

$$z^2 + z - 6 = 0,$$

Whence $z = +2$ or -3.

We now have $x + \frac{1}{x} = 2$, and $x + \frac{1}{x} = -3$.

Solving these equations gives the four values of x sought.

9. $16y^2 + \frac{1}{y^2} = 28$ 10. $\left(x + \frac{8}{x}\right)^2 + x = 42 - \frac{8}{x}.$

11. $\frac{1}{(2x-4)^2} = \frac{1}{8} + \frac{2}{(2x-4)^4}.$ SUGGESTION. Put $z \equiv (2x-4)^2$.

12. $\frac{x^2+x}{2} + \frac{2}{x^2+x} = 2.$ 13. $\frac{x^2+1}{x-1} - \frac{4(x-1)}{x^2+1} = \frac{21}{5}.$

SUGGESTION to Ex. 13. Put $z \equiv \frac{x^2+1}{x-1}.$

14. $x^4 - 2x^3 - 2x^2 + 3x - 108 = 0.$

SOLUTION. To find a function that can be used we extract the square root of the left member (§ 58). This gives $x^2 - x$ for the root, and $-3x^2 + 3x - 108$ for the remainder. Hence the given equation can be written

$$(x^2 - x)^2 - 3(x^2 - x) - 108 = 0,$$

which equation can now be solved for $x^2 - x$ first, and then for x.

15. $x^4 - 6x^3 + 5x^2 + 12x - 60 = 0.$ 16. $4x^4 + \frac{x}{2} = 4x^3 + 1.$

III. *Tartaglia's solution of the cubic by means of a quadratic equation.*

1. Given $x^3 - 6x - 9 = 0$, to find x.

SOLUTION. $(y+z)^3 - 6(y+z) - 9 = 0.$ ($y + z$ put for x.)

$y^3 + z^3 + 3yz(y+z) - 6(y+z) - 9 = 0.$ ($y + z$ cubed and arranged.)

$(y^3 + z^3) + (3yz - 6)(y+z) - 9 = 0.$ (Factoring middle terms.)

Since there are now *two* unknowns, and there was only one, x, at first, we may choose a relation between y and z expressed in an equation. Now, we see if we put $3yz-6=0$, we greatly simplify the last equation, all of the terms except y^3+z^3-9 dropping out. Then we have

$$(1)\ 3yz-6=0, \text{ or } z=\frac{2}{y}, \text{ and } (2)\ y^3+z^3-9=0.$$

Substituting $z=\frac{2}{y}$ in (2), this equation gives (using factoring method), $y^3=8$, or 1, whence $y=2$, or 1.

Then, $z=1$, or 2, and $x=y+z=3$. *Ans.*

The equation containing the other two roots can be found by dividing x^3-6x-9 by $x-3$ and setting the quotient equal to 0. (§ 105, 25).

2. $x^3-18x-35=0$. 3. $x^3-10x+24=0$.

a. A curious fact about Tartaglia's solution is that it gives a usable result only when one of the roots of the given equation is real and the other two are *imaginary*. When all the roots are real, resort may be had to the divisibility theorem (§ 23) to find the first of the three roots.

112. Solution of Harder Simultaneous Quadratics. Besides the two most common cases, or kinds of problems, in simultaneous quadratics described in §§ 108, 109, a number of others are also usually given. The number of *devices* used in solving these equations one time and another is very large.

I. *When each equation is of the form* $ax^2+by^2=c$.

1. (1) $9x^2+25y^2=225$; (2) $x^2+y^2=9$.

2. (1) $4x^2-9z^2=36$; (2) $x^2+z^2=15\frac{1}{2}$.

II. *When all the terms containing unknowns in both equations are of the second degree,* often called the **homogeneous case.**

1. $\begin{cases}(1)\ 4x^2-xy-y^2=-16,\\(2)\ 3xy+y^2=28.\end{cases}$

Solution. To solve these equations we follow the very unusual course of eliminating the *known* terms. The process may seem a little more natural if we insert z^2 as the literal part in the right members and then eliminate z^2. Notice this would make *all* the terms in the equations homogeneous.

$$
\begin{array}{lll}
(1_1) & 28x^2 - 7xy - 7y^2 = -112z^2 & \text{(Eq. (1)} \times 7.) \\
(2_1) & \qquad\quad 12xy + 4y^2 = \quad 112z^2 & \text{(Eq. (2)} \times 4.) \\
\hline
(3) & 28x^2 + 5xy - 3y^2 = \qquad 0 & \text{(Ax. ?)}
\end{array}
$$

$$(4x - y)(7x + 3y) = 0. \qquad \text{(Factoring.)}$$

Then, $4x - y = 0,$ whence $y = 4x;$

and, $7x + 3y = 0,$ whence $y = -\frac{7}{3}x.$

Now, (2) $3x(4x) + (4x)^2 = 28.$ (Putting $4x$ for y.)

$\therefore x = \pm 1,$ (Since $28x^2 = 28$.)

and $y = 4 \times \pm 1 = \pm 4$ *Ans.* (Since $y = 4x$.)

Also, (2) $3x(-\frac{7}{3}x) + (-\frac{7}{3}x)^2 = 28.$ (Putting $y = -\frac{7}{3}x$.)

$\therefore x = \pm\sqrt{-18} = \pm 3\sqrt{-2},$ (Since $-\frac{14}{9}x^2 = 28$.)

and $y = -\frac{7}{3} \times \pm 3\sqrt{-2} = \mp 7\sqrt{-2}$ *Ans.* (Since $y = -\frac{7}{3}x$.)

a. In the verification notice the upper signs go together and the lower signs together. Thus, $x = +3\sqrt{-2}$ goes with $y = -7\sqrt{-2}$.

2. $\begin{cases} 3x^2 - 4xy + 2y^2 = 17, \\ y^2 - x^2 = 16. \end{cases}$ 3. $\begin{cases} x^2 + xy + 2y^2 = 44, \\ 2x^2 - xy + y^2 = 16. \end{cases}$

III. *When each equation contains only one second degree power or product* (as x^2, y^2, or xy) *which is the same in all of the equations.* To solve such problems eliminate the second degree term and then proceed as in § 108.

1. $\begin{cases} 2y^2 + x = 30, \\ y^2 + 3x - 4y = 26. \end{cases}$ 2. $\begin{cases} x - \dfrac{1}{y} = a, \\ y - \dfrac{1}{x} = \dfrac{1}{a}. \end{cases}$

IV. *When the two equations are symmetrical* (§ 62) *in x and y.*

1. $\begin{cases} x^4 + y^4 = 17, \\ x + y = 3. \end{cases}$ Suggestion. Put $\begin{cases} x = u + v, \\ y = u - v. \end{cases}$

2. $\begin{cases} x^4 + y^4 = 82, \\ x - y = 2. \end{cases}$ 3. $\begin{cases} x^5 + y^5 = 33, \\ x + y = 3. \end{cases}$

V. *Solution first for* **function** *of the two unknowns, followed later by solution for the unknowns.*

1. $\begin{cases} (1)\ x^2 + 4y^2 - 15(x + 2y) + 80 = 0. \\ (2)\ xy = 6. \end{cases}$

SOLUTION. $(1_1)\ x^2 \quad + 4y^2 - 15(x + 2y) + 80 = 0$

$(2_1)\quad 4xy \qquad\qquad - 24 = 0$

$(3)\ (x + 2y)^2 \quad - 15(x + 2y) + 56 = 0$ (Ax. ?)

We now solve for the *function*, $x + 2y$, using factoring method.

$(3_1)\ [x + 2y - 7][x + 2y - 8] = 0,$

$(4)\ x + 2y = 7$, and $(5)\ x + 2y = 8$.

The solution is continued by combining (4) and (2), from which values of x and y are found. Then, in the same way, (5) and (2) are combined and other values of x and y found.

2. $\begin{cases} x^2 + y^2 + x + y = 18, \\ xy = 6. \end{cases}$

3. $\begin{cases} x^2 + 2xy + y + 3x = 73, \\ y^2 + 3y + x = 44. \end{cases}$

4. $\begin{cases} 9x^2 + y^2 - 63x + 21y = -86, \\ xy = 4. \end{cases}$

5. $\begin{cases} x^2 + y^2 + 4\sqrt{x^2 + y^2} = 45, \\ xy = 12. \end{cases}$

VI. *Equations containing three unknowns.*

1. $x^2 + y^2 = a;\ x^2 + z^2 = b;\ y^2 + z^2 = c.$

2. $\begin{cases} x^2 + y^2 + z^2 = 30, \\ xy + yz + zx = 17, \\ x - y - z = 2. \end{cases}$

SUGGESTION. Add 2 times (2) to (1) and extract square root of result, getting (4). Then combine (4) and (3).

3. $\begin{cases} x^2 + y^2 + z^2 = 84, \\ x + y + z = 14, \\ xy = 8. \end{cases}$

SUGGESTION. Add $2xy = 16$ to (1), and substitute $z = 14 - (x + y)$ in the resulting equation. Then solve first for $x + y$.

The General Case. Since all the problems so far undertaken have been solved, the student may get the impression that any problem in simultaneous quadratics can be solved. To show that this is not the case, we undertake to solve what appears to be a simple problem, much simpler than many we have had.

Given (1) $x^2 + y = a$, (2) $x + y^2 = b$, to find x and y.

From (2), $\qquad x = b - y^2.$

Then (1) $\qquad (b - y^2)^2 + y = a,$

or $\qquad b^2 - 2by^2 + y^4 + y = a.$

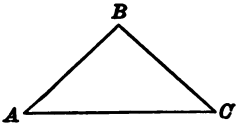
But this equation of the 4th degree cannot be solved generally.

See *American Mathematical Monthly*, Vol. X, p. 192, and Vol. VI, p. 13.

VI. THEOREMS OF GEOMETRY

113. Theorems from Geometry. *For convenience of reference* we give here a *list* of the theorems of geometry which will be used in the remaining parts of the book. If a review of geometry is desired, the pupil should be asked to prepare anew the geometrical proofs of these propositions before he uses them in the solution of algebraical problems. If a review of geometry is not desired, the geometrical exercises themselves, of course, may be omitted.

1. *The angles opposite the equal sides of an isosceles triangle are equal, and, conversely, in any triangle, the sides opposite two equal angles are equal.*



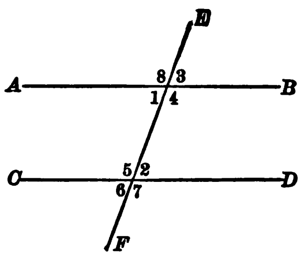
Thus, if $AB = BC$, then angle $C =$ angle A, and if angle $C =$ angle A, then $AB = BC$.

2. *An equilateral triangle is equiangular, and conversely.*

3. *If two straight lines intersect, the vertical angles formed are equal.*

4. *If two parallel lines are cut by a transversal, alternate interior angles are equal, and corresponding angles are equal, and conversely.*

Thus, if AB is parallel to CD, $\angle 1 = \angle 2$, $\angle 4 = \angle 5$, $\angle 3 = \angle 2$, etc.



5. *The opposite sides and angles of a parallelogram are equal, and the diagonals bisect each other.*

6. *The sum of the interior angles of a triangle equals two right angles.*

7. *The sum of the interior angles of an n-gon (n-sided polygon) equals $2n - 4$ right angles.*

8. *A radius perpendicular to a chord bisects it and also the subtended arc, and conversely.*

9. *Parallel chords intercept equal arcs on the circumference.*

10. *An inscribed angle, or an angle formed by a tangent and a chord, is measured by one half of the intercepted arc.*

11. *An angle formed by two chords intersecting within a circle is measured by one half of the sum of the intercepted arcs, and one formed by two secants meeting without a circle, or a secant and a tangent, or two tangents, is measured by one half of the difference of the intercepted arcs.*

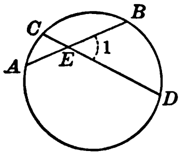


FIG. 1

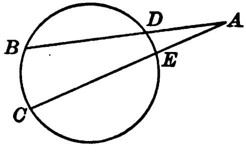


FIG. 2

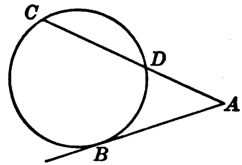
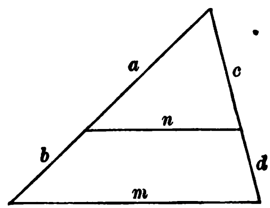


FIG. 3

Thus, $\angle 1$ (fig. 1) is measured by $\frac{1}{2}(AC + BD)$. Angle A (fig. 2) is measured by $\frac{1}{2}(BC - DE)$. Angle A (fig. 3) is measured by $\frac{1}{2}(BC - BD)$.

12. *If a line is drawn parallel to the base of a triangle, it divides the other two sides into segments which are proportional to each other and to the whole sides, and conversely.*



Thus, if n is parallel to m,

$$\frac{a}{b} = \frac{c}{d}; \quad \frac{a}{c} = \frac{b}{d}; \quad \frac{a+b}{b} = \frac{c+d}{d};$$

$$\frac{a+b}{a} = \frac{c+d}{c}.$$

13. *The bisector of an angle of a triangle, whether interior or exterior, divides the opposite side into segments which are proportional to the other two sides.*

Thus, if ABC is a triangle and CD and CD' are the bisectors of the interior and exterior angle C, we have

$$\frac{AD}{DB} = \frac{AC}{CB}; \quad \frac{AD'}{D'B} = \frac{AC}{CB}.$$

14. *Two triangles are similar (that is, their corresponding angles are equal and their corresponding sides are proportional):*

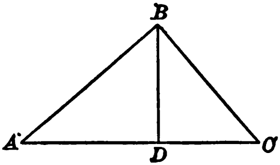
(*1*) *If they are mutually equiangular, or*

(*2*) *If an angle of one is equal to an angle of the other and the sides about these angles are proportional, or*

(*3*) *If their corresponding sides are proportional.*

15. *Two right triangles are similar if they have merely one acute angle of one triangle equal to an acute angle of the other.*

16. *If a perpendicular is dropped from the vertex of the right angle on the hypotenuse of a right-angled triangle: (1) the perpendicular is a mean proportional between the segments of the hypotenuse; (2) each leg is a mean proportional between the whole hypotenuse and its adjacent segment. Thus,*



$$\frac{AD}{DB}=\frac{DB}{DC}, \text{ and } \frac{AC}{AB}=\frac{AB}{AD}.$$

17. *If two chords intersect in a circle, their segments are reciprocally proportional. Or* (*fig. 1 under 11 above*),

$$\frac{AE}{CE}=\frac{DE}{BE}.$$

18. *If two secants intersect without a circle, they are reciprocally proportional to their external segments. Or* (*fig. 2 under 11*),

$$\frac{AB}{AC}=\frac{AE}{AD}.$$

19. *If a tangent and a secant intersect, the tangent is a mean proportional between the whole secant and its external segment. Thus* (*fig. 3, under 11*), $\frac{AC}{AB}=\frac{AB}{AD}.$

20. *Pythagorean Prop.—The square on the hypotenuse of a right triangle equals the sum of the squares on the other two sides.*

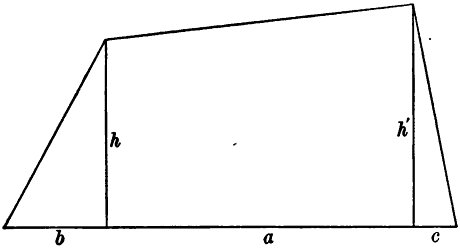
21. *One side of a regular hexagon inscribed in a circle is equal to the radius.*

22. *One side of a regular decagon inscribed in a circle of radius r is x in the proportion* $\frac{r}{x}=\frac{x}{r-x}.$

23. *Theorems about the* **areas** *of a rectangle, parallelogram, triangle, trapezoid, circle, etc.*

VII. GENERAL EXERCISE

114. General Exercise Including both Linear and Quadratic Equations and Exercises involving Geometrical Theorems.



1. To find the area of a trapezium.

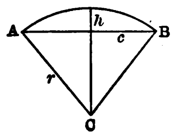
If A = area of trapezium, and a, b, c, h, h' represent measured lengths as indicated on the diagram, show that

$$A = \frac{1}{2}[a(h+h') + bh + ch'].$$

Calculate the area of a trapezium in which $a = 10$ ft., $b = 4$ ft., $c = 2$ ft., $h = 7$ ft., $h' = 11$ ft.

2. To find the area of a segment of a circle included between an arc and its chord.

Let v = number of degrees in arc AB,
c = length of chord AB,
h = sagitta,
r = radius of circle,
A = area of segment.



SUGGESTION. The area of the whole circle $= \pi r^2$, and the area of any sector is such a part of the whole circle as its angle $v°$ is of 360°. From the area of the sector must be taken the area of the triangle ABC to get the area of the segment. Thus,

$$A = \frac{v\pi r^2}{360} - \frac{c}{2}(r - h).$$

Find the area of the segment of a circle whose radius is 10 in. if the angle of the segment is 39°, its chord 6.68 in., and its sagitta 0.57 in.

3. If S = the number of right angles in the sum of the interior angles of a polygon of n sides, by the theorem of the preceding article we have the formula, $S = 2n - 4$.

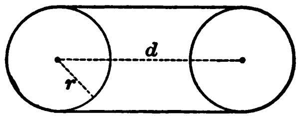
With this formula find the sum of the interior angles of polygons of 3, 4, 7, 11, 25 sides respectively.

4. If $A =$ the number of right angles in one angle of an equiangular polygon of n sides, by the preceding exercise we have the formula, $A = \frac{2n - 4}{n}$.

Find from this formula one angle of equiangular polygons of 3, 4, 5, 6, 8, 10, 12, 30, 100 sides respectively. Change the result in each case to degrees.

5. If v is the number of degrees in the vertical angle of an isosceles triangle, make a formula giving the value of b, one of the base angles of the triangle, in degrees. Find b when $v = 25°$.

6. Show that the formula for the length l of a belt passing round two equal pulleys whose radii are r ft. and the distance between whose centers is d ft., is $l = 2\pi r + 2d$.



Find l when $r = \frac{1}{2}$ and $d = 3\frac{1}{2}$.

NOTE. Pupils should search for practical formulas, like that in Ex. 6, and bring them to class, showing their use.

7. A moving picture film, 120 ft. long, is made up of a number of small equal-sized pictures. If these pictures were .1 inch less in length on the film, there would be 720 more of them. How long is each small picture?

8. Make a formula for the area S of a regular polygon of n sides one of whose sides is s and whose apothem is a. Find S when $n = 9$, $s = 1.5$, $a = 2.07$ in.

9. One of the base angles of an isosceles triangle is three times as great as the vertical angle. Find the number of degrees in each.

10. Two angles are supplementary. One has $d - 1$ degrees and the other has $6(d + 1)$ degrees. How many degrees are there in each?

11. How many sides has an equiangular polygon four of whose angles equal seven right angles?

12. Two parallel chords equally distant from the center of a circle intercept arcs whose sum is $\frac{2}{15}$ of the remaining parts of

the circumference. Find the number of degrees in each part of the circumference.

13. Two lines intersect so that $2n - 1$ degrees represents the size of one of the angles of the two pairs of vertical angles and $3n + 6$ the number of degrees in one of the angles of the other pair. Find the number of degrees in each of the four angles.

14. If two lines are cut by a transversal so that $2n - 5$ is the number of degrees in one of the interior angles and $10n - 50$ the sum of all four interior angles, find the number of degrees in each of the eight angles formed by the transversal with the two parallel lines.

15. The perimeter of a parallelogram is 360 rd. and the ratio of two adjacent sides is 2 : 5. How many rods are there in each side?

16. Two adjacent angles of a parallelogram are $2n + 10$ and $3n - 20$ degrees. How many degrees are there in each angle?

17. An inscribed angle is subtended by $2n - 10$ degrees and the rest of the circumference is denoted by $5n + 20$ degrees. How many degrees are there in the angle?

18. The arcs intercepted between two vertical angles formed by two chords which intersect within a circle are denoted by $3y - 16$ degrees and $2y + 7$ degrees. The remaining arcs of the circumference together equal $7y - 15$ degrees. How many degrees are there in the angle between the chords?

19. If A = total area, V = volume, r = radius of base, h = altitude, s = slant height, the formulas

for cylinder are $V = \pi r^2 h;\ A = 2\pi r^2 + 2\pi r h.$

for cone are $V = \frac{1}{3}\pi r^2 h;\ A = \pi r^2 + \pi r s.$

for sphere are $V = \frac{4}{3}\pi r^3;\ A = 4\pi r^2.$

Solve these equations for r; solve those that contain h for h.

Many of the results of engineering experience are put in the shape of formulas. The handling of these formulas is a simple application of algebra.

20. To find A, the aréa in square feet which the cross section of a chimney ought to have to carry the smoke, if h is the height of the chimney and F is the total number of pounds of coal burned in an hour, we have

$$A = \frac{.06\,F}{\sqrt{h}}.$$

What should be the cross section area of a chimney 60 ft. high to carry smoke of 500 lb. of coal burned in one hour?

21. To find the resistance in pounds of a locomotive and train for different speeds.

Let R = resistance in pounds per ton of 2000 lb.,
V = speed in miles per hour.

Then, $R = 3 + \frac{V}{6}.$

Find the resistance in pounds for every 2000 lb. in the weight of the whole train when it moves 36 miles an hour.

22. To find the grade resistance to the movement of a train.

Let G = grade in feet per mile,
T = weight of the train in 2000 lb. tons,
R = resistance in pounds.
Then, $R = 0.3788\,GT.$
Find R when $G = 8$, $T = 900$.

23. To find a safe load that may be put on a "pile" so that the pile will not sink under it.

Let W = safe load in pounds,
w = weight of the hammer of the pile driver used,
h = number of feet the hammer falls,
k = number of inches pile goes in at each blow, the head of the pile being in good condition.

(The "factor of safety" used in the following formula is 6, that is, the load will be only $\frac{1}{6}$ of what the pile would hold and not sink.)

Then, $$W = \frac{2\,wh}{k+1}.$$

Find W when $h = 30$ ft., $w = 2000$ lb., $k = .5$ in.

24. To find the number of bolts that should be put at the end of the cylinder of a steam engine for safety.

Let $N =$ number of bolts,
$D =$ diameter of the steam cylinder in inches,
$d =$ diameter of the bolts used in inches,
$p =$ the pressure of the steam in pounds.

Then, $$N = \frac{p}{2400}\left(\frac{D}{d}\right)^2.$$

Calculate N when $D = 36$, $p = 100$, $d = 1\frac{1}{4}$.

25. Two candles are of the same length. The one is consumed uniformly in 4 hr. and the other in 5 hr. If the candles are lighted at the same time, when will one be three times as long as the other?

26. A, B, and C bought a ship. A paid for $\frac{a}{b}$ of it, B for $\frac{m}{n}$ of it, and C paid $\$p$ for the remainder. How many dollars did A and B pay respectively?

27. If I should buy goods at a price 20 % higher than I paid and sell them for the same sum, I should gain 25 % less. What per cent did I gain?

28. A's income is $\frac{a}{b}$ of B's income. A's outgo equals $\frac{m}{n}$ of B's income, B's outgo equals $\frac{p}{q}$ of A's income. What is the ratio of their savings?

29. A dealer who buys milk at m cents a quart and sells it at n cents a quart makes a profit of p per cent. How much water per quart has he mixed with it?

30. If a monopoly trust buys a business having c dollars capital and paying d per cent dividends, how much water has been put in the capital stock if it lowers the dividend rate by p per cent and yet raises actual profits to n fold what they were?

31. At what time between a and $a+1$ o'clock is the minute hand midway between 12 and the hour hand? When is the hour

hand after a o'clock midway between b o'clock and the minute hand?

32. A teacher looks at his watch when leaving school at noon. When he comes back he finds that the hour and minute hands have changed places. What time was it when he left?

33. An army, whose length from van to rear is a mile, moves forward. An officer is sent from the rear to the van and is required to present himself to the rear again when the rear has reached the point where the van was when the army began to move. How far did the officer travel, if he did not stop at any time?

34. A, B, and C start from the same point at the same time, A going north at 3 miles per hour, B west at 4 miles per hour, and C east at 5 miles per hour. B at the end of two hours starts at such an angle as to intersect A. (1) How many hours after the starting will B intersect A, and (2) how many hours after starting must C turn and go northwesterly so as to intersect A and B when they meet?

35. By selling a horse for n dollars I gain p per cent. At what price should I sell the horse and wait r days, money being worth m per cent, to gain q per cent? Solve by both true and bank discount.

36. A, B, and C walk at rates a, b, and c $(a < b < c)$ per hour. They start from Washington at m, n, and p o'clock respectively $(m < n < p)$. When B overtakes A he is ordered by A to go back and meet C. (1) How long after A starts will A and B meet? (2) How long after A starts will B and C meet?

37. A and B run a race. A, who runs slower than B by a miles in b hours, starts first by c minutes, and they get to the n milestone together. What are their rates of running?

38. In still water a tug goes 6 miles an hour less when towing a barge than when alone. Having drawn the barge 30 miles up a stream whose current runs one mile an hour, the tug returns alone and completes the journey in $12\frac{8}{11}$ hours. What is the rate of the tug in still water?

115. Historical Notes. Though the Greeks were eminently successful in perfecting elementary geometry, very little progress was made by them in the study of either arithmetic or algebra. They studied arithmetic, strange to say, from the standpoint of geometry, representing numbers by the lengths of lines. One of their notations for numbers used the first ten letters in their alphabet to denote 1 to 10 ($\alpha = 1$, $\beta = 2$, etc.), the next eight letters to denote multiples of 10 from 20 to 90, and the last nine letters to denote the hundreds. The fact that they used letters instead of special characters for the first ten numbers probably explains in part why they made little progress in algebra.

Diophantus, of Alexandria, Egypt, is recognized as the first writer on algebra worthy of the name. It is supposed that he was of Greek descent, but he wrote (about 350 A.D.) long after the Greek learning had developed. In the first part of his "*Arithmetica*" he solved both simple and quadratic determinate equations, but he did not accept negative or surd answers for the latter. Most of his book dealt with the solution of indeterminate equations of the second degree, in which analysis he showed great skill. As a rule, no reference is made to such equations in elementary algebras. We know as little of the life of Diophantus as we do of Euclid's life. The epitaph of Diophantus is often given as a problem in algebra.

Chronologically, the next writer was **Aryabhatta**, who lived in India about 500 A.D. The arithmetical and algebraic parts of his book consisted merely of rules written in verse. The next Hindu writer was **Brahmagupta**, who lived about a century later than Aryabhatta. His algebra, also written in verse, included the solution of equations, simple and quadratic, and some indeterminate problems, all written out in full in words. These writers used credits and debts to illustrate positive and negative numbers. Our present method of representing numbers by means of figures is supposed to have originated in India about this time. Calculations made without the Arabic notation were largely performed with a wire frame, or abacus, or its equivalent. Thus, the Arabic notation is arithmetic without wires.

During the eighth century, by order of the Caliph Haroun Al-Raschid (the caliph of "The Arabian Nights"), the Hindu works on arithmetic and algebra were translated into Arabic, along with many other scientific books. In this way algebra came to be studied by a considerable number of scholars in Arabia. Prominent among the Arabian writers on algebra was **Al-Chwarizmi**, who solved quadratic equations geometrically.

In the fourth part of his book Al-Chwarizmi attempted to prove the theorems $\sqrt{a^2b}=a\sqrt{b}$, and $\sqrt{a}\sqrt{b}=\sqrt{ab}$ (see §§ 75, 66). This book is especially interesting because from it the Europeans got their knowledge of algebra and of the Arabic notation for numbers.

In the twelfth century appeared the third Hindu writer on algebra, — **Bhaskàra**. His algebra showed a great improvement over the earlier books, using abbreviations for words, and being almost symbolic. In various ways he tried to sugarcoat his treatment to make it interesting. Thus, he gave the following problem:

"The square root of half the number of bees in a swarm have flown out upon a jessamine bush; $\frac{8}{9}$ of the whole swarm have remained behind. One female bee flies about a male that is buzzing within a lotus-flower into which he was allured in the night by its sweet odor, but in which he is now imprisoned. Tell me the number of bees." (*Ans.* 72.)

His arithmetic contained a clear statement of the so-called Arabic notation for numbers, including the 0. It took several hundred years for the world to learn the use and importance of 0 in the Arabic notation. The work of Bhaskara was known to the Arabs as soon as it was published.

Algebra and the Arabic notation for numbers reached Europe chiefly through **Leonardo Fibonacci**, the son of an Italian merchant who represented his country in Barbary, receiving his education there. He became acquainted with the Arabic system of numeration and with the Arabic works on algebra. On his return to Italy, 1202 A.D., he published his work "Liber Abaci," in which he set forth the advantages of the Arabic system over the old

Roman notation with the letters I, V, X, etc., still in use among his countrymen. His treatment was taken from Al-Chwarizmi's and the other Arab works on the subject, and was written out in full in words. It had a wide circulation and helped materially in spreading a knowledge of algebra among Europeans.

Three or four centuries elapsed after the first introduction of algebra into Europe before European mathematicians took hold of the subject with the idea of perfecting it. In the sixteenth century two great results were accomplished: first, the perfecting and establishing of the symbolic notation; and second, the solving of the cubic equation.

In the discovery and publication of the solution of the cubic four persons took part. Of these Tartaglia is altogether the most interesting personality.

Tartaglia (real name Niccola Fontano) was born in Brescia, Italy, in the year 1500 A.D. In 1512 Brescia was taken by the French and many of the inhabitants were massacred in the cathedral. The boy was left for dead; but his mother found him, and nursed him back to health. His injury, however, made him a stammerer, and gave rise to the name Tartaglia, which means stammerer. His mother taught him to read and write. Being highly gifted mathematically, he was chosen to the chair of mathematics at Venice before he was thirty-five years old. Later he became an authority on the subject of gunnery, which was then interesting the European world.

Before this time, in 1505, **Scipione Ferro** had either discovered a way of solving cubics of the form $x^3 + mx = n$, or had found it in an Arabian work. Ferro explained the method to his pupil **Fiori** (or Floridus). In those days it was the custom to keep such discoveries secret in order to be able to vanquish rivals in intellectual combats and to attract students to the school of the discoverer.

About 1530 Tartaglia made known the fact that he was in possession of a method for solving a cubic of the form $x^3+px^2=q$. Fiori, hearing of it, announced that he also had a method for solving difficult problems. Tartaglia then challenged Fiori to a

Tartaglia (1500–1559)

public contest to take place Feb. 22, 1535. Following the custom then in vogue, stakes were deposited with a notary with the understanding that whoever could solve the most problems out of a collection of thirty propounded by the other, was to get the stakes. Tartaglia suspecting that Fiori's problems would depend on the solution of cubic equations, set to work to solve equations of the form $x^3 + mx = n$, and succeeded in accomplishing his purpose just ten days before the date of the algebraical duel. When the contest took place, Tartaglia solved all his opponent's problems in two hours, and by reason of his success won much fame. By 1541 he had mastered the general solution of the cubic. (See § 111, III.)

Cardan was much interested in the contest between Tartaglia and Fiori, and as he had begun writing a book on mathematics, he coaxed Tartaglia to explain his solution to him. Under the most solemn promises by Cardan to make no use of the knowledge, Tartaglia finally gave him the solution desired. But Cardan, breaking faith, some time later published the solution in his *Ars Magna*, the third earliest printed book on algebra, which appeared in 1545. This accounts for the fact that until very recently all algebras called the solution Cardan's.

Cardan, it seems, as soon as he learned the solution, taught it to his students, one of whom, **Ferrari**, succeeded in reducing the solution of the biquadratic, or equation of the fourth degree, to that of a cubic, and in this way solved the biquadratic.

After the discovery of how to solve cubic and biquadratic equations, mathematicians turned their attention to the solution of equations of still higher degrees, but all failed. Finally **Niels Henrik Abel** (1802–1829) of Norway succeeded in showing that the general solution of the fifth and higher degree equations cannot be expressed in terms of radicals.

For further matters of interest in the history of algebra, the student is referred to such works as Ball's and Cajori's histories of mathematics.

PART II. APPLICATIONS AND THEORY

CHAPTER VI

GRAPHS

I. SIMPLE GRAPHS

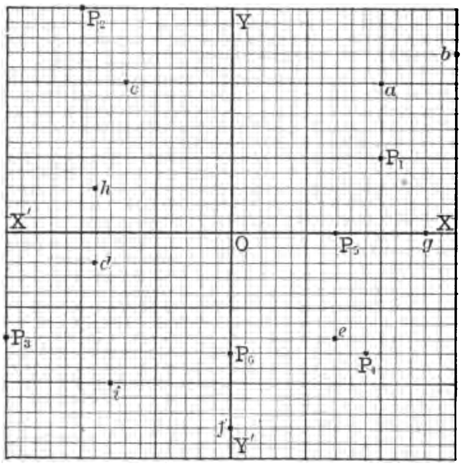
116. Graph Paper. Unit of Measure. Graph paper is paper very accurately ruled into little squares or parallelograms. Most graph paper has the centimeter (0.4 in. nearly) as the unit of measure, and divides each square centimeter into twenty-five little squares. Since 10 mm. make 1 cm., one side of the little square is thus 2 mm. in length.

117. Standard Reference Lines. Points on graph paper are located by reference to two perpendicular lines called *axes.*

118. Axes. Two centimeter lines on the graph paper are made heavier than the others and designated as the *axes*; the horizontal one as the X axis, and the vertical one as the Y axis. The point where these axes cross is called the *origin.* Other points are located by saying they are so many units (centimeters or millimeters) to the right or left of the Y axis, and so many units above or below the X axis. On the graph diagram distances measured to the right from the Y axis are positive, and those to the left are negative; those measured upward from the X axis are positive, and those downward are negative.

119. Location of Points. The distance to the right or left from the axis of Y to a point is called its **abscissa**, or x distance; the distance from the X axis to the point is called its **ordinate**, or y distance. The abscissa and ordinate of a point taken together are called the **coördinates** of the point.

1. In the figure, the abscissa of P_1 is $+2$ and the ordinate $+1$; the coördinates of P_2 are $x = -2$, $y = 3$; the coördinates of P_3 are $x = -3$, $y = -1.4$; of P_4 are $x = 1.8, y = -1.6$; of P_5 are $x = 1.4$, $y = 0$; of P_6 are $x = 0$, $y = -1.6$. For the rules for signs used see the preceding article.



2. Write the coördinates of the points located by the letters *a*, *b*, *c*, *d*, *e*, *f*, *g*, *h*, *i*, on the diagram.

3. On a piece of squared paper draw two heavy lines over two centimeter lines for axes as in the diagram just given. Now locate the following points each by a dot on the diagram, writing *a* beside the dot of the first point located, *b* beside the second, and so on.

a. $(x = 3, y = 1)$.
b. $(x = +3, y = +3)$.
c. $(x = -2, y = 1)$.
d. $(x = -1, y = 2)$.
e. $(x = 2.4, y = -1.6)$.
f. $(x = -1.2, y = -1.8)$.
g. $(x = -.6, y = 2.2)$.
h. $(x = 0, y = -2.4)$.
i. $(x = -5, y = 0)$.
j. $(x = -4.1, y = -1.6)$.

4. Make another diagram with axes and locate the following points, understanding that the first number inside the parenthesis gives the value of x or the abscissa, and the second number the value of y or the ordinate.

a. (2, 2).
b. (2, 6).
c. $(-3, 4.6)$.
d. $(-2, -4.8)$.
e. $(-3, 7.4)$.
f. $(-3, -8.1)$.
g. $(-4.9, 0)$.
h. $(7.3, -2.6)$.
i. (0, 0).
j. $(0, -6.5)$.
k. $(12, -14)$.
l. (5.5, 0).

120. Graphs. If a series of points is "plotted" (that is, located and marked on a diagram), representing values of a quantity y that changes as another quantity x changes, and these points are joined by a running line, this line is called the **graph** of the law or data that determined the points.

Often in books graphs are found constructed from statistical data. Such data can come from a great variety of sources. Thus, writers on history and economics use graphs to show to the eye quickly the changes in population, expenditures, and production. Scientists use them to show the laws of nature, engineers to show the working of machinery, and business houses to show the changes in prices, cost of production, sales, etc. In short, graphs have a wide range of uses, and the student should learn to construct and read them readily. In science and engineering graphs are often constructed mechanically, as by the thermograph, barograph, anemograph, etc.

1. **Statistical Graph.** The table gives the number of survivors at different ages out of 100,000 particular persons alive at age ten:

Age	Survivors	Age	Survivors	Age	Survivors
10	100,000	40	78,106	70	38,569
15	96,285	45	74,173	75	26,237
20	92,637	50	69,804	80	14,474
25	89,032	55	64,563	85	5,485
30	85,441	60	57,917	90	847
35	81,822	65	49,341	95	3

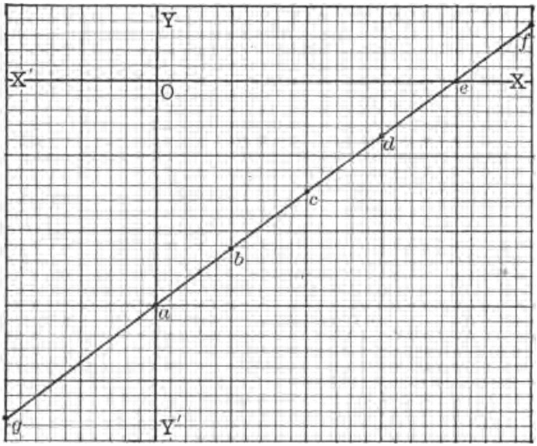
Let 1 cm. on Y axis represent 10,000 persons, and 1 cm. on X axis ten years. Then coördinates of first point are $x = 1$, $y = 10$.

2. **Graph of an Equation.** In the equation, $y = 3x + 5$, we see that y is a function of x, that is, depends on x for its value. As x changes in value, y also changes correspondingly.

We see that for every value we assign to x there will be a corresponding value for y. Such sets of corresponding values of x and y can be taken for the coördinates of points. If these points are connected with a running line, we have a graph. Evidently any equation, as $y = x^2 + 4x - 6$, will have a graph to correspond to it.

121. Graphs of Simple Equations or Equations of the First Degree (§ 35).

1. Construct the graph of the equation $3x - 4y = 12$.



TABLE

$3x - 4y = 12$

(x, y)	PT.
$(0, -3)$	a
$(1, -2.25)$	b
$(2, -1.5)$	c
$(3, -.75)$	d
$(4, 0)$	e
$(5, .75)$	f
$(-2, -4.5)$	g

EXPLANATION of how the values in the table are found. If $x = 0$ in the given equation, $3x - 4y = 12$, the equation reduces to $-4y = 12$, whence $y = -3$, as given in the table opposite a. If $x = 1$, $3x - 4y = 12$ becomes $3 - 4y = 12$, whence, transposing 3, $-4y = 9$, whence, $y = -2.25$, which is given opposite b; and in the same manner the other values of y are found. We simply *take* any convenient values for x and find the corresponding values of y, and write the pairs of corresponding values in parentheses for the coördinates of points.

Having found the coördinates of a series of points, a, b, c, d, e, f, g, we now locate these points on a diagram. The point a, whose coördinates are $(0, -3)$, is located first by a dot with the letter a beside it; then the point b is located in the same way; and so on. Last of all a line is drawn through these points.

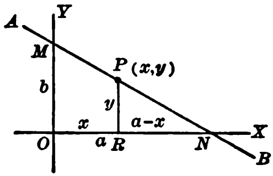
The student may now construct in the same way the graphs of the following equations. He can let $x = 0, 1, 2, 3, 4, 5, -2$ as in the preceding example.

2. $3x - 2y = 5$. 3. $3x + 5y = 15$. 4. $2x - 7y = -12$.

5. *The graph of a simple equation, or an equation of the first degree, will be found to be always a straight line,* as in Ex. 1–4.

Proof. By clearing of fractions, transposing, and dividing by the right member, we can put any equation of the first degree in the form $\frac{x}{a}+\frac{y}{b}=1$. We have to prove that the graph of this equation is a straight line.

First by letting $x=0$, we get $y=b$; then by letting $y=0$, we get $x=a$. Hence we perceive that the locus $AMNB$ (see diagram) passes through the points $(0, b)$, $(a, 0)$, or M and N.



Let P be at (x, y), and suppose that MPN is a straight line. Then PNR and MNO are similar triangles (having their angles respectively equal), and therefore their corresponding sides proportional.

Hence, (1) $$\frac{y}{b}=\frac{a-x}{a}.$$

But from given eq., $\frac{x}{a}+\frac{y}{b}=1$, we get $\frac{y}{b}=1-\frac{x}{a}$, or (2) $\frac{y}{b}=\frac{a-x}{a}$.

Now since the value of $\frac{y}{b}$ from eq. (2) which gives the *graph* equals the value of $\frac{y}{b}$ from eq. (1) of the straight line, or conversely, *we conclude that all points of the graph must lie on the straight line, and all points on the straight line must lie on the graph of the given equation.*

Hence, to locate the graph, or locus, of any first degree equation, it is sufficient to find *two* points. Letting $x=0$ and $y=0$ in turn, as above, we get the two points most easily located. A third point should be plotted as a check. If the first two points are close together, we join the *third* to one of the others.

Construct the graphs of the following equations:

6. $5x+3y=-10$. **7.** $-2x+5y=12$.

8. $6x-7y=1$. **9.** $25x+7y=29$.

10. $6x+5y=16$. **11.** $35x+17y=-86$.

12. $2x=5y$. **13.** $x=5$.

Suggestion to Ex. 13. y can have *any* value. Locate two points. The graph is a line parallel to the Y-axis, 5 units to right of YY'.

14. $y=-3$. **15.** $\frac{x}{2}-\frac{y}{3}=4.5$. **16.** $\frac{x+3y}{2}=\frac{y}{5}-1$.

17. $3x-11y=0$. **18.** $x=y$. **19.** $x=-y$.

II. GRAPHICAL SOLUTION OF SIMULTANEOUS EQUATIONS

122. Value of the Graphical Method. Whether the algebraic or the graphical method will give the answer more quickly depends on the problem and the skill of the user, but the algebraic solution is always accurate, while the graphic solution is often only approximate. One of these methods can be used to check the answer obtained by the other, though verification is also available for both. But for the student now the main value of the graphical solutions consists in the fact that they throw a strong light on the nature of simultaneous equations and on peculiarities that may occur in them.

123. Graphical Solution of Pairs of Equations containing Two Unknowns. 1. Solve (1) $\frac{x}{3}-\frac{y}{6}=\frac{1}{2}$, (2) $\frac{x}{5}-\frac{3y}{10}=\frac{1}{2}$, by the graphical method, and check with an algebraic solution.

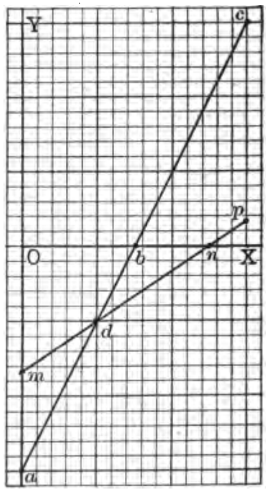
Graph Solution.

(1_1) $2x - y = 3$

(x, y)	Pt.
$(0, -3)$	a
$(1.5, 0)$	b
$(3, 3)$	c

(2_1) $2x - 3y = 5$

(x, y)	Pt.
$(0 - 1.67)$	m
$(2.5, 0)$	n
$(3, .33)$	p



The coördinates that satisfy *both* equations are those of the point d, *where the graphs cross.* The coördinates of d are $x = 1$, $y = -1$. *Answers.*

ALGEBRAIC SOLUTION.

$$\begin{array}{rl} (1_1) & 2x - y = 3 \\ (2_1) & 2x - 3y = 5 \\ \hline & 2y = -2 \\ & y = -1. \quad \textit{Ans.} \end{array}$$

(1_1) $2x-(-1)=3$. (§ 144.)

$x = 1$. *Ans.*

2. $\begin{cases} 3x + 4y = 10, \\ 4x + y = 9. \end{cases}$

3. $\begin{cases} 5x + 6y = 17, \\ 6x + 5y = 16. \end{cases}$

4. $\begin{cases} 2x + 3y = 43, \\ 10x + y = 7. \end{cases}$

5. $\begin{cases} 11x - 14y = 14, \\ 5x + 7y = 41. \end{cases}$

6. $\begin{cases} 18x - 10y = 29, \\ 14x - 15y = 24. \end{cases}$

7. $\begin{cases} 7x - 9y = -22, \\ x = -4. \end{cases}$

8. $\begin{cases} 11d + 16t = 64, \\ 7d - 12t = 13. \end{cases}$

9. $\begin{cases} 8x = 5y, \\ 13x = 8y + 1. \end{cases}$

10. $\begin{cases} 11x - 3y = 321, \\ 5x - 16y = 585. \end{cases}$

11. $\begin{cases} 7x + 2y = 76, \\ 2x - 3y = 11. \end{cases}$

SUGGESTION TO EX. 10. Since the numbers are large, the centimeter can not be used as the unit of measure. The 2-millimeter unit can be employed in this case.

124. Consistent Equations. The student may ask himself the question whether if three equations, each containing two unknowns, were taken at random, the values of x and y obtained by solving the system formed out of the first two equations would have the same values as those obtained by taking the system formed from the second and third equations, or by taking the system formed from the first and third equations.

On the other hand, if the third equation was obtained by combining the first two in some way, would the values of x and y found by solving the system formed from the first two satisfy the third equation?

EXAMPLE.

$$\begin{array}{rl} (1) & 11x - 10y = 14 \\ (2_1) & 10x + 14y = 82 \\ \hline (3) & x - 24y = -68 \end{array}$$

(2) $5x + 7y = 41$

Construct now the graphs of equations (1) and (2) and (3) to the same axes. What do you learn?

Test problems of § 44 in the same way.

125. Inconsistent Equations. Construct the graphs for the following equations on the same axes:

(1) $4x - 2y = 7$. (2) $4x - 9y = -21$. (3) $3x - 4y = -8$.

Solve the system formed from (1) and (2) algebraically and compare the answers with those on diagram obtained graphically. Do the same with the system (1) and (3), and with the system (2) and (3).

Write sets of three equations at random and see whether their graphs intersect in three points or one point.

126. Systems of Equations in which One of the Two Given Equations can be derived from the Other.

Construct the graphs for the system

$$(1)\ 2x - 5y = 11;\ (2)\ \frac{8x - 44}{5} = 4y,$$

in which (2) can be derived from (1) by first multiplying through by 4, then transposing, and then dividing through by 5.

What do you learn about these graphs?

Test your answer by making and testing other similarly constructed systems. Is then the system of marking equations, in which the equation marked (1) at the start is marked (1) with subscripts throughout, justified?

127. Systems of Equations which differ only in their Known Terms.

Construct the graphs on the same axes for the system

$$(1)\ 2x - 5y = 12,\ (2)\ 4x = 10y + 6,$$

in which, obviously, (2) can be changed into $2x - 5y = 3$.

What do you find true of these graphs?

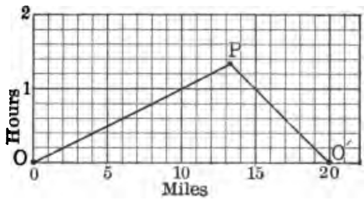
Test your answer by constructing similar systems which differ only in their known terms.

We learn then that whenever by transposing, and multiplying or dividing through, two equations can be made to differ in their known terms only, *their graphs are parallel.*

a. The graphs of three linear eqs. each having two unknowns may intersect in 0, or 1, or 2, or 3 points (not at infinity), or in every point. Explain.

128. Solution of Problems by Graphs. Any proportion problem in arithmetic can be solved graphically. The origin is always one point on the graph and the terms of the given ratio equaling $x : y$ are the respective abscissa and ordinate of another point. The answer sought is found on the diagram as the number corresponding to the third given term of the proportion. In this way graphs can be constructed giving the cost from the number of articles, meters from feet, pounds from kilograms, interest from time, circumferences from diameters, etc., and *vice versa.*

Some problems in rate, time, and distance have *two* origins, O and O'. Thus, if two men 20 mi. apart go towards each other, the one at 10 mi. and the other at 5 mi. an hour, P on the diagram gives the distance and the time each has traveled when they meet.



The student should make proportion and time-rate-distance problems like the above and solve them by the graph method.

III. GRAPHS AND GRAPHICAL SOLUTIONS OF QUADRATICS

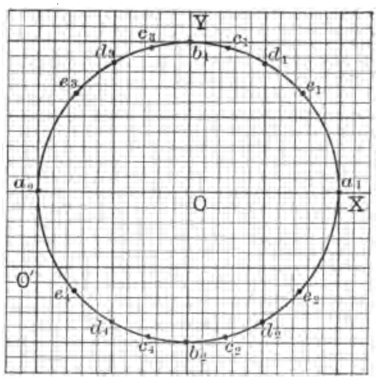
129. The Various Graphs of Equations of the Second Degree.

1. Construct the graph of the equation $x^2 + y^2 = 4$.

SOLUTION

$$x^2 + y^2 = 4$$

(x, y)	POINTS
$(\pm 2, 0)$	a_1, a_2
$(0, \pm 2)$	b_1, b_2
$(\pm .5, \pm 1.9)$	c_1, c_2, c_3, c_4
$(\pm 1, \pm 1.7)$	d_1, d_2, d_3, d_4
$(\pm 1.5, \pm 1.3)$	e_1, e_2, e_3, e_4



a. The student may not see how $(0, \pm 2)$ gives *two* points, or how $(\pm 1, \pm 1.7)$ gives *four* points. He should observe that $(0, \pm 2)$ is really the two points, $(0, 2)$ and $(0, -2)$; also that $(\pm 1, \pm 1.7)$ denotes four points, viz. $(1, 1.7)$, $(1, -1.7)$, $(-1, 1.7)$, $(-1, -1.7)$. Notice that the coördinates of each of these six points satisfy the equation $x^2 + y^2 = 4$.

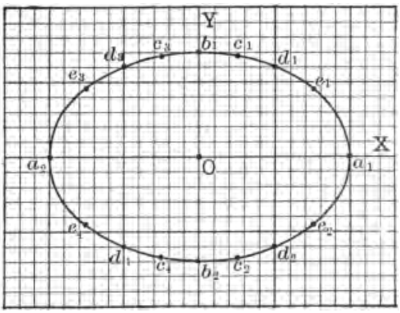
Construct now the graph of (1) $x^2 + y^2 = 9$.

From these examples we readily see that the graph of an equation of the form $x^2 + y^2 = r^2$ is a **circle** *whose radius is r.*

Hence, graph of an equation of the form $x^2 + y^2 = r^2$ can be immediately constructed by using r as radius and origin as center.

Construct the graphs for the following equations:

(2) $x^2 + y^2 = 16$. (3) $x^2 + y^2 = 12$. (4) $x^2 + y^2 = 1$.



2. Construct the graph of the equation $x^2 + 2y^2 = 4$.

SOLUTION

$x^2 + 2y^2 = 4$

(x, y)	POINTS
$(\pm 2, 0)$	a_1, a_2
$(0, \pm 1.4)$	b_1, b_2
$(\pm .5, \pm 1.4^-)$	c_1, c_2, c_3, c_4
$(\pm 1, \pm 1.2)$	d_1, d_2, d_3, d_4
$(\pm 1.5, \pm .9^+)$	e_1, e_2, e_3, e_4

b. **Meaning of Imaginary Values for x and y in finding Coördinates for Graphs.** Suppose we assign the value $x = 3$ in the equation just used, $x^2 + 2y^2 = 4$. Making this substitution, we get $y = \sqrt{-2.5}$. Thus, for $x = 3$, y is **imaginary**. This means that there are no points of the curve as far from the origin as $x = +3$ or $x = -3$. The curve stops, in fact, with the value $x = +2$ at the right, and with $x = -2$ at the left. Similarly, if we put $y = \pm 2$, we get $x = \sqrt{-4}$. We see from the diagram that there are no points of the curve above $y = +1.4$, and none below $y = -1.4$.

Construct the graphs of the following equations:

(1) $x^2 + 4y^2 = 5$. (2) $3x^2 + 4y^2 = 8$. (3) $10x^2 + y^2 = 10$.

c. These graphs, or curves, are called **ellipses**. They occur as the graphs of all equations of the form $ax^2 + by^2 = c$, that is, of all equations in which the coefficients of x^2 and y^2 are unequal, and the sign between the terms in the left member is $+$.

If the coefficients of x^2 and y^2 are close together in value, to what curve does the ellipse approximate? What happens if they are quite unequal?

The earth moves round the sun in an **elliptical** orbit, nearly circular. Halley's comet, on the other hand, moves in a very long ellipse.

(4) $x^2 + 9y^2 = 18$. (5) $10x^2 + 12y^2 = 24$. (6) $9x^2 + 12y^2 = 1$.

3. Construct the graph of the equation $x^2 - 2y^2 = 4$.

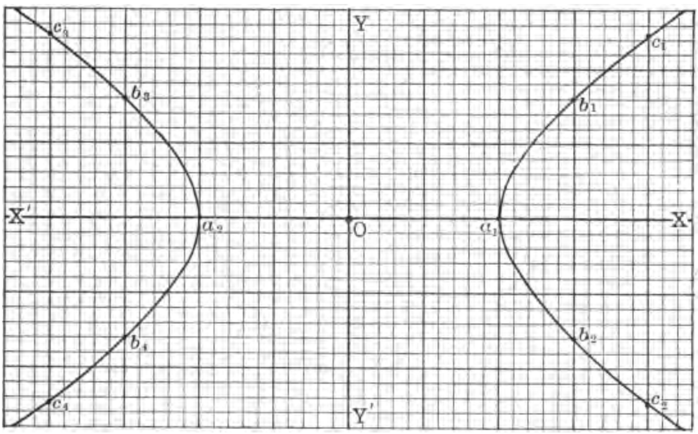
SOLUTION

$$x^2 - 2y^2 = 4$$

(x, y)	POINTS
$(\pm 2, 0)$	a_1, a_2
(0, imag.)	
(± 1, imag.)	
$(\pm 3, \pm 1.6)$	b_1, b_2, b_3, b_4
$(\pm 4, \pm 2.5^-)$	c_1, c_2, c_3, c_4
$(\pm 5, \pm 3.2^+)$	d_1, d_2, d_3, d_4

d. This curve is called an **hyperbola.** An examination of the values in the table, noting from the equation how they will continue to increase indefinitely, shows that one branch of the curve will continue out indefinitely to the northeast (to follow the customary map rules for directions), another to the southeast, another to the northwest, and a fourth to the southwest.

The *imaginary* values of y show that no points of the curve lie inside of the region inclosed by the lines $x = +2$, and $x = -2$.



Notice that the equation used is identically the same as that of Ex. 2 except that + in the ellipse equation is replaced by − in that of the hyperbola.

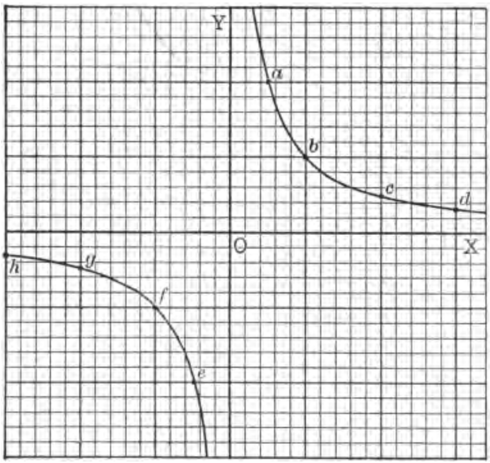
The hyperbola is the orbit of comets which come into the solar system, go round the sun, and then go out again never to return.

Construct the graphs of the following equations:

(1) $x^2 - 4y^2 = 4.$ (2) $4x^2 - y^2 = 16.$ (3) $2y^2 - 3x^2 = 6.$

(4) $x^2 - y^2 = 9.$ (5) $y^2 - x^2 = 9.$ (6) $x^2 - 10y^2 = 1.$

4. Construct the graph of the equation $xy = 1$.



SOLUTION

$xy = 1$

(x, y)	PTS.
(.5, 2)	a
(1, 1)	b
(2, .5)	c
(3, .3+)	d
. . . . . .	. .
(−.5, − 2)	e
(− 1, − 1)	f
(− 2, −.5)	g
(− 3, −.3+)	h
. . . . . .	. .

Construct the graphs of the following equations:

(1) $xy = 10.$ (2) $xy = 1.$ (3) $xy = -2.$ (4) $xy = 50.$

e. These graphs are hyperbolas, though coming from equations quite different from those in 3. They differ from the hyperbolas we have just been constructing in having their "axes" extend diagonally instead of east and west, or north and south. This is the graph for inverse proportion (see § 145). Evidently x increases as y diminishes, and *vice versa*.

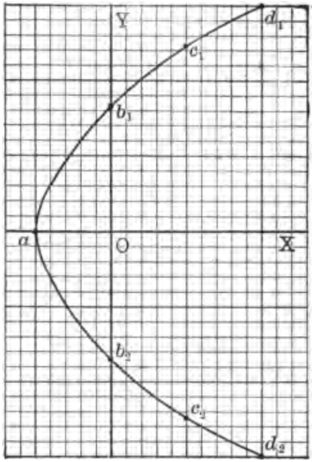
The graphs just constructed can be used to read off corresponding values for quantities which are *inversely* proportional just as we read off corresponding values for quantities which were *directly* proportional in § 128. Thus, if a man is paid $10 for doing a job of work, x is the number of days he works, and y is the wages per day, then the graph for $xy = 10$ (see eq. (1) above) will give a value for x corresponding to any assigned value of y, or a value for y corresponding to any assigned value of x.

5. Construct the graph of the equation $y^2 = 3x + 3$.

SOLUTION

$y^2 = 3x + 3$

(x, y)	PTS.
$(-1, 0)$	a
$(0, \pm 1.7)$	b_1, b_2
$(1, \pm 2.5^-)$	c_1, c_2
$(2, \pm 3)$	d_1, d_2
$(6, \pm 4.6^-)$	e_1, e_2
. . . .	. . .



f. This graph is called a **parabola**. It has apparently something like the same shape as the hyperbola, but it has only *one* branch. If x is put equal to -1.1 or any numerically larger negative number, y is imaginary. The parabola is the path of projectiles like bullets and is the curve of headlight reflectors.

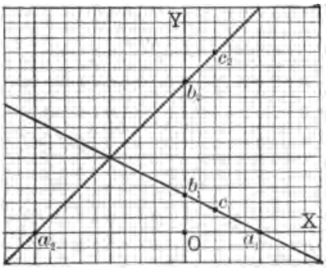
Construct the graphs of the following equations:

(1) $y^2 = 2x + 4$. (2) $y^2 = 2x - 4$. (3) $y^2 = x$.

(4) $x^2 = 2y + 4$. (5) $x^2 = 12y - 3$ (6) $x^2 = y$.

6. Construct the graph of the equation $(x - y + 2)(x + 2y - 1) = 0$, or $x^2 + xy - 2y^2 + x + 5y - 2 = 0$.

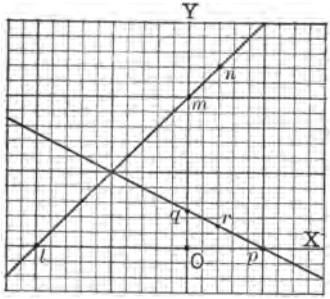
Presumably this is the equation of *the two straight lines*, $x - y + 2 = 0$, and $x + 2y - 1 = 0$, since values that satisfy either of them satisfy the given equation. (See § 121, 5.)



SOLUTION

$x^2 + xy - 2y^2 + x + 5y - 2 = 0$

(x, y)	PTS.
$(1 \text{ or } -2, 0)$	a_1, a_2
$(0, .5 \text{ or } 2)$	b_1, b_2
$(.4, .3 \text{ or } 2.4)$	c_1, c_2



SOLUTION

$x - y + 2 = 0$ $\quad$ $x + 2y - 1 = 0$

(−2, 0)	l	(1, 0)	p
(0, 2)	m	(0, .5)	q
(.4, 2.4)	n	(.4, .3)	r

g. We see from the diagrams that a_1, a_2, b_1, b_2, c_1, c_2, are all on line ln or line pq. This rule would hold if any number of points were obtained.

The pupil may have some trouble in seeing how the coördinates of a_1, a_2, b_1, b_2, c_1, c_2, were found. Notice if $y=0$, the equation reduces to $x^2+x-2=0$, whose roots by factoring are 1 and −2. Again, if $x = 0$, the equation reduces to $2y^2 - 5y + 2 = 0$, whose roots are 2 and .5. If $x = .4$, the equation reduces to $y^2 - 2.7y + .72 = 0$, whose roots are 2.4 and .3. If $x = 2$, the equation reduces to $2y^2 - 7y - 4 = 0$, whose roots are 4 and −.5. As a rule in such cases, the factoring method cannot be used, and we resort to that of § 100.

130. Effect of Change of Origin on Equations. Reverting to the figure accompanying Ex. 1, § 129, we see that if O' is taken as the origin instead of O, every abscissa will then be increased by 2 over what it was, and every ordinate by 1. The equation of the circle now becomes

$$(x-2)^2 + (y-1)^2 = 4,$$

or

$$x^2 + y^2 - 4x - 2y + 1 = 0.$$

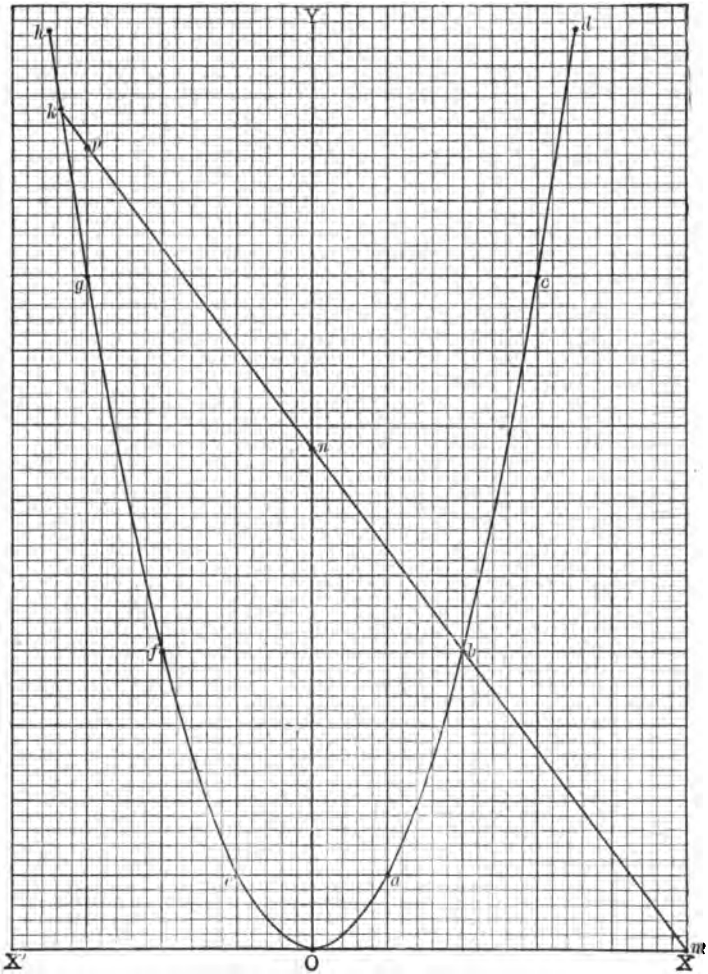
Thus we see that an equation of this form, whose coefficients of x^2 and y^2 are equal, gives a circular graph, but one in which origin is not at its center.

A similar change could be made in the equations of the other graphs without changing the graphs themselves.

131. Graphical Solution of a Quadratic Equation containing One Unknown Quantity. To construct a graph two unknowns are necessary. In the present problem one of these has to be *introduced*.

1. Given $3x^2 + 4x = 20$, to find values of x by graph method.

SOLUTION. Let (1) $y = x^2$; then substituting $y = x^2$ in the given equation, we have (2) $3y + 4x = 20$. (Solution continued on page 142.)



a. A piece of fine black thread can be laid so as to pass through the several points located on the squared paper. The position the thread takes when it passes through the points locates the graph very closely.

(1) $y = x^2$.

(x, y)	Points
(0, 0)	O
(± 1, 1)	a, e
(± 2, 4)	b, f
(± 3, 9)	c, g
(± 3.5, 12.25)	d, h
(etc., etc.)	etc.

(2) $3y + 4x = 20$.

(x, y)	Pts.
(0, 6.7)	n
(5, 0)	m
(− 3, 10.7)	p

We must now construct the graphs for equations (1) and (2). Values are assigned to y and the corresponding values of x are calculated as in Chapter IV.

Locating the several points for equation (1) on the squared paper with reference to two axes, marking them with their letters, and joining them by a running line, we have the figure as shown in the diagram, p. 141. Joining the points located by equation (2) there results the straight line mnp. These two graphs *cross* at b and k, or at $x = 2$, and $x = -3.4^-$, which are the roots sought.

Check. Solving $3x^2 + 4x = 20$ by completing the square as in § 100, we have $x = 2$, and $x = -3.33^+$.

b. The graph of $y = x^2$ is (§ 129, f) a **parabola**. The unfinished ends are known to extend on indefinitely. Have students practice making this graph.

c. Observe that the curve $y = x^2$, once constructed, can be used in the solution of any quadratic containing one unknown. It only remains in each particular problem to locate the straight line by means of its Eq. (2). Where this straight line crosses the parabola are found the values of x sought.

Construct the parabola with ink and the straight line with lead pencil. Then the lead pencil mark can be erased when one begins a new problem.

Solve the following equations by the graph method, and check by completing the square method:

2. $x^2 - 2x - 3 = 0$.
3. $3x^2 - 2 = 5x$.
4. $12x^2 + x = 6$.
5. $8x^2 - 17x = 115$.
6. $7x^2 - 39 = 8x$.
7. $10 + 16x = -6x^2$.
8. $3x^2 - 2x = 7$.
9. $7x^2 + 14x = 21$.
10. $5x^2 - 20 = 0$.
11. $7x^2 + 5x = 31$.
12. $3x^2 + 4x = 4$.
13. $5x^2 - 7x = 1$.

14. Solve by the diagram $x^2 = 7$, *i.e.* extract square root of 7.

Solution. Corresponding to $y = 7$ is $x = 2.6^+$. *Ans.*

15. Find by the diagram $\sqrt{5}$; $\sqrt{2.8}$; $\sqrt{7}$; $\sqrt{12}$; $\sqrt{.9}$; $\sqrt{6.4}$.

16. The diagram can also be used to square numbers. Thus corresponding to $x = 1.4$ is $y = 2^-$, which is the square of 1.4.

17. Find by the diagram 1.8^2; 2.5^2; 75^2 (or $.75^2$); 3.4^2.

132. Solution of Simultaneous Equations by Means of Graphs.

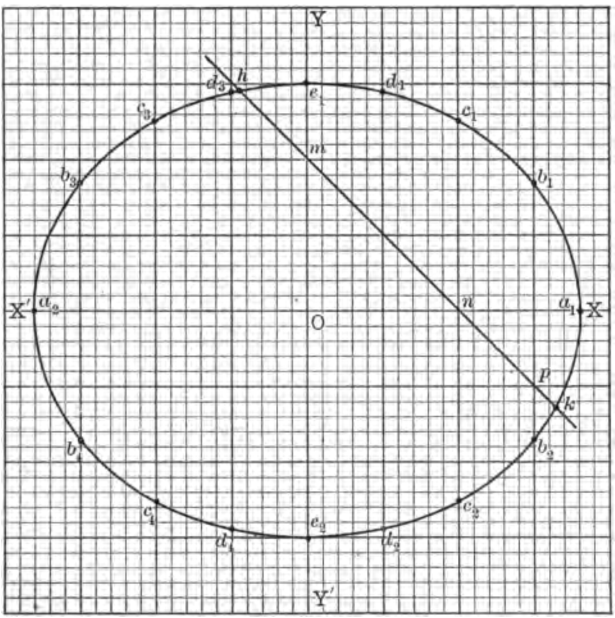
1. Solve $\begin{cases}(1)\ 9x^2+13y^2=117\\(2)\ x+y=2\end{cases}$ by the graph method.

(1) $9x^2+13y^2=117$ (2) $x+y=2$

$(0, \pm 3)$	$e_1,\ e_2$
$(\pm 1, \pm 2.9)$	$d_1,\ d_2,\ d_3,\ d_4$
$(\pm 2, \pm 2.5)$	$c_1,\ c_2,\ c_3,\ c_4$
$(\pm 3, \pm 1.7)$	$b_1,\ b_2,\ b_3,\ b_4$
$(\pm 3.6, 0)$	$a_1,\ a_2$

$(0, 2)$	m
$(2, 0)$	n
$(3, -1)$	p

Locating the several points and joining them, we have the graph as represented in the figure. Evidently the graph of $9x^2+13y^2=117$ is an ellipse whose longest axis is 7.2 and whose shortest axis is 6.



The graphs *cross* at h and k, whose coördinates are

$$\begin{cases}x=-.9,\\y=2.9,\end{cases}\ \text{and}\ \begin{cases}x=3.3-\\y=-1.3-\end{cases}$$

To check these answers, the problem is solved by the method of § 109, whence is found $\begin{cases} x = 3.3, \\ y = -1.3, \end{cases}$ and $\begin{cases} x = -.9, \\ y = 2.9. \end{cases}$

Solve the following by the graph method, and check by a purely algebraic solution. The graphs of equations like $x^2 + y^2 = 9$ can be constructed very quickly with a compass. How is the radius found in each case?

2. $\begin{cases} x^2 + y^2 = 36, \\ x + y = 8. \end{cases}$ 3. $\begin{cases} x^2 + y^2 - 40 = 0, \\ 3x - y = 6. \end{cases}$ 4. $\begin{cases} x^2 + y^2 = 37, \\ x - y = 5. \end{cases}$

5. $\begin{cases} x^2 + y^2 = 25, \\ 4x + 3y = 20. \end{cases}$ 6. $\begin{cases} x^2 + y^2 = 169, \\ 3x + 9 = 2y. \end{cases}$ 7. $\begin{cases} x^2 + y^2 = 121, \\ 5x + y = 46. \end{cases}$

8. $\begin{cases} x^2 + 2y^2 = 34, \\ x + y = 7. \end{cases}$

SUGGESTION. The graph of $x^2 + 2y^2 = 34$, as explained in § 129, 2, is an *oval*, or an *ellipse* in shape. The method of solution is precisely the same as in Ex. 1.

9. $\begin{cases} 4x^2 + y^2 = 25, \\ 2x + y = 7. \end{cases}$ 10. $\begin{cases} 9x^2 + 4y^2 = 72, \\ 5x - 3y = 1. \end{cases}$

11. $\begin{cases} 3x^2 + 4y^2 = 43, \\ 2x - y = 4. \end{cases}$ 12. $\begin{cases} 2x^2 + y^2 = 33, \\ x - 3y = 1. \end{cases}$

13. $\begin{cases} 2x^2 - xy + y^2 = 7, \\ 4x - 3y = 5. \end{cases}$ 14. $\begin{cases} 3x^2 + 2xy + 2y^2 = 35, \\ 4x - 5y = 7. \end{cases}$

15. $\begin{cases} (1)\ x^2 - 2y^2 = 1, \\ (2)\ x + 6y = 3. \end{cases}$

SOLUTION. The graph for equation (1) is (§ 129, *d*) different from the preceding ones, owing to the *minus* sign before $2y^2$. The method of solution, however, is precisely the same as before. Note that $(\pm 2, \pm 1.2)$ gives *four* points.

(1) $x^2 - 2y^2 = 1$

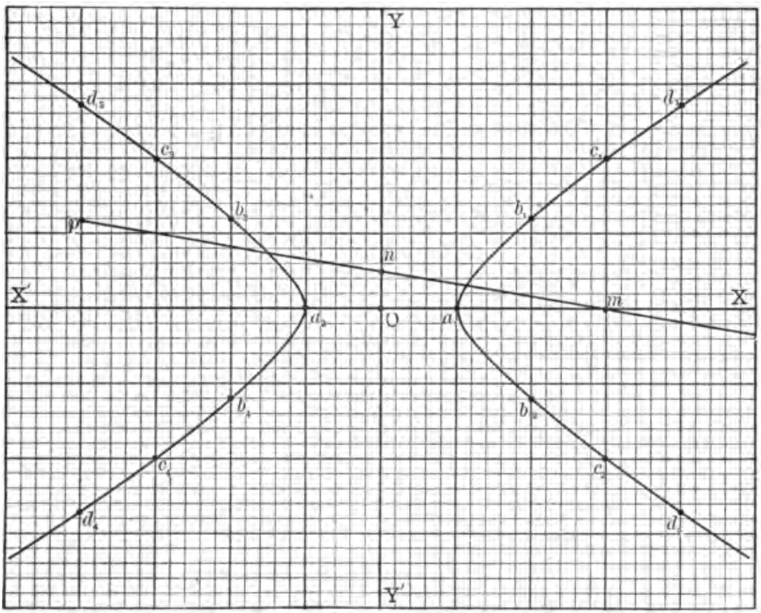
$(\pm 1,\ 0)$	$a_1,\ a_2$
$(\pm 2,\ \pm 1.2)$	$b_1,\ b_2,\ b_3,\ b_4$
$(\pm 3,\ \pm 2)$	$c_1,\ c_2,\ c_3,\ c_4$
$(\pm 4,\ \pm 2.7)$	$d_1,\ d_2,\ d_3,\ d_4$
(etc., etc.,)	etc.

(2) $x + 6y = 3$

$(3,\ 0)$	m
$(0,\ .5)$	n
$(-4,\ 1.2)$	p

Constructing the graphs through the located points, we have the accompanying diagram. The four unfinished ends of the curved graph extend out indefinitely. The curve, it is clear, consists of two parts. It is an hyperbola.

The straight line graph of equation (2) crosses the right-hand branch of the hyperbola at $x = 1.1$, $y = .3^+$, and the left-hand branch at $x = -1.5^-$, $y = .7^+$. Checking the answer by a purely algebraic solution (§ 108), we get the same results.



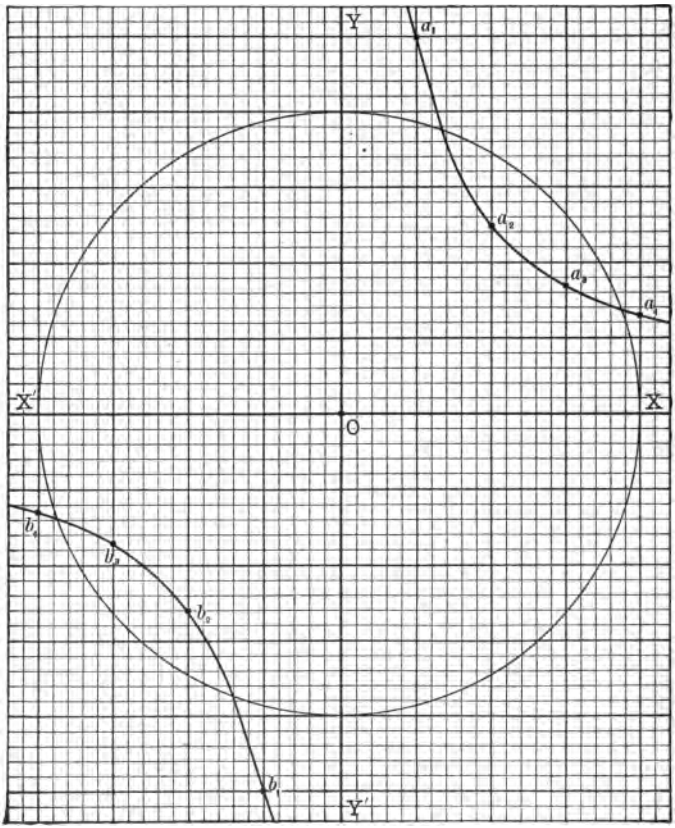
16. $\begin{cases} (1)\ x^2 + y^2 = 16, \\ (2)\ xy = 5. \end{cases}$

Solution. The graph of (1) is, of course, a circle whose radius is 4. The graph of

(2) $xy = 5$			
(1, 5)	a_1	(−1, −5)	b_1
(2, 2.5)	a_2	(−2, −2.5)	b_2
(3, 1.7)	a_3	(−3, −1.7)	b_3
(4, 1.25)	a_4	(−4, −1.25)	b_4
		etc., etc.	etc.

equation (2) is an hyperbola, but turned through an angle of 45°, as compared with that of Ex. 15.

Solve by algebra as well as by graph, being careful to get four values for x and four for y.



17. $\begin{cases} 3x^2 - y^2 = 47, \\ x + y = 9. \end{cases}$

18. $\begin{cases} x + 2y = 9, \\ 3y^2 - 5x^2 = 43. \end{cases}$

19. $\begin{cases} 2x^2 - 3xy + y^2 = 8, \\ x + y = 4. \end{cases}$

20. $\begin{cases} x^2 - 2y^2 = -7, \\ x + y = 9. \end{cases}$

21. $\begin{cases} 2y^2 - x^2 = -23, \\ x - 2y = 3. \end{cases}$

22. $\begin{cases} 3x^2 - 4y^2 = 11, \\ 4x - y = 10. \end{cases}$

23. $\begin{cases} x^2 + y^2 = 9, \\ xy = 2. \end{cases}$

24. $\begin{cases} 3x^2 + 2y^2 = 6, \\ 3xy = 1. \end{cases}$

25. $\begin{cases} 4x^2 + 9y^2 = 73, \\ xy = 4. \end{cases}$

26. $\begin{cases} (1)\ x^2 - y = 7, \\ (2)\ 3x - y = 9. \end{cases}$

Suggestion. The graph of (1) has the shape of that of § 131, but is pushed down so that the curve crosses the axis of X.

27. $\begin{cases} x^2 - 3y = 19, \\ x - y = 3. \end{cases}$

28. $\begin{cases} x - 2y^2 = 1, \\ x - 3y = 6. \end{cases}$

29. $\begin{cases} x^2 - 2y = 7, \\ xy = 3. \end{cases}$

a. An equation of the form $ax^2 + ay^2 = b$ has a circle for its graph.

An equation of the form $ax^2 + by^2 = c$ has an ellipse for its graph.

An equation of the form $ax^2 - by^2 = c$ has an hyperbola for its graph.

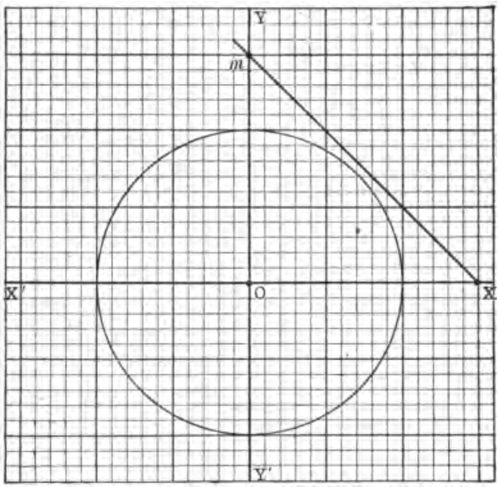
An equation of the form $xy = a$ has an hyperbola for its graph.

An equation of the form $ax^2 + by = c$ has a parabola for its graph.

An equation of the form $ax + by^2 = c$ has a parabola for its graph.

133. Graphical Significance of Imaginaries (§ 83) in the Solution of Equations. Whenever upon solving two simultaneous equations the roots come out imaginary, it will be found that the graphs of the two equations *do not intersect at all.* See § 129, *b*.

1. $\begin{cases} x^2 + y^2 = 4, \\ x + y = 3. \end{cases}$



Constructing the graphs, we have the figure as represented, the graphs not intersecting. Solving by a purely algebraic method, we get,

$$x = \frac{3 \pm \sqrt{-1}}{2}, \qquad y = \frac{3 \mp \sqrt{-1}}{2}.$$

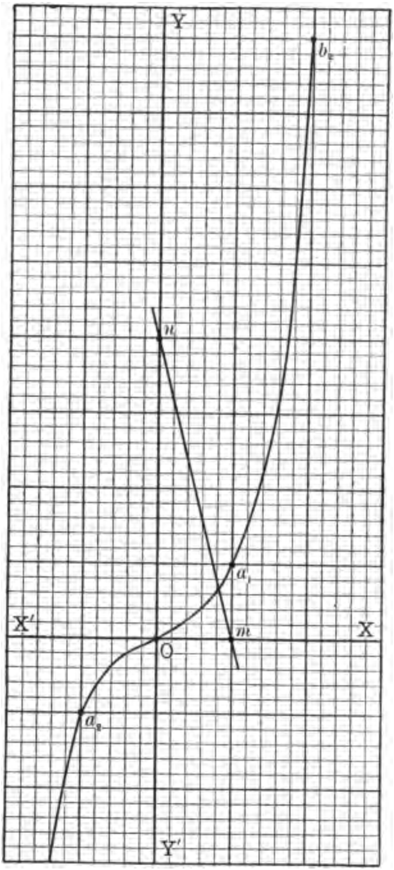
Construct the graphs in the following, solving also by algebra and verifying:

2. $\begin{cases} x^2 + y^2 = 36, \\ x - 2y = 14. \end{cases}$ 3. $\begin{cases} 16x^2 - 9y^2 = 144, \\ 6x = y. \end{cases}$ 4. $\begin{cases} xy = 5, \\ x + y = 4. \end{cases}$

134. Cubic Equations, or Equations of the Third Degree, containing One Unknown, solved by the Graph Method of § 131.

1. $x^3 + 4x - 4 = 0.$

Let, (1) $y = x^3$; then, on substituting, (2) $y + 4x = 4$.



(1) $y = x^3$.

(0, 0)	0
(1, 1)	a_1
(− 1, − 1)	a_2
(2, 8)	b_1
(− 2, − 8)	b_2
(3, 27)	c_1
(− 3, − 27)	c_2

(2) $y + 4x = 4$.

(1, 0)	m
(0, 4)	n

Constructing the two graphs from the tabulated data, we see that they intersect at $x = .8+$.

2. $3x^3 + 4x - 7 = 0$.

3. $5x^3 + 12x = 15$.

4. $4x^3 - 19x + 42 = 0$.

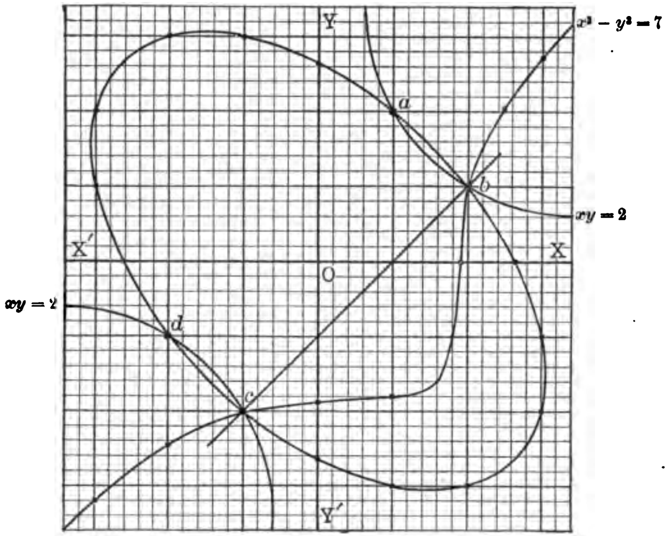
5. The graph of this article can be used to extract the cube root and to cube numbers. See § 131, 14-17.

135. Equations of any Degree in One or Two Unknowns can be solved by the graph method. However, the complexity of the shape of the graph rapidly increases when the degree of the equation goes above the second degree, with resulting difficulty in the construction.

Light is thrown on the solution of systems of equations by the use of graphs. We shall now consider an example of such graphical explanation, showing how the various graphs in a system intersect one another.

136. Comparison of the Algebraic and Graphical Solutions of the System.

$$(1)\quad x^3-y^3=7, \qquad (2)\quad x-y=1.$$



(3)	$x^2+xy+y^2=7$	[(1)÷(2)]	(5)	$x^2+2xy+y^2=9$	[(3)+(4_1)]
(1_1)	$x^2-2xy+y^2=1$		(5_1)	$x+y=\pm 3$	(Root Ax.)
(4)	$3xy=6$	(Sub. Ax.)	(2)	$x-y=1$	
(4_1)	$xy=2$			$\therefore x=2,\ y=1$; or, $x=-1,\ y=-2$.	

The graph of $x^3-y^3=7$ is the curve resembling that of § 134. The equation $x^2+xy+y^2=7$ gives the ellipse, $xy=2$ gives the hyperbola, and $x-y=1$ the straight line. Eq. (1_1) represents two coincident straight lines (§ 129, 6). All these curves intersect in (2, 1) and (− 1, − 2). The ellipse and hyperbola intersect also in (1, 2) and (− 2, − 1).

If the graphs of one pair of equations have the same points of intersection as those of another pair, the *two systems of equations* are said to be **equivalent**. The diagram shows quickly to the eye which systems of the preceding sets of equations are equivalent.

CHAPTER VII

RATIO AND PROPORTION

I. PROPORTION PROPER

137. The **ratio** of two quantities a, b is found by dividing the first by the second.

A **proportion** is an equation whose two members are ratios.

Proportions are written in two ways and read in two ways, both meaning the same thing. Thus,

either $a : b = c : d$,

or, $\frac{a}{b} = \frac{c}{d}$,

(The single colon denotes *division*. Formerly a double colon was used instead of $=$.)

may be read "a is to b as c is to d," or "the ratio of a to b equals the ratio of c to d."

138. Definitions. The first term of a ratio is called the **antecedent** and the last term the **consequent**. The first and last terms of a proportion are called the **extremes**, and the second and third terms the **means**. If the second and third terms of a proportion are the same quantity, this quantity is called a **mean proportional** between the first and last terms.

139. Fundamental Theorems about a Proportion.

1. *If four quantities are in proportion, the product of the extremes equals the product of the means.*

If $\frac{a}{b} = \frac{c}{d}$, then $ad = bc$. (By Mult. Ax.)

SUGGESTION. Both sides of the given equation are multiplied by bd, or it is cleared of fractions.

2. *Conversely. If the product of two numbers equals the product of two other numbers, the factors of either product can be made the extremes, and the factors of the other the means of a proportion.*

If $ad = bc$, then $\frac{a}{b} = \frac{c}{d}$. (By Div. Ax. Explain.)

140. Allowable Changes in the Order of the Terms of a Proportion.

1. **Alternation.** *If four numbers taken in order are in proportion, the ratio of the first to the third equals the ratio of the second to the fourth.*

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{c} = \frac{b}{d}$. $\left(\text{Mult. Ax. Multiply through by } \frac{b}{c}.\right)$

2. **Inversion.** *If four numbers taken in order are in proportion, the ratio of the second to the first equals the ratio of the fourth to the third.*

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{b}{a} = \frac{d}{c}$. (Div. Ax. Divide 1 by each member.)

141. New Proportions from a Given One.

1. **Addition.** *If four numbers are in proportion, the ratio of the sum of the first and second to the first or second equals the ratio of the sum of the third and fourth to the third or fourth.*

If $\frac{a}{b} = \frac{c}{d}$, then $\frac{a}{b} + 1 = \frac{c}{d} + 1$, or $\frac{a+b}{b} = \frac{c+d}{d}$. (Add. Ax.)

Also, $\frac{b}{a} = \frac{d}{c}$, whence $\frac{b}{a} + 1 = \frac{d}{c} + 1$, or $\frac{a+b}{a} = \frac{c+d}{c}$. (Add. Ax.)

2. **Subtraction.** *If four numbers are in proportion, the ratio of the difference of the first and second to the first or second equals the ratio of the difference of the third and fourth to the third or fourth.*

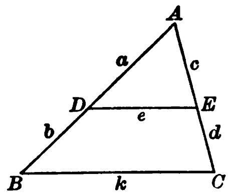
Proved in manner similar to addition.

a. **Addition and subtraction have been called respectively "composition" and "division."**

142. A Geometrical Meaning given to the Preceding Theorems. A theorem in geometry says, *If a line is drawn parallel to the base* *of a triangle, it divides the sides proportionally,* or $a:b=c:d$. By the preceding theorems we can now infer that the following proportions are also true:

$$\frac{AD}{AE}=\frac{BD}{EC};\ \frac{BD}{DA}=\frac{EC}{AE};$$

$$\frac{AB}{AD}=\frac{AC}{AE};\ \frac{AB}{BD}=\frac{AC}{EC}.$$



The two triangles ABC and ADE are similar, being mutually equiangular. Hence,

$$AB:AD=BC:DE.$$

We can see now that all these results can be summed up very simply in the single statement, *that the ratio of any two corresponding lines in the figure equals the ratio of any other two corresponding lines.*

143. Other Proportions from a Given One.

1. **Powers and Roots.** *If four numbers are in proportion, like powers or like roots of these terms are in proportion.*

If $\frac{a}{b}=\frac{c}{d}$, then $\frac{a^n}{b^n}=\frac{c^n}{d^n}$, and $\frac{a^{\frac{1}{n}}}{b^{\frac{1}{n}}}=\frac{c^{\frac{1}{n}}}{d^{\frac{1}{n}}}$. (Power and Root Axs.)

2. **Continued Proportion.** *In a series of equal ratios, the ratio of the sum of the antecedents to the sum of the consequents equals any one of the ratios.*

If $\frac{a}{g}=\frac{c}{h}=\frac{e}{k}(=r)$, then $a=gr$, $c=hr$, $e=kr$, since in each case the dividend (numerator) equals the divisor times the quotient. Then, adding these equations,

$$a+c+e=gr+hr+kr=(g+h+k)r.\ \text{(Add Ax.)}$$

$$\therefore \frac{a+c+e}{g+h+k}=r, \text{ or } \frac{a+c+e}{g+h+k}=\frac{a}{g}.\ \text{(Div. Ax. and Ax. 8. § 36.)}$$

This result easily takes on a geometrical meaning by reference to the figure of the preceding article.

Let $e=DE$, $g=AB$, $h=AC$, $k=BC$.

Then, we have, *the ratio of the perimeter of triangle ADE to the perimeter of triangle ABC equals the ratio of any two corresponding sides.*

By using a letter for each side whatever the number of sides a polygon has, this theorem is easily extended to the case of *any* similar polygons. In fact, this theorem appears in algebras mainly because it has this application in geometry.

144. Exercise in Proportion.

1. If $y : x = 7 : 2$, find what the ratio $x : y$ equals. See § 140, 2.

2. From $12\,x = 18\,y$ find the ratio of x to y. See § 139, 2.

3. From $4\,x - 9\,y = 2\,x + 5\,y$ find the ratio of x to y.

4. From $3\,x + y : y = 17 : 8$ find the ratio of x to y. (§ 139, 1.)

5. If the ratio of m to n is $\frac{7}{4}$, what is the ratio of $m + n$ to $m - n$? (§ 141.)

6. Solve $x^2 - 4 : x^2 - 9 = x^2 - 5\,x + 6 : x^2 + 4\,x + 3$. (§ 139, 1.)

7. If $a : b = b : c$, prove that $a + b : b + c = a : b$.
(141, 1, and § 140, 1.)

8. What number must be added to each term of the ratio $m : n$ so that the new ratio may equal the ratio of $p : q$? (Let $x =$ the number.)

9. Find a mean proportional between $a^2 + 2ab + b^2$ and $a^2 - 2\,ab + b^2$. (§ 138.) (Let $x =$ mean proportional.)

10. Find the fourth term of the proportion whose first three terms are $\frac{1}{a}, \frac{1}{b}, \frac{1}{c}$.

11. Find x from the proportion

$$6x + a : 4\,x + b = 3x - b : 2x - a.$$

12. Four given numbers are represented by m, n, p, q. What number added to each will give four numbers that are in proportion? (Let $x =$ the number added to each.)

13. Find two numbers whose sum, difference, and product are proportional to m, n, and p. (Let x and y be the numbers. See § 143, 2, where a, c, and e are proportional to b, d, and f.)

14. Find two numbers such that their sum is to their difference as $5:1$, and their sum is to their product as $5:4$.

15. Prove that if the terms of $a:b=c:d$ are divided by the respective terms of $a':b'=c':d'$, the quotients are in proportion.

16. If $a:b=c:d$, prove that $\dfrac{2a+3c}{2a-3c}=\dfrac{8b+12d}{8b-12d}$.

SUGGESTION. First alternate the given proportion, then multiply the first ratio by $\frac{2}{3}$, and the second ratio by $\frac{8}{12}$, then apply "addition" and "subtraction" to both ratios in turn, and finally divide the terms of one proportion by the corresponding terms of the other.

17. If $a:b=c:d$, prove that $a^2+b^2:a^2-b^2=c^2+d^2:c^2-d^2$.

18. If $a:b=c:d$, prove that $a+b:c+d=\sqrt{a^2+b^2}:\sqrt{c^2+d^2}$.

SUGGESTION. Take first $a:b=c:d$ and $a^2:b^2=c^2:d^2$ by "addition."

19. Prove that either root of $x^2-q=0$ is a mean proportional between the roots of $x^2+px+q=0$.

20. If $a:b=c:d$ show that $ab+cd$ is a mean proportional between a^2+c^2 and b^2+d^2.

SUGGESTION. State the proportion, and then multiply means and extremes together. Show that this equation reduces to the given proportion, if $a:b=c:d$. Then reverse the steps.

21. What is the ratio of the mean proportional between a and b to the mean proportional between a and d? Give this in its simplest form.

22. If $(a+b+c+d)(a-b-c+d)=(a-b+c-d)(a+b-c-d)$, prove that $a:b=c:d$.

23. If $a:b=4:5$, $d:f=5:2$, $e:c=6:7$, $d:b=7:3$, and $f:c=4:3$, $a:b:c:d:e:f=$ what continued ratio of numbers?

24. The product of two numbers is 112, and the difference of their cubes is to the cube of their difference as $31:3$. What are the numbers?

II. VARIATION

145. Variation. One quantity is said to vary directly as another, when the two are so related that the ratio of any two values of the one is equal to the ratio of the corresponding values of the other. The special symbol for variation is $\propto$, read "varies as." Practically it is little used.

Direct Variation. In ordinary scales where a weight is moved from 0 along a scale beam until the scale balances, the weight of an object w varies directly as the distance d the scale weight is moved, or $w \propto d$. Thus, if the scale weight is moved from 0 twice as far one time as another, the first object is twice as heavy as the other. The circumference c of a circle varies as its diameter d, or $c \propto d$. The graph for direct variation is a straight line. See § 128.

Inverse Variation. The daily wages w a man gets for doing a job for a fixed sum varies inversely as the number of days n he works, or $w \propto \frac{1}{n}$. The *greater* the number of days he works, the *less* money he gets per day. The number of revolutions n the driving wheels of a locomotive make in going a mile varies inversely as the diameter d of the wheels, or $n \propto \frac{1}{d}$.

Other Forms of Variation. The distance s through which a falling body moves varies as the square of the time t it is moving, or $s \propto t^2$. Thus, if the time is trebled, the distance is multiplied by 9. Again, the power of attraction a of the heavenly bodies for one another varies inversely as the square of their distance apart d, or

$$a \propto \frac{1}{d^2}.$$

In any problem in variation the symbol $\propto$ is replaced by an equality sign by introducing a constant factor c before the right member. Thus, $w \propto d$ becomes $w = cd$; $w \propto \frac{1}{n}$ becomes $w = \frac{c}{n}$; etc.

In any given physical problem c has to be found by experiment to fit the units of distance, weight, time, etc. Thus, in $s = ct^2$, if distance is measured in feet, time in seconds, and gravity is the force, $c = \frac{1}{2} g$, in which $g = 32.16$.

The graph for inverse variation is the hyperbola, since $w \cdot n = c$, where c is constant. See § 129, 4 and *e*.

146. Variation a Form of Proportion. To show that variation is a form of proportion, take the case of the diameter of a circle and its circumference.

Thus, let d_1 and d_2 be the diameters, and c_1 and c_2 the circumferences of any two circles. Then,

$$\frac{c_1}{d_1} = \frac{c_2}{d_2}.$$

(By theorem: The ratio of the circumference to the diameter is the same for all circles.)

By alternation (§ 140, 1), it follows that

$$\frac{c_1}{c_2} = \frac{d_1}{d_2}.$$

Thus, the ratio of any two circumferences equals the ratio of their diameters. This is precisely what is meant by saying the circumferences of circles vary as their diameters, or that $c \propto d$. If the circumference of a circle is doubled or trebled, its diameter likewise is doubled or trebled.

Again, since the area of a circle equals π times the square of its radius, we can say that the areas of circles vary as the squares of their radii, or $a \propto r^2$.

To show this let a_1 and a_2 be the areas of two circles, and r_1 and r_2 their respective radii. Then,

$$\frac{a_1}{a_2} = \frac{\pi r_1^2}{\pi r_2^2} = \frac{r_1^2}{r_2^2}. \quad \text{(By dividing both terms by } \pi\text{, § 29.)}$$

Since a problem in variation is a problem in proportion, three terms are always given to find the fourth. Two of these terms give a ratio, whose value is the constant c referred to in § 145.

To solve a problem in variation, (1) *First replace the symbol* $\propto$ *by* $=$ *at the same time introducing the constant factor* c *before the right member.*

(2) *Then substitute two of the given quantities which correspond in the equation found in* (1), *getting the value of* c *therefrom. Substitute this value of* c *and the remaining given quantity in the equation of* (1) *to find the fourth or required term.*

147. Exercise in solving Problems in Variation.

1. The weight W of a cube varies as the cube of its edge e, or $W \propto e^3$. If when W is 26 lb. e is 4 in., what is W when e is 5 in.?

SOLUTION. $W = ce^3$. Then $26 = c \times 64$, whence $c = \frac{13}{32}$. Then $W = \frac{13}{32} \times 125 = 50.8^-$ lb.

2. The cost C of plastering a wall varies as the product of its length l and height h, or $C \propto lh$. If it costs $11.70 to plaster a wall 21 ft. long by 12 ft. high, what will it cost to plaster a wall 37 ft. long by 11 ft. high?

3. The distance s fallen by a body from rest varies as the square of the time t during which it falls. If owing to the resistance of the atmosphere it falls only 560 ft. in 6 sec., what distance will it fall in 5 sec.?

NOTE. When the resistance of the atmosphere is taken into account, the law holds only approximately, but the error will be slight.

4. The illumination i of a book by a lamp varies inversely as the square of the distance d of the book from the light. If the illumination at 18 in. is 48 candle power, what quantity of light will fall on the book at 4 ft.?

5. If surface of sphere is given by formula $s = 4\pi r^2$, how does s vary as r changes? *Ans.* s varies as square of r.

6. If $v = \frac{4}{3}\pi r^3$ is formula for volume of a sphere, how does v vary with r? If r is doubled, what happens to v? How does r vary with v? *Ans.* As the cube root of v.

7. In $t = 2\pi\sqrt{\frac{l}{g}}$ how does t vary with l? With g?

Ans. to last. Inversely as the square root of g.

8. If T is tension on a string, W the weight of a unit length of it, l its length, and N the number of its vibrations per second, tell from the formula

$$N : N' = \frac{\sqrt{T}}{l\sqrt{W}} : \frac{\sqrt{T'}}{l'\sqrt{W'}}$$

how N varies with T; with W; with l.

148. Variation and Functionality. In § 120, 2, we developed the idea of a function gradually changing in value as the quantity on which it depends changes. We there illustrated this change by means of graphs.

We can now throw some additional light on this subject from our brief study of variation. We see in every one of the formulas given, how one quantity is dependent for its value on another quantity in the formula, and how one changes continuously as the other changes continuously.

Thus, the quantity of light falling on a page of a book varies inversely as the square of the distance from the light. As the book is moved away from the light the quantity of light falling on the book continuously decreases rapidly.

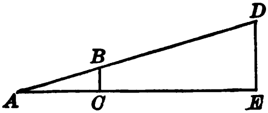
The lifting force of a simple lever varies directly with the power and also with the length of the power arm, and inversely with the length of the weight arm.

The horse power of an engine varies directly as the steam or gas pressure; it varies also directly as the length of the stroke of the piston; also directly as the area of the cylinder; also directly as the number of strokes per second.

It must be plain to the student from these illustrations that the range of application of the idea of variation or functionality is as wide as science itself, whether theoretical or applied. One branch of higher mathematics, called the calculus, deals largely with the continuous changes in values of functions.

III. THE RATIOS OF THE SIDES IN RIGHT TRIANGLES*

149. Similar Triangles. Similar triangles are defined in geometry as those having their corresponding angles equal and their corresponding sides proportional.



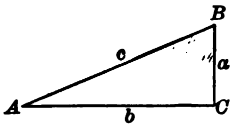
Now two right triangles are similar if they have an acute angle of one equal to an acute angle of the other.

Hence, if ABC and ADE are right triangles having the acute angle A common,

$$\frac{BC}{AB}=\frac{DE}{AD};\quad \frac{AC}{AB}=\frac{AE}{AD};\quad \frac{BC}{AC}=\frac{DE}{AE}.$$

Thus, we see that so long as two right triangles have the same-sized acute angle A, no matter how much larger one triangle is than the other, the ratio of any two sides of one triangle is equal to the ratio of the corresponding sides of the other triangle.

150. The Ratios of the Sides of a right triangle to one another are called the **trigonometrical functions** of its acute angles. These functions have been given distinctive names.



Let A be an acute angle of a right-angled triangle, with the sides and angles marked as in the diagram, a being opposite angle A, and b opposite angle B. There are six ratios that can be formed out of the three sides, viz.:

$\frac{a}{c}, \frac{b}{c}, \frac{a}{b}, \frac{b}{a}, \frac{c}{b}, \frac{c}{a}.$ Of the last two, called "secant" (sec) and "cosecant" (cosec), we will make no use.

Calling the others "sine," "cosine," "tangent," "cotangent," we now write

* The remainder of this chapter can be omitted if desired. Several reasons can be given for taking it up here if there is time. It deals with practical problems, some appearing in physics, is a preparation for learning to use the logarithmic table, and gives a good start in trigonometry. Probably the chief difficulties experienced by students in its study are algebraic in character.

(1) Sine $A = \frac{a}{c}$; or, sine $A = \frac{\text{side opposite to } A}{\text{hypotenuse}}$.

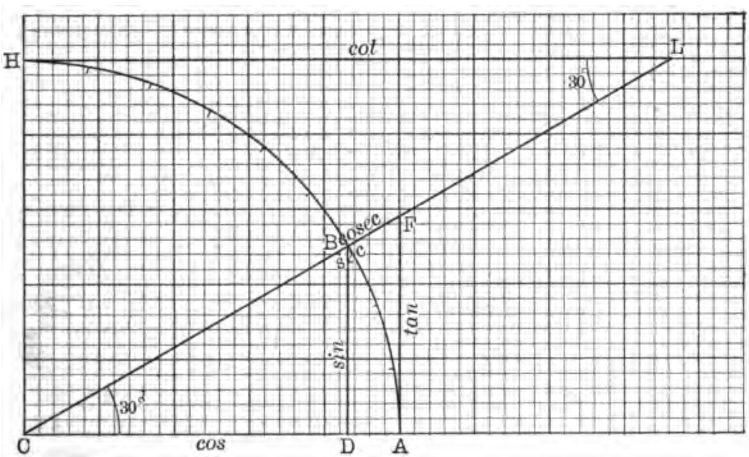
(2) Cosine $A = \frac{b}{c}$; or, cosine $A = \frac{\text{side adjacent to } A}{\text{hypotenuse}}$.

(3) Tangent $A = \frac{a}{b}$; or, tangent $A = \frac{\text{side opposite to } A}{\text{side adjacent to } A}$.

(4) Cotangent $A = \frac{b}{a}$; or, cotangent $A = \frac{\text{side adjacent to } A}{\text{side opposite to } A}$.

a. The prefix "co" in cosine and cotangent, means "complement," so that "cosine" means "complement sine," that is, "sine of complement." Notice that B (which equals $90° - A$) is the *complement* of A. Also that sine $B = \frac{b}{c} \left(\frac{\text{side opposite } B}{\text{hypotenuse}}\right) =$ cosine A; and that cosine $B = \frac{a}{c} =$ sine A.

151. Construction of a Table of Trigonometrical Functions for every 10° of angle from 0° to 90°.



We construct a quadrant of a circle (see diagram) with the lower left-hand corner of a sheet of squared paper as center and having a radius of 5 cm. We then divide the quadrant arc, with a protractor or compass, into nine equal parts of 10° each, and construct a straight line from the center through the 30° point of division of the arc, producing it until it meets the two tangent lines *AF* and *HL*. By reading directly from the squared paper, we can tell the length of all vertical distances, such as *BD* and *AF*, and of all horizontal lines, such as *CD* and *HL*, these lines belonging to the angle of 30°, as shown on the diagram. Notice that $CA = CB = CH = 5$ cm.

Now by the definitions in § 150 we have

$$\sin 30° = \frac{BD}{BC} = \frac{2.5 \text{ c.m.}}{5 \text{ cm.}} = .50.$$

$$\cos 30° = \frac{CD}{BC} = \frac{4.3 \text{ cm.}}{5 \text{ cm.}} = .86.$$

$$\tan 30° = \frac{AF}{AC} = \frac{2.9 \text{ cm.}}{5 \text{ cm.}} = .58.$$

In getting cot 30°, to avoid dividing *AC*, 5 cm., by *AF*, 2.9 cm., we use triangle *CHL*, in which angle $L = 30°$.

$$\cot 30° = \frac{HL}{HC} = \frac{8.6^{+} \text{ cm.}}{5 \text{ cm.}} = 1.7^{+}.$$

The student may now make a diagram like that just given, but whose radius is 10 cm., and construct lines not merely through the center and the 30° point, but also through the center and *all* the points of division of the arc, extending them until they meet the two tangents *AF* and *HL* (which lines will have to be made longer than in the diagram given in the book). From this diagram he may fill out the blank spaces in the following Natural Function table by getting the corresponding measurements and quotients for the angles 0°, 10°, 20°, etc. Notice that the lines corresponding to *BD* and *AF* for 0° are each 0, and that the lines corresponding to *HL* when the angle is 0° is infinite in length (∞), as also the line corresponding to *AF* when the angle is 90°.

Explanation. When filling in the values for the degrees at the left side of the table, read the functions at the top, but when filling in the values for the degrees at the right of the table, read the functions at the bottom. It will be found that
sin 10° = cos 80° (§ 150, *a*),
tan 30° = cot 60°,
etc.

Natural Function Table

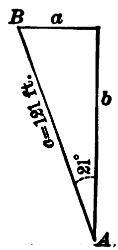
	Sin	Tan	Cot	Cos	
0°	0	0	∞	1.00	90°
10°					80°
20°					70°
30°	.50	.58	1.7	.86	60°
40°					50°
	Cos	Cot	Tan	Sin	

The student may now compare the values he has obtained with those given in the Table of Natural Functions found on page 164.

a. It is important to note that each of the expressions "sin *A*," or "cos 20°," or the like, denotes *one number*, and has to be treated as such in making algebraical transformations. In algebra each letter in "sin *A*" would denote a number, and sin *A* would signify their product; but in trigonometry these four letters are handled *as if they were a single letter*.

152. Use of the Table of Functions. If we know the values of certain "parts" (sides or angles) of any right triangle, we can find the remaining parts by means of the trigonometrical ratios to be found in the table, page 164, for all the different sizes of angles 0°–90°. Examples will make this plain.

153. Exercise in Solving Problems using the Four-Place Natural Function Table (page 164).

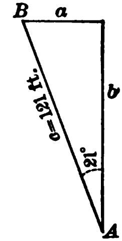


1. A man walks 121 ft. in the direction 21° west of north. How far west has he gone? How far north?

Solution. (1) To find *a*, the distance directly west.

We have $\sin A = \frac{a}{c}$ (§ 150); $a = c \times \sin A$. (Mult. Ax.)

Angle	Sine	Tangent	Cotangent	Cosine	
0°	0	0	∞	1	90°
1	.0175	.0175	57.2900	.9998	89
2	.0349	.0349	28.6363	.9994	88
3	.0523	.0524	19.0811	.9986	87
4	.0698	.0699	14.3006	.9976	86
5	.0872	.0875	11.4301	.9962	85
6	.1045	.1051	9.5144	.9945	84
7	.1219	.1228	8.1443	.9925	83
8	.1392	.1405	7.1154	.9903	82
9	.1564	.1584	6.3138	.9877	81
10	.1736	.1763	5.6713	.9848	80
11	.1908	.1944	5.1446	.9816	79
12	.2079	.2126	4.7046	.9781	78
13	.2250	.2309	4.3315	.9744	77
14	.2419	.2493	4.0108	.9703	76
15	.2588	.2679	3.7321	.9659	75
16	.2756	.2867	3.4874	.9613	74
17	.2924	.3057	3.2709	.9563	73
18	.3090	.3249	3.0777	.9511	72
19	.3256	.3443	2.9042	.9455	71
20	.3420	.3640	2.7475	.9397	70
21	.3584	.3839	2.6051	.9336	69
22	.3746	.4040	2.4751	.9272	68
23	.3907	.4245	2.3559	.9205	67
24	.4067	.4452	2.2460	.9135	66
25	.4226	.4663	2.1445	.9063	65
26	.4384	.4877	2.0503	.8988	64
27	.4540	.5095	1.9626	.8910	63
28	.4695	.5317	1.8807	.8829	62
29	.4848	.5543	1.8040	.8746	61
30	.5000	.5774	1.7321	.8660	60
31	.5150	.6009	1.6643	.8572	59
32	.5299	.6249	1.6003	.8480	58
33	.5446	.6494	1.5399	.8387	57
34	.5592	.6745	1.4826	.8290	56
35	.5736	.7002	1.4281	.8192	55
36	.5878	.7265	1.3764	.8090	54
37	.6018	.7536	1.3270	.7986	53
38	.6157	.7813	1.2799	.7880	52
39	.6293	.8098	1.2349	.7771	51
40	.6428	.8391	1.1918	.7660	50
41	.6561	.8693	1.1504	.7547	49
42	.6691	.9004	1.1106	.7431	48
43	.6820	.9325	1.0724	.7314	47
44	.6947	.9657	1.0355	.7193	46
45	.7071	1.0000	1.0000	.7071	45
	Cosine	Cotangent	Tangent	Sine	Angle



Turning to the Table on opposite page, for the value of sin A, and substituting it in the last equation, we get

$$a = 121 \times .3584 = 43.37^- \text{ ft.} \quad \textit{Ans.}$$

(2) To find B. $B = 90° - A = 90° - 21° = 69°$.

(3) To find b, the distance directly north. We have

$\sin B = \frac{b}{c}$ (§ 150). Hence, $b = c \times \sin B$. (Mult. Ax.)

Then referring to the table for the value of sin 69° (and remembering that when the angle is over 45°, the function name must be read at the bottom of the page), we get

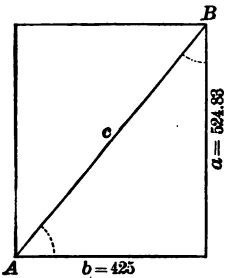
$$b = 121 \times .9336 = 112.97^- \text{ ft.} \quad \textit{Ans.}$$

2. Given $A = 32°$, $c = 259$ yd., to find a, B, and b.

3. Given $A = 42°$, $c = 12.4$ mi., to find a, B, and b.

4. Given $A = 73°$, $c = 2542$ m., to find a, B, and b.

5. Given the lengths of two sides of a rectangular block as 425 ft. and 524 ft. 10 in., to find the two angles this diagonal makes with the sides, and the length of the diagonal from one corner to the opposite one.



SOLUTION. — (1) To find A, the angle between the diagonal and a side. We have

$$\tan A = \frac{a}{b} = \frac{524.83}{425} = 1.2349.$$

Turning to the Table, we find if

$\tan A = 1.2349$ that $A = 51°$. *Ans.*

(2) To find B. $B = 90° - 51° = 39°$.

(3) To find the diagonal c.

$\sin A = \frac{a}{c}$. Hence, $c \times \sin A = a$, and $c = \frac{a}{\sin A}$. (Axs. ?)

Substituting the value of sin A found in the table, we have

$$c = \frac{524.83}{.7771} = 675.4 \text{ ft.} \quad \textit{Ans.}$$

a. This answer can be checked by geometry, using the formula $c^2 = a^2 + b^2$. Notice that c can be found with less work by trigonometry than by geometry.

6. Given $a = 36.506$, $b = 82$, to find A, B, and c.

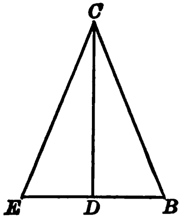
Find the remaining parts with the following conditions given.

7. $a = 40.394$, $b = 190$.

8. $a = 90.9,\ b = 225.$

9. $a = 63.028,\ b = 70.$

10. Given one side 18 ft. of a regular octagon (eight-sided equilateral and equiangular polygon), to find the perpendicular distance from its center to one side, and the distance from center to a vertex.



SOLUTION. In the figure, C is the center of the polygon, EB is one side, and CD is the perpendicular from the center on the side. Then DCB is a right triangle in which we will regard $\angle DCB$ as angle A of our formula, DB as side a, CD as side b, and CB as side c. Then, $A = \frac{1}{16}$ of 360°, or $22\frac{1}{2}°$, and the opposite side $a = 9$ ft.

(1) To find b. We have,

$$\tan A = \frac{a}{b};\ \text{therefore } b \times \tan A = a, \text{ whence } b = \frac{a}{\tan A} = \frac{9}{.4143} = 21.72.$$

EXPLANATION. The .4143, which is the tangent of $22\frac{1}{2}°$, is found by "interpolating." From the table,

$$\tan 22° = .4040,\ \tan 23° = .4245;$$
$$\tan 22\tfrac{1}{2}° = .4040 + \tfrac{1}{2}(.4245 - .4040) = .4143 \text{ (to nearest .0001).}$$

(2) To find c. We write

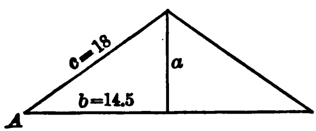
$$\sin A = \frac{a}{c};\ \therefore c = \frac{a}{\sin A}\ (\text{Axs.?}) = \frac{9}{.3827} = 23.52.$$

11. Given $A = 52\frac{1}{3}°$, $a = 35$ inches, to find b, c, and B.

Find the remaining parts in the following three problems, as in Ex. 10.

12. $A = 33\frac{3}{4}°,\ a = 25.$ **13.** $B = 39°,\ a = 72.$ **14.** $A = 48\frac{1}{4}°,\ b = 17.6.$

15. A house 29 ft. wide has a roof which measures 18 ft. from the ridge to either side of the house. Find the angle the roof makes with the horizontal, and the height of the ridge above the top story.



SOLUTION. (1) $\text{Cos}\, A = \frac{b}{c} = \frac{14.5}{18} = .8056.$

Now, .8056 lies between cos 36° = .8090 and cos 37° = .7986. The difference for 1° is 104, and the difference between 8090 and 8056 is 34. Hence, the number of *minutes* of angle to be added to 36° is $\frac{34}{104}$ of $60'$, or $20'^-$. Therefore $A = 36^\circ\ 20'^-$. *Ans.*

(2) $\sin A = \frac{a}{c}$; $\therefore a = c \times \sin A = 18 \times .5925 = 10.67$ ft. *Ans.*

EXPLANATION. To find the sine of $36^\circ\ 20'$, one gets $\sin 36^\circ$ and adds to it $\frac{20}{60}$ of the (tabular) difference between $\sin 36^\circ$ and $\sin 37^\circ$.

IMPORTANT NOTE. The student should see that while the sine and tangent *increase* as the angle increases, though not in proportion to the increase in the angle, the cosine and cotangent *decrease* as the angle increases.

Thus,

$\sin 0^\circ = .0000$	$\tan 10^\circ = .1763$	$\cos 20^\circ = .9397$	$\cot 20^\circ = 2.7475$
$\sin 30^\circ = .5000$	$\tan 30^\circ = .5774$	$\cos 40^\circ = .7660$	$\cot 40^\circ = 1.1918$
$\sin 60^\circ = .8660$	$\tan 60^\circ = 1.7321$	$\cos 60^\circ = .5000$	$\cot 60^\circ = .5774$
$\sin 90^\circ = 1.0000$	$\tan 80^\circ = 5.6713$	$\cos 80^\circ = .1736$	$\cot 80^\circ = .1763$

154. Rule for finding a Function from its Angle in the Natural Table, p. 164.

Look first for the function corresponding to the given number of degrees. Then, if there are minutes, get the "tabular difference" by subtracting the number found from that for the next higher number of degrees, and multiply this difference by the fraction of a degree denoted by the given number of minutes. If the function is a sine or a tangent, **add** *the product to the first number found in the table; but if it is a cosine or a cotangent,* **subtract** *it from that number.*

155. Rule for finding an Angle containing Minutes from its Function in the Natural Table.

First find the numbers in the table between which the given function lies, and subtract the less from the greater for the "tabular difference" for the corresponding interval. If the function is a sine or a tangent, subtract the smaller of the two numbers in the table from the given value of the function and take such a part of 60' as the difference found is of the tabular difference, and add this number of minutes to the number of degrees corresponding to the lesser tabular number; but if the function is a cosine or a cotangent, subtract the given value of the function from the larger tabular number and take such a part of 60' as the difference found is of the tabular difference, *and add this number of minutes to the number of degrees corresponding to the larger of the two tabular numbers.*

156. Rule for solving Right Triangles for Unknown Parts. In every problem two parts will always be given to find the remaining three. Of these, one, an acute angle, is always found by subtracting the other acute angle from 90°. To find the remaining parts two formulas will be needed.

To provide the formula for any given case, select that one of the formulas, (1), (2), (3), § 150, *which contains the two given parts and the one that is to be found, and solve by algebra for the unknown quantity.*

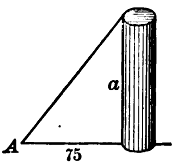
157. Exercise in solving for Unknown Parts in Right Triangles.

Find the remaining parts in the following nine problems:

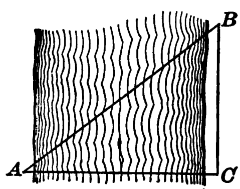
1. $A = 25° \ 40'$, $b = 84$.
2. $A = 25° \ 35'$, $c = 25.6$.
3. $b = 73$, $c = 99$.
4. $b = 1291$, $c = 1674$.
5. $a = 43.1$, $c = 56.19$.
6. $a = 29.7$, $B = 73° \ 23'$.
7. $B = 24° \ 54'$, $a = 78.44$.
8. $a = 192.5$, $b = 173.8$.
9. $a = 19.65$, $b = 27.84$.
10. $B = 17° \ 18'$, $c = 84.7$.

11. How high is a tower that casts a shadow 75 ft. in length, when the "angle of elevation" of the sun is 52° 35'?

SUGGESTION. $A = 52° \ 35'$, $b = 75$; to find a.



12. A flagstaff 90 ft. high casts a shadow 117 ft. long. Find the "altitude" of the sun, that is, the angle the line from the sun through the top of the staff makes with a horizontal line where it meets the ground. The altitude angle corresponds to angle of elevation A in the preceding exercise.

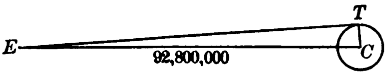


13. A distance BC is measured 124 ft. along the bank of a river from a point C opposite a tree at A on the other bank (ACB = right angle), and angle B is

measured and found to be 52° 24′. What does *AC*, the distance across the river, equal?

14. One side of a regular heptagon (seven-sided polygon) is 17 in. Find first the perpendicular on a side, and then the area of the heptagon. (See Ex. 10, § 153.)

15. The top of a lighthouse is 200 ft. above sea level, and the angle of depression to a buoy is 9° 53′. Find the horizontal distance of the buoy from the lighthouse.



16. The angle *E* at the earth's center subtended by the sun's radius is approximately 16′, and the sun's distance is 92,800,000 mi. Find the sun's radius in miles.

17. What must be the length of a ladder set at an angle of 71° 14′ with the ground to reach a window 22 ft. high?

18. The base of an isosceles triangle is 15 in., and the angle at the vertex is $46\frac{1}{2}°$. Find altitude of triangle.

19. The chord of a circle is 25 meters long, and the angle at the center subtended by it is 40° 15′. Find radius of circle.

20. Find radius of small circle in latitude 30° if it equals 3963 mi. × cos 30°. (See figure in § 151.)

21. Two forces act at *A* (figure in Ex. 5, § 153), the horizontal one *b* with force of 15 lb. and vertical one *a* with force of 19 lb. Find resultant force *c* and angle it makes with horizontal.

22. A river 1 mi. wide flows at the rate of 4 mi. an hour. A steam ferryboat with a speed of 8 mi. an hour has to run directly across. In what direction must she steer, and what time will she be in crossing?

23. The perimeter of a right triangle is 6.37 ft. and one of its acute angles is 33° 24′. Find the three sides.

NOTE. *Any oblique triangle* can be solved, if three parts are given, by properly drawing a perpendicular dividing it into two right triangles, and solving *both* triangles by the method of this chapter. See Crockett's *Elements of Plane Trigonometry*, pp. 107–110.

CHAPTER VIII

LOGARITHMS

I. NATURE OF LOGARITHMS

158. Calculations made through Use of Exponents. Whenever numbers which are the powers of one number, as, for instance, of 2, are to be multiplied, divided, raised to powers, or have their roots extracted, these operations can be performed very quickly, if a table containing its various powers has been prepared.

TABLE

NUMBER	POWER OF 2
2	2
4	2^2
8	2^3
16	2^4
32	2^5
64	2^6
128	2^7
256	2^8
512	2^9
1,024	2^{10}
2,048	2^{11}
4,096	2^{12}
8,192	2^{13}
16,384	2^{14}
32,768	2^{15}
65,536	2^{16}

1. Calculate 16×64 by use of the table.

SOLUTION. $16 = 2^4$. (From the table.)
$64 = 2^6$. (From the table.)
$\therefore 16 \times 64 = 2^4 \times 2^6 = 2^{10}$. (Law of exponents.)
But $2^{10} = 1024$. (From the table.)
$\therefore 16 \times 64 = 1024$. *Ans.*

2. Calculate $8 \times 128 \times 16$ by using the table.

3. Calculate $16,384 \div 256$ with table.

SOLUTION. $16384 = 2^{14}$. (From the table.)
$256 = 2^8$. (From the table.)
$\therefore 16384 \div 256 = 2^{14} \div 2^8 = 2^6 = 64$. (From the table.)

4. $\dfrac{64 \times 256 \times 16}{32 \times 512}$. 5. $\dfrac{8192 \times 512 \times 65536}{4096 \times 4 \times 32768}$.

6. Square 32 by use of table; also 256.

7. Extract square root of 16,384; of 4096.

8. Extract cube root of 32,768; of 4096.

9. Extract 4th root of 4096.

a. The examples just given show that calculations involving only numbers which are powers of 2 can be made very easily and expeditiously. Now, we can express *all* numbers as powers of 2 by the simple device of using *fractional exponents.* By calculations it is found that $15 = 2^{3.9}$; $144 = 2^{7.17}$. Similar expressions can be found for *any* number. Thus, we see how a table could be made with 2 for *base* which could be used to make calculations with any numbers. Moreover, some other number, as 3, could be used instead of 2 as the base in the table, all numbers being represented as powers of 3. As we shall see later, the number 10 is the base used in most tables of this kind.

159. Logarithms are Exponents. In $3^2 = 9$, we can say 2 is the logarithm of 9 to base 3. Similarly, since $3^4 = 81$, 4 is the logarithm of 81 to base 3. In logarithms three different numbers are always involved: (1) A number. (2) Its logarithm. (3) The base used.

The **logarithm of a given number** is the exponent of the power to which a base must be raised to produce this number.

A system of logarithms is a set of numbers with their logarithms all taken to the same base. Notice that the logarithm of 1 *in any system* is 0, since $a^0 = 1$ (§ 69).

SYSTEM OF LOGARITHMS WITH BASE 2

NUMBER	LOGARITHM	REASON	NUMBER	LOGARITHM	REASON
1	0	$2^0 = 1$			
2	1	$2 = 2$	$\frac{1}{2}$	−1	$2^{-1} = \frac{1}{2}$
3	1.5850	$2^{1.5850} = 3$	$\frac{1}{3}$	−1.5850	$2^{-1.5850} = \frac{1}{3}$
4	2	$2^2 = 4$	$\frac{1}{4}$	−2	$2^{-2} = \frac{1}{4}$
5	2.3223	$2^{2.3223} = 5$	$\frac{1}{5}$	−2.3223	$2^{-2.3223} = \frac{1}{5}$
8	3	$2^3 = 8$	$\frac{1}{8}$	−3	$2^{-3} = \frac{1}{8}$
etc.	etc.	etc.	etc.	etc.	etc.

The student is not supposed to know how the decimal logarithms, like 1.5850, are found. Originally they were obtained by a long process of extracting roots. (See the article "Logarithms" in the Encyclopædia Britannica.) Since logarithms are exponents, they may be interpreted as such. Thus, in the equation $2^{\frac{15850}{10000}} = 3$, we see that the 15850th power of the 10000 root of 2 equals 3, and if these operations were actually performed on 2, the result would be 3.

160. Notation and Terms. To avoid writing long exponents, such an equation as $2^{1.5850} = 3$ is changed into $\log_2 3 = 1.5850$, and is read "logarithm of 3 to base 2 equals 1.5850." The subscript indicating the base is usually omitted when 10 is the base.

The integral part of a logarithm is called its **characteristic**, and the decimal part its **mantissa**. Thus, in 1.5850, which is the logarithm of 3 to base 2, the 1 is the characteristic and the .5850 is the mantissa of the logarithm.

Express the following in the language of logarithms:

1. $2^4 = 16$.

SOLUTION. $\log_2 16 = 4$. Read 'the logarithm of 16 to base 2 is 4.'

2. $3^3 = 27$. 3. $10^3 = 1000$. 4. $4^2 = 16$. 5. $2^5 = 32$.

6. $3^{-2} = \frac{1}{9}$. 7. $10^2 = 100$. 8. $10^{-1} = \frac{1}{10}$. 9. $10^{-4} = .0001$.

Express the following equations, using the language of exponents:

10. $\log_2 8 = 3$. 11. $\log_4 64 = 3$. 12. $\log_5 25 = 2$.

13. $\log_8 4 = \frac{2}{3}$. 14. $\log_{10} .01 = -2$. 15. $\log_9 27 = \frac{3}{2}$.

16. What is the logarithm of 9 in a system *whose base is* 3? Of 81? Of 27? Of 3? Of $\frac{1}{9}$? Of $\frac{1}{27}$?

17. What is the logarithm of 256 in a system *whose base is* 16? Of 16? Of 4? Of 8? Of 64?

18. What is the logarithm of 100 in a system *whose base is* 10? Of 1000? Of 100000? Of $\frac{1}{10}$? Of .01? Of 1? Of .001?

19. What is the logarithm of 81 in a system *whose base is* 27? Of 3? Of 9? Of 243? Of $\frac{1}{27}$? Of $\frac{1}{9}$? Of $\frac{1}{81}$?

161. The Briggsian or Common System of Logarithms uses 10 for its base.

1. Since $10^4 = 10{,}000$, then, $\log 10{,}000 = 4$ (§ 160).

$10^3 = 1{,}000$,	then,	$\log 1{,}000 = 3$.	
$10^2 = 100$,	then,	$\log 100 = 2$.	
$10^1 = 10$,	then,	$\log 10 = 1$.	
$10^0 = 1$ (§ 69),		$\log 1 = 0$.	
$10^{-1} = .1$	(§ 69),	$\log .1 = -1$.	
$10^{-2} = .01$		$\log .01 = -2$.	
$10^{-3} = .001$		$\log .001 = -3$.	
etc.		etc.	

2. It is clear from the first two lines of 1 that any number between 1000 and 10,000 (as 6924.7), having four figures to the left of the decimal point, has for its logarithm 3 + a decimal, because its logarithm must lie *between* 3 and 4. Again, any number between 100 and 1000, having three figures to the left of the decimal point, has for its logarithm 2 + a decimal, because its logarithm must lie between 2 and 3; and any number between 10 and 100, having two integral orders, has for its logarithm 1 + a decimal, because its logarithm must lie between 1 and 2; and finally, any number between 1 and 10, as 5.698, having but one integral order, has for its logarithm 0 + a decimal, because its logarithm must lie between 0 and 1. Hence, generally

3. *The characteristic of the logarithm of any number greater than* 1 *is one less than the number of its integral orders.*

Thus, the characteristic of the logarithm of 729.4 is 2; of 7460 is 3; of 3.96 is 0.

4. In the common system, in which the base is 10, *the mantissas do not change when the decimal point is moved.*

To understand why this is so, take $10^{.1038} = 1.27$. Multiplying or dividing the members of this equation by $10^2 = 100$, or $10 = 10$, etc., we have, by the laws of exponents, § 17:

$10^{2.1038} = 127$	(Mult. Ax.),	or, $\log 127$	$= 2.1038.$
$10^{1.1038} = 12.7$	(Mult. Ax.),	or, $\log 12.7$	$= 1.1038.$
$10^{.1038-1} = .127$	(Div. Ax.),	or, $\log .127$	$= \bar{1}.1038.$
$10^{.1038-2} = .0127$	(Div. Ax.),	or, $\log .0127$	$= \bar{2}.1038.$
etc.			etc.

The minus signs *over* the characteristics at the right belong to the characteristics only. Thus, by regarding the characteristics only as changing signs, mantissas stay the same no matter where the decimal point in the number is changed to, and *mantissas are always positive.*

5. By examining the last two lines of the table in 1 of this article we see that any number between .001 and .01, having 2 ciphers before its first significant figures, has -3 for characteristic, since its logarithm lies between -3 and -2 and the mantissa added is positive. Again, any number between .01 and .1, having one cipher before its first significant figure, has -2 for its characteristic, since its logarithm lies between -2 and -1 and the mantissa added is positive; also, any number between .1 and 1, having *no* cipher before its first significant figure, has -1 for characteristic, since its logarithm lies between -1 and 0 and the mantissa added is positive. Hence, generally (see 3, p. 173)

6. *The characteristic of the logarithm of any number equals the number of places from unity to the highest order filled by a significant figure, positive if to left, negative if to right.*

Thus, the characteristic of the logarithm of .00468 is -3; of .3794 is -1; of .000067 is -5; of 867 is 2; of 6 is 0; etc.

162. Explanation of the Four Place Logarithmic Table. The Table (see pp. 176, 177) gives the mantissa only of all the numbers from 100 to 999. The first two figures of the numbers are found in the column marked "N"; the third is one of the ten figures at the top of the table. Thus, the mantissa of 487 is found by taking the 48 in the "N" column and the 7 from the topmost row of figures; the mantissa sought is in the 7 column opposite 48 and is .6875. The characteristic of 487 is 2 (§ 161, 3), so that log 487 = 2.6875.

At the intersection of lines and columns are found the 900 mantissas corresponding to the 900 numbers from 100 to 999, the first two figures determining the line and the last the column. A decimal point before each mantissa is understood.

The columns of numbers marked "D" are merely the differences between the mantissas. Thus, the difference between .3032 and .3054 is 22 as in the table; and between .4133 (corresponding to 259) and .4150 (corresponding to 260) is 17.

This table gives directly the mantissas of all numbers consisting of three figures, preceded or followed by any number of ciphers. Thus, the mantissa of 31 is the same as that for 310, and is .4914 (see § 161, 4); the mantissa of the logarithm of 8 is the same as that for 800; the mantissa for 320000 or for .032 is the same as that for 320.

163. Exercise in finding the Logarithms of Numbers whose Mantissas are given directly in the Table. Common fractions and mixed numbers must be reduced to decimals.

Find the logarithms of:

1. 329. *Ans.* 2.5172. See § 161, 3 and § 162.

2. 764.	**3.** 125.	**4.** 969.	**5.** 370.	**6.** 37.
7. 400.	**8.** 40.	**9.** 4.	**10.** 7.	**11.** 70.
12. 700.	**13.** 7000.	**14.** .0372.	**15.** .000561.	**16.** .000029.
17. .002.	**18.** 3680.	**19.** $\frac{1}{4}$.	**20.** $\frac{1}{25}$.	**21.** $4\frac{1}{2}$.
22. $47\frac{1}{5}$.	**23.** $12\frac{1}{2}$.	**24.** $.06\frac{1}{2}$.	**25.** $.003\frac{3}{4}$.	**26.** $\frac{7}{8}$.

N	0	D	1	D	2	D	3	D	4	D	5	D	6	D	7	D	8	D	9	D
10	0000	43	0043	43	0086	42	0128	42	0170	42	0212	41	0253	41	0294	40	0334	40	0374	40
11	0414	39	0453	39	0492	39	0531	38	0569	38	0607	38	0645	37	0682	37	0719	36	0755	37
12	0792	36	0828	36	0864	35	0899	35	0934	35	0969	35	1004	34	1038	34	1072	34	1106	33
13	1139	34	1173	33	1206	33	1239	32	1271	32	1303	32	1335	32	1367	32	1399	31	1430	31
14	1461	31	1492	31	1523	30	1553	31	1584	30	1614	30	1644	29	1673	30	1703	29	1732	29
15	1761	29	1790	28	1818	29	1847	28	1875	28	1903	28	1931	28	1959	28	1987	27	2014	27
16	2041	27	2068	27	2095	27	2122	26	2148	27	2175	26	2201	26	2227	26	2253	26	2279	25
17	2304	26	2330	25	2355	25	2380	25	2405	25	2430	25	2455	25	2480	24	2504	25	2529	24
18	2553	24	2577	24	2601	24	2625	23	2648	24	2672	23	2695	23	2718	24	2742	23	2765	23
19	2788	22	2810	23	2833	23	2856	22	2878	22	2900	23	2923	22	2945	22	2967	22	2989	21
20	3010	22	3032	22	3054	21	3075	21	3096	22	3118	21	3139	21	3160	21	3181	20	3201	21
21	3222	21	3243	20	3263	21	3284	20	3304	20	3324	21	3345	20	3365	20	3385	19	3404	20
22	3424	20	3444	20	3464	19	3483	19	3502	20	3522	19	3541	19	3560	19	3579	19	3598	19
23	3617	19	3636	19	3655	19	3674	18	3692	19	3711	18	3729	18	3747	19	3766	18	3784	18
24	3802	18	3820	18	3838	18	3856	18	3874	18	3892	17	3909	18	3927	18	3945	17	3962	17
25	3979	18	3997	17	4014	17	4031	17	4048	17	4065	17	4082	17	4099	17	4116	17	4133	17
26	4150	16	4166	17	4183	17	4200	16	4216	16	4232	17	4249	16	4265	16	4281	17	4298	16
27	4314	16	4330	16	4346	16	4362	16	4378	15	4393	16	4409	16	4425	15	4440	16	4456	16
28	4472	15	4487	15	4502	16	4518	15	4533	15	4548	16	4564	15	4579	15	4594	15	4609	15
29	4624	15	4639	15	4654	15	4669	14	4683	15	4698	15	4713	15	4728	14	4742	15	4757	14
30	4771	15	4786	14	4800	14	4814	15	4829	14	4843	14	4857	14	4871	15	4886	14	4900	14
31	4914	14	4928	14	4942	13	4955	14	4969	14	4983	14	4997	14	5011	13	5024	14	5038	13
32	5051	14	5065	14	5079	13	5092	13	5105	14	5119	13	5132	13	5145	14	5159	13	5172	13
33	5185	13	5198	13	5211	13	5224	13	5237	13	5250	13	5263	13	5276	13	5289	13	5302	13
34	5315	13	5328	12	5340	13	5353	13	5366	12	5378	13	5391	12	5403	13	5416	12	5428	13
35	5441	12	5453	12	5465	13	5478	12	5490	12	5502	12	5514	13	5527	12	5539	12	5551	12
36	5563	12	5575	12	5587	12	5599	12	5611	12	5623	12	5635	12	5647	11	5658	12	5670	12
37	5682	12	5694	11	5705	12	5717	12	5729	11	5740	12	5752	11	5763	12	5775	11	5786	12
38	5798	11	5809	12	5821	11	5832	11	5843	12	5855	11	5866	11	5877	11	5888	11	5899	12
39	5911	11	5922	11	5933	11	5944	11	5955	11	5966	11	5977	11	5988	11	5999	11	6010	11
40	6021	10	6031	11	6042	11	6053	11	6064	11	6075	10	6085	11	6096	11	6107	10	6117	11
41	6128	10	6138	11	6149	11	6160	10	6170	10	6180	11	6191	10	6201	11	6212	10	6222	10
42	6232	11	6243	10	6253	10	6263	11	6274	10	6284	10	6294	10	6304	10	6314	11	6325	10
43	6335	10	6345	10	6355	10	6365	10	6375	10	6385	10	6395	10	6405	10	6415	10	6425	10
44	6435	9	6444	10	6454	10	6464	10	6474	10	6484	9	6493	10	6503	10	6513	9	6522	10
45	6532	10	6542	9	6551	10	6561	10	6571	9	6580	10	6590	9	6599	10	6609	9	6618	10
46	6628	9	6637	9	6646	10	6656	9	6665	10	6675	9	6684	9	6693	9	6702	10	6712	9
47	6721	9	6730	9	6739	10	6749	9	6758	9	6767	9	6776	9	6785	9	6794	9	6803	9
48	6812	9	6821	9	6830	9	6839	9	6848	9	6857	9	6866	9	6875	9	6884	9	6893	9
49	6902	9	6911	9	6920	8	6928	9	6937	9	6946	9	6955	9	6964	8	6972	9	6981	9
50	6990	8	6998	9	7007	9	7016	8	7024	9	7033	9	7042	8	7050	9	7059	8	7067	9
51	7076	8	7084	9	7093	8	7101	9	7110	8	7118	8	7126	9	7135	8	7143	9	7152	8
52	7160	8	7168	9	7177	8	7185	8	7193	9	7202	8	7210	8	7218	8	7226	9	7235	8
53	7243	8	7251	8	7259	8	7267	8	7275	9	7284	8	7292	8	7300	8	7308	8	7316	8
54	7324	8	7332	8	7340	8	7348	8	7356	8	7364	8	7372	8	7380	8	7388	8	7396	8

N	0	D	1	D	2	D	3	D	4	D	5	D	6	D	7	D	8	D	9	D
55	7404	8	7412	7	7419	8	7427	8	7435	8	7443	8	7451	8	7459	7	7466	8	7474	8
56	7482	8	7490	7	7497	8	7505	8	7513	7	7520	8	7528	8	7536	7	7543	8	7551	8
57	7559	7	7566	8	7574	8	7582	7	7589	8	7597	7	7604	8	7612	7	7619	8	7627	7
58	7634	8	7642	7	7649	8	7657	7	7664	8	7672	7	7679	7	7686	8	7694	7	7701	8
59	7709	7	7716	7	7723	8	7731	7	7738	7	7745	7	7752	8	7760	7	7767	7	7774	8
60	7782	7	7789	7	7796	7	7803	7	7810	8	7818	7	7825	7	7832	7	7839	7	7846	7
61	7853	7	7860	8	7868	7	7875	7	7882	7	7889	7	7896	7	7903	7	7910	7	7917	7
62	7924	7	7931	7	7938	7	7945	7	7952	7	7959	7	7966	7	7973	7	7980	7	7987	6
63	7993	7	8000	7	8007	7	8014	7	8021	7	8028	7	8035	6	8041	7	8048	7	8055	7
64	8062	7	8069	6	8075	7	8082	7	8089	7	8096	6	8102	7	8109	7	8116	6	8122	7
65	8129	7	8136	6	8142	7	8149	7	8156	6	8162	7	8169	7	8176	6	8182	7	8189	6
66	8195	7	8202	7	8209	6	8215	7	8222	6	8228	7	8235	6	8241	7	8248	6	8254	7
67	8261	6	8267	7	8274	6	8280	7	8287	6	8293	6	8299	7	8306	6	8312	7	8319	6
68	8325	6	8331	7	8338	6	8344	7	8351	6	8357	6	8363	7	8370	6	8376	6	8382	6
69	8388	7	8395	6	8401	6	8407	7	8414	6	8420	6	8426	6	8432	7	8439	6	8445	6
70	8451	6	8457	6	8463	7	8470	6	8476	6	8482	6	8488	6	8494	6	8500	6	8506	7
71	8513	6	8519	6	8525	6	8531	6	8537	6	8543	6	8549	6	8555	6	8561	6	8567	6
72	8573	6	8579	6	8585	6	8591	6	8597	6	8603	6	8609	6	8615	6	8621	6	8627	6
73	8633	6	8639	6	8645	6	8651	6	8657	6	8663	6	8669	6	8675	6	8681	5	8686	6
74	8692	6	8698	6	8704	6	8710	6	8716	6	8722	5	8727	6	8733	6	8739	6	8745	6
75	8751	5	8756	6	8762	6	8768	6	8774	5	8779	6	8785	6	8791	6	8797	5	8802	6
76	8808	6	8814	6	8820	5	8825	6	8831	6	8837	5	8842	6	8848	6	8854	5	8859	6
77	8865	6	8871	5	8876	6	8882	5	8887	6	8893	6	8899	5	8904	6	8910	5	8915	6
78	8921	6	8927	5	8932	6	8938	5	8943	6	8949	5	8954	6	8960	5	8965	6	8971	5
79	8976	6	8982	5	8987	6	8993	5	8998	6	9004	5	9009	6	9015	5	9020	5	9025	6
80	9031	5	9036	6	9042	5	9047	6	9053	5	9058	5	9063	6	9069	5	9074	5	9079	6
81	9085	5	9090	6	9096	5	9101	5	9106	6	9112	5	9117	5	9122	6	9128	5	9133	5
82	9138	5	9143	6	9149	5	9154	5	9159	6	9165	5	9170	5	9175	5	9180	6	9186	5
83	9191	5	9196	5	9201	5	9206	6	9212	5	9217	5	9222	5	9227	5	9232	6	9238	5
84	9243	5	9248	5	9253	5	9258	5	9263	6	9269	5	9274	5	9279	5	9284	5	9289	5
85	9294	5	9299	5	9304	5	9309	6	9315	5	9320	5	9325	5	9330	5	9335	5	9340	5
86	9345	5	9350	5	9355	5	9360	5	9365	5	9370	5	9375	5	9380	5	9385	5	9390	5
87	9395	5	9400	5	9405	5	9410	5	9415	5	9420	5	9425	5	9430	5	9435	5	9440	5
88	9445	5	9450	5	9455	5	9460	5	9465	4	9469	5	9474	5	9479	5	9484	5	9489	5
89	9494	5	9499	5	9504	5	9509	4	9513	5	9518	5	9523	5	9528	5	9533	5	9538	4
90	9542	5	9547	5	9552	5	9557	5	9562	4	9566	5	9571	5	9576	5	9581	5	9586	4
91	9590	5	9595	5	9600	5	9605	4	9609	5	9614	5	9619	5	9624	4	9628	5	9633	5
92	9638	5	9643	4	9647	5	9652	5	9657	4	9661	5	9666	5	9671	4	9675	5	9680	5
93	9685	4	9689	5	9694	5	9699	4	9703	5	9708	5	9713	4	9717	5	9722	5	9727	4
94	9731	5	9736	5	9741	4	9745	5	9750	4	9754	5	9759	4	9763	5	9768	5	9773	4
95	9777	5	9782	4	9786	5	9791	4	9795	5	9800	5	9805	4	9809	5	9814	4	9818	5
96	9823	4	9827	5	9832	4	9836	5	9841	4	9845	5	9850	4	9854	5	9859	4	9863	5
97	9868	4	9872	5	9877	4	9881	5	9886	4	9890	4	9894	5	9899	4	9903	5	9908	4
98	9912	5	9917	4	9921	5	9926	4	9930	4	9934	5	9939	4	9943	5	9948	4	9952	4
99	9956	5	9961	4	9965	4	9969	5	9974	4	9978	5	9983	4	9987	4	9991	5	9996	4

164. How to find Mantissas for Numbers not given in the Table.

1. Since changing the position of the decimal point in a number does not affect the value of the mantissa of its logarithm, to find the mantissas of numbers consisting of more than three significant figures, we may think of the decimal point as located so that we have a number consisting of three integral orders. Thus, if the given number is 46,792 we may think of it as changed to 467.92; if it is .006943, we may think of it as changed to 694.3.

2. Logarithms do not increase in proportion as their numbers increase.

Thus, log 3 = .4771; log 6 = .7782; log 12 = 1.0792.

However, if we take large numbers close together, the increments of the numbers and the increments of their logarithms are very nearly proportional.

Thus, log 720 = 2.8573; log 721 = 2.8579; log 722 = 2.8585; log 723 = 2.8591.

Now, since log 721 is midway between log 720 and log 722 correct to 4 decimal places, we conclude that log 720.5 is probably about midway between log 720 and log 721, that is,

$$\log 720.5 = 2.8573 + .5 \text{ of } .0006 = 2.8576.$$

Reasoning in the same way, we infer that

$$\log 720.85 = 2.8573 + .85 \text{ of } .0006 = 2.8578.$$

Now, as the mantissas themselves are written as whole numbers and their differences are given as whole numbers, it is not necessary to write .0006, but merely 6, multiplying it by the decimal part of the given number. Then, taking the result to the nearest unit, we add it to the next lower mantissa in the table.

3. To find the mantissa of the logarithm of any number proceed as follows:

(1) *Find the mantissa corresponding to the first three figures of which the left one is significant, that is, is not* 0.

(2) *Multiply the tabular difference (column D, following the mantissa found) by the remaining figures of the number regarded as a decimal.*

(3) *Add the product, taken to the nearest unit, to the mantissa corresponding to the first three figures.*

This process is called **interpolation.**

165. Exercise in finding the Logarithms of Numbers.

1. Find the logarithm of 67.883.

SOLUTION. log 67.8 = 1.8312; .83 of 7 = 5.81 = 6 to the nearest unit; log 67.883 = 1.8318.

As soon as possible the whole operation ought to be performed mentally. Find the logarithms of the following:

2. 843.8.	3. 921.5.	4. 6.934.	5. 290.45.	6. 1764.
7. .1764.	8. .02596.	9. .20087.	10. .006784.	11. .00005.
12. 76349.	13. $11\frac{7}{8}$.	14. $\frac{5}{9}$.	15. $\frac{6}{170}$.	16. 3.1416.
17. 10007.	18. 12401.	19. 1.002.	20. $.1\frac{1}{8}$.	21. 2006.

166. How to find the Numbers corresponding to given Logarithms. Exercise. Numbers corresponding to given logarithms are often called the **antilogarithms** of the given logarithms.

1. Find antilog of 1.8142.

SOLUTION. The mantissa .8142 corresponds to 652. By the rule for characteristics, § 161, 3, there should be 2 integral orders in the answer. Hence, the answer is 65.2.

Find the antilogarithms of the following:

2. 2.6580	3. $\bar{2}.7388$	4. 0.1959

5. Find the antilogarithm of 1.7963.

The process is the exact reverse of that of finding logarithms of numbers.

(1) ***Look for that mantissa in the table which is next smaller than the given mantissa** (in this case* .7959, *which corresponds to* 625), ***and subtract this mantissa from the given one.***

(2) ***Next divide this remainder by the tabular difference*** (7) ***following the mantissa found in the table, getting*** .57, ***and annex the quotient to the first three figures already found, getting*** 62557.

(3) ***Last of all locate the decimal point by the rules for characteristic*** (§ 161, 3, 6). ***Since the given characteristic is*** 1, ***there must be two integral orders in the answer,*** i.e. ***answer is*** **62.557.**

6. Find the antilogarithm of $\bar{3}.2432$.

Solution. 2432 − 2430 = 2 ; 2 ÷ 25 = .08. *Ans.* .0017508.

7. 2.8144.	**8.** 5.7329.	**9.** $\bar{2}.5394$.	**10.** $\bar{6}.3139$.
11. 2.4774.	**12.** $\bar{3}.6121$.	**13.** 6.1920.	**14.** $\bar{1}.0100$.
15. 0.0304.	**16.** $\bar{2}.0021$.	**17.** $\bar{6}.9779$.	**18.** 0.9988.

19. 8.2463 − 10. This form is often used instead of $\bar{2}.2463$.

20. 9.3891 − 10. **21.** 7.4284 − 10. **22.** 6.3239 − 10.

II. USES OF LOGARITHMS

167. Theory of Logarithms. As logarithms are exponents, they are governed by exactly the same laws as exponents. (§ 63, § 71.)

1. Addition and subtraction *cannot be performed* by common logarithms. Thus, a^2 and a^3 cannot be added or subtracted by working with 2 and 3.

2. Multiplication is performed by adding logarithms. (§ 17.)

Example. To multiply 721 by 369.

Solution.
log 721 = 2.8579
log 369 = 2.5670
log 266000 = 5.4249
Ans. (Antilog of 5.4249.)

Explanation.
$721 = 10^{2.8579}$
$\times\ 369 = \times\ 10^{2.5670}$
$721 \times 369 = 10^{5.4249}$

Explanation. The logarithms of 721 and 369 are obtained from the table and added; then the antilogarithm of the sum is found.

By actual multiplication $721 \times 369 = 266,049$.

a. We see here that a Four Place Table gives only four figures of the answer correct. Occasionally it will give only three correct, but usually it will give four and often five right. Logarithmic tables are constructed for varying needs. For practical purposes Four and Five Place Tables are commonly employed. Results can be had as accurate as desired by using proper kind of tables. Tables have been constructed containing as many as 15 figures in their logarithms. The theory, of course, is the same for all tables.

3. *To multiply numbers together, get the logarithm of each factor, add the results, and find the antilogarithm of the sum.*

4. Multiply 763 by 298 and check by actual multiplication. Also 3.245 and 63.29; also 93.29, 29.76, and 16.48.

5. Division is performed by subtracting logarithms. (§ 71, 3.)

Example. Divide 37.69 by 2.463.

Solution.		Explanation.
	$\log 37.69 = 1.5762$	$37.69 = 10^{1.5762}$
	$\log 2.463 = 0.3914$	$2.463 = 10^{.3914}$
	$\log 15.304^{-} = 1.1848$	$37.69 \div 2.463 = 10^{1.1848}$
	Ans. (Antilog of 1.1848.)	

By actual division the quotient is 15.302+.

6. Divide 19.65 by 2.843 and check by actual division; also 25,941 by 16,713; 34.62 by 7.329; 84.63 by .6792; .6721 by .00325.

7. Raising a number to a power is performed by multiplying its logarithm by the exponent of the power and finding the antilogarithm of the product. (§ 71, 4.)

Example. Raise 17.64 to the fourth power.

Solution.		Explanation.
	$\log 17.64 = 1.2465$	$17.64 = 10^{1.2465}$
	$\log 17.64^4 = 4.9860$	$17.64^4 = 10^{4.9860}$
	$\log 96825 = 4.9860$	$96825 = 10^{4.9860}$
	Ans.	

By actual contracted multiplication $17.64^4 = 96,827^{-}$.

8. Raise 13.25 to 3d power and check by contracted multiplication; also 9.2 to 4th power; 3^9; $.17^4$; 16^{12}; $.06^5$.

9. Extracting a root of a number is performed by dividing its logarithm by the index of the root and finding the antilogarithm of quotient. (§ 71, 5.)

EXAMPLE. Calculate $(29.34)^{\frac{1}{5}}$, or $\sqrt[5]{29.34}$.

SOLUTION. $\log 29.34 = 1.4675$ EXPLANATION. $29.34 = 10^{1.4675}$

$\log 29.34^{\frac{1}{5}} = 0.2935$ $29.35^{\frac{1}{5}} = 10^{.2935}$

$\log 1.9655 = 0.2935$ $1.9655 = 10^{.2935}$

By actual contracted multiplication $1.9655^5 = 29.315$.

10. Extract the cube root of 1763 and check by the method of § 61; $\sqrt[4]{1.953}$ (check by § 59); $\sqrt[7]{19.84}$; $\sqrt[9]{85.34}$ (check by § 61).

11. Cases 7 and 9 just given can be included under one head.

EXAMPLE. $(29382)^{\frac{4}{5}}$.

SOLUTION. $\log 29382 = 4.4680$; $\log 29382^{\frac{4}{5}} = \frac{4}{5} \times 4.4680 = 3.5744 = $ antilog 3753. *Ans.* $\therefore (29382)^{\frac{4}{5}} = 3753$.

12. Calculate $41.6^{\frac{3}{4}}$; $1.908^{\frac{5}{3}}$; $3984^{\frac{2}{7}}$; $1501^{\frac{4}{9}}$; $20.6^{\frac{3}{2}}$.

168. Exercise in Multiplication and Division by Logarithms. Check by contracted multiplication and division.

1. 26.73×19.38. 2. $26.45 \times .02687 \times 3.194$. 3. $.00857 \times .00693$.

4. $\dfrac{862 \times 48.75}{7.862 \times 6.827}$

SUGGESTION. Get logarithms of factors in numerator and add; also logarithms of factors in denominator and add; subtract latter sum from the former and get antilogarithm of difference.

5. $\dfrac{89.76 \times 98.54 \times 26.63}{.005862 \times 8271}$ 6. $\dfrac{87.51 \times 445 \times 823}{3.1416 \times .045 \times .862}$.

a. Carrying and borrowing where negative characteristics enter are likely to prove confusing to the student. It is generally simpler to add 10 to the characteristic and subtract 10 from the mantissa. Explain the following:

(1) $\begin{array}{r} \bar{4}.3421 \\ +\,2.9745 \\ \hline \bar{1}.3166 \end{array}$ (2) $\begin{array}{r} 2.6789 \\ -\,\bar{4}.8541 \\ \hline 5.8248 \end{array}$ (3) $\begin{array}{r} \bar{3}.2681 \\ -\,\bar{2}.7492 \\ \hline \bar{2}.5189 \end{array}$

Solve these exercises now by adding and subtracting 10 for each logarithm with a negative characteristic.

169. Powers and Roots by Logarithms. The case of negative exponents needs attention.

1. Raise .0734 to the fourth power.

SOLUTION. $\log .0734^4 = (8.8657 - 10) \times 4 = 35.4628 - 40 = \bar{5}.4628.$

$\therefore .0734^4 = .000029.$

2. $.0634^5$. 3. $.637^4$. 4. $.007234^7$. 5. $.07^6$.

6. Extract the fifth root of .0329.

$$\log .0329 = 8.5172 - 10 = 48.5172 - 50.$$

$$5 \underline{|48.5172 - 50}$$

Antilog of $9.7034 - 10 = .5051$. *Ans.*

Notice that -50 was chosen so as to be exactly divisible by 5.

7. $.0072^{\frac{1}{4}}$. 8. $.02852^{\frac{1}{3}}$. 9. $\sqrt{.6782}$. 10. $\sqrt[4]{.8751}$.

170. Cologarithms. The remainder obtained by subtracting a logarithm from 10 is called the **cologarithm** of the number. By means of cologarithms combined multiplications and divisions can be changed into multiplications. Thus, $a \times b \div c$ can be written $a \times b \times \frac{1}{c}$. Now $\log \frac{1}{c} = \log 1 - \log c = 0 - \log c$. Instead of subtracting $\log c$ from 0, it is more convenient to subtract it from 10; then 10 will have to be taken from the final result for each cologarithm introduced. The preceding may be stated as follows:

$$\log \frac{ab}{c} = \log a + \log b + (10 - \log c) - 10.$$

1. Calculate $\frac{36.74 \times 78.91}{19.83 \times 21.34}$, using cologarithms.

SOLUTION.	$\log 36.74 = 1.5651$	colog 19.83	colog 21.34
	$\log 78.91 = 1.8972$	10.0000	10.0000
	colog $19.83 = 8.7026$	1.2974	1.3292
	colog $21.34 = 8.6708$	8.7026	8.6708
	$\log 6.85 = 0.8357$	$\therefore$ 6.85 is answer sought.	

2. Calculate $\frac{27.43 \times 168.4}{1.938 \times 247.6}$. 3. Calculate $\frac{176.4 \times 18.25}{3.1416 \times 14.3^2}$.

171. Exercise in applying Logarithms in the Solution of Practical Problems.

1. Find the circumference of a circle whose diameter is 17.63 inches, from the formula $c = \pi d$.

SUGGESTION. The logarithm of π is .4971. The student should *remember* this, as it will save him the trouble of looking it up each time.

2. Find the area of a circle whose radius is 16.72 ft., from the formula $a = \pi r^2$.

3. Find the diameter of a circle whose circumference is 3928 m.

4. Find the surface of a sphere whose radius is 362.5 in., from the formula $s = 4\pi r^2$.

5. Find the volume of a sphere whose diameter is 12 cm., from the formula $v = \frac{1}{6}\pi d^3$.

6. Find the radius of a sphere whose surface is 25.12 sq. ft.

7. Find the area of an ellipse whose longer semiaxis a is 22.18 in. and shorter b is 16.88 in., from the formula $A = \pi ab$.

8. Find the volume of a cylinder whose radius of base r is 1.677 m. and whose length l is 23.51 m., from the formula $v = \pi r^2 l$.

9. Find the volume of a cone whose radius of base r is 33.41 ft. and altitude a is 12.16 ft., from the formula $v = \frac{1}{3}\pi r^2 a$.

10. Find area of a triangle whose sides are 16.35, 18.97, and 24.77, using formula § 16, 24.

11. Perform the multiplications and divisions called for in § 153 with logarithms.

The problems of §§ 153–157 can be solved by logarithms. For this purpose special tables are constructed which contain the logarithms of the trigonometrical functions for every degree and minute from 0° to 90°. Thus, this table contains the logarithms of the numbers in the table on page 164. By the use of such a table with an ordinary table of logarithms, problems in right triangles can be solved very quickly.

172. Compound Interest and its Calculation by Logarithms. — In compound interest the amount becomes the principal at the beginning of each new interest period. Call p the principal, a the final amount, r the rate, and n the number of years.

1. *To find the amount when the principal, rate, and time are given.*

The amount at the end of one year is $p + pr$, or $p(1 + r)$, since p is the principal and pr is one year's interest. Thus, to get the amount at the end of one year always multiply the principal by $1 +$ the rate.

Now, in compound interest the principal at the beginning of the second year is $p(1+r)$. Then the amount at the end of the second year is $p(1+r)(1+r)$, or $p(1+r)^2$; and so on for the n years. Hence
$$a = p(1+r)^n.$$

Example. Find the amount of $725.15 at 5% compound interest at the end of 6 years.

Solution. $a = 725.15(1 + .05)^6$.

$$\begin{aligned}
\log 1.05 &= .0212 \\
\log 1.05^6 &= .1272 \\
\log 725.15 &= 2.8604 \\
\log a &= 2.9876 \\
a &= 971.80. \quad \textit{Ans.}
\end{aligned}$$

a. A six- or seven-place table is needed if the answer is to be obtained correct to cents.

2. *To find the cost or "present worth" of a sum payable n years hence, supposing interest to be compounded.*

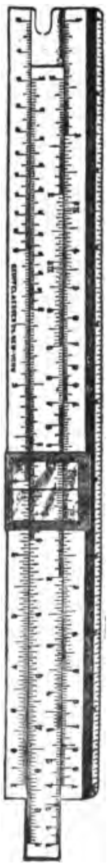
From the equation just obtained by solving for p, we get
$$p = \frac{a}{(1+r)^n}.$$

Example. Find the principal which will amount to $923 at 4% compound interest in 12 years, by substituting in the formula.

3. *To find amount when interest is compounded q times a year.* Explain derivation of formula that follows.
$$a = p\left(1 + \frac{r}{q}\right)^{qn}.$$

173. Exercise in solving Compound Interest Problems.

1. Find amount of \$933 at 5% which ran 7 yr.
2. Find amount of \$336.1 at 6% which ran 14 yr.
3. Find amount of \$182.70 at $4\frac{1}{2}$% running 11 yr.
4. Find amount of \$2673 at 3% running 22 yr.
5. Find amount of \$193.60 at 6% running 8 yr.



6. Find principal which amounts to \$775.20 in 15 yr. at 5%.
7. Find principal which amounts to \$9675 in 20 yr. at 4%.
8. Find cost or present worth of \$918 to be paid in 10 yr., allowing 5% interest.
9. Find the amount of \$225 at 5% running 75 yr.
10. Find the amount of \$700 which ran for 12 years, the interest being compounded semiannually at 6%.

Suggestion. $a = p\left(1 + \frac{r}{2}\right)^{2n}$. See § 172, 3.

11. Find amount of \$425 compounded quarterly at 7%, which ran 9 years.
12. Find what is due in 1911 on a note calling for 1¢ as principal made in the time of Julius Cæsar 50 years before Christ, if 4% compound interest is allowed.

174. Slide Rule. Rules are in common use which have for the figures given on them the left column of the logarithmic table, but instead of the mantissas being denoted by figures, these mantissas are given by their *lengths*. In using these rules one is virtually using a table of logarithms. See drawing in the margin.

III. EXPONENTIAL EQUATIONS

175. Solution of Exponential Equations. An **exponential equation** is one in which the unknown quantity is an exponent. Thus, $20^x = 75$ is an exponential equation. Such equations are solved by means of logarithms.

Let $a^x = b$ represent a simple exponential equation.

Then, $\log a^x = \log b$, (Logarithms of equals are equal.)

or, $x \times \log a = \log b$. (The logarithm of a power equals logarithm of the quantity multiplied by the exponent of the power, § 167, 7.)

$$\therefore x = \frac{\log b}{\log a}. \quad \text{(Div. Ax.)}$$

1. Solve $8^x = 45$ for x.

SOLUTION. $x = \frac{\log 45}{\log 8} = \frac{1.6532}{.9031} = 1.83+$

Solve the following equations:

2. $3^x = 29$. **3.** $10^x = 129$. **4.** $17^x = 1174$.

5. $123^x = 1684$. **6.** $2^x = 25$. **7.** $100^x = 8$.

a. The formula calls for getting the logarithms of a and b out of the table and *dividing* the latter by the former. This division can be performed by logarithms, but it is better to execute this division by the old familiar process of long division at first, and until the formula is well fixed in mind. The student may now test his divisions in the preceding problems by a logarithmic solution, and hereafter perform the divisions by logarithms.

8. $15.49^x = 6$. **9.** $127^y = 4675$. **10.** $3^x = 729$.

b. Exponential equations can have any positive value except unity for a and b. We have avoided the use of negative characteristics in the preceding problems, since they must be handled with care.

11. $12^x = .5$. SOLUTION. $x = \frac{9.6990 - 10}{1.0792} = \frac{-.3010}{1.0792} = -.28^-$.

12. $15^x = .42$. **13.** $.9^x = 25$. **14.** $.85^x = .687$.

15. $10^{2x} + 2 \times 10^x = 80$. **16.** $2^x + \frac{1}{2^x} = 5$.

176. Historical Notes. Logarithms were invented by **John Napier**, of Merchiston, Scotland, about 1614 A.D. Napier's logarithms were not logarithms of ordinary numbers but of the ratios of the legs of a right-angled triangle to the hypotenuse, and decreased as these numbers increased. Napier's logarithms are very curious indeed. To understand why they were thought of rather than those we have, it must be kept in mind *that exponents were not known or used in those days.*

Henry Briggs (later of Oxford University, England) greatly admired Napier's logarithms and was led to visit him. The scene at their meeting was impressive. When Briggs came into Napier's presence, they greeted each other, and then almost a quarter of an hour elapsed with each looking at the other and not speaking a word. Then Briggs said, "My lord, I have undertaken this long journey purposely to see your person and to know by what engine of wit or ingenuity you came first to think of this most excellent help in astronomy; but, my lord, being by you found out, I wonder nobody found it out before, when now known it is so easy."

Later, Briggs constructed tables of logarithms of numbers, as we have them, to base 10. His table was calculated to 14 decimal places, and extended from 1 to 20,000 and from 90,000 to 100,000. The gap in Briggs's table was filled later (1628) by Adrian Vlacq, of Gouda, Holland, who calculated the list of missing logarithms.

The advantage in the use of logarithms in calculations was soon recognized, and by 1630 they were in general use in Europe.

In the history of English science Napier's book comes next in importance after Newton's great work on mechanics, the "Principia." Napier's invention was not the result of a happy accident but of long continued endeavor, and it is the more remarkable because made in what was a troubled age, in a rather wild country judged by our standards, and at a time when all science was in a most crude state.

For fuller accounts of development of logarithms see "Logarithms," "Napier," "Briggs," etc., in *Encyclopædia Britannica.*

John Napier (1550-1617)

CHAPTER IX

ARITHMETICAL AND GEOMETRICAL PROGRESSIONS

I. ARITHMETICAL PROGRESSION

177. An Arithmetical Progression is a sequence of terms that increase or decrease by a common difference.

E.g. in the series 4, 7, 10, 13, 16, etc., the common difference is 3.

178. Formula for finding the *n*th Term of an Arithmetical Series.

Let a be the first term, d the common difference, n the number of terms, and l the last or nth term. Then, by definition, the series is

$$a,\ a+d,\ a+2d,\ a+3d,\ \cdots,\ l-2d,\ l-d,\ l,$$

where the dots, as usual, signify terms left out. To find the nth term, let us write the number of each term over it. Thus,

$$\begin{array}{cccccc} 1 & 2 & 3 & 4 & 5 & n \\ a, & a+d, & a+2d, & a+3d, & a+4d,\ \cdots, & a+(n-1)d. \end{array}$$

The last term is found by observing that the coefficient of d is always 1 less than the number of the term over it. Hence,

$$(1)\quad \boldsymbol{l=a+(n-1)d}.$$

179. Formula for Finding the Sum (s) of all the terms in an Arithmetical Progression. The sum can be found by writing the series *reversed* under it. Thus,

$$s=a+a+d+a+2d+a+3d+\cdots+l-d+l$$
$$s=l+l-d+l-2d+l-3d+\cdots+a+d+a$$
$$2s=a+l+a+l+a+l+a+l+\cdots+l+a+l+a.\ \text{(Add. Ax.)}$$

$\therefore 2s = n(a+l)$, since there are evidently n of the $(a+l$'s) added. Then,

$$(2)\quad s = \frac{n}{2}(a+l).$$

Make *theorems* out of formulas (1) and (2).

180. Exercise in Arithmetical Progression.

1. Find the tenth term of 2, 11, 20, ⋯

SOLUTION. In this problem $a = 2$, $d = 9$, $n = 10$. Then, using equation (1),

$$l = 2 + (10 - 1)9 = 83.$$

This answer can be checked by writing the series out. Thus, 2, 11, 20, 29, 38, 47, 56, 65, 74, 83; 83 is the 10th term.

2. Find the 12th term of 4, 6, 8, ⋯.

3. Find the 15th term of 144, 138, 132, ⋯.

SUGGESTION. Put $d = -6$.

4. Find the 25th term of $\frac{15}{8}$, $\frac{21}{8}$, $\frac{27}{8}$, ⋯.

5. Find the 17th term of .5, .3, .1, ⋯

6. Find the sum of 11 terms of 18, 21, 24,⋯

SUGGESTION. First find l as in the preceding. Then, using equation (2). find s.

7. Find the last term and sum in 5, 9, 13, etc., to 19 terms.

8. Simplify $7 + \frac{29}{4} + \frac{15}{2} +$ etc., to 16 terms.

9. Sum $2a - 5b$, $7a - 2b$, etc., to 9 terms.

10. Find the sum of 20 terms of 5, 1, -3, etc.

11. Given $a = 94$, $n = 12$, $d = -3$, to find l and s.

12. Insert 12 terms *between* 12 and 77.

SUGGESTION. Here $a = 12$, $l = 77$, $n = 14$; to find d.

13. The first term of a series is 2 and the common difference is $\frac{1}{3}$. What is the number of the term that is 10?

14. Find the sum of $-3q$, $-q$, q, etc., to p terms.

SUGGESTION. $n = p$, $d = 2q$.

15. Given $n = 7$, $d = 15$, $s = 399$, to find a and l.

SUGGESTION. Substitute these values in *both* equations (1) and (2). Then eliminate either a or l between the resulting equations.

16. Given $a = 10$, $d = 7$, $s = 582$, to find l and n.

SUGGESTION. After substituting as before and eliminating l, a quadratic equation, with n as its unknown, results.

17. Given l, a, d, to find n. SUGGESTION. Solve (1) for n.

18. Given l, a, n, to find d; given l, n, d, to find a; given s, n, l, to find a; given s, n, a, to find l; given s, l, a, to find n.

19. Given d, n, s, to find l.

SUGGESTION. Eliminate the missing letter a between (1) and (2), and solve the resulting equation for l.

20. Given a, d, n, to find s; given a, d, l, to find s; given d, n, l, to find s; given a, n, s, to find d; given a, l, s, to find d; given n, l, s, to find d; given d, n, s, to find a.

21. Given a, d, s, to find l.

SOLUTION. These four letters not appearing by themselves in either (1) or (2), we eliminate the *missing letter* n from them.

$$(2_1)\quad n = \frac{2s}{a+l}.$$

$$(1)\quad l = a + \left(\frac{2s}{a+l} - 1\right)d. \qquad (\S\ 42.)$$

$$l = a + \frac{2ds - ad - ld}{a+l}.$$

$$al + l^2 = a^2 + al + 2ds - ad - ld. \qquad (\text{Ax. ?})$$

$$l^2 + dl = a^2 - ad + 2ds. \qquad (\text{Ax. ?})$$

$$l^2 + dl + \frac{d^2}{4} = a^2 - ad + \frac{d^2}{4} + 2ds. \qquad (\text{Ax., } \S\ 100.)$$

$$l + \frac{d}{2} = \sqrt{\left(a - \frac{d}{2}\right)^2 + 2ds}. \qquad (\text{Ax. ?})$$

$$l = -\frac{d}{2} \pm \sqrt{\left(a - \frac{d}{2}\right)^2 + 2ds}. \qquad (\text{Ax. ?})$$

22. Given $a = -7$, $d = 3$, $s = 430$, to find l by substituting in the formula just found.

23. Given l, d, s, to find a; given d, l, s, to find n; given d, a, s, to find n.

24. In a potato race 100 potatoes are placed 2 ft. apart in a straight line. A runner starting from the basket picks up one potato at a time and carries it to a basket in a line with the potatoes and 10 ft. from the first potato. How far must the contestant run?

25. How many strokes does the hammer on the bell of a clock make in a 24-hour day?

26. A body falling freely falls 16.08 ft. the first second, and in each succeeding second 32.16 ft. more than in the second immediately preceding. If a stone dropped from a stationary balloon reaches the ground in 14 sec., how far does it fall the last second and how high is the balloon?

27. A stone is dropped from the top of a tower 402 ft. high. In how many seconds does it reach the ground?

28. If a bullet when fired vertically upward goes 1008 ft. the first second, how high does it rise, and how long will it be till it reaches the earth again? In this problem use 16 and 32 instead of 16.08 and 32.16.

29. A farmer agreed to pay a blacksmith for shoeing his horse $\frac{1}{4}$¢ for the first nail he drove, $\frac{1}{2}$¢ for the next, $\frac{3}{4}$¢ for the third, and so on. There were 40 nails. How much did he have to pay?

30. If a person saves \$100 and puts it at simple interest at 5% at the end of each year, how much will his property amount to at the end of 20 years?

31. A man was paid for drilling an artificial well 3.24 marks for the first meter, 3.29 marks for the second, 3.34 marks for the third, and so on. The well had to be sunk 500 meters. How much was paid for the last meter, and how much for the whole?

32. A travels uniformly 20 mi. a day; B starts 3 da. later and travels 8 mi. the first day, 12 mi. the second, and so on, in arithmetical progression. In how many days will B overtake A?

SUGGESTION. Let $x =$ number of days B travels to overtake A.

Then how many days does A travel? What is his whole distance?

With $n = x$, sum the progression and then make the equation and solve it.

33. The sum of three numbers in arithmetical progression is 12, and the sum of their squares is 66. Find them.

II. GEOMETRICAL PROGRESSION

181. A Geometrical Progression is a series in which the terms increase or decrease by a common *ratio*.

E.g. 3, 6, 12, 24, ⋯, the common ratio being 2.

182. Formula for finding the Last Term of a Geometrical Progression.

Let a be the first term, r the ratio, n the number of terms, l the last term, and s the sum of all the terms. Then, by definition, the series is,

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & & n \\ a, & ar, & ar^2, & ar^3, & ar^4, & ar^5, & \cdots, & ar^{n-1} \end{array}$$

The last term, ar^{n-1}, is found by noticing that the exponent of r is always 1 less than the number of the term (written over it in the series given above).

Hence, (1) $l = ar^{n-1}$.

183. Formula for finding the Sum of a Geometrical Progression.

To derive this sum, we write $s =$ the sum of the series, and on the next line, the first equation multiplied by r.

$$s = a + ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-2} + ar^{n-1}$$

$$rs = \quad ar + ar^2 + ar^3 + ar^4 + \cdots + ar^{n-2} + ar^{n-1} + ar^n \quad \text{(Ax. ?)}$$

$$rs - s = ar^n - a. \quad \text{(Sub. Ax. ?)}$$

$$\therefore (r-1)s = ar^n - a. \quad \text{(On factoring.)}$$

$$(2)\ s = \frac{a(r^n - 1)}{r - 1}. \quad \text{(Ax. ?)}$$

Explanation. As already stated, the equation over the long line, p. 194, is obtained by multiplying the first equation by r. Thus, r times the left member gives rs. Now, when we get the product of the right side of the first equation multiplied by r, we write the product of a and r underneath the ar of the first line, and the product of ar and r underneath ar^2 of the first line, and so on. The ar^{n-2} term of the second line comes from a missing term (indicated by dots) in the first line. When we subtract the first equation from the second, all the terms cancel except the first and last.

Again, since $l = ar^{n-1}$, $lr = ar^{n-1} \times r = ar^n$. Hence, substituting lr for ar^n in equation (2), p. 194, we have

$$(3)\quad s = \frac{lr - a}{r - 1}.$$

Make theorems out of formulas (1), (2), and (3).

184. Summation of a Geometrical Progression the Limit of whose Last Term is 0. (The last term is really an *infinitesimal*, or number smaller than any that can be named.) Using eq. (3), § 183, we have,

$$(4)\quad \text{Limit of } s = \frac{0 \times r - a}{r - 1} = \frac{-a}{r - 1} = \frac{a}{1 - r}.$$

1. Sum the series 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\cdots$ to infinity.

Solution. — Here $a = 1$, $r = \frac{1}{2}$ (2d term ÷ 1st term), and we put $l = 0$.

$$s = \frac{1}{1 - \frac{1}{2}} = 2. \quad \textit{Ans.}$$

2. Sum the series $\frac{1}{3}$, $\frac{1}{9}$, $\frac{1}{27}$, to infinity.

3. Sum the repeating decimal .282828⋯, that is find the common fraction which reduced to a decimal gives these figures.

Solution. $a = .28$, $r = .01$, $l = 0$. Then,

$$s = \frac{.28}{1 - .01} = \frac{.28}{.99} = \frac{28}{99}. \quad \textit{Ans.}$$

Proof: $28 \div 99 = .282828\cdots$.

4. Sum the repeating decimal .43782782⋯.

Suggestion. Regard .00782 as the first term, and add the answer to .43. What does r equal here?

185. Exercise in Geometrical Progression.

1. Find the 8th term of 6, 12, 24, ⋯.
2. Find the 7th term of 7, 21, 63, ⋯.
3. Find the 9th term and sum of nine terms of 1, 3, 9, ⋯.
4. Find the 10th and 16th terms of 256, 128, 64, ⋯.
5. Find the last term and sum of 8.1, 2.7, .9, ⋯ to 7 terms.
6. Find the last term and sum of $\sqrt{2}, \sqrt{6}, 3\sqrt{2}, \cdots$ to 12 terms
7. Find the last term and sum of $-\frac{2}{5}, \frac{1}{2}, -\frac{5}{8}, \cdots$ to 6 terms.
8. Sum to n terms $5, 2\frac{1}{2}, 1\frac{1}{4}, \cdots$. SUGGESTION. Here $n = n$.
9. Sum to infinity $3 + \frac{3}{2} + \frac{3}{4} + \cdots$.
10. Sum to infinity $4 + 2\frac{2}{3} + 1\frac{7}{9} + \cdots$.
11. Find the common fraction value of .3333⋯.
12. Reduce to a common fraction .3787878⋯.
13. Given $a = 8$, $r = 2$, $s = 248$, find l and n.
14. Given $r = 6$, $n = 5$, $l = 1296$, find a and s.
15. Given $r = 2$, $n = 12$, $l = 43008$, find a and s.
16. Insert 6 geometrical means between 56 and $-\frac{7}{16}$.
17. The 4th term of a geometrical progression is 160 and the ratio is 4; find the 7th and 15th terms.
18. A man deposited $500 in a savings bank that paid 4% interest, compounded annually. If the money was left 4 years, what sum stood to the depositor's credit?

SUGGESTION. $r = 1.04$, $n = 5$. The $500 is *first* term.

19. In using an air pump the pressure is reduced $\frac{1}{8}$ at each stroke and stands at 30 in. at the beginning. What is the height of the mercury after the fifth stroke?

SUGGESTION. $r = \frac{7}{8}$, $n = 6$.

20. If a rubber ball rebounds $\frac{1}{4}$ of the distance it falls each time, what will be the distance a ball moves that is thrown up with a force to carry it 75 ft.? (See § 184.)

21. If a man, whether by his example or designedly, leads a single fellow-man into the path of rectitude each year during 20 years, and each of these men in turn leads one man aright each year from the time of his change, and so for every one affected, what will be the total number influenced as a result of the first man's effort at the end of the twenty years?

22. Achilles pursued a tortoise, which was one stadium ahead of him, with a speed 12 times greater than that of the tortoise. When Achilles reached the place where the tortoise had been when he started, the tortoise was still $\frac{1}{12}$ stadium in advance of him; Achilles having traversed this distance, the tortoise was still $\frac{1}{144}$ stadium ahead of him, and so on, *ad infinitum*. Will Achilles then never catch up with the tortoise?

NOTE. This problem is a statement of Zeno's celebrated sophism. If x = number of stadia Achilles must run to catch tortoise, show that $x = 1\frac{1}{11}$. While it is perfectly allowable *to conceive* of the distance divided up in the way described, the sum of the infinite number of parts is only $1\frac{1}{11}$ stadia. Achilles will catch the tortoise in the time it takes him to run $1\frac{1}{11}$ stadia. The sophism consists in implying a finite period of time is needed for each division of $(1 + \frac{1}{12} + \frac{1}{144}$, etc.) stadia.

23. Seven old women go to Rome; each woman has 7 mules, each mule carries 7 sacks, each sack contains 7 loaves, with each loaf are 7 knives, each knife is put in 7 sheaths. What is the sum total of all? (From Leonardo's *Liber Abaci*.)

24. The population of a city is 100,000, and it increases 50 % every 4 years. What will the population be in 20 years?

25. A person has two parents, each of his parents has two parents, and so on. How many ancestors has a person, going back ten generations, counting his great-grandparents as the first generation (and assuming that each ancestor is an ancestor in only one line of descent)?

NOTE. — Before leaving the subject the pupil should be tested to see if he can distinguish between arithmetical and geometrical progression problems.

CHAPTER X

ANNUITIES. APPLICATION OF PROGRESSIONS AND LOGARITHMS

186. Annuities and their Calculation. An **annuity certain** is a sum of money payable at the end of each year, or other period, for a fixed number of periods.

One fundamental doctrine about interest should be clearly understood, viz., that in order to add sums of money due at different times, they must be reduced to the same time. \$ 100 due 1 year ago, \$ 100 due now, and \$ 100 due in 1 year are three different sums. Assuming 6 % as the rate of interest, \$100 due 1 year ago will have drawn 1 year's interest, and be worth at the present time \$106: \$100 due now is worth \$100; \$100 due in 1 year is worth the present value of \$100 due in 1 year, (see § 172, 2), or \$100 ÷ 1.06 (= \$94.34). Similarly, \$100 due 2 years ago is worth now at compound interest \$112.36, and the present value of \$100 due in 2 years is \$100 ÷ 1.06^2, or \$89.

The present value of \$ 1 due in 1 year, if $r =$ the rate per cent, is $\frac{\$1}{1+r}$. The present value of \$ 1 due in 2 years is $\frac{\$1}{(1+r)^2}$; and the present value of \$ 1 due in n years at compound interest is $\frac{\$1}{(1+r)^n}$. For convenience in what follows we will denote the present worth of \$ 1 due in 1 year by p: then the present worth of \$ 1 due in 2 years is p^2, and in n years is p^n. Thus, $p = \frac{1}{1+r}$.

1. *To find the cost of an annuity.*

If a man contracted with a bank for the latter to pay him an annuity of \$ 500 at the end of each year for 20 years from date of agreement, how much cash would the bank demand from him, if it was to pay him $3\frac{1}{2}$ % compound interest on all money remaining unpaid at any time? The bank could afford to make such a contract, because it could make a profit by lending the money obtained to its customers at a higher rate than $3\frac{1}{2}$ %.

Let C be the cost of A dollars each year for n years and p the present worth of \$1 due in 1 year at the given rate per cent. The present value, or cost of the first payment, is evidently Ap dollars; the present value of the second payment of A dollars due at the end of 2 years is Ap^2 dollars; and so on: the present value of the last payment of A dollars due at the end of n years is Ap^n dollars.

Adding these sums to get the cost of all the payments, we have,

$$C = Ap + Ap^2 + Ap^3 + \cdots + Ap^n.$$

But the terms of the right member form a geometrical progression in which $a = Ap$, $l = Ap^n$, $r = p$, $s = C$. Substituting these values in (§ 183, eq. (3)) $s = \frac{lr - a}{r - 1}$, we get

$$C = \frac{Ap^n \times p - Ap}{p - 1} = A\frac{p^{n+1} - p}{p - 1}. \qquad (1)$$

Example. Find the cost of an annuity of \$150 each year to run 12 yr. if 3 % interest is allowed on all money not paid back.

$$\log p = \log \frac{1}{1 + r} = 0 - \log (1 + .03) = \bar{1}.9872. \text{ (Since } \log 1 = 0.)$$

$$\therefore p = .971.$$

$$\log p^{n+1} = \log \frac{1}{(1 + r)^{n+1}} = 0 - (n + 1) \log (1 + .03) = \bar{1}.8336.$$

$$\therefore p^{13} = .6817.$$

Then, $$C = 150 \times \frac{.6817 - .971}{.971 - 1} = 150 \times \frac{.2893}{.029}.$$

$\log C = \log 150 + \log .2893 - \log .029.$ $\therefore C =$ \$1496.55. See § 172, *a*.

Twelve payments of \$150 each make \$1800 paid back. The cost, \$1496.55, *with its interest*, is equivalent to the total paid back.

2. *To find the cost of an annuity to begin at the end of m years and run for n years.*

Here one must find the present worth of $\$A\frac{p^{n+1}-p}{p-1}$ due in m years by multiplying this sum by p^m (§ 172, 2). If C' is cost, then

$$C' = Ap^m\frac{p^{n+1}-p}{p-1}. \tag{2}$$

187. Repayment of Interest and Principal together in Annual Payments.

To find how much must be paid in annual installments, interest and principal together, to pay back a sum of money borrowed.

This problem makes A unknown and C known in formula (1), § 186. Solving for A, we get

$$A = C\frac{p-1}{p^{n+1}-p}. \tag{3}$$

EXAMPLE. A man borrows $1800, agreeing to repay it in ten equal annual payments, such payments to include interest and principal. How much shall he pay each year if 4% is the rate of interest allowed? *Ans.* $221.94.

188. Sinking Fund. 1. *To find how much money must be secured and put at interest each year so that such amounts taken together will equal a given sum due at a specified time in the future.*

Let $S =$ the sum to be had at the end of n years, $F =$ the number of dollars to be secured each year, and r the rate per cent of interest. To save space we will write a for $1+r$. Thus a is the amount of $1 due at the end of 1 yr. at r per cent.

The F dollars secured by taxation, or otherwise, by the end of the 1st year will be loaned out and draw interest for $n-1$ years, and will amount to Fa^{n-1} dollars (§ 172, 1); the second F dollars secured by the end of the second year, will draw interest for $n-2$ years, and amount to Fa^{n-2} dollars; and so on. The last F dollars will not draw interest, being secured during the last year. Writing these amounts *in reverse order*, we have

$$S = F + Fa + Fa^2 + Fa^3 + \cdots + Fa^{n-1}.$$

The terms of the right member of this equation form a geometrical progression, in which a (the first term) $= F$, $l = Fa^{n-1}$, $s = S$, r (the ratio) $= a$. Substituting these values in the formula $s = \frac{lr - a}{r - 1}$ for the sum of a geometrical progression, we get

$$S = \frac{Fa^{n-1} \times a - F}{a - 1} = \frac{F(a^n - 1)}{r}. \quad \left(\text{Since } a - 1 = 1 + r - 1 = r.\right) \qquad (4)$$

Solving for F, as it is the unknown, we get

$$F = \frac{rS}{a^n - 1}. \qquad (5)$$

EXAMPLE. How much must be raised by a town, and put at interest at 4% each year, to pay off bonds amounting to \$30,000 which become due in 20 yr.?

$$\log 1.04^{20} = 20 \times \log 1.04 = .3400; \text{ hence, } a^n = 2.188.$$
$$\log F = \log 30000 + \log .04 - \log 1.188.$$
$$\therefore F = \$1010.23.$$

NOTE. Towns often create a sinking fund by buying up their own bonds, or those of other places.

2. *To find the amount of an unpaid annuity.*

Evidently S in formula (4) is the amount of an annuity of F dollars left unpaid until the end of the period.

3. To find the amount of an annuity which remained unpaid m years *after the last payment was due.*

Here we must find the amount A' of $\frac{F(a^n - 1)}{r}$ dollars which ran for m years. Thus

$$A' = Fa^m \frac{a^n - 1}{r}. \qquad (6)$$

189. Bonds. A **bond** is in effect a promissory note legally authorized and signed by the proper officers of the municipality or firm issuing the bonds, usually having interest coupons attached for each interest payment.

Bonds vary in value somewhat in the manner of stocks, according to differences in character of the properties bonded, to variations in the money market, etc. Thus of two issues of $1000 bonds due in 10 years at 5 %, one may sell at 104 ($1040 for a $1000 bond) and another at 96 ($960 for a $1000 bond). A, who demands greater security, may prefer the former, yielding $4\frac{1}{2}$ % on his investment, while B, who will take a greater risk for a greater return may prefer the latter, yielding about $5\frac{1}{2}$ %. Thus, in studying bonds, two rates of interest appear: one, that named in the bond, giving its rate of interest; the other, the income rate that the purchaser will realize on his money if he buys the bond at a price other than par.

The calculation of bonds, though much like that of annuities, is quite complicated and should be deferred to a later part of the curriculum. Bond tables are in common use in which the left column gives prices from above par to below par (as from $145 to $76). The other columns are headed by numbers of years, as 1–50, and the body of the table gives the interest rate realized if the bond runs for the corresponding period at the top of the column and is bought at the price opposite in the left column. (See p. 205.)

190. Exercise in Annuities.

1. Find the cost of an annual pension of $800 each year for 8 years, allowing 4 % compound interest.

SUGGESTION. Use eq. (1), § 186.

2. Find what annuity for 5 years $2220 will yield at 4 % interest by using eq. (3), § 187.

3. If a corporation sets aside $5000 each year for 10 years and puts it at compound interest at 4 %, what sum will be available at the end of the time? Which of the formulas (1) to (5) suits the problem?

4. Find the cost of a pension of $1200 each year for 15 yr. at $3\frac{1}{2}$ %.

5. A man 59 yr. of age is expected to live 15 yr. He has $15,000 with which to buy a life annuity. If money is worth 4 %, how much will he get each year?

6. A town owes \$ 150,000 due in 15 yr. What sum must be collected and set aside annually to cancel the debt when it is due, if 5 % interest can be obtained?

7. Find the cost of an annuity of \$ 400, to begin at the end of 10 yr., and run for 20 yr. after that, if money brings $3\frac{1}{2}$ %.

8. An annuity of \$ 75 to run for 12 yr. was unpaid at the end of 15 yr. What amount was due if the rate was 3 % ?

9. A city borrowed \$ 160,000 to construct a sewer system, and agreed with the lender to repay this sum, principal and interest together, in 25 equal annual installments. If 4 % was the interest paid, how much was paid each year?

10. A man borrowed \$ 750 of another, agreeing to repay it in 5 equal annual payments, principal and interest together, with $4\frac{1}{2}$ % compound interest. What should each payment be?

11. An annuity of \$ 500 was unpaid for 10 yr. How much did it amount to if the interest was compounded *semiannually* at 4 % ?

SUGGESTION. The geometric series here would be $500 + 500 \times 1.0404 + 500 \times 1.0404^2 + \cdots$, 1.0404 equaling 1.02^2.

The following exercises are to be solved by use of the tables on pp. 204 and 205.

12. Find the amount of \$ 2700 at 3 % compound interest at the end of 15 years.

13. Find the present worth of \$ 1500 due at the end of 8 years, at 4 % interest.

14. Find the amount of an unpaid annuity of \$ 350 a year for 15 years, at 5 %.

15. Find the cost of a \$ 1000 bond paying 5 % semiannual compound interest if it has 10 years to run and will realize the purchaser 4.38 %.

16. What rate of interest will a purchaser realize who pays \$ 1040 for a \$ 1000 bond, paying 5 % semiannual interest, if it has 15 years to run?

INTEREST AND ANNUITIES

Years	Present Value of		Amount of		Annual Payment
	$1 due at end of *n*th year	$1 per annum due at end of every year	$1 due at end of *n*th year	$1 per annum due at end of every year	which discharges debt of $1 and its interest in *n* years
At 3%					
1	.9709	.9709	1.0300	1.0000	1.0300
2	.9426	1.9135	1.0609	2.0300	.5226
3	.9151	2.8286	1.0927	3.0909	.3535
4	.8885	3.7171	1.1255	4.1836	.2690
5	.8626	4.5797	1.1593	5.3091	.2184
6	.8375	5.4172	1.1941	6.4684	.1846
8	.7894	7.0197	1.2668	8.8923	.1425
10	.7441	8.5302	1.3439	11.4639	.1172
15	.6419	11.9379	1.5580	18.5989	.0838
20	.5537	14.8775	1.8061	26.8704	.0672
25	.4776	17.4131	2.0938	36.4593	.0574
At 4%					
1	.9615	.9615	1.0400	1.0000	1.0400
2	.9246	1.8861	1.0816	2.0400	.5302
3	.8890	2.7751	1.1249	3.1216	.3603
4	.8548	3.6299	1.1699	4.2465	.2755
5	.8219	4.4518	1.2167	5.4163	.2246
6	.7903	5.2421	1.2653	6.6330	.1908
8	.7307	6.7327	1.3686	9.2142	.1485
10	.6756	8.1109	1.4802	12.0061	.1233
15	.5553	11.1184	1.8009	20.0236	.0899
20	.4564	13.5903	2.1911	29.7781	.0736
25	.3751	15.6221	2.6658	41.6459	.0640
At 5%					
1	.9524	.9524	1.0500	1.0000	1.0500
2	.9070	1.8594	1.1025	2.0500	.5378
3	.8638	2.7232	1.1576	3.1525	.3672
4	.8227	3.5460	1.2155	4.3101	.2820
5	.7835	4.3295	1.2763	5.5256	.2310
6	.7462	5.0757	1.3401	6.8019	.1970
8	.6768	6.4632	1.4775	9.5491	.1547
10	.6139	7.7217	1.6289	12.5779	.1295
15	.4810	10.3797	2.0789	21.5786	.0963
20	.3769	12.4622	2.6533	33.0660	.0802
25	.2953	14.0939	3.3864	47.7271	.0710

PORTION OF A PAGE OF A BOND TABLE

Rate of interest realized if purchased at price in same row of left column and held for number of years at top of column.

At 5%, (Interest Payable Semiannually)

Price	Numbers of Years to Maturity							
	2	4	6	8	10	15	20	25
120			1.50	2.25	2.70	3.29	3.59	3.76
115		1.15	2.31	2.89	3.23	3.68	3.91	4.04
110	2.42	2.37	3.16	3.55	3.79	4.10	4.25	4.34
105		3.64	4.05	4.26	4.38	4.54	4.62	4.66
104	2.92	3.91	4.24	4.40	4.50	4.63	4.69	4.73
103	3.43	4.18	4.43	4.55	4.62	4.72	4.77	4.80
102	3.95	4.45	4.62	4.70	4.75	4.81	4.84	4.87
101	4.47	4.72	4.81	4.85	4.87	4.91	4.92	4.93
100	5.00	5.00	5.00	5.00	5.00	5.00	5.00	5.00
99	5.54	5.28	5.20	5.15	5.13	5.10	5.08	5.07
98	6.08	5.56	5.40	5.31	5.26	5.20	5.16	5.14
97	6.63	5.85	5.60	5.47	5.39	5.30	5.24	5.21
96	7.18	6.14	5.80	5.63	5.53	5.40	5.33	5.29
95	7.74	6.44	6.00	5.79	5.66	5.50	5.41	5.37
90		7.97	7.07	6.63	6.37	6.03	5.86	5.76
85		9.60	8.22	7.53	7.12	6.59	6.33	6.19
80			9.44	8.50	7.94	7.21	6.85	6.56

17. If a debt of $20,000 is paid off in 20 annual intallments of principal and interest together, how much will have to be paid each year if 4% compound interest is allowed?

18. What is the present worth, or *cost*, of an annuity of $750 a year for 25 years if the rate is 4%?

19. I pay $36.84 per thousand on a life policy. What will be the total sum invested at the end of 20 yr., allowing 4% compound interest?

Suggestion. Insurance premiums are paid at the *beginning* of each year. Find amount of annuity for 19 years at end of 20 years and add to it amount of first payment running for 20 years.

20. What will a 5% $1000 bond having 25 yr. to run and costing $1050 net a purchaser?

CHAPTER XI

BINOMIAL THEOREM

I. MATHEMATICAL INDUCTION

191. Mathematical Induction. — In the proof of the binomial theorem, which follows, use is made of **mathematical induction.** Mathematical induction, like induction in natural science, proceeds from the particular to the general, but it differs from induction in natural science in that the generalization is complete and absolute in its application, while in natural science the induction is very rarely, or never, complete. To make the nature of the method clear let us consider some examples of its application.

1. Show by mathematical induction that the *sum* of n terms of the sequence

$$\frac{1}{1\times 2}, \frac{1}{2\times 3}, \frac{1}{3\times 4} \cdots \text{is } \frac{n}{n+1}$$

PROOF. Let $S_1 \equiv \frac{1}{1\times 2} = \frac{1}{2}$; $S_2 \equiv \frac{1}{1\times 2} + \frac{1}{2\times 3} = \frac{2}{3}$; etc.

We see that *the value of each S* (sum) *examined has for its numerator the same as the S's subscript and for its denominator 1 more than its numerator.* Then, by this rule we should have

$$S_n = \frac{1}{1\times 2} + \frac{1}{2\times 3} + \frac{1}{3\times 4} + \cdots + \frac{1}{n\times(n+1)} = \frac{n}{n+1}.$$

Now "S_{n+1}" means the sum of all the terms from the first to the $n+1$ *th.* This sum can be found by adding the next term to the value of S_n.

Thus, $$S_{n+1} = \frac{n}{n+1} + \frac{1}{(n+1)(n+2)} = \frac{n+1}{n+2} \quad \text{(By addition.)}$$

Hence we see that *IF* $S_n = \frac{n}{n+1}$, it *follows* that $S_{n+1} = \frac{n+1}{n+2}$.

But we *know* $S_2 = \frac{2}{3}$, whence it follows that $S_3 = \frac{3}{4}$, and so on indefinitely.

Thus, the formula $S_n = \dfrac{n}{n+1}$ holds universally for this sequence, which was to be proved.

Notice particularly the argument on the last line of p. 206 and top of this page.

This method can be used for proving many such exercises. Thus, by it we can prove that

$$1^3 + 2^3 + 3^3 + \because \text{ to } n^3 = \left(\frac{n}{2}\right)^2 (n+1)^2.$$

2. Show by mathematical induction that $x^n - y^n$ is always exactly divisible by $x - y$.

PROOF.

$$\begin{array}{l|l} x^n - y^n & x - y \\ \underline{x^n - x^{n-1}y} & \overline{x^{n-1}} \\ x^{n-1}y - y^n & \end{array}$$

or $y(x^{n-1} - y^{n-1})$

Now, evidently if $x^{n-1} - y^{n-1}$ contains $x - y$, the remainder after the first division will, and so the whole division will come out without remainder.

Thus, we learn that if the difference of two same powers of x and y, viz. $x^{n-1} - y^{n-1}$ contains $x - y$, then the difference of the next higher powers, $x^n - y^n$, will exactly contain $x - y$.

But we know that $x^2 - y^2$ contains $x - y$; hence $x^3 - y^3$ contains $x - y$; then $x^4 - y^4$ contains $x - y$ exactly; and so on indefinitely. This theorem can be proved by means of the divisibility theorem, § 23, VII.

3. Show by mathematical induction that $x^{2n} - y^{2n}$ contains $x + y$.

SUGGESTION. In this case the division will have to be carried on until two terms are found in the quotient.

4. Show by mathematical induction that $x^{2n+1} + y^{2n+1}$ contains $x + y$. (§ 21, VIII.)

SUGGESTION. Make this demonstration dependent on that of Ex. 3, getting only one term in the quotient.

a. The demonstrations by mathematical induction, as already stated, give results as universally true as by any other method. In physical science results obtained by induction always remain open to revision. Thus, while it would be found that water boils at 212° at a thousand places, we should find the rule breaking down at mountain heights. In mathematical induction all the particulars are arranged in a series like ninepins. We upset one of the nine pins and all the series goes down.

II. THE BINOMIAL THEOREM

192. The Binomial Coefficients. We saw in § 53 that the signs and exponents in raising binomials to powers gave little trouble, but that the *coefficients* were found with more difficulty.

Let us write down these coefficients for several powers. They are

for $(a+b)^1$,	1, 1
for $(a+b)^2$,	1, 2, 1
for $(a+b)^3$,	1, 3, 3, 1
for $(a+b)^4$,	1, 4, 6, 4, 1
for $(a+b)^5$,	1, 5, 10, 10, 5, 1
for $(a+b)^6$,	1, 6, 15, 20, 15, 6, 1

Pascal, an eminent French mathematician (1623–1662), observed that the coefficients in any line can be obtained from the line above it by adding the coefficient immediately above the required one to the one which precedes it.

Thus, 6 in the fourth power comes from adding 3 above it to the 3 just before this 3. The last 15 in the sixth power comes from adding 5 above it to the 10 which precedes the 5; and so on. This writing and deriving binomial coefficients is called "Pascal's triangle."

Sir Isaac Newton (1642–1727) conceived the idea of making a *formula* which would give directly the expansion of $(a+b)$ to any power, or any desired term in it, without the use of Pascal's triangle.

193. Development of Newton's Theorem for any Positive Integral Exponent.

Let us take n to denote the exponent of the desired power and use the rules of § 53 to write down the result in the same way in which we wrote down particular powers, such as the sixth, tenth, etc. In this way we get

$$(a-b)^n = a^n - \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\times 2}a^{n-2}b^2 - \frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}b^3 + \text{etc.}$$

Thus, to get any coefficient we multiply the coefficient of the preceding term by the exponent of the leading letter, and divide by 1 more than the exponent of the other letter.

Now, while we saw that this rule held as far as we tried it, we do not *know* yet that it holds always, and we seek to prove that it is universally true. Instead of passing from the fourth to the fifth power, or from the fifth to the sixth, let us now try to pass from the nth to the $(n+1)$th power. To do this we multiply both sides of the equation just written by $a-b$. Then, for the left side of the new equation we have $(a-b)^n \times (a-b) = (a-b)^{n+1}$. For the right side, we have

$$\begin{array}{llll}
a^n & -\frac{n}{1}a^{n-1}b & +\frac{n(n-1)}{1\times 2}a^{n-2}b^2 & -\frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-3}b^3+\text{etc.} \\
a & -b & & \\
\hline
a^{n+1} & -\frac{n}{1}a^n b & +\frac{n(n-1)}{1\times 2}a^{n-1}b^2 & -\frac{n(n-1)(n-2)}{1\times 2\times 3}a^{n-2}b^3+\text{etc.} \\
 & -a^n b & +\frac{n}{1}a^{n-1}b^2 & -\frac{n(n-1)}{1\times 2}a^{n-2}b^3+\text{etc.} \\
\hline
a^{n+1} & -\frac{(n+1)}{1}a^n b & +\frac{(n+1)n}{1\times 2}a^{n-1}b^2 & -\frac{(n+1)(n)(n-1)}{1\times 2\times 3}a^{n-2}b^3+\text{etc.}
\end{array}$$

Hence,

$$(a-b)^{n+1}=a^{n+1}-\frac{(n+1)}{1}a^n b+\frac{(n+1)n}{1\times 2}a^{n-1}b^2-\frac{(n+1)(n)(n-1)}{1\times 2\times 3}a^{n-2}b^3+\text{etc.}$$

Examining this result, we see that the rules of § 53 give exactly the same terms as we have here.

From this we learn that *If the binominal theorem holds for the nth power, it holds also for the terms of the* $(n+1)$*th power; that is, for the next higher power.*

Notice that we *assumed* the theorem held for the nth power, and *proved* that *if* it held for the nth power, it would hold also for the terms of the $(n+1)$th power.

To finish the demonstration we use mathematical induction. We say we know the theorem holds for the sixth power, for we got that result by actual multiplication. Therefore, by the demonstration just concluded, the theorem holds for the next higher, or seventh, power. Again, since it holds for the seventh power, by the theorem it holds for the eighth power; and so on indefinitely. Therefore, it holds for all powers.

194. To find the *r*th Term in the Binomial Development.

$$\overset{1}{(a \pm b)^n = a^n} \pm \overset{2}{\frac{n}{1}a^{n-1}b} + \overset{3}{\frac{n(n-1)}{1 \times 2}a^{n-2}b^2} \pm \overset{4}{\frac{n(n-1)(n-2)}{1 \times 2 \times 3}a^{n-3}b^3} + \text{etc.}$$

Here we have written the number of each term over it.

Suppose now we attempt to write down the rth term. To do this we notice that the last number subtracted from n in the numerator of any coefficient is always 2 less than the number of the term; that the last figure in any denominator is always 1 less than the number of the term; that the number subtracted from n in the exponent of a is always 1 less than the number of the term. Hence, the rth term in the value of $(a+b)^n$ is

$$\pm \frac{n(n-1)(n-2)\cdots(n-[r-2])}{1 \times 2 \times 3 \times \cdots r-1}a^{n-(r-1)}b^{r-1}.$$

1. Find the 6th term of $(a+b)^{13}$, by substituting in the formula for the rth term $n=13$, $r=6$.

SOLUTION. $\dfrac{13 \times 12 \times 11 \times 10 \times 9}{1 \times 2 \times 3 \times 4 \times 5}a^8b^5 = 1287\ a^8b^5$. *Ans.*

2. Find the 5th term of $(x+y)^{11}$.

3. Find the 4th term of $(x-y)^{17}$.

4. Find the 6th term of $(3x-2y)^{10}$. (See § 53, 5.)

5. Find the 12th term of $(2a-b)^{15}$.

195. Exercise in Raising Binomials to Powers by the Theorem.

1. Expand $(a+b)^{21}$, getting six terms of the answer.

2. Expand $(x^{\frac{2}{3}}+3)^6$. 3. Expand $(3x^{\frac{2}{3}}+a^{\frac{3}{2}})^7$.

4. Find the 7th term of $(2x-y)^{15}$.

5. Find $(2x-y)^{18}$ to seven terms in answer.

6. Expand $1-\dfrac{x}{3}$ to the 9th power.

7. Find the rth term of $(3x-2y)^{25}$.

8. Find the middle or 6th term of $\left(x - \frac{1}{x}\right)^{10}$.

9. Find the middle term of $(m^{\frac{1}{3}} - m^{-2})^8$.

10. Find the sum of the binomial coefficients for the 15th power.

SUGGESTION. If $(a + b)^{15}$ were expanded and then a and b each set equal to 1, we should get the sum of the coefficients, which sum evidently would equal left member, or 2^{15}.

11. Expand $\left(\sqrt{\frac{x}{2}} + \sqrt[3]{\frac{y}{3}}\right)^4$.

12. Expand $\left(\frac{\sqrt{-a}}{2} - \frac{\sqrt{-b}}{3}\right)^5$.

While the proof given in § 193 does not cover the case of fractional exponents, it is true that the binomial theorem holds good also for fractional and negative exponents.

13. Show that one gets the same result by applying the binomial theorem to $(a + b)^{\frac{1}{2}}$, as one does by extracting its root by the method of § 58.

14. Apply the binomial theorem to expand $(x + y)^{\frac{2}{3}}$ and $(x + y)^{\frac{7}{3}}$, and then form the product of these results, getting the first terms of $(x + y)^3$, or $x^3 + 3x^2y + 3xy^2 + y^3$.

15. Expand $(a + b)^{\frac{4}{5}}$ to four terms.

16. Expand $(x - 2)^{-2}$ to five terms.

17. Expand $(a - b)^{-\frac{5}{6}}$ to four terms.

CHAPTER XII

INEQUALITIES

196. Inequalities. An **inequality**, or an **inequation**, is a statement that one quantity is greater or less than another. The sign for "greater than" is $>$, and that for "less than" is $<$. The opening of the angle is towards the greater quantity. These symbols, with $=$, are called symbols of relation. By "greater than" is meant higher up in the algebraic scale.

197. Principles governing the Solution of Inequalities.

1. *It is allowable to* **transpose** *in an inequality just as in an equation.*

PROOF. Let $a \pm c > b$.

Then, $a > b \mp c$. (Equals added to or subtracted from unequals give unequals, since to do this merely slides both members of the inequation the same distance along the algebraic scale.)

2. *It is allowable to multiply or divide an inequality by the same positive number; but multiplying or dividing by a negative number reverses the sign.*

PROOF. (1) Let $a > b$, and m be a positive number.

Then, $ma > mb$. (Unequals multiplied by the same positive number give unequals, and that product is the greater which is obtained from the greater of the two unequals. This is true whatever signs a and b have.)

(2) Let m be negative.

Then, $ma < mb$. (By the preceding case and because the quantity which was greater than the other *at first* becomes less than the other *afterwards*, when the signs of both are changed.)

The pupil should test these statements with numbers expressed in the Arabic notation.

3. *Inequalities can be added, greater to greater and less to less, the former sum being greater than the latter, but it is not allowable to subtract one inequality from another.*

The first part of the theorem is evidently true, but the student may not see why he cannot subtract one inequation from another. An example or two is sufficient here, since if the rule fails in *one* instance, it does not hold.

Thus,

$$\begin{array}{r} 12 > 10 \\ 5 > 4 \\ \hline 7 > 6 \end{array} \qquad \begin{array}{r} 12 > 11 \\ 5 > 2 \\ \hline 7 \not> 9 \end{array}$$

4. *The members of an inequality both of whose terms are positive can be raised to the same power.*

198. Special Theorems in Inequalities, and Exercises.

1. To prove that the sum of the squares of any two real and unequal numbers is greater than twice their product.

PROOF. Let a and b be any two real and unequal numbers.

Then, $(a-b)^2 > 0.$ (Since a square number is always positive, that is, greater than 0.)

Or $a^2 - 2ab + b^2 > 0.$ (Squaring $a - b$.)

$\therefore a^2 + b^2 > 2ab.$ Q.E.D. (§ 197, 1.)

2. Prove that any positive fraction whose terms are unequal plus its reciprocal is greater than 2.

Thus, to prove $\frac{x}{y} + \frac{y}{x} > 2.$

SUGGESTION. Make this Ex. depend on Ex. 1, using § 197, 2.

3. Prove that the arithmetic mean of two different positive numbers is greater than the geometric mean.

Thus, to prove $\frac{a+b}{2} > \sqrt{ab}.$

SUGGESTION. Square both members, § 290, 4.

4. Show that $x^3 + y^3 > x^2y + xy^2$ when $x + y > 0$. (See § 197, 2).

5. Show that $x^3 + 1 > x^2 + x$ when $x + 1 > 0$ and $x \neq 1$.

6. Show that $x^2 + y^2 + z^2 > xy + xz + yz$ if x, y, z are unequal.

SUGGESTION. Use § 198, 1, and § 197, 3, and then divide through by 2.

7. If $2x + \frac{1}{2}x - 4 > 6$, show that $x > 4$.

8. Show that the difference between two sides of a triangle is less than the third side. If a, b, c, are the three sides, show that $c - b < a$, using § 197, 1.

9. Two of the sides, a and b, of a triangle are respectively 9 and 15 inches. Between what limits must the third side c lie?

10. If $3x - 7 - 5x > \frac{3}{2}x - 9$, what limiting value does x have?

11. If a, b, and c are positive numbers, and if $a > b$, prove that $\frac{a+c}{b+c} < \frac{a}{b}$.

12. For what values of x is $x^2 - 6x - 7 > 0$? (See § 100.)

13. A's record time for running a mile is 4 minutes and 30 seconds, and B's is 4 minutes and 40 seconds. Less than how many feet handicap shall A give B so as still to beat him?

CHAPTER XIII

REASONING IN EQUATIONS — DISCUSSION OF THE QUADRATIC — DISCUSSION OF PROBLEMS — DEFINITIONS

I. REASONING IN EQUATIONS

199. Validity of Processes in the Solution of Equations. It might be supposed that any root of any equation obtained by axiomatic processes would verify when substituted in the original equation, but such is not always the case. This does not mean that the axioms are not true, but that new equations obtained by means of them are not **equivalent** to the given equation, that is, do not have identically the same roots as the given equation.

200. Equivalent Equations. *Reversibility of the steps of a solution.* Whether a root obtained will satisfy the given form of the equation depends on whether the steps of the solution are *reversible.* Two equations such that either can be derived from the other, and which have exactly the same roots are said to be **equivalent** (see p. 43). The processes used in the solution of *simple* equations rarely give anything except equivalent equations, but in the solution of radical, quadratic, and higher equations, new roots are frequently introduced, and sometimes roots of the given equations are lost in the process of solution.

1. *Two equations are equivalent if one can be obtained from the other by adding the same quantity to both members.* (By Addition and Subtraction Axioms.)

PROOF. Let S and S' denote the two members of an equation and a any third quantity, each of these being capable of containing both known and unknown quantities.

Then, any value of the unknown which satisfies $S = S'$ evidently satisfies $\boldsymbol{S} + \boldsymbol{a} = \boldsymbol{S}' + \boldsymbol{a}$. Also any value which satisfies $\boldsymbol{S} + \boldsymbol{a} = \boldsymbol{S}' + \boldsymbol{a}$, also satisfies $\boldsymbol{S} = \boldsymbol{S}'$.

Since transposition depends on the addition and subtraction axioms, equations obtained by it are equivalent.

2. *Equations obtained by multiplying or dividing both members of an equation by the same known quantity are equivalent.*

If a is a known number, and if $\boldsymbol{S} = \boldsymbol{S}'$, then $a\boldsymbol{S} = a\boldsymbol{S}'$; also if $a\boldsymbol{S} = a\boldsymbol{S}'$, then $\boldsymbol{S} = \boldsymbol{S}'$.

3. *An equation obtained from another by squaring its members or raising them to a higher power, or extracting the same root of each member is not equivalent to the given equation.*

As before, let $\boldsymbol{S}$ and $\boldsymbol{S}'$ denote the two members.

We have $\boldsymbol{S} = \boldsymbol{S}'$,

whence $\boldsymbol{S}^2 = \boldsymbol{S}'^2$; (Sq. Ax.)

then $\boldsymbol{S}^2 - \boldsymbol{S}'^2 = 0$. (Sub. Ax.)

Factoring, $(\boldsymbol{S} + \boldsymbol{S}')(\boldsymbol{S} - \boldsymbol{S}') = 0$.

We have then $\boldsymbol{S} + \boldsymbol{S}' = 0$, whence $\boldsymbol{S} = -\boldsymbol{S}'$; (Solution by factoring.)

$\boldsymbol{S} - \boldsymbol{S}' = 0$, whence $\boldsymbol{S} = \boldsymbol{S}'$.

The last equation is identical with the given equation, but we have also the equation $\boldsymbol{S} + \boldsymbol{S}' = 0$ *in addition*, which is a new equation.

Example. Given $\sqrt{x+5} + \sqrt{x} = 1$.

$\sqrt{x+5} = 1 - \sqrt{x}$. (Sub. Ax.)

$x + 5 = 1 - 2\sqrt{x} + x$. (Sq. Ax.)

$2\sqrt{x} = -4$. (Sub. Ax.)

$4x = 16,\ x = 4$. (Sq. Ax.)

Verification. $\sqrt{4+5} + \sqrt{4} \neq 1$, or, $3 + 2 \neq 1$.

Thus 4 is not a root of the given equation. Notice that if we had started with the equation $\sqrt{x+5} - \sqrt{x} = 1$, the solution would have differed from that above only in a few signs, and the answer would have been 4, which answer would have verified. The three equations $\sqrt{x+5} + \sqrt{x} = \pm 1$, and $-\sqrt{x+5} + \sqrt{x} = 1$ have been called impossible equations, since if the signs are taken just as they stand there is no root that will satisfy them.

4. *An equation may not be multiplied or divided through by a function of x which equals 0, or some number divided by 0, when the value of x is substituted in it.*

EXAMPLE. $$\frac{y+\frac{1}{y}}{y-\frac{1}{y}}+\frac{1+\frac{1}{y}}{1-\frac{1}{y}}=2.$$

$$y+\frac{1}{y}-1-\frac{1}{y^2}+y+1-\frac{1}{y}-\frac{1}{y^2}=2\left(y-1-\frac{1}{y}+\frac{1}{y^2}\right). \quad \text{(Mult. Ax.)}$$

Clearing this equation of fractions and collecting, we get

$$4y^2+4y=8, \text{ whence } y=1 \text{ or } -2.$$

But while $y=-2$ satisfies the original equation, $y=1$ *does not*, since $\frac{2}{0}+\frac{2}{0}$ on the left side *has no meaning* if 0, as usual, stands for what is called absolute 0, and not a limit as explained in § 207, while 2 on the right side is 2. The reason why the root $y=1$ appeared is because when we first cleared of fractions, we multiplied the equation by $y-1$, and because $y=1$ we were really multiplying by 0, which is not allowable by the principle just stated.

This principle explains the fallacy in the so-called "proof" that $1=2$.

Thus, let $a=x$; then $ax-x^2\equiv a^2-x^2$;

whence, on factoring, $x(a-x)=(a+x)(a-x)$.

Then $x=a+x$, (Dividing by $a-x$.)

or, $x=x+x=2x$; (Since $a=x$.)

whence $1=2$. (Dividing by x.)

The first and second equations are true equations; the third and all the rest are false, since to get the third we divided by $a-x$, and as $a=x$ making $a-x=0$, we were dividing by zero.

We may set forth the truth, illustrated under 4, more briefly as follows:

Let R be any expression containing the unknown, and, as before, let S and S' be the two members of an equation.

Then, $S=S'$.

Whence $RS=RS'$. (Mult. Ax.)

Then, $RS-RS'=0$, or $R(S-S')=0$. (Transposing and factoring.)

Setting each factor equal to 0, we have $R=0$, and $S-S'=0$.

Thus, the new factor with its root has been introduced into the equation.

This case includes 3, above, since we can write $\boldsymbol{R} = \boldsymbol{S} + \boldsymbol{S}'$.

In conclusion, then, if at any time we cannot tell whether a root found belongs to the given equation, we always have this recourse that we can substitute it in that equation and find out. Roots should always be verified in the original equations.

201. The Analytic Method. Algebra uses the *analytic* method; arithmetic, the *synthetic*. When a problem is solved by the synthetic method, one starts with the numbers that are given, and, by combining them in a certain order, obtains the answer. In algebra, on the other hand, one treats the answer as if known, calling it x; then, forming the equation, the course is *backward* to the answer.

To illustrate, take the problem, What number is that whose square increased by 11 times the number equals 60? Putting this in the form of an equation, we have

$x^2 + 11x = 60.$ (If there is such a number as described in the problem, then this equation is true.)

Then, $4x^2 + 44x = 240.$ (If the preceding equation holds, this one holds.)

$4x^2 + 44x + 121 = 361.$ (If the preceding equation holds, this one holds.)

$2x + 11 = \pm 19.$ (If the preceding equation holds, this one holds.)

$x = 4, \text{ or } -15.$ (If the preceding equation holds, this one holds.)

Now, we assume we can reverse the reasoning. We know there is such an equation as $x = 4$, or $x = -15$; then (multiplying each member by 2, and adding 11 to each result), the fourth equation holds, then the third, then the second, then the first. In this way we see that the steps of our solutions should be reversible if we desire to draw safe conclusions.

The problem 'to find a number such that the sum of its square root and the square root of 5 more than the number is unity' gives the equation of the preceding article, $\sqrt{x+5}+\sqrt{x}=1$. By solving this equation, 4 is found for the value of x. But we are prepared to find that 4 does not satisfy the given condition, because one of the steps of the solution (squaring both members) was not reversible.

202. Exercise in Reasoning in Equations.

1. Prove that cubing the members of an equation introduces a new factor with new roots.

2. Solve $2x\sqrt{x^2+3}-2x\sqrt{x^2+2}=1$, and verify.

3. Solve $\frac{x-3}{5}-\frac{8}{x^2}=\frac{2}{x}-\frac{8-6x+x^2}{x^2}$, and verify all the roots.

4. Solve $\frac{5x-a}{x+a}+\frac{x}{x+b}=\frac{4x-2a}{x+a}+\frac{4x-a}{x+b}+1$. Verify.

5. Solve $2\sqrt[3]{x^3+5x+19}=(x+3)(\sqrt{-3}-1)$. Verify.

SUGGESTION. Solve also same equation with $x+3$ as right member.

203. Systems of Equations. We saw in §§ 44, 45, that a set of two given equations might be replaced by one new equation and one of the given equations, or by two entirely new equations.

An equation obtained by adding the members of two equations, or by adding after each equation has been multiplied through by a known quantity, can take the place of either of the given equations.

Thus, if $S=S'$ and $T=T'$, then $S+T=S'+T'$;
also if $S+T=S'+T'$, and $S=S'$, then $T=T'$.

If $S=S'$, then $aS=aS'$ and if $T=T'$ then $bT=bT'$, where a and b are supposed to be known quantities.

Then, $aS+bT=aS'+bT'$. Also if this equation and either of the given ones hold true, the remaining given one holds true.

But in simultaneous quadratic equations all kinds of changes were made in the given equations in the processes of solution. Hence the plan of verifying roots found in the original equation or equations should be regularly followed.

II. THEORY OF QUADRATIC EQUATIONS

204. Relation between the Coefficients and Roots in Quadratic Equations.

1. Every quadratic equation containing one unknown number may be reduced to the form $x^2 + px + q = 0$.

This can always be accomplished by

(a) *Clearing of fractions, when fractions appear;*

(b) *Transposing all terms to the left member;*

(c) *Collecting, and, when necessary in literal equations, factoring x^2 out of two or more x^2 terms and x out of two or more x terms.*

(d) *Dividing the equation through by the coefficient of x^2.*

Because any quadratic equation can be reduced to the form $x^2 + px + q = 0$, we will study this type of equation.

2. Solve the following eight typical equations, all except the last two by the factoring method.

(1) $x^2 - 5x + 6 = 0$. (2) $x^2 + 5x + 6 = 0$.

(3) $x^2 - x - 6 = 0$. (4) $x^2 + x - 6 = 0$.

(5) $x^2 - 4x + 4 = 0$. (6) $x^2 - 2x = 0$.

(7) $x^2 - 2x - 6 = 0$. (8) $x^2 - 2x + 6 = 0$.

3. Questions on the preceding solutions. A careful study of the solutions of the preceding equations raises the following questions:

(1) What number of roots has every quadratic?

(2) What relation exists between the coefficient of x and the sum of the two roots of the equation?

If the student has trouble answering these questions, he should study §§ 98, 99 anew.

(3) What relation exists between the known term in the equation and its two roots?

(4) If all the terms of an equation are positive and its roots are real, what can you say of the signs of the roots?

(5) If all the terms of an equation are positive except that containing x, and its roots are real, what can you say of the signs of its roots?

(6) If the known term of an equation is negative, what signs have the two roots?

(7) If the known term and the term containing x of an equation are both negative, which is numerically greater, the positive or the negative root? If the term containing x is positive instead of negative, which root is numerically greater?

(8) If the known term of an equation is absent, what value has one of the roots in every case?

(9) If the left member of the equation whose right member is zero is a perfect square, what can you say of the relative size of the two roots?

(10) If the known term is positive and greater than the square of half the coefficient of x, what will invariably be the character of the roots?

The question arises, can we *prove* the answers to the preceding questions true in general?

4. Proofs concerning the *number*, *sum*, and *product* of the roots of a quadratic equation.

To prove these theorems we will first solve the equation

$$x^2 + px + q = 0.$$

$$4x^2 + 4px = -4q. \qquad \text{(Hindu Rule.)}$$

$$4x^2 + 4px + p^2 = p^2 - 4q. \qquad \text{(Completing square. Ax.?)}$$

$$2x + p = \sqrt{p^2 - 4q}. \qquad \text{(Ax.?)}$$

$$x = -\tfrac{1}{2}p \pm \tfrac{1}{2}\sqrt{p^2 - 4q}. \qquad \text{(Ax.?)}$$

(1) Taking the sign before the radical as positive, we get one root, and taking it negative we get a second root. Thus, there are *two* roots to a quadratic equation.

(2) Prove by adding the two roots just found, viz.,

$$-\tfrac{1}{2}p + \tfrac{1}{2}\sqrt{p^2 - 4q} \text{ and } -\tfrac{1}{2}p - \tfrac{1}{2}\sqrt{p^2 - 4q},$$

that the sum of the roots of any equation equals the coefficient of x in the given equation with its sign changed.

(3) Prove by multiplying one of the two roots just found by the other, that their product is q, that is, prove *that the product of the roots is always equal to the known term in the left member.*

The student may now verify that this truth holds in Exs. (7), (8) in 2 above.

a. It must be borne in mind that the theorems in (2) *and* (3) *just proved hold true only when the coefficient of* x^2 *is unity; or when the equation has the form* $x^2 + px + q = 0$.

b. In the study of the quadratic which follows, the expression,

$$\sqrt{p^2 - 4q}, \text{ or } p^2 - 4q, \text{ (or } b^2 - 4ac),$$

takes a prominent place. The quantity $p^2 - 4q$ is called the **discriminant** of the quadratic, because by it we can tell all about the roots of the equation, as to whether they are equal, rational, irrational, imaginary, or one of them is zero; also as to their signs, in certain cases.

(4) If p and q are both positive and $p^2 > 4q$, making $p^2 - 4q$ positive, then $\sqrt{p^2 - 4q}$ is real. Then both the roots are real. We see that both roots are negative; since $-\frac{1}{2}p \pm \frac{1}{2}\sqrt{p^2 - 4q}$ is negative whether the sign before the radical is + or −, because $\frac{1}{2}\sqrt{p^2 - 4q}$ is numerically less than $\frac{1}{2}p$; it is numerically less than $\frac{1}{2}p$, because if p^2 is *diminished* by $4q$ and then the square root is extracted, the result is less than p.

(5) If p alone is negative and $p^2 > 4q$, then $-\frac{1}{2}p$ is a *positive* quantity. Also, as before, $\frac{1}{2}p > \frac{1}{2}\sqrt{p^2 - 4q}$. Hence, both roots are positive.

(6) If q is negative, $p^2 - 4q$ is positive since p^2 itself is positive, and if q is negative the term $-4q$ becomes positive. Notice also that in this case $\frac{1}{2}\sqrt{p^2 - 4q} > \frac{1}{2}p$. Hence, when the sign before the radical is positive, the root is positive; and when it is negative, the root is negative.

(7) If $q = 0$, $-\frac{1}{2}p + \frac{1}{2}\sqrt{p^2 - 4q} = 0$. Hence, one root is 0.

(8) If $p^2 - 4q = 0$, or $q = \left(\frac{p}{2}\right)^2$, the two roots are equal, each

being $-\frac{1}{2}p$. Notice that when $q=\left(\frac{p}{2}\right)^2$, q "completes the square" and makes the left member a perfect square.

(9) If q is positive and $p^2 < 4q$, both roots are *imaginary.*

TABULAR DISCUSSION OF $x^2+px+q=0$

I. $p^2>4q$	1. $p>0,\ q>0$ 2. $p<0,\ q>0$ 3. $p>0,\ q<0$ 4. $p<0,\ q<0$	Both roots negative Both roots positive Pos. root (numerically) $<$ neg. root Pos. root (numerically) $>$ neg. root
II. $p^2=4q$	1. $p>0$ 2. $p<0$	Roots both negative and equal Roots both positive and equal
III. $p^2<4q$		Roots imaginary
IV. $p=0$	1. $q<0$ 2. $q>0$	Roots equal numerically but opposite in sign Roots imaginary
V. $q<0$		Roots always real

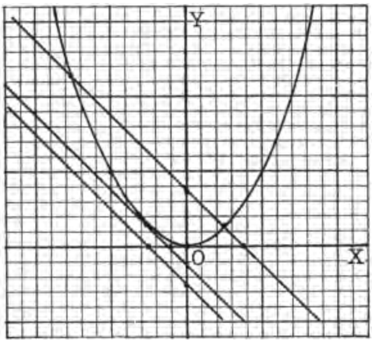
205. Exercises in telling the Character of the Roots of Quadratic Equations. If the equation is not given in the form $x^2+px+q=0$, change it to that form. First apply the test $q<0$; then, if necessary, $p^2-4q>0$ test to see if the roots are real. Then tell by the coefficients the character of the roots as explained in the preceding article.

1. $x^2+22x-75=0$.
2. $5x^2+20x=25$.
3. $x^2-6x=0$.
4. $x^2-\frac{2}{3}x=32$.
5. $4x^2+15=16x$.
6. $1+9x^2=6x$.
7. $23x+5=10x^2$.
8. $3x^2+8x+3=0$.
9. $(x-1)(x-2)=1$.
10. $2x^2-11x+16=0$.
11. $6x^2-35x-6=0$.
12. $4z^2+12z+11=0$.
13. $\frac{x^2}{9}+\frac{x}{3}+\frac{35}{4}=0$.
14. $(x+4)^2=8x+25$.

15. $\frac{2x+1}{1-2x}-\frac{5}{7}=\frac{x-8}{2}.$ 16. $\frac{9}{2x+1}+\frac{3}{x-3}=4.$

206. Graphical Explanation of real and different, real and coincident, and imaginary roots of the quadratic.

1. Solve by the method of § 131 the three equations.



$$4x^2+4x-3=0,$$
$$4x^2+4x+1=0,$$
$$4x^2+4x+2=0.$$

In the diagram in the margin which contains the solutions of all three equations we see that the roots of $4x^2+4x-3=0$, are real and distinct; the roots of $4x^2+4x+1=0$, are real and coincident, the changing of the value of the known term causing the straight line graph to move to the left until the straight line is tangent to the curve and the two roots of the first equation have *come together*; and the roots of $4x^2+4x+2=0$, are imaginary, the straight line never meeting the parabola at all.

2. Solve by the method of § 131 the following equations, using one diagram for all:

$$x^2-x+1=0,\quad x^2-2x+1=0,\quad x^2-3x+1=0,$$
$$x^2+3x+1=0,\quad x^2+2x+1=0,\quad x^2+x+1=0.$$

Explanation. Notice in this case that all the straight line graphs go through the point $(0, -1)$. The first does not intersect the parabola, the second is tangent to it, giving coincident roots, the third intersects it in one point and would intersect it in another if both were extended. The other three equations give corresponding lines on the other side of the y-axis.

a. **Parameters.** The student is asked to observe that in Ex. 1, the known term changed and the other two remained the same, which had the effect of keeping the graphs parallel. In Ex. 2, by letting the coefficient of x change, and the other terms remain the same, the graphs became rays going out from $(0, -1)$. When one of the coefficients of an equation is allowed to change in this way it is called a **parameter.**

207. The Symbolical Forms, $\frac{0}{a}, \frac{a}{0}, \frac{0}{0}$.

1. If the numerator of a fraction changes in value and approaches indefinitely closer and closer to 0, while the denominator remains the same, or is constant, the fraction also approaches indefinitely close to 0 in value. In this way we are led to write $\frac{0}{a} = 0.$

2. If the numerator of a fraction remains constant and the denominator decreases indefinitely towards 0, the value of the fraction evidently *increases* indefinitely in value. Hence with this understanding, and denoting this quotient by ∞, we can write

$$\frac{a}{0} = \infty.$$

3. If both numerator and denominator decrease indefinitely towards 0, and we have no means of knowing their relative values, then the value of the fraction $\frac{0}{0}$ is *indeterminate.*

a. We have already observed that division by 0 has no meaning. But the above "forms" are convenient and are used by mathematicians. Perhaps the best way to view them is to think of 0 as denoting the *limit* towards which a variable is approaching.

Thus, $\frac{0}{5} = 0$ is to be understood as a short form of stating that the *limit* towards which $\frac{x}{5}$ tends, when x approaches the limit 0 is 0.

To show that $\frac{0}{0}$ has different values or is "indeterminate" we may write,

$$\frac{x^2-1}{x-1} = \frac{0}{0} = x + 1 = 2 \text{ (when } x = 1\text{)};$$

$$\frac{x^2-4}{x-2} = \frac{0}{0} = x + 2 = 4 \text{ (when } x = 2\text{)}.$$

208. Further Discussion of the Quadratic.

If $ax^2 + bx + c = 0$ is solved, the roots are

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

1. If $c = 0$, $x = \frac{-b+b}{2a} = 0$; or $x = \frac{-b-b}{2a} = -\frac{b}{a}$.

2. If $a = 0$, and $-b \pm \sqrt{b^2 - 4ac} \neq 0$, $x = \infty$.

3. If $a = 0$, and $-b \pm \sqrt{b^2 - 4ac} = 0$, x is indeterminate.

To understand these results the symbol $\doteq$ should be used. Thus, $a \doteq 0$ means that a is an infinitesimal quantity which **approaches indefinitely close to 0.** The symbol $\doteq$ is read "approaches the limit."

Using this sign, we have:

(1) If $c \doteq 0$, $x \doteq 0$, or $x \doteq -\frac{b}{a}$.

(2) If $a \doteq 0$, and $-b + \sqrt{b^2 - 4ac} \neq 0$, $x \doteq \infty$.

(3) If $a \doteq 0$, and $-b \pm \sqrt{b^2 - 4ac} \doteq 0$, $x =$ indeterminate.

III. DISCUSSION OF PROBLEMS

209. The Courier Problem, often given in texts, is well adapted to illustrate what is meant by "the discussion of a problem." It gives rise to a simple equation.

1. A and B travel in the same direction at the rate of a and b miles per hour respectively. A arrives at a certain place, K, at a certain time, and B arrives at L, which is c miles from K, d hours after (*or* before) A was at K. In how many hours from the time A was at K will they be or were they together?

Solution. Let $x =$ number of hours from time A was at K until one passes the other at M.

Then, $KM = ax$; $LM = (x - d)b$;

and $ax = c + (x - d)b$.

Solving, $ax - bx = c - bd$.

$$x = \frac{c - bd}{a - b}. \quad \textit{Ans.}$$

I. Suppose $a > b$ and $c > bd$. Then x is positive, which shows one carrier overtook the other *after* A was at K, both traveling in the direction K to M.

During the d hours before B reached L he traveled bd miles. But, by supposition, $bd < c$. Thus, when A was at K, B was between K and L. Since A travels faster than B by supposition, he will catch up with B at some place to the right of where B was when A was at K. But we do not know whether M is to the right of L as represented in the diagram or to the left of it.

We have, $LM = (x-d)b = \left(\dfrac{c-bd}{a-b} - d\right)b = \dfrac{(c-ad)b}{a-b}$.

If, now, $c > ad$, then LM is positive, and M lies to the right of L as we supposed and so represented in the diagram; but if $c < ad$, then LM is negative, and M lies to the *left* of L.

Notice, if c is supposed greater than ad, it will take A more than d hours to reach L, and he will pass B at a point to the right of L; but if c is less than ad, A will catch up with B before the d hours are up, that is, before B reaches L.

II. Suppose $a > b$, but $c < bd$. This makes the value of x negative, which shows A passed B *before* he reached K.

III. Suppose $a < b$, and $c > bd$. This makes x negative and shows that B passed A at a point to the left of K. In this case B travels more rapidly than A and will reach K before A does.

IV. Suppose $a < b$, and $c < bd$. This makes x positive and shows that B overtook A to the right of K. In this case B travels more rapidly than A and was *behind* A when he arrived at K.

V. If $c = bd$, then $x = 0$. In this case A and B are together when A is at K.

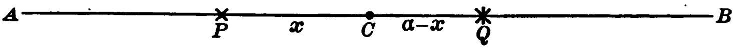
VI. If $a = b$, and $c \neq bd$, then $x = \infty$. Here A travels as fast as B and one is behind the other. The former will never catch up with the latter, or, as mathematicians say, he will catch up with him at infinity.

VII. If $a = b$, and $c = bd$, then $x = \dfrac{0}{0}$. In this case A and B are together at K, and, walking at the same rate, they *stay* together. Thus, x can have *any* value.

2. Prepare a corresponding discussion of this problem on the supposition the couriers A and B travel in opposite direction instead of the same direction.

3. Change the conditions of 1, by supposing that A arrives at K d hours after B arrived at L, both traveling in the same direction.

210. Clairaut's Problem of the Lights leads to the solution of a quadratic equation.



Two lights at P and Q are a feet apart. It is required to find the points in AB which are equally illuminated by the two lights. Let C be one such point, x feet from P.

By a law of optics the intensity with which a light shines at any outside point is inversely proportional to the square of the distance from the point to the source of the light.

Let m^2 = illumination by light P one foot from P,

and n^2 = illumination by light Q one foot from Q.

Then the illumination at C from P is $\frac{m^2}{x^2}$ and from Q is $\frac{n^2}{(a-x)^2}$.

Hence,
$$\frac{m^2}{x^2} = \frac{n^2}{(a-x)^2}.$$

$$\frac{m}{x} = \pm\frac{n}{a-x} \qquad \text{(Root Ax.)}$$

$$\therefore ma - mx = \pm nx, \text{ or } mx \pm nx = ma,$$

giving
$$x = \frac{ma}{m \pm n}.$$

1. What does the double sign in the value of x teach as regards the *number* of points of equal illumination?

2. If + in the value of x is taken, what kind of a fraction is $\frac{m}{m+n}$, proper or improper? Where then is C with reference to P and Q? Between them or on one side or the other?

3. If $-$ in the value of x is taken, and $n < m$ making the value of the fraction positive, what kind of fraction is $\frac{m}{m-n}$, proper or improper? Is the corresponding value of x greater or less than PQ? Where then will this point of equal illumination be located?

4. If $-$ in the value of x is taken, and $n > m$, what sign has value of x? In which direction from P will the corresponding point of equal illumination lie?

We see, then, from the preceding that there are always *two* points of equal illumination, one lying *between* P and Q, and the other either to the right of Q or to the left of P depending on which is the stronger light.

5. If $m=n$, $x=\frac{ma}{2m}$, or $\frac{ma}{0}$. Thus, one value of $x=\frac{1}{2}a$, locating c midway between P and Q, and the other point is at ∞.

6. If $a=0$, and $m=n$, $x=\frac{0}{0}$, which is indeterminate. Thus, if the two lights are together and one is as strong as the other, they will illuminate *every* point equally.

PART III. ADVANCED ALGEBRA

CHAPTER XIV

THEORY OF EQUATIONS AND SOLUTION OF HIGHER EQUATIONS

I. ROOTS AND COEFFICIENTS

211. Relations between the Roots and Coefficients of Equations.

In § 204, we learned that in a quadratic equation whose right member is 0, *and whose coefficient of* x^2 *is unity, the coefficient of* x *with its sign changed equals the sum of the roots, and the known term equals the product of the roots.*

Solve the following by § 100 and test the foregoing rule:

$$x^2 - 6x + 8 = 0;\ x^2 - (a + b)x + ab = 0.$$

The question arises, — what corresponding relations exist for equations of higher degrees?

In § 103 it was shown that to construct an equation whose roots are given, each root is subtracted from x, the remainders are multiplied together, and the product is set equal to 0.

1. Construct the equation whose roots are a, b, c.

SOLUTION. $(x - a)(x - b)(x - c) = 0.$

Performing the multiplications indicated and arranging the result according to the powers of x we have

$$x^3 - (a + b + c)x^2 + (ab + ac + bc)x - abc = 0. \quad \textit{Ans.}$$

(The student should perform these operations in full.)

2. Construct the equation whose roots are a, b, c, d.

SOLUTION. $(x - a)(x - b)(x - c)(x - d) = 0,$

or, $x^4 - (a + b + c + d)x^3 + (ab + ac + ad + bc + bd + cd)x^2$
$- (abc + abd + acd + bcd)x + abcd = 0.$ *Ans.*

3. In the two preceding exercises, we see that when a complete equation (§ 216, 2), such as $x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \cdots + p_n = 0$,* has 0 for its right member, and 1 for the coefficient of the highest power of the unknown:

(1) *The coefficient p_1 of the second term with its sign changed equals the* **sum** *of the roots.*

(2) *The coefficient p_2 of the third term equals the sum of all the products of* **pairs** *of the roots.*

(3) *The coefficient p_3 of the fourth term with its sign changed equals the sum of all the products of the roots taken* **three** *together.*

(4) *The coefficient p_4 of the fifth term equals the sum of all the products of the roots taken* **four** *together.*

It can be shown by taking equations of higher degrees that (1), (2), (3), (4), just given, hold generally for all equations, and that *the sixth coefficient p_5 with its sign changed equals the sum of all the products of the roots taken* **five** *together, and so on.* Observe the change of sign noted for the second, fourth, sixth, etc., coefficients.

a. The coefficients of the equations in Ex. 1–2 just given are symmetric functions (§ 62) of the roots, since any two roots can be interchanged without altering the value of the coefficients.

212. Constants and Variables. A **constant** is a quantity whose value does not change throughout a discussion. A **variable** is a quantity which is regarded as changing in value and passing through a series of values, generally a continuous series; as, for example, the values of x and y for the continuous series of points of a graph.

Constants are of two kinds: *absolute*, as $1, 6, \pi$; and *arbitrary*, as a, k, m. Usually variables are denoted by the last letters of the alphabet and arbitrary constants by the others.

* We begin here to use the convenient suffix notation, which locates the position of coefficients by their suffixes. Thus, p_1 (the second coefficient), one number, is followed by p_2 (the third coefficient), another number, and so on. Notice that there are $n+1$ terms in the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_n = 0$, and hence the last coefficient is p_n.

213. Functions and Notation for Functions. We have seen, § 148, how as one quantity changes, another depending on it, called the *dependent variable*, also changes. Now if two variables are so related that to every value of one of them there corresponds a definite value of the other, the second variable is called a **function** of the first.

Thus, the circumference of a circle is a function of the radius, and the area of a circle is another function of the radius. The distance a body falls by gravity is a function of the time.

Any expression containing x, as $3x^2 + 2x + 5$, is a function of x, since if x changes the expression also changes, and its value can be found corresponding to any given value of x.

In studying the theory of equations it is very convenient to have a notation to denote *any* "polynomial in x" (§ 9). Mathematicians commonly use "$f(x)$," or "$F(x)$" (read "function x") to represent any such function of x expressed in the different powers of x. Thus,

$$(1)\quad f(x) \equiv ax^n + bx^{n-1} + cx^{n-2} + dx^{n-3} + \cdots + kx + l;$$

or, $$(2)\quad f(x) \equiv x^n + p_1x^{n-1} + p_2x^{n-2} + p_3x^{n-3} + \cdots + p_{n-1}x + p_n,$$

in which n is a positive integer and the coefficients can have any values except that all cannot equal 0 at the same time.

Either of the equations (1) or (2), $f(x) = 0$, is called **the general equation** in one unknown. Notice that (1) $f(x) = 0$ can always be changed into (2) by dividing through by a.

Heretofore letters used have denoted numbers. Thus, $f(x)$ would have meant $f \times x$. To let $f(x)$ denote any polynomial in x is a new use for a letter followed by a parenthesis.

If $\qquad f(x) \equiv 4x^3 + 2x^2 - 4x + 7;$

then $\qquad f(a) \equiv 4a^3 + 2a^2 - 4a + 7;$

and $\qquad f(-3) \equiv 4(-3)^3 + 2(-3)^2 - 4(-3) + 7 = -71.$

We may say that by the symbol $f(\ \ )$ is meant *a form* into whose blank spaces any quantity can be placed.

Thus, if $\qquad f(\ \ ) \equiv 5(\ \ )^4 - 3(\ \ )^2 + 2,$

then $\qquad f(m) \equiv 5m^4 - 3m^2 + 2,$

and $\qquad f(p+q) = 5(p+q)^4 - 3(p+q)^2 + 2.$

II. REMAINDER THEOREM

214. The Remainder Theorem. *If $f(x)$ is divided by $x - a$, the remainder is $f(a)$.*

This theorem states in mathematical language *that if any polynomial in x is divided by $x - a$, the remainder is the same as the result obtained by substituting a for x in the given polynomial.*

Thus, if $x^2 + 7x - 11$ is divided by $x - 4$, the remainder equals the result of substituting 4 for x in $x^2 + 7x - 11$. Test this.

1. **Proof of the Remainder Theorem.** Let Q be the quotient and R the remainder when $f(x)$ is divided by $x - a$, the division being continued until x is no longer found in the remainder.

Then, $f(x) \equiv Q(x - a) + R$ (Because dividend = divisor × quotient + remainder.)

This equation holds true no matter what value x has. (§ 1)

Substituting a for x in this equation, we get,

$f(a) = R$ (Since $Q(a - a) = 0$.)

which was to be proved.

EXAMPLE. Show that the remainder obtained by dividing $3x^3 + 7x^2 - 5x + 6$ by $x - 3$ equals the number obtained by substituting 3 for x in the given dividend, or 135. Similarly, show that $2y^4 - 3y^2 - 7$ divided by $y + 2$ (or $y - (-2)$) gives for remainder $2(-2)^4 - 3(-2)^2 - 7$, or 13. Make dividends and divisors at random and test the truth of the theorem until it is well understood.

2. **Factor Theorem. Corollary of Remainder Theorem.** *If a is a root of $f(x) = 0$, then $f(x)$ is exactly divisible by $x - a$; and conversely.* (See § 23, VII.)

For, if a is a root of $f(x) = 0$, that is, if $f(a) = 0$, then the remainder when $f(x)$ is divided by $x - a$ is zero, (by the remainder theorem), and $f(x)$ is exactly divisible by $x - a$.

Also, conversely, if $f(x)$ is exactly divisible by $x - a$, then the remainder equals zero, that is, $f(a) = 0$. But, if $f(a) = 0$, then a is a root of $f(x)$.

EXAMPLE. Of the equation $x^2 - 5x - 36 = 0$, 9 is a root, since $9^2 - 5 \times 9 - 36 \equiv 0$: show by division that $x^2 - 5x - 36$ is divisible by $x - 9$.

215. Number of Roots of an Equation. *Every rational integral equation, $f(x)=0$, of the nth degree has n roots and no more.*

To prove this we have to assume the truth of what has been called *the fundamental theorem in algebra*, viz.:

Every equation of the form $ax^n+bx^{n-1}+cx^{n-2}+\cdots+kx+l=0$ (in which n is a positive integer and a, b, c, etc., may have any values, real or imaginary, except that a cannot equal 0, since then the equation would be no longer of degree n), has a root, either real or imaginary.

This theorem was proved first by Gauss in 1797. Its demonstration will be found in treatises on the theory of equations.

PROOF. If $f(x)=0$ has a root, r_1, then $f(x)$ is divisible exactly by $x-r_1$ (§ 214, 2). Calling the quotient $f_1(x)$, we have $f_1(x)=0$, since the equation is satisfied with either factor equal to 0 (§ 105). Now $f_1(x)=0$, by the fundamental theorem, also has a root. Call this root r_2, and divide $f_1(x)$ by $x-r_2$, getting $f_2(x)$. Then, $f_2(x)=0$ as before.

Evidently this process can be continued as long as x is still found in the quotient. Now $ax^n+bx^{n-1}+cx^{n-2}+\cdots+kx+l$ divided by $x-r_1$ gives a quotient of the form $ax^{n-1}+b'x^{n-2}+\cdots+k'x+l'$. This quotient divided by $x-r_2$ gives a quotient of the form $ax^{n-2}+b''x^{n-3}+c''x^{n-4}+\cdots+k''x+l''$, and so on. It is clear from this that there are n such divisors as $x-r_1$. Hence

$$f(x)\equiv a(x-r_1)(x-r_2)(x-r_3)\cdots(x-r_n)=0.$$

This equation is satisfied by $r_1, r_2, r_3, \cdots r_n$, or has n roots.

Suppose it has still another root, r_{n+1}, different from any of the preceding. Then

$$a(r_{n+1}-r_1)(r_{n+1}-r_2)(r_{n+1}-r_3)\cdots(r_{n+1}-r_n)=0.$$

But, by hypothesis, since $r_{n+1}\neq r_i\,(i=1,\cdots n)$, none of these factors can equal 0, and therefore their product cannot equal 0, (because a product cannot vanish unless one of its factors vanishes,) and the equation is not satisfied. Hence the equation can have *only* n roots.

a. Notice in the theorem at the beginning of this article that it is said $f(x)=0$ has a root. This statement applies to $f(x)$ as given in § 213, (1). There are what are called impossible equations which have no roots if the signs of their radicals are taken in certain ways. (See § 200, 3.)

III. DESCARTES'S RULE OF SIGNS

216. Descartes's Rule of Signs. A **variation** of sign in a polynomial quantity occurs when a minus term follows a plus term, or when a plus term follows a minus term.

Thus, $3x^4-7x^3+10x^2+5x-7$ has three variations of sign.

In $x-a=0$ there is one variation of sign, and evidently one positive root; in $x+a=0$, there is no variation of sign and no positive root. In $x^2-7x+12=0$, there are two variations of sign and two positive roots; in $x^2-x-12=0$, there is one variation of sign and one positive root; in $x^2+7x+12=0$, there are no variations of sign and no positive roots. Observing these relations, Descartes was led to find a proof (not given here) of the following theorem:

1. **Rule.** *The number of positive roots of the equation $f(x)=0$ cannot exceed the number of variations of sign in its left member; and the number of its negative roots cannot exceed the number of variations of sign in $f(-x)=0$.*

Thus, $6x^3-7x^2-2x+4=0$ cannot have more than two positive roots since it has only two variations of sign.

2. **Incomplete Equations.** An equation in which no power of the unknown from the highest down is missing is called a **complete** equation (§ 94). An incomplete equation can be made formally complete by inserting the missing powers each with a zero coëfficient.

Since terms with zero coefficients do not affect the equation one way or another, we can give all such terms the sign of the preceding term. In this way we see that the formally complete equations (or those with zero coefficients) will have the same number of variations as the corresponding incomplete equations.

Thus, Descartes's rule holds for incomplete as well as for complete equations.

3. **Limit of the Number of Negative Roots.** If $-x$ is put for x in any equation, its negative roots become the positive and its positive roots the negative roots of the derived equation, without any change occurring in the numerical magnitude of the roots.

Thus, $-x$ for x in $x^2-3x-10=0$, gives $x^2-(-3x)-10=0$, or $x^2+3x-10=0$. The roots of the first equation are 5 and -2, and those of the second are -5 and 2. (§ 98.)

If an equation has the signs of its roots changed by substituting $-x$ for x, and if the new equation is tested for a possible number of positive roots, the answer evidently will be the limit of the number of *negative* roots of *the original* equation.

Substituting $-x$ for x, it is clear, will change the signs of the terms containing odd powers of x only, and will leave the terms containing even powers unchanged.

217. Exercise in the Use of Descartes's Rule of Signs. The student must clearly understand, at the outset, that Descartes's rule does not tell how many positive and how many negative roots an equation has, but merely gives *limits* to these numbers.

Thus, $x^3-5x+12=0$, according to Descartes's rule, cannot have more than two positive roots. As a matter of fact, it does not have any, since one root is -3 and the others are *imaginary*.

Find the greatest number of positive roots and of negative roots possible in the following equations. Check the answers by solving, when this is feasible.

1. $x^2-5x+6=0$.

SOLUTION. This equation has two variations of sign and therefore may have two positive roots. Putting $-x$ for x in the given equation, we have $x^2-(-5x)+6=0$, or, $x^2+5x+6=0$, which has no variations of sign, and therefore no negative roots. Consequently the given equation has no negative roots. Evidently the roots of this equation are 2 and 3 (§ 98).

2. $x^2-7x+12=0$.
3. $x^2-4x+3=0$.
4. $x^2-6x+10=0$. (§ 100, § 83.)
5. $x^2+6x+10=0$.

It is known that the roots of the following equations are all real. How many are positive and how many are negative?

6. $x^4-25x^2+60x-36=0$.
7. $x^4-7x^3+17x^2-17x+6=0$.
8. $x^3-3x-2=0$.
9. $x^3+12x^2+45x+50=0$.
10. $x^3-5x-3=0$.
11. $4x^4-4x^3-13x^2+18x-6=0$.

Find the maximum number of positive roots in each of the following equations; also the maximum number of negative roots; also information concerning imaginary roots if any can be inferred.

12. $x^3+3x-5=0$.
13. $x^3-7x+11=0$.
14. $x^3-7x-36=0$.
15. $x^5-4x^2-5=0$.
16. $x^5+1=0$.
17. $x^6+3x^2-5x+1=0$.
18. $x^6+x^2+1=0$.
19. $x^5+x^2+1=0$.
20. Determine the character of the roots of $x^4+4=0$.
21. Show that $x^6-3x^2=4x-5$ has at least two imaginary roots.

IV. SYNTHETIC DIVISION

218. Synthetic Division. It frequently happens that it is necessary to calculate the value of $f(x)$ when some value is assigned to x. Thus, if we were plotting $y=f(x)$, we should have to assign values, say, 1, 2, 3, 4, ⋯ to x and find the corresponding values of $f(x)$, or y. Now, by using the Remainder Theorem, § 214, the operation can be systematized and greatly shortened.

1. To calculate $5x^3-7x^2-34x-4$ when $x=3$.

SOLUTION. By § 214 the value sought is the remainder obtained when the given quantity is divided by $x-3$. But such a division is a very special form of long division, the divisor being a *binomial* with its first term always x. Such divisions can be shortened still more than ordinary contracted divisions (§ 20, Ex. 47).

I. *Long division*

$$\begin{array}{l|l} \mathbf{5x^3}-\ \mathbf{7}x^2-\mathbf{34}x-\mathbf{4} & x-\mathbf{3} \\ \underline{5x^3-\mathbf{15}x^2} & \overline{5x^2+8x-10} \\ \qquad\quad \mathbf{8}x^2-34x & \\ \qquad\quad \underline{8x^2-\mathbf{24}x} & \\ \qquad\qquad\ -\mathbf{10}x-\ 4 & \\ \qquad\qquad\ \underline{-10x+\mathbf{30}} & \\ \qquad\qquad\qquad\ -\mathbf{34}\ \ Ans. & \end{array}$$

II. *Same division contracted*

$$\begin{array}{l|l} 5-\ 7-34-4 & -3 \\ \underline{\quad -15-24+30} & \\ \text{Quotient: } 5+\ 8-10-\mathbf{34} & \text{Remainder. } Ans. \end{array}$$

III. *Same division as* II, *but with* -3 *changed to* $+3$ *and the subtractions to additions.*

$$\begin{array}{l|l} 5-\ 7-34-4 & +3 \\ \underline{\qquad 15+24-30} & \\ \text{Quotient: } 5+\ 8-10-\mathbf{34} & \text{Remainder. } Ans. \end{array}$$

EXPLANATION. In the contracted division, II, only that part of the long division, I, is retained which is essential, namely, the part in black-faced type. Evidently the light-faced figures in I merely duplicate those already set down in the solution, and the *letters* can all be *omitted.* The products of the second term of the divisor and the several terms of the quotient are written directly under the corresponding terms of the dividend.

Notice, in the *last* form, III, that 15 comes from 3×5; $+8$ from *adding* -7 and 15; 24 from 3×8; and -30 from 3×-10.

Compare the *quotient* and the *remainder* of the long division process with the last lines of II and III, viz., $5 + 8 - 10 - 34$.

From this example we construct a rule for finding numerical values:

2. **Rule for Synthetic Division.** (1) *Arrange the polynomial according to the descending powers of x, and, omitting all the x's, write only the coefficients with their signs, supplying the coefficient $+0$ for every power of x missing; also $+0$ for the known term in case it is missing; then, in the position of divisor, place the number substituted for x.*

(2) *Multiply the coefficient of the highest power of x in the given quantity by the number substituted for x, placing the product under the second coefficient, and add this column algebraically.*

(3) *Next, multiply sum thus found by number substituted for x, placing the product under the third coefficient, and add as before. So continue till all the coefficients are used. The last sum is the value of the polynomial when the given value of x is substituted for x.*

(4) *The numbers preceding the value sought on last line are the coefficients of the quotient, whose degree is one less than that of dividend.*

Evaluate the following expressions by synthetic division, and also each time copy down the quotient obtained with the omitted powers of x reinserted in their proper places.

3. $x^3 - 7x^2 + 12x - 5$, when $x = 2$.
4. $x^3 - 3x^2 - 2x - 1$, when $x = 4$.
5. $x^4 - 6x^2 - 2x + 11$, when $x = 5$.

SUGGESTION. Write $1 + 0 - 6 - 2 + 11 \underline{|5}$.

6. $x^3 + 11x^2 - 17$, when $x = +2$; also when $x = -2$.
7. $x^5 + 3x^4 + 2x^2 - 1$, when $x = 1$; also when $x = -1$.

Find the quotients and remainders in following divisions:

8. $(x^3 - 6x^2 + 11x - 6) \div (x - 2)$.
9. $(x^4 + x^3 - x - 1) \div (x + 5)$.
10. $(4x^3 - 6x^2 - 2x - 7) \div (x - 4)$.
11. $(x^5 - x^4 + 6x) \div (x + 3)$.

V. TRANSFORMATIONS OF EQUATIONS

219. Construction of an Equation whose Roots are severally the same Multiple of those of a Given Equation. In § 216, 3, we saw the signs of all the roots of a given equation changed by substituting $-x$ for x. In the same way by substituting $\frac{y}{r}$ for x we get an equation with y for the unknown whose roots are severally r times those of the given equation, since if $\frac{y}{r} = x$, $y = rx$.

Thus, if $x = \frac{y}{3}$, $x^2 - 3x - 4 = 0$

becomes $$\left(\frac{y}{3}\right)^2 - 3\left(\frac{y}{3}\right) - 4 = 0, \text{ or } y^2 - 9y - 36 = 0.$$

The roots of the given equation, $x^2 - 3x - 4 = 0$, are evidently 4 and -1, while those of the resulting equation, $y^2 - 9y - 36 = 0$, are 12 and -3, the latter pair of roots being respectively three times the former pair.

1. Generalizing, let $ax^n + bx^{n-1} + cx^{n-2} + \cdots + kx + l = 0$ be an equation of any degree, and in this equation put $\frac{y}{r}$ for x. Then

$$a\left(\frac{y}{r}\right)^n + b\left(\frac{y}{r}\right)^{n-1} + c\left(\frac{y}{r}\right)^{n-2} + \cdots + k\left(\frac{y}{r}\right) + l = 0,$$

or $$ay^n + bry^{n-1} + cr^2y^{n-2} + \cdots + kr^{n-1}y + lr^n = 0, \quad \text{(Mult. Ax.)}$$

by raising each fraction to the power indicated, and then multiplying both members of the equation by r^n.

From this we see that the following is true generally:

2. **Rule.** *To get an equation with roots r times those of a given equation, first write the given equation in the complete equation form with 0's for the coefficients of missing terms and with 0 for its right member; then make the coefficient of the highest power of the unknown in the required equation the same as in the given equation.*

For the second coefficient of the required equation write the second coefficient of the given equation multiplied by r; for the third coefficient write the former third coefficient multiplied by r^2; for the fourth coefficient, the former fourth coefficient multiplied by r^3; and so on.

a. The word multiple of course includes *sub*multiples. Thus, a new equation whose roots are to be $\frac{1}{r}$ respectively of those of a given equation, will have for its second coefficient $\frac{1}{r}$ of that of the second coefficient in the given equation, and for its third coefficient $\frac{1}{r^2}$ of the third coefficient; and so on.

3. Construct the equations which have respectively:

Roots 3 times those of $x^3 - 6x^2 + 11x - 6 = 0$.
Roots 2 times those of $x^3 + 6x + 7 = 0$.
Roots $\frac{1}{4}$ of those of $x^4 + x^2 - 20 = 0$.
Roots -3 times those of $x^3 - 15x^2 - 14x + 2 = 0$.
Roots $\frac{2}{5}$ of those of $ax^3 - bx + c = 0$.

4. Change the equation $3x^3 - 5x^2 - 7x + 12 = 0$ into another whose coefficient of x^3 is 1 and whose other coefficients are integral, by forming the equation with roots 3 times its roots.

220. Integral and Rational Roots.

THEOREM. *If an equation of the form* $x^n + ax^{n-1} + bx^{n-2} + \cdots + kx + l = 0$, *in which* n, a, b, $\cdots$ *are integers, has rational roots, they are integers and factors of* l.

PROOF. Suppose that such an equation can have a root of the form $\frac{p}{q}$ in which p and q are integers but p is prime to q. Then,

$$\left(\frac{p}{q}\right)^n + a\left(\frac{p}{q}\right)^{n-1} + b\left(\frac{p}{q}\right)^{n-2} + \cdots + k\left(\frac{p}{q}\right) + l = 0,$$

or, $p^n + ap^{n-1}q + bp^{n-2}q^2 + \cdots + kpq^{n-1} + lq^n = 0$; (Mult. Ax.)

whence, $ap^{n-1} + bp^{n-2}q + \cdots + kpq^{n-2} + lq^{n-1} = -\frac{p^n}{q}$. (Sub. and Div. Ax's.)

But the left member of the last equation is integral, since a, b, $\cdots$, p, q, and n are all assumed to be integers, while the right member cannot be integral, because, by our first supposition, p is prime to q. Hence an equation of the form

$$x^n + ax^{n-1} + bx^{n-2} + \cdots + kx + l = 0,$$

cannot have a fraction for a root. If it has a rational root, such root must be integral and a factor of l.

Find the real roots in the following equations, testing first for the signs of the roots by Descartes's rule.

1. $x^3 - 2x^2 - 19x + 20 = 0.$

SOLUTION. By Descartes's rule this equation may have two positive roots and one negative root. We try in turn, by synthetic division, the divisors of 20, viz., ± 1, ± 2, ± 4, ± 5, ± 10, ± 20. If we write down the coefficients of the given equation, we can perform the synthetic divisions mentally. Since 1, -4, and 5 give 0 for remainder in each case, these numbers are the roots sought.

2. $x^3 - 4x^2 + x + 6 = 0.$
3. $y^3 + 3y^2 - 22y - 24 = 0.$
4. $x^3 - 19x - 30 = 0.$
5. $z^4 - 8z^3 + 3z^2 + 32z - 28 = 0.$

Find the rational roots in the following equations by first transforming them to equations whose coefficient of the highest power of the unknown is unity (§ 219, 4), and then solving for integral roots. Use Descartes's rule.

6. $2x^3 - 7x^2 + 16x - 15 = 0.$

SOLUTION. If we form the equation whose roots are twice those of the given equation, it is clear that 2 will be a factor of every term of the new equation and can be removed, thus making the coefficient of the highest power of x unity, and the rational roots integral by § 220, Theorem.

By the rule of § 219 we have for the transformed equation:

$$2x^3 - 14x^2 + 64x - 120 = 0,$$

or,

$$x^3 - 7x^2 + 32x - 60 = 0.$$

By Descartes's rule this equation can have no negative roots. Trying 1, 2, 3, 4, etc. in turn, by synthetic division, we find that 3 is a root. Then $\frac{3}{2}$ is a root of the given equation.

7. $2x^3 - 15x^2 + 43x - 45 = 0.$
8. $4x^3 - 15x^2 + 33x - 18 = 0.$

221. Decreasing all the Roots of an Equation by the Same Amount.

1. Let $x^2 - 7x - 18 = 0$ be a given equation, whose roots are 9 and -2; to find an equation whose roots are less by 3 than those of this equation; that is, whose roots are 6 and -5.

We wish the new unknown y to be equal to $x - 3$, so we put $x = y + 3$ in the given equation, y thus becoming the unknown in the new equation. Then

$$(y+3)^2-7(y+3)-18=0, \text{ or } y^2-y-30=0.$$

The roots of the last equation are evidently 6 and -5, or respectively 3 less than the roots of the given equation.

2. Let $f(x)\equiv ax^3+bx^2+cx+d=0$ be regarded as an example of an equation of any degree having any coefficients.

If, as in the preceding example, we substitute $y+m$ for x, the new equation containing y for its unknown will have its three roots each less by m than those of the given equation. We have

$$(1)\ f(y+m)=a(y+m)^3+b(y+m)^2+c(y+m)+d=0,$$

or, $$(2)\ f(y+m)=ay^3+(3am+b)y^2+(3am^2+2bm+c)y + am^3+bm^2+cm+d=0,$$

or, $$(3)\ f(y+m)=ay^3+b'y^2+c'y+d'=0,$$

by *substituting* b' for $3am+b$, c' for $3am^2+2bm+c$, and d' for am^3+bm^2+cm+d, to save writing the long expressions.

We now reverse the operation used to derive equations (1), (3), and substitute in (3) x for $y+m$, and $x-m$ for y, thus coming back to the equation $f(x)=0$ with which we started, arranged, however, in a new form. In this way we get

$$(4)\ f(x)\equiv a(x-m)^3+b'(x-m)^2+c'(x-m)+d'=0.$$

We see, now, that if $a(x-m)^3+b'(x-m)^2+c'(x-m)+d'$ is divided by $x-m$, the quotient is $a(x-m)^2+b'(x-m)+c'$ and *the remainder* is d'. If the quotient $a(x-m)^2+b'(x-m)+c'$ is divided by $x-m$, the quotient is $a(x-m)+b'$ and the remainder is c'. If the last quotient is divided by $x-m$, the resulting quotient is a and the remainder is b'. But these remainders just found, d', c', b', are the coefficients of y in the new equation (3) whose roots are to be each m less than the roots of the given equation. Now $a(x-m)^3+b'(x-m)^2+c'(x-m)+d'=0$ is, by equation (4), the same as the given equation $f(x)=0$. Hence (since, evidently, the preceding argument would apply to an equation of any degree) we learn:

3. *To find an equation whose roots are each m less than those of $f(x)=0$, divide $f(x)$ by $x-m$, then the quotient by $x-m$, and so on. The last quotient together with the several remainders taken in reverse order are the coefficients of the desired equation, the last quotient being the coefficient of the highest power of the unknown; the last remainder, the next coefficient; and so on.*

4. Reduce the roots of the equation $x^3-7x^2+2x+40=0$ by 3.

SOLUTION

$$\begin{array}{l|l} x^3 - 7x^2 + 2x + 40 & x - 3 \\ \cline{2-2} x^3 - 3x^2 & x^2 - 4x - 10 \\ \hline \quad - 4x^2 + 2x & \\ \quad - 4x^2 + 12x & \\ \hline \qquad\quad - 10x + 40 & \\ \qquad\quad - 10x + 30 & \\ \hline \qquad\qquad\quad 10 & \end{array} \qquad \begin{array}{l|l} x^2 - 4x - 10 & x - 3 \\ \cline{2-2} x^2 - 3x & x - 1 \\ \hline \quad - x - 10 & \\ \quad - x + 3 & \\ \hline \qquad - 13 & \end{array} \qquad \begin{array}{l|l} x - 1 & x - 3 \\ \cline{2-2} x - 3 & 1 \\ \hline \quad + 2 & \end{array}$$

Hence, by rule p. 242, the new equation is $x^3 + 2x^2 - 13x + 10 = 0$.

PROOF. The roots of this equation, by § 220, are 1, 2, -5; those of the given equation are 4, 5, -2, or each is 3 greater.

In the solution just given *long* division was used. This can be replaced to great advantage by synthetic division.

5. Find the equation whose roots are 2 less than the roots of $x^4 - 2x^3 - 4x^2 + x - 4 = 0$, using synthetic division.

SOLUTION

$$\begin{array}{l|l} 1 - 2 - 4 + 1 - 4 & 2 \\ \quad\; 2 + 0 - 8 - 14 & \\ \hline 1 + 0 - 4 - 7 - 18 & \\ \quad\; 2 + 4 + 0 & \\ \hline 1 + 2 + 0 - 7 & \\ \quad\; 2 + 8 & \\ \hline 1 + 4 + 8 & \\ \quad\; 2 & \\ \hline 1 + 6 & \end{array}$$

EXPLANATION

Notice that 1, 0, -4, -7 on the third line are the coefficients of the quotient of the given function of x divided by $x - 2$ and the last term -18 is the remainder. The work then shows $x^3 - 4x - 7$ divided by $x - 2$, giving on the fifth line the coefficients 1, 2, 0, and the remainder, -7. Next, $x^2 + 2x$ is divided by $x - 2$, giving on the seventh line the coefficients 1, 4, and the remainder 8. Last of all $x + 4$ is divided by $x - 2$ giving the quotient 1 and the remainder 6. Hence, the equation sought, which has for its coefficients the last quotient and the several remainders, is $x^4 + 6x^3 + 8x^2 - 7x - 18 = 0$. *Ans.*

Decrease the roots of

6. $x^2 - 7x + 12 = 0$ by 2.

7. $x^3 - 2x^2 + 8x - 7 = 0$ by 2.

8. $x^3 - 3x^2 + 2x - 22 = 0$ by 3.

9. $x^4 + 3x^3 - 4x^2 + 5x = 7$ by 3.

10. $x^3 + 4x - 8 = 0$ by 4. (§ 218, 2, (1))

11. $x^4 - 2x^2 + 1 = 0$ by 0.2.

12. Increase roots of $x^3 + 2x + 5 = 0$ by 1. SUG. Divisor $= -1$.

13. Increase the roots of $x^4 + 2x + 5 = 0$ by 2.

14. Decrease the roots of $x^3 - 7x + 4 = 0$ by 0.07.

15. Decrease the roots of $x^2 - 11x + 5 = 0$ by 0.004.

222. Making the Sum of the Roots of an Equation Zero. In the equation $x^n + p_1x^{n-1} + p_2x^{n-2} + \cdots + p_{n-1}x + p_n = 0$ the second coefficient, with its sign changed, or $-p_1$, equals the sum of the roots (§ 211, 3). Hence, if each of the n roots of this equation is increased by $\frac{p_1}{n}$, the sum of the roots of the new equation is 0, and the coefficient of x^{n-1} in this equation is zero. (Used § 111, III to remove x^2.)

223. Reciprocal Equations. If $\frac{1}{x}$ is substituted for x in any equation, the resulting equation will evidently have roots that are the reciprocals of those of the given equation. Equations that are not changed at all when this substitution is made are called *reciprocal* equations. Their roots in pairs are reciprocals of each other. See § 102, 11.

VI. COMPLEX ROOTS

224. Complex Roots. Complex roots, if they occur at all, appear in pairs in all equations, $f(x) = 0$, whose coefficients are real quantities. This can be shown both algebraically and graphically.

1. **Algebraic Proof.**

Suppose $a + bi$ (§ 83) is a root of $ax^n + bx^{n-1} + \cdots + kx + l = 0$.

Substituting $a + bi$ for x, we find that all the terms that contain an odd power of bi are imaginary, and all the other terms are real. Combining all the real terms into one quantity, denoted by P, and all the imaginary terms into another quantity, denoted by Qi, we have

$$P + Qi = 0.$$

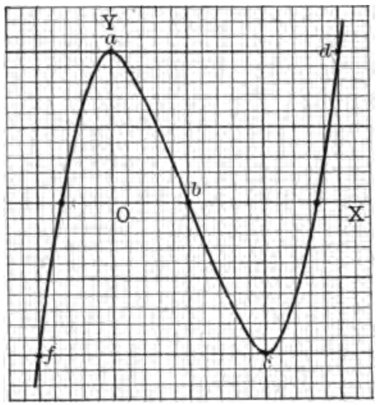
But, by § 84, $\qquad P = 0$, and $Q = 0$.

Substituting $a - bi$ for x, we get $P - Qi$, as all the P terms have the same signs as before, while every Q term has the sign opposite to that which it had before, since odd powers of negative terms are negative.

But we have just learned that $P = 0$, and $Q = 0$. Hence $P - Qi = 0$, and the equation is satisfied when $a - bi$ is substituted for x. Therefore $a - bi$ is a root.

2. Graphic Explanation.

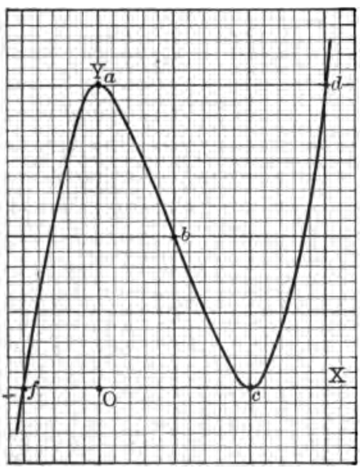
To explain graphically why imaginary roots appear in pairs, we construct the graph for the equation $y = x^3 - 3x^2 + 2$ and then the graphs for this equation modified by changing the known term.



We have, for the graph of $y = x^3 - 3x^2 + 2$ the first figure, at the right; for the graph of $y = x^3 - 3x^2 + 4$ the second figure, below; and for the graph of $y = x^3 - 3x^2 + 6$ the figure on p. 246.

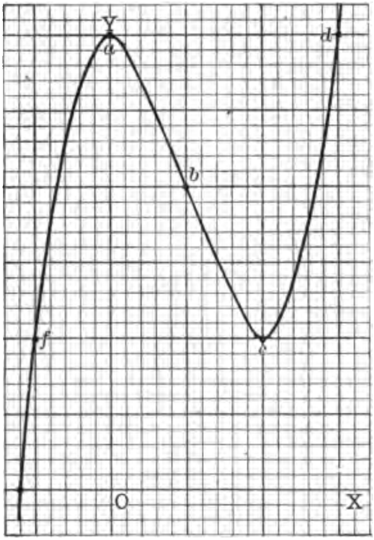
The graph of the first equation crosses the axis of X, where $y = 0$, at three points, which shows that $x^3 - 3x^2 + 2 = 0$ has three real roots, viz., $x = 2.74$, 1, and $-.74$.

The graph of the second equation crosses the axis of X at one point, $x = -1$, and comes down and just *touches* the axis of X at $x = 2$. Thus, $x^3 - 3x^2 + 4 = 0$ has two *equal* roots, $x = 2$, and $x = 2$, and the third root $x = -1$.



The graph of the third equation crosses the axis of X where $x = -1.2$, *but does not cross it again*. In this case we see the equation $x^3 - 3x^2 + 6 = 0$ has one real root $x = -1.2$, *and two imaginary roots.*

Thus, the two real roots of the first and second equations disappeared at the same time and two imaginary roots took their places.



Notice that the three graphs *are exactly the same curve*, but in different positions with respect to the axes of X and Y, the latter curves being pushed upward from the position of the first.

a. An equation of the nth degree in x can cross the axis of X in no more than n points, since an equation of the nth degree can have no more than n roots. It may have less than n real roots, in which case it will cross the axis of X in less than n points, or even in no points, all the roots being imaginary.

Evidently the number of pairs of imaginary roots is the same as the number of elbows of the curve that do not cross the axis of X.

b. We see in the example just given the two ends of the curve extending away indefinitely, the one above the axis of X to the right, and the other below the axis to the left. It is evident that there is no need of tracing these parts of the graphs in search of roots. This leads us to the next topic, limits of roots.

VII. LOCATION OF ROOTS

225. Limits of Roots. A **superior limit** to the real roots of an equation is a number greater than the greatest root. An **inferior limit** is a number less (lower in the algebraical scale) than the least root. A superior limit may often be found by grouping terms so that each group contains only one negative term, and finding a value of x which makes each group positive. An inferior limit can be found in the same way after substituting $-x$ for x in the given equation.

Example. Find limits for the roots of $x^4 - 25x^2 + 60x - 36 = 0$.

Solution. $(x^4 - 25x^2) + (60x - 36) \equiv x^2(x^2 - 25) + 12(5x - 3) = 0$.

Examining the last form of the equation, we see that $x = 5$ makes the first quantity zero and the second a positive number. Evidently any greater number than 5 would make the left member positive and such a value of x would therefore not satisfy the equation. By synthetic divisions we find that 4 is a superior limit.

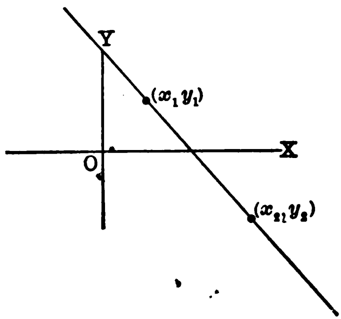
Putting $-x$ for x, and factoring as before, we have

$$x^2(x^2-25)-12(5x+3)=0.$$

Here the limit must be greater than 5. Trying 7 we see that it or any larger number makes the left member positive. Hence 4 is a **superior** limit and -7 is an **inferior** limit to the roots of the given equation.

226. Location of the Roots of an Equation between Certain Limits.

If x_1 when put for x in the equation $y=f(x)$ makes y positive, and x_2 when put for x makes y negative (*i.e.* if the two remainders from the two synthetic divisions are opposite in sign), then at least one root of $f(x)=0$ lies between x_1 and x_2.



For, the curve passing from (x_1, y_1) to (x_2, y_2) will have to cross the axis of X at least once to go from a positive to a negative value of y. Where it crosses, the value of y is 0, that is, $f(x)=0$. The value of x where the curve crosses the axis of X is the root sought.

Locate the roots of the following equation, first testing as well as possible for superior and inferior limits of the roots so as to find what range of values to assign to x.

1. $x^4-6x^2-2x+2=0$.

Solution. Writing the equation in the form

$$x^2(x^2-6)+2(1-x)=0,$$

we readily see that 3 is a superior limit; and, putting $-x$ for x, we see that -3 is an inferior limit.

Writing $y=f(x)\equiv x^4-6x^2-2x+2$, we let $x=-3, -2, -1, 0, 1, 2, 3$ in turn, finding the value of y in each case by synthetic division, and arranging the results in the usual tabular form for a graph construction.

Synthetic Divisions

I. $1 + 0 - 6 - 2 + 2 \,\lfloor -3$
$\quad -3 + 9 - 9 + 33$
$\quad -3 + 3 - 11 + 35.$ *Ans.*

II. $1 + 0 - 6 - 2 + 2 \,\lfloor -2$
$\quad -2 + 4 + 4 - 4$
$\quad -2 - 2 + 2 - 2.$ *Ans.*

III. $1 + 0 - 6 - 2 + 2 \,\lfloor -1$
$\quad -1 + 1 + 5 - 3$
$\quad -1 - 5 + 3 - 1.$ *Ans.*

IV. $1 + 0 - 6 - 2 + 2 \,\lfloor 1$
$\quad 1 + 1 - 5 - 7$
$\quad 1 - 5 - 7 - 5.$ *Ans.*

V. $1 + 0 - 6 - 2 + 2 \,\lfloor 2$
$\quad 2 + 4 - 4 - 12$
$\quad 2 - 2 - 6 - 10.$ *Ans.*

VI. $1 + 0 - 6 - 2 + 2 \,\lfloor 3$
$\quad 3 + 9 + 9 + 21$
$\quad 3 + 3 + 7 + 23.$ *Ans.*

Tabular Results

$y = x^4 - 6x^2 - 2x + 2$

	(x, y)	Pt.
I	$(-3, \ 35)$	a
II	$(-2, -2)$	b
III	$(-1, -1)$	c
	$(0, 2)$	d
IV	$(1, -5)$	e
V	$(2, -10)$	f
VI	$(3; \ 23)$	g

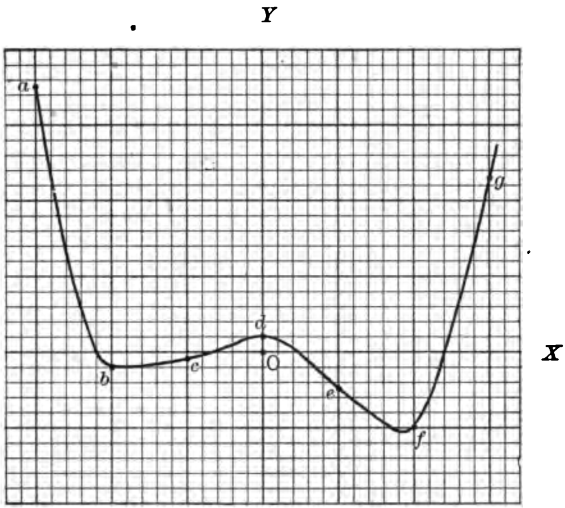
Graph

An examination of the values in the table shows that the graph for these values extends only a short distance to the right and left of the Y-axis, but a considerable distance from the X-axis. The graph will be easier to construct and more useful, if the unit for abscissas is taken longer than that for ordinates. This can be done since the numbers expressing the crossing points on the axis of X will be the same for any scale of abscissas. Let us take a millimeter unit for ordinates and a centimeter unit for abscissas. In this way we get the graph on facing page.

The graph shows that there are four places where the curve crosses the axis of X, or four real roots. It is evident also that the branch ba extends indefinitely upward to the left, and the branch fg extends indefinitely upward to the right. This makes clear the value of finding superior and inferior limits, since there is plainly no advantage in tracing the curves beyond a and g.

The graph shows that one root lies between -2 and -3; the second root between 0 and -1; the third root between 0 and $+1$; and the fourth root between 2 and 3.

a. Between a value of x that makes y positive and one that makes it negative, may be found either one root or some *odd* number of roots. Thus, between a and b in the diagram one root lies; between a and f three roots lie.



227. Approximate Solutions for the Roots of any Equations by Means of Graphs. The graphic method can evidently be used to solve any kind of equation, even the most complicated, as one containing the unknown in exponents as well as in ordinary terms, but this method may not give all the roots of all such equations. After two points are located one on each side of the axis of x and close together, these points can be located on a new diagram on a much larger scale, covering only the region about the points. This diagram will give a good approximation to the root sought, provided only one root lies in this region.

Thus, if it is known by synthetic division that a root of $x^4-6x^2-2x+2=0$ lies between $(-2.3, 2.84)$ and $(-2.1, -.81)$, a straight line joining these points crosses the axis of x at -2.14.

But the curve bends a little to the left giving -2.15. Synthetic division shows that the root is -2.15^+.

VIII. HORNER'S METHOD

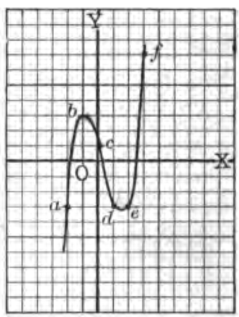
228. Horner's Method of Approximation to the Roots of Equations with Numerical Coefficients. This method is used when very precise values of the roots are sought. The method consists in first locating a root as lying between two integers, and then decreasing all the roots of the equation by the smaller integer; next, in locating the corresponding root as lying between two *tenths* in the new equation, and then decreasing all the roots of this equation by the smaller of the two numbers of tenths; then, in locating the same root as lying between two hundredths in the last derived equation and decreasing all the roots of this equation by the smaller number of hundredths; and so continuing, as far as desired. In solving for negative roots, the signs of all the roots are first changed by substituting $-x$ for x in the given equation, the process then being the same as that just described.

1. Get, to two decimal places, a root of $x^3 - 4x^2 + x + 3 = 0$.

SOLUTION. By substituting in $y = x^3 - 4x^2 + x + 3$ (using synthetic division), we find the values as set down in the table, and by locating these on the diagram, we get the corresponding graph.

$y = x^3 - 4x^2 + x + 3$

(x, y)	
$(-1, -3)$	a
$(0, 3)$	b
$(1, 1)$	c
$(2, -3)$	d
$(3, -3)$	e
$(4, 7)$	f



Examining the graph we see by § 226 that there is a root between -1 and 0, another root between 1 and 2, and a third between 3 and 4. We will solve for the last, lowering the roots by 3.

$$\begin{array}{l} 1 - 4 + 1 + 3 \,\lfloor 3 \\ \quad\; 3 - 3 - 6 \\ \hline 1 - 1 - 2 - \mathbf{3} \\ \quad\; 3 + 6 \\ \hline 1 + 2 + \mathbf{4} \\ \quad\; 3 \\ \hline 1 + \mathbf{5} \end{array}$$

The equation with the new roots is therefore $x^3 + 5x^2 + 4x - 3 = 0$. The graph shows that x probably lies between 3.3 and 3.5; so we will find the remainders, first for 0.3, and then for 0.5.

$$\begin{array}{l} 1 + 5 \quad + 4 \quad - 3 \quad \lfloor .3 \\ \quad\;\; .3 + 1.59 + 1.677 \\ \hline 1 + 5.3 + 5.59 - \mathbf{1.323} \end{array} \qquad \begin{array}{l} 1 + 5 \quad + 4 \quad - 3 \quad \lfloor .5 \\ \quad\;\; .5 + 2.75 + 3.375 \\ \hline 1 + 5.5 + 6.75 + \mathbf{.375} \end{array}$$

Thus, the sign of $x^3 + 5x^2 + 4x - 3$ changes between $x = .3$ and $x = .5$ and as $1.323 > .375$ the root is probably over .4. So we will lower the roots by .4.

$$\begin{array}{l} 1 + 5 \quad + 4 \quad - 3 \quad \lfloor .4 \\ \quad\;\; .4 + 2.16 + 2.464 \\ \hline 1 + 5.4 + 6.16 - \mathbf{.536} \\ \quad\;\; .4 + 2.32 \\ \hline 1 + 5.8 + \mathbf{8.48} \\ \quad\;\; .4 \\ \hline 1 + \mathbf{6.2} \end{array}$$

The new equation then is

$$x^3 + 6.2x^2 + 8.48x - .536 = 0.$$

Since we are now to lower the root by hundredths, those terms which contain the highest powers of x will have *small values*. We can therefore get a good approximation to value of x by dropping them and solving from the terms that remain.

Thus

$$8.48x - .536 = 0 \text{ (nearly), or } x = .06+.$$

Let us now substitute .06 and .07 in $x^3 + 6.2x^2 + 8.48x - .536$ and see if the sign changes. If so, the root of this equation lies between .06 and .07.

$$\begin{array}{l} 1 + 6.2 \quad + 8.48 \quad - .536 \,\lfloor .06 \\ \quad\;\; .06 + \;.376 + .532 \\ \hline 1 + 6.26 + 8.86 \quad - \mathbf{.004} \end{array} \qquad \begin{array}{l} 1 + 6.2 \quad + 8.48 - .536 \,\lfloor .07 \\ \quad\;\; .07 + \;.44 + .624 \\ \hline 1 + 6.27 + 8.92 + \mathbf{.088} \end{array}$$

Now we first decreased the root of the given equation by 3, then the corresponding root of the new equation by .4. We see then that if we should lower the corresponding root of the last equation by .06, the corresponding root in the new derived equation would be very small. Hence 3.46+ is a close approximation to the root sought.

a. It may be observed that from the graphic standpoint what we do by Horner's method is to carry the origin closer and closer to the position of the root sought. Thus (in the above example) the axis OY was first carried 3 units to the right; then 0.4 to the right. Evidently we could continue this process as long as desired.

229. Negative Roots. Complete Form of Solution by Horner's Method.

To find *negative* roots by Horner's method the given equation is first transformed by substituting $-x$ for x. The new equation is then solved for positive roots as in the preceding article. Such positive roots of the new equation are evidently negative roots of the given equation.

1. Find a negative root of $x^3 - 3x^2 - 2x + 5 = 0$.

SOLUTION. Substituting $-x$ for x in the given equation, we get,

$$x^3 + 3x^2 - 2x - 5 = 0.$$

This equation is to be solved for its *positive* roots.

It will be found by trial that $x = 2$ makes the left member positive and $x = 1$ makes it negative, so that a root lies between 1 and 2.

MODEL SOLUTION BY HORNER'S METHOD

1	3		−2		−5		1.33005 *Ans.*
	1		4		2		
	4		2		−3000(1)		
	1		5		2667		
	5		700(1)		−333000(2)		
	1		189		332337		
	60(1)		889		−66300000	0000(3)	
	3		198		56435247	5125	
	63		108700(2)		−9864752	4875(4)	
	3		2079				
	66		110779				
	3		2088				
	690(2)		11286700	0000(3)			
	3		349	5025			
	693		11287049	5025			
	3		349	5 050			
	696		11287399	0075(4)			
	3						
	69	9000(3)					
		5					
	69	9005					
		5					
	69	9010					
		5					
	69	9015(4)					

EXPLANATION. In the solution p. 252, the lines for the synthetic divisions are not drawn across the page, and the work is carried on *in one continuous process, that is, without rewriting the several transformed equations.* The second figure of the root .3 has to be found by trial as in § 228. A fair approximation to the second decimal figure of the root is found by dividing 333 by 1087. (See § 228.)

Also in the solution p. 252, the decimal point does not appear except in the answer. When tenths in the answer are reached, for each new transformed equation one cipher is annexed to the second coefficient, two ciphers are annexed to the third coefficient, and so on. This plan of ignoring the decimal point in the work is similar to that followed in square and cube root of arithmetical numbers. These ciphers serve to make the coefficients of the several transformed equations stand out in the solution.

The figures (1), (2), (3), (4) are also inserted in the solution to indicate the corresponding coefficients of the several transformed equations.

a. The solution shows how rapidly the decimal figures accumulate. One form of abridging the solution stops annexing ciphers. But probably the simplest way is to continue to annex the ciphers, and then decide how many places shall be discarded, and cut off *the same number* from each of the coefficients of the last derived equation. Thus, in the solution just given four places are cut off by vertical lines from each coefficient. As usual, we carry from the first right-hand figure cut off.

b. After hundredths are obtained in the root a good approximation can be found for the next figure of the root by dividing the last term of the last derived equation by the coefficient of x in the preceding term. After four or five decimal figures are found, one or two more can be obtained quite accurately by the same procedure.

Thus, $.9864752 \div 11287399 = .87$. This gives 1.3300587 as the value of the required negative root correct to seven decimal places.

2. Locate and find to five decimal places two negative roots of

$$x^3 + 3x^2 - 2x - 5 = 0.$$

230. Rule for and General Exercise in the Use of Horner's Method.

Rule. (1) *Apply Descartes's rule to find character of roots.*

(2) *Plot $y = f(x)$ to find approximate values of x.*

(3) *Select a positive root and decrease roots by its integral part; then decrease roots of resulting equation by tenths, first estimating from graph, and then locating accurately by synthetic divisions. See* § 228.

(4) *Put x^2, x^3, ⋯ terms equal to 0 in last transformed equation, and solve for x, getting approximate number of hundredths in the root. Check by synthetic divisions and decrease roots of last equation by lesser number of hundredths; and so continue. See § 229, b.*

(5) *Solve for the other positive roots in the same way.*

(6) *Solve for the negative roots, after putting $-x$ for x.*

1. Find to four decimal places the three roots of $x^3+x^2-2x-1=0$, checking the answer by § 211, 3, (1).

2. Find to four decimal places the one real root of

$$x^3+5x+3=0.$$

3. Find to 5 places of decimals the root of $x^3-3x-4=0$, which lies between 2 and 3.

4. Find to four decimal places the root lying between 0 and 1 of

$$x^3+6x^2+10x-1=0,$$

5. Find to four decimal places the three roots of

$$x^3-3x-1=0.$$

6. Find to six decimal places the root lying between 3 and 4 of

$$2x^3+x^2-15x-59=0.$$

7. Two roots of $x^3+x^2-10x+9=0$, lie between 1 and 2. Find them both to four decimal places.

8. Find to four decimal places, two roots of $x^4-12x+7=0$.

9. Find to four decimal places the four roots of

$$x^4-19x^2-23x-7=0.$$

10. By Horner's method find to three decimal places the real cube root of -3, that is, solve $x^3+3=0$. Check by ordinary solution and by logarithms.

11. Solve to four decimal places $x^4-x^3+x-2=0$.

12. Find to 5 decimal places the positive root of

$$x^4-3x^3-2x^2+x+15=0,$$

after first removing the rational root.

13. On the equation $x^3 - x^2 + 12x - 11 = 0$, depends the inscription in a circle of a regular polygon of 37 sides. Find x to two decimal places.

14. A piece of swamp land 100 ft. square is to be raised 1 ft. by ground taken from a ditch surrounding it. If the ditch is to be as wide as it is deep, what is the width to two decimal places?

15. The diameter of a water pipe 200 ft. long, which is to discharge 100 cu. ft. per second under a head of 10 ft., is given by the real root of the equation $x^5 - 38x - 101 = 0$. Find the diameter correct to 2 decimal places.

16. A sphere of ice 1 ft. in diameter floating in water sinks to a depth of x ft. given by the equation $2x^3 - 3x^2 + 0.93 = 0$. Find the depth correct to three decimal places.

17. Find the rational roots and compute to three decimal places the other real roots of (1) $x^2 + y = 2$ and (2) $x + y^2 = 6$.

231. Historical Notes. Horner's method was published first in the Transactions of the Philosophical Society of London in 1819. Professor Chrystal says "its spirit is purely arithmetical; and its beauty, which can only be appreciated when one has used it in particular cases, is of that indescribably simple kind that distinguishes the use of position in the decimal notation and the arrangement of the simple rules of arithmetic."

Karl Friedrich Gauss (1777–1855) was born in humble circumstances in Brunswick, Germany. Owing to the talent he showed, he received the aid of the Duke of Brunswick in getting a liberal education. In 1807 he was appointed Director of the Göttingen University Observatory, which position he held till his death.

Gauss stands in the front rank of eminent mathematicians, both because of the number and importance of his writings, and because of his influence on the development of mathematics through his contemporaries and in after times. He took a very active interest in the study of magnetism and showed the possibility of magnetic communication before Morse invented the telegraph.

A number of references are made to Gauss in other places.

Karl Friedrich Gauss (1777–1855)

CHAPTER XV

PERMUTATIONS AND COMBINATIONS

I. PERMUTATIONS

232. Permutations and Combinations, or the Theory of Choice, may be considered as a separate department of mathematics, which deals with units, but distinguishes between them, and may regard the order in which they appear.

In arithmetic, and heretofore in algebra, no distinction has been made between units. Thus, the apples in a basket dealt with in a problem might include all sizes and kinds, but each apple was counted as one, whatever its character. In choice such problems as the following appear: In how many ways can the three letters a, b, c be arranged? *Ans.* Six ways; viz., *abc*, *acb*, *bac*, *bca*, *cab*, *cba*. In how many ways can 4 boys be seated on a bench holding four persons? *Ans.* 24. (The method of finding the answers to this and the following questions will be explained later.) How many different baseball nines can be formed in a school of 50 boys? In how many ways can 12 guests be seated at a table by means of cards for them at each seat? How many signals can be given by a vessel having 5 differently colored electric lights with three different positions for each light?

233. Fundamental Principle used in Solving Problems in Choice. Before stating the principle, we shall give three examples to which it applies. (1) How many different couples of a boy and a girl can be formed with four boys and four girls? *Ans.* 16; because the first boy can go with each of the four girls to make four different couples, and so can each of the other three boys. (2) How many different badges of red and blue can be made out of seven shades of red and four shades of blue? *Ans.* 28; for, each shade of red will go with each shade of blue to make a badge. (3) How many different neckwear effects can be had with 4 different styles of collar, 6 different ties, and 3 different scarfpins. *Ans.* $4 \times 6 \times 3 (= 72)$.

The nearly self-evident principle used in deriving these results may be stated as follows:

Fundamental Principle. *If one thing can be done in* ***a*** *ways, and after it is done in one of these ways, a second thing can be done in* ***b*** *ways, and then a third in* ***c*** *ways, and so on, all can be done in* ***abc*** *. . . ways.*

EXAMPLE. In a school of 24 girls and 18 boys, in how many ways may a girl and a boy be chosen to take the two principal characters in a play?

234. Permutations. Each different *arrangement* of some or all of a number of things is called a **permutation.**

EXAMPLE. Nine boys appear on the playground to play base-ball. How many different "batteries" (of a pitcher and catcher) can be formed out of them? How many different teams can be formed?

SOLUTION. How many different choices are there for pitcher? After the pitcher is chosen, how many choices are there for catcher? Then, by the principle in § 233, how many batteries can be formed? *Ans.* 72. Again, after the pitcher and catcher are selected, how many choices are there for first baseman; then for second baseman, and so on. Then, by the principle in § 233, in how many ways can all the nine boys arrange themselves to play? *Ans.* $9 \times 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1 = 362{,}880 \equiv 9\,!$.

This result, 9!, is read "factorial nine," or "nine admiration." The notation $n\,!$ is defined to mean the product of all the numbers from 1 to n. It is thought that the exclamation point was used to express astonishment at the size of the result. $n\,!$ is also often written $\underline{|n}$.

235. Formulas for Permutations.

In a literary society of 25 persons, how many different sets of 4 officers, *viz.* president, vice president, secretary, and treasurer, can be elected?

SOLUTION. How many choices are there for president? After the president is elected, how many choices are there for vice president; then, for secretary; and then for treasurer? Now according to the fundamental principle, how many different sets of officers can there be?

Ans. $25 \times 24 \times 23 \times 22 = 303{,}600$.

Generalizing, let $n =$ the number of different things considered (as the 25 persons), $r =$ the number taken together at one time (as the 4 officers), and let ${}_nP_r$ be a symbol to denote the number of permutations of n things r together.

In making an arrangement of *two* things out of n, how many choices have we for first place? *Ans.* n. For second place? *Ans.* $n-1$, because one has now been taken out for first place. Then, how many arrangements are there in all, by the fundamental principle (§ 233)? Hence,

$${}_nP_2 = n(n-1).$$

Thus, with 12 men in a team playing baseball, 12×11, or 132 different batteries could be formed. With a company of 15 musicians all able to play the piano, 15×14, or 210 different piano duet combinations could be formed. Notice that *order* or arrangement *counts* here, since if A plays the upper part and B the lower, the result is different from the reverse arrangement.

If there are n things taken *three* together (as the giving of three prizes to a class of n persons), we have, by the fundamental principle, since after two things are chosen there are then $n-2$ choices for the third,

$${}_nP_3 = n(n-1)(n-2).$$

Similarly, $${}_nP_4 = n(n-1)(n-2)(n-3),$$

and so on. The student should write values of ${}_nP_5$, ${}_nP_6$, $\cdots$.

Notice now that the number subtracted from n in the last factor in each of the formulas just given is *one less* than the number of things taken together. Hence, we have generally,

$$(1)\quad {}_nP_r = n(n-1)(n-2)(n-3)\cdots(n-[r-1]).$$

Observe that n, the number of things, is always the first factor, and that there are just r factors multiplied together on the right side; that is, if the things are taken "two together" there are *two* factors, if "three together" there are *three* factors, and so on.

If $r = n$, that is, if the n things are taken *all together*, Eq. (1) becomes

$$(2)\quad {}_nP_n = n(n-1)(n-2)\cdots 1 = n!.$$

EXAMPLE. To find the number of ways a band of 12 musicians using 12 different instruments could play if each person could play any instrument and all always played.

Sometimes **repetitions** of elements occur, as in the following problem: How many different arrangements of 4 trees can there be in front of a house if a choice is allowed between elms, maples, lindens, chestnuts, and cherry trees? Observe that we suppose one can get at the nursery any number of trees of any of the kinds named. If we start to plant them how many choices are there for the first hole? for the second, if the same kind of tree can be planted as before? for the third? and so on? *Ans.* 5^4.

If ${}_nP_r' =$ the number of permutations of n things r together when repetitions are allowed,

$$(3)\quad {}_nP_r' = n \times n \times n \times \cdots \text{ to } r \text{ factors} = n^r.$$

236. Exercise in Permutations.

1. With 12 cups and 12 saucers all different, how many different combinations of a cup and a saucer can be made? See § 233.

2. If a cent and a dime are thrown, each falling heads or tails, in how many ways can they fall? (§ 233).

3. A boy has a choice of 5 routes by which to go to school and a choice of 3 routes by which to return. In how many ways may he go and return?

4. There are 6 vowels and 20 consonants in the alphabet. How many words consisting of a vowel and a consonant can be made, if the vowel always comes first? if the consonant always comes first? in all?

5. Calculate with Eq. (1), § 235, ${}_4P_2$; ${}_5P_2$; ${}_6P_2$; ${}_4P_3$; ${}_5P_3$; ${}_4P_4$; ${}_8P_2$; ${}_{20}P_3$; ${}_7P_1$; ${}_6P_6$; ${}_{81}P_3$.

6. In how many ways can a class of 10 pupils be arranged at the blackboard?

7. How many signals can be made with 7 different flags placed one above the other?

8. There are 7 messages, and four boys offer their services. If the person giving them out gives any number he likes to each boy, how many choices has he?

SUGGESTION. How many choices has he with the first message? with the second? and so on.

9. Eleven different statues are to fill 11 niches. In how many different ways can they be set up?

10. How many different tickets are needed on a railroad having 75 stations?

11. How many changes can be rung on a peal of 8 bells if all are rung each time?

12. How many different permutations are there of the 26 letters, four together?

13. If the number of permutations of n things 5 together is equal to twice the number of permutations of n things 3 together, what number is n?

SUGGESTION. Solve as a problem in equations, n being the unknown. Divide by $n(n-1)(n-2)$.

14. Twelve persons who dine together agree to change seats so that they shall not sit exactly the same at any two meals. Allowing 3 meals a day, how many years will elapse before they have to repeat the original seating?

15. How many numbers are there consisting of 10 different figures?

16. How many different strains of music, consisting of two measures of 4 quarter notes each, can be written on the natural scale if any note may occupy any position from C on the added line below to E on the topmost space, or 10 different positions?

17. How many different rigs can be sent out by a livery stable keeper who has 20 horses and 12 buggies which can be used either single or double? It is assumed that the two different ways of hitching two horses make two different teams.

18. In the Morse signaling system, using dots and dashes, how many signals can be made with 2 or less than 2 dots and dashes? with 3 or less than 3? with exactly 3 dots and 2 dashes?

19. How many basket-ball fives can be selected in a school of 50 if different positions for any set of players are regarded as making different teams?

20. How many baseball nines can be formed out of 16 men of whom 3 are pitchers, 2 are catchers, and the others can play in any of the remaining positions?

SUGGESTION. Use § 233 after getting ${}_{11}P_7$.

21. Out of 6 consonants and 3 vowels, how many words can be formed consisting of 3 consonants and 2 vowels provided the vowels always come in the second and fourth places?

22. Ten eastern gentlemen met at a party and each saluted all the rest. If 5 minutes were consumed in each greeting, how many hours were required?

23. How many signals can be made with a semaphore having 3 arms on each side, if each arm can stand in 4 different positions?

24. How many kinds of hexameter verse can there be?

SUGGESTION. Hexameter in theory consists of 6 dactyls (— ◡ ◡), but the last is always replaced by a spondee (— —), or a trochee (— ◡). The first four measures permit freely of spondees instead of dactyls. The fifth place is only occasionally spondee. How many choices are there for first measure? for second; and so on? how many in all?

II. COMBINATIONS

237. Combinations. Each different *group* out of a number of things is called a **combination**. Changing the *order of arrangement* in each group does not change the combination. Thus, a group of five persons in an automobile belonging to a touring party of twenty can seat themselves in $5 \times 4 \times 3 \times 2 \times 1$, or 120 different ways, but they form only one combination, or group, out of the 20 persons.

The distinction between permutations and combinations is well shown by the difference between a set of 4 *officers* and a *committee* of 4. In the case of the officers the order counts, since it will make a difference whether A is president and B vice president, or *vice versa;* while in a committee, all will stand on the same footing, having the same duty to perform. As another illustration, consider the number of football elevens that can be sent out from a school. From the standpoint of the captain, the number is one of permutations, since to him the order or positions in which the men play counts; while from the standpoint of the faculty, as absentees from classes, the number is only one of combinations of men out of the school.

It is convenient to calculate combinations from permutations. Thus, if the number of sets of 4 officers in a society of 30 members is $30 \times 29 \times 28 \times 27$, the number of committees of 4 persons is $\frac{30 \times 29 \times 28 \times 27}{1 \times 2 \times 3 \times 4}$, because *every* group of 4 persons makes $4!$, or 24, different sets of officers but only *one* committee. In a city council of 8 persons, there could be $\frac{8 \times 7 \times 6}{1 \times 2 \times 3}$ committees of 3, since there would be $8 \times 7 \times 6$ arrangements of 3 men each, and each committee of 3 men could be arranged in $1 \times 2 \times 3$ ways.

In general, r things taken r together can be arranged in $r!$ ways (§ 235, Eq. (2)). These $r!$ permutations reduce to *one* combination. Thus, the number of *permutations* of n things r together is $r!$ times the number of *combinations* of n things r together. Consequently if the total number of *permutations* of n things r together is divided by $r!$, the quotient is the number of *combinations* of n things r together. Hence, if ${}_nC_r$ denotes the number of combinations of n things r together, we have, by § 235, Eq. (1),

$$(4) \qquad {}_nC_r = \frac{n(n-1)(n-2)(n-3)\cdots(n-[r-1])}{r!}.$$

This value of ${}_nC_r$ can be changed *in form* by multiplying both terms by $(n-r)!$. In this way, we get

$${}_nC_r = \frac{n(n-1)(n-2)\cdots(n-[r-1])\times(n-r)\cdots 1}{r!\,(n-r)!},$$

or,

(5) $${}_nC_r = \frac{n!}{r!\,(n-r)!}$$ (Since the preceding numerator includes every factor from n to 1.)

Now, $${}_nC_{n-r} = \frac{n!}{(n-r)!\,(n-[n-r])!} = \frac{n!}{(n-r)!\,r!}.$$ (By substituting $n-r$ for r in (5).)

a. This last formula compared with (5) shows that the number of *combinations* of n things r together is equal to the number of combinations of n things $(n-r)$ together. Hence, if we have to calculate the number of combinations of, say, 80 things 73 together, we shall get the same result if we calculate the number of combinations of 80 things 7 together, and the latter calculation is easier to perform.

b. Formula (5) gives the number of ways in which n things can be divided into two *classes* of n and $n-r$ things respectively.

c. Formulas (4) and (5) both give ${}_nC_r$, but (4) as a rule will give the simpler solution.

d. **Binomial Coefficients as Combinations.** By reference to § 193 we see that the third binomial coefficient is the number of combinations of n things 2 together; the fourth coefficient gives the number of combinations of n things 3 together: and the general, rth term, coefficient gives the number of combinations of n things $r-1$ together. Why this relation exists can be seen by examining § 211, 1, 2, supposing $a = b = c, \cdots$.

238. Exercise in Combinations.

1. Calculate ${}_3C_2$; ${}_4C_2$; ${}_5C_2$; ${}_5C_3$; ${}_5C_4$; ${}_6C_2$; ${}_4C_4$; ${}_{13}C_{11}$ (see § 237, *a*); ${}_{21}C_{18}$; ${}_{75}C_4$; ${}_{28}C_4$; ${}_{100}C_{96}$; ${}_{16}P_3$; ${}_{12}P_4$; ${}_{12}C_{11}$.

2. How many different committees of 3 each can be selected out of a club of 75 members?

3. In how many ways can two men divide 16 horses so that one will get 12 and the other 4? (See § 237, *b*.)

4. In an examination 12 questions were given, of which only 8 were required. How many choices were given?

5. How many different sets of 20 councilmen can be elected out of 30 candidates?

6. How many different connections will a telephone girl have to make on a line having 95 subscribers?

7. There are 6 coins of different denominations. How many different sums can be made up with them?

8. In how many ways can a committee of 7 *with a chairman* be selected out of a council of 15 members?

SUGGESTION. How many different committees can be selected without reference to chairman? If each man in turn of any committee becomes chairman, how many different committees are there?

9. How many distinct sounds can be made on the 60 keys of a piano by striking 6 at a time, assuming that any 6 can be struck, as by a mechanical player?

10. How many basket-ball teams can be selected in a school of 300 if no attention is paid to the way the team plays when selected?

11. How many quadrilaterals can be formed out of 24 different points, no three of which lie in the same straight line? how many triangles? how many hexagons?

12. How many tetrahedrons can be formed out of 24 different points, no four of which lie in the same plane?

13. If we allow 80 as the number of well-characterized chemical elements, and assume that one atom each of any two or more can unite, how many substances can be made out of two different elements? out of 3? out of 4?

239. Permutations of n things all together where some are alike. If all the things were unlike, there would be $n!$ permutations in all (§ 235, Eq. (2)). If, now, p things are alike, one arrangement of them will not be changed by rearrangements of the p like letters. Hence, there will be $p!$ times as many arrangements originally as when p are alike. Thus, if p things of one kind are alike, q of another kind are alike, r of a third kind are alike, the number of permutations of the n things reduces to

$$\frac{n!}{p!\ q!\ r!}.$$

1. How many signals can be made by hanging 14 flags each time on a staff if 2 flags are white, 3 are black, 5 are blue, and 4 are red?

2. Galileo, in a letter dated Dec. 11, 1610, conveyed to the astronomer Kepler the nature of the light of Venus seen by him first in his telescope in the following cryptogram: "Haec immatura a me iam frustra leguntur o. y." which, by changing the order of the letters, becomes "Cynthiae figuras aemulatur mater amorum," that is, "the mother of loves has the phases of Cynthia" (the moon). In how many other ways can these 35 letters be arranged besides the form Galileo used?

III. SELECTIONS

240. Selections. Selections are made by *taking* or *leaving*.

1. The number of different selections that can be made from n articles is $2^n - 1$. For, there are two choices for each article, either to take it or leave it; the one case must be subtracted from the total in which nothing is taken at any time.

Thus, if a stall in a bazaar contains 5 articles, the purchaser may secure 1, 2, 3, 4, or 5 articles. In the case of one article, it may be *any* one, or 5 choices; in the case of 2, they may be *any* combination of 2 out of the 5; in the case of 3, they may be *any* combination of 3 out of 5; and so on. Now we get the same result by using the formula, $2^5 - 1$, as by the reasoning just given, or 31.

Thus, $$5+\frac{5\times 4}{2}+\frac{5\times 4\times 3}{1\times 2\times 3}+\frac{5\times 4\times 3\times 2}{1\times 2\times 3\times 4}+1=31.$$

But reasoning that you have the two choices *of taking or leaving* for each article gives the result much more quickly than by the other process.

EXAMPLE. How many choices has a lady at a bargain counter where there are 10 articles? *Ans.* 1023.

2. The number of different selections that can be made from $p+q+r+\cdots$ articles of which p are alike of one sort, q alike of another sort, r alike of a third sort, and so on, is

$$(p+1)(q+1)(r+1)\cdots-1.$$

For, the person selecting can take of the first sort either none, or one, or two, and so on up to p; that is, he has $p+1$ choices; and so for the other things. As before, the one case in which nothing is taken at any time is excluded.

EXAMPLE. How many selections can be made from 3 pairs of shoes all alike, 5 suits all alike, and 4 hats all alike?

IV. GENERAL EXERCISE

241. General Exercise in Permutations and Combinations.

1. With 5 suits, 6 hats, and 4 pairs of shoes in her wardrobe, in how many costumes can a lady appear?

2. From 20 Republicans and 15 Democrats how many committees can be chosen, each consisting of 3 Republicans and 2 Democrats?

3. In how many ways could 56 men be located in a sleeping car, having 14 upper and 14 lower berths, each holding 2 persons?

4. If 7 contestants enter a mile race, in how many ways can first, second, and third places be won?

5. In how many ways can an 8-oared crew be selected from 20 aspirants for a place, and in how many ways could any crew be seated?

6. Six basket-ball teams wish to arrange a series such that each team will meet every other team twice. How many games will have to be scheduled?

7. In how many ways can a baseball nine be selected from 20 players of whom 6 are outfield players (three positions), 8 are infield players, 4 are pitchers, and 2 are catchers?

8. In how many ways can a person invite one or more of six friends to dinner?

9. How many parallelograms are formed if a set of 16 parallel lines meets another set of 12 parallel lines?

10. Three positions are vacant and there are 16 applicants. In how many ways can the positions be filled?

11. In how many ways can six ladies and three gentlemen arrange a game of lawn tennis, each *side* to consist of one lady and one gentleman?

12. How many different objects can be measured on a metric balance with which go 12 different weights?

13. If there are 5 different letter boxes, in how many ways can a person deposit two letters in them?

14. In how many ways can six different beads be arranged on a string? (Notice that for each arrangement the beads can be slid along one place six times without changing the bracelet, and that any order and this order reversed also give the same bracelet, since turning it over does not make a new bracelet.)

15. Find the number of bracelets that can be formed by stringing together four pearls, five rubies, and six diamonds.

16. In a telephone exchange with 500 subscribers how many connections can be made? How many different calls can there be?

17. In a certain telephone exchange 213531 different connections can be made. How many subscribers are there?

18. How many distinct sounds can be produced on 13 keys of an octave on a piano by striking 4 at a time?

19. How many different pickets of 5 men and an officer can be made out of 100 men and 8 corporals?

20. There are q candidates for r seats. Find in how many different ways the voting may result; that is, find the number of different sets of candidates that may be elected.

CHAPTER XVI

PROBABILITY

242. Probability or Chance Defined. If all the causes that go to the producing of an effect are known, then the effect can be predicted. Very frequently, however, the causes of events are not completely known, and we are then led to talk of the *chance* or *probability* that any event will happen.

The **chance** or **probability** that an event will happen has been defined by La Placė as follows: *If an event can happen in **a** ways and fail in **b** ways, all equally likely to happen, the probability or chance that it will happen is* $\frac{a}{a+b}$, *and that it will fail is* $\frac{b}{a+b}$.

Thus, if 20 slips of paper containing the names of 7 Americans, 3 Englishmen, 6 Germans, and 4 Norwegians, members of a lodge, are put in a box and one is drawn at random to choose a delegate, the chance that an American is chosen is $\frac{7}{20}$, and the chance that an American is not chosen is $\frac{13}{20}$; the chance that an Englishman is chosen is $\frac{3}{20}$, and the chance that an Englishman is not chosen is $\frac{17}{20}$; and so on. Or, again, if it rains 73 days in a year at a certain place and does not rain 292 days, we may say the chance that it will rain on any given day in the future is $\frac{73}{365}$, or $\frac{1}{5}$, and the chance that it will not rain is $\frac{292}{365}$, or $\frac{4}{5}$.

If an event can happen in a ways and fail in b ways all equally likely to happen, then we say the **odds** in favor of its happening are as a to b, and the odds against its happening are as b to a; but the word odds implies an improper fraction ratio. Thus, the odds against rain in the illustration just given are 292 to 73, or 4 to 1.

a. Particular attention is called to the phrase "all equally likely to happen" in the definition of chance given above. In the case of the drawing of a slip, since all the slips may be supposed to be practically alike, any one slip would be as likely to be taken out as any other; but in the case of the weather, this would not be true, since certain months are likely to have more rainy days than others. For the definition to hold, then, all the events must be equally likely to happen. In this weather problem it would have been more satisfactory to take a period of a certain month rather than of a year.

D'Alembert said: "There are two possible cases with respect to each future event, one that it will occur, the other that it will not occur. Hence the chance of every event is $\frac{1}{2}$, and the definition of probability is meaningless." Evidently D'Alembert is wrong about this, since, as we have just seen in the examples given, chances are not, as a rule, the same.

EXAMPLE. If two coins are tossed simultaneously, what is the chance that they will fall both heads?

SOLUTION. It would be false reasoning to say that since they can fall both heads or both tails, or one head and one tail, that the chance that one can happen is $\frac{1}{3}$. For these three things are not all equally likely to happen, the case of head and tail being twice as likely to happen as either both heads or both tails. *Four* events here are equally likely to happen, viz., head-head, head-tail, tail-head, and tail-tail. Hence the chance that the coins will fall both heads is $\frac{1}{4}$. The chance that either one will be head and the other tail is $\frac{1}{2}$. The problem implies also that each coin is perfectly formed and exactly as likely to fall one way as the other.

243. Complementary Event. The sum of $\frac{a}{a+b}$ and $\frac{b}{a+b}$ is 1. Hence, if we know the chance that an event will happen, the chance that it will fail is found by subtracting the former from unity. Either event is called the *complementary event* to the other. The probability 1 denotes certainty of happening; the probability 0 denotes certainty of failing.

244. The Law of Averages, *that the average of a certain number of data approximately equals the average of a like number of corresponding data,* is a law of nature that deserves to rank very high in importance. If it were not for the law of averages, we might seem to be living in a world of chance instead of one of law.

It is highly desirable that every one should have a good understanding of the law of averages, of its application, and of its limitations. Perhaps the best example of the working of the law may be seen in election returns. Every one knows that as soon as an election is over, precinct returns are published to the world. Even in a national election, after only a fraction of the returns are in, people begin forecasting the result, forming their judgments from the law of averages. It is very unsafe, however, to predict results, basing the prediction on a few precincts, especially if these lie in one district, since local conditions may materially affect results there. It is only averages, the larger the scale the better, that seem to hold good. This applies to statistics of all kinds, such as the growth in population, the government's revenue, the death rate, etc.

245. Probability and the Law of Averages. The definition of probability or chance found in § 242 cannot be applied to important classes of events, such as are enumerated in statistics, since it is impossible to enumerate the different ways in which events can happen, and impossible also to have them equally likely to occur. But it is easy to know from statistics what has happened under very similar conditions in the past, and from such results what will happen in the future can be pretty safely inferred. In this way, formulas and rules obtained from a study of problems to which the definition does apply, can be carried over to problems that involve statistics. With the preceding in mind, the following definition of probability can be used:

$$\text{Probability} = \frac{\text{Number of ways a similar event actually occurred}}{\text{Number of ways the event might have happened}}.$$

246. Examples of Probability.

1. In round numbers, the population of Continental United States in 1880 was 50,000,000; in 1890 it was 62,500,000; in 1900 it was 76,000,000; in 1910 it was 92,000,000. About what will it probably be in 1920? in 1930? Calculate what it ought to have been in 1900, and compare the result with the actual figures. Do the same for 1910.

Suggestion. To calculate the population in 1920, multiply the population in 1910 by the ratio of the population in 1910 to that of 1900; to calculate the population in 1930 multiply the population in 1920 by the ratio used before, since that is the best available.

2. The total revenue of the general government in round numbers for a certain year was \$ 516,000,000, of which the internal revenue was \$ 272,000,000. The total revenue the next year with the same laws in force was \$ 567,000,000. Find an approximate value for the internal revenue the latter year.

(Actually it was \$ 295,000,000.)

3. The area of Lake Superior is 31,000 sq. mi., and it drains an area of 85,000 sq. mi. The area of Lake Erie is 10,000 sq. mi., and it drains an area of 40,000 sq. mi. Are the areas of these lakes approximately proportional to the areas they drain? (Here we find a wide variation; but still a rough proportion exists.)

4. The mean temperature of Chicago for the month of January for a period of 31 yr. was 23.8°. If the mean temperature for the month of January, 1902, was 25.2°, what was the variation from the normal? Is this a close approximation?

5. If a penny is thrown 6 times, according to the law of probability, it ought to fall heads 3 times and tails 3 times. As a matter of fact, the actual happening is likely to vary greatly from this for so small a number of times. If, however, the penny is symmetrically constructed, and is thrown 500 times, something like what number of the throws should fall heads, and what number tails?

a. The preceding problem suggests the statement that in nearly all the mathematical works that treat of this subject, the problems given deal largely with cards and dice, or with balls in an urn, or the like. Now, there is a double reason why such problems are objectionable: first, because they are not very practical; and second, because the laws of probability do not hold well for a small number of events. Even with a nearly perfect die the rules would not hold well for a small number of throws.

6. The theory of probability is applied to mortality tables. The table on next page gives the number of survivors at different ages out of 100,000 particular persons alive at the age of 10.

Age	Survivors	Age	Survivors	Age	Survivors
10	100,000	40	78,106	70	38,569
15	96,285	45	74,173	75	26,237
20	92,637	50	69,804	80	14,474
25	89,032	55	64,563	85	5,485
30	85,441	60	57,917	90	847
35	81,822	65	49,341	95	3

7. Taking figures from this table, calculate what is the chance that a person 15 years of age will live to the age of 35. *Ans.* $\frac{81822}{96285}$.

8. What is the chance that a person 40 years old will live to be 80 years old? What is the chance that a person 70 years old will live to be 90 years old?

247. Probability and the Theory of Combinations. Many solutions in probability depend on the theory of combinations.

1. In a receptacle were placed 30 names of persons to be drawn for jury service, of which 12 were to be drawn. Sixteen of the 30 belonged to men who lived in the county seat. What is the chance that all the men selected will be from the county seat?

Solution. The number of combinations of 30 things, 12 together, or the total number of different juries is $\frac{30!}{12!\ 18!}$ (§ 237, eq. (5)). The total number of juries that could be formed out of the 16 men, or the number favorable to all being from the county seat is $\frac{16!}{12!\ 4!}$. Hence, the chance that all will be from the county seat is $\frac{16!\ 18!}{30!\ 4!}$, which equals $\frac{4}{190,095}$, or one chance in about 50,000.

2. In a railroad wreck it was reported that out of 20 persons on the train 5 were injured. If one family consisted of 7 persons, what was the chance that all the injured belonged to this family?

3. From a committee of 4 Juniors and 6 Seniors, a subcommittee of 3 was chosen. Find the probability that it will consist of 2 Seniors, and 1 Junior.

Suggestion. Find the number of combinations of 10 persons, 3 together; then of 6 persons, 2 together, and 4 persons, 1 together, using § 233.

248. The Addition Rule. *The chance that one or other of several related events will happen is the* **sum** *of the chances that each will happen.*

1. Out of a gang of workmen consisting of 6 Americans, 5 Italians, and 4 Englishmen, 3 were drawn at random for a particular job. Find the chance that all will be of the same nationality.

SOLUTION :

The chance that they will all be Americans is $\frac{{}_6C_3}{{}_{15}C_3}$, or $\frac{4}{91}$ (§ 237, (4)).

The chance that they will all be Italians is $\frac{2}{91}$ (§ 237, (4)).

The chance that they will all be Englishmen is $\frac{4}{455}$ (§ 237, (4)).

The chance that they all are of the same nationality is $\frac{34}{455}$.

2. In a society consisting of 25 members of whom 15 are Protestants and 10 Catholics, 5 officers are chosen at random. What is the chance that they will be either all Protestant or all Catholic?

3. What is the chance in Ex. 2 that Protestants and Catholics will both be represented?

SUGGESTION. Get the complementary event to that of Ex. 2.

249. The Multiplication Rule. Compound Events. *The chance that two distinct events will occur simultaneously is the* **product** *of the chances that each will occur.*

1. What is the chance if two dice are thrown that both will turn up 3? *Ans.* $\frac{1}{6} \times \frac{1}{6}$, or $\frac{1}{36}$.

2. The chance that it rains in Mr. A's town on any given day is $\frac{1}{4}$. Now Mr. A, on the average, carries an umbrella one day in seven. What is the chance that Mr. A will be caught in the rain without an umbrella?

3. A regiment of soldiers enlisting is to have 1000 privates and 25 officers. What is the chance that two brothers enlisting will both become officers?

EXPECTATION

250. Expectation is a term used to describe the sum one may expect back from money put in hazard. The *mathematical expectation is always the product of the sum to be realized and the chance of getting it.*

1. In a lottery ten tickets were sold at $1.25 apiece, and 2 prizes were to be given for the two lucky tickets, one for $8 and the other for $2. What is the "expectation" of a man who purchases a ticket?

SOLUTION. His expectation on the $8 prize is $\frac{1}{10} \times \$8$, or $0.80.
His expectation on the $2 prize is $\frac{1}{10} \times \$2$, or .20.
His expectation on both prizes is $0.80 + $0.20 = $1.00.

2. In a pool at a horse race, one man bid $15 for first choice of the winner, another $10 for second choice, still another $5 for third choice, and a fourth bid $6 for the "field," that is, all the other horses that ran. The manager takes out 10 % for expenses. What should be the "expectation" of the several bidders, if the actual chances of winning of the several horses are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{6}$, $\frac{1}{12}$? *Ans.* $16.20, $8.10, $5.40, $2.70.

The first and third men had a little the best of the bargain, but taken as a whole they lost the manager's 10 %.

251. Exercise in Probability.

1. Two numbers are chosen at random. Find the chance that their sum is even.

2. A man pays $5 for one ticket in a lottery in which there are 1000 tickets. The prizes are: one of $1000, two of $500 each, 12 of $100 each, and 40 of $10 each. What is his expectation?

3. A letter is drawn at random out of each of the words choice and chance. Show that the probability that they are the same letter is $\frac{1}{6}$.

4. A, B, and C have equal claims for a prize. C says to A and B, you two draw lots first; then let the loser withdraw, and the winner draw lots with me for the prize. Is this fair? Show that the chances are $\frac{1}{4}$, $\frac{1}{4}$, $\frac{1}{2}$ respectively.

5. The common number lottery contains 90 numbers, of which 5 are drawn out. Show that the chance that all a man's numbers who possesses three tickets will be drawn is $\frac{1}{117,480}$.

6. If three dolls whose first cost are respectively \$1, \$2, and \$3 are raffled off with 100 tickets at 10¢ each, what should be the expectation of a person who has bought \$2 worth of tickets?

7. The probability that A can solve a problem is $\frac{3}{4}$, and that B can solve it is $\frac{3}{5}$. What is the probability that both will solve it? The probability that either one or the other will solve it?

8. The English House of Commons, consisting of 670 members, has (1913), 84 Irish Nationalists and 41 Independent Labor members. If a committee of five is chosen at random, what is the chance that it will contain 2 Nationalists and one Labor member?

9. If 3 points are taken at random on a circle, what is the chance that they will not lie on the same semicircle. *Ans.* $\frac{3}{4}$.

SUGGESTION. Draw any diameter and calculate the chance that all the points will lie on the same semicircle.

10. If at a live stock show four animals of equal merit are judged, first by an agricultural student and then by the regular judges, what is the chance that the student will rank the animals in the same order as the judges?

11. A man has left his umbrella in one or other of three stores which he visited. He is in the habit of leaving it once every 4 times on the average that he goes into a store. Find the chance that he left it in the first, second, and third stores respectively.

12. What is the chance that A and B, each aged 20, will both be alive at the age of 60? (See § 246, 6.) What is the chance that of A, B, and C alive at the age of 30, only one will be alive at 70? Use logarithms.

SUGGESTION. These are compound events, § 249. In the second problem find the chance that A and B are dead and C is alive, and multiply the result by 3, since either of the three alive would satisfy the conditions.

13. What is the chance that out of eight particular persons alive at the age of 10, only six shall be alive at the age of 50?

SUGGESTION. What is the chance that A will die? B? What is the chance C is alive? D? etc. What is the chance that A and B die and the rest survive? (§ 249.) How many different sets of two can be formed out of eight persons? What, then, is the chance that just two will have died? Use logarithms in making the calculation.

14. If on the average one vessel in 100 is wrecked, find the chance that of 10 vessels expected only 8 will arrive safely. Find the chance that at least 8 will arrive safely.

15. A's skill at a game is $\frac{2}{3}$ of B's. What is the chance if they play that A will win at least 2 games out of 5?

SUGGESTION. Find A's chance of winning all five games; then his chance of winning 4 and losing one; and so on.

16. A set of dominoes is numbered from double blank to double 6. If one is drawn at random, what is the probability that it contains a total of 6?

17. The English physicist, Hooke, published the discovery contained in the Latin sentence "Ut tensio, sic vis" (as the resistance, so is the power), by the cipher *c e i i i n o s s s t t u v*. If one arrangement of these letters is as likely as another, find the chance of getting the right one in 500 trials.

18. A person writes n letters and addresses n envelopes. If the letters are put into the envelopes at random, what is the chance that all go wrong?

SUGGESTION. What is the chance of missing on the first letter? After it is placed, what is the chance of missing on the second? and so on.

19. A rod is broken at random into three pieces. Show that the chance that no one of the pieces is greater than the other two together is $\frac{1}{4}$.

SUGGESTION. Calculate the chance of the complementary event, dividing the line first into four parts, getting cases that can be settled with this division; then into 8 parts, getting cases not before included; then into 16 parts; and so on.

NOTE. A standard work on Permutations, Combinations, and Probability is Whitworth's "Choice and Chance." Besides a great variety of problems this book contains a chapter on the wrongfulness and disadvantage of gambling, including a discussion showing that insurance is the reverse of gambling.

CHAPTER XVII

DETERMINANTS

252. A **determinant** may be *described* as a homogeneous quantity constructed according to certain rules out of n^2 quantities or elements, and commonly written in square array. A determinant of *the second order* consists of 2^2 quantities or elements; one of *the third order* of 3^2 quantities, and so on. Determinants originated from a study of the solution of simultaneous equations.

Solving the equations $\begin{cases} (1) \ a_1x + b_1y = c_1 \\ (2) \ a_2x + b_2y = c_2 \end{cases}$, we have

$$(2_1) \quad a_1a_2x + a_1b_2y = a_1c_2 \quad \text{(Mult. Ax.)}$$

$$(1_1) \quad a_1a_2x + a_2b_1y = a_2c_1 \quad \text{(Mult. Ax.)}$$

$$(a_1b_2 - a_2b_1)y = a_1c_2 - a_2c_1 \quad \text{(Sub. Ax.)}$$

$$y = \frac{a_1c_2 - a_2c_1}{a_1b_2 - a_2b_1}. \quad \textit{Ans.}$$

Then (§ 42),

$$x = \frac{b_2c_1 - b_1c_2}{a_1b_2 - a_2b_1}. \quad \textit{Ans.}$$

The numerators and denominators of the values of x and y are determinants.

253. Determinant Notation. In the determinant notation, the rule for writing down the values of x and y of the preceding article is simple. In this notation the quantities or "elements" involved are written, as already stated, in square array, and placed between two vertical lines.

Thus, the denominator of the values of x and y in the preceding article, $a_1b_2 - a_2b_1$ is written $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$.

In $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$, we have two horizontal *rows*, a_1b_1 and a_2b_2; two vertical *columns* a_1a_2 and b_1b_2; and two diagonals, the first a_1b_2 called the **principal diagonal** and the second a_2b_1 called the **secondary diagonal.**

The notation $\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}$ *signifies the product of the two quantities in the principal diagonal* a_1b_2 *less the product of the two quantities in the secondary diagonal* a_2b_1, *or,* $a_1b_2 - a_2b_1$.

254. Exercise in Evaluation of Determinants of Second Order.

1. $\begin{vmatrix} 3 & 2 \\ 6 & 8 \end{vmatrix}$. SOLUTION: $\begin{vmatrix} 3 & 2 \\ 6 & 8 \end{vmatrix} = 3 \times 8 - 2 \times 6 = 12.$ *Ans.*

2. $\begin{vmatrix} 4 & 2 \\ 5 & 9 \end{vmatrix}$ 3. $\begin{vmatrix} 2 & 4 \\ -3 & 2 \end{vmatrix}$ 4. $\begin{vmatrix} 3 & -2 \\ 6 & 7 \end{vmatrix}$ 5. $\begin{vmatrix} 3 & -5 \\ -6 & -4 \end{vmatrix}$ 6. $\begin{vmatrix} -1 & 0 \\ -2 & 4 \end{vmatrix}$

7. $\begin{vmatrix} x & 5 \\ 2y & 4 \end{vmatrix}$ 8. $\begin{vmatrix} a+b, & c+d \\ c-d, & a-b \end{vmatrix}$ 9. $\begin{vmatrix} x^3 & -y^3 \\ 4 & -7 \end{vmatrix}$

255. Solution of Simultaneous Equations containing Two Unknowns by Determinants.

Using notation § 253 for the values of x and y in § 252, we see

$$\begin{cases} (1) & a_1x + b_1y = c_1 \\ (2) & a_2x + b_2y = c_2 \end{cases}$$ have for their values of x and y,

$$x = \frac{\begin{vmatrix} c_1 & b_1 \\ c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}}.$$

Comparing these values of x and y with Eq.'s (1) and (2) we see *that for the denominator of each value we merely write the four coefficients of x and y in* (1) *and* (2) *as they stand, in square array inclosed by lines; to get the numerator of x we take its denominator and substitute the column* $\begin{matrix} c_1 \\ c_2 \end{matrix}$ *(of right members) for* $\begin{matrix} a_1 \\ a_2 \end{matrix}$, *the coefficients* *of x; and for the numerator of y, we take its denominator and substitute c_1, c_2, respectively, for the coefficients of y.*

Solve, using determinants and verify answers (see §§ 41–43):

1. $$\begin{cases} 4x - 7y = -2, \\ 5x + 3y = 21. \end{cases}$$

SOLUTION:

$$x = \frac{\begin{vmatrix} -2 & -7 \\ 21 & 3 \end{vmatrix}}{\begin{vmatrix} 4 & -7 \\ 5 & 3 \end{vmatrix}} = \frac{-6-(-147)}{12-(-35)} = \frac{141}{47} = 3. \quad \textit{Ans.}$$

$$y = \frac{\begin{vmatrix} 4 & -2 \\ 5 & 21 \end{vmatrix}}{\begin{vmatrix} 4 & -7 \\ 5 & 3 \end{vmatrix}} = \frac{84-(-10)}{12-(-35)} = \frac{94}{47} = 2. \quad \textit{Ans.}$$

VERIFICATION (1) $4 \times 3 - 7 \times 2 \equiv -2$; (2) $5 \times 3 + 3 \times 2 \equiv 21$.

2. $$\begin{cases} 4x + 9y = 3, \\ 3x + 7y = 2. \end{cases}$$

3. $$\begin{cases} 2x + 7y = 11, \\ 5x - 9y = 1. \end{cases}$$

4. $$\begin{cases} 5x - 3y = 20, \\ 3x - 4y = 1. \end{cases}$$

5. $$\begin{cases} 4x + 3y = 17, \\ 5x - y = 7. \end{cases}$$

6. $$\begin{cases} 6x - y = 27, \\ 8y - 3x + 36 = 0. \end{cases}$$

7. $$\begin{cases} 5x - 8y + 46 = 0, \\ 2x + 3y + 6 = 0. \end{cases}$$

8. $$\begin{cases} ax + by = 1, \\ cx + dy = 1. \end{cases}$$

9. $$\begin{cases} ax + by = 0, \\ mx + ny = 1. \end{cases}$$

256. Determinants of the Third Order.

$$\text{If the equations } \begin{cases} (1)\ a_1x + b_1y + c_1z = d_1, \\ (2)\ a_2x + b_2y + c_2z = d_2, \\ (3)\ a_3x + b_3y + c_3z = d_3 \end{cases}$$

are solved by the regular method of solution for three equations containing three unknown quantities (§ 46) by eliminating z between (1) and (2), getting (4), and then z between (1) and (3), getting (5), and lastly solving (4) and (5) by eliminating y, there results the value of x at the top of the next page.

The values of y and z are obtained by substitution in (4) and (1).

$$x = \frac{d_1b_2c_3 + d_2b_3c_1 + d_3b_1c_2 - d_1b_3c_2 - d_2b_1c_3 - d_3b_2c_1}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1}.$$

$$y = \frac{a_1d_2c_3 - a_1d_3c_2 - a_2d_1c_3 + a_3d_1c_2 + a_2d_3c_1 - a_3d_2c_1}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c}.$$

$$z = \frac{a_1b_2d_3 - a_1b_3d_2 - a_2b_1d_3 + a_3b_1d_2 + a_2b_3d_1 - a_3b_2d_1}{a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1}.$$

The numerators and denominators of these values of x, y, and z, as in the case of the roots of two equations containing two unknowns, are determinants, and can be written in the square notation as follows:

$$x = \frac{\begin{vmatrix} d_1 & b_1 & c_1 \\ d_2 & b_2 & c_2 \\ d_3 & b_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad y = \frac{\begin{vmatrix} a_1 & d_1 & c_1 \\ a_2 & d_2 & c_2 \\ a_3 & d_3 & c_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}, \quad z = \frac{\begin{vmatrix} a_1 & b_1 & d_1 \\ a_2 & b_2 & d_2 \\ a_3 & b_3 & d_3 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix}}.$$

Notice, as in the case of determinants of the second order, *that for the denominator of each value we write the nine coefficients of x, y, z in the given equations* (1), (2), (3), *as they stand, in square array; to get the numerator of x we take its denominator and substitute the column formed from d_1, d_2, d_3 (the right members) for the column of coefficients of x; and we make similar substitutions for the values of y and z.*

Comparing the denominators of the two values of x given above, we have

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} \equiv a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1.$$

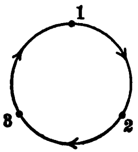
Let us now study by what rules the right side of the last equation can be derived from the left side. We observe that

(1) In each *term* of the right side only one letter comes from any one row, and also only one letter from any one column of the left side.

(2) The right member includes all possible products (3! of them, § 235, (2)) having one letter from each row and each column.

(3) The letters in each term on the right side are put in alphabetical order.

(4) The subscripts of the *positive* terms are in cyclic order and the *negative* ones are *not* in cyclic order. By cyclic order is meant arranged as in the circle in the margin so that 2 follows 1, 3 follows 2, and then 1 follows 3. If n numbers are on the circle, 1 follows n.



257. General Determinant of n^2 Elements and $n!$ terms. The rules (1)–(3) § 256 apply to determinants of any order.

A general rule for the signs of the terms of a determinant of any order can be formulated by taking account of what are called *inversions* of the subscripts, the letters themselves being in alphabetical order. If we take the natural order 1, 2, 3, 4 ⋯ as the standard, then every time any larger subscript number comes before a smaller, it is counted as an inversion. The rule for the signs of the terms then is: *If there are no inversions or an even number of inversions of subscripts, the sign of the term is* $+$; *if there is an odd number of inversions, it is* $-$.

Thus, in $a_2b_3c_1$, there are two inversions, 2 before 1, and 3 before 1, and the sign is $+$; in $a_1b_3c_2$ there is one inversion 3 before 2, and the sign is $-$; in $a_3b_2c_1$, there are three inversions, 3 before 1, 2 before 1, and 3 before 2, and the sign is $-$. In $a_4b_2c_3d_1$ there are 5 inversions and the sign is $-$.*

Find the signs in the following by counting inversions:

1. $a_2b_1c_3$; $a_3b_1c_2$; $a_1b_2c_3$; $a_1b_3c_2d_4$; $a_3b_4c_1d_2$; $a_4b_3c_2d_1$.

2. $a_1b_4c_3d_2$; $a_2b_3c_4d_1e_5$; $a_5b_4c_3d_2e_1$; $a_4b_2c_1d_3$; $a_3b_4c_2d_1$.

* In higher mathematics the elements of determinants are very often denoted by a single letter with *double subscripts*, one giving the row and the other the column of the element. Thus a_{11} would denote the element in the first row and first column, and a_{pq} the element in the pth row and qth column. To get the sign of any term in the expanded form of such a determinant, the second subscripts can all be written in natural order, 1, 2, 3 ⋯ and then the inversion of the first

258. Properties of determinants illustrated in determinants of the second order. Prove the identities of this article by "expanding" each member into the form of an ordinary quantity, and comparing the results.

1. Prove that $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix}.$$

We learn from this that *interchanging rows and columns does not alter the value of a second order determinant.*

2. Prove that $$\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv -\begin{vmatrix} b_1 & a_1 \\ b_2 & a_2 \end{vmatrix} \equiv -\begin{vmatrix} a_2 & b_2 \\ a_1 & b_1 \end{vmatrix}.$$

Thus, *interchanging the two columns or the two rows of a second order determinant changes its sign.*

3. Prove that $$\begin{vmatrix} ma_1 & mb_1 \\ a_2 & b_2 \end{vmatrix} \equiv m\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} \equiv \begin{vmatrix} ma_1 & b_1 \\ ma_2 & b_2 \end{vmatrix}.$$

Thus, *multiplying the "elements" of one row or one column of a second order determinant by a number multiplies it by this number.*

4. Prove that $$\begin{vmatrix} a_1 & b_1 \\ ma_1 & mb_1 \end{vmatrix} \equiv 0 \equiv \begin{vmatrix} a_1 & ma_1 \\ a_2 & ma_2 \end{vmatrix}.$$

Thus, *if the elements of one row of a second order determinant are respectively equal to the same multiple of the elements of another row, the determinant equals* 0. *A similar principle holds for two such columns.*

Evidently then also when two rows or two columns are *the same* the determinant is 0.

We now consider corresponding properties of any determinant.

259. Properties of Determinants.

1. *Interchanging any two adjacent rows (or columns) of a determinant changes its sign.*

subscripts can be counted as in the rule given for the sign of this term. Evidently one set of subscripts takes the place of the different letters, $a, b, c, \cdots$. Thus,

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} = \begin{matrix} a_{11}a_{22}a_{33} + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} \\ - a_{11}a_{32}a_{23} - a_{31}a_{22}a_{13} - a_{21}a_{12}a_{33}. \end{matrix}$$

For, this amounts to the inversion of two subscripts, or two letters and hence two subscripts, in each term. (See § 257.)

2. *Interchanging any two rows (or columns) of a determinant changes its sign.*

For, to get one row in another row's place by successive interchanges of adjacent rows requires, let us say, n interchanges: then to get the second row back to the first one's place will require one less, or $n-1$ interchanges. Altogether there will be $2n-1$ successive interchanges, — an *odd* number. But an odd number of changes of sign means that the sign of the original determinant has been changed.

a. Hereafter, as in the proof just given, we will often use the word row in proofs to mean either a row or column.

3. *Multiplying or dividing the elements of any one row (or column) of a determinant by the same number multiplies or divides the value of the determinant by this number.*

For, one letter and only one in each term of the "expanded" determinant will be multiplied or divided by the multiplier or divisor number, which has the effect of multiplying or dividing the value of the determinant by this number. Thus,

$$\begin{vmatrix} 3 & 6 & 9 \\ 3 & 4 & 8 \\ 2 & 2 & 5 \end{vmatrix} = 3\begin{vmatrix} 1 & 2 & 3 \\ 3 & 4 & 8 \\ 2 & 2 & 5 \end{vmatrix} = 6\begin{vmatrix} 1 & 1 & 3 \\ 3 & 2 & 8 \\ 2 & 1 & 5 \end{vmatrix} = \begin{vmatrix} 1 & 1 & 18 \\ 3 & 2 & 48 \\ 2 & 1 & 30 \end{vmatrix}.$$

Here 3 is first taken out of the first row; then 2 is taken out of the second column; then 6 is put back into the determinant by multiplying the elements of the third column by it.

4. *If two rows or columns of a determinant are identical, or become identical after removing a common factor, its value is zero.*

For suppose the value of the determinant is D. Then, interchanging the two identical rows, by 2 above, changes its sign, giving $-D$. But interchanging two identical rows leaves the determinant unchanged. Thus $D=-D$, whence $2D=0$ or $D=0$.

260. Separation of a Determinant into Two or More. Addition of Determinants having all but One Row or Column Common.

By expanding both sides it is easy to show that

$$\begin{vmatrix} a_1 + a_1' & b_1 & c_1 \\ a_2 + a_2' & b_2 & c_2 \\ a_3 + a_3' & b_3 & c_3 \end{vmatrix} \equiv \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} + \begin{vmatrix} a_1' & b_1 & c_1 \\ a_2' & b_2 & c_2 \\ a_3' & b_3 & c_3 \end{vmatrix}.$$

Thus, *the sum of two determinants having all but one column (or row) common is equal to one determinant containing the common columns, and in its remaining column elements which are the respective sums of the corresponding elements of the two given determinants.* Conversely, *one determinant can be separated into two by separating each of the elements of any column (or row) into two parts and making these parts corresponding elements of partial determinants.*

261. Changes in a Determinant which do not Alter its Value.

1. *Any even number of interchanges of rows or columns does not alter the value of a determinant.*

2. *Any row, or column, of a determinant can have added to its respective elements the corresponding elements of any other row, or column, each multiplied by the same number, without altering its value.*

For, upon separating the resulting determinant into its constituent determinants, as in § 260, one of the resulting determinants is the original one, and the other (after taking out the common factor from the elements of the added column) has two columns common, and by § 259, 4 vanishes.

Extending this idea, evidently we can add to the respective elements of any row (or column) the sums of the corresponding elements of two or more rows, each multiplied by any factor without altering its value.

This theorem can very often be used to shorten the labor of evaluating a determinant.

262. Minors in Determinants. The determinant (§ 256)

$$a_1b_2c_3 + a_2b_3c_1 + a_3b_1c_2 - a_1b_3c_2 - a_2b_1c_3 - a_3b_2c_1 \equiv$$
$$a_1(b_2c_3 - b_3c_2) - b_1(a_2c_3 - a_3c_2) + c_1(a_2b_3 - a_3b_2).$$

Hence

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix} - b_1\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix} + c_1\begin{vmatrix} a_2 & b_2 \\ a_3 & b_3 \end{vmatrix}.$$

The determinants on the right side are called "*minors*" of the determinant on the left side. The minor of the element a_1, or $\begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix}$, is obtained by *erasing* in the given determinant the row and column in which a_1 is found. The minor of b_1 is $\begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix}$ obtained by erasing in the given determinant the row and column containing b_1. Similarly the minor of b_2 is $\begin{vmatrix} a_1 & c_1 \\ a_3 & c_3 \end{vmatrix}$; and so on.

The minors of a determinant are commonly represented by capital letters. Thus, the minor of a_1 is A_1; that of c_3 is C_3; and so on. We have, then,

$$A_1 = \begin{vmatrix} b_2 & c_2 \\ b_3 & c_3 \end{vmatrix};\ B_1 = \begin{vmatrix} a_2 & c_2 \\ a_3 & c_3 \end{vmatrix};\ B_3 = \begin{vmatrix} a_1 & c_1 \\ a_2 & c_2 \end{vmatrix};\ \text{etc.}$$

What are the respective minors of a_3, c_2, a_2, c_3, c_1?

In this notation we can write the equation given above thus:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 - b_1B_1 + c_1C_1.$$

Hence the value of this determinant can be expressed in terms of a row (or column) of elements, and minors of these elements.

263. Determinants Expressed in Terms of their Minors. Any determinant can be expressed in terms of the elements and their minors from any row or column. For, an examination shows that such an expression contains all possible products of elements each taken from every row and column, but never more than one element taken from any row or column. (See § 256.)

It remains, then, to determine the signs of the several terms. In doing this, it should always be borne in mind that the principal diagonal, $a_1, b_2, c_3, d_4 \cdots$ of every determinant or any minor of it is always taken as positive. Moreover, if rows or columns are interchanged, the principal diagonal of the resulting determinant is always taken positive.

Now, any element, as b_4 (see 4th order determinant below) can be made to take the place of a_1, and its minor, unchanged in arrangement of its rows and columns, the place of A_1, by the simple device of interchanging rows so that this element falls in the first row and then interchanging columns so that the same element falls in the first column. The product of the element and its minor is then positive. *If there has been an odd number of interchanges of sign, the sign of the determinant has been changed and the product of the element and its minor is negative in the original determinant; otherwise it is positive.*

Thus, the sign of the term b_3B_3 is $-$, since *two* interchanges of rows brings b_3 to the first row, and then *one* interchange of the first two columns brings b_3 to the position of a_1.

What sign has c_3C_3? d_2D_2? a_4A_4? c_1C_1? c_2C_2? c_4C_4?

An element in jth row and kth column calls for $j+k$ interchanges. Hence, the sign of this element times its minor is $(-1)^{j+k}$.

Test this rule in the following examples:

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = a_1A_1 - b_1B_1 + c_1C_1$$
$$= -a_2A_2 + b_2B_2 - c_2C_2$$
$$= -b_1B_1 + b_2B_2 - b_3B_3$$
$$= \quad c_1C_1 - c_2C_2 + c_3C_3$$

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix} = a_1A_1 - b_1B_1 + c_1C_1 - d_1D_1$$
$$= a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4$$
$$= a_3A_3 - b_3B_3 + c_3C_3 - d_3D_3$$

1. Write in determinant form the minors of the following from the determinant of fourth order just given: a_1; a_4; c_3; d_2; b_3; b_4.

2. What sign in any order determinant has c_2C_2? d_4D_4? b_4B_4? b_2B_2? b_3B_3? a_4A_4? c_5C_5? d_6D_6?

3. Express the 4th order determinant in terms of a_4, b_4, c_4, d_4 and their minors; also in terms of d_1, d_2, d_3, d_4 and their minors.

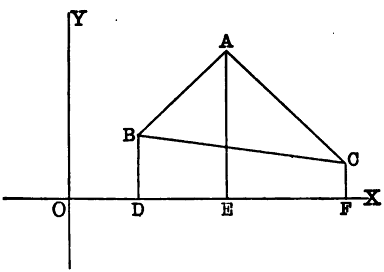
264. Exercise in the Evaluation of Determinants of the Third Order by Means of Minors.

1. $\begin{vmatrix} 3 & 6 & 9 \\ 2 & -4 & -3 \\ 1 & 7 & 6 \end{vmatrix}$. SOLUTION. $3\begin{vmatrix} -4 & -3 \\ 7 & 6 \end{vmatrix} - 6\begin{vmatrix} 2 & -3 \\ 1 & 6 \end{vmatrix} + 9\begin{vmatrix} 2 & -4 \\ 1 & 7 \end{vmatrix} =$
$3(-3) - 6 \times 15 + 9 \times 18 = 63$. *Ans.*

2. $\begin{vmatrix} 1 & 6 & 9 \\ 7 & 4 & -2 \\ 3 & -1 & -5 \end{vmatrix}$. **3.** $\begin{vmatrix} 4 & 9 & -3 \\ -2 & 1 & 2 \\ 1 & -7 & 3 \end{vmatrix}$. **4.** $\begin{vmatrix} 4 & 6 & 0 \\ 2 & 3 & 5 \\ 0 & 6 & 1 \end{vmatrix}$.

5. $\begin{vmatrix} 1 & a & -a \\ a & 2 & 3 \\ a & 1 & a \end{vmatrix}$. **6.** $\begin{vmatrix} 2 & x & x \\ 3 & -2 & 1 \\ x & -2 & 4 \end{vmatrix}$. **7.** $\begin{vmatrix} 1 & 2 & 3 \\ 3 & -1 & 4 \\ 1+x & 1-x & x \end{vmatrix}$.

8. $\begin{vmatrix} 1+x & 1 & 1 \\ 1 & 1+y & 1 \\ 1 & 1 & 1+z \end{vmatrix}$. **9.** $\begin{vmatrix} a^2+b^2 & 2ab & 1 \\ ab & a^2+b^2 & 1 \\ 1 & 1 & 1 \end{vmatrix}$.



10. The area of a triangle can be put in the form of a determinant if it is given by the coördinates of its three vertices.

Let $OE = x_1$, $EA = y_1$, $OD = x_2$, $DB = y_2$, $OF = x_3$, $FC = y_3$, and $A =$ area of triangle. Then

$$A = AEDB + ACFE - BCFD$$
$$= \tfrac{1}{2}\{(x_1 - x_2)(y_1 + y_2) + (x_3 - x_1)(y_3 + y_1) - (x_3 - x_2)(y_3 + y_2)\}$$
$$= \tfrac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$$ (as may be seen by expanding the determinant and the expression above.)

11. Calculate the area of the triangle whose coördinates are (2, 3), (6, 4), (8, 10); also that of the triangle whose coördinates are (3, − 2), (6, − 11), (4, − 9); also of triangle whose vertices are at (− 6, 4), (11, 12), (4, 0).

265. Exercise in the Solution of Simultaneous Equations Containing Three Unknowns. Solve the following equations, using the formulas of § 256 and calculating the values of the determinants as in the last article, and verify answers. (See § 255, 1.)

1. $\begin{cases} 2x+5y-3z=17, \\ 6x-2y-5z=-3, \\ 3x+7y+4z=-18. \end{cases}$

2. $\begin{cases} 2x+3y-2z=5, \\ 3x-2y+4z=16, \\ 4x+3y-z+5=0. \end{cases}$

3. $\begin{cases} 3p+4q+5r=10, \\ 4p-5q-3r=25, \\ 5p-3q-4r=21. \end{cases}$

4. $\begin{cases} 4m-12n-20p=9, \\ 8m-6n+10p=5, \\ 12m-18n-5p=13. \end{cases}$

266. Exercise in the Evaluation of Determinants of the Fourth Order. Referring to the values of a determinant of the fourth order as given in § 263, we may say that the first value given, viz., $a_1A_1 - b_1B_1 + c_1C_1 - d_1D_1$ is, as a rule, as convenient as any to use.

Calculate the values of the following, using the formula just given:

SOLUTION

1. $$\begin{vmatrix} 4 & 6 & 1 & 3 \\ 1 & 2 & 3 & 1 \\ 2 & 1 & 6 & 4 \\ 1 & 6 & 9 & 4 \end{vmatrix} = 4\begin{vmatrix} 2 & 3 & 1 \\ 1 & 6 & 4 \\ 6 & 9 & 4 \end{vmatrix} - 6\begin{vmatrix} 1 & 3 & 1 \\ 2 & 6 & 4 \\ 1 & 9 & 4 \end{vmatrix} + \begin{vmatrix} 1 & 2 & 1 \\ 2 & 1 & 4 \\ 1 & 6 & 4 \end{vmatrix} - 3\begin{vmatrix} 1 & 2 & 3 \\ 2 & 1 & 6 \\ 1 & 6 & 9 \end{vmatrix}$$

$$= 4(9) - 6(-12) + (-17) - 3(-18) = 145. \quad \text{(By § 264.)}$$

2. $\begin{vmatrix} 1 & 2 & 3 & 4 \\ 1 & 4 & 3 & 2 \\ 2 & 3 & 4 & 1 \\ 4 & 3 & 2 & 1 \end{vmatrix}$

3. $\begin{vmatrix} 2 & 6 & 3 & 9 \\ -1 & 8 & 1 & 3 \\ 2 & 6 & 1 & 8 \\ -1 & 2 & 1 & 4 \end{vmatrix}$

4. $\begin{vmatrix} 4 & 3 & 1 & 1 \\ 6 & 1 & 1 & 2 \\ 8 & 2 & 1 & 2 \\ 6 & 1 & 2 & 3 \end{vmatrix}$

5. $\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 19 & 3 & 17 \\ 33 & 39 & 8 & 38 \end{vmatrix}$ 6. $\begin{vmatrix} a & b & c & d \\ b & a & d & c \\ c & d & a & b \\ d & c & b & a \end{vmatrix}$ 7. $\begin{vmatrix} a & b & 0 & c \\ d & e & 0 & f \\ g & h & i & k \\ l & m & q & p \end{vmatrix}$

SUGGESTION TO EX. 7. Use the elements of column 3 and their minors in evaluating this determinant since two of these terms are 0.

267. Solution of Simultaneous Equations by Means of Determinants.

Let us take the system of four simultaneous equations:

$$\begin{aligned} &(1)\ a_1x + b_1y + c_1z + d_1u = f_1. \\ &(2)\ a_2x + b_2y + c_2z + d_2u = f_2. \\ &(3)\ a_3x + b_3y + c_3z + d_3u = f_3. \\ &(4)\ a_4x + b_4y + c_4z + d_4u = f_4. \end{aligned}$$

By determinants we can eliminate y, z, u in one operation by multiplying (1) by A_1, (2) by A_2, (3) by A_3, and (4) by A_4. Thus,

$$\begin{aligned} &(1_1)\ a_1A_1x + b_1A_1y + c_1A_1z + d_1A_1u = f_1A_1 \\ &(2_1)\ a_2A_2x + b_2A_2y + c_2A_2z + d_2A_2u = f_2A_2 \\ &(3_1)\ a_3A_3x + b_3A_3y + c_3A_3z + d_3A_3u = f_3A_3 \\ &(4_1)\ a_4A_4x + b_4A_4y + c_4A_4z + d_4A_4u = f_4A_4 \end{aligned}$$

Subtracting the sum of (2_1) and (4_1) from the sum of (1_1) and (3_1), we have

$$(a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4)x = f_1A_1 - f_2A_2 + f_3A_3 - f_4A_4,$$

since the coefficients of y, z, and u each equals 0, as will be explained immediately.

The coefficient of x, $a_1A_1 - a_2A_2 + a_3A_3 - a_4A_4$, is by § 263 the value of the determinant

$$\begin{vmatrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \\ a_4 & b_4 & c_4 & d_4 \end{vmatrix}$$

formed by taking the 16 coefficients of x, y, z, u in the given equations as they stand in the equations. We will call this determinant D. Notice that if $D = 0$, there is no solution. (See § 207, 2.)

The coefficients of y, z, and u, viz., $b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4$, $c_1A_1 - c_2A_2 + c_3A_3 - c_4A_4$, and $d_1A_1 - d_2A_2 + d_3A_3 - d_4A_4$ are each 0.

For, we can get $b_1A_1 - b_2A_2 + b_3A_3 - b_4A_4$ by the simple device of replacing the first column a_1, a_2, a_3, a_4 of D by the corresponding b's from the second column and then evaluating the determinant by the method of minors. But, whenever two columns of a determinant are *identical*, the value of the determinant equals 0 by § 259, 4. For like reason, coefficients of z and u are each 0.

Examining $f_1A_1 - f_2A_2 + f_3A_3 - f_4A_4$, we see that it can be obtained by replacing the a's of the first column of D by the corresponding f's in the right members.

It is clear that the method of this article can be applied to a determinant of any order. Hence we have the following:

Rule. *The value of an unknown obtained from n equations of the first degree containing n unknowns is a fraction whose denominator is the determinant of n rows and columns formed from the coefficients of the unknowns as they stand in the given equations, and whose numerator is obtained from its denominator by replacing the column of coefficients of the required unknown by the corresponding known right members of the given equations.*

268. Exercise in Solving Linear Equations. (See §§ 255, 265.)

1. $\begin{cases} x + 2y - 3z + u = 4 \\ 2x - y + 2z - 3u = 1 \\ 5x - 3y - z - 2u = 11 \\ 3x + 4y - 5z + 6u = -9 \end{cases}$

2. $\begin{cases} 2x - 3y + 4z - u = 4 \\ 4x + 2y - z + 2u = 13 \\ x - y + 2z + 3u = 17 \\ 3x + 2y - z + 4u = 20 \end{cases}$

269. Systems of Equations Containing Fewer Unknowns than Equations. Let us take the linear system

$$(1)\quad a_1x + b_1y + c_1 = 0$$
$$(2)\quad a_2x + b_2y + c_2 = 0$$
$$(3)\quad a_3x + b_3y + c_3 = 0.$$

In order that these equations may be *consistent*, that is, in order that the same values of x and y may satisfy all three equations, the values of x and y found from solving the system composed of (1) and (2) when substituted in (3) must satisfy it. Making this substitution, we have

$$a_3 \frac{\begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} + b_3 \frac{\begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix}}{\begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix}} + c_3 = 0,$$

$$\text{or, } a_3 \begin{vmatrix} -c_1 & b_1 \\ -c_2 & b_2 \end{vmatrix} + b_3 \begin{vmatrix} a_1 & -c_1 \\ a_2 & -c_2 \end{vmatrix} + c_3 \begin{vmatrix} a_1 & b_1 \\ a_2 & b_2 \end{vmatrix} = 0, \quad \text{(Mult. Ax.)}$$

$$\text{or, } \begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0. \quad \text{(By § 263.)}$$

This last equation is called the **eliminant** or **resultant** of the given system of equations. It expresses the condition that the given system is consistent.

The method here used for three equations containing two unknowns can be extended to n linear equations containing $n-1$ unknowns. Thus, *the determinant formed out of the coefficients and known terms of n linear equations containing $n-1$ unknowns must vanish in order that the given system may be consistent.*

The eliminant can often be determined by what is called Sylvester's **dialytic method** of elimination.

Let (1) $a_1x + b_1 = 0$, and (2) $a_2x^2 + b_2x + c_2 = 0$ be two consistent equations, or equations having the same value of x in both.

The condition for this consistency is easily found by substituting $x = -\frac{b_1}{a_1}$, in the second equation. This *eliminant* Sylvester found as follows. He multiplied both members of (1) by x, getting (3), and then had the three equations,

$$\begin{cases} (1) \quad a_1x + b_1 = 0 \\ (3) \quad a_1x^2 + b_1x = 0 \\ (2) \quad a_2x^2 + b_2x + c_2 = 0 \end{cases} \quad \text{whose eliminant is} \quad \begin{vmatrix} 0 & a_1 & b_1 \\ a_1 & b_1 & 0 \\ a_2 & b_2 & c_2 \end{vmatrix} = 0.$$

This method can be extended to equations of higher degrees.

270. Exercise in Shortening the Evaluation of Determinants. Before beginning the evaluation of a determinant the student should propose to himself the following question:

Are one or more of the elements of any row (see § 259, *a*) respectively equal to the corresponding elements of another row, or to the same multiples of the elements of another row? If so, substitute for the elements of one of the two rows the remainders after subtracting, or after multiplying and making them the same and then subtracting (§ 261, 2). Use, as a rule, the row or column with the most 0's to evaluate the determinant by use of its minors. See § 263 for sign of result.

1. $\begin{vmatrix} 2 & 8 & 6 \\ 1 & 2 & 3 \\ 4 & 6 & 9 \end{vmatrix}$ SOLUTION. The elements of the first row can have subtracted from them twice the corresponding elements of the second row. The elements of the third row can have subtracted from them three times the corresponding elements of the second row. In this way we get

$$\begin{vmatrix} 2-2 & 8-4 & 6-6 \\ 1 & 2 & 3 \\ 4-3 & 6-6 & 9-9 \end{vmatrix} = \begin{vmatrix} 0 & 4 & 0 \\ 1 & 2 & 3 \\ 1 & 0 & 0 \end{vmatrix} = -4\begin{vmatrix} 1 & 3 \\ 1 & 0 \end{vmatrix} = 12. \quad \textit{Ans.}$$

2. $\begin{vmatrix} 3 & 6 & 8 \\ 1 & 3 & 4 \\ 5 & 13 & 17 \end{vmatrix}$ SUGGESTION. Multiply the 2d row by 2 and subtract from the 1st row, thus getting new 1st row; multiply 2d row by 4 and subtract from 3d row for new 3d row.

3. $\begin{vmatrix} 1 & 7 & 6 \\ 2 & 8 & -2 \\ 3 & 12 & -3 \end{vmatrix}$ SUGGESTION. Multiply 1st column by 4 and subtract from 2d column for new 2d column. Then calculate by using the minors of the new 2d column elements.

4. $\begin{vmatrix} 1 & 4 & -2 \\ 2 & 8 & -6 \\ 1 & -3 & 2 \end{vmatrix}$ 5. $\begin{vmatrix} 19 & 13 & 16 \\ 12 & 6 & 9 \\ 1 & 1 & 1 \end{vmatrix}$ 6. $\begin{vmatrix} b+c & a & a \\ b & c+a & b \\ c & c & a+b \end{vmatrix}$

7. $\begin{vmatrix} 3 & 3 & 4 & 2 \\ -1 & 1 & 2 & 1 \\ -2 & 2 & 3 & 1 \\ 2 & 4 & 4 & 2 \end{vmatrix}$ 8. $\begin{vmatrix} 3 & 2 & 1 & 4 \\ 15 & 29 & 2 & 14 \\ 16 & 29 & 1 & 14 \\ 26 & 39 & 1 & 24 \end{vmatrix}$ 9. $\begin{vmatrix} a & v & g & p \\ e & a & f & p \\ v & v & a & p \\ d & c & b & a \end{vmatrix}$

10. $\begin{vmatrix} a+2b & a+3b & a+4b \\ a+3b & a+4b & a+5b \\ a+4b & a+5b & a+6b \end{vmatrix}$ 11. $\begin{vmatrix} (a+b)^2 & c^2 & c^2 \\ a^2 & (b+c)^2 & a^2 \\ b^2 & b^2 & (c+a)^2 \end{vmatrix}$

271. Miscellaneous Exercise in Determinants. Evaluate the determinants of this article in the best available way.

Solve the two following equations.

1. $\begin{cases} 3x+4y-2z=5, \\ 4x-3y+8z=-4, \\ 2x+8y-3z=5. \end{cases}$

2. $\begin{cases} 3x+y-z+2u=0, \\ -2x+3y+z-4u=21, \\ x-y+2z-3u=6, \\ 4x+2y-3z+u=12. \end{cases}$

3. A problem in the mixture of gases has the following equations. Solve the system.

$$\begin{cases} (1)\ x+y+z=a, \\ (2)\ 2x+3y+6z=b, \\ (3)\ 2x+2.5y+2.5z=c. \end{cases}$$

4. Field's process for the determination of chlorine, bromine, and iodine has the following equations. Solve them.

$$\begin{cases} (1)\ x+y+z=a, \\ (2)\ 1.31x+y+z=b, \\ (3)\ 1.637x+1.25y+z=c. \end{cases}$$

5. In § 264, 10, it was shown that if (x_1, y_1), (x_2, y_2), (x_3, y_3) are the coördinates of three points, then

$\frac{1}{2}\begin{vmatrix} x_1 & y_1 & 1 \\ x_2 & y_2 & 1 \\ x_3 & y_3 & 1 \end{vmatrix}$ equals the area of the triangle which has these points as vertices. Evidently, then, when the three points lie in a straight line, the area of the triangle equals zero, and the determinant equals zero. *Thus the determinant equal to zero is the condition that the points should lie on a straight line.*

Find whether the following sets of points lie on a straight line, or whether they form a triangle, and if the latter, what is the area of the triangle: (2, 3), (4, 0), (– 2, 9); (0, – 4), (7.5, 2), (3, – 1.6); (1, 7), (2, 6), (– 1, – 4).

6. Find the area of the quadrilateral whose vertices are at (4, 4), (3, 6), (– 2, 5), and (3, – 5) by finding the areas of the two triangles whose vertices are the first, second, and third points, and the third, fourth, and first points, and adding them.

272. Historical Notes. — Determinants were called "*resultants*" by La Place, but Gauss changed the name to the one by which they are now known. The works of Leibnitz, the co-developer with Sir Isaac Newton of the Calculus, contain the germ of the idea of determinants. The first satisfactory description of these functions was published in 1772. La Place, Lagrange, and Gauss aided in developing the subject, but the general theory of determinants was worked out first by Cauchy. Cramer gave the rule by inversions of subscripts for the signs of the terms. In later times Binet in France, Jacobi in Germany, and Cayley and Sylvester in England, had much to do with extending the boundaries of this subject.

James Joseph Sylvester (see frontispiece) was born in London in 1814 and died in the same city in 1897. In 1837 he was second "wrangler" in the "Tripos" examination at St. John's College, Cambridge, but being a Jew, and unwilling to subscribe to the Thirty-nine Articles, he could not graduate and was ineligible for a fellowship. However, he got his degree from Trinity College, Dublin. Soon after leaving Cambridge, he was appointed to the chair of Natural Philosophy at University College, London, where he was associated with his friend, the celebrated professor of mathematics, Augustus DeMorgan. In 1840 he came to America to take the chair of mathematics in the University of Virginia. Here he remained only six months because the expression of his views on slavery got him into trouble. In 1844 he became actuary of an English Insurance Company. In 1855 he was elected to the chair of mathematics in the Royal Military Academy at Woolwich, where he remained till 1870. In the year 1878 he again came to America, this time to take the chair of mathematics in the Johns Hopkins University at Baltimore, then just opened, at a salary of $5000 in gold. Johns Hopkins being chiefly a graduate school, drawing its students from colleges and universities all over the country, Sylvester was able to give a remarkable impetus to the study of higher mathematics in America. In 1883 Sylvester returned to England to take the Savilian professorship of mathematics in Oxford University, where he remained until 1893.

Sylvester published articles dealing with algebra in general, with determinants, partitions, elimination, the theory of equations, matrices, universal algebra, invariants, and reciprocants, as also studies in geometry and mechanics. He was the first editor of the *American Journal of Mathematics*, to which he contributed thirty papers, some of great length. When the history of American mathematics comes to be written, the name of Sylvester will occupy a very prominent and honorable place.

CHAPTER XVIII

LIMITS — INFINITE SERIES

I. LIMITS

273. Limits. The examples of a limit probably most familiar to the student are those of a circle as the limit of the area of an inscribed regular polygon when the number of its sides is indefinitely increased, and of its circumference as the limit of the perimeter of the polygon.

Another example is that of $\frac{2}{3}$ as the limit towards which the terms of the sequence .6, .66, .666, .6666, ⋯ tend.

Still another example is that of 2 as the limit toward which the terms of the sequence

$$1,\ 1+\tfrac{1}{2},\ 1+\tfrac{1}{2}+\tfrac{1}{4},\ 1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8},\ 1+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8}+\tfrac{1}{16},\ \cdots \text{ tend.}$$

From these examples we may get the following definition.

Definition of a Limit. *If a variable **x** assumes a given sequence of values such that the numerical difference between a constant **a**, and the variable **x** becomes and remains less than any assignable quantity, however small, then **x** is said to approach **a** as its limit.*

To express this relation we write $x \doteq a$, read "x approaches the limit a," or $\lim x = a$ (§ 208).

274. Limit of a Ratio. In § 207 it was shown that when $x = 1$,

$$\frac{x^2-1}{x-1} = \frac{0}{0} = x + 1 = 2. \quad \left(\text{Since } \frac{x^2-1}{x-1} = x + 1.\right)$$

To get a clear idea of what this means, we think of x as having a value very close to 1; then it is easy to find by calculation that $\frac{x^2-1}{x-1}$ has a value close to 2; also the closer we take x to 1, the closer does the given function of x approach to 2. Thus, by bringing x sufficiently close to 1, $\frac{x^2-1}{x-1}$ can be made to differ *as little as we please from* 2, and continue to differ by as little as we please. Hence, by the definition in the preceding article, 2 is the limit of $\frac{x^2-1}{x-1}$ as x approaches the limit 1. This truth is expressed by writing

$$\lim_{x \doteq 1} \frac{x^2-1}{x-1} = 2.$$

If the numerator x of a fraction is constant or approaches a constant a as a limit $(a \neq 0)$, while the denominator y approaches the limit 0, then the value of the fraction is said to become **infinite**. If $\frac{x}{y}$ is the fraction, then $\lim_{y \doteq 0} \frac{x}{y} = \infty$.

The symbol ∞ does not denote a limit, neither does it denote a definite number. The equation $\lim u = \infty$ should not be read "u approaches infinity," and $u = \infty$ should not be read "u equals infinity," but each equation should be read "u becomes infinite," or "u increases without limit."

Instead of a variable x approaching in value a finite quantity, as in the first example above, it may become and remain ∞.

Thus $$\lim_{x=\infty} \frac{x+1}{x} = 1$$

since $\frac{x+1}{x} = 1 + \frac{1}{x}$, and it is clear by the definition of a limit that $\lim_{x=\infty}\left(1+\frac{1}{x}\right)$ is 1 since $\lim_{x=\infty} \frac{1}{x} = 0$.

275. Theorems Concerning Limits. The explanations given for one or two of the theorems following will probably be as satisfactory to the mind of the student as the formal proofs usually given. For this reason the formal proofs are not given.

1. *The limit of the sum of two functions of x equals the sum of their limits, provided the latter do not take the form $\infty - \infty$.*

For, if x and y are two variables and a and b the respective limits towards which the variables tend, the sum of $a-x$ and $b-y$, or, $(a+b)-(x+y)$, can be made to become and remain numerically less than any assignable quantity, however small, since each can be made as small as we please.

2. *The limit of the product of a constant and a variable equals the constant times the limit of the variable.*

For, if $x=a-u$, x being the variable and a its limit, and m is a constant, then

$$\lim mx = \lim ma - \lim mu. \quad \text{(By 1, above.)}$$

Now, since by definition of a limit, u can be made as small as we please, $\lim mu = 0$. Hence $\lim mx = ma$.

3. *The limit of the product of two variables each of which approaches the limit 0 is 0.*

This may be taken as self-evident.

4. *The limit of the product of two functions of x is equal to the product of their limits.*

PROOF. Let a and b be the two limits, and x and y respectively the variables approaching them, and $a-x=u$ and $b-y=v$, whence $u \doteq 0$, and $v \doteq 0$.

Then,
$$\begin{aligned} \lim xy &= \lim (a-u)(b-v) \\ &= \lim (ab - av - bu + uv) \\ &= ab. \quad \text{(By 1, 2, and 3 above)} \end{aligned}$$

5. *The limit of the quotient of two variables equals the quotient of their limits.*

Using the notation of the preceding proof, we have

$$\frac{x}{y} = \frac{a-u}{b-v} \equiv \frac{a}{b-v} - \frac{u}{b-v}$$

$$\equiv \frac{a}{b} + \frac{av}{b(b-v)} - \frac{u}{b-v}.$$

Then, $\lim \frac{x}{y} = \frac{a}{b}$. (By 1 and 4 above.)

a. The doctrine of limits can be made to include complex number (§ 83) values as well as real values. Thus, if a is a constant and a variable $f(x)$ assumes a given sequence of values such that $|a - f(x)|$ (the absolute value, § 88, of $a - f(x)$) becomes and remains less than any assignable number, however small, then $f(x)$ is said to approach a as its limit.

In this case a is a point in Argand's diagram and $f(x)$ is a variable point approaching it.

276. Indeterminate Forms. (See § 207.) The principal indeterminate forms are $\infty - \infty$, $0 \times \infty$, $0 \div 0$, $\infty \div \infty$. It can be shown, since $\frac{1}{0} = \infty$, that these may all be made to depend on $0 \div 0$. Thus, $0 \times \infty$ may be written $0 \times \frac{1}{0}$, and $\infty \div \infty$ may be written $\frac{1}{0} \div \frac{1}{0}$.

The form $\frac{0}{0}$ will always result from putting $x = 0$ in evaluating a rational fraction if the absolute terms, or terms which do not contain x in numerator and denominator, are absent. To evaluate such a quantity, arrange the terms in numerator and denominator in ascending order; next, reduce the fraction to its lowest terms by dividing by some power of x; then put $x = 0$ in the resulting fraction.

If the lowest terms in both numerator and denominator are of the *same* degree, the limit will be finite and $\neq 0$; if the term of lowest degree is in the *numerator*, the limit will be ∞; and if the term of lowest degree is in the *denominator*, it will be 0.

Thus, $$\lim_{x \doteq 0} \frac{4x - 5x^2 + 2x^3}{3x - 7x^4} = \lim_{x \doteq 0} \frac{4 - 5x + 2x^2}{3 - 7x^3} = \frac{4}{3};$$

$$\lim_{x \doteq 0} \frac{3x^2 - 7x^3 - x^5}{5x^4 - 4x^5 - 2x^6} = \lim_{x \doteq 0} \frac{3 - 7x - x^3}{5x^2 - 4x^3 - 2x^4} = \frac{3}{0} = \infty;$$

$$\lim_{x \doteq 0} \frac{x^3 - 2x^4}{x - 5x^3} = \lim_{x \doteq 0} \frac{x^2 - 2x^3}{1 - 5x^2} = \frac{0}{1} = 0.$$

277. Exercise. Evaluate the following expressions:

1. $\lim_{x \doteq 0} \frac{2x^3 - 3x^2 - 7x}{5x^2 - 11x}$.

2. $\lim_{x \doteq 0} \frac{2x^2 + x^4}{3x^3 + x^5}$.

3. $\lim_{x = \infty} \frac{4x^2 + 5x^4}{2x + 6x^4}$.

4. $\lim_{x \doteq 0} \frac{4x^2 - 3x^3}{5x + 11x^4}$.

5. $\lim_{x = \infty} \frac{2x - 7x^5}{x - 3x^3 + 4x^4}$.

6. $\lim_{x = \infty} \frac{12x - 17x^2}{15x - 18x^3}$.

II. INFINITE SERIES

278. Definition of Series. A **series** is a sequence of terms formed in accordance with some law and connected by the signs of addition or subtraction.

If the number of its terms is *finite*, the series is called a **finite series**; and if the number of terms is *infinite*, the series is called an **infinite series.**

The symbol Σ (read "summation of" or "summation of from number below sign to number above it") is often used to denote a series, Σ being the Greek letter for S, the initial of "sum."

Thus, $$\sum u_n = u_1 + u_2 + u_3 + \cdots + u_n.$$

and $$\sum_{n=1}^{n=\infty} u_n = u_1 + u_2 + u_3 + \cdots \text{ to infinity.}$$

An expanded *determinant* is of the form $\Sigma \pm a_1 b_2 c_3 \cdots$.

279. Convergency and Divergency of Series. A series is said to be **convergent** when the sum of the first n terms approaches a fixed limit as n is increased indefinitely.

A series is said to be **divergent** if it is not convergent.

A nonconvergent series in which the sum of n terms though always finite does not approach a determinate limit is called an **oscillating** series.

1. If $\frac{1}{1-x}$ is reduced to a series by dividing the numerator by the denominator as in long division, we get

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \cdots$$

Is the series in the right member convergent or divergent?

SOLUTION. If $x = \frac{1}{2}$, the left member of this equation $= 2$, and the right member approaches the limit 2 as more and more terms are included.

If $x = 2$, the left member $= -1$, while the right member increases in value indefinitely, as more and more terms are taken.

If $x = -1$, the series becomes $1 - 1 + 1 - 1 + \cdots$, and its value is alternately 0 and 1, thus being an oscillating series.

We have here, then, an example of a series *convergent* for one value of the letter contained in it, *divergent* for another value, and *oscillating* for a third.

2. The series $1 + x + x^2 + x^3 + \cdots$ is evidently a geometrical progression whose ratio is x. Now if $x < 1$, that is, if x differs from 1 by a finite amount, $\lim x^n = 0$, because if a proper fraction is multiplied by itself in a continued product, the multiplicand is each time diminished by a finite fraction of itself, and the limit of such a product is evidently 0.

Then, by § 184, the sum of $1 + x + x^2 + x^3 + x^4 + \cdots$ to infinity is $\frac{1}{1-x}$. Now, if $x < 1$, $\frac{1}{1-x}$ is a finite number, and hence the series converges to a finite sum.

a. The equation in 1 above becomes identically true at any time, of course, by annexing to the terms of the quotient the remainder over the divisor.

280. Tests for Convergency and Divergency of Series.

1. **Comparison Test for Convergency.** *If the terms of one series after a certain finite number of terms are equal to or less than the corresponding terms of another series of positive terms which is known to be convergent, the first series is convergent.*

The truth of this theorem may be taken as apparent, though proofs are often given.

To show that the first series on p. 303 is convergent we write under it the convergent progression of § 184. (See § 279, 2.)

$$2 + 1 + \frac{1}{2^2} + \frac{1}{3^3} + \frac{1}{4^4} + \frac{1}{5^5} + \cdots \frac{1}{(n-1)^{n-1}} + \cdots$$

$$1 + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \frac{1}{2^4} + \frac{1}{2^5} + \cdots \frac{1}{2^{n-1}} + \cdots$$

Here we observe that every term of the given series after the third is less than the term below it. To show that this is true generally, we will compare the nth terms of the two series. We have, if $n > 3$,

$$\frac{1}{(n-1)^{n-1}} < \frac{1}{2^{n-1}}.$$

2. *A series with terms of different signs is convergent if the series formed from it by making all the terms positive is convergent.*

3. **Comparison Test for Divergency.** *If the terms of one series of positive terms after a certain finite number of terms are equal to or greater than the corresponding terms of a known divergent series of positive terms, then the given series is divergent.*

Thus, the harmonic* series $1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots$ is divergent. For, taking two of its terms, as, $\frac{1}{3}$ and $\frac{1}{4}$, then the next four of its terms, then the next eight of its terms, and so on, we see that

$$\frac{1}{3} + \frac{1}{4} > \frac{1}{4} + \frac{1}{4} = \frac{1}{2};\ \frac{1}{5} + \frac{1}{6} + \frac{1}{7} + \frac{1}{8} > 4 \times \frac{1}{8} = \frac{1}{2};$$

$$\frac{1}{9} + \frac{1}{10} + \cdots + \frac{1}{16} > 8 \times \frac{1}{16} = \frac{1}{2};\ \text{etc.}$$

$$\therefore\ 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \cdots > 1 + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots.$$

Now the right member of this inequation can be made greater than any finite number, however great, by taking enough terms of the series, so that the right member series is divergent.

But the left member series, or the given series, is greater than the right member series. Hence it is divergent.

* A harmonic progression is one the reciprocals of whose terms are in arithmetical progression. Thus, the reciprocals of $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4} \cdots$ are $1, 2, 3, 4, \cdots$. The word harmonic was chosen because if a set of vibrating strings of uniform tension having lengths proportional to $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}$ are sounded together, they produce harmony.

4. Ratio Test for Convergency or Divergency. *If, in the series* $u_1 + u_2 + u_3 + \cdots + u_n + u_{n+1} + \cdots$, *the ratio* $\frac{u_{n+1}}{u_n}$ *approaches a limit* k *as* n *increases without limit, then the series is convergent if* $k < 1$, *and divergent if* $k > 1$.

PROOF. Let $S = u_1 + u_2 + u_3 + u_4 + \cdots$, and suppose $\frac{u_{n+1}}{u_n} \doteq k < 1$, $\frac{u_{n+2}}{u_{n+1}} \doteq k < 1$, and so on.

Then $$S = u_1 + u_2 + u_3 + \cdots + u_n[1 + \frac{u_{n+1}}{u_n} + \frac{u_{n+2}}{u_{n+1}} \cdot \frac{u_{n+1}}{u_n} + \frac{u_{n+3}}{u_{n+2}} \cdot \frac{u_{n+2}}{u_{n+1}} \cdot \frac{u_{n+1}}{u_n} + \cdots.]$$

Now, because $\frac{u_{n+1}}{u_n} \doteq k$; $\frac{u_{n+2}}{u_{n+1}} \cdot \frac{u_{n+1}}{u_n} \doteq k^2$; etc.,

$$S \doteq u_1 + u_2 + u_3 + \cdots + u_n[1 + k + k^2 + k^3 + \cdots]$$

or, $$S \doteq u_1 + u_2 + u_3 + \cdots + u_n\left(\frac{1}{1-k}\right). \quad (\S\ 279.)$$

But we saw in § 279, 2, that $\frac{1}{1-k}$ converges to a finite number when $k < 1$. This makes the right member of the last statement a finite number. Hence S is convergent when the ratio is less than k. But when $k > 1$, $\frac{1}{1-k}$ is divergent, which makes the right member of the statement itself divergent, since S approaches $u_1 + u_2 + \cdots + \frac{u_n}{1-k}$ as its limit.

In $1 + \frac{1}{2} + \frac{1}{3} + \cdots$ (divergent), and $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \cdots$ (convergent), $k \doteq 1$. Thus, $k \doteq 1$ may give either convergent or divergent value.

1. The series $1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$ is convergent, because

$$\frac{1}{(n+1)!} \div \frac{1}{n!} = \frac{n!}{(n+1)!} = \frac{1}{n+1},$$

and $n > 0$ gives a ratio less than a number which is less than 1.

2. Series $\frac{1}{1} + \frac{2}{2} + \frac{2^2}{3} + \frac{2^3}{4} + \cdots + \frac{2^{n-1}}{n} + \cdots$ is divergent, because

$$\frac{2^{n-1}}{n} \div \frac{2^{n-2}}{n-1} = \frac{2(n-1)}{n} > 1, \text{ when } n > 2.$$

281. Power Series. The series

$$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots + a_nx^n + \cdots,$$

in which the coefficients are independent of x, is called a power series.

Evidently the ratio test for convergency is easy to apply to a power series. If a power series is convergent for $x = b$, it is convergent for every value of x less than b.

282. Convergency or Divergency of the Binomial Series. In the binomial formula § 194 we put $a = 1$, $b = x$. Then, the ratio of the $(r+2)$th term to the $(r+1)$th is

$$\frac{\dfrac{n(n-1)(n-2)(n-3)\cdots(n-r)}{1\times 2\times 3\times 4\times\cdots(r+1)}x^{r+1}}{\dfrac{n(n-1)(n-2)(n-3)\cdots(n-r+1)}{1\times 2\times 3\times 4\times\cdots\times r}x^{r}}, \text{ or } \frac{(n-r)x}{r+1}.$$

NOTE. The formula, § 194, gives the rth term, and the last factor in the numerator is $n-r+2$, and the last in the denominator is $r-1$. If the $(r+2)$th term is found, the last term in the numerator becomes $n-r$, and the last in the denominator $r+1$. The $(r+1)$th term in the denominator is found by reasoning in the same way.

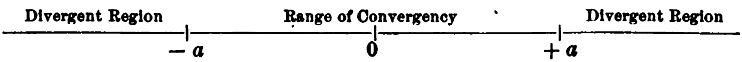
Now, $$\lim_{r=\infty}\frac{n-r}{r+1}x = \lim_{r=\infty}\frac{\dfrac{n}{r}-1}{1+\dfrac{1}{r}}x = -x.$$

hence, by the ratio test, if x is numerically less than 1, or if $-1 < x < 1$, the series is convergent.

A quantity $(a+x)^n$ can be written $a^n\left(1+\dfrac{x}{a}\right)^n$.

This expression by the proof just given is convergent if $-1 < \dfrac{x}{a} < 1$; that is, if $-a < x < a$.

If distances from 0 on a horizontal line represent numerical values, we have on it for ranges of convergency and divergency



283. Exponential Series. The power series

$$1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots$$

is called the **exponential series.**

To find whether it is convergent we use the ratio test, dividing the $(n+1)$th term by the nth. Thus,

$$\frac{x^n}{n!}\div\frac{x^{n-1}}{(n-1)!}=\frac{x}{n}<1 \text{ for all values of } n>x.$$

Thus, the exponential series is convergent for finite values of x.

Binomial expansion and passing to limits by the calculus gives

$$\lim_{m=\infty}\left(1+\frac{1}{m}\right)^{mx}$$

$$=\lim_{m=\infty}\left[1+\frac{mx}{m}+\frac{mx(mx-1)}{2!}\frac{1}{m^2}+\frac{mx(mx-1)(mx-2)}{3\,!}\frac{1}{m^3}+\cdots\right]$$

$$=1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\cdots+\frac{x^n}{n!}+\cdots. \quad (\S\,276.)$$

Thus, the left member above is an expression for the exponential series. The expression obtained by putting $x=1$ in the left member, or $\lim_{m=\infty}\left(1+\frac{1}{m}\right)^m$ is commonly denoted by the letter e.

Then, $$e^x=1+\frac{x}{1}+\frac{x^2}{2!}+\frac{x^3}{3!}+\frac{x^4}{4!}+\frac{x^5}{5!}+\cdots.$$

From this equation we see why the power series of this article is called the exponential series.

Putting $x=1$ in the last equation, we get

$$e=1+\tfrac{1}{1}+\tfrac{1}{2}+\tfrac{1}{6}+\tfrac{1}{24}+\tfrac{1}{120}+\cdots$$
$$=2.71828 \text{ (correct to five decimal places).}$$

284. Logarithmic Series. The power series

$$x-\frac{x^2}{2}+\frac{x^3}{3}-\cdots+(-1)^{n-1}\frac{x^n}{n}\pm\cdots$$

is called the **logarithmic series.**

Dividing the last term written by the one which precedes it, we have

$$\frac{x^n}{n} \div \frac{x^{n-1}}{n-1}, \text{ or } \frac{(n-1)x}{n}.$$

Then, $$\lim_{n=\infty} \frac{(n-1)x}{n} = x. \qquad (§ 275.)$$

By the ratio test this series is convergent when $-1 < x < 1$.

285. Calculation of Logarithms. The logarithmic series can be used to calculate logarithms. It is shown in the calculus that the logarithmic series converges to $\log_e(1+x)$ when $-1 < x < 1$.

Then, $$\log_e(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots.$$

To obtain a formula converging more rapidly, we proceed as follows. Substituting $-x$ for x in the formula just written, we have

$$\log_e(1-x) = -x - \frac{x^2}{2} - \frac{x^3}{3} - \frac{x^4}{4} - \cdots$$

$$\therefore \log_e(1+x) - \log_e(1-x) = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right), \quad \text{(Sub. Ax.)}$$

or, $$\log_e \frac{1+x}{1-x} = 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \cdots\right).$$

We now put $\dfrac{1+x}{1-x} = \dfrac{m+1}{m}$; whence, $x = \dfrac{1}{2m+1}$.

Then, $$\log_e \frac{m+1}{m} = 2\left(\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \frac{1}{5(2m+1)^5} + \cdots\right),$$

or,

$$\log_e(m+1) = \log_e m + 2\left(\frac{1}{2m+1} + \frac{1}{3(2m+1)^3} + \frac{1}{5(2m+1)^5} + \cdots\right).$$

By writing $m = 1$, $m = 2$, $\cdots$ in this formula, we get

$$\log_e 2 = 0 + 2\left(\frac{1}{3} + \frac{1}{81} + \frac{1}{1215} + \cdots\right) = 0.6931,$$

$$\log_e 3 = 0.6931 + 2\left(\frac{1}{5} + \frac{1}{375} + \frac{1}{15625} + \cdots\right) = 1.0986.$$

. . . .

Ordinary logarithms can be calculated from **natural logarithms** or **Napierian logarithms** (*i.e.* logarithm to base e) by means of the following theorem.

Theorem. *The logarithm of any number to base b equals the logarithm of this number to base c divided by the logarithm of b to base c.*

PROOF. Let n be any number, and x its logarithm to base b.

Then, $b^x = n$. (By definition of logarithm, § 159.)

$$\therefore x = \frac{\log_c n}{\log_c b}, \quad \text{(By § 175.)}$$

or

$$\log_b n = \frac{\log_c n}{\log_c b}. \quad \text{Q. E. D.}$$

Now, let $c = e$, and $b = 10$. Then

$$\log_{10} n = \frac{\log_e n}{\log_e 10}.$$

Also let $n = 10$, $c = 10$, and $b = e = 2.71828$. (§ 283.)

Then

$$\log_e 10 = \frac{\log_{10} 10}{\log_{10} 2.71828}$$

$$= \frac{1}{.4343} = 2.303.$$

Hence to get a common from a natural logarithm, divide the latter by 2.303 or multiply it by .4343 and *vice versa.*

Thus, $\log_{10} 2 = .6931 \times .4343$ (see value .6931 given p. 307)
$= .3010$ (as in the ordinary log table).

The number .4343 is called the *modulus* of the common system of logarithms.

III. THE FINITE DIFFERENCE METHOD

286. The Finite Difference Method finds the successive differences between the terms of the given series * (called the differences of the *first* order); then the successive differences (called the second order differences) between the terms of the series formed out of the differences of the first order; and so on with third and fourth, etc., orders of differences.

This method can be used to find any particular term and also the sum of a finite number of terms of any given series.

* The word series is here used, as it often is, to mean a *sequence*.

Thus, if given series is	1^3,		2^3,		3^3,		4^3,		5^3,		6^3,	...
or	1,		8,		27,		64,		125,		216,	...
Then, 1st order of differences is		7,		19,		37,		61,		91		
the 2d order of differences is			12,		18,		24,		30			
the 3d order of differences is				6,		6,		6				
the 4th order of differences is					0,		0					

287. To find the First Term of any Order of Differences. Let the series be $a_1, a_2, a_3, a_4, \cdots a_n$. Then by simple subtraction we get the several orders of differences as follows:

1st order $\quad a_2 - a_1, \quad a_3 - a_2, \quad a_4 - a_3, \quad a_5 - a_4, \quad a_6 - a_5, \cdots$
2d order $\quad a_3 - 2a_2 + a_1, \quad a_4 - 2a_3 + a_2, \quad a_5 - 2a_4 + a_3, \cdots$
3d order $\quad a_4 - 3a_3 + 3a_2 - a_1, \quad a_5 - 3a_4 + 3a_3 - a_2, \cdots$
4th order $\quad a_5 - 4a_4 + 6a_3 - 4a_2 + a_1, \cdots$

. . . .

The quantities pointed off by commas in the different orders of differences are called *terms*. Evidently these terms have for their coefficients the binomial coefficients, with the signs alternately + and −. Thus, the terms of the second order of differences have the coefficients for the second power of a binomial, or 1, −2, 1; the terms of the third order of differences have the coefficients for the cube of a difference, or 1, −3, 3, −1; and the terms of the fourth order of differences the coefficients of the fourth power of a binomial, or 1, −4, 6, −4, 1.

That this rule holds generally can be proved by mathematical induction as in § 191. Then if d_n denotes the first term of the nth order of differences,

$$d_n = a_{n+1} - na_n + \frac{n(n-1)}{2!}a_{n-1} + \frac{n(n-1)(n-2)}{3!}a_{n-2} - \text{, etc.}$$

Example. Find the first term of the 3d order of differences of $1^2, 2^2, 3^2, 4^2, \cdots$.

Solution. $d_3 = 16 - 3 \times 9 + \frac{3 \times 2}{2} \times 4 - \frac{3 \times 2 \times 1}{3!} \times 1 = 0.$

288. To find the *n*th Term of a Series. Let d_1, d_2, d_3, ⋯ denote the first terms of the first, second, third, ⋯ orders of differences. From the preceding article we get the left column equations below. The right column equations are obtained by substituting values from the left column after transposing.

$$d_1 = a_2 - a_1 \qquad a_2 = a_1 + d_1$$
$$d_2 = a_3 - 2a_2 + a_1 \qquad a_3 = a_1 + 2d_1 + d_2$$
$$d_3 = a_4 - 3a_3 + 3a_2 - a_1 \qquad a_4 = a_1 + 3d_1 + 3d_2 + d_3$$
$$d_4 = a_5 - 4a_4 + 6a_3 - 4a_2 + a_1 \qquad a_5 = a_1 + 4d_1 + 6d_2 + 4d_3 + d_4$$

From the values given at the right we see that the value of a_3, or the third term of the given series, has the first term of the given series for its first term and the several first terms of the different orders of differences arranged with the binomial coefficients of the second power; the fourth term a_4 has a similar value, but with the binomial coefficients of the third power; and so on.

Generalizing the result (a proof by mathematical induction can be given), we see that the nth term will have the binomial coefficients of the $(n-1)$th power. Thus,

$$a_n = a_1 + (n-1)d_1 + \frac{(n-1)(n-2)}{2!}d_2 + \frac{(n-1)(n-2)(n-3)}{3!}d_3 + \cdots.$$

Example. Find the 10th term of the series 1, 5, 15, 35, 70, 126, ⋯.

Solution.	1,	5,	15,	35,	70,	126
1st order of differences	4,	10,	20,	35,	56	
2d order of differences	6,	10,	15,	21		
3d order of differences	4,	5,	6			
4th order of differences	1,	1				
5th order of differences	0					

Substituting in the formula for a_n, we have $d_1 = 4$, $d_2 = 6$, $d_3 = 4$, $d_4 = 1$, $d_5 = 0$. Then,

$$a_{10} = 1 + 9 \times 4 + \frac{9 \times 8}{2} \times 6 + \frac{9 \times 8 \times 7}{2 \times 3} \times 4 + \frac{9 \times 8 \times 7 \times 6}{2 \times 3 \times 4} \times 1 = 715. \quad \textit{Ans.}$$

289. To find the Sum of *n* Terms of a Series, a_1, a_2, a_3, ⋯ a_n. To find the sum of this series we write down first an auxiliary series as follows:

$$0,\ a_1,\ a_1 + a_2,\ a_1 + a_2 + a_3,\ a_1 + a_2 + a_3 + a_4,\ \cdots,\ a_1 + a_2 + \cdots + a_n.$$

Getting the first order of differences of this series, we have $a_1, a_2, a_3, a_4, a_5, \cdots a_n$.

But the series $0, a_1, a_1 + a_2, \cdots$ evidently has for its *fourth* term the *sum* of the series a_1, a_2, a_3; and for its *fifth* term the *sum* of the series a_1, a_2, a_3, a_4; and so on; and for its $(n+1)$th term the *sum* of the series $a_1, a_2, a_3, a_4, a_5, \cdots a_n$. The preceding article tells us how to find the nth term of any series; hence to find the sum of n terms of the series $a_1, a_2, a_3, \cdots a_n$, we must find the $(n+1)$th term of the series $0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \cdots$.

It should be noted that the *second* order of differences of the auxiliary series is the *first* order of differences of the series whose sum is to be found; the *third* order of differences of the auxiliary series is the *second* order of the series whose sum is to be found; and so on.

In the light of the preceding statements we see that to find the sum S_n of n terms of the series $a_1, a_2, a_3, \cdots a_n$ we must make the following substitutions in the formula of § 288:

$n+1$ for n; n for $n-1$; $n-1$ for $n-2$; and so on: 0 for a_1; a_1 for d_1; d_1 for d_2; and so on.

Substituting these values, we have

$$S_n = 0 + na_1 + \frac{n(n-1)}{2!}d_1 + \frac{n(n-1)(n-2)}{3!}d_2 + \cdots.$$

EXAMPLE. Find the sum of n terms of the series of $1^2, 2^2, 3^2, 4^2, \cdots$.

SOLUTION.

$$\begin{array}{ccccccccc} 1, & & 4, & & 9, & & 16, & & 25 \\ & 3, & & 5, & & 7, & & 9 & \\ & & 2, & & 2, & & 2 & & \\ & & & 0, & & 0 & & & \end{array}$$

Here we have $a_1 = 1$, $d_1 = 3$, $d_2 = 2$, $d_3 = 0$. Then,

$$S_n = n + \frac{n(n-1)}{2} \times 3 + \frac{n(n-1)(n-2)}{6} \times 2$$

$$= (6n + 9n^2 - 9n + 2n^3 - 6n^2 + 4n) \div 6$$

$$= (2n^3 + 3n^2 + n) \div 6 = \frac{n(n+1)(2n+1)}{6}. \quad \textit{Ans.}$$

290. Exercise on Limits and Series. Evaluate the following expressions:

1. $\lim\limits_{x=0} \dfrac{4x^2 - 7x^3 + x^4}{3x^2 - 2x^4}$. 2. $\lim\limits_{x=\infty} \dfrac{2x^4 - 7x^2}{4x^4 - 5x^2}$.

3. Show that $\lim\limits_{n=\infty} \dfrac{n^2}{(n+1)^2} = 1$.

4. Prove that $1 + \frac{1}{2} + \frac{1}{2 \cdot 3} + \frac{1}{2 \cdot 3 \cdot 4} + \cdots$ is convergent by comparing it with the convergent geometric series

$$1 + \frac{1}{2} + \frac{1}{2 \cdot 2} + \frac{1}{2 \cdot 2 \cdot 2} + \cdots.$$

5. Show that $\frac{1}{3} + \frac{2}{3^2} + \frac{3}{3^3} + \frac{4}{3^4} + \cdots$ is convergent by the ratio method.

6. Show that $\frac{1}{1 \cdot 2} + \frac{1}{2 \cdot 3} + \frac{1}{3 \cdot 4} + \cdots$ converges to unity.

Suggestion. Since, *e.g.* $\frac{1}{2 \times 3} = \left(\frac{1}{2} - \frac{1}{3}\right)$, the series may be written:

$$\left(1 - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{n} - \frac{1}{n+1}\right) + \cdots.$$

Canceling terms, we get $1 - \frac{1}{n+1}$. Find the value of this expression when $n = \infty$.

7. Show that $\frac{1}{2^2} + \frac{1}{3^2} + \frac{1}{4^2} + \cdots$ is convergent.

8. Show that $\frac{1}{2} + \frac{1}{4} + \frac{1}{6} + \frac{1}{8} + \cdots$ is divergent.

Solution. $\frac{1}{2} + \frac{1}{4} > \frac{1}{4}$; $\frac{1}{6} + \frac{1}{8} > \frac{1}{4}$; $\frac{1}{10} + \frac{1}{12} + \frac{1}{14} + \frac{1}{16} > \frac{1}{4}$; $\frac{1}{18} + \frac{1}{20} + \frac{1}{22} + \frac{1}{24} + \frac{1}{26} + \frac{1}{28} + \frac{1}{30} + \frac{1}{32} > \frac{1}{4}$. Now $\frac{1}{4} + \frac{1}{4} + \frac{1}{4} + \frac{1}{4} \cdots$ is divergent. Hence, $\cdots$

9. Show that for no finite value of x is

$$\frac{1}{x} + \frac{1}{x+1} + \frac{1}{x+2} + \cdots \text{ convergent.}$$

Suggestion. Let $x = 0, 1, 2, \cdots$ and compare with a known divergent series.

10. Show that $\frac{1}{3!}+\frac{1}{5!}+\frac{1}{7!}+\cdots+\frac{1}{(2n+1)!}+\cdots$ is convergent.

11. Show that $\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{4}}+\cdots$ is divergent.

12. Find the first term of the third order of differences of the series 1, 3, 6, 10, 15, etc.

13. Find the first term of the fifth order of differences of the series $1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \cdots$.

14. Find the 9th term of the series 1, 3, 6, 10, 15, ⋯.

15. Find the nth term of the series 2, 6, 12, 20, 30, ⋯.

16. Find the 9th term of the series

$$2\times5\times7,\ 4\times7\times9,\ 6\times9\times11,\ 8\times11\times13, \text{ etc.}$$

17. Find the 16th and nth terms of the series 1×2, 3×4, 5×6, ⋯.

18. Find the sum of 10 terms of the series $1^2, 2^2, 3^2, 4^2, \cdots$.

19. Find the sum of 18 terms of the series 3, 11, 31, 69, 131, ⋯.

20. Find the sum of n terms of the series formed by the cubes of the natural numbers commencing with 1.

21. Find $\log_e 4$ by substituting in the formula of § 285; find also $\log_e 5$ in same way. Then find $\log_e 6$ by adding logs of 2 and 3. Then find $\log_e 7$ by substituting in the formula.

22. Find the Briggsian or common logarithm of 3 from its Napierian logarithm; also that of 5. Check answers by reference to the common logarithm table.

23. Find the natural logarithm of 7, 37, 1974.

291. Historical Note. The French mathematician Baron **Augustin Louis Cauchy** was born in Paris in 1789. In his youth the great mathematicians, Lagrange and La Place, were in the height of their glory. In works of wonderful breadth and rigor they had put almost the finishing touches on the theory of celestial mechanics of their day, and they had helped in the construction and introduction of the metric system. Both men took an interest in young Cauchy and persuaded him in 1813 to give up the profession of engineer and turn his energies to the advancement of *pure* mathematics. He secured a position in the École Polytechnique at Paris, which he held till 1830. When, on the accession to power of King Louis Philippe, he was required to subscribe to oaths of allegiance, he refused to do so. Soon after this, he was appointed to the chair of mathematics in the University of Turin, Italy, and later he became the tutor of the grandson of the deposed King Charles X of France. This position enabled him to travel, and thus meet many mathematicians. In 1838 he returned to France and was offered a position in the Collège de France, which he refused. In 1848 he again took a professorship in the École Polytechnique, which position he retained till his death in 1857.

Cauchy's writings (treatises and contributions to scientific journals totaling in number 789) cover the whole field of mathematics and are characterized by great clearness and rigor of treatment. They exercised a great influence over his contemporaries. In addition to his three great works, the "Course in Analysis," the "Infinitesimal Calculus," and the "Application of the Calculus to Geometry," he wrote a book on higher algebra. His main contributions include studies in the determination of the number of real and imaginary roots of an algebraic equation, studies of convergency of series, of the theory of numbers, of complex numbers, of the theory of groups and substitutions, of the theory of functions, of differential equations, and of determinants. It was Cauchy's paper in 1841 that brought determinants into general use. He cleared up difficulties in the Calculus by means of the theory of limits and the doctrine of continuity. In the application of mathematics to natural science also, his contributions were highly important. A complete edition of his works has been issued by the French government in 27 volumes.

AUGUSTIN LOUIS CAUCHY (1789–1857)

CHAPTER XIX

UNDETERMINED COEFFICIENTS — PARTIAL FRACTIONS — CONTINUED FRACTIONS

I. UNDETERMINED COEFFICIENTS

292. Undetermined Coefficients. Many times, in mathematical investigations, power series are introduced with coefficients undetermined. The problem then is to find these unknown coefficients.

EXAMPLE 1. Find a polynomial of the second degree, $ax^2 + bx + c$, which equals 1 when $x = 0$, equals 2 when $x = 1$, and equals 9 when $x = 2$.

SOLUTION. Substituting $x = 0, 1, 2$, in turn, in $ax^2 + bx + c$, and setting the results equal respectively to the numbers given in the problem, we have

(1) $$c = 1.$$

(2) $$a + b + c = 2.$$

(3) $$4a + 2b + c = 9.$$

Substituting $c = 1$ in the last two equations and solving them by the method of § 43, we have $a = 3$, $b = -2$. Hence

$$3x^2 - 2x + 1$$

is the polynomial sought, as may be easily verified.

a. Had we taken $mx + n$ instead of $ax^2 + bx + c$, we should have obtained the inconsistent equations (1) $n = 1$, (2) $m + n = 2$, (3) $2m + n = 9$, showing that the problem is impossible in this form.

But had we taken $ax^3 + bx^2 + cx + d$ instead of $ax^2 + bx + c$, we should have obtained three equations to determine four unknowns, and the problem would have had an infinite number of solutions.

EXAMPLE 2. Express $2x^2 + 3x + 5$ in terms of powers of $x + 2$, that is, in the form $a(x + 2)^2 + b(x + 2) + c$.

SUGGESTION. Expand the latter expression and arrange it according to the powers of x. Then, setting the two given quantities equal to each other, make the coefficients of x^2, x, and the known terms respectively equal to each other. $2(x+2)^2 - 5(x+2) + 7$. *Ans.*

We see, in the preceding examples, that certain conditions are given, and the problem is presented of finding whether there exists a function of a specified form that will satisfy the given conditions. This method can be used in simple multiplication, long division, factoring, and fractions. In such cases we usually know what form the answer will take. Thus, instead of dividing, we can set $\frac{x^3 - 3x - 110}{x - 5} = x^2 + Bx + C$, and find B and C by the method of this article.

We proceed now to study under what conditions the rule about equating coefficients on both sides holds true.

Theorem of Undetermined Coefficients.

If $p_0 + p_1x + p_2x^2 + p_3x^3 + \cdots = q_0 + q_1x + q_2x^2 + q_3x^3 + \cdots$ *for every value of* x *that makes these series converge, the coefficients of like powers of* x *are equal, or* $p_0 = q_0$, $p_1 = q_1$, $p_2 = q_2$, $p_3 = q_3$, $\cdots$.

PROOF. Transposing the right member to the left, and arranging, we have,

(1) $(p_0 - q_0) + (p_1 - q_1)x + (p_2 - q_2)x^2 + (p_3 - q_3)x^3 + \cdots = 0.$

For convenience, let $p_0 - q_0 = a_0$, $p_1 - q_1 = a_1$, etc. Then,

(2) $$a_0 + a_1x + a_2x^2 + a_3x^3 + \cdots = 0,$$

for every value of x that makes the left member converge.

Setting $x = 0$, we get $a_0 = 0$. Then,

(3) $$a_1x + a_2x^2 + a_3x^3 + \cdots = 0,$$

for every value of x that makes the left member converge.

Now it is assumed that x has other values than 0, so we can divide by x (§ 200, 4), getting

(4) $$a_1 + a_2x + a_3x^2 + a_4x^3 + \cdots = 0,$$

which holds true for every value of x that makes the left member converge, except possibly $x = 0$.

But it follows from this that $a_1 = 0$, since if $a_1 \neq 0$, by taking x small enough, but not equal to 0, we can make the value of

$a_2x + a_3x^2 + a_4x^3 + \cdots$ numerically less than a_1, and such a value of x would not satisfy (4).

In the same way it can be shown that $a_2 = 0$, $a_3 = 0$, and so on.

Now, since $p_0 - q_0 = a_0 = 0$, $p_1 - q_1 = a_1 = 0$, $p_2 - q_2 = a_2 = 0$, and so on, we have $p_0 = q_0$, $p_1 = q_1$, $p_2 = q_2$, $p_3 = q_3$, and so on, which was to be proved.

293. Reversion of Series. It often happens, if x is expressed as a power series in y, that we may want to express y through a power series in x, thus making y instead of x the independent variable. (§ 213.)

Suppose we have given

(1) $$x = my + ny^2 + py^3$$

in which m, n, p are given coefficients, and we desire to find a, b, c, so that

(2) $$y = ax + bx^2 + cx^3 + dx^4 \cdots.$$

To solve this problem we use the method of undetermined coefficients. By substituting the value of y from (2) in (1), we have

$$x = m(ax + bx^2 + cx^3) + n(ax + bx^2 + cx^3)^2 + p(ax + bx^2 + cx^3)^3,$$

or, $x = max + (mb + na^2)x^2 + (mc + 2\,abn + pa^3)x^3 + \cdots.$

Equating coefficients, we get

$$ma = 1;\ mb + na^2 = 0;\ mc + 2\,abn + pa^3 = 0;$$

$$\therefore a = \frac{1}{m};\ mb + \frac{n}{m^2} = 0;\ mc + \frac{2\,bn}{m} + \frac{p}{m^3} = 0;$$

$$\therefore b = -\frac{n}{m^3};\ \therefore c = \frac{2\,n^2 - mp}{m^5}.$$

Hence, $$y = \frac{x}{m} - \frac{nx^2}{m^3} + \frac{(2\,n^2 - mp)x^3}{m^5} + \cdots.$$

1. Find y in terms of x from the equation $x = y + y^2 + y^3$ by the method used above, and check the answer by substituting in the formula just obtained.

2. Find y in terms of x from $x = 1 - 2\,y + 3\,y^2$.

Suggestion. Transpose 1, getting $x - 1 = -2\,y + 3y^2$, and put $z \equiv x - 1$, whence $z = -2\,y + 3\,y^2$. Then, with $z = -2\,y + 3\,y^2$, proceed as before. Last of all, replace z by $x - 1$.

II. PARTIAL FRACTIONS

294. Application of the Theorem of Undetermined Coefficients to Decomposition of Fractions into Partial Fractions.

Partial fractions are fractions which when added together produce a given fraction. The operation of finding partial fractions is thus merely the converse of addition of fractions.

1. Given $\dfrac{x+1}{(x-2)(x-3)}$, to decompose it into two fractions.

SOLUTION. Let $\dfrac{x+1}{(x-2)(x-3)} \equiv \dfrac{A}{x-2} + \dfrac{B}{x-3}$. (In which A and B are undetermined numbers.)

Then, $x + 1 \equiv Ax - 3A + Bx - 2B$, (Mult. Ax.)

or, $x + 1 \equiv (A + B)x - (3A + 2B)$. (Arranging and factoring.)

Now, by the theorem of undetermined coefficients, since in such fractions any letter can have any value (providing no denominator becomes 0), the coefficients of x on the two sides of the equation are equal, and the terms that do not contain x are equal. Then,

$$(1)\ A + B = 1 \qquad (2)\ -3A - 2B = 1$$

Solving these equations by § 43, we get $A = -3$, $B = 4$.

Hence, $$\frac{x+1}{(x-2)(x-3)} \equiv \frac{4}{x-3} - \frac{3}{x-2}.$$

Prove the result correct by addition of fractions.

Resolve into partial fractions and prove:

2. $\dfrac{3x+8}{x^2+7x+6}$.
3. $\dfrac{x+1}{x^2-7x+12}$.
4. $\dfrac{1+x}{x-x^2}$.

5. $\dfrac{x^2}{(x+1)(x-1)(x-2)}$.

SUGGESTION. When there are three different factors each of the first degree in the denominator, there is a partial fraction for each, the letter C being used for the third numerator.

6. If $\dfrac{x^2+9x+6}{(x-1)(x^2+2x+5)} = \dfrac{A}{x-1} + \dfrac{Bx+C}{x^2+2x+5}$, find A, B, C.

a. Three cases in partial fractions need to be recognized.

I. *When all the factors of the denominator of the given fraction are of the first degree and different.*

Such problems are solved as Exs. 1–5.

II. *When a factor of the denominator of the given fraction is a prime quantity of the second degree.*

The partial fraction corresponding to it must have a numerator of the form $Bx + C$ as in Ex. 6.

III. *When a factor of the given denominator is a power.*

Suppose x^2 and $(x-2)^2$ are in the given denominator. Then a fraction must appear for each power of both factors: as $\frac{A}{x}, \frac{B}{x^2}, \frac{C}{x-2}, \frac{D}{(x-2)^2}$.

Evidently all these forms are *possible* parts of a decomposed fraction; therefore we assume they exist and give them coefficients. Any of such coefficients, of course, can be zero.

7. $\dfrac{6x^3 - 8x^2 - 4x + 1}{x^2(x-1)^2}$.

8. $\dfrac{x^2+1}{(x-1)(x^2+x+1)}$.

9. $\dfrac{x^2+x+1}{(x-1)^3}$. SUGGESTION. $\dfrac{A}{x-1} + \dfrac{B}{(x-1)^2} + \dfrac{C}{(x-1)^3}$.

10. $\dfrac{x^2-x+1}{(x^2+1)(x-1)^2}$.

11. $\dfrac{1}{x^3-1}$.

CONTINUED FRACTIONS

295. A **continued fraction** is a fraction of the form in the margin.

$$a + \cfrac{b}{c + \cfrac{d}{e + \cfrac{f}{g + \cdots}}}$$

Notice that b is a numerator and all below the line underneath it is the corresponding denominator, and so with d, f, etc.

Continued fractions are either **terminating**, or **infinite** (non-terminating). When all the numerators of a terminating continued fraction are 1 and all the signs are +, it is called a **simple** continued fraction. To save space

$$\cfrac{1}{a + \cfrac{1}{b + \cfrac{1}{c + \cdots}}} \text{ is written } \frac{1}{a+}\frac{1}{b+}\frac{1}{c+}\cdots \text{ or, } \frac{1}{a}+\frac{1}{b}+\frac{1}{c}+\cdots$$

The expressions at the right are easily distinguished from $\frac{1}{a} + \frac{1}{b} + \frac{1}{c} + \cdots$ by the position of the + signs.

296. Reduction of Rational Fractions to Simple Continued Fractions, and the Converse Operation.

1. Reduce $\frac{97}{137}$ to a simple continued fraction.

SOLUTION.

$$\begin{array}{l} 97)137(1 \\ \underline{97} \\ 40)97(2 \\ \underline{80} \\ 17)40(2 \\ \underline{34} \\ 6)17(2 \\ \underline{12} \\ 5)6(1 \\ \underline{5} \\ 1)5(5 \\ \underline{5} \end{array}$$

We begin by dividing both terms of the given fraction by 97, getting $\dfrac{1}{\frac{137}{97}}$, or, $\dfrac{1}{1+\frac{40}{97}}$. Then we divide both terms of $\frac{40}{97}$ by 40; and so continue.

$$\therefore \frac{97}{137} = \cfrac{1}{1+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{2+\cfrac{1}{1+\cfrac{1}{5}}}}}}$$

$$= \frac{1}{1+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{2+}\,\frac{1}{1+}\,\frac{1}{5}.$$

Evidently 1 *is the numerator of each fraction and the several quotients in the continued division at the left above are the integral parts of the several denominators of the contined fraction.*

The correctness of the answer found can be *checked* by calculating the value of the fraction, commencing with the lowest or last fraction and working backwards.

Check. $1 \div \frac{6}{5} = \frac{5}{6}$; $1 \div (2 + \frac{5}{6}) = \frac{6}{17}$; $1 \div (2 + \frac{6}{17}) = \frac{17}{40}$; $1 \div (2 + \frac{17}{40}) = \frac{40}{97}$; $1 \div (1 + \frac{40}{97}) = \frac{97}{137}$.

Reduce the following fractions to simple continued fractions and check the results by reducing the answers back to simple fractions. (The student will do well to use the old notation along with the new until he becomes familiar with both.)

2. $\frac{13}{89}$; $\frac{87}{128}$; $\frac{248}{379}$; $\frac{142}{518}$; $\frac{233}{177}$; .317; $\frac{613}{2368}$.

297. Approximate Values of Continued Fractions. By ignoring all but the first, all but the first two, all but the first three, and so on, of the simple fractions in the standard form of the continued fraction, approximate values of the continued fraction and of its equal simple fraction are obtained. In this way we get from $\frac{1}{1}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{1}+\frac{1}{5}\left(=\frac{97}{137}\right)$ the following values:

$$\frac{1}{1};\ \frac{1}{1}+\frac{1}{2}=\frac{2}{3};\ \frac{1}{1}+\frac{1}{2}+\frac{1}{2}=\frac{5}{7};\ \frac{1}{1}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}=\frac{12}{17};$$

$$\frac{1}{1}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{1}=\frac{17}{24};\ \frac{1}{1}+\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\frac{1}{1}+\frac{1}{5}=\frac{97}{137}.$$

$\frac{1}{1}=1.0000$

$\frac{2}{3}=.6667$

$\frac{5}{7}=.7143$

$\frac{12}{17}=.7059$

$\frac{17}{24}=.7083$

$\frac{97}{137}=.7080$

Reducing the values just found to decimals, we have the figures in the margin at the left. Examining these figures, we see that the first approximation, 1.0000, is too large; the second, .6667, is too small; the third is too large; the fourth is too small; the fifth is too large; and the last is the exact value of the given continued fraction correct to four decimal places. We see also that each new value is closer than the preceding to the true value.

In full treatments of the subject of continued fractions it is shown by using letters that what is true of the example just given is true in general, viz. *that the odd approximations to the value of a continued fraction obtained by dropping terms are all too large and the even approximations are all too small, and that each new approximation is closer to the true value than the preceding approximation.*

298. Recurring continued fractions are continued fractions in which a group of successive denominators is repeated in regular order indefinitely.

Thus, $\frac{1}{3}+\frac{1}{5}+\frac{1}{3}+\frac{1}{5}+\frac{1}{3}+\frac{1}{5}+,\ \cdots;\ \frac{\dot{1}}{a}+\frac{1}{b}+\frac{\dot{1}}{c}+,\ \cdots.$

The part in the second example between the dots, which is to be repeated, is called the *period* of the continued fraction.

An interesting truth concerning these continued fractions is *that every recurring continued fraction is a root of a quadratic, and conversely.*

Suppose, for example, that

$$x = \frac{\dot{1}}{a+}\frac{\dot{1}}{b}\cdots, \text{ or } x = \frac{1}{a+}\frac{1}{b+}\frac{1}{a+}\frac{1}{b+}\cdots.$$

Evidently, by definition, the quantity added to b in the last denominator is x itself. Then,

$$x = \cfrac{1}{a+\cfrac{1}{b+x}}; \text{ or, } x = \frac{b+x}{ab+ax+1}; \text{ or, } ax^2+abx-b=0.$$

Hence, the value of the continued fraction $\frac{\dot{1}}{a+}\frac{\dot{1}}{b}\cdots$ is a root of the quadratic equation $ax^2+abx-b=0$.

In the same way it can be shown that a continued fraction with a greater number of simple fractions in its period is likewise the root of a quadratic.

By substituting 4 for a and 6 for b in the formula found above, we have $\frac{\dot{1}}{4+}\frac{\dot{1}}{6}\cdots$ as the root of the quadratic $4x^2+24x-6=0$, and its value is thus easily found by solving the quadratic to be $\sqrt{10.5}-3$.

To show that the converse of the above theorem holds true, viz. that every root of a quadratic is a recurring continued fraction, we will apply the process which reduces the root of a quadratic to a recurring continued fraction to the result just found. Rationalizing *numerators* (see § 75, 4, and § 29), we get

$$\frac{\sqrt{10.5}-3}{1} = \frac{1}{2+\frac{2}{3}\sqrt{10.5}} = \cfrac{1}{4+\cfrac{\frac{2}{3}\sqrt{10.5}-2}{1}} = \cfrac{1}{4+\cfrac{1}{\sqrt{10.5}+3}}$$

$$= \cfrac{1}{4+\cfrac{1}{6+(\sqrt{10.5}-3)}} = \frac{\dot{1}}{4+}\frac{\dot{1}}{6}\cdots.$$

Similarly any irrational number, as $\sqrt{3}$, can be reduced to a recurring continued fraction.

Thus, since $\sqrt{3} = 1 +$ a decimal, we write

$$\sqrt{3} = 1 + \frac{\sqrt{3}-1}{1} = 1 + \frac{2}{\sqrt{3}+1}.$$

Then, since $\sqrt{3} + 1 = 2 +$ a decimal, we write

$$\sqrt{3} = 1 + \frac{2}{2 + \frac{\sqrt{3}-1}{1}} = 1 + \frac{2}{2 + \frac{2}{\sqrt{3}+1}} = 1 + \frac{2}{2 + \frac{2}{2 + \frac{\sqrt{3}-1}{1}}}$$

$$= 1 + \frac{2}{2+}\,\frac{2}{2+}\,\frac{2}{2}\cdots.$$

299. General Remarks on Continued Fractions. It can be shown that any number, commensurable or incommensurable, can be expressed in only one way by a continued fraction, either terminating or infinite. Every commensurable, or rational, number can be reduced to a terminating continued fraction.

The general theory of continued fractions has been used to prove that π and e (the base of Napierian logarithms) and integral powers of them are incommensurable.

The ancient Greeks thought the ratio of the circumference to the diameter of a circle ought to be capable of being found exactly, and the search for this value was one of the three great problems of antiquity. It was not till towards the close of the nineteenth century that π was shown to be incommensurable.

Continued fractions have been used in the solution of indeterminate equations (§ 49). Lagrange used them in the solution of equations of degrees higher than the second.

CHAPTER XX

COMPLEX NUMBERS

300. Different Kinds of Numbers and Number Systems. The different kinds of numbers can be made to appear as arising from the solution of different species of equations.

If a is a positive number, the attempt to solve $x + a = 0$ gives rise to the negative number $-a$. If a and b are integral whole numbers, the attempt to solve $ax \mp b = 0$ gives rise to the rational fraction $\pm \frac{b}{a}$. If a is a positive number not a square, the attempt to solve $x^2 - a = 0$ gives rise to the irrational number $\sqrt{a}$. If a is a positive number, the attempt to solve $x^2 + a = 0$ produces the pure imaginary $\sqrt{-a}$. The solution of an equation of the form $x^2 + ax + b = 0$ may give rise to roots of any of the preceding kinds, and also to two more kinds, viz. $a \pm \sqrt{b}$ and $a \pm \sqrt{-b}$, the first being partly rational and partly irrational, and the second, partly real and partly imaginary. The attempt to solve equations of the fifth and higher degrees (see end of § 115, p. 126), exponential equations (§ 175), etc., fails in the sense that such roots cannot always be expressed in terms of any or all of the kinds of numbers just described.

If a, b are integers the preceding can be tabulated as follows:

1. $x + a = a$ defines 0.
2. $x - a = 0$ defines the natural series of numbers.
3. $x + a = 0$ defines negative numbers.
4. $ax \mp b = 0$ defines rational fractions.
5. $x^n - a = 0$ defines surds.
6. $x^2 + a^2 = 0$ defines pure imaginaries.

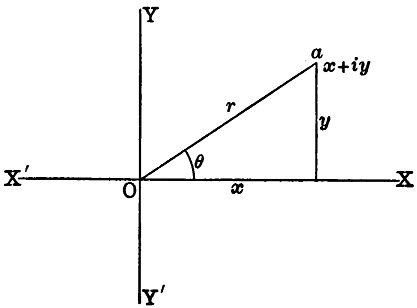
7. $a^x - b = 0$ defines one type of transcendental quantities.*

8. $x^{\frac{1}{n}} + a = 0$ defines what are called Riemannians.†

301. Addition and Subtraction of Complex Numbers was explained both algebraically and graphically in §§ 83, 86, 87. In order to explain the multiplication and division of these numbers, some trigonometrical formulas are needed. For this reason the discussion of this portion of the subject of complex numbers has been deferred until now.

For the convenient graphical explanation of the multiplication and division of complex numbers another form of them, called the *polar form*, is needed.

302. Polar Representation of Complex Numbers. If the complex number $x + iy$ denotes point a on the diagram, r denotes the length of the line from the origin to point a, and the Greek letter θ (theta) denotes the angle this line makes with the X-axis, OX, then (§ 150)



$$x = r \cos \theta,$$
$$y = r \sin \theta,$$

and

$$x + iy = r \cos \theta + ir \sin \theta$$
$$= r (\cos \theta + i \sin \theta).$$

Thus, r and θ can take the place of x and y in locating points on a diagram, and they are called the **polar coördinates** of a point.

The angle θ is called the **amplitude** of the complex number, and r is called its **modulus** (see § 88), or absolute value.

303. Reduction of Complex Numbers from One Notation to the Other.

If r and θ are given to find x and y, we have,

(1) $x = r \cos \theta$; (2) $y = r \sin \theta$.

* See footnote, § 319. † See Chrystal's Algebra, Vol. II, p. 238.

If x and y are given to find r and θ, we have,

(3) $\tan\theta = \frac{y}{x}$; (4) $r = \sqrt{x^2+y^2}$;

or, (5) $r = \frac{y}{\sin\theta}$. $\left(\text{Since } \sin\theta = \frac{y}{r}.\right)$

304. Exercise in Reducing from One Notation for Complex Numbers to the Other. Reduce the following from the ordinary to polar coördinates using the trigonometrical table found on page 164. Only angles in the first quadrant are given.

1. $3+2i$. 2. $4+6i$. 3. $7+14i$.
4. $9+0i$. 5. $1+i$. 6. $0+3i$.

Reduce the following to the ordinary notation.

7. $6(\cos 30° + i \sin 30°)$. 8. $4(\cos 19° + i \sin 19°)$.
9. $12(\cos 45° + i \sin 45°)$. 10. $10(\cos 12° 30' + i \sin 12° 30')$.
11. $15(\cos 18° 47' + 0)$. 12. $25(0 + i \sin 47° 18')$.

305. Multiplication of Complex Numbers.

Let $a + b\sqrt{-1}$ and $c + d\sqrt{-1}$ be two complex numbers.

Then, since $\sqrt{-1} \times \sqrt{-1} = -1$ by definition (see § 83),

$$(a + b\sqrt{-1})(c + d\sqrt{-1}) = (ac - bd) + (ad + bc)\sqrt{-1},$$

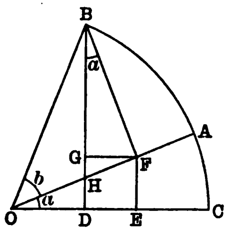
or the product of two complex numbers is a complex number.

Perform the multiplication in the following in which i is put for $\sqrt{-1}$.

1. $(3 + 4i)(2 + 7i)$. 2. $(7 - 6i)(8 + 2i)$.
3. $(12 - 5i) \times 3i$. 4. $6(43 - 12i)$.
5. $(2 - i)^3$. (§ 53.) 6. $(a - bi)^4$.
7. $(2 - 3i)(4 - 2i)(6 - i)(8 + 3i)$.

Before taking up the graphical multiplication of complex numbers, it is necessary to prove a theorem in trigonometry.

306. The Addition Theorem. This theorem is usually expressed by a formula that gives *the sine of the sum of two angles* in terms of the sine and cosine of each.



Let COB be the sum of the angles a and b (see figure), *and let radius* $OC = 1$.

Drop a perpendicular BF from B on OA, and draw BD and FE perpendicular to OC, and GF parallel to OC.

We have, $\sin(a+b) = \frac{DB}{OB} = DB$, (Definition of sine DOB, § 150, and because $OC = OB = 1$.)

$$\cos(a+b) = \frac{OD}{OB} = OD, \quad \text{(Definition of cos } DOB.)$$

and $\sin b = \frac{FB}{OB} = FB$; $\cos b = \frac{OF}{OB} = OF$. (Since $OB = 1$.)

Then, from the triangle FOE, we get

$$\sin a = \frac{EF}{OF} = \frac{EF}{\cos b}; \text{ and } \cos a = \frac{OE}{OF} = \frac{OE}{\cos b};$$

$$\therefore EF = \sin a \cos b; \quad \text{(Mult. Ax.)}$$

$$\text{and } OE = \cos a \cos b. \quad \text{(Mult. Ax.)}$$

But $\angle GBF = \angle DOH$. (Since each is the complement of one of two vertical angles at H.)

Also, from the triangle GBF, we get

$$\sin a = \frac{GF}{BF} = \frac{GF}{\sin b}; \text{ and } \cos a = \frac{BG}{BF} = \frac{BG}{\sin b};$$

$$\therefore GF = \sin a \sin b; \quad \text{(Mult. Ax.)}$$

$$\text{and } BG = \cos a \sin b. \quad \text{(Mult. Ax.)}$$

Then, since

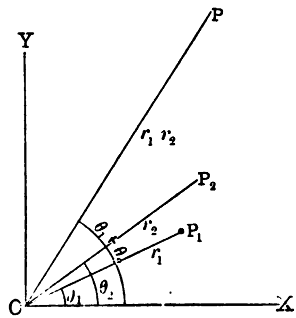
$\sin(a+b) = DB = EF + GB$, (By first eq. above and from the figure.)

(1) $\sin(a+b) = \sin a \cos b + \cos a \sin b$. (By substituting for EF and GB their values, p. 328.)

and since $\cos(a+b) = OD = OE - GF$. (From figure.)

(2) $\cos(a+b) = \cos a \cos b - \sin a \sin b$. (By substituting for OE and GF their values, p. 328.)

307. Graphical Representation of the Product of Two Complex Numbers.



To explain the geometrical significance of the product of two complex numbers, we first reduce the complex numbers to the polar form.

Let $P_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

and $P_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

be any two complex numbers locating the points P_1 and P_2 on the diagram.

Then, $P_1P_2 = r_1(\cos\theta_1 + i\sin\theta_1) \times r_2(\cos\theta_2 + i\sin\theta_2)$

$$= r_1r_2[(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2) + i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)]$$

$$= r_1r_2[\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)].$$ (By § 306, (1) and (2).)

Thus, the **modulus** r_1r_2 *of the product is the product of the moduli of the two factors, and the* **amplitude**, $\theta_1+\theta_2$, *is the sum of the amplitudes of the two factors.*

Referring to the figure we see that OP is r_1r_2, and $XOP = \theta_1 + \theta_2$. Point P is thus the product of points P_1 and P_2.

308. Exercise in the Multiplication of Complex Numbers Graphically.

Construct, graphically, with a protractor and scale, the factors and products of the following changing numbers given in the ordinary form to the polar form.

1. $3(\cos 15° + i\sin 15°) \times 5(\cos 22° + i\sin 22°)$.
2. $4(\cos 12° + i\sin 12°) \times 2(\cos 42° + i\sin 42°)$.

3. $(4+2i)(2+3i)$. 4. $(1+i)(4+4i)$.

5. $6(\cos 15° + i \sin 15°)$. 6. $4(12+16i)$.

The exercises given purposely avoid negative signs for the coordinates of both real and imaginary points. The rule given in the preceding article holds, however, for angles of all sizes.

309. Powers of a Complex Number Expressed in the Polar System. De Moivre's Theorem. We have just learned that the product of two complex numbers is a complex number whose amplitude is the *sum* of the amplitudes of the factors, and whose modulus is the *product* of the moduli of the factors. Evidently this rule can be extended so as to apply to the product of any number of factors, and in particular to any integral power of a complex number. Hence (n being integral or fractional),

$$[r(\cos a + i \sin a)]^n = r^n (\cos na + i \sin na).$$

This formula expresses what is called *De Moivre's Theorem*, which is widely used in higher mathematics.

1. Construct $[2(\cos 10° + i \sin 10°)]^3$.
2. Construct $[12(\cos 40° + i \sin 40°)]^{\frac{1}{4}}$.

310. Solution of Equations of the Form $x^n \pm m = 0$ (in which m may have any value real, imaginary, or complex) **by trigonometry.**

Let $$x^n = r(\cos a + i \sin a).$$

Then, $$x = r^{\frac{1}{n}}\left(\cos \frac{a}{n} + i \sin \frac{a}{n}\right).$$ *Ans.* (By equation of preceding article.)

Here $r^{\frac{1}{n}}$ denotes the arithmetical nth root.

Find and construct graphically an imaginary root in the following, using the table on p. 164 and the logarithm table, pp. 176, 177.

1. $x^6 = 4(\cos 20° + i \sin 20°)$. 2. $x^5 = 12$.

SUG. TO EX. 2. Write $x^5 = 12(\cos 360 + i \sin 360°)$ since $\cos 360° = \cos 0° = 1$, and $\sin 360° = \sin 0° = 0$. Extract root of coefficient by logarithms.

3. $x^8 = 27$. 4. $x^4 = 76$. 5. $x^7 = 128$.

311. Division of Complex Numbers. (See § 83, Ex. 22, 23.) Perform the following simplifications in complex numbers:

1. $\dfrac{2+3i}{1-5i}$. 2. $\dfrac{1-i}{1+2i}$. 3. $\dfrac{i}{7+5i}$. 4. $\dfrac{6+4i}{i}$.

312. Graphical Representation of Division of Complex Numbers. Since the product of two complex numbers is a complex number whose modulus is the product of the given moduli and whose amplitude equals the sum of the given amplitudes, it may be inferred that the quotient of one complex number divided by another has for its modulus the quotient of the dividend modulus divided by the divisor modulus, and for its amplitude the dividend amplitude minus the divisor amplitude.

313. The Complex Number System a Closed System. We have seen that the sum, the difference, the product, the quotient, the power, and the root of complex numbers is always a complex number, or, what we may regard as special cases of a complex number, viz., a real, or a pure imaginary. This was not true of the other kinds of numbers, as explained in § 300.

314. Historical Note. J. R. Argand, who was born in Geneva, in 1768, developed the geometrical interpretation of complex numbers and published in 1806 an account of them similar to that given in this chapter. Wessel had developed the same general idea in 1797, and the Abbé Buée in 1804, though not so fully as Argand. These results remained unnoticed by mathematicians until 1813, when Argand published an account of his studies, extending them somewhat, in *Gergonne's Annales.* He gave a proof of the existence of n roots, and no more, of every rational algebraical equation of the nth degree with real coefficients.

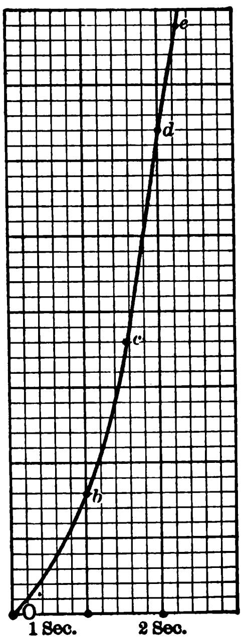
Later, in the writings of Gauss and Cauchy and many others, the Argand idea was developed into the important theory of complex numbers, which takes a very prominent place in the theory of functions, this theory of functions including a vast range of mathematical study.

CHAPTER XXI

GRAPHS OF HIGHER EQUATIONS. APPLICATIONS

315. Functional Relations Expressed by Graphs. Reread § 213, which explains that when one quantity, y, varies as another, x, the former is called a function of the latter. A graph can be used to express the continuous relationship between the two.

Thus, if $y =$ no. of feet a falling body has moved, and $x =$ no. of seconds it has fallen from rest, the formula, $s = \frac{1}{2} gt^2$, gives $y = 16.08\,x^2$ by letting $y = s$, $x = t$. The distance moved through is thus a *function* of the time. Constructing the graph $y = 16\,x^2$ (using 16 for 16.08), by taking a centimeter as the unit on the X-axis, and a millimeter as the unit on the Y-axis, we have the graph in the margin.



From this graph we can read off, as in § 131, the distance for any given time, or the time for any given distance. Thus, find the time for 10 ft.; 30 ft.; 70 ft.: also find the distance for 0.5 sec.; 1.2 sec.; 1.6 sec.

Similarly, the graph in § 131 expresses the area, y, of a square as a function of one side, x; and that in § 134 the volume, y, of a cube as a function of x, one side. In § 129 the y quantity in each case is a function of the x quantity.

316. Construction of the Graph $y = ax^n$ for Different Values of a and n. Here y is regarded as a function of x.

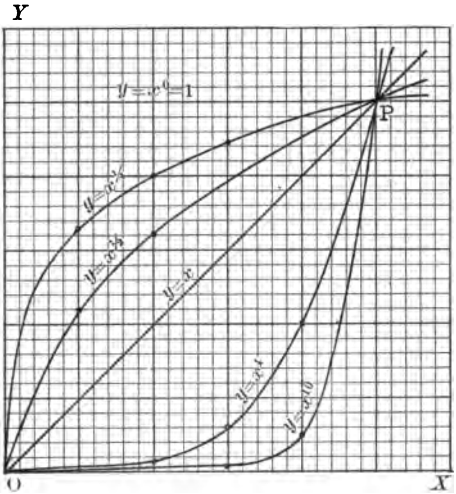
1. Construct the graphs for $y = x^2$, and $y = 2\,x^2$, on the same diagram. The graph for $y = x^2$ was given in § 131. Evidently for the equation $y = 2\,x^2$, each value of y will

be twice as great as for $y = x^2$, so that point a (Fig. § 131) will be at (1, 2), and point b at (2, 8), and so on. Thus, the graph of $y = 2\,x^2$ falls within that of $y = x^2$. Similarly, the graph of $y = 3\,x^2$ would fall within that of $y = 2\,x^2$, while that of $y = \frac{1}{2}\,x^2$ would fall without that of $y = x^2$. If the student will construct all these curves to the same axes, he will have what is called a *family* of curves, in this case the *parabola* family.

2. Construct on the same axes graphs for $y = x^3$; $y = 2\,x^3$; $y = \frac{1}{3}\,x^3$. See § 134.

a. The coefficient a in the equation $y = ax^n$ is called the parameter of its curve. Evidently changing this parameter does not change the character of the curve, but merely the scale of measurement of the ordinates.

3. The curves whose equations are $y = x^0 = 1$, $y = x^{\frac{1}{4}}$, $y = x^{\frac{1}{2}}$, $y = x$, $y = x^4$, $y = x^{10}$, are given in the margin, 5 centimeters being taken as the unit of measure. Verify that they are correctly constructed by locating points on each of the curves. Notice that all the curves except that for $y = 1$ go through $O(0, 0)$, and $P(1, 1)$, since these coördinates satisfy the equations of these curves. Notice also that all the curves except two extend on to the left of the Y-axis.



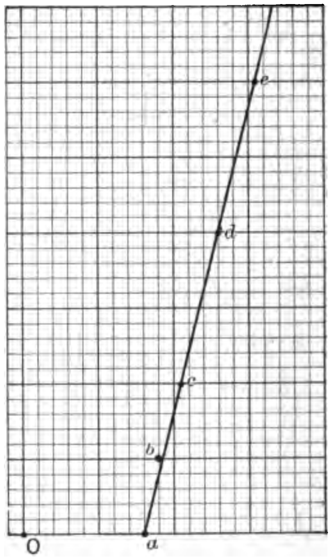
b. The graph of $y = b + ax^n$, or, $y - b = ax^n$, is closely related to that of $y = ax^n$. Evidently each value of y, Exs. 1–3, is to be increased by b.

4. Construct a graph from the equation given in Ex. 21, § 114, in which the resistance is a function of the speed.

317. Practical Applications of Graphs. Professor John Perry of London gives the following problems to show one way in which graphs can be used in engineering.

1. When the weight, y, was being lifted by a laboratory crane, the handle effort, x (the force applied at right angles to the handle), was measured and found to have the values in the table below, from which the accompanying diagram was constructed. For convenience the scale on the X-axis is $4 = 1$ cm.; and on Y-axis is $50 = 1$ cm.

(x, y)	POINT
(6.4 0)	a
(7.2, 50)	b
(8.4, 100)	c
(10.4, 200)	d
(12.4, 300)	e



The points evidently lie very nearly on a straight line. The straight line graph which will come the nearest to going through all the located points is drawn as in the figure. The equation of this straight line we may assume to be $y = ux + v$, in which u and v are unknowns to be found. Substituting the coördinates of two points on the line (say a and e) in the equation $y = ux + v$, we have

(1) $0 = 6.4\,u + v$. (2) $300 = 12.4\,u + v$.

from which (§ 42), $u = 50$, $v = -320$. Hence, the law for the relation between weight and power in this crane is expressed by the equation

$$y = 50\,x - 320.$$

Find from this formula the power when the weight is 450; also the weight when the power is 25.

2. If x is the electric power in kilowatts sent out from an electric lighting station, and y is the number of pounds of coal burned in the boiler furnaces per hour, find from the data in the margin a formula connecting x and y, first verifying the fact that the graph is approximately a straight line.

(x, y)
(349, 1121)
(291, 1020)
(228, 927)
(171, 820)
(119, 743)
(71, 652)

3. An engineer wanted to be able to state approximately the cost, y, of a steam plant to furnish x horsepower. He found that if $x = 200$, $y = £\,4200$; if $x = 120$, $y = £\,2450$; if $x = 30$, $y = £\,725$. See, first, if the graph is approximately a straight line; then find the equation between x and y; lastly, find from the formula the cost of a steam plant to furnish 160 horsepower.

a. The graphs in engineering problems like the above are usually not straight lines. If they are not, they may be compared with any graph whose equation is known that comes closest to them.

318. Interesting Graphs Derived from Equations of Higher Degrees.*

Construct the graphs to the following equations:

1. $y = x^3 - x$. 2. $y^2 = x^3 - x$. 3. $yx^2 = 81$.

4. $y^2 = x^2 + x^3$. 5. $y^2 = (x-1)(x-2)(x-3)$.

SUGGESTION TO EX. 5. Put $x = 0, \frac{1}{2}, 1, \frac{5}{4}, \frac{6}{4}, \frac{7}{4}, 2, \frac{9}{4}, \frac{5}{2}$, etc., up to 3, and from there on put x equal to whole numbers. Notice when values of y become imaginary.

6. $y^2 = (x-1)(x-2)^2$. 7. $y^2 = (x-1)^2(x-2)$.

8. $y^2 = (x-1)^3$. 9. $10\,y^2 = x^3 - 9\,x^2 + 24\,x - 16$.

10. $(x^2 + y^2 + 4)^2 - 16\,x^2 = 16$. Also right member $= 10$; also $= 20$. Cassini's oval.

* Nearly all these curves will be found discussed in Salmon's *Higher Plane Curves*, a standard work on the subject.

11. $9y^2 = (2-y)^2(x^2+y^2)$. Conchoid of Nicomedes.

12. $(x^2+y^2-x)^2 = x^2+y^2$. The Cardioide.

13. $(x^2+y^2-3x)^2 = x^2+y^2$. Pascal's Limaçon.

14. $(x^2-9)^2+(y^2-4)^2=1$.
15. $(x^2-4)^2+(y^2-1)^2=1$.

16. $(x^2-9)^2+(y^2-1)^2=16$.
17. $(x^2-4)^2+(y^2-1)^2=16$.

18. $(x^2-4)^2+(y^2-4)^2=16$.
19. $y^3=x^2+x^4$.

319. Transcendental Curves. By a transcendental * curve is meant one that cannot be expressed by an algebraic equation whose members are expressed in terms of integral powers of the independent variable x.

1. Construct the sine curve.

SUGGESTION. Take 2 centimeters to denote 90° on the X-axis. At each 9° point of these 2 centimeters erect an ordinate equal to the sine as given in the table, § 153, using 1 centimeter as unit.

2. Construct the cosine curve.

3. Construct the tangent curve.

4. Construct the graph of $y = \log_{10} x$; then find logarithms from numbers, and numbers from logarithms, by means of the graph. Check results by referring to the log table.

SUGGESTION. Let $x = .01, 0.1, 1, 3, 6, 8, 10, 20$, finding from the table in § 163 the corresponding values of y, and tabulating the results.

5. Construct the spiral of Archimedes from the equation $r = 3\theta$, in which θ is measured in radians or angles subtended by the radius of a circle.

SUGGESTION. See § 302. When $\theta = 0$, $r = 0$; when $\theta = 1(57.3°)$, $r = 3$; when $\theta = 2(114.6°)$, $r = 6$; etc.

* By a transcendental quantity, or function, is meant one which can not be expressed in terms of the unit or independent variable using the ordinary fundamental operations of algebra. Thus, π, e, $\tan x$, etc., are transcendental quantities.

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