# THE STRENGTH AND STIFFNESS OF STEEL UNDER BI-AXIAL LOADING

BY

ALBERT JOHN BECKER
B. S. in M. E. University of Michigan, 1903
M. E. University of Michigan, 1907

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THESIS

Submitted in Partial Fulfillment of the Requirements for the

Degree of

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IN ENGINEERING

IN

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# THE STRENGTH AND STIFFNESS OF STEEL UNDER BIAXIAL LOADING.*

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*This bulletin embodies the principal data of the thesis of Albert John Becker, presented in partial fulfillment of the requirements for the degree of Doctor of Philosophy in Engineering in the Graduate School of the University of Illinois, June, 1915, together with further experimental data taken by the author to extend the investigation.

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# THE STRENGTH AND STIFFNESS OF STEEL UNDER BIAXIAL LOADING.

## I. Introduction.

1. *Scope of Investigation.*—The purpose of this investigation was to determine the laws governing the strength and stiffness of mild steel when subjected to combined stress produced by two tensions at right angles to each other or by a compression combined with a tension at right angles. In order to give a satisfactory basis for comparison of results, the plan of investigation provided that the ratio between the two stresses be kept constant throughout the test of a specimen, and J. B. Johnson's tangent method of determining the "yield point" or "apparent elastic limit" was selected.

The specimens tested were drawn steel tubes of uniform size and practically of uniform thickness. These tubes were subjected to an axial load and to internal pressure. The only variable was the ratio of the circumferential stress to the axial stress. Comparison has been made only in the test results from sets of tubes cut from a single length of seamless drawn tubing. By means of strain gage readings a knowledge of the distribution of stress on the cross section was obtained; no assumptions were made except that of uniform distribution of the circumferential tensile stress throughout the thickness of the tube wall.

The investigations of strength and of stiffness were carried on simultaneously, but the results are discussed separately. The points investigated are:

(a) The change of yield-point stress of the material with increasing ratios of circumferential tensile stress to axial tensile or compressive stress.

(b) Stiffness of the material (strains accompanying stress) for increasing ratios of circumferential tensile stress to axial tensile or compressive stress.

No discussion has been given of the engineering applications, for it is realized that while these applications are important, more work is needed to establish the conclusions reached. When this has been done and all the work has been correlated, it will be a simple matter to make an application of these principles to engineering design.

2. *Acknowledgment.*—All the tests were made in the Laboratory of Applied Mechanics of the University of Illinois, under the supervision of Professors A. N. Talbot and H. F. Moore, to whom acknowledgment is made for their suggestions and criticisms and for the interest they

have shown in the progress of the investigation. Acknowledgment is also made to Mr. J. O. Draffin, research fellow in the Engineering Experiment Station, for his assistance in the conduct of the various tests. It is also desired to make an acknowledgment to the Joint Committee on Stresses in Railroad Track for the use of the new model 4-in. Berry Strain Gage.

3. *General Statement.*—When a steel bar is tested in tension or compression, certain phenomena are observed which have been incorporated as fundamental facts in the theories of the elastic behavior of bodies under stress. In such a test, both the strength and the stiffness of the material are observed, the former by noting the yield point and ultimate strength, the latter by observing the unit-strains corresponding to successive loads and computing the modulus of elasticity. Repeated experiments have shown that for material of the same composition and

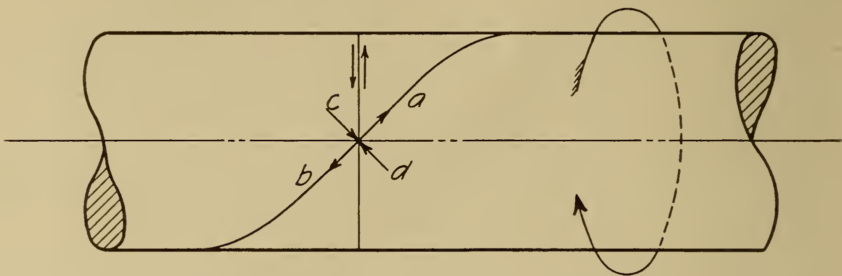


FIG. 1. ILLUSTRATION OF STRESSES PRODUCED ON OBLIQUE PLANES IN A BAR SUBJECTED TO TORSION.

treatment, these results are practically constant and can be used as a basis of design. The strength of any material of construction cannot be determined by mathematical analysis, neither can its stiffness. Poisson's ratio, modulus of elasticity, yield point, and Hooke's law are experimental results.

When an investigation of combined stress is attempted, there arises the question of the extent to which the calculations may be based upon the values obtained in the experiments in simple tension, compression, and shear. Constants determined by uni-directional loading cannot be indiscriminately applied to bi-directional loading. Theories have been evolved in which these constants are used by taking account of the interaction of the applied stresses. The analyses for these are correct from the mathematical standpoint, but the soundness of the basic assumptions can be demonstrated only by experiment.

Different combinations of simple stresses are possible, and it may be expected that the same analysis will not apply to all combinations. The presence of shearing stress in a bar subjected to simple tension and the tensile and compressive stresses accompanying the shearing stress due to a torsional load indicate that the governing conditions depend upon the relative strength of the material in shear, tension, and compression. A cast iron bar tested in torsion fails in tension on an oblique plane, because the tensile strength is less than the shearing strength. It is, therefore, logical to suppose that different stress combinations will produce failures differing in character for different materials.

4. *Combined Stress.*—Three types of stress applications are possible, uni-directional or simple stress, bi-directional or biaxial, and tri-directional. The first is illustrated by a specimen subjected to tension or compression in an ordinary testing machine. Bi-directional or biaxial stress is the application of two stresses in the same plane acting in directions at right angles to each other. Tri-directional stress is the application of three stresses at right angles to each other. The condition of biaxial stress is more important, from the point of view of the engineering applications, than that of three stresses at right angles to each other.

The possible combinations of biaxial stress are as follows:

Tension with tension.
Tension with compression.
Compression with compression.
Compression with tension.
Shear (torsion) with tension.
Shear (torsion) with compression.

These may be divided into three classes, tension with tension and compression with compression forming the first, tension with compression and compression with tension the second, and the combination of either tension or compression with torsion forming the third. The third class includes also two special cases of the second class; for a simple torque is equivalent to two equal principal stresses, one compression and the other tension, so that a torque combined with tension or compression can be reduced to the case of tension combined with compression or vice versa. This equivalence will readily be seen by considering a bar of circular cross-section subjected to torsion alone, Fig. 1. The stress on a plane at right angles to the bar is a pure shearing stress, depending in intensity upon the diameter of the bar and upon the torque. But this is not the only plane of stress. As in a bar in simple tension, so in this case there are planes on which both tensile and shearing stresses occur; there are also planes upon which no shearing stresses occur. Referring to Fig. 1, with the torque as shown by the arrow, the stress on the 45° plane CD is tension, and on plane AB at right angles to this plane, the stress is compression. This is equivalent to a biaxial loading which develops a tensile and a compressive stress at right angles to each other and each equal to the shearing stress. It should be noted that there are stresses on oblique planes which may control the strength of the material.

Applications of combined stress are to be found in the familiar examples of the steam boiler for tension combined with tension, and of the crank shaft for tension or compression combined with torque. Biaxial stresses occur in flat plates and in flat concrete slabs or girderless floors.

## II. Theories of the Strength of Materials Under Combined Stress.

5. *The Six Theories.*—The mathematical discussion of stresses and strains in a thin tube under axial load and internal pressure is given in Appendix II, page 58. It follows closely the method used by Love* in his work on the theory of elasticity, to which those who wish to investigate the subject further are referred.

Six theories have been advanced to cover the problems of the strength of material under combined stress. Two of them are empirical, one is developed from a molecular hypothesis, one from the mathematical theory of elasticity, and two from static relations of stresses. Three of these theories have found considerable favor and are given first.

6. *The Maximum Strain Theory.*—In the mathematical theory of elasticity, after the relations between stress and strain are established for simple stress, three equations of the following types are derived:

$$E\epsilon_1 = \sigma_1 - \frac{1}{m}(\sigma_2 + \sigma_3),$$

where $\sigma_1$, $\sigma_2$, and $\sigma_3$ are the three stresses at right angles to each other, $E$ is the modulus of elasticity assumed constant in all directions, $\epsilon_1$ is the unit-strain in the direction of $\sigma_1$, and $\frac{1}{m}$ is Poisson's ratio.† Stresses

*The Mathematical Theory of Elasticity, A. E. H. Love.

†A stress in any direction produces strain in that direction and also strain at right angles to that direction. The numerical ratio between the unit-strain at right angles to the direction of the force and the unit-strain in the direction of the force is called Poisson's ratio.

are considered positive if tension, and negative if compression. $E\epsilon_1$ is called by various writers the reduced stress, the true stress, or the ideal stress, but as the term stress is generally used by engineers to mean an internal resisting force which holds external forces in equilibrium it seems best to refer to it merely as $E\epsilon$. Writing two equations similar to the above for $E\epsilon_2$ and $E\epsilon_3$, the three equations for the reduced stress are obtained. The maximum strain theory takes these three equations and assumes that whatever the combination of stresses, the material will fail when the maximum strain (which will be in the direction of the greatest stress) reaches a value equal in magnitude to that at the yield-point stress in simple tension or compression. $E\epsilon$ at the yield-point stress for any combination of stresses must be the same, provided the yield-point stress is the same for tension as for compression. For ductile materials, $E$ is usually assumed to be constant and it follows that $\epsilon$ must be the same when the yield-point stress of the material is reached, no matter what combination of stresses is used. But for a brittle material, where $E$ varies, the strain $\epsilon$ must vary in an inverse ratio; that is, the product remains constant.

The maximum strain theory, or St. Venant's theory as it is sometimes called, holds that when a material is subjected to two or three stresses at right angles to each other, its strength is increased if the stresses are of like sign and that its strength is diminished if the stresses are opposite in sign. Thus two tensions or two compressions will produce an increase in the elastic strength of the material, whereas a tension combined with a compression produces a reduction in strength. For a stress ratio of one to one, both stresses tension, the material will be increased in strength 43 per cent if Poisson's ratio is 0.3, while if one stress is tension and the other compression, it will be weakened 23 per cent for the same stress ratio.

If in the equation for reduced stress given above, $\sigma_2$ and $\sigma_3$ are zero, the case is that of a bar in simple tension (compression is expressed as negative tension) and dividing both sides of the equation by $\epsilon_1$, the result is the equation of the modulus of elasticity.

For combined stress according to this theory, then, the strain accompanying a given stress is changed by the addition of another stress at right angles to the first. It is increased if the stresses have unlike signs and diminished if they have like signs. Also, the strain $\epsilon$ is the measure of $E\epsilon$ (the reduced stress) and the material will not reach the yield point until the strain $\epsilon$ reaches the value corresponding to the strain obtained in simple tension at the yield point. It should be emphasized that all elastic theory holds only within the elastic limit, or more correctly within the limit of proportionality, where $E$ remains constant for an individual stress-strain diagram. But the slight variation up to the yield point, even though the value of $E$ does change slightly, does not invalidate the theory, and the yield point is commonly taken as the limit of the discussion.

The maximum strain theory is based upon the mathematical theory of elasticity. Temperature effect is neglected and Hooke's law is assumed to hold rigidly. Herein lies its weakness, for the maximum strain theory, like the mathematical theory of elasticity, is dependent upon the accuracy of the relation assumed between stresses and strains. It has been shown* that there is a cooling of a bar of metal as the stress is increased up to the yield-point stress and it is also well known that Hooke's law is only an approximation.† A very good approximation it is, to be sure, for engineering purposes, but lack of isotropy in the materials, cold working and similar causes tend to change conditions, so that a slight deviation from Hooke's law may be observed considerably before the yield-point stress is reached. While the maximum strain theory has a good foundation, it must not be expected that the measured strains upon a body known to be not wholly isotropic, will conform exactly to this theory of stiffness.

The question of strength is quite different, for there is no assurance that the strains are the true measures of strength. Reasonable as the assumption may be, it is an assumption whose correctness must be demonstrated by experiment.

7. *The Maximum Stress Theory.*—The maximum stress theory, or Rankine's theory as it is sometimes called, virtually assumes that whatever the ratio of the stresses in the two directions and whether they are of like or opposite sign, the material will reach the yield point when, and only when, one of the stresses reaches the value corresponding to the yield-point stress in simple tension or in compression, as the case may be. It takes no account of Poisson's ratio as affecting strength and assumes that a material is neither weakened nor strengthened by the addition of a second stress at right angles to the first. If, then, this theory holds, the material should reach its yield point when the greater stress reaches the yield point stress for uni-directional loading.

8. *The Maximum Shear Theory.*—In the preceding theories failure

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*C. A. P. Turner, Trans. Am. Soc. C. E., 1902. Lawson and Capp, Inter. Assn. Test. Mat., 1912. Ew. Rasch, Inter. Assn. Test. Mat., 1909.

†Hedrick, Engineering News, Sept. 16, 1915.

by yielding is considered to take place in tension or compression, whereas the maximum shear theory, or Guest's law as it is sometimes called, holds that all failures are failures by yielding due to shear when the shearing unit-stress reaches the shearing yield-point stress. Therefore, if loads are gradually applied to two specimens developing simple stress in one and combined stress in the other but so as to keep the shearing stresses the same in each specimen, the yielding failure in the two cases will be identical.

The basic principle of the maximum shear theory, that the failure in combined stress is the result of the shearing stress reaching the shearing yield-point stress, when carried to its logical conclusion demands that when two of the principal stresses are zero the failure is still due to shear. A steel bar subjected to axial tension only must therefore fail in shear. The maximum shear in this case occurs on a 45° plane and its intensity is one-half the tensile unit-stress. If the yielding due to shearing stress occurs at the same time as yielding due to tensile stress the yield point unit-stress of the material in shear must be just one-half that in tension, but if the shearing yield-point stress is reached first—as this theory maintains—then the ratio is somewhat less than one-half.

If the stresses which are combined are a compression and a tension, the resulting maximum shearing unit-stress is one-half the sum of the tensile and compressive unit-stresses. When the tensile and compressive stresses are equal, the intensity of the shearing stress is equal to the intensity of the tensile or compressive stresses and failure will take place by shear unless the shearing yield-point stress is equal to or greater than that of either tension or compression. It seems entirely possible, then, that failure may be caused under certain conditions by shear and that in other cases its intensity may be insufficient to cause yielding, the tensile or the compressive yield-point stress being reached first.

Considering compression as negative tension, there are two kinds of elementary stress treated in mechanics—tension and shear. They are quite distinct and have different accompanying phenomena. While a definite relationship may be established between the shearing and tensile stresses, the material may fail either in tension or in shear. This is suggested by the fact that mild steel in torsion gives a square break, a shearing failure, but cast iron tested in torsion breaks along a helicoid, failing in tension because the material is weaker in tension than in shear.

This duality of conditions while not entirely overlooked, has been advanced heretofore solely to form two distinct theories of failure, but these have not been connected. The possibility that both shear and tension may govern, each within certain limits, has apparently not been mentioned in the publications and discussions on this subject. Mallock* has stated a dual law which is quite different from that discussed above. He proposes a volume extension limit and a shear limit, each dependent upon the other, and assumes that the material will fail when the limit of either is reached. This is quite distinct from the simple stresses as controlling factors in the failure of the material, but it recognizes the possibility of dual control.

The usual stress derivation for combined stress given in textbook is based upon the static equilibrium of forces and an application is made to a circular shaft in combined bending and torsion. A solution is given for the maximum normal stress and shearing stress on oblique planes, and safe working stresses are assigned. The assignment of working stresses in shear and tension fixes an arbitrary ratio of shear to tension, and the larger of the two shaft diameters determined by the two formulas is to be taken.

9. *The Internal Friction Theory.*—A short cylinder of brittle material when tested in compression fractures by shearing along a diagonal plane which, if failure be due to shear, should make an angle of 45° with the axis, since this is the plane of greatest shearing intensity. But the angles observed in experiments differ from 45°. In the attempt to explain this variation the theory of internal friction has resulted. When two particles under stress tend to slide over each other, a condition is set up similar to that of ordinary sliding friction. On the supposition that this resistance is similar to sliding friction, one of the laws governing the latter is applied; namely, that the coefficient of internal friction is independent of the load or stress. Therefore, slipping will occur along the surface of the plane inclined at an angle $\beta$ with the axis of the specimen such that $\beta = 45° - \frac{\phi}{2}$ for compression and $\beta = 45° + \frac{\phi}{2}$ for tension. $\phi$ is the angle of friction and $\tan \phi = \mu$, the coefficient of friction. If the limiting friction per unit of surface is the same for tension and for compression, then the normal stress on the surface of slipping, at the instant when yielding begins, must be the same in each case, since this is $\frac{1}{\mu}$ times the limiting friction.

It has been said that the chief difference between the internal friction theory and the maximum shear theory is that the former is based

*Proc. Royal Society of London, 1909.

upon a maximum resistance to sliding, while the latter is based upon a maximum shearing stress. If the angle of friction is zero, the internal friction theory becomes the maximum shear theory.

10. *Mohr's Theory.**

Let $k_4$ = the shearing yield-point stress.

Let $k_3$ = the stress in compression and in tension (equal) which together produce a shearing stress equal to the shearing yield-point stress, $k_4$.

Let $k_1$ = the tensile yield-point stress.

Let $k_2$ = the compressive yield-point stress.

Mohr derives the formulas:

$$k_3 = \frac{k_1 \; k_2}{k_1 + k_2} \text{ and } k_4 = \frac{1}{2} \sqrt{k_1 \; k_2}$$

The usual theory developed from the static relation of stresses gives for two equal stresses of unlike sign the following relation for the stress intensities:

Shearing stress = $\frac{1}{2}$ (tensile stress + compressive stress) which is the same as Mohr's theory when the tensile and compressive yield-point stresses are equal. Mohr's theory is an attempt to modify the shearing yield-point stress according to the tensile and compressive yield-point stresses. When these are equal this theory presents nothing new, for it then coincides with the maximum shear theory. If the yield-point stresses are different, Mohr's theory brings in a new relation regarding the shear failure in combined stress. It is virtually an acceptance of the maximum shear theory with the definition of the value of that shear at the yield-point.

11. *Wehage's Theory.*†—This theory is based upon a few experiments on cross-shaped pieces of paper submitted to tension in two directions at right angles to each other. If the material has a different yield-point stress in the two directions, the following elliptic relation is given as an empirical deduction:

$$\left(\frac{t_1}{T_1}\right)^2 + \left(\frac{t_2}{T_2}\right)^2 = 1$$

$T_1$ and $T_2$ are the yield-point or the ultimate stresses in the two directions (as, for instance, with and across the direction of rolling), and $t_1$ and $t_2$ are the applied stresses in the corresponding directions. When $T_1 = T_2$, this elliptic relation becomes a circular one.

This theory assumes that the material is *weakened* by the applica-

*Zeitschrift des Vereines Deutcher Ingenieure, 1900.
†Zeitschrift des Vereines Deutcher Ingenieure, 1905.

tion of two tensions for the reason that such stresses tend to lessen the cohesion between the fibers. The assertion is also made that a compression combined with a tension should strengthen the material by increasing this cohesion, although no formula is proposed.

12. *Graphical Presentation of Three Theories.*—A graphical presentation frequently serves to give a better idea of the working of a theory or formula and for this reason the three most important theories

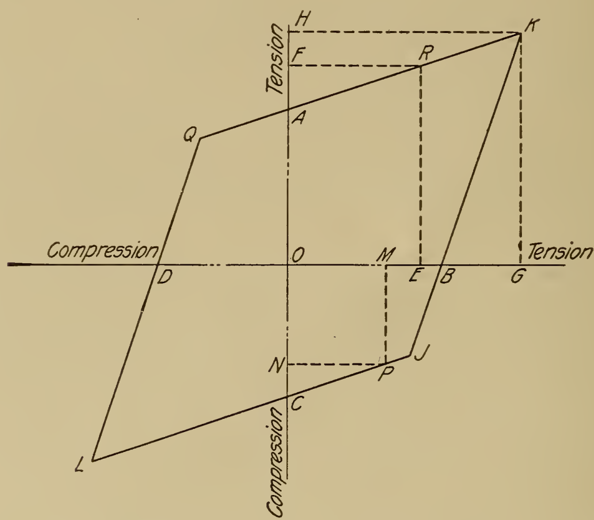



Fig. 2. Graphical Representation of Stresses According to the Maximum Strain Theory.

are represented in Fig. 2, 3, and 4, for the four combinations of simple tension and compression. To make the presentation more general, different yield-point stresses in compression and in tension have been assumed where this is possible.

Maximum Strain Theory. Let OA (Fig. 2) and OB represent the yield-point stress in simple tension and OC and OD that in compression. A tensile stress equal to OE would require a tensile stress equal to OF at right angles to cause yielding. For two equal tensile stresses the condition of yielding would not be reached until each stress attained the value OG, equal to OH. The increase in strength is OG — OB.

For a compression combined with an equal tension, yielding would occur when each stress attained the value O N, equal to O M. The other two quadrants are similar, two compressions producing the same relative effect as two tensions, and a tension and compression producing a corresponding effect to a compression and a tension.

Maximum Stress Theory. Yielding takes place in tension or in compression and since the stress in one direction is not affected by a second stress at right angles to the first the diagram will be a square. The center of the square, however, will not be the origin of co-ordinates since the tensile and compressive yield-points will in general be different. If a tensile stress OB or a compressive stress OD, Fig. 3, equal to the yield-point stress, is applied in one direction, any stress, OE less than the yield-point stress in tension, may be applied at right angles without causing further yielding. In other words a second stress acting at right

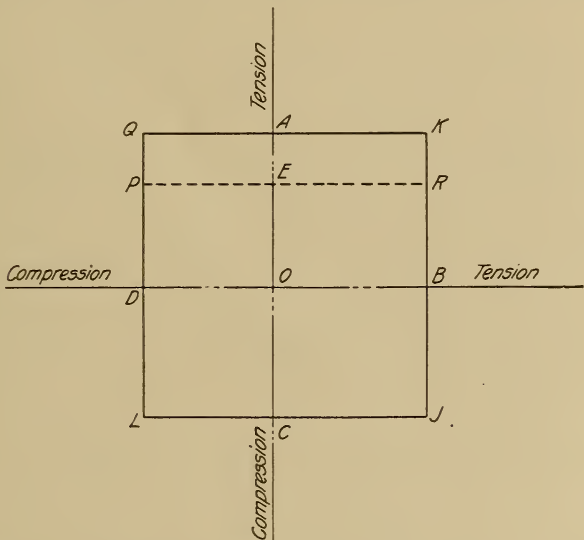



FIG. 3. GRAPHICAL REPRESENTATION OF STRESSES ACCORDING TO THE MAXIMUM STRESS THEORY.

angles to the first yield-point stress does not change the yield-point stress of the material.

Maximum Shear Theory. The first and third quadrants (Fig. 4) correspond to the maximum stress theory. This follows from the fact that the shearing stress equals one-half the difference between the greatest and the least of the three principal stresses. For biaxial loading one of the three principal stresses is zero and in the first and third quadrants the other two are of like sign, hence the shearing stress will be one-half the greatest stress. But the limiting shearing stress must be constant, therefore the greatest limiting principal stress must be constant and for like stresses (first and third quadrants) the diagram corresponds to the maximum stress theory. For a combination of tension and compression (second and fourth quadrants) the lines CB and AD are inclined at an angle of 45°, because the tensile stress plus the compressive stress is a constant and is equal to twice the shearing stress.

$$t + c = \text{constant.}$$

By setting $t$ and $c$ each equal to zero in turn, it is seen that $t$ must equal $c$, and this theory demands an equal yield-point stress for tension

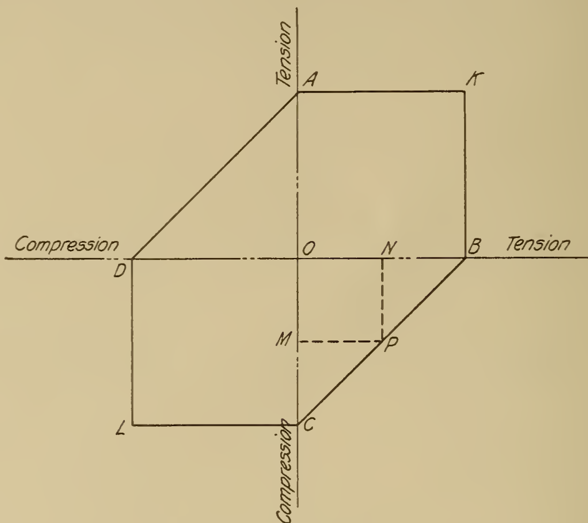



Fig. 4. Graphical Representation of Stresses According to the Maximum Shear Theory.

and compression. Two equal stresses of unlike sign will then cause yielding of the material when each stress equals ON or OM.

## III. Experimental Work.

13. *Form of Specimen.*—The selection of the type of specimen to be used in the experimental work was a problem of considerable difficulty. Specimens subjected to direct tension or compression in two directions were not considered because of complications produced by the method of application of the load. A cube subjected to compression in two directions could easily have been set up, but the friction between the bearing blocks and surfaces of the cube introduces inequalities and resistance to the change in cross section which could easily vitiate the results.*

*See Zeitschrift des Vereins Deutscher Ingenieure, 1900, p. 1530.

A large number of short square steel bars, closely spaced to form in effect a bearing block, were considered not to obviate this difficulty sufficiently. Similarly, a tension specimen held at the four edges would not be practicable. Direct stress application seemed out of the question, and recourse was first had to bending to produce stresses in two directions at right angles to each other.

The first biaxial stress experiments in this series of tests were made upon flat cross-shaped specimens subjected to cross bending to produce two compressions or two tensions at right angles to each other. The stress distribution was so far from regular that no safe comparisons could be made. Such difficulties were encountered that this form of test specimen was discarded.

After a preliminary test, thin tubes were adopted as the form of test specimen. They proved satisfactory on account of the certainty with which biaxial stress of known magnitude could be applied by means of an axial load in a testing machine and internal hydrostatic pressure producing a circumferential tension. This method gives two well defined principal stresses at right angles to each other, the stress in the third direction being small since it varies from the intensity of the hydrostatic pressure on the inside to zero on the outside. It is much easier to cover the total range of stress ratios by the use of hydrostatic pressure and axial tension or compression in the tubes, than to use torque and axial load on solid bars. The latter method is inferior to the tube tests since only a small portion of the material is carried to the yield-point stress. The experiments are more successful when as much of the specimen as possible is uniformly stressed, and the best condition is that wherein the entire specimen is uniformly stressed. This is true both on account of the pronounced yield-point effect and on account of the smallness of the strains to be measured. The thinness of the wall and the relatively large tube diameter made the stresses practically uniform throughout the tube. It may be expected that the stress-strain diagrams will show a much sharper break than for solid bar specimens and the yield point is more positively determined. There are no greater eccentricities of application of load when using the tube than when working with a solid bar, and on account of the greater diameter of the tube, this eccentricity is relatively less important.

Strains were measured by means of a Berry strain gage, using a 2-in. gage length in the cross bending tests and a 4-in. gage length in the tube tests. The accuracy and reliability of an instrument of this type has been demonstrated repeatedly and reference is made to the tests by A. N. Talbot and W. A. Slater on reinforced concrete buildings, as given in Bulletin No. 64 of the Engineering Experiment Station of the University of Illinois, to show what results may be

FIG. 5. VIEW SHOWING CROSS-BENDING TEST SPECIMEN UNDER LOAD.

achieved with such an instrument. A discussion of the strain gage and its use is given in a paper by Slater and Moore in Vol. XIII of the Proceedings of the American Society for Testing Materials.

The use of the strain gage marks a decided advance in the measurement of strains. With this instrument it was possible in these tests to take twenty-eight readings on as many gage lines for each increment of load, whereas other investigators have been able to take four at the most and often only two. The advantage of a portable instrument over an attached one is very great and the rapidity of operation and freedom from danger of jarring the instrument as well as the ability to read overlapping gage lines, as was done in these tests, marks a decided step in advance.

14. *Cross-bending Tests.*—The set-up for the bending tests is shown in Fig. 5. Two specimens were prepared from ¼-in. soft steel plate of the shape shown in the figure. Tension specimens were prepared from the portions cut away. In order to have the upper surface unobstructed for the use of the strain gage, the beam was loaded as an overhung beam with four equal loads placed symmetrically one on each projection of the cross-shaped specimen. The center part of the cross was thus subjected on the top to two tensions at right angles to each other.

Load was applied by placing known weights on the yokes at the ends of the arms of the specimen, thus giving a definite bending moment. The strains were measured over 2-in. gage lines with a Berry strain gage. Instead of a uniform stress over the center portion of the test piece, the readings showed a considerable variation. The effect of the sharp reëntrant angles at the corners in changing the lines of stress must have been considerable, for the yield point was reached first at the corners. The lines of yielding spread inward along a line making an angle of approximately 45° with the center lines. As the load was increased these lines divided, curving toward the adjacent corners, gradually changing direction and becoming parallel to the lines of symmetry of the specimen shortly before the lines from adjacent corners joined. New lines formed beside the first ones and others appeared outside the center of the cross. The latter were straight and parallel to the support. The lines are clearly shown in Fig. 6, which is from a photograph of the compression side of the first specimen tested. The lines marking the square from corner to corner and the center lines were used to lay out the specimen and must not be confused with the lines of yielding. The specimen, considered as a beam, widens abruptly for the center four inches, but the effect of this increased width in carrying stress was slight. The places of greatest stress were near each corner and to measure the maximum strain would have required a very short gage line. This stress condition is due to the form of the specimen rather than to combined stress.

15. *Tube Specimens.*—Specimens made from 6-in. tubes with

¼-in. walls were used. Four lengths of seamless drawn tubing were bought in the open market and made into test specimens. A series number was given to the specimens cut from a length of tubing and each specimen was numbered individually. The number of the test specimens

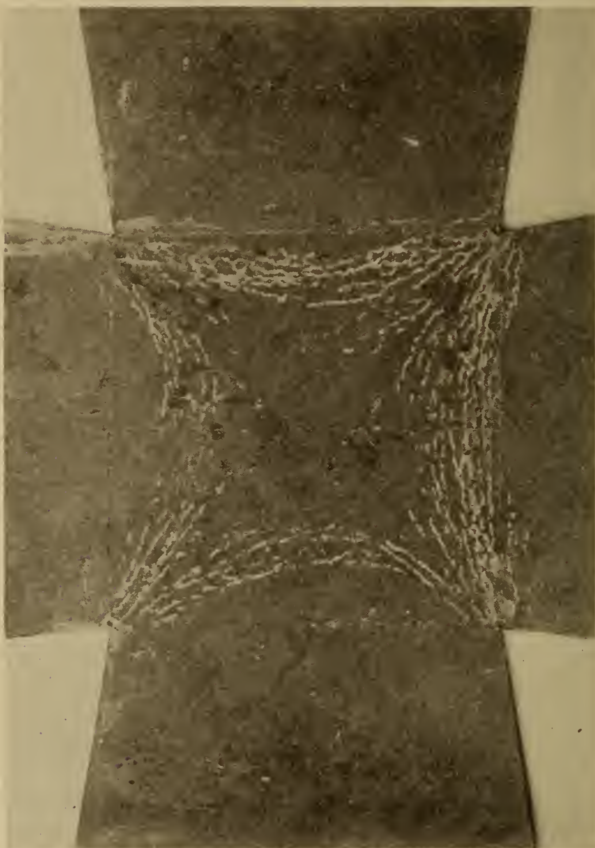


Fig. 6. View Showing Compression Side of Cross-Bending Test Specimen After Test.

in each series cut from each length of tubing and the character of the stresses applied are as follows:

| Series | Number of Specimens | Specimen Number | Character of Combined Stress |
|---|---|---|---|
| 1 | 5 | 1-2-3-4-5 | Tension with tension |
| 2 | 4 | 6-7-8-9 | Tension with tension |
| 3 | 6 | 1-2-3-4-5 | Compression with tension |
| 4 | 5 | 8-9-10 | Tension with tension |

Tube No. 6 of Series 3 and tube No. 7 of Series 4 were tested in torsion only. There was a marked difference in the physical properties of the material of the four lengths of tubing. This is shown by the stress-strain diagrams of the tensile tests made on specimens cut from the tubes. The yield point stress varied from 21,500 lb. per sq. in. to

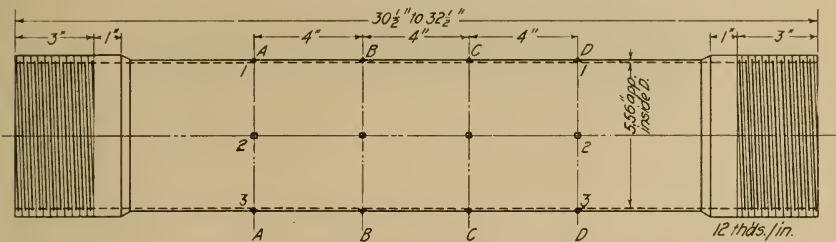



Fig. 7. Dimensions of Tube Test Specimen.

50,000 lb. per sq. in., Series 1, 2, 3, and 4 having yield-point stresses of 42,500, 21,500, 24,000, and 50,000 lb. per sq. in. respectively. The tubes were not annealed, but the first three series gave very uniform results for all gage lines, and showed a decided change at the yield point. The specimens of Series 4 showed a much greater variation. The behavior was that of hard, brittle steel of quite irregular composition.

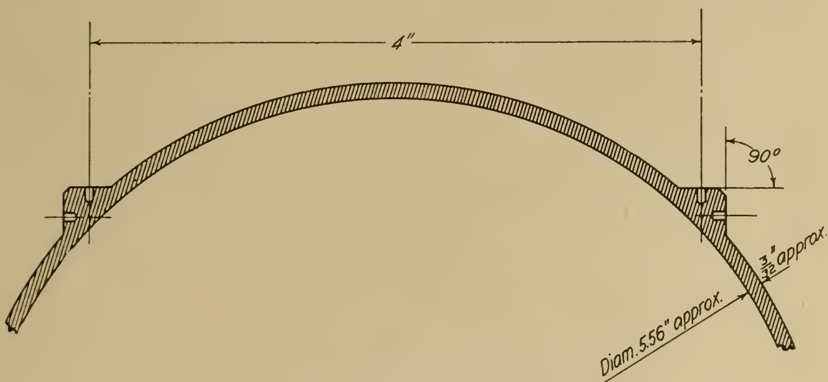



Fig. 8. Arrangement of Gage Holes on Tubes.

There was little reduction of area and the rupture was sharp and sudden, both in the tension specimens and in the one tube that broke during testing. The stress-strain diagram for the Series 4 show only qualitative results. These tubes were not suited for a test of this character, the inner and outer circumferences of the tube before machining were not concentric circles, and some gage lines gave diagrams that curved throughout, similar to the diagrams of drawn wire. There was no well-defined yield point. As the stress-strain diagrams did not give positive results, no use will be made of this series.

16. *Preparation of the Tubes.*—The test specimens were first

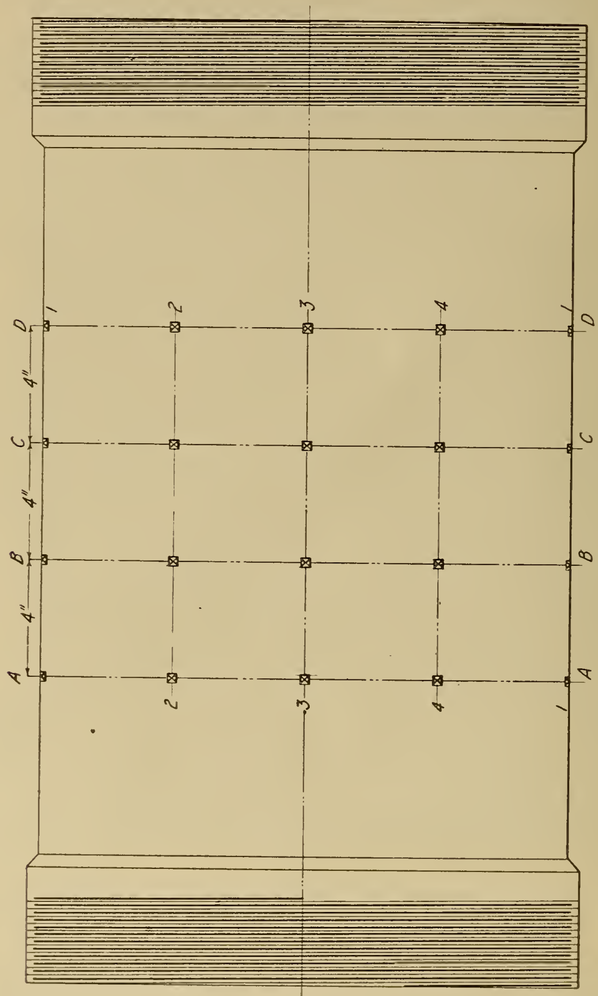



Fig. 9. Location of Gage Lines on Developed Outer Surface of a Tube.

bored out for the entire length on a horizontal boring mill and then turned to the dimensions shown in Fig 7. Each tube was threaded on the two ends with a taper thread of twelve threads per inch over a length of three inches. The tube was left full thickness for about an inch beyond the threads to furnish a bearing for packing. The remainder of the tube was turned to an approximate thickness of 3/32 in. except for four bands of 1/4-in. width spaced four inches apart along the tube. The greater part of these bands were afterwards milled off leaving four projections on each band for the gage holes. The tube was thus spanned with four circumferential gage lines each four inches long. The axial gage lines used one of the two holes so that the projections could be reduced to the smallest possible size. This gave four rows of three axial gage lines each, twelve in all, and four bands of four circumferential gage lines, sixteen in all, making it necessary to take twenty-eight readings, exclusive of the standard bar and check readings for each increment of load. The standard bar readings are necessary in tests with the strain gage to detect variations in the instrument due to temperature or jarring of the points.

Fig. 9 shows the position of the gage lines on a developed surface of a tube specimen. The circumferential bands were lettered A, B, C, and D; the axial lines were numbered 1, 2, 3, and 4. Thus an axial gage line would take two holes in the same axial line, but in two consecutive circumferential bands. It would, consequently, be called by the letters of the bands, in order, and by the number of the axial line. Thus AB 3 would be an axial gage line spanning the distance between the circumferential bands A and B and lying along the axial line 3. As soon as the tube was machined the numbering was fixed and the projections on the A band marked with small prick punch marks to identify the axial lines. In this way the readings for the thickness of the tube walls could be correlated with the strain gage readings. The gage holes were drilled by hand using a No. 54 drill. They were not reamed.

The boring of the tube caused a slight change of shape of the cross section due to the removal of the inner skin of metal, and after the outside was turned the thickness was uniformly varying, usually having two points of maximum thickness diametrically opposite, and at 90° from these, two points of minimum thickness. This renders the tube slightly elliptical (but not over 0.02 in. in 5.50 in.) and of varying thickness. While the variation in thickness was as high as 15 per cent in some cases, it apparently did not affect the averages of the readings, although the individual circumferential curves show the effect of this variation and the effect of the water pressure in making the tube more nearly cylindrical.

17. *Determination of the Thickness of Tube Walls.*—The principle of the apparatus adopted for measuring the thickness of the tube walls is that a micrometer caliper with a very deep throat. Fig. 10 shows the apparatus with the tube in position for a zero reading. A 4½ by 2½ by 7/16-in. T-bar was clamped at one end to a support and a stiff wooden bar was bolted to it. At one end of the wooden bar an Ames Dial read-


FIG. 10. VIEW OF APPARATUS USED IN MEASURING THICKNESS OF TUBE WALLS.

ing to thousandths of an inch was fastened so that the plunger rested on a steel ball (*a* Fig. 10) embedded in the stem of the T-bar. To determine the thickness of the tube wall the plunger of the dial was raised, the tube was slipped over the T-bar and rested on the steel ball. Two other steel balls ( *b* and *c* Fig. 10) were embedded in the stem of the T-bar, one on each side of the ball under the plunger at such a distance from it that the tube always swung free on the center ball and one of the others. The ball under the plunger was slightly higher than either of the others to insure a bearing on it at all times. When the plunger of the dial was in contact with the tube, the thickness of the tube was the difference between the reading then taken and the zero reading. Zero readings were obtained by suspending the tube in two fine wire slings in such a manner that its weight came on the T-bar in the same way as when the tube swung on the steel balls. With the plunger of the dial in contact with the steel ball, the initial or zero reading was taken for every position of the tube along an axial line. As the T-bar was a cantilever with two point loading, this gave slightly different zero readings for the various positions of the tube, but any error arising on account of the deflection of the apparatus was removed. With the tube in a given position and with the plunger on the ball, a reading was taken after a traverse of two axial lines. These readings were taken to detect any possible change in the apparatus and are not zero readings. They correspond to the standard bar readings when using a strain gage. A set of check readings was taken and the average of the two readings was used. Readings were taken to tenths of a division (ten-thousandths of an inch) and tube thicknesses are given in thousandths of an inch.

It is thought that this method of measurement is accurate and the check results obtained with a micrometer after the tube had been cut, have borne out this conclusion. The tube must be of relatively large diameter to apply this method, but with 6-in. tubes no difficulty was experienced.

18. *Method of Testing.*—Two steel castings were designed to fit over the ends of a tube. The stresses carried by these heads were comparatively low, for the maximum load was but 167,000 lb., and the material was about 3/4 in. thick at the thinnest part. The castings were machined all over and threaded internally, at one end to receive the tube and at the other to receive a 4-in. bar which served to apply the tension. The two threaded portions were separated by about an inch of metal which served to retain the water under pressure in the tube. These castings are shown in Fig. 11.

To withstand the water pressure, two layers of 3/8-in. hydraulic packing were used in an ordinary four-screw stuffing box. The heads were recessed to receive the packing and the gland, while the tube walls were left nearly full thickness for about an inch beyond the threads to furnish a firm bearing for the packing. After the packing was adjusted to position there were no perceptible leaks although pressures up to 1,800 lb. per sq. in. were used. Fig. 11 shows the general arrangement of the apparatus for the tension tests.

All the tests except the torsion tests were made in the 600,000-lb. Riehle machine of the Laboratory of Applied Mechanics of the University of Illinois. By using spherical seats with careful centering of the specimens in the machine, the eccentricity of loading was reduced

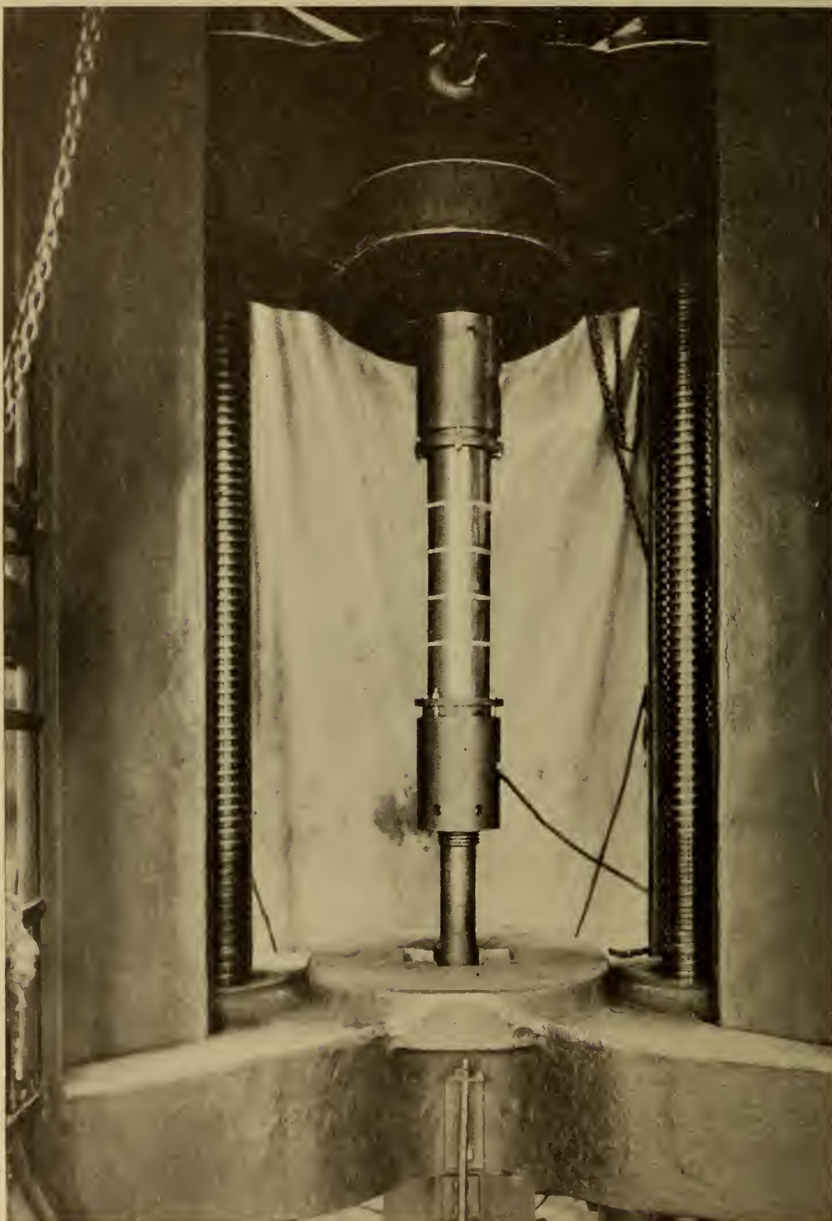


FIG. 11. VIEW SHOWING ARRANGEMENT OF APPARATUS FOR TENSION TEST.

to a minimum and the bending stresses were low as shown by the uniformity of the individual stress-strain diagrams. In the tension tests the nuts of the 4-in. bars bore directly against the spherical seats, the lower one being inverted. For the compression tests the bars were removed and the tube heads bore directly against the spherical seats and the upper spherical seat was inverted. The length of thread on the specimens tended to give a good distribution of load and the distance of the first gage hole from the end of the thin part of the wall (6½ in.) together with the thinness of the wall itself, were sufficient to insure a high degree of uniformity of stress. Holes were drilled into each head to connect into the interior of the tube; the hole in the lower head was for connection to the pump and the hole in the upper head was for the purpose of filling the tube with water. Each hole was tapped with a ½-in. pipe tap.

The torsion tests were made in a 230,000-lb. in. Olsen Torsion Machine. The heads were screwed on the specimens as in the other tests, and short steel bars threaded on one end transmitted the torque from the machine to the steel heads. Two fixed wooden clamps with 40-in. arms, one of which carried a pointer and the other a scale, were used to measure the angle of torsion over a known gage length.

19. *Character and Sequence of the Tests.*—Three tests in each of Series 1 and 2 were first made. These were the tests in direct tension, the tests with the ratio of tensile stresses equal to 0.475 and with this ratio equal to 0.92. The results of these tests were worked up before the remainder of the tests in the series were made, so that the other ratios could be chosen to the best advantage. As the difference in the strength of the steel in the direction of drawing and across it would complicate the problem, and as it was not intended to raise the question of the variation in strength in different directions throughout the specimen, the highest ratio of circumferential stress to axial stress used was made less than 1.0, being 0.92.

The average area of the inner cross section of the tubes was about 24.50 sq. in., and the axial load due to the water pressure was 2,450 lb. per 100 lb. per sq. in. water pressure. To produce a ratio of circumferential tension to axial tension equal to 0.50 required a machine load of $4 \times 2{,}450 - 1 \times 2{,}450 = 7{,}350$ lb. per 100 lb. per sq. in. water pressure, since the water pressure acts with the machine load. For axial compression combined with circumferential tension, the two quantities would be added instead of subtracted, since the water pressure tends to reduce the machine load. Dividing the net axial load (9,800 lb. per

100 lb. per sq. in. water pressure) by the cross-sectional area of the tube gives the unit axial stress. A slight error is introduced by using the inside diameter of the tube rather than the mean diameter, for in order that the circumferential tension shall be exactly twice the axial tension when the water pressure alone is acting, the mean diameter must be used to compute the axial tension. This error is about 1½ per cent, which represents the variation of the circumferential tensions from the mean. The stress ratios for Series 3 and 4 were planned complete and carried out as planned. The stress ratios used in the four series are given in Table 1.

The strain gage used was a 4-in. Berry strain gage made for the Joint Committee on Stresses in Railroad Track and loaned by that Committee. It has invar steel sides and shows a negligible correction for temperature. Two standard bars were used to detect any variation of the instrument due to jarring or striking the fixed point. All data have been corrected for variation in the standard bar readings. To avoid variation due to change of temperature of the tubes, they were usually

TABLE 1.
OUTLINE OF TEST SPECIMENS AND TESTS.

| Series No. | Tube No. | Ratio of Circumferential to Axial Stress | Stress Combination |
|---|---|---|---|
| 1 | 5 | 0.00 | Axial tension only |
| | 1 | 0.24 | Tension with tension |
| | 2 | 0.475 | " " " |
| | 4 | 0.69 | " " " |
| | 3 | 0.92 | " " " |
| 2 | 9 | 0.00 | Axial tension only |
| | 7 | 0.475 | Tension with tension |
| | 8 | 0.92 | " " " |
| | 6 | 0.92 | " " " |
| 3 | 4 | 0.00 | Axial compression only |
| | 2 | 0.20 | Compression with tension |
| | 5 | 0.30 | " " " |
| | 3 | 0.60 | " " " |
| | 1 | 0.90 | " " " |
| | 6 | 1.00 | Torsion only |
| 4 | 9 | 0.00 | Axial tension only |
| | 10 | 0.30 | Tension with tension |
| | 8 | 0.50 | " " " |
| | 11 | 0.80 | " " " |
| | 7 | 1.00 | Torsion only |

filled with water in the evening and by the time the test began the next day the tube and water were at a temperature that scarcely changed during the entire test.

20. *Test Operations.*—The initial load in all cases was small, producing an average axial unit-stress of approximately 4,000 lb. per sq. in. This load was applied after the specimen had been carefully centered and the spherical seats tried. A load sheet was prepared for each test which gave the required machine loads, the approximate yield point, the water pressure, and unit-stress. When the load was increased the water pressure was increased first and then the machine load.

The record of a test was a combination of the ordinary record and a graphical one. Co-ordinate paper was used and was divided into a series of rectangles, one for each standard bar and gage line. Along one side of this rectangle the instrument reading was noted and this reading was then plotted against the machine load. In this way the progress of the test was very evident and any doubtful reading was checked. When the nature of the curve is well known, it is advisable to see that the results for any gage line that do not show some systematic sequence of plotted points are checked to insure their accuracy. If this is not done false breaks may sometimes be obtained in the curve. If the error is experimental, the check reading will correct it, and if the stress suddenly departs from the straight line law, the check reading will be a repetition of the first reading and will give greater confidence in the result. Though but few errors were discovered and corrected, the result justifies the method employed. Whenever there are variations from the straight line in the stress-strain diagram, these are indications of a change in the rate of taking stress. As the load changes, the distribution of stress over a given cross section often changes, so that at one point there may be a rapid increase in the elongations for one increment of load, while in an adjoining gage line the change is slight. The next load increment may bring about a complete reversal of the conditions shown by the previous instrument readings.

Whatever variation occurs in one gage line, usually it is reflected in one or more of the others, so that the average takes out all these peculiarities. This is especially true of the circumferential readings.

It will be seen that the circumferential gage line readings give the correct unit-strain, the chord length being used and not the arc length. Circumferential readings are subject to the tendency of the tube to become truly cylindrical under water pressure. For low water pressures

TABLE 2.

DATA OF TUBES.

| Tube No. | Inside Diameter Inches | Location | Tube Walls. Average Thickness, Inches | Sectional Area, Sq. In. |
|---|---|---|---|---|
| | | Series 1. | | |
| | | AB | 0.089 | 1.590 |
| 1 | 5.564 | BC | 0.088 | 1.570 |
| | | CD | 0.086 | 1.527 |
| | | AB | 0.087 | 1.543 |
| 2 | 5.558 | BC | 0.087 | 1.543 |
| | | CD | 0.088 | 1.570 |
| | | AB | 0.087 | 1.543 |
| 3 | 5.561 | BC | 0.087 | 1.543 |
| | | CD | 0.087 | 1.543 |
| | | AB | 0.085 | 1.508 |
| 4 | 5.563 | BC | 0.084 | 1.481 |
| | | CD | 0.083 | 1.472 |
| | | AB | 0.091 | 1.623 |
| 5 | 5.554 | BC | 0.092 | 1.641 |
| | | CD | 0.092 | 1.641 |
| | | Series 2. | | |
| | | AB | 0.083 | 1.467 |
| 6 | 5.579 | BC | 0.083 | 1.467 |
| | | CD | 0.080 | 1.422 |
| | | AB | 0.091 | 1.623 |
| 7 | 5.588 | BC | 0.091 | 1.623 |
| | | CD | 0.091 | 1.623 |
| | | AB | 0.094 | 1.678 |
| 8 | 5.560 | BC | 0.094 | 1.678 |
| | | CD | 0.094 | 1.678 |
| | | Series 3. | | |
| | | AB | 0.106 | 1.891 |
| 1 | 5.573 | BC | 0.106 | 1.891 |
| | | CD | 0.112 | 2.001 |
| | | AB | 0.114 | 2.040 |
| 2 | 5.561 | BC | 0.111 | 1.981 |
| | | CD | 0.111 | 1.981 |
| | | AB | 0.108 | 1.934 |
| 3 | 5.566 | BC | 0.107 | 1.912 |
| | | CD | 0.106 | 1.889 |
| | | AB | 0.116 | 2.076 |
| 4 | 5.581 | BC | 0.114 | 2.041 |
| | | CD | 0.112 | 2.004 |
| | | AB | 0.094 | 1.688 |
| 5 | 5.622 | BC | 0.094 | 1.688 |
| | | CD | 0.096 | 1.724 |
| | | AB | 0.084 | 1.509 |
| 6 | 5.634 | BC | 0.084 | 1.509 |
| | | CD | 0.083 | 1.490 |

this was sufficient in some cases to change the stress from a tension to a compression or vice versa.

21. *Diagrams and Tables.*—Stress-strain diagrams representing the general average results of the axial and circumferential gage lines are given in Fig. 17, 18, and 19, while sample diagram showing the average results at different sections of the tube for both the tension-tension and the compression-tension experiments are given in Fig. 13 to 16. Diagrams of the experimental results of Series 1 and 2 are to be found in Fig. 22, and those of Series 3 in Fig. 23. A comparison of the theories of the strength of materials under combined stress is made in Fig. 24, while Fig. 26 and 27 illustrate some of the work of other investigators. An outline of the test specimens and tests and the principal data of the tubes are given in Tables 1 and 2. Table 3 is given as a sample of the data for a single tube, tube No. 4 of Series 1. These data have been reduced and corrected for standard bar readings. All the original and reduced data as well as the stress-strain diagrams are on file at the Laboratory of Applied Mechanics of the University of Illinois.

## IV. Discussion of Results.

22. *The Criterion of Strength.*—There are three possible stress limits any one of which may be the criterion of the strength of material —limit of proportionality, yield point, and rupture or ultimate strength. It is recognized that there may be a sharp distinction between the laws governing ductile materials and the laws governing brittle materials, for such a distinction is observed in the stress-strain diagrams and in compression and torsion failures. Since this discussion is limited to ductile materials, conditions will be treated only as they apply to such materials.

It would appear at first thought that the limit of proportionality would be the proper basis upon which to determine the relative strength of material. The mathematical theory of elasticity is based upon Hooke's law generalized, engineering practice bases its computations largely upon this same law, and several investigators have used the limit of proportionality (which they called the elastic limit) as their criterion, notably Hancock* and Turner.†

The limit of proportionality, or p-limit, is defined as the stress at which the constancy of the ratio of stress to strain ceases; that is, the modulus of elasticity is a constant up to this stress. It is often stated

*American Society for Testing Materials, 1905, '06, '07, '08.
†Engineering, London, February 5, 1909.

## Table 3.

### Test Data of Axial Gage Lines Tube No. 4, Series 1.

Ratio of Circumferential Tension to Axial Tension 0.69.

Inside Diameter of Tube 5.563.

AB Gage Lines

*Average Thickness of Tube .085 in. Area of Section 1.508 sq. in*

| Water Pressure lb. per sq. in. | | Axial Load due to W. Pressure | Machine Load pounds | Total Axial Load pounds | Av. Axial Unit Stress lb. per sq. in. | Reading on Gage Line | | | | Differences | | | | Av. Diff. | Av Unit Elongation |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Gage Rdg. | Corrected | | | | | AB1 | AB2 | AB3 | AB4 | AB1 | AB2 | AB3 | AB4 | | |
| 100 | 100 | 2430 | 5100 | 7500 | 4950 | 80.1 | 10.9 | 28.0 | 62.5 | 0 | 0 | 0 | 0 | 0. | 0 |
| 300 | 300 | 7290 | 14100 | 21400 | 14200 | 77.0 | 5.9 | 2..9 | 58.1 | 3.1 | 5.0 | 6.1 | 4.4 | 4.7 | .00024 |
| 500 | 500 | 12150 | 23050 | 35200 | 23300 | 72.5 | 0.0 | 17.1 | 527 | 7.6 | 10.9 | 10.9 | 9.8 | 9.8 | 00049 |
| 700 | 700 | 17000 | 32100 | 49100 | 32500 | 66.1 | 94.0 | 11.9 | 478 | 14.0 | 16.9 | 16.1 | 147 | 15.4 | .00077 |
| 900 | 800 | 21900 | 41100 | 63000 | 41700 | 61.0 | 86.9 | 5.0 | 41.9 | 19.1 | 24.0 | 23.0 | 20.6 | 21.7 | 00109 |
| 950 | 950 | 23100 | 43400 | 66500 | 44000 | 60.3 | 84.1 | 3.5 | 409 | 198 | 26.8 | 24.5 | 21.6 | 23.2 | 00116 |
| 1000 | 1000 | 24300 | 45600 | 69900 | 46200 | 59.0 | 82.0 | 1.8 | 39.7 | 21.1 | 28.9 | 26.2 | 22.8 | 24.8 | 00124 |
| 1050 | 1050 | 25500 | 47900 | 73400 | 48500 | 57.3 | 79.0 | 99.0 | 379 | 22.1 | 319 | 29.0 | 24.6 | 26.9 | 00135 |
| 1100 | 1090 | 26500 | 50100 | 76600 | 50900 | 55.2 | 76.4 | 97.7 | 36.1 | 24.9 | 34.5 | 30.3 | 26.4 | 29.0 | .00145 |
| 1150 | 1140 | 27700 | 52400 | 80100 | 53100 | 53.1 | 73.3 | 95.3 | 340 | 27.0 | 37.6 | 327 | 28.5 | 31.5 | 00158 |
| 1200 | 1190 | 29200 | 54500 | 83500 | 55300 | 50.9 | 69.0 | 93.0 | 32.1 | 29.2 | 41.9 | 35.0 | 30.4 | 341 | 00171 |
| 1250 | 1240 | 30100 | 56800 | 86900 | 57600 | 47.0 | 65.3 | 88.0 | 28.2 | 33.1 | 456 | 40.0 | 34.3 | 38.3 | 00192 |
| 1300 | 1290 | 31300 | 59100 | 90400 | 60000 | 30.9 | 57.9 | 74.7 | 16.1 | 49.2 | 530 | 53.3 | 46.4 | 50.5 | C0253 |

BC Gage Lines

*Average Thickness of Tube .084 in Area of Section 1.481 sq. in.*

| Water Pressure lb per sq. in. | | Axial Load due to W Pressure | Machine Load pounds | Total Axial Load pounds | Av Axial Unit Stress lb. per sq. in. | Reading on Gage Line | | | | Differences | | | | Av Diff. | Av Unit Elongation |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Gage Rdg. | Corrected | | | | | BC1 | BC2 | BC3 | BC4 | BC1 | BC2 | BC3 | BC4 | | |
| 100 | 100 | 2430 | 5100 | 7500 | 5060 | 92.1 | 79.5 | 35.0 | 63.5 | 0 | 0 | 0 | 0 | 0 | 0 |
| 300 | 300 | 7290 | 14100 | 21400 | 14400 | 87.0 | 748 | 29.9 | 58.4 | 5.1 | 4.7 | 5.1 | 5.1 | 5.0 | .00025 |
| 500 | 500 | 12150 | 23050 | 35200 | 23800 | 81.9 | 69.0 | 24.7 | 53.6 | 10.2 | 10.5 | 10.3 | 9.9 | 10.2 | .00051 |
| 700 | 700 | 17000 | 32100 | 49100 | 33100 | 76.0 | 63.9 | 19.1 | 480 | 16.1 | 15.6 | 15.9 | 15.5 | 15.8 | .00079 |
| 900 | 900 | 21900 | 41100 | 63000 | 42500 | 68.9 | 57.2 | 13.0 | 42.9 | 23.2 | 22.3 | 22.0 | 20.6 | 22.0 | .00110 |
| 950 | 950 | 23100 | 43400 | 66500 | 44800 | 67.2 | 55.0 | 11.0 | 41.2 | 24.9 | 24.5 | 24.0 | 22.3 | 23.9 | 00120 |
| 1000 | 1000 | 24300 | 45600 | 69900 | 47200 | 65.0 | 53.5 | 103 | 40.0 | 27.1 | 26.0 | 24.7 | 23.5 | 25.4 | .00127 |
| 1050 | 1050 | 25500 | 47900 | 73400 | 49400 | 62.5 | 51.5 | 8.4 | 38.5 | 29.6 | 28.0 | 26.6 | 25.0 | 27.3 | 00137 |
| 1100 | 1090 | 26500 | 50100 | 76600 | 51900 | 59.9 | 49.2 | 6.9 | 36.6 | 32.2 | 30.3 | 28.1 | 26.9 | 29.4 | .00147 |
| 1150 | 1140 | 27700 | 52400 | 80100 | 54100 | 56.3 | 47.4 | 4.8 | 34.3 | 35.8 | 32.1 | 30.2 | 29.2 | 31.8 | .00159 |
| 1200 | 1190 | 29200 | 54500 | 83500 | 56400 | 52.3 | 44.8 | 1.5 | 303 | 39.8 | 34.7 | 33.5 | 33.2 | 35.3 | 00177 |
| 1250 | 1240 | 30100 | 56800 | 86900 | 58600 | 478 | 41.5 | 97.3 | 23.1 | 44.3 | 38.0 | 37.7 | 40.4 | 40.1 | 00201 |
| 1300 | 1290 | 31300 | 59100 | 90400 | 61100 | 30.0 | 30.0 | 73.8 | 91.0 | 62.1 | 49.2 | 61.2 | 72.5 | 61.3 | .00307 |

CD Gage Lines

*Average Thickness of Tube .083 in. Area of Section 1.472 sq. in*

| Water Pressure lb. per sq. in. | | Axial Load due to W. Pressure | Machine Load pounds | Total Axial Load pounds | Av. Axial Unit Stress lb. per sq. in. | Reading on Gage Line | | | | Differences | | | | Av. Diff. | Av Unit Elongation |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Gage R'd'g. | Corrected | | | | | CD1 | CD2 | CD3 | CD4 | CD1 | CD2 | CD3 | CD4 | | |
| 100 | 100 | 2430 | 5100 | 7500 | 5100 | 68.2 | 44.9 | 95.0 | 40.9 | 0 | 0 | 0 | 0 | 0 | 0 |
| 300 | 300 | 7290 | 14100 | 21400 | 14600 | 64.1 | 39.5 | 89.8 | 34.9 | 4.1 | 54 | 5.2 | 6.0 | 5.2 | .00026 |
| 500 | 500 | 12150 | 23050 | 35200 | 23900 | 59.0 | 347 | 84.0 | 30.9 | 92 | 10.2 | 11.0 | 10.0 | 10.1 | .00051 |
| 700 | 700 | 17000 | 32100 | 49100 | 33300 | 53.0 | 29.0 | 78.9 | 25.0 | 15.2 | 15.9 | 16.1 | 15.9 | 15.8 | .00079 |
| 900 | 900 | 21900 | 41100 | 63000 | 42900 | 47.3 | 23.2 | 74.0 | 19.4 | 20.9 | 21.7 | 21.0 | 21.5 | 21.3 | .00107 |
| 950 | 950 | 23100 | 43400 | 66500 | 45100 | 46.1 | 22.0 | 73.0 | 18.3 | 22.1 | 22.9 | 22.0 | 22.6 | 22.4 | .00112 |
| 1000 | 1000 | 24300 | 45600 | 69900 | 47500 | 43.9 | 20.0 | 71.3 | 17.0 | 24.3 | 24.9 | 23.7 | 23.9 | 24.2 | .00121 |
| 1050 | 1050 | 25500 | 47900 | 73400 | 49700 | 41.0 | 17.1 | 69.3 | 15.2 | 27.2 | 27.8 | 25.7 | 25.7 | 26.6 | .00133 |
| 1100 | 1090 | 26500 | 50100 | 76600 | 52200 | 38.8 | 14.1 | 67.0 | 13.9 | 29.4 | 30.8 | 28.0 | 270 | 29.0 | .00145 |
| 1150 | 1140 | 27700 | 52400 | 80100 | 54400 | 35.4 | 11.1 | 65.0 | 11.8 | 32.8 | 33.7 | 30.0 | 29.1 | 31.4 | .00157 |
| 1200 | 1190 | 29200 | 54500 | 83500 | 56800 | 320 | 7.4 | 62.5 | 9.0 | 36.2 | 37.5 | 32.5 | 31.8 | 34.5 | .00173 |
| 1250 | 1240 | 30100 | 56800 | 86900 | 59000 | 26.9 | 4.0 | 55.0 | 4.5 | 41.3 | 407 | 40.0 | 36.4 | 39.6 | .00198 |
| 1300 | 1290 | 31300 | 59100 | 90400 | 61500 | 8.0 | 86.0 | 47.2 | 90.0 | 60.2 | 58.9 | 47.8 | 50.9 | 54.5 | .00273 |

that the distinction between yield point and p-limit is very slight and that it really makes no material difference which is used. But a glance at the stress-strain diagrams in Fig. 13 to 16, will show that in some cases the modulus of elasticity changes and that the diagram consists of a broken line instead of a straight line nearly up to the yield point. This fact, due to the lack of isotropy in the material and to the mechanical work done upon it, makes it difficult to get consistent results by using the p-limit as a criterion. When the material has been cold worked, the stress-strain diagram often curves away from a straight line slowly and the exact point of departure is not easily located. Special treatment of the material usually affects the yield point in the same way in different specimens, but not the p-limit.

The use of rupture or ultimate strength as a criterion of the strength of ductile materials still persists in the case of simple stresses, and specifications ordinarily require that the ultimate strength of the material shall have a certain value. But this is an indirect measure of the toughness rather than of the strength, and in the best specifications the yield point (or elastic limit as it is frequently but incorrectly called) is specified as well. Conditions at rupture give no indication of those existing at the yield point and whatever value a knowledge of the conditions attending rupture in a ductile material may have, no conclusions can be drawn from them which may safely be applied to the period preceding the yield point. As engineering design deals principally with stresses within the yield-point stress, rupture cannot be considered as the criterion, even though Bridgman* in his tests on thick cylinders uses it and decries the use of the yield point. When the distribution of stress is unknown and no extensometers are used to measure strains, rupture is the only criterion available.

For ductile material that has not been worked cold, the stress-strain diagram shows a very decided change in character when the material passes the yield point. When the material has been cold-rolled or cold-drawn, the yielding is more gradual and the curve, instead of breaking sharply, departs more gradually from a straight line. If the specimen of the cold-rolled or cold-drawn material is tested in simple tension with an extensometer, and the load is slowly but steadily applied, the roll of the curve is apparent a short time before the yield point is registered by the drop of the beam.

As all the investigations hereinafter described were made with instruments to measure the strains, some criterion must be adopted that

*Phil. Mag., July, 1912.

is applicable to a stress-strain diagram. The first deviation from a straight line (p-limit) is an indefinite point to locate, and, after considering everything that has been noted above, the method proposed by the late J. B. Johnson was adopted. This is called by him the "apparent elastic limit," although it is here taken as the yield point. This method empirically locates a point at which there is evidently some plastic action and furnishes a very convenient method for comparison of results. It is defined as the unit-stress at which "the rate of deformation is 50 per cent greater than it is at zero stress." Fig. 12 shows the application to a stress-strain diagram. Let O B E be a stress-strain diagram drawn in the usual manner. Then A O B is the angle determining the slope

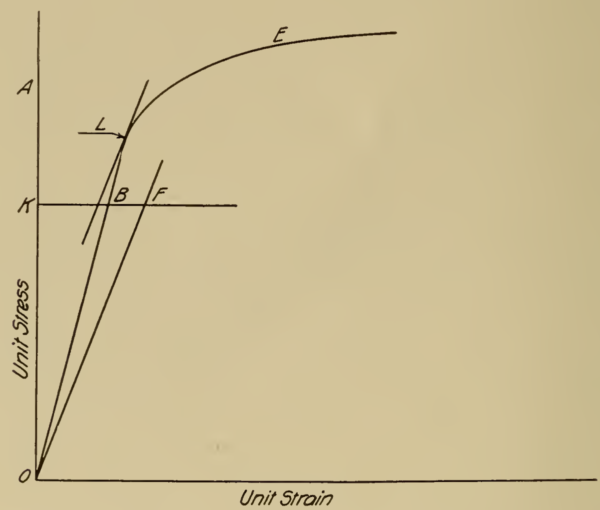
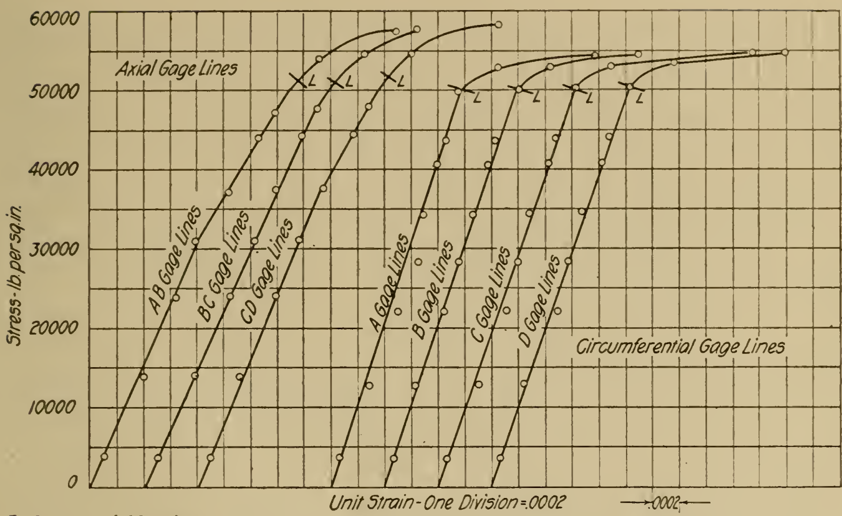



FIG. 12. STRESS-STRAIN DIAGRAM SHOWING JOHNSON'S APPARENT ELASTIC LIMIT.

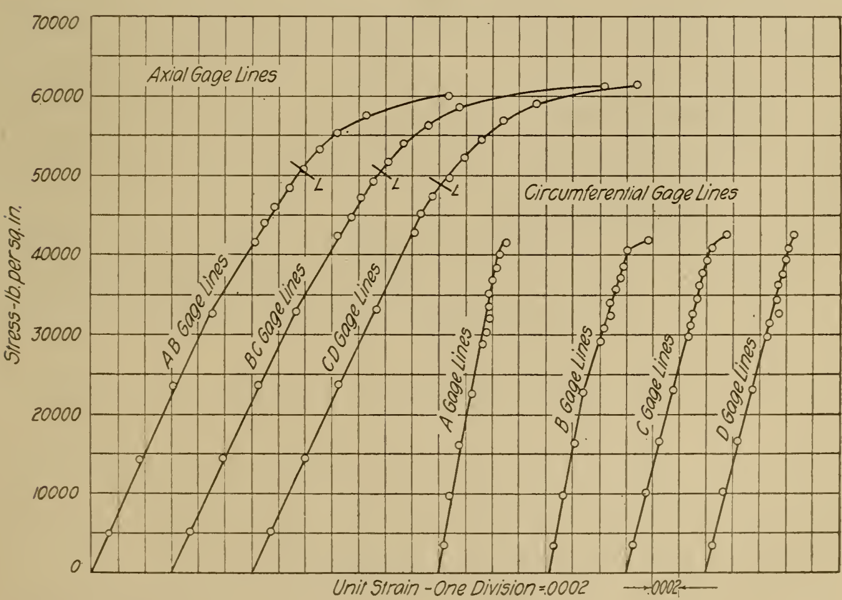
at zero stress. At any point K lay off horizontally a distance K F equal to 1.50 times K B. Then O F is the slope 50 per cent greater than the slope at zero stress. A parallel to O F drawn tangent to the curve B E, locates the point of tangency L and the corresponding stress is the yield-point stress.

23. *Strength.*—In the tabulation of the results of the tests of tubes under biaxial stress, the average of the strains measured on the four gage lines intersected by any cross section was taken as the strain at that section of the tube. Thus, the strains for the axial gage lines, A B1, A B2, A B3, and A B4 of a tube (for notation see page 23 and




L denotes yield point

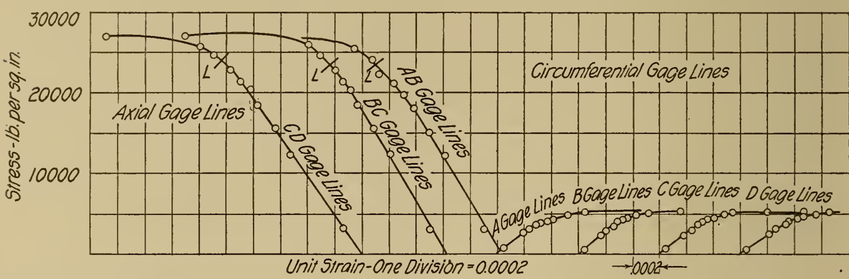
FIG. 13. STRESS-STRAIN DIAGRAMS FOR TUBE NO. 3, SERIES 1. RATIO OF CIRCUMFERENTIAL TO AXIAL TENSION, 0.94.




L denotes yield point

FIG. 14. STRESS-STRAIN DIAGRAMS FOR TUBE NO. 4, SERIES 1. RATIO OF CIRCUMFERENTIAL TO AXIAL TENSION, 0.69.

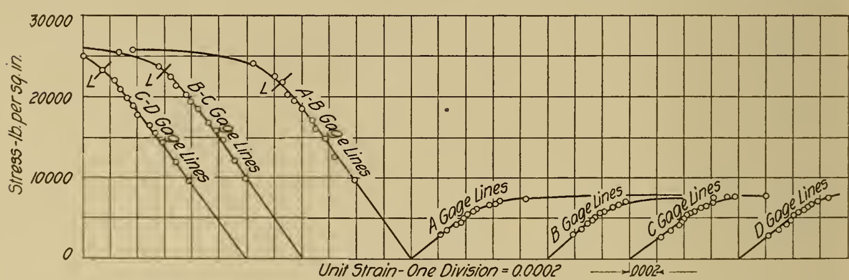
Fig. 9) were averaged, and this average is taken as the strain of the A B gage lines of that tube. Likewise for the B C and C D gage lines. For the circumferential gage lines, 1–2–A, 2–3–A, 3–4–A, and 4–1–A were averaged; that is, the four gage lines made a complete traverse of the circumference. There are then three sets of average results for the




L denotes yield point

FIG. 15. STRESS-STRAIN DIAGRAMS FOR TUBE NO. 2, SERIES 3. RATIO OF CIRCUMFERENTIAL TENSION TO AXIAL COMPRESSION, 0.20.

axial gage lines and four for the circumferential gage lines of each tube. The curves formed from these average results (see Fig. 13 to 16 for samples) were then used to obtain the general average results for each

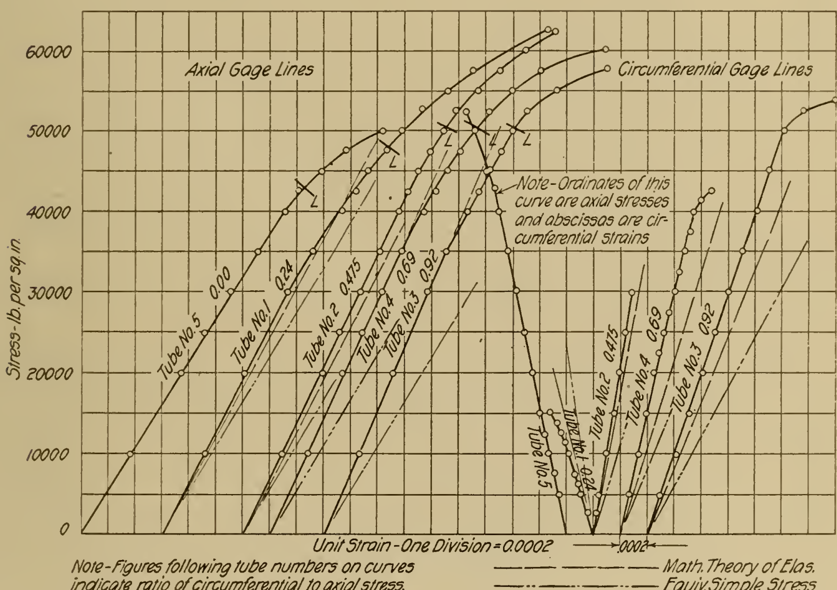



L denotes yield point

FIG. 16. STRESS-STRAIN DIAGRAMS FOR TUBE NO. 5, SERIES 3. RATIO OF CIRCUMFERENTIAL TO AXIAL TENSION, 0.30.

of the tubes. That is, the general average results for the circumferential strains represent the average obtained from all the circumferential gage lines in any one tube, and the general average results for the axial strains the average obtained from all the axial gage lines. The only exception is in the case of tube No. 1, Series 1, where the averages of the A B gage lines are omitted in the general average. Each general average curve represents the average results of twelve axial gage lines or sixteen circumferential gage lines. These general average stress-strain diagrams are given in Fig. 17 to 19.

The yield-point stresses are quite uniform for the different sets of gage lines and in close agreement with those of the general average curves. Because of this uniformity, the use of the general average curves as a basis of comparison seems justified. The circumferential strains are plotted with the apparent circumferential tensile stresses as ordinates, except in the case of the tubes where no internal pressure was applied. In these cases the ordinates are the axial stresses, so that it




L denotes yield point

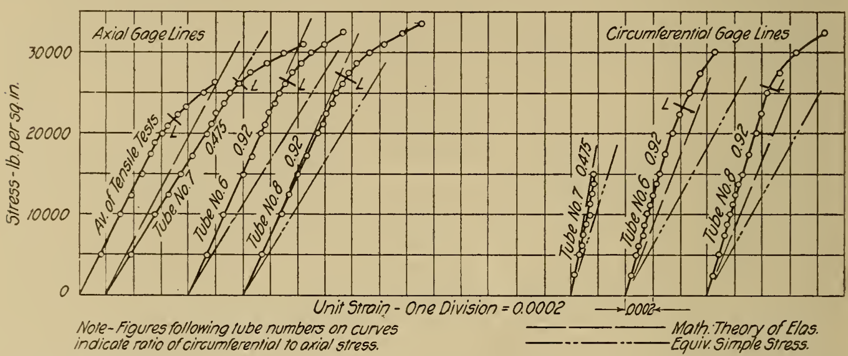
FIG. 17. STRESS-STRAIN DIAGRAMS SHOWING GENERAL AVERAGES FOR SERIES 1. TENSION WITH TENSION.

is easy to determine Poisson's ratio, which is given in Fig. 19 by the ratio of abscissas, corresponding to the same stress, on the two curves of tube 4, such as $r$ to $r'$.

If diagrams are drawn having the yield-point unit-stresses as ordinates and the ratio of the circumferential tension to axial tension or axial compression as abscissas, a comparison can be made with the results reached by the different theories. For the combination of tension with tension, the maximum stress theory and the maximum shear theory demand that the yield-point stress shall be constant for all ratios.

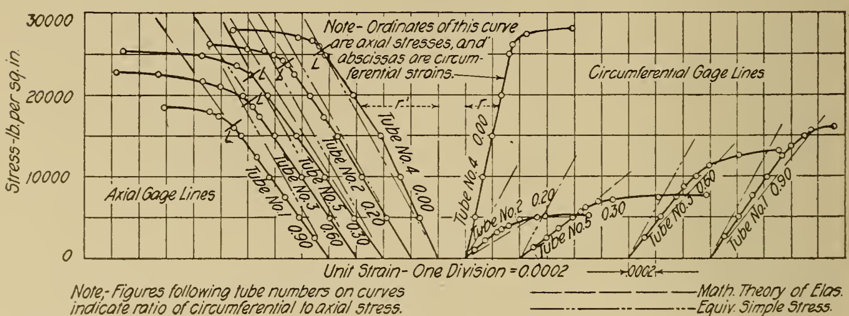
Mohr's theory and the internal friction theory have the same requirements; Wehage's theory demands a reduction in the yield-point stress and the maximum strain theory demands an increase in proportion to the increase of stress ratio.

For the combination of compression with tension the maximum stress theory demands that the yield-point stress shall be constant for all




L denotes yield point

FIG. 18. STRESS-STRAIN DIAGRAMS SHOWING GENERAL AVERAGES FOR SERIES 2. TENSION WITH TENSION.




L denotes yield point

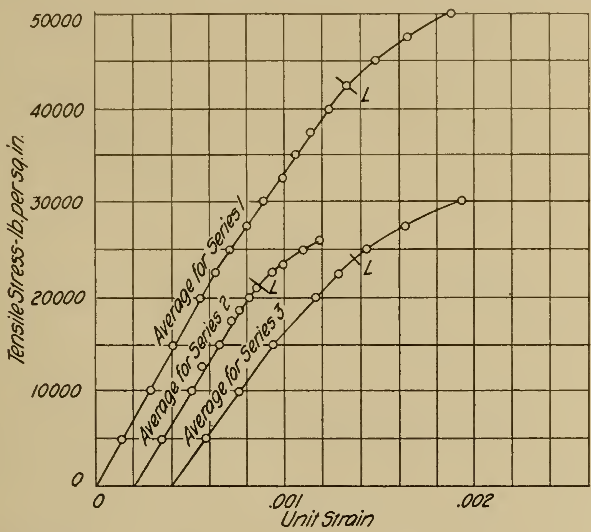
FIG. 19. STRESS-STRAIN DIAGRAMS SHOWING GENERAL AVERAGES FOR SERIES 3. COMPRESSION WITH TENSION.

stresses, while the maximum strain theory, the internal friction theory, the maximum shear theory, and Mohr's theory demand a decrease in the yield-point stress as the stress ratio increases.

What is the law that governs? Referring to the stress-strain curves of the general averages of the axial gage lines, Fig. 17, 18, and 19, it will be seen that for Series 1 and 2 as the stress ratio increases the yield-point stress rises unmistakably until the value of the stress ratio of

0.50 is reached. Beyond this the yield-point stress remains constant, no matter what the stress ratio. For Series 3 the yield-point stress steadily diminishes as the stress ratio increases.

Since for the case of compression combined with tension all the




L denotes yield point

Fig. 20. Tension Tests of Small Specimens From Tubes of Series 1, 2 and 3. Results of Ten Tests for Each Series.

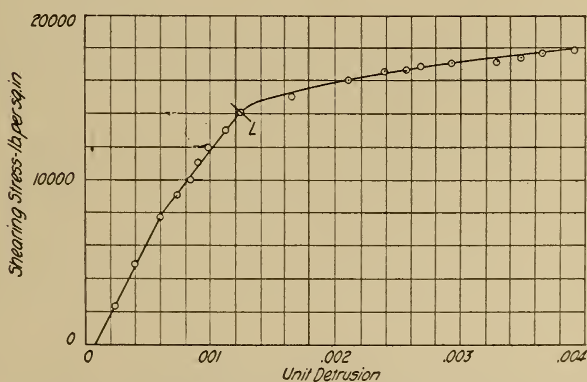



Fig. 21. Torsion Test of Tube No. 6, Series 3.

theories except one demand a decrease of the yield-point stress as the stress ratio increases, while for tension combined with tension the maximum strain theory is the only one which calls for the increase that has been observed. Series 1 and 2 will be discussed first and the results of Series 3 compared with the conclusions drawn from the results of Series 1 and 2.

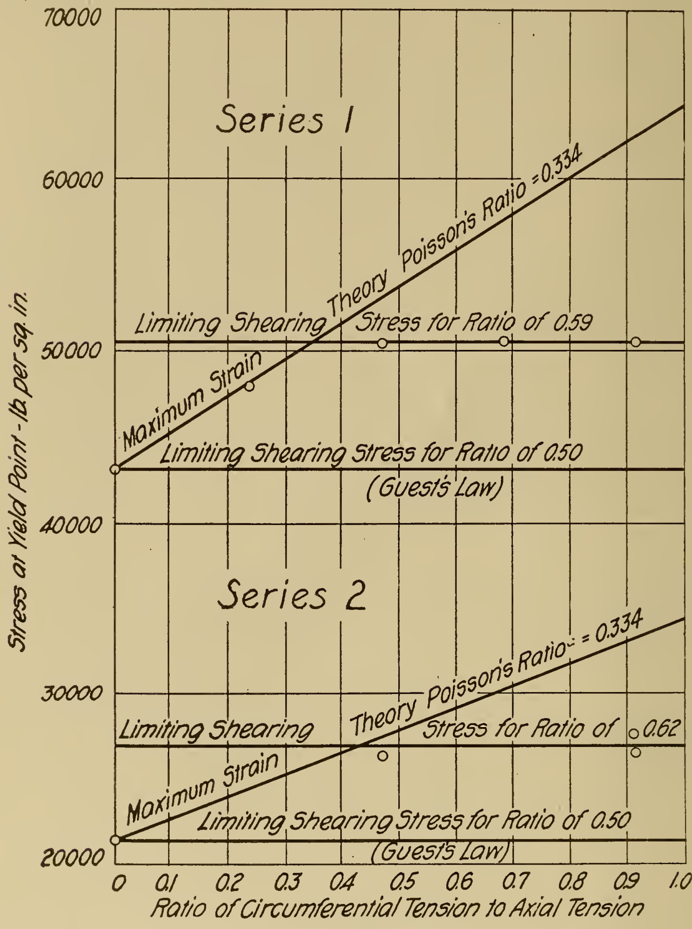



FIG. 22. DIAGRAM GIVING YIELD POINT STRESSES AND STRESS RATIOS FOR SERIES 1 AND 2.

In Fig. 22 the yield-point stresses of Series 1 and 2 are plotted against the ratio of circumferential tension to axial tension. The line of the maximum strain theory is then drawn through the yield-point stress determined in simple tension (stress ratio zero), the inclination being determined by Poisson's ratio (0.334). For Series 1 the yield-point stress was taken from the test of tube No. 5 and for Series 2 the average of the tension tests of twenty small specimens cut from tubes of Series 2 was taken.*

The determination of Poisson's ratio is discussed on page 46. A line of constant yield-point stress is drawn which best fits the experimental points for stress ratios of 0.50 or above. It is seen that the line of the maximum strain theory fits the experimental points up to its intersection with the line of constant yield-point stress, and that thereafter the line of constant yield-point stress well fits the points. This line of constant yield-point stress may also be a line of constant shearing stress. If, as Fig. 22 seems to indicate, tension ceases to be a governing factor and the shearing stress becomes dominant, two things must be true for the line of constant yield-point stress:

(a) The shearing unit-stress must actually reach the shearing yield-point stress as determined by tests in pure shear, and

(b) Since the shear is one-half the maximum principal stress, this maximum principal stress must remain a constant.

The first condition is important only in so far as showing that the shearing yield-point stress must be greater than one-half the tensile yield-point stress; otherwise the shear would be dominant at all times. The latter is the contention of the maximum shear theory. Counting compression a negative tension and with the principal unit-stresses numbered in the order of their magnitude, $p_1$, $p_2$, $p_3$, the criterion for shearing stress is:

$$\text{Shearing unit-stress} = \tfrac{1}{2}\,(p_1 - p_3),$$

but as the third principal stress is zero, this reduces to $\tfrac{1}{2}\,p_1$. The water pressure inside the tube does not constitute a third principal stress (compressive), for all the readings of the strains were taken on the outside of the specimen where the third principal stress was undoubtedly zero, if the atmospheric pressure is neglected.

The maximum shear theory carried to its logical conclusion requires that the yield-point stress of the material subjected to two stresses of like sign at right angles shall not vary from that reached in simple tension, for the shear is the determining factor at all times. If the theory holds in this form, a horizontal line drawn through the

*The tension test of tube No. 9, Series 2, the first test made, did not furnish the necessary data on account of an unexpectedly low yield point. It is thought that the use of the yield-point stress obtained from the average curve for the specimens from the tubes of Series 2 (see Fig. 20) is justified because the break of the curve of the small specimens from the tubes of Series 1 agrees closely with the break in the curve obtained from tube No. 5 of that series (axial load only), 42,500 lb. per sq. in. and 43,000 lb. per sq. in. respectively. The yield-point stress obtained from the average curve of the specimens from the tubes of Series 2 (21,500 lb. per sq. in.) has been taken as the yield-point stress in simple tension of Series 2 and the value of Poisson's ratio obtained from Series 1 has been used for Series 2.

yield-point stress in simple tension should pass through all the points. Instead, it touches only the initial point. Experiments* have shown that for ductile material the ratio of the shearing yield-point stress, obtained by torsion tests, to the tensile yield-point stress varies with the material, but usually lies between 0.55 and 0.65, in the majority of

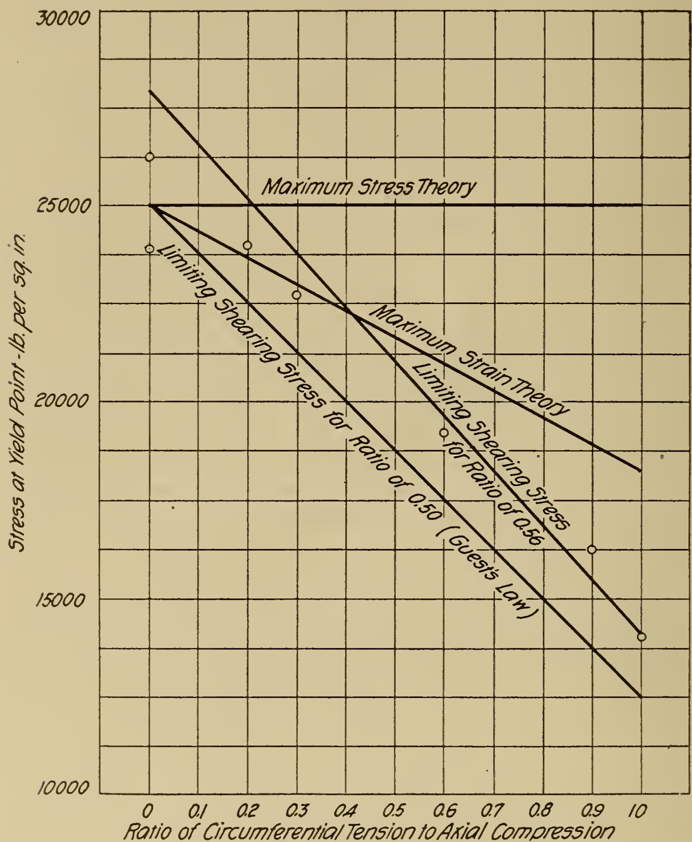



FIG. 23. DIAGRAM GIVING YIELD POINT STRESSES AND STRESS RATIOS FOR SERIES 3.

tests ranging near 0.60, which is the commonly accepted value. A few tests show a ratio less than 0.50, but they are relatively small in number. With a ratio of 0.60, the shearing yield-point stress line would lie above the line through the yield-point stress in simple tension by an amount equal to 0.20 of the latter stress. The exact location of the line will

*L. B. Turner, Engineering, London, February 5, 1909.

vary with the material, but as long as the ratio of the yield-point stresses is above 0.50, there is the hiatus between this condition and that demanded by the above form of the maximum shear theory.

The horizontal line through the experimental points in Fig. 22 is evidently the limit of the shearing strength. It corresponds to a ratio of shearing yield-point stress to tensile yield-point stress of 0.59 for Series 1 and 0.62 for Series 2, which values agree well with the majority of experiments.

These tests indicate that there are two laws covering the case of combined stress when the stresses are both tension and act in two directions at right angles. Apparently the point at which the change in law occurs depends upon the ratio of the yield-point stress in shear to that in tension and the change from one law to the other may occur at different ratios of the principal stresses for different materials. It is important to establish this ratio of yield-point stresses, for if it is not approximately constant the use of combined stress formulas will require a knowledge of such a ratio for all materials.

Before discussing Series 3, the two laws just referred to will be applied to the other combinations of stress and a comparison made with the maximum stress theory, the maximum strain theory, and the maximum shear theory. Assuming the ratio of the shearing and tensile yield-point stresses to be 0.60 and the tensile and compressive yield-point stresses equal, the co-ordinates of the rectangle A B C D (Fig. 24) represent the maximum stress theory, the rhombus Q K J L the maximum strain theory, and the figure A $K_1$, B C $L_1$, D A the maximum shear theory. The line A M $K_2$, N B represents the two laws in the tension-tension quadrant, while B R S C represents them in the tension-compression quadrant. The lines M $K_2$ and $K_2$N are parallel to the axes and at such a distance from them that the ordinate of M $K_2$ and the abscissa of $K_2$N are each 1.20 times O A or O B, the tensile yield-point stress. R S is parallel to B C and at such a distance from it that one-half the sum of the ordinate and abscissa of any point between R and S is equal to 0.60 of O B or O C. The construction of the other two quadrants is such that the figure is symmetrical about the bisectors of the quadrants. The diagrams, Fig. 22, showing the comparison of theory and experiment for Series 1 and 2 correspond to A M $K_2$ in the tension-tension quadrant.

In Fig. 23 the yield-point stresses of Series 3 are plotted as ordinates and the stress ratios of circumferential tensile stress to axial compressive stress as abscissas. Before discussing the various theories in connection with the experimental results, the starting points of the theoretical lines must be fixed. The maximum shear theory demands the same yield-point stress in tension and in compression; the maximum strain theory and the maximum stress theory do not. From the average curve of ten specimens cut from a ten-inch remnant of the original tubing from

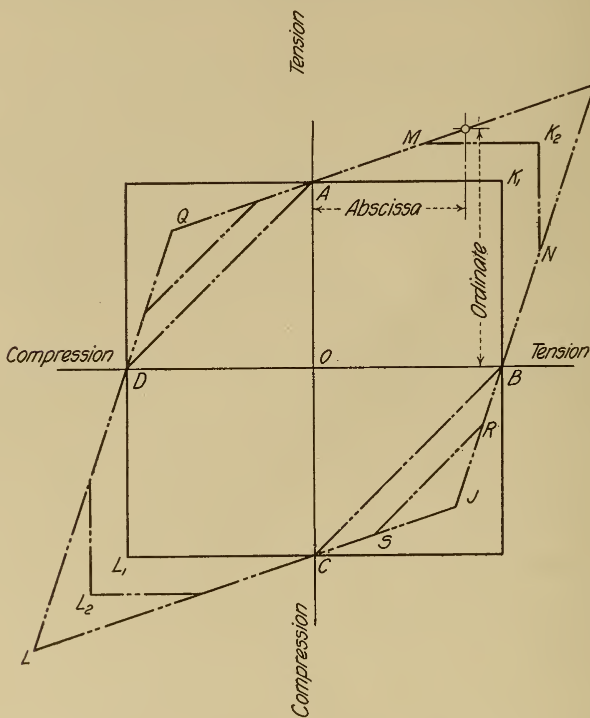



FIG. 24. REPRESENTATION OF YIELD POINT STRENGTHS FOR COMBINED STRESSES ACCORDING TO THE MAXIMUM STRESS THEORY, THE MAXIMUM STRAIN THEORY, AND THE MAXIMUM SHEAR THEORY.

which the tube specimens of Series 3 were cut, (Fig. 20) the tensile yield-point stress was found to be 24,000 lb. per sq. in. The compressive yield-point stress obtained from tube No. 4 (no internal water pressure) was 26,250 lb. per sq. in. With the demand of the maximum shear theory for equal yield-point stress in tension and in compression it seems correct to take as the initial point of the line of that theory the average of these values, or 25,100 lb. per sq. in. The lines of the maximum strain theory and of the maximum stress theory were also drawn through this average value of the yield-point stresses and compressive yield-point stresses The use of this average value is believed to be justified by the observed fact that the yield-point stress of low-carbon steel in tension is found in nearly all cases to have the same numerical value as the yield-point stress in compression. In determining the line for the maximum strain theory a value of Poisson's ratio of 0.395 was used. This value was obtained from the test of tube No. 4, Series 3 (Fig. 19). A line of constant shearing stress has been drawn through the yield-point stress (14,000 lb. per sq. in.) obtained from the torsion test, since simple torsion produces tensile and compressive stresses of equal intensities and hence corresponds to a stress ratio of unity (see Fig. 21). The lines of internal friction theory and of Mohr's theory practically coincide with the maximum shear theory.

An inspection of Fig. 23 shows that for Series 3 as well as for Series 1 and 2, the experimental results follow the maximum strain theory up to a certain stress ratio and then follow a line of constant shear which is the maximum shear developed. The ratio of the shearing yield-point stress from the torsion test to the average of the tensile and compressive yield-point stresses is 0.56. The question of the neglect of water pressure as a third stress does not enter in this series, for taking the stresses in the order of their magnitude the compression due to the water pressure becomes intermediate between the circumferential tension and the axial compression, so that the maximum shearing stress is equal to one-half the sum of the axial compressive stress and the circumferential stress. This series leads to the same conclusions as the other two, although the ratio of the shearing and tensile yield-point stresses is somewhat lower.

The net result of this investigation as it affects the strength of steel under combined stress in two directions at right angles to each other—biaxial loading—is that instead of a single law, whatever its nature, as has heretofore been assumed, there are two distinct laws governing the strength of the material, each law dominant within its limits. These two laws are the maximum strain theory and the maximum shear theory; the first governs until the shearing yield-point stress of the material is reached, after which the shear theory holds. The exact point of the change from one law to the other depends upon the ratio of the shearing yield-point stress to the yield-point stress in simple tension and compression.

24. *Stiffness.*—Although strains have been measured in many tests

heretofore made, no attempt seems to have been made to determine the law of stiffness. It has been taken for granted that the deductions of the mathematical theory of elasticity, as embodied in St. Venant's theory, hold, or else no attention has been paid to strains except as related to the strength of the material in the determination of the yield point or so-called elastic limit. The weakness of the mathematical theory of elasticity lies in its generalization of Hooke's law and the neglect of the temperature changes, so that the strains obtained in tests of isotropic materials will only closely approximate the computed values. The effect of shear in producing strain has been neglected and is small before the yield-point stress is reached, but the variation of shearing strength in different directions throughout the specimen, the possibility of a change in Poisson's ratio with increasing stress, the possibility of a different Poisson's ratio and modulus of elasticity with and across the direction of rolling or drawing, enter to complicate the problem. The material experimented upon is not the isotropic substance assumed in the theory. Lines have been drawn on the stress-strain curves of the general averages of the axial gage lines, Fig. 17, 18, and 19, giving the strains as computed by the mathematical theory of elasticity using the values of Poisson's ratio* and modulus of elasticity obtained from tests in simple tension and in compression. These lines agree quite closely with the observed values except in the case of tube No. 1 of Series 1, and tube No. 7 of Series 2, the former showing lower strains and the latter greater strains than the computed values. Apparently within the range of application of the mathematical theory of elasticity, where $E$ is constant, the strains follow the theory with sufficient exactness to say that the theory holds. Lines have also been drawn to represent the strains corresponding to a simple tensile or compressive stress equal to the greater principal stress.

For the circumferential lines there have been drawn on the stress-

*The values of Poisson's ratio for the tubes tested in simple compression and in simple tension were obtained by dividing the circumferential unit-strain taken from the general average curves of these tubes (which is the same as the diametral unit-strain) by the corresponding axial unit-strain. For Series 1 this ratio for tube No. 5 is 0.334; for Series 3, obtained from tube No. 4, it is 0.395. The modulus of elasticity of Series 1 is 27,200,000 lb. per sq. in., and for Series 3 it is 29,500,000 lb. per sq. in. An examination of the axial and circumferential stress-strain diagrams of tube No. 5, Series 1, Fig. 17, and of tube No. 4, Series 3, Fig. 19, shows that in the first case (tension) Poisson's ratio remains practically constant, diminishing about 6 per cent after the yield-point stress has been passed, but that in the second case (compression) this ratio increases to almost 0.50 after the yield-point stress has been passed. There is no reason why Poisson's ratio should be constant for all kinds of steel, and it may well be that tension and compression tests on the same material will show different results. It is not known what the effect of the hollow specimen is in changing this ratio for tension or compression tests, but it is thought that the method used to obtain Poisson's ratio is accurate and reliable. It is to be noted that for the compression tests the value of both Poisson's ratio and the modulus of elasticity are higher than for the tension tests.

strain curves of the general averages, Fig. 17, 18, and 19, lines giving the strains computed by the mathematical theory of elasticity and also the strains accompanying simple tensile stresses equal to the circumferential stresses. The values of Poisson's ratio and the modulus of elasticity are taken the same as for the axial lines. The lines of the mathematical theory of elasticity do not fit as well as in the case of the axial strains. It can be seen that to fit the experimental points of Series 1 and 2, it is necessary to use a higher value for both the modulus of elasticity and Poisson's ratio, the latter requiring the greater change. An increase in Poisson's ratio will increase the strains of tube No. 1 and lower those of the other tubes of these two series. An increase of the modulus of elasticity will diminish all the strains proportionally. It will be recalled that the value of Poisson's ratio obtained in compression tests was high, 0.395. For Series 3, where Poisson's ratio is higher than for Series 1 and 2, the modulus of elasticity alone need be increased. With a higher modulus the computed circumferential curves fit the experimental curves quite closely except for tube No. 5. There is a strong probability that both Poisson's ratio and the modulus of elasticity vary in the two directions with and across the direction of drawing. It is scarcely probable that the law changes, and the close agreement between the computed and observed values for the axial strains gives strong support to the belief that all the strains follow the requirements of the mathematical theory of elasticity. The indications are that the modulus of elasticity and Poisson's ratio may be different in different directions throughout the steel, in much the same way that Bauschinger has shown that the shearing strength of rolled steel varies in different directions.

In Series 1 and 2, Fig. 17 and 18, for the tubes tested with a stress ratio of 0.92, the yield-point stress in the circumferential direction was practically the same as the yield-point stress in the axial direction, but the circumferential curves show a more sudden yielding of the material. In Series 3, Fig. 19, for the tube tested with a stress ratio of 0.90, the circumferential yield-point stress was lower than that in an axial direction. All the circumferential stress-strain curves of Series 3 show a sharp, sudden break when the yield-point stress in the axial direction is reached, no matter what the circumferential stress was. Granting that for Series 3 the value of Poisson's ratio increases to 0.50 above the yield-point stress, this is not sufficient to account for the great increase in the strains. It must be that the shearing stresses, which have passed the shearing yield-point stress, produce shearing strains of sufficient magnitude to account for this increase in the circumferential strains. This explanation is more strongly suggested by the stress-strain curves of Series 1 and 2, where Poisson's ratio remains nearly constant. The circumferential stress-strain diagrams that continue to show an increase in strain after the axial yield-point stress has been passed are those from the tubes whose axial yield-point stresses lie on the line of constant shear of Fig. 22. Those that do not show an increase at this time are from the tubes whose axial yield-point stresses lie on the line of the maximum strain theory.

That the shearing strains accompanying the axial stress can affect the circumferential strains is shown by the stress-strain diagrams for the circumferential lines of tube No. 4, Series 1, Fig. 14. The circumferential stress-strain diagram continues straight for a short distance after the yield-point stress has been passed in the axial direction, the circumferential stress corresponding to the axial yield-point stress being 34,500 lb. per sq. in., approximately. This is during the stage intermediate between the elastic and plastic conditions. When the axial curve breaks sharply, the circumferential curve changes direction also. If Poisson's ratio were the only factor, all the diagrams, with the possible exception of those of tubes No. 3, 6, and 8, where a high stress ratio was used, would show diminishing strains with increasing stress after the yield-point stress in the axial direction had been passed. This means that the tendency to reduce the diameter of the tube, due to the rapidly increasing axial strains, would be greater than the tendency to increase the diameter produced by the increase of the water pressure. But the curves of tubes No. 2 and 4 show an increasing strain (increasing tube diameter) even though the circumferential stresses were well below the yield-point stress of the material. These two tubes are the ones whose yield-point stresses lie on the line of constant shear, Fig. 22, and without the assistance of the shearing strains in producing circumferential strains, the curves of these two tubes would show a diminishing circumferential strain as the circumferential stress increased after the axial yield-point stress had been passed. The shear which causes yielding in an axial direction is on a different plane from that causing yielding in a circumferential direction. The former shear acts along a plane which passes through the direction line of the circumferential tension and cuts the axis of the tube at an angle of 45°. The latter shear acts on a plane which passes through the direction line of the axial tension and is parallel to the axis of the tube making an angle of 45° with the direction line of the circumferential stress. These shearing stresses are of different magnitudes, according to the ratio of the stresses, and each is equal to one-half the principal stress cut by its plane at an angle of 45°.

Apparently the strains follow the requirements of the mathematical theory of elasticity for all stress ratios, but there may be different values of Poisson's ratio and the modulus of elasticity for the axial and circumferential directions. After the yield-point stress in one direction has been passed the shearing strains have a considerable influence upon the deformations in the second direction.

25. *Comparison With the Methods and Results of Other Investigations.*—Attention is called to several points of difference between the method of investigation here recorded and the methods used by others. The greatest difference lies in the use of a portable extensometer to measure strains, the strain gage, whereby a large number of measurements were taken, both along the specimen and around it. Previous investigations used a fixed extensometer which measured strains along one or two gage lines, or, in some cases, used no strain measurements. No assumptions of uniform stress distribution were made, in the present series, for the strain gage records the variations and the gage length can be varied to suit the needs. With readings taken on a large number of gage lines for every load increment, a certain positiveness of result is attained which is impossible with attached instruments and few gage lines. Local effects are thus minimized. Another difference lies in the larger size of the specimens tested and in the smaller ratio of thickness of tube wall to diameter. Because of the form of specimen and the method of applying load, the stress was nearly uniform throughout the specimen; there was no "helping" effect by understressed material, no point of maximum stress to be located. The use of Johnson's apparent elastic limit method for determining yield-point stress gives a definite point for comparison.

An attempt was made to keep a definite ratio between circumferential and axial stresses throughout the test of each tube, so that comparison might be made later for these ratios. As far as possible, it was intended with a set of specimens cut from a given length of tubing to cover the entire range of stress ratio within tension-tension or compression-tension quadrants. The experiments reported by others and referred to in this section show generally a haphazard ratio of stresses, and the loads used were such that a definite stress was produced in one direction and then the other stress was increased until yielding took place.

The earliest important investigation of this subject was that reported by J. J. Guest* in 1900. The tests were made upon small steel, copper, and brass tubes about 1¼ in. outside diameter and varying in thickness from 0.025 in. to 0.034 in. Tests were made in combined torsion and

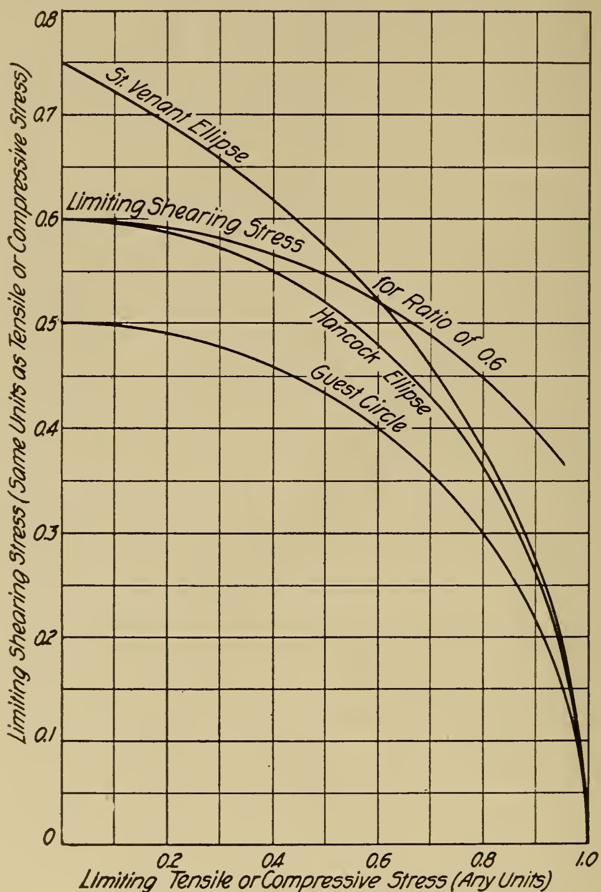



FIG. 25. RELATION BETWEEN SHEARING STRESSES DUE TO TORSION AND THE TENSILE OR COMPRESSIVE STRESSES DUE TO AXIAL LOAD OR BENDING.

axial tension, in torsion and circumferential tension, and in axial and circumferential tension. The strains were measured by a two-point extensometer, and although it was attached to the outside of the tube, the full hydrostatic pressure was counted as a third principal stress (compressive). Other than the tests on tube No. 1 of Guest's investi-

*Phil. Mag., 1900.

gation, there are but two tests where the stress ratio of circumferential tensile stress to axial tensile stress is 0.50 or less, and test No. 1 and one of the others follow the maximum strain theory closely. The yield-point stress was used as the basis of comparison, and each test was carried just beyond the yield point. Criticism may be made of the repeated use of the same specimen, since the yield-point stress is raised by repeated loading beyond the yield-point stress of the first test. It is not stated whether the tubes were annealed between tests. The results are taken to justify the maximum shear theory, and in the main they do within the field investigated, since the majority of the tests had a stress ratio between 0.50 and 1.00 within which limits the shear theory undoubtedly holds. The tests also show that the maximum shear developed is greater than one-half the yield-point stress in simple tension.

Following Guest comes the work of C. A. M. Smith,* W. A. Scoble,† E. L. Hancock,‡ and Wm. Mason* on bars and tubes in torsion and tension or compression and on small tubes in compression and internal pressure. All these results are used to justify the maximum shear theory which demands that the shearing yield-point stress is equal to one-half the yield-point stress in simple tension. With one exception, however, that of Scoble's tests reported in 1906, the maximum shear developed is greater than one-half the yield-point strength in tension, which, as noted above, was also found in Guest's tests. The majority of these tests—like Guest's—are in the region where the stress ratio is greater than 0.50. These tests cover the entire four quadrants of combined stress.

The tests of Professors Smith and Hancock will be shown on diagrams similar to Fig. 25 (Fig. 26 and 27), in which the ordinates represent the shearing stress due to torque and the abscissas represent the tensile or compressive stress due to axial load or bending. The diagram of Fig. 25 will be discussed before the tests are taken up. The shearing stress is plotted to twice the scale of the tensile or compressive stress. If a circle with a radius equal to the tensile yield-point stress is drawn with O as a center, it will represent the relation between the shearing and the direct stress which produces a combined stress causing yielding required by the maximum shear theory. It will be observed that the shearing yield-point stress must therefore equal exactly one-half the tensile yield-point stress. A circle with radius equal to the shearing yield-point stress obtained from tests in simple torsion (0.6 the tensile yield-

*Inst. Mech. Engrs., 1909.
†Phil. Mag., 1906
‡Am. Soc. for Testing Materials, 1905, 6, 7, 8.

point stress) is shown; also, the ellipse representing the St. Venant or maximum strain theory, beginning at the tensile yield-point stress. The two laws as advanced in this bulletin require that the maximum strain theory hold to the intersection of the St. Venant ellipse and the circle for limiting shearing stress for ratio of 0.6, and that then the shear

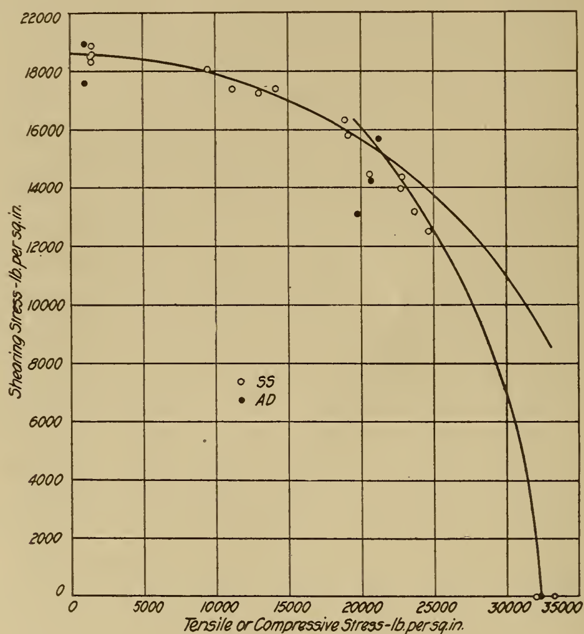



FIG. 26. RESULTS OF TESTS, BY C. A. M. SMITH.

shall govern. Hancock's ellipse has been added to show how closely he came to the results here advanced.

The results of Smith's and Hancock's tests have been plotted in Fig. 26 and 27. Both compression and tension have been plotted on the same side of the diagrams (symmetry permitting this), and the results of the different tests have been changed proportionally in order to compare them with a single set of theoretical curves. A comparison of the two laws herein proposed with the experimental results of these investigations show that the experimental results fit these laws better than the maximum shear theory which the tests were taken to prove.

Fig. 26 shows the results of C. A. M. Smith's tests on S. S. and

A. D. steel. Professor Smith maintains that the shearing yield-point stress of steel is one-half the tensile yield-point stress within small limits and quotes Turner's tests* to prove his point. His own tests do not bear out his contention and Turner's tests show considerable variation, averaging about 0.54 for this ratio. Professor Smith's tests are examples of careful work, but the interpretation of the tests as an unqualified endorsement of the maximum shear law cannot be accepted.

Mr. Scoble's tests seem to indicate that the shearing yield-point stress is lower than half the tensile yield-point stress. This result may possibly be accounted for by the way the shearing yield-point stress was located. This stress was taken at the intersection of the straight line of the elastic portion of the stress-strain diagram with a line drawn through the diagram beyond the yield point. Since a stress-strain curve for torsion breaks more quickly than a tension curve, it may be that the determination of the shearing yield-point stresses are affected by this. Scoble's method of measuring the bending moment by means of the deflection of the beam may be in error, for the law of deflection under the combined stress would be influenced by the very law he was seeking to determine.

The results of Professor Hancock's tests† are shown in Fig. 27. The curves of the maximum shear theory and the maximum strain theory have been drawn as well as his ellipse. Hancock used the p-limit as his criterion. He alone of these investigators realized the shortcomings of the maximum shear theory and endeavored to remedy them by fitting an ellipse to the experimental results. The ellipse fits quite closely, but while it is a close approximation, it does not fit the results as closely as do the curves representing the two laws herein proposed. His ellipse is empirical, while the combination of the maximum strain theory with the maximum shear theory has a foundation in the theory of the strength of materials.

Since torsion combined with compression or tension can be resolved into a case of tension combined with compression, Smith's and Hancock's tests fall in the fourth quadrant and show the applicability of the two laws there.

Mason's tests on tubes in compression and internal pressure show that the maximum shearing stress developed is greater than the shearing stress developed in simple compression. The average of all his

*Engineering, London, February 5, 1909.

†Proceedings of the American Society for Testing Materials, 1908.

tests in which a constant stress ratio of one to one was used, gives this maximum shearing stress as 0.60 of the compressive yield-point stress. As all the tests had the one stress ratio, it is not possible to make a comparison with theories, but the point thus located falls on the line of the two laws.

Minor inconsistencies are to be expected in experimental work of this nature, both on account of the variation in the material tested and

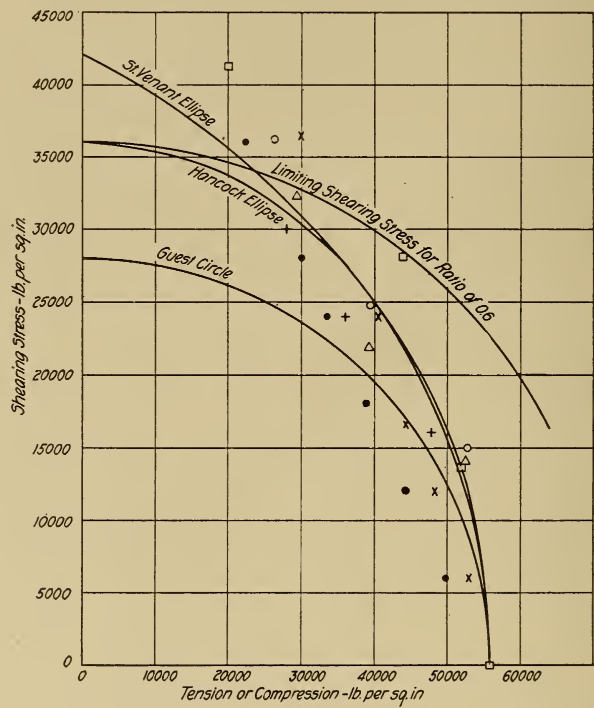



Fig. 27. Results of Tests, by E. L. Hancock.

on account of the apparatus used. The number of tests made by these investigators is insufficient to establish completely any theory, but a careful study of the published data will lead to the conclusion that the two laws, the theory here advanced, conform more closely to the experimental results than any single law.

26. *Summary and Conclusions.*—The following summary deals with the method of investigation and with the deductions which have been made from the data. As this is the first investigation of combined stress wherein a portable strain measuring instrument—such as the strain

gage—has been used, it is felt that considerable emphasis may be laid upon this fact. The size of the specimen is much larger than any heretofore used. These conditions tend to give more trustworthy results.

The experimental conclusions are:

1. The use of a portable strain measuring instrument is a decided advantage since it makes it possible to take measurements on a large number of gage lines for each increment of load, obviating to a large extent the effect of local variations in the test specimen.
2. The use of large tubes with thin walls gives quite uniform stress distribution, the yield-point stress is more positively determined, and the effect of eccentricity of loading is less than with solid bars on account of the larger diameter of the tube.
3. With large tubes the thickness of the tube walls can be accurately determined.
4. Flat plates in cross bending give uneven distribution of stress and are not satisfactory for biaxial loading tests.

The deductions which have been made from the experimental data are:

5. With increasing values of the ratio of the biaxial stresses the yield-point strength follows the maximum strain theory until the value of the shearing stress reaches the shearing yield point, then the shearing stress controls according to a maximum shear theory. There are thus two independent laws each dominant within proper limits instead of some single law as has heretofore been assumed.
6. Because these two laws govern the strength of ductile materials under biaxial loading, the ratio for simple stresses of the shearing yield-point stress to the tensile yield-point stress is important.
7. The stiffness follows the requirements of the mathematical theory of elasticity for all stress ratios, but the values of Poisson's ratio and the modulus of elasticity may be different in the two directions, with and across the rolling and drawing of the steel.
8. The results of the tests reported by previous investigators conform better to the two laws of strength than to any single law.

## APPENDIX I.

### Bibliography.

Basquin, O. H., The Circular Diagram of Stress and Its Application to the Theory of Internal Friction. Proc. Western Society of Engineers, November, 1912, page 815.

Bridgman, P. W., Breaking Tests under Hydrostatic Pressure and Conditions of Rupture. Philosophical Magazine, July, 1912, page 63.

Crawford, W. J., The Elastic Strength of Flat Plates, an Experimental Research. Proc. Royal Society of Edinburgh, 1911-12, page 348.

Cook and Robertson, The Strength of Thick Hollow Cylinders under Internal Pressure. Engineering, London, December 15, 1911, page 787.

Editorial in Engineering, London, November 29, 1912, page 745.

Guest, J. J., On the Strength of Ductile Materials under Combined Stress. Phil. Mag., July, 1900, page 69.

Gulliver, G. H., Deformations in Metals. Proc. Inst. Mech. Engrs., February, 1905, page 141.

On the Cohesion of Steel and on the Relation Between the Yield Points in Tension and Compression. Proc. Royal Society of Edinburgh, 1907-8, page 374.

On the Effect of Internal Friction in Cases of Compound Stress. Proc. Royal Society of Edinburgh, 1908-9.

Internal Friction in Loaded Materials. International Association for Testing Materials, 1909, VIII.

A New Experimental Method of Investigating Certain Systems of Stress. Proc. Royal Society of Edinburgh, 1909-10, page 38.

Hancock, E. L., Results of Tests of Materials Subjected to Combined Stress. American Society for Testing Materials, 1905-6-7-8. Philosophical Magazine, October, 1906; February, 1908.

Hatt, W. K., Moduli of Elasticity for Compound Stress. Proc. Purdue Society of C. E., 1897.

Lanza, G., Applied Mechanics, 1910.

Lawson and Capp, Thermo-Electric Indication of Strain as a Testing Method. International Association for Testing Materials, 1912, $IX_8$.

Love, A. E. H., The Mathematical Theory of Elasticity, 1904.

Malaval, M., Notes on the Strength of Cylindrical Tubes Stressed to Rupture. International Association for Testing Materials, 1912, $X_5$.

Mallock, A., The Extension of Cracks in an Isotropic Material. Proc. Royal Society of London, 1909, page 26.

Some Unclassified Mechanical Properties of Solids and Liquids. Proc. Royal Society of London, 1912, page 466.

Mason, Wm., Mild Steel Tubes in Compression and under Combined Stress. Inst. Mech. Engrs., 1909, page 1205.

Matsumura, T., and Hamabe, G., Tests on Combined Bending and Torsional Strength of Cast Iron. Memoirs of the College of Engineering, Kyoto Imperial University, February, 1915.

Mesnager, M., Deformations des Metaux. Comptes Rendues, Vol. 126, 1898, page 515.

MOHR, O., Welche Umstände bedingen die Elastizitäts grenze und den Bruch eines Materials? Zeitschrift des Vereins Deutcher Ingenieure, 1900, page 1530.

RASCH, EW., Method for Determining Elastic Strength by Means of Thermo-Electric Measurements. International Association for Testing Materials, 1909, $VII_3$.

SCOBLE, W. A., The Strength and Behavior of Ductile Materials under Combined Stress. Philosophical Magazine, December, 1906, page 533.

Ductile Materials under Combined Stress. Philosophical Magazine, 1910, page 116.

Tests of Brittle Materials under Combined Stress. Philosophical Magazine, 1910, page 908.

Report on Combined Stress. British Assn. for the Adv. of Science, 1913, page 168.

SLATER and MOORE, Use of the Strain Gage in the Testing of Materials. American Society for Testing Materials, Vol. 13, 1913, page 1019.

SMITH, C. A. M., Compound Stress Experiments. Inst. Mech. Engrs., 1909, page 1237.

Some Experiments on Solid Steel Bars under Combined Stress. Engineering, London, August 20, 1909.

The Strength of Pipes and Cylinders. Engineering, London, March 5, 1909, page 327.

Guest's Law of Combined Stress. Engineering, London, April 23, 1909, page 545.

TALBOT and SLATER, Tests of Reinforced Concrete Buildings. Bulletin 64, Engineering Experiment Station, University of Illinois.

TURNER, C. A. P., Thermo-Electric Measurement of Stress. Trans. Am. Soc. C. E., 1902, page 26.

WEHAGE, H., Die Zulässige Anstrengung eines Materials bei Belastung nach Mehreren Richtungen. Zeitschrift des Vereins Deutcher Ingenieure, July, 1905, page 1077.

WILLIAMS, I., A Determination of Poisson's Ratio. Philosophical Magazine, 1912, page 886.

## APPENDIX II. MATHEMATICAL TREATMENT.

1. *Stresses and Strains.*—The analysis of stress and strain in elastic materials known as the mathematical theory of elasticity embodies the most complete and elaborate theory of the action of elastic bodies under stress. The following brief presentation of the mathematical theory of elasticity as it applies to the problem of the investigation follows largely the treatment of Love.* It will be desirable to outline briefly the work leading up to the derivation of the general equations of the mathematical theory of elasticity connecting stress and strain before taking up the derivation of the equations of stress and strain in a cylinder under internal pressure and an axial load.

In the theory of elasticity the relations between three sets of magnitudes must be considered.

1. The displacements of the points of the strained body. If the ordinary rectangular system of coordinates is used for reference, the displacement $s$ of a point due to the strain is resolved into components $u, v, w$ parallel respectively to the $X$, $Y$ and $Z$ axes.

2. The strain components. Let $\epsilon_1$, $\epsilon_2$, $\epsilon_3$ denote the strains in the directions of the $X$, $Y$, $Z$ axes, respectively; then

$$\epsilon_1 = \frac{\partial u}{\partial x}, \epsilon_2 = \frac{\partial v}{\partial y}, \epsilon_3 = \frac{\partial w}{\partial z} \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

The components of shearing strain are defined as follows:

$$\epsilon_{23} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}, \epsilon_{31} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}, \epsilon_{12} = \frac{\partial v}{\partial x} + \frac{\partial u}{\partial y} \quad \ldots\ldots\ldots (2)$$

Here $\epsilon_{23}$ denotes the shearing strain in the plane $YZ$, etc. Along with the strain components may be included the components of the rotation of an element of the body. If the displacement involves a rotation $\omega$ of the element as a whole and this rotation be resolved into $X$, $Y$ and $Z$ components, then these components are given by the relations

$$\omega_1 = \frac{\partial w}{\partial y} - \frac{\partial v}{\partial z}, \omega_2 = \frac{\partial u}{\partial z} - \frac{\partial w}{\partial x}, \omega_3 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \quad \ldots\ldots\ldots (3)$$

3. The stress components. The six stress components may be denoted by $\sigma_1$, $\sigma_2$, $\sigma_3$; $\sigma_{23}$, $\sigma_{31}$, $\sigma_{12}$. $\sigma_1$ is the stress in the direction of the $X$-axis on a plane perpendicular to the $X$-axis; similarly for $\sigma_2$ and $\sigma_3$. $\sigma_{23}$ is the stress in the direction of the $Y$-axis over a plane perpendicular to the $Z$-axis; therefore it is a shearing stress.

*The Mathematical Theory of Elasticity, A. E. H. Love, 1904.

The stress components must satisfy certain conditions of equilibrium, which are expressed by three equations of the following type (assuming that the body forces, such as gravity, may be neglected, and that the body is at rest).

$$\frac{\partial \sigma_1}{\partial x} + \frac{\partial \sigma_{12}}{\partial y} + \frac{\partial \sigma_{13}}{\partial z} = 0 \quad \ldots\ldots\ldots\ldots\ldots\ldots (4)$$

The six strain components $\epsilon_1$, $\epsilon_2$, etc., and the six stress components are connected by certain relations. Hooke's law, the linear relation between stress and strain, is the basis of these relations. Each stress component is taken as a linear function of the six strain components; thus

$$\left.\begin{array}{l} \sigma_1 = a_1\epsilon_1 + a_2\epsilon_2 + a_3\epsilon_3 + a_4\epsilon_{23} + a_5\epsilon_{31} + a_6\epsilon_{12} \\ \sigma_2 = b_1\epsilon_1 + b_2\epsilon_2 + b_3\epsilon_3 + b_4\epsilon_{23} + b_5\epsilon_{31} + b_6\epsilon_{12} \end{array}\right\} \ldots\ldots\ldots (5)$$

etc.

A consideration of the work done in deforming a body leads to the conclusion that there must exist a so-called strain-energy function $V$, such that

$$\sigma_1 = \frac{\partial V}{\partial \epsilon_1}, \ \sigma_2 = \frac{\partial V}{\partial \epsilon_2}, \text{ etc.}$$

It follows that the function $V$ must be a homogeneous quadratic function of the six strain components and must have therefore 21 terms. The number of coefficients apparently 36 in eq. (5) is thereby reduced to 21 by relations of the form $a_2 = b_1$, $a_3 = c_1$, $a_4 = d_1$, etc.; that is,

$V$ is the symmetric determinant of the quadric $\sum_{1}^{6} c_{ij}\, \epsilon_{ij}$.

If the body is isotropic, these 21 coefficients can be reduced to two. Denoting by $\lambda$ one of these remaining coefficients and by $\mu$ one-half the difference between the two coefficients, the following relations between stresses and strains are established:

$$\left.\begin{array}{l} \sigma_1 = \lambda\Delta + 2\mu\epsilon_1 \\ \sigma_2 = \lambda\Delta + 2\mu\epsilon_2 \\ \sigma_3 = \lambda\Delta + 2\mu\epsilon_3 \end{array}\right\} \ldots\ldots\ldots\ldots\ldots\ldots (6)$$

where $\sigma_1$, $\sigma_2$ and $\sigma_3$ are the stresses along the $X$, $Y$ and $Z$ axes respectively and $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$ are the corresponding strains. $\Delta$ is the dilatation and is equal to the sum of $\epsilon_1$, $\epsilon_2$ and $\epsilon_3$.

Let $\frac{1}{m}$ = Poisson's ratio

$E$ = the modulus of elasticity.

Applying the relations between stress and strain to a bar in simple tension, the following relation between Poisson's ratio and the modulus of elasticity is established. Both $\frac{1}{m}$ and $E$ are to be determined from tests in simple tension or compression.

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \qquad (7)$$

$$\frac{1}{m} = \frac{\lambda}{2(\lambda + \mu)} \qquad (8)$$

$G =$ the shearing modulus of elasticity $= \mu$.

$$G = \frac{1}{2}\left(\frac{m}{m+1}\right)E \qquad (9)$$

The values of $\lambda$ and $\mu$ may now be established in terms of $E$ and $\frac{1}{m}$.

$$\lambda = \frac{Em}{(m+1)(m-2)} \qquad (10)$$

$$\mu = \frac{Em}{2(m+1)} \qquad (11)$$

Adding the second and third equations of (6)

$$\sigma_2 + \sigma_3 = 2\lambda\,\Delta + 2\mu\,(\epsilon_2 + \epsilon_3)$$
$$= 2\lambda\,\Delta + 2\mu\,(\Delta - \epsilon_1)$$

$$\Delta = \frac{\sigma_2 + \sigma_3 + 2\mu\epsilon_1}{2(\lambda + \mu)}$$

$$\lambda\,\Delta = \frac{1}{m}(\sigma_2 + \sigma_3 + 2\mu\epsilon_1)$$

$$\sigma_1 - 2\mu\,\epsilon_1 = \frac{1}{m}(\sigma_2 + \sigma_3) + \mu\epsilon_1\left(\frac{\lambda}{\lambda + \mu}\right)$$

$$\left.\begin{aligned} \sigma_1 &= \frac{1}{m}(\sigma_2 + \sigma_3) + E\epsilon_1 \\ \text{Similarly}\quad \sigma_2 &= \frac{1}{m}(\sigma_1 + \sigma_3) + E\epsilon_2 \\ \sigma_3 &= \frac{1}{m}(\sigma_1 + \sigma_2) + E\epsilon_3 \end{aligned}\right\} \qquad (12)$$

Rearrangement of Eq. (12) gives

$$\left.\begin{aligned} E\epsilon_1 &= \sigma_1 - \frac{1}{m}(\sigma_2+\sigma_3) \\ E\epsilon_2 &= \sigma_2 - \frac{1}{m}(\sigma_1+\sigma_3) \\ E\epsilon_3 &= \sigma_3 - \frac{1}{m}(\sigma_1+\sigma_2) \end{aligned}\right\} \qquad (13)$$

These are the three fundamental equations connecting stress and strain. $E\epsilon_1$, $E\epsilon_2$ and $E\epsilon_3$ are called by various writers the reduced stresses, the true stresses, or the ideal stresses.

2. *Stresses and Strains in a Thin Tube.*—For bodies of cylindrical form it is convenient to use cylindrical coordinates $r$, $\theta$ and $z$ instead of the rectangular system $x$, $y$, $z$. The $z$ coordinate is measured parallel to the axis of the tube, $r$ denotes the radial distance from the axis, and $\theta$ the angle of an axial plane from some chosen initial plane.

Denoting by $u$, $v$ and $w$ the displacement components, as before ($u$ radial, $w$ axial and $v$ perpendicular to a radius $r$) the three strain components $\epsilon_r$, $\epsilon_\theta$, $\epsilon_z$ are given by the relations

$$\epsilon_r = \frac{\partial u}{\partial r},\ \epsilon_\theta = \frac{1}{r}\frac{\partial v}{\partial \theta} + \frac{u}{r},\ \epsilon_z = \frac{\partial w}{\partial z} \qquad (14)$$

and the corresponding stress components are given by the equations

$$\left.\begin{aligned} \sigma_r &= \lambda\Delta + 2\mu\epsilon_r \\ \sigma_\theta &= \lambda\Delta + 2\mu\epsilon_\theta \\ \sigma_z &= \lambda\Delta + 2\mu\epsilon_z \end{aligned}\right\} \qquad (15)$$

Expressions for the shearing strain and stress components may be deduced, but they are not needed in the present investigation.

In the case of a hollow cylinder under internal pressure, conditions of symmetry require that the displacement $v$ shall be zero; hence the expression for $\epsilon_\theta$ in (14) reduces to $\epsilon_\theta = \frac{u}{r}$. Furthermore, it is permissible in the case under consideration to assume a condition of plane strain, in which all points in a cross section of the cylinder experience the same displacement $w$ in the $z$ direction. With this assumption, $w = az$, where $a$ is a constant, and therefore $\epsilon_z = a$. With these simplifying assumptions, the expression for the dilatation takes the form

$$\Delta = \frac{\partial \mu}{\partial r} + \frac{u}{r} + a$$

If now general expressions for the displacements $u$ and $w$ are found, eq. (14) will give the strain components and eq. (15) the stress components. The conditions of equilibrium must, however, be satisfied, that is relations analogous to (4) must be introduced. It is possible, however, to eliminate the stress components by the aid of eq. (6) or eq. (15) and thus to express the equilibrium conditions in terms of displacements only. Thus from (4) and (3) may be derived the relation

$$(\lambda+2\mu)\frac{\partial\Delta}{\partial x}-2\mu\left(\frac{\partial\omega_3}{\partial y}-\frac{\partial\omega_2}{\partial z}\right)=0 \quad \ldots\ldots\ldots\ldots (16)$$

with two similar; and in cylindrical coordinates a similar process leads to the relation

$$(\lambda+2\mu)\frac{\partial\Delta}{\partial r}-2\mu\left(\frac{1}{r}\frac{\partial\omega_3}{\partial\theta}-\frac{\partial\omega_2}{\partial z}\right)=0 \quad \ldots\ldots\ldots\ldots (17)$$

In the case under consideration the rotation components $\omega_2$ and $\omega_3$ must be zero ($\omega_2$ may have a small finite value at the extreme ends of the tube), hence (17) reduces to

$$(\lambda+2\mu)\frac{\partial}{\partial r}\left(\frac{\partial u}{\partial r}+\frac{u}{r}+a\right)=0 \quad \ldots\ldots\ldots\ldots (18)$$

Integration of this equation leads to the following relation for the displacement:

$$u=Cr+\frac{D}{r}$$

in which $C$ and $D$ are constant.

Introducing this expression for $u$ in the expression for $\epsilon_r$, $\epsilon_\theta$ and $\Delta$, the result is

$$\epsilon_r=\frac{\partial u}{\partial r}=C-\frac{D}{r^2}$$

$$\epsilon_\theta=\frac{u}{r}=C+\frac{D}{r^2}$$

$$\Delta=2C+a$$

Hence the relations (15) become

$$\sigma_r=\lambda(2C+a)+2\mu\left(C-\frac{D}{r^2}\right)$$

$$=2C(\lambda+\mu)-2\mu\frac{D}{r^2}+\lambda a \quad \ldots\ldots\ldots\ldots (20)$$

$$\sigma_\theta\,\lambda=(2C+a)+2\mu\left(C+\frac{D}{r^2}\right)$$

$$=2C(\lambda+\mu)+2\mu\frac{D}{r^2}+\lambda a \quad \ldots\ldots\ldots\ldots (21)$$

$$\sigma_z=\lambda(2C+a)+2\mu a$$

$$=2C\lambda+(\lambda+2\mu)a \quad \ldots\ldots\ldots\ldots (22)$$

To determine the constants $C$ and $D$, we have the conditions
$\sigma_r = -p_1$ the internal pressure, when $r = r_1$, the internal radius
$\sigma_r = -p_0$ the external pressure, when $r = r_0$, the external radius
From (20)

$$-p_1 = 2C(\lambda+\mu) - 2\mu\frac{D}{r^2} + \lambda a$$

$$-p_0 = 2C(\lambda+\mu) - 2\mu\frac{D}{r^2} + \lambda a$$

whence

$$2\mu D = (p_1 - p_0)\frac{r_1^2 r_0^2}{r_0^2 - r_1^2} \qquad (23)$$

$$2C(\lambda+\mu) = \frac{p_1 r_1^2 - p_0 r_0^2}{r_0^2 - r_1^2} - \lambda a \qquad (24)$$

Equations (23) and (24) may be written

$$2\mu D = T \qquad (23a)$$

$$2C(\lambda+\mu) = S - \lambda a \qquad (24a)$$

It will be observed that $S = \frac{p_1 r_1^2 - p_0 r_0^2}{r_0^2 - r_1^2}$ gives the mean intensity of tensile stress in a cross-section of a closed tube due to the internal fluid pressure $p_1$, with external pressure $p_0$. In the test the axial stress was in part applied by the testing machine; hence its value may be denoted by $kS$, where $k$ is a constant that becomes equal to 1 when the axial stress is one-half of the hoop tension. In the test an axial stress was applied by the testing machine and this must be added to the axial stress due to internal pressure. Hence the *total* axial stress may be taken as $kS$. Putting $kS$ for $\sigma_z$ in (22) and combining with (24a), we have two equations for the determination of $a$ and $C$, namely:

$$kS = 2\lambda C + (\lambda + 2\mu)a$$

$$2C(\lambda+\mu) = S - \lambda a$$

From these the following results are readily obtained:

$$a = \frac{S}{E}\left(k - \frac{2}{m}\right) \qquad (25)$$

$$C = \frac{S}{E}\left[1 - \frac{1}{m}(1+k)\right] \qquad (26)$$

Also from (23a)

$$D = \frac{T}{2\mu} = \frac{T}{E}\frac{1+m}{m} \qquad (27)$$

The strain component $\epsilon_\theta$ is now found.

$$\epsilon_\theta = C + \frac{D}{r^2} = \frac{1}{E}\left\{S\left[1 - \frac{1}{m}(1+k)\right] + \frac{T}{r^2}\frac{1+m}{m}\right\} \ldots\ldots\ldots (28)$$

The value of $\epsilon_\theta$ at the outer surface of the cylinder, where the strain was measured, is found by taking $r = r_0$. Substituting now the proper expressions for $S$ and $T$, and putting $r = r_0$, (28) becomes

$$(\epsilon_\theta)_0 = \frac{1}{E}\left\{\frac{p_1 r_1^2 - p_0 r_0^2}{r_0^2 - r_1^2}\left[1 - \frac{1}{m}(1+k)\right] + (p_1 - p_0)\frac{r_1^2}{r_0^2 - r_1^2}\frac{1+m}{m}\right\} \ldots (29)$$

Since $p_0$ is small compared with $p_1$, the terms $p_1 r_1^2 - p_0 r_0^2$ and $(p_1 - p_0)\, r_2$ may be considered equal. With this approximation (29) becomes

$$(\epsilon_\theta)_0 = \frac{1}{E}\left\{\frac{p_1 r_1^2 - p_0 r_0^2}{r_0^2 - r_1^2}\left(2 - \frac{k}{m}\right)\right\} \ldots\ldots\ldots\ldots (30)$$

If we consider a closed cylindrical tube with internal hydrostatic pressure $p_1$ and external pressure $p_0$, the net load producing axial tension is

$$\pi(p_1 r_1^2 - p_0 r_0^2)$$

and the area of the cross section of the tube is

$$\pi(r_0^2 - r_1^2)$$

Denoting the load by $P$ and the area by $A$, we have

$$S = \frac{p_1 r_1^2 - p_0 r_0^2}{r_0^2 - r_1^2} = \frac{P}{A}$$

$$(\epsilon_\theta)_0 = \frac{P}{EA}\left(2 - \frac{k}{m}\right) \ldots\ldots\ldots\ldots (31)$$

$$\epsilon_z = a = \frac{P}{EA}\left(k - \frac{2}{m}\right) \ldots\ldots\ldots\ldots (32)$$

In the test the axial load applied was $kP = L$; hence

$$(\epsilon_\theta)_0 = \frac{L}{EA}\left(\frac{2m - k}{k\,m}\right) \ldots\ldots\ldots\ldots (33)$$

$$\epsilon_z = \frac{L}{EA}\left(\frac{k\,m - 2}{k\,m}\right) \ldots\ldots\ldots\ldots (34)$$

The corresponding values of $E\epsilon$ (reduced stresses) are

$$E\,(\epsilon_\theta)_0 = \frac{L}{A}\frac{2m - k}{k\,m} \ldots\ldots\ldots\ldots (35)$$

$$E\epsilon_z = \frac{L}{A}\frac{km - 2}{k\,m} \ldots\ldots\ldots\ldots (36)$$

The actual stresses are

$$\sigma_z = k\,S = \frac{k\,P}{A} = \frac{L}{A}$$

$$\sigma_\theta = S + \frac{T}{r^2}$$

$$\sigma_r = S - \frac{T}{r^2}$$

In the preceding discussion the results have been obtained in terms of $p_1$ and $p_0$ the absolute internal and external fluid pressures. Evidently $p_0$ is the pressure of the atmosphere. In the test the internal pressure $p_1$ was measured by the gage, and no account was taken of the external pressure $p_0$. This procedure is justified by the following results:

Let $p' = p_1 - p_0$ = internal gage pressure.

Then $$S = \frac{p_1 r_1^2 - p_0 r_0^2}{r_0^2 - r_1^2} = \frac{p_1 r_1^2 - p_0 r_0^2 - p_0 r_1^2 + p_0 r_1^2}{r_0^2 - r_1^2}$$

$$= (p_1 - p_0)\frac{r_1^2}{r_0^2 - r_1^2} - p_0\frac{r_0^2 - r_1^2}{r_0^2 - r_1^2} = \frac{p' r_1^2}{r_0^2 - r_1^2} - p_0$$

and $$T = (p_1 - p_0)\frac{r_0^2 r_1^2}{r_0^2 - r_1^2} = p'\frac{r_0^2 r_1^2}{r_0^2 - r_1^2}$$

From (38) the hoop tension at the outer surface is

$$(\sigma_\theta)_0 = S + \frac{T}{r_0^2} = \frac{p' r_1^2}{r_0^2 - r_1^2} - p_0 + \frac{p' r_1^2}{r_0^2 - r_1^2} = 2\frac{p' r_1^2}{r_0^2 - r_1^2} - p_0$$

Since $p_0$ is entirely negligible in comparison with $S$, we may take

$$(\sigma_\theta)_0 = 2S = \frac{2L}{k\,A} \quad \ldots\ldots\ldots\ldots\ldots\ldots (40)$$

For the corresponding stress at the inner surface, we have

$$(\sigma_\theta)_1 = S + \frac{T}{r_1^2} = p'\frac{(r_1^2 + r_0^2)}{r_0^2 - r_1^2} = 2S + p' = \frac{2L}{k\,A} + p' \ldots\ldots (41)$$

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## VITA

The candidate was born October 6, 1877, in Evansville, Indiana. He received his elementary and high school education in the schools of that city, graduating from the high school in 1894. From 1894 to 1899 he was employed in the blacksmith and fitting departments of a plow works in Evansville.

In the fall of 1899 he entered the Engineering Department of the University of Michigan, receiving the degree of B. S. in M. E. in 1903, and the M. E. degree in 1907. During the last year of attendance he was assistant to Professor M. E. Cooley. From June 1903 to September 1904 he was with the Kalamazoo Gas Co., Kalamazoo, Michigan, as Assistant Superintendent in charge of the enlargement and remodeling of the plant.

In September 1904 he went to the University of North Dakota as instructor in Mechanical Engineering, and was advanced to Assistant Professor in 1906. One year later (1907) he was made Professor of Applied Mathematics, which position he still holds. During the past year he has been on leave of absence to carry on graduate work at the University of Illinois.

He is a member of Sigma Xi, the Society for the Promotion of Engineering Education, and of the American Society of Mechanical Engineers.