Islanders are stated to be worn, Mr. Ball declared he could not doubt the golden ornaments were worn in a similar manner. The Sandwich Island articles to which he alluded formed a part of the fine collection made in Cook's voyages, and deposited in the Museum of the University. He trusted he would be able to make many of the weapons and ornaments therein contained useful in throwing light on Irish antiquities. He referred to several curious instances, where the use of hypothesis had misled antiquaries, and where observations of existing people had set their opinions aside. He mentioned that he had recently proved, that an article long existing in the University Museum, and known as the best example of an old form of a trumpet, had, by the discovery of its remaining parts, proved to be a chemical instrument for burning gas, or inflammable vapour; and he concluded by stating, that the article figured in the seventeenth volume of the Transactions of the Royal Irish Academy, as an astronomical instrument of the ancient Irish, proved to be a piece of chain armour. These two last mistakes he gave as examples of a want of exactness of observation, and of the mischief of hypothesis.

The Secretary read a paper by Professor Young of Belfast, on Diverging Infinite Series, and on certain Errors in Analysis connected therewith.

The subject of diverging series is one of considerable perplexity in analysis, and has given occasion to theories of explanation involving views and statements entirely opposed to the general principles of algebraical science. It has, for instance, been affirmed of such series—when they present themselves as developments of finite expressions—that, though algebraically true, they may, nevertheless, be arithmetically false. By some they are considered to justify conclusions palpably erroneous and absurd, as, for example, that

$$1 + 2 + 4 + \ldots = -1,$$

while by others they are regarded as meaningless results, and have thus been altogether rejected from analysis.

It is impossible to avoid the occurrence of these series: they present themselves at a very early stage of algebra, in the form of geometrical progressions and binomial developments; and thenceforward are continually met with by the analyst up to the remotest applications of the integral calculus. The existing vagueness and indecision, as to the proper mode of interpreting such series, is thus a matter of some concern, as calculated to retard the progress of science, to diminish our confidence in some of the truths of analysis, and to give currency to results involving error and contradiction.

In the present communication it will be my endeavour to ascertain the causes of the perplexities and discrepancies above adverted to, and to discover the legitimate interpretation of diverging infinite series; from which it will, I think, follow that certain expressions received into analysis as the sums of several of these, are erroneous. The fact that Poisson, Cauchy, Abel, and indeed most of the modern continental writers, reject diverging infinite series, and pronounce them to have no sums, does not render such an endeavour the less necessary; inasmuch as the analytical operations, in virtue of which finite values have been attributed to extensive classes of these series by Euler and subsequent investigators, remain, I believe, unimpugned. Widely different methods appear to concur in furnishing the same numerical results for such series; as, for instance, the method of definite integrals, and that deduced from the differential theorem, both so frequently applied by Euler to effect the summations of series of this kind; and the numerical results obtained by him have often, apparently, been verified by later computers; some of whom have employed methods quite distinct from those of Euler; as, for instance, Horner, who arrived at Euler's results by aid of considerations drawn from the theory of continued fractions.*

So long, therefore, as the admitted operations of analysis thus conduct to conclusions—and conclusions, too, mutually confirmatory of one another, though arrived at by very different paths—we are surely not authorized in summarily rejecting them as meaningless or absurd, merely on account of any inherent difficulties involved in them. The only ground for such rejection, that can generally be considered as sufficiently cogent by analysts, must be errors in the reasoning by which those conclusions are reached. In attempting, therefore, now to point out the existence of these errors, it will be perceived that I proceed on the assumption that nothing has as yet been advanced, by the rejectors of diverging infinite series, against the *reasonings* of Euler, Lacroix, and others, in reference to this matter; more especially that the method of definite integrals, and that depending on the differential theorem, have not as yet been shewn to be erroneous. I may be wrong in this supposition; if so, I should feel most anxious to withdraw this Paper, rather than obtrude upon the attention of the Academy the discussion of a topic already disposed of—and, doubtless, in a more complete and satisfactory manner—elsewhere.

I.—As noticed above, the first step in the general theory of series occurs under the head of geometrical progression; the form of the series proposed for summation being

$$a + ax + ax^2 + ax^3 + \&c. \qquad (1)$$

where it is to be observed that the "&c." implies the endless progression of the terms beyond ax^3, according to the law exhibited in the terms which precede; excluding, however, every thing in the form of supplement or correction. The general expression for the sum of n terms of this series is known to be

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* Annals of Philosophy: July, 1826, p. 50.

$$s = \frac{a}{1-x} - \frac{ax^n}{1-x}. \qquad (2)$$

Now it is customary to write the development of $\frac{a}{1-x}$ as follows, viz.

$$\frac{a}{1-x} = a + ax + ax^2 + ax^3 + \&c. \qquad (3)$$

and then to commit the mistake of confounding this with the series (1) above; overlooking the fact that the "&c." in the one, except under particular restrictions as to the value of x, is very different, as to the meaning involved in it, from that in the other.

If we dispense with the "&c." in the series (1), we may write that series thus:

$$a + ax + ax^2 + ax^3 + \ldots + ax^{\infty}, \qquad (4)$$

the sum of which will be truly expressed by the formula (2), by making n infinite; as that formula is perfectly general. But this same formula gives for $\frac{a}{1-x}$ the development

$$\frac{a}{1-x} = a + ax + ax^2 + ax^3 + \ldots + ax^{\infty} + \frac{ax^{\infty}}{1-x}, \qquad (5)$$ *

shewing that the "&c." in (3) differs from that in (1) by a quantity which is infinitely great, whenever x is not a proper function: except in the single case of $x = -1$. When x *is* a proper fraction, the two series become identical by the evanescence of $\frac{ax^{\infty}}{1-x}$.

It thus appears that $\frac{a}{1-x}$ is *not* the fraction which generates the series (1), x being unrestricted: what this fraction really generates is exhibited in (5) above, an equation which is always true, whatever arithmetical value we assign to x; and to obtain the general expression for the sum of (1), we

* As the exponent in this last expression is infinite, it seems unnecessary to write it $\infty + 1$.

must connect to $\frac{a}{1-x}$ the correction $-\frac{ax^{\infty}}{1-x}$: a correction which is ambiguous as to sign, when x is negative.

When x is > 1, the series, omitting this correction, is ∞; the correction itself is also ∞, and opposite in sign: it is the difference of these two infinites which is the finite undeveloped expression.

There is thus no discrepancy between a geometrical series and the expression which generates it: nor is it the case that by connecting the two by the sign of equality, we shall have an equation algebraically true, but in certain cases arithmetically false, as has been frequently affirmed of late. The reverse of this affirmation is the more correct statement; inasmuch as by interposing the sign of equality between $\frac{a}{1-x}$ and the series (1), instead of the series (5), we have an equation algebraically false, though, within certain limits, arithmetically true: this last circumstance arising from the fact that the omitted correction, which renders the equation algebraically defective, would have vanished of itself, between the arithmetical limits adverted to, had it been introduced. Thus, the series noticed at the commencement of this paper, viz.

$$1 + 2 + 4 + 8 + 16 + \&c.,$$

and which is intended to represent the development of $\frac{1}{1-2}$, arises from expressing the general development of $\frac{1}{1-x}$ in the defective form

$$1 + x + x^2 + x^3 + x^4 + \ldots + x^{\infty},$$

instead of, in the accurate form,

$$1 + x + x^2 + x^3 + x^4 + \ldots + x^{\infty} + \frac{x^{\infty}}{1-x},$$

which defective form introduces arithmetical error only when x exceeds unit. When $x = 2$, the error arising from this defect is infinitely great; the true form giving, in that case,

$$-1 = 1 + 2 + 4 + 8 + 16 + \ldots - 2^{\infty} = \infty - \infty,$$

which involves no error or contradiction.

It hence appears that when the geometrical development is a converging series, for an arithmetical value of the common ratio, no error can arise from the omission of the supplementary correction, which is always necessary for the completion of the *algebraic* form of that development; but that when the arithmetical value of the ratio is such as to render the series divergent, the algebraic error necessarily introduces an arithmetical error infinitely great: the correction of the algebraic form furnishes, in such a case, the expression $\infty - \infty$, that is the difference of two infinites, for the finite undeveloped numerical value: and in this there is nothing inexplicable or peculiar.

We see, therefore, that in passing from the convergent to the divergent state of a geometrical series, we have no occasion for any new principle, such, for instance, as the *sign of transition*, introduced by Dr. Peacock, in the discussion of this subject, in his very valuable and instructive Report on Analysis, presented at the third meeting of the British Association. If there only be strict algebraic accuracy between the finite expression and its developed form, there will necessarily be equally strict *numerical* accuracy, whatever arithmetical values be given to the arbitrary symbols: a truth which must indeed universally hold in all the results of analysis.

II.—The developments of the binomial theorem, as well as those considered above, have also been the source of much perplexity and misinterpretation, when they have assumed a divergent form. In contemplating these developments, the fact has been overlooked, that although, when interminable, they each involve an infinite series, whose terms succeed one another, according to a certain uniform law, yet that series alone is not the complete algebraical equivalent of the undeveloped expression: a supplementary function of the symbols employed is always necessary to such completeness. This has already been seen in the development of $\frac{1}{1-x}$ or $(1-x)^{-1}$, which is a particular case of the binomial development: besides the series, the supplementary expression $\frac{x^{\infty}}{1-x}$ is necessary to the complete algebraical equivalence of the two members of the equation. And it is plain, from the nature of common division, that a like supplementary addition must be made to the infinite series furnished by the development of $\frac{1}{(1-x)^n}$ or $(1-x)^{-n}$. In the extraction of roots, too, as in $(1-x)^{\frac{1}{2}}$, $(1-x)^{\frac{1}{3}}$, &c., it is equally plain that, however far the extraction be extended, we approach no nearer to the actual exhaustion or annihilation of the algebraic remainder; and therefore we are not authorized to dismiss this remainder and to account it zero, when general *algebraic* accuracy is to be exhibited; although, as in geometrical series, we may do this in those particular numerical cases in which the remainder, if retained, would vanish. It thus appears that, calling the remainder after n terms, whether n be finite or infinite, $f(x)$, the ordinary binomial series, to n terms, will be the complete development, not of $(1-x)^{\frac{1}{m}}$, but of $(1-x-f(x))^{\frac{1}{m}}$; and therefore that, if this series be equated to $(1-x)^{\frac{1}{m}}$ merely, it will require a supplemental correction to produce strict algebraical equivalence; which correction must be such as to vanish for those numerical values of x, which cause $f(x)$ to vanish.

These values are all those which render the series divergent: for, as well known, we can, in every such case, approach by the series alone as near to the numerical value of the undeveloped expression as we please. It is thus only when the series ceases to be convergent, that the correction adverted to has any arithmetical existence, adjusting the equality of the two sides of the equation, and precluding the inconsistency so frequently affirmed to have place between them.

From these simple considerations, it is easy to explain and reconcile such results as

$$(a^2-x)^{\frac{1}{2}} = a - \frac{x}{2a} - \frac{x^2}{2.4a^3} - \frac{3x^2}{2.4.6a^5} - \frac{3.5x^4}{2.4.6.8a^7} - \&c.$$

for all arithmetical values of x; the "&c." being regarded as comprehending all that is necessary to render the second member of the equation a complete algebraical equivalent of the first. When x exceeds a^2, the series becomes divergent; and the first member of the equation becomes imaginary: and since it is impossible that any imaginary quantity can enter the series, it follows that it is in the supplementary correction under the "&c." that such quantity must occur, when in that correction a value greater than a^2 is given to x.

From what has now be shewn, it may, I think, be legitimately inferred—as far, at least, as geometrical and binomial series are concerned—

1. That whenever any such series becomes divergent for particular arithmetical values, what has been called above the supplementary correction becomes arithmetically effective, and cannot be disregarded without arithmetical error.

2. And that so far from such series being, as usually affirmed, always algebraically true, though sometimes arithmetically false, on the contrary, they are always algebraically false, though sometimes arithmetically true:—true in those cases, namely, and in those only, in which the proper algebraic correction becomes evanescent.

III.—Let us now pass to the consideration of other classes of diverging series.

There are two ways of investigating the differential of $\sin x$, or of $\sin mx$: one by proceeding, as Lagrange has done, by actual algebraic development; and the other by employing the method of limits, independently of development. According to Lagrange, we must proceed upon the assumption that

$$\sin mx = \text{A}x + \text{B}x^2 + \text{C}x^3 + \&\text{c}.$$

justifying this assumption on the ground that x and $\sin mx$ vanish together; which can be considered valid only so long as $m = \infty$ is excluded. In fact, whether we seek the development of $\sin mx$ after the manner of Lagrange, or by the theorem of Maclaurin, it is essential to the very nature of the investigation that the unknown coefficients A, B, C, &c. be all assumed to be finite. We cannot conclude, therefore, from Lagrange's reasoning, that $\dfrac{d\sin mx}{dx} = m\cos mx$, when m is infinite: and similar considerations forbid the conclusion that $\dfrac{d\cos mx}{dx} = -m\sin mx$, in like circumstances. The method of limits equally militates against such a conclusion; thus, if the function were $\sin x$, we should have

$$\sin(x+h) - \sin x = 2\sin\tfrac{1}{2}h\cos(x+\tfrac{1}{2}h),$$

or

$$\frac{\sin(x+h) - \sin x}{h} = \frac{\sin\frac{1}{2}h}{\frac{1}{2}h}\cos(x+\tfrac{1}{2}h);$$

and since $\dfrac{\sin\frac{1}{2}h}{\frac{1}{2}h} = 1$, in the limit, or when $h = 0$, we should safely infer that $\dfrac{d\sin x}{dx} = \cos x$. But, by proceeding in like manner with $\sin mx$, we should have

$$\frac{\sin(mx+mh) - \sin mx}{h} = m\frac{\sin\frac{1}{2}mh}{\frac{1}{2}mh}\cos(mx+\tfrac{1}{2}mh),$$

from which, if m be infinite, it could not be inferred that $\dfrac{d\sin mx}{dx} = m\cos mx$; since we have no right to affirm that $\dfrac{\sin\frac{1}{2}mh}{\frac{1}{2}mh}$ tends to 1, as h diminishes, and finally terminates in that value when $h = 0$; nor that, in like circumstances, $\cos(mx + \frac{1}{2}mh) = \cos mx$. We have nothing to justify the assertion that $\dfrac{\sin\frac{1}{2}h}{\frac{1}{2}h}$ and $\dfrac{\sin\frac{1}{2}mh}{\frac{1}{2}mh}$ are the same at the limits when m is infinite: and it should create no surprise if conclusions, deduced from this assumption, prove to be absurd.

Bearing this in remembrance, let us take the series

$$\frac{x}{2} = \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \&c.$$

first given by Euler, and which is known to be rigorously true for all values of x below π.*

From this series the following results have been deduced by differentiation, and they have been pretty generally received into analysis:

$$\tfrac{1}{2} = \cos x - \cos 2x + \cos 3x - \cos 4x + \&c.$$
$$0 = -\sin x + 2\sin 2x - 3\sin 3x + 4\sin 4x - \&c.$$
$$0 = -\cos x + 2^2\cos 2x - 3^2\cos 3x + 4^2\cos 4x - \&c.$$

and, generally,

$$0 = \cos x - 2^{2n}\cos 2x + 3^{2n}\cos 3x - 4^{2n}\cos 4x + \&c.$$
$$0 = \sin x - 2^{2n+1}\sin 2x + 3^{2n+1}\sin 3x - 4^{2n+1}\sin 4x + \&c.$$

so that putting $x = 0$ in the first of these, and $x = \frac{\pi}{2}$ in the second, we have

$$0 = 1 - 2^{2n} + 3^{2n} - 4^{2n} + \&c.$$
$$0 = 1 - 3^{2n+1} + 5^{2n+1} - 7^{2n+1} + \&c.$$

results which are all inadmissible; because, from the outset, it is assumed that

$$\frac{d\sin mx}{dx} = m\cos mx, \text{ and } \frac{d\cos mx}{dx} = -m\sin mx;$$

though m be infinite.

In reference to the preceding results, Abel justly asks: "Peut-on imaginer rien de plus horrible que de débiter

$$0 = 1 - 2^{2n} + 3^{2n} - 4^{2n} + \&c.$$

où n est un nombre entier positif?"†

* It will be shewn, towards the close of this Paper, that it is true for all values up to π inclusive.

† Œuvres Completes, tome ii. p. 266.

It is plain that, however far such a series as this be ex-extended, a supplementary correction is always necessary to complete the equation; which correction must be infinite in value if the series be infinitely extended: and the analytical considerations offered above fully accord with this statement, the contrary of which could never have been entertained had not analysis seemed to justify the strange conclusion. All that analysis really authorizes us in saying, in reference to the extreme cases here considered, is—as the French analysts express it—that "la méthode ordinaire est en défaute."

Having mentioned the name of Abel in connexion with this subject, it may not be out of place to notice here, that that distinguished genius seemed inclined to trace the erroneous results above to another cause: "On applique aux séries infinies toutes les opérations, comme si elles étaient finies; mais cela est-il bien permis? Je crois que non. Où est il démontré qu'on obtient la différentielle d'un série infinie en en prenant la différentielle de chaque terme?" And he then adduces the result,

$$\tfrac{1}{2} = \cos x - \cos 2x + \cos 3x - \&c.$$

which he pronounces to be "résultat tout faux."*

But I submit that no such results of differentiation can ever be absurd, unless the absurdity attaches to one or more of the individual terms.

In the former part of this paper the examination was restricted to those classes of diverging series which arise from the development of fractions into geometrical series, and from the expansion of a binomial: but it is plain that the reasonings, in reference to the former developments, equally apply to those which arise from any fraction $\frac{f(x)}{\phi(x)}$; and the reasoning, in reference to the latter, equally applies to any root or power of (x). And, in what is shewn above, we see how divergent

* Œuvres, tome ii. p. 268.

trigonometrical series, arising from differentiating convergent forms, are to be understood.

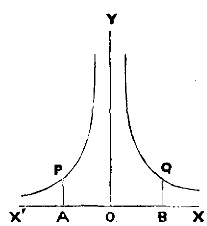
IV.—It remains now to be noticed that in some of the more advanced parts of analysis—especially in the doctrine of definite integrals—conclusions have been reached which seem to contradict the proposition endeavoured to be established in this Paper, viz. that convergent infinite series have no finite sum. But all such conclusions will be found upon examination to originate in mistake. I proceed to examine the more important of these.

The following has been recently offered, by a very cautious writer, in support of the statement that "1 + 2 + 4 + &c. *ad infinitum*, is an algebraic representative of — 1, though it only gives the notion of infinity to any attempt to conceive its arithmetical value":

$$\int x^{-2}dx = -x^{-1},\ \int_a^b x^{-2}dx = a^{-1} - b^{-1},\ \text{which is finite;}$$

$$\int_{-m}^{0} x^{-2}dx = +\infty,\ \int_0^m x^{-2}dx = +\infty,\ \int_{-m}^{+m} x^{-2}dx = -\frac{2}{m}.$$

If, then, we construct the curve whose equation is $y = x^{-2}$, and if $OA = -m$, $OB = +m$, we find the areas PAOY... and QOBY... both positive and infinite, which agrees with all our notions derived from the theory of curves. Again, if we attempt to find the area PYQB, by summing PAOY and YOQB, we find an infinite and positive result, which still is strictly intelligible. But if we want to find the area by integrating at



once from P to Q, we find, as above, $-\frac{2}{m}$, a *negative result*, for the sum of two positive infinite quantities. The integral then, y being infinite between the limits, takes an algebraic character, standing in much the same relation to the required arithmetical result, which must have been observed in divergent series. " Thus, &c.," as quoted above.* The analogy thus apparently established is traceable to an oversight, of very easy detection, in the preceding integrations; which, in the correct form, will stand as follow :

$$\int_{-m}^{0} x^{-2}dx = +\infty - \frac{1}{m}$$

$$\int_{0}^{m} x^{-2}dx = +\infty - \frac{1}{m}$$

∴ adding,

$$\int_{-m}^{m} x^{-2}dx = 2\infty - \frac{2}{m}.$$

Or thus,

$$\int_{a}^{b} x^{-2}dx = \left(-\infty + \frac{1}{a}\right) - \left(-\infty + \frac{1}{b}\right)$$

$$\therefore \int_{-m}^{m} x^{-2}dx = \left(+\infty - \frac{1}{m}\right) - \left(-\infty + \frac{1}{m}\right)$$

$$= 2\infty - \frac{2}{m}.$$

But errors of a much more important kind occur in all the applications of definite integrals to the summation of diverging series: a mode of summation first, I believe, adopted by Euler, and very generally employed by subsequent analysts. A single example of this method will be sufficient to shew the character of the errors adverted to; which, though so glaring as almost to obtrude themselves upon the attention, have not hitherto, so far as I know, been noticed by any writer. Any one of the examples given by Euler (Institutiones Calc. Diff.), and afterwards by Lacroix (Traite du Calcul. &c., tome iii.), will answer the present purpose: I shall take that at page 573 of the English edition of the smaller work of Lacroix, viz.

$$s = 1.t - 1.2t^2 + 1.2.3t^3 - \&c. \qquad (6)$$

which, Sir John Herschel remarks, is such that " however

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* De Morgan's Differential and Integral Calculus, p. 571.

small a value we attribute to t, the series must always diverge after a certain number of terms."*

The reasoning by which a finite sum is determined for s, when $t = 1$, is as follows:

$$\frac{sdt}{t} = 1.dt - 1.2tdt + 1.2.3t^2dt - \&c. \qquad (7)$$

$$\therefore \int \frac{sdt}{t} = t - 1.t^2 + 1.2t^3 - \&c. \qquad (8)$$

$$= t - st \qquad (9)$$

$$\therefore \frac{sdt}{t} = (1 - s)dt - tds,$$

or,

$$\frac{ds}{dt} + \frac{1+t}{t^2}s = \frac{1}{t};$$

and from this is found, for s, the definite integral

$$s = \frac{1}{t}e^{\frac{1}{t}} \int_0^t e^{-\frac{1}{t}} dt;$$

from which it is inferred that "if $t = 1$, or the above integral be taken from $t = 0$ to $t = 1$, we have the expression for the value of the series

$$1 - 1.2 + 1.2.3 - \&c."$$

Now several objections lie against the preceding reasoning: in the first place it is assumed, in the final step, that s vanishes, for $t = 0$, notwithstanding that "however small a value we attribute to t the series must always diverge," and thus at length furnish terms infinitely great: and in the next place it is assumed—and the assumption is somewhat similar to that

* If, however, t be indefinitely near to zero, the "certain number of terms" adverted to in the text, will be indefinitely great; that is, the divergency will be indefinitely postponed: the series therefore cannot be considered as divergent up to the limit $t = 0$; yet, as the statement in the text seems to imply this, I have considered it to be comprehended in the hypothesis; although, as I have shewn, the point is of no moment in the matter under discussion.

already animadverted upon at page 35—that the series (7) is strictly the differential of the series (8) which involves the term $1.2.3\ldots(n-1)t^n$, n being infinitely great, and for the differential of which the calculus seems to make no provision. But, waiving these objections, the deduction (9) is palpably erroneous, and altogether fatal to the final conclusion. For the series s is evidently coextensive with the series (8), and so, of course, is st; that is, if (8) contain n terms, so also must st: if therefore a new term t be prefixed to $-st$, in order that $t-st$ may commence with the same terms as the series (8), the series $t-st$ will contain $n+1$ terms; that is, however great n may be, $t-st$ will contain, besides the whole of the series (8), an additional term still more remote: so that if n be infinite, and we assume, as above, that the two series are equal, we commit an error infinitely great. And this is the error, thus introduced, which will be found to vitiate all Euler's processes for summing divergent series by definite integrals: an error which obviously has no existence for the convergent cases of those series; since the additional term, noticed above, is, in such cases, not infinite, but zero. We may safely infer, therefore, that the results so often quoted in analysis, viz.

$$1-1+1.2-1.2.3+\ldots\ldots = \cdot596347362324$$
$$1-1.2+1.2.3-\ldots\ldots\ldots = \cdot621449624236$$
$$1-1.2.3+1.2.3.4.5\ldots\ldots = \cdot343279002556$$
$$\&c. \qquad\qquad \&c.$$

all involve errors infinitely great; and this, as it ought to be, is quite consistent with the common-sense view of diverging infinite series.

V.—There is another method of investigation by which these erroneous results appear to be established: the method suggested by the well-known differential theorem. But, as in the processes already considered, so here, that theorem will be found upon examination to be applicable only to convergent series. This will be manifest from what follows.

The differential theorem may be satisfactorily established by conducting the investigation thus:

Let

$$a - bx + cx^2 - dx^3 + \&c. = s$$

$$\therefore \quad - bx + cx^2 - dx^3 + \&c. = s - a \qquad (10)$$

$$\therefore - b + cx + dx^2 + ex^3 - \&c. = \frac{s-a}{x}. \qquad (11)$$

Consequently, by adding these two equations together, and representing the numerical differences $b-c$, $c-d$, $d-e$, &c. by Δ, Δ', Δ'', &c., there will result the equation

$$- b - \Delta.x + \Delta'.x^2 - \Delta''.x^3 + \&c. = \frac{x+1}{x}(s-a) \qquad (12)$$

$$\therefore -bx - \Delta.x^2 + \Delta'.x^3 - \Delta''.x^4 + \&c. = (x+1)(s-a) = s'$$

$$\therefore \; s = \frac{s'}{x+1} + a;$$

that is,

$$s = a - \frac{bx}{x+1} + \frac{x}{x+1}\left[0 - \Delta.x + \Delta'.x^2 - \Delta''.x^3 + \&c.\right] \qquad (13)$$

And by treating the series within the brackets as the original was treated, and so on, we shall finally obtain the transformation

$$s = a - \frac{bx}{x+1} - \frac{\Delta.x^2}{(x+1)^2} - \frac{\Delta^2.x^3}{(x+1)^3} - \&c.$$

or putting $a = 0$, and dividing by $-x$, we have

$$b - cx + dx^2 - ex^3 + \&c. =$$

$$\frac{b}{x+1} + \frac{\Delta.x}{(x+1)^2} + \frac{\Delta^2.x^2}{(x+1)^3} + \frac{\Delta^3.x^3}{(x+1)^4} + \&c.$$

which is the usual form of the theorem.

Now the preceding reasoning is inadmissible except the proposed series be convergent; that is, except rx^n approaches to zero as n approaches to infinity, rx^n standing generally for the n^{th} term of (10). For in (12), which results from the sum of (10) and (11), this n^{th}, or final term, is regarded as zero, and is neglected; inasmuch as it is by this term that the series (10) extends beyond the series (11) to the right; a fact which is of no moment when this term merges in zero, but of infinite consequence when it merges in infinity. In such a case therefore, a numerical error, of infinite amount, is committed at this step of the reasoning. Again, if the series within the brackets at (13), have its terms, like those of the original, tending to infinity, another numerical error of infinite amount comes to be introduced; and so on. In fact, just as in the method of definite integrals, before discussed, it is assumed, at each step of the reasoning, that terms infinitely great are excluded; and not only so, but that the terms ultimately diminish to zero. In the contrary case, therefore, the differential theorem is altogether inapplicable, leading to results which are equally inadmissible, whether the terms of the series increase without limit, or remain stationary in value: forming what has been called a neutral series. In this latter case the error committed will be finite; in the former it will be infinite. That an error is really committed in the application of this theorem to neutral series, will be more explicitly shewn presently.

Notwithstanding the imperfections noticed above, it should create no surprise that, in the applications of the differential theorem to particular diverging series, we so often obtain the algebraic function whose development really gives rise to the series, although no numerical approximation to the diverging series itself. The function, whose development gives rise to the series, being represented by $f(x)$, the series itself may be represented by $f(x) - \phi(x)$, agreeably to what has already been shewn in the former part of this Paper: it is the neglect of the function $\phi(x)$, in the particular application considered, that introduces the infinite numerical error into (13); leading us to conclude that, for the proposed value of x, $f(x) = \text{s}$, instead of $f(x) - \phi(x) = \text{s}$. Now if there exist a convergent case of s, that is a case in which $\phi(x) = 0$, the differential theorem will compute it, furnishing the proper function of x, $f(x)$, which accurately expresses the series in all its convergent cases, and of which the development *gives rise* to the series in its general form. When no such function $f(x)$ really exists, then it is only to the numerical value of an approximate function that our computation tends in particular numerical cases; as, for instance, in such a case as that considered at p. 39.

It may be worth while to notice here, as an immediate inference from the differential theorem, that when a series, proceeding according to the powers of x, and extending to infinity, has its coefficients such that their differences at length become zero, that series is always the development of a rational fraction whose denominator is some integral power of $(1 \pm x)$.

There is, I think, a mistake committed in always attributing this theorem to Euler. It was published by Stirling, in his Methodus Differentialis, so early as 1730; and I believe no mention of it occurs in the writings of Euler till long after this date.

VI.—As far as I know, there is but one other general analytical principle that has been affirmed to give countenance to doctrines opposed to those attempted to be established in the present Paper: the principle, namely, that when an algebraic expression, for continuous numerical values of the variable, approaches continuously to a certain finite numerical value, this value properly expresses the ultimate, or limiting state of that expression. In virtue of this principle, it has been stated* that, "Poisson would admit $1^2 - 2^2 + 3^2 - 4^2 + \ldots = 0$, since there is no question that, g being less than unity, the mere arithmetical computer might establish, to any number of decimal places, the identity of $1^2 - 2^2 g + 3^2 g^2 - \ldots$ and $(1-g)(1+g)^{-3}$."† But I submit that the series here

* Transactions of the Cambridge Philosophical Society, Part II. 1844.

† In order that the series $1^2 - 2^2 g + 3^2 g^2 - \ldots$ may become convergent after n terms, there must evidently exist the condition

$$\left(\frac{n+1}{n}\right)^2 g < 1, \text{ whence } g < \left(\frac{n}{n+1}\right)^2;$$

proposed exceeds the powers of computation more and more as g approaches to 1; involving at length terms infinitely great, and thus tending to no finite limit. In other words, however many terms of this series be summed, the results would diverge more and more from zero as g approaches to 1; and would actually become infinite when g reaches this limit. The conclusion, therefore, that $1^2-2^2+3^2- \ldots = 0$ is, as in the other instances discussed in this Essay, erroneous to an infinite extent: and it thus affords one more example of the truth of the doctrine here advanced.

The general analytical principle announced above has been misapplied, or improperly neglected, in many important inquiries connected with series. It may not be uninstructive to advert more particularly to some instances of this.

At page 267 of the second volume of his works, Abel has the following remark: "On peut démontrer rigoureusement qu'on aura, pour toutes les valeurs de x inférieures à π,

$$\frac{x}{2} = \sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \&c.$$

Il semble qu'on pourrait conclure que la même formule aurait lieu pour $x = \pi$; mais cela donnerait

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and as $\left(\frac{n}{n+1}\right)^2$ is itself less than 1 for every finite value of n, however great, it follows that g may approach so near to 1 as to postpone the point of convergency beyond any finite limit; which is tantamount to saying that this point can never actually be reached. The series, therefore, cannot tend to merge into zero as g approaches to 1; so that zero is *not* the limit to which the series continuously approaches as g approaches continuously to 1; and therefore the general principle stated in the text does not countenance the conclusion that $1^2-2^2+3^2- \ldots = 0$.

I cannot help regarding the criterion of convergency proposed by Cauchy (Cours d'Analyse, p. 152) as open to objection; since, according to it, we should pronounce a series to be convergent under circumstances in which the point of convergency would be postponed beyond any finite limits: moreover, what security have we that *neutrality* may not have place before divergency commences?

$$\frac{\pi}{2} = \sin \pi - \tfrac{1}{2} \sin 2\pi + \tfrac{1}{3} \sin 3\pi - \&c. = 0,$$

résultat absurde."

Now the formula, agreeably to the general principle here affirmed to be in fault, does really comprehend the limiting case $x = \pi$, as well as all the cases up to this; for when x reaches this limit all the signs of the series become *plus;* and as it is known that

$$1 + \tfrac{1}{2} + \tfrac{1}{3} + \tfrac{1}{4} + \&c. = \infty,$$

the series presents a particular case of $0 \times \infty$; which it is wrong to declare to be 0, in contradiction of its legitimate interpretation, $\frac{\pi}{2}$, on the left. This error has led Abel into other mistakes of consequence: thus, at page 90 of his first volume, he says that the function

$$\text{“} \sin \phi - \tfrac{1}{2} \sin 2\phi + \tfrac{1}{3} \sin 3\phi - \&c.$$

a la propriété remarquable pour les valeurs $\phi = \pi$ et $\phi = -\pi$ d'être discontinue." And at page 71 the same erroneous view has induced him to animadvert upon a certain principle of Cauchy, which the true interpretation of the matter would have tended to confirm.

Fourier, Poisson, and many other modern analysts, have also made similar mistakes in their general investigations respecting series. Thus, to quote Professor Peacock as to the views of the former,

$$\text{“} \cos x = \frac{4}{\pi}\left[\frac{2}{1.3} \sin 2x + \frac{4}{3.5} \sin 4x + \frac{6}{5.7} \sin 6x + \&c.\right]$$

a very singular result, which is, of course, true only between the limits 0 and π, excluding those limits."*

The series is, however, true including the limits: for when $x = 0$, the signs are all *plus;* and, as it is easily shewn that

$$\frac{2}{1.3} + \frac{4}{3.5} + \frac{6}{5.7} + \&c. = \infty,$$

* Proceedings of the Third Meeting of the British Association, p. 257.

we here again have a case of $0 \times \infty$, correctly interpretable by the left hand member of the equation; that is, the right hand member, when $x = 0$, is accurately 1. When $x = \pi$, the signs of the series all become *minus:* therefore the true value in that case is -1.

Before concluding this subject it may be proper to observe, that the investigation, whence the series for $\frac{x}{2}$ is usually deduced, is deficient in generality. Whenever logarithms are employed in connexion with imaginary quantities, the imaginary forms of the logarithms, as well as the real, ought always to be introduced into the investigation : hence the logarithmic expression, from which the series alluded to is derived, should be written thus:

$$\log u = u - u^{-1} - \frac{u^2 - u^{-2}}{2} + \frac{u^3 - u^{-3}}{3} - \frac{u^4 - u^{-4}}{4} + \&c. + 2k\pi\sqrt{-1}$$

By substituting in this $e^{x\sqrt{-1}}$ for u, and then dividing the result by $2\sqrt{-1}$, we shall have the correct and general form,

$$\frac{x}{2} = \sin x - \frac{\sin 2x}{2} + \frac{\sin 3x}{3} - \frac{\sin 4x}{4} + \&c. + k\pi,$$

where k is any whole number, positive or negative, determinable in any particular case, so as to conform to the first member of the equation : regarding that first member, x not exceeding π, as indifferently either $\frac{x}{2}$, or $k\pi + \frac{x}{2}$.

I have here used the limited logarithmic forms of Euler, and not the more general ones furnished by Mr. Graves's theory of imaginary logarithms,* since these limited forms are sufficient for all the *real* values in the general result.

It now merely remains to be shewn that, as briefly stated at page 43, the differential theorem is inapplicable, not only when the proposed series is divergent, but also when it ceases to be convergent, and becomes what Hutton has called a neu-

* Philosophical Transactions, Part I. 1829.

tral series. Thus—although the contrary has often been affirmed—we cannot legitimately infer from this theorem, without the aid of an additional principle, that

$$1 - 1 + 1 - 1 + 1 - 1 + \&c. = \tfrac{1}{2}.$$

For, as already shewn, the series within the brackets at (13) is deficient by a quantity, which in this case is ± 1. Introducing this, (13) gives for s the ambiguous result $\frac{1}{2} \pm \frac{1}{2}$; that is, 1 or 0. The additional principle adverted to, and which is absolutely essential to the received conclusion, is that already stated at page 44; or, as Dr. Whewell briefly expresses it, "that what is true *up to* the limit, is true *at* the limit."

The differential theorem, therefore, can never be employed with success to sum either a divergent or a neutral series; or to convert either into a convergent series.

There has been supposed to exist a perfect analogy between $1-1+1-1+$ &c., as the limiting case of $1-g+g^2-g^3+$ &c., and $1^2-2^2+3^2-4^2+$ &c., as the limiting case of $1-2^2g+3^2g^2-4^2g^3+$ &c., and that, in consequence of this analogy, we have as much right to affirm that $1^2-2^2+3^2-4^2+$ &c. is accurately expressed by 0, the limiting case of $(1-g)(1+g)^{-3}$, the fraction which generates $1^2-2^2g+3^2g^2-4^2g^3+$ &c., as that $1-1+1-1+$ &c. is accurately expressed by $\frac{1}{2}$, the limiting case of $\frac{1}{1+g}$, the fraction which generates $1-g+g^2-g^3+$ &c. But there is a total absence of analogy between these two instances: the series $1-g+g^2-g^3+$ &c. presents a series of convergent cases from $g=0$, up to $g=1$; and whatever rule or formula enables us to find the summation in all cases must necessarily enable us to find it in the extreme positive limits 0 and 1; for no values, short of those limits, can be the first and last of the admissible cases. But this rule or formula of summation, whatever it be, is constructed conformably to certain hypotheses; viz. that the convergent series expressed by it, commences, *in all cases*, with a finite quantity, such that the terms of the series, by continual diminution, tend to zero.

The circumstances are very different with respect to $1^2 - 2^2g + 3^2g^2 - 4^2g^3 +$ &c. As observed in the foot-note at p. 44, the commencement of convergency, in the limiting case, is at a term infinitely distant from the origin of the proposed series, and infinitely great. What analogy can there be between the *general* converging series—if it may be so called—of which this is a limiting case, and ordinary convergent series? And can it be affirmed, of any one of its cases, that the terms necessarily tend to zero? The answers to these questions will, I think, destroy all idea of analogy in such examples as those adduced above.

I have been compelled, in several parts of the present Paper, to dissent from certain doctrines and opinions promulgated by some very distinguished writers on analysis. In developing the principles and views here submitted to the Royal Irish Academy, I could not easily avoid a reference to these. I trust, however, that I have done so in no captious or uncandid spirit: I have only been anxious to arrive at truth in an inquiry of acknowledged perplexity, and of interest, perhaps, in the estimation of some, sufficient to justify the attempt. There are one or two points of analytical delicacy involved in this inquiry, which may perhaps be open to further discussion: if I have myself fallen into error in my treatment of these, I hope I shall be indulged with the same candour and consideration which I have endeavoured to exercise towards others.

Professor Mac Cullagh made a communication on the subject of Total Reflexion.

In the case of total reflexion the vibrations which take place in the rarer medium are in general elliptical, and when this medium is a crystal, the equations by which the ellipse of vibration is determined are very complicated. The projection