of the divisions of the scale was 3′.98; with the modifications since introduced into the reading part of the apparatus, the scale divisions have nearly the same value as in the instrument for the measurement of the declination, so that the readings may be made with certainty to less than the tenth of a minute. The present value of the inclination in Dublin is about 70° 50′; and the mean deflection produced by the iron bar in its actual position being about 19°, it follows from (7) that the changes of inclination are inferred with the same degree of precision, very nearly, as the observed changes of angle.

The last test to which the instrument was subjected, was, to employ it for some time in the regular observation of inclination changes, for which it is destined; and to ascertain how far the mean results of the observations of successive weeks agreed in exhibiting the law of the diurnal variation. The instrument was accordingly observed for five successive weeks, every second hour during the day and night, and the means calculated, omitting those days in which the series was broken by changes of adjustment during experiment. The curves now laid before the Academy represent the projected results of the observations of each of these weeks, together with that of the mean of the whole. An inspection of them is sufficient to show that the curves of the separate weeks accord with one another, and with the mean, as nearly as can be expected in the results of such limited series, the discordances being only such as are due to the known irregularities in the direction of the earth's magnetic force.

A communication from the President was read, containing some remarks supplementary to the account which he had given at a former meeting, of his Researches respecting Fluctuating Functions, (see Proceedings, June 22nd, 1840).

degree, with the minute changes induced by the variations of the earth's force. It remains for future examination to determine how far such permanent changes, if they occur, may impair the accuracy of the results.

The following general observations are extracted, on the nature and history of this branch of analysis:—

Lagrange appears to have been the first who was led (in connexion with the celebrated problem of vibrating cords) to assign, as the result of a species of interpolation, an expression for an arbitrary function, continuous or discontinuous in form, between any finite limits, by a series of sines of multiples, in which the coefficients are definite integrals. Analogous expressions, for a particular class of rational and integral functions, were derived by Daniel Bernouilli, through successive integrations, from the results of certain trigonometric summations, which he had characterized in a former memoir as being *incongruously true*. No further step of importance towards the improvement of this theory seems to have been made, till Fourier, in his researches on Heat, was led to the discovery of his well known theorem, by which any arbitrary function of any real variable is expressed, between finite or infinite limits, by a double definite integral. Poisson and Cauchy have treated the same subject since, and enriched it with new views and applications; and through the labours of these and, perhaps, of other writers, the theory of the development or transformation of arbitrary functions, through functions of determined forms, has become one of the most important and interesting departments of modern algebra.

It must, however, be owned that some obscurity seems still to hang over the subject, and that a further examination of its principles may not be useless or unnecessary. The very existence of such transformations as in this theory are sought for and obtained, appears at first sight paradoxical; it is difficult at first to conceive the possibility of expressing a perfectly arbitrary function through any series of sines or cosines; the variable being thus made the subject of known and determined operations, whereas it had offered itself originally as the subject of operations unknown and undetermined. And even after this first feeling of paradox is removed, or relieved, by the consideration that the number of the operations of known form is infinite, and that the operation of arbitrary form reappears in another part of the expression, as performed on an auxiliary variable; it still requires attentive consideration to see clearly how it is possible that none of the values of this new variable should have any influence on the final result, except those which are extremely nearly equal to the variable originally proposed. This latter difficulty has not, perhaps, been removed to the complete satisfaction of those who desire to examine the question with all the diligence its importance deserves, by any of the published works upon the subject, A conviction, doubtless, may be attained, that the results are true, but something is, perhaps, felt to be still wanting for the full rigour of mathematical demonstration. Such has, at least, been the impression left on the mind of the present writer, after an attentive study of the reasonings usually employed, respecting the transformations of arbitrary functions.

Poisson, for example, in treating this subject, sets out, most commonly, with a series of cosines of multiple arcs; and because the sum is generally indeterminate, when continued to infinity, he alters the series by multiplying each term by the corresponding power of an auxiliary quantity which he assumes to be less than unity, in order that its powers may diminish, and at last vanish; but, in order that the new series may tend indefinitely to coincide with the old one, he conceives, after effecting its summation, that the auxiliary quantity tends to become unity. The limit thus obtained is generally zero, but becomes on the contrary infinite when the arc and its multiples vanish; from which it is inferred by Poisson, that if this arc be the difference of two variables, an original and an auxiliary, and if the series be multiplied by any arbitrary function of the latter variable, and integrated with respect thereto, the effect of all the values of that variable will disappear from the result, except the effect of those which are extremely nearly equal to the variable originally proposed.

Poisson has made, with consummate skill, a great number of applications of this method; yet it appears to present, on close consideration, some difficulties of the kind above alluded to. In fact, the introduction of the system of factors, which tend to vanish before the integration, as their indices increase, but tend to unity, after the integration, for all finite values of those indices, seems somewhat to change the nature of the question, by the introduction of a foreign element. Nor is it perhaps manifest that the original series, of which the sum is indeterminate, may be replaced by the convergent series with determined sum, which results from multiplying its terms by the powers of a factor infinitely little less than unity; while it is held that to multiply by the powers of a factor infinitely little greater than unity would give an useless or even false result. Besides there is something unsatisfactory in employing an apparently arbitrary contrivance for annulling the effect of those terms of the proposed series which are situated at a great distance from the origin, but which do not themselves originally tend to vanish as they become more distant therefrom. Nor is this difficulty entirely removed, when integration by parts is had recourse to, in order to show that the effect of these distant terms is insensible in the ultimate result; because it then becomes necessary to differentiate the arbitrary function; but to treat its differential coefficient as always finite is to diminish the generality of the inquiry.

Many other processes and proofs are subject to similar or different difficulties; but there is one method of demonstration employed by Fourier, in his separate Treatise on Heat, which has, in the opinion of the present writer, received less notice than it deserves, and of which it is proper here to speak. The principle of the method here alluded to may be called the *Principle of Fluctuation*, and is the same which was enunciated under that title in the remarks prefixed to this paper. In virtue of this principle (which may thus be considered as having been indicated by Fourier, although not expressly stated by him), if any function, such as the sine or cosine of an infinite multiple of an arc, changes sign infinitely often within a finite extent of the variable on which it depends, and has for its mean value zero; and if this, which may be called a *fluctuating function*, be multiplied by any arbitrary but finite function of the same variable, and afterwards integrated between any finite limits; the integral of the product will be zero, on account of the mutual destruction or neutralization of all its elements.

It follows immediately from this principle, that if the factor by which the fluctuating function is multiplied, instead of remaining always finite, becomes infinite between the limits of integration, for one or more particular values of the variable on which it depends; it is then only necessary to attend to values in the immediate neighbourhood of these, in order to obtain the value of the integral. And in this way Fourier has given what seems to be the most satisfactory published proof, and (so to speak) the most natural explanation of the theorem called by his name; since it exhibits the actual process, one might almost say the interior mechanism, which, in the expression assigned by him, destroys the effect of all those values of the auxiliary variable which are not required for the result. So clear, indeed, is this conception, that it admits of being easily translated into geometrical constructions, which have accordingly been used by Fourier for that purpose.

There are, however, some remaining difficulties connected with this mode of demonstration, which may perhaps account for the circumstance that it seems never to be mentioned, nor alluded to, in any of the historical notices which Poisson has given on the subject of these transformations. For example, although Fourier, in the proof just referred to, of the theorem called by his name, shows clearly that in integrating the product of an arbitrary but finite function, and the sine or cosine of an infinite multiple, each successive positive portion of the integral is destroyed by the negative portion which follows it, if infinitely small quantities be neglected, yet he omits to show that the infinitely small outstanding difference of values of these positive and negative portions, corresponding to a single period of the trigonometric function introduced, is of the second order; and, therefore, a doubt may arise whether the infinite number of such infinitely small periods, contained in any finite interval, may not produce, by their accumulation, a finite result. It it also desirable to be able to state the argument in the language of limits, rather than in that of infinitesimals; and to exhibit, by appropriate definitions and notations, what was evidently foreseen by Fourier, that the result depends rather on the *fluctuating* than on the *trigonometric* character of the auxiliary function employed.

The same view of the question had occurred to the present writer, before he was aware that indications of it were to be found among the published works of Fourier; and he still conceives that the details of the demonstration to which he was thus led may be not devoid of interest and utility, as tending to give greater rigour and clearness to the proof and the conception of a widely applicable and highly remarkable theorem.

Yet, if he did not suppose that the present paper contains something more than a mere expansion or improvement of a known proof of a known result, the Author would scarcely have ventured to offer it to the Transactions* of the Royal Irish Academy. It aims not merely to give a more perfectly satisfactory demonstration of Fourier's celebrated theorem

* Sir William Hamilton's Essay on Fluctuating Functions, will be found in the Second Part of volume xix. of the Transactions of the Academy.

than any which the writer has elsewhere seen, but also to present that theorem, and many others analogous thereto, under a greatly generalized form, deduced from the principle of fluctuation. Functions more general than sines or cosines, yet having some correspondent properties, are introduced throughout; and constants, distinct from the ratio of the circumference to the diameter of a circle, present themselves in connexion therewith. And thus, if the intention of the writer have been in any degree accomplished, it will have been shown, according to the opinion expressed in the remarks prefixed to this paper, that the development of the important principle above referred to gives not only a new clearness, but also (in some respects) a new extension, to this department of science.

DONATIONS.

Memorie dell' Imperiale Regio Instituto del Regno Lombardo-Veneto. Vols. 1–5.

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