ON THE MANNER

IN WHICH

ALGEBRAIC FUNCTIONS

OF THE PRINCIPAL VARIABLE,

ARE IN CERTAIN CASES

INTRODUCED INTO THE INTEGRALS

OF

LINEAR DIFFERENTIAL EQUATIONS

THAT HAVE CONSTANT COEFFICIENTS.

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LET $V = 0$; V being a linear function of $\varphi.x$, $\frac{d.\varphi.x.}{dx}$, $\frac{d^2.\varphi.x}{dx^2}$,............ $\frac{d^n.\varphi.x}{dx^n}$, and $\varphi.x$ being an unknown function of x, which it is the business of the integration to determine. This function is in its nature an exponential one. For, let α be any root of the equation $V = 0$; V being what V becomes, when the exponent of the characteristic d of each term is transported to the right of the suffixed quantity, so as, for instance, to change any term $k. \frac{d^r \varphi.x}{dx^r}$ into $k \frac{(d. \varphi. x)^r}{dx^r}$, and of course $\varphi.x$ or $\frac{d^o. \varphi. x}{dx^o}$ into unity. $Bc^{\alpha x}$ will satisfy the given equation; B being any arbitrary constant, and c being the well known transcendental 2,718, &c. whose hyperbolic logarithm is unity. For, substitute this quantity for $\varphi.x$ in the function V; any term of it, as $k. \frac{d^r. \varphi.x}{dx^r}$, will become $k. \frac{d^r. Bc^{\alpha x}}{dx^r} = k B c^{\alpha x}. \alpha^r$; and the entire function V will therefore become $Bc^{\alpha x} \acute{V}_{\alpha}$; $\acute{V}_{\alpha}$ being what V becomes when for $\frac{d. \varphi. x}{dx}$ we write α, and being consequently, by the supposition, equal to nothing. This value of $\varphi.x$ will therefore render equal to nothing $V, = B c^{\alpha x} \acute{V}_{\alpha}$; and will consequently satisfy the given equation; and it will in like manner be satisfied by $\acute{B}c^{\acute{\alpha} x}$, $\ddot{B}c^{\ddot{\alpha} x}$, &c. if $\acute{\alpha}$, $\ddot{\alpha}$, &c. be also roots of the equation, $\acute{V} = o$, and $\acute{B}$, $\ddot{B}$, &c. be other arbitrary constants. It appears then that the quantity $\varphi.x$ should generally consist of n terms of the form $Bc^{\alpha x}$, and is therefore, as I said above, in its nature exponential. If any roots of the equation, $\acute{V} = o$, should be impossible, the impossibility may be removed by the following artifice. Let, for example, $\beta \pm \lambda \sqrt{-1}$ be roots of the aforesaid equation. The corresponding parts of $\varphi.x$ will be $F c^{(\beta + \lambda \sqrt{-1}) x} + \acute{F} c^{(\beta - \lambda \sqrt{-1}) x}$ Feign F and $\acute{F}$ to be respectively equal to $\frac{B - \acute{B} \sqrt{-1}}{2}$, and $\frac{B + \acute{B} \sqrt{-1}}{2}$, B and $\acute{B}$ being new arbitraries; and the part of $\varphi.x$ just mentioned will become

$$\frac{Bc^{\beta x} \cdot \left\{ c^{\lambda x \sqrt{-1}} + c^{-\lambda x \sqrt{-1}} \right\}}{2} + \frac{\grave{B}c^{\beta x} \cdot \left\{ c^{\lambda x \sqrt{-1}} - c^{-\lambda x \sqrt{-1}} \right\}}{2\sqrt{-1}}$$

$= Bc^{\beta x} \cdot \cos \lambda x + \grave{B}c^{\beta x} \cdot \sin \lambda x$; an expression delivered from the impossible form. But this is a mere artifice of analysis, and does not alter the nature of the function $\varphi. x$, which is still composed of quantities of the form $Bc^{\alpha x}$, though concealed under a certain disguise. It is however well known that cases occur, in which quantities of the forms $Fc^{\alpha x} \cdot x^r$, or $\grave{F}x^r$, F and $\grave{F}$ being arbitrary, form a part of the expression for $\varphi.x$; and, when we consider the heterogeneity of exponential and algebraic functions, it may be perhaps thought not altogether unworthy of our attention to examine in what manner this is brought about. I have endeavoured in the present paper to give a brief explanation of this difficulty, if such it shall be considered by any.

The cases alluded to, in which quantities of either of the above-mentioned forms are included in $\varphi.x$ are those, in which any root of the equation, $\acute{V} = o$, is equal to nothing, or in which two or more roots of this equation are equal to each other. First, I shall suppose it to have a root; not actually equal to nothing, but evanescent, and represented by $h\,\delta\alpha$, h being a finite coefficient. The corresponding term of $\varphi.x$ will be

$Bc^{h x . \delta \alpha} = B. \left\{ 1 + h\,x.\,\delta\alpha + h^2\,x^2.\,\frac{\delta\alpha^2}{1.\,2} + \&c. \right\}$, which ultimately is reduced to its first term. It appears then that a single root of the equation, $\acute{V} = o$, being equal to nothing will introduce into $\varphi.x$ an arbitrary term independent of x. We have next to consider the case of two roots being equal to zero; but as this is included in the more general case of two roots being equal to each other, I shall proceed to this last; and suppose that the equation, $\acute{V} = o$, has two roots represented by $\alpha + h\,\delta\,\alpha$ and $\alpha + h'\,\delta\,\alpha$, which are of course ultimately equal to each other. The terms of $\varphi.x$ corresponding to these roots are $Bc^{[\alpha + h\,\delta\,\alpha]\,x}$ and $\acute{B}c^{[\alpha + h'\,\delta\,\alpha]\,x}$; which being developed according to the powers of the indefinitely small variation $\delta\,\alpha$, and added together give $(B+\acute{B})c^{\alpha\,x}+(B\,h+\acute{B}\,h')\,c^{\alpha\,x}x\,\delta\,\alpha+(B\,h^2+\acute{B}\,h'^2)\,c^{\alpha\,x}x^2\frac{\delta\,\alpha^2}{1.2}+$ &c. (1) Instead of the arbitrary constants B and $\acute{B}$, we may feign two others A and A', so that $B + \acute{B} = A$, and $B\,h + \acute{B}\,h' = \frac{A'}{\delta\,\alpha}$. In fact, this condition will be fulfilled, if we make $B = \frac{\frac{A'}{\delta\,\alpha} - A\,h'}{h - h'}$ and $\acute{B} = \frac{A\,h - \frac{A'}{\delta\,\alpha}}{h - h'}$. Substituting these values for B and $\acute{B}$ in the function (1), we have $A\,c^{\alpha\,x} + A'\,c^{\alpha\,x}\,x + A'\,c^{\alpha\,x}.(h + h')\,x^2.\frac{\delta\,\alpha}{1.\,2} +$ &c. which is ultimately reduced to its two first terms. By pursuing a similar analysis it will readily appear, that if there be i roots indefinitely near to α, and represented by $\alpha + h\,\delta\,\alpha$, $\alpha + h'\,\delta\,\alpha$, $\alpha + h''\,\delta\,\alpha$, &c.; and if we develope the i terms, introduced by them into $\varphi.x$, of the form $B\,c^{[\alpha + h\,\delta\,\alpha]\,x}$ according to the powers of $\delta\,\alpha$; we may equate the coefficients of $c^{\alpha\,x}$, $c^{\alpha\,x}\,x\,\delta\,\alpha$, $c^{\alpha\,x}\,x^2\,\delta\,\alpha^2$, ... $c^{\alpha\,x}\,x^{(i-1)}\,\delta\,\alpha^{(i-1)}$, to

i new arbitrary constants represented by A, $\frac{A'}{\delta\alpha}$, $\frac{A''}{\delta\alpha^2}$, ... $\frac{A'''\ \&c.}{\delta\alpha^{(i-1)}}$; the remainder of the developement of $c^{h\delta\alpha}$, $c^{h'\delta\alpha}$ &c. will ultimately disappear, leaving the i terms $A\,c^{\alpha x}$, $A'c^{\alpha x}.\,x$, $A''\,c^{\alpha x}.\,x^2$ $A'''\,c^{\alpha x}.\,x^3$, ... $A'''\,^{\&c.}\,c^{\alpha x}.\,x^{(i-1)}$, as part of the expression of $\varphi.x$.

It appears then that the form of the function $\varphi.x$ is in its nature, as I stated in the beginning of this paper, *exponential*; and that the algebraic functions, that occur in it in certain cases, are no more than the remains, or, if I may use the expression, the skeletons of exponential ones; the rest of which have disappeared, in consequence of the evanescent factors, by which they were multiplied.