American Mathematical Series

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GENERAL EDITOR

ANALYTIC GEOMETRY OF SPACE

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NEW YORK
HENRY HOLT AND COMPANY
1914

Norwood Press
J. S. Cushing Co. — Berwick & Smith Co.
Norwood, Mass., U.S.A.

PREFACE

In this book, which is planned for an introductory course, the first eight chapters include the subjects usually treated in rectangular coördinates. They presuppose as much knowledge of algebra, geometry, and trigonometry as is contained in the major requirement of the College Entrance Examination Board, and as much plane analytic geometry as is contained in the better elementary textbooks. In this portion, proofs of theorems from more advanced subjects in algebra are supplied as needed. Among the features of this part are the development of linear systems of planes, plane coördinates, the concept of infinity, the treatment of imaginaries, and the distinction between centers and vertices of quadric surfaces. The study of this portion can be regarded as a first course, not demanding more than thirty or forty lessons.

In Chapter IX tetrahedral coördinates are introduced by means of linear transformations, under which various invariant properties are established. These coördinates are used throughout the next three chapters. The notation is so chosen that no ambiguity can arise between tetrahedral and rectangular systems. The selection of subject matter is such as to be of greatest service for further study of algebraic geometry.

In Chapter XIII a more advanced knowledge of plane analytic geometry is presupposed, but the part involving Plücker's numbers may be omitted without disturbing the continuity of the subject. In the last chapter extensive use is made of the calculus, including the use of partial differentiation and of the element of arc.

The second part will require about fifty lessons.

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ANALYTIC GEOMETRY OF SPACE

CHAPTER I

COÖRDINATES

1. Rectangular coördinates. The idea of rectangular coördinates as developed in plane analytic geometry may be extended to space in the following manner.

Let there be given three mutually perpendicular planes (Fig. 1) XOY, YOZ, ZOX, intersecting at O, the **origin.** These planes will be called **coördinate planes.** The planes ZOX, XOY intersect in $X'OX$, the X-axis; the planes XOY, YOZ intersect in $Y'OY$, the Y-axis; the planes YOZ, ZOX intersect in $Z'OZ$, the Z-axis. Distances measured in the directions $X'OX$, $Y'OY$, $Z'OZ$, respectively, will be considered positive; those measured in the opposite directions will be regarded as negative. The **coördinates** of any point P are its distances from the three coördinate planes. The distance from the plane YOZ is denoted by x, the distance from the plane ZOX is denoted by y, and the distance from the plane XOY is denoted by z. These three numbers x, y, z are spoken of as the x-, y-, z-coördinates of P, respectively. Any point P in space has three real coördinates. Conversely, any three real numbers x, y, z, taken as x-, y-, and z-coördinates, respectively, determine a point P; for if we lay off a distance $OA=x$ on the X-axis, $OB=y$ on the Y-axis, $OC=z$ on

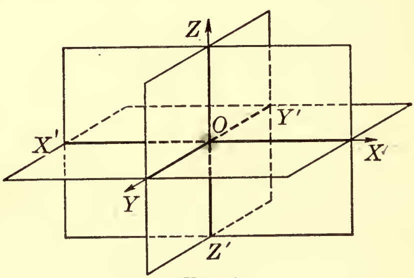

FIG. 1.

the Z-axis, and draw planes through A, B, C parallel to the coördinate planes, these planes will intersect in a point P whose coördinates are x, y, and z.

It will frequently be more convenient to determine the point P whose coördinates are x, y, and z, as follows: Lay off the distance $OA = x$ on the X-axis (Fig. 2). From A lay off the distance $AD = y$ on a parallel to the Y-axis. From D lay off the distance $DP = z$ on a parallel to the Z-axis.

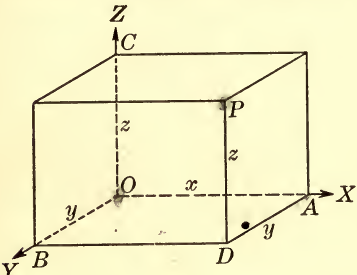

FIG. 2.

The eight portions of space separated by the coördinate planes are called **octants**. If the coördinates of a point P are a, b, c, the points in the remaining octants at the same absolute distances from the coördinate planes are $(-a, b, c)$, $(a, -b, c)$, $(a, b, -c)$, $(-a, -b, c)$, $(-a, b, -c)$, $(a, -b, -c)$, $(-a, -b, -c)$.

Two points are symmetric with regard to a plane if the line joining them is perpendicular to the plane and the segment between them is bisected by the plane. They are symmetric with regard to a line if the line joining them is perpendicular to the given line and the segment between them is bisected by the line. They are symmetric with regard to a point if the segment between them is bisected by the point.

The problem of representing a figure in space on a plane is considered in descriptive geometry, where it is solved in several ways by means of projections. In the figures appearing in this book a particular kind of parallel projection is used in which the X-axis and the Z-axis are represented by lines perpendicular to each other in the plane of the paper; the Y-axis is represented by a line making equal angles with the other two. Distances parallel to the X-axis or to the Z-axis are represented correctly to scale, but distances parallel to the Y-axis will be foreshortened, the amount of which may be chosen to suit the particular drawing considered. It will usually be convenient for the student, in drawing figures on cross section paper, to take a unit on the Y-axis $1/\sqrt{2}$ times as long as the unit on the other axes.

EXERCISES

1. Plot the following points to scale, using cross section paper: (1, 1, 1), (2, 0, 3), (−4, −1, −4), (−3, −4, 1), (4, 4, −1), (−7, 2, 3), (−1, 5, −5), (−4, 2, 8), (3, −4, −1), (2, 1, −3), (−1, 0, 0), (4, −2, 2), (0, 0, 2), (0, −1, 0), (−3, 0, 0), (0, 0, 0).

2. What is the locus of a point for which $x = 0$?

3. What is the locus of a point for which $x = 0,\ y = 0$?

4. What is the locus of a point for which $x = a,\ y = b$?

5. Given a point (k, l, m), write the coördinates of the point symmetric with it as to the plane XOY; the plane ZOX; the X-axis; the Y-axis; the origin.

2. Orthogonal projections. The orthogonal projection of a point on a plane is the foot of the perpendicular from the point to the plane. The orthogonal projection on a plane of a segment PQ of a line* is the segment $P'Q'$ joining the projections P' and Q' of P and Q on the plane.

The orthogonal projection of a point on a line is the point in which the line is intersected by a plane which passes through the given point and is perpendicular to the given line. The orthogonal projection of a segment PQ of a line l on a second line l' is the segment $P'Q'$ joining the projections P' and Q' of P and Q on l.

For the purpose of measuring distances and angles, one direction along a line will be regarded as positive and the opposite direction as negative. A segment PQ on a directed line is positive or negative according as Q is in the positive or negative direction from P. From this definition it follows that $PQ = -QP$.

The angle between two intersecting directed lines l and l' will be defined as the smallest angle which has its sides extending in the positive directions along l and l'. We shall, in general, make no convention as to whether this angle is to be considered positive or negative. The angle between two non-intersecting directed lines l and l' will be defined as equal to the angle between two intersecting lines m and m' having the same directions as l and l', respectively.

* We shall use the word *line* throughout to mean a straight line.

THEOREM I. *The length of the projection of a segment of a directed line on a second directed line is equal to the length of the given segment multiplied by the cosine of the angle between the lines.*

Let PQ (Figs. 3 *a*, 3 *b*) be the given segment on l and let $P'Q'$ be its projection on l'. Denote the angle between l and l' by θ. It is required to prove that

$$P'Q' = PQ \cos\theta.$$

Through P' draw a line l'' having the same direction as l. The angle between l' and l'' is equal to θ. Let Q'' be the point in

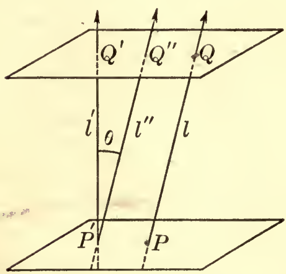

FIG. 3 *a*.

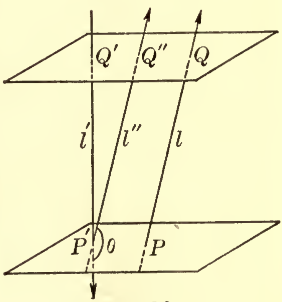

FIG. 3 *b*.

which l'' meets the plane through Q perpendicular to l'. Then the angle $P'Q'Q''$ is a right angle. Hence, by trigonometry, we have

$$P'Q' = P'Q'' \cos\theta.$$

But $$P'Q'' = PQ.$$

It follows that $$P'Q' = PQ\cos\theta.$$

It should be observed that it makes no difference in this theorem whether the segment PQ is positive or negative. The segment $PQ = r$ will always be regarded as positive in defining cosines.

THEOREM II. *The projection on a directed line l of a broken line made up of segments P_1P_2, P_2P_3, $\cdots$, $P_{n-1}P_n$ of different lines is the sum of the projections on l of its parts, and is equal to the projection on l of the straight line P_1P_n.*

For, let P'_1, P'_2, P'_3, $\cdots$, P'_{n-1}, P'_n be the projections of P_1, P_2, P_3, $\cdots$, P_{n-1}, P_n, respectively. The sum of the projections is equal to $P'_1P'_n$; that is,

$$P'_1P'_2 + P'_2P'_3 + \cdots + P'_{n-1}P'_n = P'_1P'_n.$$

But $P'_1P'_n$ is the projection of P_1P_n. The theorem therefore follows.

COROLLARY. *If P_1, P_2, $\cdots$, P_{n-1} are the vertices of a polygon, the sum of the projections on any directed line l of the segments P_1P_2, P_2P_3, $\cdots$, $P_{n-1}P_1$ formed by the sides of the polygon is zero.*

Since in this case P_n and P_1 coincide, it follows that P'_1 and P'_n also coincide. The sum of the projections is consequently zero.

EXERCISES

1. If O is the origin and P any point in space, show that the projections of the segment OP upon the coördinate axes are equal to the coördinates of P.

2. If the coördinates of P_1 are x_1, y_1, z_1 and of P_2 are x_2, y_2, z_2, show that the projections of the segment P_1P_2 upon the coördinate axes are equal to $x_2 - x_1$, $y_2 - y_1$, $z_2 - z_1$, respectively.

3. If the lengths of the projections of P_1P_2 upon the axes are respectively 3, -2, 7 and the coördinates of P_1 are $(-4, 3, 2)$, find the coördinates of P_2.

4. Find the distance from the origin to the point $(4, 3, 12)$.

5. Find the distance from the origin to the point (a, b, c).

6. Find the cosines of the angles made with the axes by the line joining the origin to each of the following points.

$(1, 2, 0)$	$(1, 1, 1)$	$(-7, 6, 2)$
$(0, 2, 4)$	$(1, -4, 2)$	(x, y, z)

3. Direction cosines of a line. Let l be any directed line in space, and let l' be a line through the origin which has the same direction. If α, β, γ (Fig. 4) are the angles which l' makes with the coördinate axes, these are also, by definition (Art. 2), the angles which l makes with the axes. They are called the direction angles of l and their cosines are called **direction cosines.** The latter will be denoted by λ, μ, ν, respectively.

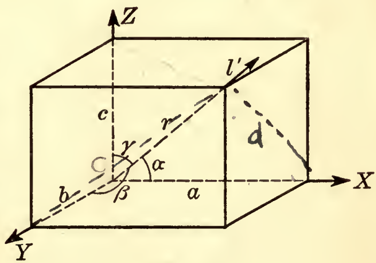

FIG. 4.

Let $P \equiv (a, b, c)$ be any point on l' in the positive direction from the origin and let $OP = r$. Then, from trigonometry, we have

$$\lambda = \cos\alpha = \frac{a}{r}, \qquad \mu = \cos\beta = \frac{b}{r}, \qquad \nu = \cos\gamma = \frac{c}{r}.$$

But r is the diagonal of a rectangular parallelopiped whose edges are

$$OA = a, \qquad OB = b, \qquad OC = c.$$

Hence, we obtain $$r = \sqrt{a^2 + b^2 + c^2}.$$

In this equation, as in the formulas throughout the book, except when the contrary is stated, indicated roots are to be taken with the positive sign.

By substituting this value of r in the above equations, we obtain

$$\lambda = \cos\alpha = \frac{a}{\sqrt{a^2 + b^2 + c^2}},$$

$$\mu = \cos\beta = \frac{b}{\sqrt{a^2 + b^2 + c^2}},$$

$$\nu = \cos\gamma = \frac{c}{\sqrt{a^2 + b^2 + c^2}}.$$

By squaring each member of these equations and adding the results, we obtain

$$\lambda^2 + \mu^2 + \nu^2 = 1, \tag{1}$$

hence we have the following theorem.

THEOREM. *The sum of the squares of the direction cosines of a line is equal to unity.*

If λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2 are the direction cosines of two like directed lines, we have

$$\lambda_1 = \lambda_2, \ \mu_1 = \mu_2, \ \nu_1 = \nu_2.$$

If the lines are oppositely directed, we have

$$\lambda_1 = -\lambda_2, \ \mu_1 = -\mu_2, \ \nu_1 = -\nu_2.$$

4. Distance between two points. Let $P_1 \equiv (x_1, y_1, z_1)$, $P_2 \equiv (x_2, y_2, z_2)$ be any two points in space. Denote the direction cosines of the line P_1P_2 (Fig. 5) by λ, μ, ν and the length of the segment P_1P_2 by d. The projection of the segment P_1P_2 on each of the axes is equal to the sum of the projections of P_1O and OP_2, that is

$$\lambda d = x_2 - x_1,\ \mu d = y_2 - y_1,\ \nu d = z_2 - z_1.$$

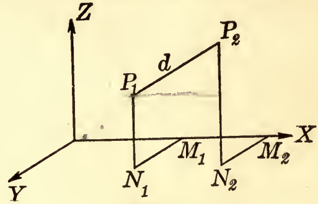

FIG. 5.

By squaring both members of these equations, adding, and extracting the square root, we obtain

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}. \tag{2}$$

EXERCISES

1. Find the distance between (3, 4, − 2) and (− 5, 1, − 6).

2. Show that the points (− 3, 2, − 7), (2, 2, − 3), and (− 3, 6, − 2) are vertices of an isosceles triangle.

3. Show that the points (4, 3, − 4), (− 2, 9, − 4), and (− 2, 3, 2) are vertices of an equilateral triangle.

4. Express by an equation that the point (x, y, z) is equidistant from (1, 1, 1) and (2, 3, 4).

5. Show that $x^2 + y^2 + z^2 = 4$ is the equation of a sphere whose center is the origin and whose radius is 2.

6. Find the direction cosines of the line P_1P_2, given :

(a) $P_1 \equiv (0, 0, 0)$, $P_2 \equiv (2, 3, 5)$.
(b) $P_1 \equiv (1, 1, 1)$, $P_2 \equiv (2, 2, 2)$.
(c) $P_1 \equiv (1, -2, 3)$, $P_2 \equiv (4, 2, -1)$.

7. What is known about the direction of a line if (a) $\cos\alpha = 0$? (b) $\cos\alpha = 0$ and $\cos\beta = 0$? (c) $\cos\alpha = 1$?

8. Show that the points (3, − 2, 7), (6, 4, − 2), and (5, 2, 1) are on a line.

9. Find the direction cosines of a line which makes equal angles with the coördinate axes.

5. Angle between two directed lines. Let l_1 and l_2 be two directed lines having the direction cosines λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2, respectively. It is required to find an expression for the cosine of the angle between l_1 and l_2. Through O (Fig. 6) draw two

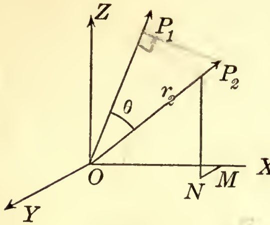

FIG. 6.

lines OP_1 and OP_2 having the same directions as l_1 and l_2, respectively. Let $OP_2 = r_2$ and let the coördinates of P_2 be

$$x_2 = OM, \qquad y_2 = MN, \qquad z_2 = NP_2.$$

The projection of OP_2 on OP_1 is equal to the sum of the projections of the broken line $OMNP_2$ on OP_1 (Art. 2).

Hence $$OP_2 \cos\theta = OM\,\lambda_1 + MN\,\mu_1 + NP_2\,\nu_1.$$

But $OP_2 = r_2,\quad OM = x_2 = r_2\lambda_2,\quad MN = y_2 = r_2\mu_2,\quad NP = z_2 = r_2\nu_2.$

Hence, we obtain

$$r_2 \cos\theta = r_2\lambda_1\lambda_2 + r_2\mu_1\mu_2 + r_2\nu_1\nu_2,$$

or $$\cos\theta = \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2. \tag{3}$$

The condition that the two given lines are perpendicular is that $\cos\theta = 0$. Hence we have the following theorem:

THEOREM. *Two lines l_1 and l_2 with direction cosines λ_1, μ_1, ν_1 and λ_2, μ_2, ν_2, respectively, are perpendicular if*

$$\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 = 0. \tag{4}$$

The square of the sine of θ may be found from (1) and (3). Since $\sin^2\theta = 1 - \cos^2\theta$, it follows that

$$\begin{aligned}\sin^2\theta &= (\lambda_1^2 + \mu_1^2 + \nu_1^2)(\lambda_2^2 + \mu_2^2 + \nu_2^2) - (\lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2)^2 \\ &= (\lambda_1\mu_2 - \lambda_2\mu_1)^2 + (\mu_1\nu_2 - \mu_2\nu_1)^2 + (\nu_1\lambda_2 - \nu_2\lambda_1)^2. \end{aligned} \tag{5}$$

6. Point dividing a segment in a given ratio. Let $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ be two given points (Fig. 7). It is required to find the point $P \equiv (x, y, z)$ on the line P_1P_2 such that $P_1P : PP_2 = m_1 : m_2$. Let λ, μ, ν be the direction cosines of the line P_1P_2. Then (Art. 2, Th. I) we have

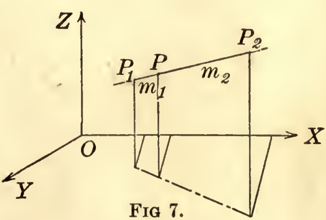

FIG 7.

$$P_1P\,\lambda = x - x_1 \text{ and } PP_2\,\lambda = x_2 - x.$$

Hence $$P_1P\,\lambda : PP_2\,\lambda = x - x_1 : x_2 - x = m_1 : m_2.$$

On solving for x we obtain

$$x = \frac{m_2 x_1 + m_1 x_2}{m_1 + m_2}. \tag{6}$$

Similarly,

$$y = \frac{m_2 y_1 + m_1 y_2}{m_1 + m_2},$$

$$z = \frac{m_2 z_1 + m_1 z_2}{m_1 + m_2}.$$

It should be noticed that if m_1 and m_2 have the same sign, P_1P and PP_2 are measured in the same direction so that P lies between P_1 and P_2. If m_1 and m_2 have opposite signs, P lies outside the segment P_1P_2. By giving m_1 and m_2 suitable values, the coördinates of any point on the line P_1P_2 can be represented in this way. In particular, if P is the mid-point of the segment P_1P_2, $m_1 = m_2$, so that the coördinates of the mid-point are

$$x = \frac{x_1 + x_2}{2}, \qquad y = \frac{y_1 + y_2}{2}, \qquad z = \frac{z_1 + z_2}{2}.$$

EXERCISES

1. Find the cosine of the angle between the two lines whose direction cosines are $\frac{1}{\sqrt{14}}$, $\frac{2}{\sqrt{14}}$, $\frac{3}{\sqrt{14}}$ and $\frac{2}{\sqrt{30}}$, $\frac{-1}{\sqrt{30}}$, $\frac{5}{\sqrt{30}}$.

2. Find the direction cosines of each of the coördinate axes.

3. The direction cosines of a line are proportional to 4, -3, 1. Find their values.

4. The direction cosines of two lines are proportional to 6, 2, -1 and -3, 1, -5, respectively. Find the cosine of the angle between the lines.

5. Show that the lines whose direction cosines are proportional to 3, 6, 2; -2, 3, -6; -6, 2, 3 are mutually perpendicular.

6. Show that the points (7, 3, 4), (1, 0, 6), (4, 5, -2) are the vertices of a right triangle.

7. Show that the points (3, 7, 2), (4, 3, 1), (1, 6, 3), (2, 2, 2) are the vertices of a parallelogram.

8. Find the coördinates of the intersection of the diagonals in the parallelogram of Ex. 7.

9. Show by two different methods that the three points (4, 13, 3), (3, 6, 4), (2, -1, 5) are collinear.

10. A line makes an angle of 75° with the X-axis and 30° with the Y-axis. How many positions may it have? Find, for each position, the cosine of the angle it makes with the Z-axis.

11. Determine the coördinates of the intersection of the medians of the triangle with vertices at (1, 2, 3), (2, 3, 1), (3, 1, 2).

12. Prove that the medians of any triangle meet in a point twice as far from each vertex as from the mid-point of the opposite side. This point is called the *center of gravity* of the triangle.

13. Prove that the three straight lines joining the mid-points of opposite edges of any tetrahedron meet in a point, and are bisected by it. This point is called the *center of gravity* of the tetrahedron.

14. Show that the lines joining each vertex of a tetrahedron to the point of intersection of the medians of the opposite face pass through the center of gravity.

15. Show that the lines joining the middle points of the sides of any quadrilateral form a parallelogram.

16. Show how the ratio $m_1 : m_2$ (Art. 6) varies as P describes the line P_1P_2.

7. Polar Coördinates. Let OX, OY, OZ be a set of rectangular axes and P be any point in space. Let $OP = \rho$ have the direction angles α, β, γ. The position of the line OP is determined by α, β, γ and the position of P on the line is given by ρ, so that the position of the point P in space is fixed when ρ, α, β, γ are known. These quantities ρ, α, β, γ are called the **polar coördinates** of P. As α, β, γ are direction angles, they are not independent, since by equation (1)

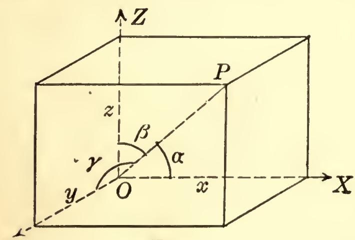

FIG. 8.

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

If the rectangular coördinates of P are x, y, z, then (Art. 3)

$$x = \rho \cos \alpha, \qquad y = \rho \cos \beta, \qquad z = \rho \cos \gamma.$$

8. Cylindrical coördinates. A point is determined when its directed distance from a fixed plane and the polar coördinates of its orthogonal projection on that plane are known. These coördinates are called the **cylindrical coördinates** of a point. If the point P is referred to the rectangular axes x, y, z, and the fixed plane is taken as $z = 0$ and the x-axis for polar axis, we may write (Fig. 9)

$$x = \rho \cos\theta, \qquad y = \rho \sin\theta, \qquad z = z,$$

in which ρ, θ, z are the cylindrical coördinates of P.

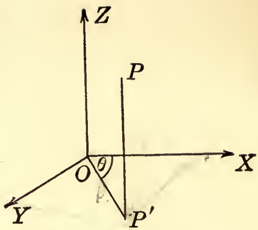

FIG. 9.

9. Spherical coördinates. Let OX, OY, OZ, and P be chosen as in Art. 7, and let P' be the orthogonal projection of P on the plane XOY. Draw OP. The position of P is defined by the distance ρ, the angle $\phi = ZOP$ which the line OP makes with the z-axis, and the angle θ (measured by the angle XOP') which the plane through P and the z-axis makes with the plane XOZ. The numbers ρ, ϕ, θ are called the **spherical coördinates** of P. The length ρ is called the radius vector, the angle ϕ is called the co-latitude, and θ is called the longitude.

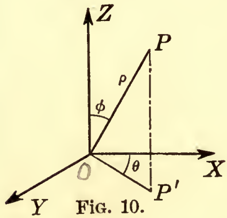

FIG. 10.

If $P \equiv (x, y, z)$, then, from the figure (Fig. 10),

$$OP' = \rho \cos(90 - \phi) = \rho \sin\phi.$$

Hence
$$x = \rho \sin\phi \cos\theta,$$
$$y = \rho \sin\phi \sin\theta,$$
$$z = \rho \cos\phi.$$

On solving these equations for ρ, ϕ, θ, we find

$$\rho = \sqrt{x^2 + y^2 + z^2}, \qquad \phi = \text{arc cos}\frac{z}{\sqrt{x^2 + y^2 + z^2}}, \qquad \theta = \text{arc tan}\frac{y}{x}.$$

EXERCISES

1. What locus is defined by $\rho = 1$?

2. What locus is defined by $\alpha = 60°$?

3. What locus is defined by $\theta = 30°$?

4. What locus is defined by $\phi = 45°$?

5. Transform $x^2 + y^2 + z^2 = 4$ to: (a) polar coördinates, (b) spherical coördinates, (c) cylindrical coördinates.

6. Transform $x^2 + y^2 = z^2$ into spherical coördinates; into cylindrical coördinates.

7. Express the distance between two points in terms of their polar coördinates.

CHAPTER II

PLANES AND LINES

10. Equation of a plane. A plane is characterized by the properties:

(*a*) It contains three points not on a line.

(*b*) It contains every point on any line joining two points on it.

(*c*) It does not contain all the points of space.

THEOREM. *The locus of the points whose coördinates satisfy a linear equation*

$$Ax + By + Cz + D = 0 \tag{1}$$

with real coefficients is a plane.

We shall prove this theorem on the supposition that $C \neq 0$. Since A, B, C are not all zero, a proof for the case in which $C = 0$ can be obtained in a similar way.

It is seen by inspection that the coördinates $\left(0, 0, -\frac{D}{C}\right)$, $\left(0, 1, -\frac{(B+D)}{C}\right)$, $\left(1, 0, -\frac{(A+D)}{C}\right)$ satisfy the equation. These three points are not collinear, since no values of m_1, m_2 other than zero satisfy the simultaneous equations (Art. 6)

$$m_1 = 0, \qquad m_2 = 0, \qquad m_1 A + m_2 B = 0.$$

Let $P_1 \equiv (x_1, y_1, z_1)$ and $P_2 \equiv (x_2, y_2, z_2)$ be any two points whose coördinates satisfy (1). The coördinates of any point P on the line P_1P_2 are of the form

$$x = \frac{m_1 x_2 + m_2 x_1}{m_1 + m_2}, \qquad y = \frac{m_1 y_2 + m_2 y_1}{m_1 + m_2}, \qquad z = \frac{m_1 z_2 + m_2 z_1}{m_1 + m_2}.$$

The equation (1) is satisfied by the coördinates of P if

$$m_2(Ax_1 + By_1 + Cz_1 + D) + m_1(Ax_2 + By_2 + Cz_2 + D) = 0,$$

but since the coördinates of P_1 and P_2 satisfy (1), we have

$$Ax_1 + By_1 + Cz_1 + D = 0, \qquad Ax_2 + By_2 + Cz_2 + D = 0,$$

hence the coördinates of P satisfy (1) for all values of m_1 and m_2.

Finally, not all the points of space lie on the locus defined by (1), since the coördinates $\left(0,\ 0,\ -\frac{(D+C)}{C}\right)$ do not satisfy (1). This completes the proof of the theorem.

11. Plane through three points. Let $(x_1,\ y_1,\ z_1)$, $(x_2,\ y_2,\ z_2)$, $(x_3,\ y_3,\ z_3)$ be the coördinates of three non-collinear points. The condition that these points all lie in the plane

$$Ax + By + Cz + D = 0$$

is that their coördinates satisfy this equation, thus

$$\begin{aligned} Ax_1 + By_1 + Cz_1 + D &= 0,\\ Ax_2 + By_2 + Cz_2 + D &= 0,\\ Ax_3 + By_3 + Cz_3 + D &= 0. \end{aligned}$$

The condition that four numbers A, B, C, D (not all zero) exist which satisfy the above four simultaneous equations is

$$\begin{vmatrix} x & y & z & 1 \\ x_1 & y_1 & z_1 & 1 \\ x_2 & y_2 & z_2 & 1 \\ x_3 & y_3 & z_3 & 1 \end{vmatrix} = 0. \tag{2}$$

This is the required equation, for it is the equation of a plane, since it is of first degree in x, y, z (Art. 10). The plane passes through the given points, since the coördinates of each of the given points satisfy the equation.

12. Intercept form of the equation of a plane. If a plane intersects the X-, Y-, Z-axes in three points A, B, C, respectively, the segments OA, OB, and OC are called the intercepts of the plane. Let A, B, C all be distinct from the origin and let the lengths of the intercepts be a, b, c, so that $A \equiv (a,\ 0,\ 0)$, $B \equiv (0,\ b,\ 0)$, $C \equiv (0,\ 0,\ c)$. The equation (2) of the plane determined by these three points (Art. 11) may be reduced to

$$\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1. \tag{3}$$

This equation is called the **intercept form** of the equation of a plane.

EXERCISES

1. Find the equation of the plane through the points (1, 2, 3), (3, 1, 2), (5, − 1, 3).

2. Find the equation of the plane through the points (0, 0, 0), (1, 1, 1), (2, 2, − 2). What are its intercepts?

3. Prove that the four points (1, 2, 3), (2, 4, 1), (− 1, 0, 1), (0, 0, 5) lie in a plane. Find the equation of the plane.

4. Determine k so that the points (1, 2, − 1), (3, − 1, 2), (2, − 2, 3), (1, − 1, k) shall lie in a plane.

5. Find the point of intersection of the three planes, $x + y + z = 6$, $2z - y + 2x = 0$, $x - 2y + 3z = 4$.

13. The normal form of the equation of a plane. Let ABC (Fig. 11) be any plane. Let OQ be drawn through the origin perpendicular to the given plane and intersecting it at P'. Let the direction cosines of OQ be λ, μ, ν and denote the length of the segment OP' by p.

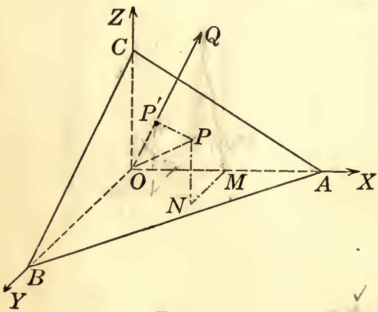

FIG. 11.

Let $P \equiv (x, y, z)$ be any point in the given plane. The projection of P on OQ is P' (Art. 2). Draw OP and the broken line $OMNP$, made up of segments $ON = x$, $MN = y$, and $NP = z$, parallel to the X-, Y-, and Z-axes, respectively. The projections of OP and $OMNP$ on OQ are equal (Art. 2, Th. II). The projection of the broken line is $\lambda x + \mu y + \nu z$, the projection of OP is OP' or p, so that

$$\lambda x + \mu y + \nu z = p. \tag{4}$$

This equation is satisfied by the coördinates of every point P in the given plane. It is not satisfied by the coördinates of any other point. For, if P_1 is a point not lying in the given plane, it is similarly seen, since the projection of OP_1 on OQ is not equal to p, that the coördinates of P_1 do not satisfy (4).

Hence, (4) is the equation of the plane. It is called the **normal form** of the equation of the plane. The number p in this equation is positive or negative, according as P' is in the positive or negative direction from O on OQ.

14. Reduction of the equation of a plane to the normal form. Let

$$Ax + By + Cz + D = 0 \tag{5}$$

be any equation of first degree with real coefficients. It is required to reduce this equation to the normal form. Let $Q \equiv (A, B, C)$ be the point whose coördinates are the coefficients of x, y, z in this equation. The direction cosines of the directed line OQ are (Art. 3)

$$\lambda = \frac{A}{\sqrt{A^2+B^2+C^2}},\quad \mu = \frac{B}{\sqrt{A^2+B^2+C^2}},\quad \nu = \frac{C}{\sqrt{A^2+B^2+C^2}}. \tag{6}$$

If we transpose the constant term of (3) to the other member of the equation, and divide both numbers by $\sqrt{A^2+B^2+C^2}$, we obtain

$$\frac{A}{\sqrt{A^2+B^2+C^2}}x + \frac{B}{\sqrt{A^2+B^2+C^2}}y + \frac{C}{\sqrt{A^2+B^2+C^2}}z = \frac{-D}{\sqrt{A^2+B^2+C^2}}. \tag{7}$$

The plane determined by (7) is identical with that determined by (5) since the coördinates of a point will satisfy (7) if, and only if, they satisfy (5). By subtituting from (6) in (7) and comparing with (4), we see that the locus of the equation is a plane perpendicular to OQ and intersecting OQ at a point P' whose distance from O is

$$p = \frac{-D}{\sqrt{A^2+B^2+C^2}}. \tag{8}$$

In these equations, the radical is to be taken with the positive sign. The coefficients of x, y, z are proportional to λ, μ, ν in such a way that the direction cosines of the normal to the plane are fixed when the signs of A, B, C are known. But the plane is not changed if its equation is multiplied by -1, hence the position of the plane alone is not sufficient to determine the direction of the normal. In order to define a positive and a negative side of a plane we shall first prove the following theorem:

THEOREM. *Two points P_1, P_2 are on the same side or on opposite sides of the plane $Ax + By + Cz + D = 0$, according as their coördinates make the first member of the equation of the plane have like or unlike signs.*

For, let $P_1 \equiv (x_1, y_1, z_1)$, $P_2 \equiv (x_2, y_2, z_2)$ be two points not lying on the plane. The point $P \equiv (x, y, z)$ in which the line P_1P_2 intersects the plane is determined (Art. 6) by the values of m_1, m_2 which satisfy the equation

$$m_1(Ax_2 + By_2 + Cz_2 + D) + m_2(Ax_1 + By_1 + Cz_1 + D) = 0.$$

If $Ax_1 + By_1 + Cz_1 + D$ and $Ax_2 + By_2 + Cz_2 + D$ have unlike signs, then m_1 and m_2 have the same sign, and the point P lies between P_1 and P_2. If $Ax_1 + By_1 + Cz_1 + D$ and $Ax_2 + By_2 + Cz_2 + D$ have the same sign, then the numbers m_1, m_2 have opposite signs, hence the point P is not between P_1 and P_2.

When all the terms in the equation

$$Ax + By + Cz + D = 0$$

are transposed to the first member, a point (x_1, y_1, z_1) will be said to be on the positive side of the plane if $Ax_1 + By_1 + Cz_1 + D$ is a positive number; the point will be said to be on the negative side if this expression is a negative number. Finally, the point is on the plane if the expression vanishes. It should be observed that the equation must not be multiplied by -1 after the positive and negative sides have been chosen.

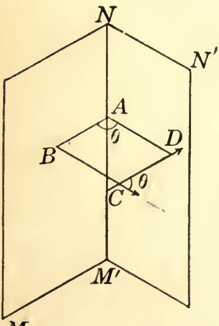

FIG. 12.

15. Angle between two planes. The angle between two planes is equal to the angle between two directed normals to the planes; hence, by Arts. 5 and 14, we have at once the following theorem:

THEOREM. *The cosine of the angle θ between two planes*

$$Ax + By + Cz + D = 0,$$
$$A'x + B'y + C'z + D' = 0$$

is defined by the equation

$$\cos\theta = \frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2}\sqrt{A'^2 + B'^2 + C'^2}}. \qquad (9)$$

In particular, the condition that the planes are perpendicular is

$$AA' + BB' + CC' = 0. \qquad (10)$$

The conditions that the planes are parallel are (Art. 3)

$$\frac{A}{A'}=\frac{B}{B'}=\frac{C}{C'}. \tag{11}$$

The equations (11) are satisfied whether the normals have the same direction or opposite directions. From the definition of the angle between two planes it follows that in the first case the two planes are parallel and in the second case they make an angle of 180 degrees with each other. We shall say, however, that the planes are parallel in each case.

16. Distance to a point from a plane. Let $P \equiv (x_1, y_1, z_1)$ be a given point and $Ax + By + Cz + D = 0$ be the equation of a given plane. The distance to P from the plane is equal to the distance from the given plane to a plane through P parallel to it.

The equation

$$Ax + By + Cz - (Ax_1 + By_1 + Cz_1) = 0$$

represents a plane, since it is of first degree with real coefficients (Art. 10). It is parallel to the given plane by Eqs. (11). It passes through P since the coördinates of P satisfy the equation. When the equations of the planes are reduced to the normal form, they become, respectively,

$$\frac{A}{\sqrt{A^2+B^2+C^2}}x+\frac{B}{\sqrt{A^2+B^2+C^2}}y + \frac{C}{\sqrt{A^2+B^2+C^2}}z=\frac{-D}{\sqrt{A^2+B^2+C^2}},$$

$$\frac{A}{\sqrt{A^2+B^2+C^2}}x+\frac{B}{\sqrt{A^2+B^2+C^2}}y + \frac{C}{\sqrt{A^2+B+C^2}}z=\frac{Ax_1+By_1+Cz_1}{\sqrt{A^2+B^2+C^2}}.$$

The second members of these two equations represent the distances of the two planes from the origin, hence the distance from the first plane to the second, which is equal to the distance d to P from the given plane is found by subtracting the former from the latter.

The result is
$$d = \frac{Ax_1 + By_1 + Cz_1 + D}{\sqrt{A^2 + B^2 + C^2}}. \tag{12}$$

The direction to P from the plane, along the normal, is positive or negative according as the expression in the numerator of the second member is positive or negative (Art. 14), that is, according as P is on the positive or negative side of the plane.

EXERCISES

1. Reduce the equation $3x - 12y - 4z - 26 = 0$ to the normal form.

2. Write the equation of a plane through the origin parallel to the plane $x + 2y = 6$.

3. What is the distance from the plane $3x + 4y - z = 5$ to the point $(2, 2, 2)$?

4. Find the distance between the parallel planes

$$2x - y + 3z = 4, \quad 2x - y + 3z + 5 = 0.$$

5. Which of the points $(4, 3, 1)$, $(1, -4, 3)$, $(3, 5, 2)$, $(-1, 2, -2)$, $(5, 4, 6)$ are on the same side of the plane $5x - 2y - 3z = 0$ as the point $(1, 6, -3)$?

6. Find the coördinates of a point in each of the dihedral angles formed by the planes

$$3x + 2y + 5z - 4 = 0, \quad x - 2y - z + 6 = 0.$$

7. Show that each of the planes $25x + 39y + 8z - 43 = 0$ and $25x - 39y + 112z + 113 = 0$ bisect a pair of vertical dihedral angles formed by the planes $5x + 12z + 7 = 0$ and $3y - 4z - 6 = 0$. Which plane bisects the angle in which the origin lies?

8. Find the equation of the plane which bisects that angle formed by the planes $3x - 2y + z - 4 = 0$, $2x + y - 3z - 2 = 0$, in which the point $(1, 3, -2)$ lies.

9. Find the equations of the planes which bisect the dihedral angles formed by the planes $A_1x + B_1y + C_1z + D_1 = 0$, $A_2x + B_2y + C_2z + D_2 = 0$.

10. Find the equation of the locus of a point whose distance from the origin is equal to its distance from the plane $3x + y - 2z = 11$.

11. Write the equation of a plane whose distance from the point $(0, 2, 1)$ is 3, and which is perpendicular to the radius vector of the point $(2, -1, -1)$.

12. Show that the planes $2x - y + z + 3 = 0$, $x - y + 4z = 0$, $3x + y - 2z + 8 = 0$, $4x - 2y + 2z - 5 = 0$, $9x + 3y - 6z - 7 = 0$, and $7x - 7y + 28z - 6 = 0$ bound a parallelopiped.

13. Write the equation of a plane through $(1, 2, -1)$, parallel to the plane $x - 2y - z = 0$, and find its intercepts.

14. Find the equation of the plane passing through the points (1, 2, 3), (2, -3, 6) and perpendicular to the plane $4x + 2y + 3z = 1$.

15. Find the equation of the plane through the point (1, 3, 2) perpendicular to the planes

$$2x + 3y - 4z = 2, \quad 4x - 3y - 2z = 5.$$

16. Show that the planes $x + 2y - z = 0$, $y + 7z - 2 = 0$, $x - 2y - z - 4 = 0$, $x + 3y + z = 4$, and $3x + 3y - z = 8$ bound a quadrilateral pyramid.

17. Find the equation of the locus of a point which is 3 times as far from the plane $3x - 6y - 2z = 0$ as from the plane $2x - y + 2z = 9$.

18. Determine the value of m such that the plane $mx + 2y - 3z = 14$ shall be 2 units from the origin.

19. Determine k from the condition that $x - ky + 3z = 2$ shall be perpendicular to $3x + 4y - 2z = 5$.

17. Equations of a line. Let $A_1x + B_1y + C_1z + D_1 = 0$ and $A_2x + B_2y + C_2z + D_2 = 0$ be the equations of two non-parallel planes. The locus of the two equations considered as simultaneous is a line, namely, the line of intersection of the two planes (Art. 10). The simultaneous equations

$$A_1x + B_1y + C_1z + D_1 = 0,$$
$$A_2x + B_2y + C_2z + D_2 = 0$$

are called the **equations of the line.**

The locus represented by the equations of two parallel planes, considered as simultaneous, will be considered later (Art. 33).

18. Direction cosines of the line of intersection of two planes. Let λ, μ, ν be the direction cosines of the line of intersection of the two planes

$$L_1 \equiv A_1x + B_1y + C_1z + D_1 = 0,$$
$$L_2 \equiv A_2x + B_2y + C_2z + D_2 = 0.$$

Since the line lies in the plane $L_1 = 0$, it is perpendicular to the normal to the plane. Hence, (Arts. 5, 14)

$$\lambda A_1 + \mu B_1 + \nu C_1 = 0.$$

Similarly, $$\lambda A_2 + \mu B_2 + \nu C_2 = 0.$$

By solving these two equations for the ratios of λ, μ, ν, we obtain

$$\frac{\lambda}{B_1C_2 - B_2C_1} = \frac{\mu}{C_1A_2 - C_2A_1} = \frac{\nu}{A_1B_2 - A_2B_1}. \qquad (13)$$

The denominators in these expressions are, therefore, proportional to the direction cosines. In many problems, they may be used instead of the direction cosines themselves, but, in any case, the actual cosines may be determined by dividing these denominators by the square root of the sum of their squares. It should be observed that the equations of a line are not sufficient to determine a positive direction on it.

19. Forms of the equations of a line. If λ, μ, ν are the direction cosines of a line, and if $P_1 \equiv (x_1, y_1, z_1)$ is any point on it, the distance d from P_1 to another point $P \equiv (x, y, z)$ on the line satisfies the relations (Art. 4)

$$\lambda d = x - x_1, \ \mu d = y - y_1, \ \nu d = z - z_1.$$

By eliminating d, we obtain the equations

$$\frac{x - x_1}{\lambda} = \frac{y - y_1}{\mu} = \frac{z - z_1}{\nu}, \tag{14}$$

which are called the **symmetric form** of the equations of the line.

Instead of the direction cosines themselves, it is frequently convenient to use, in these equations, three numbers a, b, c, proportional, respectively, to λ, μ, ν. The equations then become

$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}. \tag{15}$$

They may be reduced to the preceding form by dividing the denominator of each member by $\sqrt{a^2 + b^2 + c^2}$ (Art. 3).

If the line (15) passes through the point $P_2 \equiv (x_2, y_2, z_2)$, the coördinates of P_2 satisfy the equations, so that

$$\frac{x_2 - x_1}{a} = \frac{y_2 - y_1}{b} = \frac{z_2 - z_1}{c}.$$

On eliminating a, b, c between these equations and (15), we obtain

$$\frac{x - x_1}{x_2 - x_1} = \frac{y - y_1}{y_2 - y_1} = \frac{z - z_1}{z_2 - z_1}. \tag{16}$$

These equations are called the **two-point form** of the equations of a line.

20. Parametric equations of a line. Any point on a line may be defined in terms of a fixed point on it, the direction cosines of the line, and the distance d of the variable point from the fixed one. Thus, by Art. 4

$$x = x_1 + \lambda d, \quad y = y_1 + \mu d, \quad z = z_1 + \nu d. \tag{17}$$

If λ, μ, ν are given and (x_1, y_1, z_1) represents a fixed point, any point (x, y, z) on the line may be defined in terms of d. To every real value of d corresponds a point on the line, and conversely. These equations are called **parametric equations** of the line, the parameter being the distance.

It is sometimes convenient to express the coördinates of a point in terms of a parameter κ which is defined in terms of d by a linear fractional equation of the form

$$d = \frac{\alpha + \beta\kappa}{\gamma + \delta\kappa},$$

in which α, β, γ, δ are constants satisfying the inequality

$$\alpha\delta - \beta\gamma \neq 0.$$

By substituting these values of κ in (17) and simplifying, we obtain equations of the form

$$x = \frac{a_1 + b_1\kappa}{a_4 + b_4\kappa}, \quad y = \frac{a_2 + b_2\kappa}{a_4 + b_4\kappa}, \quad z = \frac{a_3 + b_3\kappa}{a_4 + b_4\kappa}, \tag{18}$$

in which a_1, b_1, etc., are constants. Equations (18) are called the parametric equations of the line in terms of the parameter κ.

It should be observed that the denominators in the second members of equations (18) are all alike. Each value of κ for which $a_4 + b_4\kappa \neq 0$ determines a definite point on the line. As $a_4 + b_4\kappa$ approaches zero, the distance of the corresponding point from the origin increases without limit. To the value determined by $a_4 + b_4\kappa = 0$ we shall say that there corresponds a unique point which we shall call the **point at infinity** on the line.

EXERCISES

1. Find the points in which the following lines pierce the coördinate planes:

(a) $x + 2y - 3z = 1, \quad 3x - 2y + 5z = 2.$

(b) $x + 3y + 5z = 0, \quad 5x - 3y + z = 2.$

(c) $x + 2y - 5 = 0, \quad 2x - 3y + 2z = 7.$

2. Write the equations of the line $x + y - 3z = 6$, $2x - y + 2z = 7$ in the symmetric form, the two-point form, the parametric form.

3. Show that the lines $4x + y - 3z = 0$, $2x - y + 2z + 6 = 0$, and $8x - y + z = 1$, $10x + y - 4z + 1 = 0$ are parallel.

4. Write the equations of the line through $(3, 7, 3)$ and $(-1, 5, 6)$. Determine its direction cosines.

5. Find the equation of the plane passing through the point $(2, -2, 0)$ and perpendicular to the line $z = 3$, $y = 2x - 4$.

6. Find the value of k for which the lines $\frac{x-3}{2k} = \frac{y+1}{k+1} = \frac{z+3}{3}$ and $\frac{x-1}{3} = \frac{y+5}{1} = \frac{z+2}{k-2}$ are perpendicular.

7. Do the points $(2, 4, 6)$, $(4, 6, 2)$, $(1, 3, 8)$ lie on a line?

8. For what value of k are the points $(k, -3, 2)$, $(2, -2, 3)$, $(6, -1, 4)$ collinear?

9. Is there a value of k for which the points $(k, 2, -2)$, $(2, -2, k)$, and $(-2, 1, 3)$ are collinear?

10. Show that the line $\frac{x-2}{3} = \frac{y+2}{-1} = \frac{z-3}{4}$ lies in the plane $2x + 2y - z + 3 = 0$.

11. In equations (18) show that, as κ approaches infinity, the corresponding point approaches a definite point as a limit. Does this limiting point lie on the given line?

21. Angle which a line makes with a plane. Given the plane

$$Ax + By + Cz + D = 0$$

and the line
$$\frac{x - x_1}{a} = \frac{y - y_1}{b} = \frac{z - z_1}{c}.$$

The angle which the line makes with the plane is the complement of the angle which it makes with the normal to the plane. The direction cosines of the normal to the plane are proportional to A, B, C and the direction cosines of the line are proportional to a, b, c, hence the angle θ between the plane and the line is determined (Art. 5) by the formula

$$\sin\theta = \frac{aA + bB + cC}{\sqrt{A^2 + B^2 + C^2}\sqrt{a^2 + b^2 + c^2}}. \qquad (19)$$

EXERCISES

1. Show that the planes $2x - 3y + z + 1 = 0$, $5x + z - 1 = 0$, $4x + 9y - z - 5 = 0$ have a line in common, and find its direction cosines.

2. Write the equations of a line which passes through (5, 2, 6) and is parallel to the line $2x - 3z + y - 2 = 0$, $x + y + z + 1 = 0$.

3. Find the angle which the line $x + y + 2z = 0$, $2x - y + 2z - 1 = 0$ makes with the plane $3x + 6z - 5y + 1 = 0$.

4. Find the equation of the plane through the point $(2, -2, 0)$ and perpendicular to the line $x + 2y - 3z = 4$, $2x - 3y + 4z = 0$.

5. Find the equation of the plane determined by the parallel lines $\frac{x+1}{3} = \frac{y-2}{2} = \frac{z}{1}$, $\frac{x-3}{3} = \frac{y+4}{2} = \frac{z-1}{1}$.

6. For what value of k will the two lines $x + 2y - z + 3 = 0$, $3x - 2z + 1 = 0$; $2x - y + z - 2 = 0$, $x + y - z + k = 0$ intersect?

7. Find the equation of the plane through the points $(1, -1, 2)$ and (3, 0, 1), parallel to the line $x + y - z = 0$, $2x + y + z = 0$.

8. Show that the lines $\frac{x-2}{3} = \frac{y+1}{3} = \frac{z}{-2}$ and $\frac{x-3}{-1} = \frac{y+4}{3} = \frac{z+2}{2}$ intersect, and find the equation of the plane determined by them.

9. Find the equation of the plane through the point (a, b, c), parallel to each of the lines, $\frac{x-x_1}{l_1} = \frac{y-y_1}{m_1} = \frac{x-z_1}{n_1}$; $\frac{x-x_2}{l_2} = \frac{y-y_2}{m_2} = \frac{z-z_2}{n_2}$.

10. Find the equation of the plane through the origin and perpendicular to the line $3x - y + 4z + 5 = 0$, $x + y - z = 0$.

11. Find the value of k for which the lines $\frac{x-3}{2k} = \frac{y+1}{k+1} = \frac{z-3}{5}$; $\frac{x-1}{3} = \frac{y+5}{1} = \frac{z+2}{k-2}$ are perpendicular.

12. Find the values of k for which the planes $kx - 5y + (k+6)z + 3 = 0$ and $(k-1)x + ky + z = 0$ are perpendicular.

13. Find the equations of the line through the point (2, 3, 4) which meets the Y-axis at right angles.

22. Distance from a point to a line. Given the line

$$\frac{x-x_1}{\lambda} = \frac{y-y_1}{\mu} = \frac{z-z_1}{\nu}$$

and the point $P_2 \equiv (x_2, y_2, z_2)$ not lying on it. It is required to find the distance between the point and the line.

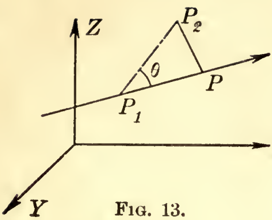

FIG. 13.

Let $P_1 \equiv (x_1, y_1, z_1)$ (Fig. 13) be any point on the line; let P be the foot of the perpendicular from P_2 on the line; θ the angle between the given line and the line P_1P_2; let d be the length of the segment P_1P_2. We have (Fig. 13)

$$P_2P^2 = P_1P_2^2 \sin^2\theta = d^2 - d^2\cos^2\theta.$$

The direction cosines of the line P_1P_2 are $\dfrac{x_2-x_1}{d}$, $\dfrac{y_2-y_1}{d}$, $\dfrac{z_2-z_1}{d}$, from which (Art. 5)

$$\cos\theta = \lambda\frac{x_2-x_1}{d} + \mu\frac{y_2-y_1}{d} + \nu\frac{z_2-z_1}{d}.$$

Hence,

$$PP_2^2 = d^2 - d^2\cos^2\theta = (x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2 - (\lambda(x_2-x_1) + \mu(y_2-y_1) + \nu(z_2-z_1))^2. \quad (20)$$

23. Distance between two non-intersecting lines. Given the two lines

$$\frac{x-x_1}{\lambda_1} = \frac{y-y_1}{\mu_1} = \frac{z-z_1}{\nu_1} \text{ and } \frac{x-x_2}{\lambda_2} = \frac{y-y_2}{\mu_2} = \frac{z-z_2}{\nu_2}$$

which do not intersect. It is required to find the shortest distance between them. Let λ, μ, ν be the direction cosines of the line on which the distance is measured. Since this line is perpendicular to each of the given lines, we have, by Art. 5, Equations (4) and (5),

$$\frac{\lambda}{\mu_1\nu_2 - \nu_1\mu_2} = \frac{\mu}{\nu_1\lambda_2 - \nu_2\lambda_1} = \frac{\nu}{\lambda_1\mu_2 - \mu_1\lambda_2} = \frac{\pm 1}{\sin\theta},$$

where θ is the angle between the given lines.

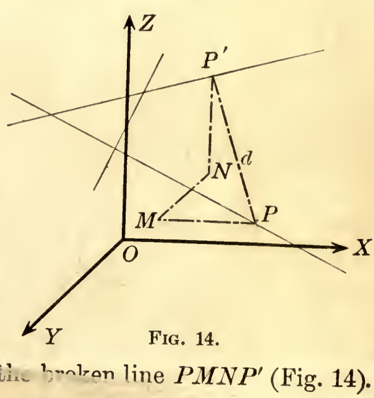

FIG. 14.

The length d of the required perpendicular is equal to the projection on the common perpendicular of the segment PP', and is equal to the projection of the broken line $PMNP'$ (Fig. 14).

$$d = \lambda(x_1 - x_2) + \mu(y_1 - y_2) + \nu(z_1 - z_2)$$

or

$$d = \pm \begin{vmatrix} x_1 - x_2 & \lambda_1 & \lambda_2 \\ y_1 - y_2 & \mu_1 & \mu_2 \\ z_1 - z_2 & \nu_1 & \nu_2 \end{vmatrix} \frac{1}{\sin\theta}. \tag{21}$$

EXERCISES

1. Find the distance from the origin to the line $\frac{x-1}{2} = \frac{y-3}{4} = \frac{z-2}{1}$.

2. Find the distance from (1, 1, 1) to $x + y + z = 0$, $3x - 2y + 4z = 0$.

3. Find the perpendicular distance from the point $(-2, 1, 3)$ to the line $x + 2y - z + 3 = 0$, $3x - y + 2z + 1 = 0$.

4. What are the direction cosines of the line through the origin and the point of intersection of the lines $x + 2y - z + 3 = 0$, $3x - y + 2z + 1 = 0$; $2x - 2y + 3z - 2 = 0$, $x - y - z + 3 = 0$.

5. Determine the distance of the point (1, 1, 1) to the line $x = 0$, $y = 0$ and the direction cosines of the line on which it is measured.

6. Find the distance between the lines $\frac{x}{2} = \frac{y+2}{-2} = \frac{z-1}{1}$ and $\frac{x-1}{4} = \frac{y-3}{2} = \frac{z+1}{-1}$.

7. Find the equations of the line along which the distance in Ex. 6 is measured.

8. Find the distance between the lines $2x + y - z = 0$, $x - y + 2z = 3$ and $x + 2y - 3z = 4$, $2x - 3y + 4z = 5$.

9. Express the condition that the lines $\frac{x - x_1}{l_1} = \frac{y - y_1}{m_1} = \frac{z - z_1}{n_1}$, $\frac{x - x_2}{l_2} = \frac{y - y_2}{m_2} = \frac{z - z_2}{n_2}$ intersect.

24. System of planes through a line. If

$$L_1 \equiv A_1x + B_1y + C_1z + D_1 = 0,$$
$$L_2 \equiv A_2x + B_2y + C_2x + D_2 = 0$$

are the equations of two intersecting planes, the equation $k_1L_1 + k_2L_2 = 0$ is, for all real values of k_1 and k_2, the equation of a plane passing through the line $L_1 = 0$, $L_2 = 0$. For, $k_1L_1 + k_2L_2 = 0$ is always of the first degree with real coefficients, and is therefore the equation of a plane (Art. 10); this plane passes through the line $L_1 = 0$, $L_2 = 0$, since the coördinates of every point on the line satisfy $L_1 = 0$ and $L_2 = 0$ and consequently satisfy the equation

$k_1L_1 + k_2L_2 = 0$. Conversely, the equation of any plane passing through the line can be expressed in the form $k_1L_1 + k_2L_2 = 0$, since k_1 and k_2 can be so chosen that the plane $k_1L_1 + k_2L_2 = 0$ will contain any point in space. Since any plane through the given line is determined by the line and a point not lying on it, the theorem follows.

To find the equations of the plane determined by the line $L_1 = 0$, $L_2 = 0$, and a point P_1 not lying on it, let the coördinates of P_1 be (x_1, y_1, z_1). If P_1 lies in the plane $k_1L_1 + k_2L_2 = 0$, its coördinates must satisfy the equation of the plane; thus

$$k_1(A_1x_1 + B_1y_1 + C_1z_1 + D_1) + k_2(A_2x_1 + B_2y_1 + C_2z_1 + D_2) = 0.$$

On eliminating k_1 and k_2 between this equation and $k_1L_1 + k_2L_2 = 0$, we obtain

$$0 = (A_2x_1 + B_2y_1 + C_2z_1 + D_2)(A_1x + B_1y + C_1z + D_1) \\ - (A_1x_1 + B_1y_1 + C_1z_1 + D_1)(A_2x + B_2y + C_2z + D_2),$$

as the equation of the plane determined by the line $L_1 = 0$, $L_2 = 0$, and the point P_1.

It will be convenient to write the above equation in the abbreviated form

$$L_2(x_1)L_1(x) - L_1(x_1)L_2(x) = 0.$$

The totality of planes passing through a line is called a **pencil** of planes. The number k_2/k_1 which determines a plane of the pencil is called the parameter of the pencil.

If, in the equation

$$k_1L_1 + k_2L_2 = 0,$$

k_1 and k_2 are given such values that the coefficient of x is equal to zero, the corresponding plane is perpendicular to the plane $x = 0$. Since this plane contains the line, it intersects the plane $x = 0$ in the orthogonal projection of the line $L_1 = 0$, $L_2 = 0$. Similarly, if k_1 and k_2 are given such values that the coefficient of y is equal to zero, the corresponding plane is perpendicular to the plane $y = 0$ and will cut the plane $y = 0$ in the projection of $L_1 = 0$, $L_2 = 0$ on that plane; if the coefficient of z is made to vanish, the plane will contain the projection of the given line upon the plane $z = 0$. The three planes of the system $k_1L_1 + k_2L_2 = 0$ obtained in this way are called the three **projecting planes** of the line $L_1 = 0$, $L_2 = 0$ on the coördinate planes.

Since two distinct planes passing through a line are sufficient to determine the line, two projecting planes of a line may always be employed to define the line. If the line is not parallel to the plane $z = 0$, its projecting planes on $x = 0$ and $y = 0$ are distinct and the equations of the line may be reduced to the form (Fig. 15)

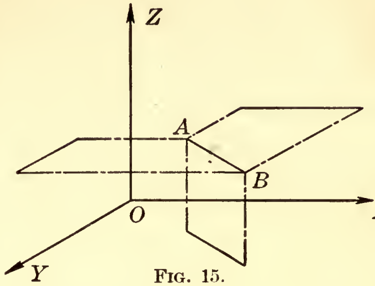

FIG. 15.

$$x = mz + a, \quad y = nz + b. \tag{22}$$

If the line is parallel to $z = 0$, the value of k for which the coefficient of x is made to vanish will also reduce the coefficient of y to zero, so that the projecting planes on $x = 0$ and on $y = 0$ coincide. This projecting plane $z = c$ and the projecting plane on $z = 0$ may now be chosen to define the line. If the line is not parallel to the X-axis, the equations of the line may be reduced to (Fig. 16)

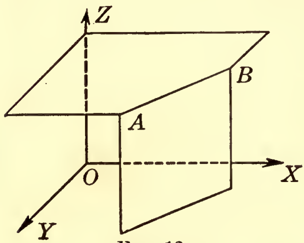

FIG. 16.

$$x = py + c, \quad z = c. \tag{23}$$

Finally, if the line is parallel to the X-axis, its equations may be reduced to (Fig. 17)

$$y = b, \quad z = c. \tag{24}$$

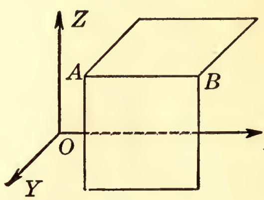

FIG. 17.

If the planes $L_1 = 0$, $L_2 = 0$ are parallel but distinct, so that

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} \neq \frac{D_1}{D_2},$$

then every equation of the form $k_1L_1 + k_2L_2 = 0$, except when $\frac{k_1}{k_2} = \frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2}$, defines a plane parallel to the given ones. Conversely, the equation of any plane parallel to the given ones can be written in the form $k_1L_1 + k_2L_2 = 0$ by so choosing $k_1 : k_2$

that the plane will pass through a given point. In this case the system of planes $k_1L_1 + k_2L_2 = 0$ is called a **pencil of parallel planes.**

Two equations,

$$L_1 \equiv A_1x + B_1y + C_1z + D_1 = 0,$$
$$L_2 \equiv A_2x + B_2y + C_2z + D^2 = 0,$$

will represent the same plane when, and only when, the coefficients A_1, B_1, C_1, D_1 are respectively proportional to A_2, B_2, C_2, D_2; thus, when

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2}.$$

These conditions may be expressed by saying that every determinant of order two formed by any square array in the system

$$\begin{Vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \end{Vmatrix}$$

shall vanish.

In this case multipliers k_1, k_2 can be found such that the equation $k_1L_1 + k_2L_2 = 0$ is identically satisfied.

Conversely, if multipliers k_1, k_2 can be found such that the preceding identity is satisfied, then the equations $L_1 = 0$, $L_2 = 0$ define the same plane.

EXERCISES

1. Write the equation of a plane through the line $7x + 2y - z - 3 = 0$, $3x - 3y + 2z - 5 = 0$ perpendicular to the plane $2x + y - 2z = 0$.

2. What is the equation of the plane determined by the line $2x - 3y - z + 2 = 0$, $x - y + 4z = 3$ and the point $(3, 2, -2)$?

3. Determine the equation of the plane passing through the line $x + 2z = 4$, $y - z = 8$ and parallel to the line $\frac{x-3}{1} = \frac{y+4}{1} = \frac{z-7}{2}$.

4. Does the plane $x + 2y - z + 3 = 0$ have more than one point in common with the line $3x - y + 2z + 1 = 0$, $2x - 3y + 3z - 2 = 0$?

5. Determine the equations of the line through (1, 2, 3) intersecting the two lines $x + 2y - 3z = 0$, $y - 4z = 4$ and $2x - y + 3z = 3$, $3x + y + 2z + 1 = 0$.

25. Application in descriptive geometry. A line may be represented by the three orthogonal projections of a segment of the line, each drawn to scale. Consider the XZ-plane (elevation, or vertical plane) as the plane of the paper, and the XY-plane as turned about the X-axis until it coincides with the XZ-plane. The projections in the XY-plane are thus drawn to scale on the same paper as projections on the XZ-plane, but points are distinguished by different symbols, as P', P_1. The XY-plane is called the plan or horizontal plane. Finally, let the YZ-plane be turned about the Z-axis until it coincides with the XZ-plane, and let figures in the new position be drawn to scale. This is called the end or profile plane. Thus, in the figure (Fig. 18), a segment PQ, wherein $P = (7, 4, 8)$, $Q = (13, 9, 12)$, may be indicated by the three segments $P'Q'$, P_1Q_1, P_pQ_p.

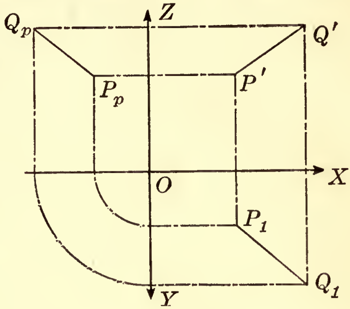

FIG. 18.

EXAMPLE. Find the equations of the projecting planes of the line

$$2x + 3y - 4z = 5, \quad x - 4y + 5z = 6.$$

Here, $\quad L_1 = 2x + 3y - 4z - 5, \quad L_2 = x - 4y + 5z - 6,$

$$k_1L_1 + k_2L_2 = (2k_1 + k_2)x + (3k_1 - 4k_2)y + (-4k_1 + 5k_2)z + (-5k_1 - 6k_2) = 0.$$

If $k_2 = -2k_1$, the coefficient of x disappears; thus the equation of the plane projecting the given line on the plane $x = 0$ is

$$11y - 14z + 7 = 0.$$

If $\dfrac{k_2}{k_1} = \dfrac{3}{4}$, the coefficient of y vanishes; the projecting plane on $y = 0$ is fonnd to be $11x - z = 38$.

Finally, if $\dfrac{k_2}{k_1} = \dfrac{4}{5}$, the projecting plane on $z = 0$ is found. Its equation is $14x - y = 49$.

EXERCISES

Find the equations of the projecting planes of each of the following lines:

1. $x + 2y - 3z = 4, \qquad 2x - 3y + 4z = 5.$

2. $2x + y + z = 0, \qquad x - y + 2z = 3.$

3. $x + y + z = 4, \qquad x - y + 3z = 4.$

4. $A_1x + B_1y + C_1z + D_1 = 0, \quad A_2x + B_2y + C_2z + D_2 = 0.$

26. Bundles of planes. The plane $L_3 = A_3x + B_3y + C_3z + D_3 = 0$ will belong to the pencil determined by the planes $L_1 = 0$, $L_2 = 0$, assumed distinct, when three numbers k_1, k_2, k_3, not all zero, can be found such that the equation $k_1L_1 + k_2L_2 + k_3L_3 = 0$ is identically satisfied for all values of x, y, z. This condition requires that the four equations $k_1A_1 + k_2A_2 + k_3A_3 = 0$, $k_1B_1 + k_2B_2 + k_3B_3 = 0$, $k_1C_1 + k_2C_2 + k_3C_3 = 0$, $k_1D_1 + k_2D_2 + k_3D_3 = 0$ are satisfied by three numbers k_1, k_2, k_3, not all zero; hence, that the four equations

$$|A_1B_2C_3| = 0, \qquad |B_1C_2D_3| = 0, \qquad |C_1D_2A_3| = 0, \qquad |D_1A_2B_3| = 0$$

are all satisfied, wherein we have written for brevity,

$$|A_1B_2C_3| \equiv \begin{vmatrix} A_1 & B_1 & C_1 \\ A_2 & B_2 & C_2 \\ A_3 & B_3 & C_3 \end{vmatrix}, \text{ etc.}$$

These simultaneous conditions may be expressed by saying that every determinant of order three formed by the elements contained in any square array in the system

$$\begin{Vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \end{Vmatrix}$$

shall vanish.

Conversely, if these conditions are satisfied, then three constants k_1, k_2, k_3 can be found such that the equation $k_1L_1 + k_2L_2 + k_3L_3 = 0$ is identically satisfied, and the three planes $L_1 = 0$, $L_2 = 0$, $L_3 = 0$ belong to the same pencil.

Let
$$\begin{aligned} L_1 &\equiv A_1x + B_1y + C_1z + D_1 = 0, \\ L_2 &\equiv A_2x + B_2y + C_2z + D_2 = 0, \\ L_3 &\equiv A_3x + B_3y + C_3z + D_3 = 0 \end{aligned}$$

be the equations of three planes not belonging to a pencil. If we solve these three equations for (x, y, z), we find for the coördinates of the point of intersection of the three planes, in case $|A_1B_2C_3| \neq 0$,

$$x = -\frac{|D_1B_2C_3|}{|A_1B_2C_3|}, \qquad y = -\frac{|A_1D_2C_3|}{|A_1B_2C_3|}, \qquad z = -\frac{|A_1B_2D_3|}{|A_1B_2C_3|}. \quad (25)$$

If $|A_1B_2C_3| = 0$, but not all the determinants in the numerators of (25) are zero, no set of values of x, y, z will satisfy all three equations. In this case, the line of intersection of any two of the planes is parallel to the third. For, if $L_1 = 0$ and $L_2 = 0$ intersect, the direction cosines of their line of intersection are proportional (Art. 18) to

$$B_1C_2 - B_2C_1, \qquad C_1A_2 - C_2A_1, \qquad A_1B_2 - A_2B_1.$$

The condition that this line is parallel to the plane $L_3 = 0$ is (Art. 21)

$$A_3(B_1C_2 - B_2C_1) + B_3(C_1A_2 - C_2A_1) + C_3(A_1B_2 - A_2B_1) = 0,$$

which is exactly the condition $|A_1B_2C_3| = 0$. The proof for the other lines and planes is found in the same way.

If at least one of the determinants $|A_1B_2C_3|$, $|D_1B_2C_3|$, $|A_1D_2C_3|$, and $|A_1B_2D_3|$ is not zero, the system of planes

$$k_1L_1 + k_2L_2 + k_3L_3 = 0$$

is called a **bundle**. If $|ABC| \neq 0$, all the planes of the bundle pass through the point (25), since the coördinates of this point satisfy the equation of every plane of the bundle. Conversely, the equation of every plane passing through the point (25) can be expressed in this form. This point is called the **vertex** of the bundle. If $|ABC| = 0$, all the planes of the bundle are parallel to a fixed line (such as $L_1 = 0$, $L_2 = 0$). In this case, the bundle is called a **parallel bundle**.

27. Plane coördinates. The equation of any plane not passing through the origin may be reduced to the form

$$ux + vy + wz + 1 = 0. \tag{26}$$

When the equation is in this form, the position of the plane is fixed when the values of the coefficients u, v, w (not all zero) are known; and conversely, if the position of the plane (not passing through the origin) is known, the values of the coefficients are fixed. Since the numbers (u, v, w) determine a plane definitely, just as (x, y, z) determine a point, we shall call the set of numbers (u, v, w) the **coördinates of the plane** represented by equation (26). Thus, the plane (3, 5, 2) will be understood to mean the plane whose equation is $3x + 5y + 2z + 1 = 0$. Similarly, the equation of the plane $(2, 0, -1)$ is $2x - z + 1 = 0$.

If u, v, w are different from zero, they are the negative reciprocals of the intercepts of the plane (u, v, w) on the axes (Art. 12).

If $u = 0$, the plane is parallel to the X-axis; if $u = 0$, $v = 0$, the plane is parallel to the XY-plane. The vanishing of the other coefficients may be interpreted in a similar way.

28. Equation of a point. If the point (x_1, y_1, z_1) lies in the plane (26), the equation

$$ux_1 + vy_1 + wz_1 + 1 = 0 \tag{27}$$

must be satisfied. If x_1, y_1, z_1 are considered fixed and u, v, w variable, (27) is the condition that the plane (u, v, w) passes through the point (x_1, y_1, z_1). For this reason, equation (27) is called the **equation of the point** (x_1, y_1, z_1) in plane coördinates.

Thus,
$$u - 5v + 2w + 1 = 0$$
is the equation of the point $(1, -5, 2)$; similarly,
$$3u + w + 1 = 0$$
is the equation of the point $(3, 0, 1)$.

If equation (27) is multiplied by any constant different from zero, the locus of the equation is unchanged. Hence, we have the following theorem:

THEOREM. *The linear equation*

$$Au + Bv + Cw + D = 0 \quad (D \neq 0)$$

is the equation of the point $\left(\frac{A}{D}, \frac{B}{D}, \frac{C}{D}\right)$ *in plane coördinates.*

Thus, $u - 5v - 3w - 2 = 0$ is the equation of the point $\left(\frac{-1}{2}, \frac{5}{2}, \frac{3}{2}\right)$.

The condition that the coördinates (u, v, w) (v, u, w) of a plane satisfy two linear equations

$$ux_1 + vy_1 + wz_1 + 1 = 0, \qquad ux_2 + vy_2 + wz_2 + 1 = 0$$

is that the plane passes through the two points (x_1, y_1, z_1) and (x_2, y_2, z_2) and therefore through the line joining the two points. The two equations are called the **equations of the line** in plane coördinates.

EXERCISES

1. Plot the following planes and write their equations: $(1, 2, \frac{1}{3})$, $(3, -\frac{1}{2}, -\frac{1}{4})$, $(-1, \frac{1}{4}, -\frac{1}{3})$.

2. Find the volume of the tetrahedron bounded by the coördinate planes and the plane $(-\frac{1}{2}, -\frac{1}{3}, -\frac{1}{5})$.

3. What are the coördinates of the planes whose equations are

$$7x + 5y - z + 1 = 0, \qquad x - 6y + 11z + 5 = 0, \qquad 9x - 4 = 0?$$

4. Find the angle which the plane $(2, 6, 5)$ makes with the plane $(-1, \frac{1}{3}, 2)$.

5. Write the equations of the points $(1, 1, 1)$, $(2, -1, \frac{1}{3})$, $(6, -2, 1)$.

6. What are the coördinates of the points whose equations are

$$2u - v - 3w + 1 = 0, \qquad u + 2w - 3 = 0, \qquad w - 2 = 0?$$

7. Find the direction cosines of the line

$$3u - v + 2w + 1 = 0, \qquad u + 5v + 2w - 1 = 0.$$

8. What locus is determined by three simultaneous linear equations in (u, v, w)?

9. Write the equation satisfied by the coördinates of the planes whose distance from the origin is 2. What is the locus of a plane which satisfies this condition?

29. Homogeneous coördinates of the point and of the plane. It is sometimes convenient to express the coördinates x, y, z of a point in terms of four numbers x', y', z', t' by means of the equations

$$\frac{x'}{t'} = x, \quad \frac{y'}{t'} = y, \quad \frac{z'}{t'} = z.$$

A set of four numbers (x', y', z', t'), not all of which are zero, that satisfy these equations are said to be the **homogeneous coördinates** of a point. If the coördinates (x', y', z', t') are given, the point is uniquely determined (for the case $t' = 0$, compare Art. 32), but if (x, y, z) are given, only the ratios of the homogeneous coördinates are determined, since (x', y', z', t') and (kx', ky', kz', kt') define the same point, k being an arbitrary constant, different from zero.

Similarly, if the coördinates of a plane are (u, v, w), four numbers (u', v', w', s'), not all of which are zero, may be found such that

$$\frac{u'}{s'} = u, \quad \frac{v'}{s'} = v, \quad \frac{w'}{s'} = w.$$

The set of numbers (u', v', w', s') are called the **homogeneous coördinates** of the plane.

Where no ambiguity arises, the accents will be omitted from the homogeneous coördinates.

30. Equation of a plane and of a point in homogeneous coördinates.

If, in the equation

$$Ax + By + Cz + D = 0$$

($D \neq 0$, and A, B, C are not all zero) the homogeneous coördinates of a point are substituted, we obtain, after multiplying by t, the equation of the plane in homogeneous coördinates

$$Ax + By + Cz + Dt = 0.$$

The homogeneous coördinates of this plane are (A, B, C, D).

If, in the equation

$$Au + Bv + Cw + D = 0$$

($D \neq 0$, and A, B, C are not all zero) the homogeneous coördinates of a plane are substituted, we obtain, after multiplying by s, the equation of the point in homogeneous coördinates

$$Au + Bv + Cw + Ds = 0.$$

The homogeneous coördinates of this point are (A, B, C, D).

31. Equation of the origin. Coördinates of planes through the origin. The necessary and sufficient condition that the plane whose equation is $ux + vy + wz + st = 0$ shall pass through the origin is $s = 0$. We see then that $s = 0$ is the equation of the origin, and that $(u, v, w, 0)$ are the homogeneous coördinates of a plane through the origin. Since $s = 0$, it follows from Art. 29 that the non-homogeneous coördinates of such a plane cease to exist.

32. The plane at infinity. Let (x, y, z, t) be the homogeneous coördinates of a point. If we assign fixed values (not all zero) to x, y, z and allow t to vary, the corresponding point will vary in such a way that, as $t \doteq 0$, one or more of the non-homogeneous coördinates of the point increases without limit. If $t = 0$, the non-homogeneous coördinates cease to exist, but it is assumed that there still exists a corresponding point which is said to be at infinity. It is also assumed that two points at infinity coincide if, and only if, their homogeneous coördinates are proportional.

The equation of the locus of the points at infinity is $t = 0$. Since this equation is homogeneous of the first degree in x, y, z, t, it will be said that $t = 0$ is the equation of a plane. This plane is called the **plane at infinity**.

33. Lines at infinity. Any finite plane is said to intersect the plane at infinity in a line. This line is called the infinitely distant line in the plane. The equations of the infinitely distant line in the plane $Ax + By + Cz + Dt = 0$ are $Ax + By + Cz = 0$, $t = 0$.

THEOREM. *The condition that two finite planes are parallel is that they intersect the plane at infinity in the same line.*

If the planes are parallel, their equations may be written in the form (Art. 15)

$$Ax + By + Cz + Dt = 0, \quad Ax + By + Cz + D't = 0. \tag{28}$$

It follows that they both pass through the line

$$Ax + By + Cz = 0, \quad t = 0. \tag{29}$$

Conversely, the equations of any two finite planes through the line (29) may be written in the form (28). The planes are therefore parallel.

34. Coördinate tetrahedron. The four planes whose equations in point coördinates are

$$x = 0, \quad y = 0, \quad z = 0, \quad t = 0$$

will be called the four coördinate planes in homogeneous coördinates. Since the planes do not all pass through a common point, they will be regarded as forming a tetrahedron, called the coördinate tetrahedron. The coördinates of the vertices of this tetrahedron are

$$(0, 0, 0, 1), \quad (0, 0, 1, 0), \quad (0, 1, 0, 0), \quad (1, 0, 0, 0).$$

The coördinates of the four faces in plane coördinates are

$$(0, 0, 0, 1), \quad (0, 0, 1, 0), \quad (0, 1, 0, 0), \quad (1, 0, 0, 0).$$

The equations of the vertices are $u = 0$, $v = 0$, $w = 0$, $s = 0$.

EXERCISES

1. Find the non-homogeneous coördinates of the following points and planes:

(*a*) $7x + 3y + 3z - 4 = 0$,
(*b*) $10x - 3y + 15 = 0$,
(*c*) $x - 2 = 0$,
(*d*) $9u - v - 3w + 2 = 0$,
(*e*) $u + v - w - 7 = 0$,
(*f*) $2w + 11 = 0$.

2. Determine the coördinates of the infinitely distant point on the line

$$3x + 2y + 5t = 0, \quad 2x - 10z + 3t = 0.$$

3. Show that if $L_1(u) \equiv A_1u + B_1v + C_1w + D_1s = 0$, and $L_2(u) \equiv A_2u + B_2v + C_2w + D_2s = 0$ are the equations of two points, the equation of any point on the joining line may be written in the form $k_1L_1 + k_2L_2 = 0$.

4. Show that the planes $x + 2y + 7z - 3t = 0$, $x + 3y + 6z = 0$, $x + 4y + 5z - 2t = 0$ determine a parallel bundle. Find the equation of the plane of the bundle through the points $(2, -1, 1, 1)$, $(2, 5, 0, 1)$.

35. System of four planes. The condition that four given planes

$$\begin{aligned} L_1 &\equiv A_1x + B_1y + C_1z + D_1t = 0, \\ L_2 &\equiv A_2x + B_2y + C_2z + D_2t = 0, \\ L_3 &\equiv A_3x + B_3y + C_3z + D_3t = 0, \\ L_4 &\equiv A_4x + B_4y + C_4z + D_4t = 0 \end{aligned}$$

all pass through a point is that four numbers (x, y, z, t), not all zero, exist which satisfy the four simultaneous equations. The condition is, consequently, that the determinant

$$\begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix}$$

is equal to zero. If this condition is not satisfied, the four planes are said to be **independent**. When the given planes are independent, four numbers k_1, k_2, k_3, k_4 can always be found such that the equation

$$k_1L_1 + k_2L_2 + k_3L_3 + k_4L_4 = 0$$

shall represent any given plane. For, let $ax + by + cz + d = 0$ be the equation of the given plane. The two equations will represent the same plane if their coefficients are proportional, that is, if numbers k_1, k_2, k_3, k_4, not all zero, can be found such that

$$\begin{aligned} a &= k_1A_1 + k_2A_2 + k_3A_3 + k_4A_4, \\ b &= k_1B_1 + k_2B_2 + k_3B_3 + k_4B_4, \\ c &= k_1C_1 + k_2C_2 + k_3C_3 + k_4C_4, \\ d &= k_1D_1 + k_2D_2 + k_3D_3 + k_4D_4. \end{aligned}$$

Since the planes are independent, the determinant of the coefficients in the second members of these equations is not zero, and the numbers k_1, k_2, k_3, k_4 can always be determined so as to satisfy these equations.

These results, together with those of Arts. 24, 26, may be expressed as follows: The necessary and sufficient condition that a system of planes have no point in common is that the matrix* formed by their coefficients is of rank four; the planes belong to a bundle when the matrix is of rank three; the planes belong to a pencil when the matrix is of rank two; finally, the planes all coincide when the matrix is of rank one. We shall use the expression "rank of the system of planes" to mean the rank of the matrix of coefficients in the equations of the planes.

EXERCISES

1. Determine the nature of the following systems of planes:

(*a*) $2x - 5y + z - 3t = 0,\ x + y + 4z - 5t = 0,\ x + 3y + 6z - t = 0.$

(*b*) $3x + 4y + 5z - 5t = 0,\ 6x + 5y + 9z - 10t = 0,\ 3x + 3y + 5z - 5t = 0,\ x - y + 2z = 0.$

(*c*) $2x + 4y = 0,\ 5x + 7y + 2z = 0,\ 3x + 4y - 2z + 3t = 0,\ x = 0.$

(*d*) $2x + 5y + 3z = 0,\ 7y - 5z + 4t = 0,\ x - y + 4z = 8t.$

2. Show that the line $x + 3y - z + t = 0,\ 2x - y + 2z - 3t = 0$ lies in the plane $7x + 7y + z - 3t = 0$.

3. Determine the conditions that the planes

$$x = cy + bz,\quad y = ax + cz,\quad z = bx + ay$$

shall have just one common point; a common line; are identical.

4. Prove that the planes $2x - 3y - 7z = 0,\ 3x - 14y - 13z = 0,\ 8x - 31y - 33z = 0$ have a line in common, and find its direction cosines.

5. Show that the planes $3x - 2y - t = 0,\ 4x - 2z - 2t = 0,\ 4x + 4y - 5z = 0$ belong to a parallel bundle.

* Any rectangular array of numbers

$$\begin{Vmatrix} A_1 & B_1 & C_1 & D_1 & \cdots & M_1 \\ A_2 & B_2 & C_2 & D_2 & \cdots & M_2 \\ A_3 & B_3 & C_3 & D_3 & \cdots & M_3 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_n & B_n & C_n & D_n & \cdots & M_n \end{Vmatrix}$$

is called a matrix. Associated with every matrix are other matrices obtained by suppressing one or more of the rows or one or more of the columns of the given matrix, or both; in particular, associated with every square matrix, that is, one in which the number of rows is equal to the number of columns, is a determinant whose elements are the elements of the matrix. Conversely, associated with every determinant is a square matrix, formed by its elements. We shall use the word rank to define the order of the non-vanishing determinant of highest order contained in any given matrix. The rank of the determinant is defined as the rank of the matrix formed by the elements of the determinant.

CHAPTER III

TRANSFORMATION OF COÖRDINATES

The coördinates of a point, referred to two different systems of axes, are connected by certain relations which will now be determined. The process of changing from one system of axes to another is called a **transformation of coördinates.**

36. Translation. Let the coördinates of a point P with respect to a set of rectangular axes OX, OY, OZ be (x, y, z) and with respect to a set of axes $O'X'$, $O'Y'$, $O'Z'$, parallel respectively to the first set, be (x', y', z'). If the coördinates of O', referred to the axes OX, OY, OZ are (h, k, l) we have (Fig. 19)

$$x = x' + h, \quad y = y' + k, \quad z = z' + l. \tag{1}$$

For, the projection on OX of OP is equal to the sum of the projections of OO' and $O'P$ (Art. 2), but the projection of OP is x, of OO' is h, and of $O'P$ is x'; hence $x = x' + h$. The other formulas are derived in a similar way. Since the new axes can be obtained from the old ones by moving the three coördinate planes parallel to the X-axis a distance h, then parallel to the Y-axis a distance k, and parallel to the Z-axis a distance l, without changing their directions, the transformation (1) is called a **translation of axes.**

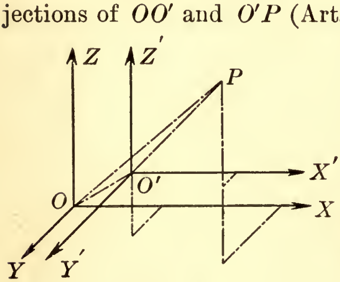

Fig. 19.

37. Rotation. Let the coördinates of a point P, referred to a set of rectangular axes OX, OY, OZ, be x, y, z, and referred to another rectangular system OX', OY', OZ' having the same origin, be x', y', z'. Let $x' = OL'$, $y' = L'M'$, $z' = M'P$ (Fig. 20); and let the direction cosines of OX', referred to OX, OY, OZ, be λ_1, μ_1, ν_1; those of OY' be λ_2, μ_2, ν_2, and of OZ' be λ_3, μ_3, ν_3.

We shall show that

$$\begin{aligned} x &= \lambda_1 x' + \lambda_2 y' + \lambda_3 z', \\ y &= \mu_1 x' + \mu_2 y' + \mu_3 z', \\ z &= \nu_1 x' + \nu_2 y' + \nu_3 z'. \end{aligned} \tag{2}$$

For, the projection of OP (Fig. 20) on the axis OX is x. The sum of the projections of OL', $L'M'$, and $M'P$ is $\lambda_1 x' + \lambda_2 y' + \lambda_3 z'$.

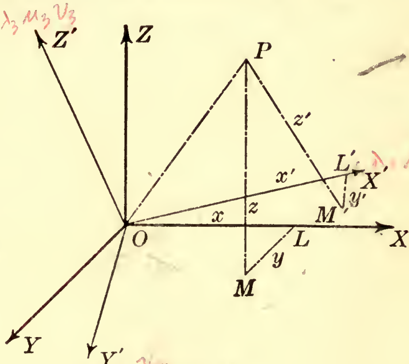

FIG. 20.

That these two expressions are equal follows from Art. 2. The second and third equations are obtained in a similar way.

The direction cosines of OX, OY, and OZ, with respect to the axes OX', OY', OZ' are $\lambda_1, \lambda_2, \lambda_3$; μ_1, μ_2, μ_3; ν_1, ν_2, ν_3, respectively. If we project OP and $OL = x$, $LM = y$, and $MP = z$ on OX', OY', and OZ', we obtain

$$\begin{aligned} x' &= \lambda_1 x + \mu_1 y + \nu_1 z, \\ y' &= \lambda_2 x + \mu_2 y + \nu_2 z, \\ z' &= \lambda_3 x + \mu_3 y + \nu_3 z. \end{aligned} \tag{2'}$$

The systems of equations (2) and (2′) are expressed in convenient form by means of the accompanying diagram.

	x'	y'	z'
x	λ_1	λ_2	λ_3
y	μ_1	μ_2	μ_3
z	ν_1	ν_2	ν_3

Since λ_1, μ_1, ν_1; λ_2, μ_2, ν_2; λ_3, μ_3, ν_3 are the direction cosines of three mutually perpendicular lines, we have the six relations

$$\begin{aligned} \lambda_1^2 + \mu_1^2 + \nu_1^2 &= 1, & \lambda_1\lambda_2 + \mu_1\mu_2 + \nu_1\nu_2 &= 0, \\ \lambda_2^2 + \mu_2^2 + \nu_2^2 &= 1, & \lambda_2\lambda_3 + \mu_2\mu_3 + \nu_2\nu_3 &= 0, \\ \lambda_3^2 + \mu_3^2 + \nu_3^2 &= 1, & \lambda_3\lambda_1 + \mu_3\mu_1 + \nu_3\nu_1 &= 0. \end{aligned} \tag{3}$$

We have seen that $\lambda_1, \lambda_2, \lambda_3$; μ_1, μ_2, μ_3; ν_1, ν_2, ν_3 are also the direction cosines of three mutually perpendicular lines. It follows that

$$\begin{array}{ll} \lambda_1^2+\lambda_2^2+\lambda_3^2=1, & \lambda_1\mu_1+\lambda_2\mu_2+\lambda_3\mu_3=0, \\ \mu_1^2+\mu_2^2+\mu_3^2=1, & \mu_1\nu_1+\mu_2\nu_2+\mu_3\nu_3=0, \\ \nu_1^2+\nu_2^2+\nu_3^2=1, & \nu_1\lambda_1+\nu_2\lambda_2+\nu_3\lambda_3=0. \end{array} \tag{4}$$

It will next be shown that

$$\begin{array}{lll} \lambda_1=\epsilon(\mu_2\nu_3-\nu_2\mu_3), & \lambda_2=\epsilon(\mu_3\nu_1-\nu_3\mu_1), & \lambda_3=\epsilon(\mu_1\nu_2-\nu_1\mu_2), \\ \mu_1=\epsilon(\nu_2\lambda_3-\lambda_2\nu_3), & \mu_2=\epsilon(\nu_3\lambda_1-\lambda_3\nu_1), & \mu_3=\epsilon(\nu_1\lambda_2-\lambda_1\nu_2), \\ \nu_1=\epsilon(\lambda_2\mu_3-\mu_2\lambda_3), & \nu_2=\epsilon(\lambda_3\mu_1-\mu_3\lambda_1), & \nu_3=\epsilon(\lambda_1\mu_2-\mu_1\lambda_2), \end{array} \tag{5}$$

where $\epsilon=\pm 1$. From the first and third equations of the last column of (4) we obtain

$$\frac{\lambda_1}{\mu_2\nu_3-\nu_2\mu_3}=\frac{\lambda_2}{\mu_3\nu_1-\nu_3\mu_1}=\frac{\lambda_3}{\mu_1\nu_2-\nu_1\mu_2}.$$

If we denote the value of these fractions by ϵ, solve for λ_1, λ_2, and λ_3 and substitute in the first of equations (4), we obtain

$$\epsilon^2[(\mu_2\nu_3-\nu_2\mu_3)^2+(\mu_3\nu_1-\nu_3\mu_1)^2+(\mu_1\nu_2-\nu_1\mu_2)^2]=1.$$

OY OZ

Since the lines OY' and OZ' are perpendicular, the coefficient of ϵ^2 is unity (Art. 5, Eq. (5)). It follows that $\epsilon^2=1$ or $\epsilon=\pm 1$. The first three of equations (5) are consequently true. The other equations may be verified in a similar way.

It can now be shown that

$$\begin{vmatrix} \lambda_1 & \lambda_2 & \lambda_3 \\ \mu_1 & \mu_2 & \mu_3 \\ \nu_1 & \nu_2 & \nu_3 \end{vmatrix}=\epsilon=\pm 1. \tag{6}$$

For, expand the determinant by minors of the elements of the first row, and substitute for the cofactors of $\lambda_1, \lambda_2, \lambda_3$ their values from (5). The value of the determinant reduces to

$$\frac{1}{\epsilon}(\lambda_1^2+\lambda_2^2+\lambda_3^2)=\frac{1}{\epsilon}=\epsilon.$$

It will be shown in the next Article that if $\epsilon=1$, the system of axes $O\text{–}X'Y'Z'$ can be obtained by rotation from $O\text{–}XYZ$. If $\epsilon=-1$, a rotation and reflection are necessary.

38. Rotation and reflection of axes. Having given three mutually perpendicular directed lines, forming the trihedral angle $O\text{–}XYZ$ (Fig. 21), and three other mutually perpendicular directed lines through O, forming the trihedral angle $O\text{–}X'Y'Z'$, we shall show that the trihedral angle $O\text{–}XYZ$ can be revolved in such a way that OX and OZ coincide in direction with OX' and OZ', respectively. OY will then coincide with OY' or will be directed oppositely to it.

Let NN' be the line of intersection of the planes XOY and $X'OY'$. Denote the angle ZOZ' by θ, the angle XON by ϕ, and the angle NOX' by ψ. Let the axes $O\text{–}XYZ$ be revolved as a rigid body about OZ through the angle ϕ, so that OX is revolved into the position ON. Denote the new position of OY by OY_1, so that the angle $YOY_1 = \phi$. The trihedral angle $O\text{–}XYZ$ is thus revolved into $O\text{–}NY_1Z$. Now let $O\text{–}NY_1Z$ be revolved about ON through an angle θ, so that OZ takes a position OZ', and OY_1, a position OY_2. Then the angle $ZOZ' = \text{angle } Y_1OY_2 = \theta$. The trihedral angle $O\text{–}NY_1Z$ is thus brought into the position $O\text{–}NY_2Z'$. Finally, let the trihedral angle in this last position be revolved about OZ' through an angle ψ, so that ON is revolved into OX'. By the same operation OY is revolved into a direction through O perpendicular to OX' and to OZ'. It either coincides with OY' or is oppositely directed. In the first case the trihedral $O\text{–}XYZ$ has been rotated into the trihedral $O\text{–}X'Y'Z'$. In the second case the rotation must be followed by changing the direction of the Y-axis. This latter operation is called **reflection on the plane $y' = 0$.** It cannot be accomplished by means of rotations.

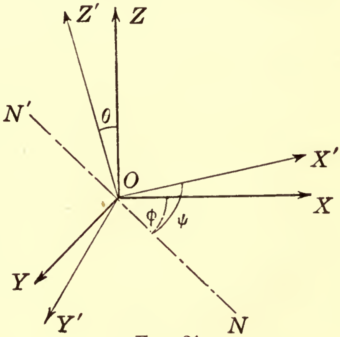

FIG. 21.

In case the trihedral $O\text{–}XYZ$ can be rotated into $O\text{–}X'Y'Z'$, the number ϵ (Art. 37) is positive; otherwise, it is negative. For, during a continuous rotation of the axes, the value of ϵ (Eq. (6)) cannot change discontinuously. If, after the rotation, the trihedrals coincide, we have, in that position, $\lambda_1 = \mu_2 = \nu_3 = 1$ and the other cosines are zero, so that (Eq. (6)) $\epsilon = 1$. If, however, at the end of the rotation, OY and OY' are oppositely directed, $\lambda_1 = \nu_3 = 1$, $\mu_2 = -1$, and $\epsilon = -1$.

39. Euler's formulas for rotation of axes. Let the coördinates of a point P referred to $O\text{–}XYZ$ be (x, y, z), referred to $O\text{–}NY_1Z$ be (x_1, y_1, z_1), referred to $O\text{–}NY_2Z$ be (x_2, y_2, z_2), and referred to $O\text{–}X'Y'Z'$ be (x', y', z'), (Fig. 21).

In the first rotation, through the angle ϕ, z remains fixed. Hence, from plane analytic geometry,

$$z = z_1, \quad x = x_1 \cos\phi - y_1 \sin\phi, \quad y = x_1 \sin\phi + y_1 \cos\phi.$$

In the rotation through the angle θ, x_1 remains fixed. Hence we have

$$x_1 = x_2, \quad y_1 = y_2 \cos\theta - z_2 \sin\theta, \quad z_1 = y_2 \sin\theta + z_2 \cos\theta.$$

Finally, if $O\text{–}X'Y'Z'$ can be obtained from $O\text{–}XYZ$ by rotation, z_2 remains fixed, and we have

$$z_2 = z', \quad x_2 = x' \cos\psi - y' \sin\psi, \quad y_2 = x' \sin\psi + y' \cos\psi.$$

On eliminating x_2, y_2, z_2; x_1, y_1, z_1, the final result is obtained, namely:

$$x = x'(\cos\phi\cos\psi - \sin\phi\sin\psi\cos\theta) - y'(\cos\phi\sin\psi + \sin\phi\cos\psi\cos\theta) + z' \sin\phi\sin\theta.$$

$$y = x'(\sin\phi\cos\psi + \cos\phi\sin\psi\cos\theta) - y'(\sin\phi\sin\psi - \cos\phi\cos\psi\cos\theta) - z' \cos\phi\sin\theta.$$

$$z = x' \sin\psi\sin\theta + y' \cos\psi\sin\theta + z' \cos\theta.$$

If $O\text{–}X'Y'Z'$ cannot be obtained from $O\text{–}XYZ$ by rotation, the sign of y' should be changed. These formulas are known as **Euler's formulas.**

40. Degree of an equation unchanged by transformation of coordinates. If in an equation $F(x, y, z) = 0$ the values of x, y, z are replaced by their values in any transformation of axes the degree of F cannot be made larger, since x, y, z are replaced by linear expressions in x', y', z'. But the degree of the equation cannot be made smaller, since by returning to the original axes and to the original equation, it would be made larger, which was just seen to be impossible.

EXERCISES

1. Transform the equation $x^2 - 3yz + y^2 - 6x + z = 0$ to parallel axes through the point $(1, -1, 2)$.

2. By means of equations (2) show that the expression $x^2 + y^2 + z^2$ is unchanged by rotation of the axes. Interpret geometrically.

3. Show that the lines $x = \frac{y}{4} = \frac{z}{2}$; $\frac{x}{2} = \frac{y}{-1} = z$; $\frac{x}{2} = y = \frac{z}{-3}$ are mutually perpendicular. Write the equations of a transformation of coördinates to these lines as axes.

4. Translate the axes in such a way as to remove the first degree terms from the equation $x^2 - 2y^2 + 6z^2 - 16x - 4y - 24z + 37 = 0$.

5. Show that the equation $ax + by + cz + s = 0$ may be reduced to $x = 0$ by a transformation of coördinates.

6. Find the equation of the locus $11x^2 + 10y^2 + 6z^2 - 8yz + 4zx - 12xy - 12 = 0$ when lines through the origin whose direction cosines are $\frac{1}{3}, \frac{2}{3}, \frac{2}{3}$; $\frac{2}{3}, \frac{1}{3}, -\frac{2}{3}$; $-\frac{2}{3}, \frac{2}{3}, -\frac{1}{3}$ are taken as new coördinate axes.

7. Show that if $O\text{-}X'Y'Z'$ can be obtained from $O\text{-}XYZ$ by rotation, and if OY can be made to coincide with OX by a revolution of 90 degrees, counterclockwise, as viewed from the positive end of the Z-axis, then OY' can be revolved into OX' by rotating counterclockwise through 90 degrees as viewed from the positive Z'-axis.

8. Derive from Ex. 7 a necessary and sufficient condition that $O\text{-}X'Y'Z'$ can be obtained from $O\text{-}XYZ$ by rotation.

CHAPTER IV

TYPES OF SURFACES

41. Imaginary points, lines, and planes. In solving problems that arise in analytic geometry, it frequently happens that the values of some of the quantities x, y, z which satisfy the given conditions are imaginary. Although we shall not be able to plot a point in the sense of Art. 1, when some or all of its coördinates are imaginary, it will nevertheless be convenient to refer to any triad of numbers x, y, z, real or imaginary, as the coördinates of a point. If all the coördinates are real, the point is real and is determined by its coördinates as in Art. 1; if some or all of the coördinates are imaginary or complex, the point will be said to be imaginary. Similarly, a set of plane coördinates u, v, w will define a real plane if all the coördinates are real; if some or all of the coördinates are imaginary, the plane will be said to be imaginary.

A linear equation in x, y, z, with coefficients real or imaginary, will be said to define a plane, and a linear equation in u, v, w, with coefficients real or imaginary, will be said to define a point.

The equations of any two distinct planes, considered as simultaneous, will be said to define a line. It follows that if (x_1, y_1, z_1) and (x_2, y_2, z_2) are any two points on the line, then the coördinates of any other point on the line can be written in the form $k_1x_1 + k_2x_2$, etc. The line is also determined by the equations of any two distinct points on it.

The line joining two imaginary points is real if it also contains two real points. If $P \equiv (a + ik, b + il, c + im)$ is an imaginary point, the point $P' \equiv (a - ik, b - il, c - im)$, whose coördinates are the respective conjugates of those of P, is called the point conjugate to P. The line joining any two conjugate points is real; thus the equations of the line PP' are $lx - ky + bk - al = 0$, $(bm - cl)x + (ck - am)y + (al - bk)z = 0$. The line of intersection of two imaginary planes is real if through it pass two distinct real planes. The line of intersection of two conjugate planes is real.

From the preceding it follows that no imaginary line can contain more than one real point, and through an imaginary line cannot pass more than one real plane. If a plane passes through an imaginary point and not through its conjugate, the plane is imaginary. If a point lies in an imaginary plane and not in its conjugate, the point is imaginary.

One advantage of using the form of statement suggested in this Article is that many theorems may be stated in more general form than would otherwise be possible. We may say, for example, that every line has two (distinct or coincident) points in common with any given sphere.

With these assumptions the preceding formulas will be applied to imaginary elements as well as to real ones. No attempt will be made to give to such formulas a geometric meaning when imaginary quantities are involved.

In the following chapters, in all discussions in which it is necessary to distinguish between real and imaginary quantities, it will be assumed, unless the contrary is stated, that given points, lines, and planes, and the coefficients in the equations of given surfaces, are real.

EXERCISES

1. Show that the point $(2+i,\ 1+3i,\ i)$ lies on the plane $x-2y+5z=0$.

2. Find the coördinates of the points of intersection of the line whose parametric equations are (Art. 20) $x = 1 + \frac{3}{13}d$, $y = -2 + \frac{4}{13}d$, $z = 5 - \frac{12}{13}d$, with the sphere $x^2 + y^2 + z^2 = 1$.

3. Show that the line of intersection of the planes $x + iy = 0$, $(1+i)x + (3-2i)y = 0$ is real.

4. Find the coördinates of the point of intersection of the line through $(3, 2, -2)$ and $(4, 0, 3)$ with the plane $x + 3y + (1-2i)z + 1 = 0$.

5. Find the equation of the plane determined by the points $(5+i,\ 2,\ -2-i)$, $(4+2i,\ -1+2i,\ 0)$, $(i,\ 1+2i,\ 1+3i)$.

6. Determine the points in which the sphere $(x-1)^2 + y^2 + (z+2)^2 = 1$ intersects the X-axis.

42. Loci of equations. The locus defined by a single equation among the variables x, y, z is called a surface. A point $P \equiv (x_1, y_1, z_1)$ lies on the surface $F = 0$ if, and only if, the coördinates of P satisfy the equation of the surface. We have seen, for example, that the locus of a linear equation is a plane. Moreover, the locus of the equation

$$x^2 + y^2 + z^2 = 1$$

is a sphere of radius unity with center at the origin.

The locus of the real points on a surface may be composed of curves and points, or there may be no real points on the surface; for example, the locus of the real points on the surface

$$x^2 + y^2 = 0$$

is the Z-axis; the locus of real points on the surface

$$x^2 + y^2 + z^2 = 0$$

is the origin; the surface

$$x^2 + y^2 + z^2 + 1 = 0$$

has no real points.

If the equation of a surface is multiplied by a constant different from zero, the resulting equation defines the same surface as before; for, if $F = 0$ is the equation of the surface and k a constant different from zero, the coördinates of a point P will satisfy the equation $kF = 0$ if, and only if, they also satisfy the equation $F = 0$.

The locus of two simultaneous equations is the totality of the points whose coördinates satisfy both equations. If $F(x, y, z) = 0$, $f(x, y, z) = 0$ are the equations of two surfaces, then the locus of the simultaneous equations $F = 0$, $f = 0$ is the curve or curves in which these surfaces intersect. Every point on the curve of intersection may be imaginary.

The locus of three simultaneous equations is the totality of the points whose coördinates satisfy the three simultaneous equations.

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EXERCISES

1. Find the equation of the locus of a point whose distance from the Z-axis is twice its distance from the XY-plane.

2. Discuss the locus defined by the equation $x^2 + z^2 = y^2$.

3. Find the equation of the locus of a point the sum of the squares of whose distances from the points $(1, 3, -2)$, $(6, -4, 2)$ is 10.

4. Find the equation of the locus of a point which is three times as far from the point $(2, 6, 3)$ as from the point $(4, -2, 4)$.

5. Find the equations of the locus of a point which is 5 units from the XY-plane and 3 units from the point $(3, 7, 1)$.

6. Find the equations of the locus of a point which is equidistant from the points $(2, 3, 7)$, $(3, -4, 6)$, $(4, 3, -2)$.

7. Find the coördinates of the points in which the line $x = -4$, $z = 2$ intersects the cylinder $y^2 = 4x$.

43. Cylindrical surfaces. It was seen in Art. 42 that the locus of a single equation $F(x, y, z) = 0$ is a surface. We shall now discuss the types of surfaces which arise when the form of this equation is restricted in certain ways.

THEOREM. *If the equation of a surface involves only two of the coördinates x, y, z, the surface is a cylindrical surface whose generating lines are parallel to the axis whose coördinate does not appear in the equation.*

Let $f(x, y) = 0$ be an equation containing the variables x and y but not containing z. If we consider the two equations $f(x, y) = 0$, $z = 0$ simultaneously, we have a plane curve $f(x, y) = 0$ in the plane $z = 0$. If $(x_1, y_1, 0)$ is a point of this curve, $f(x_1, y_1) = 0$. The coördinates of any point on the line $x = x_1$, $y = y_1$ are of the form x_1, y_1, z. But these coördinates satisfy the equation $f(x_1, y_1) = 0$ independently of z, hence every point of the line lies on the surface $f(x, y) = 0$. It is therefore generated by a line moving parallel to the Z-axis and always intersecting the curve $f(x, y) = 0$ in the XY-plane. The surface is consequently a cylindrical surface. In the same way it is shown that $\phi(x, z) = 0$ is the equation of a cylindrical surface whose generating elements are parallel to the Y-axis, and that $F(y, z) = 0$ is the equation of a cylindrical surface whose generating elements are parallel to the X-axis.

44. Projecting cylinders. A cylinder whose elements are perpendicular to a given plane and intersect a given curve is called the projecting cylinder of the given curve on the given plane.

The equation of the projecting cylinder of the curve of intersection of two surfaces $F(x, y, z) = 0$, $f(x, y, z) = 0$ on the plane $z = 0$ is independent of z (Art. 43). The equations of this cylinder may be obtained by eliminating z between the equations of the curve.

If F and f are polynomials in z, the elimination may be effected in the following way, known as Sylvester's method of elimination. Since the coördinates of points on the curve satisfy $F=0$ and $f=0$, they satisfy

$$F=0,\ zF=0,\ z^2F=0,\ \cdots,\quad f=0,\ zf=0,\ z^2f=0,\ \ldots,$$

simultaneously. If we consider these equations as linear equations in the variables z, z^2, z^3, $\cdots$, and eliminate z and its powers, we obtain an equation $R(x, y)=0$, which is the equation required. The following example will illustrate the method.

Given the curve

$$z^2+3\,xz+x+y=0,\quad 2\,z^2+3\,z+x+y^2=0.$$

The equation of its projecting cylinder on $z=0$ is found by eliminating z between the given equations and

$$z^3+3\,xz^2+(x+y)z=0,\quad 2\,z^3+3\,z^2+(x+y^2)z=0.$$

The result is

$$\begin{vmatrix} 1 & 3\,x & x+y & 0 \\ 0 & 1 & 3\,x & x+y \\ 2 & 3 & x+y^2 & 0 \\ 0 & 2 & 3 & x+y^2 \end{vmatrix}=0,$$

which simplifies to

$$(y^2-2\,y-x)^2=9\,(1-2\,x)\,(xy^2+x^2-x-y).$$

The equations of the projecting cylinders on $x=0$ and on $y=0$ may be found in a similar manner.

45. Plane sections of surfaces. The equation of the projecting cylinder of the section of a surface $F(x, y, z)=0$ by a plane $z=k$ parallel to the XY-plane may be found by putting $z=k$ in the equation of the surface. The section of this cylinder $F(x, y, k)=0$ by the plane $z=0$ is parallel to the section by $z=k$. Since parallel sections of a cylinder, by planes perpendicular to the elements, are congruent, we have the following theorem:

Theorem. *If in the equation of a surface, we put $z=k$ and consider the result as the equation of a curve in the plane $z=0$, this curve is congruent to the section of the surface by the plane $z=k$.*

46. Cones. A surface such that the line joining an arbitrary point on the surface to a fixed point lies entirely on the surface is a cone. The fixed point is the vertex of the cone.

THEOREM. *If the equation of a surface is homogeneous in x, y, z, the surface is a cone with vertex at the origin.*

Let $f(x, y, z) = 0$ be the equation of the surface. Let f be homogeneous of degree n in (x, y, z), and let $P_1 \equiv (x_1, y_1, z_1)$ be an arbitrary point on the surface, so that $f(x_1, y_1, z_1) = 0$. The origin lies on the surface, since $f(0, 0, 0) = 0$. The coördinates of any point P on the line joining P_1 to the origin are (Art. 6)

$$x = kx_1, \ y = ky_1, \ z = kz_1, \text{ where } k = \frac{m_2}{m_1 + m_2}.$$

But the coördinates of P satisfy the equation, since

$$f(x, y, z) = f(kx_1, ky_1, kz_1) = k^n f(x_1, y_1, z_1) = 0$$

for every value of k. Thus, every point of the line OP_1 lies on the surface, which is therefore a cone with the vertex at the origin.

EXERCISES

1. Describe the loci represented by the following equations:

(a) $x^2 + y^2 = 4$.

(b) $y^2 = x$.

(c) $y = \sin x$.

(d) $\frac{x^2}{4} + \frac{y^2}{9} = 1$.

(e) $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

(f) $x(x-1)(x-1)(x-3) = 0$.

2. Describe as fully as possible the locus of the equation $4x^2 + y^2 = 25z^2$.

3. Show that the section of the surface $x^2 + y^2 = 9z$ by the plane $z = 4$ is a circle. Find the coördinates of its center and the length of its radius.

4. Find the equation of the projection upon the plane $z = 0$ of the curve of intersection of the surfaces

$$yz + 1 = 0, \ (x^2 + y^2 - 1)z + 2y = 0.$$

5. Show that the section of the surface $x^2z^2 + a^2y^2 = r^2z^2$ by the plane $z = k$ is an ellipse. Find its semi-axes. By giving k a series of values, determine the form of the surface.

6. Show that if the equation of a surface is homogeneous in $x - h$, $y - k$, $z - l$, the surface is a cone with vertex at (h, k, l).

7. By using homogeneous coördinates, show that the cylinder $f(x, y, t) = 0$ can be considered a cone with vertex at $(0, 0, 1, 0)$.

47. Surfaces of revolution. The surface generated by revolving a plane curve about a line in its plane is called a surface of revolution. The fixed line is called the **axis of revolution.** Every point of the revolving curve describes a circle, whose plane is perpendicular to the axis of revolution, whose center is on the axis and whose radius is the distance of the point from the axis.

To determine the equation of the surface generated by revolving a given curve about a given axis, take the plane of the given curve for the XY-plane and the axis of revolution for the X-axis. Let the equation of the given curve in $z=0$ be $f(x, y)=0$. Let $P_1 \equiv (x_1, y_1, 0)$, Fig. 22, be any point on the curve, so that $f(x_1, y_1)=0$

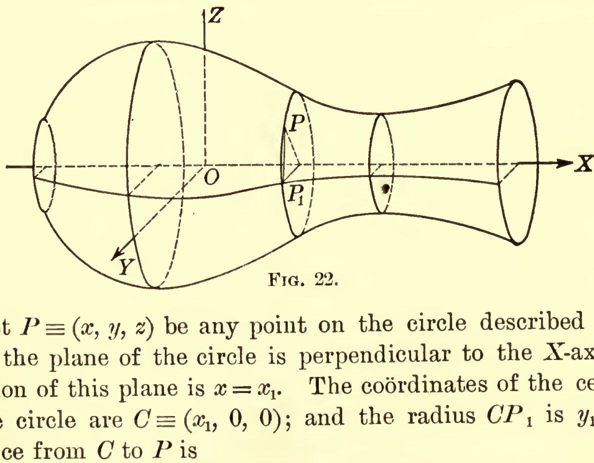

FIG. 22.

and let $P \equiv (x, y, z)$ be any point on the circle described by P_1. Since the plane of the circle is perpendicular to the X-axis, the equation of this plane is $x=x_1$. The coördinates of the center C of the circle are $C \equiv (x_1, 0, 0)$; and the radius CP_1 is y_1. The distance from C to P is

$$y_1 = \sqrt{(x_1 - x_1)^2 + (y-0)^2 + (z-0)^2} = \sqrt{y^2+z^2}.$$

On substituting

$$x_1 = x, \quad y_1 = \sqrt{y^2+z^2}$$

in the equation $f(x_1, y_1)=0$ we obtain, as the condition that the point P lies on the surface,

$$f(x, \sqrt{y^2+z^2})=0,$$

which is the desired equation.

In the same way it may be seen that the equation of the surface of revolution obtained by revolving the curve $f(x, y)=0$ about the Y-axis is

$$f(\sqrt{x^2+z^2}, y)=0.$$

EXERCISES

1. What is the equation of the surface generated by revolving the circle $x^2 + y^2 = 25$ about the X-axis? about the Y-axis?

2. Obtain the equation of the surface generated by revolving the line $2x + 3y = 15$ about the X-axis. Show that the surface is a cone. Find its vertex. What is the equation of the section made by the plane $x = 0$? Find the equation of the cone generated by revolving the line about the Y-axis.

3. Why is the resulting equation of the same degree as that of the generating curve in Ex. 1, but twice the degree of the given curve in Ex. 2? Formulate a general rule.

4. What is the equation of the surface generated by revolving the line $y = a$ about the X-axis? about the Y-axis?

5. If the curve $f(x, y) = 0$ crosses the x-axis at the point $(x_1, 0, 0)$, describe the appearance of the surface

$$f(x, \sqrt{y^2 + z^2}) = 0 \text{ near the point } (x_1, 0, 0).$$

6. Find the equation of the surface generated by revolving the following curves about the X-axis and about the Y-axis. Draw a figure of each surface.

(*a*) $\frac{x^2}{4} + \frac{y^2}{9} = 1$. (*c*) $y^2 = 8x$. (*e*) $y = \sin x$.

(*b*) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$. (*d*) $x^2 + (y - 1)^2 = 4$. (*f*) $y = e^x$.

CHAPTER V

THE SPHERE

48. The equation of the sphere. The equation of the sphere having its center at (x_0, y_0, z_0) and radius r is

$$(x-x_0)^2+(y-y_0)^2+(z-z_0)^2=r^2, \qquad (1)$$

or

$$x^2+y^2+z^2-2\,x_0x-2\,y_0y-2\,z_0z+x_0^{\,2}+y_0^{\,2}+z_0^{\,2}-r^2=0.$$

Any equation of the form

$$a(x^2+y^2+z^2)+2\,fx+2\,gy+2\,hz+k=0, \quad a\neq 0 \qquad (2)$$

may be written in the form

$$\left(x+\frac{f}{a}\right)^2+\left(y+\frac{g}{a}\right)^2+\left(z+\frac{h}{a}\right)^2=\frac{f^2+g^2+h^2-ak}{a^2}. \qquad (3)$$

If $f^2+g^2+h^2-ak>0$, this is seen, by comparing with (1), to be a sphere with center at $\left(-\frac{f}{a},\ -\frac{g}{a},\ -\frac{h}{a}\right)$ and radius

$$\frac{\sqrt{f^2+g^2+h^2-ak}}{a}.$$

If the expression under the radical sign vanishes, the center is the only real point lying on the sphere, which in this case has a zero radius, and is called a **point sphere.** If the expression under the radical is negative, no real point lies on the locus, which is called an **imaginary sphere.**

49. The absolute. We shall now prove the following theorem:

THEOREM I. *All spheres intersect the plane at infinity in the same curve.*

In order to determine the intersection of the sphere and the plane at infinity, we first write the equation of the sphere in homogeneous coördinates:

$$a(x^2+y^2+z^2)+2\,fxt+2\,gyt+2\,hzt+kt^2=0, \quad a\neq 0.$$

The equations of the curve of intersection of this sphere with the plane at infinity are

$$t = 0, \quad x^2 + y^2 + z^2 = 0. \tag{4}$$

Since these equations are independent of the coefficients a, f, g, h, k which appear in the equation of the sphere, the theorem follows.

The curve determined by equations (4) is called the **absolute.** Since the homogeneous coördinates of a point cannot all be zero (Art. 29), there are no real points on the absolute.

The equation of any surface of second degree which contains the absolute may be written in the form

$$a(x^2 + y^2 + z^2) + (kx + ly + mz + nt)t = 0.$$

If $a \neq 0$, this is the equation of a sphere (Art. 48). If $a = 0$, the locus of the equation is two planes of which at least one is $t = 0$. In the latter case also, we shall call the surface a sphere, since its equation is of the second degree and it passes through the absolute. When it is necessary to distinguish it from a proper sphere, it will be called a composite sphere. With this extended definition, we have at once the following theorem:

THEOREM II. *Every surface of the second degree which contains the absolute is a sphere.*

Any plane

$$ux + vy + wz + st = 0,$$

other than $t = 0$, intersects the absolute in two points whose coördinates may be found by solving the equation of the plane as simultaneous with the equations of the absolute. Any circle in this plane is the intersection of the plane with a sphere. Since the absolute lies on the sphere, the circle must pass through the two points in which its plane intersects the absolute. These two points are called the **circular points** in the plane.

Evidently all the planes parallel to the given one will contain the same circular points. The reason for the designation circular points is seen from the fact that any conic lying in any real transversal plane and passing through the circular points is a circle, as will now be shown. Since the equations of the absolute are not changed by displacement of the axes, it is no restriction to take $z=0$ for the equation of the transversal plane. The coördinates of the points in which the plane $z=0$ meets the curve $t=0$, $x^2+y^2+z^2=0$ are $(1, i, 0, 0)$, $(1, -i, 0, 0)$. A conic in the plane $z=0$ has an equation in homogeneous coördinates of the form

$$Ax^2+By^2+2\,Hxy+2\,Gxt+2\,Fyt+Ct^2=0.$$

If the points $(1, i, 0, 0)$, $(1, -i, 0, 0)$ lie on this curve,

$$A=B,\ \ H=0.$$

But these are exactly the conditions that the conic is a circle. Conversely, it follows at once that every circle in the plane $z=0$ passes through the two circular points in that plane. A conic in an imaginary plane will be defined as a circle if it passes through the circular points of the plane.

If the two circular points in a plane coincide, the plane is said to be tangent to the absolute. Such a plane is called an **isotropic plane**. The condition that the plane $ux+vy+wz+st=0$ is isotropic is found, by imposing the condition that its intersections with the absolute coincide, to be

$$u^2+v^2+w^2=0. \tag{5}$$

This equation is the equation of the absolute in plane coördinates.

EXERCISES

1. Write the equation of a sphere, given

(*a*) center at $(0, 0, 0)$ and radius r,

(*b*) center at $(-1, 4, 2)$ and radius 6,

(*c*) center at $(2, 1, 5)$ and radius 4.

2. Determine the center and radius of each of the following spheres:

(*a*) $x^2+y^2+z^2+7\,x+2\,y+z+5=0.$

(*b*) $x^2+y^2+z^2+2\,x+4\,y-6\,z+14=0.$

(*c*) $2(x^2+y^2+z^2)-x-2\,y+5\,z+5=0.$

(*d*) $x^2+y^2+z^2+fx=0.$

3. Find the points of intersection of the absolute and the plane

$$2\,x-y+2\,z+15\,t=0.$$

4. Find the coördinates of the points of intersection of the line $x=-2+\frac{2}{3}d$, $y=3-\frac{2}{3}d$, $z=-2+\frac{1}{3}d$ with the sphere $x^2+y^2+z^2+1=0$.

5. Show that $x^2+y^2+z^2=0$ is the equation of a cone.

6. Find the distance of the point $(1, 0, i)$ from the origin.

7. Show that the radius of the circle in which $z = 2$ intersects the sphere $x^2 + y^2 + z^2 = 1$ is imaginary.

8. Prove that, if (x_1, y_1, z_1) is any point exterior to the sphere $(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$, the expression $(x_1 - x_0)^2 + (y_1 - y_0)^2 + (z_1 - z_0)^2 - r^2$ is the square of the segment on a tangent from (x_1, y_1, z_1) to the point of contact on the sphere.

50. Tangent Plane. Let $P \equiv (x_1, y_1, z_1)$ be any point on the sphere

$$a(x^2 + y^2 + z^2) + 2fx + 2gy + 2hz + k = 0.$$

The plane passing through P perpendicular to the line joining P to the center of the sphere is the tangent plane to the sphere at P. It is required to find its equation. The coördinates of the center are $\left(-\frac{f}{a}, -\frac{g}{a}, -\frac{h}{a}\right)$. The equations of the line joining the center to P are (Art. 19)

$$\frac{x - x_1}{-\frac{f}{a} - x_1} = \frac{y - y_1}{-\frac{g}{a} - y_1} = \frac{z - z_1}{-\frac{h}{a} - z_1}.$$

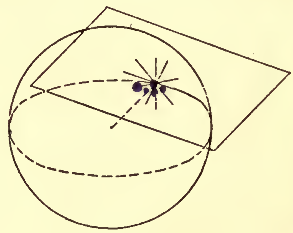
FIG. 23.

The equation of the plane passing through P and perpendicular to this line is

$$\left(\frac{f}{a} + x_1\right)(x - x_1) + \left(\frac{g}{a} + y_1\right)(y - y_1) + \left(\frac{h}{a} + z_1\right)(z - z_1) = 0.$$

If we expand the first member of this equation + mult. by a and add to it $a(x_1^2 + y_1^2 + z_1^2) + 2fx_1 + 2gy_1 + 2hz_1 + k$, which is equal to zero since the point (x_1, y_1, z_1) lies on the sphere, we obtain

$$a(x_1x + y_1y + z_1z) + f(x + x_1) + g(y_1 + y) + h(z_1 + z) + k = 0, \quad (6)$$

which is the required equation of the tangent plane.

51. The angle between two spheres. The angle between two spheres at a point P_1 on their curve of intersection is defined as equal to the angle between the tangent planes to the spheres at P_1.

To determine the magnitude of this angle, let the coördinates of P_1 be (x_1, y_1, z_1) and let the equations of the spheres be

$$a(x^2 + y^2 + z^2) + 2fx + 2gy + 2hz + k = 0,$$
$$a'(x^2 + y^2 + z^2) + 2f'x + 2g'y + 2h'z + k' = 0.$$

The equations of the tangent planes to these spheres at P_1 are

$$a(x_1x + y_1y + z_1z) + f(x + x_1) + g(y + y_1) + h(z + z_1) + k = 0.$$
$$a'(x_1x + y_1y + z_1z) + f'(x + x_1) + g'(y + y_1) + h'(z + z_1) + k' = 0.$$

Since the angle θ between the spheres is equal to the angle between these planes, we have (Art. 15)

$$\cos\theta = \frac{(ax_1 + f)(a'x_1 + f') + (ay_1 + g)(a'y_1 + g') + (az_1 + h)(a'z_1 + h')}{\sqrt{(ax_1+f)^2 + (ay_1+g)^2 + (az_1+h)^2}\sqrt{(a'x_1+f')^2 + (a'y_1+g')^2 + (a'z_1+h')^2}}.$$

Since (x_1, y_1, z_1) lies on both spheres, this relation reduces to

$$\cos\theta = \frac{2ff' + 2gg' + 2hh' - ak' - a'k}{2\sqrt{f^2 + g^2 + h^2 - ak}\sqrt{f'^2 + g'^2 + h'^2 - a'k'}}. \qquad (7)$$

Since this expression is independent of the coördinates of P_1, we have the following theorem:

THEOREM. *Two spheres intersect at the same angle at all points of their curve of intersection.*

If $\theta = 90$ degrees, the spheres are said to be orthogonal. The condition that two spheres are orthogonal is

$$2ff' + 2gg' + 2hh' - ak' - a'k = 0. \qquad (8)$$

52. Spheres satisfying given conditions. The equation of a sphere is homogeneous in the five coefficients a, f, g, h, k. Hence the sphere may be made to satisfy four conditions, as, for example, to pass through four given points, or to intersect four given spheres at given angles. If the given conditions are such that $a = 0$, the sphere is composite (Art. 49).

EXERCISES

1. Prove that the point $(-3, 1, -4)$ lies on the sphere $x^2 + y^2 + z^2 + 6x + 24y + 8z = 0$ and write the equation of the tangent plane to the sphere at that point.

2. Find the angle of intersection of the spheres $x^2 + y^2 + z^2 + x + 6y + 2z + 9 = 0$, $x^2 + y^2 + z^2 + 5x + 3z + 4 = 0$.

3. Find the equation of the sphere with its center at $(1, 3, 3)$ and making an angle of 60 degrees with the sphere $x^2 + y^2 + z^2 = 4$.

4. Determine the equation of the sphere which passes through the points $(0, 0, 0)$, $(0, 0, 3)$, $(0, 2, 0)$, $(1, 2, 1)$.

5. Determine the equation of the sphere which passes through the points $(1, 3, 2)$, $(3, 2, -5)$, $(-1, 2, 3)$, $(4, 5, 2)$.

6. Write the equation of the sphere passing through the points $(2, 2, -1)$, $(3, -1, 4)$, $(1, 3, -2)$ and orthogonal to the sphere

$$x^2 + y^2 + z^2 - 3x + y + z = 0.$$

7. Write the equation of the sphere inscribed in the tetrahedron $x = 0$, $y = 0$, $5x + 12z + 3 = 0$, $3x - 12y + 4z = 0$.

53. Linear systems of spheres. Let

$$S \equiv a(x^2 + y^2 + z^2) + 2fx + 2gy + 2hz + k = 0,$$
$$S' \equiv a'(x^2 + y^2 + z^2) + 2f'x + 2g'y + 2h'z + k' = 0$$

be the equation of two spheres. The equation

$$\lambda_1 S + \lambda_2 S' = 0,$$

or
$$(a\lambda_1 + a'\lambda_2)(x^2 + y^2 + z^2) + 2(f\lambda_1 + f'\lambda_2)x + 2(g\lambda_1 + g'\lambda_2)y + 2(h\lambda_1 + h'\lambda_2)z + k\lambda_1 + k'\lambda_2 = 0$$

also represents a sphere for all values of λ_1 and λ_2. Every sphere of the system $\lambda_1 S + \lambda_2 S' = 0$ contains the curve of intersection of $S = 0$ and $S' = 0$ (Art. 42). In particular, if $a\lambda_1 = -a'\lambda_2$, the sphere $\lambda_1 S + \lambda_2 S' = 0$ is composite; it consists of the plane at infinity (which intersects all the spheres of the system in the absolute) and the plane

$$2(a'f - af')x + 2(a'g - ag')y + 2(a'h - ah')z + a'k - ak' = 0, \quad (9)$$

which intersects all the spheres of the system in a fixed circle, common to $S = 0$ and $S' = 0$. The plane (9) is called the **radical plane** of the given system of spheres.

It will now be shown that the radical plane is the locus of the centers of the spheres intersecting $S = 0$ and $S' = 0$ orthogonally. For this purpose let

$$a_0(x^2 + y^2 + z^2) + 2f_0x + 2g_0y + 2h_0z + k_0 = 0 \quad (10)$$

be the equation of a sphere. It will be orthogonal to S if (Art. 51)

$$2f_0f + 2g_0g + 2h_0h - a_0k - ak_0 = 0,$$

and to S' if

$$2f_0f' + 2g_0g' + 2h_0h' - a_0k' - a'k_0 = 0.$$

If we eliminate k_0 between these two equations, we have

$$2(a'f-af')f_0+2(a'g-ag')g_0+2(a'h-ah')h_0-(a'k-ak')a_0=0, \quad (11)$$

which is exactly the condition that the center $\left(\frac{f_0}{-a_0}, \frac{g_0}{-a_0}, \frac{h_0}{-a_0}\right)$ of the orthogonal sphere lies in the radical plane (9). Conversely, if a_0, f_0, g_0, h_0 are given numbers which satisfy (9), a value of k_0 can be found such that the corresponding sphere (10) is orthogonal to every sphere of the system $\lambda_1 S+\lambda_2 S'=0$.

Again, if

$$S'' \equiv a''(x^2+y^2+z^2)+2f''x+2g''y+2h''z+k''=0$$

is a sphere whose center does not lie on the line joining the centers of S and S', every sphere of the system

$$\lambda_1 S+\lambda_2 S'+\lambda_3 S''=0 \quad (12)$$

passes through the points of intersection of the spheres $S=0$, $S'=0$, $S''=0$.

Every sphere of the system (12) determined by values of $\lambda_1, \lambda_2, \lambda_3$ for which

$$\lambda_1 a+\lambda_2 a'+\lambda_3 a''=0$$

is composed of two planes of which one is the plane at infinity and the other passes through the line

$$2(a'f-af')x+2(a'g-ag')y+2(a'h-ah')z+a'k-ak'=0, \quad (13)$$
$$2(a''f-af'')x+2(a''g-ag'')y+2(a''h-ah'')z+a''k-ak''=0.$$

This line is called the **radical axis** of the system of spheres (12). By comparing equations (13) with (11) and the equation analogous to (11) for $S''=0$, it may be shown that the radical axis is the locus of centers of the spheres which intersect all the spheres of the system (12) orthogonally.

Now let

$$S'''=a'''(x^2+y^2+z^2)+2f'''x+2g'''y+2h'''z+k'''=0$$

be the equation of a sphere whose center is not in the plane determined by the centers of $S=0$, $S'=0$, $S''=0$. The condition that a sphere of the system

$$\lambda_1 S+\lambda_2 S'+\lambda_3 S''+\lambda_4 S'''=0$$

is composite, is that λ_1 λ_2 λ_3 and λ_4 satisfy the relation

$$\lambda_1 a+\lambda_2 a'+\lambda_3 a''+\lambda_4 a'''=0.$$

The sphere orthogonal to all the spheres of the system is in this case uniquely determined by equations analogous to ~~(10)~~. 9
The center of this orthogonal sphere is called the **radical center** of the system. Through the radical center passes one plane of every composite sphere of the system.

EXERCISES

1. Prove that the center of any sphere of the system $\lambda_1 S + \lambda_2 S' = 0$ lies on the line joining the center of $S = 0$ to the center of $S' = 0$.

2. Prove that the line joining the centers of the spheres $S = 0$ and $S' = 0$ is perpendicular to the radical plane of the system $\lambda_1 S + \lambda_2 S' = 0$.

3. Show that the radical axis of the system $\lambda_1 S + \lambda_2 S' + \lambda_3 S'' = 0$ is perpendicular to the plane of centers of the spheres belonging to the system.

4. Determine the equation of the system of spheres orthogonal to the system $\lambda_1 S + \lambda_2 S' + \lambda_3 S'' = 0$.

5. Show that two point spheres are included in the system $\lambda_1 S + \lambda_2 S' = 0$.

6. Show that any sphere of the system $\lambda_1 S + \lambda_2 S' = 0$ is the locus of a point, the ratio of whose distances from the centers of the two point spheres of the system is constant.

7. If $S = 0$, $S' = 0$, $S'' = 0$, $S''' = 0$, $S'''' = 0$ are the equations of five spheres which do not belong to a linear system of four or less terms, show that the equation of any sphere in space can be expressed by the equation $S \equiv \Sigma \lambda_i S^{(i)} = 0$.

54. Stereographic projection. Let O be a fixed point on the surface of a sphere of radius r, and let π be the plane tangent to the sphere at the opposite end of the diameter passing through O. The intersection with π of the line joining O to any point P_1 on the surface is called the **stereographic projection** of P_1 (Fig. 24).

To determine the equations connecting the coordinates of P_1 and its projection, take the plane π for the plane $z = 0$, and the diameter of the sphere through O for Z-axis. The equation of the sphere is

$$x^2 + y^2 + z^2 - 2rz = 0.$$

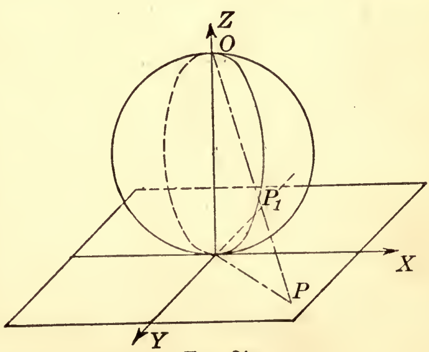

FIG. 24.

The equations of the line joining $O \equiv (0, 0, 2r)$ to $P_1 \equiv (x_1, y_1, z_1)$ on the sphere are (Art. 19)

$$\frac{x}{x_1} = \frac{y}{y_1} = \frac{z - 2r}{z_1 - 2r}.$$

To determine the coördinates $(x, y, 0)$ of P, the point in which OP_1 intersects π, we make the equations of the line simultaneous with $z = 0$. On solving for x, y, z we obtain

$$x = \frac{2rx_1}{2r - z_1}, \quad y = \frac{2ry_1}{2r - z_1}, \quad z = 0.$$

These equations can be solved for x_1, y_1, z_1 by making use of the fact that, since P_1 lies on the sphere,

$$x^2_1 + y^2_1 + z^2_1 - 2rz_1 = 0.$$

The results are

$$x_1 = \frac{4r^2x}{x^2 + y^2 + 4r^2}, \quad y_1 = \frac{4r^2y}{x^2 + y^2 + 4r^2}, \quad z_1 = \frac{2r(x^2 + y^2)}{x^2 + y^2 + 4r^2}. \qquad (14)$$

THEOREM I. *The stereographic projection of a circle is a circle.*

Let the equation of the plane of the given circle on the sphere be

$$Ax + By + Cz + D = 0.$$

The condition that P_1 lies on this circle is consequently

$$Ax_1 + By_1 + Cz_1 + D = 0.$$

If we substitute from (14) in this equation, we obtain as the equation of the stereographic projection,

$$4Ar^2x + 4Br^2y + 2Cr(x^2 + y^2) + D(x^2 + y^2 + 4r^2) = 0, \qquad (15)$$

which represents a circle in the XY-plane.

In particular, if the plane of the given circle passes through O, the stereographic projection of the circle is composite. The condition that the plane

$$Ax + By + Cz + D = 0$$

passes through O is

$$2rC + D = 0.$$

If this condition is satisfied, the equation of the circle of projection is, in homogeneous coördinates,

$$t(Ax + By + Dt) = 0.$$

The points of the line $t = 0$ correspond only to the point O itself. The line

$$Ax + By + Dt = 0$$

is the line of intersection of the plane of the circle and the plane of projection. We have consequently the following theorem:

THEOREM II. *The circles on the sphere which pass through the center of projection are projected stereographically into the lines in which their planes intersect the plane of projection.*

The angle between two intersecting curves is defined as the angle between their tangents at the point of intersection. We shall prove the following theorem:

THEOREM III. *The angle between two intersecting curves on the sphere is equal to the angle between their stereographic projections.*

It will suffice if we prove the theorem for great circles. For, let C'_1 and C'_2 be any two curves whatever on the sphere having a point P' in common. The great circles whose planes pass through the tangents to C'_1 and C'_2 at P' are tangent to C'_1 and C'_2, respectively, at P'. Let C_1, C_2, and P_1 be the stereographic projections of C'_1, C'_2, and P'. The stereographic projections of the great circles are tangent to C_1 and C_2, respectively, at P_1 so that the angle between them is the angle between C_1 and C_2. If, then, the theorem holds for great circles, it holds for all intersecting curves.

The condition that a circle is a great circle is that its plane

$$Ax + By + Cz + D = 0$$

passes through the center $(0, 0, r)$ so that

$$Cr + D = 0.$$

The equation (15) of the stereographic projection reduces to

$$C(x^2 + y^2) + 4r(Ax + By - rC) = 0.$$

The angle between two great circles is equal to the angle between their planes, since the tangents to the circles at their common points are perpendicular to the line of intersection of their planes. The angle θ between the planes

$$Ax + By + Cz - Cr = 0$$

and

$$A'x + B'y + C'z - C'r = 0$$

is defined by the formula (Art. 15)

$$\cos\theta = \frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2}\sqrt{A'^2 + B'^2 + C'^2}}. \tag{16}$$

The tangents to the projections

$$C(x^2 + y^2) + 4\,r(Ax + By - rC) = 0,$$
$$C'(x^2 + y^2) + 4\,r(A'x + B'y - rC') = 0$$

of the given circles, at the point (x_1, y_1) in which they intersect, are

$$(Cx_1 + 2\,rA)\,x + (Cy_1 + 2\,rB)\,y + 2\,rAx_1 + 2\,rBy_1 - 4\,r^2C = 0,$$
$$(C'x_1 + 2\,rA')x + (C'y_1 + 2\,rB')\,y + 2\,rA'x_1 + 2\,rB'y_1 - 4\,r^2C' = 0.$$

The angle ϕ between these circles is given by the formula

$$\cos\phi = \frac{(Cx_1 + 2\,rA)(C'x_1 + 2\,rA') + (Cy_1 + 2\,rB)(C'y_1 + 2\,rB')}{\sqrt{(Cx_1 + 2\,rA)^2 + (Cy_1 + 2\,rB)^2}\sqrt{(C'x_1 + 2\,rA')^2 + (C'y_1 + 2\,rB')^2}}.$$

By expanding this expression and making use of the fact that (x_1, y_1) lies on both circles, we may simplify the preceding equation to

$$\cos\phi = \frac{AA' + BB' + CC'}{\sqrt{A^2 + B^2 + C^2}\sqrt{A'^2 + B'^2 + C'^2}}. \tag{17}$$

From (16) and (17) we have $\cos\theta = \cos\phi$. We may consequently choose the angles in such a way that $\theta = \phi$, which proves the proposition.

The relation established in Theorem III makes stereographic projection of great importance in map drawing.

CHAPTER VI

FORMS OF QUADRIC SURFACES

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55. Definition of a quadric. The locus of an equation of the second degree in x, y, z is called a **quadric surface.** In this chapter certain standard types of the equation will be considered. It will be shown later that the equation of any non-composite quadric may, by a suitable transformation of coördinates, be reduced to one of these types.

56. The ellipsoid. The locus of the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$$

is called the **ellipsoid.** Since only the second powers of the variables x, y, z appear in the equation, the surface is symmetrical as to each coördinate plane, as to each coördinate axis, and as to the origin.

The coördinates of the points of intersection of the ellipsoid with the X-axis are found by putting $y=z=0$ to be $(\pm a, 0, 0)$. Its intersections with the Y-axis are $(0, \pm b, 0)$, and with the Z-axis are $(0, 0, \pm c)$. These six points are called the **vertices.** The segments of the coördinate axes included between the vertices are called the **axes** of the ellipsoid. The point of intersection of the axes is called the **center.** The segments from the center to the vertices are the **semi-axes**; their lengths are a, b, c. We shall suppose the coördinate axes are so chosen that $a \geqq b \geqq c > 0$. The segment joining the vertices on the X-axis is then known as the **major axis**; that joining the vertices on the Y-axis as the **mean axis**; that joining the vertices on the Z-axis as the **minor axis.**

The section of the ellipsoid by the plane $z=k$ is an ellipse whose equations are

$$\frac{x^2}{a^2\left(1-\frac{k^2}{c^2}\right)}+\frac{y^2}{b^2\left(1-\frac{k^2}{c^2}\right)}=1, \quad z=k.$$

The semi-axes of this ellipse are $a\sqrt{1-\frac{k^2}{c^2}}$, $b\sqrt{1-\frac{k^2}{c^2}}$. As $|k|$ increases from 0 to c, the axes of the ellipses of section decrease. If $|k| = c$, the ellipse reduces to a point. If $|k| > c$, the ellipse of section is imaginary, since its axes are imaginary. The real part of the surface therefore lies entirely between the planes $z = c$ and $z = -c$.

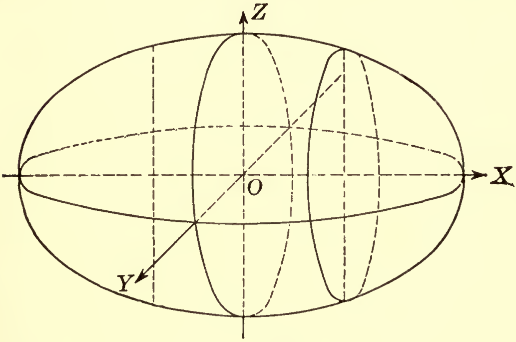
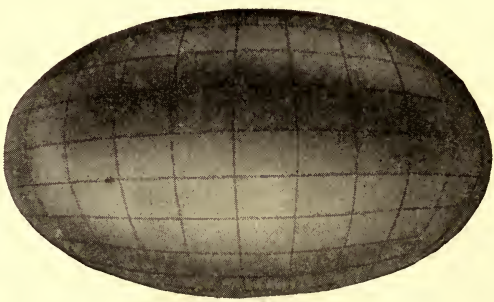


FIG. 25.

In the same manner, it is seen that the plane $y = k'$ intersects the surface in a real ellipse if $|k'| < b$, that the ellipse reduces to a point if $|k'| = b$, and that it becomes imaginary if $|k'| > b$. Finally, it is seen that the section $x = k''$ is a real ellipse, a point, or an imaginary ellipse, according as $|k''|$ is less than, equal to, or greater than a. The ellipsoid, therefore, lies entirely within the rectangular parallelopiped formed by the planes $x = a$, $y = b$, $z = c$; $x = -a$, $y = -b$, $z = -c$, and has one point on each of these planes (Fig. 25).



If $a = b > c$, the ellipsoid is a surface of revolution (Art. 47) obtained by revolving the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{c^2} = 1$$

about its minor axis. This surface is called an **oblate spheroid.**

If $a > b = c$, the ellipsoid is the surface of revolution obtained by revolving the same ellipse about its major axis. It is called a **prolate spheroid.**

If $a = b = c$, the surface is a sphere.

57. The hyperboloid of one sheet. The surface represented by the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is called an **hyperboloid of one sheet.** It is symmetric as to each of the coördinate planes, as to each of the coördinate axes, and as to the origin.

The section of the surface by the plane $z = k$ is an ellipse whose equations are

$$\frac{x^2}{a^2\left(1 + \frac{k^2}{c^2}\right)} + \frac{y^2}{b^2\left(1 + \frac{k^2}{c^2}\right)} = 1, \quad z = k.$$

This ellipse is real for every real value of k. The semi-axes are

$$a\sqrt{1 + \frac{k^2}{c^2}}, \quad b\sqrt{1 + \frac{k^2}{c^2}},$$

which are the smallest when $k = 0$, and increase without limit as $|k|$ increases. For no value of k does the ellipse reduce to a point.

The plane $y = k'$ intersects the surface in the hyperbola

$$\frac{x^2}{a^2\left(1 - \frac{k'^2}{b^2}\right)} - \frac{z^2}{c^2\left(1 - \frac{k'^2}{b^2}\right)} = 1, \quad y = k'.$$

If $|k'| < b$, the transverse axis of the hyperbola is the line $z = 0$, $y = k'$, and the conjugate axis is $x = 0$, $y = k'$; the lengths of the semi-axes are $a\sqrt{1 - \frac{k'^2}{b^2}}$, $c\sqrt{1 - \frac{k'^2}{b^2}}$. As $|k'|$ increases from zero to b, the semi-axes decrease to zero. When $|k'| = b$, the equation cannot be put in the above form, but becomes $\frac{x^2}{a^2} - \frac{z^2}{c^2} = 0$ and the hyperbola is composite; it consists of the two lines

$$\frac{x}{a} + \frac{z}{c} = 0,\ y = b; \quad \frac{x}{a} - \frac{z}{c} = 0,\ y = b;$$

when $k' = -b$, the hyperbola consists of the lines

$$\frac{x}{a} + \frac{z}{c} = 0,\ y = -b;\ \frac{x}{a} - \frac{z}{c} = 0,\ y = -b.$$

These four lines lie entirely on the surface. If $|k'| > b$, the transverse axis of the section is $x = 0$, $y = k'$ and the conjugate axis is $z = 0$, $y = k'$. The lengths of the semi-axes are

$$a\sqrt{\frac{k'^2}{b^2} - 1},\ c\sqrt{\frac{k'^2}{b^2} - 1}.$$

They increase without limit as k' increases.

The plane $x = k''$ intersects the surface in the hyperbola

$$\frac{y^2}{b^2\left(1 - \frac{k''^2}{a^2}\right)} - \frac{z^2}{c^2\left(1 - \frac{k''^2}{a^2}\right)} = 1,\ x = k''.$$

If $|k''| < a$, the transverse axis of this hyperbola is $z = 0$, $x = k''$. The section on the plane $x = a$ consists of the two lines

$$\frac{y}{b} + \frac{z}{c} = 0, \quad x = a;\ \frac{y}{b} - \frac{z}{c} = 0, \quad x = a.$$

The section on the plane $x = -a$ consists of the lines

$$\frac{y}{b} + \frac{z}{c} = 0, \quad x = -a;\ \frac{y}{b} - \frac{z}{c} = 0, \quad x = -a.$$

If $|k''| > a$, the line $y = 0$, $x = k''$ is the transverse axis and $z = 0$, $x = k''$ is the conjugate axis. As $|k''|$ increases, the lengths of the semi-axes increase without limit. The form of the surface is indicated in Fig. 26.

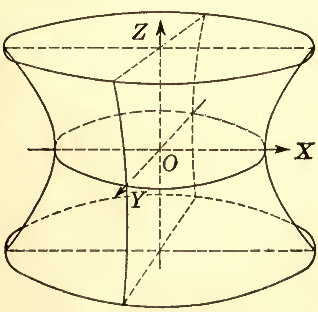


FIG. 26.

If $a = b$, the hyperboloid is the surface of revolution obtained by revolving the hyperbola

$$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1, \quad y = 0$$

about its conjugate axis.

58. The hyperboloid of two sheets. The locus of the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

is called an **hyperboloid of two sheets.** It is symmetric as to each of the coördinate planes, the coördinate axes, and the origin.

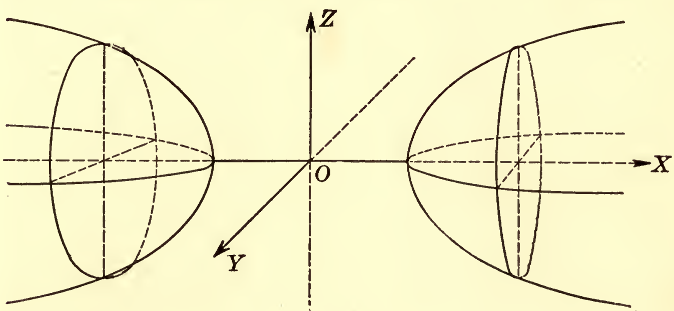


FIG. 27.

The plane $z = k$ intersects the surface in the hyperbola

$$\frac{x^2}{a^2\left(1 + \frac{k^2}{c^2}\right)} - \frac{y^2}{b^2\left(1 + \frac{k^2}{c^2}\right)} = 1, \quad z = k.$$

The transverse axis is $y = 0$, $z = k$, for all values of k. The lengths of the semi-axes are $a\sqrt{1 + \frac{k^2}{c^2}}$, $b\sqrt{1 + \frac{k^2}{c^2}}$. They are smallest for $k = 0$, namely a and b, and increase without limit as $|k|$ increases. The hyperbola is not composite for any real value of k.

The plane $y = k'$ intersects the surface in the hyperbola

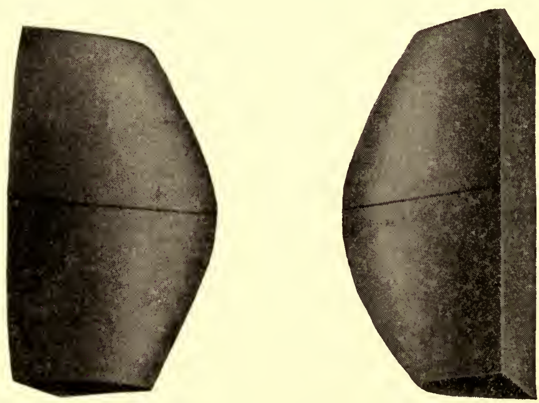
$$\frac{x^2}{a^2\left(1 + \frac{k'^2}{b^2}\right)} - \frac{z^2}{c^2\left(1 + \frac{k'^2}{b^2}\right)} = 1, \quad y = k'.$$

The transverse axis is $z=0$, $y=k'$. The conjugate axis is $x=0$, $y=k'$. If $k'=0$, the lengths of the semi-axes are a and c; they increase without limit as k' increases.

The plane $x=k''$ intersects the surface in the ellipse

$$\frac{y^2}{b^2\left(\frac{k''^2}{a^2}-1\right)}+\frac{z^2}{c^2\left(\frac{k''^2}{a^2}-1\right)}=1, \quad x=k''.$$

This ellipse is imaginary if $|k''|<a$. If $|k''|=a$, the semi-axes are zero; they increase without limit as k'' increases.



If $b=c$, the hyperboloid of two sheets is the surface of revolution obtained by revolving the hyperbola

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1, \quad z=0$$

about its transverse axis.

59. The imaginary ellipsoid. The surface defined by the equation

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1$$

is called an **imaginary ellipsoid.** Since the sum of the squares of three real numbers cannot be negative, there are no real points on it.

work

EXERCISES

1. By translating the axes of coördinates, show that the surface defined by the equation $2x^2 + 3y^2 + 4z^2 - 4x - 6y + 16z + 16 = 0$ is an ellipsoid. Find the coördinates of the center and the lengths of the semi-axes.

2. Classify and describe the surface $x^2 + y^2 - 4x - 3y + 10z = 20 - z^2$.

3. Show that the surface $2x^2 - 3z^2 - 5z = 7 - 2y^2$ is a surface of revolution. Find the equations of the generating curve.

4. On the hyperboloid of one sheet $x^2 + y^2 - z^2 = 1$, find the equations of the two lines which pass through the point $(1, 0, 0)$; through $(-1, 0, 0)$.

5. Classify and plot the loci defined by the following equations:

(*a*) $9x^2 + 16y^2 + 25z^2 = 1$,
(*b*) $4x^2 - 9y^2 - 16z^2 = 25$,
(*c*) $4x^2 - 16y^2 + 9z^2 = 25$,
(*d*) $x^2 + y^2 - 4z^2 = 25$,
(*e*) $x^2 + 4y^2 + z^2 = 9$,
(*f*) $x^2 + 4y^2 + 9z^2 + 8 = 0$.

60. The elliptic paraboloid. The locus of the equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2nz$$

is called an **elliptic paraboloid**. The surface is symmetric as to the planes $x = 0$ and $y = 0$ but not as to $z = 0$. It passes through the origin, and lies on the positive side of $z = 0$ if n is positive and on the negative side if n is negative. In the following discussion it will be assumed that n is positive. If n is negative, it is necessary only to reflect the surface on the plane $z = 0$.

The section of the paraboloid by the plane $z = k$ is an ellipse whose semi-axes are $a\sqrt{2nk}$ and $b\sqrt{2nk}$, respectively. If $k < 0$, the ellipse is imaginary. If $k = 0$, the ellipse reduces to a point, the origin. As k increases, the semi-axes of the ellipse increase without limit.

The section of the paraboloid by the plane $y = k'$ is the parabola

$$\frac{x^2}{a^2} = 2nz - \frac{k'^2}{b^2}, \quad y = k'.$$

For all values of k' these parabolas are congruent. As k' increases, the vertices recede from the plane $y = 0$ along the parabola

$$\frac{y^2}{b^2} = 2nz, \quad x = 0.$$

The sections by the planes $x = k''$ are the congruent parabolas

$$\frac{y^2}{b^2} = 2\,nz - \frac{k''^2}{a^2}, \quad x = k''.$$

Their vertices describe the parabola

$$\frac{x^2}{a^2} = 2\,nz, \quad y = 0.$$

The form of the surface is indicated by Fig. 28.

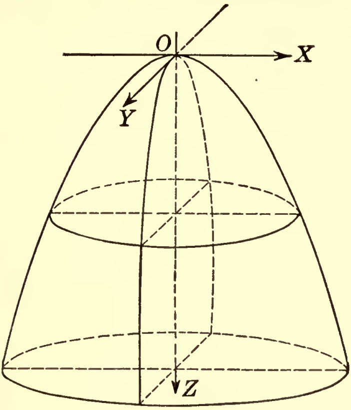
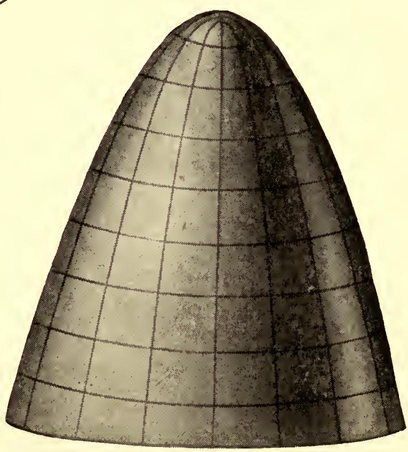


FIG. 28.



If $a = b$, the paraboloid is the surface of revolution generated by revolving the parabola $\dfrac{x^2}{a^2} = 2\,nz, \quad y = 0$ about the Z-axis.

61. The hyperbolic paraboloid. The surface defined by the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2\,nz$$

is called an **hyperbolic paraboloid.** The surface is symmetric as to the planes $x = 0$ and $y = 0$, but not as to $z = 0$.

As before, let it be assumed that $n > 0$. The plane $z = k$ intersects the surface in the hyperbola

$$\frac{x^2}{a^2\,2\,nk} - \frac{y^2}{b^2\,2\,nk} = 1, \; z = k.$$

If $k > 0$, the line $x = 0$, $z = k$ is the transverse axis and $y = 0$, $z = k$ is the conjugate axis. If $k < 0$, the axes are interchanged. The lengths of the semi-axes increase without limit as $|k|$ increases.

When $k=0$, the section of the paraboloid consists of the two lines

$$\frac{x}{a}+\frac{y}{b}=0,\ z=0;\ \frac{x}{a}-\frac{y}{b}=0,\ \ z=0.$$

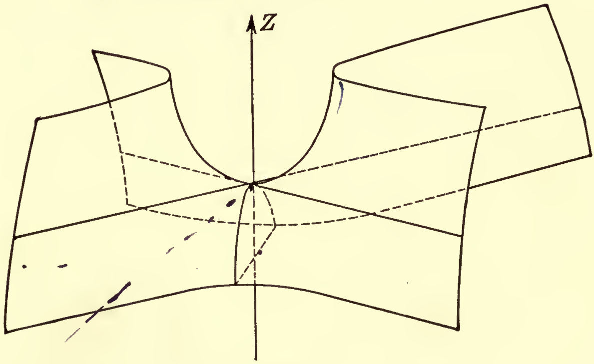


FIG. 29.

The section of the surface by the planes $y=k'$ are the congruent parabolas

$$\frac{x^2}{a^2}=2\,nz+\frac{k'^2}{b^2},\ \ y=k'.$$

The vertices of these parabolas describe the parabola

$$\frac{y^2}{b^2}=-2\,nz,\ x=0.$$

The sections by the planes $x=k''$ are congruent parabolas whose vertices describe the parabola

$$\frac{x^2}{a^2}=2\,nz,\ y=0.$$

62. The quadric cones. The cone (Art. 46)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$$

is called the **real quadric cone.** Its vertex is at the origin. The section of the cone by the plane $z = c$ is the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \ z = c.$$

The cone is therefore the locus of a line which passes through the origin and intersects this ellipse.

If $a = b$, the surface is the right circular cone generated by revolving the line $\frac{x}{a} = \frac{z}{c}$, $y = 0$ about the Z-axis.

The equation

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0$$

represents an imaginary quadric cone. There are no real points on it except the origin.

63. The quadric cylinders. The cylinders (Art. 43) whose equations are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1; \ \frac{x^2}{a^2} - \frac{y^2}{b^2} = 1; \ \frac{x^2}{a^2} + \frac{y^2}{b^2} = -1; \ y^2 = 2px$$

are called **elliptic, hyperbolic, imaginary,** and **parabolic cylinders,** respectively, since the sections of them by the planes $z = k$ are congruent ellipses, hyperbolas, imaginary ellipses, and parabolas, respectively.

64. Summary. The surfaces discussed will be enumerated again for reference.

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$	Ellipsoid.	(Art. 56)
$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$	Hyperboloid of one sheet.	(Art. 57)
$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$	Hyperboloid of two sheets.	(Art. 58)
$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = -1.$	Imaginary ellipsoid.	(Art. 59)
$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 2nz.$	Elliptic paraboloid.	(Art. 60)

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2\,nz.$ Hyperbolic paraboloid. (Art. 61)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0.$ Real quadric cone. (Art. 62)

$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0.$ Imaginary quadric cone. (Art. 62)

$\frac{x^2}{a^2} \pm \frac{y^2}{b^2} = \pm 1;\ y^2 = 2\,px.$ Quadric cylinders. (Art. 63)

EXERCISES

Classify the following surfaces:

1. $4\,z^2 - 6\,x^2 + 2\,y^2 = 3.$

2. $x^2 + 3\,y^2 + 5\,x + 2\,y + 7 = 0.$

3. $x^2 + 3\,y^2 + 4\,x - 2\,z = 0.$

4. $4\,x^2 + 4\,y^2 - 3\,z^2 = 0.$

5. $2\,z^2 - x^2 - 3\,y^2 - 2\,x - 12\,y = 15.$

6. $x^2 - 2\,y^2 - 6\,y - 6\,z = 0.$

7. Find the equation of and classify the locus of a point which moves so that (a) the sum of its distances, (b) the difference of its distances from two fixed points is constant. Take the points $(\pm\, a, 0, 0)$.

8. Find and classify the equation of the locus of a point which moves so that its distance from $(a, 0, 0)$ bears a constant ratio to its distance (a) from the plane $x = 0$; (b) from the Z-axis.

9. Show that the locus of a point whose distance from a fixed plane is always equal to its distance from a fixed line perpendicular to the plane is a quadric cone.

10. A line moves in such a way that three points fixed on it remain in three fixed planes at right angles to each other. Show that any other point fixed on the line describes an ellipsoid. (SUG. Find the direction cosines of the line in terms of the coördinates of the point chosen, and substitute in formula (1), Art. 3.)

CHAPTER VII

CLASSIFICATION OF QUADRIC SURFACES

65. Intersection of a quadric and a line. The most general form of the equation of a quadric surface is (Art. 55)

$$F(x, y, z) = ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy \\ + 2lx + 2my + 2nz + d = 0. \qquad (1)$$

We shall suppose, unless the contrary is stated, that the coefficients are all real, and that the coefficients of the second-degree terms are not all zero.

To determine the points of intersection of a given line (Art. 20)

$$x = x_0 + \lambda r, \quad y = y_0 + \mu r, \quad z = z_0 + \nu r \qquad (2)$$

with the quadric (1), substitute the values of x, y, z from (2) in $F(x, y, z)$ and arrange in powers of r. The result is

$$Qr^2 + 2Rr + S = 0, \qquad (3)$$

in which

$$Q \equiv a\lambda^2 + b\mu^2 + c\nu^2 + 2f\mu\nu + 2g\nu\lambda + 2h\lambda\mu, \qquad (4)$$

$$R \equiv (ax_0 + hy_0 + gz_0 + l)\lambda + (hx_0 + by_0 + fz_0 + m)\mu + (gx_0 + fy_0 + cz_0 + n)\nu$$

$$\equiv \frac{1}{2}\left(\frac{\partial F}{\partial x_0}\lambda + \frac{\partial F}{\partial y_0}\mu + \frac{\partial F}{\partial z_0}\nu\right),$$

$$S \equiv F(x_0, y_0, z_0).$$

The roots in r of equation (3) are the distances from the point $P_0 \equiv (x_0, y_0, z_0)$ on the line (2) to the points in which this line intersects the quadric.

If $Q \neq 0$, equation (3) is a quadratic in r. If $Q = 0$, but R and S are not both zero, (3) is still to be considered a quadratic, with one or more infinite roots. If $Q = R = S = 0$, (2) is satisfied for all values of r and the corresponding line (2) lies entirely on the quadric. We have, consequently, the following theorems:

THEOREM I. *Every line which does not lie on a given quadric surface has two (distinct or coincident) points in common with the surface.*

finite or infinite

THEOREM II. *If a given line has more than two points in common with a given quadric, it lies entirely on the quadric.*

For, if (3) is satisfied by more than two values of r, it is satisfied for all values.

66. Diametral planes, center. Let P_1 and P_2 be the points of intersection of the line (2) with the quadric. The segment P_1P_2 is called a chord of the quadric.

THEOREM I. *The locus of the middle point of a system of parallel chords of a quadric is a plane.*

Let r_1 and r_2 be the roots of (3) so that $P_0P_1 = r_1$ and $P_0P_2 = r_2$. The condition that P_0 is the middle point of the chord P_1P_2 is

$$P_0P_1 + P_0P_2 = 0,$$

or

$$r_1 + r_2 = 0.$$

Hence, from (4), we have

$$(ax_0 + hy_0 + gz_0 + l)\lambda + (hx_0 + by_0 + fz_0 + m)\mu + (gx_0 + fy_0 + cz_0 + n)\nu = 0. \qquad (5)$$

If, now, λ, μ, ν are constants, but x_0, y_0, z_0 are allowed to vary, the line (2) describes a system of parallel lines. The locus of the middle points of the chords on these lines is given by (5). Since (5) is linear in x_0, y_0, z_0, this locus is a plane.

Such a plane is called a **diametral plane**.

$(ax + hy + gz + l)\lambda + (hx + by + fz + m)\mu + (gx + fy + cz + n)\nu$

THEOREM II. *All the diametral planes of a quadric have at least one (finite or infinite) point in common.*

For all values of λ, μ, ν the plane (5) passes through the intersection of the planes

$$\begin{aligned} ax + hy + gz + l &= 0, \\ hx + by + fz + m &= 0, \\ gx + fy + cz + n &= 0. \end{aligned} \qquad (6)$$

In discussing the locus determined by (6), it will be convenient to put, for brevity,

$$D \equiv \begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix}, \quad N \equiv \begin{vmatrix} a & h & l \\ h & b & m \\ g & f & n \end{vmatrix}, \quad M \equiv \begin{vmatrix} a & g & l \\ h & f & m \\ g & c & n \end{vmatrix}, \quad L \equiv \begin{vmatrix} h & g & l \\ b & f & m \\ f & c & n \end{vmatrix}. \qquad (7)$$

If $r^2 = 0$ r is on the surface and two coincident points

If $D \neq 0$, the planes (6) intersect in a single finite point (Art. 26)

$$x = -\frac{L}{D}, \quad y = \frac{M}{D}, \quad z = -\frac{N}{D}. \tag{7}$$

If this point (x_0, y_0, z_0) does not lie on the surface, it is called the **center** of the quadric. It is the middle point of every chord through it. If the point (x_0, y_0, z_0) does lie on the surface, it is called a **vertex** of the quadric. In either case the system of planes (5) is a bundle with vertex at

$$\left(-\frac{L}{D}, \frac{M}{D}, -\frac{N}{D}\right).$$

If $D = 0$, but L, M, N are not all zero, the planes (6) intersect in a single infinitely distant point, the homogeneous coördinates of which are found, by making (6) homogeneous and solving, to be $(L, -M, N, 0)$. The system of planes (5) is a parallel bundle. The quadric is, in this case, said to be non-central.

If the system of planes (6) is of rank two (Art. 35), the planes determine a line; the diametral planes (5) constitute a pencil of planes through the line. If this line is finite and does not lie on the quadric, it is called a **line of centers**; if it is finite and does lie on the quadric, it is called a **line of vertices.** If the system is of rank one, the diametrical planes coincide. If each point of this plane does not lie on the quadric, it is called a **plane of centers**; if every point of the plane lies on the quadric, it is called a **plane of vertices.**

EXAMPLE. Find the center of the quadric

$$x^2 + 4y^2 - z^2 + 4xy + 4yz + 2zx + 2x + 4y - 2z + d = 0.$$

The equations (6) for determining the center are

$$x + 2y + z + 1 = 0, \quad x + 2y + z + 1 = 0, \quad x + 2y - z - 1 = 0,$$

from which $x + 2y = 0$, $z + 1 = 0$. This line is a line of centers unless $d = -1$, in which case it is a line of vertices.

EXERCISES

1. Find the coördinates of the points in which the line $x = 1 + \frac{2}{3}r$, $y = -2 - \frac{2}{3}r$, $z = -1 + \frac{r}{3}$ intersects the quadric $x^2 + 3y^2 - 4z^2 + 4z - 2y - 45 = 0$.

Find which of the following quadrics have centers. Locate the center when it exists.

2. $x^2 - 2y^2 + 6z^2 + 12xz - 11 = 0.$

3. $2x^2 + y^2 - z^2 - 2xz + 4xy + 4yz + 2y - 4z - 4 = 0.$

4. $xy + yz + zx - x + 2y - z - 9 = 0.$

5. $2x^2 + 5y^2 + z^2 - 4xy - 2x - 4y - 8 = 0.$

6. $x^2 - xz - yz - z = 0.$

7. $x^2 + y^2 + z^2 - 2yz + 2xz - 2xy - x + y - z = 0.$

8. $x^2 + 4y^2 + z^2 - 4yz - 2xz + 4xy + 10x + 5y - 7z + 15 = 0.$

9. Show that any plane which passes through the center of a quadric is a diametral plane.

10. Let P_1 and P_2 be two points on an ellipsoid, and let O be its center. Prove that if P_1 is on the diametral plane of the system of chords parallel to OP_2, then P_2 is on the diametral plane of the system of chords parallel to OP_1.

67. Equation of a quadric referred to its center. If a quadric has a center (x_0, y_0, z_0), its equation, referred to its center as origin, may be obtained in the following way:

If we translate the origin to the center by putting

$$x = x' + x_0, \qquad y = y' + y_0, \qquad z = z' + z_0,$$

the equation $F(x, y, z) = 0$ is transformed into

$$ax'^2 + by'^2 + cz'^2 + 2fy'z' + 2gz'x' + 2hx'y' + 2(ax_0 + hy_0 + gz_0 + l)x' + 2(hx_0 + by_0 + fz_0 + m)y' + 2(gx_0 + fy_0 + cz_0 + n)z' + S = 0$$

wherein, as in Eq. (4), $S = F(x_0, y_0, z_0)$.

Since (x_0, y_0, z_0) is the center, it follows from (6) that

$$\begin{aligned} ax_0 + hy_0 + gz_0 + l &= 0, \\ hx_0 + by_0 + fz_0 + m &= 0, \\ gx_0 + fy_0 + cz_0 + n &= 0, \end{aligned} \tag{8}$$

so that the coefficients of x', y', z' are zero, and the equation has the form (after dropping the accents)

$$ax^2 + by^2 + cz^2 + 2gyz + 2fzx + 2hxy + S = 0. \tag{9}$$

The function $S \equiv F(x_0, y_0, z_0)$ may be written in the form

$$S \equiv F(x_0, y_0, z_0) \equiv$$
$$x_0(ax_0 + hy_0 + gz_0 + l) + y_0(hx_0 + by_0 + fz_0 + m) + z_0(gx_0 + fy_0 + cz_0 + n) + lx_0 + my_0 + nz_0 + d.$$

Hence, from (8) we have

$$S = lx_0 + my_0 + nz_0 + d. \tag{10}$$

By eliminating x_0, y_0, z_0 from (8) and (10) we obtain the relation

$$\begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d-S \end{vmatrix} = 0.$$

This equation may be written in the form

$$\begin{vmatrix} a & h & g \\ h & b & f \\ g & f & c \end{vmatrix} S = \begin{vmatrix} a & h & g & l \\ h & b & f & m \\ g & f & c & n \\ l & m & n & d \end{vmatrix}.$$

Denote the right-hand member of this equation by Δ. The coefficient of S is D (Eq. 7). Hence

$$DS = \Delta,$$

or, if $D \neq 0$,

$$S = \frac{\Delta}{D}. \tag{11}$$

If $D \neq 0$ and $\Delta = 0$, it follows from (9) and (11) that the quadric is a cone (Art. 46). The vertex of the quadric is the vertex of the cone.

If $\Delta = 0$ and $S \neq 0$, then $D = 0$. Since (x_0, y_0, z_0) was assumed to be a finite point, it follows that $L = M = N = 0$ so that the surface has a line or plane of centers.

If $\Delta = 0$ and $S = D = 0$, then from (9) the surface is composite. Every point common to the component planes is a vertex.

The determinant Δ is called the **discriminant** of the given quadric. If $\Delta = 0$, the quadric is said to be **singular**. If $\Delta \neq 0$, the quadric is **non-singular**.

68. Principal planes. A diametral plane which is perpendicular to the chords it bisects is called a **principal plane**.

THEOREM. *If the coefficients in the equation of a quadric are real, and if the quadric does not have the plane at infinity as a component, the quadric has at least one real, finite, principal plane.*

The condition that the diametral plane (5)

11

$$(a\lambda + h\mu + g\nu)x + (h\lambda + b\mu + f\nu)y + (g\lambda + f\mu + c\nu)z + l\lambda + m\mu + n\nu = 0$$

is perpendicular to the chords it bisects is (Art. 14)

$$\frac{a\lambda + h\mu + g\nu}{\lambda} = \frac{h\lambda + b\mu + f\nu}{\mu} = \frac{g\lambda + f\mu + c\nu}{\nu}. \tag{12}$$

If we denote the common value of these fractions by k, equations (12) may be replaced by

$$\begin{aligned} (a-k)\lambda + h\mu + g\nu &= 0, \\ h\lambda + (b-k)\mu + f\nu &= 0, \\ g\lambda + f\mu + (c-k)\nu &= 0. \end{aligned} \tag{13}$$

The condition that these equations in λ, μ, ν have a solution other than 0, 0, 0 is

$$\begin{vmatrix} a-k & h & g \\ h & b-k & f \\ g & f & c-k \end{vmatrix} = 0, \tag{14}$$

or, developed and arranged in powers of k,

$$k^3 - (a+b+c)k^2 + (ab+bc+ca-f^2-g^2-h^2)k - D = 0, \tag{15}$$

where D has the same meaning as in (7). This equation is called the **discriminating cubic** of the quadric $F(x, y, z) = 0$.

To each real root, different from zero, of the discriminating cubic corresponds, on account of (13), (12), and (5), a real finite principal plane. Our theorem will consequently be proved if we show that equation (15) has at least one real root different from zero. The proof will be given in the next article.

69. Reality of the roots of the discriminating cubic. We shall first prove the following theorem:

THEOREM I. *The roots of the discriminating cubic are all real.*

Let k_1 be any root of (15) and let λ_0, μ_0, ν_0 (not all zero) be values of λ, μ, ν that satisfy (13) when $k = k_1$. If k is a complex number, λ_0, μ_0, ν_0 may be complex. Let

$$\lambda_0 = \lambda_1 + i\lambda'_1, \quad \mu_0 = \mu_1 + i\mu'_1, \quad \nu_0 = \nu_1 + i\nu'_1,$$

where $i = \sqrt{-1}$ and $\lambda_1, \lambda'_1, \mu_1, \mu'_1, \nu_1, \nu'_1$ are real.

Substitute k_1 and these values of λ_0, μ_0, ν_0 for k, λ, μ, ν in (13), multiply the resulting equations by $\lambda_1 - i\lambda'_1$, $\mu_1 - i\mu'_1$, $\nu_1 - i\nu'_1$, respectively, and add. The result is

$$\begin{aligned}(\lambda_1^2 + \lambda'^2_1 + \mu_1^2 + \mu'^2_1 + \nu_1^2 + \nu'^2_1)k_1 = (\lambda_1^2 + \lambda'^2_1)a + (\mu_1^2 + \mu'^2_1)b \\ + (\nu_1^2 + \nu'^2_1)c + 2(\mu_1\nu_1 + \mu'_1\nu'_1)f + 2(\nu_1\lambda_1 + \nu'_1\lambda'_1)g \\ + 2(\lambda_1\mu_1 + \lambda'_1\mu'_1)h.\end{aligned}$$

The coefficient of k_1 is real and different from zero. The number in the other member of the equation is real. Hence k_1 is real. Since k_1 is any root of (15), the theorem follows.

THEOREM II. *Not all the roots of the discriminating cubic are equal to zero.*

The condition that all the roots of (15) are zero is

$$a + b + c = 0,\ ab + bc + ca - f^2 - g^2 - h^2 = 0,\ D = 0.$$

Square the first member of the first equation, and subtract twice the first member of the second from it. The result is

$$a^2 + b^2 + c^2 + 2f^2 + 2g^2 + 2h^2 = 0.$$

Since these numbers are real, it follows that

$$a = b = c = f = g = h = 0;$$

but if these conditions are satisfied, the equation of the quadric contains no term in the second degree in x, y, z, which is contrary to hypothesis (Art. 65).

70. Simplification of the equation of a quadric. Let the axes be transformed in such a way that a real, finite principal plane of the quadric $F(x, y, z) = 0$ is taken as $x = 0$. Since the surface is now symmetric with respect to $x = 0$ (Art. 68), the coefficients of the terms of first degree in x must all be zero. Hence the equation has the form

$$ax^2 + by^2 + cz^2 + 2fyz + 2my + 2nz + d = 0.$$

Moreover, $a \neq 0$, since otherwise $x = 0$ would not be a principal plane (Art. 69).

Now let the planes $y = 0$, $z = 0$ be rotated about the X-axis through the angle θ defined by $\tan 2\theta = \dfrac{2f}{b-c}$. This rotation reduces the coefficient of yz to zero, and the equation has the form

$$a'x^2 + b'y^2 + c'z^2 + 2\,m'y + 2\,n'z + d' = 0, \tag{16}$$

wherein $a' \neq 0$, but any of the other coefficients may be equal to zero.

71. Classification of quadric surfaces. Since the equation of a quadric can always be reduced to the form (16), a complete classification can be made by considering the possible values of the coefficients.

I. Let both b' and c' be different from zero. By translation of the axes in such a way that $\left(0, \dfrac{-m'}{b'}, \dfrac{-n'}{c'}\right)$ is the new origin, the equation reduces to

$$a'x^2 + b'y^2 + c'z^2 = d''.$$

If $d'' \neq 0$, divide by d'' and put

$$\frac{d''}{a'} = \pm a^2,\quad \frac{d''}{b'} = \pm b^2,\quad \frac{d''}{c'} = \pm c^2,$$

the signs being so chosen that a, b, c are real. This gives the following four types:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1. \quad \text{Ellipsoid.} \qquad \text{(Art. 56)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \quad \text{Hyperboloid one sheet.} \qquad \text{(Art. 57)}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \quad \text{Hyperboloid two sheets.} \qquad \text{(Art. 58)}$$

$$-\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1. \quad \text{Imaginary ellipsoid.} \qquad \text{(Art. 59)}$$

If $d'' = 0$, the reduced forms are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 0. \quad \text{Imaginary cone.} \qquad \text{(Art. 62)}$$

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 0. \quad \text{Real cone.} \qquad \text{(Art. 62)}$$

II. Let $c' = 0$, $b' \neq 0$.

If $n' \neq 0$, by a translation of axes, the equations may be reduced to

$$a'x^2 + b'y^2 + 2\,n'z = 0.$$

This equation takes the form

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\,nz. \quad \text{Elliptic paraboloid.} \qquad \text{(Art. 60)}$$

or

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2\,nz. \quad \text{Hyperbolic paraboloid.} \qquad \text{(Art. 61)}$$

according as a' and b' have the same or opposite signs.

If $n'=0$, the equation may be reduced to

$$a'x^2+b'y^2+d''=0.$$

If $d''\neq 0$, this may be written in the form

$$\frac{x^2}{a^2}\pm\frac{y^2}{b^2}\pm 1=0. \quad \text{Quadric cylinder.} \qquad \text{(Art. 63)}$$

and if $d''=0$, $\quad \dfrac{x^2}{a^2}\pm\dfrac{y^2}{b^2}=0.$ Pair of intersecting planes.

III. Let $b'=c'=0$. Equation (16) is in this case

$$a'x^2+2\,m'y+2\,n'z+d'=0.$$

If m and n are not both zero, since the plane $2\,m'y+2\,n'z+d'=0$ is at right angles to $x=0$, we may rotate and translate the axes so that this plane is the new $y=0$. The equation of the surface becomes

$$x^2=2\,my. \quad \text{Parabolic cylinder.} \qquad \text{(Art. 63)}$$

If m' and n' are both zero, we have,

if $\quad d'\neq 0,\quad x^2+k^2=0.$ Two parallel planes.

if $\quad d'=0,\quad x^2=0.$ One plane counted twice.

72. Invariants under motion. A function of the coefficients of the equation of a surface, the value of which is unchanged when the axes are rotated and translated (Arts. 36 and 37), is called an **invariant under motion** of the given surface. It will be shown that the expressions

$$I\equiv a+b+c,$$

$$J\equiv bc+ca+ab-f^2-g^2-h^2,$$

$$D\equiv\begin{vmatrix} a & h & g\\ h & b & f\\ g & f & c\end{vmatrix},\qquad \Delta\equiv\begin{vmatrix} a & h & g & l\\ h & b & f & m\\ g & f & c & n\\ l & m & n & d\end{vmatrix},$$

formed from the coefficients of the equation (1) of a quadric are invariants under motion.

73. Proof that I, J, and D are invariants. When the axes are translated (Art. 36), the coefficients of the terms in the second degree in the equation of a quadric are unchanged. Hence I, J, and D are unchanged.

Since the equations of rotation (Art. 37) are linear and homogeneous in x, y, z, x', y', z', the degree of any term is not changed by these transformations, so that a term of the first degree cannot be made to be of the second, nor conversely. Suppose the expression

$$f(x, y, z) \equiv ax^2 + 2\,hxy + by^2 + 2\,gxz + 2\,fyz + cz^2$$

is transformed by a rotation into

$$f'(x', y', z') = a'x'^2 + 2\,h'x'y' + b'y'^2 + 2\,g'x'z' + 2\,f'y'z' + c'z'^2.$$

Now consider the function

$$\phi(x, y, z) \equiv f(x, y, z) - k(x^2 + y^2 + z^2).$$

The expression $x^2 + y^2 + z^2$ is the square of the distance of a point (x, y, z) from the origin, and will therefore remain of the same form $x'^2 + y'^2 + z'^2$ by the transformation of rotation (Art. 37).

If, then, $f(x, y, z)$ is changed into $f'(x', y', z')$, $\phi(x, y, z)$ will be changed into

$$\phi'(x', y'\, z') = f'(x', y', z') - k(x'^2 + y'^2 + z'^2).$$

If k has such a value that ϕ is the product of two linear factors in x, y, z, then, for the same value of k, the expression ϕ' will be the product of two linear factors in x', y', z'. The condition that ϕ is the product of two factors is that its discriminant vanishes, that is

$$\begin{vmatrix} a-k & h & g \\ h & b-k & f \\ g & f & c-k \end{vmatrix} = 0,$$

which, developed in powers of k, is exactly the equation of the discriminating cubic (Art. 68)

$$k^3 - Ik^2 + Jk - D = 0.$$

Similarly, the condition that ϕ' is the product of two linear factors is

$$k^3 - I'k^2 + J'k - D' = 0,$$

where I', J', and D' are the expressions I, J, and D formed from the coefficients of $f'(x', y', z')$.

These two equations have the same roots, hence the coefficients of like powers of k must be proportional. But the coefficient of k^3 is unity in each, hence,

$$I' = I, \qquad J' = J, \qquad D' = D,$$

that is, I, J, D are invariants.

From the theorem just proved the following is readily obtained:

THEOREM. *When the axes are transformed in such a way that the coefficients of xy, yz, and zx are all zero, the coefficients of x^2, y^2, and z^2 are the roots of the discriminating cubic.*

For, if the equation of the quadric has been reduced to

$$a'x^2 + b'y^2 + c'z^2 + 2\,l'x + 2\,m'y + 2\,n'z + d' = 0,$$

the discriminating cubic is

$$k^3 - (a' + b' + c')k + (a'b' + b'c' + c'a')k - a'b'c' = 0.$$

The roots of this equation are a', b', and c'. This proves the proposition.

From the theorem just proved, the following criteria immediately follow:

If two roots of the discriminating cubic are equal and different from zero, the quadric is a surface of revolution, and conversely.

If all three roots of the discriminating cubic are equal and different from zero, the quadric is a sphere.

If $\Delta \neq 0$, and a root of the discriminating cubic is zero, the quadric is non-central.

If two roots of the discriminating cubic are equal to zero, the terms of second degree in the equation of the quadric form a perfect square.

74. Proof that Δ is invariant. It will first be proved that Δ is invariant under rotation. The reasoning is similar to that in Art. 73. Let

$$F(x, y, z) = ax^2 + by^2 + cz^2 + 2\,fyz + 2\,gzx + 2\,hxy + 2\,lx + 2\,my + 2\,nz + d = 0$$

be the equation of the given quadric. Let this equation be transformed by a rotation into

$$F(x', y', z') = a'x'^2 + b'y'^2 + c'z'^2 + 2f'y'z' + 2g'z'x' + 2h'x'y' + 2l'x' + 2m'y' + 2n'z' + d' = 0.$$

This rotation transforms the expression

$$\Phi(x, y, z) = F(x, y, z) - k(x^2 + y^2 + z^2 + 1)$$

into $$\Phi'(x', y', z') = F'(x', y', z') - k(x'^2 + y'^2 + z'^2 + 1).$$

The discriminants of Φ and Φ' are, respectively,

$$\begin{vmatrix} a-k & h & g & l \\ h & b-k & f & m \\ g & f & c-k & n \\ l & m & n & d-k \end{vmatrix} \text{ and } \begin{vmatrix} a'-k & h' & g' & l' \\ h' & b'-k & f' & m' \\ g' & f' & c'-k & n' \\ l' & m' & n' & d'-k \end{vmatrix}.$$

The roots of the quartic equations in k obtained by equating these discriminants to zero are equal; since a value of k which makes $\Phi = 0$ singular also makes $\Phi' = 0$ singular and conversely (Art. 67). Hence, since the coefficient of k^4 in each equation is unity, the constant terms are equal; that is, $\Delta = \Delta'$. Hence, Δ is invariant under rotation.

In order to prove that Δ is invariant under translation, let the axes be translated to parallel axes through (x_0, y_0, z_0). The equation of the quadric becomes (cf. Art. 67)

$$\begin{aligned} F'(x', y', z') = {} & ax'^2 + by'^2 + cz'^2 + 2fx'y' + 2gy'z' + 2hz'x' \\ & + 2(ax_0 + hy_0 + gz_0 + l)x' + 2(hx_0 + by_0 + fz_0 + m)y' \\ & + 2(gx_0 + fyc + cz_0 + n)z' + S = 0, \end{aligned}$$

where $S = F(x_0, y_0, z_0)$. The discriminant of $F'(x', y', z')$ is

$$\begin{vmatrix} a & h & g & ax_0+hy_0+gz_0+l \\ h & b & f & hx_0+by_0+fz_0+m \\ g & f & c & gx_0+fy_0+cz_0+n \\ ax_0+hy_0+gz_0+l & hx_0+by_0+fz_0+m & gx_0+fy_0+cz_0+n & S \end{vmatrix}.$$

Multiply the first column by x_0, the second by y_0, the third by z_0, and subtract their sum from the last column. In the resulting determinant, multiply the first row by x_0, the second by y_0, the third by z_0, and subtract their sum from the last row. Finally divide the first row and column each by x_0, the second row and column each by y_0, and the third row and column each by z_0.

The resulting determinant is Δ. Hence $\Delta' = \Delta$, so that Δ is invariant under translation. Since Δ is invariant under both translation and rotation, it is invariant under motion.

75. Discussion of numerical equations. In order to determine the form and position of a quadric with a given numerical equation, it is advisable to determine the standard form (Art. 71) to which the equation of the given quadric may be reduced, and the position in space of the coördinate axes for which the equation has this standard form. For this purpose the roots k_1, k_2, k_3 of the discriminating cubic and the value of the discriminant Δ should first be computed.

A. If all the roots k_1, k_2, k_3 are different from zero, the three principal planes may be determined as in Art. 68. If these planes are taken as coördinate planes, the equation reduces to (Art. 67, Eq. 11; Art. 73)

$$k_1x^2 + k_2y^2 + k_3z^2 + \frac{\Delta}{k_1k_2k_3} = 0.$$

B. If one root k_3 is zero, two finite principal planes may be determined as before. Let these be taken as $x = 0$ and $y = 0$. At least one intersection of the new Z-axis with the surface is at infinity. If this axis does not lie on the surface, and does meet the surface in one finite point, the axes should be translated to this point as origin. The equation of the surface now has the form

$$k_1x^2 + k_2y^2 + 2n''z = 0.$$

Since

$$\Delta = \begin{vmatrix} k_1 & 0 & 0 & 0 \\ 0 & k_2 & 0 & 0 \\ 0 & 0 & 0 & n'' \\ 0 & 0 & n'' & 0 \end{vmatrix},$$

it follows that

$$n'' = \sqrt{\frac{-\Delta}{k_1k_2}}.$$

If the new Z-axis lies on the quadric, or if it has no finite point in common with it, any point on the new Z-axis may be chosen for origin and the equation takes the form

$$k_1x^2 + k_2y^2 + S = 0,$$

where (Art. 67)

$$S = lx_0 + my_0 + nz_0 + d,$$

and (x_0, y_0, z_0) are the old coördinates of the new origin.

C. If two roots of the discriminating cubic are zero, the terms of the second degree in the original equation form a perfect square, so that the equation of the surface, referred to the original axes, is of the form

$$(\alpha x + \beta y + \gamma z)^2 + 2\,lx + 2\,my + 2\,nz + d = 0,$$

or $$(\alpha x + \beta y + \gamma z + \delta)^2 + 2(l - \alpha\delta)x + 2(m - \beta\delta)y + 2(n - \gamma\delta)z + d - \delta^2 = 0. \quad (17)$$

If the planes $$\alpha x + \beta y + \gamma z + \delta = 0,$$ $$2(l - \alpha\delta)x + 2(m - \beta\delta)y + 2(n - \gamma\delta)z + d - \delta^2 = 0$$

are not parallel, we may choose δ so that they are perpendicular. The first term of (17) is proportional to the square of the distance of the point (x, y, z) from the plane

$$\alpha x + \beta y + \gamma z + \delta = 0.$$

The remaining terms of (17) are proportional to the distance to the second plane. If these planes, with the appropriate value of δ, are chosen as $x = 0$, $y = 0$, the equation reduces to

$$(\alpha^2 + \beta^2 + \gamma^2)y^2 + 2\sqrt{(l - \alpha\delta)^2 + (m - \beta\delta)^2 + (n - \gamma\delta)^2}\; x = 0.$$

If the two planes are parallel, δ may be so chosen that

$$l - \alpha\delta = 0, \quad m - \beta\delta = 0, \quad n - \gamma\delta = 0.$$

The equation now becomes

$$(\alpha^2 + \beta^2 + \gamma^2)y^2 + d^2 - \delta^2 = 0,$$

wherein $\alpha x + \beta y + \gamma z + \delta = 0$ is the new $y = 0$.

EXAMPLE 1. Discuss the equation

$$x^2 - 2\,y^2 + 6\,z^2 + 12\,xz - 16\,x - 4\,y - 36\,z + 62 = 0.$$

The equations determining the center are $x + 6\,z - 8 = 0$, $2\,y + 2 = 0$, $6\,x + 6\,z - 18 = 0$, from which the coördinates of the center are $(2, -1, 1)$. The invariants are $I = 5$, $J = -44$, $D = 60$, $\Delta = 1800$. Hence, the discriminating cubic is

$$k^3 - 5\,k^2 - 44\,k - 60 = 0.$$

Its roots are $k_1 = 10$, $k_2 = -2$, $k_3 = -3$. The transformed equation is

$$10\,x^2 - 2\,y^2 - 3\,z^2 + 30 = 0.$$

$-12h + = 0$
$6l - 4h = 1$

The direction cosines of the new axes through $(2, -1, 1)$ are found, as in Art. 68, by giving k the values $10, -2, -3$, to be

$$\frac{2}{\sqrt{13}},\ 0,\ \frac{3}{\sqrt{13}};\ 0,\ 1,\ 0;\ \frac{-3}{\sqrt{13}},\ 0,\ \frac{2}{\sqrt{13}}.$$

The surface is an hyperboloid of one sheet.

EXAMPLE 2. Discuss the quadric

$$11x^2 + 10y^2 + 6z^2 - 8yz + 4zx - 12xy + 72x - 72y + 36z + 150 = 0.$$

The discriminating cubic is

$$k^3 - 27k^2 + 180k - 324 = 0.$$

Its roots are 3, 6, 18. $\Delta = -3888$. The surface is an ellipsoid.
The equations for finding the center are

$$11x - 6y + 2z + 36 = 0, \quad -6x + 10y - 4z - 36 = 0,$$
$$2x - 4y + 6z + 18 = 0.$$

The coördinates of the center are $(-2, 2, -1)$. The direction cosines of the axes are

$$\tfrac{1}{3},\ \tfrac{2}{3},\ \tfrac{2}{3};\quad \tfrac{2}{3},\ \tfrac{1}{3},\ -\tfrac{2}{3};\quad -\tfrac{2}{3},\ \tfrac{2}{3},\ -\tfrac{1}{3}.$$

The equation of the ellipsoid referred to its axes is

$$3x^2 + 6y^2 + 18z^2 = 12.$$

EXAMPLE 3. Discuss the quadric

$$3x^2 - y^2 + 2z^2 + 6yz - 4zx - 2xy - 14x + 4y + 20z + 21 = 0.$$

The discriminating cubic is

$$k^3 - 4k^2 - 13k + 19 = 0.$$

Its roots are approximately $1.2, 5.7, -2.9$. $\Delta = 0$. The surface is a cone.
The equations for finding the vertex are

$$3x - y - 2z - 7 = 0, \quad -x - y + 3z + 2 = 0, \quad -2x + 3x + 2z + 10 = 0.$$

The coördinates of the vertex are $(1, -2, -1)$. The direction cosines of the axes are approximately

$$.8,\ .4,\ .5;\ .6,\ -4,\ -.7;\ 0,\ .6,\ -.4.$$

The equation of the cone referred to its axes is approximately

$$1.2x^2 + 5.7y^2 - 2.9z^2 = 0.$$

EXAMPLE 4. Discuss the quadric

$$4x^2 + y^2 + z^2 - 2yz + 4xz - 4xy - 8x + 4z + 7 = 0.$$

This equation may be written in the form

$$(2x - y + z + \delta)^2 = (8 + 4\delta)x - 2\delta y - (4 - 2\delta)z - 7 + \delta^2.$$

If $\delta = -1$, the planes $2x - y + z - 1 = 0$ and $4x + 2y - 6z - 6 = 0$ are perpendicular. If we take these planes as $y' = 0$ and $x' = 0$, the equation of the surface reduces to $6y^2 = \sqrt{56}\,x$. The surface is a parabolic cylinder.

Start.

EXERCISES

Discuss the quadrics:

1. $3x^2 + 2y^2 + z^2 - 4xy - 4yz + 2 = 0.$

2. $x^2 - y^2 + 2z^2 - 2yz + 4xz + 4xy - 2x - 4y - 1 = 0.$

3. $z^2 + x + y + 1 = 0.$

4. $2x^2 + 5y^2 + 2z^2 + 2yz + 6xy + 2y + 4z + 2 = 0.$

5. $x^2 + y^2 - 2z^2 + 2yz + 2zx + 2xy - 4x - 2y + 2z = 0.$

6. $3x^2 + 2y^2 + 2z^2 + 2yz + 6x + 6y - z + 9 = 0.$

7. $2xy + 6zx + 14x + 1 = 0.$

8. $\sqrt{x} + \sqrt{y} + \sqrt{z} = 0.$

9. $x^2 + y^2 + 9z^2 - 6yz + 6zx - 2xy - x + y - 3z = 0.$

$4y^2 + 4y^2 + z^2 - 4yz - 4xz + 8xy - 6y + 6z - 3 = 0$

10. $x^2 + y^2 + 9z^2 - 6yz + 6xz - 2xy - 4x + 2y + 6z + 5 = 0.$

11. $3x^2 + 2y^2 + 4yz - 2zx - 4x - 8z - 8 = 0.$

12. $x^2 - y^2 - 2z^2 - 4yz + 2xy - 2y + 2z = 0.$

13. $x^2 - 6yz + 3zx + 2xy + x - 13z = 0.$

14. $x^2 - 2y^2 + z^2 - 4zx - 12xy + 4y + 4z - 9 = 0.$

15. $x^2 + 2y^2 + 2z^2 + 2xy - 2x - 4y - 4z = 0.$

16. $3x^2 + y^2 + z^2 + yz - 3zx - 2xy + 2x + 4y + 2z = 0.$

17. For what values of c is the surface

$$5x^2 + 3y^2 + cz^2 + 2xz + 15 = 0$$

a surface of revolution?

18. Determine d in such a way that

$$x^2 + y^2 + 5z^2 + 2yz + 4xz - 4xy + 2x + 2y + d = 0$$

is a cone.

Parabol cyl. new origin $2x + 2y - 2z - 1 =$

$2x - y + 2z - 2 = 0,$ $\lambda = 0$ $\frac{2}{3}\ -\frac{1}{3}\ \frac{2}{3};$

$\frac{2}{3}\ \frac{2}{3}\ \frac{2}{3};$ $-\frac{1}{3}\ \frac{2}{3}\ \frac{2}{3}$ $5y^2 + 4x = 0$

CHAPTER VIII

SOME PROPERTIES OF QUADRIC SURFACES

76. Tangent lines and planes. If the two points of intersection of a line and a quadric coincide at a point P_0, the line is called a **tangent line** and P_0 the point of tangency. If the surface is singular, it is supposed in this definition that P_0 is not a vertex.

THEOREM. *The locus of the lines tangent to the quadric at P_0 is a plane.*

Let the equation of the quadric be

$$F(x, y, z) \equiv ax^2 + by^2 + cz^2 + 2\,fyz + 2\,gzx + 2\,hxy \\ + 2\,lx + 2\,my + 2\,nz + d = 0, \qquad (1)$$

and let the equation of any line through $P_0 \equiv (x_0, y_0, z_0)$ be (Art. 20)

$$x = x_0 + \lambda r, \quad y = y_0 + \mu r, \quad z = z_0 + \nu r. \qquad (2)$$

Since P_0 lies on the quadric, $F(x_0, y_0, z_0) = 0$. Hence, one root of equation (3), Chapter VII, which determines the intersections of the line (2) with the quadric (1), is zero. The condition that a second root is zero is $R = 0$, or

$$\lambda(ax_0 + hy_0 + gz_0 + l) + \mu(hx_0 + by_0 + fz_0 + m) \\ + \nu(gx_0 + fy_0 + cz_0 + n) = 0. \qquad (3)$$

If we substitute in (3) the values of λ, μ, ν from (2), we obtain

$$(x - x_0)(ax_0 + hy_0 + gz_0 + l) + (y - y_0)(hx_0 + by_0 + fz_0 + m) \\ + (z - z_0)(gx_0 + fy_0 + cz_0 + n) = 0, \qquad (4)$$

which must be satisfied by the coördinates of every point of every line tangent to the quadric at P_0. Conversely, if (x, y, z) is any point distinct from P_0, whose coördinates satisfy (4), the line determined by (x, y, z) and P_0 is tangent to the surface at P_0. Since (4) is of the first degree in (x, y, z), it is the equation of a plane. This plane is called the **tangent plane** at P_0.

The equation (4) of the tangent plane may be simplified. Multiply out, transpose the constant terms to the second member, and add $lx_0 + my_0 + nz_0 + d$ to each member of the equation. The second member is $F(x_0, y_0, z_0)$, which is equal to zero, since P_0 lies on the quadric. The equation of the tangent plane thus reduces to the form

$$\begin{aligned} axx_0 + byy_0 + czz_0 + f(yz_0 + zy_0) + g(zx_0 + xz_0) + h(xy_0 + yx_0) \\ + l(x + x_0) + m(y + y_0) + n(z + z_0) + d = 0. \end{aligned} \tag{5}$$

This equation is easily remembered. It may be obtained from the equation of the quadric by replacing x^2, y^2, z^2 by xx_0, yy_0, zz_0; $2yz, 2zx, 2xy$ by $yz_0 + zy_0, zx_0 + xz_0, xy_0 + yx_0$; and $2x, 2y, 2z$ by $x + x_0, y + y_0, z + z_0$, respectively.

77. Normal forms of the equation of the tangent plane. The equation of the tangent plane to the central quadric

$$ax^2 + by^2 + cz^2 = 1 \tag{6}$$

at the point (x_0, y_0, z_0) on it is

$$axx_0 + byy_0 + czz_0 = 1.$$

Let the normal form of the equation of this plane (Art. 13) be

$$\lambda x + \mu y + \nu z = p, \tag{7}$$

so that

$$\frac{\lambda}{p} = ax_0, \quad \frac{\mu}{p} = by_0, \quad \frac{\nu}{p} = cz_0.$$

Since (x_0, y_0, z_0) lies on the quadric, we have

$$ax_0^2 + by_0^2 + cz_0^2 = 1,$$

from which

$$\frac{\lambda^2}{a} + \frac{\mu^2}{b} + \frac{\nu^2}{c} = p^2. \tag{8}$$

Conversely, if this equation is satisfied, the plane (7) is tangent to the quadric (6).

By substituting the value of p from (8) in (7), we have

$$\lambda x + \mu y + \nu z = \sqrt{\frac{\lambda^2}{a} + \frac{\mu^2}{b} + \frac{\nu^2}{c}},$$

which is called the normal form of the equation of the tangent plane to the central quadric (6).

It follows from (8) that the necessary and sufficient condition that the plane

$$ux + vy + wz = 1$$

is tangent to the quadric (5) is that

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{w^2}{c} = 1. \tag{9}$$

This equation is called the **equation of the quadric (6) in plane coördinates.**

Again, if

$$ax^2 + by^2 = 2\,nz \tag{10}$$

is the equation of a paraboloid (Arts. 60 and 61), it is proved in a similar way that the normal form of the equation of the tangent plane to the paraboloid is

$$\lambda x + \mu y + \nu z = -\frac{n}{2\,\nu}\left(\frac{\lambda^2}{a} + \frac{\mu^2}{b}\right) \tag{11}$$

and that the condition that the plane

$$ux + vy + wz = 1$$

is tangent to the paraboloid is

$$\frac{u^2}{a} + \frac{v^2}{b} + \frac{2\,w}{n} = 0. \tag{12}$$

Equation (12) is the **equation of the paraboloid in plane coördinates.**

78. Normal to a quadric. The line through a point P_0 on a quadric, perpendicular to the tangent plane at P_0, is called the **normal** to the surface at P_0.

It follows from equation (4) that the equations of the normal at P_0 to the quadric $F(x, y, z) = 0$ are

$$\frac{x - x_0}{ax_0 + hy_0 + gz_0 + l} = \frac{y - y_0}{hx_0 + by_0 + fz_0 + m} = \frac{z - z_0}{gx_0 + fy_0 + cz_0 + n}. \tag{13}$$

EXERCISES

1. Show that the point $(1, -2, 1)$ lies on the quadric $x^2 - y^2 + z^2 + 4\,yz + 2\,zx + xy - x + y + z + 12 = 0$. Write the equations of the tangent plane and the normal line at this point.

2. Show that the equation of the tangent plane to a sphere, as derived in Art. 76, agrees with the equation obtained in Art. 50.

3. Prove that the normals to a central quadric $ax^2 + by^2 + cz^2 = 1$, at all points on it, in a plane parallel to a principal plane, meet two fixed lines, one in each of the other two principal planes.

4. Prove that, if all the normals to the central quadric $ax^2 + by^2 + cz^2 = 1$ intersect the X-axis, the quadric is a surface of revolution about the X-axis.

5. Prove that the tangent plane at any point of the quadric cone $ax^2 + by^2 + cz^2 = 0$ passes through the vertex.

6. Prove that the locus of the point of intersection of three mutually perpendicular tangent planes to the central quadric $ax^2 + by^2 + cz^2 = 1$ is the concentric sphere $x^2 + y^2 + z^2 = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. This sphere is called the **director sphere** of the given central quadric.

7. Prove that through any point in space pass six normals to a given central quadric, and four normals to a given paraboloid.

79. Rectilinear generators. The equation of the hyperboloid of one sheet

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1$$

may be written in the form

$$\left(\frac{x}{a} + \frac{z}{c}\right)\left(\frac{x}{a} - \frac{z}{c}\right) = \left(1 + \frac{y}{b}\right)\left(1 - \frac{y}{b}\right),$$

or

$$\frac{\frac{x}{a} + \frac{z}{c}}{1 + \frac{y}{b}} = \frac{1 - \frac{y}{b}}{\frac{x}{a} - \frac{z}{c}}, \tag{14}$$

or also

$$\frac{\frac{x}{a} + \frac{z}{c}}{1 - \frac{y}{b}} = \frac{1 + \frac{y}{b}}{\frac{x}{a} - \frac{z}{c}}. \tag{15}$$

Let the value of each member in (14) be denoted by ξ, so that by clearing of fractions we have

$$\frac{x}{a} + \frac{z}{c} = \xi\left(1 + \frac{y}{b}\right), \quad \left(1 - \frac{y}{b}\right) = \xi\left(\frac{x}{a} - \frac{z}{c}\right). \tag{16}$$

For each value of ξ, these equations define a line. Every point on such a line lies on the surface, since its coördinates satisfy

(14). Moreover, through each point of the surface passes a line of the system (16) since the coördinates of each point on the surface satisfy (14) and consequently satisfy (16). The system of lines (16), in which ξ is the parameter, is called a **regulus** of lines on the hyperboloid. Any line of the regulus is called a **generator.**

Similarly, by equating each member of (15) to η, we obtain the system of lines whose equations are

$$\frac{x}{a}+\frac{z}{c}=\eta\left(1-\frac{y}{b}\right),\quad 1+\frac{y}{b}=\eta\left(\frac{x}{a}-\frac{z}{c}\right),$$

in which η is the parameter. This system of lines constitutes a second regulus lying on the surface. The two reguli will be called the ξ regulus and the η regulus, respectively. Through every point P of the surface passes one, and but one, generator belonging to each regulus. Moreover, any plane that contains a generator of one regulus contains a generator of the other regulus also. The equation of any plane through a generator of the ξ regulus, for example, may be written in the form (Art. 24)

$$\frac{x}{a}+\frac{z}{c}-\xi\left(1+\frac{y}{b}\right)=\eta\left[\left(1-\frac{y}{b}\right)-\xi\left(\frac{x}{a}-\frac{z}{c}\right)\right].$$

Since this equation may also be written in the form

$$\frac{x}{a}+\frac{z}{c}-\eta\left(1-\frac{y}{b}\right)=\xi\left[\left(1+\frac{y}{b}\right)-\eta\left(\frac{x}{a}-\frac{z}{c}\right)\right],$$

it follows that this plane also passes through a generator of the η regulus. Every such plane is tangent to the surface at the point of intersection P of the generators in it, since every line in the plane through P has its two intersections with the surface coincident at P.

EXAMPLE. The equations of the reguli on the hyperboloid

$$\frac{x^2}{4}+\frac{y^2}{9}-z^2=1$$

are
$$\frac{x}{2}+z=\xi\left(1+\frac{y}{3}\right),\quad 1-\frac{y}{3}=\xi\left(\frac{x}{2}-z\right);$$

and
$$\frac{x}{2}+z=\eta\left(1-\frac{y}{3}\right),\quad 1+\frac{y}{3}=\eta\left(\frac{x}{2}-z\right).$$

The point (2, 6, 2) lies on the surface. The values of ξ and η which determine the generators through this point are $\xi = 1$, $\eta = -3$. Hence, the equations of these generators are

$$\frac{x}{2} + z = 1 + \frac{y}{3},\ 1 - \frac{y}{3} = \frac{x}{2} - z, \text{ and } \frac{x}{2} + z = -3\left(1 - \frac{y}{3}\right),\ 1 + \frac{y}{3} = -3\left(\frac{x}{2} - z\right).$$

The equation of the plane determined by these lines is

$$3x + 4y - 12z - 6 = 0.$$

This is the equation of the tangent plane at (2, 6, 2) (Art. 76).

It is similarly seen that the equation

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 2nz$$

of the hyperbolic paraboloid may also be written in the forms

$$\frac{\frac{x}{a} + \frac{y}{b}}{2nz} = \frac{1}{\frac{x}{a} - \frac{y}{b}} = \xi,$$

and

$$\frac{\frac{x}{a} - \frac{y}{b}}{2nz} = \frac{1}{\frac{x}{a} + \frac{y}{b}} = \eta.$$

Hence, on this surface also, there is a ξ regulus and an η regulus The generators of the ξ regulus are parallel to the fixed plane $\frac{x}{a} - \frac{y}{b} = 0$; those of the η regulus, to the fixed plane $\frac{x}{a} + \frac{y}{b} = 0$. By writing the above equations in homogeneous coördinates, it is seen that the line $\frac{x}{a} + \frac{y}{b} = 0$, $t = 0$ in the plane at infinity belongs to the ξ regulus; and the line $\frac{x}{a} - \frac{y}{b} = 0$, $t = 0$ to the η regulus. Hence the plane at infinity is tangent to the paraboloid.

The hyperboloid of one sheet and the hyperbolic paraboloid are sometimes called ruled quadrics, since the reguli on them are real. It will be shown (Art. 115), that on every non-singular quadric there are two reguli; but, on all the quadrics except these two, the reguli are imaginary.

80. Asymptotic cone. The cone whose vertex is the center of a given central quadric, and which contains the curve in which the quadric intersects the plane at infinity, is called the **asymptotic cone** of the given quadric.

If the equation of the quadric is

$$ax^2 + by^2 + cz^2 = t^2,$$

the equation of its asymptotic cone is

$$ax^2 + by^2 + cz^2 = 0.$$

For, this equation is the equation of a cone with vertex at the center $(0, 0, 0, 1)$ of the given quadric (Art. 46). Its curve of intersection with the plane at infinity coincides with the curve of intersection

$$ax^2 + by^2 + cz^2 = t^2,\ t = 0,$$

of the given surface with that plane.

EXERCISES

1. Show that the quadric $xy = z$ is ruled. Find the equations of its generators.

2. Show that $x^2 - 2z^2 + 5y - x + 8z = 0$ is a ruled quadric.

3. Prove that, for all values of k, the line $x + 1 = ky = -(k + 1)z$ lies on the surface $yz + zx + xy + y + z = 0$.

4. Prove that $(y + mz)(x + nz) = z$ represents an hyperbolic paraboloid which contains the X-axis and the Y-axis.

5. Show that every generator of the asymptotic cone of a central quadric is tangent to the surface at infinity. From this property derive a definition of an asymptotic cone.

6. Show that every generator of the asymptotic cone of an hyperboloid of one sheet is parallel to a generator of each regulus on the surface.

81. Plane sections of quadrics.

THEOREM I. *The section of a quadric by a finite plane, which is not a component of the surface, is a conic.*

For, let π be any given finite plane, and let the axes be chosen so that the equation of this plane is $z = 0$. Let the equation of the quadric, referred to this system of axes, be

$$ax^2 + by^2 + cz^2 + 2fyz + 2gzx + 2hxy + 2lx + 2my + 2nz + d = 0. \quad (17)$$

If, when $z = 0$, (17) vanishes identically, the given quadric is composite and $z = 0$ is one component; otherwise, the locus defined in the XY-plane by putting $z = 0$ in (17) is a conic.

THEOREM II. *The sections of a quadric by a system of parallel planes are similar conics and similarly placed.*

Let the axes be chosen so that the equations of the given system of parallel planes is $z = k$, and let (17) be the equation of the given quadric. The equation of the projecting cylinder of the section by the plane $z = k$ is

$$ax^2 + 2\,hxy + by^2 + 2\,(l + gk)x + 2\,(m + fk)y + ck^2 + 2\,nk + d = 0.$$

The curves in which these cylinders intersect $z = 0$, and consequently (Art. 45) the curves of which they are the projections, are similar and similarly placed, since the coefficients of x^2, xy, and y^2 in the above equation are independent of k.*

The equations of the section of the surface by the plane at infinity are found by making (17) homogeneous in x, y, z, t and putting $t = 0$. They are

$$ax^2 + by^3 + cz^2 + 2\,fyz + 2\,gzx + 2\,hxy = 0,\ t = 0.$$

The locus of these equations is called the infinitely distant conic of the quadric. This conic consists of two lines if the first member of the first equation is the product of two linear factors. The condition for factorability is

$$D = 0.$$

EXERCISES

1. Find the semi-axes of the ellipse in which the plane $z = 1$ intersects the quadric $x^2 + 4\,y^2 - 3\,z^2 + 4\,yz - 2\,x - 4\,y = 1$.

2. Show that the planes $z = k$ intersect the quadric $2\,x^2 - y^2 + 3\,z^2 + 4\,xz - 2\,yz + 4\,x + 2\,y = 0$ in hyperbolas. Find the equations of the locus of the centers of these hyperbolas.

3. Show that the curve of intersection of the sphere $x^2 + y^2 + z^2 = r^2$ and the ellipsoid $\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1$ lies on the cone

$$\left(\frac{1}{a^2} - \frac{1}{r^2}\right)x^2 + \left(\frac{1}{b^2} - \frac{1}{r^2}\right)y^2 + \left(\frac{1}{c^2} - \frac{1}{r^2}\right)z^2 = 0.$$

Find the values of r for which this cone is composite. Show that each component of the composite cones intersects the ellipsoid in a circle.

* Cf. Salmon, "Conic Sections," 6th edition, p. 222.

82. Circular sections. We shall prove the following theorem :

THEOREM I. *Through each real, finite point in space pass six planes which intersect a given non-composite, non-spherical quadric in circles. If this quadric is not a surface of revolution nor a parabolic cylinder, these six planes are distinct; two are real and four are imaginary. If the quadric is a surface of revolution or a parabolic cylinder, four of the planes are real and coincident and two are imaginary.*

Two proofs will be given, based on different principles.

Proof I. Since parallel sections of a quadric are similar, it will suffice if we prove this theorem for planes through the origin. The planes through any other point, parallel to the planes of the circular sections through the origin, also intersect the quadric in circles.

Let the axes be chosen in such a way that the equation of the quadric is (Art. 70)

$$k_1x^2 + k_2y^2 + k_3z^2 + 2\,lx + 2\,my + 2\,nz + d = 0, \tag{18}$$

where k_1, k_2, k_3 are the roots of the discriminating cubic (Art. 73). The condition that a plane intersects this quadric in a circle is that its conics of intersection with the given quadric and with a sphere coincide.

The curve of intersection of the quadric (18) with the sphere

$$k\,(x^2 + y^2 + z^2) + 2\,lx + 2\,my + 2\,nz + d = 0 \tag{19}$$

coincides with the intersection of either of these surfaces with the cone

$$(k_1 - k)\,x^2 + (k_2 - k)\,y^2 + (k_3 - k)\,z^2 = 0.$$

This cone is composite if the first member of its equation is factorable, that is, if k is equal to k_1, k_2, or k_3.

It follows that each of the six planes

$$\sqrt{k_1 - k_3}\,x = \pm\sqrt{k_3 - k_2}\,y$$
$$\sqrt{k_1 - k_2}\,x = \pm\sqrt{k_2 - k_3}\,z$$
$$\sqrt{k_2 - k_1}\,y = \pm\sqrt{k_1 - k_3}\,z$$

intersects the quadric (18) in a conic which lies on the sphere (19) and is consequently a circle.

If $k_1 > k_2 > k_3$, the six planes are distinct. The planes

$$\sqrt{k_1 - k_2}\, x = \pm \sqrt{k_2 - k_3}\, z$$

are real. The others are imaginary.

If $k_1 = k_2 \gtrless k_3$, the last four planes coincide with $z = 0$. The other two are imaginary. If $k_1 = k_2 \neq 0$, the quadric (18) is a surface of revolution (Art. 73). If $k_1 = k_2 = 0$, it is a parabolic cylinder (Art. 75).

If the equation of the surface is in the form (17), and k_1, k_2, k_3 are the roots of its discriminating cubic, it follows from the discussion in Article 73, that the equations of the planes of the circular sections through the origin are

$$ax^2 + by^2 + cz^2 + 2\,fyz + 2\,gzx + 2\,hxy - k_1(x^2 + y^2 + z^2) = 0,$$
$$ax^2 + by^2 + cz^2 + 2\,fyz + 2\,gzx + 2\,hxy - k_2(x^2 + y^2 + z^2) = 0,$$
$$ax^2 + by^2 + cz^2 + 2\,fyz + 2\,gzx + 2\,hxy - k_3(x^2 + y^2 + z^2) = 0.$$

Proof II. It was shown (Art. 49) that a plane section of a quadric is a circle if it passes through the circular points of its plane. The conic in which the quadric meets the plane at infinity has four points of intersection with the absolute. Any plane other than the plane at infinity which passes through two of these points will meet the quadric in a conic through the circular points of the plane; hence the section is a circle.

The coördinates of the points of intersection may be found by making the equations

$$ax^2 + by^2 + cz^2 + 2\,fyz + 2\,gzx + 2\,hxy = 0, \quad x^2 + y^2 + z^2 = 0$$

simultaneous. Since both equations have real coefficients and the second is satisfied by no real values of the variables, it follows that the four points P_1, P_2, P_3, P_4 consist of two pairs of conjugate imaginary points, or of one pair counted twice.

In the first case, let P_1, P_2 be one pair of conjugate points, and P_3, P_4 the other. The lines P_1P_2, P_3P_4 are real (Art. 41), while the lines P_1P_3, P_2P_4, P_1P_4, P_2P_3 are imaginary. The pairs of lines P_1P_2, P_3P_4; P_1P_3, P_2P_4; P_1P_4, P_2P_3 constitute composite conics passing through all four of the points P_1, P_2, P_3, P_4.

In the second case, let $P_2 \equiv P_4$ and $P_1 \equiv P_3$. The lines P_1P_2 and P_3P_4 coincide, and the lines P_1P_3, P_2P_4 are tangents to both curves, which have double contact with each other at these points.

In either case the equations of the lines P_iP_k can be found as follows. Through the points of intersection of (17) and the absolute passes a system of conics

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy-k(x^2+y^2+z^2)=0,\ t=0. \quad (19')$$

A conic of this system will consist of two straight lines through the four points of intersection if its equation is factorable, that is,

$$\begin{vmatrix} a-k & h & g \\ h & b-k & f \\ g & f & c-k \end{vmatrix}=0;$$

thus k must be a root of the discriminating cubic (Art. 73). Let k_1, k_2, k_3 be the roots of this equation. The equations of the pairs of lines are then

$$ax^2+by^2+cz^2+2fyz+2gzx+2hxy-k_1(x^2+y^2+z^2)=0,\ t=0, \quad (20)$$

with similar expressions for k_2 and k_3. From Art. 41 it follows that for one of the roots k_i the two factors of the first member of the quadratic equation (20) are real, but the factors for each of the others are imaginary when the roots k_i are all distinct.

If u, v are the two linear factors of (20), then the line $u=0$, $t=0$ will pass through one pair of points and $v=0$, $t=0$ will pass through the other. A plane of the pencil $u+pt=0$ will cut the quadric in a circle. Since a plane is determined by a line and a point not on the line, the theorem follows.

In case two roots of the discriminating cubic are equal and different from zero, the quadric is one of revolution; the two conics in the plane at infinity now have double contact.

If $k_1>k_2>k_3$, the planes determined by the second root are real.

83. Real circles on types of quadrics. The above results will now be applied to the consideration of the real planes of circular section for the standard forms of the equation of the quadric (Chap. VI).

(a) For the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

the roots of the discriminating cubic are $1/a^2$, $1/b^2$, $1/c^2$.

Let $a > b > c > 0$. Since parallel sections of the surface are similar, it follows that the equations of the real planes of circular section are

$$c\sqrt{a^2-b^2}\,x \pm a\sqrt{b^2-c^2}\,z + d = 0, \tag{21}$$

where d is a real parameter.

The circle in which a plane (21) intersects the ellipsoid is real if the plane intersects the ellipsoid in real points, that is, if it is not more distant from the center than the tangent planes parallel to it. The condition for this is (Arts. 76 and 16) $|d| \leqq ac\sqrt{a^2-c^2}$. If $|d| > ac\sqrt{a^2-c^2}$, the circles are imaginary.

If $|d| = ac\sqrt{a^2-c^2}$, the circles are point circles. The four planes determined by these two values of d are the tangent planes to the ellipsoid at the points

$$\left(a\sqrt{\frac{a^2-b^2}{a^2-c^2}},\ 0,\ \pm c\sqrt{\frac{b^2-c^2}{a^2-c^2}}\right).$$

Each of these points is called an **umbilic.**

The two systems of planes (21) are also the real planes of circular section of the imaginary cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=0,$$

and of the imaginary ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1.$$

(*b*) The equations of the real planes of circular section of the hyperboloids of one and two sheets

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=\pm 1,$$

and of the real cone

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0,$$

where $a > b > 0$, are found to be

$$c\sqrt{a^2-b^2}\,y \pm b\sqrt{a^2+c^2}\,z + d = 0.$$

On the hyperboloid of one sheet and the real cone, the radii of the circles are real for all values of d. On the hyperboloid of two sheets, the circles are real only if $|d| \geq bc\sqrt{b^2+c^2}$. The coördinates of the umbilics on the hyperboloid of two sheets are

$$\left(0,\ \pm b\sqrt{\frac{a^2-b^2}{b^2+c^2}},\ c\sqrt{\frac{a^2+c^2}{b^2+c^2}}\right).$$

(c) The real planes of circular section of the elliptic paraboloid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=2\,nz,\ a>b>0,\ n>0$$

and the real or imaginary elliptic cylinders

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=\pm 1,\ a>b>0$$

are determined by

$$\pm\sqrt{a^2-b^2}\,y+bz+d=0.$$

On the real elliptic cylinder, the circles are real, and on the imaginary cylinder they are imaginary, for all values of d. On the elliptic paraboloid, the circles are real if $d \leq \frac{bn}{2}(a^2-b^2)$. The coördinates of the umbilics on the elliptic paraboloid are

$$\left[0,\ \pm bn\sqrt{a^2-b^2},\ \frac{n}{2}(a^2-b^2)\right].$$

(d) For the hyperbolic paraboloid

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=2\,nzt$$

and the hyperbolic cylinder

$$\frac{x^2}{a^2}-\frac{y^2}{b^2}=t^2$$

the equations of the planes of the circular sections are

$$bx \pm ay + dt = 0.$$

The circles in these planes are all composite. For, the planes

$$bx+ay+dt=0$$

intersect these surfaces in the fixed infinitely distant line

$$bx + ay = 0, \quad t = 0$$

and in a rectilinear generator which varies with d. Similarly, the planes

$$bx - ay + dt = 0$$

intersect them in the line

$$bx - ay = 0, \quad t = 0$$

and in a variable generator.

Also on the hyperbolic cylinder

$$x^2 = 2\,myt$$

the real circles are all composite, since the planes $x = dt$ intersect the surface in the fixed line $x = t = 0$, and in a variable generator.

We have, therefore, the following theorem:

THEOREM II. *On the hyperbolic paraboloid, the hyperbolic cylinder, and the parabolic cylinder, the real circular sections are composite. The components of each circle are an infinitely distant line and a rectilinear generator which intersects it.*

EXERCISES

1. Find the equations of the real circular sections of the surface $4\,x^2 + 2\,y^2 + z^2 + 3\,yz + xz = 1$.

2. Find the equations of the real circular sections of the surface $2\,x^2 + 5\,y^2 + 3\,z^2 + 4\,xy = 1$.

3. Find the radius of a circular section through the origin in Ex. 2.

4. Find the equations of the real planes through $(1, -3, 2)$ which intersect the ellipsoid $2\,x^2 + y^2 + 4\,z^2 = 1$ in circles.

5. Find the conditions which must be satisfied by the coefficients of the equation $F(x, y, z) = 0$ of a quadric if the planes $z = k$ intersect it in circles.

6. Show that the centers of the circles in Ex. 5 lie on a line. Find the equations of this line.

7. Find the second system of real planes cutting circles from the quadric in Ex. 5.

8. Find the conditions which must be satisfied by the coefficients if the plane $Ax + By + Cz + D = 0$ intersects the quadric $F(x, y, z) = 0$ in circles.

9. Find the coördinates of the center and the radius of the circle in which the plane $x = 2\,z + 5$ intersects the cone $3\,x^2 + 2\,y^2 - 2\,z^2 = 0$.

10. Show that, for all values of λ, the equation of the planes of the circular sections of the quadrics

$$(a+\lambda)x^2+(b+\lambda)y^2+(c+\lambda)z^2=1$$

are the same. The quadrics of this system are said to be concyclic.

84. Confocal quadrics. The system of surfaces represented by the equation

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=1,\ a>b>c>0, \tag{22}$$

in which k is a parameter, is called a system of **confocal quadrics.** The sections of the quadrics of the system by the principal planes $x=0$, $y=0$, $z=0$ are confocal conics.

If $k>-c^2$, the surface (22) is an ellipsoid; if $-c^2>k>-b^2$, the surface is an hyperboloid of one sheet; if $-b^2>k>-a^2$, the surface is an hyperboloid of two sheets; if $-a^2>k$, the surface is an imaginary ellipsoid.

If $k>-c^2$, but approaches $-c^2$ as a limit, the minor axis of the ellipsoid approaches zero as a limit, and the ellipsoid approaches as a limit the part of the XY-plane within the ellipse

$$\frac{x^2}{a^2-c^2}+\frac{y^2}{b^2-c^2}=1. \tag{23}$$

If $-c^2>k>-b^2$, the surface is an hyperboloid of one sheet. As k approaches $-c^2$, the surface approaches the part of the XY-plane exterior to the ellipse (23). As k approaches $-b^2$, the surface approaches that part of the XZ-plane which contains the origin and is bounded by the hyperbola

$$\frac{x^2}{a^2-b^2}-\frac{z^2}{b^2-c^2}=1. \tag{24}$$

If $-b^2>k>-a^2$, the surface is an hyperboloid of two sheets. As k approaches $-b^2$, the hyperboloid approaches that part of the plane $y=0$ which does not contain the origin. As k approaches $-a^2$, the real part of the surface approaches the plane $x=0$, counted twice.

The ellipse (23) in the XY-plane and the hyperbola (24) in the XZ-plane are called the focal conics of the system (22).

The vertices of the focal ellipse are

$$(\pm\sqrt{a^2-c^2},\ 0,\ 0).$$

The foci are

$$(\pm\sqrt{a^2-b^2},\ 0,\ 0).$$

On the focal hyperbola the vertices are $(\pm\sqrt{a^2-b^2}, 0, 0)$ and the foci are $(\pm\sqrt{a^2-c^2}, 0, 0)$. Hence, on the focal conics, the vertices of each are the foci of the other.

85. Confocal quadrics through a point. Elliptic coördinates.

THEOREM I. *Three confocal quadrics pass through every point P in space. If P is real, one of these quadrics is an ellipsoid, one an hyperboloid of one sheet, and the third an hyperboloid of two sheets.*

If $P\equiv(x_1, y_1, z_1)$ lies on a quadric of the system (22), the parameter k satisfies the equation

$$(k+a^2)(k+b^2)(k+c^2)-x_1^2(k+b^2)(k+c^2)-y_1^2(k+c^2)(k+a^2) \\ -z_1^2(k+a^2)(k+b^2)=0. \qquad (25)$$

Since this is a cubic equation in k, and each of its roots determines a quadric of the system through P, there are three quadrics of the system (22) which pass through P.

Let P be real.

If $k \doteq +\infty$, the first member of (25) becomes positive.
If $k=-c^2$, it is $-z_1^2(-c^2+a^2)(-c^2+b^2)$, which is negative.
If $k=-b^2$, it is $-y_1^2(-b^2+c^2)(-b^2+a^2)$, which is positive.
If $k=-a^2$, it is $-x_1^2(-a^2+b^2)(-a^2+c^2)$, which is negative.

Hence the roots of (25) are real. One is greater than $-c^2$, one lies between $-c^2$ and $-b^2$, and the third between $-b^2$ and $-a^2$.

Denote these roots by k_1, k_2, k_3. Hence, we have

$$k_1>-c^2>k_2>-b^2>k_3>-a^2.$$

Then, of the three quadrics

$$\begin{aligned}\frac{x^2}{a^2+k_1}+\frac{y^2}{b^2+k_1}+\frac{z^2}{c^2+k_1}&=1,\\ \frac{x^2}{a^2+k_2}+\frac{y^2}{b^2+k_2}+\frac{z^2}{c^2+k_2}&=1,\\ \frac{x^2}{a^2+k_3}+\frac{y^2}{b^2+k_3}+\frac{z^2}{c^2+k_3}&=1\end{aligned} \qquad (26)$$

which pass through P, the first is an ellipsoid, the second an hyberboloid of one sheet, and the third an hyperboloid of two sheets.

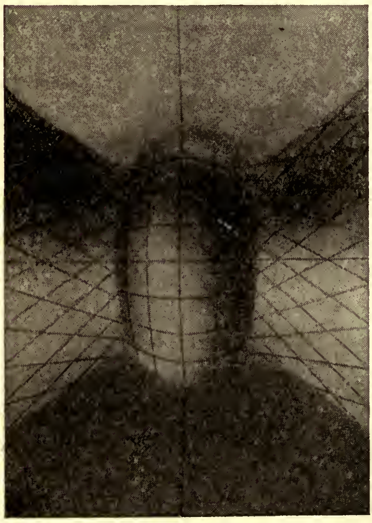
THEOREM II. *The three quadrics of a confocal system which pass through P intersect each other at right angles.*

For, the equations of the tangent planes to the first two quadrics (26) are

$$\frac{x_1x}{a^2+k_1}+\frac{y_1y}{b^2+k_1}+\frac{z_1z}{c^2+k_1}=1,$$
$$\frac{x_1x}{a^2+k_2}+\frac{y_1y}{b^2+k_2}+\frac{z_1z}{c^2+k_2}=1.$$

These planes are at right angles if

$$\frac{x_1^2}{(a^2+k_1)(a^2+k_2)}+\frac{y_1^2}{(b^2+k_1)(b^2+k_2)}+\frac{z_1^2}{(c^2+k_1)(c^2+k_2)}=0.$$



That this condition is satisfied is seen by substituting the coördinates of P in (26), subtracting the second equation from the first, and removing the factor k_2-k_1, which was seen to be different from zero. The proof for the other pairs may be obtained in the same way.

The three roots k_1, k_2, k_3 of equation (25) are called the **elliptic coördinates** of the point P. To find the expressions for the rectangular coördinates of P in terms of the elliptic coördinates, we substitute the coördinates (x_1, y_1, z_1) of P in (26) and solve for x_1^2, y_1^2, z_1^2. The result is

$$x_1^2=\frac{(a^2+k_1)(a^2+k_2)(a^2+k_3)}{(a^2-b^2)(a^2-c^2)},$$
$$y_1^2=\frac{(b^2+k_1)(b^2+k_2)(b^2+k_3)}{(b^2-a^2)(b^2-c^2)}, \tag{27}$$
$$z_1^2=\frac{(c^2+k_1)(c^2+k_2)(c^2+k_3)}{(c^2-a^2)(c^2-b^2)}.$$

It is seen at once from these equations that k_1, k_2, and k_3 are the elliptical coördinates, not only of P, but also of the points symmetric with P as to the coördinate planes, axes, and origin.

86. Confocal quadrics tangent to a line.

THEOREM I. *Any line touches two quadrics of a confocal system.*

The points of intersection of a given line with a quadric of the system (22) are determined by the equation (Art. 65)

$$\left(\frac{\lambda^2}{a^2+k}+\frac{\mu^2}{b^2+k}+\frac{\nu^2}{c^2+k}\right)r^2+2\left(\frac{x_0\lambda}{a^2+k}+\frac{y_0\mu}{b^2+k}+\frac{z_0\nu}{c^2+k}\right)r \\ +\left(\frac{x_0^{\;2}}{a^2+k}+\frac{y_0^{\;2}}{b^2+k}+\frac{z_0^{\;2}}{c^2+k}-1\right)=0.$$

The condition that this line is tangent is

$$\left(\frac{x_0\lambda}{a^2+k}+\frac{y_0\mu}{b^2+k}+\frac{z_0\nu}{c^2+k}\right)^2 \\ -\left(\frac{\lambda^2}{a^2+k}+\frac{\mu^2}{b^2+k}+\frac{\nu^2}{c^2+k}\right)\left(\frac{x_0^{\;2}}{a^2+k}+\frac{y_0^{\;2}}{b^2+k}+\frac{z_0^{\;2}}{c^2+k}-1\right)=0.$$

When expanded and simplified, this equation reduces to

$$k^2+[(b^2+c^2)\lambda^2+(c^2+a^2)\mu^2+(a^2+b^2)\nu^2-(x_0\mu-y_0\lambda)^2 \\ -(y_0\nu-z_0\mu)^2-(z_0\lambda-x_0\nu)^2]\,k+[b^2c^2\lambda^2+c^2a^2\mu^2+a^2b^2\nu^2 \\ -(x_0\mu-y_0\lambda)c^2-(y_0\nu-z_0\mu)b^2-(z_0\lambda-x_0\nu)a^2]=0.$$

Since this equation is quadratic in k, the theorem follows.

THEOREM II. *If two confocal quadrics touch a line, the tangent planes at the points of contact are at right angles.*

Let k_1 and k_2 be the parameters of the quadrics, and let $P'\equiv(x', y', z')$, $P''\equiv(x'', y'', z'')$ be the points of tangency of the line with the given quadrics. The equations of the tangent planes at P' and P'' are (Art. 76), respectively,

$$\frac{x'x}{a^2+k_1}+\frac{y'y}{b^2+k_1}+\frac{z'z}{c^2+k_1}=1, \quad \frac{x''x}{a^2+k_2}+\frac{y''y}{b^2+k_2}+\frac{z''z}{c^2+k_2}=1.$$

These planes are at right angles, if

$$\frac{x'x''}{(a^2+k_1)(a^2+k_2)}+\frac{y'y''}{(b^2+k_1)(b^2+k_2)}+\frac{z'z''}{(c^2+k_1)(c^2+k_2)}=0. \qquad (28)$$

Since the line through P' and P'' is tangent to both quadrics, it lies in the tangent planes at both points. Hence P' and P'' lie in both planes, so that

$$\frac{x'x''}{a^2+k_1}+\frac{y'y''}{b^2+k_1}+\frac{z'z''}{c^2+k_1}=1, \quad \frac{x'x''}{a^2+k_2}+\frac{y'y''}{b^2+k_2}+\frac{z'z''}{c^2+k_2}=1.$$

By subtracting one of these equations from the other, it is seen that (28) is satisfied. The planes are therefore at right angles.

87. Confocal quadrics in plane coördinates. The equation of the system (22) in homogeneous plane coördinates (Art. 77) is

$$a^2u^2+b^2v^2+c^2w^2-s^2+k(u^2+v^2+w^2)=0.$$

Since this equation is of the first degree in k, we have the following theorem:

THEOREM. *An arbitrary plane* (u_1, v_1, w_1, s_1) *is tangent to one and only one quadric of a confocal system.*

The (imaginary) planes whose homogeneous coördinates satisfy the two equations

$$a^2u^2+b^2v^2+c^2w^2-s^2=0, \quad u^2+v^2+w^2=0$$

are exceptional. They touch all the quadrics of the system. Hence, all the quadrics of a confocal system touch all the planes common to the quadric $k=0$ and the absolute.

EXERCISES

1. Prove that the difference of the squares of the perpendicular from the center on two parallel tangent planes to two given confocal quadrics is constant. This may be used as a definition of confocal quadrics.

2. Prove that the locus of the point of intersection of three mutually perpendicular planes, each of which touches one of three given confocal quadrics, is a sphere.

3. Write the equation of a quadric of the system (22) in elliptic coördinates. Derive from (27) a set of parametric equations of this quadric, using elliptic coördinates as parameters.

4. Discuss the system of confocal paraboloids

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}=2\,nz+kn^2.$$

5. Discuss the confocal cones

$$\frac{x^2}{a^2+k}+\frac{y^2}{b^2+k}+\frac{z^2}{c^2+k}=0.$$

CHAPTER IX

TETRAHEDRAL COÖRDINATES

88. Definition of tetrahedral coördinates. It was pointed out in Art. 34 that the four planes $x=0$, $y=0$, $z=0$, and $t=0$, which do not all pass through a point, may be considered as forming a tetrahedron which was called the coördinate tetrahedron. We shall now show that a system of coördinates may be set up in which the tetrahedron determined by any four given non-concurrent planes is the coördinate tetrahedron. A system of coördinates so determined will be called a system of tetrahedral coördinates.

Let the equations of the four given non-concurrent planes (referred to a given system of homogeneous coördinates) be

$$A_ix + B_iy + C_iz + D_it = 0, \qquad i = 1, 2, 3, 4. \tag{1}$$

Since these planes do not all pass through a point, the determinant

$$T = \begin{vmatrix} A_1 & B_1 & C_1 & D_1 \\ A_2 & B_2 & C_2 & D_2 \\ A_3 & B_3 & C_3 & D_3 \\ A_4 & B_4 & C_4 & D_4 \end{vmatrix} \tag{2}$$

does not vanish.

Let the coördinates (x, y, z, t) of any point P in space be substituted in the first members of (1) and denote the values of the resulting expressions by x_1, x_2, x_3, x_4, respectively, so that

$$\begin{aligned} x_1 &= A_1x + B_1y + C_1z + D_1t, \\ x_2 &= A_2x + B_2y + C_2z + D_2t, \\ x_3 &= A_3x + B_3y + C_3z + D_3t, \\ x_4 &= A_4x + B_4y + C_4z + D_4t. \end{aligned} \tag{3}$$

We shall call the four numbers x_1, x_2, x_3, x_4 determined by these equations the **tetrahedral coördinates** of P. The four planes (1) are called the **coördinate planes.** Their equations in tetrahedral coördinates are $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, and $x_4 = 0$, respectively.

Since the four planes (1) do not all pass through a point, the coördinates x_1, x_2, x_3, x_4 cannot all be zero for any point in space. When (x, y, z, t) are given, the values of x_1, x_2, x_3, x_4 are uniquely determined by (3). Conversely, since the determinant (2) does not vanish, equations (3) can be solved for x, y, z, t so that, when x_1, x_2, x_3, x_4 are given, one and only one set of values of x, y, z, t can be found. Since (x, y, z, t) and (kx, ky, kz, kt) represent the same point (Art. 29), it follows from (3) that (x_1, x_2, x_3, x_4) and (kx_1, kx_2, kx_3, kx_4) represent the same point, k being an arbitrary constant, different from zero.

89. Unit point. A system of tetrahedral coördinates is not completely determined when the positions of its coördinate planes are known. For, since the equations

$$k(Ax + By + Cz + Dt) = 0, \qquad k \neq 0,$$

and

$$Ax + By + Cz + Dt = 0$$

represent the same plane (Art. 24), it follows that if k_1, k_2, k_3, k_4 are four arbitrary constants different from zero, the equations

$$x'_i = k_i(A_i x + B_i y + C_i z + D_i t), \qquad i = 1, 2, 3, 4 \tag{4}$$

define a system of tetrahedral coördinates having the same coördinate planes as (3) but such that

$$x'_i = k_i x_i, \qquad i = 1, 2, 3, 4.$$

The point whose tetrahedral coördinates with respect to a given system are all equal, so that $x_1 : x_2 : x_3 : x_4 = 1 : 1 : 1 : 1$, is called the unit point of the system.

THEOREM I. *Any point P, not lying on a face of the coördinate tetrahedron, may be taken as unit point.*

For, by substituting the coördinates (x, y, z, t) of P in (4) values of k_1, k_2, k_3, k_4 may be found such that $x_1' = x_2' = x_3' = x_4'$, so that P is the unit point.

Since the ratios $k_1 : k_2 : k_3 : k_4$ are fixed when the unit point has been chosen, we have the following theorem:

THEOREM II. *The system of tetrahedral coördinates is determined when the coördinate planes $x_1 = 0$, $x_2 = 0$, $x_3 = 0$, $x_4 = 0$ and the unit point $(1, 1, 1, 1)$ have been chosen.*

EXERCISES

In the following problems, the equations in homogeneous coördinates of the coördinate planes of the given system of tetrahedral coördinates are

$$x - y + 2t = 0, \qquad x + 2y - 2z + t = 0,$$
$$3x + 3y + 2z + 2t = 0, \qquad x - 3y + z + 2t = 0.$$

The homogeneous coördinates of the unit point are $(-1, 2, -1, 1)$.

1. Find the tetrahedral coördinates of the points whose homogeneous rectangular coördinates are (x, y, z, t), $(0, 0, 0, 1)$, $(1, 1, 1, 1)$, $(5, 1, -2, 1)$, $(3, 1, 1, 0)$, $(0, 1, -1, 0)$.

2. Find the rectangular coördinates of the points whose tetrahedral coördinates are $(-1, 1, 4, 3)$, $(1, 2, -1, -5)$, $(0, 0, 1, 3)$, (x_1, x_2, x_3, x_4).

3. Write the equation of the surface $x_1 + 2x_2 - 2x_3 - x_4 = 0$ in rectangular coördinates. Show that the locus is a plane.

4. Write the equation of the plane $5x + y + z - t = 0$ in tetrahedral coördinates.

5. Write the equation of the surface $x_1x_2 + x_3x_4 = 0$ in rectangular coördinates.

6. Solve Exs. 1 and 2 when the point whose rectangular coördinates are $(3, 1, -2, 2)$ is taken as unit point.

7. Why may not a point lying in a face of the tetrahedron of reference be taken as unit point?

90. Equation of a plane. Plane coördinates. From the equation

$$ux + vy + wz + st = 0 \tag{5}$$

of a plane in homogeneous rectangular coördinates, the corresponding equations in tetrahedral coördinates can be found by solving equations (3) for x, y, z, t and substituting in (5). The resulting equation is linear and homogeneous in x_1, x_2, x_3, x_4 of the form

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0, \tag{6}$$

with constant coefficients u_1, u_2, u_3, u_4. Conversely, any equation of the form (6) defines a plane. For, if x_1, x_2, x_3, x_4 are replaced by their values from (3), the resulting equation is

$$ux + vy + wz + st = 0,$$

wherein

$$\begin{aligned} u &= A_1u_1 + A_2u_2 + A_3u_3 + A_4u_4, \\ v &= B_1u_1 + B_2u_2 + B_3u_3 + B_4u_4, \\ w &= C_1u_1 + C_2u_2 + C_3u_3 + C_4u_4, \\ s &= D_1u_1 + D_2u_2 + D_3u_3 + D_4u_4. \end{aligned} \tag{7}$$

The coefficients u_1, u_2, u_3, u_4 in (6) are called the **tetrahedral coördinates of the plane** (compare Arts. 27 and 29). It follows from equations (7) and (2) that, if u_1, u_2, u_3, u_4 (not all zero) are given, the plane is definitely determined, and that, if the plane is given, its tetrahedral coördinates (u_1, u_2, u_3, u_4) are fixed except for an arbitrary multiplier, different from zero.

91. Equation of a point. Let (x_1, x_2, x_3, x_4) be the coördinates of a given point. The condition that a plane whose coördinates are (u_1, u_2, u_3, u_4) passes through the given point is, from (6)

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0. \tag{8}$$

This equation, which is satisfied only by the coördinates of the planes which pass through the given point, is called the equation of the point (x_1, x_2, x_3, x_4) in plane coördinates (cf. Art. 28).

It should be noticed that, in the equation (6) of a plane, (u_1, u_2, u_3, u_4) are constants and (x_1, x_2, x_3, x_4) are variables. In the equation (8) of a point (x_1, x_2, x_3, x_4) are constants and (u_1, u_2, u_3, u_4) are variables.

92. Equations of a line. The locus of the points whose coördinates satisfy two simultaneous linear equations

$$\begin{aligned} u'_1x_1 + u'_2x_2 + u'_3x_3 + u'_4x_4 &= 0, \\ u''_1x_1 + u''_2x_2 + u''_3x_3 + u''_4x_4 &= 0 \end{aligned} \tag{9}$$

is a line (Art. 17). The two simultaneous equations are called the equations of the line in point coördinates.

Similarly, the locus of the planes whose coördinates satisfy two simultaneous linear equations

$$\begin{aligned} x'_1u_1 + x'_2u_2 + x'_3u_3 + x'_4u_4 &= 0, \\ x''_1u_1 + x''_2u_2 + x''_3u_3 + x''_4u_4 &= 0 \end{aligned} \tag{10}$$

is a line (Art. 28). These two simultaneous equations are called the equations of the line in plane coördinates.

EXERCISES

1. Write the equations and the coördinates of the vertices and of the faces of the coördinate tetrahedron.

2. Write the equations in point and in plane coördinates of the edges of the coördinate tetrahedron.

3. Find the equations of the following points: (1, 1, 1, 1), (3, − 5, 7, − 1), (− 1, 6, − 4, 2), (7, 2, 4, 6).

4. Write the coördinates of the following planes:

$x_1 + x_2 + x_3 + x_4 = 0,\ 7\,x_1 - x_2 - 3\,x_3 + x_4 = 0,\ x_1 + 9\,x_2 - 5\,x_3 - 2\,x_4 = 0.$

5. Write the equations of the line $x_1 + x_2 = 0,\ x_3 - 7\,x_4 = 0$ in plane coördinates. 1 −1 0 0 $u_1 - u_2$

SUG. Write the equations of two points on the line. 0 0 7 1 $7u_3 + u_4$

6. Find the coördinates of the point of intersection of the planes (1, 2, 7, 3), (1, 3, 6, 0), (1, 4, 5, 2).

93. Duality. We have seen that any four numbers x_1, x_2, x_3, x_4, not all zero, are the coördinates of a point and that any four numbers u_1, u_2, u_3, u_4, not all zero, are the coördinates of a plane. The condition that the point (x_1, x_2, x_3, x_4) lies in the plane (u_1, u_2, u_3, u_4), or that the plane (u_1, u_2, u_3, u_4) passes through the point (x_1, x_2, x_3, x_4) is

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0.$$

This equation remains unchanged if x_1, x_2, x_3, x_4 and u_1, u_2, u_3, u_4 are interchanged.

The equations (9) and (10) of a line are simply interchanged if point and plane coördinates are interchanged.

From the above observations, the following important principle, called the principle of duality, may be deduced; namely, that if we interchange x_1, x_2, x_3, x_4 and u_1, u_2, u_3, u_4 in the proof of a theorem concerning the incidence of points, lines, and planes, or concerning point and plane coördinates, we obtain at once the proof of a second theorem. The theorem so derived is called the dual of the first. It is obtained from the given one by interchanging the words point and plane in the statement.

In the next two Articles we shall write side by side for comparison the proofs of several theorems and their duals.

The symbols (x), (x'), (u), etc., will be used as abbreviations for (x_1, x_2, x_3, x_4), (x'_1, x'_2, x'_3, x'_4), (u_1, u_2, u_3, u_4) etc., respectively.

94. Parametric equations of a plane and of a point.

Let (x'), (x''), (x''') be three given non-collinear points. The equation of the plane determined by them is found, by the same method as that employed in Art. 11, to be

$$\begin{vmatrix} x_1 & x_2 & x_3 & x_4 \\ x'_1 & x'_2 & x'_3 & x'_4 \\ x''_1 & x''_2 & x''_3 & x''_4 \\ x'''_1 & x'''_2 & x'''_3 & x'''_4 \end{vmatrix} = 0. \quad (11)$$

Let (x) be any point in the plane (11). From the form (11) of the equation of the plane it follows that there exist four numbers p, l_1, l_2, l_3, not all zero, such that

$$px_i = l_1 x'_i + l_2 x''_i + l_3 x'''_i, \quad i = 1, 2, 3, 4. \quad (13)$$

In particular, we have $p \neq 0$, since otherwise it would follow that (x'), (x''), and (x''') are collinear (Art. 95), which is contrary to hypothesis. Conversely, every point (x) whose coördinates are expressible in the form (13), $p \neq 0$ lies in the plane (11) since its coördinates satisfy the equation of the plane.

Equations (13) are called the parametric equations of the plane (11), and l_1, l_2, l_3 are called the homogeneous parameters of the points of the plane.

Let (u'), (u''), (u''') be three given non-collinear planes. The equation of the point determined by them is found, by the same method as that employed in Art. 11, to be

$$\begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ u'_1 & u'_2 & u'_3 & u'_4 \\ u''_1 & u''_2 & u''_3 & u''_4 \\ u'''_1 & u'''_2 & u'''_3 & u'''_4 \end{vmatrix} = 0. \quad (12)$$

Let (u) be any plane through the point (12). From the form (12) of the equation of the point it follows that there exist four numbers p, l_1, l_2, l_3, not all zero, such that

$$pu_i = l_1 u'_i + l_2 u''_i + l_3 u'''_i, \quad i = 1, 2, 3, 4. \quad (14)$$

In particular, we have $p \neq 0$ since otherwise it would follow that (u'), (u''), and (u''') are collinear (Art. 95), which is contrary to hypothesis. Conversely, every plane (u) whose coördinates are expressible in the form (14), $p \neq 0$ passes through the point (12) since its coördinates satisfy the equation of the point.

Equations (14) are called the parametric equations of the point (12), and l_1, l_2, l_3 are called the homogeneous parameters of the planes through the point.

The system of points (13) is said to form a **plane field.** The equation of the points of this plane field is found by substituting the values of x_1, x_2, x_3, x_4 from (13) in the equation $\Sigma u_i x_i = 0$ of a point. The resulting equation

$$l_1 \Sigma x'_i u_i + l_2 \Sigma x''_i u_i + l_3 \Sigma x'''_i u_i = 0$$

is the equation, in plane coordinates, of the plane field (13).

The system of planes (14) is said to form a **bundle of planes.** The equation of the planes of the bundle is found by substituting the values of u_1, u_2, u_3, u_4 from (14) in the equation $\Sigma u_i x_i = 0$ of a plane. The resulting equation

$$l_1 \Sigma u'_i x_i + l_2 \Sigma u''_i x_i + l_3 \Sigma u'''_i x_i = 0$$

is the equation, in point coördinates, of the bundle of planes (14).

95. Parametric equations of a line. Range of points. Pencil of planes.

THEOREM. *If* (x) *is any point on the line determined by two given distinct points* (x') *and* (x''), *every determinant of order three in the matrix*

$$\left\|\begin{matrix} x_1 & x_2 & x_3 & x_4 \\ x'_1 & x'_2 & x'_3 & x'_4 \\ x''_1 & x''_2 & x''_3 & x''_4 \end{matrix}\right\| \qquad (15)$$

is equal to zero.

For, the points (x), (x'), (x'') and any fourth point (x''') are coplanar. Their coördinates consequently satisfy (11). Since (11) is satisfied for all values of x'''_1, x'''_2, x'''_3, x'''_4, it follows that the coefficient of each of these variables is equal to zero, that is, that all the determinants of order three in (15) are equal to zero.

THEOREM. *If* (u) *is any plane through the line determined by two given distinct planes* (u') *and* (u''), *every determinant of order three in the matrix*

$$\left\|\begin{matrix} u_1 & u_2 & u_3 & u_4 \\ u'_1 & u'_2 & u'_3 & u'_4 \\ u''_1 & u''_2 & u''_3 & u''_4 \end{matrix}\right\| \qquad (16)$$

is equal to zero.

For, the planes (u), (u'), (u'') and any fourth plane (u''') are concurrent. Their coördinates consequently satisfy (12). Since (12) is satisfied for all values of u'''_1, u'''_2, u'''_3, u'''_4, it follows that the coefficient of each of these variables is equal to zero, that is, that all the determinants of order three in (16) are equal to zero.

Conversely, if the determinants of order three in (15) are all equal to zero, the points (x), (x'), and (x'') are collinear, since they are coplanar with any fourth point (x''') whatever.

It follows from the above theorem that there exist three numbers p, l_1, l_2, not all zero, such that

$$px_i = l_1 x'_i + l_2 x''_i,\ i = 1, 2, 3, 4. \quad (17)$$

In particular, we have $p \neq 0$, since otherwise the coördinates of the points (x') and (x'') would be proportional so that the points would coincide.

Equations (17) are called the parametric equations of the line determined by (x') and (x''). The coefficients l_1 and l_2 are called the homogeneous parameters of the points on the line.

The system of points (17) is said to form a **range of points.** The equation of the points of this range is found, by substituting from (17) in the equation

$$\Sigma u_i x_i = 0$$

of a point, to be

$$l_1 \Sigma x'_i u_i + l_2 \Sigma x''_i u_i = 0.$$

Conversely, if the determinants of order three in (16) are all equal to zero, the planes (u), (u'), and (u'') are collinear, since they have a point in common with any fourth plane (u''') whatever.

It follows from the above theorem that there exist three numbers p, l_1, l_2, not all zero, such that

$$pu_i = l_1 u'_i + l_2 u''_i,\ i = 1, 2, 3, 4. \quad (18)$$

In particular, we have $p \neq 0$, since otherwise the coördinates of the planes (u') and (u'') would be proportional so that the planes would coincide.

Equations (18) are called the parametric equations of the line determined by (u') and (u''). The coefficients l_1 and l_2 are called the homogeneous parameters of the planes through the line.

The system of planes (18) is said to form a **pencil of planes** (Art. 24). The equation of the planes of this pencil is found, by substituting from (18) in the equation

$$\Sigma u_i x_i = 0$$

of a plane, to be

$$l_1 \Sigma u'_i x_i + l_2 \Sigma u''_i x_i = 0.$$

EXERCISES

1. Prove the following theorems analytically. State and prove their duals.

(*a*) A line and a point not on it determine a plane.

(*b*) If a line has two points in common with a plane, it lies in the plane.

(*c*) If two lines have a point in common, they determine a plane.

(*d*) If three planes have two points in common, they determine a line.

2. Write the parametric equations of the plane determined by the points $(1, 7, -1, 3)$, $(2, 5, 4, 1)$, $(10, -1, -3, -5)$. Find the coördinates of this plane.

3. Write the parametric equations of the point determined by the planes $(-5, 3, 4, 1)$, $(7, -5, 3, 2)$, $(6, -4, -3, 1)$. Find the coördinates of this point.

4. Write the equation, in plane coördinates, of the field of points in the plane $x_1 + 2x_2 - x_3 - x_4 = 0$.

SUG. First find the coördinates of three points in the plane.

5. Find the parametric equations of the pencil of planes which pass through the two points $u_1 - 5u_2 + 3u_3 - u_4 = 0$, $7u_1 + 2u_2 - u_3 - u_4 = 0$.

6. Prove that the points $(1, 2, -3, -1)$, $(3, -2, 5, -2)$, $(1, -6, 11, 0)$ are collinear. Find the parametric equations of the line determined by these points and the equation in plane coördinates of the range of points on this line.

96. Transformation of point coördinates. Let (x_1, x_2, x_3, x_4) be the coördinates of a point referred to a given system of tetrahedral coördinates, so that

$$x_i = a_{i1}x + a_{i2}y + a_{i3}z + a_{i4}t, \qquad i = 1, 2, 3, 4, \tag{19}$$

in which the determinant of the coefficients *

$$A \equiv |a_{11} \quad a_{22} \quad a_{33} \quad a_{44}| \neq 0.$$

Let the coördinates of the same point, referred to a second system of tetrahedral coördinates, be

$$x'_i = a'_{i1}x + a'_{i2}y + a'_{i3}z + a'_{i4}t, \qquad i = 1, 2, 3, 4, \tag{20}$$

in which

$$A' \equiv |a'_{11} \quad a'_{22} \quad a'_{33} \quad a'_{44}| \neq 0.$$

* The symbol $|a_{11} \quad a_{22} \quad a_{33} \quad a_{44}|$ will be used for brevity to denote

the determinant $\begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix}.$

It is required to determine the equation connecting the two sets of coördinates (x_1, x_2, x_3, x_4) and (x'_1, x'_2, x'_3, x'_4). For this purpose solve equations (20) for x, y, z, t. The results are

$$A'x = \Sigma A'_{i1}x'_i,\quad A'y = \Sigma A'_{i2}x'_i,\quad A'z = \Sigma A'_{i3}x'_i,\quad A't = \Sigma A'_{i4}x'_i,$$

in which A'_{ik} is the cofactor of a'_{ik} in the determinant A'. Substitute these values of x, y, z, t in (19) and simplify. The result is of the form

$$\begin{aligned} x_1 &= \alpha_{11}x'_1 + \alpha_{12}x'_2 + \alpha_{13}x'_3 + \alpha_{14}x'_4,\\ x_2 &= \alpha_{21}x'_1 + \alpha_{22}x'_2 + \alpha_{23}x'_3 + \alpha_{24}x'_4,\\ x_3 &= \alpha_{31}x'_1 + \alpha_{32}x'_2 + \alpha_{33}x'_3 + \alpha_{34}x'_4,\\ x_4 &= \alpha_{41}x'_1 + \alpha_{42}x'_2 + \alpha_{43}x'_3 + \alpha_{44}x'_4, \end{aligned} \tag{21}$$

wherein

$$A'\alpha_{ik} = a_{i1}A'_{k1} + a_{i2}A'_{k2} + a_{i3}A'_{k3} + a_{i4}A'_{k4}, \qquad i, k = 1, 2, 3, 4. \tag{22}$$

The determinant

$$T \equiv \begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{vmatrix}$$

is called the determinant of the transformation (21). This determinant is different from zero, for if we substitute in it the values of the α_{ik} from (22), we have at once * †

$$T \equiv \frac{1}{A'^4}|\, a_{11} \quad a_{22} \quad a_{33} \quad a_{44}\,|\;|\,A'_{11} \quad A'_{22} \quad A'_{33} \quad A'_{44}\,| \equiv \frac{AA'^3}{A'^4} \neq 0.$$

* The product of two determinants of order four

$$A \equiv |\, a_{11} \quad a_{22} \quad a_{33} \quad a_{44}\,| \text{ and } B \equiv |\, b_{11} \quad b_{22} \quad b_{33} \quad b_{44}\,|$$

is also a determinant of order four

$$C \equiv |\, c_{11} \quad c_{22} \quad c_{33} \quad c_{44}\,|,$$

in which

$$c_{ik} = a_{i1}b_{k1} + a_{i2}b_{k2} + a_{i3}b_{k3} + a_{i4}b_{k4}, \qquad i, k = 1, 2, 3, 4.$$

This theorem can easily be verified by substituting these values of c_{ik} in C and expressing C as the sum of determinants, every element of each being the product of an element of A and an element of B. Of the sixty-four determinants in the sum, forty vanish identically, having all the elements of one column proportional to the elements of another. Each of the remaining twenty-four determinants has B as a factor. When the factor B is removed, the resulting expression is the expansion of the determinant A.

† The determinant $|\, A'_{11} \quad A'_{22} \quad A'_{33} \quad A'_{44}\,|$ whose elements are the cofactors of the elements of A' is equal to A'^3, as is seen immediately by multiplying it by A' by the preceding rule, and simplifying the result.

Since $T \neq 0$, the system (21) can be solved for x'_1, x'_2, x'_3, x'_4 in terms of x'_1, x'_2, x'_3, x'_4. The results are

$$
\begin{aligned}
Tx'_1 &= \beta_{11}x_1 + \beta_{21}x_2 + \beta_{31}x_3 + \beta_{41}x_4,\\
Tx'_2 &= \beta_{12}x_1 + \beta_{22}x_2 + \beta_{32}x_3 + \beta_{42}x_4,\\
Tx'_3 &= \beta_{13}x_1 + \beta_{23}x_2 + \beta_{33}x_3 + \beta_{43}x_4,\\
Tx'_4 &= \beta_{14}x_1 + \beta_{24}x_2 + \beta_{34}x_3 + \beta_{44}x_4,
\end{aligned}
\tag{23}
$$

in which β_{ik} is the cofactor of α_{ik} in the determinant T.

The transformations (21) and (23) are said to be inverse to each other.

97. Transformation of plane coördinates. Let

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0 \tag{24}$$

be the equation of a given plane, referred to the system of tetrahedral coördinates determined by (19). Let the equation of the same plane, referred to the system (20), be

$$u'_1x'_1 + u'_2x'_2 + u'_3x'_3 + u'_4x'_4 = 0. \tag{25}$$

If, in (24), we replace x_1, x_2, x_3, x_4 by their values from (21), we obtain, after rearranging the terms,

$$
\begin{aligned}
(\alpha_{11}u_1 + \alpha_{21}u_2 + \alpha_{31}u_3 + \alpha_{41}u_4)x'_1 + (\alpha_{12}u_1 + \alpha_{22}u_2 + \alpha_{32}u_3 + \alpha_{42}u_4)x'_2 \\
+ (\alpha_{13}u_1 + \alpha_{23}u_2 + \alpha_{33}u_3 + u_{43}u_4)x'_3 + (\alpha_{14}u_1 + \alpha_{24}u_2 + \alpha_{34}u_3 \\
+ \alpha_{44}u_4)x'_4 = 0.
\end{aligned}
\tag{26}
$$

Since equations (25) and (26) are the equations of the same plane, their coefficients are proportional, hence

$$pu'_i = \alpha_{1i}u_1 + \alpha_{2i}u_2 + \alpha_{3i}u_3 + \alpha_{4i}u_4, \qquad i = 1, 2, 3, 4, \tag{27}$$

where $p \neq 0$ is a factor of proportionality. If we solve equations (27) for u_1, u_2, u_3, u_4, we have

$$\sigma u_i = \beta_{i1}u'_1 + \beta_{i2}u'_2 + \beta_{i3}u'_3 + \beta_{i4}u'_4, \quad i = 1, 2, 3, 4, \tag{28}$$

in which $\sigma \neq 0$ and the β_{ik} have the same meaning as in (23).

Since, when x_1, x_2, x_3, x_4 are subjected to a transformation (21), u_1, u_2, u_3, u_4 are subjected simultaneously to the transformation (28), the systems of variables (x) and (u) are called **contragredient**.

EXERCISES

1. Prove that the four planes determined by equating to zero the second members of equations (23) are the faces of the coördinate tetrahedron of the system (x'_1, x'_2, x'_3, x'_4).

2. State and prove the dual of the theorem in Ex. 1 for the second members of equations (27).

3. By means of equations (21) and (23) find the coördinates in each system of the unit point of the other system.

4. Determine the equations of a transformation of coördinates in which the only change is that a different point is chosen as unit point.

98. Projective transformations. Equations (21) were derived as the equations connecting the coördinates of a given arbitrary point referred to two systems of tetrahedral coördinates. We shall now give these equations another interpretation, entirely distinct from the preceding one, but equally important.

Let there be given a system of equations (21) with determinant T not equal to zero. Let P' be a given point and let its coördinates, in a given system of tetrahedral coördinates, be (x'_1, x'_2, x'_3, x'_4). By substituting the coördinates of P' in the second members of (21), we determine four numbers x_1, x_2, x_3, x_4, which we consider as the coördinates (in the same system of coördinates as those of P') of a second point P. To each point P' in space corresponds, in this way, one and only one point P. Moreover, when the coördinates of P are given, the coördinates of P' are fixed by (23), so that to each point P corresponds one and only one point P'. It is useful to think of the point P' as actually changed into P by the transformation (21) so that, by means of (21), the points of space change their positions.

A transformation determined by a system of equations of the type (21), with determinant T not equal to zero, is called a **projective transformation.** The projective transformation (23) is called the inverse of (21). If, by (21), P' is transformed into P, then, by (23), P is transformed into P'.

By (21), the points of the plane (u') are transformed into the points of the plane (u) determined by (28). Equations (28) are called the equations of the transformation (21) in plane coördinates.

99. Invariant points. The points which remain fixed when operated on by a given projective transformation (21) are called the **invariant points** of the transformation. To determine these points, put $x_i = px'_i$ in (21). The condition on p in order that the resulting equations

$$\begin{aligned} (\alpha_{11}-p)x'_1+\alpha_{12}x'_2+\alpha_{13}x'_3+\alpha_{14}x'_4&=0,\\ \alpha_{21}x'_1+(\alpha_{22}-p)x'_2+\alpha_{23}x'_3+\alpha_{24}x'_4&=0,\\ \alpha_{31}x'_1+\alpha_{32}x'_2+(\alpha_{33}-p)x'_3+\alpha_{34}x'_4&=0,\\ \alpha_{41}x'_1+\alpha_{42}x'_2+\alpha_{43}x'_3+(\alpha_{44}-p)x'_4&=0 \end{aligned} \tag{29}$$

have a set of solutions (not all zero) in common is that

$$D(p)=\begin{vmatrix} \alpha_{11}-p & \alpha_{12} & \alpha_{13} & \alpha_{14}\\ \alpha_{21} & \alpha_{22}-p & \alpha_{23} & \alpha_{24}\\ \alpha_{31} & \alpha_{32} & \alpha_{33}-p & \alpha_{34}\\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44}-p \end{vmatrix}=0. \tag{30}$$

Let p_1 be a root of $D(p)=0$. If p_1 is substituted for p in (29), the points (x') whose coördinates satisfy the resulting equations are invariant points of the given transformation.

If $D(p_1)$ is of rank three, equations (29) determine a single invariant point when $p=p_1$ (Art. 35). If $D(p_1)$ is of rank two, equations (29) determine a line when $p=p_1$. Each point of this line is an invariant point of the transformation. If $D(p_1)$ is of rank one, equations (29) determine a plane of invariant points when $p=p_1$. If all the elements of $D(p_1)$ are zero, every point in space remains fixed. In this last case, the transformation is called the identical transformation.

100. Cross ratio. The cross ratio of four numbers k_1, k_2, k_3, k_4 is defined by the equation

$$\sigma=\frac{k_1-k_2}{k_1-k_4}:\frac{k_3-k_2}{k_3-k_4}.$$

$\frac{1-3}{1-4} \Big/ \frac{3-2}{3-4}$

The cross ratio of four collinear points P_1, P_2, P_3, P_4, or of four collinear planes $\pi_1, \pi_2, \pi_3, \pi_4$, is equal to the cross ratio of the ratios of their homogeneous parameters (equations (17) or (18)). If the parameters of the given points or planes are, respectively, l_1, l_2; l'_1, l'_2; l''_1, l''_2; l'''_1, l'''_2, it follows that their cross ratio is

$$\sigma=\frac{l_2l'_1-l_1l'_2}{l_2l'''_1-l_1l'''_2}:\frac{l''_2l'_1-l'_2l''_1}{l''_2l'''_1-l'''_2l''_1}.$$

If $\sigma = -1$, the four given points or planes are said to be **harmonic.**

An important property of the cross ratio is stated in the following theorem:

THEOREM. *The cross ratio of four points (or planes) is equal to the cross ratio of any four points (or planes) into which they can be projected.*

In the projective transformation (21), let the points (x') and (x'') of equation (17) be projected into (y') and (y''), respectively. It follows that the point of the range (17) whose parameters are l_1 and l_2 is projected into a point (y) of the range determined by (y') and (y'') such that

$$y_i = l_1 y'_i + l_2 y''_i, \qquad i = 1, 2, 3, 4.$$

Since the parameters of the points are unchanged, the cross ratio is unchanged. Similarly for a set of four planes through a line.

Conversely, two ranges of points, or pencils of planes, are projective if the cross ratio of any four elements in the first is the same as that of the corresponding elements in the second.

EXERCISES

1. Let $A \equiv (1, 0, 0, 0)$, $B \equiv (0, 1, 0, 0)$, $C \equiv (0, 0, 1, 0)$, $D \equiv (0, 0, 0, 1)$, $E \equiv (1, 1, 1, 1)$. Find the equations of a projective transformation which interchanges these points as indicated, determine the roots of $D(p) = 0$, and find the configuration of the invariant elements when

(*a*) A is transformed into A, B into B, C into C, D into E, E into D.
(*b*) A is transformed into B, B into A, C into D, D into C, E into E.
(*c*) A is transformed into B, B into C, C into A, D into D, E into E.
(*d*) A is transformed into B, B into C, C into D, D into E, E into A.

2. Show that a projective transformation can be found that will transform five given points A, B, C, D, E, no four of which are in one plane, into five given points A', B', C', D', E', respectively, no four of which lie in one plane. Show that the transformation is then uniquely fixed.

3. A non-identical projective transformation that coincides with its own inverse is called an **involution.** Find the condition that the transformation (21) is an involution.

4. Show that the transformations $x_1 = x'_1$, $x_2 = x'_2$, $x_3 = \pm x'_3$, $x_4 = -x'_4$ are involutions. Find the invariant points in each case.

5. If P, P' are any two distinct corresponding points in either involution of Ex. 4, prove the following statements:

(*a*) The line PP' contains two distinct invariant points M, M'.

(*b*) The points $(PP'MM')$ are harmonic.

6. Find the invariant points of the transformation $x_1 = x'_2$, $x_2 = x'_3$, $x_3 = x'_4$, $x_4 = x'_1$. Show that the points of space are arranged in sets of four which are interchanged among themselves.

7. Interpret the equations (Art. 36) of a translation of axes as the equations of a projective transformation. Find the invariant elements.

8. Interpret the equations (Art. 37) of a rotation of axes as the equations of a projective transformation. Show how this transformation can be effected.

9. Find the cross ratio of the four points on the line (17) whose parameters are (0, 1), (1, 1), (1, 5), (4, 3).

CHAPTER X

QUADRIC SURFACES IN TETRAHEDRAL COÖRDINATES

101. Form of equation. Since the equation $F(x, y, z, t) = 0$ may be transformed into an equation in tetrahedral coördinates by means of equation (3) of Art. 88, it follows that the equation of a quadric surface in tetrahedral coördinates is of the form

$$A \equiv \Sigma a_{ik}x_i x_k = a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 + 2\,a_{12}x_1x_2 + 2\,a_{13}x_1x_3 + 2\,a_{14}x_1x_4 + 2\,a_{23}x_2x_3 + 2\,a_{24}x_2x_4 + 2\,a_{34}x_3x_4 = 0. \quad a_{ik} = a_{ki}. \tag{1}$$

Conversely, any equation of this form will represent a quadric surface, since by replacing each x_i by its value from (3), Art. 88, the resulting equation $F(x, y, z, t) = 0$ is of the form discussed in Chapters VI, VII, and VIII.

102. Tangent lines and planes. Let (x) and (y) be any two points in space. The coördinates of any point (z) on the line joining (x) to (y) are of the form (Art. 95)

$$z_i = \lambda x_i + \mu y_i, \quad i = 1, 2, 3, 4. \tag{2}$$

If (z) lies on the quadric $A = 0$, then

$$\lambda^2 A(x) + 2\,\lambda\mu A(x, y) + \mu^2 A(y) = 0, \tag{3}$$

wherein

$$A(x, y) = A(y, x) \equiv (a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + a_{41}y_4)x_1 + (a_{21}y_1 + a_{22}y_2 + a_{32}y_3 + a_{42}y_4)x_2 + (a_{31}y_1 + a_{32}y_2 + a_{33}y_3 + a_{34}y_4)x_3 + (a_{41}y_1 + a_{42}y_2 + a_{43}y_3 + a_{44}y_4)x_4 = \frac{1}{2}\sum \frac{\partial A}{\partial x_i} y_i = \frac{1}{2}\sum \frac{\partial A}{\partial y_i} x_i. \tag{4}$$

If (y) lies on $A = 0$, then $A(y) = 0$ and one root of (3) is $\lambda = 0$. If (y) is so chosen that both roots of (3) are $\lambda = 0$, we must have $A(x, y) = 0$. If (x) is regarded as variable, and $A(x, y)$ is not identically zero, the equation $A(x, y) = 0$ defines a plane. The line joining any point in this plane to the fixed point (y) on the quadric A touches the surface at the point (y) (Art. 76). The line is a tangent line and the plane $A(x, y) = 0$ is a tangent plane to $A = 0$ at (y).

EXERCISES

1. Find the equation of the tangent plane to $x_1^2 + x_2^2 + x_3^2 - a^2x_4^2 = 0$ at the point $(0, 0, a, 1)$.

2. Show that equation (4) vanishes identically if

$$A = ax_1^2 + bx_2^2 + cx_3^2 = 0 \text{ and } (y) = (0, 0, 0, 1).$$

3. Determine the coördinates of the points in which the line $x_1 + 2x_2 + x_4 = 0$, $x_3 - 2x_4 = 0$ meets the surface $x_4^2 - x_1x_2 + x_2x_3 + 4x_3^2 = 0$.

4. Show that the line $x_4 = 0$, $x_1 - 3x_2 = 0$ touches the surface

$$x_4^2 - 3x_1^2 + 5x_2^2 + x_4(x_1 + 5x_2) + x_3x_4 = 0.$$

103. Condition that the tangent plane is indeterminate. If equation (4) is satisfied identically, the coefficient of each x_i must vanish. Thus we have the four equations

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + a_{41}y_4 &= 0,\\ a_{12}y_1 + a_{22}y_2 + a_{32}y_3 + a_{42}y_4 &= 0,\\ a_{13}y_1 + a_{23}y_2 + a_{33}y_3 + a_{43}y_4 &= 0,\\ a_{14}y_1 + a_{24}y_2 + a_{34}y_3 + a_{44}y_4 &= 0. \end{aligned} \tag{5}$$

If these equations are multiplied by y_1, y_2, y_3, y_4, respectively, and the products added, the result is $A(y) = 0$, hence if the coördinates of a point (y) satisfy all the equations (5), the point lies on the surface $A = 0$. From (3) it follows that the line joining any point in space to a point (y) satisfying equations (5) will meet the surface $A = 0$ in two coincident points at (y). If (x) is any other point on the surface A, so that $A(x) = 0$, it follows from (3) that every point on the line joining (x) to (y) lies on the surface. The surface A is in this case singular and (y) is a vertex (Arts. 66 and 67).

Conversely, if $A(x) = 0$ is singular, with a vertex at (y), the two intersections with the surface of the line joining (y) to any point in space coincide at (y). The coefficient $A(x, y)$ is identically zero and the coördinates of (y) satisfy (5). Since these coördinates are not all zero, it follows that the determinant

$$\Delta \equiv \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \tag{6}$$

vanishes. Conversely, if $\Delta = 0$, then four numbers y_1, y_2, y_3, y_4 can be found such that the four equations (5) are satisfied. The point (y) lies on $A(x) = 0$ and in the plane $A(x, y) = 0$. The line joining (y) to any point (x) will have two coincident points in common with $A(x) = 0$ at (y); that is, (y) is a vertex of the quadric A. We thus have the following theorem:

THEOREM. *The necessary and sufficient condition that a quadric surface is singular is that the determinant Δ vanishes.*

The determinant Δ is called the **discriminant of the quadric** A. If it does not vanish, the quadric will be called non-singular. Unless the contrary is stated, it will be assumed throughout this chapter that the surface is non-singular.

104. The invariance of the discriminant. In Chapter VII certain invariants under motion were considered. We shall now prove the following theorem which will include that of Art. 74 as a particular case.

THEOREM I. *If the equation of a quadric surface is subjected to a linear transformation* (Art. 96), *the discriminant of the transformed equation is equal to the product of the discriminant of the original equation and the square of the determinant of the transformation.*

Let $A(x) \equiv \sum_{i=1}^{4} \sum_{k=1}^{4} a_{ik} x_i x_k = 0$ be the equation of a given quadric, and let

$$x_i = \alpha_{i1}x'_1 + \alpha_{i2}x'_2 + \alpha_{i3}x'_3 + \alpha_{i4}x'_4, \quad i = 1, 2, 3, 4$$

define a linear transformation of non-vanishing determinant T. If these values of x_i are substituted in $A(x)$, the equation becomes

$$A'(x') = \sum_{i=1}^{4} \sum_{k=1}^{4} a'_{ki} x'_i x'_k = 0, \quad a'_{ik} = a'_{ki},$$

in which

$$a'_{ik} = \sum_{l=1}^{4} \sum_{m=1}^{4} a_{lm} \alpha_{li} \alpha_{mk}.$$

If we now put

$$r_{lk} = \sum_{m=1}^{4} a_{lm}\alpha_{mk},$$

it follows that

$$a'_{ik} = \sum_{l=1}^{4} \alpha_{li} r_{lk}.$$

If we form the discriminant Δ' of $A'(x')$, we may write

$$\Delta' = \begin{vmatrix} \alpha_{11}r_{11} + \alpha_{21}r_{21} + \alpha_{31}r_{31} + \alpha_{41}r_{41} & \alpha_{11}r_{12} + \alpha_{21}r_{22} + \alpha_{31}r_{32} + \alpha_{41}r_{42} \cdots \\ \alpha_{12}r_{11} + \alpha_{22}r_{21} + \alpha_{32}r_{31} + \alpha_{42}r_{41} & \alpha_{12}r_{12} + \alpha_{22}r_{22} + \alpha_{32}r_{32} + \alpha_{42}r_{42} \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{vmatrix}.$$

This determinant may be expressed as the product of two determinants T and R (Art. 96, footnote), thus

$$\begin{vmatrix} \alpha_{11} & \alpha_{12} & \alpha_{13} & \alpha_{14} \\ \alpha_{21} & \alpha_{22} & \alpha_{23} & \alpha_{24} \\ \alpha_{31} & \alpha_{32} & \alpha_{33} & \alpha_{34} \\ \alpha_{41} & \alpha_{42} & \alpha_{43} & \alpha_{44} \end{vmatrix} \cdot \begin{vmatrix} r_{11} & r_{12} & r_{13} & r_{14} \\ r_{21} & r_{22} & r_{23} & r_{24} \\ r_{31} & r_{32} & r_{33} & r_{34} \\ r_{41} & r_{42} & r_{43} & r_{44} \end{vmatrix},$$

the columns in the first factor being associated with the columns in the second to form the elements of the rows in the product.

Similarly, the second factor may be expressed in the form

$$\begin{vmatrix} a_{11}\alpha_{11} + a_{12}\alpha_{21} + a_{13}\alpha_{31} + a_{14}\alpha_{41} & a_{11}\alpha_{12} + a_{12}\alpha_{22} + a_{13}\alpha_{32} + a_{14}\alpha_{42} \cdots \\ a_{21}\alpha_{11} + a_{22}\alpha_{21} + a_{23}\alpha_{31} + a_{24}\alpha_{41} & a_{21}\alpha_{12} + a_{22}\alpha_{22} + a_{23}\alpha_{32} + a_{24}\alpha_{42} \cdots \\ \cdots & \cdots \\ \cdots & \cdots \end{vmatrix},$$

which is the product of Δ by T, the elements of the rows in the first factor being multiplied by the elements of the columns in the second, hence

$$\Delta' = T^2\Delta.$$

On account of this relation, the discriminant is said to be a **relative invariant** under linear transformation of tetrahedral coördinates. Moreover, the following theorem will now be proved.

THEOREM II. *Any sth minor of Δ' may be expressed as a linear function of the sth minors of Δ.*

The method of proof will be sufficiently indicated by consideration of the minor

$$\begin{vmatrix} a'_{11} & a'_{12} \\ a'_{12} & a'_{22} \end{vmatrix} \text{ of } \Delta'.$$

This determinant, when written in full,

$$\begin{vmatrix} \alpha_{11}r_{11} + \alpha_{21}r_{21} + \alpha_{31}r_{31} + \alpha_{41}r_{41} & \alpha_{11}r_{12} + \alpha_{21}r_{22} + \alpha_{31}r_{32} + \alpha_{41}r_{42} \\ \alpha_{12}r_{11} + \alpha_{22}r_{21} + \alpha_{32}r_{31} + \alpha_{42}r_{41} & \alpha_{12}r_{12} + \alpha_{22}r_{22} + \alpha_{32}r_{32} + \alpha_{42}r_{42} \end{vmatrix}$$

may be expressed as the sum of sixteen determinants, four of which vanish identically. The remaining ones may be arranged in pairs, by combining the determinant formed by the ith term of the first column and the kth term in the second with that formed by the kth term in the first column and the ith in the second. Every such pair is equivalent to the product of a second minor of Δ and a second minor of T. If $i = 2$, $k = 3$, for example, we have

$$\begin{aligned} &\begin{vmatrix} \alpha_{21}r_{21} & \alpha_{31}r_{32} \\ \alpha_{22}r_{21} & \alpha_{32}r_{32} \end{vmatrix} + \begin{vmatrix} \alpha_{31}r_{31} & \alpha_{21}r_{22} \\ \alpha_{32}r_{31} & \alpha_{22}r_{22} \end{vmatrix} \\ &\qquad = \begin{vmatrix} \alpha_{21} & \alpha_{31} \\ \alpha_{22} & \alpha_{32} \end{vmatrix} r_{21}r_{32} - \begin{vmatrix} \alpha_{21} & \alpha_{31} \\ \alpha_{32} & \alpha_{22} \end{vmatrix} r_{31}r_{22} \\ &\qquad = \begin{vmatrix} \alpha_{21} & \alpha_{31} \\ \alpha_{22} & \alpha_{32} \end{vmatrix} \cdot \begin{vmatrix} r_{21} & r_{31} \\ r_{22} & r_{32} \end{vmatrix}. \end{aligned}$$

In this way it is seen that every second minor of Δ' is a linear function of the second minors of the determinant R, the coefficients not containing r_{ik}.

By replacing each r_{ik} by its value $\sum_{m=1}^{4} a_{im}\alpha_{mk}$ and repeating the same process, it may be seen that each second minor of R may be expressed as a linear function of the second minors of Δ, the coefficients not containing any a_{ik}. The same reasoning may be applied to the first minors of Δ'. This proves the proposition.

As a corollary we have the further proposition:

THEOREM III. *The rank of the discriminant of the equation of a quadric surface is not changed by any linear transformation with non-vanishing determinant.*

For, it follows from Th. II that the rank of Δ' is not greater than that of Δ. Neither can it be less, since by the inverse transformation the minors of Δ may be expressed linearly in terms of those of Δ'.

We may now conclude: if the discriminant Δ is of rank four, the quadric $A(x) = 0$ is non-singular (Art. 103). If Δ is of rank three, $A = 0$ is a non-composite cone, for if we take its vertex (Art. 103) as the vertex $(0, 0, 0, 1)$ of the tetrahedron of reference, the equation $A = 0$ reduces to

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2\,a_{12}x_1x_2 + 2\,a_{13}x_1x_3 + 2\,a_{23}x_2x_3 = 0.$$

The line joining any point on the surface to $(0, 0, 0, 1)$ lies on the surface, which is therefore a cone (Art. 46). Since by hypothesis Δ is of rank three, we have

$$|\,a_{11}a_{22}a_{33}\,| \neq 0,$$

hence the cone is non-composite. If Δ is of rank two, the quadric is composite, for if we take two vertices as $(0, 0, 0, 1)$ and $(0, 0, 1, 0)$, the equation reduces to

$$a_{11}x_1^2 + a_{22}x_1^2 + 2\,a_{12}x_1x_2 = 0,$$

which is factorable. Since by hypothesis Δ is of rank two, $a_{11}a_{22} - a_{12}^2$ is not zero, hence the two components do not coincide. If Δ is of rank one, the equation may be reduced to the form $x_1^2 = 0$, which represents a plane counted twice.

105. Lines on the quadric surface.

THEOREM. *The section of a quadric surface made by any of its tangent planes consists of two lines passing through the point of tangency.*

For, let (y) by any point on a quadric surface $A = 0$, and (z) any point on the tangent plane at (y), so that $A(y) = 0$, $A(y, z) = 0$. If (z) is on the curve of intersection of $A(x) = 0$, $A(x, y) = 0$, then $A(z) = 0$ and (3) is identically satisfied, hence every point of the line joining (y) to (z) lies on the surface. Since the section of a quadric made by any plane is a conic (Art. 81) and one component of this conic is the line joining (y) to (z), the residual component in the tangent plane is also a straight line.

The second line also passes through (y), since every line lying in the tangent plane and passing through (y) has two coincident points of intersection with the surface at (y).

106. Equation of a quadric in plane coördinates. Let the plane

$$u_1x_1 + u_2x_2 + u_3x_3 + u_4x_4 = 0 \tag{7}$$

be tangent to the given quadric A, and let (y) be its point of tangency. Since $A(x, y) = 0$ is also the equation of the tangent plane at (y), the equation $\Sigma u_ix_i = 0$ must differ from $A(x, y) = 0$ by a constant factor k (Art. 24), hence

$$\begin{aligned} a_{11}y_1 + a_{21}y_2 + a_{31}y_3 + a_{41}y_4 &= ku_1, \\ a_{12}y_1 + a_{22}y_2 + a_{32}y_3 + a_{42}y_4 &= ku_2, \\ a_{13}y_1 + a_{23}y_2 + a_{33}y_3 + a_{43}y_4 &= ku_3, \\ a_{14}y_1 + a_{24}y_2 + a_{34}y_3 + a_{44}y_4 &= ku_4. \end{aligned} \tag{8}$$

Moreover, since (y) lies in the tangent plane, we have

$$u_1y_1 + u_2y_2 + u_3y_3 + u_4y_4 = 0. \tag{9}$$

On eliminating y_1, y_2, y_3, y_4 and k between (8) and (9), we obtain as a necessary condition that the plane (u) shall be tangent to the surface,

$$\Phi(u) = \begin{vmatrix} a_{11} & a_{21} & a_{31} & a_{41} & u_1 \\ a_{12} & a_{22} & a_{32} & a_{42} & u_2 \\ a_{13} & a_{23} & a_{33} & a_{43} & u_3 \\ a_{14} & a_{24} & a_{34} & a_{44} & u_4 \\ u_1 & u_2 & u_3 & u_4 & 0 \end{vmatrix} = 0. \tag{10}$$

Conversely, if the coördinates of a plane (u) satisfy (10), and if also $\Delta \neq 0$, then the plane is tangent to the quadric $A = 0$. For, if (10) is satisfied, five numbers y_1, y_2, y_3, y_4, k, not all zero, can be found which satisfy (8) and (9). In particular, $k \neq 0$, for otherwise, since $\Delta \neq 0$, it would follow from (8) that $y_1 = y_2 = y_3 = y_4 = 0$, contrary to the hypotheses. Since u_1, u_2, u_3, u_4 are not all zero, it follows from (8) that y_1, y_2, y_3, y_4 are not all zero, and hence are the coördinates of a point. By solving (8) for u_1, u_2, u_3, u_4 and substituting in (9), we obtain $A(y) = 0$, hence the point (y) lies on the quadric A. From (4) and (7) it follows that the plane (7) is tangent to A at the point (y).

The equation $\Phi(u)=0$ is of the second degree in u_1, u_2, u_3, u_4. It is the equation of the quadric in plane coördinates.

By duality it follows that any equation of the second degree in plane coördinates, whose discriminant is not zero, is the equation of a quadric surface in plane coördinates.

If Δ is of rank three, so that $A=0$ is the equation of a cone, the equation $\Phi(u)=0$ reduces to $(\Sigma k_i u_i)^2=0$, $\Sigma k_i u_i=0$ being the equation of the vertex of the cone. If Δ is of rank less than three, $\Phi(u)=0$ vanishes identically. The equation $\Phi(u)=0$ was in fact derived simply by imposing the condition that the section of the quadric by the plane (u) should be composite.

EXERCISES

1. If the equation $A(x)=0$ contains but three variables, show that it represents a singular quadric.

2. Calculate the discriminant of $x_4^2+x_1^2+x_2^2-x_3^2=0$.

3. Show that the discriminant of $\Phi(u)=0$ contains the discriminant of $A(x)=0$ as a factor.

4. Given $A(x)=ax_1^2+bx_2^2+cx_3^2+dx_4^2=0$, determine the form of the equation $\Phi(u)=0$.

5. When the equation $\Phi(u)=0$ is given, show how to obtain the equation $A(x)=0$.

6. Given $A(x)=ax_1^2+bx_2^2+2\,cx_3x_4=0$, find $\Phi(u)=0$.

7. Find the discriminant of

$$A(x)=x_1^2-x_2^2-x_1x_3-x_2x_3+x_1x_4+x_2x_4+x_3x_4=0,$$

and determine the form of $\Phi(u)=0$.

8. Given $\Phi(u)=u_1^2-2\,u_1u_2+u_2^2+2\,u_1u_3+2\,u_1u_4-2\,u_2u_3-2\,u_2u_4+u_3^2+u_4^2+2\,u_3u_4=0$, find $A(x)=0$ and interpret geometrically.

9. Find the two lines lying in the tangent plane $x_1=0$ to the quadric $x_1x_2+x_3^2-x_4^2=0$.

10. Write the equation of a quadric passing through each vertex of the tetrahedron of reference.

11. Write the equation of a quadric touching each of the coördinate planes (use dual of method of Ex. 10).

12. Write the equation of a quadric which touches each edge of the tetrahedron of reference.

13. What locus is represented by the equation $\Sigma a_{ik}u_iu_k=0$ when the discriminant is of rank three? of rank two? of rank one?

14. Show that through any line two planes can be drawn tangent to a given non-singular quadric.

107. Polar planes. When the coördinates z_1, z_2, z_3, z_4 of any point (z) in space are substituted in $A(x, z) = 0$, the resulting equation defines a plane called the **polar plane** of (z) as to the quadric A.

Let (y) be any point in the polar plane of (z), so that $A(y, z) = 0$. Since the expression

$$A(y, z) = A(z, y)$$

is symmetric in the two sets of coördinates y_1, y_2, y_3, y_4 and z_1, z_2, z_3, z_4, it follows that (z) lies in the polar plane of (y). Hence we have the following theorem:

THEOREM. *If the point (y) lies on the polar plane of (z), then (z) lies on the polar plane of (y).*

Any two points (y), (z), each of which lies on the polar plane of the other, are called **conjugate points** as to the quadric $A(x) = 0$.

Dually, any two planes are said to be conjugate if each passes through the pole of the other.

108. Harmonic property of conjugate points. We shall prove the following theorem.

THEOREM. *Any two conjugate points (x), (y) and the two points in which the line joining them intersects the quadric constitute a set of harmonic points.*

The coördinates of the points (z) in which the line joining the conjugate points (x), (y) as to the quadric A are obtained by putting $z_i = \lambda x_i + \mu y_i$ and substituting these values in $A(z) = 0$. The values of the ratio $\lambda : \mu$ are roots of the equation (Art. 102)

$$\lambda^2 A(x) + 2\,\lambda\mu A(x, y) + \mu^2 A(y) = 0.$$

Since $A(x, y) = 0$, if one root is $\lambda_1 : \mu_1$, the other is $-\lambda_1 : \mu_1$. The coördinates of the points (x), (y) and the two points of intersection are therefore of the form

$$x_i,\ \ y_i,\ \ \lambda_1 x_i + \mu_1 y_i,\ \ \lambda_1 x_i - \mu_1 y_i,\ \ i = 1, 2, 3, 4,$$

hence, the four points are harmonic (Art. 100).

Dually, any two conjugate planes (u), (v) and the two tangent planes to the quadric through their line of intersection determine a set of harmonic planes.

109. Locus of points which lie on their own polar planes. The condition that a point (y) lies on its own polar plane $A(x, y) = 0$ as to $A(x)=0$ is $A(y, y) = A(y) = 0$, that is, that the point lies on the quadric. We therefore have the theorem:

THEOREM. *The locus of points which lie on their polar planes as to a quadric $A(x) = 0$ is the quadric itself.*

Since when (y) is a point on $A(x) = 0$, $A(x, y) = 0$ is the equation of the tangent plane to $A(x) = 0$ at (y), it follows that the polar plane of any point on the surface is the tangent plane at that point.

A point which lies on its own polar plane will be said to be **self-conjugate**. Dually, a plane which passes through its own pole will be said to be self-conjugate.

110. Tangent cone. If from a point (y) not on the quadric A all the tangent lines to the surface are drawn, these lines define a cone, called the **tangent cone** to A from (y).

THEOREM. *The tangent cone to a quadric from any point not on the surface is a quadric cone.*

Let (x) be any point in space. The coördinates of the points (z) in which the line joining (x) to (y) meets the quadric A are of the form

$$z_i = \lambda x_i + \mu y_i,$$

in which $\lambda : \mu$ are roots of the quadric equation

$$\lambda^2 A(x) + 2\,\lambda\mu A(x, y) + \mu^2 A(y) = 0.$$

The two points of intersection will be coincident if

$$[A(x, y)]^2 = A(x) A(y). \tag{11}$$

If now (y) is fixed and (x) is any point on the surface defined by (11), then the line joining (x) to (y) will be tangent to $A = 0$. Since this equation is of the second degree in x, the theorem follows.

The curve of intersection of the tangent cone from (y) and the quadric is found by considering (11) and $A(x) = 0$ simultaneous. The intersection is evidently defined by

$$[A(x, y)]^2 = 0, \quad A(x) = 0.$$

This locus is the conic of intersection of the quadric and the polar plane of the point (y), counted twice.

If (y) is a point on the surface, then $A(y) = 0$ and the tangent cone reduces to the tangent plane to $A = 0$ at (y), counted twice.

111. Conjugate lines as to a quadric. We shall now prove the following theorem.

THEOREM. *The polar plane of every point of the line joining any two given points (y), (z) passes through the line of intersection of the polar planes of (y) and (z).*

The polar planes of (y) and of (z) are $A(x, y) = 0$ and $A(x, z) = 0$. The coördinates of any point of the line joining (y) and (z) are of the form $\lambda y_i + \mu z_i$; and the polar plane of this point is $A(x, \lambda y + \mu z) = 0$. Since this equation is linear in $\lambda y_i + \mu z_i$, it may be rewritten in the form

$$\lambda A(x, y) + \mu A(x, z) = 0,$$

which proves the theorem.

From Art. 107 it follows that the polar plane of every point of the second line passes through the first. Two such lines are called conjugate as to the quadric. If from P, any point on the quadric, the transversal to any pair of conjugate lines is drawn, it will meet the quadric again in the harmonic conjugate of P as to the points of intersection with the conjugate lines, since its intersections with these lines are conjugate points (Arts. 107, 108).

EXERCISES

1. Determine the equation of the polar plane of (1, 1, 1, 1) as to the quadric $$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

2. Find the equation of the line conjugate to $x_1 = 0$, $x_2 = 0$ as to the quadric $$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0.$$

3. Show that any four points on a line have the same cross ratio as their four polar planes.

4. Find the tangent cone to $x_1x_2 - x_3x_4 = 0$ from the point (1, 2, 1, 3).

5. If a line meets a quadric in P and Q, show that the tangent planes at P and Q meet in the conjugate of the line.

6. Show that the quadrics $x_1^2 + x_2^2 + x_3^2 - kx_4^2 = 0$, $x_1^2 + x_2^2 + x_3^2 - lx_4^2 = 0$ are such that the polar plane of $(0, 0, 0, 1)$ is the same for both. Interpret this fact geometrically.

7. Write the equation of a quadric containing the line $x_1 = 0$, $x_2 = 0$. How many conditions does this impose upon the equation?

8. Write the equation of a quadric containing the line $x_1 = 0$, $x_2 = 0$ and the line $x_3 = 0$, $x_4 = 0$.

9. Show that through any three lines, no two of which intersect, passes one and but one quadric.

112. Self-polar tetrahedron. Associated with every tetrahedron $P_1P_2P_3P_4$ is a tetrahedron $\pi_1\pi_2\pi_3\pi_4$ formed by the polar planes of its vertices, π_1 of P_1, π_2 of P_2, π_3 of P_3, and π_4 of P_4. Conversely, it follows from Art. 107 that the plane $P_1P_2P_3$ is the polar plane of the point $\pi_1\pi_2\pi_3$, etc.

Two tetrahedra $P_1P_2P_3P_4$, $\pi_1\pi_2\pi_3\pi_4$, such that the faces of each are the polar planes of the vertices of the other as to a given quadric, are called **polar reciprocal** tetrahedra. If the two tetrahedra coincide, so that the plane π_1 is identical with the plane $P_2P_3P_4$, etc., the tetrahedron is called a **self-polar** tetrahedron.

To determine a self-polar tetrahedron choose any point P_1 not on $A(x)$ and determine its polar plane π_1. In this polar plane choose any point P_2 not on $A(x)$ and determine its polar plane π_2. This plane passes through P_1 (Art. 107). On the line of intersection of $\pi_1\pi_2$ choose a third point P_3 not on $A(x)$ and determine its polar plane π_3. The plane π_3 passes through P_1 and P_2.

Finally, let P_4 be the point of intersection of $\pi_1\pi_2\pi_3$. The polar plane π_4 of P_4 passes through points $P_1P_2P_3$. Hence the tetrahedron $P_1P_2P_3P_4 = \pi_1\pi_2\pi_3\pi_4$ is a self-polar tetrahedron.

113. Equation of a quadric referred to a self-polar tetrahedron.

THEOREM. *The necessary and sufficient condition that the equation of a quadric contains only the squares of the coördinates is that a self-polar tetrahedron is chosen as tetrahedron of reference.*

If the tetrahedron of reference is a self-polar tetrahedron, the polar plane of the vertex $(0, 0, 0, 1)$ is $x_4 = 0$. But the equation $A(x, y) = 0$ of the polar plane of $(0, 0, 0, 1)$ is $a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4 = 0$, hence $a_{41} = a_{42} = a_{43} = 0$. Since the polar plane of

(0, 0, 1, 0) is $x_3 = 0$, it follows further that $a_{13} = a_{23} = 0$, and since the polar plane of (0, 1, 0, 0) is $x_2 = 0$, that $a_{12} = 0$. But if these conditions are all satisfied, then the polar plane of (1, 0, 0, 0) is $x_1 = 0$, and the equation of the quadric has the form

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 = 0.$$

Conversely, if the equation of a quadric has the form

$$a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + a_{44}x_4^2 = 0,$$

the tetrahedron of reference is a self-polar tetrahedron. Since $\Delta \neq 0$, the coefficients a_{ii} are all different from zero.

If the coefficients in the equation of a quadric are real numbers, it follows from equation (4) that the polar plane of a real point is a real plane, hence from Art. 89 the equation of the quadric can be reduced to the form $\Sigma a_{ii}x^2 = 0$ by a real transformation of coördinates, that is, one in which all the coefficients in the equations of transformation are real numbers.

By a suitable choice of a real unit point the equation of the quadric may further be reduced to the form

$$x_1^2 + x_2^2 \pm x_3^2 \pm x_4^2 = 0.$$

114. Law of inertia. The equation of a quádric having real coefficients may thus be reduced by a real transformation to one of the three forms

$$\begin{array}{ll} (a) & x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \\ (b) & x_1^2 + x_2^2 + x_3^2 - x_4^2 = 0, \\ (c) & x_1^2 + x_2^2 - x_3^2 - x_4^2 = 0. \end{array}$$

THEOREM. *The equation of any real non-singular quadric may be reduced by a real transformation to one and only one of the types* (*a*), (*b*), (*c*).

A quadric of type (*a*) contains no real points, as the sum of the squares of four real numbers can be zero only when all the numbers are zero. If the equation is of type (*b*), the surface contains real points, but no real lines, for a real line lying on the surface would cut every real plane in a real point, but the section of (*b*) by $x_4 = 0$ is the conic $x_1^2 + x_2^2 + x_3^2 = 0$, which contains no real points. If the equation of a quadric can be reduced to type (c), the surface contains real points and real lines. The line $x_1 - x_3 = 0$, $x_2 - x_4 = 0$, for example, lies on the surface. Any real plane through it will intersect the quadric in this line and another real line. If the equation of a quadric can be reduced to one of those forms by a real transformation, it can evidently not be reduced to either of the others, since real lines and real points remain real lines and real points.

The theorem of this Article is known as the law of inertia of quadric surfaces. It states that the numerical difference between the number of positive terms and the number of negative terms is a constant for any particular equation independently of what real transformation is employed.

By a transformation which may involve imaginary coefficients the equation of any quadric may be reduced to the form $\Sigma x_i^2 = 0$. For this purpose it is necessary only to replace x_i by $\frac{x_i}{\sqrt{a_{ii}}}$ in the equation $\Sigma a_{ii} x_i^2 = 0$ of Art. 113.

115. Rectilinear generators. Reguli. If in the equation $\Sigma x_i^2 = 0$, the transformation

$$x_1 = x'_1 + x'_2, \quad x_2 = i(x'_1 - x'_2), \quad x_3 = i(x'_3 + x'_4), \quad x_4 = (x'_3 - x'_4)$$

is made, it is seen that the equation of any quadric can also be written in the form

$$x_1 x_2 - x_3 x_4 = 0. \tag{12}$$

If the quadric is of type (c), its equation can be reduced to (12) by a real transformation. In the other cases the transformation is imaginary.

The line of intersection of the planes

$$k_1 x_1 = k_2 x_3, \quad k_1 x_4 = k_2 x_2 \tag{13}$$

lies on the quadric for every value of $k_1 : k_2$, since the coördinates of any point (y) on (13) are seen by eliminating $k_1 : k_2$ to satisfy (12). Conversely, if the coördinates of any point (y) on the quadric are substituted in (13), a value of $k_1 : k_2$ is determined such that the corresponding line (13) lies on the quadric and passes through (y).

No two lines of the system (13) intersect, for if $k_1x_1 = k_2x_3$, $k_1x_4 = k_2x_2$, and $k'_1x_1 = k'_2x_2$, $k'_1x_4 = k'_2x_3$ are the two lines, the condition that they intersect is

$$\begin{vmatrix} k_1 & 0 & k_2 & 0 \\ k'_1 & 0 & k'_2 & 0 \\ 0 & k_2 & 0 & k_1 \\ 0 & k'_2 & 0 & k'_1 \end{vmatrix} = -(k_1k'_2 - k_2k'_1)^2 = 0.$$

But this condition is not satisfied unless $k_1 : k_2 = k'_1 : k'_2$, that is, unless the two lines coincide, hence:

THEOREM. *Through each point on the quadric* (12) *passes one and but one line of the system* (13), *lying entirely on the surface.*

A system of lines having this property is called a **regulus** (Art. 79).

In the same way it is shown in the system of lines

$$l_1x_1 = l_2x_4, \quad l_1x_3 = l_2x_2 \tag{14}$$

is a regulus lying on the same quadric (12). Those two reguli will be called the k-regulus and the l-regulus, respectively. It was seen that no two lines of the same regulus intersect. It will now be shown that every line of each regulus intersects every line of the other. Let

$$k_1x_1 = k_2x_3, \quad k_1x_4 = k_2x_2$$

be a line of the k-regulus and

$$l_1x_1 = l_2x_4, \qquad l_1x_3 = l_2x_2$$

be a line of the l-regulus. The condition that these lines intersect is that

$$\begin{vmatrix} k_1 & 0 & k_2 & 0 \\ 0 & k_2 & 0 & k_1 \\ l_1 & 0 & 0 & l_2 \\ 0 & l_2 & l_1 & 0 \end{vmatrix} = 0.$$

But this equation is satisfied identically; hence the lines intersect for all values of $k_1 : k_2$ and $l_1 : l_2$.

116. Hyperbolic coördinates. Parametric equations. Each value of the ratio $k_1 : k_2$ uniquely determines a line of the k-regulus; each value of $l_1 : l_2$ uniquely determines a line of the l-regulus. These two lines intersect; their point of intersection lies on the quadric;

through this point passes no other line of either regulus. Thus, a pair of values $k_1 : k_2$ and $l_1 : l_2$ fixes a point on the surface. Conversely, any point on the surface fixes the line of each system passing through it, and consequently a pair of values of $k_1 : k_2$ and $l_1 : l_2$. These two numbers are called **hyperbolic coördinates** of the point.

From equations (13), (14) the relations between the coördinates x_1, x_2, x_3, x_4 of a point on the surface and the hyperbolic coördinates $k_1 : k_2$, $l_1 : l_2$ are

$$x_1 = k_2 l_2, \quad x_2 = k_1 l_1, \quad x_3 = k_1 l_2, \quad x_4 = k_2 l_1. \tag{15}$$

These equations are called the parametric equations of the quadric (12). Since the equation of any non-singular quadric can be reduced to the form (12) by a suitable choice of tetrahedron of reference, it follows that the general form of the parametric equation of a quadric surface, referred to any system of tetrahedral coördinates, may be written in the form

$$x_i = \alpha_{1_i} k_2 l_2 + \alpha_{2_i} k_1 l_1 + \alpha_{3_i} k_1 l_2 + \alpha_{4_i} k_2 l_1, \quad i = 1, 2, 3, 4.$$

117. Projection of a quadric upon a plane. Given a quadric surface A and a plane π. If each point P of A is connected with a fixed point O on A but not on π, the line OP will intersect π in a point P', called the **image** of P. Conversely, if any point P' in π is given, the point P of which it is the image is the residual point in which OP intersects A. If P describes a locus on A, P' will describe a locus on π, and conversely. This process is called the projection of A upon π.

Through O pass two generators g_1 and g_2 of A, one of each regulus. These lines intersect π in points O_1, O_2, which are singular elements in the projection, since any point of g_1 has O_1 for its image, and any point of g_2 has O_2 for its image. The tangent plane to A at O contains the lines g_1, g_2, hence it intersects the plane π in the line O_1O_2. Any point P' of O_1O_2 will be the image of O. The line O_1O_2 will be called a singular line.

The tangent lines to A at O form a pencil in the tangent plane; any line of this pencil is fixed if its point of intersection with O_1O_2 is known. If a curve on A passes through O, the point in which its tangent cuts O_1O_2 will be said to be the image of the point O on that curve. The generators of the regulus to which g_1 belongs all intersect g_2; each, with O, determines a plane passing through g_2, and the intersections of these planes with π is a pencil of lines passing through O_2. Similarly for the other regulus and O_1. The two reguli on A have for images the pencils of lines in π with vertices at O_1, O_2.

118. Equations of the projection. Let O, O_1, O_2 be three vertices of the tetrahedron of reference; take for fourth vertex the point of contact O' of the other tangent plane through O_1O_2. If

$O=(0, 0, 0, 1)$, $O_1=(0, 0, 1, 0)$, $O'=(1, 0, 0, 0)$, $O_2=(0, 1, 0, 0)$,

the equation of the surface may be written

$$A = x_1x_4 - x_2x_3 = 0.$$

Let ξ_1, ξ_2, ξ_3 be the coördinates of a point in the image plane, referred to the triangle of intersection of $x_1 = 0$, $x_2 = 0$, $x_3 = 0$ and the image plane π or $\Sigma\alpha_ix_i = 0$. Any point of the line joining $(0, 0, 0, 1)$ to (y_1, y_2, y_3, y_4) on A will have coördinates of the form

$$ky_1,\ ky_2,\ ky_3,\ ky_4 + \lambda,$$

wherein $k\Sigma\alpha_iy_i + \alpha_4\lambda = 0$ for the point in which the line pierces the plane π.

Moreover, since $\xi_i = ky_i$ $(i = 1, 2, 3)$ and $y_1y_4 - y_2y_3 = 0$,

$$y_4 = \frac{y_2y_3}{y_1} = k\,\frac{\xi_2\xi_3}{\xi_1}.$$

Hence, a point (y) on A and its image (ξ) in π are connected by the equations

$$\rho y_1 = \xi_1^2, \quad \rho y_2 = \xi_1\xi_2, \quad \rho y_3 = \xi_1\xi_3, \quad \rho y_4 = \xi_2\xi_3. \qquad (16)$$

If $\xi_1 = 0$, then $y_1 = 0$, $y_2 = 0$, $y_3 = 0$, so that any point of the line O_1O_2 corresponds to O. If $\xi_1 = 0$ and $\xi_2 = 0$, all the y_i vanish, but if we allow a point to approach O_1 in π along the line $\xi_1 - \tau\xi_2 = 0$, then the corresponding point on A is

$$\rho y_1 = \tau^2\xi_2^2, \quad \rho y_2 = \tau\xi_2^2, \quad \rho y_3 = \tau\xi_2\xi_3, \quad \rho y_4 = \xi_2\xi_3,$$

from which the factor ξ_2 can be removed. If now ξ_2 is made to vanish, the point on A is defined by

$$y_1 = 0, \quad y_2 = 0, \quad y_3 - \tau y_4 = 0.$$

Thus, to the point O_1 correspond all the points of the generator g_1, but in such manner that to a direction $\xi_1 - \tau\xi_2 = 0$ through O_1 corresponds a definite point $(0, 0, \tau, 1)$ on g_1. To the line $\xi_1 - \tau\xi_2 = 0$ as a whole corresponds the line

$$y_1 - \tau y_2 = 0, \quad y_3 - \tau y_4 = 0,$$

that is, a generator of the regulus g_1. A plane section cut from A by the plane $\Sigma u_i x_i = 0$ has for image in π the conic whose equation is

$$u_1\xi_1^2 + u_2\xi_1\xi_2 + u_3\xi_1\xi_3 + u_4\xi_2\xi_3 = 0.$$

It passes through O_1, O_2.

EXERCISES

1. Prove that if the image curve C' is a conic not passing through O_1 nor O_2, then the curve C on A has a double point at O, intersects each generator of each regulus in two points, and is met by an arbitrary plane in four points.

2. If C' is a conic through O_1 but not O_2, then C passes through O, intersects each generator g_1 in two points and each generator g_2 in one point; it is met by a plane in three points.

3. By means of equations (16), show that C of Ex. 1 lies on another quadric surface, and find its equation.

4. By means of equations (16), show that C of Ex. 2 lies on another quadric, having a line in common with A. Find the equation of the surface and the equations of the line common to both.

119. Quadric determined by three non-intersecting lines. Let the equations of three straight lines l, l', l'', no two of which intersect, be respectively

$$\Sigma u_i x_i = 0,\ \Sigma v_i x_i = 0;\ \Sigma u'_i x_i = 0,\ \Sigma v'_i x_i = 0;\ \Sigma u''_i x_i = 0,\ \Sigma v''_i x_i = 0.$$

It is required to find the locus of lines intersecting l, l', l''.

Let (y) be a point on l'' so that

$$\Sigma u''_i y_i = 0, \quad \Sigma v''_i y_i = 0. \tag{17}$$

The equation of the plane determined by (y) and l is

$$\Sigma u_i y_i \Sigma v_i x_i - \Sigma u_i x_i \Sigma v_i y_i = 0, \tag{18}$$

and of the plane determined by (y) and l' is

$$\Sigma u'_i y_i \Sigma v'_i x_i - \Sigma u'_i x_i \Sigma v'_i y_i = 0. \tag{19}$$

The planes (18) and (19) intersect in a line which intersects l, l', l''. Moreover, the equations of every line which intersects the given lines may be written in this form. If we eliminate y_1, y_2, y_3, y_4 from (17), (18), (19), we obtain a necessary condition that a point (x) lies on such a line. The equation is

$$\begin{vmatrix} u_1(vx)-v_1(ux) & u_2(vx)-v_2(vx) & u_3(vx)-v_3(ux) & u_4(vx)-v_4(vx) \\ u'_1(v'x)-v'_1(u'x) & — & — & — \\ u''_1 & u''_2 & u''_3 & u''_4 \\ v''_1 & v''_2 & v''_3 & v''_4 \end{vmatrix} = 0, \qquad (20)$$

wherein (ux) is written for Σu_1x_1, etc.

Since this equation is of the second degree, the locus is a quadric. The skew lines l, l', l'' all lie on it, hence it cannot be singular. The common transversals of l, l', l'' belong to one regulus, and l, l', l'' themselves are three lines of the other regulus.

If $x_1=0$, $x_2=0$ is chosen for l', and $x_3=0$, $x_4=0$ for l'', the equation becomes

$$\begin{vmatrix} x_2 & -x_1 & 0 & 0 \\ 0 & 0 & x_4 & -x_3 \\ u_1 & u_2 & u_3 & u_4 \\ v_1 & v_2 & v_3 & v_4 \end{vmatrix} = 0.$$

If we write

$$u_kv_i - u_iv_k = u_{ki},$$

this assumes the form

$$u_{13}x_1x_3 + u_{24}x_2x_4 + u_{23}x_2x_3 + u_{14}x_1x_4 = 0.$$

The pencil of planes $k_1x_1 + k_2x_2 = 0$ is associated with the pencil $k_1x_3 + k_2x_4 = 0$ in such a way that associated planes pass through the same point of the line l. Two pencils of planes associated in this way are projective (Art. 100).

The locus of the intersection of corresponding planes of two projective pencils of planes whose axes do not intersect is a non-singular quadric containing the axes of both pencils.

Dually, the lines joining the corresponding points of two projective ranges generate a quadric surface. The lines containing the given ranges of points belong to the other regulus of the quadric. For this reason it is sometimes convenient to consider the generators of one regulus as directrices of the other regulus.

120. Transversals of four skew lines. Lines in hyperbolic position. We can now solve the problem of determining the number of lines in space which intersect four given skew lines l_1, l_2, l_3, l_4 by proving the following theorem:

THEOREM. *Four skew lines have at least two (distinct or coincident) transversals. If they have more than two, they all belong to a regulus.*

Any three of the lines, as l_1, l_2, l_3, determine a quadric on which l_1, l_2, l_3 lie and belong to one regulus. The common transversals of l_1, l_2, l_3 constitute the generators of the other regulus. The line l_4 either pierces this quadric in two points P_1, P_2, or lies entirely on the surface. In the first case, through each of the points P_1, P_2 passes one generator of each regulus, hence one line meeting l_1, l_2, l_3. But P_1, P_2 are on l_4, hence through P_1, P_2 passes one line meeting all four of the given lines. In the second case, l_4 belongs to the same regulus as l_1, l_2, l_3.

Four lines which belong to the same regulus are said to be in **hyperbolic position.**

EXERCISES

1. Write the equations of the quadric determined by the lines

$$x_1 + x_2 = 0,\ x_3 + x_4 = 0;\ 2x_1 + x_2 - x_3 = 0,\ x_2 + x_3 - 2x_4 = 0;$$
$$x_1 - x_2 - x_3 + x_4 = 0,\ x_1 + 2x_2 + 3x_3 + 4x_4 = 0.$$

2. Find the equations of the two transversals of the four lines

$$x_1 = 0,\ x_2 = 0;\ x_3 = 0,\ x_4 = 0;\ x_1 + x_3 = 0,\ x_2 + x_4 = 0;$$
$$x_1 + x_4 = 0,\ x_2 - x_3 = 0.$$

3. When a tetrahedron is inscribed in a quadric surface, the tangent planes at its vertices meet the opposite faces in four lines in hyperbolic position.

4. State the dual of the theorem in Ex. 3.

5. Find the polar tetrahedron of the tetrahedron of reference as to the general quadric $A = 0$.

121. The quadric cone. It has been seen (Art. 104) that the surface $A = \Sigma a_{ik}x_ix_k = 0$ represents a proper cone if and only if the discriminant is of rank three. In this case there is one point (y) whose coördinates satisfy the four equations

$$a_{i1}y_1 + a_{i2}y_2 + a_{i3}y_3 + a_{i4}y_4 = 0, \quad i = 1, 2, 3, 4. \qquad (21)$$

The point (y) is the vertex of the cone.

The equation of the polar plane (Art. 107) of any point (z) with regard to the cone is

$$(\Sigma a_{i1} z_i) x_1 + (\Sigma a_{i2} z_i) x_2 + (\Sigma a_{i3} z_i) x_3 + (\Sigma a_{i4} z_i) x_4 = 0. \qquad (22)$$

On rearranging the equation in the form

$$(\Sigma a_{1i} x_i) z_1 + (\Sigma a_{2i} x_i) z_2 + (\Sigma a_{3i} x_i) z_3 + (\Sigma a_{4i} x_i) z_4 = 0, \qquad (23)$$

it is seen that the coördinates of the vertex (y) will make the coefficient of every coördinate z_i vanish, hence:

Theorem I. *The polar plane of any point in space with regard to a quadric cone passes through the vertex. The polar plane of the vertex itself is indeterminate.*

Moreover, the polar plane of all points on the line joining any point (z) to the vertex will coincide with the polar plane of (z), since the coördinates of any point on the line joining the vertex (y) to the point (z) are of the form $k_1 y_i + k_2 z_i$. On substituting these values in (23) and making use of (21) we obtain (22) again. In particular, if (z) lies on the surface, the whole line (y) (z) is on the surface; the polar plane is now a tangent plane to the cone along the whole generator passing through (z). Hence:

Theorem II. *Every tangent plane to the cone passes through the vertex and touches the surface along a generator.*

If the vertex of the cone is chosen as the vertex (0, 0, 0, 1) of the tetrahedron of reference, then from (22), $a_{14} = a_{24} = a_{34} = a_{44} = 0$, hence the equation of the surface is independent of x_4. Conversely, if the equation of a quadric does not contain x_4, then $\Delta = 0$ and the surface is a proper or composite cone with vertex at (0, 0, 0, 1). The equation of any quadric cone with vertex at (0, 0, 0, 1) is of the form

$$K = \Sigma a_{ik} x_i x_k = 0, \qquad i, k = 1, 2, 3.$$

The equation of the tangent plane to K at a point (z) is

$$(a_{11} z_1 + a_{12} z_2 + a_{13} z_3) x_1 + (a_{21} z_1 + a_{22} z_2 + a_{23} z_3) x_2 \\ + (a_{31} z_1 + a_{32} z_2 + a_{33} z_3) x_3 = 0.$$

If a plane $\Sigma u_i x_i = 0$ coincides with this plane, then

$$\begin{aligned} a_{11}z_1 + a_{12}z_2 + a_{13}z_3 &= lu_1, \\ a_{21}z_1 + a_{22}z_2 + a_{23}z_3 &= lu_2, \\ a_{31}z_1 + a_{32}z_2 + a_{33}z_3 &= lu_3, \\ u_4 &= 0. \end{aligned}$$

Moreover, the point (z) must lie in the plane $\Sigma u_i x_i = 0$, hence $\Sigma u_i z_i = 0$. If z_1, z_2, z_3, l are eliminated from these equations, the resulting equations are

$$u_4 = 0, \quad \begin{vmatrix} a_{11} & a_{12} & a_{13} & u_1 \\ a_{21} & a_{22} & a_{23} & u_2 \\ a_{31} & a_{32} & a_{33} & u_3 \\ u_1 & u_2 & u_3 & 0 \end{vmatrix} = 0, \tag{24}$$

which define the cone in plane coördinates.

If the vertex of the cone $A = 0$ is at the point (k), then $\Phi(u) = (\Sigma k_i u_i)^2 = 0$ (Art. 106). If $k_i \neq 0$, the section of $A = 0$ by the plane $x_i = 0$ is a conic whose equation in plane coördinates is obtained by equating to zero the first minor of $\phi(u)$ corresponding to a_{ii}. The first minor of any element a_{ii} of the principal diagonal equated to zero, together with $\Phi(u) = 0$, will, if $k_i \neq 0$, define the given cone.

122. Projection of a quadric cone upon a plane. Given a point O on a cone K, but not at its vertex. To project the cone from O upon a plane π not passing through O, connect every point P on K with O. The point P' in which OP cuts π is called the projection of P upon π. Let g be the generator of K through O, and O' the point in which g pierces π. Let l be the line of intersection of π and the tangent plane along g. The point O on K corresponds to any point of l, and to O' in π correspond all the points of g. With these exceptions there is one-to-one correspondence between the points of π and of K. A curve defined on either will uniquely determine a curve on the other.

Let K be defined by $x_1x_3 - x_2^2 = 0$, π by $x_3 = 0$, and $O \equiv (0, 0, 1, 0)$.

If $P' \equiv (\xi_1, \xi_2, 0, \xi_4)$, the coördinates of $P \equiv (x_1, x_2, x_3, x_4)$ are seen, as in Art. 108, to be connected with those of P' by the equations

$$\rho x_1 = \xi_1^2, \quad \rho x_2 = \xi_1\xi_2, \quad \rho x_3 = \xi_2^2, \quad \rho x_4 = \xi_1\xi_4.$$

EXERCISES

1. Show that

$$4\,x_1^2+6\,x_1x_2+8\,x_2^2+9\,x_3^2+12\,x_3x_4+4\,x_4^2=0$$

represents a cone. Find the coördinates of its vertex.

2. Find a value of k such that the equation

$$x_1^2-5\,x_1x_2+6\,x_2^2+4\,x_3^2-kx_3x_4+x_4^2=0$$

represents a cone.

3. Write the equations of the cone of Ex. 1 in plane coördinates.

4. In equations (24), replace u_i by x_i and interpret the resulting equations.

5. Prove that if the two lines of intersection of a quadric and a tangent plane coincide, the surface is a cone.

6. What locus on the cone K has for its projection in π a conic:

(a) not passing through O'?
(b) passing through O', not touching l?
(c) touching l at O'?

7. State some properties of the projection upon π of a curve on K which passes k times through O, has k' branches at the vertex, and intersects g in n additional points.

CHAPTER XI

LINEAR SYSTEMS OF QUADRICS

In this chapter we shall discuss the equation of a quadric surface under the assumption that the coefficients are linear functions of one or more parameters.

123. Pencil of quadrics. If

$$A \equiv \Sigma a_{ik}x_i x_k = 0, \quad B \equiv \Sigma b_{ik}x_i x_k = 0$$

are the equations of two distinct quadric surfaces, the system

$$A - \lambda B \equiv \Sigma (a_{ik} - \lambda b_{ik}) x_i x_k = 0, \tag{1}$$

in which λ is the parameter, is called a **pencil** of quadrics.

Every point which lies on both the given quadrics lies on every quadric of the pencil, for if the coördinates of a point satisfy the equations $A = 0$, $B = 0$, they also satisfy the equation $A - \lambda B = 0$ for every value of λ.

Through any point in space not lying on the intersection of $A = 0$, $B = 0$ passes one and but one quadric of the pencil. If (y) is the given point, its coördinates must satisfy the equation (1), hence

$$A(y) - \lambda B(y) = 0.$$

If this value of λ is substituted in (1), we obtain the equation

$$A(y)\,B - B(y)\,A = 0$$

of the quadric of the pencil (1) through the point (y).

124. The λ-discriminant. The condition that a quadric $A - \lambda B = 0$ of the pencil (1) is singular is that its discriminant vanishes, that is,

$$|a_{ik} - \lambda b_{ik}| \equiv \begin{vmatrix} a_{11} - \lambda b_{11} & a_{12} - \lambda b_{12} & a_{13} - \lambda b_{13} & a_{14} - \lambda b_{14} \\ a_{12} - \lambda b_{12} & a_{22} - \lambda b_{22} & a_{23} - \lambda b_{23} & a_{24} - \lambda b_{24} \\ a_{13} - \lambda b_{13} & a_{23} - \lambda b_{23} & a_{33} - \lambda b_{33} & a_{34} - \lambda b_{34} \\ a_{14} - \lambda b_{14} & a_{24} - \lambda b_{24} & a_{34} - \lambda b_{34} & a_{44} - \lambda b_{44} \end{vmatrix} = 0. \tag{2}$$

This determinant will be called the **λ-discriminant.** If it is identically zero, the pencil (1) will be called a singular pencil. If the pencil is not singular, equation (2) may be written in the form

$$\Delta\lambda^4 + 4\,\Theta\lambda^3 + 6\,\Phi\lambda^2 + 4\,\Theta'\lambda + \Delta' = 0. \tag{3}$$

If $\Delta \neq 0$, this equation is of the fourth degree in λ. If $\Delta = 0$, the equation will still be considered to be of the fourth degree, with one or more infinite roots. It follows at once from equation (3) that in any non-singular pencil of quadrics there are four distinct or coincident singular quadrics. If in (3), λ is put equal to zero, Δ' results. But from (2), this is the discriminant of $A = 0$. Similarly, Δ is the discriminant of $B = 0$. Let β_{ik} be the cofactor of b_{ik} in Δ. From (2) and (3) we obtain

$$-4\,\Theta = a_{11}\beta_{11} + a_{22}\beta_{22} + \cdots + a_{34}\beta_{34}.$$

If $\Theta = 0$, $A = 0$ is said to be **apolar** to $B = 0$. Similarly, if $\Theta' = 0$, $B = 0$ is said to be apolar to $A = 0$. A geometric interpretation of this property will be given later (Art. 149).

125. Invariant factors. If the equations of the quadrics of a non-singular pencil are transformed by a linear substitution such that $A = 0$ is transformed into $A' = 0$ and $B = 0$ into $B' = 0$, then $A - \lambda B = 0$ becomes $A' - \lambda B' = 0$. Moreover, if T is the determinant of the transformation of coördinates, then (Art. 104)

$$|\, a'_{ik} - \lambda b'_{ik} \,| = T^2 \,|\, a_{ik} - \lambda b_{ik} \,|.$$

From this formula we have at once

THEOREM I. *If $(\lambda - \lambda_1)^{k_0}$ is a factor of $|\, a_{ik} - \lambda b_{ik} \,|$, it is also a factor of $|\, a'_{ik} - \lambda b'_{ik} \,|$ and conversely.*

Hence the numerical value and multiplicity of every root of the λ-discriminant is invariant under any linear transformation of coördinates. Moreover, by a proof similar to that of Theorem II, Art. 104, we obtain the following theorem:

THEOREM II. *Every sth minor of the transformed λ-discriminant is a linear function of the sth minors of the original λ-discriminant and conversely.*

From the two theorems I and II we obtain at once

THEOREM III. *If* $(\lambda-\lambda_1)^k$ *is a factor of all the sth minors of* $|a_{ik}-\lambda b_{ik}|$, *then it is also a factor of all the sth minors of* $|a'_{ik}-\lambda b'_{ik}|$ *and conversely.*

Let $(\lambda-\lambda_1)^{k_0}$ be a factor of the λ-discriminant,

$(\lambda-\lambda_1)^{k_1}$ of all its first minors,

$(\lambda-\lambda_1)^{k_2}$ of all its second minors, etc.,

k_s being the highest exponent of the power of $(\lambda-\lambda_1)$ that divides all the sth minors, and k_r being the first exponent of the set that is zero.

Let also

$$L_1=k_0-k_1,\quad L_2=k_1-k_2,\quad \cdots,\quad L_r=k_{r-1}. \tag{4}$$

From Theorem III we have:

THEOREM IV. *The expressions*

$$(\lambda-\lambda_1)^{L_1},\quad (\lambda-\lambda_1)^{L_2},\quad \cdots,\quad (\lambda-\lambda_1)^{L_r}$$

are independent of the choice of the tetrahedron of reference.

These expressions are called **invariant factors or elementary divisors to the base** $\lambda-\lambda_1$ of the λ-discriminant.

We shall next prove the following theorem:

THEOREM V. *The exponent of each invariant factor is at least unity.*

Let
$$|a_{ik}-\lambda b_{ik}|=(\lambda-\lambda_1)^{k_0}F(\lambda),$$
where $F(\lambda)$ is not divisible by $(\lambda-\lambda_1)$.

Then
$$\frac{d}{d\lambda}|a_{ik}-\lambda b_{ik}|=(\lambda-\lambda_1)^{k_0-1}f(\lambda),$$

where $f(\lambda)$ is not divisible by $(\lambda-\lambda_1)$. But the derivative of $|a_{ik}-\lambda b_{ik}|$ with respect to λ may be expressed as a linear function of the first minors,* and is consequently divisible by $(\lambda-\lambda_1)^{k_1}$ at least.

* If the elements of a determinant $|abcd|$ are functions of a variable, it follows from the definition of a derivative that the derivative of the determinant as to the variable may be expressed as the sum of determinants of the form

$$|a'bcd|+|ab'cd|+|abc'd|+|abcd'|,$$

in which a'_1 is the derivative of a_1, etc.

If these determinants are expanded in terms of the columns which contain the derivative, it follows that the derivative of the given determinant is expressible as a linear function of its first minors.

Hence
$$k_1 \leqq k_0 - 1 \text{ or } L_1 \geqq 1.$$

The proof in the other cases may be obtained in a similar way.

126. The characteristic. It is now desirable to have a symbol to indicate the arrangement of the roots in a given λ-discriminant. There may be one, two, three, or four distinct roots. If $k_0 = 1$ for any root λ_1, then $L_1 = 1$, and no other L_s appears for that factor. If $k_0 = 2$, then L_1 may be 1 or 2, according as the same factor is contained in all the first minors or not. If all the exponents L_s associated with the same root are enclosed in parentheses (L_1, L_2, $\cdots$), and all the sets for all the bases in brackets, the configuration is completely defined. This symbol is called the **characteristic** of the pencil (1). *E.g.*, suppose

$$|a_{ik} - \lambda b_{ik}| = (\lambda - \lambda_1)^2 (\lambda - \lambda_2)^2,$$

and that $\lambda - \lambda_2$ is also a factor of all the first minors, but that $\lambda - \lambda_1$ is not. The characteristic is [2(11)]. If $\lambda - \lambda_1$ is also a factor of all first minors so that $L_1 = 1$, $L_2 = 1$ to the base $\lambda - \lambda_1$, the symbol has the form [(11)(11)].

From (4) it is seen that $L_1 + L_2 + \cdots + L_r = k_0$, that is, that the sum of the exponents for any one root is equal to the multiplicity of that root. Since the sum of the multiplicities of all the roots is equal to four, we have the following theorem:

THEOREM. *The sum of the exponents in the characteristic is always equal to four.*

EXERCISES

1. Express the minor $\begin{vmatrix} a'_{12} - \lambda b'_{12} & a'_{13} - \lambda b'_{13} \\ a'_{23} - \lambda b'_{23} & a'_{33} - \lambda b'_{33} \end{vmatrix}$ of $|a'_{ik} - \lambda b'_{ik}|$ in terms of the second minors of $|a_{ik} - \lambda b_{ik}|$.

2. Find the invariant factors and characteristic of each of the following forms:

$$(a) \begin{vmatrix} 1-\lambda & 0 & 0 & 0 \\ 0 & 0 & \lambda & 0 \\ 0 & \lambda & 1 & 0 \\ 0 & 0 & 0 & \lambda \end{vmatrix}; \qquad (b) \begin{vmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 1 & 0 \\ 0 & 1 & 0 & \lambda \\ 0 & 0 & \lambda & 1 \end{vmatrix};$$

$$(c) \begin{vmatrix} 0 & 0 & \lambda & 0 \\ 0 & 0 & 0 & \lambda \\ \lambda & 0 & 0 & 1 \\ 0 & \lambda & 1 & 0 \end{vmatrix}; \qquad (d) \begin{vmatrix} 0 & \lambda & 0 & 0 \\ \lambda & 0 & 1 & 0 \\ 0 & 1 & \lambda & 0 \\ 0 & 0 & 0 & 1-\lambda \end{vmatrix}.$$

127. Pencil of quadrics having a common vertex. If the λ-discriminant is identically zero, the discussion in Arts. 124–126 does not apply. In case all the quadrics have a common vertex, we may proceed as follows. If the common vertex is taken as (0, 0, 0, 1), the variable x_4 will not appear in the equation. We then form the λ-discriminant of order three of the equation in x_1, x_2, x_3. If this discriminant is not identically zero, we determine its invariant factors and a characteristic such that the sum of the exponents is three.

Similarly, if the quadrics have a line of vertices in common, we form the λ-discriminant of order two, and a corresponding characteristic; if the quadrics have a plane of vertices in common, the λ-discriminant is of order one.

128. Classification of pencils of quadrics. The principles developed in the preceding Articles will now be employed to classify pencils of quadrics and to reduce their equations to the simplest forms. When the equation of the pencil is given, the characteristic is uniquely determined. It will be assumed that for any given pencil $A - \lambda B = 0$, the λ-discriminant has been calculated and the form of its characteristic obtained. For convenience, the cases in which $A = 0$ and $B = 0$ coincide will be included in the classification, although in this case $A - \lambda B = 0$ does not constitute a pencil as defined in Art. 123.

Since any two distinct quadrics of a pencil are sufficient to define the pencil, we shall always suppose that the quadric $B = 0$ is so chosen that the λ-discriminant has no infinite roots.

129. Quadrics having a double plane in common. By taking the plane for $x_1 = 0$, the equation reduces to

$$\lambda_1 x_1^2 - \lambda x_1^2 = 0,$$
$$A = \lambda_1 x_1^2, \qquad B = x_1^2,$$

and the cl racteristic is [1].

130. Quadrics having a line of vertices in common. Let $x_1 = 0$, $x_2 = 0$ be the equations of the line of vertices. Every quadric consists of a pair of planes passing through this line, and the equation of the pencil has the form

$$A - \lambda B = a_{11}x_1^2 + 2\,a_{12}x_1x_2 + a_{22}x_2^2 - \lambda(b_{11}x_1^2 + 2\,b_{12}x_1x_2 + b_{22}x_2^2) = 0.$$

Three cases appear:

(a) The λ-discriminant has two distinct roots λ_1, λ_2.

(b) The λ-discriminant has a double root λ_1, but not every first minor vanishes for $\lambda = \lambda_1$.

(c) The λ-discriminant is of rank zero for $\lambda = \lambda_1$.

In case (a), $A - \lambda_1 B$ is a square and $A - \lambda_2 B$ is another square. Let the tetrahedron of reference be so chosen that

$$A - \lambda_1 B = x_2^2, \quad A - \lambda_2 B = x_1^2.$$

If we solve these equations for A and B, we may, after a suitable change of unit point, write A, B in the form

$$A = \lambda_1 x_1^2 + \lambda_2 x_2^2 = 0, \quad B = x_1^2 + x_2^2.$$

In case (b) we have the relation

$$(a_{11}b_{22} - a_{22}b_{11})^2 = 4(a_{11}b_{12} - a_{12}b_{11})(a_{12}b_{22} - a_{22}b_{12}),$$

which is the condition that $A = 0$, $B = 0$ have a common factor. By calling this common factor x_1, and the other factor of $B = 0$ (which is by hypothesis distinct from the first) $2\,x_2$, we may put

$$A - \lambda_1 B = x_1^2, \quad B = 2\,x_1 x_2.$$

Solving for A, B, we have

$$A = x_1^2 + 2\,\lambda_1 x_1 x_2, \quad B = 2\,x_1 x_2.$$

In case (c), we have $A - \lambda_1 B \equiv 0$, hence we may write at once

$$\lambda_1(x_1^2 + x_2^2) - \lambda(x_1^2 + x_2^2) = 0.$$

The invariant factors are $\lambda - \lambda_1$, $\lambda - \lambda_1$.

In this case we have then the following types:

[11]	$A = \lambda_1 x_1^2 + \lambda_2 x_2^2,$	$B = x_1^2 + x_2^2,$
[2]	$A = 2\,\lambda_1 x_1 x_2 + x_1^2,$	$B = 2\,x_1 x_2,$
[(11)]	$A = \lambda_1(x_1^2 + x_2^2),$	$B = x_1^2 + x_2^2.$

131. Quadrics having a vertex in common. Let the common vertex be taken so that the equation of the pencil contains only three variables, x_1, x_2, x_3. It will first be assumed that the λ-discriminant is not identically zero.

Suppose $|a_{ik} - \lambda b_{ki}| = 0$ has at least one simple root λ_1. The expression $A - \lambda_1 B$ is the product of two distinct linear factors, hence the quadric $A - \lambda_1 B = 0$ consists of two distinct planes, both passing through the point $(0, 0, 0, 1)$. Let the line of intersection of the planes be taken for $x_3 = 0$, $x_2 = 0$, so that the expression $A - \lambda_1 B$ does not contain x_1. It follows that

$$a_{11} - \lambda_1 b_{11} = 0, \quad a_{12} - \lambda_1 b_{12} = 0, \quad a_{13} - \lambda b_{13} = 0.$$

By means of those relations a_{11}, a_{12}, a_{13} can be eliminated from the λ-discriminant. The result may be written in the form

$$|a_{ik} - b_{ik}| = \begin{vmatrix} b_{11}(\lambda - \lambda_1) & b_{12}(\lambda - \lambda_1) & b_{13}(\lambda - \lambda_1) \\ b_{12}(\lambda - \lambda_1) & a_{22} - \lambda b_{22} & a_{23} - \lambda b_{23} \\ b_{13}(\lambda - \lambda_1) & a_{23} - \lambda b_{23} & a_{33} - \lambda b_{33} \end{vmatrix}.$$

Since λ_1 was assumed to be a simple root of $|a_{ik} - \lambda b_{ik}|$, it follows that $b_{11} \neq 0$. The equation of the pencil now has the form

$$\begin{aligned} \lambda_1(b_{11}x_1^2 + 2\, b_{12}x_1x_2 + 2\, b_{13}x_1x_3) + a_{22}x_2^2 + a_{33}x_3^2 + 2\, a_{23}x_2x_3 \\ - \lambda(b_{11}x_1^2 + 2\, b_{12}x_2x_1 + 2\, b_{13}x_1x_3 + b_{22}x_2^2 + 2\, b_{23}x_2x_3 + b_{33}x_3^2) = 0. \end{aligned}$$

If we make the substitution

$$y_1 = x_1 + \frac{b_{12}x_2 + b_{13}x_3}{b_{11}}, \quad y_2 = x_2, \quad y_3 = x_3,$$

then replace y_1, y_2, y_3 by x_1, x_2, x_3, the equation of the pencil takes the form

$$\lambda_1 x_1^2 + \phi(x_2, x_3) - \lambda(x_1^2 + f(x_2, x_3)) = 0,$$

in which $\phi(x_2, x_3)$ and $f(x_2, x_3)$ are homogeneous quadratic functions of x_2, x_3. The above transformation may be interpreted geometrically as follows: Since $b_{11} \neq 0$, the quadric $B = 0$ does not pass through the point $(1, 0, 0, 0)$. The polar plane

$$b_{11}x_1 + b_{12}x_2 + b_{13}x_3 = 0$$

of the point $(1, 0, 0, 0)$ as to B is consequently not a tangent plane to B at this point. The transformation makes this polar plane the new x_1, changes the unit point, and leaves $x_2 = 0$, $x_3 = 0$ unchanged.

The expression $\phi(x_2, x_3) - \lambda f(x_2, x_3)$ may now be classified according to the method of Art. 130, and the associated functions of x_1, x_2, x_3 are obtained by adding $\lambda_1 x_1^2$ to $\phi(x_2, x_3)$, x_1^2 to $f(x_2. x_3)$.

Next suppose that $|a_{ik} - \lambda b_{ik}| = 0$ has no simple root. It has, then, a triple root which we shall denote by λ_1. If $\lambda - \lambda_1$ is not a factor of all the first minors, the quadric $A - \lambda_1 B = 0$ consists of two distinct planes. Let the tetrahedron of reference be chosen in such a way that these two planes are taken as $x_2=0$, $x_3=0$, so that the equation of the quadric has the form

$$A-\lambda_1 B=2(a_{23}-\lambda_1 b_{23})x_2x_3=0,$$

wherein $a_{23}-\lambda_1 b_{23}\neq 0$, but

$a_{11}-\lambda_1 b_{11}=0,\quad a_{22}-\lambda_1 b_{22}=0,\quad a_{33}-\lambda_1 b_{33}=0,\quad a_{12}-\lambda_1 b_{12}=0,$ $a_{13}-\lambda_1 b_{13}=0$, and

$$|a_{ik}-\lambda b_{ik}|=\begin{vmatrix} b_{11}(\lambda-\lambda_1) & b_{12}(\lambda-\lambda_1) & b_{13}(\lambda-\lambda_1) \\ b_{12}(\lambda-\lambda_1) & b_{22}(\lambda-\lambda_1) & a_{23}-\lambda b_{23} \\ b_{13}(\lambda-\lambda_1) & a_{23}-\lambda b_{23} & b_{33}(\lambda-\lambda_1) \end{vmatrix}.$$

Since $(\lambda-\lambda_1)^3$ is a factor of this determinant and $a_{23}-\lambda_1 b_{23}\neq 0$, it follows that $b_{11}=0$, and $b_{13}b_{12}=0$, that is, either $b_{13}=0$ or $b_{12}=0$. Since it is simply a matter of notation which factor is made to vanish, let $b_{13}=0$. Then $b_{12}\neq 0$, since $|a_{ik}-\lambda b_{ik}|\not\equiv 0$. Geometrically, this means that the plane $x_2=0$ touches $B=0$ along the line $x_2=0$, $x_3=0$. The plane $x_3=0$ intersects the cone $B=0$ in the line $x_2=0$, $x_3=0$ and in one other line. By a further change of coördinates, if necessary, the tangent plane to $B=0$ along this second line may be taken for $x_1=0$.

We then have

$$B=2\,b_{12}x_1x_2+b_{33}x_3^2,$$

but since

$$A=\lambda_1 B+2(a_{23}-\lambda_1 b_{23})x_2x_3,$$

we may, by a suitable choice of unit point, write the equation of the pencil in the form

$$A-\lambda B=\lambda_1(2\,x_1x_2+x_3^2)+2\,x_2x_3-\lambda(2\,x_1x_2+x_3^2)=0.$$

If $\lambda-\lambda_1$ is also a factor of all the first minors of the λ-discriminant, but not of all its second minors, $A-\lambda_1 B$ is a square and represents a plane counted twice. This plane may be chosen for $x_2=0$ so that

$$A-\lambda_1 B=(a_{22}-\lambda_1 b_{22})\,x_2^2.$$

Since $(\lambda-\lambda_1)^3$ is a factor of the λ-discriminant, we must also have

$$b_{11}b_{33}-b_{13}^2=0.$$

Geometrically, this condition expresses that $x_2=0$ is a tangent plane to the cone $B=0$. We may now write

$$B=2\,b_{12}x_1x_2+b_{33}x_3^2,\quad A=\lambda_1 B+(a_{22}-\lambda_1 b_{22})x_2^2.$$

Hence, by a suitable choice of unit point, the equation of the pencil may be reduced to

$$\lambda_1(2\,x_1x_2+x_3^2)+x_2^2-\lambda(2\,x_1x_2+x_3^2)=0.$$

If $\lambda-\lambda_1$ is also a factor of all the second minors of $|a_{ik}-\lambda b_{ik}|$, the equation of $B=0$ is a multiple of that of $A=0$ and the equation of the pencil may be written in the form

$$\lambda_1(x_1^2+x_2^2+x_3^2)-\lambda(x_1^2+x_2^2+x_3^2)=0.$$

We have thus far supposed, in this Article, that the λ-discriminant did not vanish identically. It may happen that the determinant $|a_{ik}-\lambda b_{ik}|$ is identically zero even though the quadrics of the pencil do not have a line of vertices in common. In this case every quadric of the pencil consists of a pair of planes. Let $A=\phi(x_1, x_2)$, $B=f(x_2, x_3)$. Since $|a_{ik}-\lambda b_{ik}|$ is identically zero, it follows that

$$a_{11}(b_{22}b_{33}-b_{23}^2)=0,\quad b_{33}(a_{11}a_{22}-a_{12}^2)=0,$$

and hence that $a_{11}=0$, $b_{33}=0$, as otherwise the quadrics would have a line of vertices in common, contrary to hypothesis.

By an obvious change of coördinates, we may write the equation of the pencil in the form $2\,x_1x_2-\lambda 2\,x_2x_3=0$. This is called the singular case in three variables. Its characteristic will be denoted by the symbol $\{3\}$. Collecting all the preceding results of the present Article, we have the following types of pencils of quadrics with a common vertex.

[111]	$\lambda_1x_1^2+\lambda_2x_2^2+\lambda_3x_3^2$	$x_1^2+x_2^2+x_3^2$
[21]	$\lambda_1x_1^2+2\,\lambda_2x_2x_3+x_2^2$	$x_1^2+2\,x_2x_3$
[1(11)]	$\lambda_1x_1^2+\lambda_2(x_2^2+x_3^2)$	$x_1^2+x_2^2+x_3^2$
[3]	$\lambda_1(2\,x_1x_2+x_3^2)+2\,x_2x_3$	$2\,x_1x_2+x_3^2$
[(21)]	$\lambda_1(2\,x_1x_2+x_3^2)+x_2^2$	$2\,x_1x_2+x_3^2$
[(111)]	$\lambda_1(x_1^2+x_2^2+x_3^2)$	$x_1^2+x_2^2+x_3^2$
$\{3\}$	$2\,\lambda_1x_1x_2$	$2\,x_2x_3$

EXERCISES

1. Determine the invariant factors for each pencil in the above table.

2. Determine the nature of the locus $A = 0$, $B = 0$ for each pencil in the above table.

3. Find the invariant factors and the characteristic of the pencils of quadric cones defined by

(*a*) $A = 3x_1^2 + 9x_2^2 + 4x_2x_3 - 2x_1x_3 - 6x_1x_2 = 0,$
$B = 5x_1^2 + 8x_2^2 - 2x_3^2 - 6x_1x_3 - 14x_1x_2 = 0.$

(*b*) $A = 5x_1^2 + 3x_2^2 + 2x_3^2 + 4x_2x_3 - 2x_1x_3 + 2x_1x_2 = 0,$
$B = 9x_1^2 - x_2^2 + x_3^2 - 4x_2x_3 + 14x_1x_3 + 42x_1x_2 = 0.$

(*c*) $A = 5x_1^2 - 5x_2^2 + x_3^2 + 6x_2x_3 + 10x_1x_3 - 4x_1x_2 = 0,$
$B = 10x_1^2 + 2x_2^2 + 10x_3^2 - 10x_2x_3 + 24x_1x_3 - 16x_1x_2 = 0.$

(*d*) $A = 2x_1^2 + 2x_2^2 - 2x_2x_3 - 2x_1x_3 = 0,$
$B = x_1^2 + 3x_2^2 + x_3^2 - 4x_2x_3 - 2x_1x_3 = 0.$

4. Find the form of the intersection of $A = 0$, $B = 0$ in each of the pencils of Ex. 3.

5. Write the equations of each of the pencils in Ex. 3 in the reduced form.

132. Quadrics having no vertex in common. As in the preceding case, we shall suppose, except when the contrary is stated, that $|a_{ik} - \lambda b_{ik}|$ is not identically zero. If $(\lambda - \lambda_1)$ is a simple factor of the λ-discriminant, then $A - \lambda_1 B = 0$ is the equation of a cone. By choosing its vertex as (1, 0, 0, 0) and proceeding exactly as in Art. 131, the equation may be reduced to the form

$$\lambda_1 x_1^2 + \phi(x_2, x_3, x_4) - \lambda(x_1^2 + f(x_2, x_3, x_4)) = 0.$$

By this process the variable x_1 has been separated and the functions $\phi(x_2, x_3, x_4)$, $f(x_2, x_3, x_4)$ can be reduced by the methods of Art. 131, not including the singular case.

The only new cases that arise are those in which the roots of $|a_{ik} - \lambda b_{ik}| = 0$ are equal in pairs or in which all four are equal.

Consider first the case in which there are two distinct double roots λ_1 and λ_2, neither of which is a root of all the first minors of the λ-discriminant. The quadrics $A - \lambda_1 B = 0$, $A - \lambda_2 B = 0$ are cones having distinct vertices. Let the vertex of the first be taken as (0, 0, 0, 1) and that of the second as (0, 0, 1, 0). The equation of the former does not contain x_4. Hence, we have

$$a_{14} - \lambda_1 b_{14} = 0,\ a_{24} - \lambda_1 b_{24} = 0,\ a_{34} - \lambda_1 b_{34} = 0,\ a_{44} - \lambda_1 b_{44} = 0.$$

When those values of a_{14} are substituted in $|a_{ik} - \lambda b_{ik}| = 0$, $\lambda - \lambda_1$ is seen to be a factor. The condition that $(\lambda - \lambda_1)^2$ is a factor is that either $b_{44} = 0$ or that $\lambda - \lambda_1$ is a factor of the minor corresponding to $a_{44} - \lambda b_{44}$. But in the latter case $\lambda - \lambda_1$ is a factor of all the first minors, contrary to hypothesis, hence $b_{44} = 0$.

Proceeding in the same way with the factor $\lambda - \lambda_2$, it is seen that

$$a_{13} - \lambda_2 b_{13} = 0,\ a_{23} - \lambda_2 b_{23} = 0,\ a_{33} - \lambda_2 b_{33} = 0,\ a_{34} - \lambda_2 b_{34} = 0$$

and also that $b_{33} = 0$. Hence the vertices of both cones lie on the quadric $B = 0$. Let the tangent plane to $B = 0$ at $(0, 0, 0, 1)$ be taken as $x_2 = 0$, and the tangent plane to $B = 0$ at $(0, 0, 1, 0)$ be taken as $x_1 = 0$. Since $B = 0$ is non-singular, b_{13} in the transformed equation does not vanish, hence the plane $x_2 = 0$ intersects the cone $A - \lambda_1 B = 0$ in the line $x_1 = x_2 = 0$ and in another line. Let the tangent plane along this second line be taken as $x_3 = 0$; that is, make the transformation

$$\begin{aligned} y_1 &= x_1, \quad y_2 = x_2, \\ 2\, b_{13}(\lambda_2 - \lambda_1) y_3 &= (a_{11} - \lambda_1 b_{11}) x_1 + 2\,(a_{12} - \lambda_1 b_{12}) x_2 + 2\, b_{13}(\lambda_2 - \lambda_1) x_3, \\ y_4 &= x_4. \end{aligned}$$

The equation of the cone has now the form

$$A - \lambda_1 B = (a_{22} - \lambda_1 b_{22}) x_2^2 + 2\,(a_{13} - \lambda_1 b_{13}) x_1 x_3 = 0.$$

Similarly, the plane $x_1 = 0$ intersects the cone $A - \lambda_2 B = 0$ in the line $x_1 = 0$, $x_2 = 0$ and in another line. Make a further transformation by choosing the tangent plane to $A - \lambda_2 B = 0$ along this line for the new x_4, thus

$$\begin{aligned} y_1 &= x_1, \quad y_2 = x_2, \quad y_3 = x_3, \\ 2\, b_{24}(\lambda_1 - \lambda_2) y_4 &= (a_{12} - \lambda_2 b_{12}) x_1 + (a_{22} - \lambda_2 b_{22}) x_2 + 2\, b_{24}(\lambda_1 - \lambda_2) x_4. \end{aligned}$$

The equation of the second cone now has the form

$$A - \lambda_2 B = (a_{11} - \lambda_2 b_{11}) x_1^2 + 2(a_{24} - \lambda_2 b_{24}) x_2 x_4 = 0.$$

By a suitable choice of unit point the equation of the pencil may be reduced to

$$\lambda_1(x_1^2 + 2\, x_2 x_4) + \lambda_2(x_2^2 + 2\, x_1 x_3) - \lambda(x_1^2 + x_2^2 + 2\, x_1 x_3 + 2\, x_2 x_4) = 0.$$

If the invariant factors are $(\lambda - \lambda_1)$, $(\lambda - \lambda_1)$, $(\lambda - \lambda_2)^2$, the quadric $A - \lambda_1 B = 0$ is a pair of distinct planes and as before $A - \lambda_2 B = 0$ is a cone having its vertex on the quadric $B = 0$. Let the line of intersection of the two planes of $A-\lambda_1 B=0$ be taken as $x_3=0$, $x_4=0$, and let the vertex of $A-\lambda_2 B=0$ be at $(0, 0, 1, 0)$ as before. Since this vertex lies on $A-\lambda_2 B=0$ and on $B=0$, it lies on every quadric of the pencil, in particular, therefore, on $A-\lambda_1 B=0$. Thus, one of the planes of the pair constituting $A-\lambda_1 B=0$ is the plane $x_4=0$. The other may be taken as $x_3=0$ so that

$$A-\lambda_1 B=(a_{34}-\lambda_1 b_{34})x_3x_4=0.$$

The plane $x_4=0$ is not tangent to $A-\lambda_2 B=0$, since otherwise the discriminant $|a_{ik}-\lambda b_{ik}|$ would vanish identically. Hence we may choose for $x_1=0$, and $x_2=0$ any pair of planes conjugate to each other and each conjugate to $x_4=0$ as to the cone $A-\lambda_2 B=0$. The equation of the cone $A-\lambda_2 B=0$ is now

$$A-\lambda_2 B=(a_{11}-\lambda_2 b_{11})x_1^2+(a_{22}-\lambda_2 b_{22})x_2^2+(a_{44}-\lambda_2 b_{44})x_4^2=0.$$

From these two equations we may reduce the equation of the pencil to the form

$$2\lambda_2 x_3x_4+\lambda_1(x_1^2+x_2^2+x_4^2)-\lambda(2x_3x_4+x_1^2+x_2^2+x_4^2)=0.$$

If $(\lambda-\lambda_2)$ is also a factor of all the first minors, so that the invariant factors are $(\lambda-\lambda_1)$, $(\lambda-\lambda_1)$, $(\lambda-\lambda_2)$, $(\lambda-\lambda_2)$, the quadrics $A-\lambda_1 B=0$ and $A-\lambda_2 B=0$ both consist of non-coincident planes. These four planes do not all pass through a common point, since in that case all the quadrics of the pencil would have a common vertex at that point, contrary to the hypothesis. We may consequently take

$$A-\lambda_1 B=(a_{33}-\lambda_1 b_{33})x_3^2+(a_{44}-\lambda_1 b_{44})x_4^2=0,$$
$$A-\lambda_2 B=(a_{11}-\lambda_2 b_{11})x_1^2+(a_{22}-\lambda_2 b_{22})x_2^2=0.$$

By a suitable choice of unit point the equation of the pencil assumes the form

$$\lambda_1(x_1^2+x_2^2)+\lambda_2(x_3^2+x_4^2)-\lambda(x_1^2+x_2^2+x_3^2+x_4^2)=0.$$

The remaining cases to consider are those in which $|a_{ik}-\lambda b_{ik}|$ has a fourfold factor $(\lambda-\lambda_1)^4$. Suppose first that $\lambda-\lambda_1$ is not a factor of all the first minors. The quadric $A-\lambda_1 B=0$ is a cone with vertex on $B=0$. Its vertex may be taken as $(1, 0, 0, 0)$, and the tangent plane to $B=0$ at this point as $x_2=0$. Since $A-\lambda_1 B=0$ is a cone with vertex at $(1, 0, 0, 0)$ we have

$$a_{11}-\lambda_1 b_{11}=0,\quad a_{12}-\lambda_1 b_{12}=0,\quad a_{13}-\lambda_1 b_{13}=0,\quad a_{14}-\lambda_1 b_{14}=0.$$

Since (1, 0, 0, 0) lies on $B=0$, we have $b_{11}=0$, and since the tangent plane at (1, 0, 0, 0) is $x_2=0$, it follows that $b_{13}=0$, $b_{14}=0$. The λ-discriminant now has the form

$$\begin{vmatrix} 0 & b_{12}(\lambda-\lambda_1) & 0 & 0 \\ b_{12}(\lambda-\lambda_1) & a_{22}-\lambda b_{22} & a_{23}-\lambda b_{23} & a_{24}-\lambda b_{24} \\ 0 & a_{23}-\lambda b_{23} & a_{33}-\lambda b_{33} & a_{34}-\lambda b_{34} \\ 0 & a_{24}-\lambda b_{24} & a_{34}-\lambda b_{34} & a_{44}-\lambda b_{44} \end{vmatrix}.$$

Since $(\lambda-\lambda_1)^4$ is a factor and $b_{12}\neq 0$, it follows that

$$\begin{vmatrix} b_{33} & b_{34} \\ b_{34} & b_{44} \end{vmatrix}(\lambda-\lambda_1)^2=\begin{vmatrix} a_{33}-\lambda b_{33} & a_{34}-\lambda b_{34} \\ a_{34}-\lambda b_{34} & a_{44}-\lambda b_{44} \end{vmatrix}.$$

The section of the pencil of quadrics $A-\lambda B=0$ by the plane $x_2=0$ is the pencil of composite conics

$$a_{33}x_3^2+a_{44}x_4^2+2\,a_{34}x_3x_4-\lambda(b_{33}x_3^2+b_{44}x_4^2+2\,b_{34}x_3x_4)=0, \quad x_2=0.$$

The characteristic of this pencil of composite conics is [2]; it consists (Art. 130) of pairs of lines through (1, 0, 0, 0) all of which have one line g in common. The plane $x_2=0$ cuts the cone $A-\lambda_1B=0$ in the line g counted twice, and g is defined by one of the factors of $b_{33}x_3^2+2\,b_{34}x_3x_4+b_{44}x_4^2$, since it is common to all the conics of the pencil. The tangent plane $x_2=0$ to $B=0$ therefore contains the line g and another line g'. Through the line g', which passes through the vertex of the cone $A-\lambda_1B=0$, can be drawn two tangent planes to the cone. One of them is $x_2=0$. Choose the other for $x_3=0$. The plane $x_3=0$ will touch the cone $A-\lambda_1B=0$ along a line g''. The plane containing the two generators g, g'' of the cone is next chosen as $x_4=0$. The equation of the cone $A-\lambda_1B=0$ now has the form

$$A-\lambda_1B=2(a_{23}-\lambda_1b_{23})x_2x_3+(a_{44}-\lambda_1b_{44})x_4^2=0.$$

The plane $x_3=0$ contains the generator g' of $B=0$, hence it is tangent to $B=0$, and intersects $B=0$ in a line g_1 of the other regulus. The plane $x_4=0$ contains the generator g of $B=0$, hence meets the surface in another line g_2. The lines g, g' are of opposite systems, hence g_1, g_2 belong to different reguli and intersect. The plane of g_1, g_2 may be taken as the plane $x_1=0$. The quadric $B=0$ now has the equation

$$B=2\,b_{12}x_1x_2+2\,b_{34}x_3x_4=0.$$

By means of this equation and the equation of the cone $A-\lambda_1 B = 0$ it is seen that the equation of the pencil may be reduced, by a suitable choice of unit point, to

$$\lambda_1(2\, x_1x_2 + 2\, x_3x_4) + 2\, x_2x_3 + x_4^2 - \lambda(2\, x_1x_2 + 2\, x_3x_4) = 0.$$

Now suppose $\lambda - \lambda_1$ is also a factor of all the first minors, but not of all the second minors. The surface $A - \lambda_1 B = 0$ consists of a pair of planes which may be taken for $x_3 = 0$ and $x_4 = 0$, so that

$$A - \lambda_1 B = 2(a_{34} - \lambda_1 b_{34})x_3x_4 = 0,$$

and

$$A - \lambda B \equiv 2(a_{34} - \lambda_1 b_{34})x_3x_4 + (\lambda_1 - \lambda)\, B.$$

If the λ-discriminant is calculated and the factor $(\lambda - \lambda_1)^2$ removed, it is seen that in order for $|a_{ik} - \lambda b_{ik}|$ to have the further factor $(\lambda - \lambda_1)^2$ the expression $b_{11}b_{22} - b_{12}^2$ must vanish. Hence $b_{11}x_1^2 + 2\, b_{12}x_1x_2 + b_{22}x_2^2$ either vanishes identically, or is a square of a linear expression.

In the first case, $b_{11} = 0$, $b_{12} = 0$, $b_{22} = 0$, so that the line $x_3 = 0$, $x_4 = 0$ lies on the quadric $B = 0$. The plane $x_3 = 0$ passes through this line and intersects $B = 0$ in a second line g'. Similarly, $x_4 = 0$ intersects $B = 0$ in $x_3 = 0$ and in another line g''. Another tangent plane through g' may be taken as $x_2 = 0$, and the plane of g'' and the second line in $x_2 = 0$ as $x_1 = 0$. The equation of $B = 0$ is

$$B = 2\, b_{13}x_1x_3 + 2\, b_{24}x_2x_4 = 0,$$

and the equation of the pencil may be reduced to the form

$$\lambda_1(2\, x_1x_3 + 2\, x_2x_4) + 2\, x_3x_4 - \lambda\,(2\, x_1x_3 + 2\, x_2x_4) = 0.$$

In case $b_{11}x_1^2 + 2\, b_{12}x_1x_2 + b_{22}x_2^2$ is a square, not identically zero, the line $x_3 = 0$, $x_4 = 0$ touches $B = 0$ but does not lie on it. Let the point of tangency be taken as $(0, 1, 0, 0)$ so that $b_{12} = 0$, $b_{22} = 0$. If we now remove the factor $(\lambda - \lambda_1)^3$ from the λ-discriminant and then put λ equal to λ_1, the result is $b_{23}b_{24}(a_{34} - \lambda_1 b_{34})$. This expression is equal to zero, since $(\lambda - \lambda_1)^4$ is a factor of the λ-discriminant. But $a_{34} - \lambda_1 b_{34} \neq 0$, as otherwise A would be identical with B; hence either $b_{23} = 0$ or $b_{24} = 0$. Let the notation be such that $b_{24} = 0$. Then the section of the quadric $B = 0$ by the plane $x_3 = 0$ consists of two lines through $(0, 1, 0, 0)$.

Let L be the harmonic conjugate of the line $x_3 = 0$, $x_4 = 0$ with regard to these two lines, and let P be any point on the conic $x_4 = 0$, $B = 0$. If the plane determined by P and L is chosen for $x_1 = 0$ and the tangent plane to $B = 0$ at P is taken for $x_2 = 0$, the equation of $B = 0$ becomes

$$B = b_{11}x_1^2 + 2\,b_{23}x_2x_3 + b_{44}x_4^2 = 0,$$

and the equation of the pencil has the form

$$\lambda_1(x_1^2 + x_4^2 + 2\,x_2x_3) + 2\,x_3x_4 - \lambda\,(x_1^2 + x_4^2 + 2\,x_2x_3) = 0.$$

Now suppose that $\lambda - \lambda_1$ is a factor of all the second minors, but not of all the third minors, so that $A - \lambda_1 B = 0$ is a plane counted twice. Let this plane be taken as $x_4 = 0$.

$$A - \lambda_1 B = (a_{44} - \lambda_1 b_{44})x_4^2 = 0.$$

By substituting these values in the λ-discriminant, it is seen that the determinant $|b_{11}b_{22}b_{33}|$ must also vanish if $\lambda - \lambda_1$ is to be a fourfold root. This means that the section of the quadric $B = 0$ by the plane $x_4 = 0$ consists of two lines, hence that $x_4 = 0$ is a tangent plane to $B = 0$. Let planes through these two lines be taken as $x_1 = 0$, $x_2 = 0$. The remaining generators in $x_1 = 0$ and in $x_2 = 0$ belong to opposite reguli and therefore intersect. The plane determined by them is now to be taken as $x_3 = 0$. The equation of $B = 0$ is $2\,b_{12}x_1x_2 + 2\,b_{34}x_3x_4 = 0$, hence the equation of the pencil may be reduced to the form

$$\lambda_1(2\,x_1x_2 + 2\,x_3x_4) + x_4^2 - \lambda(2\,x_1x_2 + 2\,x_3x_4) = 0.$$

If finally $\lambda - \lambda_1$ is a factor of all the third minors, the two equations $A = 0$, $B = 0$ differ only by a constant factor. If $B = 0$ is reduced to the sum of squares by referring it to any self-polar tetrahedron, the equation of the pencil becomes

$$\lambda_1(x_1^2 + x_2^2 + x_3^2 + x_4^2) - \lambda(x_1^2 + x_2^2 + x_3^2 + x_4^2) = 0.$$

Thus far it has been assumed that the λ-discriminant did not identically vanish. Now suppose $|a_{ik} - \lambda b_{ik}| \equiv 0$ so that all the quadrics of the pencil are singular. By hypothesis they do not have a common vertex. In the singular pencil two distinct composite quadrics cannot exist, for, if $A = 0$, $B = 0$ were composite, we could choose $A \equiv 2\,x_1x_2$, $B \equiv 2\,x_3x_4$, since the quadrics of the pencil do not have a common vertex. But the λ-discriminant of the pencil $A - \lambda B = 0$ is not identically zero, contrary to hypothesis, hence the pencil does not contain two distinct composite quadrics. The quadrics $A = 0$, $B = 0$ may therefore be chosen as cones. Let the vertex of $A = 0$ be taken as (0, 0, 0, 1) and the vertex of $B = 0$ as (1, 0, 0, 0).

Let g, g' be generators of $A = 0$, $B = 0$ which intersect, but such that the tangent planes along each of them does not pass through the vertex of the other cone. The plane g, g' can be taken as $x_3 = 0$, the tangent plane to $A = 0$ along g as $x_1 = 0$, and the tangent plane to $B = 0$ along g' as $x_4 = 0$.

The equations of the singular quadrics $A = 0$, $B = 0$ are now of the form

$$A = a_{11}x_1^2 + 2\,a_{12}x_1x_2 + 2\,a_{13}x_1x_3 + a_{33}x_3^2 = 0,$$
$$B = b_{33}x_3^2 + 2\,b_{24}x_2x_4 + 2\,b_{34}x_3x_4 + b_{44}x_4^2 = 0,$$

and the λ-discriminant is

$$|a_{ik} - \lambda b_{ik}| = \begin{vmatrix} a_{11} & a_{12} & a_{13} & 0 \\ a_{12} & 0 & 0 & -\lambda b_{24} \\ a_{13} & 0 & a_{33} - \lambda b_{33} & -\lambda b_{34} \\ 0 & -\lambda b_{24} & -\lambda_{34} & -\lambda b_{44} \end{vmatrix}.$$

Since this expression vanishes identically, the coefficient of each power of λ must be equal to zero. These conditions are $a_{11} = 0$, $b_{44} = 0$, $a_{12}b_{34} - b_{24}a_{13} = 0$. The last condition expresses that the planes $a_{12}x_2 + a_{13}x_3 = 0$ and $b_{24}x_2 + b_{34}x_3 = 0$ are coincident. By transforming the equation of this plane to $x_2 = 0$, the equation of the pencil reduces to

$$2\,x_1x_2 + ax_3^2 - \lambda(2\,x_2x_4 + x_3^2) = 0.$$

This case is called the singular case in four variables. The characteristic will be denoted by the symbol [{3} 1].

The determination of the invariant factors and the form of the characteristic for each of the above pencils is left as an exercise for the student. The properties of the curve of intersection will be developed in Chapter XIII, but in each case the curve is described in the following table for reference. The table includes only those forms which do not have common double point.

133. Forms of pencils of quadrics.

CHARACTERISTIC	SIMPLIFIED FORMS OF A AND B	CURVE OF INTERSECTION OF $A=0$ AND $B=0$
[1111]	$A=\lambda_1x_1^2+\lambda_2x_2^2+\lambda_3x_3^2+\lambda_4x_4^2$ $B=x_1^2+x_2^2+x_3^2+x_4^2$	A general space quartic of the first species.
[112]	$A=\lambda_1x_1^2+\lambda_2x_2^2+2\,\lambda_3x_3x_4+x_3^2$ $B=x_1^2+x_2^2+2\,x_3x_4$	A nodal quartic.
[11(11)]	$A=\lambda_1x_1^2+\lambda_2x_2^2+\lambda_3(x_3^2+x_4^2)$ $B=x_1^2+x_2^2+x_3^2+x_4^2$	Two conics which intersect at two distinct points.
[13]	$A=\lambda_1x_1^2+\lambda_2(2\,x_2x_3+x_4^2)+2\,x_3x_4$ $B=x_1^2+2\,x_2x_3+x_4^2$	A cuspidal quartic.
[1(21)]	$A=\lambda_1x_1^2+\lambda_2(2\,x_2x_3+x_4^2)+x_3^2$ $B=x_1^2+2\,x_2x_3+x_4^2$	Two conics which touch each other.
[1(111)]	$A=\lambda_1x_1^2+\lambda_2(x_2^2+x_3^2+x_4^2)$ $B=x_1^2+x_2^2+x_3^2+x_4^2$	A conic counted twice. At each point of this conic the quadrics are tangent.
[22]	$A=\lambda_1(x_1^2+2\,x_2x_4)+\lambda_2(x_2^2+2\,x_1x_3)$ $B=x_1^2+x_2^2+2\,x_2x_4+2\,x_1x_3$	A generator and a space cubic. The generator and the cubic intersect in distinct points.
[2(11)]	$A=\lambda_1(x_1^2+x_2^2+x_4^2)+2\,\lambda_2x_3x_4$ $B=x_1^2+x_2^2+x_4^2+2\,x_3x_4$	Two intersecting generators, and a conic which intersects each generator. The three points of intersection are distinct.
[(11)(11)]	$A=\lambda_1(x_1^2+x_2^2)+\lambda_2(x_3^2+x_4^2)$ $B=x_1^2+x_2^2+x_3^2+x_4^2$	Four generators which intersect at four points.

CHARACTERISTIC	SIMPLIFIED FORMS OF A AND B	CURVE OF INTERSECTION OF $A=0$ AND $B=0$
[4]	$A=\lambda_1(2x_1x_2+2x_3x_4)+2x_2x_3+x_4^2$ $B=2x_1x_2+2x_3x_4$	A generator and a space cubic. The generator touches the cubic.
[(22)]	$A=\lambda_1(2x_1x_3+2x_2x_4)+2x_3x_4$ $B=2x_1x_2+2x_2x_4$	Three generators, one counted twice. This generator intersects each of the others.
[(31)]	$A=\lambda_1(x_1^2+x_4^2+2x_2x_3)+2x_3x_4$ $B=x_1^2+x_4^2+2x_2x_3$	Two intersecting generators and a conic which touches the plane of the generators at their point of intersection.
[(211)]	$A=\lambda_1(2x_1x_2+2x_3x_4)+x_1^2$ $B=2x_1x_2+2x_3x_4$	Two intersecting generators each counted twice. The quadrics touch at each point of each generator.
[(1111)]	$A=\lambda_1(x_1^2+x_2^2+x_3^2+x_4^2)$ $B=x_1^2+x_2^2+x_3^2+x_4^2$	The quadrics coincide.
[{3}1]	$A=2x_1x_2+ax_3^2$ $B=2x_2x_4+x_3^2$	A conic and a generator counted twice. The vertices of the cones all lie on this generator.

EXERCISES

1. Derive the invariant factors of each of the above systems of quadrics.

2. Find the equations of each conic and each rectilinear generator of intersection of the quadrics of the above pencils.

3. Determine the invariant factors; find the equations of the curve of intersection, and write the equations in the reduced form of the pencils determined by

$$(a)\quad \begin{aligned} A &= x_1^2-x_2^2+2x_3^2+2x_4^2+5x_3x_4=0,\\ B &= 3x_1^2-x_2^2+x_3^2-8x_4^2-2x_1x_2-2x_3x_4=0. \end{aligned}$$

(b) $A = x_1^2 + x_2^2 + 4x_3^2 + x_4^2 + 4x_1x_2 + 6x_2x_3 + 4x_1x_3 = 0,$
$B = x_2^2 + 3x_3^2 + x_4^2 + 2x_1x_3 + 2x_2x_3 = 0.$

(c) $A = 3x_1^2 - x_2^2 - 2x_3^2 + 2x_4^2 + 2x_1x_2 - 4x_1x_3 = 0,$
$B = 4x_1^2 - x_2^2 + 2x_3^2 + 3x_4^2 + 2x_1x_2 + 2x_1x_4 + 4x_3x_4 = 0.$

(d) $A = 3x_1^2 + 2x_2^2 - x_3^2 - x_4^2 + 4x_1x_2 - 2x_3x_4 = 0,$
$B = 3x_1^2 - x_2^2 - x_3^2 - x_4^2 + x_1x_2 - 2x_3x_4 - 3x_2x_4 - 3x_1x_4 = 0.$

4. To what type does a pencil of concentric spheres belong? A pencil of tangent spheres?

134. Line conjugate to a point. The equation of the polar plane of a point (y) with respect to any quadric of the pencil (1) is

$$\Sigma a_{ik}y_ix_k - \lambda\Sigma b_{ik}y_ix_k = 0.$$

As λ varies, this system defines a pencil of planes (Art. 24). The axis of the pencil, namely the line

$$\Sigma a_{ik}y_ix_k = 0, \quad \Sigma b_{ik}y_ix_k = 0$$

is said to be **conjugate** to the point (y) as to the pencil of quadrics.

Let (y) describe a line, two points of which are (y') and (y''). It is required to find the locus of the conjugate line. Since

$$y_i = \mu_1 y'_i + \mu_2 y''_i, \quad i = 1, 2, 3, 4$$

(Art. 95), the line conjugate to (y) is, by definition,

$$\mu_1\Sigma a_{ik}y'_ix_k + \mu_2\Sigma a_{ik}y''_ix_k = 0, \quad \mu_1\Sigma b_{ik}y'_ix_k + \mu_2\Sigma b_{ik}y''_ix_k = 0.$$

As (y) describes the line joining (y') to (y'') the ratio $\mu_1 : \mu_2$ takes all possible values. If between these equations $\mu_1 : \mu_2$ is eliminated, the resulting equation defines the quadric surface

$$\Sigma a_{ik}y'_ix_k \cdot \Sigma b_{ik}y''_ix_k - \Sigma a_{ik}y''_ix_k \cdot \Sigma b_{ik}y'_ix_k = 0. \tag{5}$$

From the method of development it follows (Art. 119) that all the lines of the system belong to one regulus (Art. 115).

The polar planes, with respect to a given quadric of the pencil, of two fixed points (y'), (y'') on the given line intersect in the line

$$\Sigma a_{ik}y'_ix_k - \lambda\Sigma b_{ik}y'_ix_k = 0, \quad \Sigma a_{ik}y''_ix_k - \lambda\Sigma b_{ik}y''_ix_k = 0.$$

If between these equations λ is eliminated, the resulting equation defines the same quadric (5). From Art. 115 it follows that this second system of lines constitutes the other regulus on the surface.

135. Equation of the pencil in plane coördinates. Let $A-\lambda B=0$ be the equation of a non-singular pencil of quadrics. The equation

$$\begin{vmatrix} a_{11}-\lambda b_{11} & a_{12}-\lambda b_{12} & a_{13}-\lambda b_{13} & a_{14}-\lambda b_{14} & u_1 \\ a_{12}-\lambda b_{12} & a_{22}-\lambda b_{22} & a_{23}-\lambda b_{23} & a_{24}-\lambda b_{24} & u_2 \\ a_{13}-\lambda b_{13} & a_{23}-\lambda b_{23} & a_{33}-\lambda b_{33} & a_{34}-\lambda b_{34} & u_3 \\ a_{14}-\lambda b_{14} & a_{24}-\lambda b_{24} & a_{34}-\lambda b_{34} & a_{44}-\lambda b_{44} & u_4 \\ u_1 & u_2 & u_3 & u_4 & 0 \end{vmatrix}=0 \qquad (6)$$

expresses the condition that the section of a quadric of the pencil by a plane (u) is composite (Art. 106). For a given value λ_1 of λ, (6) is the equation of the quadric $A-\lambda_1 B=0$ in plane coördinates, if it is non-singular. If $A-\lambda_1 B=0$ is a cone, (6) is the equation of its vertex counted twice. If $A-\lambda_1 B=0$ is composite, (6) vanishes identically.

Equation (6) is called the equation of the pencil in plane coördinates. Arranged in powers of λ, it is of the form

$$\Phi_1(u)\lambda^3+3\,\Psi_1(u)\lambda^2+3\,\Psi_2(u)\lambda+\Phi_2(u)=0. \qquad (7)$$

If $\Phi_1(u)\neq 0$, the equation is of the third degree in λ. When (7) is not identically zero, it will be said to be a cubic in any case, even if it has one or more infinite roots. Hence we have the following theorem:

THEOREM. *Every plane intersects three distinct or coincident quadrics of a non-singular pencil in composite conics.*

The coefficient of each power of λ in (7) is homogeneous and of the second degree in u_1, u_2, u_3, u_4 (if it is not identically zero), hence, when equated to zero, it defines a quadric in plane coördinates. Since the pencil is non-singular, we may, without loss of generality, assume that the quadrics $A=0$, and $B=0$ are non-singular (Art. 128). The equation $\Phi_2(u)=0$ is seen, by putting $\lambda=0$ in (6), to be the equation of $A=0$ in plane coördinates. An analogous statement holds for $\Phi_1(u)=0$ and $B=0$. The geometric meaning of the other coefficients will be discussed later (Art. 149).

EXERCISES

1. Write the equation in plane coördinates of the pencil of quadrics
$x_1^2 - x_2^2 + x_3^2 + 3\,x_4^2 - 6\,x_1x_4 + 4\,x_3x_4 - \lambda(2\,x_2x_4 + x_1^2 + x_2^2 + x_3^2) = 0.$

2. Determine the equations of the three quadrics of the pencil of Ex. 1 which touch the plane $x_4 = 0$.

3. Determine equation (7) for the pencil

$$a(2\,x_1x_2 + 2\,x_3x_4) + x_1^2 - \lambda(2\,x_1x_2 + 2\,x_3x_4) = 0.$$

Show that (7) vanishes identically for each of the planes $x_1 = 0$, $x_3 = 0$, $x_4 = 0$, and interpret the fact geometrically.

136. Bundle of quadrics. If $A = \Sigma a_{ik}x_ix_k = 0$, $B = \Sigma b_{ik}x_ix_k = 0$, $C = \Sigma c_{ik}x_ix_k = 0$ are three given quadrics which do not belong to the same pencil, the system defined by the equation

$$\lambda_1 A + \lambda_2 B + \lambda_3 C = 0, \tag{8}$$

in which $\lambda_1, \lambda_2, \lambda_3$ are parameters, is called a **bundle** of quadrics. The three given quadrics $A = 0$, $B = 0$, $C = 0$ intersect in at least eight distinct or coincident points,* through each of which pass all the quadrics of the bundle. These eight points cannot be taken at random, for in order that a quadric shall pass through eight given points, the coördinates of each point must satisfy its equation, thus giving rise to eight linear homogeneous equations among the coefficients in the equation of the quadric. If the eight given points are chosen arbitrarily, these eight equations are independent and the system of quadrics determined by them is a pencil.

It is seen that seven given arbitrarily chosen points determine a bundle of quadrics passing through them. Since all the quadrics of the bundle have at least one fixed eighth point in common, we have the following theorem:

THEOREM I. *All the quadric surfaces which pass through seven independent points in space pass through a fixed eighth point.*

* Three algebraic surfaces whose equations are of degrees m, n, p, respectively, intersect in at least mnp distinct or coincident points. If they have more than mnp points in common, then they have one or more curves in common. For a proof of this theorem see Salmon: Lessons Introductory to Modern Higher Algebra, Arts. 73, 78. We shall assume the truth of this theorem.

These points are called eight **associated points**. If the coördinates of any fixed arbitrarily chosen point (y) are substituted in (8), the condition that (y) lies on the quadric furnishes one linear relation among the λ_i. Hence through (y) pass all the quadrics of a pencil and therefore a proper or composite quartic curve lying on every quadric of the pencil. This quartic curve passes through the eight associated points of the bundle.

If (y) is chosen on the line joining any two of the eight associated points, every quadric of the pencil passing through it will contain the whole line, since each quadric of the pencil contains three points on the line (Art. 65, Th. II). The residual intersection is a proper or composite cubic curve passing through the other six of the associated points and cutting the given line in two points.

137. Representation of the quadrics of a bundle by points of a plane. Let $\lambda_1, \lambda_2, \lambda_3$ be regarded as the coördinates of a point in a plane, which we shall call the λ-plane. To each point of the λ-plane corresponds a definite set of values of the ratios $\lambda_1 : \lambda_2 : \lambda_3$ and hence a definite quadric of the bundle (1) and conversely, so that the quadrics of the bundle and the points of the λ-plane are in one to one correspondence. To the points of any straight line in the λ-plane correspond the quadrics of a pencil contained in the bundle. The line will be said to correspond to the pencil. Since any two lines intersect in a point, it follows that any two pencils of quadrics contained in the bundle have one quadric in common.

138. Singular quadrics of the bundle. Those values of $\lambda_1, \lambda_2, \lambda_3$ which satisfy the equation

$$|\lambda_1 a_{ik} + \lambda_2 b_{ik} + \lambda_3 c_{ik}| = 0 \tag{9}$$

will define singular quadrics of the bundle. Unless special relations exist among the coefficients a_{ik}, b_{ik}, c_{ik}, none of these cones will be composite, for in that case all of the first minors of (9) must vanish, thus giving rise to three independent conditions among the $\lambda_1, \lambda_2, \lambda_3$, which are not satisfied for arbitrary values of the coefficients. It follows further that, under the same conditions, no two cones contained in the bundle have the same vertex. For, if $K=0$, $L=0$ were two cones having the same vertex, then every cone of the pencil $\lambda_1 K + \lambda_2 L = 0$ would have this point for a vertex. By choosing this point as vertex (0, 0, 0, 1) of the tetrahedron of reference, the pencil could be expressed in terms of the three variables x_1, x_2, x_3. The discriminant of this pencil equated to zero would be a cubic in $\lambda_1 : \lambda_2$ whose roots define composite cones which were shown above not to exist for arbitrary values of a_{ik}, b_{ik}, c_{ik}.

It follows from (9) that the points in the λ-plane determined by values of $\lambda_1, \lambda_2, \lambda_3$ which define cones of the bundle of (8) lie on a quartic curve C_4. Every point of this curve defines a cone of the bundle, and conversely. Each cone has a vertex, and it was just shown that no two cones have the same vertex. We have therefore the following theorem:

THEOREM. *The vertices of the cones in a general bundle describe a space curve J. The points of J are in one to one correspondence with the points of the curve C_4 in the λ-plane.*

The four points in which any line in the λ-plane intersects C_4 correspond to the four singular quadrics of the pencil which corresponds to the line. If P is any point on the quartic curve, the tangent line to C_4 at P defines a pencil of quadrics in which one singular quadric is counted twice; if the residual points of intersection of the tangent line and C_4 are distinct from each other and from the point of contact, the characteristic of the pencil is [211]. All the quadrics of the pencil pass through the vertex of the cone corresponding to the point of contact.

139. Intersection of the bundle by a plane. If the quadrics of the bundle (8) are not all singular, the equation

$$\begin{vmatrix} s_{11} & s_{12} & s_{13} & s_{14} & u_1 \\ s_{12} & s_{22} & s_{23} & s_{24} & u_2 \\ s_{13} & s_{23} & s_{33} & s_{34} & u_3 \\ s_{14} & s_{24} & s_{34} & s_{44} & u_4 \\ u_1 & u_2 & u_3 & u_4 & 0 \end{vmatrix} = 0, \tag{10}$$

wherein $s_{ik} = \lambda_1 a_{ik} + \lambda_2 b_{ik} + \lambda_3 c_{ik}$, is called the equation of the bundle in plane coördinates. If the coördinates of a given plane (u) are substantiated in (10), the resulting equation, if it does not vanish identically, is homogeneous of degree three in $\lambda_1, \lambda_2, \lambda_3$ and is consequently the equation of a cubic curve C_3 in the λ-plane. Equation (10) is the condition that the section of the quadric $(\lambda_1, \lambda_2, \lambda_3)$ by the plane (u) shall be composite. Every such composite conic in the plane (u) has at least one double point. It will now be shown that the locus of the point of tangency to (u) of the quadrics of the bundle which are touched by (u) is a cubic curve.

The equation of any plane (u) may be reduced to $x_4 = 0$ by a suitable choice of coördinates. Let $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$ be any set of values of $\lambda_1, \lambda_2, \lambda_3$ which satisfy (10) when we have replaced u_1, u_2, u_3, each, by zero and u_4 by 1.

The section of the quadric $\bar{\lambda}_1 A + \bar{\lambda}_2 B + \bar{\lambda}_3 C = 0$ by the plane $x_4 = 0$ is a composite conic having at least one double point $(y_1, y_2, y_3, 0)$. The coördinates of (y) must satisfy the relations

$$\bar{\lambda}_1 \Sigma a_{ik} y_k + \bar{\lambda}_2 \Sigma b_{ik} y_k + \bar{\lambda}_3 \Sigma c_{ik} y_k = 0, \text{ for } i = 1, 2, 3.$$

If from these three equations $\bar{\lambda}_1, \bar{\lambda}_2, \bar{\lambda}_3$ are eliminated, the result is the equation of the locus of the point of contact (y). Since the resulting equation is of degree three in the homogeneous variables y_1, y_2, y_3, the locus is a cubic curve. It is called the Jacobian of the net of conics in the given plane.

140. The vertex locus J. The **order** of a space curve is defined as the number of its (real and imaginary) intersections with a given plane.

We shall now prove the following theorem:

THEOREM. *The vertex locus J of a general bundle is of order six.*

For, the condition that the vertex of a cone of the bundle lies in a given plane (u) is that the corresponding point in the λ-plane lies on each of the curves (9) and (10). The theorem will follow if it is shown that these curves have contact of just the first order at each of the common points so that their twelve intersections coincide in pairs.

Let the given plane be taken as $x_4 = 0$. The equation of a cone of the bundle having its vertex in this plane can be reduced to

$$x_2^2 + x_3^2 + x_4^2 = 0,$$

and that of the bundle to the form

$$\lambda_1 A + \lambda_2 B + \lambda_3 (x_2^2 + x_3^2 + x_4^2) = 0.$$

The point in the λ-plane corresponding to the cone is $(0, 0, 1)$. It lies on $C_4(\lambda)$ and on $C_3(\lambda)$. It is to be shown that $C_4(\lambda)$, $C_3(\lambda)$ have the same tangent at $(0, 0, 1)$, but that they do not have contact of higher than the first order. In (9) put $c_{22} = c_{33} = c_{44} = 1$ and all the other $c_{ik} = 0$, and develop in powers of λ_3. The result may be written in the form

$$(a_{11}\lambda_1 + b_{11}\lambda_2)\lambda_3^3 + \left\{ \begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{12} & \phi_{22} \end{vmatrix} + \begin{vmatrix} \phi_{11} & \phi_{13} \\ \phi_{13} & \phi_{33} \end{vmatrix} + \begin{vmatrix} \phi_{11} & \phi_{14} \\ \phi_{14} & \phi_{44} \end{vmatrix} \right\} \lambda_3^2 + \cdots = 0,$$

wherein $\phi_{ik} = a_{ik}\lambda_1 + b_{ik}\lambda_2 = \phi_{ki}$.

Similarly in (10) put $u_1 = u_2 = u_3 = 0$, $u_4 = 1$, $c_{ik} = 0$, and develop in powers of λ_3. The result is

$$(a_{11}\lambda_1 + b_{11}\lambda_2)\lambda_3^2 + \left\{ \begin{vmatrix} \phi_{11} & \phi_{12} \\ \phi_{12} & \phi_{22} \end{vmatrix} + \begin{vmatrix} \phi_{11} & \phi_{13} \\ \phi_{13} & \phi_{33} \end{vmatrix} \right\} \lambda_3 + \cdots = 0.$$

These curves both pass through the point $(0, 0, 1)$ and have the same tangent $a_{11}\lambda_1 + b_{11}\lambda_2 = 0$ at that point. By making the two equations simultaneous, it is seen that they do not have contact of order higher than the first unless

$$\phi_{11}\phi_{44} \equiv \phi_{14}^2,$$

which is not satisfied unless particular relations exist among the coefficients a_{ik}, b_{ik}.

141. Polar theory in a bundle.

THEOREM. *The polar planes of a point (y) with regard to all the quadrics of a bundle pass through a fixed point (y').*

For, the polar plane of the point (y) with regard to a quadric of the bundle $\lambda_1 A + \lambda_2 B + \lambda_3 C = 0$ has the equation

$$\lambda_1 \Sigma a_{ik} x_i y_k + \lambda_2 \Sigma b_{ik} x_i y_k + \lambda_3 \Sigma c_{ik} x_i y_k = 0.$$

For all values of λ_1, λ_2, λ_3 this plane passes through the point (y') of intersection of the three planes

$$\Sigma a_{ik} x_i y_k = 0, \quad \Sigma b_{ik} x_i y_k = 0, \quad \Sigma c_{ik} x_i y_k = 0. \tag{11}$$

From the theorem that if the polar plane of (y) passes through (y'), then the polar plane of (y') passes through (y), it follows that all the points in space are arranged in pairs of points (y), (y')

conjugate as to every quadric of the bundle. Since the coördinates y_1, y_2, y_3, y_4 and y'_1, y'_2, y'_3, y'_4 appear symmetrically in the equations

$$\Sigma a_{ik} y'_i y_k = 0, \quad \Sigma b_{ik} y'_i y_k = 0, \quad \Sigma c_{ik} y'_i y_k = 0$$

defining the correspondence between (y) and (y'), the correspondence is called **involutorial.**

By solving the equations defining the correspondence for y'_1, y'_2, y'_3, y'_4 we obtain

$$\sigma y_1' = \begin{vmatrix} \Sigma a_{2k} y_k & \Sigma a_{3k} y_k & \Sigma a_{4k} y_k \\ \Sigma b_{2k} y_k & \Sigma b_{3k} y_k & \Sigma b_{4k} y_k \\ \Sigma c_{2k} y_k & \Sigma c_{3k} y_k & \Sigma c_{4k} y_k \end{vmatrix}$$

and similar expressions for y'_2, y'_3, y'_4. If we denote the second members of the respective equations by $\phi_i(y)$, then replace both y_i and y'_i by x_i and x'_i, respectively, the equations defining the involution may be written in the form

$$\sigma x'_i = \phi_i(x), \quad \rho x_i = \phi_i(x'). \tag{12}$$

If (y) describes a plane $\Sigma u_i x_i = 0$, the equation of the locus of (y') may be obtained by eliminating the coördinates of (y) from (11) and the equation $\Sigma u_i y_i = 0$. The result is

$$\begin{vmatrix} u_1 & u_2 & u_3 & u_4 \\ \Sigma a_{1k} x_k & \Sigma a_{2k} x_k & \Sigma a_{3k} x_k & \Sigma a_{4k} x_k \\ \Sigma b_{1k} x_k & \Sigma b_{2k} x_k & \Sigma b_{3k} x_k & \Sigma b_{4k} x_k \\ \Sigma c_{1k} x_k & \Sigma c_{2k} x_k & \Sigma c_{3k} x_k & \Sigma c_{4k} x_k \end{vmatrix} = 0. \tag{13}$$

Hence, if (y) describes a plane, (y') describes a cubic surface. Similarly, if (y') describes a plane, (y) describes a cubic surface.

If (y') describes a line l, the point (y) to which it corresponds describes a curve of order three. For, corresponding to each intersection of the locus of (y) with the plane $\Sigma u_i x_i = 0$ there is a point of intersection of l and the cubic surface, image of the plane. But l intersects the surface (13) in three points, hence $\Sigma u_i x_i = 0$ intersects the locus of (y) in three points; that is, the locus is a curve of order three. Similarly, if (y) describes a straight line, (y') will describe a curve of order three.

The vertex locus J lies on the surface (13) for all positions of the plane $\Sigma u_i x_i = 0$. For, let (y') be any point on J. Since (y') is the vertex of a cone belonging to the bundle, its polar plane with respect to this cone is indeterminate (Art. 121). Hence there exists a set of values of λ_1, λ_2, λ_3, not all zero, for which this plane is indeterminate. It follows that the matrix

$$\left\| \begin{matrix} \Sigma a_{1k}x_k & \Sigma a_{2k}x_k & \Sigma a_{3k}x_k & \Sigma a_{4k}x_k \\ \Sigma b_{1k}x_k & \Sigma b_{2k}x_k & \Sigma b_{3k}x_k & \Sigma b_{4k}x_k \\ \Sigma c_{1k}x_k & \Sigma c_{2k}x_k & \Sigma c_{3k}x_k & \Sigma c_{4k}x_k \end{matrix} \right\|$$

is of rank at most two. Thus, in the equation of the cubic surface (13), the coefficient of each u_i vanishes when the coördinates of any point J are substituted in it; hence the equation is satisfied for all values of (u_1, u_2, u_3, u_4).

Any two planes $\Sigma u_i x_i = 0$, $\Sigma v_i x_i = 0$ intersect in a line; their image surfaces intersect in a composite curve of order nine, consisting of J and the cubic curve, image of the line. If the point (y) is the vertex of a cone belonging to the bundle, the three polar planes of (y) determined by (11) belong to a pencil. Let l be the axis of this pencil. Every point of the line l corresponds to (y) in the correspondence (11), since it is involutorial.

As (y) describes J, its corresponding line l describes a ruled surface R. The image of a cubic surface $\Sigma u_i \phi_i = 0$ in the involution (12) is the plane $\Sigma u_i y_i = 0$ and a residual surface of order eight. As this residual surface is the locus of l, we conclude that the ruled surface R is of order eight.

142. Some special bundles. While it would lead beyond the scope of this book to give a complete classification of bundles of quadrics, like that for pencils of quadrics as developed in Arts. 131–133, still it is desirable to mention a few particular cases. It was seen (Art. 138) that in the general bundle there are no composite quadrics. But bundles containing composite quadrics may be constructed; for example, the bundle

$$\lambda_1 A + \lambda_2 B + \lambda_3 x_1 x_2 = 0$$

evidently contains the composite quadric $x_1 x_2 = 0$. If $x_1 = 0$ intersects the curve of intersection of $A = 0$, $B = 0$ in four points, and if $x_2 = 0$ intersects it in four points, so that no component of the curve lies in either plane $x_1 = 0$, $x_2 = 0$, then these two sets of four points constitute eight associated points.

Every point of the line $x_1=0$, $x_2=0$ is a vertex of a composite cone of the bundle. The locus J consists of this line and of a residual curve of order five. The image curve $C_4(\lambda)$ in the λ-plane has a double point corresponding to the composite quadric, as may be seen as follows. The equation of $C_4(\lambda)=0$ now has the form

$$\lambda_3^2\phi_2(\lambda_1, \lambda_2)+\lambda_3\phi_3(\lambda_1, \lambda_2)+\phi_4(\lambda_1, \lambda_2)=0,$$

in which ϕ_2, ϕ_3, ϕ_4 do not contain λ_3. Hence the point $\lambda_1=0$, $\lambda_2=0$ is a double point on $C_4(\lambda)=0$; it corresponds to the quadric $x_1x_2=0$. The points of $C_4(\lambda)$ are now in one to one correspondence with the curve of order five, forming one part of J, and the double point is associated with the whole line $x_1=0$, $x_2=0$.

Similarly, bundles of quadrics may be constructed having eight associated double points lying on two, three, four, five, or six pairs of planes. In the last case the equation of the bundle may be written in the form

$$\lambda_1(x_1^2-x_4^2)+\lambda_2(x_2^2-x_4^2)+\lambda_3(x_3^2-x_4^2)=0.$$

The eight associated points are $(\pm 1, \pm 1, \pm 1, 1)$. The curve J consists of the six edges of a tetrahedron and $C_4(\lambda)$ is composed of the four sides of a quadrilateral. Its equation is

$$\lambda_1\lambda_2\lambda_3(\lambda_1+\lambda_2+\lambda_3)=0.$$

In this case the equations (12) of the involution (y), (y') have the simple form

$$y'_i=\frac{\sigma}{y_i}, \qquad i=1, 2, 3, 4,$$

in which σ is constant.

Bundles of quadrics exist having a common curve and one or more distinct common points. The spheres through two fixed points furnish an example.

EXERCISES

1. Show that (1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0), (0, 0, 0, 1), (1, 1, 1, 1), (1, 1, -1, -1), (1, -1, 1, -1), (1, -1, -1, 1) are eight associated points.

2. Prove that if P is a given point and l a given line through it, there is one and only one quadric of the bundle to which l is tangent at P.

3. Determine the characteristic of the pencil of quadrics in a general bundle corresponding to:

(*a*) A tangent to $C_4(\lambda)$.
(*b*) A double tangent to $C_4(\lambda)$.
(*c*) An inflexional tangent to $C_4(\lambda)$.

4. What is the general condition under which $C_4(\lambda)$ may have a double point?

5. Determine the nature of the bundle

$$\lambda_1(x_1^2 - x_2x_3) + \lambda_2(x_1^2 + x_2^2 + x_3^2 - 4\,x_4^2) + \lambda_3(x_1^2 - x_3^2) = 0$$

and of the involution of corresponding points (y), (y').

6. If three quadrics have a common self-polar tetrahedron, the twenty-four tangent planes at their eight intersections all touch a quadric.

7. Write the equation of a bundle of quadrics passing through two given skew lines and a given point.

8. If four of the eight common tangent planes of three quadrics meet in a point, the other four all meet in a point.

9. Show that the cubic curve, image of an arbitrary line, intersects the locus of vertices J in 8 points.

10. Show that the surface R of Art. 141 contains J as a threefold curve.

143. Webs of quadrics. If $A = \Sigma a_{ik}x_ix_k = 0$, $B = \Sigma b_{ik}x_ix_k = 0$, $C = \Sigma c_{ik}x_ix_k = 0$, $D = \Sigma d_{ik}x_ix_k = 0$ are four quadrics not belonging to the same bundle, the linear system

$$\lambda_1 A + \lambda_2 B + \lambda_3 C + \lambda_4 D = 0 \tag{14}$$

is called a **web** of quadrics. Through any point in space pass all the quadrics of a bundle belonging to the web, through any two independent points a pencil, and through any three independent points, a single quadric of the web.

144. The Jacobian surface of a web. The polar planes of a point (y) with regard to the quadrics of a web form a linear system

$$\lambda_1\Sigma a_{ik}x_iy_k + \lambda_2\Sigma b_{ik}x_iy_k + \lambda_3\Sigma c_{ik}x_iy_k + \lambda_4\Sigma d_{ik}x_iy_k = 0. \tag{15}$$

If the point (y) is chosen arbitrarily, this plane may, by giving λ_1, λ_2, λ_3, λ_4 suitable values, be made to coincide with any plane in space, unless there are particular relations among the coefficients $a_{ik}, b_{ik}, c_{ik}, d_{ik}$. Thus an arbitrary plane is the polar plane of (y) with regard to some quadric of the web. There exists a locus of points (y) whose polar planes with regard to all the quadrics of a web pass through a fixed point (y'). This locus is called the Jacobian of the web. Since the equations connecting (y) and (y') are symmetrical, it follows that (y') also lies on the Jacobian. A pair of points (y), (y') such that all the polar planes of each pass through the other are called conjugate points on the Jacobian.

To determine the equation of the Jacobian, we impose the condition that the four polar planes of (y)

$$\Sigma a_{ik}x_iy_k = 0, \quad \Sigma b_{ik}x_iy_k = 0, \quad \Sigma c_{ik}x_iy_k = 0, \quad \Sigma d_{ik}x_iy_k = 0$$

pass through a point. The result is

$$K_4 = \begin{vmatrix} \Sigma a_{1i}y_i & \Sigma a_{2i}y_i & \Sigma a_{3i}y_i & \Sigma a_{4i}y_i \\ \Sigma b_{1i}y_i & \Sigma b_{2i}y_i & \Sigma b_{3i}y_i & \Sigma b_{4i}y_i \\ \Sigma c_{1i}y_i & \Sigma c_{2i}y_i & \Sigma c_{3i}y_i & \Sigma c_{4i}y_i \\ \Sigma d_{1i}y_i & \Sigma d_{2i}y_i & \Sigma d_{3i}y_i & \Sigma d_{4i}y_i \end{vmatrix} = 0. \tag{16}$$

The condition that a point (y) is the vertex of a cone contained in the web is that its coördinates satisfy the equations

$$\lambda_1\Sigma a_{ik}y_i + \lambda_2\Sigma b_{ik}y_i + \lambda_3\Sigma c_{ik}y_i + \lambda_4\Sigma d_{ik}y_i = 0, \ k = 1, 2, 3, 4 \tag{17}$$

for some values of $\lambda_1, \lambda_2, \lambda_3, \lambda_4$.

By eliminating $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ we obtain equation (16). This gives the theorem:

THEOREM. *The Jacobian surface is the locus of the vertices of the cones contained in the web of quadrics.*

Now let (y) be a point whose polar planes not only pass through a point but have a line l in common. At such a point (y), not only is $K_4 = 0$, but all the first minors of (16) are zero. Since equations (17) are in this case not independent conditions on $\lambda_1, \lambda_2, \lambda_3, \lambda_4$, there exists at least a pencil of quadrics in the web which have (y) for vertex. Three quadrics of this pencil are composite (Art. 131) and their lines of vertices all pass through (y). The polar planes of the points of the line of vertices of any composite quadric of the pencil have a point on l in common. Since the polar planes of the points of l pass through (y), the line l lies on $K_4 = 0$. There are as many lines l on $K_4 = 0$ as there are sets of values of λ_1, λ_2, λ_3, λ_4 that will make all the first minors of (16) vanish. In the general web there are ten such sets.*

Three lines l pass through each point (y), and three points (y) lie on each line l.

145. Correspondence with the planes of space. The polar plane of a fixed poin (y) with regard to any quadric Q of the web will be called the associated plane of (y) as to Q. When Q describes a pencil, its associated plane will describe a pencil; when Q describes a bundle, its associated plane will describe a bundle. The quartic curve of intersection of two quadrics of the web corresponds to the line of intersection of their associated planes, and to every set of eight associated points of a bundle of quadrics in the web corresponds one point, the vertex of the bundle of associated planes. Through any two points a straight line can be drawn, hence through any two sets of eight associated points within the web can be passed a pencil of quadrics belonging to the web. Since through any three points a plane can be passed, it follows that a quadric of the web can be found which passes through any three sets of eight associated points in the web.

146. Web with six basis points. The maximum number of distinct basis points a web can have without having a basis curve is six. Let 1, 2, 3, 4, 5, 6 designate the six basis points of a web having six basis points. All the quadrics of the web through an arbitrary point P belong to a bundle, and hence have eight associated points (Art. 136) in common, the eighth point P' being fixed when 1, 2, 3, 4, 5, 6 and P are given. Between $P \equiv (\xi)$ and $P' \equiv (\xi')$ exists a non-linear correspondence.

We shall now prove the following theorem:

THEOREM I. *In the case of a web with six distinct basis points, the Jacobian surface $K_4 = 0$ is also the locus of points (ξ) such that $(\xi) \equiv (\xi')$.*

* Salmon: Lessons Introductory to Modern Higher Algebra, Lesson XIX. The configuration of these lines on the Jacobian has been studied by Reye. See Crelle's Journal, Vol. 86 (1880).

In order to prove this we shall prove the following theorems:

THEOREM II. *The quadrics of a bundle of the web which pass through the vertex of a given cone of the web have, at this vertex, a common tangent line.*

THEOREM III. *Conversely, if all the quadrics of a bundle have a common tangent line at a given point, a cone belonging to the bundle has its vertex at the point.*

To prove Theorem II, let the vertex of the given cone be $(1, 0, 0, 0)$, so that its equation $C=0$ does not contain x_1. Let $A=0$, $B=0$ be any two non-singular quadrics of the bundle passing through the point, so that $a_{11}=0$, $b_{11}=0$. The equation of the tangent plane to the quadric $\lambda_1 A+\lambda_2 B+\lambda_3 C=0$ at $(1, 0, 0, 0)$ is

$$\lambda_1(a_{12}x_2+a_{13}x_3+a_{14}x_4)+\lambda_2(b_{12}x_2+b_{13}x_3+b_{14}x_4)=0.$$

But these planes all contain the line

$$a_{12}x_2+a_{13}x_3+a_{14}x_4=0, \quad b_{12}x_2+b_{13}x_3+b_{14}x_4=0,$$

which proves the proposition.

To prove Theorem III, let $x_1=0$, $x_2=0$ be the equations of the line, and $(0, 0, 0, 1)$ the common point. We may then take

$$\begin{aligned} A &= 2\,a_{14}x_1x_4+\phi(x_1, x_2, x_3)=0, \\ B &= 2\,b_{24}x_2x_4+\psi(x_1, x_2, x_3)=0, \\ C &= 2\,c_{14}x_1x_4+2\,c_{24}x_2x_4+f(x_1, x_2, x_3)=0, \end{aligned}$$

wherein ϕ, ψ, f contain only x_1, x_2, x_3.

In the bundle

$$\lambda_1 A+\lambda_2 B+\lambda_3 C=0,$$

the quadric corresponding to $\lambda_1=-c_{14}b_{24}$, $\lambda_2=-a_{14}c_{24}$, $\lambda_3=a_{14}b_{24}$ is a cone with vertex at $(0, 0, 0, 1)$ since the equation of the quadric does not contain x_4.

Since at the vertex of every cone two associated points coincide, and conversely, at every coincidence is the vertex of a cone, the proposition of Theorem I follows.

The ten pairs of planes determined by the six basis points 1, 2, 3, 4, 5, 6 taken in groups of three, as, for example, the pair of planes (123), (456), are composite quadrics of the web. The line of vertices of each pair lies on $K_4=0$. The surface $K_4=0$ also contains the fifteen lines joining the basis points by twos, since through any point of such a line five lines can be drawn to the six basis points, and a quadric cone of the web is fixed by these five lines.

If the basis points are taken for vertices of the tetrahedron of reference, the unit point, and the point (a_1, a_2, a_3, a_4), the equation of $K_4=0$ is found to be

$$\begin{vmatrix} a_1x_2x_3x_4 & x_1 & a_1 & 1 \\ a_2x_3x_4x_1 & x_2 & a_2 & 1 \\ a_3x_4x_1x_2 & x_3 & a_3 & 1 \\ a_4x_1x_2x_3 & x_4 & a_4 & 1 \end{vmatrix} = 0.$$

This surface is known as the **Weddle surface.***

If in (17) the values of y_1, y_2, y_3, y_4 are eliminated, the resulting equation $\Delta(\lambda)=0$ of degree four in the λ_i will define those values for which the equation $\lambda_1A+\lambda_2B+\lambda_3C+\lambda_4D=0$ is a cone of the web. The vertex of this cone is a point $(\xi)=(\xi')$. Let $\lambda_1, \lambda_2, \lambda_3, \lambda_4$ be considered as the tetrahedral coördinates of a plane. To each plane (λ) corresponds a quadric of the web (14) and conversely. A linear equation with given coefficients $a\lambda_1+b\lambda_2+c\lambda_3+d\lambda_4=0$ determines a point in the λ-space (Art. 91). By making this equation and (14) simultaneous, we define a bundle whose basis points are the points (x) whose coördinates satisfy the equations

$$\frac{A}{a}=\frac{B}{b}=\frac{C}{c}=\frac{D}{d}.$$

Of the eight associated points so determined, the given points 1, 2, 3, 4, 5, 6 are six. Either of the remaining points $P\equiv(\xi)$, $P'\equiv(\xi')$ will uniquely determine the other and also uniquely determine the point (a, b, c, d) in the λ-space. The equation $a\lambda_1+b\lambda_2+c\lambda_3+d\lambda_4=0$ thus defines a one to two correspondence between the points of the λ-space and the points P and P'. For

* First discussed in the Cambridge and Dublin Mathematical Journal, Vol. 5 (1850), p. 69.

points of K, P and P' coincide. The locus of the corresponding point (a, b, c, d) is called the **Kummer surface.***

We have thus proved the following theorem:

THEOREM IV. *The points of the Weddle surface and the points of the Kummer surface are in one to one correspondence.*

EXERCISES

1. Show that all the quadrics having a common self-polar tetrahedron form a web.

2. Determine the Jacobian of the web of Ex. 1.

3. Determine under what conditions the Jacobian of a web will have a plane as component.

4. Find the Jacobian of the web defined by the spheres passing through the origin $x = 0$, $y = 0$, $z = 0$.

5. Show that the Jacobian of a web having two basis lines is indeterminate.

6. Discuss the involution of conjugate points (y), (y') for the web of Ex. 4.

7. Show that the spheres cutting a given sphere orthogonally define a web.

8. Show that the equation of the quadric determined by the lines joining the points $(1, 0, 0, 0)$, (a_1, a_2, a_3, a_4); $(0, 1, 0, 0)$, $(0, 0, 1, 0)$; $(1, 1, 1, 1)$, $(0, 0, 0, 1)$ is

$$x_4x_1(a_2 - a_3) + (a_3x_2 - a_2x_3) + x_1(a_4x_3 - a_4x_2) = 0.$$

147. Linear systems of rank *r*. The linear system of quadrics

$$\lambda_1F_1 + \lambda_2F_2 + \cdots + \lambda_rF_r = 0, \tag{19}$$

wherein

$$F_j = \Sigma a_{ik}^{(j)}x_ix_k, \qquad j = 1, 2, \cdots r$$

is said to be of rank r, if the matrix

$$\left\| \begin{matrix} a_{11}^{(1)} & a_{22}^{(1)} & a_{33}^{(1)} \cdots a_{34}^{(1)} \\ a_{11}^{(2)} & a_{22}^{(2)} & a_{33}^{(2)} \cdots a_{34}^{(2)} \\ \cdot & \cdot & \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \\ \cdot & \cdot & \cdot \quad \cdot \\ a_{11}^{(r)} & a_{22}^{(r)} & a_{33}^{(r)} \cdots a_{34}^{(r)} \end{matrix} \right\| \tag{20}$$

* First discussed by E. E. Kummer in the Monatsberichte der k. preussischen Akademie der Wissenschaften, Berlin, 1863.

is of rank r, that is, if there does not exist a set of values of λ_1, λ_2, $\cdots$, λ_r, not all zero, such that the expression

$$\lambda_1 F_1 + \lambda_2 F_2 + \cdots + \lambda_r F_r$$

is identically zero. All the quadrics in space form a linear system of rank ten, since the equation of any quadric may be expressed linearly in terms of the ten quadrics, x_1^2, x_2^2, $\cdots$, x_3x_4 for which the matrix (20) is of rank ten.

All the quadrics in space whose coefficients satisfy $10-r$ independent homogeneous linear equations form a linear system of rank r. For, if $\Sigma b_{ik}x_ix_k = 0$ is the equation of any quadric whose coefficients satisfy the given conditions, then all the coefficients b_{ik} can be expressed linearly in terms of the coefficients of r quadrics belonging to the system. Thus

$$b_{ik} = \lambda_1 a_{ik}^{(1)} + \lambda_2 a_{ik}^{(2)} + \cdots + \lambda_r a_{ik}^{(r)}, \quad i, k = 1, 2, 3, 4, \qquad (21)$$

wherein

$$\Sigma a_{ik}^{(1)} x_i x_k = 0, \quad \cdots, \quad \Sigma a_{ik}^{(r)} x_i x_k = 0$$

are fixed quadrics belonging to the system.

Conversely, $10-r$ independent homogeneous linear conditions may be found which are satisfied by the coefficients in the equations of the quadrics $F_1 = 0$, $F_2 = 0$, $\cdots$, $F_r = 0$, and consequently by the coefficients in the equations of all the quadrics of the linear system (19) of rank r.

148. Linear systems of rank r in plane coördinates. The system of quadrics

$$\lambda_1 \Phi_1 + \lambda_2 \Phi_2 + \cdots + \lambda_r \Phi_r = 0,$$

wherein $\Phi_l \equiv \Sigma \beta_{ik}^{(l)} u_i u_k$, is called a linear system of rank r in plane coördinates if there does not exist a set of values λ_1, λ_2, $\cdots$, λ_r for which the given equation is satisfied identically. These systems may be discussed in the same manner as that considered in the preceding article.

149. Apolarity. Let $F = \Sigma a_{ik} x_i x_k = 0$ be the equation of a quadric in point coördinates and $\Phi \equiv \Sigma \beta_{ik} u_i u_k = 0$ be the equation of a quadric in plane coördinates. If the equation

$$\Sigma a_{ik}\beta_{ik} \equiv a_{11}\beta_{11} + a_{22}\beta_{22} + a_{33}\beta_{33} + a_{44}\beta_{44} + 2\,a_{12}\beta_{12} + 2\,a_{13}\beta_{13} + 2\,a_{14}\beta_{14} + 2\,a_{23}\beta_{23} + 2\,a_{24}\beta_{24} + 2\,a_{34}\beta_{34} = 0 \qquad (22)$$

is satisfied by the coefficients in the equations of the two quadrics, $F=0$ is said to be apolar to $\Phi=0$, and $\Phi=0$ is said to be apolar to $F=0$. It should be noticed that in this definition the equation $F=0$ is given in point coördinates, and that of $\Phi=0$ in plane coördinates. It should also be noticed that if $F=0$ and $\Phi=0$ are two given apolar quadrics, and if $\Sigma\alpha_{ik}u_iu_k=0$ is the equation of $F=0$ in plane coördinates, and $\Sigma b_{ik}x_ix_k=0$ is the equation of $\Phi=0$ in point coördinates, then it does not necessarily follow that $\Sigma\alpha_{ik}b_{ik}=0$ because $\Sigma a_{ik}\beta_{ik}=0$.

In order to show the significance of the condition (22) of apolarity, we shall prove the following theorem:

THEOREM I. *The expression $a_{ik}\beta_{ik}$ is a relative invariant.*

Let the coördinates of space be subjected to the linear transformation

$$x_i=\alpha_{i1}x'_1+\alpha_{i2}x'_2+\alpha_{i3}x'_3+\alpha_{i4}x'_4, \quad i=1, 2, 3, 4$$

of determinant $T\neq 0$. The coördinates of the planes of space undergo the transformation (Art. 97)

$$u_i=A_{i1}u'_1+A_{i2}u'_2+A_{i3}u'_3+A_{i4}u'_4, \quad i=1, 2, 3, 4.$$

The equation $F(x)=0$ goes into $\Sigma a'_{ik}x'_ix'_k=0$,
wherein (Art. 104)

$$a_{ik}'=\Sigma_l\Sigma_m a_{lm}\alpha_{li}\alpha_{mk}$$

and $\Phi=0$ is transformed in $\Sigma\beta'_{ik}u_iu'_k=0$, wherein

$$\beta_{ik}=\Sigma_l\Sigma_m\beta_{lm}A_{li}A_{mk}.$$

The proof of the theorem consists in showing (Art. 104) that

$$\Sigma a'_{ik}\beta'_{ik}=T^n\Sigma a_{ik}\beta_{ik}.$$

In the first member, replace a'_{ik}, β'_{ik} by their values from the above equations, and collect the coefficients of any term $a_{lm}\beta_{lm}$ in the result. We find

$$\Sigma_l\Sigma_k\alpha_{li}A_{li}\cdot\alpha_{mk}A_{mk}=\Sigma_l\alpha_{li}A_{li}\cdot\Sigma_k\alpha_{mk}A_{mk}=T^2,$$

hence

$$\Sigma a'_{ik}\beta'_{ik}=T^2\Sigma a_{ik}\beta_{ik},$$

which proves the proposition.

The vanishing of this relative invariant may be interpreted geometrically by means of the following theorem:

THEOREM II. *If $F=0$, $\Phi=0$ are apolar quadrics, there exists a tetrahedron self-polar as to $\Phi=0$ and inscribed in $F=0$.*

This theorem should be replaced by others in the following exceptional cases in which no such tetrahedron exists.

(*a*) If $F=0$ is a plane counted twice. In this case (22) is the condition that the coördinates in this plane satisfy $\Phi=0$.

(*b*) If $\Phi=0$ is the equation of the tangent planes to a proper conic C and if $F=0$ intersects the plane of $C=0$ in a line counted twice, (22) is the condition that this line touches C.

We shall consider first the special cases (*a*) and (*b*).

Let $$F\equiv(u_1x_1+u_2x_2+u_3x_3+u_4x_4)^2.$$

Then $a_{ik}=u_iu_k$ and (22) reduces at once to $\Phi=0$.

In case (*b*), let the plane of C be taken as $x_4=0$ and the line of intersection of $F=0$ with $x_4=0$ be taken as $x_1=x_4=0$. Then

$$\Phi\equiv\beta_{11}u_1^2+\beta_{22}u_2^2+\beta_{33}u_3^2+2\,\beta_{12}u_1u_2+2\,\beta_{23}u_2u_3+2\,\beta_{31}u_3u_1=0,$$
and $$F\equiv a_{11}x_1^2+2\,a_{14}x_1x_4+2\,a_{24}x_2x_4+2\,a_{34}x_3x_4+2\,a_{44}x_4^2=0,$$

where $a_{11}\neq 0$. Hence (22) reduces to $\beta_{11}=0$, that is, to the condition that $x_1=x_4=0$ touches C.

To prove Theorem II, excluding cases (*a*) and (*b*), we must consider various cases. First suppose $\Phi=0$ is non-singular. Choose a point P_1 on $F=0$, not on the intersection $F=0$, $\Phi=0$, and find its polar plane π_1 as to $\Phi=0$. In π_1 take a point P_2 on $F=0$, not on $\Phi=0$, and find its polar plane π_2 as to $\Phi=0$. On the line $\pi_1\pi_2$ choose a point P_3 on $F=0$, not on $\Phi=0$, and find its polar plane π_3. If the point of intersection of π_1, π_2, π_3 is called P_4, then $P_1P_2P_3P_4\equiv\pi_1\pi_2\pi_3\pi_4$ is taken for the tetrahedron of reference; we may, by proper choice of the unit plane, reduce the equation of $\Phi=0$ to $u_1^2+u_2^2+u_3^2+u_4^2=0$. Equation (22) now has the form $a_{11}+a_{22}+a_{33}+a_{44}=0$. Since three of the vertices P_1, P_2, P_3 were chosen on $F=0$, three coefficients $a_{ii}=0$, hence the fourth must also vanish, which proves the proposition for this case.

It should be observed that if $F=0$, $\Phi=0$ define the same quadric, equation (22) cannot be satisfied since their equations may be reduced simultaneously to

$$F = x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \qquad \Phi = u_1^2 + u_2^2 + u_3^2 + u_4^2 = 0.$$

Now let $\Phi = 0$ be the equation of the tangent planes to a proper conic c. Take the plane of C as $x_4 = 0$, so that

$$\beta_{14} = \beta_{24} = \beta_{34} = \beta_{44} = 0.$$

If $F = 0$ is composite and x_4 is one component, equation (22) is identically satisfied. In this case we may take three vertices of a triangle in $x_4 = 0$ self-polar as to the conic C and any point on $F = 0$ not on $x_4 = 0$ as vertices of a tetrahedron self-polar to $\Phi = 0$ and inscribed in $F = 0$. If $F = 0$ consists of $x_4 = 0$ counted twice, (22) expresses the condition that the plane belongs to $\Phi = 0$, whether $\Phi = 0$ is singular or not. This is the exceptional case (a).

If $x_4 = 0$ is not a component of $F = 0$, (22) has the form

$$a_{11}\beta_{11} + a_{22}\beta_{22} + a_{33}\beta_{33} + 2\,a_{12}\beta_{12} + 2\,a_{13}\beta_{13} + 2\,a_{23}\beta_{23} = 0,$$

which is the condition that the section C' of $F = 0$ by the plane $x_4 = 0$ is apolar to C.

It follows by the theorem for apolar conics analogous to Theorem II that a triangle exists which is inscribed in C' and is self-polar to C. A tetrahedron having the vertices of this triangle for three of its vertices and a fourth vertex on $F = 0$ but not on $x_4 = 0$ satisfies the condition of the theorem (dual of Th. I, Art. 121).

If $\Phi = 0$ is the equation of two distinct points, (22) expresses the condition that these points are conjugate as to $F = 0$. This is also the condition that a tetrahedron exists which is inscribed in $F = 0$ and is self-polar to $\Phi = 0$. If $\Phi = 0$ is the equation of a point counted twice, (22) expresses that the point lies on $F = 0$. This is the dual of the exceptional case (a).

In each of the above cases, the tetrahedron which satisfies the conditions of the theorem can be chosen in an infinite number of ways, hence we have the following theorem.

Theorem III. *If one tetrahedron exists which is inscribed in $F = 0$ and is self-polar as to $\Phi = 0$, then an infinite number of such tetrahedra exist.*

By duality we have the following theorems:

THEOREM IV. *If $F=0$, $\Phi=0$ are apolar quadrics, there exists a tetrahedron self-polar as to $F=0$ and circumscribed to $\Phi=0$.*

THEOREM V. *If one tetrahedron exists which is circumscribed to $\Phi=0$ and is self-polar as to $F=0$, then an infinite number of such tetrahedra exist.*

Moreover, both the exceptional cases of Theorem II have an immediate dual interpretation; they will not be considered further.

With the aid of these results we can now give an interpretation to the vanishing of the coefficients Θ and Θ' of equation (3), Art. 124, and of $\Psi_1(u)$, $\Psi_2(u)$ of equation (7), Art. 135. If $B=0$ in (1) is non-singular, let its equation in plane coördinates be $\Sigma\beta_{ik}u_iu_k=0$. Since β_{ik} is the first minor of b_{ik} in the discriminant of $B=0$, it follows at once from equation (3) that $\Theta'\equiv\Sigma a_{ik}\beta_{ik}$. Hence $\Theta'=0$ is the condition that $A=0$ is apolar to $B=0$. If $B=0$ is a cone, it is similarly seen that $\Theta'=0$ is the condition that the vertex of the cone $B=0$ lies on $A=0$. If $B=0$ is composite, Θ' is identically zero, independently of A, since the discriminant of $B=0$ is of rank two, hence all the coefficients β_{ik} vanish. An analogous discussion holds for $\Theta=0$.

The surface $\Psi_1(u)=0$ (Art. 135) may be defined as the envelope of a plane which intersects $A=0$ in a conic which is apolar to the conic in which it intersects $B=0$. For particular singular quadrics this definition will not always apply.

Let an arbitrary plane of $\Psi_1(u)=0$ be taken as $x_4=0$. It follows from equation (7) that

$$|a_{11}b_{22}b_{33}|+|b_{11}a_{22}b_{33}|+|b_{11}b_{22}a_{33}|=0. \tag{23}$$

Let the sections of $A=0$, $B=0$ by $x_4=0$ be C, C', respectively. If C' is not composite, it is seen by writing the equation of C' in line coördinates that (23) is the condition that C is apolar to C'. If C' is a pair of distinct lines, (23) is the condition that their point of intersection lies on C. If C' is a line counted twice, (23) is satisfied identically for all values of a_{ik}, since all the first minors of the discriminant of C' vanish.

An analogous discussion holds for $\Psi_2(u)=0$.

150. Linear systems of apolar quadrics. Since equation (22) is linear in the coefficients of $F=0$, from Art. 147 we may state the following theorem:

THEOREM I. *All the quadrics apolar to a given quadric form a linear system of rank nine.*

Conversely, since the coefficients of the equations of all the quadrics of a linear system of rank nine satisfy a linear condition which may be written in the form of equation (27), we have the further theorem:

THEOREM II. *All the quadrics of any linear system of rank nine are apolar to a fixed quadric.*

From the condition that a plane counted twice is apolar to a quadric (Art. 149), it follows that this fixed quadric is the envelope of the double planes of the given linear system.

If a quadric $F=0$ is apolar to each of r quadrics

$$\Phi_1 = \Sigma\beta_{ik}^{(1)}u_iu_k = 0, \quad \Phi_2 = \Sigma\beta_{ik}^{(2)}u_iu_k = 0, \cdots,$$
$$\Phi_r = \Sigma\beta_{ik}^{(r)}u_iu_k = 0,$$

the coefficients in its equation satisfy the r conditions

$$\Sigma a_{ik}\beta_{ik}^{(1)} = 0, \quad \Sigma a_{ik}\beta_{ik}^{(2)} = 0, \cdots, \quad \Sigma a_{ik}\beta_{ik}^{(r)} = 0. \tag{24}$$

It follows that if a quadric is apolar to each of the given quadrics, it is apolar to all the quadrics of the linear system

$$\lambda_1\Phi_1 + \lambda_2\Phi_2 + \cdots + \lambda_r\Phi_r = 0.$$

The conditions that this linear system is of rank r are equivalent to the conditions that the corresponding equations (24) are indedendent. Hence:

THEOREM III. *All the quadrics apolar to the quadrics of a linear system of rank r in plane coördinates form a linear system of rank $10-r$ in point coördinates and dually.*

EXERCISES

1. Find the equation of the quadric in plane coördinates to which all the quadrics through a point are apolar.

2. How many double planes are there in a general linear system of rank seven in point coördinates?

3. Show that all the pairs of points in a linear system of rank six in plane coördinates lie on a quadric surface.

4. Show that all the spheres in space form a linear system and find its rank.

5. Find the system apolar to the system in Ex. 5.

6. Show that a system of confocal quadrics (Art. 84) is a linear system of rank two in plane coördinates. Determine the characteristic and the singular quadrics of the system (Art. 133).

7. Show that, if the matrix (20) is of rank $r' < r$, the system of quadrics (19) is a linear system of rank r'.

CHAPTER XII

TRANSFORMATIONS OF SPACE

151. Projective metric. In order to characterize a transformation of motion, either translation, or rotation, or both, or a transformation involving motion and reflection, as a special case of a projective transformation, it will first be shown under what circumstances orthogonality is preserved when a new system of coördinates is chosen.

If the new axes can be obtained from the old ones by motion and reflection, the plane $t = 0$ must evidently remain fixed, and the expression $x^2 + y^2 + z^2$, which defines the square of the distance from the point $(0, 0, 0, 1)$ to the point $(x, y, z, 1)$, must be transformed into itself or into $(x - at)^2 + (y - bt)^2 + (z - ct)^2$, according as the point $(0, 0, 0, 1)$ remains fixed or is transformed into the point $(a, b, c, 1)$. It will be shown that, conversely, any linear transformation having this property is a motion or a motion and a reflection.

152. Pole and polar as to the absolute. We shall first point out the following relation between the direction cosines of a line and the coördinates of the point in which it pierces the plane at infinity.

THEOREM I. *The homogeneous coördinates of the point in which a line meets the plane at infinity are proportional to the direction cosines of the line.*

The equations of a line through the given finite point (x_0, y_0, z_0, t_0) and having the direction cosines (λ, μ, ν) are

$$\frac{t_0x - x_0t}{\lambda} = \frac{t_0y - y_0t}{\mu} = \frac{t_0z - z_0t}{\nu}. \tag{1}$$

The point $(x, y, z, 0)$ in which the line pierces the plane at infinity is given by the equations

$$\frac{t_0x}{\lambda} = \frac{t_0y}{\mu} = \frac{t_0z}{\nu},$$

from which the theorem follows.

We shall now establish the following theorems concerning poles and polars as to the absolute.

THEOREM II. *The necessary and sufficient condition that a plane and a line are perpendicular is that the line at infinity in the plane is the polar of the point at infinity on the line as to the absolute.*

The absolute was defined (Art. 49) as the imaginary circle in the plane at infinity defined by the equations

$$x^2 + y^2 + z^2 = 0, \qquad t = 0. \tag{2}$$

The polar line as to the absolute of the point $(\lambda, \mu, \nu, 0)$ in which the line (1) intersects the plane at infinity is

$$\lambda x + \mu y + \nu z = 0, \qquad t = 0. \tag{3}$$

The equation of any plane through this line is of the form

$$\lambda x + \mu y + \nu z + kt = 0. \tag{4}$$

These planes are all perpendicular to the line (1). Conversely, the equation of any plane perpendicular to the line (1) is of the form (4); the plane will therefore intersect the plane at infinity in the line (3).

THEOREM III. *The necessary and sufficient condition that two lines are perpendicular is that their points at infinity are conjugate as to the absolute.*

The condition that two lines are perpendicular is that each lies in a plane perpendicular to the other, that is, that each intersects the polar line of the point at infinity on the other as to the absolute.

Finally, since two planes are perpendicular if each contains a line perpendicular to the other, we have the following theorem:

THEOREM IV. *The necessary and sufficient condition that two planes are perpendicular is that their lines at infinity are conjugate as to the absolute.*

A tangent plane to the absolute is conjugate to any plane passing through the point of contact; in particular, it is conjugate to itself. It should be observed that the equation of a tangent plane to the absolute cannot be reduced to the normal form, hence we cannot speak of the direction cosines of such a plane.

Consider the pencil of planes passing through any real line. We may choose two perpendicular planes of the pencil as $x=0$, $y=0$, and write the equation of any other plane of the pencil in the form

$$y = mx.$$

The equations of the two tangent planes to the absolute which pass through this line are $y = ix$ and $y = -ix$. By using the usual formula to obtain the tangent of the angle ϕ between $y = ix$ and $y = mx$, we obtain

$$\tan \phi = \frac{m-i}{1+im} = \frac{m-i}{i(m-i)} = \frac{1}{i} = -i$$

independent of m. For this reason tangent planes to the absolute are called **isotropic planes.** The cone having its vertex at (a, b, c) and passing through the absolute has an equation of the form

$$(x-a)^2 + (y-b)^2 + (z-c)^2 = 0.$$

If we employ the formula of Art. 4 for the distance between two points, we see that the distance of any point of the cone from the vertex is equal to zero. For this reason the cone is called a minimal cone. Moreover, if P_1 and P_2 are any two points on the same generator, since

$$VP_2 - VP_1 = P_2P_1,$$

we conclude that the distance between any two points on any line that intersects the absolute is zero. For this reason these lines are called **minimal lines.** They have no direction cosines (Art. 3).

153. Equations of motion. Let an arbitrary point P be referred to a rectangular system of coördinates x, y, z, t and to a tetrahedral system x_1, x_2, x_3, x_4, with the restriction that $x_4 = 0$ is the equation of the plane at infinity $t = 0$. The equations connecting the two systems of coördinates are

$$\begin{aligned} \sigma x &= \lambda x_1 + \lambda' x_2 + \lambda'' x_3 + h x_4, \\ \sigma y &= \mu x_1 + \mu' x_2 + \mu'' x_3 + h' x_4, \\ \sigma z &= \nu x_1 + \nu' x_2 + \nu'' x_3 + h'' x_4, \\ \sigma t &= x_4. \end{aligned} \tag{5}$$

Divide the first three equations of (5) by the last, member by member, and replace the non-homogeneous coördinates $\frac{x}{t}$, etc., by

x', y', z' and $\frac{x_1}{x_4}$, etc., by x'_1, x'_2, x'_3. If P is any point not in the plane at infinity, we shall prove the following theorem:

THEOREM I. *The most general linear transformations of the form* (5) *that will transform the expression*

$$x'^2 + y'^2 + z'^2 \text{ into } x'^2_1 + x'^2_2 + x'^2_3$$

are the rotations and reflections about the point $x' = 0'$, $y' = 0$, $z' = 0$.

If we substitute the values of x', y', z' in the expression $x'^2 + y'^2 + z'^2$, we obtain

$$(\lambda x'_1 + \lambda' x'_2 + \lambda'' x'_3 + h)^2 + (\mu x'_1 + \mu' x'_2 + \mu'' x'_3 + h')^2 + (\nu x'_1 + \nu' x'_2 + \nu'' x'_3 + h'')^2.$$

If this is equal to $x'^2_1 + x'^2_2 + x'^2_3$ for all finite values of x'_1, x'_2, x_3', we have the following relations

$$\begin{aligned} &\lambda^2 + \mu^2 + \nu^2 = \lambda'^2 + \mu'^2 + \nu'^2 = \lambda''^2 + \mu''^2 + \nu''^2 = 1, \\ &\lambda\lambda' + \mu\mu' + \nu\nu' = \lambda'\lambda'' + \mu'\mu'' + \nu'\nu'' = \lambda''\lambda + \mu''\mu + \nu''\nu = 0, \qquad (6) \\ &h\lambda + h'\mu + h''\nu = 0,\ h\lambda' + h'\mu' + h''\nu' = 0,\ h\lambda'' + h'\mu'' + h''\nu'' = 0. \end{aligned}$$

Since the determinant $|\lambda\mu'\nu''|$ is not zero, it follows that $h = h' = h'' = 0$. The formulas (6) which do not contain h, h', h'' are exactly the relations among the coefficients to define a rotation or a rotation and reflection about the origin (Art. 37). This proves the proposition.

By similar reasoning we may prove the theorem:

THEOREM II. *Transformations that will transform*

$$x'^2 + y'^2 + z'^2 \text{ into } (x'_1 - a)^2 + (x'_2 - b)^2 + (x'_3 - c)^2$$

consist of motion or of motion and reflection.

154. Classification of projective transformations. The equations of any projective transformation (Art. 98) are of the form

$$\begin{aligned} kx'_1 &= a_{11}x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4, \\ kx'_2 &= a_{21}x_1 + a_{22}x_2 + a_{23}x_3 + a_{24}x_4, \\ kx'_3 &= a_{31}x_1 + a_{32}x_2 + a_{33}x_3 + a_{34}x_4, \\ kx'_4 &= a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + a_{44}x_4. \end{aligned} \qquad (7)$$

We shall now consider the problem of classifying the existing types of such transformations and of reducing their equations to the simplest form.

The invariant points of the transformation (7) are determined by those values of k which satisfy the equation

$$D(k) = \begin{vmatrix} a_{11}-k & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22}-k & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33}-k & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44}-k \end{vmatrix} = 0. \qquad (8)$$

The classification will depend fundamentally on the invariant factors (Art. 125) of this determinant.

In equation (7), (x) and (x') are regarded as different points, referred to the same system of coördinates. In order to simplify the equations, we shall refer both points to a new system of coördinates. To do this both (x) and (x') are to be operated upon by the same transformation

$$x_i = \Sigma p_{ik} y_k, \qquad x'_i = \Sigma p_{ik} y'_k.$$

We shall use the symbols (x), (y) to indicate coördinates of the same point, referred to two different systems of coördinates, while equations between (x) and (x') or between (y) and (y') will define a projective transformation between two different points, referred to the same system of coördinates.

Let k_1 be a root of $D(k)=0$. The four equations

$$\begin{aligned}(a_{11}-k_1)x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4 &= 0,\\ a_{21}x_1 + (a_{22}-k_1)x_2 + a_{23}x_3 + a_{24}x_4 &= 0,\\ a_{31}x_1 + a_{32}x_1 + (a_{33}-k_1)x_3 + a_{34}x_4 &= 0,\\ a_{41}x_1 + a_{42}x_2 + a_{43}x_3 + (a_{44}-k_1)x_4 &= 0\end{aligned}$$

are therefore consistent and determine at least one point invariant under the transformation.

Let $$\sum_{k=1}^{4} \beta_{ik} x_k = 0 \qquad (i = 2, 3, 4)$$

be the equations of three planes passing through this invariant point but not belonging to the same pencil, and let

$$\sum_{k=1}^{4} \beta_{1k} x_k = 0$$

be the equation of any plane not passing through the invariant point. If now we put

$$y_i = \Sigma\beta_{ik}x_k, \qquad i = 1, 2, 3, 4,$$

and solve the equations for the x_i,

$$x_i = \sum_{k=1}^{4}\gamma_{ik}y_k, \text{ and put also } x'_i = \Sigma\gamma_{ik}y'_k,$$

then substitute these values in the members of (7), the new equations, when solved for y'_i, will be of the form

$$\begin{aligned}
y'_1 &= k_1y_1 + b_{12}y_2 + b_{13}y_3 + b_{14}y_4,\\
y'_2 &= \qquad\quad\; b_{22}y_2 + b_{23}y_3 + b_{24}y_4,\\
y'_3 &= \qquad\quad\; b_{32}y_2 + b_{33}y_3 + b_{34}y_4,\\
y'_4 &= \qquad\quad\; b_{42}y_2 + b_{43}y_3 + b_{44}y_4.
\end{aligned}$$

Without changing the vertex (1, 0, 0, 0), the planes $y_2 = 0$, $y_3 = 0$, $y_4 = 0$ may be replaced by others by repeating this same process on the last three equations; in this way we may replace the coefficients b_{32}, b_{42} by 0; by a further application to the variables y_3, y_4 we may replace b_{43} by 0.

Referred to the system of coördinates just found, the equations of the projective transformation (7) are

$$\begin{aligned}
x'_1 &= k_1x_1 + c_{12}x_2 + c_{13}x_3 + c_{14}x_4,\\
x'_2 &= \qquad\quad\; c_{22}x_2 + c_{23}x_3 + c_{24}x_4,\\
x'_3 &= \qquad\qquad\qquad\quad\;\; c_{33}x_3 + c_{34}x_4,\\
x'_4 &= \qquad\qquad\qquad\qquad\qquad\;\; c_{44}x_4,
\end{aligned} \tag{9}$$

in which c_{22}, c_{33}, c_{44} are all roots of $D(k) = 0$.

Equations (9) represent the form to which the equations of any projective transformation may be reduced. The further simplification depends upon the values of the coefficients, that is, upon the characteristic (Art. 127) of $D(k)$.

If $c_{34} \neq 0$ and $c_{33} \neq c_{44}$, make the further transformation

$$x_1 = y_1, \quad x_2 = y_2, \quad x_3 = y_3 + \frac{c_{34}y_4}{c_{44} - c_{33}}, \quad x_4 = y_4.$$

On making this substitution we reduce the equations of (9) to a form in which the coefficient c_{34} is replaced by zero.

In any case, if $i<k$ and $c_{ii}\neq c_{kk}$, we may always remove the term c_{ik} by replacing x_i by $x_i+\frac{c_{ik}x_k}{c_{kk}-c_{ii}}$ in both members of the equation. If $c_{ii}=c_{kk}$ and $c_{ik}\neq 0$, by a change of unit point, c_{ik} may be replaced by unity; thus, if $c_{33}=c_{44}$ and $c_{34}\neq 0$, by writing $c_{34}x_4=y_4$, we obtain the equations

$$\begin{aligned} y'_3 &= c_{33}y_3 + y_4, \\ y'_4 &= c_{33}y_4. \end{aligned}$$

These two types of transformations will reduce the equations to their simplest form in every case in which $D(k)=0$ has no root of multiplicity greater than two.

If $D(k)=0$ has one simple root k_1 and a triple root k_2, the preceding method can be applied to reduce the equations of the transformation to

$$\begin{aligned} x'_1 &= k_1x_1, \\ x'_2 &= k_2x_2+a_{23}x_3+a_{24}x_4, \\ x'_3 &= k_2x_3+a_{34}x_4, \\ x'_4 &= k_2x_4. \end{aligned}$$

In case $a_{24}=0$, the preceding method can be applied again; thus, if $a_{34}\neq 0$, $a_{23}\neq 0$, each may be replaced by unity; if coefficients a_{23}, a_{24}, a_{34} are zero, the transformation is already expressed in its simplest form. If $a_{24}=0$, either or both of the coefficients a_{23} and a_{34}, if not zero, may be replaced by unity by a transformation of the type just discussed.

If $a_{24}\neq 0$, $a_{34}\neq 0$, replace x_2 by the substitution

$$x_2=y_2+\frac{a_{24}y_3}{a_{34}}.$$

In the transformed equation, the new a_{24} is zero. In the same way, if $a_{24}\neq 0$, $a_{34}=0$, but $a_{23}\neq 0$, put

$$x_3=y_3-\frac{a_{24}y_4}{a_{23}},$$

and the same result will be accomplished. Finally, if $a_{24}\neq 0$, but $a_{34}=0$, $a_{23}=0$, put

$$x_1=y_1, \quad x_2=y_3, \quad x_3=y_2, \quad x_4=y_4 \tag{10}$$

in both members of the equation. Now $a_{24}=0$, and the complete reduction can be made as before.

If $D(k)=0$ has a fourfold root k_1, equations (9) reduce to

$$\begin{aligned}
x'_1 &= k_1x_1 + a_{12}x_2 + a_{13}x_3 + a_{14}x_4,\\
x'_2 &= \qquad\quad\; k_1x_2 + a_{23}x_3 + a_{24}x_4,\\
x'_3 &= \qquad\qquad\qquad\;\; k_1x_3 + a_{34}x_4,\\
x'_4 &= \qquad\qquad\qquad\qquad\qquad\; k_1x_4.
\end{aligned}$$

By transformations analogous to those in the preceding case, the coefficients a_{13}, a_{14}, and a_{24} may be reduced to zero, and the coefficients a_{12}, a_{23}, and a_{34} to zero or to unity.

This completes the problem of reduction. The determination of the locus of the invariant points and the characteristic of $D(k)$ in the various cases is left as an exercise for the student. The results are collected in the following table.

155. Standard forms of equations of projective transformations.

CHARACTERISTIC	EQUATIONS		LOCUS OF INVARIANT POINTS
[1111]	$x'_1=k_1x_1,$ $x'_3=k_3x_3,$	$x'_2=k_2x_2$ $x'_4=k_4x_4$	Four distinct points.
[112]	$x'_1=k_1x_1,$ $x'_3=k_3x_3+x_4,$	$x'_2=k_2x_2$ $x'_4=k_3x_4$	Two distinct, two coincident points.
[11(11)]	$x'_1=k_1x_1,$ $x'_3=k_3x_3,$	$x'_2=k_2x_2$ $x'_4=k_3x_4$	Two distinct points and a line.
[13]	$x'_1=k_1x_1,$ $x'_3=k_2x_3+x_4,$	$x'_2=k_2x_2+x_3$ $x'_4=k_2x_4$	One distinct, three coincident points.
[1(21)]	$x'_1=k_1x_1,$ $x'_3=k_2x_3+x_4,$	$x'_2=k_2x_2$ $x'_4=k_2x_4$	A point and a line.
[1(111)]	$x'_1=k_1x_1,$ $x'_1=k_2x_3,$	$x'_2=k_2x_2$ $x'_2=k_2x_4$	A point and a plane.
[22]	$x'_1=k_1x_1+x_2,$ $x'_3=k_2x_3+x_4,$	$x'_2=k_1x_2$ $x'_4=k_2x_4$	Two pairs of coincident points.
[2(11)]	$x'_1=k_1x_1+x_2,$ $x'_3=k_2x_3,$	$x'_2=k_1x_2$ $x'_4=k_2x_4$	Two coincident points and a line.
[(11)(11)]	$x'_1=k_1x_1,$ $x'_3=k_2x_3,$	$x'_2=k_1x_2$ $x'_4=k_2x_4$	Two lines.
[4]	$x'_1=k_1x_1+x_2,$ $x'_3=k_1x_3+x_4,$	$x'_2=k_1x_2+x_3$ $x'_4=k_1x_4$	Four coincident points.

CHARACTERISTIC	EQUATIONS		LOCUS OF INVARIANT POINTS
[(22)]	$x'_1 = k_1x_1 + x_2,$ $x'_3 = k_1x_3 + x_4,$	$x'_2 = k_1x_2$ $x'_4 = k_1x_4$	A line.
[(13)]	$x'_1 = k_1x_1,$ $x'_3 = k_1x_3 + x_4,$	$x'_2 = k_1x_2 + x_3$ $x'_4 = k_1x_4$	A line.
[(112)]	$x'_1 = k_1x_1,$ $x'_3 = k_1x_3 + x_4,$	$x'_2 = k_1x_2$ $x'_4 = k_1x_4$	A plane.
[(1111)]	$x'_1 = k_1x_1,$ $x'_3 = k_1x_3,$	$x'_2 = k_1x_2$ $x'_4 = k_1x_4$	All points of space; the identical transformation.

EXERCISES

1. In type [1111] obtain the necessary and sufficient condition that the transformation obtained by applying the given transformation p times is the identity.

2. In [1(111)] show that the line joining any point P to its image P' always passes through the invariant point.

3. In Ex. 2, let O be the invariant point, and let a line PP' intersect the invariant plane in M. Show that the cross ratio of $OMPP'$ is constant. This transformation is called perspectivity. If the points $OMPP'$ are harmonic, it is called central involution.

4. In [(11)(11)] show that the line joining any point P to its image P' meets both invariant lines, and that the cross ratio of P, P' and these points of intersection is constant.

5. Discuss the duals of the types of transformations of Art. 155.

156. Birational transformations. Besides the projective transformations, we have already met (Arts. 141, 146) with certain non-linear transformations in which corresponding to an arbitrary point (x) is a definite point (x') and conversely. These are all particular illustrations of a class of transformations which will now be considered.

Let

$$x'_i = \phi_i(x_1, x_2, x_3, x_4), \quad i = 1, 2, 3, 4 \tag{11}$$

be four rational integral functions of x_1, x_2, x_3, x_4, all of the same degree. When x_1, x_2, x_3, x_4 are given, the values of x'_1, x'_2, x'_3, x'_4 are uniquely determined, hence corresponding to a point (x) is a definite point (x'). If the equations (11) can be solved rationally for x_1, x_2, x_3, x_4 in terms of x'_1, x'_2, x'_3, x'_4,

$$x_i = \psi_i(x'_1, x'_2, x'_3, x'_4), \quad i = 1, 2, 3, 4, \tag{12}$$

in which all the functions ψ_i are of the same degree, then to a point (x') also corresponds a definite point (x). In this case the transformation defined by (11) is called **birational**; that defined by (12) is called the inverse of that defined by (11).

When the point (x') describes the plane $\Sigma u'_i x'_i = 0$, the corresponding point (x) describes the surface

$$u'_1\phi_1(x) + u'_2\phi_2(x) + u'_3\phi_3(x) + u'_4\phi_4(x) = 0. \tag{13}$$

This surface will be said to correspond to the plane (u'). If the u'_i are thought of as parameters, we may say: corresponding to all the planes of space are the surfaces of a web defined by (13). In the same way it is seen that, corresponding to the planes $\Sigma u_i x_i = 0$ of the system (x), are the surfaces of the web

$$u_1\psi_1(x') + u_2\psi_2(x') + u_3\psi_3(x') + u_4\psi_4(x') = 0. \tag{14}$$

Three planes (u') which do not belong to a pencil have one and only one point in common, hence three surfaces of the web (13), which do not belong to a pencil, determine a unique point (x) common to them all, whose coördinates are functions of the coördinates of (u').

This fact shows that in the case of non-linear transformations the web defined by (13) cannot be a linear combination of arbitrary surfaces of given degree. For if the ϕ_i are non-linear, any three of them intersect in more than one point, but it was just seen that of the points of intersection there is just one point whose coördinates depend upon the particular surfaces of the web chosen. The remaining intersections are common to all the surfaces of the web. They are called the **fundamental points** of the system (x) in the tranformation (11). When the coördinates of a fundamental point are substituted in (9), the coördinates of the corresponding point (x') all vanish. For the fundamental points the correspondence is not one to one. The fundamental points of (x') are the common basis points of the surfaces $\psi_i(x') = 0$.

157. Quadratic transformations. We have seen (Art. 98) that if the ϕ_i are linear functions, the transformation (11) is projective, and that no point is common to all four planes $\phi_i(x) = 0$. The simplest non-linear transformations are those in which the ϕ_i are quadratic. We shall consider the case in which all the quadrics of the web have a conic c in common.

Let the equations of the given conic be

$$\Sigma u_i x_1 = 0, \qquad f(x) = 0.$$

Any quadric of the system

$$\Sigma u_i x_i (\lambda_1 x_1 + \lambda_2 x_2 + \lambda_3 x_3 + \lambda_4 x_4) + \lambda_5 f(x) = 0$$

will pass through this conic. Among the quadrics of this system those passing through an arbitrary point P define a web. Any two quadrics $H_1 = 0$, $H_2 = 0$ of this web intersect in a space curve consisting of the conic c and a second conic c' which passes through P. The planes of c and of c' constitute a composite quadric belonging to the pencil determined by $H_1 = 0$ and $H_2 = 0$, and the conics c, c' lie on every quadric of the pencil. Hence c, c' intersect in two points, as otherwise the line of intersection of the two planes would have at least three points on every quadric of the pencil, which is impossible.

Any third quadric $H_3 = 0$ of the web but not of the pencil determined by $H_1 = 0$, $H_2 = 0$ passes through c and P. The plane of c' intersects $H_3 = 0$ in a conic c'' passing through P and the two points common to c, c' and in just one other point. The position of this fourth point of intersection depends on the choice of the bundle $H_1 = 0$, $H_2 = 0$, $H_3 = 0$. We have thus proved that the web of quadrics defined by a conic and a point P has the necessary property mentioned in Art. 156 possessed by the web determined by a birational transformation.

Let the equations of the conic c be

$$x_4 = 0, \quad e_1 x_1^2 + e_2 x_2^2 + e_3 x_3^2 = 0.$$

If P is not on the plane $x_4 = 0$, it may be chosen as vertex $(0, 0, 0, 1)$ of the tetrahedron of reference. The equation of the web has the form

$$\lambda_1 x_1 x_4 + \lambda_2 x_2 x_4 + \lambda_3 x_3 x_4 + \lambda_4 (e_1 x_1^2 + e_2 x_2^2 + e_3 x_3^2) = 0.$$

In analogy with equation (11) we may now put

$$x'_1 = x_1x_4,\ x'_2 = x_2x_4,\quad x'_3 = x_3x_4,\ x'_4 = e_1x_1^2 + e_2x_2^2 + e_3x_3^2. \quad (15)$$

The most general form of the transformation of this type may be obtained by replacing the x'_i by any linear functions of them with non-vanishing determinant.

In the derivation of equations (12) it makes no difference whether the conic c is proper or composite, hence three cases arise, according as $e_1 = e_2 = e_3 = 1$ or $e_1 = e_2 = 1$, $e_3 = 0$ or $e_1 = 1$, $e_2 = e_3 = 0$. The equations are

$$x'_1 = x_1x_4,\quad x'_2 = x_2x_4,\quad x'_3 = x_3x_4\quad x'_4 = x_1^2 + x_2^2 + x_3^2. \qquad (a)$$
$$x'_1 = x_1x_4,\quad x'_2 = x_2x_4,\quad x'_3 = x_3x_4\quad x'_4 = x_1^2 + x_2^2. \qquad (b)$$
$$x'_1 = x_1x_4,\quad x'_2 = x_2x_4,\quad x'_3 = x_3x_4\quad x'_4 = x_1^2. \qquad (c)$$

Now let P approach a point K on the conic c. If c is composite, suppose its factors are distinct and that K lies on only one of them. In the limit the line KP is tangent to all the quadrics of the web determined by c and P. But the tangent to c at K is also tangent to all these quadrics at K. Hence the plane of these two lines is a common tangent plane to all the quadrics of the web at $K \equiv P$.

Let P be taken as $(1, 0, 0, 0)$, the common tangent plane at P as $x_2 = 0$, and let the equations of the conic be reduced to $x_4 = 0$, $x_1x_2 + ex_3^2 = 0$. The equation of the web has the form

$$\lambda_1x_2x_4 + \lambda_2x_3x_4 + \lambda_3x_4^2 + \lambda_4(x_1x_2 + ex_3^2) = 0.$$

The two cases, according as $e = 1$ or $e = 0$, give rise to the transformations

$$x'_1 = x_2x_4,\ x'_2 = x_3x_4,\ x'_3 = x_4^2,\ x'_4 = x_1x_2 + x_3^2, \qquad (d)$$
$$x'_1 = x_2x_4,\ x'_2 = x_3x_4,\ x'_3 = x_4^2,\ x'_4 = x_1x_2 \qquad (e)$$

of this type.

Finally, let c be composite and let the point K which P approaches lie on both components of c. Since all the quadrics through c have in this case the plane of c for common tangent plane at K, the point P must approach c in such a way that the line KP approaches the plane of c as a limiting position. The conics in which the quadrics of the web are intersected by any plane through P and K have two points in common at K and one at P. Hence in the limit, all these conics must have three intersections coincident at $K = P$.

Let the equations of c be $x_4 = 0$, $x_2^2 + ex_3^2 = 0$, and the coördinates of P be (1, 0, 0, 0). The equations of the system of rank five of quadrics through c is

$$\lambda_1 x_1 x_4 + \lambda_2 x_2 x_4 + \lambda_3 x_3 x_4 + \lambda_4 x_4^2 + \lambda_5 (x_2^2 + ex_3^2) = 0.$$

The section of this system by any plane through P, different from $x_4 = 0$, will consist of a system of conics touching each other at P. The required web belongs to this system and satisfies the condition that its section by any plane through P other than $x_4 = 0$ is a system of conics having three intersections coincident at (1, 0, 0, 0).

The equations of the section by the plane $x_3 = 0$ are

$$\lambda_1 x_1 x_4 + \lambda_2 x_2 x_4 + \lambda_4 x_4^2 + \lambda_5 x_2^2 = 0, \quad x_3 = 0.$$

All these conics touch each other at P. Let $\lambda'_1, \lambda'_2, \lambda'_4, \lambda'_5$ be the parameters of one conic, and $\lambda_1, \lambda_2, \lambda_4, \lambda_5$ of another contained in this system. The equations of the lines from (1, 0, 0, 0) to the two remaining intersections of these two conic are

$$(\lambda_1\lambda'_5 - \lambda_5\lambda'_1)x_2^2 + (\lambda_1\lambda'_2 - \lambda_2\lambda'_1)x_2x_4 + (\lambda_1\lambda'_4 - \lambda_4\lambda'_1)x_4^2 = 0.$$

One of these remaining points is also at P if $\lambda_1\lambda'_5 - \lambda_5\lambda'_1 = 0$. Hence all the quadrics of the web satisfy a relation of the form $\lambda_5 + k\lambda_1 = 0$. It is no restriction to put $k = 1$. It can now be shown that the conics cut from the quadrics of the web $\lambda_5 + \lambda_1 = 0$ by any plane $a_4x_4 + a_2x_2 + a_3x_3 = 0$ through P have three coincident points in common at P.

The equation of the web is

$$\lambda_1 x_2 x_4 + \lambda_2 x_3 x_4 + \lambda_3 x_4^2 + \lambda_4 (x_2^2 + ex_3^2 - x_1 x_4) = 0.$$

The two birational transformations defined by webs of quadrics of this type are

$$x'_1 = x_2x_4,\ x'_2 = x_3x_4,\ x'_3 = x_4^2,\ x'_4 = x_2^2 + x_3^2 - x_1x_4. \qquad (f)$$
$$x'_1 = x_2x_4,\ x'_2 = x_3x_4,\ x'_3 = x_4^2,\ x'_4 = x_2^2 - x_1x_4. \qquad (g)$$

The inverse transformations of forms $(a) \cdots (g)$ are also quadratic. For this reason transformations of this type are called quadratic-quadratic.

158. Quadratic inversion. A geometric method of constructing some of the preceding types of birational transformations will now be considered. Given a quadric A and a point O. Let P be any point in space, and P' the point in which the polar plane of P as to A cuts the line OP. The transformation defined by having P' correspond to P is called **quadratic inversion.** If O does not lie on the quadric $A = 0$, let $O \equiv (0, 0, 0, 1)$ and let the equation of $A = 0$ be

$$e_1x_1^2 + e_2x_2^2 + e_3x_3^2 - x_4^2 = 0.$$

If $P \equiv (y_1, y_2, y_3, y_4)$, the coördinates of P' are

$$y'_1 = y_1y_4, \ y'_2 = y_2y_4, \ y'_3 = y_3y_4, \ y'_4 = e_1y_1^2 + e_2y_2^2 + e_3x_3^2,$$

which include forms (a), (b), (c). If O lies on A, we may take

$$A \equiv x_2^2 + e_3x_3^2 - x_1x_4 = 0, \ \ O \equiv (0, 0, 0, 1).$$

The coördinates of P' in this case are functions of y_1, y_2, y_3, y_4 defined by (f) and (g). The quadratic-quadratic transformations (a), (b), (c), (f), (g) can therefore be generated in this manner.

159. Transformation by reciprocal radii. If, for the quadric $A = 0$ (Art. 158) we take the sphere

$$x^2 + y^2 + z^2 = k^2t^2 \tag{16}$$

and for O the center of this sphere, the equations of the transformation assume the form

$$x' = k^2xt, \ y' = k^2yt, \ z' = k^2zt, \ t' = x^2 + y^2 + x^2. \tag{17}$$

On account of the relation

$$OP \cdot OP' = k^2 \tag{18}$$

existing between the segments from O to any pair of corresponding points P, P', it is called the transformation by **reciprocal radii.** Any plane not passing through O goes into a sphere passing through O and the circle in which the given plane meets the sphere (16), which is called the sphere of inversion.

The fundamental elements are the center O, the plane at infinity, and the asymptotic cone of the sphere of inversion.

A plane $ax+by+cz+dt=0$ not passing through the origin $(d\neq 0)$ is transformed into a sphere

$$ak^2xt+bk^2yt+ck^2zt+d(x^2+y^2+z^2)=0$$

passing through the origin.

A plane passing through the origin is transformed into a composite sphere consisting of the given plane and the plane at infinity. We shall say that planes through the origin are transformed into themselves.

A sphere

$$a(x^2+y^2+z^2)+2\,fxt+2\,gyt+2\,hzt+mt^2=0 \tag{19}$$

not passing through the origin $(m\neq 0)$ is transformed into the sphere

$$m(x^2+y^2+z^2)+2\,fk^2xt+2\,gk^2yt+2\,hk^2zt+ak^4=0. \tag{20}$$

The factor $x^2+y^2+z^2$ can be removed from the transformed equation. A sphere passing through the origin $(m=0)$ is transformed into a composite sphere consisting of a plane and the plane at infinity.

If any surface passes through the origin, its image is seen to be composite, one factor being the plane at infinity. The plane at infinity is the image of the center O, which is a fundamental point.

In particular, the sphere (19) will go into itself if $m=ak^2$; but this is exactly the condition that the sphere (19) is orthogonal to the sphere of inversion, hence we may say:

THEOREM I. *The spheres which are orthogonal to the sphere of inversion go into themselves when transformed by reciprocal radii.*

We shall now prove the following theorem:

THEOREM II. *Angles are preserved when transformed by reciprocal radii.*

Let $\quad A_1x+B_1y+C_1z+D_1t=0,\ A_2x+B_2y+C_2z+D_2t=0$

be any two planes. The angle θ at which they intersect is defined by the formula (Art. 15)

$$\cos\theta=\frac{A_1A_2+B_1B_2+C_1C_2}{\sqrt{(A_1^2+B_1^2+C_1^2)(A_2^2+B_2^2+C_2^2)}}. \tag{21}$$

These planes go into the spheres

$$D_1(x^2+y^2+z^2)+A_1k^2xt+B_1k^2yt+C_1k^2zt=0,$$
$$D_2(x^2+y^2+z^2)+A_2k^2xt+B_2k^2yt+C_2k^2zt=0.$$

Since the angle of intersection of two spheres is the same for every point of their curve of intersection (Art. 51) and both spheres pass through O, we may determine the angle at which the spheres intersect by obtaining the angle between the tangent planes at O. These tangent planes are

$$A_1x+B_1y+C_1z=0,\ A_2x+B_2y+C_2z=O,$$

hence the angle between them is defined by (21). Since the angle of intersection of any two surfaces at a point lying on both is defined as the angle between their tangent planes at this common point, the proposition is proved.

160. Cyclides. Since lines are transformed by reciprocal radii into circles passing through O, a ruled surface will be transformed into a surface containing an infinite number of circles. A quadric has two systems of lines, hence its transform will contain two systems of circles, and every circle of each system will pass through O. Moreover, the quadric contains six systems of circular sections lying on the planes of six parallel pencils (Art. 82). Hence the transform will also contain six additional systems of circles, not passing through O, but so situated that each system lies on a pencil of spheres passing through O.

By rotating the axes (Art. 37), we may reduce (Art. 70) the equation of any quadric not passing through O to the form

$$ax^2+by^2+cz^2+t^2+2\,lxt+2\,myt+2\,nzt=0 \qquad (22)$$

without changing the form of the equation of the sphere of inversion. By transforming this surface by reciprocal radii, we obtain

$$(x^2+y^2+z^2)^2+2\,k^2(x^2+y^2+z^2)(lx+my+nz)t$$
$$+k^4(ax^2+by^2+cz^2)t^2=0.$$

This surface is called the nodal cyclide; it contains the absolute as a double curve and has a double point at the point O.*

* A point P on a surface is called a double point or node when every line through P meets the surface in two coincident points at P. A curve on a surface is called a double curve when every point of the curve is a double point of the surface.

If the given quadric is a cone with vertex at P, its image will have a double point at O and another at P'. The circles which are the images of the generators of the cone pass through O and P'.

The equation of the cone may be taken as

$$a(x-ft)^2+b(y-gt)^2+c(z-ht)^2=0 \tag{23}$$

and the equation of the transform is

$$(af^2+bg^2+ch^2)(x^2+y^2+z^2)^2-2\,k^2t(x^2+y^2+z^2)(afx+bgy+chz)$$
$$+k^4(ax^2+by^2+cz^2)t^2=0.$$

This surface has a node at the origin and at the transform $(f,\ g,\ h,\ f^2+g^2+h^2)$ of the vertex of the cone (23). It is called a binodal cyclide.

If, in equation (22), $b=c$, so that the given quadric is a surface of revolution, the transformed equation may be written in the form

$$[x^2+y^2+z^2+k^2(lx+my+nz)t+\frac{b}{2}\,k^4t^2]^2+(a-b)k^4x^2t^2$$
$$-k^4\left(lx+my+nz+\frac{b}{2}\,k^2t\right)^2t^2=0.$$

It has a node at O and at the points in which the line $x=0$, $2\,lx+2\,my+2\,nz+k^2bt=0$ intersects the sphere $x^2+y^2+z^2+2\,lxt+2\,myt+2\,nzt+bk^2t^2=0$. It is called the trinodal cyclide.

Finally, if the cone (21) is one of revolution, the resulting cyclide has four nodes, and is called a cyclide of Dupin. If the center of inversion is within the cone, so that no real tangent planes can be drawn to the cone through the line OP, the surface is called a spindle cyclide; if the center is outside the cone, the resulting surface is called a horn cyclide.

The generating circles of a cone of revolution intersect the rectilinear generators at right angles. Since both the lines and the circles are transformed into circles and angles are preserved by the transformation, we have the following theorem:

THEOREM III. *Through each point of a cyclide of Dupin pass two circles lying entirely on the surface. These circles meet each other at right angles.*

A particular case of the spindle cyclide is obtained by taking the axis of the cone through the center of inversion. The resulting cyclide is in this case a surface of revolution. It may be generated by revolving a circle about one of its secants. If the points of intersection of the circle and the secant are imaginary, the cyclide is called the ring cyclide. It has the form of an anchor ring. In this case all the nodes of the cyclide are imaginary.

EXERCISES

1. If A consists of a pair of non-parallel planes and O is taken on one of them, show that the quadratic inversion reduces to the linear transformation defined in Art. 155, Ex. 3 as central involution.

2. Obtain the transform of the ellipsoid

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=t^2$$

with regard to the sphere $x^2+y^2+z^2=1$. How many systems of circles lie on the resulting surface? Show that four minimal lines pass through O and lie on the surface.

3. Show that the transform of the paraboloid $ax^2+by^2=2z$ by reciprocal radii is a cubic surface. How many systems of circles lie on this surface? How many straight lines?

4. Discuss the transform of a quadric cone by reciprocal radii when the center of the sphere of inversion lies on the surface but is not at the vertex.

5. Show that a surface of degree n passing k times through the center of inversion is transformed by reciprocal radii into a surface of degree $2(n-k)$, having the absolute as an $(n-k)$-fold curve.

6. Show that the center of an arbitrary sphere is not transformed into the center of the transformed sphere by reciprocal radii.

7. Given the transformation

$$x'_1=(x_1-x_3)x_2,\quad x'_2=(x_1-x_2)x_3,\quad x'_3=(x_1-x_2)x_4,\quad x'_4=x_2x_3.$$

Find the equations of the inverse transformation and discuss the basis points in (x).

8. Given the transformation

$$x'_1=x_1x_2,\quad x'_2=x_2x_3,\quad x'_3=x_3x_1,\quad x'_4=x_4(x_1+x_2+x_3).$$

Find the equations of the inverse transformation. Discuss the basis points in the web of quadrics $\lambda_1x_1x_2+\lambda_2x_2x_3+\lambda_3x_3x_1+\lambda_4x_4(x_1+x_2+x_3)=0$.

CHAPTER XIII

CURVES AND SURFACES IN TETRAHEDRAL COÖRDINATES

I. Algebraic Surfaces

161. Number of constants in the equation of a surface. The locus of the equation

$$f(x) = \Sigma \frac{n!}{\alpha!\,\beta!\,\gamma!\,\delta!}\, a_{\alpha\beta\gamma\delta} x_1{}^{\alpha} x_2{}^{\beta} x_3{}^{\gamma} x_4{}^{\delta} = 0, \tag{1}$$

wherein α, β, γ, δ are positive integers (or zero) satisfying the equation $\alpha + \beta + \gamma + \delta = n$, is called an algebraic surface of degree n.

If the equation is arranged in powers of one of the variables, as x_4, thus

$$u_0 x_4{}^n + u_1 x_4{}^{n-1} + \cdots + u_n = 0, \tag{2}$$

in which u_i is a homogeneous polynomial of degree i in x_1, x_2, x_3, the number of constants in the equation can be readily calculated. For we may write

$$u_i = \phi_0 x_3{}^i + \phi_0 x_3{}^{i-1} + \cdots + \phi_i,$$

ϕ_k being a homogeneous polynomial in x_1, x_2, of degree k and consequently containing $k + 1$ constants. The number of constants in u_i is therefore

$$1 + 2 + \cdots + i + 1 = \frac{(i+1)(i+2)}{2} = \frac{(i+2)!}{i!\,2!}.$$

This number of constants in u_i is now to be summed for all integral values of i from 0 to n. By induction the sum is readily found to be

$$1 + \frac{2 \cdot 3}{2} + \frac{3 \cdot 4}{2} + \cdots + \frac{(n+1)(n+2)}{2} = \frac{(n+3)!}{n!\;3!}$$

which is the number of homogeneous coefficients in the equation of the surface. The number of independent conditions which the surface can satisfy is one less than this or

$$\frac{(n+3)!}{n!\,3!} - 1 = \frac{n^3 + 6\,n^2 + 11\,n}{6}.$$

162. Notation. It will be convenient to introduce the following symbols:

$$\Delta_y f(x) \equiv y_1 \frac{\partial f(x)}{\partial x_1} + y_2 \frac{\partial f(x)}{\partial x_2} + y_3 \frac{\partial f(x)}{\partial x_3} + y_4 \frac{\partial f(x)}{\partial x_4} = \sum y_i \frac{\partial f(x)}{\partial x_i}$$

$$\Delta_y^2 f(x) \equiv \sum y_i y_k \frac{\partial^2 f}{\partial x_i\, \partial x_k}. \tag{3}$$

$$\Delta_y^r f(x) \equiv \sum \frac{r\,!}{a\,!\,b\,!\,c\,!\,d\,!}\, y_1^a y_2^b y_3^c y_4^d \frac{\partial^r f(x)}{\partial x_1^a \partial x_2^b \partial x_3^c \partial x_4^d},$$

wherein $1 \overline{\gtrless} r \overline{\gtrless} n$ and a, b, c, d are positive integers (or zero), satisfying the condition $a + b + c + d = r$.

EXERCISES

Let $f(x) = a_{4000}x_1^4 + a_{0400}x_2^4 + a_{0040}x_3^4 + a_{0004}x_4^4 + 6\,a_{2200}x_1^2x_2^2 + 6\,a_{0220}x_2^2x_3^2 + 6\,a_{0202}x_2^2x_4^2 + 6\,a_{0022}x_3^2x_4^2 + 6\,a_{2020}x_1^2x_3^2 + 6\,a_{2002}x_1^2x_4^2$.

1. Find $\Delta_y^r f(x)$ for $r = 1, 2, 3, 4$.

2. Show that $\Delta_y\,[\Delta_y^2 f(x)] = \Delta_y^3 f(x)$.

3. Show that $\frac{1}{3\,!}\Delta_y^3 f(x) = \Delta_x f(y)$.

4. Show that $\Delta_y^2 f(x) = \Delta_x^2 f(y)$.

5. Show that $\Delta_x f(x) = 4 f(x)$; $\Delta_x^2 f(x) = 12 f(x)$; $\Delta_x^3 f(x) = 24 f(x)$.

163. Intersection of a line and a surface. If (y), (x) are any two points in space, the coördinates of any point (z) on the line joining them are of the form $z_i = \lambda y_i + \mu x_i$ (Art. 95). If (z) lies on $f(x) = 0$, then $f(\lambda y + \mu x) = 0$. By Taylor's theorem for the expansion of a function of four variables, we have, since $\Delta_x^{n+k} f(y) = 0$ for all positive integral values of k,

$$f(\lambda y + \mu x) \equiv \lambda^n f(y) + \lambda^{n-1}\mu \Delta_x f(y) + \cdots$$
$$+ \frac{\lambda^{n-r}\mu^r}{r\,!}\Delta_x^r f(y) + \cdots + \frac{\mu^n}{n\,!}\Delta_y^n f(y) = 0. \tag{4}$$

This equation may also be written in the form

$$f(\lambda y + \mu x) \equiv \mu^n f(x) + \mu^{n-1}\lambda \Delta_y f(x) + \cdots$$
$$+ \frac{\mu^{n-r}\lambda^r}{r\,!}\Delta_y^r f(x) + \cdots + \frac{\lambda^n}{n\,!}\Delta_y^n f(x) = 0, \tag{5}$$

which is equivalent to the preceding one.

If these equations are identically satisfied, the line joining (y) to (x) lies entirely on the surface. If they are not identically satisfied, they are homogeneous, of degree n in λ, μ and consequently determine n intersections of the line and the surface. If we define the **order** of a surface as the number of points in which it is intersected by a line, we have the following theorem.

THEOREM. *The order of a surface is the degree of its equation in point coördinates.*

164. Polar surfaces. In (4) let the point (y) be fixed but let (x) vary in such a way that the equation

$$\Delta_y^r f(x) = 0 \tag{6}$$

is satisfied.

This equation defines a surface of order $n - r$ called the **rth polar surface** of the fixed point (y) with regard to the given surface $f(x) = 0$. When $r = n - 1$, the surface (6) is a plane. It is called the **polar plane** of the point (y) as to $f(x) = 0$; when $r = n - 2$, the resulting quadric defined by (6) is called the polar quadric, etc.

In the identities (4) and (5) the coefficients of like powers of λ, μ are equal, that is,

$$\frac{1}{r!} \Delta_y^r f(x) \equiv \frac{1}{(n-r)!} \Delta_x^{n-r} f(y).$$

From this identity we have the following theorem:

THEOREM I. *If (x) lies on the rth polar of (y), then (y) lies on the $(n - r)$th polar of (x).*

If in (4), the two points (y), (x) are coincident, then

$$f(\lambda x + \mu x) = (\lambda + \mu)^n f(x) \equiv \lambda^n f(x) + \lambda^{n-1}\mu\Delta_x f(x) + \cdots$$
$$+ \frac{\lambda^{n-r}\mu^r}{r!} \Delta_x^r f(x) + \cdots.$$

By expanding $(\lambda + \mu)^n$ by the binomial theorem and equating coefficients of like powers of λ, μ in the preceding identity, we obtain

$$\Delta_x^r f(x) \equiv \frac{n!}{(n-r)!} f(x),$$

which is called the generalized Euler theorem for homogeneous functions. From this identity we have the following theorem:

THEOREM II. *The locus of a point which lies on any one and therefore on all its own polar surfaces is the given surface* $f(x) = 0$.

From the definition of $\Delta_y^r f(x)$ (Art. 162) it follows that if $s < r < n$,

$$\Delta_y^r f(x) = \Delta_y^s [\Delta_y^{r-s} f(x)].$$

From this identity we have the theorem:

THEOREM III. *The sth polar surface of the* $(r-s)$*th polar surface of* (y) *with respect to* $f(x) = 0$ *coincides with the rth polar surface of* (y).

EXERCISES

1. Determine the coördinates of the points in which the line joining (1, 0, 0, 0) to (0, 0, 0, 1) intersects the surface

$$x_1^3 + 2x_2^3 - x_3^3 - 4x_4^3 + 4x_1^2x_4 - x_1x_4^2 + 5x_2^2x_3 - 6x_1x_2x_3 = 0.$$

2. Determine a so that two intersections of the line joining (0, 1, 0, 0) to (0, 0, 1, 0) with the surface

$$x_1^4 + x_2^4 + x_3^4 + x_4^4 + ax_2^3x_3 + 2(a-1)x_2^2x_3^2 + 4x_2x_3^3 + 6x_1x_2x_3x_4 = 0$$

coincide.

3. Show that any line through (1, 0, 0, 0) has two of its intersections with the surface

$$3x_2^2x_3^2 + x_4^4 + 6x_1^2x_2^2 + 12x_3^2x_4^2 + 4x_2x_3^3 + 24x_1x_2x_3x_4 = 0$$

coincident at (1, 0, 0, 0).

4. Prove the theorems of Art. 164 for the surface of Ex. 3 by actual differentiation.

165. Tangent lines and planes. A line is said to touch a surface at a point P on it if two of its intersections with the surface coincide at P. In equation (4) let (y) now be a fixed point on the given surface so that $f(y) = 0$. One root of (4) is now $\mu = 0$, and one of the intersections (x) coincides with (y).

The condition that a second intersection of the line $(y)(x)$ coincides with (y) is that μ^2 is a factor of (4), that is, that (x) is a point in the plane

$$\Delta_x f(y) \equiv x_1\frac{\partial f(y)}{\partial y_1} + x_2\frac{\partial f(y)}{\partial y_2} + x_3\frac{\partial f(y)}{\partial y_3} + x_4\frac{\partial f(y)}{\partial y_4} = 0. \qquad (7)$$

All the lines which touch $f(x) = 0$ at (y) lie in the plane (7) and every line through (y) in this plane is a tangent line. This plane is called the **tangent plane** of (y). The plane (7) is also the polar plane of (y); hence we have the theorem:

THEOREM. *The polar plane of a point P on the surface is the tangent plane to the surface at P.*

From Art. 164, Theorem III it also follows that the tangent plane to $f(x)=0$ at a point (y) on it is also the tangent plane at (y) to all the polar surfaces of (y) with regard to $f(x)=0$.

166. Inflexional tangents. A line is said to have contact of the second order with a surface at any point P on it if three of its intersections with the surface coincide at P.

Let (y) be a given point on the surface, so that $f(y)=0$. The condition that the line $(y)(z)$ has contact of the second order at (y) is that μ^3 is a factor of (4), that is, that (z) lies on the tangent plane and on the polar quadric of (y). Hence (z) lies on the intersection of

$$\Delta_x f(y)=0, \quad \Delta_x^2 f(y)=0.$$

Since $\Delta_x f(y)=0$ is the tangent plane of the quadric $\Delta_x^2 f(y)=0$ at (y), the conic of intersection of the tangent plane and polar quadric consists of two lines, each of which has contact of the second order with $f(x)=0$ at the point (y). These two lines are called the **inflexional tangents** to the surface at the point P. The section of the surface by an arbitrary plane through either of these lines has an inflexion at (y), the given line being the inflexional tangent.

167. Double points. A point P is said to be a **double point** or **node** on a surface if every line through the point has two intersections with the surface coincident at P. If (y) is a double point on $f(x)=0$, equation (4) has μ^2 as factor for all positions of (z), that is, $\Delta_z f(y)=0$ is an identity in z_1, z_2, z_3, z_4. It follows that if (y) is a double point, its coördinates satisfy the four equations

$$\frac{\partial f(y)}{\partial y_1}=0, \quad \frac{\partial f(y)}{\partial y_2}=0, \quad \frac{\partial f(y)}{\partial y_3}=0, \quad \frac{\partial f(y)}{\partial y_4}=0. \qquad (8)$$

Conversely, if these conditions are satisfied, it follows, since $nf(y)\equiv\Delta_y f(y)$, that equation (4) has the double root $\mu^2=0$ and

(y) is a double point. Hence the necessary and sufficient condition that $f(x) = 0$ has a double point at (y) is that the coördinates of (y) satisfy equations (8). Unless the contrary assumption is stated, it will be assumed that $f(x) = 0$ has no double points.

EXERCISES

1. Impose the condition that the point (0, 0, 0, 1) lies on the surface $f(x) = 0$ and find the equation of the tangent plane to the surface at that point.

2. Determine the condition that the surface of Ex. 1 has a double point at (0, 0, 0, 1).

3. Show that the point (1, 1, 1, 1) lies on the surface of Ex. 1, Art. 164, and determine the equation of the tangent plane at that point.

4. Find the equations of the inflexional tangents of the surface of Ex. 1, Art. 164, at the point (1, 1, 1, 1).

5. Show that the lines through a double point on a surface $f(x) = 0$ which have three intersections with the surface coincident at the double point form a quadric cone.

6. Show that there are six lines through a double point on a surface $f(x) = 0$ which have four points of intersection with the surface coincident at the double point.

7. Prove that the curve of section of a surface by any tangent plane has a double point at the point of tangency, and the inflexional tangents are the tangents at the double point.

168. The first polar surface and tangent cone. If in equation (7), the coördinates x_1, x_2, x_3, x_4 are regarded as fixed, and y_1, y_2, y_3, y_4 as variable, the locus of the equation is the first polar of the point (x).

THEOREM I. *The first polar surface of any point in space passes through all the double points of the given surface.*

For, if $f(x) = 0$ has one or more double points, the coördinates of each must satisfy the system of equations (8) and also (7).

THEOREM II. *The points of tangency of the tangent planes to the surface from a point (x) lie on the curve of intersection of the given surface and the first polar of (x).*

For, if (y) is the point of tangency of a tangent plane to $f(x) = 0$ which passes through the given point (x), the coördinates of (y) satisfy $f(y)=0$ and $\Delta_x f(y)=0$. Conversely, if (y) is a non-multiple point on this curve, it follows that the tangent plane at (y) passes through the given point (x).

Since the line joining (x) to any point (y) on the curve defined in Theorem II lies in the tangent plane at (y), it follows that it is a tangent line. The locus of these lines is a cone called the **tangent cone** from (x) to the surface $f(x)=0$. To obtain the equation of this cone we think of (x) as fixed in (4) and impose the condition on (y) that two of the roots of the equation in $\lambda : \mu$ shall be coincident. Hence we have the following theorem:

THEOREM III. *The equation of the tangent cone from any point* (x) *is obtained by equating the discriminant of* (4) *to zero.*

169. Class of a surface. Equation in plane coördinates. A point (x) lies on the surface $f(x)=0$ if its coördinates satisfy the equation of the surface. Similarly, a plane (u) touches the surface if its coördinates satisfy a certain relation, called the equation of the surface in plane coördinates. The **class** of a surface is the dual of its order; it is defined as the number of tangent planes to the surface that pass through an arbitrary line and is equal to the degree of the equation of the surface in plane coördinates.

THEOREM. *The class of an algebraic surface of order* n, *having* δ *double points and no other singularities, is* $n(n-1)^2 - 2\delta$.

Let $f(x)=0$ be of order n, and let $P_1=(y)$, $P_2=(z)$ be two points on an arbitrary line l. The point of tangency of every tangent plane to $f(x)=0$ that passes through l lies on the surface $f(x)=0$, on the polar of (y) and on the polar of (z), so that its coördinates satisfy the equations

$$f(x)=0, \quad \Delta_y f(x)=0, \quad \Delta_z f(x)=0.$$

These surfaces are of orders n, $n-1$, $n-1$, respectively, and have $n(n-1)^2$ points in common; if $f(x)=0$ has no double points, each of these points is a point of tangency of a plane through the line l, tangent to the given surface. If $f(x)=0$ has a double point, $\Delta_y f(x)=0$ and $\Delta_z f(x)=0$, both pass through it, hence the number of remaining sections is reduced by two.

If the plane $u_1x_1+u_2x_2+u_3x_3+u_4x_4=0$ is tangent to $f(x)=0$

at (y), then this plane and that defined by equation (7) must be identical, hence

$$\rho u_1 = \frac{\partial f(y)}{\partial y_1}, \quad \rho u_2 = \frac{\partial f(y)}{\partial y_2}, \quad \rho u_3 = \frac{\partial f(y)}{\partial y_3}, \quad \rho u_4 = \frac{\partial f(y)}{\partial y_4}. \qquad (9)$$

Moreover, (y) lies in the given plane and also on the given surface, hence

$$u_1 y_1 + u_2 y_2 + u_3 y_3 + u_4 y_4 = 0, \quad f(y) = 0. \qquad (10)$$

If between (9) and (10) the coördinates of (y) are eliminated, the resulting equation will be the equation of the given surface in plane coördinates. If $f(x) = 0$ has double points, the resulting equation will be composite in such a way that the equation of each double point appears as a double factor.

EXERCISES

1. Determine the equation of the tangent cone to the surface

$$x_1^3 + x_2^3 + x_3^3 + x_4^3 = 0$$

from the point (1, 0, 0, 0).

2. Write the equation of the surface of Ex. 1 in plane coördinates.

3. Write the equation of the surface

$$x_1x_2x_3 + x_1x_3x_4 + x_1x_2x_4 + x_2x_3x_4 = 0$$

in plane coördinates.

4. Write the equation of the surface $x_1^2x_3 - x_2^2x_4 = 0$ in plane coördinates.

170. The Hessian. The locus of the points of space whose polar quadrics are cones is called the **Hessian** of the given surface $f(x) = 0$. The equation of the polar quadric of a point (x) may be written in the form

$$\sum \frac{\partial^2 f(x)}{\partial x_i \partial x_k} y_i y_k = 0, \qquad (11)$$

in which y_1, y_2, y_3, y_4 are the current coördinates. This quadric will be a cone if its discriminate vanishes (Art. 103), hence if we put for brevity

$$f_{ik} \equiv \frac{\partial^2 f(x)}{\partial x_i \partial x_k},$$

the equation of the Hessian may be written in the form

$$H = \begin{vmatrix} f_{11} & f_{12} & f_{13} & f_{14} \\ f_{12} & f_{22} & f_{23} & f_{24} \\ f_{13} & f_{23} & f_{33} & f_{34} \\ f_{14} & f_{24} & f_{34} & f_{44} \end{vmatrix} = 0. \qquad (12)$$

It is of order $4(n-2)$.

It will now be shown that the Hessian may also be defined as the locus of double points on first polar surfaces of the given surface. The equation of the first polar of (y) as to $f(x)=0$ is

$$\sum y_i \frac{\partial f(x)}{\partial x_i} = 0.$$

If this surface has a double point, the coördinates of the double point make each of its first partial derivatives vanish, by (8), thus

$$\begin{aligned}
y_1 \frac{\partial^2 f(x)}{\partial x_1^2} + y_2 \frac{\partial^2 f}{\partial x_1 \partial x_2} + y_3 \frac{\partial^2 f}{\partial x_1 \partial x_3} + y_4 \frac{\partial^2 f}{\partial x_1 \partial x_4} &= 0,\\
y_1 \frac{\partial^2 f}{\partial x_1 \partial x_2} + y_2 \frac{\partial^2 f}{\partial x_2^2} + y_3 \frac{\partial^2 f}{\partial x_2 \partial x_3} + y_4 \frac{\partial^2 f}{\partial x_2 \partial x_4} &= 0,\\
y_1 \frac{\partial^2 f}{\partial x_1 \partial x_3} + y_2 \frac{\partial^2 f}{\partial x_2 \partial x_3} + y_3 \frac{\partial^2 f}{\partial x_3^2} + y_4 \frac{\partial^2 f}{\partial x_3 \partial x_4} &= 0,\\
y_1 \frac{\partial^2 f}{\partial x_1 \partial x_4} + y_2 \frac{\partial^2 f}{\partial x_2 \partial x_4} + y_3 \frac{\partial^2 f}{\partial x_3 \partial x_4} + y_4 \frac{\partial^2 f}{\partial x_4^2} &= 0.
\end{aligned} \tag{13}$$

The condition that these equations in y_1, y_2, y_3, y_4 are consistent is that their determinant is equal to zero, that is, that (x) lies on the Hessian.

171. The parabolic curve. The curve of intersection of the given surface with its Hessian is called the **parabolic curve** on the surface.

THEOREM. *At any point of the parabolic curve the two inflexional tangents to the surface coincide.*

For, let (x) be a point on the parabolic curve. Since (x) lies on the Hessian, its polar quadric is a cone. This cone passes through (x) (Art. 164). The inflexional tangents are the lines which the cone has in common with its tangent plane at (x) (Art. 166). These lines coincide (Art. 121).

172. The Steinerian. It was just seen that the polar quadric of any point on the Hessian is a cone. Let (x) be a point on H, and (y) the vertex of its polar quadric cone. As (x) describes H, (y) also describes a surface, called the **Steinerian** of $f(x) = 0$. The polar quadric of (x) is given by equation (11). If (y) is the vertex of the cone, its coördinates satisfy (13). The equation of the

Steinerian may be obtained by eliminating x_1, x_2, x_3, x_4 from these four equations (13). As the equations (13) were obtained by imposing the condition that the first polar of (x) has a double point, we may also define the Steinerian as the locus of a point whose first polar surface has a double point (lying on the Hessian).

EXERCISES

1. Prove that the Hessian and the Steinerian of a cubic surface coincide.

2. Prove that if a point of the Hessian coincides with its corresponding point on the Steinerian, it is a double point of the given surface, and conversely.

3. Determine the equation of the Hessian of the surface

$$a_1x_1^3 + a_2x_2^3 + a_3x_3^3 + a_4x_4^3 + a_5(x_1 + x_2 + x_3 + x_4)^3 = 0.$$

4. Determine the order of the Steinerian of a general surface of order n.

II. Algebraic Space Curves

173. Systems of equations defining a space curve. A curve which forms the complete or partial intersection of two algebraic surfaces is called an **algebraic curve**; if the curve does not lie in a plane, it is called a **space curve.**

If a given curve C forms the complete intersection of two surfaces $F_1 = 0$, $F_2 = 0$, so that the points of C, and no other points, lie on both surfaces, then the equations of these surfaces, considered as simultaneous, will be called the equations of the given curve.

If the intersection of the surfaces $F_1 = 0$ and $F_2 = 0$ is composite, and C is one component, the equations $F_1 = 0$, $F_2 = 0$ are satisfied not only by the points of C but also by the points of the residual curve. If a surface $F_3 = 0$ through C can be found which has no points of intersection with the residual curve except those on C, the simultaneous equations $F_1 = 0$, $F_2 = 0$, $F_3 = 0$ are satisfied only by the points of C and are called the equations of the curve.

If the surfaces $F_1 = 0$, $F_2 = 0$, $F_3 = 0$ through C have one or more points in common which do not lie on C, then a fourth surface $F_4 = 0$ can be found through C which does not contain these residual points, but may intersect the residual curve of $F_1 = 0$, $F_2 = 0$ in other points not on $F_3 = 0$; in this case the simultaneous equations $F_1 = 0$, $F_2 = 0$, $F_3 = 0$, $F_4 = 0$ are called the equations of the curve. In this way a system of equations can be found which are simultaneously satisfied by points of C and by no others.

As an illustration, consider the composite intersection of the quadric surfaces

$$x_1x_3 - x_2^2 = 0, \quad x_2x_4 - x_3^2 = 0.$$

It consists of a space curve and the line $x_2 = 0$, $x_3 = 0$. The surface $x_1x_4 - x_2x_3 = 0$ also contains the space curve since it contains every point common to the quadrics except points on the line $x_2 = 0$, $x_3 = 0$. These three surfaces are sufficient to define the curve. The surface $x_1x_4(x_1 - x_4) - x_2^3 + x_2x_3x_4 = 0$ also contains the given curve. It does not, however, with the two given surfaces constitute a system whose equations define the given curve. All three equations are satisfied, not only by the coördinates of the points of the curve, but by the coördinates of the point (1, 0, 0, 1) which does not lie on the curve, since it does not lie on the surface $x_1x_4 - x_2x_3 = 0$. The surface $x_1x_4(x_1 + x_4) - x_2^3 - x_3^3 = 0$ passes through the curve but not through the point (1, 0, 0, 1). The curve is therefore completely defined by regarding the four equations

$$x_1x_3 - x_2^2 = 0, \qquad x_2x_4 - x_3^2 = 0,$$
$$x_1x_4(x_1 - x_4) - x_2^3 + x_2x_3x_4 = 0, \qquad x_1x_4(x_1 + x_4) - x_2^3 - x_3^3 = 0$$

as simultaneous.

174. Order of an algebraic curve. Let $F_\mu = 0$, $F_{\mu'} = 0$ be two surfaces of orders μ, μ', respectively, and let C be their (proper or composite) curve of intersection. Any plane that does not contain C (or a component of it) intersects C in $\mu\mu'$ points. For, any such plane intersects $F_\mu = 0$ in a curve of order μ, and intersects $F_{\mu'}$ in a curve of order μ'. These coplanar curves have precisely $\mu\mu'$ points in common.*

It can in fact be shown that every algebraic curve, whether defined as the complete intersection of two surfaces or not, is intersected by any two planes, neither of which contains the curve or a component of it, in the same number of points.† We

* See, *e.g.*, Fine: College Algebra (1905), p. 519.

† Halphen: Jour. de l'école polytechnique, Vol. 52 (1882), p. 10.

shall assume, without proof, the truth of this statement. The number of points in which an arbitrary plane intersects an algebraic curve is called the order of the curve (Art. 140).

175. Projecting cones. If every point of a space curve is joined by a line to a fixed point P in space, a cone is defined, called the projecting cone of the curve from the point P. If the point P lies at infinity, the projecting cone from P is a cylinder (Art. 44). Except in metrical cases to be discussed later we shall make no distinction between cylinders and cones.

For an arbitrary point P an arbitrary generator of the projecting cone intersects the curve in only one point. It may happen, however, for particular positions of the point P, that every generator meets the curve in two or more points. If in this case P does not lie on the curve or if P lies on the curve and every generator through P intersects the curve in two or more points distinct from P, the curve is called a conical curve.

Let P be a point not on the curve, such that an arbitrary generator of the projecting cone from P meets the curve in just one point. The order of the projecting cone is the number of generators in an arbitrary plane through its vertex. Each generator contains one point on the curve, hence the order of the projecting cone is equal to the order of the curve. If P is on the curve, the order of the projecting cone is one less than the order of the curve.

THEOREM. *To find the equation of the projecting cone of the simple or composite curve defined by the complete intersection of two surfaces, from a vertex of the tetrahedron of reference, eliminate between the equations the variable which does not vanish at that vertex.*

Let the equations of the given surfaces be $F_\mu = 0$ and $F_{\mu'} = 0$ and let it be required to project the curve of intersection of these surfaces from the point $(0, 0, 0, 1)$.

Let (y) be any point of space. The coördinates of any point (x) on the line joining $(0, 0, 0, 1)$ to (y) are of the form

$$x_1 = \lambda y_1, \quad x_2 = \lambda y_2, \quad x_3 = \lambda y_3, \quad x_4 = \lambda y_4 + \sigma.$$

The points in which this line intersects $F_\mu = 0$, $F_{\mu'} = 0$ are defined by

$$\left.\begin{aligned} F_\mu(x) &= F_\mu(\lambda y_1,\ \lambda y_2,\ \lambda y_3,\ \lambda y_4 + \sigma) \\ &= \lambda^\mu F_\mu\left(y_1,\ y_2,\ y_3,\ y_4 + \frac{\sigma}{\lambda}\right) = 0, \\ F_{\mu'}(x) &= F_{\mu'}(\lambda y_1,\ \lambda y_2,\ \lambda y_3,\ \lambda y_4 + \sigma) \\ &= \lambda^{\mu'} F_{\mu'}\left(y_1,\ y_2,\ y_3,\ y_4 + \frac{\sigma}{\lambda}\right) = 0, \end{aligned}\right\} \tag{14}$$

respectively. The condition that the line intersects both surfaces in the same point is that these equations have a common root in $\frac{\sigma}{\lambda}$, hence the equation of the projecting cone is obtained by eliminating $\frac{\sigma}{\lambda}$ between these two equations (cf. Art. 44). If $\frac{\sigma}{\lambda}$ is eliminated from (14), y_4 is also eliminated and the resulting surface is identical with that obtained by eliminating x_4 between the equations of the given surfaces.

If the curve of intersection is composite, the projecting cone is composite, one component belonging to each component curve.

A method for determining the projecting cone from any point P in space may be deduced by similar reasoning, but the process is not quite so simple.

EXERCISES

1. Show that the intersection of the surfaces

$$x_1x_2 - x_3x_1 + x_4^2 - x_2x_3 = 0, \quad x_1x_3^2 - x_1x_2x_4 + x_2(x_4^2 - x_2x_3) = 0$$

is composite.

2. Represent each component curve of Ex. 1 completely by two or more equations.

3. Find the equation of the projecting cone of the curve

$$x_2^2 + x_3^2 + x_4^2 + 2\,x_1x_4 = 0, \quad x_4^2 + 2\,x_2x_4 - x_1^2 + 2\,x_3^2 = 0$$

from the point (0, 0, 0, 1).

4. Find the equation of the projecting cone of the curve

$$x_1^2 + 4\,x_3^2 - x_4^2 = 0, \quad x_1^2 - 2\,x_2^2 + 2\,x_3^2 - 3\,x_4^2 = 0$$

from the point (0, 0, 0, 1).

5. Find the equation of the projecting cone of the curve

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \quad a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = 0$$

from the point (0, 0, 0, 1).

6. Show by means of elimination that, if (0, 0, 0, 1) does not lie on the curve $F_\mu = 0$, $F_{\mu'} = 0$, the order of the projecting cone from (0, 0, 0, 1) is $\mu\mu'$, provided the curve is not conical from (0, 0, 0, 1).

7. Find the equation of the projecting cone of the curve

$$x_1^2 + 2x_2^2 - x_3^2 = 0, \quad x_4^2 - x_2x_3 + x_1^2 = 0$$

from the point (1, 1, 1, 1).

176. Monoidal representation. If a non-composite space curve C_m of order m is defined as the complete or partial intersection of two surfaces $F_\mu = 0$, $F_{\mu'} = 0$, other surfaces on which the curve lies can be obtained from the given ones by algebraic processes. Among such surfaces we have already discussed the projecting cone from a given point P. We shall now show how to obtain the equation of a surface which contains C_m and has at P a point of multiplicity one less than the order of the surface. Such a surface is called a **monoid**.

In determining the equation of a monoid through C_m, we shall assume that neither the complete intersection of $F_\mu = 0$ and $F_{\mu'} = 0$ nor any component of it is a conical curve from P. We shall also assume that P does not lie on this curve of intersection.

Let P be chosen as (0, 0, 0, 1) and let the equations $F_\mu = 0$, $F_{\mu'} = 0$ be arranged in powers of x_4 (Art. 161).

$$F_\mu \equiv u_0x_4^{\mu} + u_1x_4^{\mu-1} + \cdots + u_\mu = 0,$$
$$F_{\mu'} \equiv v_0x_4^{\mu'} + v_1x_4^{\mu'-1} + \cdots + v_{\mu'} = 0,$$

wherein u_i, v_i are homogeneous functions of x_1, x_2, x_3 of degree i. Let the notation be so chosen that $\mu' \geqq \mu$. The equation

$$v_0x_4^{\mu'-\mu}F_\mu - u_0F_{\mu'} = 0$$

contains x_4 to at most the power $\mu' - 1$. The surface represented by it passes through the curve C_m, since the equation is satisfied by the coördinates of every point which satisfy $F_\mu = 0$ and $F_{\mu'} = 0$.

The equation

$$v_{\mu'}F_\mu - u_\mu F_{\mu'} = 0$$

is divisible by x_4. If this factor is removed, the resulting equation is of degree at most $\mu' - 1$ in x_4, and determines a surface which passes through C_m.

If either of these equations contains x_4 to the first but to no higher power, the surface determined by it is of the type required. If not, the two equations cannot both be independent of x_4 nor can they coincide, since in that case the curve $F_\mu = 0$, $F_{\mu'} = 0$ would be conical from (0, 0, 0, 1).

By applying this same process to the two equations just obtained, we may obtain two new ones which contain x_4 to at most the power $\mu' - 2$.

Continuing in this way with successive partial elimination, we obtain finally an equation of the form

$$M \equiv x_4\phi_{n-1}(x_1, x_2, x_3) - \phi_n(x_1, x_2, x_3) = 0,$$

in which ϕ_{n-1} and ϕ_n are homogeneous functions, not identically zero, of degree $n-1$ and n, respectively, in x_1, x_2, x_3. The surface $M = 0$ is of order n and has an $(n-1)$-fold point at (0, 0, 0, 1). It is consequently a monoid. The surface $\phi_n = 0$ is a cone; it is called the superior cone of the monoid. If $n > 1$, $\phi_{n-1} = 0$ is the equation of another cone, called the inferior cone of the monoid.

Let $f_m(x_1, x_2, x_3) = 0$ be the equation of the projecting cone from (0, 0, 0, 1). The equations

$$f_m(x_1, x_2, x_3) = 0, \quad x_4\phi_{n-1}(x_1, x_2, x_3) - \phi_n(x_1, x_2, x_3) = 0$$

are said to constitute a monoidal representation of the curve C_m.

The advantage of this representation is that the residual intersection, if any, of the two surfaces $M = 0$, $f_m = 0$ consists of lines common to the three cones

$$f_m = 0, \quad \phi_{n-1} = 0, \quad \phi_n = 0.$$

For, let P be a point common to $f_m = 0$, $M = 0$, but not lying on C_m, nor at the vertex (0, 0, 0, 1). The generator of $f_m = 0$ passing through P intersects C_m in some point P'. Since this generator has P, P' and $n-1$ points at (0, 0, 0, 1) in common with $M = 0$, it lies entirely on the monoid (Art. 164). For every point of this line, that is, independently of the value of x_4, the equation $x_4\phi_{n-1} - \phi_n = 0$ must be satisfied; hence the given generator lies on the inferior cone and on the superior cone.

It follows at once from the above discussion that if any generator of $f_m = 0$ intersects C_m in two points P, Q, it lies entirely on the monoid and forms a part of the residual intersection. Such a line is called a double generator of the projecting cone, since, in tracing the curve, the generator takes the position determined by P on C_m and also the position, coincident with the first, determined by Q. Every such line counts for two intersections of $M = 0$ and $f_m = 0$.

Each of these bisecants of the curve is said to determine an **apparent double point** of C_m from $(0, 0, 0, 1)$; the curve appears from $(0, 0, 0, 1)$ to have a double point on each of these lines.

It can be proved* and will here be assumed that the number of apparent double points of a given curve is the same for every point not lying on it, except the vertices of the cones, if any, on which C_m is a conical curve. This number will be denoted by h.

We shall now show that if a point P which does not lie on C_m, nor on any line that intersects C_m in more than two points, nor at the vertex of a cone (if any) of bisecants to C_m, is chosen for the vertex, then the order of the monoid from P is at least half the order of C_m.

The complete intersection of the projecting cone $f_m = 0$ and the monoid $x_4\phi_{n-1} - \phi_n = 0$ is a curve of order mn. The curve C_m is one component of order m, and the h bisecants of C_m through $(0, 0, 0, 1)$ together form a component of order $2h$. If the number of residual intersecting lines is denoted by k, then

$$mn - m - 2h = k, \quad k \geqq 0.$$

The h bisecants of C_m and the k residual lines are components of the intersection of $\phi_{n-1} = 0$, $\phi_n = 0$. Hence

$$n(n-1) = h + k = h(n-1) - h,$$

from which

$$(m-n)(n-1) = h \leqq \frac{m}{2}(n-1),$$

and

$$n \geqq \frac{m}{2},$$

which proves the proposition.

177. Number of intersections of algebraic curves and surfaces.

THEOREM. *Any surface of order μ which does not contain a given non-composite curve of order m intersects it in $m\mu$ points.*

* Noether: Zur Grundlegung der Theorie der algebraischen Raumkurven, Abhandlungen der k. preussischen Akademie der Wissenschaften für 1882.

Let C_m be the given curve and $F_\mu = 0$ be the equation of the given surface. Choose (0, 0, 0, 1) not on $F_\mu = 0$, and let the monoidal equations of C_m be $f_m = 0$, $x_4\phi_{n-1} - \phi_n = 0$. The complete intersection of $f_m = 0$, $x_4\phi_{n-1} - \phi_n = 0$ consists of C_m and of $m(n-1)$ lines through (0, 0, 0, 1). As $F_\mu = 0$ does not pass through (0, 0, 0, 1), it cannot contain any of these lines. Hence $F_\mu = 0$, $f_m = 0$, $M_n = 0$ have no common component. They consequently intersect in $mn\mu$ points. Of these, $m\mu(n-1)$ points are where the residual lines intersect $F_\mu = 0$. The remaining $m\mu$ points lie on C_m. If C_m has $m\mu + 1$ points on $F_\mu = 0$, it lies on the surface, for the three surfaces $f_m = 0$, $M_n = 0$, $F_\mu = 0$ have now $mn\mu + 1$ points in common, and therefore all contain a common curve. Since the lines do not lie on $F_\mu = 0$, and $f_m = 0$, $M_n = 0$ have no other component curve except C_m, it follows that C_m must lie on $F_\mu = 0$.

EXERCISES

1. Show that a plane or any proper quadric is a monoid.

2. Write the equation of a monoid of order three.

3. Show that the only curve of order one is a line.

4. Show that the only irreducible curve of order two is a conic.

5. Show that a composite curve of order two exists which does not lie in a plane. How many apparent double points has this curve?

6. Show that a bundle of quadrics pass through a proper space cubic curve.

7. Write a monoidal representation of a space cubic curve.

8. Show that every irreducible curve of order four lies on a quadric surface.

9. Discuss the statements of Exs. 6 and 8 for the case of composite cubics and composite quartics.

178. Parametric equations of rational curves. Since a space curve is defined as the complete or partial intersection of two surfaces, the coördinates of its points are functions of a single variable. The expressions for the coördinates of a point as functions of a single variable may not be rational. A curve which possesses the property that all its coördinates can be expressed as rational functions of a single variable is called a **rational curve.** By definition the equations of such a curve can be written parametrically in the form

$$x_i = f_i(t) = a_{i0}t^m + a_{i1}t^{m-1} + \cdots + a_{im}, \quad i = 1, 2, 3, 4.$$

Since the variables x_i are homogeneous, it is no restriction to suppose that the polynomials $f_i(t)$ have no common factor. To every value of t corresponds a unique point (x) on the curve, but it may happen that more than one value of t will define the same point (x) on the curve. If, for example, the functions $f_i(t)$ can be expressed in the form

$$f_i(t) \equiv F_i(\phi(t), \psi(t)),$$

in which F_i are homogeneous rational functions, of the same order, of the two polynomials $\phi(t)$, $\psi(t)$, then $f_i(t)$ will define the same point for every value of t that satisfies the equation

$$\phi(t) = s\psi(t),$$

where s is given. In this case the coördinates of the points on the curve are rational functions of s.

Conversely, it will now be shown that if to each point (x) of the curve correspond n values of t $(n \geqq 1)$, then t may be replaced by a new variable, in terms of which the correspondence between it and the point (x) on the curve is one to one.

Let $t_1, t_2, \cdots, t_n$ all correspond to the same point (x). The expressions

$$f_i(t)f_k(t_1) - f_i(t_1)f_k(t) \qquad i, k = 1, 2, 3, 4$$

vanish for $t = t_1, t_2, \cdots, t_n$, hence they have a common factor of order n, whose coefficients contain t_1,

$$\phi_0(t_1)t^n + \phi_1(t_1)t^{n-1} + \cdots + \phi_n(t_1).$$

If t_1 is replaced by t_2, the expression must have the same factor, hence the function

$$\phi_0(t_2)t^n + \phi_1(t_2)t^{n-1} + \cdots + \phi_n(t_2)$$

has the same roots. Similarly for $t_3, \cdots, t_n$. It follows that the ratios of the coefficients

$$\phi_0 : \phi_1 : \cdots \phi_n$$

have the same values for $t_1, t_2, \cdots, t_n$. These ratios cannot be constant for every point (x) on the curve, since in that case $t_1, \cdots, t_n$ would be independent of (x), contrary to hypothesis. If we now put

$$\frac{\phi_i(t)}{\phi_k(t)} = s, \qquad i \neq k = 1, \cdots n$$

and eliminate t between this equation and $x_i = f_i(t)$, the resulting equations may be written in the form

$$x_i = b_{i0}s^p + b_{i1}s^{p-1} + \cdots + b_{ip},$$

in which $\qquad np = m.$

When the correspondence between (x) and t is one to one, the order of the curve $x_i = f_i(t)$ is m. For, to each point of intersection of the curve with an arbitrary plane $\Sigma u_i x_i = 0$ corresponds a root of the equation $\Sigma u_i f_i(t) = 0$, and conversely.

179. Tangent lines and developable surface of a curve. Let C be the intersection of two algebraic surfaces $F = 0$, $F' = 0$ and let P be an arbitrary point on C. The line t of intersection of the tangent planes to $F = 0$ and $F' = 0$ at P has two points in common with each of the surfaces coincident at P (Art. 165), and hence with C. The line is called the tangent line to the curve C at the point P. The locus of the tangent lines to C is a ruled surface. This surface is called the developable surface of C. Its equation may be found by eliminating the coördinates y_1, y_2, y_3, y_4 of P between the equations of C and of the tangent planes, thus:

$$F(y) = 0,\ F'(y) = 0,\ \sum x_i \frac{\partial F(y)}{\partial y_i} = 0,\ \sum x_i \frac{\partial F'(y)}{\partial y_i} = 0.$$

EXAMPLE. The intersection of the surfaces

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0,\quad a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = 0$$

is a quartic curve. The equation of the developable surface of this quartic is obtained by eliminating y_1, y_2, y_3, y_4 between the equations

$$y_1^2 + y_2^2 + y_3^2 + y_4^2 = 0,\quad a_1y_1^2 + a_2y_2^2 + a_3y_3^2 + a_4y_4^2 = 0,$$
$$x_1y_1 + x_2y_2 + x_3y_3 + x_4y_4 = 0,$$
$$a_1x_1y_1 + a_2x_2y_2 + a_3x_3y_3 + a_4x_4y_4 = 0.$$

If we write a_{ik} for $a_i - a_k$, the result may be written in the form

$$\begin{aligned}&4\, a_{12}a_{13}a_{42}a_{43}(a_{13}x_1^2 + a_{23}x_2^2 + a_{43}x_4^2)(a_{21}x_1^2 + a_{23}x_3^2 + a_{34}x_4^2)x_2^2x_4^2\\ &-[a_{23}a_{14}^2x_1^2x_4^2 + a_{24}a_{13}^2x_1^2x_3^2 + a_{34}a_{12}^2x_1^2x_2^2 + a_{22}a_{34}^2x_3^2x_4^2 + a_{31}a_{24}^2x_2^2x_4^2\\ &+a_{23}(a_{12}a_{34} + a_{13}a_{24})x_2^2x_3^2]^2 = 0.\end{aligned}$$

The number of tangents to the curve C_m which meet an arbitrary line is called the **rank** of the curve. From this definition it follows that the rank is equal to the order of the developable surface. It is the same number for every line not on the surface (Art. 163).

180. Osculating planes. Equation of a curve in plane coördinates. Every plane through the tangent line to C at P contains the line and has therefore two points in common with C at P.

Such a plane is called a **tangent plane.** Among the tangent planes there is one having three intersections with C at P. This plane is called the **osculating plane** to C at P. The number of osculating planes to C which pass through an arbitrary point in space is called the **class** of C. This number is the same for every point in space.* If C is the intersection of $F=0$ and $F'=0$, we can obtain two equations which must be satisfied by the coördinates of the osculating planes of C by eliminating two of the variables, as x_3, x_4, between the equations $F=0$, $F'=0$, and the equation of the plane $\Sigma u_i x_i = 0$, then imposing the condition that the resulting homogeneous equation in the other two variables has a triple root.

EXAMPLE. The two surfaces $x_1^2 + 2\,x_2x_4 = 0$, $x_2^2 + 2\,x_1x_3 = 0$ intersect in the line $x_1 = 0$, $x_2 = 0$ and a space cubic curve. If between the first equation and $\Sigma u_i x_i = 0$ we eliminate x_4, we find

$$u_4x_1^2 - 2\,u_1x_1x_2 - 2\,u_2x_2^2 - 2\,u_3x_2x_3 = 0.$$

Now if we eliminate x_3 between this result and the second given equation, we obtain

$$u_4x_1^3 - 2\,u_1x_1^2x_2 - 2\,u_2x_1x_2^2 + u_3x_2^3 = 0.$$

Finally, if this cubic has three equal roots, its first member must be a cube. Hence

$$2\,u_1^2 + 3\,u_4u_2 = 0,\ \ 2\,u_2^2 + 3\,u_1u_3 = 0.$$

A system of two or more equations in plane coördinates (Art. 173) which are satisfied by the coördinates of the osculating planes of C, and by no others, is said to define the curve C in plane coördinates. To a curve C defined in this way may be applied a discussion dual to that given in Arts. 174–179.

EXERCISES

1. Find a system of parametric equations of the rational curve

$$x_1x_2 - x_3x_4 = 0,\ \ x_2x_3 = x_1^2 - x_2^2.$$

2. Write the equation of the developable surface of the cubic curve lying on the surfaces

$$x_1^2 + 2\,x_2x_4 = 0,\ \ x_2^2 + 2\,x_1x_3 = 0.$$

3. Find two equations satisfied by the coördinates of the osculating planes of the curve

$$x_1x_2 - x_3x_4 = 0,\ \ x_2^2 = x_3^2 + x_4^2.$$

* See reference in Art. 176.

4. Define the dual of the projecting cone of a curve and show how its equation may be obtained.

5. Derive the dual of a monoidal representation of a curve.

6. Define the dual of an apparent double point.

7. What is the dual of the rank of a space curve?

181. Singular points, lines, and planes. A point P on a curve is called an actual double point if two of the points of intersection of C with any plane through P coincide at P. If the two tangent lines to C at P are distinct, the point is called a **node.** If the two tangents at P coincide, the point is called a **cusp** or **stationary point.** Curves may have higher point singularities, for example, a curve may pass through the same point three or more times, etc., but such singularities will not be considered here.

A plane is said to be a **double osculating plane** if it is the osculating plane at two points on the curve. A plane having four points of intersection with the curve coincident at P is called a **stationary plane.**

A line is called a **double tangent** if it touches the curve in two distinct points. If a tangent line has three coincident points in common with the curve, it is called a **stationary** or an **inflexional tangent.** The point of contact is called a **linear inflexion.**

182. The Cayley-Salmon formulas. We shall now collect, for the purpose of pointing out certain relations existing among them, the following numbers associated with a given space curve. We shall assume that these numbers are fixed when the curve is given, and are independent of the arbitrarily chosen plane, line, or point that may be used to determine them.

Given a space curve C. Let

$m =$ its order (Art. 140).

$n =$ its class (Art. 180).

$r =$ its rank (Art. 179).

$H =$ the number of its nodes (Art. 181).

$h =$ the number of its apparent double points (Art. 176).

$g =$ the number of lines of intersection of two of its osculating planes which lie in a given plane (dual of h).

$G =$ number of double osculating planes (Art. 181).

$\alpha =$ the number of its stationary planes (Art. 181).

$\beta =$ the number of its stationary points (Art. 181).
$v =$ the number of its linear inflexions (Art. 181).
$\omega =$ the number of its actual double tangents (Art. 181).
$x =$ the number of points lying in a given plane, through which pass two distinct tangents to C.
$y =$ the number of planes passing through a given point, which contain two distinct tangents to C.

These numbers are connected by certain equations called the Cayley-Salmon formulas; they are derived from the analogous equations, known as Plucker's formulas, connecting the characteristic numbers of plane curves. Let $\mu =$ order, $\nu =$ class, $\delta =$ number of double points, $\tau =$ number of double tangents, $\kappa =$ number of cusps, $\iota =$ number of inflexions, of an algebraic plane curve. Plucker's formulas are *

$$\nu = \mu(\mu - 1) - 2\,\delta - 3\,\kappa; \quad \iota = 3\,\mu(\mu - 2) - 6\,\delta - 8\,\kappa;$$
$$\mu = \nu\,(\nu - 1) - 2\,\tau - 3\,\iota; \quad \kappa = 3\,\nu\,(\nu - 2) - 6\,\tau - 8\,\iota.$$

Those in the second line are the duals in the plane of those in the first line.

Let the given space curve C be projected, from an arbitrary point P not lying on it, upon an arbitrary plane not passing through P. The plane curve then has the following characteristic numbers:

$$\mu = m,\ \nu = r,\ \delta = h + H,\ \tau = y + \omega,\ \kappa = \beta,\ \iota = n + v.$$

By substituting in the Plücker formulas, we obtain

$$\begin{aligned} r &= m(m-1) - 2(h + H) - 3\,\beta; \\ n + v &= 3\,m(m-2) - 6(H + h) - 8\,\beta; \\ m &= r(r-1) - 2(y + \omega) - 3(n + v); \\ \beta &= 3\,r(r-2) - 6(y + \omega) - 8(n + v). \end{aligned} \tag{15}$$

By duality in space, that is, by taking the section of the developable surface by an arbitrary plane, we have

$$\begin{aligned} r &= n(n-1) - 2\,(G + g) - 3\,\alpha; \\ m + v &= 3\,n(n-2) - 6(G + g) - 8\,\alpha; \\ n &= r(r-1) - 2(x + \omega) - 3(m + v); \\ \alpha &= 3\,r(r-2) - 6(x + \omega) - 8(m + v). \end{aligned} \tag{16}$$

* Salmon: Higher Plane Curves, 3d edition (1879). See p. 66.

Of these eight equations, six are independent. One relation exists among the first set of four, and one relation among the second set.

The **genus** of a curve is the difference between the sum of its apparent and actual double and stationary points and the maximum number of double points which a non-composite plane curve of the same order may have. If the genus of the space curve C is denoted by p, we have

$$p=\frac{(m-1)(m-2)}{2}-(H+h+\beta)=\frac{(n-1)(n-2)}{2}-(G+g+\alpha)=$$
$$\frac{(r-1)(r-2)}{2}-(y+\omega+n+v)=\frac{(r-1)(r-2)}{2}-(x+\omega+m+v).$$

183. Curves on non-singular quadric surfaces. It has been seen (Art. 115) that the equation of any non-singular quadric surface may be reduced to the form

$$x_1x_2-x_3x_4=0, \tag{17}$$

and that through each point of the surface passes a generator of each regulus of the two systems

$$x_1-\lambda x_4=0, \quad x_3-\lambda x_2=0, \tag{18}$$
$$x_3-\mu x_1=0, \quad x_2-\mu x_4=0. \tag{18$'$}$$

The coördinates of the point of intersection of the generator $\lambda=$ constant with the generator $\mu=$ constant are (Art. 115)

$$\rho x_1=\lambda, \quad \rho x_2=\mu, \quad \rho x_3=\lambda\mu, \quad \rho x_4=1. \tag{19}$$

Consider the locus of the points whose parameters λ, μ satisfy a given equation $f(\lambda, \mu)=0$, algebraic, and of degree k_1 in λ and of degree k_2 in μ. The curve $f(\lambda, \mu)=0$ meets an arbitrary generator $\mu=$ constant in k_1 points, and an arbitrary generator $\lambda=$ constant in k_2 points. It will be designated by the symbol $[k_1, k_2]$. The order of the curve is k_1+k_2, since the plane determined by any two generators of different reguli meets the curve in k_1+k_2 points on these two lines, and nowhere else.

By replacing λ, μ in $f(\lambda, \mu)=0$ by their values, we see from (17), (18), (18′) that the curve is the intersection of the two surfaces

$$f\left(\frac{x_1}{x_4}, \frac{x_2}{x_4}\right)=0, \quad x_1x_2-x_3x_4=0.$$

The second is a monoid of order two (Art. 176) and the first is a cone with vertex at $(0, 0, 1, 0)$, a $(2-1)$-fold point on the monoid. Thus, these equations constitute a particular monoidal representation of the curve. The equations of the image (Art. 118) of the given cone on the plane $x_3 = 0$ are

$$f\left(\frac{x_1}{x_4}, \frac{x_2}{x_4}\right) = 0, \quad x_3 = 0.$$

The two generators to the quadric through the vertex of the cone $f = 0$ meet the plane in the points $(1, 0, 0, 0)$, $(0, 1, 0, 0)$. The former is a k_2-fold point on the plane curve, and the latter a k_1-fold point.

THEOREM I. *Two curves of symbols* $[k_1, k_2]$, $[k'_1, k'_2]$ *on the same non-singular quadric intersect in* $k_1k'_2 + k_2k'_1$ *points.*

Let C, C' be the given curves of symbols $[k_1, k_2]$, $[k'_1, k'_2]$, respectively, and let the equation of the quadric be reduced to the form (17) in such a way that the point $(0, 0, 1, 0)$ does not lie on either curve, and that the generators $x_1 = 0$, $x_4 = 0$; $x_2 = 0$, $x_4 = 0$ through $(0, 0, 1, 0)$ do not pass through a point of intersection of the given curves. Project the curves from $(0, 0, 1, 0)$. Their images on $x_3 = 0$ are of orders $k_1 + k_2$, $k'_1 + k'_2$, respectively; they intersect in $(k_1 + k_2)(k'_1 + k'_2)$ points. Of these points, $k_1k'_1$ coincide at $(0, 1, 0, 0)$ and $k_2k'_2$ at $(1, 0, 0, 0)$. They are the projections of the points in which the curves meet the generators passing through $(0, 0, 1, 0)$, the vertex of the projecting cone, and are therefore apparent, not actual, intersections of the space curves. The remaining

$$(k_1 + k_2)(k'_1 + k'_2) - k_1k'_1 - k_2k'_2 = k_1k'_2 + k_2k'_1$$

intersections of the plane curves are projections of the actual intersections of the space curves, hence the theorem is proved.

THEOREM II. *The number of apparent double points of a curve of symbol* $[k_1, k_2]$ *on a quadric is*

$$h = \tfrac{1}{2}(k_1^2 + k_2^2 - k_1 - k_2).$$

Through an arbitrary point O on the surface pass only two lines which meet the curve in more than one point, namely, the two generators passing through O. The generator $\mu =$ constant through O meets the curve in k_1 points, consequently counts for $\frac{k_1}{2}(k_1-1)$ bisecants through O. Similarly, the generator $\lambda=$ constant, which passes through (0, 0, 1, 0), meets the curve in k_2 points and counts for $\frac{k_2}{2}(k_2-1)$ bisecants. The number of apparent double points is the sum of these two numbers.

184. Space cubic curves.*

THEOREM I. *Through any six given points in space, no four of which lie in a plane, can be passed one and only one cubic curve.*

Let $P_1, \cdots, P_6$ be the given points. The five lines connecting P_1 to each of the remaining points uniquely determine a quadric cone having P_1 as vertex. Similarly, the lines joining P_2 to each of the other points define a quadric cone having P_2 as vertex. These two cones intersect in a composite curve of order four, one component of which is the line P_1P_2, since it lies on both cones. The residual is a curve of order three. This curve cannot be composite, for if it were, at least one component would have to be a straight line common to both cones. But that would require that the cones touch each other along P_1P_2, which would count for two. The residual intersection would in that case be a conic passing through $P_3, \cdots, P_6$. But this is impossible as it was assumed that the points $P_3, \cdots, P_6$ do not lie in a plane. No other cubic curve can be passed through the given points, for every such curve would have seven intersections with the two cones (the vertex counting for two). Hence it would lie on their curve of intersection, which is impossible, since the complete intersection is of order four.

THEOREM II. *A space cubic curve lies on all the quadrics of a bundle.*

For, let $P_1, \cdots, P_7$ be seven given points of the curve. Every quadric through these points has $2\cdot 3+1$ points in common with the curve and consequently contains the curve (Art. 177). But through the given points pass all the quadrics of a bundle (Art. 136), which proves the theorem.

Not all the quadrics of this bundle can be singular, for if so, at

* Unless otherwise stated, it will be assumed in the following discussion that the curve is non-composite.

least one of them would be composite (Art. 131) and still contain the curve. This is impossible, since the given curve is not a plane curve.

The symbol (Art. 183) of a space cubic curve on a non-singular quadric is [2, 1] or [1, 2], since such symbols as [0, 3] and [3, 0] simply define three generators belonging to the same regulus.

The forms of $f(\lambda, \mu)$ corresponding to these symbols are

$$(a_0\lambda^2 + 2\,a_1\lambda + a_2)\mu + (b_0\lambda^2 + 2\,b_1\lambda + b_2) = 0, \tag{20}$$

$$(a'_0\mu^2 + 2\,a'_1\mu + a'_2)\lambda + (b'_0\mu^2 + 2\,b'_1\mu + b'_2) = 0. \tag{20'}$$

Conversely, every irreducible equation of this form will define a cubic curve on the quadric.

Since these equations have six homogeneous coefficients, five independent linear conditions are sufficient to determine a curve of either system. Hence through any five points on a given non-singular quadric can be drawn two cubics, one of each symbol. Some of these cubics may be composite.

From the formula of Art. 183 it follows that on a given non-singular quadric two cubics having the same symbol intersect in four points, while two cubics having different symbols intersect in five points.

THEOREM III. *Every space cubic curve is rational.*

Let the parametric equations of a non-singular quadric through the given cubic be reduced to the form (19). The equations of the curve in λ, μ are of the form (20) or (20′). In (20), let $\lambda = t$, solve for μ in terms of t, and substitute the values of λ and of μ in terms of t in (19).

The resulting equations reduce to the form

$$x_i = a_{i0}t^3 + a_{i1}t^2 + a_{i2}t + a_{i3}, \qquad i = 1, 2, 3, 4. \tag{21}$$

These are the parametric equations of the curve (Art. 178). Since the curve is by hypothesis of order three, to each value of t corresponds a definite point on the curve, and conversely.

Since the cubic (21) does not lie in a plane, the determinant $|\,a_{ik}\,| \neq 0$. The parametric equations, referred to the tetrahedron defined by

$$x_i = a_{i0}x'_1 + a_{i1}x'_2 + a_{i2}x'_3 + a_{i3}x'_4, \qquad i = 1, 2, 3, 4,$$

are, after dropping the primes,

$$x_1 = t^3, \quad x_2 = t^2, \quad x_3 = t, \quad x_4 = 1. \tag{22}$$

From (22), the intersections of the curve with the plane $\Sigma u_i x_i = 0$ are defined by the roots of the equation

$$u_1 t^3 + u_2 t^2 + u_3 t + u_4 = 0. \tag{23}$$

The condition that the plane is an osculating plane is that the roots of (23) are all equal (Art. 180). It follows that the coördinates of the osculating plane at the point whose parameter is t may be expressed in the form

$$u_1 = 1, \quad u_2 = -3t, \quad u_3 = 3t^2, \quad u_4 = -t^3.$$

These equations are called the parametric equations of the cubic curve in plane coördinates.

The condition that the osculating plane at the point whose parameter is t passes through a given point (y) in space is that t is a root of the equation

$$y_4 t^3 - 3y_3 t^2 + 3y_2 t - y_1 = 0. \tag{24}$$

Since this equation is a cubic in t, it follows that the cubic curve is of class three.

We shall now prove the following theorem:

THEOREM IV. *The points of contact of the three osculating planes to a cubic curve through an arbitrary point P lie in a plane passing through P.*

Let $\Sigma a_i x_i = 0$ be the plane passing through the points of osculation of the three planes passing through any given point $P \equiv (y)$. The parameters of the points of osculation of the three osculating planes through (y) are the roots of (24). The roots of (24) must also satisfy the equation

$$a_1 t^3 + a_2 t^2 + a_3 t + a_4 = 0,$$

hence

$$\frac{a_1}{y_4} = \frac{a_2}{-3y_3} = \frac{a_3}{3y_2} = \frac{a_4}{-y_1}.$$

From these conditions it follows that $\Sigma a_i y_i = 0$, so that (y) lies in the plane of the points of osculation.

By the method of Art. 179 the equation of the developable surface of the cubic curve is found to be

$$(x_1x_4 - x_2x_3)^2 - 4(x_2^2 - x_1x_3)(x_3^2 - x_2x_4) = 0.$$

This is also the condition that equation (24) has two equal roots. From this equation it follows that the rank of the cubic curve is four (Art. 179).

It was stated without proof in Art. 133 that the basis curve of a pencil of quadrics of characteristic [22] is a cubic and a bisecant; it was also stated that the basis curve of a pencil of characteristic [4] is a cubic curve and a tangent to it. We shall now prove these statements.

It was shown in Art. 132 that the [22] pencil of quadrics is defined by the two surfaces

$$x_1^2 + 2\,x_2x_4 = 0, \quad x_2^2 + 2\,x_1x_3 = 0.$$

These quadrics intersect in the line $x_1 = 0$, $x_2 = 0$ and the space cubic whose parametric equations can be found by putting $x_4 = 1$, $x_1 = 2\,t$ in the equations of the surfaces, in the form

$$x_1 = 2\,t, \quad x_2 = -2\,t^2, \quad x_3 = -t^3, \quad x_4 = 1.$$

It intersects the line $x_1 = 0$, $x_2 = 0$ in the two points (0, 0, 0, 1), (0, 0, 1, 0).

Similarly, it was seen that a pencil of characteristic [4] is defined by the surfaces

$$x_1x_2 + x_3x_4 = 0, \quad 2\,x_2x_3 + x_4^2 = 0.$$

The basis curve of this pencil consists of the cubic

$$x_1 = 1, \quad x_2 = 2\,t^3, \quad x_3 = -t, \quad x_4 = 2\,t^2$$

and of the line $x_2 = 0$, $x_4 = 0$ which touches it at (0, 1, 0, 0).

If in the parametric equation (20) of a cubic we replace λ by $\frac{x_1}{x_4}$, and μ by $\frac{x_2}{x_4}$, we determine as the projecting cone from (0, 0, 1, 0) a cubic cone with a double generator. It follows that the projecting cone of the cubic is intersected by a plane in a nodal or cuspidal plane cubic curve. We shall now prove the converse theorem.

THEOREM V. *Any nodal or cuspidal plane cubic curve is the projection of a space cubic.*

Let the plane of the cubic be taken as $x_3 = 0$, and the node or cusp at (0, 1, 0, 0). The equation of the curve is of the form

$$x_2(a_0x_1^2 + 2\,a_1x_1x_4 + a_2x_4^2) + b_0x_1^2x_4 + 2\,b_1x_1x_4^2 + b_2x_4^3 = 0.$$

By dividing this equation by x_4^3 and replacing $x_1 : x_4$ by λ, $x_2 : x_4$ by μ, we obtain equation (20) of a space cubic curve of which the given curve is the projection.

Theorem VI. *Any plane nodal cubic curve has three points of inflexion lying on a line.*

If a space cubic is projected from any point (y) upon a plane, the osculating planes from (y) will be cut by the plane of projection in the inflexional tangents of the image curve and the points of osculation will project into the points of inflexion. From the theorem that the points of osculation lie in a plane through (y) it follows that the points of inflexion of the plane cubic lie on a line.

EXERCISES

1. Show that any space cubic curve and a line which touches it or intersects it twice form the basis curve of a pencil of quadrics.

2. Show that a composite cubic curve exists, through which only one quadric surface can pass.

3. Prove that the osculating planes to a cubic curve at its three points of intersection with a given plane (u) intersect at a point in (u).

4. Show that if a cubic curve has an actual double point or a trisecant it must lie in a plane.

5. Obtain all the Cayley-Salmon numbers for the proper space cubics.

6. Where must the vertex of the projecting cone be taken, in order that the plane projection of a proper space cubic shall have a cusp?

7. Show that the projection of a space cubic upon a plane from a point on the curve is a conic.

8. Show that the cubic curve through the six basis points of a web of quadrics determined by six basis points lies entirely on the Weddle surface (Art. 146).

9. Show that a cubic through any six of eight associated points (Art. 136) will have the line joining the other two for bisecant (or tangent).

185. Metric classification of space cubic curves. The space cubic curves are metrically classified according to the form of their intersection with the plane at infinity. If the three intersections are real and distinct, the curve is called a cubical hyperbola. It has three rectilinear asymptotes and lies on three cylinders all of which are hyperbolic. If the points at infinity are all real and two are coincident, the curve is called a cubical hyperbolic parabola. It has one asymptote, and lies on one parabolic cylinder and on one hyperbolic cylinder. If all three of the points of intersection are coincident, the plane at infinity is an osculating plane. The curve is called a cubical parabola. It has no rectilinear asymptote and lies on a parabolic cylinder. Finally, two of the points of intersection may be imaginary. The curve is now called a cubical ellipse. It has one rectilinear asymptote and lies on one elliptic cylinder. An interesting particular case of the cubical ellipse is the curve called the horopter curve on account of its part in the theory of physiological optics. If one looks with both eyes at a point P in space, the eyes are turned so that the two images fall on corresponding points of the retinae. The locus of the points in space whose images fall on corresponding points is a horopter curve through the point P.

186. Classification of space quartic curves.*

THEOREM I. *Every space quartic curve lies on at least one quadric surface.*

For, through any nine points on the curve a quadric surface can be passed. This surface must contain the curve, since it has $2 \times 4 + 1$ points in common with it (Art. 177).

If a quartic curve lies on two different quadrics $A = 0$, $B = 0$, it is called a quartic of the first kind. A quartic of the first kind is the basis curve of a pencil $A - \lambda B = 0$ of quadrics. Not all the quadrics of this pencil are singular, since in every singular pencil are some composite quadrics. Composite quadrics are impossible in this case, since the curve does not lie in a plane. The symbol of the curve on any non-singular quadric on which it lies is [2, 2], since each generator of one quadric will intersect the other quadric defining the curve in two points.

A quartic having the symbol [1, 3] cannot lie on two different quadrics, nor can it lie on a quadric cone, since every generator would have to cut the curve in the same number of points. The [1, 3] curve is called a quartic of the second kind.

It follows from Arts. 132 and 184 that except in the cases of the characteristics [1111], [112], [13], the basis curve of a pencil

* See footnote of Art. 184.

of quadrics is composite. It will now be shown that in these three cases the basis curve is not composite, that in the case [1111] the basis curve has no double point, that in the case [112] it has a node, and that in the case [13] it has a cusp or stationary point (Art. 181). That the basis curve is not composite may be seen as follows: If it were, one component would have to be a line or a conic. It cannot be a line, for the line would have to lie on every quadric of the pencil, hence pass through the vertex of every cone contained in the pencil. From the equations of the pencils having these characteristics (Art. 133) it is seen that in each case there is at least one cone whose vertex does not lie on the basis curve. Moreover, one component cannot be a conic, for the quadric of the pencil determined by an arbitrary point P in the plane of the conic would contain the plane of the conic, and hence be composite; but pencils having these characteristics have no composite quadrics. It will now be shown that the basis curve of the pencil [1111] has no actual node or cusp. It will be called the non-singular quartic curve of the first kind. Suppose the basis curve had a node at O. The projecting cone to the curve from O is of order two. The quadric of the pencil through an arbitrary point P on the projecting cone contains the line OP, since it has three points in common with it. This quadric and the cone must coincide, since they have a quartic curve and a straight line in common. Hence the cone would belong to the pencil, but this is impossible, since no cone of the pencil [1111] has its vertex on the basis curve.

From the equation of the pencil of characteristic [112] it follows that the vertex (0, 0, 0, 1) of the cone

$$(\lambda_1 - \lambda_3)x_1^2 + (\lambda_2 - \lambda_3)x_2^2 + x_3^2 = 0$$

of the pencil lies on the basis curve. This point is an actual double point on the curve, since every plane through it has two points of intersection with the curve coincident at that point. All the quadrics of the pencil touch the plane $x_3 = 0$ at (0, 0, 0, 1); every plane through either of the distinct lines $(\lambda_1 - \lambda_3)x_1^2 + (\lambda_2 - \lambda_3)x_2^2 = 0$, in which $x_3 = 0$ intersects the cone has three intersections with the curve coincident at (0, 0, 0, 1). These two lines are tangents at the node.

Finally, the vertex of the cone

$$(\lambda_1 - \lambda_2)x_1^2 + 2\,x_3x_4 = 0$$

of the [13] pencil is a double point on the basis curve. The tangent lines $x_1 = 0$, $x_3 = 0$ coincide. The double point is a cusp.

The parametric equation of a quartic of symbol [2, 2] has the form

$$(a_0\lambda^2 + 2\,a_1\lambda + a_2)\,\mu^2 + 2(b_0\lambda^2 + 2\,b_1\lambda + b_2)\mu + c_0\lambda^2 + 2\,c_1\lambda + c_2 = 0. \quad (25)$$

The quartic defined by (25) is the intersection of the quadric $x_1x_2 - x_3x_4 = 0$ (Art. 183) and the quadric

$$a_0x_3^2 + 2\,a_1x_2x_3 + a_2x_2^2 + 2\,b_0x_1x_3 + 4\,b_1x_1x_2 + 2\,b_2x_2x_4 + c_0x_1^2 + 2\,c_1x_1x_4 + c_2x_4^2 = 0. \quad (25')$$

If the quartic of intersection has a double point or cusp, we may take the double point as (0, 0, 0, 1), and a cone with vertex at that point for one of the quadrics passing through it. The parametric equation (25) now has the form

$$(2\,a_1\lambda + a_2)\mu^2 + 2(b_1\lambda^2 + 2\,b_1\lambda)\mu + c_0\lambda^2 = 0. \quad (26)$$

If in (26) we put $\lambda = \mu t$, solve for t, and put the values of μ and $\lambda = \mu t$ in equations (19), we obtain a set of parametric equations of the singular quartic curve of the first kind, of the form

$$x_i = a_{i0}t^4 + a_{i1}t^3 + a_{i2}t^2 + a_{i3}t + a_{i4}, \quad i = 1, 2, 3, 4; \quad (27)$$

hence the nodal and cuspidal quartics are rational.

A quartic of the second kind can be expressed parametrically in terms of the parameter which appears to the third degree in its parametric equation, hence the quartics of the second kind are also rational. Rational curves will be discussed later (Art. 188).

THEOREM II. *Through a quartic curve of the second kind and any two of its trisecants can be passed a non-composite cubic surface.*

For, through nineteen points in space a cubic surface can be passed (Art. 161). Choose thirteen on the quartic curve, one on the trisecant g, one on the trisecant g', not on the curve, and four others in space, not in a plane nor on the quadric on which the quartic lies. The quartic curve and the lines g and g' must lie on the non-composite cubic surface determined by these nineteen points as well as on the quadric containing the regulus of trisecants, hence together they form the complete intersection of the cubic and the quadric.

187. Non-singular quartic curves of the first kind. Two quartic curves of the first kind lying on the same quadric intersect in eight points (Art. 183); these points are eight associated points defining a bundle (Art. 136), since they lie on three distinct quadrics not having a curve in common.

The number of apparent double points of a non-singular quartic C_4 of the first kind is two. For each bisecant of C_4 through an arbitrary point P is a generator of the quadric of the pencil having C_4 for basis curve which passes through P. Conversely, each generator of every quadric through C_4 is a bisecant.

Of the Cayley-Salmon numbers we now have $m=4$, $h=2$, $\beta=0$, $H=0$. It also follows from the definition that $G=v=\omega=0$, hence from the formulas of Art. 182 we have

$$m=4,\ n=12,\ r=8,\ H=0,\ h=2,\ G=0,\ g=38,\ \alpha=16,\ \beta=0,$$
$$v=0,\ \omega=0,\ x=16,\ y=8,\ p=1.$$

THEOREM I. *Through any bisecant of a non-singular space quartic curve of the first kind can be drawn four tangent planes to the curve, besides those having their point of contact on the given bisecant.*

Let the given bisecant be taken as $x_1=0$, $x_4=0$ and the quadric of the pencil containing it as $x_1x_2-x_3x_4=0$. Let another quadric of the pencil be determined by (25′). Any plane of the pencil $x_1=mx_4$ intersects C_4 in two points on $x_1=0$, $x_4=0$ and in two other points determined by the roots of the quadratic equation in $x_2:x_4$

$$x_2^2(a_0m^2+2\,a_1m+a_2)+2\,x_2x_4(b_0m^2+2\,b_1m+b_2)$$
$$+\,x_4^2(c_0m^2+2\,c_1m+c_2)=0.$$

The planes determined by values of m which make the roots of this equation equal are tangent planes. The condition on m is

$$4(b_0m^2+2\,b_1m+b_2)^2-4(a_0m^2+2\,a_1m+a_2)(c_0m^2+2\,c_1m+c_2)=0. \quad (28)$$

Since this equation is of the fourth degree, the theorem is established.

THEOREM II. *An arbitrary tangent to a non-singular quartic of the first kind intersects four other tangents at points not on the curve.*

This is a particular case of Theorem I, since a tangent is a bisecant whose points of intersection with the curve coincide.

THEOREM III. *The cross ratio of the four tangent planes through any bisecant is the same number for every bisecant of the curve.*

Two cases are to be considered, according as the two given bisecants intersect on C_4 or not. Let g, g' be two bisecants through a point P on C_4, but not lying on the same quadric of the pencil. Let the equation of the quadric of the pencil through C_4 which contains g be reduced to the form $x_1x_2 - x_3x_4 = 0$ in such a way that the equations of g are $x_1 = 0$, $x_4 = 0$ and the points of intersection of g' with C_4 are $(0, 0, 1, 0)(0, 0, 0, 1)$. In (25′) we now have $a_0 = 0$, $c_2 = 0$, and also in (28). The points of intersection not on g' of a plane $x_1 = nx_2$ and C_4 are determined by the roots of the equation

$$2(c_1n^2 + b_2n)x_2^2 + (c_0n^2 + 4\,b_1n + a_2)x_2x_3 + 2(b_0n + a_1)x_3^2 = 0.$$

The parameters n_1, n_2, n_3, n_4 of the four tangent planes are roots of the equation

$$(c_0n^2 + 4\,b_1n + a_2)^2 - 16(b_0n + a_1)(c_1n^2 + b_2n) = 0.$$

The cross ratio of the four roots of this equation is equal to the cross ratio of the roots of (28) (when $a_0 = c_2 = 0$), since the two equations can be shown to have the same invariants.*

To prove the theorem when g, g' intersect at P on C_4 and lie on the same quadric through C_4, consider any third bisecant g'' of C_4 through P. The cross ratios on g and on g' are each equal to that on g''.

This completes the proof of the first case.

To prove the theorem when the two bisecants do not intersect on C_4, consider a third bisecant connecting a point of intersection on the first with a point of intersection on the second. The cross ratio on each of the given lines is equal to that on the transversal.

This cross ratio is called the modulus of the quartic curve.

* Burnside and Panton: Theory of Equations, 3d edition, p. 148, Ex. 16. It will be found that I and J have the same values for each equation.

The projecting cone of C_4 from a point on it is a cubic cone. The section of this cone made by a plane not passing through the vertex is a cubic curve. Conversely, any plane cubic curve is the projection of a space quartic curve of the first kind. Consider the cubic curve in the plane $x_3 = 0$. It is no restriction to choose the triangle of reference with the two vertices (1, 0, 0, 0), (0, 1, 0, 0) on the curve. The most general cubic equation in x_1, x_2, x_4, but lacking the terms x_1^3, x_2^3, may be written in the form

$$2\,a_1x_2^2x_1 + a_2x_2^2x_4 + 2\,b_0x_1^2x_2 + 4\,b_1x_1x_2x_4 + 2\,b_2x_2x_4^2 + c_0x_1^2x_4 + 2\,c_1x_1x_4^2 + c_2x_4^3 = 0.$$

But this is exactly the result of projecting (Art. 175) from the point (0, 0, 1, 0) the curve (25) for the case $a_0 = 0$, that is, when the quartic curve passes through (0, 0, 1, 0).

From Theorem III it now follows that the cross ratio of the four tangents to any non-singular cubic curve from a point on it, not counting the tangent at the point, is constant.

It was seen that every non-singular quartic lies on four quadric cones whose vertices (Art. 133) are the vertices of the tetrahedron self-polar as to the pencil of quadric surfaces on which the curve lies (Art. 112). Let t, t' be two distinct tangents of C_4 which intersect in a point P. The plane π determined by t, t' touches C_4 in the points of contact T, T' of t, t', respectively. The following properties will now be proved:

(1) The line $l \equiv TT'$ is a generator of a quadric cone on which C_4 lies.

(2) The plane π is a tangent plane to this cone along l.

(3) The point P lies in the face of the self-polar tetrahedron opposite to the vertex through which l passes.

The plane π cuts the pencil of quadric surfaces on which C_4 lies in a pencil of conics touching each other at T and T'. One conic of this pencil consists of the line l counted twice, hence l is a generator of a cone of the pencil and π is its tangent plane. Moreover, l is the polar line of P as to the pencil of conics, hence the vertex of the cone and the point P are conjugate points. Thus P lies in that face of the self-polar tetrahedron which is opposite the vertex of the cone.

If π approaches a stationary plane (Art 181), then T, T', P

approach coincidence, and the tangents t, t' both approach l. This occurs at every point in which C_4 intersects the faces of the self-polar tetrahedron. We have thus the following theorem:

THEOREM V. *The points of contact of the sixteen stationary planes* ($\alpha = 16$) *of a non-singular quartic curve of the first kind lie in the faces of the common self-polar tetrahedron. The planes belonging to the points in each face pass through the opposite vertex.*

Referred to the self-polar tetrahedron, the equations of the quartic are (Art. 133)

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0, \quad a_1x_1^2 + a_2x_2^2 + a_3x_3^2 + a_4x_4^2 = 0.$$

The equation of the developable was derived in Art. 179.

The section of the developable surface by the plane $x_4 = 0$ is the quartic curve ($a_{ik} \equiv a_i - a_k$),

$$a_{24}a_{13}^2x_1^2x_3^2 + a_{34}a_{12}^2x_1^2x_2^2 + a_{23}(a_{12}a_{34} + a_{13}a_{23})x_2^2x_3^2 = 0$$

counted twice. It is a double curve on the developable. It is the locus in the plane $x_4 = 0$ of the points of intersection of tangents to C_4. A similar locus lies in each of the other faces of the self-polar tetrahedron. Since the Cayley-Salmon number x is 16, the entire locus of intersecting tangents to C_4 is these four curves.

Since the points of intersection of C_4 with the faces of the self-polar tetrahedron are the points of contact of the sixteen stationary planes, the coördinates of these points are

$$(\pm\sqrt{a_{23}},\ \pm\sqrt{a_{31}},\ \pm\sqrt{a_{12}},\ 0),\ (\pm\sqrt{a_{24}},\ \pm\sqrt{a_{41}},\ 0,\ \pm\sqrt{a_{12}}),$$
$$(\pm\sqrt{a_{34}},\ 0,\ \pm\sqrt{a_{41}},\ \pm\sqrt{a_{13}}),\ (0,\ \pm\sqrt{a_{34}},\ \pm\sqrt{a_{42}},\ \pm\sqrt{a_{23}}).$$

EXERCISES

1. Find the locus of a point P such that the two bisecants to C_4 from P coincide.

2. How many generators of each quadric through C_4 are tangent to the curve?

3. By the method of Art. 180 find the equations of the stationary planes.

4. Show that any plane containing three points of contact of stationary planes will pass through a fourth. How many distinct planes of this kind are there?

5. Find the locus of a point P such that the plane projection of C_4 from P will be a quartic curve with one double point and one cusp; two cusps.

188. Rational quartics. The parametric equations of any rational quartic may be written in the form

$$x_i = a_{i0}t^4 + 4\,a_{i1}t^3 + 6\,a_{i2}t^2 + 4\,a_{i3}t + a_{i4}, \quad i = 1, 2, 3, 4.$$

The parameters of the points of intersection of the curve with any plane $\Sigma u_i x_i = 0$ are the roots of the equation

$$t^4\Sigma u_i a_{i0} + 4\,t^3\Sigma u_i a_{i1} + 6\,t^2\Sigma u_i a_{i2} + 4\,t\Sigma u_i a_{i3} + \Sigma u_i a_{i4} = 0.$$

Let t_1, t_2, t_3, t_4 be the roots of this equation. From the formulas expressing the coefficients in terms of the roots we have at once

$$\begin{aligned} (t_1 + t_2 + t_3 + t_4)\,\Sigma a_{i0}u_i + 4\,\Sigma a_{i1}u_i &= 0,\\ (t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4)\,\Sigma a_{i0}u_i - 6\,\Sigma a_{i2}u_i &= 0,\\ (t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4)\,\Sigma a_{i0}u_i + 4\,\Sigma a_{i3}u_i &= 0,\\ t_1t_2t_3t_4\Sigma a_{i0}u_i - \Sigma a_{i4}u_i &= 0. \end{aligned} \tag{29}$$

If we eliminate $u_1 : u_2 : u_3 : u_4$ from these four equations, we obtain as the condition that $t_1, \cdots, t_4$ are the parameters of four coplanar points, the equation

$$\begin{aligned} 12\,A_4t_1t_2t_3t_4 &+ 3\,A_3(t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4)\\ &+ 2\,A_2(t_1t_2 + t_1t_3 + t_1t_4 + t_2t_3 + t_2t_4 + t_3t_4)\\ &+ 3\,A_1(t_1 + t_2 + t_3 + t_4) + 12\,A_0 = 0, \end{aligned} \tag{30}$$

in which $A_0 = |\,a_{11}a_{22}a_{33}a_{44}\,|$, $A_1 = |\,a_{10}a_{22}a_{33}a_{44}\,|$, etc. If $t_1 = t_2 = t_3 = t_4$ in (29), the corresponding point will be a point of contact of a stationary plane. Hence there are four points of contact of stationary planes. These four points are defined by the equation

$$A_4t^4 + A_3t^3 + A_2t^2 + A_1t + A_0 = 0. \tag{31}$$

THEOREM. *If a quartic curve has a double point, the parameters of the points of contact of the stationary planes are harmonic.*

Let P be the double point and let t_1, t_2 be the values of the parameter at P. Since P is coplanar with any other two points on the curve, equation (30) is satisfied independently of the values of t_3 and t_4. Thus t_1, t_2 must satisfy the conditions

$$\begin{aligned} 12\,A_4t_1t_2 + 3\,A_3(t_1 + t_2) + 2\,A_2 &= 0,\\ 3\,A_3t_1t_2 + 2\,A_2(t_1 + t_2) + 3\,A_1 &= 0,\\ 2\,A_2t_1t_2 + 3\,A_1(t_1 + t_2) + 12\,A_0 &= 0. \end{aligned} \tag{32}$$

These equations are compatible only when the determinant vanishes, thus

$$\begin{vmatrix} 12\,A_4 & 3\,A_3 & 2\,A_2 \\ 3\,A_3 & 2\,A_2 & 3\,A_1 \\ 2\,A_2 & 3\,A_1 & 12\,A_0 \end{vmatrix} = 0.$$

But this is the condition that the roots of (31) are harmonic.*

The condition that the double point is a cusp is $t_1 = t_2$. In this case equations (32) are replaced by the quadratic equations

$$6\,A_4t^2 + 3\,A_3t + A_2 = 0, \quad 3\,A_3t^2 + 4\,A_2t + 3\,A_1 = 0,$$
$$A_2t^2 + 3\,A_1t + 6\,A_0 = 0.$$

But these are the conditions that (31) has a triple root. Hence, on a cuspidal cubic, three of the points of contact of stationary planes coincide at the cusp. There is in this case only one proper stationary plane.

Three points on C_4 are collinear if their parameters t_1, t_2, t_3 satisfy (30) for all values of t_4. The necessary conditions are

$$\begin{aligned} &12\,A_4t_1t_2t_3 + 3\,A_3(t_1t_2 + t_2t_3 + t_3t_1) + 2\,A_2(t_1 + t_2 + t_3) + 3\,A_1 = 0, \\ &3\,A_3t_1t_2t_3 + 2\,A_2(t_1t_2 + t_2t_3 + t_3t_1) + 3\,A_1(t_1 + t_2 + t_3) + 12\,A_0 = 0. \end{aligned} \tag{33}$$

If the curve has a double point, the parameters t_1, t_2 of the double point satisfy these conditions for every value of t_3. If it does not have a double point, the equations (33) are satisfied, for any given value of t_3, by the parameters of the other points on the trisecant through t.

If the equations resulting from (33) by putting $t_1 = t_2 = t_3$ have a common solution t', the curve has a linear inflexion at the point whose parameter is t'. The condition that these equations in t' have a common solution is exactly the condition that (31) has a double root. In particular, if (31) is a square, the curve has two distinct linear inflexions.

EXERCISES

1. Obtain the Cayley-Salmon numbers for:

(*a*) the nodal quartic.

(*b*) the cuspidal quartic.

(*c*) the general quartic of the second kind.

(*d*) the quartic having a linear inflexion.

(*e*) the quartic having two linear inflexions.

* When the roots of a quartic equation are harmonic, the invariant J vanishes. See Burnside and Panton: Theory of Equations, 4th edition, Vol. 1, p. 150.

2. Show that every [1, k] curve on a quadric is rational and can have no actual double point.

3. Show that every rational quartic is nodal, cuspidal, or a quartic of the second kind.

4. Show that if a rational quartic does not have a cusp or a linear inflexion, its parametric equations can be written in the form

$$x_1 = (t+1)^4, \quad x_2 = (t+a)^4, \quad x_3 = t^4, \quad x_4 = 1.$$

Find the values of a for which the curve is nodal.

5. Prove that if a quartic has a single linear inflexion, its equations can be written in the form

$$x_1 = t^4, \quad x_2 = t^3, \quad x_3 = (t+1)^4, \quad x_4 = 1,$$

and if it has two distinct linear inflexions, in the form

$$x_1 = t^4, \quad x_2 = t^3, \quad x_3 = t, \quad x_4 = 1.$$

6. Show that the equations of a cuspidal quartic can be written in the form

$$x_1 = t^4, \quad x_2 = t^3, \quad x_3 = t^2, \quad x_4 = 1.$$

7. Show that the tangents at the points of contact of the stationary planes of a rational quartic are in hyperbolic position (Art. 120).

8. Show that through any point P on a rational quartic curve pass three osculating planes to the curve besides the one at P, and that the plane of the points of contact passes through P.

9. Determine the number of generators of a quadric surface which are tangent to a [1, 3] curve lying on it.

10. Determine the number of generators of a quadric surface which are tangent to a nodal quartic curve lying on it.

11. Find the parametric equations in plane coördinates of the curves of Ex. 5.

CHAPTER XIV

DIFFERENTIAL GEOMETRY

In this chapter we shall consider some of the properties of curves and surfaces which depend on the form of the locus in the immediate neighborhood of a point on it. Since the properties to be determined involve distances and angles, we shall use rectangular coördinates.

I. Analytic Curves

189. Length of arc of a space curve. The locus of a point whose coördinates are functions, not all constant, of a parameter u

$$x = f_1(u), \quad y = f_2(u), \quad z = f_3(u) \tag{1}$$

is a **space curve.** The length of arc of such a curve is defined as the limit (when it exists) of the perimeter of an inscribed polygon as the lengths of the sides uniformly approach zero. Curves for which no such limit exists will be excluded from our discussion.

By reasoning similar to that in plane geometry it is seen that the length of arc s from the point whose parameter is u_1 to the point whose parameter is u is

$$s = \int_{u_1}^{u} \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\, du. \tag{2}$$

This equation defines s as a function of u. If the function so defined is not a constant, equation (2) also defines u as a function of s. In this case we may write (1) in the form

$$x = x(s), \quad y = y(s), \quad z = z(s), \tag{3}$$

in which s is the parameter.

Unless the contrary is stated, we shall suppose that s is the parameter in each case, and that x, y, z are analytic functions of

s in the interval under consideration. In the neighborhood of $(x(s), y(s), z(s))$, to which we shall refer as the point s, we have

$$\begin{aligned} x_1 &= x + x'\Delta s + \frac{x''}{2}(\Delta s)^2 + \dots, \\ y \; \cancel{x_1} &= y + y'\Delta s + \frac{y''}{2}(\Delta s)^2 + \dots, \\ z_1 &= z + z'\Delta s + \frac{z''}{2}(\Delta s)^2 + \dots, \end{aligned} \tag{4}$$

in which $x' = \dfrac{dx}{ds}, \quad x'' = \dfrac{d^2x}{ds^2}$, etc.

It follows from equation (2) that

$$x'^2 + y'^2 + z'^2 = 1. \tag{5}$$

By differentiating equation (5) we obtain

$$x'x'' + y'y'' + z'z'' = 0. \tag{6}$$

We have thus far supposed that the second member of (2) was not a constant. If the second member of (2) is a constant, we have

$$\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2 = 0. \tag{7}$$

Curves for which this condition (7) is satisfied are called minimal curves. They will be discussed presently. It will be supposed, except when the contrary is stated, that the curve under consideration is not a minimal curve.

190. The moving trihedral. The tangent line to the curve at the point $P \equiv (x, y, z)$ on it may be defined as the limiting position of a secant as two intersections of the line with the curve approach P.

From (4) the equations of the tangent at P are

$$\frac{X - x}{x'} = \frac{Y - y}{y'} = \frac{Z - z}{z'}. \tag{8}$$

Let λ, μ, ν be the direction cosines of the tangent, the direction in which s increases being positive. From (8) and (5) we have

$$\lambda = x', \quad \mu = y', \quad \nu = z'. \tag{9}$$

The plane through $P \equiv (x, y, z)$ perpendicular to the tangent line is called the **normal plane.** Its equation is

$$x'(X-x)+y'(Y-y)+z'(Z-z)=0. \tag{10}$$

The osculating plane at P is the limiting position of a plane through the tangent line at P and a point P' on the curve, as P' approaches P. We shall now determine the equation of the osculating plane.

The equation of any plane through P is

$$A(X-x)+B(Y-y)+C(Z-z)=0.$$

It contains the tangent (8) if

$$Ax'+By'+Cz'=0,$$

and will be satisfied for powers of Δs up to the third (Eqs. (4)) if

$$Ax''+By''+Cz''=0.$$

By eliminating A, B, C, we obtain, as equation of the osculating plane at P,

$$\begin{vmatrix} X-x & Y-y & Z-z \\ x' & y' & z' \\ x'' & y'' & z'' \end{vmatrix}=0. \tag{11}$$

The line of intersection of the osculating plane and the normal plane is called the **principal normal.** From (10) and (11) its equations are found to be

$$\frac{X-x}{x''}=\frac{Y-y}{y''}=\frac{Z-z}{z''}. \tag{12}$$

If λ_1, μ_1, ν_1 are the direction cosines of the principal normal, and if we put

$$\frac{1}{\rho}=\sqrt{x''^2+y''^2+z''^2}, \tag{13}$$

we have

$$\lambda_1=\rho x'', \quad \mu_1=\rho y'', \quad \gamma_1=\rho z''. \tag{14}$$

The plane through P perpendicular to the principal normal is called the **rectifying plane.** From (12) its equation is

$$x''(X-x)+y''(Y-y)+z''(Z-z)=0. \tag{15}$$

The intersection of the rectifying plane and the normal plane is called the **binormal**. From equations (10) and (15) its equations are

$$\frac{X-x}{y'z''-y''z'}=\frac{Y-y}{z'x''-z''x'}=\frac{Z-z}{x'y''-x''y'}. \tag{16}$$

If λ_2, μ_2, ν_2 are the direction cosines of the binormal, we have the relations

$$\lambda_2=\rho(y'z''-y''z'),\quad \mu_2=\rho(z'x''-z''x'),\quad \nu_2=\rho(x'y''-x''y'). \tag{17}$$

The trirectangular trihedral whose edges extend in the positive directions from P along the tangent, principal normal, and binormal is called the moving trihedral to the curve at $P\equiv(x, y, z)$.

From (9), (14), and (17), we have

$$\begin{vmatrix} \lambda & \mu & \nu \\ \lambda_1 & \mu_1 & \nu_1 \\ \lambda_2 & \mu_2 & \nu_2 \end{vmatrix} = 1. \tag{18}$$

It follows at once (Arts. 37, 38) that the positive directions of the coördinate axes can be brought into coincidence with the positive directions of the moving trihedral at the point P by motion alone, without reflexion. Moreover, we have (Art. 37)

$$\begin{aligned} \lambda &= \mu_1\mu_2 - \nu_1\mu, & \mu &= \nu_1\lambda_2 - \lambda_1\nu_2, & \nu &= \lambda_1\mu_2 - \lambda_2\mu_1, \\ \lambda_1 &= \mu_2\nu - \nu_2\mu, & \mu_1 &= \nu_2\lambda - \lambda_2\nu, & \nu_1 &= \lambda_2\mu - \lambda\mu_2, \\ \lambda_2 &= \nu_1\mu - \mu_1\nu & \mu_2 &= \lambda_1\nu - \nu_1\lambda, & \nu_2 &= \lambda\mu_1 - \lambda_1\mu. \end{aligned} \tag{19}$$

191. Curvature. The curvature of a space curve is defined, like that of a plane curve, as the limit, if it exists, of the ratio of the measure of the angle between two tangents to the length of arc of the curve between their points of contact, as the points approach coincidence.

Let θ be the angle between the tangents to the curve at P and P'. The direction cosines of the tangent at P are x', y', z' (9), those at P' are

$$x'+x''\Delta s+\cdots,\quad y'+y''\Delta s+\cdots,\quad z'+z''\Delta s+\cdots.$$

From Art. 5, we have

$$\sin^2\Delta\theta=\{(y'z''-y''z')^2+(z'x''-x'z'')^2+(x'y''-x''y')^2\}(\Delta s)^2+\cdots,$$

the remaining terms all containing higher powers of Δs. From (5) and (6) the coefficient of $(\Delta s)^2$ reduces to $x''^2 + y''^2 + z''^2$. Since

$$\lim_{\Delta s \doteq 0} \frac{\sin \Delta\theta}{\Delta\theta} = 1,$$

we have, on account of (13),

$$\frac{d\theta}{ds} = \frac{1}{\rho} \tag{20}$$

as the expression for the curvature at P. The reciprocal of the curvature is called the radius of curvature.

If $\frac{1}{\rho} = 0$ at a point P on the curve, the tangent at P has three points of intersection with the curve coincident at P; hence P is a linear inflexion.

192. Torsion. The torsion of a space curve is defined as the limit, if it exists, of the ratio of the angle between two osculating planes to the length of arc between their points of osculation, as the points approach coincidence. The reciprocal of the torsion is called the radius of torsion and is denoted by σ.

In order to find the value of σ, let $\Delta\tau$ be the angle between the osculating planes at the points whose parameters are s and $s + \Delta s$. By a process similar to that of Art. 191 we obtain

$$\sin^2 \Delta\tau = \{(\mu_2\nu'_2 - \nu_2\mu'_2)^2 + (\nu_2\lambda'_2 - \lambda_2\nu'_2)^2 + (\lambda_2\mu'_2 - \mu_2\lambda'_2)^2\}(\Delta s)^2 + \cdots,$$

the remaining terms all containing higher powers of Δs. By differentiating (17) we have

$$\lambda'_2 = \frac{\rho'}{\rho}\lambda_2 + \rho(y'z''' - z'y'''), \quad \mu'_2 = \frac{\rho'}{\rho}\mu_2 + \rho(z'x''' - x'z'''),$$

$$\nu'_2 = \frac{\rho'}{\rho}\nu_2 + \rho(x'y''' - y'x'''). \tag{21}$$

It follows that

$$\mu_2\nu'_2 - \nu_2\mu'_2 = \rho^2 x' \begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix},$$

with similar expressions for $\nu_2\lambda'_2 - \lambda_2\nu'_2$ and for $\lambda_2\mu'_2 - \mu_2\lambda'_2$.

If we substitute these values in the above expression for $\sin\Delta\tau$, pass to the limit, take the square root, and assign opposite signs to the two members, we obtain the result

$$\frac{d\tau}{ds}=\frac{1}{\sigma}=-\rho^2\begin{vmatrix} x' & y' & z' \\ x'' & y'' & z'' \\ x''' & y''' & z''' \end{vmatrix}, \tag{22}$$

which is the formula required. Expand the determinant of equation (22) in terms of the elements of the second row, replace x'', y'', z'' by their values from (14), and the cofactors of these numbers by their values from (21), and put $\lambda_1\lambda_2+\mu_1\mu_2+\nu_1\nu_2$ equal to zero, since the principal normal and binormal are orthogonal. By performing these operations we simplify (22) to the form

$$\frac{1}{\sigma}=\lambda_1\lambda'_2+\mu_1\mu'_2+\nu_1\nu'_2. \tag{23}$$

193. The Frenet-Serret formulas. The nine equations

$$\begin{aligned}
&\lambda'=\frac{\lambda_1}{\rho}, && \mu'=\frac{\mu_1}{\rho}, && \nu'=\frac{\nu_1}{\rho},\\
&\lambda'_1=-\left(\frac{\lambda}{\rho}+\frac{\lambda_2}{\sigma}\right), && \mu'_1=-\left(\frac{\mu}{\rho}+\frac{\mu_2}{\sigma}\right), && \nu'_1=-\left(\frac{\nu}{\rho}+\frac{\nu_2}{\sigma}\right),\\
&\lambda'_2=\frac{\lambda_1}{\sigma}, && \mu'_2=\frac{\mu_1}{\sigma}, && \nu'_2=\frac{\nu_1}{\sigma},
\end{aligned} \tag{24}$$

are called the **Frenet-Serret formulas**.

The first three follow at once by replacing λ, μ, ν and λ_1, μ_1, ν_1 by their values from (9) and (14).

To derive the last three, differentiate the identities

$$\lambda_2^2+\mu_2^2+\nu_2^2=1,\quad \lambda\lambda_2+\mu\mu_2+\nu\nu_2=0$$

with respect to s and substitute for λ', μ', ν' their values from (24) which we have just established. The results are

$$\lambda_2\lambda'_2+\mu_2\mu'_2+\nu_2\nu'_2=0,\quad \lambda\lambda'_2+\mu\mu'_2+\nu\nu'_2=0.$$

From these equations we obtain, after simplifying by means of (19),

$$\lambda'_2=\delta\lambda_1,\quad \mu'_2=\delta\mu_1,\quad \nu'_2=\delta\nu_1,$$

δ being a factor of proportionality. To determine its value, substitute these values of $\lambda'_2, \mu'_2, \nu'_2$ in (23). Since $\lambda_1^2 + \mu_1^2 + \nu_1^2 = 1$, we find $\delta = \frac{1}{\sigma}$. The last three equations of (24) are thus established.

To find the values λ'_1, differentiate the identity $\lambda_1 = \mu_2\nu - \nu_2\mu$ (19) and substitute for $\mu', \nu', \mu'_2, \nu'_2$ their values from (24). By (19) the result reduces to the form $\lambda'_1 = -\left(\frac{\lambda}{\rho} + \frac{\lambda_2}{\sigma}\right)$. The values of μ'_1, ν'_1 are found in the same way.

194. The osculating sphere. The sphere which has contact of the third order with a curve at a point P is called the osculating sphere of the curve at P. To determine the center and radius of the osculating sphere at $P \equiv (x, y, z)$, denote the coördinates of the center by (x_2, y_2, z_2) and the radius by R.

The equation of the sphere is

$$(X - x_2)^2 + (Y - y_2)^2 + (Z - z_2)^2 = R^2.$$

This equation must be satisfied by the coördinates defined by (4) to terms in $(\Delta s)^3$ inclusive. From these conditions we obtain the following equations

$$\begin{aligned} &(x - x_2)^2 + (y - y_2)^2 + (z - z_2)^2 = R^2, \\ &(x - x_2)x' + (y - y_2)y' + (z - z_2)z' = 0, \\ &(x - x_2)x'' + (y - y_2)y'' + (z - z_2)z'' + 1 = 0, \\ &(x - x_2)x''' + (y - y_2)y''' + (z - z_2)z''' = 0. \end{aligned} \tag{25}$$

By solving the last three equations for $x - x_2$, $y - y_2$, $z - z_2$ and simplifying by means of (21), (22), and (24) we find

$$x_2 = x + \rho\lambda_1 - \rho'\sigma\lambda_2,\ y_2 = y + \rho\mu_1 - \rho'\sigma\mu_2,\ z_2 = z + \rho\nu_1 - \rho'\sigma\nu_2. \tag{26}$$

If we substitute these values of x_2, y_2, z_2 in the first of equations (25) and simplify, we obtain

$$R^2 = \rho^2 + \sigma^2\rho'^2. \tag{27}$$

THEOREM. *The condition that a space curve lies on a sphere is*

$$\rho + \sigma(\sigma'\rho' + \sigma\rho'') = 0.$$

If a given curve lies on a sphere, the sphere is the osculating sphere at all points of the curve so that x_2, y_2, z_2 and R are constants. Conversely, if these quantities are constants, the curve lies on a sphere.

To determine the condition that the coördinates of the center are constant, differentiate equations (26) and simplify by means of (24). Since λ_2, μ_2, ν_2 are not all zero, the condition is

$$\rho + \sigma(\sigma'\rho' + \sigma\rho'') = 0.$$

By differentiating (27) we see that R is also constant if this equation is satisfied. This proves the proposition.

195. Minimal curves. We have thus far excluded from discussion those curves (Art. 189)

$$x = f_1(u), \quad y = f_2(u), \quad z = f_3(u),$$

for which

$$\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2 = 0. \tag{28}$$

Such curves we called **minimal curves.** A few of their properties will now be derived.

From (28) we may write

$$\frac{dx}{du} + i\frac{dy}{du} = -t\frac{dz}{du}$$

and

$$t\left(\frac{dx}{du} - i\frac{dy}{du}\right) = \frac{dz}{du}$$

in terms of a parameter t. From these equations we deduce

$$\frac{\dfrac{dx}{du}}{\dfrac{1-t^2}{2}} = \frac{\dfrac{dy}{du}}{\dfrac{i(1+t^2)}{2}} = \frac{\dfrac{dz}{du}}{t}.$$

If we denote the value of these fractions by $\phi(u)$, solve for $\frac{dx}{du}, \frac{dy}{du}, \frac{dz}{du}$ and integrate, assuming that $\phi(u)$ is integrable, we find that the equations of a minimal curve may be written in the form

$$x = \frac{1}{2}\int(1-t^2)\phi(u)du, \quad y = \frac{i}{2}\int(1+t^2)\phi(u)du, \quad z = \int t\phi(u)du, \tag{29}$$

in which t is a constant or a function of u. If t is constant, the locus (29) is a line. For, let k be defined by $k = \int^u \phi(u)du$.

In terms of k we obtain

$$x=\frac{1-t^2}{2}k+x_1, \quad y=\frac{i}{2}(1+t^2)k+y_1, \quad z=tk+z_1,$$

wherein x_1, y_1, z_1 are constants of integration. The locus of the point (x, y, z) is the minimal line through the point (x_1, y_1, z_1)

$$\frac{x-x_1}{\frac{1-t^2}{2}}=\frac{y-y_1}{\frac{i(1+t^2)}{2}}=\frac{z-z_1}{t}.$$

The equation of the locus of the minimal lines through any point (x_1, y_1, z_1) is found by squaring the terms of these equations and adding numerators and denominators, respectively, to be the cone

$$(x-x_1)^2+(y-y_1)^2+(z-z_1)^2=0,$$

having its vertex at (x_1, y_1, z_1) and passing through the absolute. This is identical with the equation of the point sphere (Art. 48).

If t is not constant, but a function of u, we may take t as the parameter. Let $u=\psi(t)$, and let $\phi(u)du=\phi(\psi(t))\psi'(t)dt$ be replaced by $F(t)dt$. Equations (29) have the form

$$x=\frac{1}{2}\int(1-t^2)F(t)dt, \quad y=\frac{i}{2}\int(1+t^2)F(t)dt, \quad z=\int tF(t)dt. \tag{30}$$

Let $f(t)$ be defined by $\frac{d^3f}{dt^3}=F(t)$. By integrating equations (30) by parts we have

$$\begin{aligned} x&=\frac{(1-t^2)}{2}\frac{d^2f(t)}{dt^2}+t\frac{df(t)}{dt}-f(t)+x_1,\\ y&=\frac{i(1+t^2)}{2}\frac{d^2f(t)}{dt^2}-it\frac{df(t)}{dt}+if(t)+y_1,\\ z&=t\frac{d^2f(t)}{dt^2}-\frac{df(t)}{dt}+z_1,\end{aligned} \tag{31}$$

x_1, y_1, z_1 being constants. The equations of any non-rectilinear minimal curve may be expressed in this form.

EXERCISES

1. The curve

$$x=a\cos\phi, \quad y=a\sin\phi, \quad z=a\phi$$

is called a circular helix. Find the parametric equations of the curve in terms of the length of arc.

2. At an arbitrary point of the helix of Ex. 1 find the direction cosines of the tangent, principal normal, and binormal. Also find the values of ρ and σ.

3. Find the radius of the osculating sphere at an arbitrary point of the space cubic $x = t$, $y = t^2$, $z = t^3$.

4. Show that the equations of a curve, referred to the moving trihedral of a point P on it, may be written in the form

$$x = s - \frac{s^3}{6\rho^2} + \cdots, \quad y = \frac{s^2}{2\rho} - \left(\frac{d\rho}{ds}\right)\frac{s^3}{6\rho^2} + \cdots, \quad z = -\frac{s^3}{6\rho\sigma} + \cdots,$$

s being the length of arc from P.

5. Discuss the equations (31) of a minimum curve in each of the following cases:

(*a*) $f(t)$ a quadratic function of t.
(*b*) $f(t)$ a cubic function of t.

II. Analytic Surfaces

196. Parametric equations of a surface. The locus of a point (x, y, z) whose coördinates are analytic real functions of two independent real variables u, v

$$x = f_1(u, v), \quad y = f_2(u, v), \quad z = f_3(u, v), \tag{32}$$

such that not every determinant of order two in the matrix

$$\left\| \begin{matrix} \dfrac{\partial f_1}{\partial u} & \dfrac{\partial f_2}{\partial u} & \dfrac{\partial f_3}{\partial u} \\[2ex] \dfrac{\partial f_1}{\partial v} & \dfrac{\partial f_2}{\partial v} & \dfrac{\partial f_3}{\partial v} \end{matrix} \right\| \tag{33}$$

is identically zero, is called an analytic surface. The locus defined by those values of u, v for which the matrix (33) is of rank less than two is called the Jacobian of the surface. Points on the Jacobian will be excluded in the following discussion.

The reason for the restriction (33) is illustrated by the following example.

EXAMPLE. Consider the locus

$$x = u + v, \quad y = (u + v)^2, \quad z = (u + v)^3.$$

For any given value t, any pair of values u, v which satisfy the equation $u + v = t$ define the point (t, t^2, t^3). The locus of the equations is a space cubic curve. In this example the matrix (33) is of rank one.

The necessary and sufficient condition that u, v enter f_1, f_2, f_3 in such a way that x, y, z can be expressed as functions of one variable is that the matrix is of rank less than two.

197. Systems of curves on a surface. If in (32) u is given a constant value, the resulting equations define a curve on the surface. If u is given different values, the corresponding curve describes a system of curves on the surface. Similarly, we may determine a system of curves $v = \text{const.}$ The two systems of curves $u = \text{const.}$, $v = \text{const.}$ are called the **parametric curves** for the given equations of the surface; the variables u, v are called the **curvilinear coördinates** on the surface.

Any equation of the form

$$\phi(u, v) = c \tag{34}$$

determines, for a given value of c, a curve on the surface. The parametric equations of the curve may be obtained by solving (34) for one of the variables and substituting its value in terms of the other in (32). If we now give to c different values, equation (34) determines a system of curves on the surface.

If $\phi(u, v) = c$, $\psi(u, v) = c'$ are two distinct systems of curves on the surface, such that

$$\frac{\partial \phi}{\partial u}\frac{\partial \psi}{\partial v} - \frac{\partial \phi}{\partial v}\frac{\partial \psi}{\partial u} \not\equiv 0,$$

by putting $\phi(u, v) = u'$, $\psi(u, v)' = v$ and solving for u, v we may express x, y, z in terms of u', v'. This process is called the transformation of curvilinear coördinates.

198. Tangent plane. Normal line. The tangent plane to a surface at a point P on it is the plane determined by the tangents at P to the curves on the surface through P.

The equations of the tangent lines to the curves $u = \text{const.}$ and $v = \text{const.}$ at $P \equiv (x, y, z) = (u, v)$ are (Art. 190)

$$\frac{X - x}{\frac{\partial x}{\partial v}} = \frac{Y - y}{\frac{\partial y}{\partial v}} = \frac{Z - z}{\frac{\partial z}{\partial v}},$$

$$\frac{X - x}{\frac{\partial x}{\partial u}} = \frac{Y - y}{\frac{\partial y}{\partial u}} = \frac{Z - z}{\frac{\partial z}{\partial u}}.$$

The plane of these two lines is

$$\begin{vmatrix} X-x & Y-y & Z-z \\ \dfrac{\partial x}{\partial u} & \dfrac{\partial y}{\partial u} & \dfrac{\partial z}{\partial u} \\ \dfrac{\partial x}{\partial v} & \dfrac{\partial y}{\partial v} & \dfrac{\partial z}{\partial v} \end{vmatrix} = 0. \tag{35}$$

Let $v = \phi(u)$ be the equation of any other curve on the surface through (u, v). The equations of its tangent lines are

$$\frac{X-x}{\dfrac{\partial x}{\partial u}+\dfrac{\partial x}{\partial v}\dfrac{d\phi}{du}} = \frac{Y-y}{\dfrac{\partial y}{\partial u}+\dfrac{\partial y}{\partial v}\dfrac{d\phi}{du}} = \frac{Z-z}{\dfrac{\partial z}{\partial u}+\dfrac{\partial z}{\partial v}\dfrac{d\phi}{du}}.$$

This line lies in the plane (35) independently of the form of $\phi(u)$, hence (35) is the equation of the tangent plane.

The normal is the line perpendicular to the tangent plane at the point of tangency. Its equations are

$$\frac{X-x}{\dfrac{\partial y}{\partial u}\dfrac{\partial z}{\partial v}-\dfrac{\partial z}{\partial u}\dfrac{\partial y}{\partial v}} = \frac{Y-y}{\dfrac{\partial z}{\partial u}\dfrac{\partial x}{\partial v}-\dfrac{\partial x}{\partial u}\dfrac{\partial z}{\partial v}} = \frac{Z-z}{\dfrac{\partial x}{\partial u}\dfrac{\partial y}{\partial v}-\dfrac{\partial y}{\partial u}\dfrac{\partial x}{\partial v}}.$$

We shall denote the direction cosines of the normal by $\bar{\lambda}, \bar{\mu}, \bar{\nu}$. Their values are

$$\begin{aligned} \bar{\lambda} &= \frac{1}{D}\left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right), \\ \bar{\mu} &= \frac{1}{D}\left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right), \\ \bar{\nu} &= \frac{1}{D}\left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right), \end{aligned} \tag{36}$$

wherein

$$D^2 = \left(\frac{\partial y}{\partial u}\frac{\partial z}{\partial v}-\frac{\partial z}{\partial u}\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial u}\frac{\partial x}{\partial v}-\frac{\partial x}{\partial u}\frac{\partial z}{\partial v}\right)^2 + \left(\frac{\partial x}{\partial u}\frac{\partial y}{\partial v}-\frac{\partial y}{\partial u}\frac{\partial x}{\partial v}\right)^2. \tag{37}$$

If $D = 0$, the tangent plane (35) is isotropic (Art. 152), and the formula for determining the direction cosines of the normal fails. We shall limit our discussion to the case in which $D \neq 0$.

The equation of the tangent plane may be written in the form

$$\bar{\lambda}(X-x)+\bar{\mu}(Y-y)+\bar{\nu}(Z-z) = 0.$$

199. Differential of arc. Let $\phi(u, v)=0$ be the equation of a curve on the surface (32). The differential of the length of arc of this curve is given by the formula (Art. 189)

$$ds^2 = dx^2 + dy^2 + dz^2,$$

in which

$$dx = \frac{\partial x}{\partial u} du + \frac{\partial x}{\partial v} dv,\quad dy = \frac{\partial y}{\partial u} du + \frac{\partial y}{\partial v} dv,\quad dz = \frac{\partial z}{\partial u} du + \frac{\partial z}{\partial v} dv,$$

and the differentials du, dv satisfy the equation

$$\frac{\partial \phi}{\partial u} du + \frac{\partial \phi}{\partial v} dv = 0.$$

If we substitute these values for dx, dy, dz in the expression for ds we obtain

$$ds^2 = E du^2 + 2\, F du dv + G dv^2, \tag{38}$$

in which

$$E = \left(\frac{\partial x}{\partial u}\right)^2 + \left(\frac{\partial y}{\partial u}\right)^2 + \left(\frac{\partial z}{\partial u}\right)^2,$$

$$F = \frac{\partial x}{\partial u}\frac{\partial x}{\partial v} + \frac{\partial y}{\partial u}\frac{\partial y}{\partial v} + \frac{\partial z}{\partial u}\frac{\partial z}{\partial v}, \tag{39}$$

$$G = \left(\frac{\partial x}{\partial v}\right)^2 + \left(\frac{\partial y}{\partial v}\right)^2 + \left(\frac{\partial z}{\partial v}\right)^2.$$

Since the expression $\phi(u, v)$ does not enter explicitly in the equation (39), the expression for ds has the same form and the coefficients E, F, G have the same values for all the curves passing through P, but the value of $dv : du$ depends upon the curve chosen.

The coefficients E, F, G are called the **fundamental quantities of the first order**. From (37) and (39) it follows that

$$D^2 = EG - F^2.$$

Let C be a curve on the surface through (u, v) and let ds be the element of arc on C. The direction cosines λ, μ, ν of the tangent to C are

$$\lambda = \frac{dx}{ds} = \frac{\partial x}{\partial u}\frac{du}{ds} + \frac{\partial x}{\partial v}\frac{dv}{ds},\qquad \mu = \frac{dy}{ds} = \frac{\partial y}{\partial u}\frac{du}{ds} + \frac{\partial y}{\partial v}\frac{dv}{ds},$$

$$\nu = \frac{dz}{ds} = \frac{\partial z}{\partial u}\frac{du}{ds} + \frac{\partial z}{\partial v}\frac{dv}{ds}.$$

If we replace ds by its value from (38), divide numerator and denominator of each equation by du, and replace $dv:du$ by k, we have

$$\lambda = \frac{\dfrac{\partial x}{\partial u} + k\dfrac{\partial x}{\partial v}}{\sqrt{E + 2Fk + Gk^2}},$$

$$\mu = \frac{\dfrac{\partial y}{\partial u} + k\dfrac{\partial y}{\partial v}}{\sqrt{E + 2Fk + Gk^2}}, \qquad (40)$$

$$\nu = \frac{\dfrac{\partial z}{\partial u} + k\dfrac{\partial z}{\partial v}}{\sqrt{E + 2Fk + Gk^2}}.$$

It follows from these equations that at a given point on the surface the tangent line to a curve passing through the point is uniquely determined when the value of the ratio $dv:du = k$ is known, since $\dfrac{\partial x}{\partial u}$, $\dfrac{\partial x}{\partial v}$, etc., are fixed when the point (u, v) is given.

200. Minimal curves. Each factor of the expression

$$Edu^2 + 2\,Fdudv + Gdv^2,$$

when equated to zero, determines a system of curves on the surface. Let $\phi(u, v)du + \psi(u, v)dv$ be such a factor. By equating to zero and integrating we obtain an equation of the form $f(u, v,) = c$, in which c is a constant of integration, which determines a system of curves on the surface.

The two systems of curves determined in this way are minimal curves (Art. 195), since the differential of arc of every curve of each system satisfies the condition

$$ds^2 = Edu^2 + 2\,Fdudv + Gdv^2 = 0.$$

This equation determines, at (u, v), two values of the ratio $dv : du = k$ which define two imaginary tangents to minimal curves. The two tangents coincide at points for which $D = 0$.

In the succeeding discussion we shall assume that minimal curves are excluded.

201. Angle between curves. Differential of surface. The angle between the tangents to the curves $u = \text{const.}$, $v = \text{const.}$ is determined from Art. 198 by the formula (Art. 5)

$$\cos\omega = \frac{F}{\sqrt{EG}}, \text{ from which } \sin\omega = \frac{\sqrt{EG - F^2}}{\sqrt{EG}} = \frac{D}{\sqrt{EG}}.$$

The curvilinear quadrilateral whose vertices are determined by (u, v), $(u + \Delta u, v)$, $(u, v + \Delta v)$, $(u + \Delta u, v + \Delta v)$ is approximately a parallelogram such that the lengths of the adjacent sides are, from (38), $\sqrt{E}du$, $\sqrt{G}dv$, and the included angle is ω.

Hence we have in the limit for the differential of surface

$$dS = \sin\omega\sqrt{EG}dudv = Ddudv.$$

Let C, C' be two given curves on the surface through a point P. We shall denote the differentials of u, v, s on C by du, dv, ds and the differentials of u, v, s on C' by $\delta u, \delta v, \delta s$. The direction cosines λ, μ, ν of the tangent to C are determined by replacing k in (40) by $dv : du$; similarly the direction cosines λ', μ', ν' of the tangent to C' are determined by replacing k by $\delta v : \delta u$.

If θ is the angle between the tangents to C and C' at (u, v),

$$\cos\theta = \lambda\lambda' + \mu\mu' + \nu\nu' = \frac{Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v}{ds\delta s}. \qquad (41)$$

From (41) we have at once the following theorem:

THEOREM. *The condition that two directions determined by the ratios $dv : du$, $\delta v : \delta u$ are orthogonal is*

$$Edu\delta u + F(du\delta v + dv\delta u) + Gdv\delta v = 0. \qquad (42)$$

202. Radius of normal curvature. Meusnier's theorem. Let ψ be the angle which the principal normal to C makes with the normal to the surface. Let λ_1, μ_1, ν_1 denote the direction cosines of the principal normal and ds the differential of arc along C. We have, from (14)

$$\cos\psi = \lambda_1\bar{\lambda} + \mu_1\bar{\mu} + \nu_1\bar{\nu} = \rho\left(\bar{\lambda}\frac{d^2x}{ds^2} + \bar{\mu}\frac{d^2y}{ds^2} + \bar{\nu}\frac{d^2z}{ds^2}\right),$$

ρ being the radius of curvature of C at (u, v).

But

$$\frac{d^2x}{ds^2}=\frac{\partial^2 x}{\partial u^2}\left(\frac{du}{ds}\right)^2+2\,\frac{\partial^2 x}{\partial u\partial v}\,\frac{du}{ds}\frac{dv}{ds}+\frac{\partial^2 x}{\partial v^2}\left(\frac{dv}{ds}\right)^2+\frac{\partial x}{\partial u}\frac{d^2u}{ds^2}+\frac{\partial x}{\partial v}\frac{d^2v}{ds^2},$$

with similar expressions for $\frac{d^2y}{ds^2}$, $\frac{d^2z}{ds^2}$. Substitute these values for the second derivatives in the equation for $\cos\psi$. Since the normal to the surface is perpendicular to the tangents to the curves $u=\text{const.}$, $v=\text{const.}$, we have the relations

$$\bar\lambda\frac{\partial x}{\partial u}+\bar\mu\frac{\partial y}{\partial u}+\bar\nu\frac{\partial z}{\partial u}=0,\quad \bar\lambda\frac{\partial x}{\partial v}+\bar\mu\frac{\partial y}{\partial v}+\bar\nu\frac{\partial z}{\partial v}=0.$$

If we replace ds by its value from (38), the equation for $\cos\psi$ may be reduced to

$$\frac{\cos\psi}{\rho}=\frac{Ldu^2+2\,Mdudv+Ndv^2}{Edu^2+2\,Fdudv+Gdv^2},\tag{43}$$

wherein

$$L\equiv\bar\lambda\frac{\partial^2 x}{\partial u^2}+\bar\mu\frac{\partial^2 y}{\partial u^2}+\bar\nu\frac{\partial^2 z}{\partial u^2},$$

$$M\equiv\bar\lambda\frac{\partial^2 x}{\partial u\partial v}+\bar\mu\frac{\partial^2 y}{\partial u\partial v}+\bar\nu\frac{\partial^2 z}{\partial u\partial v},\tag{44}$$

$$N\equiv\bar\lambda\frac{\partial^2 x}{\partial v^2}+\bar\mu\frac{\partial^2 y}{\partial v^2}+\bar\nu\frac{\partial^2 z}{\partial v^2}.$$

The quantities L, M, N are called the **fundamental quantities of the second order** for the given surface.

The second member of equation (43) depends only on (u, v) and the ratio $dv:du=k$. Consider the plane section of the surface determined by the normal to the surface and the tangent to C. Such a section is called a normal section. Let the radius of curvature of this normal section at (u, v) be denoted by R. From (43) we have

$$\frac{1}{R}=\frac{Ldu^2+2\,Mdudv+Ndv^2}{Edu^2+2\,Fdudv+Gdv^2},\tag{45}$$

and hence

$$R\cos\psi=\rho.\tag{46}$$

The result expressed in equation (46) may be stated in the following form, known as Meusnier's theorem:

THEOREM. *The center of curvature of any point of a curve on a surface is the projection on its osculating plane of the center of curvature of the normal section tangent to the curve at the point.*

203. Asymptotic tangents. Asymptotic curves. The two tangents to the given surface at (u, v) defined by the equation

$$Ldu^2 + 2\,Mdudv + Ndv^2 = 0 \tag{47}$$

are called the **asymptotic tangents** at P.

From (45) we have at once the following theorem:

THEOREM I. *If the curve C on the surface is tangent to an asymptotic tangent at (u, v), then either the osculating plane to C coincides with the tangent plane to the surface or C has a linear inflexion at (u, v).*

The two systems of curves defined by the factors of (47) are called the **asymptotic curves** of the surface. They have the property that their tangents are the asymptotic tangents to the surface. We have the further theorems:

THEOREM II. *If a straight line lies on a surface, it coincides with an asymptotic tangent at each of its points, hence the line is an asymptotic curve.*

THEOREM III. *The osculating plane at each point of a real asymptotic curve, not a straight line, coincides with the tangent plane to the surface at that point.*

204. Conjugate tangents. The equations of the tangent planes at $P \equiv (x, y, z)$ and at $P' \equiv (x + \Delta x, y + \Delta y, z + \Delta z)$ on the surface are (Art. 198)

$$\begin{aligned} \bar\lambda(X - x) + \bar\mu(Y - y) + \bar\nu(Z - z) = 0,\\ (\bar\lambda + \Delta\bar\lambda)(X - x - \Delta x) + (\bar\mu + \Delta\bar\mu)(Y - y - \Delta y)\\ + (\bar\nu + \Delta\bar\nu)(Z - z - \Delta z) = 0. \end{aligned} \tag{48}$$

Let P' approach P along a curve whose tangent at P is determined by $k = dv : du$. We shall now determine the limiting position of the line of intersection of the planes. If we subtract the first of equations (48) from the second, member by member, and pass to the limit, we have

$$d\bar{\lambda}(X-x)+d\bar{\mu}(Y-y)+d\bar{\nu}\,(Z-z)+\bar{\lambda}dx+\bar{\mu}dy+\bar{\nu}dz=0.$$

But $\bar{\lambda}dx+\bar{\mu}dy+\bar{\nu}dz=0$, since the normal to the surface at P is perpendicular to every tangent at P. Hence the limiting position of the line of intersection passes through P, since it lies in the tangent plane at P and in the plane $d\bar{\lambda}(X-x)+d\bar{\mu}(Y-y)+d\bar{\nu}(Z-z)=0$ through P. Let the point (X, Y, Z) on the line of intersection be denoted by $X=x+\delta x=x+\frac{\partial x}{\partial u}\delta u+\frac{\partial x}{\partial v}\delta v$, etc. (Art. 199). We have

$$d\bar{\lambda}\left(\frac{\partial x}{\partial u}\delta u+\frac{\partial x}{\partial v}\delta v\right)+d\bar{\mu}\left(\frac{\partial y}{\partial u}\delta u+\frac{\partial y}{\partial v}\delta v\right)+d\bar{\nu}\left(\frac{\partial z}{\partial u}\delta u+\frac{\partial z}{\partial v}\delta v\right)=0.$$

If we replace $\bar{\lambda}$, $\bar{\mu}$, $\bar{\nu}$ by their values from (36) and simplify, this equation reduces to

$$Ldu\delta u+M(du\delta v+dv\delta u)+Ndv\delta v=0, \tag{49}$$

which determines $\delta v:\delta u$ linearly in terms of $dv:du$.

Since equation (49) is symmetric in $dv:du$ and $\delta v:\delta u$, it follows that if a point P'' approaches P in the direction determined by $\delta v:\delta u$, the limiting position of the line of intersection of the tangent planes at P and P'' is determined by $dv:du$.

Two tangents determined by $dv:du$, $\delta v:\delta u$ which satisfy (49) are called **conjugate tangents.**

THEOREM. *The necessary and sufficient condition that a tangent coincides with its conjugate is that it is an asymptotic tangent.*

For, if in (49) we put $\delta v:\delta u=dv:du$, we obtain (47). Conversely, if $dv:du$ satisfies (47) and $\delta v:\delta u$ is conjugate to it, then $dv:du=\delta v:\delta u$.

205. Principal radii of normal curvature. In order to determine the maximum and minimum values of R in equation (45) at a given point (u, v) put $dv:du=k$ and differentiate R as a function of k. The derivative vanishes for values of k determined by the equation

$$(FN-GM)k^2-(GL-EN)k+(EM-FL)=0. \tag{50}$$

If this equation is not identically satisfied, the two roots k_1, k_2 are real and distinct, since the part under the radical may be expressed as the sum of two squares.

$$(GL-EN)^2-4(FN-GM)(EM-FL)$$
$$\equiv 4\frac{D^2}{E^2}(EM-FL)^2+\left[EN-GL-\frac{2F}{E}(EM-FL)\right]^2.$$

One root will determine the tangent $dv:du$ such that the normal section through it will have a maximum radius of curvature R_1 and the other will determine the normal section having the minimum radius of curvature R_2.

The tangents at (u, v) determined by the roots of (50) are called the tangents of principal curvature, and the corresponding radii R_1, R_2 are called the principal radii of curvature. To determine the values of R_1 and R_2 we have from (45) and (50)

$$\frac{L+kM}{E+kF}=\frac{M+kN}{F+kG}=\frac{1}{R}.$$

By eliminating k between these equations, we obtain the quadratic equation

$$(LN-M^2)R^2-(EN-2FM+GL)R+EG-F^2=0, \tag{51}$$

whose roots are R_1 and R_2.

The expression $\dfrac{1}{R_1}+\dfrac{1}{R_2}$ is called the **mean curvature** of the surface at (u, v); the expression $\dfrac{1}{R_1}\cdot\dfrac{1}{R_2}$ is called the **total curvature** of the surface at (u, v). From (51) we have

$$\frac{1}{R_1}+\frac{1}{R_2}=\frac{EN-2FM+GL}{EG-F^2}, \tag{52}$$
$$\frac{1}{R_1R_2}=\frac{LN-M^2}{EG-F^2}.$$

206. Lines of curvature. If in (50) we put $k=dv:du$, we obtain

$$(EM-FL)du^2-(GL-EN)dudv+(FN-GM)dv^2=0. \tag{53}$$

The two factors of this equation determine two systems of curves called lines of curvature of the surface. If the two directions at

(u, v) of the lines of curvature are denoted by $dv : du$ and $\delta v : \delta u$, then, from (53)

$$\frac{dv\delta u + \delta v du}{du\delta u} = \frac{GL - EN}{FN - GM}, \quad \frac{dv\delta v}{du\delta u} = \frac{EM - FL}{FN - GM},$$

from which

$$\begin{aligned} Edu\delta u + F(dv\delta u + du\delta v) + Gdv\delta v &= 0, \\ Ldu\delta u + M(dv\delta u + du\delta v) + Ndv\delta v &= 0. \end{aligned} \tag{54}$$

From the first of these equations we have, by (41), the following theorem:

THEOREM I. *The two lines of curvature at a point on the surface are orthogonal.*

From the second equation we have, by (49), the further theorem:

THEOREM II. *The tangents to the lines of curvature at a point on the surface are conjugate directions.*

Conversely, if two systems of curves on the surface are orthogonal and conjugate, their equations satisfy (53) and (54), hence they are lines of curvature.

The normals to the surface at the points of a given curve C on it generate a ruled surface. The ruled surface is said to be developable if the limit of the ratio of the distance between the normals to two points P, P' on C to the arc PP' approaches zero as P' approaches P.

It should be noticed that in particular a cone satisfies the condition of being a developable surface. A cylinder is regarded as a limiting case of a cone, and is included among developable surfaces.

THEOREM III. *The condition that the normals to a surface at the points of a curve on it describe a developable is that the curve is a line of curvature.*

Let $P \equiv (x, y, z)$ and $P' \equiv (x + \Delta x, y + \Delta y, z + \Delta z)$ be two points on the given curve C. The equations of the normals at P and P' are (Art. 20)

$$X = x + \bar{\lambda} r, \quad Y = y + \bar{\mu} r, \quad Z = z + \bar{\nu} r,$$
$$X = x + \Delta x + (\bar{\lambda} + \Delta\bar{\lambda})r', \quad Y = y + \Delta y + (\bar{\mu} + \bar{\mu}\Delta)r',$$
$$Z = z + \Delta z + (\bar{\nu} + \Delta\bar{\nu})r'.$$

The ratio of the distance Δl to the arc Δs is (Art. 23)

$$\frac{\Delta l}{\Delta s}=\frac{\Delta x(\bar\mu\Delta\bar\nu-\bar\nu\Delta\bar\mu)+\Delta y(\bar\nu\Delta\bar\lambda-\bar\lambda\Delta\bar\nu)+\Delta z(\bar\lambda\Delta\bar\mu-\bar\mu\Delta\bar\lambda)}{\Delta s\sqrt{(\bar\mu\Delta\bar\nu-\bar\nu\Delta\bar\mu)^2+(\bar\nu\Delta\bar\lambda-\bar\lambda\Delta\bar\nu)^2+(\bar\lambda\Delta\bar\mu-\bar\mu\Delta\bar\lambda)^2}}.$$

Divide numerator and denominator of the second member of this equation by Δs^2, and pass to the limit as $\Delta s \doteq 0$. Using the differential notation to indicate lim $\Delta x : \Delta s$, etc., we have

$$\lim_{\Delta s\doteq 0}\frac{\Delta l}{\Delta s}=\frac{dx(\bar\mu d\bar\nu-\bar\nu d\bar\mu)+dy(\bar\nu d\bar\lambda-\bar\lambda d\bar\nu)+dz(\bar\lambda d\bar\mu-\bar\mu d\bar\lambda)}{\sqrt{(\bar\mu d\bar\nu-\bar\nu d\bar\mu)^2+(\bar\nu d\bar\lambda-\bar\lambda d\bar\nu)^2+(\bar\lambda d\bar\mu-\bar\mu d\bar\lambda)^2}}. \quad (55)$$

Both numerator and denominator of the second member of this equation vanish for those values of $\bar\lambda$, $\bar\mu$, $\bar\nu$ which satisfy the equations

$$\frac{d\bar\lambda}{\bar\lambda}=\frac{d\bar\mu}{\bar\mu}=\frac{d\bar\nu}{\bar\nu}=k$$

and the limiting value of the ratio $\frac{\Delta l}{\Delta s}$ is indeterminate. The denominator cannot vanish for any other values of $\bar\lambda$, $\bar\mu$, $\bar\nu$.

Since $$\bar\lambda^2+\bar\mu^2+\bar\nu^2=1,$$

we have, by differentiating,

$$\bar\lambda d\bar\lambda+\bar\mu d\bar\mu+\bar\nu d\bar\nu=0,$$

which reduces, under the condition that $d\bar\lambda=k\bar\lambda$, etc., to

$$k(\bar\lambda^2+\bar\mu^2+\bar\nu^2)=k=0.$$

Since $k=0$, we have $d\bar\lambda=d\bar\mu=d\bar\nu=0$. Hence the normal to the surface has a constant direction for all points of the curve C. The surface generated by the normal is in this case a cylinder.

If the denominator of (55) is not zero, the condition that the surface generated by normals to the surface along C is a developable is that the numerator of the second member of (55) is zero, that is, that

$$dx(\bar\mu d\bar\nu-\bar\nu d\bar\mu)+dy(\bar\nu d\bar\lambda-\bar\lambda d\bar\nu)+dz(\bar\lambda d\bar\mu-\bar\mu d\bar\lambda)=0.$$

If we substitute for $\bar\lambda$, $\bar\mu$, $\bar\nu$ their values from (36) and for dx, dy, dz their values $\frac{\partial x}{\partial u}du+\frac{\partial x}{\partial v}dv$, etc., we can reduce this equation to (53), which proves the theorem.

207. The indicatrix. Let the lines of curvature be chosen for parametric curves. In (54), $dv = 0$ and $\delta u = 0$, but $du \neq 0$, $\delta v \neq 0$, hence $F = 0$, $M = 0$.

Let C' be a curve making an angle θ with $u =$ cons. and let R be the radius of normal curvature in the direction of C'. Along $u =$ cons., $ds = \sqrt{G}dv$, hence from (41),

$$\cos\theta = \sqrt{G}\frac{dv}{ds}, \quad \sin\theta = \sqrt{E}\frac{du}{ds}.$$

From (45) and (52) we now have the formula

$$\frac{1}{R} = \frac{\cos^2\theta}{R_1} + \frac{\sin^2\theta}{R_2}.$$

This equation is known as Euler's formula for the radius of curvature of normal sections. It is intimately connected with the shape of the surface about P.

Let the surface be referred to the tangents of principal curvature and normal at P as X, Y, Z axes.

Let x, y be taken as parameters. The equation in x, y, z has the form

$$z = \left(\frac{\partial z}{\partial x}\right)x + \left(\frac{\partial z}{\partial y}\right)y + \frac{1}{2}\left(\frac{\partial^2 z}{\partial x^2}\right)x^2 + \left(\frac{\partial^2 z}{\partial x \partial y}\right)xy + \frac{1}{2}\left(\frac{\partial^2 z}{\partial y^2}\right)y^2 + \cdots.$$

Since $z = 0$ is the equation of the tangent plane at the origin, $\left(\frac{\partial z}{\partial x}\right) = 0$ and $\left(\frac{\partial z}{\partial y}\right) = 0$. Since the X and Y axes are the tangents of principal curvature at the origin,

$$\left(\frac{\partial^2 z}{\partial x^2}\right) = \frac{1}{R_1}, \quad \left(\frac{\partial^2 z}{\partial x \partial y}\right) = 0, \quad \left(\frac{\partial^2 z}{\partial y^2}\right) = \frac{1}{R_2},$$

hence, neglecting powers of x and y higher than the second, the approximate equation of the surface for points near (0, 0, 0) is

$$2z = \frac{x^2}{R_1} + \frac{y^2}{R_2}. \tag{56}$$

If $\frac{1}{R_1}$ and $\frac{1}{R_2}$ are both different from zero, the surface defined by (56) is a paraboloid. If one of them is zero and the other finite, the surface is a parabolic cylinder. If both are zero, the surface is the tangent plane to the given surface. This last case will not be considered further.

The section of the quadric (56) by a plane $z=$ cons. is called the indicatrix of the given surface at a point P.

If R_1 and R_2 have the same sign, the section is an ellipse for a plane on one side of the tangent plane, and is imaginary for a plane on the other side. In the neighborhood of P the surface lies entirely on one side of the tangent plane. Such a point P is called an elliptic point on the surface.

If R_1 and R_2 have opposite signs, the paraboloid (56) is hyperbolic and the section by any plane $z=$ cons. on either side of the tangent plane is a real hyperbola. The point P is in this case called a hyperbolic point on the surface.

If $\dfrac{1}{R_1}$ or $\dfrac{1}{R_2}$ is zero, the section $z=$ cons. consists of two parallel lines for a plane on one side of the tangent plane, and is imaginary for a plane on the other side. It follows from (52) that at such points $LN-M^2=0$, and from (47) that the two asymptotic tangents coincide. The point P is in this case called a parabolic point on the surface.

In all three cases, the directions of the asymptotic tangents to the surface at a point P are the directions of the asymptotes of the indicatrix. At an elliptic point the asymptotic tangents are imaginary; at a hyperbolic point they are real and distinct; at a parabolic point they are coincident. Moreover, conjugate tangents on the surface are parallel to conjugate diameters on the indicatrix. The asymptotic tangents are self-conjugate.

EXERCISES

1. Find the equation of the tangent plane and the direction cosines of the normal to the surface $x=u\cos v$, $y=u\sin v$, $z=u^2$ at the point (u, v).

2. Determine the differential equation of the asymptotic lines on the surface defined in Ex 1.

3. Show that the parametric curves in Ex. 1 are orthogonal.

4. Find the lines of curvature on the surface $x=a(u+v)$, $y=b(u-v)$, $z=uv$.

5. Prove that if $E:F:G=L:M:N$ for every point of a surface, the surface is either a sphere or a plane.

ANSWERS

Page 3. Art. 1

2. The YZ-plane. 3. The Z-axis.

4. A line parallel to the Z-axis through $(a, b, 0)$.

5. $(k, l, -m)$, $(k, -l, m)$, $(k, -l, -m)$, $(-k, l, -m)$, $(-k, -l, -m)$.

Page 5. Art. 2

3. $(-1, 1, 9)$. 4. 13. 5. $\sqrt{a^2+b^2+c^2}$.

6. $\frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}}, 0;\ \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}};\ \frac{-7}{\sqrt{89}}, \frac{6}{\sqrt{89}}, \frac{2}{\sqrt{89}}.$

$0, \frac{1}{\sqrt{5}}, \frac{2}{\sqrt{5}};\ \frac{1}{\sqrt{21}}, \frac{-4}{\sqrt{21}}, \frac{2}{\sqrt{21}};$

$\frac{x}{\sqrt{x^2+y^2+z^2}}, \frac{y}{\sqrt{x^2+y^2+z^2}}, \frac{z}{\sqrt{x^2+y^2+z^2}}.$

Page 7. Art. 4

1. $\sqrt{89}$. 4. $\sqrt{(x-1)^2+(y-1)^2+(z-1)^2}=\sqrt{(x-2)^2+(y-3)^2+(z-4)^2}$.

6. (a) $\frac{2}{\sqrt{38}}, \frac{3}{\sqrt{38}}, \frac{5}{\sqrt{38}}$. (b) $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

(c) $\frac{3}{\sqrt{41}}, \frac{4}{\sqrt{41}}, \frac{-4}{\sqrt{41}}$.

7. (a) Parallel to the YZ-plane. (b) Parallel to the Z-axis.

(c) Parallel to the X-axis. 9. $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

Page 9. Art. 6

1. $\frac{\sqrt{105}}{14}$. 2. 1, 0, 0; 0, 1, 0; 0, 0, 1. 3. $\frac{4}{\sqrt{26}}, \frac{-3}{\sqrt{26}}, \frac{1}{\sqrt{26}}$.

4. $\frac{-11}{\sqrt{1435}}$. 8. $(\frac{5}{2}, \frac{9}{2}, 2)$. 10. Two. $\pm\frac{\sqrt{\sqrt{3}-1}}{2}$. 11. $(2, 2, 2)$.

Page 11. Art. 9

1. Sphere of radius 1, center at origin.

2. Cone of revolution, with X-axis for axis.

3. Plane through Z-axis, making angle of 30° with X-axis.

4. Cone of revolution, with Z-axis for axis.

5. (a) $\rho = 2$; (b) $\rho = 2$; (c) $\rho^2 + z^2 = 4$. **6.** $\phi = 45°$, $\rho^2 = z^2$.

7. $\sqrt{\rho_1{}^2 + \rho_2{}^2 - 2\rho_1\rho_2(\cos\alpha_1\cos\alpha_2 + \cos\beta_1\cos\beta_2 + \cos\gamma_1\cos\gamma_2)}$.

Page 14. Art. 12

1. $3x + 4y + 2z = 17$. **2.** $x - y = 0$. X- and Y- intercepts zero.

3. $-4x + 3y + z = 5$. **4.** $k = 2$. **5.** $(-3, 4, 5)$.

Page 18. Art. 16

1. $\frac{3}{13}x - \frac{12}{13}y - \frac{4}{13}z = 2$. **2.** $x + 2y = 0$.

3. $\dfrac{7}{\sqrt{26}}$. **4.** $\dfrac{9}{\sqrt{14}}$. **5.** $(4, 3, 1)$; $(1, -4, 3)$.

7. $25x + 39y + 8z - 43 = 0$. **8.** $5x - y - 2z - 6 = 0$.

9. $\dfrac{A_1x + B_1y + Cz_1 + D_1}{\sqrt{A_1{}^2 + B_1{}^2 + C_1{}^2}} = \pm\dfrac{A_2x + B_2y + C_2z + D_2}{\sqrt{A_2{}^2 + B_2{}^2 + C_2{}^2}}$.

10. $14(x^2 + y^2 + z^2) = (3x + y - 2z - 11)^2$.

11. $2x - y - z + 3 + 3\sqrt{6} = 0$. **13.** $x - 2y - z + 2 = 0$; $-2, 1, 2$.

14. $21x - 9y - 22z + 63 = 0$. **15.** $3x + 2y + 3z - 15 = 0$.

17. $11x - y + 16z - 63 = 0$ and $17x - 13y + 12z - 63 = 0$.

18. $m = \pm 6$. **19.** $k = -\frac{3}{4}$.

Page 21. Art. 20

1. (a) $\left(0, \dfrac{11}{4}, \dfrac{3}{2}\right)$; $\left(\dfrac{11}{14}, 0, \dfrac{-1}{14}\right)$; $\left(\dfrac{3}{4}, \dfrac{1}{8}, 0\right)$.

(b) $\left(0, \dfrac{-5}{9}, \dfrac{1}{3}\right)$; $\left(\dfrac{5}{12}, 0, \dfrac{-1}{12}\right)$; $\left(\dfrac{1}{3}, \dfrac{-1}{9}, 0\right)$.

(c) $\left(0, \dfrac{5}{2}, \dfrac{29}{4}\right)$; $\left(5, 0, \dfrac{-3}{2}\right)$; $\left(\dfrac{29}{7}, \dfrac{3}{7}, 0\right)$.

4. $\dfrac{x-3}{4} = \dfrac{y-7}{2} = \dfrac{z-3}{-3}$; $\dfrac{4}{\sqrt{29}}, \dfrac{2}{\sqrt{29}}, \dfrac{-3}{\sqrt{29}}$. **5.** $x + 2y + 2 = 0$.

6. $k = \frac{1}{2}$. **7.** Yes. **8.** $k = -2$. **9.** No. **11.** Yes.

Page 23. Art. 21

1. $\dfrac{-1}{3\sqrt{3}}, \dfrac{1}{3\sqrt{3}}, \dfrac{5}{3\sqrt{3}}$.

2. $2x + y - 3z + 6 = 0$, $x + y + z - 13 = 0$.

3. $\text{arc sin}\,\dfrac{16}{\sqrt{29}\sqrt{70}}$. **4.** $x + 10y + 7z + 18 = 0$.

5. $8x + y - 26z + 6 = 0$. **6.** $k = -1$. **7.** $x + 2z = 5$.

8. $3x - y + 3z - 7 = 0$. **9.** $\begin{vmatrix} x-a & y-b & z-c \\ l_1 & m_1 & n_1 \\ l_2 & m_2 & n_2 \end{vmatrix} = 0$.

10. $3x - 7y - 4z = 0$. **11.** $k = \frac{3}{4}$.

12. $k = 2$ and $k = 3$. **13.** $2x - z = 0$, $y = 3$.

Page 25. Art. 23

1. $\sqrt{\frac{38}{21}}$. **2.** $\sqrt{3}$. **3.** 0. **4.** $\frac{7}{\sqrt{113}}, 0, \frac{-8}{\sqrt{113}}$.

5. $\sqrt{2}$; $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0$. **6.** $\frac{1}{\sqrt{5}}$. **7.** $15x + 43 = 0$, $12y = 6z + 13$.

8. $\frac{\sqrt{14}}{2}$. **9.** $\begin{vmatrix} x_1 - x_2 & l_1 & l_2 \\ y_1 - y_2 & m_1 & m_2 \\ z_1 - z_2 & n_1 & n_2 \end{vmatrix} = 0$.

Page 28. Art. 24

1. $61x - 52y + 35z - 93 = 0$. **3.** $x + 5y - 3z - 44 = 0$.

2. $12x - 17y + 3z + 4 = 0$. **4.** Yes.

5. $7x + 12y - 13z + 8 = 0$, $x - 3y + 4z - 7 = 0$.

Page 29. Art. 25

1. $7y - 10z - 3 = 0$, $7x - z - 22 = 0$, $10x - y - 31 = 0$.

2. $y - z + 2 = 0$, $x + z = 1$, $x + y = 1$.

3. $y - z = 0$, $x + 2z = 4$, $x + 2y = 4$.

4. $(A_1B_2 - A_2B_1)y + (A_1C_2 - A_2C_1)z + (A_1D_2 - A_2D_1) = 0$.
$(B_1A_2 - B_2A_1)x + (B_1C_2 - B_2C_1)z + (B_1D_2 - B_2D_1) = 0$.
$(C_1A_2 - C_2A_1)x + (C_1B_2 - C_2B_1)y + (C_1D_2 - C_2D_1) = 0$.

Page 33. Art. 28

1. $x + 2y + \frac{z}{3} + 1 = 0$; $3x - \frac{y}{2} - \frac{z}{4} + 1 = 0$; $-x + \frac{y}{4} - \frac{z}{3} + 1 = 0$.

2. 5. **3.** $(7, 5, -1)$; $\left(\frac{1}{5}, \frac{-6}{5}, \frac{11}{5}\right)$; $\left(\frac{9}{4}, 0, 0\right)$.

4. $\text{arc cos}\, \frac{3\sqrt{10}}{\sqrt{299}}$.

5. $u + v + w = 1$, $6u - 3v + w + 3 = 0$, $6u - 2v + w + 1 = 0$.

6. $(2, -1, -3)$; $\left(\frac{-1}{3}, 0, \frac{-2}{3}\right)$; $\left(0, 0, \frac{-1}{2}\right)$. **7.** $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}$.

8. A plane. **9.** $4(u^2 + v^2 + w^2) = 1$. A sphere.

Page 35. Art. 34

1. (*a*) $\left(\frac{-7}{4}, \frac{-3}{4}, \frac{-3}{4}\right)$. (*d*) $\left(\frac{9}{2}, \frac{-1}{2}, \frac{-3}{2}\right)$.

(*b*) $\left(\frac{2}{3}, \frac{-1}{5}, 0\right)$. (*e*) $\left(\frac{-1}{7}, \frac{-1}{7}, \frac{1}{7}\right)$.

(*c*) $\left(\frac{-1}{2}, 0, 0\right)$. (*f*) $\left(0, 0, \frac{2}{11}\right)$.

2. $(-10, 15, -2, 0)$. **4.** $7x + 9y + 54z - 59t = 0$.

Page 37. Art. 35

1. (*a*) Parallel bundle. Rank 3. (*b*) Rank 4. (*c*) Rank 4.

(*d*) Parallel bundle. Rank 3.

3. The determinant $\begin{vmatrix} -1 & c & b \\ a & -1 & c \\ b & a & -1 \end{vmatrix}$ is of rank 3; of rank 2; of rank 1.

4. $\frac{-59}{\sqrt{3867}}, \frac{5}{\sqrt{3867}}, \frac{-19}{\sqrt{3867}}$.

Page 43. Art. 40

1. $x^2 - 3yz + y^2 - 4x - 8y + 4z + 4 = 0$.

3. $x = \frac{x'}{\sqrt{21}} + \frac{2y'}{\sqrt{6}} + \frac{2z'}{\sqrt{14}}$,

$y = \frac{4x'}{\sqrt{21}} - \frac{y'}{\sqrt{6}} + \frac{2z'}{\sqrt{14}}$,

$z = \frac{2x'}{\sqrt{21}} + \frac{y'}{\sqrt{6}} - \frac{3z'}{\sqrt{14}}$.

4. New equation is $x^2 - 2y^2 + 6z^2 = 49$.

Translation is $x = x' + 8$, $y = y' - 1$, $z = z' + 2$.

6. $3x^2 + 6y^2 + 18z^2 = 12$.

Page 45. Art. 41

2. $\left(\frac{28 \pm 6i}{13}, \frac{-6 \pm 8i}{13}, \frac{5 \mp 24i}{13}\right)$. **4.** $\left(\frac{17 - 4i}{5}, \frac{6 + 8i}{5}, -4i\right)$.

5. $(13 + 9i)x + (3 + 4i)y + (16 - 7i)z = 23 + 64i$.

6. $(1 \pm i\sqrt{3}, 0, 0)$.

Page 46. Art. 42

1. $x^2 + y^2 = 4z^2$. **3.** $x^2 + y^2 + z^2 - 7x + y + 30 = 0$.

4. $8(x^2 + y^2 + z^2) - 68x + 48y - 66z + 275 = 0$.

5. $z=5$, $(x-3)^2+(y-7)^2+(z-1)^2=9$.

6. $2x-14y-2z+1=0$, $4x-18z+33=0$. 7. $(-4, \pm 4i, 2)$.

Page 49. Art. 46

3. Center at $(0, 0, 4)$; radius 6. 4. $x^2-y^2=1$. 5. $r, \dfrac{rk}{a}$.

Page 51. Art. 47

1. $x^2+y^2+z^2=25$. 2. $9(y^2+z^2)=(15-2x)^2$. Vertex $(\frac{15}{2}, 0, 0)$; $x=0$, $9(y^2+z^2)=225$; $4(x^2+z^2)=9(5-y)^2$.

4. $y^2+z^2=a^2$; $y=a$.

6. (a) $\dfrac{x^2}{4}+\dfrac{y^2}{9}+\dfrac{z^2}{9}=1$; $\dfrac{x^2}{4}+\dfrac{y^2}{9}+\dfrac{z^2}{4}=1$.

(b) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}-\dfrac{z^2}{b^2}=1$; $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}+\dfrac{z^2}{a^2}=1$.

(c) $y^2+z^2=8x$; $y^4=64(x^2+z^2)$.

(d) $(x^2+y^2+z^2-5)^2=16-4x^2$; $x^2+(y-1)^2+z^2=4$.

(e) $y^2+z^2=\sin^2 x$; $y=\sin\sqrt{x^2+z^2}$.

(f) $y^2+z^2=e^{2x}$; $y=e^{\sqrt{x^2+z^2}}$.

Page 54. Art. 49

1. (a) $x^2+y^2+z^2=r^2$. (b) $x^2+y^2+z^2+2x-8y-4z=15$.
(c) $x^2+y^2+z^2-4x-2y-10z+14=0$.

2. (a) Center $\left(-\dfrac{7}{2}, -1, -\dfrac{1}{2}\right)$; radius $\dfrac{\sqrt{34}}{2}$.

(b) Center $(-1, -2, 3)$; radius 0.

(c) Center $\left(\dfrac{1}{4}, \dfrac{1}{2}, \dfrac{-5}{4}\right)$; radius $\dfrac{i\sqrt{10}}{4}$.

(d) Center $\left(-\dfrac{f}{2}, 0, 0\right)$; radius $\dfrac{f}{2}$.

3. $(-4\pm 3i, 2\pm 6i, 5, 0)$.

4. $\left(\dfrac{2\pm 2i\sqrt{2}}{3}, \dfrac{1\mp 2i\sqrt{2}}{3}, \dfrac{-2\pm\sqrt{2}\,i}{3}\right)$. 6. 0.

Page 56. Art. 52

1. $y=1$.

2. Arc cos $\dfrac{\sqrt{10}}{2}$. The spheres have no real point in common.

3. $x^2+y^2+z^2-2x-6y-6z+10=0$ and $x^2+y^2+z^2-2x-6y-6z-6=0$.

4. $x^2 + y^2 + z^2 + x - 2y - 3z = 0.$

5. $2x - 3y + z + 5 = 0.$ The sphere is composite.

6. $10(x^2 + y^2 + z^2) + 71x - 68y - 89z - 185 = 0.$

7. $4232(x^2 + y^2 + z^2) - 276x + 276y + 1932z + 225 = 0.$

Page 69. Art. 59

1. Center $(1, 1, -2)$; semi-axes $\frac{\sqrt{10}}{2}, \frac{\sqrt{15}}{3}, \frac{\sqrt{5}}{2}.$

2. Sphere; center $\left(2, \frac{3}{2}, -5\right)$; radius $\frac{\sqrt{205}}{2}.$

3. $y = 0,\ 2x^2 = 3z^2 + 5z + 7.$ Rotated about the Z-axis.

4. $x = 1,\ y = z;\ x = 1,\ y = -z;\ x = -1,\ y = z;\ x = -1,\ y = -z.$

5. (*a*) Ellipsoid. (*b*) Hyperboloid of two sheets. (*c*) Hyperboloid of one sheet. (*d*) Hyperboloid of revolution of one sheet. (*e*) Ellipsoid. (*f*) Imaginary ellipsoid.

Page 73. Art. 64

1. Hyperboloid of one sheet. **2.** Imaginary cylinder.

3. Elliptic paraboloid. **4.** Real cone. **5.** Hyperboloid of two sheets.

6. Hyperbolic paraboloid. **7.** $\frac{x^2}{b^2} + \frac{y^2 + z^2}{b^2 - a^2} = 1.$

8. (*a*) $(1 - r^2)x^2 + y^2 + z^2 - 2ax + a^2 = 0.$

(*b*) $(1 - r^2)x^2 + (1 - r^2)y^2 + z^2 - 2ax + a^2 = 0.$

Page 76. Art. 66

1. $\left(\frac{-8 \pm \sqrt{109}}{3}, \frac{5 \mp \sqrt{109}}{3}, \frac{-17 \pm \sqrt{109}}{6}\right).$ **2.** $(0, 0, 0).$

3. $(-\frac{2}{3}, \frac{3}{5}, -\frac{2}{15}).$ **4.** $(-1, 2, -1).$ **5.** $(\frac{3}{2}, 1, 0).$

6. Vertex $(0, -1, 0).$ **7.** Plane of centers $2(x - y + z) - 1 = 0.$

8. Non-central.

Page 89. Art. 75

1. Hyperboloid of two sheets. Center $(0, 0, 0)$. Direction cosines of axes $\frac{2}{3}, -\frac{2}{3}, \frac{1}{3};\ \frac{2}{3}, \frac{1}{3}, -\frac{2}{3};\ \frac{1}{3}, \frac{2}{3}, \frac{2}{3}.$ $5x^2 + 2y^2 - z^2 + 2 = 0.$

2. Hyperboloid of one sheet. Center $(1, \frac{2}{3}, -\frac{2}{3})$. Direction cosines of axes

$$\frac{2 + 2\sqrt{5}}{2\sqrt{15 + 4\sqrt{5}}}, \frac{\sqrt{5} - 1}{2\sqrt{15 + 4\sqrt{5}}}, \frac{5 + \sqrt{5}}{2\sqrt{15 + 4\sqrt{5}}};\ \frac{2 - 2\sqrt{5}}{2\sqrt{15 - 4\sqrt{5}}}, \frac{-\sqrt{5} - 1}{2\sqrt{15 - 4\sqrt{5}}},$$

$$\frac{5 - \sqrt{5}}{2\sqrt{15 - 4\sqrt{5}}};\ \frac{-3}{\sqrt{29}}, \frac{4}{\sqrt{29}}, \frac{2}{\sqrt{29}}. \quad \frac{5 + \sqrt{5}}{2}x^2 + \frac{5 - \sqrt{5}}{2}y^2 - 3z^2 = \frac{10}{3}.$$

3. Parabolic cylinder. New origin on $z=0$, $x+y+1=0$. Direction cosines of axes $\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0;\ 0,\ 0,\ 1;\ \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}, 0.\qquad y^2+\sqrt{2}\,x=0.$

4. Two imaginary planes. Line of vertices is $2x+3y=0$, $y+2z+2=0$. Direction cosines of axes $\frac{1}{\sqrt{10}}, 0, \frac{-3}{\sqrt{10}};\ \frac{3}{\sqrt{35}}, \frac{5}{\sqrt{35}}, \frac{1}{\sqrt{35}};\ \frac{3}{\sqrt{14}}, \frac{-2}{\sqrt{14}}, \frac{1}{\sqrt{14}}.\qquad 2x^2+7y^2=0.$

5. Hyperbolic paraboloid. New origin $(\frac{1}{4}, \frac{5}{12}, \frac{5}{6})$. Direction cosines of axes $\frac{1}{2\sqrt{3-\sqrt{6}}}, \frac{1}{2\sqrt{3-\sqrt{6}}}, \frac{\sqrt{6}-2}{2\sqrt{3-\sqrt{6}}};\ \frac{1}{2\sqrt{3+\sqrt{6}}}, \frac{1}{2\sqrt{3+\sqrt{6}}}, \frac{-\sqrt{6}-2}{2\sqrt{3+\sqrt{6}}};\ \frac{-1}{\sqrt{2}}, \frac{1}{\sqrt{2}}, 0.\qquad \sqrt{6}\,x^2-\sqrt{6}\,y^2+\sqrt{2}\,z=0.$

6. Ellipsoid of revolution. Center $\left(-1, \frac{-13}{6}, \frac{4}{3}\right)$. Direction cosines of axis of revolution $0, \frac{1}{\sqrt{2}}, \frac{-1}{\sqrt{2}}.\qquad 3x^2+3y^2+z^2=\frac{7}{6}.$

7. Hyperbolic cylinder. Line of centers $x=0$, $y+3z+7=0$. Direction cosines of axes

$$\frac{\sqrt{10}}{\sqrt{20}}, \frac{1}{\sqrt{20}}, \frac{3}{\sqrt{20}};\ \frac{-\sqrt{10}}{\sqrt{20}}, \frac{1}{\sqrt{20}}, \frac{3}{\sqrt{20}};\ 0, \frac{-3}{\sqrt{10}}, \frac{1}{\sqrt{10}}.$$

$\sqrt{10}\,x^2-\sqrt{10}\,y^2+1=0.$

8. Real cone of revolution. Vertex $(0, 0, 0)$. Direction cosines of axis of revolution $\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.\qquad 2x^2+2y^2-z^2=0.$

9. Two parallel planes. Plane of centers $2x-2y+6z-1=0$. Direction cosines of new X-axis $\frac{1}{\sqrt{11}}, \frac{-1}{\sqrt{11}}, \frac{3}{\sqrt{11}}.\quad 11x^2=\frac{1}{4}.$

10. Parabolic cylinder. Origin on $3x-2y=\frac{509}{242}$, $x-6z=\frac{773}{242}$. Direction cosines of axes

$$\frac{28}{\sqrt{1298}}, \frac{-17}{\sqrt{1298}}, \frac{-15}{\sqrt{1298}};\ \frac{-1}{\sqrt{11}}, \frac{1}{\sqrt{11}}, \frac{-3}{\sqrt{11}};\ \frac{6}{\sqrt{118}}, \frac{9}{\sqrt{118}}, \frac{1}{\sqrt{118}}.$$

$$11y^2=\frac{\sqrt{5192}}{11}x.$$

11. Real cone. Vertex $(0, 2, -2)$. Direction cosines of axes, .67, $-$ .56, $-$ .48; .71, .67, .23; .19, $-$ .50, $-$ 85. $3.72x^2+2.68y^2-1.40z^2=0.$

12. Hyperbolic paraboloid. Origin $\left(\frac{35}{72}, \frac{13}{72}, \frac{-1}{72}\right)$. Direction cosines of axes $\frac{1}{\sqrt{28-10\sqrt{7}}}, \frac{\sqrt{7}-2}{\sqrt{28-10\sqrt{7}}}, \frac{\sqrt{7}-3}{\sqrt{28-10\sqrt{7}}}; \frac{-1}{\sqrt{28+10\sqrt{7}}},$

$$\frac{\sqrt{7}+2}{\sqrt{28+10\sqrt{7}}}, \frac{\sqrt{7}+3}{\sqrt{28+10\sqrt{7}}}; \frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}.$$

$$(-1+\sqrt{7})x^2-(1+\sqrt{7})y^2 = \tfrac{4}{3}\sqrt{3}\,z.$$

13. Hyperboloid of one sheet. Center $\left(\frac{5}{6}, \frac{-7}{4}, \frac{5}{18}\right)$. Direction cosines of axes .21, $-$ .65, .69; .91, .41, .10; .36, $-$ .64, $-$ .68.

$$3.09\,x^2 + 1.59\,y^2 - 3.67\,z^2 = \tfrac{25}{18}.$$

14. Hyperboloid of one sheet. Center $\left(\frac{2}{15}, \frac{3}{5}, \frac{-26}{15}\right)$. Direction cosines of axes $-$ .77, .56, .28; .14, $-$ .31, .94; .63, .76, .13. $6.17\,x^2 + .71\,y^2 - 6.88\,z^2 = \frac{169}{15}$.

15. Ellipsoid. Center (0, 1, 1). Direction cosines of axes

$$\frac{2}{\sqrt{10+2\sqrt{5}}}, \frac{1+\sqrt{5}}{\sqrt{10+2\sqrt{5}}}, 0; \frac{-2}{\sqrt{10-2\sqrt{5}}}, \frac{-1+\sqrt{5}}{\sqrt{10-2\sqrt{5}}}, 0; 0, 0, 1.$$

$$\frac{3+\sqrt{5}}{2}x^2 + \frac{3-\sqrt{5}}{2}y^2 + 2\,z^2 = 4.$$

16. Ellipsoid. Center $\left(\frac{-9}{2}, \frac{-7}{2}, -6\right)$. Direction cosines of axes .83, $-$ .33, $-$ .44; .26, .95, $-$ .22; .49, .07, .87. $4.20\,x^2 + .59\,y^2 + .20\,z^2 = \frac{35}{2}$.

17. $\frac{7}{2}, 5 \pm 2\,i$. **18.** $\frac{-29}{28}$.

Page 92. Art. 78

1. $x + 10\,y - 3\,z + 22 = 0,\ \frac{x-1}{1} = \frac{y+2}{10} = \frac{z-1}{-3}$.

Page 96. Art. 80

1. $y = \xi,\ z = \xi x;\ x = \eta,\ z = \eta y$.

Page 97. Art. 81

1. $\sqrt{5}, \frac{\sqrt{5}}{2}$. **2.** $x + z + 1 = 0,\ y + z - 1 = 0$. **3.** a, b, c.

Page 103. Art. 83

1. $x+y-z=d$ and $x-y+2z=p$. **2.** $x-(2\pm\sqrt{6})y=d$.

3. $\dfrac{1}{\sqrt{3}}$. **4.** $y+3\pm\sqrt{2}(z-2)=0$. **5.** $a=b,\ h=0$.

6. $ax+gz+l=0,\ ay+fz+m=0$. **7.** $2gx+2fy+(c-a)z=d$.

8. $(b-k)A^2+(a-k)B^2-2hAB=0,$

$(c-k)B^2+(b-k)C^2-2fBC=0,$

$(a-k)C^2+(c-k)A^2-2gCA=0,$

k being a root of the discriminating cubic.

9. $(-1,0,-3)$. $\dfrac{\sqrt{30}}{2}$.

Page 108. Art. 87

3. $k_i=\text{cons.}$ $i=1,2,3,4$. For parametric equations, substitute this value of k_i, in Eqs. (27).

Page 111. Art. 89

1. $(-6x+6y-12t,\ x+2y-2z+t,\ 6x+6y+4z+4t,\ -x+3y-z-2t)$, $(-12,1,4,-2)$, $(12,-2,-20,1)$, $(18,-6,-16,1)$, $(12,-3,-28,1)$, $(3,2,1,2)$.

2. $(-373,179,92,283)$, $(-500,181,145,344)$, $(-153,61,38,107)$, $(-37x_1-96x_2-9x_3+156x_4,\ 11x_1+24x_2-3x_3-60x_4,\ 8x_1+48x_2-6x_3-36x_4,\ 31x_1+60x_2+3x_3-108x_4)$.

3. $15x+5y+11z+16t=0$.

4. $197x_1+468x_2+57x_3-792x_4=0$.

5. $6x^2-15y^2+2z^2+3yz-zx-3xy+17xt+9yt-6zt+10t^2=0$.

6. $(22x-22y+44t,\ 12x+24y-24z+12t,\ 33x+33y+22z+22t,\ 66x-198y+66z+132t)$, $(22,6,11,66)$, $(22,12,55,33)$, $(33,36,44,33)$, $(22,18,77,33)$, $(22,-48,-11,264)$, $(-97121,36427,22804,66851)$, $(296167,-115487,-64346,-205981)$, $(-185625,71181,42570,128403)$, $(814x_1-6912x_2-297x_3-61776x_4,\ -242x_1+1728x_2-99x_3+23760x_4,\ -176x_1+3456x_2-198x_3+14256x_4,\ -682x_1+4320x_2+99x_3+42768x_4)$.

Page 113. Art. 92

1. Vertices: $u_1=0,\ (1,0,0,0);\ u_2=0,\ (0,1,0,0);$
$u_3=0,\ (0,0,1,0);\ u_4=0,\ (0,0,0,1).$
Faces: $x_1=0,\ (1,0,0,0);\ x_2=0,\ (0,1,0,0);$
$x_3=0,\ (0,0,1,0);\ x_4=0,\ (0,0,0,1).$

2. $x_1=0,\ x_2=0;\ u_3=0,\ u_4=0.$
$x_1=0,\ x_3=0;\ u_2=0,\ u_4=0.$
$x_1=0,\ x_4=0;\ u_2=0,\ u_3=0.$

$x_2 = 0,\ x_3 = 0;\ u_1 = 0,\ u_4 = 0.$
$x_2 = 0,\ x_4 = 0;\ u_1 = 0,\ u_3 = 0.$
$x_3 = 0,\ x_4 = 0;\ u_1 = 0,\ u_2 = 0.$

3. $u_1 + u_2 + u_3 + u_4 = 0,\ 3\,u_1 - 5\,u_2 + 7\,u_3 - u_4 = 0,$
$-u_1 + 6\,u_2 - 4\,u_3 + 2\,u_4 = 0,\ 7\,u_1 + 2\,u_2 + 4\,u_3 + 6\,u_4 = 0.$

4. $(1, 1, 1, 1),\ (7, -1, -3, 1),\ (1, 9, -5, 2).$

5. $u_1 - u_2 = 0,\ 7\,u_3 + u_4 = 0.$ **6.** $(-9, 1, 1, 0).$

Page 117. Art. 95

2. $px_1 = l_1 + 2\,l_2 + 10\,l_3,$
$px_2 = 7\,l_1 + 5\,l_2 - l_3,$ $(176, -175, 40, 363).$
$px_3 = -l_1 + 4\,l_2 - 3\,l_3,$
$px_4 = 3\,l_1 + l_2 - 5\,l_3.$

3. $pu_1 = -5\,l_1 + 7\,l_2 + 6\,l_3,$
$pu_2 = 3\,l_1 - 5\,l_2 - 4\,l_3,$ $(21, 32, 1, 5).$
$pu_3 = 4\,l_1 + 3\,l_2 - 3\,l_3,$
$pu_4 = l_1 + 2\,l_2 + l_3.$

5. $pu_1 = l_1 + 7\,l_2,\ pu_2 = -5\,l_1 + 2\,l_2,\ pu_3 = 3\,l_1 - l_2,\ pu_4 = -l_1 - l_2.$

6. $px_1 = l_1 + 3\,l_2,\ px_2 = 2\,l_1 - 2\,l_2,\ px_3 = -3\,l_1 + 5\,l_2,\ px_4 = -l_1 - 2\,l_2.$
$l_1(u_1 + 2\,u_2 - 3\,u_3 - u_4) + l_2(3\,u_1 - 2\,u_2 + 5\,u_3 - 2\,u_4) = 0.$

Page 120. Art. 97

3. $(\alpha_{11} + \alpha_{12} + \alpha_{13} + \alpha_{14},\ \alpha_{21} + \alpha_{22} + \alpha_{23} + \alpha_{24},\ \alpha_{31} + \alpha_{32} + \alpha_{33} + \alpha_{34},$ $\alpha_{41} + \alpha_{42} + \alpha_{43} + \alpha_{44}).$ $(\beta_{11} + \beta_{21} + \beta_{31} + \beta_{41},\ \beta_{12} + \beta_{22} + \beta_{32} + \beta_{42},$ $\beta_{13} + \beta_{23} + \beta_{33} + \beta_{43},\ \beta_{14} + \beta_{24} + \beta_{34} + \beta_{44}).$

4. $x_1 = k_1x_1',\ x_2 = k_2x_2',\ x_3 = k_3x_3',\ x_4 = k_4x_4'.$

Page 122. Art. 100

1. (*a*) $x_1 = x_1' - x_4',\ x_2 = x_2' - x_4',\ x_3 = x_3' - x_4',\ x_4 = -x_4'.$
$D(p) \equiv (1 + p)(1 - p)^3.$ Invariant points are $(1, 1, 1, 2)$ and all the points of $x_4 = 0.$

(*b*) $x_1 = x_2',\ x_2 = x_1',\ x_3 = x_4',\ x_4 = x_3'.$ $D(p) \equiv (p^2 - 1)^2.$

Every point on each of the lines

$$x_1 + x_2 = 0,\ x_3 + x_4 = 0;\ x_1 - x_2 = 0,\ x_3 - x_4 = 0.$$

(*c*) $x_1 = x_3',\ x_2 = x_1',\ x_3 = x_2',\ x_4 = x_4'.$ $D(p) \equiv (1 - p)^2(p^2 + p + 1).$

The points $(1, \omega, \omega^2, 0),\ (1, \omega^2, \omega, 0),\ \omega^3 = 1,$ and every point of the line $x_1 = x_2 = x_3.$

(*d*) $x_1 = -x_4',\ x_2 = x_1' - x_4',\ x_3 = x_2' - x_4',\ x_4 = x_3' - x_4'.$
$D(p) \equiv p^4 + p^3 + p^2 + p + 1.$ $(\theta, 1 + \theta, -\theta^2(1 + \theta), -\theta^2),\ \theta^5 = 1,\ \theta \neq 1.$

3. $\dfrac{\alpha_{ik}}{\beta_{ki}} = \text{cons.}\quad i, k = 1, 2, 3, 4.$

4. In case $x_3 = x_3'$, the point (0, 0, 0, 1) and all the points of the plane $x_4 = 0$.

In case $x_3 = -x_3'$, every point of each of the lines $x_1 = 0,\ x_2 = 0;\ x_3 = 0,\ x_4 = 0$.

6. $(1, 1, 1, 1)$, $(1, -1, 1, -1)$, $(1, i, -1, -i)$, $(1, -i, -1, i)$.

7. All the points in the plane at infinity.

9. $\frac{17}{16}$.

Page 125. Art. 102

1. $x_3 - ax_4 = 0$.

3. $(2 \pm 2i\sqrt{127},\ -3 \mp i\sqrt{127},\ 8,\ 4)$.

Page 131. Art. 106

2. $\Delta = -1$.

4. $\dfrac{u_1^2}{a} + \dfrac{u_2^2}{b} + \dfrac{u_3^2}{c} + \dfrac{u_4^2}{d} = 0$.

6. $bcu_1^2 + cau_2^2 + 2\,abu_3u_4 = 0$.

7. $\Delta = \frac{1}{4}$. $\Phi(u) \equiv u_2^2 - u_1u_2 + u_1u_3 - u_2u_3 - u_1u_4 + u_2u_4 - 2\,u_3u_4 = 0$.

8. $A(x) \equiv 0$.

9. $x_1 = 0,\ x_3 - x_4 = 0$ and $x_1 = 0,\ x_3 + x_4 = 0$.

10. $a_{ii} = 0$. $i = 1, 2, 3, 4$.

12. $a_{ik}^2 = a_{ii} \cdot a_{kk}$, $i, k = 1, 2, 3, 4$.

13. A conic; two distinct points; two coincident points.

Page 134. Art. 111

1. $x_1 + x_2 + x_3 + x_4 = 0$. 2. $x_3 = 0,\ x_4 = 0$.

4. $(2\,x_1 + x_2 - 3\,x_3 - x_4)^2 + 4(x_1x_2 - x_3x_4) = 0$.

7. $x_1\Sigma u_ix_i - x_2\Sigma v_ix_i = 0$. Three.

8. $a_{13}x_1x_3 + a_{14}x_1x_4 + a_{23}x_2x_3 + a_{24}x_2x_4 = 0$.

Page 141. Art. 118

3. $a_{11}x_1^2 + a_{22}x_2^2 + a_{33}x_3^2 + 2\,a_{12}x_1x_2 + 2\,a_{23}x_2x_3 + 2\,a_{13}x_1x_3 = 0$.

4. $a_{11}x_1^2 + a_{22}x_2^4 + 2\,a_{12}x_1x_2 + 2\,a_{23}x_2x_3 + 2\,a_{13}x_1x_3 = 0$.
$x_1 = 0,\ x_2 = 0$.

Page 143. Art. 120

1. $8\,x_1^2 + x_2^2 - 5\,x_3^2 - 2\,x_4^2 + 9\,x_1x_2 + 5\,x_1x_3 + 18\,x_1x_4 + 13\,x_2x_4 - 7\,x_3x_4 = 0$.

2. $i(x_1 + x_3) - (x_2 + x_4) = 0$, $i(x_1 + x_4) - (x_2 - x_3) = 0$, and
$i(x_1 + x_3) + (x_2 + x_4) = 0$, $i(x_1 + x_4) + (x_2 - x_3) = 0$.

5. Equations of faces $\Sigma a_{ik}x_k = 0$, $i = 1, 2, 3, 4$.

Page 146. Art. 122

1. $(0, 0, 2, -3)$. 2. $k = \pm 4$.

3. $72\,u_1^2 + 36\,u_2^2 + 23\,u_3^2 - 54\,u_1u_2 = 0$; $2\,u_3 - 3\,u_4 = 0$.

6. (*a*) A quartic curve with double point at O.
   (*b*) A cubic curve passing through O.
   (*c*) A plane section of K, not passing through O.

Page 150. Art. 126

1. $\Sigma \begin{vmatrix} a_{ik} - \lambda b_{ik} & a_{il} - \lambda b_{il} \\ a_{mk} - \lambda b_{mk} & a_{ml} - \lambda b_{ml} \end{vmatrix} \begin{vmatrix} \alpha_{i1} & \alpha_{i3} \\ \alpha_{m1} & \alpha_{m3} \end{vmatrix} \begin{vmatrix} \alpha_{k2} & \alpha_{k3} \\ \alpha_{l2} & \alpha_{l3} \end{vmatrix}$

2. (*a*) $\lambda - 1$, $\lambda^2\ \lambda$, [1(21)]. (*c*) λ^2, λ^2 [(22)].
   (*b*) λ^4 [4]. (*d*) λ^3, $\lambda - 1$ [31].

Page 156. Art. 131

2. [111]. Four distinct lines.
   [21]. Two distinct and two coincident lines.
   [1(11)]. Two pairs of coincident lines.
   [3]. Three coincident lines and one distinct line.
   [(21)]. Four coincident lines.
   [(111)]. A quadric cone.
   {3}. A plane and a line.

3. (*a*) $\lambda - 1$, $\lambda - \frac{1}{2}$, $\lambda - \frac{1}{3}$ [111].
   (*b*) $\lambda + \frac{1}{41}$, $\lambda - \frac{1}{5}$, $\lambda - \frac{1}{5}$. [1(11)].
   (*c*) $(\lambda - 1)^2$, $\lambda - \frac{1}{2}$. [21].
   (*d*) $(\lambda - 1)^3$. [3].

4. (*a*) Four distinct lines.
   (*b*) Two pairs of coincident lines.
   (*c*) Two distinct and two coincident lines.
   (*d*) Three coincident lines and one distinct line.

5. (*a*) $x_1^2 + \dfrac{x_2^2}{2} + \dfrac{x_3^2}{3} - \lambda\,(x_1^2 + x_2^2 + x_3^2) = 0$.

   (*b*) $\dfrac{-x_1^2}{41} + \dfrac{(x_2^2 + x_3^2)}{5} - \lambda(x_1^2 + x_2^2 + x_3^2) = 0$.

   (*c*) $\dfrac{x_1^2}{2} + 2\,x_2x_3 + x_2^2 - \lambda(x_1^2 + 2\,x_2x_3) = 0$.

   (*d*) $2\,x_1x_2 + x_3^2 + 2\,x_2x_3 - \lambda(2\,x_1x_2 + x_3^2) = 0$.

Page 164. Art. 133

2. [11(11)]. $\sqrt{\lambda_1 - \lambda_3}x_1 \pm \sqrt{\lambda_3 - \lambda_2}x_2 = 0$, $x_1^2 + x_2^2 + x_3^2 + x_4^2 = 0$.
   [1(21)]. $\sqrt{\lambda_2 - \lambda_1}x_1 \pm x_3 = 0$, $x_1^2 + 2\,x_2x_3 + x_4^2 = 0$.
   [1(111)]. $x_1 = 0$, $x_2^2 + x_3^2 + x_4^2 = 0$.

[22]. $x_1 = 0,\ x_2 = 0.$

[2(11)]. $x_1+ix_3=0,\ x_4=0;\ x_1-ix_2=0,\ x_4=0;\ x_3=0,\ x_1^2+x_2^2+x_4^2=0.$

[(11)(11)]. $x_1 + ix_2 = 0,\ x_3 + ix_4 = 0;\ x_1 + ix_2 = 0,\ x_3 - ix_4 = 0;$ $x_1 - ix_2 = 0,\ \ x_3 + ix_4 = 0;\ x_1 - ix_2 = 0,\ x_3 - ix_4 = 0.$

[4]. $x_2 = 0,\ x_4 = 0.$

[(22)]. $x_2 = 0,\ x_3 = 0;\ x_1 = 0,\ x_4 = 0;\ x_3 = 0,\ x_4 = 0;$ the last one counted twice.

[(31)]. $x_1+ix_4=0,\ x_3=0;\ x_1-ix_4=0,\ x_3=0;\ x_4=0,\ x_1^2+2\,x_2x_3=0.$

[(211)]. $x_1 = 0,\ x_3 = 0;\ x_1 = 0,\ x_4 = 0.$

[{3}1]. $x_2 = x_3 = 0;\ x_1 - ax_4 = 0,\ 2\,x_2x_4 + x_3^2 = 0.$

3. (*a*) $(\lambda - \frac{1}{2})^2,\ (\lambda - \frac{1}{2})^2.\ \ x_1 - x_2 = 0,\ x_3 + 2\,x_4 = 0;$
$x_1 - x_2 - \sqrt{3}(x_3 + 2\,x_4) = 0,\ \ \sqrt{3}(x_1 + x_2) + 2\,x_3 + x_4 = 0;$
$x_1 - x_2 + \sqrt{3}(x_3 + 2\,x_4) = 0.\ \sqrt{3}(x_1 + x_2) - 2\,x_3 - x_4 = 0.$
$x_1x_3 + x_2x_4 + 2\,x_3x_4 - \lambda(2\,x_1x_3 + 2\,x_2x_4) = 0.$

(*b*) $\lambda - 1,\ \lambda - 1,\ (\lambda + 1)^2.\ x_1 + x_3 = 0,\ x_1 - x_2 + ix_4 = 0;\ x_1 + x_3 = 0,$ $x_1 - x_2 - ix_4 = 0;$
$x_1 + x_3 + 4\,x_2 = 0,\ x^2{}_4 + (x_1 - x_2)^2 - 24\,x_2^2 - 16\,x_2x_3 = 0.$
$x_1^2 + x_2^2 + x_4^2 - 2\,x_3x_4 - \lambda(x_1^2 + x_2^2 + x_4^2 + 2\,x_3x_4) = 0.$

(*c*) $\lambda + 3,\ \ \lambda - 1,\ \ \lambda - 1,\ \lambda - 1.\ \ x_1 + 2\,x_3 + x_4 = 0,\ \ 5\,x_1^2 - x_2^2 + 6\,x_3^2$ $+ 4\,x_1x_3 + 2\,x_1x_2 = 0.$
$-3\,x_1^2 + x_2^2 + x_3^2 + x_4^2 - \lambda(x_1^2 + x_2^2 + x_3^2 + x_4^2) = 0.$

(*d*) $\lambda - 1,\ \ \lambda - 1,\ \lambda^2.\ \ x_1 + x_2 = 0,\ \ x_1 + x_3 + x_4 = 0;\ \ x_1 + x_2 = 0,$ $x_1 - x_3 - x_4 = 0;\ x_2 + x_4 = 0,\ 3\,x_1^2 + x_2^2 - x_3^2 + 4\,x_1x_2 + 2\,x_2x_3 = 0.$
$x_1^2 + x_2^2 + x_4^2 - \lambda(x_1^2 + x_2^2 + x_4^2 + 2\,x_3x_4) = 0.$

4. [1(111)]. [2(11)].

Page 167. Art. 135

1. $(\lambda^3 + 2\,\lambda^2 + 4\,\lambda + 1)u_1^2 + (3\,\lambda^2 + 7\,\lambda - 10)u_2^2 + (\lambda^3 + 2\,\lambda^2 + 9\,\lambda + 6)u_3^2$ $+ (\lambda^2 - 1)(\lambda - 1)u_4^2 - 6(\lambda^2 - \lambda)u_1u_2 + 12(\lambda + 1)^2u_1u_3 + 6(\lambda^2 - 1)u_1u_4$ $+ 4(\lambda^2 - \lambda)u_2u_3 + 2\,\lambda(\lambda - 1)^2u_2u_4 - 4(\lambda^2 - 1)\,u_3u_4 = 0.$

2. $2\,x_2^2 - 3\,x_4^2 + 6\,x_1x_4 + 2\,x_2x_4 - 4\,x_3x_4 = 0,$ twice.
$2\,x_1^2 + 2\,x_3^2 + 3\,x_4^2 - 6\,x_1x_4 + 2\,x_2x_4 + 4\,x_3x_4 = 0.$

3. $2(u_1u_2 + u_3u_4)\lambda^3 + (u_2^2 - 6\,au_1u_2 - 6\,au_3u_4)\lambda^2$ $+ (6\,a^2u_1u_2 + 6\,a^2u_3u_4 - 2\,au_2^2)\lambda + a^2u_2^2 - 2\,a^3u_1u_2 - 2\,a^3u_3u_4 = 0.$

Page 174. Art. 142

3. (*a*) [211]. (*b*) [22]. (*c*) [31].

4. All the quadrics of the bundle touch a fixed line at a fixed point.

5. The quadrics touch $x_3 = 0,\ x_2 - 2\,x_4 = 0$ at $(0, 2, 0, 1)$, and $x_3 = 0,$ $x_2 + 2\,x_4 = 0$ at $(0, 2, 0, -1)$; they have four basis points in the plane $x_2 - x_3 = 0$, at the points

$(2, 2, 2, \sqrt{3}),\ (2, 2, 2, -\sqrt{3}),\ (-2, 2, 2, \sqrt{3}),\ (-2, 2, 2, -\sqrt{3}).$

$$y'_1 = 4\,y_3^2y_4, \quad y'_2 = 4\,y_1y_4\,(2\,y_3 - y_2),$$
$$y'_3 = 4\,y_1y_3y_4, \quad y'_4 = y_1\,(2\,y_3^2 + 2\,y_2y_3 - y_2^2).$$

8. $\lambda_1(ux)(ux')[(u'''x)(u''x'') - (u''x)(u'''x')]$
$+ \lambda_2(u''x)(u''x')[(u'x)(ux') - (ux)(u'x')]$
$+ \lambda_3[(u'x)(u'''x)(ux')(u''x') - (ux)(u''x)(u'x')(u'''x')] = 0.$
(For notation, see Art. 119.)

Page 180. Art. 146

2. $y_1y_2y_3y_4 = 0.$ 3. The plane counted twice is a quadric of the web.

4. $(x^2 + y^2 + z^2)t^2 = 0.$

6. Any point on $x^2 + y^2 + z^2 = 0$ is conjugate to any point on $t = 0$.

Page 187. Art. 150

1. $(\Sigma u_i x'_i)^2 = 0.$ 2. 8. 4. 5.

5. $\lambda_1(u^2 - w^2) + \lambda_2(v^2 - w^2) + \lambda_3 uv + \lambda_4 vw + \lambda_5 wu = 0.$

6. [1111]. $(a^2 - c^2)u^2 + (b^2 - c^2)v^2 = s^2;$
$(a^2 - b^2)u^2 - (b^2 - c^2)v^2 = s^2;$
$(a^2 - b^2)u^2 + (a^2 - c^2)v^2 + s^2 = 0;$
$u^2 + v^2 + w^2 = 0.$

Page 196. Art. 155

1. $k_1{}^p = k_2{}^p = k_3{}^p = k_4{}^p.$

Page 205. Art. 160

2. $(x^2 + y^2 + z^2)^2 = \left(\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2}\right)t^2.$ Eight.

3. $2\,z(x^2 + y^2 + z^2) = (ax^2 + by^2)t.$ Eight. Fifteen.

7. $x_1 = x_2'(x_1' + x_4')(x_2' + x_4'),\qquad x_2 = x_2'x_4'(x_1' + x_4'),$
$x_3 = x_2'x_4'(x_2' + x_4'),\qquad x_4 = x_3'x_4'(x_2' + x_4').$
$(1, 0, 0, 0)$, $(0,1,0,0)$; the line $x_1 = 0$, $x_2 = 0$. Touch at $(0, 0, 0, 1)$.

8. $x_1 = x_1'x_3'(x_1'x_2' + x_2'x_3' + x_3'x_1'),\qquad x_2 = x_1'x_2'(x_1'x_2' + x_2'x_3' + x_3'x_1'),$
$x_3 = x_2'x_3'(x_1'x_2' + x_2'x_3' + x_3'x_1'),\qquad x_4 = x_1'x_2'x_3'x_4'.$
$(1, 0, 0, 0)$, $(0, 1, 0, 0)$, $(1, 0, 0, 0)$. Four coincident at $(0, 0, 0, 1)$.

Page 207. Art. 162

1. $\Delta_y f(x) = 4\,y_1(a_{4000}x_1^3 + 3\,a_{2200}x_1x_2^2 + 3\,a_{2020}x_1x_3^2 + 3\,a_{2002}x_1x_4^2)$
$+ 4\,y_2(3\,a_{2200}x_1^2x_2 + a_{0400}x_2^3 + 3\,a_{0220}x_2x_3^2 + 3\,a_{0202}x_2x_4^2)$
$+ 4\,y_3(3\,a_{2020}x_1^2x_3 + 3\,a_{0220}x_2^2x_3 + a_{0040}x_3^3 + 3\,a_{0022}x_3x_4^2)$
$+ 4\,y_4(3\,a_{2002}x_1^2x_4 + 3\,a_{0202}x_2^2x_4 + 3\,a_{0022}x_3^2x_4 + a_{0004}x_4^3).$

$$\begin{aligned}\Delta_y^2 f(x) = {} & 12\, y_1^2(a_{4000}x_1^2 + a_{2200}x_2^2 + a_{2020}x_3^2 + a_{2002}x_4^2)\\ & + 12\, y_2^2(a_{2200}x_1^2 + a_{0400}x_2^2 + a_{0220}x_3^2 + a_{0202}x_4^2)\\ & + 12\, y_3^2(a_{2020}x_1^2 + a_{0220}x_2^2 + a_{0040}x_3^2 + a_{0022}x_4^2)\\ & + 12\, y_4^2(a_{2002}x_1^2 + a_{0202}x_2^2 + a_{0022}x_3^2 + a_{0004}x_4^2)\\ & + 48\, y_1y_2a_{2200}x_1x_2 + 48\, y_1y_3a_{2020}x_1x_3 + 48\, y_1y_4a_{2002}x_1x_4\\ & + 48\, y_2y_3a_{0220}x_2x_3 + 48\, y_2y_4a_{0202}x_2x_4 + 48\, y_3y_4a_{0022}x_3x_4.\end{aligned}$$

$$\begin{aligned}\Delta_y^3 f(x) = {} & 24(y_1^3a_{4000}x_1 + y_2^3a_{0400}x_2 + y_3^3a_{0040}x_3 + y_4^3a_{0004}x_4\\ & + 3\, y_1^2y_2a_{2200}x_2 + 3\, y_1y_2^2a_{2200}x_1 + 3\, y_1^2y_3a_{2020}x_3\\ & + 3\, y_1y_3^2a_{2020}x_1 + 3\, y_1^2y_4a_{2002}x_4 + 3\, y_1y_4^2a_{2002}x_1\\ & + 3\, y_2^2y_3a_{0220}x_3 + 3\, y_2y_3^2a_{0220}x_2 + 3\, y_2^2y_4a_{0202}x_4\\ & + 3\, y_2y_4^2a_{0202}x_2 + 3\, y_3^2y_4a_{0022}x_4 + 3\, y_3y_4^2a_{0022}x_3).\end{aligned}$$

$\Delta_y^4 f(x) \equiv 24\, f(y)$.

Page 209. Art. 164

1. $(1, 0, 0, 1)$, $(1, 0, 0, -1)$, $(4, 0, 0, -1)$.
2. 4, $\dfrac{2359 + 131\sqrt{17}}{376}$, $\dfrac{2359 - 131\sqrt{17}}{376}$.

Page 211. Art. 167

1. $a_{000\,n} = 0,\quad a_{100\,n-1}x_1 + a_{010\,n-1}x_2 + a_{001\,n-1}x_3 = 0.$
2. $a_{000\,n} = 0,\quad a_{100\,n-1} = 0,\quad a_{010\,n-1} = 0,\quad a_{001\,n-1} = 0.$
3. $2(x_1 - x_3) + 5(x_2 - x_4) = 0.$
4. $2(x_1 - x_3) + 5(x_2 - x_4) = 0,\ \ 4\,x_1 + 32\,x_2 - 36\,x_4 \pm \sqrt{1042}(x_2 - x_4) = 0.$

Page 213. Art. 169

1. $(x_2^3 + x_3^3 + x_4^3)^2 = 0.$
2. $u_1^{\frac{3}{2}} + u_2^{\frac{3}{2}} + u_3^{\frac{3}{2}} + u_4^{\frac{3}{2}} = 0.$
3. $u_1^{\frac{1}{2}} + u_2^{\frac{1}{2}} + u_3^{\frac{1}{2}} + u_4^{\frac{1}{2}} = 0.$
4. $u_1^2u_3 + u_2^2u_4 = 0.$

Page 215. Art. 172

3. $\dfrac{1}{a_1x_1} + \dfrac{1}{a_2x_2} + \dfrac{1}{a_3x_3} + \dfrac{1}{a_4x_4} + \dfrac{1}{a_5(x_1 + x_2 + x_3 + x_4)} = 0.$
4. $4(n-2)^3.$

Page 218. Art. 175

2. $x_1 = 0,\ x_4^2 - x_2x_3 = 0;\ x_2 - x_3 = 0,\ x_2 - x_4 = 0;$
   $x_1x_2 - x_1x_3 + x_4^2 - x_2x_3 = 0,\ x_3^2 - x_2x_4 - x_2^2 + x_2x_3 = 0,$
   $x_2^2 + x_2x_3 + 2\,x_2x_4 + x_3x_4 - x_1x_2 = 0.$
3. $(x_1^2 + x_2^2 - x_3^2)^2 - 4(x_1 - x_2)(x_1^3 + x_2^3 + x_2x_3^2 - 2\,x_1x_3^2) = 0.$
4. $x_1^2 + x_2^2 + 5\,x_3^2 = 0.$
5. $(a_1 - a_4)x_1^2 + (a_2 - a_4)x_2^2 + (a_3 - a_4)x_3^2 = 0.$
7. $(x_1^2 - 2\,x_2^2 + x_3^2 + 2\,x_4^2 - 2\,x_2x_3)^2$
   $- 2(x_1 - 3\,x_2 + 2\,x_4)[(x_1^2 + x_4^2 - x_2x_3)(2\,x_1 + 4\,x_2 - 2\,x_3)$
   $-(x_1^2 + 2\,x_2^2 - x_3^2)(2\,x_1 - x_2 - x_3 + 2\,x_4)] = 0.$

Page 225. Art. 180

1. $x_1 = t(t^2-1)$, $x_2 = t^2-1$, $x_3 = (t^2-1)^2$, $x_4 = t$.

2. $(4\,x_3x_4 - x_1x_2)^2 - 4(x_2^2 + 2\,x_1x_3)(x_1^2 + 2\,x_2x_4) = 0.$

3. $12(u_2^2 - u_3^2)(u_2^2 - u_4^2) - 12(u_1u_2 - u_3u_4)^2$
$+ (u_1^2 + 2\,u_2^2 - u_3^2 - u_4^2)^2 = 0.$
$36(u_2^2 - u_3^2)(u_2^2 - u_4^2)(u_1^2 + 2\,u_2^2 - u_3^2 - u_4^2)$
$+ 18(u_1^2 - 4\,u_2^2 + 2\,u_3^2 + 2\,u_4^2)(u_1u_2 - u_3u_4)^2$
$-(u_1^2 + 2\,u_2^2 - u_3^2 - u_4^2)^3 = 0.$

Page 234. Art. 184

5. $m=3$, $n=3$, $r=4$, $H=0$, $h=1$, $G=0$, $g=1$, $\alpha=0$, $\beta=0$,
$v=0$, $\omega=0$, $x=0$, $y=0$, $p=0$.

6. On the developable of the given curve.

Page 241. Art. 187

1. The four quadric cones on which C_4 lies.

3. Eight. Four of each regulus.

4. 16 stationary planes.
24 planes tangent to C_4 at each of two stationary points.
96 planes tangent at one and passing through two other stationary points.
116 planes through four distinct stationary points.

5. The developable surface of C_4. The four quartic curves in which the faces of the self-polar tetrahedron intersect the developable surface.

Page 243. Art. 188

1. (*a*) $m=4$, $n=6$, $r=6$, $H=1$, $h=2$, $G=0$, $g=6$, $\alpha=4$, $\beta=0$,
$v=0$, $\omega=0$, $x=6$, $y=4$, $p=0$.
(*b*) $m=4$, $n=4$, $r=5$, $H=0$, $h=2$, $G=0$, $g=2$, $\alpha=1$, $\beta=1$,
$v=0$, $\omega=0$, $x=2$, $y=2$, $p=0$.
(*c*) $m=4$, $n=6$, $r=6$, $H=0$, $h=3$, $G=0$, $g=6$, $\alpha=4$, $\beta=0$,
$v=0$, $\omega=0$, $x=6$, $y=4$, $p=0$.
(*d*) $m=4$, $n=5$, $r=6$, $H=0$, $h=3$, $G=0$, $g=4$, $\alpha=2$, $\beta=0$,
$v=1$, $\omega=0$, $x=5$, $y=4$, $p=0$.
(*e*) $m=4$, $n=4$, $r=6$, $H=0$, $h=3$, $G=0$, $g=3$, $\alpha=0$, $\beta=0$,
$v=2$, $\omega=0$, $x=4$, $y=4$, $p=0$.

4. -1, 2, $\frac{1}{2}$. **9**. Four. **10**. Four. Two of each regulus.

11. $u_1 = t^3 - 3\,t^2 - 2$, $u_2 = 4\,t(t+1)^2$, $u_3 = -t^3$, $u_4 = t^3(t+1)^2$.
$u_1 = 1$, $u_2 = -2\,t$, $u_3 = 2\,t^3$, $u_4 = -t^4$.

Page 253. Art. 195

1. $x = a \cos \frac{s}{a\sqrt{2}}, \ y = a \sin \frac{s}{a\sqrt{2}}, \ z = \frac{s}{\sqrt{2}}.$

2. Tangent $-\frac{1}{\sqrt{2}} \sin \frac{s}{a\sqrt{2}}, \ \frac{1}{\sqrt{2}} \cos \frac{s}{a\sqrt{2}}, \ \frac{1}{\sqrt{2}}.$

   Principal normal $-\cos \frac{s}{a\sqrt{2}}, \ -\sin \frac{s}{a\sqrt{2}}, \ 0.$

   Binormal $\frac{1}{\sqrt{2}} \sin \frac{s}{a\sqrt{2}}, \ -\frac{1}{\sqrt{2}} \cos \frac{s}{a\sqrt{2}}, \ \frac{1}{\sqrt{2}}.$

$$\rho = 2a, \ \sigma = -2a.$$

3. $R = \frac{1}{12}\left(\frac{1+4t^2+9t^4}{1+9t^2+9t^4}\right)^{\frac{1}{2}}(36(1+4t^2+9t^4)+(486t^7+567t^5+90t^3-6t)^2)^{\frac{1}{2}}$

5 4. (a) No curve. (b) A cubic curve.

Page 267. Art. 207

1. $2u \cos v \, x + 2u \sin v \, y = z + u^2,$

   $\frac{2u \cos v}{\sqrt{1+4u^2}}, \ \frac{2u \sin v}{\sqrt{1+4u^2}}, \ \frac{-1}{\sqrt{1+4u^2}}.$

2. $du^2 + u^2 dv^2 = 0.$

4. $(u + \sqrt{u^2 + a^2 + b^2})(v + \sqrt{v^2 + a^2 + b^2}) = c.$

   $u + \sqrt{u^2 + a^2 + b^2} = c'(v^2 + \sqrt{v^2 + a^2 + b^2}).$

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