Rob 501 - Mathematics for Robotics
Recitation #4

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1 Linear Operator and Matrix Representation

Let $\mathcal{X}$ and $\mathcal{Y}$ be two vector space over the same field $\mathcal{F}$. A mapping $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$ is a linear operator if $\forall \alpha_1, \alpha_2 \in \mathcal{F}$ and $\forall x_1, x_2 \in \mathcal{X}$: $\mathcal{L}(\alpha_1 x_1 + \alpha_2 x_2) = \alpha_1 \mathcal{L}(x_1) + \alpha_2 \mathcal{L}(x_2)$.

Note: A linear map must satisfy $\mathcal{L}(0) = 0$.

A matrix representation of a linear operator $\mathcal{L} : \mathcal{X} \to \mathcal{Y}$, with a basis $U = \{u^1, u^2, \ldots, u^m\}$ for the space $(\mathcal{X}, \mathcal{F})$ and a basis $V = \{v^1, v^2, \ldots, v^n\}$ for the space $(\mathcal{Y}, \mathcal{F})$, is an $n \times m$ matrix A such that

$$[L(x)]_V = A\,[x]_U,$$

where $[x]_U$ is the coordinates of x expressed in the basis U, $[\mathcal{L}(x)]_V$ is the coordinates of $\mathcal{L}(x)$ expressed in the basis V, and the i-th column of A is $[\mathcal{L}(u^i)]_V$.

Ex: Let $(\mathbb{P}^n, \mathbb{R})$ be the space of polynomials of degree less than or equal to n over the field $\mathbb{R}$.
Define the map $\mathcal{L} : \mathbb{P}^2 \to \mathbb{P}^3$ as:

$$(\mathcal{L}(p))\,(x) = \int_0^x p(s)\mathrm{d}s$$

1. Is $\mathcal{L}$ a linear operator?

2. Find the matrix representation A_1 of $\mathcal{L}$ with respect to the bases $U_1 = \{1, x, x^2\}$ for $\mathbb{P}^2$ and $V_1 = \{1, x, x^2, x^3\}$ for $\mathbb{P}^3$.

3. Change the basis in $\mathbb{P}^2$ to $U_2 = \left\{1,\ x,\ \frac{1}{2}(3x^2 - 1)\right\}$. Find the new matrix representation A_2.

4. Keep the basis U_1 in $\mathbb{P}^2$, but change the basis in $\mathbb{P}^3$ to $V_2 = \left\{1,\ x,\ \frac{1}{2}(3x^2 - 1),\ \frac{1}{2}(5x^3 - 3x)\right\}$. Find the new matrix representation A_3.

2 Similarity Transform

1. Let $A,\ B \in \mathbb{C}^{n \times n}$. A and B are similar if there exists an invertible matrix $P \in \mathbb{C}^{n \times n}$ such that $B = PAP^{-1}$.

2. If A is similar to a diagonal matrix, A is said to be diagonalizable.

   Ex: Check whether the following matrices are diagonalizable. If they are, calculate A^{100}.
   (1) $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, (2) $A = \begin{bmatrix} 1 & 1 \\ 4 & 1 \end{bmatrix}$.