Rob 501 Handouts
Background Material on Logic and Proofs

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Introduction to Mathematical Arguments

Sources:

- Various Proof Techniques
  http://math.kennesaw.edu/~plaval/math4381/induction.pdf
- Factorization into primes using the Second Principle of Induction [aka Strong Induction].
  http://courses.engr.illinois.edu/cs173/sp2009/lectures/lect_18.pdf
- How to negate statements with quantifiers
  http://www.math.binghamton.edu/dennis/478.f07/EleAna.pdf

1.3 Induction and Other Proof Techniques

The purpose of this section is to study the proof technique known as **mathematical induction**. Before we do so, we will quickly review the other proof techniques used in mathematics.

1.3.1 Review of Proof techniques Other than Induction

Direct Proofs

We derive the result to prove by combining logically the given assumptions (if any), definitions, axioms and known theorems.

Example 64 *Prove that the sum of two odd integers is even.*
Recall that an integer n is even if $n = 2k$ and it is odd if $n = 2k + 1$ for some integer k. We start with two odd integers we call a and b. This means that there exist integers k_1 and k_2 such that $a = 2k_1 + 1$ and $b = 2k_2 + 1$. Now,

$$\begin{aligned} a + b &= 2k_1 + 1 + 2k_2 + 1 \\ &= 2k_1 + 2k_2 + 2 \\ &= 2(k_1 + k_2 + 1) \end{aligned}$$

If k_1 and k_2 are integers, $k_1 + k_2 + 1$ is also an integer. Hence, $a + b$ is even.

Proof by Contrapositive

Suppose that P and Q are two statements. "If P then Q" is equivalent to its contrapositive "if not Q then not P ". Instead of proving one, the other can be proven.

Example 65 *Prove that if n^2 is even, so is n.*
Since a number is odd, the contrapositive of this statement is "if n is odd so is n^2. We prove that instead.

$$\begin{aligned} n \text{ odd} &\implies n = 2k + 1 \text{ for some integer } k \\ &\implies n^2 = (2k+1)^2 \\ &\implies n^2 = 4k^2 + 4k + 1 \\ &\implies n^2 = 2\left(2k^2 + 2k\right) + 1 \\ &\implies n^2 \text{ is odd since } 2k^2 + 2k \text{ is an integer} \end{aligned}$$

Proof by Contradiction

We prove that under the given assumptions, assuming a statement is true leads to some contradiction. Hence, the statement cannot be true. In general, what we assume to be true is the negation of what we have to prove. Since the negation of what we have to prove leads to a contradiction, hence cannot be true. It follows that the result to prove must be true. We show a classical example.

Example 66 *Show $\sqrt{2}$ is irrational.*
We do a proof by contradiction. We assume the opposite of what we want to prove, that is $\sqrt{2}$ is rational (there are only two possibilities, either a number is rational or it is irrational). We will show this leads to a contradiction. Thus, $\sqrt{2}$ cannot be rational, hence it must be irrational. So, suppose that $\sqrt{2}$ is rational, that is $\sqrt{2} = \dfrac{m}{n}$ where m and n are integers with no common factors. This means that $m = n\sqrt{2}$ or $m^2 = 2n^2$. Thus m^2 is even, it follows that m is even (see above). If m is even, then $m = 2k$ for some integer k so that $m^2 = 4k^2$ but $m^2 = 2n^2$ hence $2n^2 = 4k^2$ or $n^2 = 2k^2$ thus n^2 is even,hence n is even. It follows that both m and n are even which is a contradiction since m and n were supposed to have no common factors.

Proof by Exhaustion

We divide the result to prove into cases and prove each one separately.

Proof by Construction

We prove an object having certain properties exists by constructing an example of an object with the required properties.

1.3.2 First Principle of Mathematical Induction

Proofs by induction are often used when one tries to prove a statement made about natural numbers or integers. Here are examples of statements where induction would be used.

- For every natural number n, $1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$
- If $x > -1$, and n is a natural number, then $(1+x)^n \geq 1 + nx$

The principle of mathematical induction, states the following:

Theorem 67 (Induction) *Let $P(n)$ denote a statement about natural numbers with the following properties:*

1. *The statement is true when $n = 1$ i.e. $P(1)$ is true.*
2. *$P(k+1)$ is true whenever $P(k)$ is true for any positive integer k.*

Then, $P(n)$ is true for all $n \in \mathbb{N}$.

Remark 68 *The case $n = 1$ is called the **base case**.*

Remark 69 *The principle of mathematical induction is also true if instead of starting at 1, we start at any integer n_0. In other words, if we prove that $P(n_0)$ is true and $P(k+1)$ is true whenever $P(k)$ is true , $k \geq n_0$, then $P(n)$ will be true for all $n \in \mathbb{Z}$ such that $n \geq n_0$.*

Remark 70 *When doing a proof by induction, it is important to write explicitly what the statement $P(n)$ is so we know what we have to prove for a given n. Before proving $P(1)$, write clearly what $P(1)$ says. Similarly, when we assume $P(k)$ true and want to deduce $P(k+1)$, write clearly what both $P(k)$ and $P(k+1)$ say so we know what we are assuming and what we need to prove.*

The theorem can easily be proven if we assume an important result about $\mathbb{N}$. This result is called the well ordering principle, which we will take as an axiom.

Axiom 71 (Well Ordering Principle) *Every nonempty subset of $\mathbb{N}$ has a smallest element. In other words, if $A \subseteq \mathbb{N}$ and $A \neq \varnothing$ then there exists $n \in A$ such that $n \leq k$ for all $k \in A$.*

Example 72 $S = \{1, 3, 5, 7, 9\}$ *is a subset of $\mathbb{N}$. Its smallest element is 1.*

Remark 73 *Can we say the same about subsets of positive real numbers?*

We can now prove theorem 67 using a proof by contradiction.
Proof of the Principle of Mathematical Induction. Suppose the hypotheses of theorem 67 are true but the conclusion is false. That is, for some k, $P(k)$ is false. let $A = \{k \in \mathbb{N} : P(k) \text{ is false}\}$. Then $A \subseteq \mathbb{N}$ and $A \neq \varnothing$. So, it has a smallest element, call it k_0. In particular, $P(k_0 - 1)$ is true since k_0 is the smallest number for which $P(k)$ is false. But by the hypotheses of the theorem, if $P(k_0 - 1)$ is true, so should $P(k_0)$. Which means $k_0 \notin A$, which is a contradiction. So, $P(n)$ must be true for all $n \in \mathbb{N}$. ■

We illustrate this principle with some examples which we state as theorems.

Theorem 74 *If n is a natural number, then* $1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$

Proof. *We do a proof by induction (though a nice direct proof also exists). Let $P(n)$ denote the statement that $1 + 2 + 3 + ... + n = \dfrac{n(n+1)}{2}$. We would like to show that $P(n)$ is true for all n. $P(1)$ states that $1 = \dfrac{1(1+1)}{2}$ which is true. This establishes that $P(1)$ is true. Next, we assume that $P(k)$ holds for some natural number k. We wish to prove that $P(k+1)$ also holds. We begin by writing what $P(k)$ and $P(k+1)$ represent so that we know what we are assuming and what we have to prove. $P(k)$ says that $1{+}2{+}3{+}...{+}k = \dfrac{k(k+1)}{2}$. $P(k+1)$ says that $1 + 2 + 3 + ... + k + (k+1) = \dfrac{(k+1)(k+2)}{2}$. Now,*

$$
\begin{aligned}
1 + 2 + 3 + ... + k + (k+1) &= (1 + 2 + 3 + ... + k) + k + 1 \\
&= \frac{k(k+1)}{2} + k + 1 \text{ by assumption} \\
&= \frac{k(k+1)}{2} + \frac{2(k+1)}{2} \\
&= \frac{(k+1)(k+2)}{2}
\end{aligned}
$$

So, we see that $P(k+1)$ holds. Therefore, by induction, $P(n)$ holds for all n. ■

Remark 75 *Proving that $P(1)$ is true is essential. Consider the statement $n+1=n$ for all $n \geq 0$. This is obviously false. However, if we do not bother to check whether $P(1)$ is true and we assume that $P(k)$ is true then we can prove that $P(k+1)$ is also true. $P(k+1)$ says that $n+2=n+1$.*

$$\begin{aligned} n+2 &= n+1+1 \\ &= n+1 \text{ by assumption since } n+1=n \end{aligned}$$

Thus we would have proven that $n+1=n$.

Theorem 76 (Bernoulli's inequality) *If $x>-1$, and n is a natural number, then $(1+x)^n \geq 1+nx$*
Proof. *We do a proof by induction. Let $P(n)$ be the statement that $x \geq -1$, and n is a natural number, then $(1+x)^n \geq 1+nx$.*

- *$P(1)$ would be the statement $1+x \geq 1+x$, which is obviously true.*
- *Assume $P(k)$ is true, that is $(1+x)^k \geq 1+kx$. we wish to prove that $P(k+1)$ is also true, that is $(1+x)^{k+1} \geq 1+(k+1)x$.*

$$\begin{aligned} (1+x)^{k+1} &= (1+x)^k (1+x) \\ &\geq (1+kx)(1+x) \text{ by assumption and since } x>-1 \\ &\geq 1+(k+1)x+kx^2 \\ &\geq 1+(k+1)x \end{aligned}$$

Thus, $P(k+1)$ holds. It follows by induction that $P(n)$ holds for every n.

■

Sometimes, it is not easy to deduce that $P(k+1)$ is true knowing that $P(k)$ is true, especially if we do not have a relationship between $P(k)$ and $P(k+1)$. In such cases, another form of mathematical induction can be used.

1.3.3 Second Principle of Mathematical Induction

Theorem 77 (Second Principle of Mathematical Induction) *Let $P(n)$ denote a statement about natural numbers with the following properties:*

1. *The statement is true when $n=1$ i.e. $P(1)$ is true.*
2. *$P(k)$ is true whenever $P(j)$ is true for all positive integers $1 \leq j < k$.*

Then, $P(n)$ is true for every natural number.

Example 78 *Consider $f:\mathbb{N}\to\mathbb{R}$ defined by $f(1)=0$, $f(2)=\frac{1}{3}$, and for $n>2$ by $f(n)=\frac{n-1}{n+1}f(n-2)$. By computing values of $f(n)$ for $n=3,4,5,6$, give a conjecture as to what a direct formula for f might be. Prove your conjecture by induction.*

- $f(3)=\frac{2}{4}f(1)=\frac{2}{4}0=0$
- $f(4)=\frac{3}{5}f(2)=\frac{3}{5}.\frac{1}{3}=\frac{3}{15}=\frac{1}{5}$
- $f(5)=\frac{4}{6}f(3)=0$
- $f(6)=\frac{5}{7}f(4)=\frac{5}{7}.\frac{1}{5}=\frac{1}{7}$
- *It seems that* $f(n)=\begin{cases} 0 \text{ if } n \text{ is odd} \\ \frac{1}{n+1} \text{ if } n \text{ is even}\end{cases}$ *We need to prove this.*
- *Proof of the conjecture. We can see from the computations that the conjecture is true for $n=1,2$. Suppose that $n>2$. Suppose our conjecture holds for all $k<n$. We need to prove the conjecture also holds for n. If n is odd, then $f(n)=\frac{n-1}{n+1}f(n-2)$. Since n is odd, so is $n-2$. Because $n-2<n$, the conjecture is true for $n-2$, so $f(n-2)=0$ hence $f(n)=0$. If n is even, then $f(n)=\frac{n-1}{n+1}f(n-2)$. $n-2$ is also even and the conjecture holds for it. So, $f(n-2)=\frac{1}{n-1}$. Therefore*

$$\begin{aligned} f(n) &= \frac{n-1}{n+1}f(n-2) \\ &= \frac{n-1}{n+1}\frac{1}{n-1} \\ &= \frac{1}{n+1} \end{aligned}$$

The conjecture is proven.

1.3.4 Exercises

1. Prove by induction that $(1+x)^n \geq 1+nx+\frac{n(n-1)}{2}x^2$ when $x\geq 0$.
2. Prove by induction that $(1-x)^n \leq 1-nx+\frac{n(n-1)}{2}x^2$ when $0\leq x<1$.
3. Prove by induction that if $a_1,a_2,...a_n$ are all non-negative, then $(1+a_1)(1+a_2)...(1+a_n)\geq 1+a_1+a_2+...+a_n$.

4. Prove by induction that $\dfrac{1}{1\times 2}+\dfrac{1}{2\times 3}+...+\dfrac{1}{n\left(n+1\right)}=\dfrac{n}{n+1}$.

5. Use mathematical induction to show that the identities below are valid for any $n\in\mathbb{N}$.

   (a) $1+3+5+...+\left(2n-1\right)=n^2$.

   (b) $1^2+2^2+3^2+...+n^2=\dfrac{n\left(n+1\right)\left(2n+1\right)}{6}$.

   (c) $1^3+2^3+3^3+...+n^3=\left[\dfrac{1}{2}n\left(n+1\right)\right]^2$.

   (d) $2+2^2+2^3+...+2^n=2\left(2^n-1\right)$.

   (e) $x^{n+1}-y^{n+1}=\left(x-y\right)\left(x^n+x^{n-1}y+x^{n-2}y^2+...+xy^{n-1}+y^n\right)$.

6. Use mathematical induction to establish the identities below for the given values of n. If no value is specified, you also need to find the smallest value of n that will work.

   (a) $2^n>n$ for all $n\in\mathbb{N}$.

   (b) $2^n>n^2$ for all $n\in\mathbb{N}$ such that $n\geq 5$.

   (c) $n!>2^n$ for all $n\in\mathbb{N}$ such that $n\geq 4$.

   (d) $n!>2^{n-1}$.

7. In the questions below, f is a function with domain $\mathbb{N}$. Use the given information to find a formula for $f\left(n\right)$ then use mathematical induction to prove your formula is correct.

   (a) $f\left(1\right)=\dfrac{1}{2}$, and for $n>1$, $f\left(n\right)=\left(n-1\right)f\left(n-1\right)-\dfrac{1}{n+1}$.

   (b) $f\left(1\right)=1$, $f\left(2\right)=4$, and for $n>2$, $f\left(n\right)=2f\left(n-1\right)-f\left(n-2\right)+2$.

8. Let $f:\mathbb{N}\rightarrow\mathbb{R}$ be defined recursively by:

$$\begin{aligned} f\left(1\right) &= 1 \\ f\left(2\right) &= 2 \\ f\left(n+2\right) &= \frac{1}{2}\left[f\left(n+1\right)+f\left(n\right)\right] \end{aligned}$$

   Use mathematical induction to prove that $1\leq f\left(n\right)\leq 2$ for every $n\in\mathbb{N}$.

Factoring Primes Using Strong Induction, Plus Other Examples

Strong induction

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This lecture presents proofs by "strong" induction, a slight variant on normal mathematical induction.

1 A geometrical example

As a warm-up, let's see another example of the basic induction outline, this time on a geometrical application. *Tiling* some area of space with a certain type of puzzle piece means that you fit the puzzle pieces onto that area of space exactly, with no overlaps or missing areas. A right triomino is a 2-by-2 square minus one of the four squares. (See pictures in Rosen pp. 277-278.) I then claim that

Claim 1 *For any positive integer n, a $2^n \times 2^n$ checkerboard with any one square removed can be tiled using right triominoes.*

> Proof: by induction on n.
>
> Base: Suppose $n = 1$. Then our $2^n \times 2^n$ checkerboard with one square remove is exactly one right triomino.
>
> Induction: Suppose that the claim is true for some integer k. That is a $2^k \times 2^k$ checkerboard with any one square removed can be tiled using right triominoes.
>
> Suppose we have a $2^{k+1} \times 2^{k+1}$ checkerboard C with any one square removed. We can divide C into four $2^k \times 2^k$ sub-checkerboards

P, Q, R, and S. One of these sub-checkerboards is already missing a square. Suppose without loss of generality that this one is S. Place a single right triomino in the middle of C so it covers one square on each of P, Q, and R.

Now look at the areas remaining to be covered. In each of the sub-checkerboards, exactly one square is missing (S) or already covered (P, Q, and R). So, by our inductive hypothesis, each of these sub-checkerboards minus one square can be tiled with right triominoes. Combining these four tilings with the triomino we put in the middle, we get a tiling for the whole of the larger checkerboard C. This is what we needed to construct.

2 Strong induction

The inductive proofs you've seen so far have had the following outline:

Proof: We will show $P(n)$ is true for all n, using induction on n.

Base: We need to show that $P(1)$ is true.

Induction: Suppose that $P(k)$ is true, for some integer k. We need to show that $P(k+1)$ is true.

Think about building facts incrementally up from the base case to $P(k)$. Induction proves $P(k)$ by first proving $P(i)$ for every i from 1 up through $k-1$. So, by the time we've proved $P(k)$, we've also proved all these other statements. For some proofs, it's very helpful to use the fact that P is true for all these smaller values, in addition to the fact that it's true for k. This method is called "strong" induction.

A proof by strong induction looks like this:

Proof: We will show $P(n)$ is true for all n, using induction on n.

Base: We need to show that $P(1)$ is true.

Induction: Suppose that $P(1)$ up through $P(k)$ are all true, for some integer k. We need to show that $P(k+1)$ is true.

The only new feature about this proof is that, superficially, we are assuming slightly more in the hypothesis of the inductive step. The difference is actually only superficial, and the two proof techniques are equivalent. However, this difference does make some proofs much easier to write.

3 Postage example

Strong induction is useful when the result for $n = k-1$ depends on the result for some smaller value of n, but it's not the immediately previous value (k). Here's a classic example:

Claim 2 *Every amount of postage that is at least 12 cents can be made from 4-cent and 5-cent stamps.*

For example, 12 cents uses three 4-cent stamps. 13 cents of postage uses two 4-cent stamps plus a 5-cent stamp. 14 uses one 4-cent stamp plus two 5-cent stamps. If you experiment with small values, you quickly realize that the formula for making k cents of postage depends on the one for making $k-4$ cents of postage. That is, you take the stamps for $k-4$ cents and add another 4-cent stamp. We can make this into an inductive proof as follows:

> Proof: by induction on the amount of postage.
>
> Base: If the postage is 12 cents, we can make it with three 4-cent stamps. If the postage is 13 cents, we can make it with two 4-cent stamps. plus a 5-cent stamp. If it is 14, we use one 4-cent stamp plus two 5-cent stamps. If it is 15, we use three 5-cent stamps.
>
> Induction: Suppose that we have show how to construct postage for every value from 12 up through k. We need to show how to construct $k+1$ cents of postage. Since we've alread proved base cases up through 15 cents, we'll assume that $k+1 \geq 16$.
>
> Since $k+1 \geq 16$, $(k+1)-4 \geq 12$. So by the inductive hypothesis, we can construct postage for $(k+1)-4$ cents using m 4-cent stamps and n 5-cent stamps, for some natural numbers m and n. In other words $(k+1)-4) = 4m+5n$.

But then $k+1 = 4(m+1) + 5n$. So we can construct $k+1$ cents of postage using $m+1$ 4-cent stamps and n 5-cent stamps, which is what we needed to show.

Notice that we needed to directly prove four base cases, since we needed to reach back four integers in our inductive step. It's not always obvious how many base cases are needed until you work out the details of your inductive step.

4 Nim

In the parlour game Nim, there are two players and two piles of matches. At each turn, a player removes some (non-zero) number of matches from one of the piles. The player who removes the last match wins.[1]

Claim 3 *If the two piles contain the same number of matches at the start of the game, then the second player can always win.*

Here's a winning strategy for the second player. Suppose your opponent removes m matches from one pile. You remove m matches from the other pile. Let's prove that this strategy works.

Proof by induction on the number of matches (n) in each pile.

Base: If both piles contain 1 match, the first player has only one possible move: remove the last match from one pile. The second player can then remove the last match from the other pile and thereby win.

Induction: Suppose that the second player can win any game that starts with two piles of n matches, where n is any value from 1 through k. We need to show that this is true if $n = k+1$.

So, suppose that both piles contain $k+1$ matches. A legal move by the first player involves removing j matches from one pile,

[1] Or, in some variations, loses. There seem to be several variations of this game.

> where $0 \leq j \leq k+1$. The piles then contain $k+1$ matches and $k+1-j$ matches.
>
> The second player can now remove j matches from the other pile. This leaves us with two piles of $k+1-j$ matches. If $j = k+1$, then the second player wins. If $j < k+1$, then we're now effectively at the start of a game with $k+1-j$ matches in each pile. Since $1 \leq k+1-j \leq k$, we know by the induction hypothesis that the second player can win the game.

The induction step in this proof uses the fact that our claim $P(n)$ is true for a smaller value of n. But since we can't control how many matches the first player removes, we don't know how far back we have look in the sequence of earlier results $P(1) \ldots P(k)$. Our previous proof about postage can be rewritten so as to avoid strong induction. It's less clear how to rewrite proofs like this Nim example.

5 Prime factorization

The "Fundamental Theorem of Arithmetic" from lecture 8 (section 3.4) states that every positive integer n, $n \geq 2$, can be expressed as the product of one or more prime numbers. Let's prove that this is true.

Recall that a number n is prime if its only positive factors are one and n. n is composite if it's not prime. Since a factor of a number must be no larger than the number itself, this means that a composite number n always has a factor larger than 1 but smaller than n. This, in turn, means that we can write n as ab, where a and b are both larger than 1 but smaller than n.[2]

> Proof by induction on n.
>
> Base: 2 can be written as the product of a single prime number, 2.
>
> Induction: Suppose that every integer between 2 and k can be written as the product of one or more primes. We need to show

[2]We'll leave the details of proving this as an exercise for the reader.

that $k+1$ can be written as a product of primes. There are two cases:

Case 1: $k+1$ is prime. Then it is the product of one prime, i.e. itself.

Case 2: $k+1$ is composite. Then $k+1$ can be written as ab, where a and b are integers such that $1 < a, b < k+1$. By the induction hypothesis, a can be written as a product of primes $p_1 p_2 \ldots p_i$ and b can be written as a product of primes $q_1 q_2 \ldots q_j$. So then $k+1$ can be written as the product of primes $p_1 p_2 \ldots p_i q_1 q_2 \ldots q_j$.

In both cases $k+1$ can be written as a product of primes, which is what we needed to show.

Again, the inductive step needed to reach back some number of steps in our sequence of results, but we couldn't control how far back we needed to go.

How to Negate Statements

Example 1.18. Recall that in the proof of Theorem 1.3, we wanted to show that $A \cap \bigcup_\alpha A_\alpha = \bigcup_\alpha (A \cap A_\alpha)$, which means that $A \cap \bigcup_\alpha A_\alpha \subseteq \bigcup_\alpha (A \cap A_\alpha)$ and $\bigcup_\alpha (A \cap A_\alpha) \subseteq A \cap \bigcup_\alpha A_\alpha$; that is,

$$x \in A \cap \bigcup_\alpha A_\alpha \implies x \in \bigcup_\alpha (A \cap A_\alpha) \quad \text{and} \quad x \in \bigcup_\alpha (A \cap A_\alpha) \implies x \in A \cap \bigcup_\alpha A_\alpha,$$

which is to say, we wanted to prove that

$$x \in A \cap \bigcup_\alpha A_\alpha \iff x \in \bigcup_\alpha (A \cap A_\alpha).$$

We can prove this quick and simple using $\iff$:

$$\begin{aligned} x \in A \cap \bigcup_\alpha A_\alpha \iff x \in A \text{ and } x \in \bigcup_\alpha A_\alpha &\iff x \in A \text{ and } x \in A_\alpha \text{ for some } \alpha \\ &\iff x \in A \cap A_\alpha \text{ for some } \alpha \\ &\iff x \in \bigcup_\alpha (A \cap A_\alpha). \end{aligned}$$

Just make sure that if you use $\iff$, the expression to the immediate left and right of $\iff$ are indeed equivalent.

1.2.3. Negations and logical quantifiers.

We already know that a statement and its contrapositive are always equivalent: "if P, then Q" is equivalent to "if not Q, then not P". Therefore, it is important to know how to "not" something, that is, find the **negation**. Sometimes the negation is obvious.

Example 1.19. The negation of the statement that $x > 5$ is $x \leq 5$, and the negation of the statement that x is irrational is that x is rational. (In both cases, we are working under the unstated assumptions that x represents a real number.)

But some statements are not so easy especially when there are **logical quantifiers**: "for every" = "for all" (sometimes denoted by $\forall$ in class, but not in this book), and "for some" = "there exists" = "there is" = "for at least one" (sometimes denoted by $\exists$ in class, but not in this book). The equal signs represent the fact that we mathematicians consider "for every" as another way of saying "for all", "for some" as another way of saying "there exists", and so forth. Working under the assumptions that all numbers we are dealing with are real, consider the statement

$$\text{For every } x,\ x^2 \geq 0. \tag{1.7}$$

What is the negation of this statement? One way to find out is to think of this in terms of set theory. Let $A = \{x \in \mathbb{R}\,;\, x^2 \geq 0\}$. Then the statement (1.7) is just that $A = \mathbb{R}$. It is obvious that the negation of the statement $A = \mathbb{R}$ is just $A \neq \mathbb{R}$. Now this means that there must exist some real number x such that $x \notin A$. In order for x to not be in A, it must be that $x^2 < 0$. Therefore, $A \neq \mathbb{R}$ just means that there is a real number x such that $x^2 < 0$. Hence, the negation of (1.7) is just

$$\text{For at least one } x,\ x^2 < 0.$$

Thus, the "for every" statement (1.7) becomes a "there is" statement. In general, the negation of a statement of the form

"For every x, P" is the statement "For at least one x, not P."

Similarly, the negation of a "there is" statement becomes a "for every" statement. Explicitly, the negation of

"For at least one x, Q" is the statement "For every x, not Q."

For instance, with the understanding that x represents a real number, the negation of "There is an x such that $x^2 = 2$" is "For every x, $x^2 \neq 2$".

EXERCISES 1.2.

1. In this problem all numbers are understood to be real. Write down the contrapositive and converse of the following statement:

$$\text{If } x^2 - 2x + 10 = 25, \text{ then } x = 5,$$

   and determine which (if any) of the three statements are true.
2. Write the negation of the following statements, where x represents an integer.
   (a) For every x, $2x + 1$ is odd.
   (b) There is an x such that $2^x + 1$ is prime.[8]
3. Here are some more set theory proofs to brush up on.
   (a) Prove that $(A^c)^c = A$.
   (b) Prove that $A = A \cup B$ if and only if $B \subseteq A$.
   (c) Prove that $A = A \cap B$ if and only if $A \subseteq B$.

1.3. What are functions?

In high school we learned that a function is a "rule that assigns to each input exactly one output". In practice, what usually comes to mind is a formula, such as

$$p(x) = x^2 - 3x + 10.$$

In fact, Leibniz who in 1692 (or as early as 1673) introduced the word "function" [**221**, p. 272] and to all mathematicians of the eighteenth century, a function was always associated to some type of analytic expression "a formula". However, because of necessity to problems in mathematical physics, the notion of function was generalized throughout the years and in this section we present the modern view of what a function is; see [**118**] or [**137, 138**] for some history.

1.3.1. (Cartesian) product. If A and B are sets, their **(Cartesian) product**, denoted by $A \times B$, is the set of all 2-tuples (or ordered pairs) where the first element is in A and the second element is in B. Explicitly,

$$\boxed{A \times B := \{(a, b)\,;\, a \in A,\ b \in B\}.}$$

We use the adjective "ordered" because we distinguish between ordered pairs, e.g. $(e, \pi) \neq (\pi, e)$, but as sets we regard then as equal, $\{e, \pi\} = \{\pi, e\}$. Of course, one can also define the product of any finite number of sets

$$A_1 \times A_2 \times \cdots \times A_m$$

as the set of all m-tuples $(a_1, \ldots, a_m)$ where $a_k \in A_k$ for each $k = 1, \ldots, m$.

Example 1.20. Of particular interest is m-dimensional Euclidean space

$$\mathbb{R}^m := \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{(m \text{ times})},$$

which is studied in Section 2.8.

[8]A number that is not prime is called composite.