MATHEMATICS

PRACTICALLY APPLIED

TO THE

USEFUL AND FINE ARTS.

BY

BARON CHARLES DUPIN,

MEMBER OF THE INSTITUTE OF THE ACADEMY OF SCIENCES

ADAPTED TO THE STATE OF THE ARTS IN ENGLAND,

BY GEORGE BIRKBECK, Esq., M.D.

PRESIDENT OF THE LONDON MECHANICS INSTITUTION, &c &c

GEOMETRY OF THE ARTS.

LONDON:

PRINTED FOR CHARLES TAIT, 63, FLEET STREET

AND WILLIAM TAIT, 78, PRINCES STREET,

EDINBURGH.

1827.

LONDON:
PRINTED BY S. AND R. BENTLEY, DORSET STREET.

PREFACE.

AMONGST the important results of the recent attempts to extend Science to the labouring classes, may be ranked the elementary treatises published by Baron Dupin. Possessing an extraordinary fund of scientific information, as well as of practical knowledge collected during a period of twenty years, in the workshops and manufacturing establishments of the most enlightened nations of Europe, combined with a singular degree of clearness, elegance, and ingenuity, in mathematical and physical expositions, this distinguished individual might have continued to delight and instruct inquirers of the highest description, by works classical and profound, but without having witnessed the occurrences alluded to, he might never have directed his attention and his efforts to this most interesting object, the improvement of humble and neglected intellect.

The value of this new species of instruction, as Dupin has correctly termed it, was felt by him on his first visit to Glasgow, where it then alone prevailed. His notice and commendation of what he had observed, awakened the attention of several enlightened friends to the extension of education in Edinburgh; of which the foundation of the School of Arts was a consequence. It was not, however, until his return to France, after a second visit to London, where public attention had been recently and successfully directed to this most promising branch of education, that he became induced or encouraged to imitate in Paris what he had witnessed and applauded in the institutions of this country. The constituted authorities there, immediately seconded his patriotic intentions, by permitting him to occupy an establishment, prepared and furnished, as it might have seemed, for such a purpose; and the first course of lectures, delivered in the *Conservatoire des Arts et Metiers*, on the application of Geometry to the arts and manufactures,—the work now offered in his own language to the British student,—was published immediately, for the assistance of the provinces, already prepared by the decided success of this attempt, for similar undertakings. This singularly happy specimen of the method and of the utility of combining science with mechanical proceedings, contributed it is probable, even more than the eloquent and enthusiastic recommendations of an individual so justly celebrated as Baron Dupin, to the rapid progress of the measure in France.

If in this career of improvement, Great Britain should appear to have been outstripped by her active and ingenious neighbours, it ought in justice to the people, to be recorded, that all that has been here effected, has been effected by themselves; that not one name in connection with the government of this country, the splendid name of Huskisson alone excepted, has lent its influence, or its encouragement, to accelerate the progress of Mechanics' Institutions; and that in France, public functionaries of every description, local and general, have vied with each other in extending this intellectual impulse. Even the illustrious individual nearest to the throne of that great empire, has publicly, and at the instant of its commencement, declared himself to be friendly to the diffusion of information amongst the working classes; and his mind was vividly affected by contemplating the new sources of prosperity, which, through these institutions, might be enjoyed by his future subjects.* In proof of this interest, Dupin, in his last eloquent communication on the subject, observes, that "when the Dauphin travelled through Lorraine, the magistrates of Metz presented to him the former pupils of the *École Polytechnique*, who lecture gratuitously to the workmen of that great city, on perspective, and on geometry, and mechanics applied to the arts. This enlightened Prince expressed the high satisfaction which he felt in becoming acquainted with the important services con-

* Exposé fait à la Société d'Encouragement pour l'Industrie Nationale, sur les progrès du nouvel enseignément de la géométrie et de la méchanique, appliquées aux arts et metiers en faveur de la classe industrielle. *Géométrie*, 16e *Leçon*, 1825.

ferred by distinguished members of that school, which enjoys the honour of his protection."*

As might be inferred from the indifference of the official men in this country, the public acts of its government contain no notice of these proceedings, or indeed advert at all to the interesting and national subject of popular or universal education. It is consolatory, however, to find that the same silence has not every where been maintained. In the speech of the King of the Netherlands on opening the States General in the Session of 1826, we find a strong expression of the value of the present efforts to facilitate the diffusion of useful knowledge, and a direct allusion to that particular branch which has here been allowed to proceed without the slightest discoverable notice. "Public instruction," says his Majesty, "*is becoming more and more adequate to the wants of society.* The indigent class can now every where enjoy it gratuitously. *In some towns a beginning has been made to give to the working classes scientific instruction,* with a view to increase their practical knowledge."

The official encouragement bestowed upon this plan in France, combined with the natural zeal of Frenchmen, has been followed by its extension to a degree somewhat astonishing. In the report

* Effets de l'Enseignement Populaire sur les Prosperités de la France, par le Baron Charles Dupin. Discours prononcé dans la séance d'ouverture du cours normal de géométrie et de mechanique, appliquées, le 29 Nov. 1826, au Conservatoire des Arts et Metiers.—*Revue Encyclopédique, 97e Livraison.*

made to the Society for the Encouragement of National Industry, just referred to, Dupin has thus detailed its progress :—" You know, gentlemen," says he, " that our most formidable rivals in manufactures, the English and the Scotch, have for some years recognized all the advantages of instruction in the sciences applied to the arts and trades, with regard to the working classes: they have established schools of this description in several of their great manufacturing towns.

" They commenced with Glasgow ; and this town soon experienced from it the most favourable effects. The example of this advantage derived from instruction by the labouring classes, once made evident to the eyes of commerce and industry, numerous imitations were soon produced. Edinburgh and London first provided instruction for the mechanic, and afterwards Liverpool, Manchester, Birmingham, Newcastle, and Aberdeen, did the same. This movement proceeded with so much rapidity, that from the first of January, to the first of July, thirty-one towns in Great Britain have established these new schools.

" If France had remained without imitating this example, or even without endeavouring to surpass it, our mechanics would soon have found themselves inferior, both theoretically and practically, to the same class in England and Scotland, and we should have been still less than ever, in a condition to sustain against our rivals the competition of commerce.

"Assured of this truth, I have considered it to be a duty to attempt in France, according to my feeble means, the extension of teaching geometry and mechanics applied to all the arts; instruction which, by a deplorable fatality, is at once the most necessary and the most abstruse."

After mentioning the extensive opportunities which he has enjoyed, of collecting valuable practical materials for the composition of a normal course, adapted, when published, for repetition by intelligent professors of mathematics, without the necessity of visiting, as he had done, the manufactories and public works in France, and other countries; and after enumerating the conductors of workshops and manufactories, who soon began publicly to communicate these lectures to the men whom they employed, Dupin proceeds "to speak of the same species of education, applied on a greater scale in the principal towns in France.

"His excellency the minister of the marine and of the colonies, desiring to contribute to the progress of the arts of production in our seaports, both commercial and military, has directed the professors of hydrography to lecture twice a-week when the shops are closed, on the application of geometry, and mechanics to the arts; forming courses similar to those which have been delivered in the Conservatoire of Paris.

"Thus, by this single act, which places the name of M. Le Comte de Chabrol, amongst those of the greatest benefactors to the industry of France, the manufacturing population of forty-four sea-ports receive gratuitous instruction: and amongst these ports we mention with pride, towns such as Marseilles, Bourdeaux, Rouen, Nantes, Hâvre, Caen, Dunkirk, Bayonne, Brest, Toulon, Lorient, Cherbourg, &c.

"In all these ports, the civil and military authorities have assisted with emulation in giving all their efforts to aid the benevolence of the minister of marine.

"The officers of the navy, the commissioners, the commissary-generals and directors and the commissioners of classes, have solicited and have seconded the mayors, the sub-prefects, and the prefects; they have rivalled these officers in zeal and emulation in procuring for their respective ports, all the means which could accommodate the professor; a large room, fire, light, &c.

"I shall content myself with a single fact, to show to the manufacturing classes of France what may be expected from the mechanics of our ports.

"The town of La Rochelle contains only 18,000 inhabitants; nevertheless, the provisional course opened this year for the working classes, at the commencement was attended by three hundred students; and six weeks afterwards this number was increased by eighty persons, from the town and its environs, to whom the professor, with a zeal worthy of the highest eulogium, delivered a preparatory course, in order that they might join those who were more advanced.

"At Nevers, a town containing 12,000 inhabitants, a similar course of instruction, which commenced in January, this year, has had an auditory of two hundred persons; which is in the same proportion with the former attendance at La Rochelle.

"I shall now speak of the courses which will immediately open in the towns of the interior.

"Thanks to the enlightened care of M. le Baron de Rambaud, Mayor of Lyons, M. Tabarand, formerly a military engineer, will teach in the second town of the kingdom, Geometry and Mechanics, as applied to the arts.

"M. Le Comte de Turmel, the Mayor of Metz, has just published a very remarkable prospectus of a course of instruction, gratuitous like all those of which I have hitherto spoken, and of which I have still to speak, which is to be delivered in the evening, by three officers of artillery, Messrs. Bergery, Poncelet, and Lemoyne, formerly pupils at the *Ecole Polytechnique*. At Nevers, where the first experiment has fully succeeded, Messrs. Morin, Boucaumont, &c., both students at the *Ecole Polytechnique*, will commence courses, not only of Geometry and Mechanics applied to the arts, but also of Natural Philosophy and Chemistry.

"At Versailles, thanks to the united attentions of M. Le Comte Destouches, Prefect, of M. Le Maire, and of M. Polonceau, engineer-in-chief *des ponts et chaussées*, Geometry and Mechanics applied to the arts, will also be taught by a former student at the *Ecole Polytechnique*. The same will occur at St.

Etienne, which will owe this service to M. Blavier, a young professor of the school of mines.

"Permit me, gentlemen, to call your attention for a moment to the course of instruction given at St. Lôo: you will see that this little town offers a very fine example to the greatest cities of France.

"In a proclamation published by M. Le Chevalier Clément, mayor of the town, on the occasion of the coronation of his Majesty, we read as follows:—

"'The 30th of May, (the day following the rejoicings of the town on the occasion of the coronation of his Majesty, fixed for the 29th of May,) will open for the manufacturing classes, who have so much reason to bless the new reign, a gratuitous course of arithmetic, of practical geometry, and linear perspective, applied to the arts and manufactures. These lectures will be given in a temporary room, which will be pointed out to the apprentices in the professions and manufacturing arts, when they apply at the house of the mayor to be entered, where the tickets of admission will be delivered to them.'—

"During this autumn, a course of geometry and mechanics applied to the arts, has been given not merely to the apprentices, but to persons of all professions in the town of St. Lôo.

"At Clermont, the chief place in the department of Puy-de-Dôme, a rich and populous town, a prefect, known by his scientific statistical labours, M. le Comte D'Allonville, has founded a school of practical geometry and linear perspective, on the excellent method given by M. Francœur.

"In the month of August, M. le Comte D'Allonville, whilst presiding at the distribution of prizes at this school, informed the public, that M. Darlay, Professor in the Royal College of Clermont, intended to open a gratuitous course, on geometry and mechanics applied to the arts, in the evening, for the benefit of the working classes.

"M. Petit, engineer *des ponts et chaussées*, is engaged in establishing the same plan of instruction in the manufacturing town of Louviers: and several great manufacturers have promised to employ all their interest in performing the same service for the towns of Elbœuf and Sedan.

"Some professors and engineers, animated by a generous desire to promote the public good, have proposed to deliver lectures at Cimoges, Poitiers, Tonnerre, Aix, Strasbourg, Rennes, Douay, Valence, &c. In all places these offers have been received with a just and lively gratitude.

"It thus appears, that on the 26th of October, 1824, instruction was afforded to the working classes at a single point only in France, and that point the capital.

"On the 26th of October, 1825, this instruction, every where gratuitous, is offered to all the mechanics of fifty-nine towns: the population of which amounts to 2,040,000."

The farther progress of these valuable arrangements for multiplying popular instruction, Dupin has exhibited in the following passage:—"This winter (Dec. 1826), thanks to the benevolence and to the effectual assistance of a great number of municipal councils, mayors, prefects, and sub-prefects, equally friends to useful knowledge, *ninety-eight* towns are endeavouring to rival each other in their zeal for imparting the new instruction to the working classes."*

It is probable, that in each of these institutions, this work and the two excellent volumes on mechanics, which were afterwards published, form the bases of the lectures on the respective subjects. That they will require some adaptation to the leading pursuits of the particular places into which they may be introduced, by extending the discussion of topics of considerable local interest, and curtailing others which may have but slight connection with the operations of the inhabitants, need not be questioned. That they may, however, with the greatest advantage, be made the text books of the courses delivered in this country, as well as in France, I have no hesitation in asserting; and no one, I think, can doubt, after perusing the following lessons, that if they were illustrated, and occasionally expanded, with distinctness and intelligence, although nothing whatever should be done in the way of adaptation or improvement, many important benefits must result to the attentive and competent student.

Where a professor, who is capable of preparing for himself an elementary treatise, may undertake courses of Geometry and Mechanics with a view to their practical application, it is not my wish, or, I ven-

* Revue Encyclopédique, Livraison 97^e, page 61.

ture confidently to believe, the wish of the eloquent and patriotic author of this work, that he should be compelled servilely to copy it, or indeed to do more than select from it, and make it his model, so far as his approbation of its* contents may render it desirable. Accordingly, I observe with great, satisfaction, the original character of the work of M. Bergery, printed for the assistance of the members of the new school established at Metz. Although both the school and the treatise have confessedly sprung from the advice and the example of Dupin, yet the plan is in many respects very different; and a part of almost every one of the thirty lessons, entitled Geometrical Laws of Nature, contains a much more extensive application to natural phenomena, than the original work: this part indeed is executed with singular clearness and ability. In regard to energy and enthusiasm, M. Bergery certainly is not inferior to M. Dupin, as the following quotation from his introductory discourse will demonstrate. After having exhibited in powerful language, the social, moral, commercial, and political advantages conferred by knowledge, he says, "If all these motives are not sufficiently powerful, I would thus address you:—You are Frenchmen; France is dear to you; you cannot, without having your hearts pained, see her trampled upon and enslaved. Alas! such is the lot which awaits her, which awaits every nation, if our indus-

* Géométrie appliquée à l'Industrie, à l'usage des artistes et des ouvriers. Sommaire des leçons publiques données dans l'hôtel de ville de Metz, par M. C. L. Bergery; 8vo. Metz, 1825.

try fails to make an immense progress. A Colossus lifts itself up near us, and threatens us; the riches and the power of England increase in an appalling manner; in a manner which makes us tremble for the future. England increases thus, because in it industry is supreme; because there the thirst after positive knowledge is extreme; because there its light has been diffused, even amongst the most simple workmen. These workmen not only attend courses of lectures, but they assemble together in classes, where one of them reads aloud instructive dissertations; they have also libraries, and they are provided with periodical works, which explain to them the processes of their trades, the nature of their machines, and every thing belonging to other professions, which may render them more skilful in their own. On this account nothing amongst us can give an idea of the rapidity, the economy, or the perfection that the English have introduced into their manufactories. This truth is painful, I am sensible: it will, without doubt, afflict those good Frenchmen, whose patriotism leads them to believe us to be superior in every thing: it will, I know, offend those, who influenced by vain national pride, obstinately refuse to see the elevation of our eternal rivals. But the truth must be proclaimed: the fatal security in which we live demands it; it must also be known to all; for on this our safety depends." In a very spirited concluding discourse, delivered on the 14th of April, 1826, M. Bergery having referred to the subjects which had been discussed, and to those which will afterwards be considered, notices the laborious duties of the teachers; but immediately adds, "We shall be amply rewarded for our fatigues, if you continue to second our efforts by your assiduity, by unrelaxing attention, and by your progress. Yes, by your progress; for in fact many amongst you have profited by our lessons much beyond our hopes. There are workmen, I can assert, because I have indisputable authority for the assertion—there are workmen, who since the opening of these courses, have improved their instruments, and have constructed others much more perfect than before: workmen who have already carried into their occupations, that geometrical spirit which has simplified their proceedings, and which leads to that precision without which the arts cannot produce any thing either good or beautiful. For these exertions they will be rewarded, or rather they are rewarded already: for already they work with more intelligence, with more pleasure, and consequently produce with greater celerity, more beautiful results."

Whilst this species of popular instruction continues slowly to extend through Great Britain, some minds of adequate qualifications and powers, may at length be influenced to attempt an equally clear, and still more complete exposition of the application of Geometry to the Arts, than is to be found in the following pages. That this has yet been effected by any British author, no well-informed or competent mathematician will venture to assert: nor, whatever may be their respective merits—and considerable merit they do certainly possess—will Dr. Olinthus Gregory, and Mr. Lees, the intelligent authors of "Mathematics for Practical Men," and the "Elements of Arithmetic, Algebra and Geometry, for the use of Students in the Edinburgh School of Arts," contend, that this publication has been rendered superfluous by any of those which have preceded it. The applications of Geometry to practice, unceasing in the lessons of Dupin, are only occasional in the useful volume of Gregory; and the Geometry of Lees is little more than a judicious abridgment (partly by the introduction of symbols) of the propositions and demonstrations constituting the elements of Euclid. In a nation, however, abounding with successful cultivators of mathematical knowledge, and with the most refined cultivators of the arts, both useful and elegant—in which, indeed, many of them have originated, and all of them have been improved—appropriate scientific talents united with sufficient practical skill, may appear, and if my wishes be answered, soon will appear, to surpass and therefore to supersede this work, which with every feeling of satisfaction and confidence, I now offer to the acceptance, to the criticism, and to the competition of my countrymen.

GEORGE BIRKBECK.

London, June 15th, 1827.

GEOMETRY OF THE ARTS.

First Lesson.

Right lines—Angles—Perpendicular and Oblique lines.

The object of Geometry is to measure extension and ascertain its relations.

Extension has three dimensions, *viz.* length, breadth, and thickness.

All the bodies in nature, as well as all those formed by art, have these three dimensions.

They are also found in every portion of space, whether it be a vacuum or be filled by a body.

The surface of a body consists of all the points which separate the portions of space occupied by the body, and of the intervening portions of space.

A surface, consequently, has both length and breadth, but it has not thickness. The points constituting the thickness of a body cannot form any part even of its surface.

The continued succession of points, which separate two portions of the surface of a body, is called a line. A geometrical line has only length, and neither breadth nor thickness.

The space filled by a body, at any given time, has all the dimensions of that body. A complete idea may be formed of this by making a mould round the body and withdrawing it, when the mere sight of the mould gives us an exact notion of the space the body occupied. An empty box encloses a portion of space, and the figure of this portion is precisely the same as that of the interior of the box.

Consequently, all the geometrical properties of the dimensions of a body also belong to the space it fills. The geometrical properties of any surface, and those of the space occupied at any given moment by it, are the same.*

This is the reason why a purely theoretical geometrician never considers any particular body nor its individual surface, in order to ascertain the relations belonging to the dimensions either of the body or of the surface. He imagines, in space itself, the form and the surface of a body, which are sufficient for him. At first this species of abstraction presents some difficulties, but it exercises the mind and strengthens the imagination; in the result, it gives great powers of conception both in pure geometry, and in geometry applied to the arts. It is of much con-

* The observations in the text would lead us, apparently, to a clearer comprehension of geometrical definitions than is usually obtained. As the geometer always considers *figured* space, his doctrines, even when most abstract, do not relate, as is sometimes supposed, to imaginary points and lines, but to the portions of space included within them. A surface is the extended limit of a solid, a line is the boundary of a surface, and a point is the termination of a line. Space itself has neither boundaries, limits, nor terminations; all bodies have: and it is the relations as to form or position of what is included in these bounds, or constitutes these limits and terminations, which the geometer considers. If we place together two pieces of polished marble, or well-planed wood, closely adjusted to one another, so as to bring into contact every point of their surfaces, the almost invisible line of separation between them will give us an idea of a geometrical line, and the termination of that line has been denominated by geometricians, a point. Now it is not this point, nor the line itself, nor the surface as a surface, which is ever considered in geometry; but the properties or relations, as to form of the marble or wood (and they may be taken in this respect as the representatives of all similarly formed bodies),—of which the point, the line, and the surface are the respective boundaries.

sequence, therefore, gradually to accustom students to such abstraction. There is one essential difference indeed, to which they must attend, between bodies, as they are considered by the geometrician, and as they actually exist.

In pure geometry, there is nothing to prevent our imagination from supposing that bodies may enter one within the other, so that they may at the same moment occupy wholly, or in part, the same portion of space. But it is not thus in the arts. The matter of two bodies cannot be at the same moment in the same portion of space. When this appears to take place, we must conclude that the matter of one only occupies the vacuities in the other; as for example, when water enters into a sponge. Hereafter it will be seen that these considerations are essential to the formation of correct ideas of the effects and the movements of machines.

If we suppose, that a body gradually diminishes in length, breadth, and thickness, it will approach more and more to an ideal limit, which is the *point* of geometricians; to form which each of these dimensions is reduced to zero.

In the arts, the term *point* is very often given to those portions of a surface, or a solid, the dimensions of which are very small. In writing, for example, there are dots or points; in geometrical designs, in chalk drawings, in miniature painting, in engraving, &c. there are dotted lines, or lines composed of points.

The termination of a sharp body, in the dimension of its length, is also called a point: such as the point of a needle: it is almost imperceptible, and in this respect approaches a point as it is understood by geometricians.

Students must accustom themselves to distinguish the various modes of considering a point, both as it is understood in pure geometry and in the arts.

In order to facilitate the study of geometry, lines are first treated of, then surfaces, and afterwards bodies; which are called *solids*, if they have a form of their own, and *volumes*, in relation to the space which they occupy, if they have no such form, and must be contained by some vessel, or be inclosed by some resisting boundaries; as, for example, wine in bottles, the water in rivers, in lakes, in the sea, &c.

Geometry supposes that bodies are solids; or at least that their form is not subject to change, except, according to some rule and within some limit, at the moment they become the object of investigation.

The most simple of all lines, and that which is the most frequently employed in the arts, is the right line.

A right line is the shortest distance between any two points: or that which we trace by always proceeding in the same direction.

As there are not between any two points, two directions, of which, each may be shorter than the other, so there cannot be drawn two distinct right lines between any two points. When two right lines terminate in the same two points, they form, consequently, only one right line. If two such lines are drawn on different bodies, and the two bodies are brought together, so that any two points of one line coincide with any two points of the other line, the two lines adapt themselves in every point as if they formed only one line. Of this property of right lines much use is made in the arts.

1st. *To ascertain that a line, already drawn, is a right line, by means of another line, which is known to be a perfect right line:*—It is only necessary, in fact, to apply the latter at any two points to the former, and see if all the other points of both correspond. If they do not, the line examined is not a right line, and it may be made so, or rectified, by the help of the perfect line.

2d. *To trace right lines:*—This is done by using bodies having one or more rectilinear edges, such as common rulers or squares. The ruler or square is laid on a surface, to which the right line, represented by the instrument, applies exactly at every point; and unless it do so, it is not possible to trace a right line on the surface. With a pencil, or any pointed or cutting instrument, a line is then drawn, which every where coincides with the ruler or bevel, and is a right line.

In this manner the glazier, by means of a ruler and his diamond, cuts the pane of glass which he is to fix, straight, or in a right line.

When we have to draw a line through two given points, the ruler must be placed equally distant from both, and as near as is required, by the thickness of the instrument, which is to draw the line: the ruler must then be held firm and immoveable, till the line is drawn, taking care that the pencil or pen always remains in contact with the ruler.

When students begin to draw geometrical figures, they are obliged to take both care and time to draw a right line, even with a pencil. They find it still more difficult to draw a right line with ink, because they must give it an equal breadth through its whole length. When the breadth is unequal and too great, the figure ceases to be correct. Students should accustom themselves, therefore, to give the lines they trace no greater breadth, than is necessary to make them distinct.

These observations apply only to lines executed in works of art; of geometrical right lines it is necessary to repeat what has already been said of points. The geometrician supposes lines have only length, without breadth; but all lines executed in the arts, even those which represent the ideal lines of geometricians, have breadth as well as length.

The term line is very often applied in business to straight excavations or elevations, considerable in length, and not very deep or high, which, on this account, seem to have some resemblance to the ideal line of geometricians. Such are the temporary *lines* thrown up, either by the besiegers or the besieged, when a fortress is invested.

In writing and printing, the term *line* is applied to a series of words running across the page in the same direction; the height of the letters being very small compared to the length of the line.

Ropemakers call a cord, which is of small diameter compared to its length, *a line*; and such a line, or cord, is one of the instruments of practical geometry employed in the arts. A cord stretched by the two ends, making allowance for the weight, takes the form of a right line. If such a cord, which has been rubbed with chalk or other substance answering the same purpose, be pulled tight over a surface on which a right line is to be drawn, and it be then lifted up in the middle, so as to make it, when let go, strike the surface with a little force, it will form the desired right line.

It is proper to distinguish carefully, in regard to lines as well as points, between the ideal limit of the geometrician and the perceptible line of the workman. In many cases, it will be evident, that the progress of the arts depends, in some measure, on the operations of the workman being directed, as much as possible, by the geometrical or ideal limit, the nature and properties of which, therefore, it is of much consequence that the student should know.

Before he advances to this part of the subject, however, he must obtain a notion of a *plane surface*, or that surface which can be made by a right line.

In whatever manner a right line may be placed on a plane, if two points of the line are in contact with it, every other point of the right line will also touch it.

In the arts, a plane may be employed to generate a right line, or a right line may be used to make a plane. This subject will be explained in detail, in the sixth lesson, when surfaces will be more especially treated of.

In general, the lines necessary in works of art, are traced on a plane surface, previously prepared; such as, for short lines, a sheet of paper or parchment, or even a tablet of ivory. For more extended lines, a large surface is prepared, the nicely planked floor of a model chamber, for example (called in his Majesty's Dock-yards a *mould loft*) such as is used by ship builders; house-carpenters and bricklayers very often draw their plans on the plane surface of a wall; engineers form models of bridges on plaster floors; and none of these artists can expect to be correct, unless the surface, supposed to be a plane, has been previously examined, and unless a right line, placed over it in every direction, touches it exactly from one end to the other.

The straight-edge.—The right line, used in practice for ascertaining the correctness of a plane surface, is called a straight-edge, and there are few problems of practical geometry more difficult than to form a perfect straight-edge. The following method of constructing one is in use in England, and is added by the Editor.

The straight-edge is usually made on a thin flat bar of steel, or rather three bars. They should be about one-eighth of an inch thick, two inches broad, and about two feet six inches long, or nearly three feet. It would be imprudent to make them of greater length, as they would be liable to bend.

These bars are to be nicely planished; and one edge of each is then to be made as straight as possible, by the common means of filing and planing. They are then to be made perfect, by grinding them mutually and reciprocally, with each other; fine emery, rendered fluid with oil, being added to promote abrasion. They are, finally, to be finished with a species of loam, carefully washed to render it perfectly clear from any coarse silicious matter.

In general, it has been thought sufficient to grind two bars together to produce a perfect straight-edge, but the necessity of having three, and of repeatedly changing them at proper intervals, until each edge is perfectly straight, will appear evident, by the following considerations:—

Let A and B, fig. 13, double, pl. 1, be two steel bars, prepared for grinding, and let us suppose the edge *a*, of A, to be slightly convex, and the edge *b*, of the bar B, to be nearly straight, or slightly concave. If A be moved on B, in an horizontal direction from right to left alternately, and a proper supply of oil and emery be kept between the bars, the convex bar A will grind the edge *b*, of the lower bar B, into a concavity corresponding to its own convexity, as shown by the dotted lines. The two curves being parallel, and perfectly coincident, the form of the edges will remain unchanged, and, however long the grinding may be continued, a perfect straight-edge will never be obtained.

Let us now take a third bar C, one edge of which may be concave as at *d*, or convex, as shown at *e*. If the convex edge *e*, is applied to the concave edge *b*, of the bar B, and the grinding be continued, a similar effect will be produced, namely, one concave, and one convex edge, as before, each exactly corresponding, and being coincident with the other.

But if all the three edges are mutually and reciprocally ground together, the two convex edges *e* and *a* of the bars C, and A, will, by the grinding, be mutually cut down, and destroy each other's convexity. Ultimately, by a very gentle approximation, a perfect straight-edge will be produced on all the three. We have supposed that two out of the three bars are convex; but if the same number of bars, being concave, are submitted to the same operation, the same effect will follow. If all the three are either convex or concave, by this method, they may be formed into perfect straight-edges.

Measures of length.—The right line, being the shortest distance between any two points, is very convenient as a measure of distance.

The ordinary dimensions of all bodies are measured by right lines; as, for example, the length, breadth, and height of a pile of wood, of a house, a ship, &c.

In order to compare these various dimensions, something must be taken as unity; and it must be ascertained how often the measure is contained, or repeated, in the object measured. If it is repeated once, twice, thrice, or, in short, any exact number of times, there is no difficulty; but there is a difficulty when a portion of the object, less than the length assumed as unity, is to be measured. The plan, in this case, is to divide the measure itself into a certain number of equal parts, as 10, 12, 36, 100, or 1000; and to ascertain how many of the tenths, twelfths, hundredths, thousandths, of the measure are contained in the remaining portion of the line to be measured.

A *scale* is a right line, AB, fig. 1, pl. 1, on which is marked a certain number of the lengths assumed as unity, and of its subdivisions. To form and divide such a scale with precision, is one of the most important operations of art, and geometry teaches us how to do it very exactly. This subject will be treated of in the fifth lesson.

It is very convenient for artisans to have with them a right line, already divided according to the system of measures adopted in a country. Formerly, in France, the foot and the ell were used; now, the *metre* is the measure graduated on rules. In England, we have the yard and the foot, divided into inches.

Workmen are apt, by ill-directed economy or parsimony, to buy rules or measures, on account of their low price, which are not accurately divided, or which are subject to warp or change by the effect of time, or rub off at the ends. It cannot be too strongly recommended to them, however, to submit to other privations, and buy good rules and good instruments of every description. The perfection which such instruments will enable them to give their work will indemnify them, and with interest, for the outlay. The importance of this recommendation will cause it to be again adverted to.

After having considered the properties of a single right line, we must consider several right lines in their relation of posit[illegible]

Let us su[illegible] that the right line, ABX, fig. 2, pl. 1, turns round the fixed point A, and takes successively the positions AC, AD, AE, &c. In this movement it will gradually recede further from the original position ABX. This separation, BAC, BAD, or BAE, of one line from another, is called an angle. The point A, whence the two lines AB AC proceed, is called the *apex* or *summit* of the angle, and the lines AB AC, are its sides.

To point out the angle formed by the sides AB AC, it is sometimes described as the angle A, but more frequently the angle BAC, placing the letter A, which belongs to the apex, between B and C, which designate the two sides respectively.

The line AX, fig. 2, continuing to turn round the point A, will reach the position AM, directly opposite to AB;

if it continue to turn, it will approach towards AB, on the contrary side, till at length, after making a complete circle, it will return to AB.

It is evident that the right line, AX, has made a demi-revolution when it reaches AM from AB. In fact, if the part of the circle, BAME, were to be folded by the line MB over the underneath part, the one would cover the other precisely, and would coalesce with it.

In manœuvring troops, after having placed them in a right line, facing one way, it is very often necessary to make them turn to the other side. They are commanded to face about, which is effected by each man turning on his heels, A, fig. 3, pl. 1. To render the movement easy, one foot, B, is placed behind the other, fig. 4, and at the same time the man turns on his heels. Each of the feet performs half a circle, fig. 5. The foot which was placed behind then comes in front, and is brought back to a line with the other, fig. 6. If the soldier were to face about once more, he would find himself fronting the original direction, and he would have made a complete circle.

Let us consider the angles formed by the right line AC, with the right line DAB, fig. 7, pl. 1. There are here two angles; BAC, a small one, and CAD, a large one: their sum is always equal to half a revolution of AC, or from AB to AD.

The angle BAC, therefore, is required in order that the angle DAC may form a complete half revolution, and DAC is required in the same manner for BAC to form a half revolution. For this reason, BAC is called the supplement, or the supplementary angle to DAC, just as DAC is the supplement or the supplementary angle to BAC.

Let us suppose that the angle BAC, is enlarged by the line AC separating from AB; we shall then see that the supplementary angle is diminished. As the smaller angle BAC, is always increasing, and the larger angle, DAC, diminishing, there will be a time when they will be equal, as in fig. 8. Each of these equal angles is what is called a right angle. Thus we see, that a right angle is made by a line performing the half of a semi-revolution, or it is the quarter of a complete revolution.

Right angles, such as BAC or DAC, fig. 8, pl. 1, or angles formed by a quarter of a revolution, are required to be made, or measured, almost every moment, in order to carry into effect a great number of practical operations.

When a body of troops, drawn up in line, in the direction AB, fig. 8, are required to face in a direction perpendicular, or at right angles to it, they turn round the point A. If they were to revolve completely round in the same direction, they would face as at first; by making only a quarter of a revolution, they face perpendicularly to their former position. Of course they can be commanded to face either to the right or the left.

Now let us suppose two other right lines, MON and OL, figs. 9 and 10, pl. 1, the position of OL being such that the two angles NOL and MOL are equal; these two angles are equal to the two former angles, BAC and CAD of fig. 8, which are right angles.

To demonstrate this, place the right line, DAB, fig. 8, on MON, fig. 9, in such a manner that they coincide exactly at every point, as must be the case with two right lines, and that the point A falls on the point O, the side AC will then coincide exactly with the side OL. Let us suppose it possible that AC, fig. 9, has some other position, and falls to the left of OL. It is evident, as the angles CAB, CAD, are equal to each other, that MOL, which is greater by COL than the first angle, and NOL, which is less by COL than the second angle, cannot be equal to each other. On the contrary, if AC, fig. 10, fall to the right of OL, the angles BAC, DAC, being equal to each other, and MOL being smaller than DAC, and NOL larger than BAC, cannot be equal to each other. Consequently AC cannot fall either to the right or the left of OL, and must fall directly on it. The right angles, formed in the one case by the right lines, AC, BD, and on the other, by the two different right lines, OL, MN, are always equal to each other.

This is the first principle on which the use of the square is founded. A square may be formed of two parts of the right line AB, AC, fig. 11, pl. 1, fixed firmly together at A, so as to form a right angle. When it is desired, from the point O, fig. 12, to draw a line, OL to form two right angles with MON, one side of the square AC is placed along ON, so that the point A comes as near as possible to the point O, and the right line OL, being drawn by the common method, is the required right line.

If workmen employ an incorrect square, all their work will be affected by its incorrectness; and therefore it is of great importance that they should be able to ascertain the accuracy of their instruments. This may be easily done.

To prove squares.—To prove the square BAC, fig. 11. begin by drawing very exactly on a plane surface the right line MON, fig. 13, pl. 1; then place the side of the square AC as nearly as possible on the line ON, and draw OL along AB. When this is done, turn the square and place it in the direction of B′A′C′, putting A′C′ over OM, and notice what is the direction of the second side A′B′. If it fall precisely on the line OL, which has been before traced, the instrument is correct; if it do not reach to OL, the instrument is incorrect, and the angle it makes is less than a right angle. If it fall beyond OL, the instrument is also incorrect, but the angle is larger than it ought to be.

We shall hereafter see by what means workmen may rectify an incorrect instrument.

There is another sort of instrument used by shipwrights and other artisans, called a bevel, represented by XYZ, fig. 14, which is very convenient for drawing and transferring all kinds of angles. It is composed of two rulers, attached to, and turning on, the same pivot, in such a manner that angles of every size, from the smallest to the largest, may be formed by them. The two rulers are fixed so closely together, that they do not move one over the other, without some little exertion, and they remain at the angle to which they are set, unless some effort is made to alter them. After this explanation, it will be seen that it is very easy to draw any angle BAC, fig. 14, pl. 1, setting out from the point O, fig. 15, and assuming a side OL of the new angle LOM, which is to be made equal to the angle BAC.

The bevel is adjusted so that the two sides, XY, YZ, coincide with the lines AC, AB, fig. 14. It is then moved to fig. 15, taking care not to alter the angle; XY is then placed on OL, and, by tracing the right line, OM, according to the direction of the side YZ, the angle MOL will be equal to BAC.

Superposition.—It is essential to remark, that the method in use to form angles, or to ascertain if they are correct, is to place squares or bevels on the figures, or the figures on one another. These means are frequently employed, both in practical operations and in a great number of geometrical demonstrations. When two figures, placed one over the other, are adjusted to each other —when all the parts of both appear in the same lines,— they have the same form and size; they are perfectly equal to one another, and one figure is made of the same form and size as another, when it is drawn on this principle. Thus tailors and habit-makers place patterns on cloth, which is to be cut out exactly of the same form as the patterns, which represent a certain shape to be given, or a part to be covered.

When a line AC, fig. 16, pl. 1, makes with DAB the two right angles BAC, CAD, AC is perpendicular to DAB. Consequently a line, AC, is always drawn perpendicular to another right line, DAB, when the square, XYZ, is placed with the side YZ along AB, and another right line, AC, is drawn along the side XY. Other means, however, of drawing perpendiculars will be pointed out.

Let us fold into two fig. 17, pl. 1, in such a manner that the right line ABE, shall be the folding line; the angles ABD, ABC, being equal, the line BC will be placed on BD, and the angle CBE will exactly cover DBE. The two latter angles are therefore equal, as well as the two former. When two right lines, therefore, intersect each other,—if among the four angles which they form there be one right angle,—the other three must also be right angles, and, in that case, each half of the line AB, BE, is perpendicular to the other right line.

It is essential to prove, that from any point B, fig. 18, pl. 1, only one perpendicular can be drawn to the given right line DAC.

Let us suppose that the two perpendiculars BA, BD, could be drawn to the right line DAC, from the point B. Prolong BA, so that A*b* equals AB, and draw the right line D*b*, and then fold the part DAC*b*, over the part DACB. The angles *b*AC, BAC, being equal, A*b* is placed directly on AB, and the point *b* on the point B, consequently D*b* will also be placed on DB, and the angle AD*b* is equal to the right angle ADB. Thus D*b* will form part of the perpendicular DB, and two right lines, *b*AB, *b*DB, may consequently be drawn between the points *b* and B, which is absurd.

Having established these preliminary principles of right angles, let us proceed to oblique angles.

When the right line CD, fig. 19, forms two unequal angles with the right line ACB, one will be smaller, and the other larger, than the right angle ACE ; the smaller is called an acute, the larger an obtuse angle.

It is evident that these two angles will occupy the same space around C, on one side of AB, as the two right angles ACE, BCE. The sum therefore of the acute angle BCD, and of the obtuse angle ACD, will be equal to two right angles.

In fact, it is easy to see that the acute angle, BCD, is a right angle, minus DCE, and the obtuse angle, ACD, is equal to a right angle plus DCE; their sum therefore must be equal to two right angles.

Let us now suppose DC produced to CF, and let us compare the two new angles ACF, BCF, with the two former angles.

The angle ACD, plus BCD, formed by CD, and the right line AB, equals two right angles, consequently BCD equals two right angles, minus ACD. The angle ACD, plus the angle ACF, formed by AC on the right line DCF, equals two right angles, consequently ACF equals two right angles, minus ACD. Consequently, also BCD and ACF, each being equal to two right angles, less ACD, are equal to one another. By the same means the equality of the angles, ACD, BCF, which, like the two former, are opposite to one another at the vertex, may be demonstrated.

When two right lines cross each other, they form four angles; and consequently, *first*, the adjacent angles, taken in pairs, are equal to two right angles; *second*, the opposite angles at the vertex are equal.

We may now compare perpendicular and oblique lines together.

If from any point D. fig. 20, pl. 1, a right line, DE, is drawn to the right line AB, and the angles AED, DEB, are not right angles, the line DE is not perpendicular to AB, it is an oblique line. If we draw DC perpendicular to AB, the angle AED is obtuse, and the angle BED in face of DC is acute.

Let us prolong DC to *d*, so that CD shall be equal to C*d*, draw E*d*, and fold up the figure, making the fold on the line AB, as if it were a hinge. C*d* will then be placed on CD, and the point *d* on the point D, for the angles, BCD, BC*d*, are equal; E*d* therefore will be equal to ED. The broken line DE*d* is longer than the right line D*d*, connecting the same points D*d*, whence the half of DE*d*, or the oblique line DE, is longer than the half of DC*d*, the perpendicular DC.

It is a general property, therefore, of a right line, DC, fig. 20, perpendicular to another right line, AB, to be shorter than any oblique line drawn from the extremity D of the perpendicular, to the right line AB. The right lines DC, DE, measure the distances from the point D to the right line AB, and it results from this, that the shortest distance from any point to a right line, is a line drawn perpendicularly from the point to the right line.

This is a remarkable property of elementary geometry, and one of the most useful in its practical applications.

Workmen have frequently to ascertain the shortest distances, the surfaces which have the least extent, and the volumes which are the least considerable, to answer some stated purpose; but to find them is not an easy task. Questions of this description, on which much of the economy of labour depends will occupy a great part of these lessons, and care will be taken to make the principles of them plain and comprehensible.

Having drawn DB, fig. 21, pl. 1, perpendicular to AC, let us suppose that BA is equal to BC; then the oblique lines drawn from D to A, and from D to C, are equal. In fact, if we fold the part BDC on BDA, the perpendicular BD, serving as the hinge, the two right angles ABD, CBD, being equal, BC will fall on BA, and C on A, and DC will be equal to DA. Consequently, two oblique lines equally distant from the perpendicular are equal.

Application of this principle to verify perpendicular lines.—Draftsmen, ship-builders, house-carpenters, masons, &c. frequently make use of this property of geometry, when they wish to ascertain if one line is perpendicular to another, and they cannot have recourse to the bevel or square. They measure very exactly two portions, BA, BC, fig. 21, equal to each other, setting out from the line BD, the position of which they desire to ascertain They then measure with a rule, or some other instrument, the distance between the points A and D or the length of the oblique line AD; they transfer the measure to DC, setting out from D; if it terminate exactly in C, the two oblique lines AD, DC, are equal, and BD is perpendicular to AC.

When it is required to verify the position of a perpendicular, DB or ABC, care must be taken not to draw the oblique line D*a* too near the perpendicular; for if it is very near to B, a considerable deviation in B from the perpendicular would only produce a slight alteration in the length of the oblique line D*b*, and a mistake might easily be committed. It would also be inconvenient to make the oblique lines at too great a distance, the best positions for them being those in which AB, BC, are in length equal to BD.

By precautions of this nature, conducted on such principles in each particular case, we can give to plans, buildings and machines, that degree of precision and correctness, which are necessary when the arts of industry are brought to a state of perfection.

It is not enough to have demonstrated that oblique lines are always longer than perpendiculars; it must also be shown that they are longer in proportion as they diverge from the perpendicular.

Let OD, fig. 22, be perpendicular to OB, it will be found that of the two oblique lines, DC, DB, the shorter is that nearer the perpendicular. Let us draw CK perpendicular to CD, DC will of course be shorter than DK, and proportionally still shorter than DB.

This property is of frequent application in mechanics. Let us suppose that it is required to bring a body B, fig. 23, pl. 1, towards AC, perpendicular to BM. Let us suppose that this body has two ropes BA, BC, fixed to it, one of which is drawn at the point A, the other at the point C, in order to lessen the distance between these points and the body. It must gradually advance towards M, the cords forming the lines AB', CB', and afterwards AB'', CB'', becoming less and less oblique, and shorter and shorter. On the contrary, if it were required to push B from AC, bars of iron or pieces of firm wood would be used, and the force applied at A and C; the bars would receive a direction more and more oblique, and would become longer and longer, both between B and A, and B and C.

SECOND LESSON

Parallel lines, and their combination with perpendicular and oblique lines.

Two right lines are parallel when, however much they may be extended in either direction, they will never meet.

From any point A, fig. 1 and fig. 2, pl. 2, a right line, AB may be drawn, which, however prolonged, would never meet another right line CD, and only one such right line can be drawn from any one point A.

From the point A draw AC perpendicular to CD, and AB perpendicular to AC, AB will then be parallel to CD. For if the two lines AB, CD, met in any one point, from that point two perpendiculars might be let fall on the right line AC, which has been already shown to be absurd.

Let us now demonstrate that every other line AE, fig. 2, will meet CD. However small may be the angle BAE, it will be conceived that in making AE turn round A, to separate it from AB, the angle BAE must be repeated a sufficient number of times to cover all the space occupied by the quarter of a revolution BAC. Take any number of points, C1, C2, C3, C4, all at an equal distance, CA, from each other, and erect the perpendiculars, C1D1, C2D2, C3D3. These perpendiculars, will divide the space BACC1, C2, C3, into parallel bands, each of which will have the same superficies as ABCD. A greater number of parallel bands can always be made, than there are of the small angles, BAE; EAE1; E1AE2; E2AE3; in the right angle BAC1. The space, therefore, occupied by any single band, BACD, is always less than the space included in one angle BAE, however small it may be. Such a condition requires that the right line, AE, should, when prolonged, intersect CD: for if it did not, it would be necessary that the space BAE, a part of BACD, should be greater than BACD, which would be absurd.

Thus, whenever two right lines, AB, CD, are parallel, if one, is perpendicular to a third line, AC, the other is also perpendicular to the same line.

In drawing plans, and in making cabinet and joinery work, this property of parallel lines is frequently made use of. An instrument is employed, called a T square, because it is formed of two pieces at right angles, MN, OP, fig. 3, pl. 2, and has the shape of this letter. The thicker and projecting branch or stock MN, is placed along the side AD, of a plank ABCD; the other part OP being perpendicular to MN, it follows that all the right lines, AB, EF, traced along the branch OP, are parallel lines.

Besides the common T square, there is another in use in Britain, which seems to have several advantages. It has a moveable blade, and an arc of a circle divided into degrees, so that it is useful for drawing parallel lines, both at right angles with the stock and at any degree of obliquity. Fig. 4, represents such a T square, the moveable blade of which, AB, can be set to any angle, and fixed to the arc G, by the thumb-screw D. The action of the thumb-screw D, and of a clamp E, may be more distinctly seen at fig. 4 double, as likewise the screw F, upon which the blade turns as a centre. The screw F being withdrawn, permits the blade to be taken out, and the edges repaired, if they should receive any injury from wear or accident. The arc G, being divided into degrees, permits the blade to be placed at any angle where it may be fixed by the thumb-screw D.

When troops are to be ranged in columns, that is to say, parallel masses, AB, CD, &c. fig. 5, guides or flugel men, A, C, E, G, are placed in a right line, and at equal distances; each body of troops is then formed into a line perpendicular to the right line ACEG; and the officers are then certain that the different masses or columns are parallel to each other.

In writing and in printing, the letters are ranged in lines always equi-distant and parallel. The letters, individually, consist of parts which are right lines.—the strokes of the *m*, and *n*, for example, which are also equi-distant and parallel. The only difference remarked in them is, that these parallels are perpendicular to the lines in roman type; they are inclined to the right in the running hand and in italics, and inclined to the left in some old black letter.

In music, parallel lines at equal distances, fig. 6, pl. 2, are used, the notes being made either in the form of dots ●, or small open circles ○; and they are either simple, or are distinguished by strokes parallel to one another. The notes are so grouped, that the same length of time is necessary to sing or chaunt the sounds of each group or bar. This is called a measure; and the measures are separated by right lines, perpendicular to the first parallels, which are consequently parallel to each other.

The five parallel lines are very often traced at the same time, by means of a ruling pen with five points, at equal distances from each other: the pen is held, so that the five points are in a line perpendicular to the ruler; and in this manner the five lines are drawn equally distant at every point, and consequently parallel.

Parallel lines, at equal distances, are of frequent occurrence in the arts. The ploughman forms his furrows in parallel lines. When he harrows his field, drawing the harrow in a direction perpendicular to the furrows, the prongs of the harrow being equi-distant, describe parallel right lines; the points of the instrument, in consequence, act equally in every part, and break down the lumps of earth which have been turned up by the ploughshare.

When an engraver wishes to give us an idea of level and uniform surfaces, he represents the parts of them which are more or less in the shade, by stronger or weaker lines, but which are always parallel and at equal distances from one another.

When he is to represent the heavens, or a plane surface, one part of which is more distant than the other from the spectator, he also employs parallel right lines. He may make them also at equal distances, provided those nearest the spectator are either deeper or broader than the others. He may also make all the lines of an equal breadth and equal depth, but separating more and more from each other, as the parts of space which they represent are less in the shade, or farther from the spectator. Even these deviations from parallelism, are subject to precise geometrical rules; and artists, who wish to work in an enlightened manner, ought to make themselves acquainted with them.

It may now be demonstrated that two parallel right lines are at an equal distance from each other, at every point in their length.

Having drawn the two parallels AB, CD, fig. 7, pl. 2, and the right lines, AC, MN, perpendicular to them, let us take H in the middle of AM, and draw HK perpendicular to the two parallel lines; fold up the figure on the line HK, so that the part of it to the left hand folds over the part to the right. The right angles KHA and KHM, on the one hand, and the right angles HKC HKN, on the other, being equal to one another, HA will be on the line HM, and KC on KN. Moreover the angles, HAC, HMN, being right angles, and, consequently, equal to each other, AC will cover MN; the point C will fall on the point N; and consequently the perpendicular AC is equal to the perpendicular MN.

Thus, all the perpendiculars, such as AC, MN, fig. 7, which measure the distance between two parallel lines, at different points, are equal to each other. They all represent the shortest distance between these parallels.

AC, MN, perpendiculars to the right line AB, are also parallels; whence the right lines AM, CN, which are perpendicular to them, are also equal to eachother.

Consequently, when we have two parallels, AB, CD, and two other right lines AC, MN, parallel to each other, but perpendicular to the two former, the portions of the two first right lines, comprised within the second lines, are equal to each other; and the portions of the two latter, comprised within the former, are also equal to each other.

Application to rail-roads:—Rail-roads are constructed of two *rails* or wheel-tracks, which may be either grooves below, or rods above the surface of the road, but fixed perfectly straight and being perfectly united.; in which or on which the wheels of carts or waggons ought to move with precision, the two wheels on the right side moving on the right hand rail, and the wheels on the left side moving on the left hand rail When one of these rails or grooves is straight, the other ought to be at a distance from it, equal at every point, to the wheels of the same axle-tree. The two rails or grooves are, therefore, parallel. Rail-roads possess many advantages over the roads now generally in use, and are much more economical for the conveyance of goods. For heavy goods, which require to be sent expeditiously, they are even, in some cases, cheaper than canals, or other water conveyance. They derive most of these advantages from the wheel-ways having no inequalities, and from their geometrical properties, being rectilinear and parallel.

Let us suppose that the line CD, fig. 7, is made to approach towards AB, while it never ceases to be perpendicular to AC; it will be always parallel to AB, to which all its parts will equally approximate.

The equality of distance in all their points, presented by parallel lines, when they approach to or recede from each other, is of considerable importance in mechanics. For example, it is applied to the movements of jennies and mules, for making cotton-thread.

Application of parallels to the movements of mules and jennies for spinning cotton:—Let the reader imagine a frame or sort of chariot moving in the direction CD, fig. 7, pl. 2, which can advance or recede parallel to AB, by means of castors or wheels of a very small diameter, moving in the two parallel wheel-tracks AC, MN. The cotton threads extend from AM, where they are placed at equal distances from each other, and proceed to bobbins, also placed at equal distances, in the direction of the right line CN, on which they are rolled. When the chariot CN moves towards AM, the distance of each point of CN from the right line AM, is equally diminished; and, in consequence, all the threads are equally rolled on the bobbins,—they turning round with a rapidity proportioned to the motion of the frame,—without ever ceasing to be all equally extended. When the chariot returns towards CN from AM, the threads are all equally lengthened. Thus the principle of the moving part of those useful machines for spinning cotton, which are now well known throughout Europe, and are so common in England, is that of the *equality of parallels comprised between parallels.* Besides having the great advantage of spinning 40, 50, 60, or even many more threads by the movement of one frame or chariot, all the threads are spun of an equal thickness; which could never have been effected in spinning each one separately, and without the geometrical means here brought into notice.

Hitherto we have only compared parallels with perpendiculars, let us now compare them with oblique lines. Draw AB, CD, fig. 8, pl. 2, oblique with respect to EACF; if the two angles, EAB, ECD, called corresponding angles, are equal, the two right lines, AB, CD, are parallel.

If they are not parallel, by producing them they will meet at some point or other, either above or below EACF: let us see if this be possible.

Produce BA and DC to *b* and *d*, and take the figure BACD, which turn upside down, so that A is placed where C, and C where A now is.

But the angle BAF, which equals EA*b*, is equal to DCF, which equals EC*d*; the side AB, therefore, when the figure is reversed, will place itself on C*d*, and the side CD, will be placed on A*b*. If, therefore, the two lines *b*AB, *d*CD, were to meet at any point on one side of AC, it would be necessary that they should meet at a second point, on the other side of AC; but this is impossible, for there would then be two right lines, which would meet each other in two points.

Thus it is an invariable rule, when two right lines, *b*AB, *d*CD, forming acute equal angles, a, a', a'', a''', with the oblique line EACF, and consequently the obtuse equal angles, o, o', o'', o''', these lines are parallel.

The converse is equally true; that is to say, when the lines are parallel every oblique line intersects them, so as to form with them four acute angles, equal to one another, and four obtuse angles, also equal to one another.

To convince the student of this, it is only necessary to observe that the right line *d*CD, fig. 8, pl. 2, drawn from the point C, so that the angles a'', and a''', are equal to a; and a', is parallel to *b*AB. More than one line parallel to *b*AB, cannot be drawn from the point C, it is therefore the right line by which a, a', a'', a'', are equal, as well as o, o', o'', o'''.

In the arts, when it is necessary to draw one right line parallel to another, these two properties of parallel lines are frequently had recourse to.

For this purpose a ruler, and a triangular instrument *xyz*, fig. 9, pl. 2, made of wood, of glass, or of metal, are employed by the artist. This triangle has two sides *xz*, *yz*, which are at right angles; and is therefore called the draftsman's square.

Let it be supposed that it is required to draw through the point A, fig. 9, a right line parallel to CD. We begin by placing the square *xyz*, so that one of its sides, *xy*, lies precisely in the direction CD. Then the ruler is placed against the side *xy* of the instrument, and being either held very firmly on the surface on which the parallel is to be drawn; or fixed in its place by weights; the square is then moved by the other hand along the ruler, till the side *xy* comes as near as possible to the point A, taking into consideration the thickness of the pencil or chalk with which the line is to be drawn. The line drawn along *xy*, will be necessarily parallel to CD, since the corresponding acute angles formed by the ruler and the two lines AB, CD, are equal to each other.

With the side *yz* of the square, we can at any time trace lines perpendicular to the ruler; which is much easier, than to draw perpendicular lines by means of two oblique lines with an equal inclination. For this purpose, however, the square must be correctly made, and nothing is more rare. Even in towns and cities, where the arts have been much improved, there are only a small number of persons who construct instruments sufficiently accurate for a good draftsman.*

* This remark of M. Dupin's cannot apply to England, though it is a proof of the state of the arts in France; and coincides with what has often been publicly related of them. Almost all the common necessary and *useful arts* are, it is said, a century behind in France, while *science* is fully as far advanced, if not further, than in England. It is at the same time necessary to caution the English workman, against buying incorrect instruments, for such there are: they cost little, but that little is thrown away.—TR.

Let us now examine the application of the properties just pointed out to the construction and movement of bodies.

Having a body of an invariable form, ABCD, fig. 10, pl. 2, let us suppose that it is moved forward in such a manner, that all its points situated in the right line A*mnp* move in the direction of A*mnpa*: every other point, B, or C, or D, of the figure ABCD will also move along a right line, B*b*, C*c*, D*d*, parallel to A*a*. In fact the body not changing its form during the motion, each point, B, C, D, always remains at the same distance from the right line A*a*; consequently it describes a right line parallel to A*mnpa*.

This geometrical property is very often made use of by workmen of all descriptions.

The movement of drawers in their respective frames is on this principle. The drawers, fig. 11, pl. 2, of tables, bureaus, cupboards, or other furniture, are guided in their movement by a frame, of which the right-angled joinings represent so many parallel right lines, A*a*, B*b*, D*d*, C*c*. When the drawer is pushed in or drawn out, if the piece of furniture is well made, that is to say if the parallelism in every part is exactly preserved, the drawer is adjusted at every point, as it moves backwards and forwards, and is no where impeded in its motion. The parallels being always comprised between the same parallels, and being in consequence equal, represent the distance of the various points of such a drawer in its various positions.

Artisans have frequently occasion to draw a line in the middle, between two other lines, and parallel to them: for this purpose, as well as for many others, they will find the following instrument, the invention of Mr. Palmer, of great use.

In fig. 12, pl. 2, A is a square bar of hard wood, having the two sliding cheeks B D, fixed tight to it. The cheek B is fixed on one end of the bar, while the cheek D slides upon it, but may be made fast at any required point by the thumb-screw C. At the cheek B, a common scribing point is fixed in the bar, and with this, and the sliding cheek D, it forms the common gage, and may be used to draw lines parallel to the edge of any piece of work. With the addition of F and G, two brass arms of equal length, this instrument bisects any solid, and draws a line along its centre. One end of each is centered in the two sliding cheeks parallel to A; the other ends are united by the screw H, which it formed into a sharp point beneath, to mark with. It is evident from what has been said of obliques and parallels, that this point will always be in the centre, between the two cheeks BD. When the two sides of the solid to be marked are not parallel, unscrew C, and as the gage is drawn along, press the two cheeks B, D, towards each other, so that they may always be in contact with the sides of the solid, and the point H will pass along the centre of the solid, as correctly as if its sides were parallel. Because the obliques F, G, are of equal length, the scribing point always preserves the same distance from the two cheeks B and D, both of which are always in contact with the two sides. When the cheeks B, D, are brought together, the two arms F G, lodge in grooves made in the cheeks to receive them.

Application to the movement of pistons in pumps:—The above explanation gives us to understand how a piston, which fits exactly into the barrel of a pump, the figure of which is represented by parallel right lines, moves in it with precision, meeting no obstacle, and yet not being loose, when the barrel of the pump and the piston are made correctly. As the piston ascends and descends alternately, each point in its outline describes a right line parallel to the axis of the barrel; and all the parallels thus described must be placed exactly on the inner surface of the barrel. In making the pistons of steam-engines, in particular, the least defect in parallelism and the least deviation, produces serious inconveniences and a great loss of power.

Application of this principle in stretching the warp and in weaving.—To prepare the warp, a certain number of threads are first of all extended parallel to each other, one end of each being fastened to a piece of list, and the other rolled up round a piece of wood. The part unrolled from the wood forms a series of right lines parallel to each other, and placed in the same plane. In order that the cloth to be wove may not be too loose in one place and too tight in another, an instrument is used called a comb, or reed, which is made of very thin and straight pieces, kept at equal distances from, and parallel to, one another by an appropriate edging. One of the threads of the warp is passed through each of the intervals between the teeth of the comb, which keeps them at equal distances from one another. By this double system of parallel right lines, one of which serves to regulate the other, when the comb is made very correctly, the weaver is enabled to make his web of great length, as well as breadth, and perfectly equal in all its parts.

The fineness and beauty of the celebrated cachemire shawls, made by the Indians, are well known, and are truly wonderful. But, as they have not the same accurate instruments as the Europeans, for preserving the parallelism and equal distances of the threads, it is not possible for them to weave shawls, which for the uniform nature of the texture can be compared to those made in Europe; although the weavers of Europe have not yet been engaged above a quarter of a century in this, to them, new branch of industry.

It is of some consequence to make the reader aware that the superiority obtained by the Europeans, in an art long practised and carried to great perfection in India, has been produced by an approach to the precision of ideal geometry in the parallelism of right lines, which, in this case, are represented by very fine threads.

There will be frequent opportunities for remarking a similar fact; and it will be seen that the progress of industry in most arts, requires the precision and rigour of geometrical conception and construction to be introduced into the workshop. This it is, and it cannot be too often repeated, which makes it more than ever necessary for workmen to be perfectly acquainted with geometry and its application to the arts.

The properties of parallel lines are very often employed to make another body or figure exactly equal to some given figure.

Let us suppose for example, that we had to make a figure *abcd*, fig. 12, pl. 2, which should be exactly equal to ABCD, already made. Draw B*b*, C*c*, D*d*, equal and parallel to A*a*, and then draw the lines *ab*, *bc*, *cd*, *da*; they will, necessarily, be equal and parallel to AB, BC, CD, DA, and the two figures will be equal.

This principle is applied both in civil and naval architecture.—When it is necessary to make a piece of wood, stone, or iron, with a projection, which is to fit exactly into a hollow or cavity, the property of parallel lines, just mentioned, is called into use. Let us suppose, for example, that, in the opening represented by ABCDEF, fig. 13, pl. 2, it is required to fit the piece of wood XY, it will be sufficient to draw the right lines, A*a*, B*b*, C*c*, D*d*, E*e*, F*f*, from the points ABCDEF, equal and parallel; then trace the line *abcdef*, and shape XY according to this line.

These means are employed to make, with the help of thin planks, the moulds or drafts of the principal parts or pieces out of which a ship is to be built according to some given plan. Shipwrights give the name of moulding to this operation. By its accuracy, derived from the properties of parallel lines, the vessel is constructed to a great nicety, of the same form and shape as she was originally planned in the conception of the naval architect.

The solidity of the vessel also, depends on the precision with which the same process is applied, in adapting all the hollow and projecting pieces, such as those represented in fig. 14, pl. 2, which are to fit into one another. As they are closely fitted, the possibility of their moving one within the other, when the vessel is exposed to a heavy sea, is lessened; and this movement or play, as it is called, is, as we shall hereafter see, one of the most dangerous causes of destruction.

Method of projection.—Application of parallel lines to *plan-drawing, and to descriptive Geometry.*—I have mentioned the method of constructing one figure equal to another by the use of parallel lines; and the same means are employed as a general method of representing or of describing bodies. This is the object of designing in descriptive geometry.

The object to be represented, is transferred to a plane, called the plane of projection, such as a table, a plank, an extended sheet of paper, &c. From every point of the object itself, lines are drawn parallel to any direction, which may be chosen on account of its convenience. It will be readily conceived, that each point of the body represented may thus be transferred to the plane of projection, by following the parallel direction which has been chosen. The new position of the point on the plane is the projection of the point.

If all the points of a right line, or of a curve, are thus projected, they will form, on the plane of projection, a new right line or a curve, which will be the projection of the primitive right line or curve.

This species of projection, or design, is made use of to represent objects in civil, military, or naval architecture, in carpentry, in sculpture, in making plans for machines, &c.

A single representation of objects is not enough to determine exactly their form and size. For this purpose we must have two plans; and it is found convenient to suppose one of them horizontal and the other vertical. To make the vertical plan, the object to be represented is projected by parallel lines which are horizontal; to make the horizontal plan, the object is projected by parallel lines which are vertical.

The horizontal projection is, properly speaking, the plan of an object; the vertical projection is called the elevation of an object.

The student should be thoroughly aware of the importance, and of the necessity of knowing and practising with great exactness the art of linear projection. He should be able to draw both the plan and elevation of every object of art which is either to be represented or made. To the products of every art an exact and precise form should be given, either according to models—or according to some rules and dimensions settled beforehand.

In the course of these lessons, the student will be informed of the proper mode of proceeding in the principal cases he is likely to meet with in practice, but this instruction will not be sufficient. The artisan should, in addition, have recourse to a master, to teach him the art of projection in its greatest extent, with all its methods and all its resources.

Application of projection in Mechanics.—Parallel lines and perpendicular lines serve in the art of projection, not only to represent the form of a body supposed to be immoveable at any given moment, but also to represent the path which each one of its points follows, or ought to follow, whenever it is set in motion. This application of geometry is therefore of great importance in mechanics. It enables us to represent by lines what is not in fact permanent; and to fix in a durable manner the trace of those movements, the nature of which is, to disappear at the very moment of their existence.

Let us suppose, for example, that I fire a musket or a cannon ball towards any given point. The centre of the ball describes a certain line, which is not marked either before or after the passage of the ball; but which may, however, be represented on a surface, either as it was, or as it will be. Such a representation is of great use in many cases; as for example, to calculate the effect of firing from a battery against a fortification. According as this line, directed to the highest part of the glacis of a fortification, enters within the place where its defenders are situated, or passes above it, at such a distance as not to touch them, the battery will be well or ill placed for the assailants, and there will or will not be any danger for the besieged. The line described by the centre of the ball is projected on a plane, on which are marked in relief the respective positions of the battery and the fortifications; and the military engineer is thus enabled to form a correct opinion of what may be expected or dreaded from the effects of the battery.

The succession of points passed through by the centre of the moon in her course round the earth,—by the centre of the earth and the other planets in their course round the sun, and the succession of points passed through by the centre of comets, are also all represented on planes by lines. To acquire a correct knowledge of the lines thus described by the several planets, comets, &c. which compose our system, required many thousand years; and it now forms one of the most noble discoveries ever perfected by the genius of man, and by the observations of several successive generations.

The machines intended to supply the wants of society and to assist the operations of man, are all constructed on the supposition that certain parts of them will perform certain determinate motions and no others. It is not enough therefore to represent the parts of any machine in any one particular position; the motion of these parts must also be represented. By employing the method of projection, by parallel and perpendicular lines, this also is effected. We can thus give an exact account beforehand of the effects which will be produced by the different parts of any machine, when they are put in motion.

Already we see a number of important applications, which may be made of parallel and perpendicular lines, simple and easy as the theory of them appears. By them we represent, and by them we fashion objects of almost every form, such as furniture, buildings, and machines; by them too we represent not only the forms of bodies, but all their movements. There is, therefore, perhaps no part of the science with which the student should be more familiarly acquainted.

One of the most useful applications of parallel lines is that for reducing to the standard of parallel right lines the figure of curved lines, or of measuring curved lines by them.

Let any curved line, MABCDN, fig. 15, pl. 2, be given; it is transferred to a principal right line or axis, *m*, *n*, by means of a succession of other parallel right lines A*a*, B*b*, C*c*, D*d*, &c. In general the latter are drawn at equal distances from one another.

Application to drawing curve lines:—The advantage of this geometrical property is, that it permits us to write and calculate, if this mode of expression be allowable, the form of the least regular curves. We have an example of this in ship-building.

The fleetness with which a vessel moves through the water, depends, other things being equal, on the form of her bottom or part beneath the water, which meets resistance from the fluid. It is necessary that the surface of this part should be closely connected, and smooth at every point, offering, whatever may be the plan of the architect, no sudden irregularities in any direction. To plan and construct a ship the most rigorous and exact geometrical principles are accordingly adopted; and parallel and perpendicular lines are in this case generally had recourse to.

Every vessel has its right, or starboard side, precisely the same as its left, or larboard side; or both sides, though each is a very irregular curve, have exactly the same form and dimensions, and are equal and similar throughout. To represent them a horizontal line is drawn, MN, fig. 16, pl. 2, which reaches from the stern-post to the stem. On this right line, divided into equal parts, MA, AB, BC, perpendiculars are erected; and on them are marked the points which indicate the extreme breadth of the ship or the situation of the water lines.

It is supposed that the floating vessel, without leaning either to one side or the other, sinks gradually down, and the line formed by the water on her surface, as she sinks, is marked at every foot or other convenient distance. These lines are called water lines, and on their smoothness, if I may so speak, or their gradual continuity, offering no abruptness nor interruption, either in the fore and aft, or horizontal, or in the perpendicular direction, depends, above all, the beauty of form in a ship. There are certain dimensions, or relations of length, breadth, and depth, which are proper for every description of vessel, as she is intended chiefly, to sail fast, to carry a great burden, or to navigate in deep or shallow waters; but these dimensions being observed, so as to make the vessel answer the principal intention,—for all vessels must be able both to sail at a certain rate and to carry a certain quantity of stores,—the excellence of her form depends on its smoothness. The curves which are to constitute the water lines are determined by half breadths, marked to the right and left of the axis or keel, on parallel lines. When those half breadths are numbered for every parallel, and for every water line, and they may be made as numerous as the builder pleases, the plan on which the ship is to be built may be carried into execution with the greatest nicety.*

* It would be difficult to find a more striking example of the utility of the application of the mathematical sciences to the practical arts, than is to be found in the success of the French nation in ship-building. They are not a maritime people. One of their ambitious sovereigns, however, resolved to make them so, and employed men of science to build ships. He and the subsequent sovereigns of France encouraged them in ascertaining mathematically the best form for ships, and in applying the mathematical sciences to their construction. The consequence has been that the French ships, particularly of their royal navy, are in general equal, if not superior, as to form, to any other ships of the whole globe. We are a maritime people, possessing a more extensive sea-coast, and more familiar with the ocean than any other nation. In the practical and merely manual part of building ships, as well as in managing them, we are superior to our neighbours. That we in general overtook and captured the finer-formed vessels of the French, was a consequence of the superior skill of our sailors; but the superiority of those vessels, as to form, was so great, that most of the ships at present in our navy have been modelled after captured French ships. Now this superiority was altogether derived from the plan of constructing their ships on mathematical principles. Such is, however, now the progress of scientific instruction in this country, that there is every rea-

We have another example of the application of these lines in the construction of roads and canals. From the line MN, fig. 17, pl. 2. taken as an axis,—it being, for example, the level of the water of the canal, or any other line parallel to this level,—draw the perpendiculars, A*a*, B*b*, C*c*, to the ground, the figure of which is determined by the curve line passing through the points *a*, *b*, *c*, *d*. To determine the heights M*m*, A*a*, B*b*, C*c*, an instrument to be hereafter described, when treating of hydraulic machines, called a theodolite or a level, is used.

Afterwards, what are called cross sections or profiles are formed, by drawing from the points A, B, C, D, horizontal lines, at right angles with M, N; and taking each of these horizontal lines for a new axis, perpendiculars are let fall from it to the surface of the earth. Their length is measured, and a figure is formed for each new axis with the perpendiculars, and that curve of the earth which corresponds to them.

These operations are indispensable, to know exactly the quantity of ground which must be dug away, in those parts which are too elevated, in order to carry it to the spots which are too low, and thus transform the primitive shape of the surface, into that which is proper for the road or canal to be constructed.

When it is required to determine exactly the form of the bottom of a lake, a river, a pool, or a roadstead, the surface of the water, or what represents it, is divided by two series of horizontal and parallel lines equally distant; those of one series being at right angles with those of the other. Perpendiculars are then let fall from the points where the parallels drawn in one direction, are intersected by the parallels drawn in another direction, down to the bottom of the river or lake; if we then connect by curve lines the lower extremities of the perpendiculars, thus let fall from the same horizontal line, we shall form a profile

son to hope, on this point, as on others involving not contention, but generous emulation, that we shall not be surpassed by our enlightened rivals. Tr.

of the bottom of the roadstead, river, lake, or harbour By letting fall a perpendicular from every point, and connecting all the extremities by curve lines in both directions, we may obtain a number of profiles sufficient to give us an idea of the form of the bottom.

In practice, when it is required to survey any river or roadstead, certain points are selected, both on different sides of the river and in the direction of its course: a boat is then rowed from point to point in both directions, describing, in fact, the horizontal lines just mentioned, on the surface of the water. As the boat moves onward, or frequently stopping in its progress for the purpose, a skilful leadsman keeps continually sounding, or letting fall, by a line and a lead, a perpendicular from the surface of the water to the bottom of the river. The depth is marked at every spot, and an accurate register kept of every sounding. By laying down these various depths on paper, making proper allowance for the rise and fall of the tide, the motion of the boat, and numberless other things which practice teaches, it is plain that we can draw a tolerably correct representation of the bottom of a river, or a part of the sea shore, or even of the great ocean itself, as far as the bottom can be reached by a plumb line. The more accurately we can determine the situation of the floating boat, and the more numerous we make the soundings, the more accurate is the knowledge we acquire of the surface of the earth, concealed from our view by the water. That knowledge does not in fact extend beyond the number of points precisely ascertained; but its importance is very great to the art of navigation. By pursuing the geometrical method here pointed out, we have acquired an accurate knowledge of the figure of a large portion of the earth which is continually covered with water. Not only does that knowledge enable the hardy navigator fearlessly to venture close to the land, almost at times rubbing the side of his vessel against the rocks, or stirring up the mud with her keel, but it enables him, when no land is in view,—when neither the sun nor the stars are to be seen, to ascertain, in many cases with tolerable precision, the situation of his vessel. He determines by his soundings, over what part of that hidden surface, the situation of which has been previously ascertained, his vessel is at any moment floating.

In place of using these means of representing the form of a portion of land, covered or not covered by water, it is sometimes preferred to employ curve lines, of such a nature that the vertical heights are equal for each curve; forming subsequently a series of horizontal curves. Generally it is assumed that the curves, which succeed each other, are at equal distances, measuring the distance vertically; consequently in vertical projection or in elevation, the horizontal sections are all represented by parallels equally distant from one another. This manner of representing a portion of the land, has the great advantage of showing to the eye on a plane, such as a sheet of paper, the complete form of a portion of the earth's surface in its various points.

To ascertain this form is useful, not only in hydrography, or in the description of places covered constantly or occasionally with water; it also serves in topography, to describe with precision and minuteness, and in detail, the parts of valleys, mountains, &c. It is very frequently employed both by the military and civil engineer, in planning roads, bridges, and fortifications.

When an aqueduct or a bridge is to be constructed, the piles employed, being all of the height of some line of level previously determined, MN, fig. 18, pl. 2, this line is divided, generally, into equal parts, MA, AB, BC, CD. From each point of the division the perpendiculars A*a*, B*b*, C*c*, D*d*, are let fall, and extended till they reach the earth; and they represent the height or length which the piles of the bridge or aqueduct must be of.

I shall not dilate at greater length, on the innumerable applications which may be made of this method of representing the forms of space, by the assistance of parallels. The student must be sensible of its importance; and of the facility and rapidity with which it may be applied. He should familiarise himself with it by frequent exercise; and by representing with the rigour of geometry, many objects, referring them to some axis and some parallels. This species of mathematical representation or design, must gradually become common in every sort of workshop.

The works, in French, of M. Francœur on *Linear Design*, of M. Lacroix on *Plane and Curved Surfaces*, and of M. Monge, on Descriptive Geometry, may be consulted with advantage. MM. Hachette and Vallée have also published very good treatises on this subject, containing some excellent matter, not to be found in other works. M. Francœur's book has been translated into English; and the student may also study with advantage Ferguson's *Art of Drawing in Perspective*, and other similar works.

THIRD LESSON.

On the Circle.

A circle is a plane surface, of which the boundary called the *circumference* is, in all its points, equally distant from a single point called the *centre.*

All the right lines drawn from the centre to the circumference, measuring equal distances, are equal to each other. These lines are called radii; and thus all the radii of a circle are equal to one another.

When two radii are directly opposite to each other, the one to the right, the other to the left of the centre, the single right line which they form is called the diameter of the circle.

Thus, in the circle ABDE, fig. 1, plate 3, C being the centre, CA, CB, CD, CE, are radii, all being equal to one another. If the two radii CA, CD, form a right line ACD, this line is a diameter of the circle.

Every diameter DA, fig. 1, pl. 3, divides the circle into two equal parts.

To be convinced of this it is only necessary to double the part DAB over the part DAE, turning it on the diameter DA, as on a hinge. If any point of the circumference DAB fell within any point of the circumference DAE, it would be nearer the centre than this point; if any point in DAB fell outside of any point in DAE it would be further from the centre. But this cannot be the case, for all the points of the circumference ABDEA are equally distant from the centre. The part DBA will therefore fall in every point on the part DAE, and the two portions of the circle separated by the diameter DA are equal to each other.

Every right line, *mn*, fig. 2, pl. 3, terminating at both ends in the circumference of a circle, is called a *chord* (cord). Every portion of the circumference of a circle *mqn*, is called an *arc* (bow). The chord is also called the subtense of an arc. The part *pq* of the radius *Cpq* included between the chord and the *arc*, and perpendicular to the chord is called the *sine* (arrow).

These names are borrowed from the use among the ancients of a piece of wood bent, by a piece of cord, nearly into the form of a portion of the circumference of the circle fig. 3, pl. 3, which they called an *arc*, and which was intended to propel the arrows *(sines)* placed on the middle of the *chord*, and in a direction perpendicular to it. This is one instance of practice having preceded science and supplied it with terms.

The radius *Cpq*, fig. 2, perpendicular to the chord *mn*, divides both the chord and the arc into two equal parts.

Let us draw the radii *Cm*, *Cn*, which are oblique lines forming equal angles with the perpendicular *Cp*. Therefore, *first*, *mp* = *np*. The chords *mq nq* are also oblique lines equal to one another, and if *Cqn* is folded on *Cqm*, the point *n* will fall on the point *m*, and the arc *nsq* on the arc *mrq*; for no point of the former arc can fall either within or without the latter, unless it be nearer to or further from the centre C. Therefore, *secondly*, the two arcs *mrq*, *nsq* are equal.

Application to linear design.—The property of the circle just demonstrated is very usefully applied in the art of design, and in most of the arts in which exact measures are to be taken and combined together.

It serves in the first place to divide an arc of a circle *mqn*, fig. 4, pl. 3, in two equal parts. Take a pair of compasses and open them more than the half of *mn*; placing one leg of the compasses in *m*, with the other describe an arc of a circle *rst*; then, fixing one point in *n*, describe another arc *vsu*, taking care that the compasses are neither opened nor closed during the operation. The point *s*, where these two arcs intersect each other, will be equally distant from *m* and *n*; and therefore it will fall on a line perpendicular to the right line *mn*, which passes through the middle of this line, and through the centre of the circle. This perpendicular will divide the chord *mn*, as well as the arc *mqn*, into two equal parts.

If the exact position of the centre is not known, draw on the side.

of the above-mentioned centre two arcs, *abc*, *dbe*, with the same opening of the compasses, the one having *m* for its centre and the other *n*; the point *b*, like the point *s* will be on the perpendicular, which divides into two equal parts the chord *mn* and its arc *mqn*.

By these means, and knowing the position of only three points. *m*, *n*, *o*, fig. 5, pl. 3, in the circumference of a circle. we can determine the place of its centre, the length of the radii, and consequently we can describe the whole circumference.

For this purpose, *first* draw, in the manner just pointed out, *qn* through the middle of *mn*, and perpendicular to it: *second*, draw *rb* through the middle of *no*, and perpendicular to it. From the point C. where the perpendiculars C*q*, C*r*, intersect each other, draw the obliques, C*m*, C*n*. C*o*, they will all be equal lines. C*m*, C*n*. C*o*, therefore, will be three radii of the circle sought, of which C will be the centre.

When the chords AB, DE, FG, fig. 6, pl. 3, of a circle are parallel the arcs AD and BE, DF and EG, contained between them, are equal to one another.

To demonstrate this proposition draw from the centre C the radius C*lmnp* perpendicular to all the chords: it will divide each of them into two equa parts. Comparing the length of the arcs which correspond with these chords we find that the arc *p*A is equal to *p*B, *p*D to *p*E, pF to *p*G, which requires that the arc AD should be equal to BE, and DF to EG.

A right line X*p*Y, fig. 6, perpendicular to C*p*, and drawn from its extremity, is wholly outside of the circle, which it only touches at the point *p*. This line is the tangent of the circle; and no other right line, at the point *p*, can pass between the circle and the tangent X*p*Y.

The radius being in fact perpendicular to the right line, X*p*Y, the termination *p* of the perpendicular is nearer the centre C, placed on this perpendicular than any other point X or Y. For the distance of any other point XY from the point C will be measured by an oblique line, necessarily longer than the perpendicular C*p*: consequently, all the points o X*p*Y, except *p*, are beyond the circle.

In the arts we find frequent use made of the properties of the circle in relation to right lines, which form tangents to it.

The circle, for example, may be made to turn round its own centre C, supposed to be fixed. Supposing the tangent XY, also to remain fixed during the movement; *first* the circle will never pass beyond XY; *secondly*, it will always touch XY in one point, *p*, placed at a distance from the centre C, equal to the radius C*p*. Consequently, when a fixed right line touches a circle in a single point, if the circle be fixed on an axis, it may be turned round without any necessity ever to make an effort to remove it from the right line, or to displace this right line.

Application to turning, in which a moveable body is exposed to a fixed instrument.—The turner uses this geometrical property in order to cut a plane surface into a circular form. He makes the plane turn round the fixed point C, as the centre of the circle. He places a cutting instrument in the direction of the tangent XY which acting at the point *p*, every part of the plane separated by the instrument, is at a greater distance from the centre C than the length C*p*. All the points of the surface of the piece thus cut are at the same distance, C*p* from the centre, and thus the form given is that of a circle.

Application to grinding and polishing stones.—The same geometrical property may be traced in the use of grindstones, and of stones for polishing the rectilinear surfaces of the products of art. The instrument to be ground on the surface to be polished is held, either by the hand or by some apparatus for this purpose, against a stone of a circular form. If the centre of the stone has been well ascertained, and the circumference is very exact, when it is turned, its surface always remains in contact with the object to be ground or polished.

No other figure but the circle has this property; and other figures, on being turned round, would sometimes be at a distance from the objects held in a fixed position, and would sometimes strike against them or force them to a distance.

In place of supposing the circle moveable, and the tangent XY fixed, we may suppose the circle fixed, and the tangent XY moveable; and, placing the tangent so that it shall always be kept at a distance from the centre C equal to the radius, it will continue to touch the circumference of the circle.

This principle is applied to make bodies circular which remain fixed; but in this case the instrument is made to revolve round the centre. One side of it is represented by the tangent XY, and the cutting part by the point *p*.

Application to wheel-carriages.—Let us suppose that the tangent XY remaining fixed, the circle is made to turn round above and on it, so that each point of the circumference is successively placed without sliding either forwards or backwards on a new point of the tangent; we shall have the movement of wheel carriages, which, it is hardly necessary to observe, is of the greatest importance in the arts.

In this movement the right line XY never ceases to be the tangent to a circle, for it always touches the circumference in only one point. Thus the centre of the circle will always remain distant from the right line XY, the exact length of the radius C*p*. When wheels roll, therefore, over a right line XY, the centre of the wheel moves in another right line parallel to the road or right line XY. Of course, if the right line is horizontal, the centre of the wheel moves also in a horizontal line.

If any other curved figure were made to roll on a horizontal line, one point, whether central or not, would rise and fall alternately; and the motion produced by a wheel of such a shape, would be neither regular nor agreeable. For this reason, the figure of a circle is given to all the wheels of carriages destined either to carry goods or passengers; and it may be added, that in proportion as they are true geometrical circles, the motion is equal, easy, and regular.

Application to parallel movements.—The property of the circle which we are at present considering, supplies the artisan with an easy method of making any point move parallel to a given right line. It is only necessary to fasten this point to the centre of a circle, and make it roll forward on its tangent.

Let us draw the line *xy*, fig. 6, pl. 3, parallel to XY, at a distance equal to two radii *Cp*, or to the diameter of the circle *pCq*. Then will *xy* touch the extremity *q* of the diameter *pq*, and like XY, it will be the tangent of the circle. If the circle is now made to turn round on X*p*Y, it will not cease also to touch *xqy*, because the distance of the two parallels is every where the same.

Application to the construction of machines.—When it is required to cause a ruler or rectilinear frame of any description to move exactly parallel to some given right line, wheels or castors are employed of an equal diameter, and truly circular, which are placed between the right line or plane serving as the basis, and the ruler or frame to be moved. It is then only necessary to pull or push the frame or the ruler, on the wheels or castors, according to the nature of the machine of which they are to form a part.

By attending to these principles, we see how much geometry may contribute to enable every description of artisan to form right lines by means of circles, and circles by means of right lines; to produce rectilinear motion by circular movements, and circular movements by rectilinear motion. Workmen themselves will know how to apply these principles, on numerous occasions, and it is, therefore, only necessary here to point them out, leaving the application to the skilful hand of the artist. In teaching, however, instructors would do well, after making the student comprehend the extent of these principles, to supply him with more examples of their application to the business of life than have yet been furnished.

Having examined the properties of circles with relation to right lines, let us now proceed to examine the properties of circles in relation to one another.

Let us suppose that two circles AB, fig. 7, pl. 3, are placed in such a manner that the distance AB, between their centres is equal to AO plus BO, or to the sum of their radii. The point O, it is evident, is on the circumference of both, and no other point, P, can be at the same time on both these circumferences.

If we draw the right lines AP, BP, the right line AO, plus BO, will necessarily be shorter than the broken line AP, plus BP, whence AP, and BP, cannot be equal to, and must be greater than the radii, AO, and BO.

The two circles are consequently tangents to each other.

Application in transmitting the rotatory motion from one axis to another.—The first circle, fig. 7, may be made to revolve without ceasing to touch the second circle, whether it be fixed, or turning either in the same direction as the former, or in an opposite direction. In this movement, the two circles will continually touch, without either intruding on the other.

In the arts we very often find one wheel put in motion, by another on this principle; either by the friction of the circumference of the two wheels; or by teeth of an equal size placed at equal distances on both wheels and working one within the other. It must however be remembered, that if one of the wheels turns from left to right, the other will turn from right to left, or the two will move in contrary directions. In fig. 7, pl. 3, this contrary motion is designated by arrows.

If three circles, A, B, C, fig. 7, pl. 3, are in contact, so that the first turns the second, and the second the third; the second turning in a different direction from the first, and the third contrary to the second, the first and third will move in the same direction. It is necessary, therefore to have three circles in contact to transmit a circular motion in the same direction, from one centre to another.

Bands thrown around circles.—When it is necessary to transmit a circular movement to a considerable distance, in place of employing very large circles or wheels, or a great many of them, two only are used, and an endless cord is passed around both. The band may either not be crossed, as in fig. 8, pl. 3, or crossed, as in fig. 9. The bands are so put on, that the parts of them, *mn*, *pq*, which are not in contact with the two wheels, are in a right line. Either of the two wheels may be made to turn, without any change taking place in the length or the direction of the circular parts of the band, *pAm*, and *qBn*; or in the length and direction of the parts, *mn*, *pq*, that are in a right line. If, at the commencement, therefore, the adhesion of the band to the circumference is sufficiently great, when motion is given to a circle or wheel, to go round with it and transmit the motion to the second circle, it will, as long as the first is moved, and without any difficulty, continue to transmit its motion in the same manner.

If, in consequence of use, or of changes in the heat and moisture of the atmosphere, the band stretches and becomes too long, a third wheel, D, fig. 10, pl. 3, is employed, which, disturbing the rectilinear part, *pq*, brings it in the direction *pr*, *rq*, and keeps it tight, notwithstanding its elongation. For this purpose, the difference of length between the right line, *pq*, and the bent line *prq*, must be equal to the elongation of the band. This method is frequently practiced in the construction of machines.

It must be carefully remarked of these bands, that their effects are different as they are crossed or not. With the bands crossed, fig. 9, pl. 3, the two circles move in an opposite direction; but when the bands are not crossed, as in fig. 8 and 10, the wheels turn in the same direction.

These are only a few examples of the movements of machines having the form of circles and right lines; in the course of these lessons, many more applications of the combined movement of right lines and circles will be pointed out.

Of the motion of one circle within another.—If we cut a circle out of a plane surface, we shall have for the part cut out, a circle in projection, and in the rest of the plane, a circumference which is sunk or hollow. Let us suppose that the projecting circle is made to revolve on its centre, all the points in its circumference always remaining at the same distance from its centre, will be always in contact with some point in the circumference of the sunk circle. The circumference in relief will, in fact as it revolves, touch in all its points the sunk circumference.

The circle is the only figure which possesses this property. Every other figure so cut out of a plane surface, and made to turn round an axis, will have some of the points in its outline, farther from, or nearer to, the central point, and these parts sometimes projecting beyond the parts of a corresponding sunk figure on the plane, and sometimes not extending to them, will leave a void space betwixt the figure in relief, and the hollow figure.

Whenever it is necessary, therefore, to close up any portion of a plain surface, while a certain part of the surface must revolve on its own axis, this part must be made in the form of a circle. For this reason, valves, corks of bottles, of flasks, and a great variety of stoppers, are made circular.

Application to steam boxes.—In the construction of steam engines, an ingenious use is made of this property of the circle, always to touch the circumference of a box which incloses it, while it revolves on its own centre. In describing circular steam boxes, this use of circular figures will be more minutely explained.

Division of the circle, and application to measuring angles.—Before explaining this division, it is necessary to make the student acquainted with an essential principle.

If two arcs of a circle AMB, DNE, fig. 11, pl. 3 are equal to each other, the chords AB, DE, which belong to these arcs, will also be equal to each other.

To demonstrate this, let us place the arc, DNE on AMB, the point D being on the point A. The two arcs preserving the same centre will exactly cover each other, the point E, will fall on the point B, and the right line or chord, DE, will be precisely the same as the chord AB.

Reciprocally the chords AB, DE, being equal, if the second chord is placed on the first, the two arcs AMB, DNE, will correspond through their whole extent, and of course will be equal. Therefore, if in a circle, fig. 12, pl. 3, we draw a number of chords, AB, BC, CD, DE, ... all equal to one another, the arcs corresponding to these chords will also be equal to one another, and the circumference of the circle will be divided into as many equal parts as there have been chords drawn. We now proceed to describe,

The most simple means of dividing the circle.

1st. Into *two* equal parts. It is only necessary to draw through the centre the diameter AB; fig. 13, pl. 3.

2d. Into *three* equal parts. The circle must be divided into six equal parts, and lines drawn from every second point of the division, fig. 15, pl. 3.

3d. Into *four* equal parts. Draw a second diameter DE, fig. 13, pl. 3, perpendicular to the first AB.

This operation may be readily performed by taking a greater opening in the compasses than the radius of the circle, and describing from the point A, as a centre, with this opening as the radius, the two arcs, *mFn*, *pGq*; and from the point B, as a centre, the arcs *rFs*, *tGu*. The right line FDCEG, is the perpendicular required.

4th. Into *five* equal parts. The circle is divided into ten equal parts, and every second point only of the division is taken, fig. 14, pl. 3.

To divide the circle into ten equal parts, the radius AC, fig. 14, pl. 3, is divided into two unequal parts, AM, MC; so that the larger part, MC, contains the smaller part, AM, as many times repeated, as the radius itself contains the larger part MC. The larger part MC, will be a chord, which, repeated ten times, will extend 'round the whole circumference. The demonstration of this method, as well as that of the division of the circle into six equal parts, is derived from the properties of triangles.

5th. Into *six* equal parts, fig. 15, pl. 3. It is only necessary to take as the chord the radius of the circle.

A perpendicular, drawn through the middle of each chord, dividing its arc into equal parts, will enable the student to divide the circumference into *eight* equal parts; if the circle be already divided into *four* equal parts, fig. 13, pl. 3,—and into *twelve* equal parts, if already divided into *six* equal parts, fig. 15, pl. 3, and so on, with all subsequent divisions.

The fifteenth part of the circumference is equal to the sixth part, minus the tenth part.

These very simple operations being continually practised in constructing machines, and in various branches of industry, it is essential that artisans should become familiar with them, and comprehend the principles on which they depend.

Having pointed out the rigorous methods which geometry supplies for determining these problems, let us now refer to a method by approximation, which in many cases answers very well.

The radius of a circle being taken equal to 10,000, the following, neglecting fractions, is the length of the chord, which belongs to each portion of the circumference: thus the chord of the

Semi-circum is	20,000	Eighth . .	7654
Third part .	17,232	Ninth . .	6840
Fourth . .	14,145	Tenth . .	6180
Fifth . .	11,746	Eleventh .	5524
Sixth . .	10,000	Twelfth .	5176
Seventh . .	8,672		

This small table will render it very easy to find the opening of the compasses necessary to divide the circle into as many equal parts as is required, from the half of the circumference, to its twelfth part.

By the means mentioned above for finding the middle of an arc, the opening of the compasses may be immediately obtained, which corresponds.

to the 14th, 16th, 18th, 20th, 22nd, 24th, 28th part, &c. of the circumference,

or to half the 7th, 8th, 9th, 10th, 11th, 12th, 14th part, &c.

An easy method of dividing an arc into two equal parts, has been pointed out; but a method has long been sought, in vain, to divide an arc, into three equal parts, on strict geometrical principles.

Application of arcs to measure angles.—As angles may be enlarged or lessened, one of them may be taken as unity, and employed to measure or represent all other angles, expressing, in figures, the number of times they contain this angle, or its subdivisions; on the same principle as any extension may be taken as the standard for linear measure, as described in the first lesson.

In place of taking an angle itself, ABC, fig.16, pl. 3, for unity, it has been found more convenient to take the arc, AB, included between the sides of the angle, and described from the point C, as a centre.

It is easy to see, that if a succession of radii, CA, CB, CD, CE, be drawn at such distances that the angles ACB, BCD, DCE, are equal, they may be placed one over the other, and their *arcs*, AB, BD, DE, applying themselves to each other in every point, will also be equal.

If we take two, three, or four of the angles, equal to unity, to form a single angle, we must also take two, three, or four times the arc corresponding to them, to have the arc corresponding to the new angle. The same arithmetical figures or numbers, consequently, will represent the number of times which both the new angle and the new arc, whatever they may be, contain respectively, the unit, or the assumed measure of angles, and its corresponding arc.

Without changing these numbers, therefore, we may take at will, as a standard, either the angle or the arc; and in practice, it has been found more convenient to adopt the latter. The following is the method of proceeding.

The circle is divided into four equal parts, which gives, consequently, four quarters of the circumference, measuring the four right angles, which include all the space round the centre C.

Each fourth part or quarter is then divided into ninety equal parts, which are called degrees.

The circumference of the circle, therefore, contains four times 90 or 360 degrees. This division appears, at first view, rather capricious, and does not accord with the decimal division, or the division by 10, 100, 1000, 10000, &c. It has some advantages, however; the principal of which is, that it can be divided into a great number of equal parts, expressed in round numbers. Thus, the *semi-circumference* equals 180 degrees, the *third* part of the circumference, 120, the *fourth* part 90, the *fifth* part 72, the *sixth* part 60, the *eighth* part 45, the *tenth* part 36, the *twelfth* part 30, the *fifteenth* part 24, the *twentieth* part 18, the *twenty-fourth* 15, the *thirtieth* part 12, the *thirty-sixth* part 10, &c. Without carrying this division any further, the student will comprehend from this specimen, the advantages of the ancient manner of dividing the circle into 360 parts or degrees.

In order to measure the parts of an angle which are smaller than degrees, each degree is divided into 60 equal parts, called *minutes*.

For still more delicate measures, the *minute* is again divided into sixty equal parts, called *seconds*, the second is again divided into sixty *thirds* or into tenths, and the third into sixty *fourths*.

The following marks, placed above the figures, are employed to signify degrees $^{\circ}$, minutes $'$, seconds $''$, thirds $'''$, fourths $''''$. For example, fifteen degrees, forty-five minutes, fifty-three seconds, thirty-seven thirds, and twenty one fourths, are thus written, $15^{\circ}\ 45'\ 53''\ 37'''\ 21''''$.

The circumference of the circle contains 21,600 minutes, or 1,296,000 seconds, 77,760,000 thirds, and 4,665,600,000 fourths.

The second is less, therefore, than the millionth part of the circumference, and the fourth is not equal to the four billionth part.

Application to Geography.—Geographers have made an important application of this division of the circle to the measurement of the surface of the earth, which is usually expressed in degrees, minutes, seconds, &c.

They have ascertained that lines, drawn on the surface of the earth, from north to south, and from east to west, till they meet, are very nearly circles; and they have divided these circles into degrees, minutes, seconds, &c.

The length of these separate parts, according to the ancient division of the circle, still adopted in Great Britain, is as follows:—

The circumference of the earth, measured on a meridian or circle, passing through both poles, is

	40,000,000 metres,	43,745,410 yards.
1 degree equals .	111,111 —	
1 minute — .	1,852 —	
1 second — .	34 —	

1 third is half a metre and a fraction.*

According to the modern division of the circle, now adopted in France,

1 degree equals	100,000 metres.
1 minute —	1,000 —
1 second —	10 —
1 third . .	1 *decimetre*, or the tenth part of a metre.
1 fourth . .	1 *millemetre.*

The division of the circumference of the circle, says Mr. C. L. Bergery,† one of those persons who have already

* The French *metre* is equal to 39.371 English inches. The statement of M. Dupin has been made in round numbers, because this number of metres is the basis of the present system of French measures. The actual circumference of the globe, measured on the meridian of Paris, is, according to M. Malte Brun, 39,999,867 metres, giving as the length of the circumference in English yards, 43,745,410. A *degree* of the French modern circle is the 100th part of a quarter of a circle, a minute is the 100th part of this degree, a second the 100th part of the minute, &c.; consequently one centesimal, or modern French degree, is equal to 54′.

† In his "*Géométrie appliquée à l'Industrie,*" Mentz, 1826.

in France, followed the path marked out by M. Dupin, though perfectly arbitrary, is of great assistance in the arts and sciences. The greater number of instruments used to ascertain the figure of any portion of the earth, have a circle, or a part of a circle belonging to them, divided into degrees, which render the operations easy and exact. By means of graduated circles, we trace the course of the heavenly bodies, discover their movements, measure their distances and their size, and predict the moment of an eclipse, or the return of the comets. The needle of the mariner's compass, always directed towards the north, points out to the navigator, on a graduated circle, whether he pursues his proper course, or how much he departs from it. Geography only represents to us with so much precision the position and relative distances of all the points of the earth, by making use of two graduated circles, the degrees of one being longitude, and of the other latitude. It may, indeed, be said, that without the graduation of the circle, which, simple as it appears, was slowly invented, the arts and sciences, and consequently civilization, would be far inferior to their present state.

Application of the division of the circle to the construction of machines.—The division of the circle into equal parts, is an indispensable operation in many of the arts, and particularly in the construction of machines; as, for example, to form cogged and toothed wheels for communicating motion, and fluted cylinders, for the operations of spinning cotton, wool, &c. As cogged wheels are made with greater or less accuracy, the movements transmitted by them, are more or less smooth, and performed with more or less friction. In practice, a great deal of precision is, in fact, obtained: but it is only by making machinery with geometrical truth, that stoppages, jirks, loss of force, &c. &c., which are always the consequence of irregularity, and incorrectness of form, are avoided or prevented. The student and the practical mechanic, will both see, therefore, that it is of great importance, carefully to ascertain that the teeth of wheels, and the flutes of cylinders, which are to work into one another, divide the surface of the circle on which they are made, into parts exactly equal. Such exactness economises power, and gives beauty and regularity to the work performed by the machine, which cannot be equalled by the unassisted hand of man.

To measure angles.—A great number of instruments are in use, on which the division of the circle in degrees and minutes, is marked.

The most simple of these instruments is the protractor, which is a semi-circle, made of brass or horn, the circumference of which is divided into degrees. If it be made of brass, the space *mnp*C, fig. 17, pl. 3, is empty; or the instrument consists of a diameter and a circular rim. C, the centre, is sometimes indicated by a notch; and two other small notches, *m, p*, allow two other points of the diameter, traced on a plane surface to be seen, which is otherwise concealed by the side *m*C*p*, or the part of the instrument corresponding to the diameter. When the instrument is made of horn, there is no necessity for these notches, on account of the substance itself being semi-transparent, and thus allowing a design or figure to be seen through it.

The protractor serves to transfer angles from one position to another, by measuring their openings, as for example, that of XOY, fig. 17, the height and one side of the new angle being given.

If we have to draw a line, CY, which is to proceed from a given point, O, in CX, and include a certain angle, for example, an angle of 55°, with CX, we place the diameter *m*C*p* on CX, and the point C, on the point O. A point, H, corresponding to 55° of the protractor, is then marked on the paper, or other plane surface, and the right line, CHY, drawn through C and H, will make, with OX, an angle of 55°.

A protractor, somewhat different from the one described by Baron Dupin, has been constructed by Mr. R. Christie, secretary to the London Mechanics Institution, intended to lay down both the sides and angles of geometrical figures to any assigned dimensions, and to solve trigonometrical problems. The translator presents it to the notice of the reader, because protractors with a vernier are described in some books to be much wanted.

This instrument, fig. 19, pl. 3, consists of a semi-circular limb, the three sides of a triangle, and four verniers.

The limb is divided into degrees and minutes by the vernier B.

The radii or sides CA and CB, may be divided into any convenient number of equal parts, according to the size of the instrument: in one of about four inches radius, each of the sides is divided into 200, and subdivided by its respective vernier into 2000.

The side AD similarly divided and subdivided, is nearly twice the length of the radius; it turns on the joint A to form an acute angle, and it will form a right angle with CA by resting against a projection at A.

At the centre of each joint AEC, there is a small hole through which the angles of any triangle formed on the instrument may be marked on the paper.

The mode of using this protractor, is similar to that of using any other furnished with a vernier, as far as regards laying down or measuring angles. It may also be applied to the solution of problems in trigonometry, by adjusting the instrument to the given parts of a triangle, when the required parts will be pointed out on the parts of the instrument corresponding to them.

The *graphometer* is an instrument similar to the protractor, and is in use amongst land-surveyors. It also consists of a semi-circumference, divided into degrees, but it is larger, and is placed on a moveable stand or frame, with three legs. It has small pieces of brass, with holes in them, called sights, fixed at the extremities of the graduated semi-circumference, which permit the eye to see in a right line perpendicular to the plane of the circle. Sometimes also, a compass box, and a magnetic needle, are placed in the centre. The observer, placing himself behind one of the sights, and looking through both, the graphometer is turned round, till the line of vision passing through the two sights, is in the direction of some particular and previously selected object. The instrument has, in addition, an index on another diameter, which moves round its centre, and which has also a sight at each end: this index is turned round to the point.

where, on looking through its two sights, some other particular object is seen. The angle formed by two right lines passing from the centre of the graphometer to these two objects, is thus measured on the circumference of the instrument by the moveable index, and is contained between it and the diameter; the vertex of the angle being at the centre of the instrument. The angles measured by this instrument are generally horizontal, and it is chiefly used in land-surveying.

There are various other instruments for measuring angles: and the teacher would certainly do well to show them all to his pupils, and explain their properties and uses. *Quadrants* contain only the fourth part of a circle; *sextants* contain only a sixth; *octants* only an eighth. These instruments are employed in geographical and astronomical observations, for measuring the earth, and determining by the celestial phenomena, the relative position of different spots. To the navigator, they are essential. By their means, when the degree of steadiness requisite to use them can be obtained, and when the heavens are not obscured, he can ascertain the position of his vessel. When to these he adds good time-keepers,—which are now so accurately constructed that they go for many months, and in all climates, without varying a second,—the skilful mariner can in general tell, as well as the traveller on land, what is his position, and his distance from the end of his journey.

Entire circles, called *repeating circles*, are also used. They are so named, because the observations are repeated on them in such a manner, that the different errors made in the different observations, partly compensate one another, and diminish the sum total of the errors.

Independently of the faults inherent in these instruments, they have all a source of error in the inequalities of the graduated division of the circle. The hand of man can never attain that rigorous precision which belongs to the conception of the geometrician. It can, however, so far lesson the errors of execution, that they become imperceptible; even if we endeavour to find them by means of instruments that seem capable of detecting the smallest inaccuracy

Instruments for dividing the circle correctly and commodiously, have been constructed, and are generally in use. A great number of circles, having the same centre, but each of a different diameter from all the others, are drawn on a plane and perfectly level surface; such, for example, as a brass or steel plate. Setting out from the smallest circle, and proceeding towards the largest, they are divided,

the 1st, 2d, 3d, 4th, 5th circles,
into 3, 4, 5, 6, 7 equal parts.

This first division must be made with great care, and ought to be proved several times by one of the methods already pointed out.

Let us suppose that another circle, or a part of a circle, is to be divided into equal parts; it must be placed in such a manner that its axis shall be the same as those of all the circles already drawn and graduated. Though this is a matter of difficulty, the operation cannot be correctly performed, unless the centre of the circle to be graduated, is placed directly over the common centre of the circles already graduated. M. Gambey,* a celebrated French mechanician, found means to surmount this difficulty, and by a simple application of parallels, to divide a circle accurately, although it is not precisely concentric, with the first graduated circles.

Let ACB, fig. 18, pl. 3, be the surface on which an arc of a circle, AB, is to be drawn, and graduated to correspond with the graduation of the concentric circles on the plate. A frame at right angles, CMNPQ, is held in such a manner, that the sides, CM, PQ, are always directed towards the centre C, of the piece ACB, which

* The reader should, perhaps, be reminded that the division of a circle or scale, is not so simple and easy as he might suppose from the text. In *Nicholson's* Operative Mechanic, will be found a description of Mr. Ramsden's instrument for dividing the circle, which is the best yet known. TR.

is to be divided, and so that they can only be moved in a direction parallel to their first position. When the graduated plate is turned round a certain distance, let us suppose 50°, the side OCA, passes to O*ca*, CB passes to *cb*, and the angle *acb* equals 50°. But during this movement, the frame carried to *cmnpq*, has not changed its direction, and the line *pq*, is always in a right line with the centre of the arc *c*. The indicator Q, marks, therefore, on the piece ACB, a succession of points all equally distant from C, or an arc of a circle, having C for its centre; and when the plate turns a degree, the indicator Q, also moves a degree over the piece to be divided.

FOURTH LESSON.

On the various forms which may be given to the products of art, by the right line and circle.

AMONG the plane figures terminated by right lines, some are regular, and some irregular; some are simple, and others complex. We shall confine our attention to such as are most frequently made use of in the arts.

Two right lines, whether parallel or not, cannot enclose, completely, any portion of space; to effect this, three lines, not parallel, are at least necessary.

A rectilinear triangle is a portion of a plane surface inclosed by three right lines. We distinguish in a triangle, ABC, fig. 1, pl. 4, its three sides, AB, BC, CA, its three angles, and the three summits of these angles, A, B, C.

It is a remarkable property of triangles, that the sum of their angles, whatever may be the size and form of the triangle, is equal to two right angles. This property is of considerable utility in the arts.

To demonstrate it, let us prolong the side AB to BE, fig. 2, pl. 4, and draw BD parallel to AC. The two parallels AC, BD, being intersected by the two right lines, ABE, BC, we shall have, 1st. the angle CAB, equal to the angle CBD. 2d. the angle ACB, equal to the angle CBD: whence the three angles A, C, B, of the triangle ACB, equal the sum of the angles, ABC, CBD, DBE, which occupy the whole space on one side of the right line ABE, or of two right angles.

Whenever we know, therefore, two of the angles of a triangle, we may find the third by a simple operation of addition and subtraction.

Let us suppose, for example, that of these two angles one shall be, 37°, the other 49° ; adding them together, we have 86°, which being subtracted from 180°, or the sum of two right angles, leaves 94°, which is the third angle of the triangle.

Since the sum of the three angles of a triangle is equal to two right angles, it is plain that if two of its angles were right angles, the other would be reduced to nothing : a triangle, therefore, can have only one right angle.

On the same principle, a triangle, ABC, fig. 1, pl. 4, cannot have more than one angle an obtuse angle, or greater than a right angle. A triangle, having one angle greater than a right angle, is called an obtuse angled triangle.

A triangle may have three *acute* angles, A, B, C, fig. 2, pl. 4 ; it is then called an *acute* angled triangle.

A right angled triangle, ABC, fig. 8, pl. 4, is a triangle, of which one of the angles, B, is a right angle. The *hypotheneuse*, AC, is the longest side, and is opposite to the right angle.

Let us now compare the sides of triangles with one another. The right line being the shortest distance between any two points, it follows, that in a triangle, any one side is shorter than the sum of the two other sides.

Of the two sides, AB, AC, of a triangle, fig. 1, pl. 4, the longest side, CB, is opposite to the largest angle A.

Let us take $Ab = AB$, and $Ac = AC$, and draw B*b*, C*c*, the angles AB*b*, A*b*B, AC*c*, A*c*C, will be equal. ABC also, is greater than AB*b*, and ACB is smaller than AC*c*, whence the angle ABC is greater than ACB.

The *equilateral* triangle ABC, fig. 3, pl. 4, is a triangle, of which all the sides are equal to one another.

The triangle ABC, fig. 4, pl. 4, has two of its sides, AC, BC, equal to each other, and is therefore called symmetrical.

In considering the two equal sides, CB, CA, as oblique lines, equal with respect to the base AB, the perpendicular CD, will fall on the middle of this base, and divide the triangle into two equal parts. The symmetry of these triangles justifies the appellation of symmetrical, given to triangles which have two sides equal.

In order to comply with the laws of symmetry, houses and public edifices are, in general, covered with roofs, the profile of which is a symmetrical triangle. Of ancient Greek temples, and of the houses in Italy, the roofs are in the form of an obtuse angle, fig. 5; of ancient Gothic buildings, of steeples, and of the houses in the north of Europe, the roofs are generally in the form of an acute angle, fig. 6, pl. 4.

An instrument for raising heavy bodies is, or rather formerly was, sometimes used, (for it is now supplanted in England by more efficacious machines,) called a *crab*, fig. 7, pl. 4. It is composed of two pieces of timber equal in length, united at the end C, and separated at the other end by the cross piece AB. The rope by which the weight is to be raised, passes over a pulley, fixed at C, and is wound up by a windlass placed near AB. The triangle ABC, is symmetrical, and the perpendicular or rope, drawn from C to the base AB, divides it into two equal parts.

In the arts, when it is necessary to complete a triangle, some parts of which only are known, the following is the method of proceeding.

First.—When the three sides, 1, 2, 3, fig. 9, pl. 4, are known.

The right line AB, is first drawn equal to side 3, in the position in which the angle is to be constructed. From the point A, as a centre, and with an opening in the compasses equal to side 2, describe the arc of the circle mCn; also from the point B, as a centre, with an opening of the compasses equal to side 1, draw the arc pCq. Through the point C, where these two arcs intersect each other, the right lines, CA, CB, are drawn to the points A and B; ABC is the triangle required.

SECOND.—When two sides, 1 and 2, and the angle *a*, fig. 10, are known.

In a suitable position, draw the line AB equal to side 2; then with any instrument which measures angles, such as the protractor, the compasses, etc. draw the line AC so that the angle BAC equals *a*, the angle given. AC is taken equal to side 1, and the right line BC, being drawn, we have the required triangle.

THIRD.—When one side only, 1, and the two angles, *a*, *b*, the summits of which are at the two extremities of the given line, are known, fig. 11, pl. 4.,—

The line AB is drawn equal to 1; with an instrument for laying down angles, draw the right lines AC and BC, making them form with AB the angles *a* and *b*; ABC is the triangle required.

All these operations are very simple, but the student should repeat them frequently with his ruler and compasses, that he may be able to execute them with both facility and correctness, when he is required to do so in the exercise of his profession.

Three methods have just been explained of constructing a triangle. 1st. With the three sides given; 2nd. with two sides and the angle between them given; and 3rd. with two angles and the side included between them given; and in each case we have seen that these data are sufficient.

Consequently, 1st, when the three sides of any two triangles are respectively equal, the triangles are equal; it is, in fact, the same triangle constructed with the same elements in different places.

2nd. When two triangles have two sides, and the angles included between them respectively equal to one another, the two triangles are equal.

3rd. When two triangles have two angles, and the side included between them equal, they also are equal.

Thus the two triangles ABC, *abc*, fig. 8, pl. 4, are equal; *first*, if AB equal *ab*, BC equal *bc*, and AC equal *ac*; *second*, if AB equal *ab*, BC equal *bc*, and if the angle B equal the angle *b*, B being comprised between AB and BC, and *b* between *ab*, and *bc*; *third*, if AB equal *ab*, if the angle A equal *a*, and if the angle B equal the angle *b*.

These three conditions of equality being frequently of service in the arts, as well as in geometrical demonstration, and in mechanical science, they ought to be well known, and constantly present to the minds of workmen.

If any one of the three conditions, according to which two triangles are equal to one another, is not rigorously fulfilled, the two triangles will not be equal. There will be some angle or some side in the one, not equal to the corresponding angle or side in the other. In practice it is of much consequence to be readily aware what are the conditions indispensable to accurate construction; it prevents a multitude of mistakes, and when errors are committed, points out speedy methods of correcting them.

Figures of four sides, or quadrilaterals, are next to be treated of. Some figures, ABCD, fig. 12, pl. 4, are completely inclosed by four right lines; they have four *angles* and four *summits*, A, B, C, D. The right lines, AC, BD, which unite the opposite summits, are called diagonals.

In geometry all figures of four sides are called quadrilaterals. Some of them are distinguished, in consequence of their forms being more or less regular, by particular names.

A *trapezium* ABCD, fig. 13, pl. 4, is a figure of four sides, two of which AB, CD are parallel.

A trapezium is *rectangular*, fig. 14, when a third side BC, is perpendicular to the two parallel sides.

A trapezium is symmetrical when the two non-parallel sides, AD, BC are, in relation to the two parallel sides, equally oblique.

In some buildings the upper part of the roof is formed of a symmetrical triangle, MDC, fig. 15, while the lower part is formed of a symmetrical trapezium, ABCD. This sort of roof (a hip curved roof) is called in France a *mansarde*, from the name of the architect, M. Mansard, to whom we are indebted for it. The vertical line, MEF, is the line of symmetry both of the triangle and of the trapezium.

The *parallelogram*, fig. 16, is a figure, which has its opposite sides parallel to each other.

This figure is of constant recurrence in the arts; we meet with it frequently in different parts of machines; i serves to produce what is called the parallel motion, etc.

According to the properties of parallels, demonstrated in the second lesson, the opposite angles of a parallelogram, A and C, fig. 16, on the one hand, and D and B on the other, are equal to each other, two being acute, and two obtuse angles. If we add one of the acute angles to one of the obtuse angles, the sum will be equal to two right angles.

Let us extend the side DC, fig. 16, to E, the right lines AD, BC, being parallel, the angle ADC will equal BCE, and DCB *plus* BCE will be equal to two right angles.

As it has been proved that parallels included between other parallels are equal—(*second lesson*) it follows that the opposite sides of a parallelogram are equal to one another: thus AB is equal to CD, and AD to BC.

The point, O, the intersection of the two diagonals, is the middle of both.

AOC, DOB, fig. 16, pl. 4, being diagonals, the triangles ABO, DCO, are equal; for, *first*, AB=DC; *second*, the angle ODC equals the angle OBA, *third*, the angle OCD=OAB, according to the properties of parallels; consequently, OB=OD, and OA=OC.

Of two diagonals, AC, DB, fig. 17, pl. 4, the longest, AC, is opposite to the largest angles, B and D.

Let us draw DE, CF perpendicular to the sides AB, CD; these perpendiculars will be equal. But EB is shorter than AF, whence the oblique DB is shorter than the oblique AC.

The parallelogram, ABCD, fig. 18, pl. 4, the four sides of which are equal, is called a *lozenge*. In consequence of its regularity and beauty, it is very frequently employed in the ornamental arts.

When two sides of a parallelogram are at right angles, all the angles are right angles.

If the angle A of the parallelogram ABCD, for example, fig. 19, pl. 4, is a right angle, the side AD is perpendicular to AB. BC is also perpendicular to AB; and the two angles A and B, as well as their angles D and C, are right angles.

Figure 19, pl. 4, is called a *rectangular* parallelogram, or simply a rectangle. In it the diagonals AC, BD, are equal.

To prove this, it is sufficient to remark that the two right-angled triangles ADC, DAB, are equal. *First*, the right angle D equals the right angle A; *second*, the side AD is common to the two triangles, and, consequently, equal for both: *third*, the side DC of the angle D in the first triangle, equals the side AB of the angle A in the second, whence the third side AC of ADC equals the third side BD of DAB; or AC, BD are the two diagonals.

A *square*, ABCD, fig. 20, pl. 4, has its four sides and four angles equal.

In summing up the properties of four sided figures, the following deserve to be engraved on the memories of the workman and youthful artist.

In the square, the four angles are equal, and are right angles; the four sides are equal to one another, and the two diagonals to each other.

In the rectangle the four angles are equal, and are right angles; it has two long sides equal to each other, two short sides, equal to each other, and two diagonals also equal to each other.

In the lozenge the four sides are equal to one another; it has two obtuse angles equal to each other, and two acute angles equal to each other. The diagonals are unequal.

In the parallelogram there are two long sides and two large angles equal to each other, and two short sides and two small angles also equal to each other. The diagonals are unequal, the longer being opposite the larger angle, and the shorter opposite the smaller angle.

Symmetry of four sided figures. By doubling these figures into two parts, so that both shall be equal to one another, we can prove that — 1st. The trapezium, with oblique equal sides, fig. 15, pl. 4, is symmetrical in relation to the right line, EF, which passes through the middle of its two bases; 2d. The rectangle, fig. 19, pl. 4, is symmetrical in relation to any right line drawn through the middle of two opposite sides; 3d. The lozenge, fig. 18, pl. 4, is symmetrical in relation to each of its diagonals; 4th. The square, fig. 20, pl. 4, is symmetrical in relation to its two diagonals, and to any right lines passing through the middle of two opposite sides. This symmetry of four sided figures is of great importance in the arts and in mechanics.

We know that in all triangles the sum of their angles is equal to two right angles. But all figures of four sides, ABCD, fig. 12, pl. 4, may be divided into two triangles, ABC, ACD, the sum of the three angles of each being equal to two right angles. The six angles of these two triangles are equal therefore to the four angles of the figure ABCD. *Consequently, in four sided figures, the sum of their angles equals twice two, or four right angles.*

If we have a five-sided figure, ABCDE, fig. 21, pl. 4, we can from one summit, A, draw two right lines, AC, AD, to the summits C and D, and we shall divide the figure into three triangles; the sum of the nine angles will be equal to the five angles of ABCDE.

Thus, in *five-sided figures, the sum of the angles equals three times two, or six right angles.*

Following the same method, we shall see that the sum of the angles, in figures of

	3,	4,	5,	6,	7,	8,	sides,
is equal to	2,	4,	6,	8,	10,	12,	right angles.

Relation of the circle to figures terminated by right lines. Around the three summits of a triangle, ABC, fig. 22, we can always describe a circle in the following manner:— From the middle of AB draw *mo* perpendicular to AB; and from the middle of BC *no* perpendicular to BC. The point *o*, where these perpendiculars meet, is equally distant from the three summits A, B, C, and is therefore the centre of a circle which passes through these three points.

A triangle thus placed within the circumference of a circle, is said to be inscribed within it.

In a right-angled triangle, fig. 23, pl. 4; or a triangle having one angle, B, right angled, the centre of the circle O, which passes through the summits of the triangle, is in the middle of the side AC, opposite to the right angle, and which has before been called the hypothenuse.

The following is the most simple method of demonstrating this truth:—

In the rectangle ABCD, fig. 25, pl. 4, the two diagonals are equal; and, consequently, their halves, OA, OB, OC, OD, which may be taken as radii of a circle, are also equal; consequently, we may always inscribe a rectangle, fig. 25, as well as a square, fig. 26, in a circle.

To give a demonstration of this, independent of the properties of rectangular figures, draw MO from the middle of AB, fig. 23, perpendicular to AB, and from N, the middle of BC, draw NO perpendicular to it. The point O, where these lines meet, is the summit of two equal triangles, AMO, BMO, in which the acute angles corresponding to AMO, BMO, are designated by the figures 1, 2. The sum of these angles is equal to a right angle. But in the large, right-angled triangle ACB, the angle A and the angle C are together equal to a right angle; whence the angles 1, 1, 1, 1, are all equal; and so are the angles 2, 2, 2, 2. The four angles 1, 1, 2, 2, round the point O, are 1 plus 2, and 1 plus 2, or equal to two right angles; whence AO and OC are in a right line; consequently, also, the point O, equally distant from A, B, C, is on the hypotheneuse, AC.

Any triangle, ABC, fig. 25, being given, if we construct a triangle, ADC, equal to it, we form a rectangle inscribed in a circle, having its centre on the middle of AC. The diameter, therefore, of the circle inscribed about the summits, A, B, C, of the right angled triangle ABC, is the long side AC of the triangle.

It follows from this, that every four-sided figure ABCD, fig. 24, pl. 4, of which two opposite angles, B, D, are right angles, may be inscribed in a circle, which passes about the four summits of the figure.

In fact, the diagonal AC, divides this figure into two right angled triangles, both inscribed in a circle having AC for its diameter.

Figures which have more than four sides, have received Greek names, signifying the number of their angles and sides. Thus a

	Pentagon,	hexagon,	heptagon,	octagon, &c.
has	5,	6,	7,	8 sides respectively.

A *polygon*, as the word expresses, is a figure of several angles and sides; and of these, regular polygons, in particular, deserve our attention, as they are of frequent and important use in the arts.

Regular polygons have all their sides and all their angles equal.

According to this definition, if we find a point O, fig. 27, pl. 4, equally distant from the three summits, A, B, C of the regular polygon ABCDEF, it is equally distant from all the other summits, and thus OA = OB = OC = OD, &c.

The symmetrical triangles AOB, BOC having their bases AB, BC, and their symmetrical sides OA, OB, OC, equal to one another, are equal. The symmetrical angles equal the half of B, for the two middle angles, being added together, form the angle B. The triangle OCD, is equal to OCB, because OC is common; CD = BC, as the sides of a regular polygon; and the angle OCD = OCB, because one of these angles is the half of their sum. It may, in like manner, be demonstrated that the triangles ODE, OEF, are equal to the former, and consequently symmetrical; whence their symmetrical sides OA, OB, OC, are equal. The point O, is consequently equally distant from all the summits of the regular polygon, and is therefore the centre of a circle, which is inscribed about them all.

Since we can always describe a circle round three summits, we can also always describe a circle about a regular polygon, whatever may be the number of its sides.

Conversely, a circle being given, we can always inscribe a polygon in it of any number of sides we please.

It is only necessary for this purpose, to divide this circumference into as many equal parts as the polygon is to have sides, and to connect by right lines, the successive points of division.

In the third lesson, the relation between the length of the radii of a circle, and the distance between the points of division, or the length of the sides of any polygon, has been stated. The student will, therefore, have no difficulty on this subject.

Application of regular polygons to regular fortifications. Military engineers employ regular polygons in constructing regular fortifications; the number of sides to be given to the polygons depending on the size of the place to be fortified. The equilateral triangle and the square are not used by them, except for field-work. The *pentagon, hexagon,* and *heptagon* are employed to enclose small places and citadels. Figures, with a greater number of sides, are adapted for the fortification of large towns.

Application of the preceding forms, in paving, glazing, mosaic, and generally to all inlaid work.—The object generally proposed in all these branches of industry, is to cover or fill a given space with figures terminated by right lines. This object may be attained by an almost infinite variety of figures, according to the infinite combination of right lines, which may be traced on a plane surface.

If it is required that all the figures shall be regular, and have the same number of sides, the problem is very limited, and can only be solved by the following figures:—

1st. By equilateral triangles, the summits of which, six and six, meet at the same point, fig. 27, pl. 4.

2d. By squares, the summits of which meet, four and four, at the same point, fig. 29.

3d. By hexagons, the summits of which meet, three and three, at the same point, fig. 28.

The following table will demonstrate these statements. The angles of polygons,

of	3,	4,	5,	6,	7	sides
are	60°	90°	108°	120°	$128°\frac{4}{7}$,	
those of	8	9	10	11	12	sides
are	135°	140°	144°	$147°\frac{3}{11}$	150°	

Now six times 60°, 4 times 90°, and 3 times 120°, make 360°. Neither of the other numbers of degrees dividing 360 by a round number, we cannot fill the space around any given point, by the angles of any other regular polygons, except those of three, four, and six sides.*

It may be remarked, that in filling the space round a point, fig. 27, with six equilateral triangles, the six outer sides form a regular hexagon, inscribed in a circle, having for its radii the interior sides of the triangles;—therefore, *the sides of a hexagon are equal to the radius of the circle in which it is inscribed:*—a proposition very useful in the arts.

The multiplicity of objects to be treated of in these lessons, will not allow us to examine, in detail, many figures, more or less regular, which, when combined together, produce a pleasing effect. To form and study them, will exercise and correct both the taste and the imagination of students.

When we have to lay down a pavement, a piece of mosaic, or some inlaid work,† on which persons are to

* The following table, taken from Lieutenant-Colonel Pasley's "*Complete Course of Practical Geometry,*" may, perhaps, afford the student some valuable additional information on polygons. Tr.

No. of sides.	Name of the Polygon.	Angle at the Centre.	Angle of the Polygon.	Angle at the base of the elementary triangle
3	Trigon	120° 00′	60° 00′	30° 00′
4	Square	90 00	90 00	45 00
5	Pentagon	72 00	108 00	54 00
6	Hexagon	60 00	120 00	60 00
7	Heptagon	51 $25\frac{5}{7}$	128 $34\frac{2}{7}$	64 $17\frac{1}{7}$
8	Octagon	45 00	135 00	67 30
9	Nonagon	40 00	140 00	70 00
10	Decagon	36 00	144 00	72 00
11	Undecagon	32 $43\frac{7}{11}$	147 $16\frac{4}{11}$	73 $38\frac{2}{11}$
12	Dodecagon	30 00	150 00	75 00

† It may be, perhaps, worthy of remark that the uncarpetted floors of most French houses, belonging to opulent people, as well as of many public rooms, such as libraries, galleries, &c. are formed either of mosaic, or inlaid wood work, while the floors of most

walk, it is of some consequence that too many summits should not meet in any one point; for in putting the foot or any great weight on this point, it will be liable to give way; which would destroy the texture and solidity of the work.

For this reason, equilateral triangles, the six summits of which meet at the same point, are seldom or never employed. In general also, squares, which meet by fours at the same point, are carefully avoided.

When a space is to be covered with regular four sided pieces, equal to one another, the squares or rectangles are ranged in straight lines; and the joints of the squares of one row, are placed directly in the middle of the square of the next row. On this principle stones are generally employed in masonry, of the shape, and placed in the manner, pointed out in fig. 30, pl. 4.

The Romans frequently made the bricks and stones they used for building walls, of a lozenge shape; and from the resemblance of this sort of masonry to net work, fig. 31, they called it net work, *opus reticulatum*.

The form of a hexagon, for paving, or for making the tiled floors of apartments, has many advantages, fig. 28.

Bees construct their cells of the form of a regular hexagon, which enables them, with any given quantity of wax, to inclose the largest possible space for the apartment of each member of the hive.

In very remote antiquity, the human race constructed very large and very solid buildings with enormous blocks of stone, cut into the form of irregular polygons, and many of the monuments then erected still exist in Italy, Sicily, and Greece. These structures have been called *cyclopean*; they are represented by fig. 32.

The advantage of this species of building is, that we can profit by the natural form of the pieces or blocks of stones destined for the erection of such monuments;

of the houses of the middle classes are paved or covered with tiles. Hence, perhaps, the importance given to these subjects in Baron Dupin's lectures. Tr.

cutting and fitting them to each other, so as to lose the smallest possible quantity of their whole mass.

In the celebrated jetty or breakwater, constructed at Plymouth, for the protection of that harbour against the violence of the sea, the top and the slope of the inner side, on the upper part, are covered with large blocks of marble, cut and fitted to one another as in the cyclopean construction. The connexion of the blocks with one another, prevents the sea from moving any single block, and makes each contribute to the solidity of the whole.

Of figures terminated by right lines, and portions of the circle.—If figures, composed only of right lines, offer to us a great variety of form, it may be easily imagined how much this variety must be augmented, by combining right lines and portions of the circle together.

The most simple of these consists of a semi-circle and its diameter. Such is the figure of the graphometer and the protractor, mentioned in the last lesson, and represented in fig. 17, pl. 3.

Such also was the form of the theatres among the ancients, and such is the form of amphitheatres employed for public meetings, lecture rooms, &c. among the moderns. The orator,* or the professor, takes his place at the centre, C, fig. 33, and the auditors are ranged on banks, in the form of semi-circles, placed at equal distances from each other, all having the point C, for their common centre, and AB for their diameter.

If from the extremities of the diameter AB, fig. 34, pl. 4, we draw lines perpendicular to the diameter, they will be tangents at A and B, to the semi-circle AMB. If we then draw, at a certain distance, the right line CF, parallel to AB, we shall have a figure very frequently met

* This remark applies to the Chamber of Deputies in France, but not to either the House of Peers or Commons in England. In them every member rises in his place, wherever that may be, to speak. In the Chamber of Deputies, an elevated pulpit, called the Tribune, is placed in the centre of a building, having the form of an amphitheatre, and every member who addresses the Chamber, ascends the tribune for that purpose. Tr.

with in works of art; it is that of arches, and door-ways of a full curve, or semi-circular arch.

If on the top of the rectangle ABEF, fig. 35, with AB, for a radius, we draw, *first*, from the point A, as the centre, the arc BM, and *second*, from B, as a centre, the arc AM, we shall form what is called an arch of *third point*.

Arches of a semi-circular form, were used in the architecture of the Greeks, and are used generally in modern architecture: arches of the third point belong to Gothic architecture. These two descriptions of architecture, are indebted to these different geometrical forms for those distinct characteristics which are essential to each. Each has fair claims to esteem and admiration, and each merits to be particularly studied, as well for the excellence of its forms and proportions, as for the boldness and solidity of its buildings.

If we describe a semi-circle on EF, fig. 34, as a diameter, we shall have an outline AMBFNE, which represents an arena, such as was appropriated by the ancients to public horse races, and therefore called *hippodromes*. The posts round which the horses were to turn, were situated at C and *c*, of the circular parts of the building.

The moderns use, in their bridges and public buildings, elliptical arches, which are formed of the arcs of several circles. In the arch, fig, 36. there are three arcs, D, E, F, having their three centres at O, P, Q. (See Lesson 14th.)*

In a species of Gothic, or rather Moorish, architecture, the arches are formed of two short arcs of a small circle, BD, GF, fig. 37, prolonged by two right lines, DE, FE, which form an obtuse angle. In England, there are many Gothic edifices constructed in this style, which are not less remarkable for the elegance of their form, than for the boldness of the structure;—the Chapel of Henry the Seventh, at Westminster; Trinity College, Cambridge;

* This kind of arch is called, from its shape, a *basket-handle arch* —*voute en anse de panier*.

and of the Palace, at Windsor, may be mentioned as examples.

Architectural profiles.—Architects have found means, by some very simple combinations of right lines and parts of the circle, called mouldings, of ornamenting the profiles of their buildings. Such mouldings are also frequently used by the house-carpenter, the joiner, the cabinet-maker, and the machinist; who ought to understand their geometrical principles.

The most simple of these mouldings is the *fillet*, composed of two parallel lines situated near to each other, and terminated at one end by a perpendicular: AB, fig. 38, is such a moulding. A succession of them, placed one above another, fig. 39, represents the capital of the Grecian Doric order of architecture; called also the *Pœstum order*, from the celebrated temple found at that place, surrounded by superb columns of this description.

The fillet is generally placed in connexion with the edifice, at its exterior extremity by means of a quarter of a circle situate below it, to which it is a tangent; and at its interior extremity, by insertion into the vertical face of the wall, pilaster, or column, which is to be ornamented.

Generally also, the fillet is surmounted by a projecting half-circle, fig. 38, called a band or corona.

The quarter of a circle is also employed separately in relief, or convex, A*m*B, fig. 40, and also sunk, or concave, A*m*B, fig. 41.

Two quarters of a circle AMB BND, fig. 42, having the same radius, and their centres O, P, on the same vertical line, form the *talon* or *cimatium*: it is also termed an ogee.

Two quarters of a circle, AMB BND, fig. 43, having the same radius, and their centres on the same horizontal line, form the *doucine* or *cimatium*, but distinguished as *cima recta*.

Such are the very simple elements out of which architects have composed all that delightful variety of cornices, friezes, bases, and capitals, which we see in edifices both ancient and modern.

It must not be supposed that the combination of these forms is entirely arbitrary, and may be made by chance, as it were, or by the unreflecting caprice of an ill-regulated imagination. The art of ornamenting the profiles of edifices, is indebted for its perfection to the faithful observation of the laws of simplicity, variety, and contrast. In place of scattering ornaments with a too lavish hand, they ought to be so collected into groupes, that the eye can easily comprehend them; and they ought to be separated from one another by a considerable space that is both plain and uniform, that the eye may be neither fatigued nor confused. In each group, the most delicate mouldings should be opposed to the most massive; and forms consisting of right lines, to those composed of circles; in order that each may bring out or contrast with those which surround it. Such are the principal rules of this part of the art of ornamenting buildings, —rules which not the great architects alone of Greece and Italy discovered and practised with amazing success,—but which were also employed, with not less art and skill, in the fine monuments which still remain to us of ancient Egypt, in the Gothic edifices of the Middle Ages, and in the mosques and palaces which the Moors erected in Spain, when they cultivated in that country the arts and sciences, then almost unknown, or annihilated, in every other part of Europe.

An application of geometry to architecture, much more important than exterior decoration, is in the conception and drawing the plans of buildings. The forms adopted by architects may almost all be reduced to the right line and circle. In some rare cases, in which they require more complicated forms, they take parts of different circles: as for example, in constructing the elliptical arch.

When architects have occasion to construct an edifice in a spot entirely free, they would be inexcusable were they not to adopt regular forms, the simplicity, uniformity, and symmetry of which gratify the eye, and display that spirit of wisdom and order which man has, in all enlightened ages, exhibited in public and national monuments.

The form most generally adopted is a square or a rectangle, because these figures admit most easily of those subdivisions which are necessary for accommodation in the interior. These forms have no other inconvenience than that of not allowing of circular lines or apartments, without a loss of space, or without leaving some abrupt corners of an irregular shape, which ought carefully to be avoided. Some use, however, may be made of such corners, for concealing objects that ought not to be exposed to view, or for constructing in them concealed staircases.

In towns, where space is very valuable, the architect is obliged to profit by the very smallest spot of ground, and to plan, as well as he can, a suite of regular apartments, in a building of an irregular form. In such cases, the art of combining together geometrical figures, may be of great use to builders, and suggest to them the forms best adapted to any particular situation.

Some architects think they are rendering their pupils skilful by encouraging them to form plans of edifices, which would cost millions of money, and which might be built without any difficulty, on some imaginary plains. By this means they give their scholars a taste for ridiculous grandeur, and inspire them with ideas of magnificence for which the people must afterwards dearly pay. It would be much better to exercise the inventive minds of young men, in forming plans of buildings suitable to those irregularities of form which we always meet with in large cities, where the houses are closely and irregularly pressed against one another.

FIFTH LESSON.

Equal, symmetrical, and proportional figures.

Two figures are equal when they are precisely of the same form, and of the same size; so that, one being placed on the other, every point in one coincides, or is blended with the corresponding point in the other.*

To make one object exactly equal to another, is of great importance in the practical arts, and geometrical science furnishes the workman with many means of doing it.

In engraving, in sculpture, and in many other branches, both of the useful and ornamental arts, for example, we must make moulds or models, or execute work according to models, so that it shall be precisely the same as some given object.

In the second lesson it has been mentioned, that by means of parallel lines of the same length, we can easily

* The following definitions may, perhaps, make the subject matter of this, and subsequent lessons, clearer to the reader:—

EQUIVALENT figures are such as have equal surfaces as to quantity.

EQUAL figures are such as, when applied to each other, coincide in all their points.

Figures are similar, when they have their angles equal each to each, and their *homologous* sides,—sides which have a corresponding position, in the two figures, or which lye adjacent to equal angles,—proportional.

Figures may be equivalent, though very dissimilar. Equal figures are always similar, but similar figures may be very unequal.

TR.

construct one figure equal to any other, and so place it, that the corresponding lines in the two figures shall be parallel.

This method is liable to error, in proportion as the parallels increase in length and are distant from one another. To these causes of error, we must add the greater or less imperfection of compasses, rulers, and instruments employed to measure distance; and the greater or less degree of fineness in the points of pencils, pens, or other instruments used for drawing lines.

The means by which the geometrician satisfies himself, in many cases, that two figures are equal to one another, may also enable the artisan to make one figure equal to another. The geometrician sometimes places one of two figures on the top of another, and observes that neither, in any point whatever, projects beyond the other.

Let it be supposed, that we are required to make any figure whatever, such as ABCD, fig. 1, pl. 5, on any extended surface whatever, MNPQ, such as a piece of cloth, a flat sheet of metal, &c. Place the figure ABCD, on MNPQ, in such a manner, that it will form, or fall on *abcd*, in the double fig. 1, pl. 5; then cutting MNPQ, according to the sides *ab*, *bc*, *cd*, we shall have *abcd* exactly equal to ABCD.

Very often, in place of immediately cutting out the second figure by the first, the outline *abcd*, is traced with a pencil, with chalk, or with ink, taking the edges of the pattern figure as guides; which being removed, the required figure is more easily cut out.

By this means many workmen, such as shipwrights, stone masons, braziers, tinsmiths, tailors, &c. &c. fashion a piece of stuff into any given form.

When the figure to be imitated is not cut out of, but only marked on a surface, the above method cannot of course be employed. In this case, if the figure is of little or no value, all the prominent points *a*, *b*, *c*, *d*, being pierced through, they may be transferred to the other surface, MNPQ, and be afterwards connected by lines. Very often, the outlines are pierced by some pointed instrument, at short and equal distances, in their whole length; then, with a little bag, containing finely powdered charcoal, we strike on the model, as it lies over the other surface, MNPQ, and the fine charcoal dust passing through the small holes, traces, very accurately, the outline of the figure to be made.

When the model cannot, or must not be perforated, a sheet of transparent paper is laid over it, and its exact form, or the parts to be made, are drawn or copied on the paper. The paper may be transferred to another surface, and pierced, or the figure may be cut out, or otherwise copied, as is most convenient.

Symmetry of figures.—Two figures, *abcd*, *a'b'c'd'*, fig. 1, repeated, pl. 5, are symmetrical when their corresponding points *a* and *a'*, *b* and *b'*, *c* and *c'*, &c. are placed on parallel lines, all of which are accurately divided in the middle by a perpendicular line MN. If we were to double the frame MNPQ, on MNP'Q', it is plain that *a* will fall on *a'*, *b* on *b'*, &c.—so that the figure *abcd*, could it be impressed on the surface MNP'Q', would produce the figure *a'b'c'd'*, which is symmetrical to the former. By means of parallels, therefore, and a perpendicular, dividing them in the middle, we can always make a figure, *a'b'c'd'*, symmetrical with another, *abcd*.

Production of equal or symmetrical figures, by engraving, printing, lithography, &c.—The object of these arts is to form, on a surface of wood, metal, or stone, or on any other substance, certain figures, of which the impression may be afterwards transferred exactly to some other surfaces. It must be observed, that the figure produced is reversed, in regard to the plate by which it is made, for the right of the plate is imprinted on the left, and the left on the right of the paper, or other material, employed to receive the impression; we must, therefore, write or draw the figure reversed, on the plate, if we wish that the impression shall represent an object in its natural position. For this reason, types are cast reversed, and placed by proceeding from the right to the left, in order that, when applied to the paper, the letters they produce may be in their natural direction, and be read from left to right. Thus, mere impression does not produce copies *equal* to the figure of the plate, but *symmetrical*.

Production of equal figures by stereotyping.—Matrices are engraved, composed, or drawn, and by means of them, impressions are made on plates, which are again employed in the ordinary mode, to print writing, music, drawings, &c.. At the first impression, the objects pass from the left to the right, and at the second, repass from the right to the left. In stereotyping, therefore, the printed objects are identical, or are equal on the primitive matrix, and on the copies taken from the intermediate plate. On this principle, the matrix for stereotyping has the letters placed or cut on it in a natural position, so that the letters made by it are in a reversed position; but the impression obtained from them is in a natural position. In lithography and in copper-plate engraving, the object to be produced is drawn or written in a natural direction, on paper or on prepared cards; the figure or writing is then transferred to the stone or copper in a reversed position, and of course shows, in the copies taken from the stone, the engraved object in its natural position.

Geometry supplies us with other means of making one figure equal to another.

Let us suppose ABCDEFGA, fig. 1, pl. 5, to consist of any number of sides. If from the summit or vertex A, of this polygon, whether regular or irregular, we draw right lines to all the other summits, we shall divide the polygon into triangles, and as it is easy to construct one triangle equal to another, by making successively the triangle *abc* equal to ABC, *acd* equal to ACD, *ade* equal to ADE, we shall, in the end, form completely the figure *abcdefg*, fig. 1, repeated equal to ABCDEFG.

We may reproduce any figure, ABCDEFGA, merely by using compasses to measure the length of its sides, and a protractor or other instrument, to take the size of the angles. Let us first draw the side *ab* equal to AB, then placing the centre of the protractor at B, and moving the base or diameter to the side AB, we can take off precisely the number of degrees and fractions of degrees which measure the angle ABC. The protractor is then placed on *b*, the point where it is proposed to construct the new figure, and the number of degrees transferred from the other figure is marked off. Let *m* be the point corresponding to the number of degrees on the protractor—marking *m* on the paper with the point of the compass, and drawing the right line *bmc* equal to BC, a second side of the figure is obtained. Afterwards, placing the protractor on C, the angle BCD may be marked off and transferred to *bcd*, and thus we may proceed with all the other angles, till the whole be completed. If the operation be well performed, when the last right line *ga* is drawn, it ought to terminate at the point *a*, and have the length of GA. But when the sides of the polygon are numerous, it is nearly impossible to make a figure so very correct. The least error in any one angle influences the whole, for the position of any one side depends on the direction of the preceding sides. Any error also in the length of a side, enlarges or lessens the figure, by carrying out or bringing in too far all the sides of the polygon.

These sources of error show how very much some methods of operating, which are rigorously correct in theory, may give rise to inaccuracies in practice. It is only by a good selection of different methods, that it is possible to unite simplicity and ease of execution with precision.

By the following methods we may ascertain if one figure be made exactly equal to another :—

If we construct successively the triangles *abc*, *acd*, fig. 1 repeated, only taking into consideration the comparison between them and those to which they are equal, we shall hardly be able to avoid very considerable errors. In fact, the number of errors committed in each angle, increasing as the number of angles increases, multiplies the chance of error. It may therefore happen that the sum of the angles constituting *bag*, will differ very considerably from BAG, though each of the particular angles *bac*, *cad*, which it includes, may differ very little from the corresponding angles, BAC, CAD.

We have the following geometrical methods of ascertaining the equality of the single angle and of the whole.

1st. The employment of parallel lines, because two angles having their sides parallel are equal.

2nd. By measuring with compasses, and thus ascertaining that AB is equal to *ab*, AG equal to *ag*, and BG equal to *bg*.

3rd. Draw the third side BG, *bg* of the two triangles ABG, *abg*, and see if the point *a* is at the same distance from BG, as the point *a* is from *bg*; that is to say, see if the perpendiculars AZ *az*, drawn from A to BG, and from *a* to *bg*, are equal to one another.

The verification of the angles ABG *abg* being finished, draw the lines AC *ac*, AD *ad*, in these angles, in order to form within them smaller equal angles. Make AC equal to *ac*, AD equal to *ad*, and AE equal to *ae*, and draw the sides *bc*, *cd*, *de*, *ef*, and the second figure will be produced.

This latter part of the operation may be proved, by ascertaining with a pair of compasses, or any other instrument, that CD is equal to *cd*, and DE equal to *de*, and that the angle ABC, equals the angle *abc*, and BCD equals *bcd*. As soon as the existence of an error is ascertained, the whole of the preceding operations must be repeated and examined, to discover its source and rectify it.

The method of squares.—There is a method employed by artists for making one figure equal to another, which consists in dividing two spaces into an equal number of corresponding squares, fig. 2, pl. 5.

The figure to be copied is first divided into a number of bands or stripes, by two series of parallel lines, each series being at right angles to the other. The four sides are numbered in order to distinguish them more easily. A similar division is made on the surface, on which a figure, equal to the former, is to be made: then the prominent or essential points found marked in each square of the original, are inserted exactly in the corresponding square of the intended copy.*

We first examine if there is anything in the space OI, OI. Within the vertical band, I. II, I. II, we find the summit A, which is on the line No. 4. 4. We take on this line an opening of the compasses, equal to the distance of this point from I.I, and place it in the new figure in I, I, at *a*. We afterwards observe that the point B is in the square II. III, 6. 7. We measure the distance from B to the lines II, II, and to 6, 6; and transferring these distances to the

* In extensive topographical surveys, whether for the purposes of war, or ascertaining the value of land, the space of which a map is to be drawn, is frequently divided into squares; one person surveys a square, marked on two lines 1, I, another a square marked 2, II. a third surveys a square marked 3, III, &c., and all the surveys, being afterwards collected, form the map of the whole space.

new figure, it gives us the point *b*. In the same way we may find the place of every other point *c*, *d*, *e*, and draw the polygon *abcde—a* equal to ABCDE—A.

In this method there are three sources of error,—1st. the parallelism, and equal distance of the lines which form the square; 2nd. in drawing the lines, with respect to their being all perfectly straight at every point, and of an equal breadth throughout; 3d. in measuring the position of each point.

These observations will make the artisan sensible how much the most simple methods are liable to error, and how necessary it is in practice to acquire that skill, care, patience, and good judgment, that fineness of tact and perception, which no theories or methods can give. Correctness is the characteristic of art in its greatest perfection. It is not surprising, therefore, that ages of attention, and numerous trials, should have been required to construct, correctly, any machine, of which the principles are well known, and the form theoretically established, but the successful working of which depends on each part being made with great precision. Nations little advanced in those arts which depend on precision, find it very difficult to overtake people who have preceded them, even when they are at liberty to borrow their improvements; for every advance already made, serves the more skilful people as a means,—by making more correct instruments, &c.—of hastening their progress, and diminishing the chances of error in their operations. An extensive and correct theory, judiciously applied to practice, can alone place those nations on an equality, which are not so at present; and can alone enable those who use it, to excel the competitors who now surpass them in manufacturing skill. To accelerate this application, is the great object of our present plan of instruction.*

* The reader should, perhaps, be reminded here, that theory will not give precision in execution. There can be no doubt, judging from the admirable elementary works which have been published, in France and Germany, both on the sciences and arts, that the

Proportional figures.—It is not only necessary to know how to render figures symmetrical, or equal to one another, but we must also often make figures of exactly the same form as others, but differing in size. This operation is geometrically performed by means of proportional lines, and similar triangles.

Let us suppose, that the right line AF, fig. 3, pl. 5, is divided into equal parts, AB, BC, CD, DE, and that from each point of the division, the parallel lines A*a*, B*b*, C*c*, D*d*, E*e*, &c. are drawn in any direction.

These parallels will be all at equal distances from each other. If we draw A 1, B 2, C 3, D 4, at right angles with the parallels, we shall have a succession of equal triangles, AB 1, BC 2, CD 3, for these triangles have their corresponding angles equal, and moreover, one side equal; that is, AB, equal to BC, equal to CD, equal to DE; whence the perpendiculars A 1, B 2, C 3, D 4, which are the corresponding sides of these triangles, and measure the distances between the consecutive parallels, are also equal to one another.

Let us draw the line *mnopqr*, in a different direction to AF, the parts *mn*, *no*, *op*, *pq*, *qr*, will be equal to one another.

If we draw perpendicular to the parallels, the lines *m* 1,

theory of them is, and has long been, as well, if not better known in those countries than in this. Nor can there be any doubt, particularly as to Germany, that the theoretical knowledge of art is as much diffused, or was up to a late period, there as here. Our superiority in manufactures arises chiefly from the practical skill of our workmen. They already possess that in a high degree, which both theory and practice are to give our competitors. The remarks of M. Dupin prove, that having preceded them, we must be much wanting to ourselves if now, or at any subsequent time, we allow them to surpass us. That one nation which chooses to borrow the arts of another, does not need to go through those ages of study and costly trials, of which M. Dupin speaks, shows us how closely the interest of all men is united; and that we can only profit by the knowledge of other nations, when they are more enlightened than ourselves. We must, therefore, be emulous, not jealous of their success.

TR.

n 2, *o* 3, these lines being equally distant, *m* 1, will be equal to *n* 2, equal to *o* 3. Moreover the triangles, *mn* 1, *no* 2, *op* 3, have their sides parallel, and consequently their angles equal, whence they are equal. The corresponding sides, *mn*, *no*, *op*, are of course also equal.

When an oblique line, therefore, AF, fig. 3, pl. 5, is divided into equal parts by a succession of parallels, A*a*, B*b*, C*c*, D*d*, these parallels also divide into equal parts every other right line, *mr*, which intersects them.

This principle is made use of to divide a given right line into as many equal parts as we wish.

Let us suppose that it is required, for example, to divide into five equal parts, the line AF, fig. 4. From the point A, draw another right line AX, in any direction; then with any opening of the compasses, mark the points 1, 2, 3, 4, 5, equally distant from one another. Draw through the points 5, and F, the right line F5, and then through the points 1, 2, 3, 4, the lines B1, C2, D3, E4, parallel to F5. The line AF, will be divided into five equal parts, for the five parts of this right line will be comprised between parallels equally distant from one another.

By these methods, the *scales* are generally divided, which are employed to draw architectural plans, whether naval, military, or civil. The importance of having an exact division is very great, for all the lines to be drawn by the *scale*, will be affected by its incorrectness. If any part of the scale be incorrect, all the parts measured or drawn by it will also be incorrect; and the same being repeated a great many times, may give rise to some new errors of no trifling magnitude.

In order to form a good division of a scale, it is necessary that the divisions A 1, 1.2, 2.3. &c. should not be smaller than AB, CD, DE. It is necessary also to place the points of the compasses very exactly on the line AX, fig. 4, pl. 5, the direction of which must, in the first instance, be scrupulously examined. The mark made by the compasses should be as fine as possible, in order that the least possible error may arise from its size; finally, when the parallels are drawn, care must be taken that the middle of the line, whether drawn by a pencil or with ink, passes exactly though the corresponding points of division, and that the parallelism is as correct as possible. When all these conditions are attended to, we may trust to the success of our operation.

The division of AF, fig. 4, may be verified by the compasses, to see if the parts AB, BC, CD, are rigorously equal.

Important minute divisions of the scale.—Very often it is required to divide the unit of the scale AM, fig. 5, pl. 5, into so many parts that each division cannot be marked exactly and distinctly on the small right line AM. In this case draw the parallels M*m*, N*n*, O*o*, &c. equally distant from one another; then draw the perpendiculars MF and A*f*, and the oblique line AF. The lengths of B*b*, C*c*, D*d*, E*e*, are to one another as 1, 2, 3, 4; they represent the divisions of MA into as many equal parts as there are equal spaces between the parallels M*m*, N*n*, O*o*, &c. For example, if MA represents one yard, and there are ten parallels to MA, all equally distant from one another, the parts B*b*, C*c*, D*d*, E*e*, will be respectively equal to 1, 2, 3, 4, tenths of a yard. With a scale constructed on this principle, instead of placing the points of the compasses on the same line MA, they are placed according to the variation of the numbers on N*n*, O*o*, P*p*, &c., which is attended also with this great advantage, that the scale lasts longer.

Verification of the plan of models of machines, or of other productions of art.—When we have to examine and prove the plan of any machine or object, which has been made according to a scale, the first thing to be done is always to ascertain if the scale itself be correct. If it be incorrect, the plan is decided, without further examination, to be badly executed; if correct, there may be other sources of error which must be examined.

To return to the division of right lines by parallels: let us suppose that AF, fig. 5, is intersected by the parallels A*m*, B*n*, F*r*, at unequal distances from one another, the parts AB, BF, comprised between these parallels, will not be equal to each other. The same will be the case with *mn*, *nr*, and every other right line, *mr*, intersected by these parallels.

But if BF is greater than AB, *nr* will be greater than *mn*, and the number of times *nr* contains *mn*, so many times will BF contain AB.

If, for example, BF contains AB four times, dividing BF into four equal parts, BC, CD, DE, EF, and drawing the parallels C*o*, D*p*, E*q*, we shall divide *nr* into as many parts *no*, *op*, *pq*, *qr*, equal to *mn*, as there are parts BC, CD, DE, EF equal to AB; whence BF will contain AB as many times as *nr* contains *mn*.

This relation between BF and AB, and between *nr* and *mn*, is expressed in the following manner.

BF divided by AB equals *nr* divided by *mn*.

$$\frac{BF}{AB} = \frac{nr}{mn}.$$

Or BF is to AB as *nr* is to *mn*.

Or BF : AB : : *nr* : *mn*.

This is called geometrical proportion: in it there are always two relations, or terms equal to each other, $\frac{BF}{AB}$ and $\frac{nr}{mn}$: thus the geometrical ratio of two quantities is the first quantity divided by the second; the inverse ratio is the second divided by the first.

A proportional BF : AB : : *nr* : *mn* has four terms, the first and last of which are called the extremes, the two others the means or middle terms.

It is a fundamental property of proportionals, that the product of the two extreme terms is equal to the product of the two mean terms.

To demonstrate this, we observe that the proportion BF : AB : : *nr* : *mn*, $\frac{BF}{AB}$ and $\frac{nr}{mn}$ being equal, if I multiply these two ratios both by AB and *mn*, the products will be equal. But BF divided by AB and multiplied by AB, and then by *mn* is BF multiplied by *mn*, or the product of the extremes. In like manner, *nr* divided by *mn*, and multiplied first by AB and then by *mn* will be merely *nr* multiplied by AB; or the product of the mean terms, whence the product of the extreme equals the product of the mean terms.

The use of geometrical proportion is almost infinite, both in geometry and arithmetic; as well as in its applications to other sciences, to commerce, and to various arts.

The following is the arithmetical method of expressing geometrical proportions by numbers.

Suppose fig. 5, pl. 5, to have been made by means of a scale, we can represent each term of the proportional BF : AB : : *nr* : *mn*, by the number of times that these portions of the right line contain the unit of the scale.

If, for example, BF = 30, AB = 5. *nr* = 24, *mn* = 4, we shall have two identical proportions.

BF : AB : : *nr* : *mn*.
30 : 5 : : 24 : 4.

In this manner we may express the ratios and proportions of lines by the ratios and proportions of numbers; and conversely the ratios and proportions of the latter by the former.

If we divide 30 by 5, we shall have a quotient which expresses the first ratio, 6. If we divide 24 by 4, we shall have a quotient which expresses the second ratio, 6. The two being equal, there is a proportional. If we divide 5 by 30, we have a *sixth* as the quotient, and if we divide 4 by 24, we have also a *sixth* as the quotient; thus, when two ratios are equal, the inverse ratios are also equal.

The proportional 30 : 5 : : 24 : 4 gives then at the same time $\frac{30}{5} = \frac{24}{4}$ and also $\frac{5}{30} = \frac{4}{24}$.

If we multiply by 24 the two terms of equality, $\frac{5}{30} = \frac{4}{24}$ we shall have $\frac{5}{30} \times 24 = 4$.

Now 5 and 24 are the mean, and 30 and 4 are the extreme terms: thus one extreme term equals the sum of the mean terms, divided by the other extreme term.

It may in like manner be also demonstrated, that each of the mean terms equals the product of the two extreme terms divided by the other mean term.

When we know, therefore, three terms of a geometrical proportion, the fourth may be immediately found by the rule just pointed out. It is the rule of three; so named because with the three terms of a proportional given, we may find the fourth. The rule of three is continually used in calculations of finance, of trade, and of business; and geometry also has its rule of *three*.

If we have three lines (A) (B) (C), fig. 6, pl. 5, it is easy to find a fourth, D, so that we should have as follows.

(A) : (B) : : (C) : (D.)

Begin by making (C) = PR at the end of A = OP. From the extremity of O draw the right line O M in any direction; from the point O, take the length O Q = (B), draw PQ, and draw RS parallel to PQ. Then we have

OP : OQ : : PR : QS

Or (A) : (B) : : (C) : D.

When the two mean terms are equal to one another, the length or the number which represents them is what is called the mean proportional between the extremes. Thus, in the proportion

2 : 4 : : 4 : 8

4 is the mean proportional between the two extremes 2 and 8.

In geometry, two lines being given in length, their mean proportional is easily found. This will be hereafter explained.

Similar triangles.—If two triangles ABC, *abc*, fig. 7, pl. 5, have their corresponding sides parallel, these sides are proportional, and the triangles are similar. Thus

AB : *ab* : : BC : *bc* : : AC : *ac*.

To demonstrate this, let us move *abc* without changing the direction of its sides, so that the point *b* falls on A; produce *ac* and BC till they meet in a point *m*, and we shall have AC = *cm*, Cm = *bc*, because they are parallels, comprised between parallels.

But AC and *cm*, Cm, and *bc*, being parallel, we have

AB : *ab* : : *cm* = AC : *ac*

AB : *ab* : : BC : C*m* = *bc*

Whence AB : *ab* : : AC : *ac* : : BC : *bc*.

If two triangles ABC, *abc*, fig. 8, are so placed and formed, that AB shall be perpendicular to *ab*, BC to *bc*, AC to *ac*, these two triangles are similar.

In fact, changing nothing of the triangle abc, let us make it turn, to the extent of a right angle, round the point a, then ac will place itself in ac' in a position parallel to AC; it will be the same for ab', and $b'c'$; whence the triangle $a'b'c'$ will have its sides parallel to those of ABC, and the two triangles will be similar. Consequently ABC and abc are also similar.

When the sides of two triangles are proportional, their corresponding angles are equal, and the triangles are similar. Let us suppose that the two triangles ABC $a'b'c'$ fig. 7, pl. 5, have no other relation than this.

AB : $a'b'$: : AC : $a'c'$: : BC : $b'c'$.

Let us imagine a second triangle abc having the side $ab = a'b'$, and its three sides parallel, respectively to AB, BC, and AC, we shall then have A : ab : : AC : ac : : BC : bc. Whence

$$a'c' = \frac{AC}{AB} a'b' \ldots ac = \frac{AC}{AB} ab \ldots$$

$$b'c' = \frac{BC}{AB} a'b' \ldots bc = \frac{BC}{AB} ab \ldots$$

If $a'b' = ab$, $a'c'$ must be equal to ac and $b'c' = bc$.

The two triangles abc, $a'b'c'$, therefore, have their three sides respectively equal and they are consequently equal: thus the angles $a = a = A$, $b' = b = B$, $c' = c = C$.

Thus, whenever the sides of two triangles are proportionals, the angles opposite the proportional sides are equal, and the triangles are similar.

When two triangles, ABC, abc, have the sides AB, AC, proportional to ab, ac, and the angle $A = a$, the two triangles are similar; for if we place the angle a on A, the proportional AB : ab : : AC : ac requires that AC and ac should be parallel; then all the three sides are parallel.

If we draw, from the point O, fig. 6, pl. 5, three right lines OPR, OQS, OTU, cutting the two parallel lines PTQ, RUS, we shall have successively, in consequence of the similar triangles,

OPT, ORU; 1st. . . . OT : OU : : PT : RU,
OQT, OSU; 2d. . . . OT : OU : : QT : SU,
and finally, PT : RU : : QT : SU.

That is to say, that the parts PT, QT, RU, SU of the two parallels intersected by the three right lines proceeding from the same point, are *proportionals*. This principle is also true conversely.

We may now extend our researches, and demonstrate that two polygons having their corresponding sides parallel and proportional, are similar polygons.

Let ABCDEFGA *abcdefga*, fig. 9, pl. 5, be polygons, the corresponding sides of which are proportional and parallel. Thus AB : ab : : BC : bc— : : m : 1. The corresponding angles being formed by parallel lines taken in pairs will be equal; therefore the angle $b = B$. Draw the lines AC, *ac*; the two triangles ABC, *abc* will be similar, for they have the angle B equal to *b* included between two proportional sides. Whence AB : *ab* : : BC : *bc* : : AC : *ac* : : m : 1. Then draw AD and *ad*, the triangles ACD and *acd* will be similar for the same reason, for AC : *ac* : : CD : *cd* : : m : 1, and the angles ACD, *acd* are equal, their sides being parallel. Whence AD is parallel to *ad*.

By continuing the reasoning now begun, we shall at length finish by resolving polygons into similar triangles.

Consequently, if we know how to make similar triangles, we may gradually and successively form polygons similar to others, however complicated they may be.

The Sector, or compass of proportion, is represented by fig. 10, pl. 5, and is employed to facilitate proportional reduction of figures. It is composed of two rulers equal to one another, and equally graduated.

To reduce the dimensions of any figure from the ratio of a given line E, to the ratio of a given line F, we take on the side AB, the length $AM = E$. The number of the degrees corresponding to the point M is noticed, as also the point N, where the same number is found on the other graduated branch. With a pair of common compasses extended to the length of F, and placing one of their points on M, we open or shut the sector, until the distance between M and N equals F, it is then evident that every length A 1, A 2, A 3, on the two branches, must correspond to the distances 1.1, 2.2, 3.3, so that we have the following proportions

E : F : : AM : MN : : A1 1.1 : : A2 : : 2.2 : : A3 : 3.3...

With a pair of common compasses, therefore, we may immediately take the reduced lengths 1.1, 2.2, 3.3, which correspond to the lengths A1, A2, A3. . .

When we have not a sector at hand, we may make one by drawing two lines AB, AC, fig. 11, pl. 5, in the following manner. Draw first AB = E, from the point B as a centre, with an opening of the compasses, BC = F, describe the arc *mCn*; and from the point A, as a centre, describe the arc BDC. From the point C, where this latter arc intersects *mCn*, draw AC. If it is required to diminish, in the ratio of E to F, any length whatever, say Ag, from the point A as a centre, describe the arc *gkh*, the distance of the points *g*, *h*, is the reduced length, for we have

$$E : F :: AB : BC :: Ag : gh.$$

Of similar regular polygons.—Regular polygons of the same number of sides, are similar. The sides being equal to each other are proportionals, and the angles not depending on the length of the sides, but on the number of them, are the same in both polygons.

The perimeters (*contour*) of similar polygons, are to each other as mere sides or lines.

In proportion as the sides of a polygon are multiplied, it approaches to the circle in which it is inscribed.

We may therefore consider circles as similar figures, or figures, of which, the lines similarly placed, are proportional.

The circumferences of circles are to one another as their radii.

If in two circles we inscribe two regular polygons, having the same number of sides, *abcdefa*, ABCDEFA, fig. 12, pl. 5, the ratio of the proportional lines in the two figures will be, 1st. that of the radii of the two circles, 2d. the sides of the two polygons; 3d. the perimeters of the two polygons; 4th. the circumferences of the two circles.

If we draw a diameter AB, in the circle, fig. 13, pl. 5, and if from any point C, in this diameter, we draw CP perpendicular to it, and also draw the right lines AP and BP, we shall form a triangle APB, having a right angle at P. This rectangular triangle, is similar to each of the triangles APC, PBC, of which it consists.

In fact, the acute angle A, is common to the two right angled triangles APB, APC; the other acute angle equals a right angle, less the angle A; whence the three angles of the two triangles are respectively equal, and the two triangles are similar.

In the same manner, the acute angle B, is common to the two rectangular triangles APB, PCB; whence they are similar, which gives the following proportions.

AB : AP : : AP : AC
AB : BP : : BP : BC
AC : CP : : CP : CB.

Consequently, 1st.—In a rectangular triangle ABP, the short side to the *left* AP, is a mean proportional between the hypothenuse AB, and that portion AC of this hypothenuse which is to the *left* of the perpendicular PC.

2nd. The short side to the *right* BP, is a mean proportional between the hypothenuse AB, and the portion BC of this hypothenuse to the *right* of the perpendicular.

3rd. The perpendicular CP, is a mean proportional between the two parts CA, CB, of the hypothenuse.

Finally, the hypothenuse being a diameter of the circle, and CP the semi chord perpendicular to the diameter; AP, BP, being two other chords drawn through the terminations of the diameter, we have—

1st. The chord on the *left* AP, a mean proportional between the diameter AB, and the part of this diameter AC to the *left* of the half chord, perpendicular to the diameter:

2nd. The chord on the *right* BP, a mean proportional between the diameter AB, and the part BC, of this diameter, and to the *right* of the half chord perpendicular to the diameter:

3rd. The half chord CP, a mean proportional between the two parts of the diameter, placed on its right and left.

These geometrical properties are of great use in estimating the effects and the movements of machines.

SIXTH LESSON.

On the surface of plane figures, terminated by straight or circular lines.

When we wish to measure surfaces terminated by right lines, or even by curves, we take as the unit of our measure, a simple figure, equally easy to construct and to subdivide; viz. a square, the side of which is equal to the unit of length.

Let us first explain how we can measure a large square with a small one; or, in other words, ascertain how many times the large square contains the smaller one.

We can form, in the large square, as many parallel bands, as one side of the small square is contained times in one side of the large square, and these parallel bands will have the side of the small square for their breadth, and the side of the large square for their length. But as many times as the side of the smaller square, is contained in the side of the larger, so many times will the smaller square be contained in each parallel band. If, for example, the side of the larger contains the side of the small square ten times, the large square is divided into ten parallel bands, having the small side for their breadth: and ten times the length of this side is their length. Each band, therefore, will contain ten times the surface of the small square; and ten times ten will be the number of small squares contained in the large one.

By the same reasoning, it may be shown, that taking the side of any square for the unit of length, it will be contained in another square, having its side of an equal length once; if the side be twice as long, four times; if three times as long, nine times, &c., as in the following table:—

1...once one	=	1	6...six times six	=	36
2...twice two	=	4	7...seven times seven	=	49
3...three times three	=	9	8...eight times eight	=	64
4...four times four	=	16	9...nine times nine	=	81
5...five times five	=	25	10...ten times ten	=	100

The numbers 1, 4, 9, 16, 25, 36, &c., are called the squares of 1, 2, 3, 4, 5, 6, because they represent the number of squares, the side of which is equal to the unit of length contained in the superfices of the squares, the sides of which are respectively as 1, or 2, or 3, or 4. The numbers 1, 2, 3, 4, &c. representing the unit of the length of each square, are called the roots of these squares.

If the square to be measured is smaller than that assumed as the unit of length, the measure itself must be divided. For example, its sides may be divided into ten equal parts, and one hundred smaller equal squares being thus formed, each of them may be taken as the unit of measure. If this second unit is too large, it is again divided, if necessary, into a hundred equal parts, which will be one hundred times one hundred, or the ten thousandth part of the primitive unit; and so on to any extent. *(See Vol. the Second, Lesson on Measures.)*

After having determined the superfices of a square taken by itself, let us consider squares combined two and two, and see in what manner geometry can represent their sum, or their difference; or in other words, how we can construct one square equal in surface to the sum or the difference of any two given squares.

Let ABCD, fig. 1, pl. 6, and *mnpq*, fig. 2, be two given squares. Let us construct a right angled triangle, of such a nature that the right angle Y, fig. 3, shall be between the two sides XY = *mn*, and

YZ = AB. If we construct squares, having XY and YZ for their sides, we shall have XY*ab* = *mnpq*, and YZ*cd* = ABCD. If we construct the large square XZ*ef*, on the large side XZ, it will be equal to the sum of the two given squares.

It has been shown, in the second lesson, that in a right angled triangle, XYZ, fig. 3, pl. 6, if we let fall the perpendicular YU, from the right angle Y to the longest side, we shall have *

$$XU : XY :: XY : XZ, \text{ whence } XY \times XY = XY^2 = XU \times XZ,$$

$$ZU : ZY :: ZY : XZ, \text{ whence } ZY \times ZY = ZY^2 = ZU \times XZ.$$

Whence XY^2 plus ZY^2, or, in other words, the sum of the two squares XY*ab*, ZY*cd*, is equal to XU, plus ZU; that is to XZ, multiplied by XZ, which is the measure of the square XZ*ef*. Thus the large square equals the sum of the other two squares.

In any right angled triangle, therefore, the square constructed upon the longest side, is equal to the sum of the squares constructed on the other two sides.

If we had occasion to find a square, equal to the difference between two given squares, we should construct a right angled triangle, having its longest side equal to XZ, fig. 3, the side of the larger square, and one of its other sides equal to XY, the side of the smaller square. The third side of the triangle will be the side of a square, which will be equal to the difference of the squares of the other two sides; for if it be added to the small square, their sum will be equal to the largest square.

As we know that 3 times 3 = 9, 4 times 4 = 16, and 5 times 5 = 25; and that 9, plus 16 = 25, we see that 3, 4, and 5, represent the sides of a right angled triangle. Artizans make use of this property, to draw a right line, YZ, perpendicular to another line, XY. They divide XY into three parts; then taking YZ = 4, and XZ = 5, of these parts they construct the triangle XYZ, in which YZ is the required perpendicular.

Let us now proceed to measure the surfaces of figures, which differ from the square in form; beginning with those which most nearly resemble it.

* To represent the square, one side of which equals AB, we write AB^2, and the superfices of the square ABCD, is consequently AB^2, which expresses AB multiplied by AB.

The surface of a rectangle is equal to the base, multiplied by the height.

To prove this, divide MQ, fig. 4, pl. 6, into parts, each of which is equal to the side AB, of the square ABCD, taken as the unit of measure. Through the points of division, draw right lines parallel to MN, which will divide the rectangle into bands, each of which is equal in length to MN, and of the same breadth as the square. Each band will contain the superficies of as many squares, ABCD, as MN contains AB. Wherefore, MN being represented by numbers, the number of the squares ABCD, contained in the rectangle MNPQ, when AB is the unit, is represented by the base MN, multiplied by the height MQ.

There is often a necessity in the arts, to find a square, the surface of which, is equal to that of a rectangle, MNPQ, and it may be done in this manner.

Place the two sides MN, MQ, end to end, so as to form one continued line, fig. 5, and on it, as a diameter, draw a semicircle. At the point M, erect MR, perpendicular to the diameter QN, and produce it, till it reaches the circumference of the semicircle. We shall then have according to lesson 5th, page 9.

$$QM : MR :: MR : MN, \text{ whence } QM \times MN = MR^2.$$

Thus the square constructed on MR, will be equal to the rectangle MNPQ, for these surfaces are measured by the same lines.

The surface of a parallelogram, LMNO, *fig.* 6, pl. 6, is equal to its base, multiplied by its height.

To demonstrate this proposition, through the points M, N, draw MQ, NP, perpendicular to MN, and extend them to OLQ. The two triangles MQL, NPO, are equal; for MQ = NP, they being parallels situated between parallels, and the corresponding angles equal. Whence the rectangle MNPQ, compared with the parallelogram MNOL, contains in addition the triangle LMQ, and contains less than it, the equal triangle ONP; the surface of the parallelogram, therefore, like that of the rectangle, is measured by the base MN, multiplied by the height PN.

The multiplication table, or square of multiplication, gives us, in numbers, the surface of a rectangle, or of a parallelogram, the two sides of which do not exceed ten.

1	2	3	4	5	6	7	8	9	10
2	4	6	8	10	12	14	16	18	20
3	6	9	12	15	18	21	24	27	30
4	8	12	16	20	24	28	32	36	40
5	10	15	20	25	30	35	40	45	50
6	12	18	24	30	36	42	48	54	60
7	14	21	28	35	42	49	56	63	70
8	16	24	32	40	48	56	64	72	80
9	18	27	36	45	54	63	72	81	90
10	20	30	40	50	60	70	80	90	100

The second column indicates the surface of rectangles or parallelograms, which having 2 for their height, have as their base 1, 2, 3, 4, &c.; the third column indicates the surface of rectangles or parallelograms, which, having 3 for their height, have 1, 2, 3, 4, &c., for their base, and so on with the others. The use of this table is too well known to require any further exemplification; the knowledge of it is indispensable for every multiplication.

The surface of a triangle ABC, fig. 7, pl. 6, is equal to half of its base, multiplied by its height.

Draw CD parallel to AB, and AD parallel to BC, the new triangle ACD, is equal to the former ABC: but ABCD forms a parallelogram, the surface of which equals AB, the base of the triangle ABC, × by its height CE; whence the half of this product equals the surface of the triangle.

Since we can always divide any figure whatever, into triangles terminated by right lines, we can immediately ascertain the superficies of any polygon, regular or irregular. Taking for each triangle, the half of the product of its base, multiplied by its height, the sum of all the products, or of all the triangles contained in the figure, will be the measure sought of the whole superficies. This application makes the study of triangles of great importance, particularly in land-surveying. Let us begin the application of this principle, by measuring the trapezium.

The surface of a trapezium equals the half of the sum of its two bases multiplied by its height.

The trapezium ABCD, fig. 8, pl. 6, the height of which is *mn*, is divided by the diagonal AC, into two triangles ABC, ACD, the first of which is measured by $\frac{1}{2}$ of AB $\times$ *mn*; the second by $\frac{1}{2}$, DC $\times$ *mn*. The sum of these two products will be the half of AB, plus $\frac{1}{2}$ CD, multiplied by *mn*; and which is thus written $\frac{1}{2}$ (AB + CD) *mn*.

By means of this theorem, we may immediately find a square equal to the trapezium.

AB + CD, fig. 8, is measured and represented by the single right line MN, fig. 5. MQ is taken $= \frac{1}{2}$ *mn*; then describe the semicircle QRN, and the perpendicular MR, will be the side of the required square.

The surface of a regular polygon equals the half of its perimeter or outline, multiplied by the distance of its centre from one of its sides.

If through the centre, O, fig. 9, pl. 6, of the regular polygon ABCD, we draw right lines to its summits, we shall divide it into the equal triangles, AOB, BOC, COD....Let O*m* be the distance from the centre to each side of the polygon, and consequently the height of these triangles; we shall then have as the measure of one, and all of them, $\frac{1}{2}$ AB $\times$ O*m*, and the total superficies, will be $\frac{1}{2}$ (AB + BC + CD . . . .) O*m*, or $\frac{1}{2}$ (ABCD . . .) O*m*.

The difference between a regular polygon and the circle in which it is inscribed, decreases, as the number of its sides augments; and the difference becomes less than any assignable quantity, if we multiply sufficiently, the number of sides. We may thus regard the circle as a regular polygon, having so great a number of sides, that the perpendicular Om, does not differ by any appreciable quantity from the radius OA.

The surface of the circle, therefore, is equal to its circumference, multiplied by the half of its radius, or to its semi-circumference, multiplied by the radius.

Impossibility of squaring the circle.—By means of the method pointed out, fig. 5, for finding a square equal to a rectangle, it would always be easy to find a square, the superficies of which should be equal to any given circle, provided we could find a right line, expressed in numbers, exactly equal in length to the circumference of a circle, the radius of which was known. But there cannot be any measure of such a right line, and, therefore, the problem of finding a square equivalent to a circle, (called squaring the circle) is one of which a rigorous solution is impossible. Students should be careful, therefore, not to waste their faculties in attempting what cannot succeed.

We can give an approximation to the measure of the circumference, and of the surface of a circle, by numbers, taking

For the radius	100,	1,000,	10,000,	100,000, &c.
The circumference will be	628,	6,283,	62,831,	628,313, &c.
And the surface will be	314,	3,141,	31,415,	314,156, &c.

If, in place of considering the whole superficies of the circle, we confine ourselves to a *sector*, AOB, fig. 9, the arc of which shall be the half the third, the fourth, or any other part of the circumference, it will be seen that this part is also the half, the third, the fourth, &c., of the surface of the circle. To measure it, we must multiply the length of the arc AnB, included between the sides OA, OB, by half the radius. If we cut off from this product, that of $\frac{1}{2}$ AB $\times$ On = surface of the triangle OAB, we shall have the superficies of the *segment* AnB.

Comparison of the superficies of similar figures. First, of triangles.

The relation of the superficies of two similar triangles, is equal to the relation of the square of the corresponding or homologous lines.

Let the bases of the two triangles AOB, *aob*, fig. 11, pl. 6, be each equal to half their height: a square, ABCD, *abcd*, constructed upon their bases, as one of the sides, will be equal to their surface. If we diminish or augment the height proportionably, the base remaining the same, we shall form all sorts of similar triangles XAB, *xab*, which retaining the same base, the surfaces will augment or diminish, in the same proportion. The relation, therefore, of their superfices being originally represented by the squares ABCD, *abcd*, of their bases, will be represented by them in all cases.

All similar figures may be divided into the same number of similar triangles, which are to one another as the squares of two corresponding lines.

The surfaces, therefore, of similar figures, terminated by right lines, are to one another, as the squares constructed on the two corresponding or homologous lines.

Thus the two polygons ABCDEFA, *abcdefa*, fig. 12, pl. 6, being similar, their surfaces are, as the squares ABMN, *abmn*, constructed on the two corresponding sides AB, *ab*.

In the same manner, it may be demonstrated, that the surfaces of circles, which are similar figures, are in proportion to the squares constructed with their radii, or with their diameters, as sides.

These proportions are very convenient in practice. The surface of a circle, the radius of which is equal to the unit of measure, cannot be expressed with much precision, even by approximation, except by very complicated numbers. But the relation of surfaces, may often be expressed with great simplicity.

There are two important properties possessed by the surface of regular polygons and circles, which are to be noticed at present; though the demonstration of them will not be given, because it depends on scientific methods, which the student could not easily comprehend.

AMONG ALL THE FIGURES WHICH HAVE A GIVEN NUMBER OF SIDES, WITH AN EQUAL EXTENT OF OUTLINE

(CONTOUR) THE SURFACE OF THE REGULAR POLYGON IS THE LARGEST.

WITH AN EQUAL OUTLINE, THE MORE NUMEROUS ARE THE SIDES OF THE REGULAR POLYGON, THE LARGER IS ITS SURFACE.

WITH EQUAL OUTLINES, ALL FIGURES TERMINATED BY ANY NUMBER OF SIDES WHATEVER, WHETHER RIGHT OR CURVED LINES, HAVE LESS SURFACE THAN THE CIRCLE.

Applications.—To be acquainted with these properties, is of importance, economically, in several arts. The quantity of lead, for example, required for glazing any given space in the Gothic method, will be the least possible, if the *panes*, having a given number of sides, are regular figures.

When we have, also for example, to make pipes for conducting water, gas, &c., and these pipes are to allow the passage of a volume of fluid previously determined, if they are made circular, the quantity of wood, or metal required for these pipes, is the least possible.

In architecture, the height and dimensions of an edifice being given, and, consequently, the extent of its external walls, the space which can be enclosed by any given quantity of masonry, will be large, in proportion as the form of the building approaches that of a regular polygon, and in proportion as the sides of that polygon are numerous.

Let us now consider the indefinite surface of the *plane* on which we have drawn the various figures just measured. When any two points of a right line are in the same plane, the whole of the line is in the plane. This property serves in the arts to construct plane surfaces and to survey them.

Application to the manufacture of porcelain.—If the workman wishes, for example, to mould the clay into a plane surface, he uses two parallel guides, or a frame-work, MN PQ, fig. 13, pl. 6. A straight piece ST, being supported by the two guides MN, PQ, is advanced progressively, and every part of the clay which projects above the plane, passing through MN, and PQ is either removed or compressed. It is not indispensable that the frame MNPQ should be constructed of straight parallel pieces; it will be sufficient that the sides, if produced, should meet at some point.

Application to cutting piles.—Saws for cutting piles according to a horizontal plane,—the depth of which beneath the water is given,—are regulated in their movement by two guides MN,PQ, fig. 13, both equally distant from the horizontal plane according to which the head of all the piles is to be fashioned; the saw itself is a transverse right line *st*, represented by its parallel ST. This parallel being kept at an invariable distance from the saw by the rectangular frame ST*ts*, and resting on MN, and PQ, the saw describes a plane *mnpq* parallel to MNPQ.

Joiners and carpenters to smooth their work make use of *planes*. They begin by making the edge of a plank exactly rectilineal by means of their plane, the wooden part of it (or the plane itself) being in a right line, and the *iron* cutting away whatever is too projecting, till the contact is equal throughout, between the wood of the plane and the plank. They then plane the sides, proceeding from the prepared side to the other, tracing, in fact by the instrument, a succession of right lines intermediate between those of the edges.

The sawyer or the carpenter marks, above and below, on the piece of wood, or on the plank, the side of which he wishes to smooth or reduce, the outline of the *plane* he is to form; and then directing the strokes of his axe or the movement of his saw by these lines, the piece gradually assumes the desired shape.

Hitherto we have only considered one plane at a time, and the lines drawn on this plane. Let us now examine a plane in relation to lines, all the parts of which are not included in it, and let us compare several planes with one another. A right line may be either perpendicular, oblique, or parallel, to any given plane.

Let AB, fig. 14, pl. 6, be the shortest line which can be drawn from the point A to the plane MNPQ. It will be, consequently, the shortest line which can be drawn from the point A to any right line drawn on the plane; and it will therefore be perpendicular to all the right lines BE,BF, drawn on the plane, proceeding from the point B, the termination of this perpendicular. The right line AB is said to be perpendicular to the plane MNPQ.

A perpendicular, therefore, drawn from any point to a plane is, *first*, the shortest distance from the point to the plane, and *secondly*, it is perpendicular to all the lines in the plane drawn through its termination.

Consequently, if we take a carpenter's square and make it revolve on one of the sides of the right angle, the other side necessarily describes a plane.

This geometrical principle comes into use in constructing many optical instruments, particularly some used in the arts of navigation and astronomy.

AB, fig. 14, pl. 6, being perpendicular to the plane MN PQ, every line AD, AF, drawn from the point A to one of the lines DBF, in the plane, is oblique to the line AB and to the plane. Thus for the plane, as for the right line, the oblique lines AD, AF, are all longer than the perpendicular AB, and longer in proportion as they deviate from this perpendicular.

Let us suppose that from the point A, all the oblique lines possible have been drawn to the right line DBF, being in the plane which passes through the termination of the perpendicular; each point D, F,...of the right line DBF, will, if the plane MNPQ, be made to revolve, describe a circle in it, and all the points of each circle will be at the same distance from any point A of the perpendicular.*

The perpendicular to the plane of the circle, which passes through its centre, is called the axis of the circle; and consequently this axis is perpendicular to all the radii.

The axis, or the axle-tree of any wheel, is perpendicular

* If we prolong BA until *Ba* = AB, the points A and *a* will be equally distant from every point D, E, of the plane, and of the circles described.

to the plane of this wheel, and, consequently, when the wheel turns on its axis, each part of it moves without quitting this plane. The wheel, therefore, in relation to surrounding objects, does not change its position, but its several points take successively the position of one another.

By this geometrical principle, the motion of mill-stones is regulated. Two stones being placed on the same axis, the planes of both are consequently parallel; one remains stationary while the other, moving with the axis, is fixed on it. The moveable stone, turning so that its lower surface moves on itself, is always at the same distance from the upper surface of the fixed stone; and if this distance be such that the grains of wheat or of other plants cannot pass without being crushed, they will be equally ground at every point between the two stones.

From this example we may learn the great advantage, I may even say the necessity, for precision in constructing machinery. If the parallelism of the stones were not perfect,—if the axis of the revolving stones were not rigorously perpendicular to the plane of both; if when in motion it could lean either to the right or to the left, the two contiguous faces of the stones would be no longer in the same plane, nor always at an equal distance from each other. In the parts where the stones were too close together, the wheat would be too much crushed, it would be heated and spoiled; in the other parts it would not be ground at all, and the stones would move apart from each other. In this case precision is something more than a luxury, or than an intellectual pleasure, it is a condition imposed by the nature of the operation, and is indispensable to its success.

Application to turning.—The properties just pointed out are employed in the arts to describe circles, by means of the turning lathe. In it there are two fixed points between which is placed the article to be turned. When we hold a cutting instrument immovably against a revolving body, the instrument cuts away all the parts which project too much, and describes a circle, having for its axis the right line which passes between the two fixed points, and which has moreover its centre on this right line.

If we suppose that the edge of the instrument advances by degrees, remaining perpendicular to the right line, all the circles successively described by the instrument will be in the same plane, and perpendicular to the right line which passes between the two fixed points. We may use a turning lathe, therefore, to produce a plane.

In fact, in the manufacture of machines, when it is necessary to make metallic plates, or the ends of cylinders, with a plane surface, very true, so that they can be adjusted with great accuracy to each other, this principle is acted on.

Bramah's machine for forming a plane surface.—Mr. Bramah causes a horizontal wheel, provided with a great number of cutting instruments, to revolve round a vertical and fixed axis. These instruments do not all project an equal distance from the plane of the wheel, but are grouped by fives or sixes, projecting gradually, further and further. The piece of wood, or other substance, to be reduced to a level, or made a perfect plane surface, is placed on a movable horizontal frame, which can be made to advance and pass under the revolving wheel. The least projecting of each group of cutting instruments, first removes the most prominent parts of the wood forming one surface, which is further reduced by the next group of projecting instruments, and the piece, as it continues to advance, is smoothed and levelled at every point, by each successive group of the cutting instruments. When all the narrow chisels in the whole surface of the wheel, have performed their operations, a plane fixed on the wheel, at the same level, with the most projecting of them, passes over the wood, which has been grooved so to speak by the chisels, and cutting down all the remaining inequalities, finishes by rendering the piece a perfect plane.

Two lines AB, CD, fig. 15, pl. 6, perpendicular to the same plane, MNPQ, are parallel to each other.

To demonstrate this, draw the right line BD, in the plane, and through the terminations B, D, and through the middle O, of BD, draw the perpendicular EF. Making OE = OF, the two points, B, D, will be equally distant from E and F. Moreover, every point A, C, of the lines AB, CD, perpendicular to the plane MNPQ, is equally distant from the points E and F. In fact, if we draw FD, and ED, these two oblique lines being equally distant from the perpendicular OD, towards EF, are equal. In the same manner, CE, CF, being also two oblique lines, equally distant from the perpendicular CD, of the plane, are also equal; AE and AF are equal, for the same reason. Thus, the perpendiculars AB, CD, belong only to the plane, which contains all those points, which are equally distant from the two fixed points E and F; whence AB, CD, perpendicular to the same right line BD, are in the same plane, and are, therefore, parallel.

The horizontal plane is that of the surface of tranquil water at the place where we are, and the perpendicular to this horizontal plane, is called the vertical; consequently, all the perpendiculars to the same horizontal plane are parallel.

It must be remarked, however, that all the lines which are perpendicular to the plane of the visible horizon, are not verticals. There is only one which can have the direction of the plumb line. Every vertical, if produced, would pass through the centre of the earth, and there cannot be, at the same time, *two* different right lines, which pass through this point, and any point in the plane of the horizon; because right lines, having two points in common, coincide, and form only one. Horizontal lines are not required to pass through any given point; it is only necessary that they should be at right angles with the plumb line. It follows, from what has just been said, of every vertical which is perpendicular to the plane of the horizon, passing through the centre of the earth, that no two such lines are exactly parallel to each other. A ship's mast may be taken as perpendicular to the plane of the horizon, at every moment, while she is circumnavigating the globe. But the various perpendiculars it forms, with the horizon at different points of her course, so far from being parallels, will form angles with each other, at every degree of inclination, till, in the neighbourhood of New Holland, for example, the perpendicular it forms, will be only a continuation, in the opposite direction, of the perpendicular it formed to the horizon, when the vessel was in the Thames. Thus, although all perpendiculars to the plane of the horizon appear to us parallels; such, for example, as the sides of our houses, strictly speaking, they are not; any more than the curved surface of the earth is, as it appears to us, a perfect plane. In fact, all verticals are the radii of a circle, having its centre at the centre of the earth; being all perpendicular to the curved surface of the horizon.

The plumb line is a cord held at one end, and having at the other end, a piece of lead. When, in a state of rest, this line takes a direction perpendicular to the place where the person is. It may, therefore, serve to ascertain in any spot, if any given plane is horizontal. It is only necessary to place one side of a square or level, in the direction of the plumb line, and ascertain if the other side applies exactly to the plane in every possible direction. Two positions are sufficient to verify the plane; because two right lines are sufficient to determine the position of any plane.

Having the position of a horizontal plane, we may reciprocally obtain a vertical in drawing a perpendicular to this plane. But this operation is not so easy as the former.

Vertical planes are those which are vertical through their whole surface. If, from any point of such a plane, a vertical line is drawn, as it is parallel to the first vertical placed in this plane, it must also be wholly in it.

Two vertical planes necessarily intersect each other in a vertical right line, because the vertical drawn from any point where these two planes intersect each other, must be entirely in both of them.

In a great number of arts, and particularly in those which relate to the construction of buildings, frequent use is made of horizontal and vertical planes, as well as vertical lines.

In our houses, the floors, the ceilings, the upper and lower surfaces of the bricks and stones of the walls, are all horizontal planes.

External and partition walls are vertical planes, and the edges of the walls of the door posts, of the windows, &c., are verticals, they being at the same time, in two vertical planes.

In the designs of descriptive geometry, of stone masonry, of carpentry, and of architecture, generally, it is supposed that the first design is on a horizontal plane, the second on a vertical plane; if the latter be the plane of the outside of the edifice, it is the elevation; if it divides or traverses the building, it is the section.

When a right line passes through two points A, C, fig. 16, pl. 6, equally distant from a plane MNPQ, all the other points of this right line AC, are at the same distance from this plane.

If, setting out from AC, we draw the parallels AB, CD, EF, perpendicular to MNPQ, drawing on this plane, the right line BFD, we shall have AB = EF = CD, whatever may be the position of the point E.

The whole of the right lines proceeding from the point A, fig. 16, perpendicular to AB, form a plane; whence all the points of this plane, have AB, as the measure of their distance from the plane MNPQ. Thus, two planes perpendicular to the same right line, are at every point, at the same distance from each other, and the lines AB, CD, being perpendicular to one, are also perpendicular to the other; measuring the shortest distance between these planes.

Two planes, NPQM, NPRS, fig. 17, pl. 6, which meet, intersect each other in a right line NP.

If, from the two points of intersection N, P, we draw a right line, it will necessarily be altogether on the two planes, which contain these two points. It will, therefore, be common to both planes.

It may be supposed, that the plane NPQM, is inclined, more or less, towards NPRS, we shall then have an angle, larger or smaller, contained between NPQM, and NPRS. To measure this angle, we proceed as follows.

Draw, fig. 17, CA, in the first plane, and CB in the second, perpendicular to NP, the right line common to both planes. The angle formed by the two planes, is represented by the angle formed by these two right lines.

Let us suppose that the plane NPQM, revolves round NP, as an axis. Each of the points in this plane, will describe a circle ; and the plane itself will have passed through the whole of the space around the axis, when each of its points shall have described the whole circumference of a circle. If we divide, into equal parts, the space thus passed through, each point, will have described, in each equal part, the same number of degrees. This number may be employed to measure the angle itself, of planes revolving round NP.

Mathematical instrument makers construct instruments for the use of mariners, geographers, and astronomers, adapted to measure the angle, which one plane makes with another, and, in general, they are executed on the principle just pointed out. An arc of a circle, graduated, AB, fig. 17, pl. 6, is placed in a plane, determined by the legs of the cross staff, CA, CB, perpendicular to the planes, the inclination of which is to be measured. One extremity B, is fixed on one of the planes, and the point A, where the arc traverses the other plane, indicates the number of degrees of inclination between the two planes.

To ascertain the direction of planes, they are in general referred to some horizontal plane ; and the intersection of the inclined plane with the horizontal plane is called the direction of the inclined plane. Consequently, if we conceive, at right angles to this direction, 1st. a horizontal line, 2d. a right line placed in the inclined plane, the angle which they form with one another, will represent the angle formed by the two planes.

The inclined line AC, fig. 17, pl. 6, which has just been determined, is more inclined than any other line placed on the inclined plane NPQM.

To demonstrate this, draw the horizontal line XOY, parallel to the direction NP of the inclined plane, and COA perpendicular to the two parallels, and CO will measure the distance between them.

To descend, therefore, from the points XOY of the inclined plane, which are all situated at the same height, to the points P, C, N, which are also all on the same level, the shortest road, or the line of greatest inclination is OA, perpendicular to the two parallels XOY, PCN.

When we come to speak of curved surfaces, it will be seen that horizontal lines and lines of the greatest inclination have been advantageously employed to represent on planes the forms and proportions of these surfaces.

Two planes are perpendicular to each other when they form to the right and left, angles which are equal to each other. These angles, if measured by perpendicular right lines, are right angles.

When a right line is perpendicular to a plane, all other planes formed by this right line, are also perpendicular to this plane.

Let AB, fig. 18, pl. 6, be perpendicular to MNPQ and FGDE, a plane formed by AB. Draw AC on MNPQ perpendicular to GD; the angle BAC which measures the inclination of the two planes will be a right angle. The two planes, therefore, will be perpendicular to each other.

When two planes, parallel to each other, are intersected by a third plane, the two right lines of intersection are parallel. In fact, if they were not they would meet at some point, whence the first and second planes of which they are a part, would also meet, and could not consequently be parallel.

Two parallel right lines, included between two parallel planes, are equal. If we draw through these two right lines, a third plane, it will intersect the two other planes, according to two other parallel lines, which will include the two former; but parallels contained between parallels are equal.

Two right lines ABC, DEF, fig. 19, pl. 6, intersected by three parallel planes NP, QR, ST, are divided into proportional parts.

To demonstrate it, draw A*ef*, parallel to DEF, the points E, F, *e*, *f*, being the points of intersection of the right lines, and the planes

QR, ST, we shall have Ae = DE ; ef = EF. But the two right lines ABC, Aef, are both in the same plane, which intersects the two planes QR, ST, in the direction of two parallel right lines Be, Cf. We shall therefore have

AB : BC : : Ae : ef : : DE : EF.

We come now to treat of solid angles, such as OABC, fig. 20, pl. 6, formed by the three right lines OA, OB, OC, meeting at the point O, and representing three portions of the planes AOB, BOC, COA. This angle, as will be seen, presents three common angles AOB, BOC, COA, and three angles formed by the planes taken in pairs. Descriptive geometry teaches the means of ascertaining the angles formed with planes, by angles formed with lines, and conversely. The consideration of these will be reserved for the next lesson.

SEVENTH LESSON.

*Solids terminated by plane surfaces.**

WE have already examined the properties of right lines and circles, and the figures which may be formed by them; and, following a similar method, we are now to treat, first, of solids terminated by plane surfaces, and afterwards of those terminated by circular forms.

Two bodies are said to be equal, when it may be supposed that they have issued from the same mould, such as the copies of busts and of *bas reliefs*, modelled by the sculptor or the figure maker.

* At the end of this lesson, M Dupin recommends, "that teachers in explaining it, should make use of small figures of prisms and pyramids, which are *equal, similar, symmetrical,* &c. The subsequent lessons ought also to be explained, by showing the pupils *cylinders, cones, spheres,* &c., both in relief, and in well-executed sections." It was thought more advisable, to place this recommendation at the beginning, than at the end of this lesson; in order that those who teach themselves, may be in time aware of the difficulty attaching to this part of the subject, and endeavour to procure for themselves some solid geometrical figures. Mr. Donne, of Bristol, constructed for sale, a set of figures, with descriptions, to enable the student to use them, which have been strongly recommended by Dr. Beddoes, in his "Demonstrative Evidence;" and by Mr. Edgeworth in his Practical Education; and Professor Leslie, of Edinburgh, was long in the habit of accompanying his mathematical instructions, by visible solid representations of the geometrical properties he explained. Tr.

Two bodies are said to be symmetrical in form and position, when the corresponding points of both may be joined by parallel right lines, the middle of which is on a plane, called the Plane of Symmetry, perpendicular to the plane of these figures.

Application.—In the arts, it is continually required to make bodies which are symmetrical, in relation to some other bodies; and to make objects which are composed of two symmetrical parts. Regular buildings, of every description, constructed on a single plan, such as palaces, temples, &c., are examples.

In such works of art as houses, churches, &c., which are to remain immovable, symmetry is only an object of luxury or of good taste; but in a multitude of bodies, destined to perform certain movements with equal facility, both to the right and to the left, it is an object of necessity. Nature, we see, on this principle, has given to most animals two symmetrical sides, united by a plane, lying in the direction of their customary progressive movements. On the same principle also, the naval architect gives to all his vessels two sides, the larboard and starboard, which are symmetrical, in relation to the plane, which lies in the direction of the ship's forward movement. Carriages are made symmetrical, in relation to a plane lying in the same direction, on an analogous principle. (This subject will be more fully explained in the second volume on Machinery.)

A *bar* is a solid of indefinite length, its faces or sides being planes, the boundaries or edges of which are parallel right lines. A *prism* is formed by dividing the bar crosswise, by two parallel planes. Each section, called a *base*, is a polygon, the number of its sides being equal to the number of the faces of the prism, or the bar. The prism is right angled or oblique, as the two bases are perpendicular or oblique, in relation to the edges. It is truncated when the bases are not parallel.

The *right angled prism* is symmetrical, in relation to a plane which intersects all the edges through the middle, at right angles; the edges being the perpendiculars, which determine the conditions of symmetry.

Some truncated *prisms* are symmetrical, in relation to a plane, which passes in like manner through the middle of all the edges, at right angles.

The *triangular prism* has three faces, fig. 1, pl. 7, and two triangular bases; and there are as many varieties in the form of this prism, as there are varieties in the form of triangles.

Application to Optics.—Prisms of this description, made of glass or crystal, are employed to decompose light, the different rays of which, separate from one another in penetrating one face of the prism to enter it, and in penetrating another to come out. We then see, on any surface which is placed to receive the light transmitted through the prism, the seven primitive colours, in the following order, *viz.*, red, orange, yellow, green, blue, indigo, violet. The figure formed by these separated rays, is called the Solar Spectrum.

Application to Architecture.—Architects employ the right angled triangular prism, with symmetrical bases, ABCDEF, fig. 7, pl. 7, to form the roofs, with two faces, pediments, or gable ends, of regular buildings. Truncated symmetrical prisms, fig. 8, are used for roofs of buildings with four fronts. As this figure is regular, and easily measured, stones are piled or heaped up in this shape, so that the quantity they contain may be almost instantly determined. For the same reason, bomb shells and cannon balls are piled up in artillery depots, and fortified towns in the same form.

In the construction of machinery also, the triangular prism, with symmetrical bases, is employed as a fixed guide, on which frames or carriages, the motion of which must be rigorously rectilineal, are made to slide.

The quadrangular prism, fig. 2, pl. 7, as its name indicates, has four faces or sides, and each of its bases is a quadrangle. When this quadrangle is a parallelogram, the prism is called a Parallelopipedon. When all its sides form right angles with each other, it is a rectangular parallelopipedon. If the bases are squares, it is a square parallelopipedon. When all the faces of the parallelopipedon are squares, it is called a Cube. Playing dice are of this shape.

Right angled prisms, with symmetrical bases, have their planes of symmetry parallel to the edges; and they pass respectively through the axis of symmetry of each base.

When the base is a rectangle, the prism has three planes of symmetry, respectively parallel to the six faces taken two and two. When the base is a lozenge, the prism has three planes of symmetry: 1st. the plane equally distant from both bases: 2d. and 3d., the planes which pass through the parallel diagonals of the lozenge bases.

In the cube there are twelve planes of symmetry, three parallel to the sides or faces, six which pass through the diagonals of the sides, and three through those of the cube.

In each of these prisms, the planes of symmetry pass through a remarkable point which is the centre of the prism; they intersect each other two and two in the direction of the lines which are the diameters or axes of the prism. This point and these lines possess some properties of importance in mechanism; which will be explained in the second volume.

The cabinet-maker, the carpenter, the smith, and a number of other artizans, make frequent use of symmetrical quadrangular prisms. The joists and beams of our houses, the rafters, and almost every other piece in the roofs, are prisms of this description. Formerly these pieces were of the form of square prisms; but since the strength and stress of timber have been better known, the advantages of having the pieces thin in the direction in which they have little to sustain or resist, and thick in the direction in which they have much to support, have been recognised: and they are now generally so made.

Pilasters and square pillars are rectangular parallelopipedons.

Crystals.—Nature presents to us, in the crystals of different substances, geometrical figures as varied as they are precise and regular; such as triangular and quadrangular prisms, hexagons, octagons, &c. The study of the forms of these crystals and the classification of substances according to them, is one of the most beautiful applications of geometry which has ever been made. It has even afforded considerable information on the nature of the substances of which the crystals are composed. By dexterously dividing the crystals according to the joinings or unions of the several faces of their primitive forms, geometry has enabled us to classify and arrange all their varieties; and to prove that, even in the midst of the greatest apparent irregularities, the forms of nature are constant and invariable.

Let us now point out the method of making of a solid, of any figure whatever, a right angled prism.

Near the object to be formed into a prism, a cord is stretched parallel to the direction in which the edges are to lie, and which, to facilitate the operation, we will suppose is horizontal. One side of a square held horizontally is placed against the cord. With a plumb line carried along the other side of the square, we mark upon the object a succession of points which belong to the base of the prism to be constructed. With an axe or saw, or other convenient instrument, the object is then fashioned in the direction of the vertical plane which passes through the points thus marked. On this plane the polygon which is to form the base, is traced. Setting out from each summit of this polygon, we make a series of notches, the bottom of each of which is in a direction perpendicular to the base; and these form the edges of the prism. From each edge to the next, the projecting pieces are cut away, and the substance smoothed by the means pointed out in the sixth lesson. To prove the accuracy of the operation, we must satisfy ourselves that the edges are perpendicular to the plane of the base, and consequently to the sides of the base which meet each edge. As an additional proof, it may be ascertained if all the edges are, at every part, at an equal distance from one another, which is indispensable; and, if taken pairwise, that they are all exactly in the same plane; which may be perceived by examining with the eye, whether each edge completely conceals all the points of the one which immediately precedes or follows it. To form the second base, draw on the face of the prism, by means of a common square, a series of lines perpendicular to the edges; the last of which must terminate exactly at the point whence the first commenced. Such is the method employed by house carpenters and shipwrights.

When the first face of the prism has been fashioned, and the contiguous faces are to be formed, the angles which these faces ought to have in relation to one another, or to the base, are measured and set off by the square or bevel.

On the face which is to be formed, notches are cut at certain distances from each other, and to such a depth that one branch of the bevel will just touch the bottom, while the other is applied to the surface already prepared; both branches being held perpendicular to the edge which divides the prepared face from the face to be completed. The bottom of the notch then forms of course, a part of the latter. After having drawn, at proper distances, the lines which are to serve as guides, it is only requisite to cut away the substance between them in order to form the new face of the prism.

In geometry, figures, capable of entering exactly one within the other, both solid and hollow, are represented by lines which in their extent and position, have no difference whatever. But in practice, the difference between the same figures is very great. The construction of prisms supplies us, as they are solid or hollow, with an example. The means of making a solid prism, by gallopers, squares, bevels, and cutting instruments, has just been explained. Let us now suppose that we have a hollow or sunk prism to cut out or make, such, for example, as common chests or boxes, which have the geometrical figures of a rectangular parallelopipedon.

We begin by reducing the planks to the necessary thickness, and being properly squared and of the requisite length and breadth, each of them will form a face or side of the hollow prism to be made. Two of these planks are cut according to the length and breadth of the box, and two according to the length and height, and two according to the breadth and height. They are then fastened together at the edges by nails or glue. Sometimes one of the sides is made to move on hinges, and is fastened with a lock. If the planks have been exactly squared when they are thus united, they will necessarily form a parallelopipedon. It must, however, be remarked, that the different planks may be united by beveling both the edges to the angle of 45, as in fig. 3, pl. 7, A*a*, B*b*; or by lapping one over another as in fig. 4.

When the box is to be made of such a size that the breadth of a plank is not sufficient for its sides, two, three, or more are united together side-wise; and if the work is of a rough kind, cross pieces are nailed over all the planks on one or both sides. Common packing-cases are of this description.

If the work is to be of a better kind, the planks are joined by cutting the edge of one into several tongues in relief, BDQP, fig. 5, pl. 7, and by cutting in the other plank corresponding grooves, BDMN, so that the pieces in relief are exactly adapted to the hollows.

The tongue, fig. 5, has the geometrical form of a rectangular prism in relief, while the groove has the same form, but hollowed out. Both may be fashioned by a plane made for the purpose.

The *tenon* and *mortice*, fig. 6, pl. 7, are also two rectangular prisms, one solid, the other hollow, which, like the tongues and grooves, are so made that they fit exactly into each other. When two prisms are to be united at right angles, this method is used. The tenon may be cut by means of a saw. The mortice on the contrary is made by the chisel, and requires more time to make than the tenon. It is another example of the difficulty, different in both cases, experienced by the workman in making similar solid or hollow prisms.

The practical arts of the carpenter and joiner offer, besides the examples just mentioned, several other ingenious and simple applications of geometrical figures, terminated by planes; some being solid and others hollow, and which fit exactly into one another.

Carpenters have very often to construct, or rather to shape out the figure of a prism by pieces of wood which form the edges of it, as in building roofs. For example, fig. 7, pl. 7, represents the wood-work of a roof having the form of a triangular prism, placed on the top of a quadrangular prism, or a wooden house of a rectangular form. To build such a house, a carpenter must be able to solve many geometrical problems, which he will easily perform by the principles already laid down in these lessons. He should, for example, be able to ascertain the dimensions, and draw every part of the work of its proper form, with its angles rigorously exact; and he should be able to transfer them accurately to the pieces of wood he is to fashion. It is of great importance for a house carpenter to be acquainted with the principles of geometry, in order that he may apply them generally, and not be stopped by an unusual occurrence. In such a case, if he is ignorant of these principles, he has no other resource, but to operate, as it were, by chance, feeling his way, and repeating his labour before the required task is accurately accomplished.

Geometry is even of still greater importance to the shipwright, because he has to construct more complicated and better-combined forms, the excellence of which depends on their being rigorously executed.

A *pyramid* is a figure which, in appearance, is more simple than the prism, for it has fewer faces or sides, but is in relief more complicated, because its faces are not parallel.

Pyramids, figs. 9, 10, 11, 12, 20, pl. 7, are composed of plane triangular faces having their summits at the same point, their base forming a plane polygon. The base of a pyramid is, therefore, a polygon, and the common summit of all the triangular faces is its summit.

A symmetrical pyramid has for its base, a symmetrical polygon, and its summit is in the plane of symmetry.

A regular pyramid has for its base, a regular polygon; and its summit as also the centre of the base, must be situated on a right line, perpendicular to the plane of the base. Thus, if we suppose the plane to be horizontal, the summit of the pyramid ought to be in a plumb line, with the centre of the base; the plumb line, in this position, represents the axis of the pyramid.

A triangular pyramid, OABC, fig. 12, has for its base a triangle, ABC. A quadrangular pyramid, ABCDE, fig. 19, has a quadrilateral figure BCDE, for its base.

The roofs of towers or steeples, which are triangular or square, are pyramids, having for their base, the triangle or the square formed by the cornice of the steeple or tower, figs. 9, and 10.

Obelisks are regular pyramids, employed in general for public monuments. Let us suppose, that such an obelisk is to be cut out of a stone quarry, and that it is lying down, its axis being horizontal, and its base vertical.

A vertical plane is cut in the rock or stone, on which is drawn the square BCDE, fig. 11, pl. 7, which square is to be the base of the obelisk. The upper face ACD, and the two contiguous faces, ACB, ADE, are then cut out, taking great care that the angles formed by the faces ACD, ACB, ADE, with the plane of the base, are perfectly equal to those of the intended obelisk. This operation is verified by ascertaining that the summit, A, is on a right line, AO, perpendicular to the plane of the base, and passing through its centre O. For this purpose, taking OM in the plane of the base, and AN, parallel and equal to OM, we find, by means of a square, if the right line MN, which ought to be parallel to OA, is at right angles in two directions, with both AN and OM. If this be the case, the axis OA, will be perpendicular to two right lines, drawn through the point O, in the plane of the base; and, consequently, perpendicular to this plane. These circumstances being ascertained, and the errors they point out, being rectified, it will only be necessary to work out the lower face, ABE, the plane of which is determined by the edges AB and AE.

Let us now suppose, that it is required to cut a triangular pyramid of any form, out of a block of stone or wood, and that we know the figures of the base, and the angles formed by the plane of the base, with the plane of each of the three other faces.

We first mark and fashion one surface into a complete plane, according to the method pointed out in the sixth lesson; then, by means of a bevel, the two branches of which are placed at right angles to the sides of the base, three plane surfaces, ABO, BCO, ACO, fig. 12, pl. 7, are marked out, forming quadrangles with the base. These are the three faces of the pyramid.

Very often the position of the summit is only given by the point *m*, fig. 12, where the perpendicular O*m*, terminates in the base, and by the height O*m*. Then, when the base has been marked, it is placed on a level, and two heights, NP, QR, are measured by the plumb line equal to O*m*, the points Q, N, being taken on a level, with the plane of the base, OR is then drawn equal to *m*Q, and OP to *m*N; and the point O, where the two horizontal lines OR, OP, ought to meet, is the summit of the pyramid. The summit being known and marked out, the block of stone or wood is cut away, by first making notches in the direction of the right lines, OA, OB, OC, and afterwards cutting away the substance contained between these right lines.

In certain cases, it would be much more simple to begin by laying off on the base, by means of a small geometrical sketch, the angles of the three sides, and then to work them out, without troubling one's self to find out the position of the summit.

It would be sufficient, for example, fig. 13, pl. 7, from the foot *m*. of the perpendicular O*m*, let fall from the summit to the base, to draw the lines *mn*, *mp*, *mq*, perpendicular, respectively, to AB, BC, CA, then to construct separately, the right angled triangles, O*mn*, O*mp*, O*mq*, the angles O*nm*, O*pm*, O*qm*, will be those formed with the base by the three faces of the required pyramid.

The elements necessary to form a triangle, have made us acquainted with the conditions necessary to constitute two triangles equal. The same conditions are applicable to pyramids. Two triangular pyramids are equal; 1st. when the three faces of one, are equal to the three faces of the other: 2nd. when two faces, and the plane angle, which they include, are equal in both pyramids: 3rd. when one face, and the three plane angles belonging to it, are equal: and 4th. when the six edges or lines, are equal in both pyramids, &c.

An acquaintance with all the properties, and the means of calculating the surface of pyramids, is of considerable importance in topographical operations, when places, the position of which is to be ascertained, are not in the same plane. The position of each place or point observed, is in this case referred to the position of three other points, forming a triangle, taken as a base. With such instruments as the graphometer, the repeating circle, or the theodolite, we measure the angle, which the visual ray directed from each summit of the triangle taken as a base to the observed object, forms either with one side, or with the plane of the base. The three visual angles, united with the three sides of the base, form a pyramid, the summit of which is the position of the point observed. These complicated operations are only performed by those who belong to the learned professions, such as hydrographers, or geographers, or land surveyors, who have extensive districts to measure, for the purpose of valuing the land.

When a body is terminated on all sides by plane surfaces, these surfaces are terminated by right lines, forming plane polygons; and we can decompose, as has been shown, all these polygons into triangles.

If we take, therefore, a point O, in the interior of a body, ABC,...fig. 21, pl. 7, we may regard it at our pleasure; 1st. as the summit of as many polygonal pyramids, as there are polygon faces to the body; or, 2nd. as the summit of as many triangular pyramids as we can draw triangles on these faces. In both cases, the whole of these pyramids will represent the whole body.

Measure of solids terminated by plane surfaces.—The square is employed to measure surfaces; to measure volumes the cube is employed, or a solid terminated on all sides by squares.

To take the cube, or measure the solid contents of any body, is to determine how many times it contains the cube, which is assumed as the unit of measure. Let us begin, by showing how we measure the volume of a large cube, by a small one.

Let us suppose, for example, that the side of the large cube C, fig. 14, contains ten times the side of the small cube *c*, and let us divide the large cube, in a direction parallel to one of its faces, into ten portions, of equal thickness. This will be the thickness of the small cube. The bases of these portions, containing ten times ten times one of the faces of the small cube, each portion will contain ten times ten small cubes. The ten portions, therefore, will contain in all, ten times ten times ten of the small cube; a multiplication which is thus expressed 10^3. Following out the same reasoning, and calculating that twice 2, multiplied by 2, make 8; and that 3 times 3, multiplied by 3, make 27, and so on it will be seen, that if the sides of the large cube contain the side of the small one,

1, 2, 3, 4, 5, 6, 7, 8, 9, 10
times, there will be
1, 8, 27, 64, 125, 216, 343, 514, 729, 1000
small cubes in the large one.

In the abbreviated method of speaking, we say that 8 is the cube of 2, 27 the cube of 3, 64 the cube of 4, &c., which denotes the number of cubes contained in a large cube, the side of which contains the side of the small cube, 2, 3, 4....times.

The volume of a quadrangular prism equals its base multiplied by its height.

Let us suppose the prism, fig. 15, pl. 7, to be rectangular. Let us divide it, parallel to its base, into as many portions as its height contains the unit of measure repeated; or, in other words, contains the side of the small cube taken as the unit. As many times as the base of each portion contains the base of the unit cube, so many times will the small cube be contained in the portion. The whole number of small cubes, therefore, equals the number indicated by the surface of the base of the prism multiplied by the number contained in its height. That is, the volume equals the base multiplied by the height.

Two prisms having the same rectangular base and the same height, one, AG, being perpendicular, fig. 16, the other, Ag, oblique, are equal in volume.

To prove it, I observe that the two triangular prisms ABEF *ef*, DCHG*hg*, are equal. In fact, they have the same height, AE = DH, and their bases AE*e*, DH*h*, are two equal triangles, for AE = DH, and the two other sides are respectively parallel. But if we add to the parallelopipedon ABCDEFGH, the triangular prism DCHG*hg*, and cut off its equal ABEF*ef*, we shall have the quadrangular oblique prism ABCD *efgh*, whence this latter has the same volume as the rectangular prism of the same base and height.

It may be easily shown that the prisms ABCDEFGH, ABCD*efgh*, fig. 15, are of the same volume as every other prism, having the same height, and of which the bases shall be parallelograms of the same surface as the rectangular base ABCD.

The volume of a right-angled triangular prism equals its base, multiplied by its height.

In fact every quadrangular prism ABCDEFGH, fig. 17, pl. 7, may be divided into two triangular prisms of equal volume, and this equality will be preserved, whatever may be the inclination given to the edges of the parallelopipedon, provided its base and height be not altered. But the surface of the base ABC, or ADC, of the triangular prisms, is the half of the surface of ABCD, the base of the parallelopipedon; whence the volume of the triangular prism is equal to its base multiplied by its height.

The volume of any polygonal prism ABCD, *abcd*, fig. 18, equals the base multiplied by the height.

Such a prism may in fact be divided into as many triangular prisms as its base ABCD can contain of the triangles ABC, ACD....As they are all of the same height as the whole prism, their whole volume will be equal to the sum of the triangular bases ABC, ACD, ADE,...multiplied by the height.

Of the solid contents of pyramids.—Let us begin with the triangular pyramid.

The volume of a triangular pyramid is the third part of its base multiplied by its height.

To demonstrate this, let us take any triangular prism, AF, fig. 19, pl. 7, and divide it by the plane ACE, which passes through the side AC of the base, and through the angle E. We shall have first a triangular pyramid ABCE having the same height and the same base as the prism. There remains also a quadrangular pyramid, of which ACFD is the base, and E the summit. Let us divide it into two triangular pyramids by a plane AEF, and we shall have the reverse pyramid ADEF, of which DEF is the base, and A the summit, and which of course, has the same base and height as the assumed prism AF. If we compare the third pyramid ACFE to ADEF, we shall see that they are equal in volume, because, making the triangles ADF = ACF their bases, they have the same summit E. We see, therefore, that the volume of every triangular prism is equivalent to that of three pyramids having the same base and the same height; and also, therefore, that the base of each pyramid multiplied by its height, which is the volume of the prism, is equal to three times the volume of this pyramid.

The volume of any pyramid whatever, fig. 20, is the third part of the base multiplied by the height.

To demonstrate this, let us divide the base into triangles ABC, ACD, ADE...each of which will be the base of a triangular pyramid, having O for its summit. Each of these triangular pyramids will have for its measure the surface of the triangles ABC, ACD...multiplied by the third of their common height. Consequently the whole pyramid will be measured by the whole of the base by the third of the height.

Cubage of a body terminated by any number of plane faces, fig. 21, pl. 7. Take any point of such a body, O, for the summit of the pyramids which have the faces of the body for their bases. The surface of each face, multiplied by the third of the distance to the summit O, will be the volume of the corresponding pyramid, and the sum of the products of all the pyramids will be the volume of the whole body. In order to carry this method easily into practice, it would be necessary that we should be able to place ourselves in the interior of such bodies, and measure directly the distance of each face. Unless this can be done, we should enter into very complicated geometrical operations, very ill adapted to the rapidity and simplicity required in the operations of practical men. Fortunately there is another method more easy and more expeditious of accomplishing the same thing.

Before explaining this method, let us measure the volume of the truncated triangular prism ABCDEF, fig. 22. We may decompose it into three pyramids, the first having ABC for its base, BE for its height, and consequently for its volume, the base ABC multiplied by the third of BE. The second pyramid having ACF for its base, and its summit at E, is equivalent to the pyramid having its summit at B, and ACF for its base; or, which is the same thing, having ABC for its base and its summit at F. The third pyramid ADFE is equivalent to the pyramid ADEB, which is equivalent to ABCF; whence we have the truncated prism ABCDEF equivalent in volume to the three pyramids, having ABC for their common base, and their respective summits in D, E, F, at the extremity of the three edges.

If the three edges are perpendicular to the base, we shall have for the volume of the three pyramids, and consequently for that of the truncated prism, the surface ABC $\times \frac{1}{3}$ (AD + BE + CF.)

It is required to find the volume of the truncated prism MNODEF, fig. 23, included between two planes, MNO, DEF, oblique with regard to the edges of the prism. Supposing ABC to be perpendicular to the edges, we shall have

Volume ABCDEF = surface ABC × $\frac{1}{3}$ (AD + BE + CF)
Volume ABCMNO = surface ABC × $\frac{1}{3}$ (AM + BN + CO)
and finally,
Volume MNODEF = surface ABC × $\frac{1}{3}$ (DM + EN + OF)

By these principles we can readily determine the volume of any body terminated by any number of plane surfaces. We decompose it into prisms and truncated triangular prisms, the volume of which may be immediately ascertained; and the sum of the volumes, of all of them, is the volume of the body.

In the same manner it may be demonstrated with equal ease, that the volume of every prism or truncated quadrangular prism, ABCDEFGH, fig 24, having its edges perpendicular to the base ABCD, may be measured by the surface of this base multiplied by the fourth part of the sum of the four edges AE, BF, CG, DH.

Let us decompose successively, the quadrangular prism, into two triangular prisms, ABCDEFG, ABCDEHG, and then into ABDEFH, BCDFGH; and

We shall have, the volume of the two former prisms = $\frac{1}{2}$ surface ABCD × $\frac{1}{3}$ (AE + BF + CG + AE + DH + CG); and the volume of the two latter prisms =

$\frac{1}{2}$ surface, ABCD × $\frac{1}{3}$ (AE + BF + DH + BF + CG + BH,)

Taking the sum of these two products, we shall have twice the volume of the quadrangular prism

$\frac{1}{2}$ surface, ABCD × $\frac{1}{3}$ (3AE + 3BE + 3CG + 3DH).

Whence the simple volume of the quadrangular prism is

$\frac{1}{4}$ surface ABCD (AE + BF + CG + DH.)

Application to measure the Tonnage of Ships.—We have seen in the second lesson, that the bottom of a ship is divided into horizontal sections by the horizontal planes of the different water lines, at equal distances from each other. It is also divided vertically by other planes, also at equal distances. These two series of planes divide the volume of the vessel's bottom into rectangular prisms, truncated on both sides and of an equal base. The whole volume of these truncated prisms is obtained by multiplying the base common to them all, by the fourth part of the four edges of each prism. But each edge is common to four prisms,* whence the whole volume of the ship's bottom will be equal to one of the rectangles; that is to say, the product of the distance of the planes of the water line multiplied by the distance of the vertical planes, and by the sum of all the edges which are horizontal lines, and are placed at the same time both on each vertical plane, and on each water line. This method being at once simple and easy, though it is only an approximation, may be employed to calculate the volume of every other body.†

* Except the edges of the sides which are common to only two prisms, and ought not, therefore, to be taken, each one more than $\frac{2}{4}$ or $\frac{1}{2}$ times. There may be four edges, which belong only to one prism, and of which we must take only the fourth part, or the quarter, to add to the sum of all the edges, which belong to four prisms.

† The method by which the tonnage or volume of British ships is ascertained, is in practice essentially different from the method described in the text. The Act of Parliament, passed in the 6th year of his present Majesty, prescribes the following rule. "The length shall be taken on a straight line, along the rabbet of the keel, from the back of the main-stern post, to a perpendicular line, from the fore part of the main stern, under the bowsprit, from which, substracting three-fifths of the breadth, the remainder shall be esteemed the just length of the keel to find the tonnage; and the breadth shall be taken from the outside of the outside plank, in the broadest part of the ship, whether that shall be above or below the main wales, exclusive of all manner of doubling planks that may be wrought upon the side of the ship; then, multiplying the length of the keel, by the breadth so taken, and that product by half the breadth, and dividing the whole by ninety-four, the quotient shall be deemed the true contents of the tonnage." If the vessel be afloat, the following is the rule. "Drop a plumb line over the stern of the ship, and measure the distance between such line, and the after part of the stern post, at the load water-mark; then measure from the top of the plumb line, in a parallel direction with the water, to a perpendicular point, immediately over the load water-mark, at the fore part of the main stern; subtracting from such measurement, the above distance, the remainder will be the ship's extreme length, from which is to be deducted, three inches for every foot of the load draught of water, for the rake abaft, also three-fifths of the ship's breadth for the rake forward, the remainder shall be esteemed the just length of the

Two symmetrical bodies are equal in volume.

If we decompose two such bodies into truncated triangular prisms, having for their edges the parallel lines, which determine their symmetry, for each truncated prism, MNODEF, fig. 24, pl. 7, placed on one side of the plane of symmetry, ABC, we shall have on the other side a truncated prism, *mnodef*; so that DM = *dm*, EN = *en*, FO = *fo*; and the two prisms will be equal in volume. Whence, the sum of all the truncated prisms, of the first body, is equal to the sum of all the corresponding prisms, of the second body. Thus, when two bodies, with plane surfaces, are symmetrical, they are always equal to each other in volume. This being true, whatever may be the number of faces, will also be true, when there are so many faces, and they are so small, that the body may be regarded as terminated by curved, and not by plane surfaces.

The plane of symmetry, therefore, of any body whatever, divides this body into two parts of equal volumes.

Of similar solids.—Two pyramids ABCD *abcd*, fig. 25, are similar, when their corresponding edges AB and *ab*, BC and *bc*, CD and *cd*, AD and *ad* are parallel.

It is evident, in fact, that the triangles formed by the corresponding faces of these two pyramids, having their sides parallel, are similar. The three plane angles, therefore, which each summit of the two pyramids respectively form, are also equal. Moreover, the three edges, forming each solid angle, being parallel, if we move *abcd*, pa-

keel, to find the tonnage." The remainder of the operation is similar to that for finding the tonnage, when the ship is not afloat. We do not profess to have any acquaintance with the principles on which this mode of measurement is founded. We can draw no comparison, therefore, between it, and that described in the text. It seems, however, entirely to overlook one element of dimensions, namely, the height or depth of the ship's hold; and we know of no secret by which the volume of any body, the exact figure of which is not known, can be accurately determined, unless the dimensions in each of the three directions be taken. There are various taxes levied on shipping, in proportion to their tonnage, and as our method of determining that, takes no account of *depth*, it has long been the practice to build British vessels disproportionately deep, to the destruction of the beauty of their form, and to their injury in many essential particulars, in order that they may carry as much, while they measure and pay as little as possible. Tr.

rallel to itself, in such a manner, that the point a, is placed on A, ab will fall on AB, ac on AC, and ad on AD; whence the planes abc and ABC, abd and ABD, acd and ACD, will correspond, one with another; the two solid angles, therefore, a and A, of the two pyramids, will also be equal. In the same manner, it may be demonstrated, that the solid angles, B and b, C and c, D and d, are equal; and thus, all the conditions required in order that these two figures may be similar, are fulfilled by the single condition of the two pyramids having their corresponding sides parallel.

If two pyramids, not having their sides parallel, have their edges proportionals, they will also be similar.

The three sides of each of their corresponding faces being proportionals, these faces will be similar; the plane angles, and consequently, also the solid angles which they form, three and three, will be equal. Thus, all the conditions of proportion will be fulfilled.

Two solids, terminated by plane faces, are similar when their corresponding edges are proportionals, and their corresponding angles, whether plane or solid, equal to one another.

We can always, in fact, decompose these solids into pyramids, the sides of which will be proportionals, and, consequently, the corresponding angles will be equal.

The volumes of the similar pyramids ABCDE *abcde*, fig. 26, pl. 7, are proportionals to the cubes of the corresponding edges.

In fact, the volume of each pyramid equals its base, multiplied by the third part of its height; or the bases, BCDEF, *bcdef*, being similar figures, are proportionals to the square constructed on one of their sides. Of these bases, we have, therefore, fig. 26, the surfaces,

$$BCDEF : bcdef :: BCMN : bcmn.$$

Let us now, on BCMN, and on *bcmn*, as bases, construct cubes, and we shall have, for the volumes of the two cubes,

$$BC^3 = BC^2 \times BC \text{ and } bc^3 = bc^2 \times bc.$$

But $$BC : bc :: \tfrac{1}{3} AH : \tfrac{1}{3} ah.$$

Therefore $$BC^3 : bc^3 :: BC^2 \times \tfrac{1}{3} AH : bc^2 \times \tfrac{1}{3} ah.$$

In this last proportion, the two latter terms represent the volume of the two pyramids, and the two former terms represent the volume of the two cubes.

The volumes of similar solids, terminated by any number of plane faces, are as the cubes of the corresponding lines.

We can, in fact, decompose them into any number of similar pyramids, all having the same relation r, to that of their corresponding sides. But two pyramids, of which the corresponding sides are to each other, as 1 is to r, are, to one another in volume, as 1 is to the cube of r. By adding together, on the one hand, all the small pyramids, and on the other, all the pyramids which are r^3 times more voluminous, the volumes will be to each other as : : $1 : r^3$.

EIGHTH LESSON.

Cylinders.

When a right line is made to move along a curve ABCD...*fig.* 3, pl. 8, while it remains parallel to any given direction it *generates* a cylinder, and hence it has been called, the *generator* of the cylinder. Each right line, A*a*, B*b*, C*c*, representing a different position of the generator, is an edge of the cylinder.

From this definition, we see, 1st. that there are as many different kinds of cylinders as species of curves, ABCD, by which the movement of the generating line can be directed; 2nd. that with the same curve, ABCD, fig. 1 and 2, we can form an infinite number of different cylinders, according as we give a different inclination to the generating line A*a*, B*b*.

A complete right line, in the conception of the geometrician, extends infinitely in both directions; so the edges of a cylinder, for it to be complete, ought also to extend infinitely in both directions.

But, in the arts, cylinders have always a termination to their edges in both directions; thus for the workman, every cylinder has two ends.

When a cylinder is terminated at one end by a plane surface, ABCD, fig. 1, this is called its base. If it is terminated at both ends by plane surfaces, parallel to each other, it is said to have two bases. It is right-lined, fig. 1; or oblique, fig. 2; as its edges are perpendicular or oblique to the planes of the bases.

Sometimes one of the planes which terminates a cylinder is not parallel to the other, as in fig. 3, which is a cylinder terminated by the plane surfaces ABCD, MNPQ. It is then supposed that the plane MNPQ, has divided the cylinder with parallel bases, ABCD, *abcd*, and the part ABCDMNPQ, as also the part *abcd*MNPQ, is each called a *truncated* cylinder.

A cylinder, with a circle for its base, is called a *circular cylinder*. In the arts, it is this figure which is generally and exclusively called a cylinder, because it is exclusively used in most branches of industry.

The right line O*o*, fig. 4, drawn through the centres of the two circles, which are the bases of the circular cylinder, is the axis of this cylinder. It passes through the centre of all the circles, formed by intersecting the cylinder by planes parallel to the two bases.

According to the properties of parallels, already demonstrated in the second lesson, the surface of the cylinder is exactly the same whether generated; 1st. by the movement of a right line along the curve ABCD, fig. 3, taking successively, the parallel positions A*a*, B*b*, C*c*, D*d*; or 2nd. by the movement of the curve ABCD...fig. 4, which takes successively the parallel positions, ABCD, A′B′C′D, A″B″C″D″, &c., along a right line, so that the same point of the curve, A, for example, takes, successively, the positions A′, A″, A‴,...of the edge A*a*.

In the arts, both these methods are employed to generate right lined circular cylinders. As it is required to give the cylinder perfect continuity, either in the direction of its edge, or in the direction of its base, so the former or the latter method is preferred.

1st. *Making cylinders by edges.*—When it is required to give a cylinder perfect continuity in the direction of its edges, a regular polygon of a great number of sides, ABCDE, is inscribed in a circle, or the polygon is circumscribed about a circle; as many plane faces or parallellograms, $ABba$, $BCcb$, as there are sides of the base, are then formed with precision along the whole length. Afterwards, with an axe, a plane, a draw-knife, a saw, or any other instrument proper for the purpose, following the longitudinal direction of the parallel right lines, Aa, Bb, Cc, &c., the projecting edges are cut away, and the cylinder is rounded. By this means, we are certain of fulfilling the condition of forming the surface by rectilineal and parallel edges. But we are not so sure that the surface which they represent has, in every part, a circle for its outline, because the plane, the axe, or whatever instrument is used, gives continuity in the rectilineal direction of the edges, and not in the direction of the circular outline.

Application to mast-making.—The masts of ships, particularly the top and top gallant masts, require to be made with a continued or smooth surface, in the direction of their length; in order that the strap or collar which passes round each mast and yard connecting one with the other, may slide up and down, without resistance; and such masts are made in the manner just pointed out.

2nd. *Making cylinders by equal and parallel curves.*—When it is necessary to ensure, above all things, smoothness and continuity in the direction of the base of the cylinder, or perpendicular to the length of the edges, it is done by the process of turning. The turning chisels describe, successively, a great number of circles ABC, A′B′C′, A″B″C″, ... fig. 4, pl. 8, so that the whole of them represents a cylinder. We are then sure that the surface is perfectly circular and continuous in the transverse direction; but, in general, we cannot be so certain of obtaining continuity in the longitudinal direction.

Application to making pike and gun-spunge handles.—In the English arsenals, I saw the following means employed to turn cylindrical surfaces. A piece of wood of the form of a prism (a poll or staff) previously prepared, of four or eight faces, is forced into the hollow of a circular plane, and as it moves forward, the plane makes it round. By this means, a cylindrical surface, exactly circular, is formed, if the prism is perfectly straight; but more or less bent, if the wood be crooked in any part.

When we wish to produce a surface which shall be perfectly cylindrical, we must be certain of its smoothness and continuity in both directions: which is accomplished by making the turning chisel move by means of a guide, in a direction parallel to the axis of the cylinder, so that its edge always remains at the same distance from the axis. We are then certain that all the circles are exactly equal, and that the edges are perfectly rectilineal.

Application to trellis and cage work.—Both means of forming cylinders are used for making open cylindrical work, such as that of cages and trellis work. It may be made of iron wire or iron bars, bars of wood, or even cords extended in right lines, representing the edges of the cylinder. Circles of the same substance, being all of the same size, and having exactly the same curvature, represent curves parallel and equal to the bases of the cylinder. The lines and circles are fastened together, by wire, or by soldering, at every point where they cross each other, and a cylindrical surface is thus correctly formed, as we see in various works of art, such as cages, baskets, columns in trellis work, &c.

We can make cylinders of a certain size, by joining together, side by side, a great number of smaller cylinders, and uniting them on the outside by circles, or circular collars. Fascines, made for military operations, are of this description; and so also are those collections of pikes, or other weapons, which are thus disposed, either for ornament or use.

There are some arts, in which the principal object consists of making cylindrical surfaces, by doubling or rolling plane continued surfaces. (See 10th Lesson. Developable Surfaces.)

Thus the cooper, or person who makes wooden measures, takes a number of small planks or staves, equally smooth, and of an equal size throughout, and he makes them into the form and dimensions of the measures in use. In France, the cylindrical measure formerly employed for grain, was called a *boisseau* (bushel), and thus the person who now makes measures, is called a *boisselier*.

That he may be certain the measures are made cylindrical, he gives them a solid bottom, which is a plane, similar to that of casks: very often, he puts an iron hoop (*circle*) round the top, and one or two pieces of iron across it (*diameters*,) to prevent the measure losing its cylindrical shape, and altering its capacity.

The copper and tin-smith, and the brazier, who work with very thin sheets of iron, copper, or tin, very often make cylindrical bodies, which, of all curved surfaces, are the easiest to form. The funnels of common stoves, spouts, &c., &c., are cylinders. In general, the workman is told what is to be the diameter and the length of each pipe or funnel: they are immediately aware what will be the circumference, and multiplying that by the length, they can tell what quantity of sheet copper, iron, or tin, will be required.

It is necessary to add, 1st. to the circumference of the tube, a breadth equal to the overlaping of the two parts of each sheet, the edges of which are to be brought together to form the cylinder; and 2nd. to each length of tube, a quantity equal to the extent which the pieces enter, one within another, at their ends.

Boilers of steam engines must be mentioned as among the most important works of a cylindrical form, made by smiths; but they have not a circular base, fig. 5, pl. 8. To fasten the sheets of copper or iron together, which are to form the boiler, cylindrical nails or rivets are employed, which fit holes made in the copper or iron exactly, and being riveted over them, completely prevent the escape of the steam. The cylindrical holes in the sheets of metal, are made by four or five punches, fixed into one frame, and at equal distances from one another. The frame can be alternately raised and forced down by means of a powerful machine. The sheet of iron to be punched for the cylindrical rivets, is placed on a support, which is fixed, while the punches are descending, in order that they all may pierce the iron at the required distances. When the frame with the punches is raised, after piercing the holes, the support on which the sheet is laid moves onward, so that the punches, on the instrument again falling, may pierce the following four or five holes at a proper distance from the former.

Such are the means employed to bring together, with precision and firmness, not only all the metallic sheets of which steam-engine boilers are made; but also the sheets of iron, of which the exterior surface of iron boats, and the iron tanks, for holding water, are made,—which have been recently introduced into the naval service.

Of these iron tanks, which, in shape, are cubes, or truncated rectangular prisms, we may remark, that their edges are blunted or rounded, by portions of the sheet iron being bent at the edges, into the form of a quarter of a right lined circular cylinder.

The plumber and the organ builder make their pipes in a cylindrical form; and they may be either made out of sheets bent into the required shape, as is done by the tinsmith and the brazier, or they may be drawn out, in the manner of wire.

Manufacture of cylinders by drawing.—At the arsenal at Chatham, the following method is employed to make hollow leaden cylinders of a given thickness and diameter:

Let ABCD, fig. 6, pl. 8, be a solid cylinder, having for its diameter the interior diameter of the hollow cylinder to be made. A cylinder of lead, thicker and shorter than the one required, is first cast around the solid cylinder, or around a mould of the same diameter. The solid cylinder, ABCD, is thrust into the hollow cylinder; and the whole is then passed through a circular wire-drawing plate, which is lessened every time the cylinders are passed through it; or they are passed, at every successive operation, through smaller holes. By this means the hollow cylinder is lengthened, and diminished in thickness, always retaining for its interior diameter, the diameter of ABCD. By degrees, the hollow cylinder is reduced to the required

thickness. When the solid cylinder, ABCD, is correctly made, this method ensures the complete continuity of the hollow cylinders in both directions.

Metal wires of all dimensions, as well as round bars of iron, are cylinders, made by reducing the metal to the proper diameter, by the process of *wire-drawing:* that is, they are dragged forcibly through circular holes in strong iron plates; the metal being passed through holes less and less in diameter, in order, gradually, to reduce the size of the bar or wire, at each operation.

Cylinders are also cast in moulds.—The pipes, for example, laid down beneath our streets, to convey water and gas to our houses; and the cylinders of common pumps, and air pumps, are made in this manner.

Making cylinders by boring.—Pipes which serve some common purpose, such as conducting water or gas, and which are not required to be very exact, may be made sufficiently correct by casting; but for pipes which must be made with greater, and even mathematical precision, such as those of the better sort of pumps; and for such cylindrical bodies as cannon, howitzers, and mortars, we are obliged to have recourse to more rigorous methods, such as the operation of boring, which will be hereafter explained. (See 12th Lesson. Surfaces of Revolution.)

Making cylinders by sawing.—Finally, cylinders may be cut out by a saw: 1st. keeping the substance fixed which is to be sawed, and making the saw advance parallel to any given direction, while, in its motion, it keeps in some line previously determined; this is the operation of common sawing: or, 2nd. the cylinders may be cut out, by giving an ascending and descending motion to the saw, without its moving backwards or forwards; giving at the same time to the piece of wood, the necessary curvilinear motion. Cylindrical surfaces are fashioned in this manner at saw mills.

Construction of cylinders by architects.—When architects wish to form a cylindrical surface, such as the arch over a door way, a vault, the arch of a bridge, &c., they begin by constructing in wood a cylindrical surface, the exterior part of which is identical with that hollow cylindrical surface which they mean to build. They construct at intervals, a polygon, ABCD, fig. 7, pl. 8, inscribed within the circumference of the arch, giving to the polygon, such a number of sides, that it may form, with the circumference, segments easy to fill, without requiring too much wood. In fact, they fill these segments by pieces of wood, on which they place, side by side, the thick planks, the ends of which are represented in fig. 7. The upper surface of these planks, forms the cylindrical surface, on which the masons place the stones, called the arch-stones.

Measure of the surface of cylinders.—We may consider the surfaces of cylinders to be composed of as many edges as our eyes can distinguish, when drawn as close as possible to each other; and, therefore, as prisms, terminated by a great number of extremely narrow faces. The outline of the base is in this case a polygon, which for us is equal to the polygon which serves as the base of the prism.

If the cylinder is rectilineal, its surface, not including the bases, equals the perimeter of one of its bases, multiplied by its height.

The total surface of the rectilineal circular cylinder and its bases, equals the circumference of one base, multiplied by the length of one edge, plus, the length of one radius of the bases.

In the prism ABCD, *abcd*, fig. 8, we can divide the longitudinal surface in the direction of the edge A*a*, and turn back every face successively, B*bc*C, C*cd*D, &c., to bring it into the same plane as A*ab*B. We shall then have a plane figure, composed of the parallels A*a*, B*b*, C*c*, &c., fig. 9, and of the sides AB, BC, CD, DE, *ab*, *bc*, *cd*, *de*, &c., perpendicular to the parallels; which requires that ABCDE, *abcde*, should be two right lines parallel to each other, and perpendicular to the edges A*a*, B*b*, &c. The rectangle thus formed, fig. 9, is what is called the developement of the surface of the prism; and the surface of the prism can be developed, because it can be executed, without its being necessary, that any parts of the surfaces *AabB*, *BbcC*, should be either lengthened or shortened to remain side by side, and form one continued plane surface. A whole lesson will be devoted to developable surfaces, among which, cylinders must be placed, as they may be considered as prisms, having an infinite number of sides.

Let us make, in the right lined cylinder, fig. 8, two oblique and parallel sections MNPQ, *mnpq*, and let it be required to measure the cylindrical surface comprised between these two sections. It is evident, that the portions of the edges M*m*, N*n*, P*p*, Q*q*, being parallel right lines, comprised between two parallel planes, are all equal. If we regard the cylinder, therefore, as a prism with a great number of faces, the surface of the parallelograms representing each face, will be—

Surface M*mn*N = AB multiplied by ... M*m*;
Surface N*np*P = BC multiplied by N*n* = M*m*;
Surface P*pq*Q = CD multiplied by P*p* = M*m*, &c.

Whence, finally the surface MNPQ...*mnpq* = ABCD, multiplied by M*m*; that is to say, equals the circumference of the base ABCD, multiplied by the length of one of the portions of the edges, included between the two parallel planes.

If it were required to measure the surface of the truncated cylinder ABCD...MNPQ, fig. 8, we should be obliged to develope the cylindrical surface, by marking each edge AM, BN, CP,...according to its length, and determine, on the developement of the cylinder, fig. 9, the surface ABCD—MNPQ.

Supposing that the cylinder was a prism of a great number of equal sides, we should have, if we made AB = BC = CD...

Surface of the truncated cylinder ABCD...MNPQ... = AB (AM + BN + CP + DQ), that is to say, the breadth of one of its sides or faces, multiplied by the sum of the edges of these faces.

Measure of the volume of cylinders.—If we regard the cylinder as a prism, composed of a great number of faces, we shall see that its volume equals the surface of its base, multiplied by its height.

The base of a right lined circular cylinder being a circle, has for its surface, the circumference multiplied by the half of its radius.

The volume of the cylinder, therefore, is equal to the circumference of the base, multiplied by the half of the radius of the base, and by the height of the cylinder.

Prisms, whether oblique or right lined, of the same base and the same height, are equal in volume, whence, also, cylinders, whether oblique or right lined, of the same base and height, are also equal in volume.

We can determine very easily the volume of a truncated right lined circular cylinder. Let ABC, fig. 10, be the circle, which serves as the base of the cylinder, and O*o* the axis, the volume of the truncated cylinder ABC*ef*, equals the surface of the base, multiplied by the axis O*o*, that is to say, equals the volume of a right lined cylinder, having O*o*, for its height.

In order to demonstrate it, let us imagine the right lined cylinder ABC*amcn*, the upper base of which has its centre in *o*, the two volumes *amne*, *cmnf*, are equal. It may be first remarked, that *o* being the centre of the circle *amnc*, the diameter *mon*, divides the circle into two equal parts.

Around *mn*, as a hinge, let us turn the two right angles of the volume *mnae*, the half circle *mna*, will then fall on the half circle *mnc*; all the parts of the edges, such as *ae*, &c. will coalesce with the edges *fc*, &c., and, therefore, the plane *mne*, will fall altogether on the plane *mnf*. The two volumes will, therefore, be included between three surfaces which coalesce; and they are, consequently, of the same volume. But the right lined cylinder possesses *mnae* more, and *mncf* less, than the truncated cylinder ABC*ef*. The two cylinders are, therefore, equal in volume, and the measure of one is also the measure of the other.

In the same manner as there are sectors of a circle, AOB, fig. 11, so there are also, sectors of a cylinder, which have the sector of the circle for their base, and which are terminated on one side AB*ab*, by the surface of the cylinder itself, and on the two other sides, by two planes A*ao*O, B*bo*O, which pass through the axis O*o*, of the cylinder.

A segment of a cylinder has for its base, a segment of a circle, ABC, fig. 12, pl. 8, and for its outline, 1st. the cylindrical part ACB*acb*; 2nd. a plane AB*ba*, parallel to the axis, having the form of a parallelogram.

Application of the properties of cylinders to determine shadows—When the rays of the sun reach us, they are so nearly parallel, that it would be difficult, with the most delicate instruments, to detect the least difference in the direction of two solar rays, falling even at a considerable distance from each other; as, for example, at the opposite extremities of a large building. For this reason, in the arts, the rays of light emanating from the sun, are regarded as exactly parallel.

When a door, a window, or an arch, having the form of an arc of a circle, ABCDE, fig. 13, pl. 8, is illuminated by the rays of the sun, A*a*, B*b*, C*c*, D*d*, E*e*, these rays being right lines parallel to each other, passing through the circumference of a circle, will form a cylinder or prism, of which ABCDE is the base. This cylinder separates all the part of the space enlightened by the sun, within the door, the window, or the arch, from the part placed in the shade.

The consideration of cylinders, their figure, and their position, is, therefore, of the highest importance, when it is necessary to determine what points will be illuminated, and what placed in the shade, in architecture, in painting, and generally in all the arts of design. In the following lessons, the means of solving, geometrically, the chief questions relative to shadows, will be given.

Application of the properties of the cylinder to descriptive geometry.—One of the most useful applications of the properties of the cylinder, is to represent, by its surface, on a plane, the design or projection of any curved lines whatever.

Let us suppose that we have, in space, a curve ABCDE, fig. 14, which is to be represented on the plane of projection MNPQ. From each point of this curve, we draw a perpendicular to the plane. The succession of points, *a*, *b*, *c*, *d*, *e*, or the termination of the perpendiculars in the plane, will form a curve, which will be the geometrical representation, or, as it is called, the projection of the curve ABCDE.

In general, every curve is projected on two planes, MNPQ, PQRS, perpendicular to each other, so that the lines of projection, A*a*, B*b*, C*c*, perpendicular to the first plane, are parallel to the second, and the lines of projection A*a'*, B*b*, C*c*, perpendicular to the second plane are parallel to the first. The two projections, *abcde*, *a'b'c'd'e'*, as we shall see, in treating of the intersection of surfaces, will be sufficient to determine, completely, the curve ABCDE, which they represent.

With a plane we can construct cylinders, and conversely, with cylinders we can construct or make planes.

Application of cylinders in husbandry.—By means of a cylinder, made to roll on a road, recently constructed or repaired, on a grass plat, or on a field lately ploughed up, the projecting parts are flattened down to a common level, and the earth is thus smoothed into a plane surface.

Pastry cooks use a piece of wood, which is generally of the figure of a cylinder, and called a roller, to spread out their paste, which is done by rolling it, pressing on it, and pushing it with both hands. Thus, the paste is flattened and transformed into thin leaves, terminated above and below by plane surfaces.

Combination of cylinders—Flattening rollers.—In place of using only a single cylinder to produce plane surfaces, it is found more advantageous to employ two combined cylinders, the axes of which are parallel. Let AB, *ab*, fig. 15, pl. 8, be the axes of the two cylinders so placed, that of themselves they neither approach to, nor separate from each other, but may be made to do so to any degree, it is the pleasure of the workman. The two axes being placed parallel to each other, and the cylinders being made as true as possible, they are at the same distance from each other, at every point. If we now cause a plate of metal, or any other substance which can be flattened, to pass between these two cylinders, it will be reduced to the thickness of the shortest distance, whatever that may be between the two cylinders.

After having caused the metal to pass once between the cylinders, if they are made to approach each other a little, and the plate be again made to pass between them, it will be reduced in thickness by a quantity equal to the distance, the cylinders have been made to approach each other. Following out this system, the plate may be reduced to any required dimensions; and by means of machines of this kind, called flatteners, metal is generally formed into sheets.

Application in Paper-making.—In the arts, we find this property of cylinders very often called into use. In making paper, for example, two cylinders covered with woollen cloth, roll out the substance of the paper, and reduce it into a continued sheet, giving it any desired length. Paper of this description is called endless.

Application in Printing.—The types necessary to print off one sheet of paper, are placed on cylinders of a considerable diameter, which are in contact with other cylinders, covered with leather, and which supply ink in proper quantity to the types. A clean sheet of paper is then made to pass between the cylinders bearing the types, of which it receives the impression. This method of printing, which allows each sheet to be struck off with extreme rapidity, is of great utility, particularly in printing newspapers, which require to be distributed within a short time, after the matter is collected and composed, however numerous may be the copies required.*

* The Printing Presses, as they still are erroneously called, we believe, of the description alluded to in the text, which are used in this country, differ materially from M. Dupin's account. To use them, the matter to be printed is set up or composed by the printers, with types in the usual method, and put into *forms*, just as for common press-work. The types are not placed on a cylinder, as described in the text, but the form for printing one side of a sheet, is put into a flat frame, being, geometrically speaking, a plane. This frame, which is of the size of the sheet of paper to be printed, when the steam-engine, or whatever moving power is employed, is set to work, moves backwards and forwards with considerable rapidity, in another frame fixed to the ground, which is of the same width between the interior parts as the exterior parts, or extreme width of the former, but three times as long. As it moves forward, the types in it being then charged with ink, it passes under, and in contact with two cylinders, which revolve above it, also with considerable rapidity. To the upper part of one of these cylinders, the sheet of paper is applied, which transmits it, and spreads it, in conjunction with the other, over the moving frame. This takes place as the revolving cylinder passes over the moving frame, and the impression of the types is accurately taken off by the pressure of the cylinder on the frame. By the backward motion of the frame, the sheet is thrown off from it; and is then carried away by a boy. Another boy constantly supplies the upper part of the cylinders with blank sheets of paper, and impressions may be taken off as rapidly as the frame can be made to move backwards and forwards, and the cylinders to revolve. In this machine, therefore, the paper, not the types, is applied to the revolving cylinder.

The types in the frame are inked in its forward movement, by means of smaller revolving cylinders placed above it, and touching it before it reaches the printing cylinders. When the ink is applied, these small cylinders are raised up and receive a fresh supply, while the frame returns beneath, and no longer in contact with them.

For great dispatch, two such frames, fixed to the ground, as the one above described, and two such moveable frames and pairs of cylinders are employed; the types for printing one side of the paper being placed in one frame, and those for printing the other side in the other. As the paper is printed on one side, it is transmitted to the other machine, and printed on the other side. When great dispatch is not required, one machine only is employed, and after the necessary number of copies are printed on one side, the *form* for

Cylinders are also employed for printing designs of every description on cloth; the designs being engraved on copper cylinders.

printing the other is placed in the frame, and the sheets printed on the second side.

Although not successfully practised, an attempt has been made to print with types in cylindrical forms; and as this occurred in an establishment of singular enterprize and ingenuity, about the time of M. Dupin's last visit to the metropolis, it is probable that he saw the machine, and concluded that it was the representative of the whole process of printing by power. Stereotype plates were cast upon types in the usual manner, and by a curious contrivance, were bent so as to apply to the surface of the cylinder, by which the impression was to be conveyed to the paper. This method, provided that it admitted of equal excellence as to execution, which, in fact, did not appear to be the case, must obviously be adapted only to a peculiar class of publications.

It may not be out of place here, nor uninteresting to those who love to trace the connection which exists between all the events of the moral world, as well as between all the physical phenomena of the universe, if it be remarked, that the great improvement in printing just mentioned, took place at a time when communication had become surprizingly rapid between every part of this country, with a great probability of the rapidity being still further increased. With this rapidity of communication, let us suppose that some alarming event occurs, exaggerated by rumours, like the murder of Mr. Perceval, or an assemblage of a riotous mob. The report of such an event, also exaggerated as it proceeded, would be transmitted by the thousand vehicles which daily and hourly leave the metropolis, and might spread fear and apprehension and dismay throughout the kingdom. It may, indeed, be supposed, with our present rapidity of communication, were there no circumstances accompanying it, tending to check the propagation of false rumours, and to spread correct information, that an alarming event, occurring in the metropolis, magnified by rumour with its thousand tongues, might shake the kingdom to its centre. So general a panic might be created, such general anxiety and terror might be produced, that it may be even doubted if our present rapidity of communication would not be incompatible with the safe existence of society.

But there are counterbalancing circumstances which arise in the natural course of things, without the agency or appointment of governments, which, perhaps, have never even taken the view just

Lithographic printing.—In lithographic presses, only one cylinder is employed. After the design has been drawn on the stone and inked, the sheet of paper destined

brought under notice. A certain portion of the community make it their business to collect information. Depending on the favour and support of the public, which would soon cast them aside were they habitually to deceive, they have the strongest of all possible motives for sifting the information they collect, and for propagating only what is true. By their attention being directed to this object, they acquire a sort of tact in discriminating;—they are acquainted with circumstances of which other men are ignorant, which enable them to appreciate the correctness of all reports; and in the vast majority of cases, the information they obtain and send abroad, is substantially correct. Newspapers which are published by this class of men, grow up in society, unwilled of governments, which very often do all they can to prevent or suppress them. In the philosophical point of view, however, just considered, tending to spread correct information, and checking the propagation of false rumours, newspapers are necessary, when communication is rapid, even for the preservation of society; and are fully as important and essential a part of social economy as gaols and judges. A government may, therefore, just as well deny justice or criminal law to its subjects, as public journals.

But if such vehicles of information be essential to preserve the existence of society, the invention which sends them forth to the world complete, in a short hour or two, when formerly twice or thrice as much time was required, must be proportionally useful. Partly in consequence of this improvement in printing, and partly in consequence of other simultaneous improvements in the management of newspapers, those published in the morning, are now ready, even on nights when the Parliament sits late, to be sent from London by the morning coaches. The most correct information which can be obtained, is thus transmitted, in company with the most speedy and earliest travellers. The exaggerated rumours of conversation, as they are spread by the coaches, are accompanied by correct accounts. Society, instead of being agitated by false rumours, remains calm and tranquil at every point. Those who have heard of the numberless false reports and apprehensions which got abroad in Great Britain, in the year 1745, or who know how much France was agitated as long as Buonaparte was in existence, by stories of his coming back, will be sensible of the value to society, of all the means which tend to check the propagation of false rumours. They will see at to receive the impression, is laid on it, and a cylinder being rolled over the whole, with an equal pressure on every part, gives to the impression equality and beauty.

Copper-plate printing.—To print from copper plate engravings, the plate, which is a plane, and the paper on which the impression is to be left, both pass together between two cylinders, which press them one against the other.

Application of double cylinders to the manufacture of bar iron.—According to the method formerly in use, and still practised, generally, on the continent of Europe, in manufacturing iron, a large mass of cast-iron, called a *loupe*, was violently heated; it was then placed on an anvil, and struck by very heavy hammers, which forced out the impure matter, and the foreign substances which the mass contained. By this process, iron was reduced into the form, more or less perfect, of prisms or bars. The English, within a few years, have substituted double cylinders, which perform the work with great regularity, for the rude labours of the hammer. Let the reader imagine two pair of cylinders, notched in such a manner as to form openings, the profile of which is a succession of lozenge shaped holes, gradually decreasing in size, as in fig. 16, pl. 8; or a succession of rectangles, gradually diminishing, as in fig. 17. The mass of iron being violently heated, and sufficiently *squared* by the hammer, is made to pass between the cylinders, moving successively through the openings 1, 2, 3,...which gradually lessen its size, and reduce the heated mass to square or flat bars. This method is very advantageous, on account of its drawing out equally, all the fibres of the iron. At length, this method has also been introduced into France, but, unfortunately, is not yet sufficiently practised.

once, that newspapers are essential parts of social order; and that they, and the improvements adverted to in printing, are necessary to render harmless that rapid communication, which is, in many respects, so extremely beneficial. Tr.

Application of cylinders to carding.—A very successful application of double cylinders has also been made for carding cotton and wool, and also to separate hemp and flax.

Two cylinders, fig. 8, are placed perfectly parallel to each other, and carding pins are fixed in a regular manner at every part of the surface of both the cylinders, so that the points of one, lock in with the points of the other. When cotton, wool, hemp, or flax, is made to pass between these cylinders, which revolve either in contrary directions, or in the same direction, but with different degrees of velocity, the filaments of these substances must necessarily extend themselves in a parallel direction, forming, when they issue from between the cylinders, a flat band, called a carding or sliver.

Application of cylinders to spinning cotton, hemp, &c.—A right lined circular cylinder, with a smooth surface, AB, fig. 19, is combined with a grooved cylinder, CD. The cotton threads are first dragged slowly through one pair of cylinders; and quicker between another pair, parallel to the first. By this means, the cotton between the two pairs of cylinders, is forced to extend proportionally to the difference in the velocity of the two pairs of cylinders. As the cotton is thus drawn out, the thread is rendered finer, which is one of the great advantages of our present spinning machines.

The manufacture of grooved cylinders, is one of the most delicate operations of art: it requires very great precision. The least defect of parallelism in the grooves, and the least inequality in the diameter of the cylinder, would be sufficient to produce, in spinning very fine threads, such differences as would make them lose all the advantages of the strength and regularity, which are compatible with their fineness.

To make grooved cylinders, a species of machine is employed, that is adapted to divide the circle into equal parts, according to the method described in the third lesson. After the number of grooves which the cylinder is to have, has been determined, and it has been placed on the circle of division which supplies this number, the first groove is formed, by making a cutting instrument move backwards and forwards, along a guide, placed parallel to the axis of the cylinder. When the first groove is completed, the instrument indicating the divisions of the circle is moved one point; the cylinder is, by this means, placed in a proper position for cutting the second groove, which is made in the same manner; and so on, till the whole are completed.

Cylinders are very often combined in a different manner. A solid cylinder is made to enter or move in a hollow cylinder; such, for example, is the movement of the pistons of pumps, fig. 20, pl. 8; the cork of a bottle is a solid cylinder; such, also, is the movement of the two parts of a sheath or case, fig. 21, and of a round tobacco or snuff-box, fig. 22, &c.

Hollow cylinders, which shut exactly within one another, are the geometrical forms of opera glasses, and of common achromatic telescopes, or spying glasses. They can be drawn out at pleasure, as at AB, fig. 23, or closed up, as at *ab*. It is evident that the easy and precise movement of such instruments, the parts of which slide within one another, depends on the perfect manner in which the inner and outer surface of each hollow cylinder is executed.

By lodging the ends of cylinders one within another, those long lines of tubes or pipes are laid down, in England, which carry both water and the material of light, beneath the streets, into the houses and shops of the people. As iron is very sensibly affected by an augmentation or diminution of temperature, if the pipes were laid down a considerable distance, and were so united that the ends of each could not move one within another, they would break.

To obviate this inconvenience, one end of each pipe is terminated by a cylinder, ABCD, fig. 24, pl. 8, larger than the body of the pipe, CF. Into this large part, the smaller end of the following pipe, *mn*, enters, in such a manner, that the two pipes can move a little one within the other, notwithstanding the solder or lead by which they are united; and thus, without permitting their contents to escape, they admit of the elongations and contractions occasioned by changes of temperature.

NINTH LESSON.

Conical Surfaces.

The surface of a cone, SABCDE, fig. 1, pl. 9, is described by a right line, which always passes through the same point, S, and along a curve, ABCDE. The right lines SA, SB, SC....are the *edges* or sides, and the point S, is the summit or vertex of the cone.

In the example given, fig. 1, in which the summit S, and the curve ABCDE, are in the same plane, the surface of the cone is the same as the surface of the plane. Thus, when a horse is driven round a circle, during the preparatory exercises of breaking in, the rope which passes to the horse, from the man or the post in the centre, and which is kept constantly in a right line, describes a cone, SABCD, fig. 3, provided the summit be out of the plane of the curve ABCD, described by the point where the line is fastened to the horse. But when the rope is horizontal, the cone becomes a plane, and the vertex S, is in the plane of the circle *abcd*, described by the horse. The sides S*a*, S*b*, S*c*....are in this case the radii of a circle.

Geometry considers the cone, fig. 1, as a surface extended on both sides without limit, like the right lines which constitute its edges. It considers two cones, formed by the parts of the same edges, which are on the opposite sides of the summit, as forming only one surface, and the summit is, for this reason, called the centre of the cone.

We find some examples of these complete or double cones in the arts. The hour-glass, fig. 2, that ancient instrument, which is, even to this day, employed on board ships to measure time, is formed of two cones, placed in this position. In any portion of time, an hour, for example, though such glasses are made to measure various periods, taken as unit, all the sand runs from the upper to the under cone, and a unit is reckoned for every time the glass is turned.

In the arts, cones are always of limited extent, and in general, only a part SABCD, fig. 1, is considered.

When the cone is terminated by a plane area, ABCDEF, fig. 1, this is called its base. In the present lesson, we shall suppose that every cone is terminated by a plane base.

The regular cone, or right lined circular cone, the most simple of all cones, is that of which the base, ABCDEF, is a circle, fig. 3, and of which the summit S, is situated on the axis SO, of the circle. The right line SO, is also the axis of the cone.

The oblique circular cone, fig. 4, has also a circle for its base, but all its edges are not equal in length to one another, and the right line SO, drawn from the summit to the centre of the base, is not perpendicular to the plane of the base.

In the regular cone, the edges SA, SB, SC, fig. 3, being oblique lines equally distant from SO, which is perpendicular to the plane of the circle, are equal to one another. All the edges of this cone, therefore, are equal to one another, and form the same angle with the axis.

Let us now suppose, that on a cone, the work of art, we draw so many very fine edges, that they offer to our senses, only the appearance of one continued surface, covered with lines, the distances between which are too small to be perceptible. The surface thus composed of small plane triangles, formed by the edges, does not differ, so to speak, from a geometrical cone; if we substitute one of these surfaces for the other, the errors, if there be any, will be so small that they will escape our notice, and may be neglected in practice.

A cone may, consequently, always be regarded as a pyramid having a great number of triangular faces, the breadth of which is extremely small, and the height of which is the same as the length of the edges.

Thus, all the measures already given, both for the surface and for the volume of pyramids, (Seventh Lesson) will also serve to measure cones.

The right lined circular cone being a regular pyramid; *first*, the whole surface of its faces, or the curved surface of the right lined circular cone, equals the circumference of its base, multiplied by the half of one edge; *second*, the whole surface of the circumference, and of the base of the right cone, is equal to the circumference of the base, multiplied by the half of one edge, *plus* the half of the radius of the base.

The volume of any cone whatever, equals the third of its height, multiplied by the surface of its base.

If we divide the cone by a plane parallel to its base, we form a truncated cone, the surface and volume of which will be measured, like the surface and volume of a truncated pyramid.

The surface of a truncated regular cone, equals the half sum of the circumference of the two bases, multiplied by the length of one of the edges, comprised between the bases.

If we divide a pyramid by a plane parallel to the base, fig. 7, the small pyramid thus obtained is similar to the large one. This being true however numerous may be the faces of the large pyramid, is equally true of the cone; as well as all the consequences derived from it. *First*, therefore, when we divide a cone by a plane parallel to its base, a small cone is separated, similar to the large one. *Secondly*—When two cones are similar, the curved surface of them is proportional to the square of the corresponding lines of the two cones; for example, to the square of the edges. *Thirdly*—The surfaces of the bases are also proportional to the square of the corresponding lines. *Fourthly*—The volumes of similar cones are proportional to the cubes of the corresponding lines.

Let us construct a truncated cone, ABC....*abc*....fig. 7, pl. 9, detaching a small cone from a large one, by a section parallel to the base. It is clear that we may ascertain the volume of the truncated cone, by calculating separately, the volume of the small cone, and subtracting it from the volume of the large one. The volumes of each part of the cone being equal to the product of its base, multiplied by the third of its height, the operation is not at all difficult.

When a cone is not right lined and circular, or even when the cone is merely not right lined, its surface cannot be measured by these rules.

In such cases, the surface of the cone must be decomposed into a number of triangles, sufficiently numerous to ensure the degree of exactness desired; and the triangles are to be produced on a plane surface, one by the side of another. The triangles SAB, SBC, SCD, of the figures 3 and 5, pl. 9, are produced in this manner, in S′A′B′, S′B′C, S′C′D′, of figures 4 and 6. It is by this means made evident, that the curved surface of the cone equals the plane surface S′A′B′C′...which is measured by the methods pointed out in the sixth lesson.

Having now stated the essential measures for the surface and for the volume of the cone, let us see what use is made of cones in the arts.

The architect and the builder construct the roofs of circular towers, of right lined circular cones, fig. 8, having for their axis the axis of the tower itself. Guns are cast, fig. 9, of the form of a series of truncated cones, the largest base of which is towards the breech of the gun. The hat-maker fashions his materials into the form of a truncated cone, with either a flat or curved rim. In the variations given to the dimensions of the part of this truncated cone, and to the dimensions of the border, consists the almost infinite diversity, in the shape of hats and bonnets, which proves both the capriciousness and the fertility of fashion. Figures 10, 11, and 12, will, perhaps, assist the memory of the student.

Organ builders terminate the lower part of their cylindrical tubes by a truncated cone, ABST, fig. 13. Those tubes, the sound of which imitates that of trumpets,—the whole of them being called the *trumpet*, (one is represented at fig. 14), have the form, through their whole extent, of a truncated cone.

Architects at times, for the sake of solidity and strength, employ columns which are of a greater diameter, from the base to a third of the height, than in the part above; the diameter gradually diminishing from that point upwards, to the part which immediately supports the capital. When columns are to be made of such a height that they cannot be formed of a single block, they are divided by a series of parallel planes; and the different pieces into which the column is divided, are regarded as truncated cones, fig. 15. Each of these pieces, called a *drum*, is worked as if it were a truncated cone.

The mast-maker makes ships' masts of a form similar to that of a column, inasmuch as he gradually reduces their diameters from the heel to the head.

Cones are made by several methods, analogous to those employed in making cylinders.

We may first form a regular polygon, ABCD, figs. 3 and 5, of a great number of sides; and each of the faces, SAB, SBC, SCD, may then be worked out, according to the method explained in the lesson relative to plane surfaces.

If, in place of a complete cone, we have only a truncated right lined circular cone, ABCD...*abcd*, ... fig. 16, pl. 9, we must first make the two plane faces ABCD,...*abcd*,...perfectly parallel. On these two planes, two points are marked, O and *o*, which are in a right line, perpendicular to the two planes. Through the two points O, *o*, draw the parallel right lines OA, *oa*, having for their length, that of the radii of the two circles ABCDE, *abcde,* which must be drawn.

The two circumferences are then to be divided into the same number of equal parts, and through the points of division A, B, C, D, ...*a*, *b*, *c*, *d*,...draw perpendiculars to the radius, in order to form two regular polygons, circumscribed about two circles. The plane trapezium faces, having for their superior and inferior bases the sides of the two polygons I. II, 2.1, II. III, 3. 2, III. IV, 4.3,... are then worked out. By this means, we form a sort of truncated pyramid, surrounding the cone. Cutting away the edges, I. 1, II. 2, III. 3, IV. 4,...with a plane, or any other suitable instrument, until the new faces thus formed touch the two circles, we make a truncated pyramid, having twice as many sides as the first, and approaching much nearer the figure of a cone. Continuing in the same manner to cut down the edges, we approach nearer and nearer to the true figure of a cone, and at length obtain that degree of exactness which is required for the particular object.

This is only an approximating method, as will be evident, and we must have recourse to other proceedings, to form a cone of perfect continuity.

Conical surfaces may be made by turning, if we cause a cutting instrument, P, fig. 17, to glide along a rectilineal guide, NM, fixed parallel to the intended edge, AS. In every position of the instrument, it will describe a circle, having for its axis the right line which passes through the points, on which the block to be turned revolves. The whole of the circles thus described, will form the surface of the cone, SABC, fig. 17. In this manner, the boy's top, SAC, fig. 18, is made; and few young gentlemen, perhaps, are sensible, when they follow the amusement of spinning tops, of the mathematical properties of their plaything.

We can form a right circular cone, by making a generating right-line turn round an axis, SO, fig. 3, always making it preserve the same angle with the axis. (See the Eleventh Lesson.)

By the definition of a cone, we see that one may at any time be produced by a movable right line always passing through a point, taken as the vertex.

Application to tracing likenesses.—An instrument, called a *likeness tracer*, ABCD, fig. 19, pl. 9, is employed to copy profiles, with great exactness. It consists of a rectilinear staff, which may be turned round the fixed point S, one end of it resting on the profile ABCD,...the other end being provided with a pencil, which touches a piece of extended paper, the plane of which is parallel to that of the profile. The curve *abcd*, described by the pencil, is similar to the profile ABCD.

To demonstrate this, let us draw OS*o*, fig. 19, perpendicular to the two parallel planes of the profile and the portrait; O and *o*, being the two points where this perpendicular meets the two planes. Let us suppose the staff which traces the portrait in any one of its positions, in AS*a*, for example. Draw OA, *oa*; the two right angled triangles ASO, *a*S*o*, are similar. In fact, the angle ASO, is equal to the angle *a*S*o*, because they are two opposite angles at the vertex. Moreover AO, *ao*, are parallel, whence the triangles ASO, *a*S*o*, are similar; and

SO : S*o* : : SA : S*a* : : OA : *oa*.

In the same manner, it may be demonstrated that,

SO : S*o* : : SA : S*a* : : SB : S*b* : : SC : S*c* : : SD : S*d*

SO : S*o* : : OA : *oa* : : OB : *ob* : : OC : *oc* : : OD : *od*

Or the lines OA and *oa*, OB and *ob*, OC and *oc*, are parallel two and two: consequently, the figures ABCDEF....*abcdef*....are similar figures, the corresponding lines of which are parallel and proportional to the distances of the fixed point S, form the planes of the profile and of the portrait. Whence, finally, the profile ABCD, and its portrait *abcd*, are similar.

Nature herself, by means of the rays which emanate from every luminous point, traces conical surfaces, similar to those of the likeness-tracer. These rays enter the ball of the eye at the pupil, and cross each other at a point S, fig. 22, to reach a surface PQ, called the retina. On this surface Nature produces a picture, which gives the forms, and even preserves the colours of external objects. The impression produced on the retina is transmitted to the optic nerve, by which it is transmitted to the brain, the seat of the understanding.*

* There seems to be a trifling assumption in this passage, which the translator thinks it necessary for him to notice if not to correct. The physiologists, who have examined the organ of vision, have ascertained, beyond a doubt, that there is such an impression made,

Thus the admirable phenomenon of vision is effected in man, and in the greater number of animals, by means of conical surfaces traced in space, and in our eye; the agent being rays of light scattered in every direction, by bodies luminous, either in their own nature, or from reflecting the light of luminous bodies.

or picture painted, on the retina, as is described in the text. They have also ascertained that the optic nerve, in a healthy state, and the brain, are both necessary to correct vision. They have proved too, that when the optic nerve and the brain are in a sound state, when individuals are awake, and when the above described impression is made on the retina, vision, or a sensation of sight, is produced; or we see the object, the image of which is impressed on the retina. But there is nothing in the connection between the picture on the retina, and our consciousness of perceiving the external object, to warrant the assertion, that this impression or this picture, is *transmitted* to the optic nerve, and subsequently *transmitted* to the brain. It is quite plain that the *image* is not transmitted, for that is altogether an optical phenomenon, and neither the nerve nor the brain has, as far as we know, any facility for either receiving or transmitting optical impressions. The picture on the retina is the ultimate physical circumstances which physiologists have yet traced when vision occurs; and between that and actual sensation and perception,—or the mental act of seeing, all is darkness. We know that there are impressions on the retina; we know that in ordinary cases when they are received, vision follows; but we know not how the bodily impression is connected with the mental perception, and to account for it, by further impressions on the optic nerve, by impressions on the brain, and by this being the seat of intelligence, only serves to keep our ignorance out of view, and attempts to explain the incomprehensible connection between the impression on the retina, and the sensation of sight, by another equally incomprehensible connection, viz. an impression on the brain with this sensation, of which connection, however, we have no knowledge whatever. There are in every direction some bounds to our knowledge, which, although they continually recede as investigation proceeds, can never be passed;—which equally exist, though their extent is different, for the most ignorant savage and the most enlightened philosopher; and at which every man feeling himself suddenly arrested, and being unable to explain the connection between certain phenomena, feels the sentiment of wonder, and is compelled to reverence a Power, the ways of which, he is thus made sensible, are inscrutable. Tr.

All the luminous points visible in the heavens during a beautiful night; all the objects which make up a delightful landscape, when seen on a clear and serene day, are painted in our eye, in all their due proportions and forms, their colours and their flitting shadows, by means of cones, in the manner just described.

Camera obscura.—Art imitates nature, constructing a chamber similar to the interior of the eye, allowing the light only to enter by a glass or lens, similar to the pupil of the eye, S, fig. 22, pl. 9. The light forms on the sides of this chamber, as on the retina, *abcd*, a representation of the colours, forms, and movements of the objects that are in the front of the lens. If we receive this transmitted light on paper, we can draw the outline which it forms, and reproduce the same colours, the same lights and shades.

The rays, which emanate from a single point S, fig. 20, pl. 2, and meet an opaque surface *abcdef*, cannot proceed onwards; the rays which touch the outline of this surface proceeding onwards, separate the space beyond the opaque body into two parts; one illuminated, and the other deprived of light by it. The space deprived of light is called the shadow of the opaque body. Thus, when an opaque surface or body is placed before a luminous point, its shadow is limited by a conical surface, having the luminous point for its vertex.

Silhouettes.—This property of the luminous rays has been turned to account in drawing portraits on a plane surface, similar to any given profiles. The profile to be imitated, *abcde*, fig. 20, is placed in a plane parallel to that on which the portrait is to be drawn. A light, such as a candle or lamp, at a convenient distance, becomes the vertex of a cone, having for its base the profile to be copied. The cone is continued to the plane of the portrait, so as to trace on it another base, ABCD... similar to the former, and marked by the outline which is the limit to the shadow of the profile; this base is the *silhouette* of the profile.

The student will observe, that the figures 19 and 20, one representing the *physionatrace*, and the other the transmitted shadow, have the same letters of reference; which is done designedly, because the demonstration given of figure 19 applies also to fig. 20, and leads to the same consequences.

*Ombres Chinoises.**—The property of conical surfaces to reproduce on a given plane the exact profile of a single figure, or a group of figures, has also been turned to good account to produce a pleasing amusement. A single light illuminates some cards cut into figures, or real figures, and transmits the lights and shadows on to a curtain which cannot be seen through, but which allows so much light to pass in the illuminated parts as to render the parts placed in the shade quite distinct to the eye of the spectator. These parts are the bases of conical surfaces, having for their vertex the lamp or any other luminous point placed behind the curtain, the edges passing by the profile of the persons or figures whose position and form are to be reproduced.

If the same object, AB, fig. 21, the shadow of which, MN, is carried to the curtain RR, moves from the luminous point S, towards *ab*; the shadow formed by *ab* is diminished, and is represented by *mn*. The position of the luminous point remaining the same; as the object represented is moved towards the curtain, the extent of the shadow is diminished, while as it is moved from the curtain, the shadow is enlarged more and more. If the object remains fixed, the shadow may be enlarged by removing the luminous point nearer to the curtain, or diminished by removing it further off.

This variety produced in the size of shadows, which preserve the same form, and the diversity of scenes resulting from the motion of these shadows, and changing the objects represented, give a great interest to this species of

* The French name is preserved, because it is well known to the English reader, and we have no substitute for it.

amusement. The properties of conical surfaces enable us to reduce to exact geometrical forms the desired effects, and to establish the proportions suited to this optical game. Let us now refer to a much more important application of the conical form of shadows than that of the *ombres chinoises*.

Principle of Perspective.—When all the visual rays possible are directed from a fixed point, S, fig. 22, pl. 9, on the curve ABCD, these rays form a cone, SABCD. If we ascertain the section *abcd*, made in this cone by a plane, MN, the figure *abcd* will form in the plane MN the representation, or, as it is called, the perspective of the figure, ABCD. The same effect, in respect of form, will be produced in the eye; that is to say, the perspective will produce on the retina the same image as ABCD; for the right lines S*a* and SA, S*b* and SB, S*c* and SC, coalesce and are the same.

Perspective, therefore, has for its aim to produce a representation of objects, so that, when viewed from any point, S, the representation shall affect our vision like the objects themselves. Our mind receiving the same images, whether proceeding from the objects or the representation of them, we find it difficult to distinguish between them; or, rather, we enjoy a resemblance obtained by the efforts of art. Such is the source of the intellectual pleasure which the spectator experiences at the sight of well executed perspective.

If the eye of the spectator were not placed at the point of view S, the cone S*abcd*, would be changed; and it would no longer produce on the retina an image perfectly similar to that produced by the object itself. Such is the disagreeable effect experienced, more or less, when the eye is placed in any other position than the point of sight; a point thus named, because it is that from which we ought to see the perspective, to enjoy its proper effects.

The perspective of curves produces cones, that of polygons produces pyramids, by the whole of the visual rays or right lines drawn from the eye to the outline of the curve or polygon.

If we look at a regular polygon, to which the plane of the picture is parallel, and the visual ray, drawn through the centre of the polygon, is perpendicular to the plane, the perspective will be similar to the polygon, and the image painted on the retina will be a similar regular polygon. But if we draw the perspective of the polygon, and remove the point of sight, the image on the retina will be regular no longer. The polygon will appear shorter vertically, and lengthened in some other direction.

Thus, when the object to be represented is not in a plane parallel to the plane of the picture, the perspective generally differs in form from the object to be represented. These differences are of infinite variety. There are, however, very important general rules, which serve to abridge the labour necessary to place objects in perspective,—an art which must be frequently practised by many artists, such as architects, landscape painters, house and ornamental painters, sculptors, when they execute *bas reliefs*, &c.

If two right lines AB, CD, fig. 23, pl. 9, are parallel to the plane of a picture MN, the perspective representation of them, on the picture, will also be two parallel right lines.

In fact, if we draw the visual rays SaA, SbB, ScC, the lines AB, *ab*, as well as CD, *cd*, will be parallel: AB and CD are parallel to one another, whence the two perspective lines *ab*, *cd*, will be also parallel, and consequently will never meet.

Let us now suppose that the lines AB, CD, EF, fig. 24, are parallel to one another, but are not parallel to the plane of the picture MN.

From the point of sight S, draw to the picture MN a right line, SO, parallel to the right lines AB, CD, EF, which are to be placed in perspective. Then draw the visual rays SA, SB, which will cross the picture in *a* and *b*. These two rays are in a plane which passes through S, and through AB, and, consequently, also through SO, parallel to AB. The three points *a*, *b*, O, therefore, which are all in this plane, and on the picture, are in a right line. Whence *ab* produced, will pass through O. The same thing may be demonstrated of *ca*, *ef*, &c.

The lines *ab*, *cd*, *ef*, therefore, the perspectives of the parallels AB, CD, EF, produced if necessary, pass always through the same point O, when AB, CD, EF, are not parallel to the plane of the picture.

The point O is very remarkable, and is called the *point of concurrence*, of the perspective of the parallels AB, CD, EF.

When objects on which there are many parallel lines are to be represented in perspective, it is advantageous to ascertain the point of concurrence of all the lines in each direction. In this manner, we gain one point of perspective for each of them; and, by obtaining another point, we are enabled to form a complete representation of them.

Application to Architecture.—In particular, when an architectural design is to be placed in perspective, it will be found very advantageous to determine the point of concurrence. The greater number of lines drawn by architectural draftsmen are parallels, either to the vertical plane, which lies in the direction of the front of the edifice to be represented, or to the vertical plane at right angles with it; and of these lines, some are vertical, others horizontal.

The plane of a picture, on which the perspective is to be drawn, is almost always vertical, fig. 25, pl. 9. In this case, all the lines which are vertical in the building, are also vertical in the perspective. The horizontal lines, which are parallel to the plane of the front, have a point of concurrence, O, which ought to be ascertained. The point of concurrence *o*, of the horizontal lines, which are perpendicular to the plane of the front, must also be determined; we have then only one single point more to fix for the horizontal and vertical lines. The method of projection supplies us with an easy method of doing this, which will be pointed out in treating of the intersection of surfaces.

When we know that some of the lines are parallel, and they are seen in perspective, we ought immediately to examine if these lines, when prolonged, would meet at a single point properly placed, which is their point of concurrence in the picture.

When an edifice is drawn in perspective, on a vertical plane, fig. 25, which, as already stated, is the most usual case, both in plan drawing and in painting, the points of concurrence of all the possible groupes of horizontal parallel lines are placed on the horizontal plane, which passes through the point of sight. This is, in effect, the only plane which can be drawn through this point, parallel to the horizontal lines. Thus, on the one hand, the point of concurrence for the perspective of the horizontal lines, parallel to the front of the building, and on the other, the point of concurrence for the perspective of the horizontal lines, perpendicular to the front, are both placed on a level with the point of sight. Consequently, at this height, the horizontal lines of both directions are put in perspective in the direction of a horizontal line, O*o*, fig. 25, situated at the same height as the point of sight.

It will be easily perceived, fig. 25, that the parts above and below the windows, which are in a right line in the building, are also in a right line in the perspective. This results, in fact, from a property of the different parts of right lines, whether separated or not; for if these parts be joined, though only by an imaginary line, they will form a continued right line; the perspective of which is a single right line, including, consequently, the representation of all the portions of a right line, which were to be placed in perspective.

Application to painting.—In pictures in which several persons are represented, the artist is careful not to place them all in the same plane, nor in the same attitude. Were he to do this, they would appear either all of the same height, or lessening, according to some regular law; so that if they were all in a standing posture, and equal in stature, not only would all their feet be placed in the same right line, but all their knees, all their arms, all their hands, and all their heads, would be respectively in the same right line; and all these right lines concurring in one and the same point, would produce an insupportable monotony.

In order to avoid this, which would be ruinous to painting, the artist is careful to place his figures at different distances from the spectators; and imagines several planes parallel to the canvass. On the first, the nearest to the spectator, objects are painted with the greatest relative dimensions. They are relatively less on the second plane, still smaller on the third, being gradually lessened in size as the imagined planes are farther from the spectator. On the first plane, or very near it, the principal persons of the picture are generally placed, whose superior dimensions, naturally, most attract attention.

According to the situation of the plane, on which the figures are placed, the perspective of them ought to be of certain dimensions. If the painter does not ascertain this precisely, his picture will be false, and the persons will not appear at different distances as he wishes to represent them. If he has placed their heads properly, and put their eyes in a suitable direction, the figures, which ought to be looking at each other, will appear to be looking another way, &c.

Painters may, and in fact do, commit many other faults against perspective; particularly when they are to represent the human body or limbs, or animals, which are not situated in a plane parallel to the plane of the picture; and which, on this account, are very often reduced in length.

These *foreshortenings* are the most difficult part of the art of drawing; and, in general, artists only succeed in them by placing models in the very position in which they desire to represent any object. They place themselves, in relation to these models, at that point where the spectator will be placed in relation to the picture.

The few principles now laid down may suffice, in a multitude of cases, to enable the student to ascertain the correctness or incorrectness of the perspective of such objects as fall under his notice. It very often happens that both architects and painters are not properly acquainted with the laws of perspective, and consequently apply them incorrectly. When geometrical knowledge is more widely spread, many gross faults, which at present offend the discrimination of only a small number of *connoisseurs*, will be perceived by the public at large; and artists, finding more enlightened critics, will not be able to err with impunity. They will be compelled to study more profoundly the applications of geometry to perspective; and thus, by enlightening the great mass of the people, though they may neither practise the arts, nor purchase the products of the artists, we secure for our country more skilful artists and more admirable specimens of their skill. In the best days of Greece and Italy, it is probable that almost every citizen could judge as correctly as the artists themselves, of the magnificent works which adorned the temples and cities of these favoured countries. When the public is an enlightened judge, accuracy of execution is a general talent. Works of art are then formed or constructed with that exactness of proportion in all their parts, which is as indispensable to perfection in the fine arts, as in the structure of those machines which, though intended to perform some service, would not move at all, were not every part constructed with geometrical precision.

Application of perspective to drawing the plans of machines, and other products of industry.—When machines, or other works of art, are to be represented, it is sometimes done in perspective. This method has an advantage over projection, inasmuch as it renders visible many of the parts which in that method conceal one another. For example, in representing the projection of a building by parallel lines, it is customary to make the vertical plane of projection either parallel to the front, or at right angles with it. In the former case, the small sides are not visible; in the latter, the front itself is not seen. Perspective, as we see in fig. 25, pl. 9, has the advantage of showing us at the same moment two sides of a building.

The method of projection enables us to place any object in perspective, rigorously correct. The object and the point of view being given in horizontal and vertical projections, and also the plane of the picture, we can obtain the perspective of any point, by drawing a right line from this point to the point of view, and ascertaining the point of intersection between the line and the plane of the picture. (See Lesson 13th. The professor who teaches from these lessons should illustrate this by some simple examples, as the perspective of a square or cube, with the necessary figures.)

To sketch a building, a machine, or any product of industry at sight, in perspective, has this advantage, that we can represent it as the eye perceives it, without the necessity of making any alteration by our minds in its appearance. Workmen and students should accustom themselves, therefore, to both species of design; at least, if they do not practise them, they should be able to conceive and judge of their effects; to practise them might, perhaps, require more time than they have to spare. For more detailed instruction, they must consult books which treat especially of these subjects.

Application to theatrical decorations.—In order to increase the illusion, which is the object aimed at in theatrical performances, and facilitate scenic representation, a large picture, representing, generally, a landscape, containing some fine buildings, is stretched across the bottom, and constitutes the back-ground of the stage. On each side, in the direction of two lines which recede from each other in advancing from the bottom towards the spectator, a series of narrow and tall pictures, called the side scenes, is placed. They are parallel to each other, and to the scene constituting the back-ground. On them are painted trees, single columns, or elaborate pictures, parts of which are on each side scene, and which are continued from one to another. This is, however, an imperfect method; for, in fact, the lines which represent on these side scenes fractions of the same right line, though, when seen from one point of view, appear to form only one line, yet they are not, and do not appear to be in the same direction when seen from any other part of the theatre. Notwithstanding this defect, a well drawn and well coloured scenic perspective so much resembles reality, that it procures for the spectators in many parts of the theatre a very agreeable illusion.

Conical projections applied to geography.—In order to represent the most remarkable objects of the terrestrial and celestial globe, a system of conical projections is sometimes employed, similar to perspective.*

Though in mechanics so much use is not made of cones combined two and two, three and three, &c. as of cylinders combined in this manner, yet they are, in some circumstances, made use of with advantage.

Regular cones, combined as in fig. 26, pl. 9, are made use of to transmit, by means of friction, the rotation of one axis to another, the two axes not being parallel. Regular cones, with teeth, fig. 27, are employed for the same purpose.

Architects, in order to form large columns, divide them into truncated cones, or drums, which are grooved, when the columns are to be fluted. To make fluted columns, much precision is necessary. If there be one thing more capable than another of giving us a correct idea of the rare skill acquired by the Athenian workmen who were employed in constructing those buildings which still constitute the glory of that industrious city, it is the perfection with which the drums forming the parts of

* One of the poles of the earth is represented as the summit of a cone, having for its base each of the curve lines, which are to be traced on the most distant hemisphere. The intersection of this cone and the plane of the *equator* is the polar projection of this curve.

the largest columns were fluted on their conical surfaces, and with which these different truncated cones were so adapted to each other, that the grooves, or fluting, when put together, were in complete unbroken continuity from the capital to the base of the column.

Precision in making the teeth of conical wheels, is not an object merely of luxury and ostentation, as perhaps making fluted columns, may be considered. On this precision depends both the facility of transmitting motion, and the economy of doing it, as will be hereafter explained, in giving an account of teethed wheels, in the second volume of this course.

TENTH LESSON.

The Developement of Surfaces.

EVERY surface which can be unrolled, spread out, or developed on a plane, in such a manner that no part of it, in performing the operation, requires to be lengthened, shortened, opened, or doubled up, is called a *developable* surface.

We have already examined two important species of developable surfaces—cylinders and cones; and we have seen that these surfaces may be represented by, or spread out on a plane, without breaking, and without doubling one part over another. We have also seen, conversely, that a part of a plane may be made, without breaking or doubling into the form of a cylinder or a cone, the figure and dimensions of which are given.

We have moreover seen, that a cylinder may always be considered as a prism composed of a great number of plane faces or sides, each having the form of a parallelogram; and that a cone may be considered as a pyramid with a great number of faces, each face having the form of a very narrow triangle.

In the same manner we may consider every developable surface, fig. 1, pl. 10, as composed of plane faces or sides, *a*A*b*, *b*B*c*, *c*C*d*, terminated by right lines A*a*, B*b*, C*c*, which are called edges.

If we wish to develope this surface, or to reduce it to a plane surface, we begin by turning the face *uAb*, round the edge A*b*, till it is in the same plane as the second face *b*B*c*; these two faces are then turned round the edge B*c*, till they are in the same plane as the third side *c*C*d*. If we perform a similar operation for each of the sides, till we bring them all into the plane of the last one, we shall then have completely developed the surface.

The difference between the cone and the surface most generally developable, is, that in the cone all the angular faces have their summit at the same point, while, in any other developable surface, the summits A, B, C,...of the sides *u*A*b*, *b*B*c*, *c*C*d*,...are not at the same point.

In the same manner as the geometricians consider cones to be formed of two parts, or consist of two cones, (fig. 1, Ninth Lesson) so they also consider developable surfaces as having two parts, or to consist of two surfaces; the first such as we have just described, and the second formed by producing the edges A*a*′, B*b*′, C*c*′,...beyond the curve ABCD...generally called the curve of inflection. In the arts, it is in general sufficient to consider only one part of developable surfaces.

Application.—When we desire to preserve any valuable objects, we in general inclose them in some substance less valuable; and this envelope, in most cases, is a flexible and plane sheet of some common material, such as paper, pasteboard, skins, cloth, tin, sheet-iron, &c. &c. such as sheaths of instruments, scabbards, tents, wrappers, boxes of all descriptions, the envelopes employed by confectioners, grocers, apothecaries, &c. &c.

All these envelopes, in whatever manner they may be doubled and redoubled, are developable surfaces. It is only necessary to remark, that the substances employed, being generally susceptible of compression or extension, may differ in places from the precise and rigorous forms of the developable surface, such as we, following the example of goometers, have here defined it.

Application to hangings and draperies.—This latter remark may be extended to the hangings and draperies with which apartments, and the interior of public buildings, are decorated. They are composed of materials which, in their foldings, differ from geometrical forms. If they constituted developable surfaces rigorously geometrical, we should have only rectilineal foldings, stiff and hard outlines, destitute of grace and variety; such, almost, as the foldings and the outlines of Etruscan draperies.

The Greeks appear to have been the first people, whose graceful and fertile imagination enabled them to perceive the beauty of those combinations, which might be obtained by attending to the two distinct properties possessed by draperies, of folding into developable surfaces composed of rectilineal edges, and of falling in uniform curves, varying by gradations regulated by the principles of good taste. In the decoration of edifices, taste itself is subject to rules, which may be reduced to general principles.

Returning now to surfaces developable according to strict principles, we shall see how very extensively they are used in the arts, and what advantages industry may derive from the geometrical solution of questions connected with them.

Let it be required to construct a developable surface (fig. 2, pl. 10,) passing through two curves ABCDEF, *abcdef*, which are not in the same plane. For this purpose, we suppose that the curve ABCDEF is a polygon, having a great number of sides, AB, BC, CD, DE;...we take a ruler, perfectly straight, one end of which, laid on its flat side, is placed on AB, and made to turn round AB till the other end of the ruler meets the curve *abcdef* in two points, very near to each other, *a* and *b*; and we then draw the right lines A*a*, B*b*. This being done, place the ruler so that the large flat side may touch BC and B*b*, and mark the point *c*, where this plane side meets the curve, and draw C*c*. In the same manner the position of D*d*, E*e*, F*f*, are determined; and thus we may produce the developable surface ABCDEF *abcdef*, which will differ very little from that which passes rigorously by the two curves ABCDEF, *abcdef*. (See Thirteenth Lesson.)

Sawing wood into a curved or circular shape.—In ship-building, it often happens that we must saw some of the timbers and planks, producing a surface of which the outline *abc*, (fig. 2, pl. 10,) is traced on the lower part of the piece, and the outline ABC on the upper part. If we wish to do this, without bending the saw, so as to make it lose its plane or developable surface, it will be necessary that the right line formed by the teeth of the saw should be drawn so as to fall successively on the right lines or edges A*a*, B*b*, C*c*,...fig. 2, and then the saw will cut through the wood, describing at the same time a developable surface.

Application of developable surfaces to stone-cutting.——Masons very frequently employ developable surfaces, which are generally cylinders or cones. To construct arches having a complicated form, the figure of all and each of the stones which are to form the arch is ascertained, as will be explained in the Thirteenth Lesson, treating of the intersection of surfaces. In order that the edifice should have the greatest possible solidity, all the stones, as they mutually support one another, ought to touch each other through the whole of that part of their surfaces which is concealed from view, and called the joints. It is necessary, therefore, that the surfaces to be joined should be determined with precision, in order that they may apply exactly to one another. This object is easily obtained, if the faces of the joinings are made developable. We can then fashion, either in pasteboard, in thin plank, or any other convenient substance, the pattern of each developable surface; apply it to the face of the joint, and ascertain if the pattern will touch this face in every part, according to the direction of the edges.

Perhaps no more striking illustration can be given of the necessity of having the surfaces of joints rigorously parallel in every part of an edifice, than what occurred in building the Pantheon at Paris.* In this edifice, a vast and lofty dome was to be supported by four groups of elegant columns. In order to give the columns the appearance of consisting only of a single stone, the drums, or truncated conical pieces of which the shafts were formed, were hollowed out towards the centre, so that the edges of every two pieces might unite closely all round, and not leave the least visible separation. The aspect of these columns, when first erected, was beautiful; they appeared a *chef-d'œuvre* of art: but when the immense weight of the arch was laid on them, the edges of the drums, which alone were in contact, not having sufficient surface to support the pressure, split and crumbled away, and the whole dome settled down till all the surfaces of each joining came into contact. The architect found it necessary to erect some large massy pillars in the centre of each group of columns which supported the dome, and the beauty of the structure disappeared. It would have been preserved, had the joinings of the drums been surfaces applied to each other in every point. Geometry supplies us with the means of doing this, in the most simple as well as in the most complicated cases.

Let us draw very correctly the curvilinear edges AB, BC, CD, DA, *ab*, *bc*, *cd*, *da*, fig. 3, pl. 10, of an arch stone. We can determine for each face of the joint a developable surface, which passes through both AB and *ab*, one for BC and *bc*, one for CD and *cd*, and one for DA and *da*. By doing the same for the adjacent stones, we may be sure that the faces in contact will apply to each other in every point. When we know the position and figure of AB and *ab*, of BC and *bc*, it will be very easy to employ the method already given, fig. 2, to determine each developable surface.

When a large superficies is to be covered with the leaves of any very thin and flexible substance, these

* Formerly the Church of St. Genevieve. "The dome of the Pantheon," says Mr. Scott, "towers above all in light graceful pride; it arrests the eye of the spectator by the boldness of its elevation, and detains it by the gracefulness of its construction."—(*Visit to Paris.*) Tr.

leaves are bent or fashioned according to some developable surfaces, by the following method. The workmen or artists draw on the superficies to be covered fig. 4, the curve lines ABCDE, *abcde*, *a'b'c'd'e*, *a''b''c''d''e''*, which are every where at a distance from each other, equal to the size of the leaves to be employed. They then bend the leaves, so that they are adapted to the lines ABCDE, *abcde*, and then to the lines *a'b'c'd'e'*, *a''b''c''d''e''*. They place these leaves in succession, uniting them by glue, or paste, or solder, or by bringing the edges over each other, and then fastening them together.

Application to covering domes and cupolas, and coppering ships.—It was by following this method, that the magnificent cupola of the *Halle au Bled** (Corn Market) was covered with sheets of copper.

By following the same method, the bottoms of ships are sheathed or coppered, with thin narrow sheets of this metal, fig. 7, ABCDEF. The edges of the copper sheets are cut straight, or in a right line, though their edges very often cover one another in a direction not corresponding to this line. But by bringing the edges one over the other, so that the doublings are not equal at all the angles, nor rectilinear for all the sides, the same effect is produced as if each sheet had been cut into such a shape as would have allowed the edges of all to have come just into contact, supposing them all to have been soldered together, while the whole of them exactly covered the ship's bottom.

The means adopted by shipwrights is practicable, because the surface of the ship's bottom is very large in relation to the size of each sheet of copper; and because the copper may be extended a little in any part, so as to be adapted at every point to the two directions of the curva-

* This is a very large circular building in the middle of one of the most populous and frequented parts of Paris, in which, alone, all the wholesale buying and selling of corn and flour for the consumption of the metropolis takes place. Tr.

ture of the ship's bottom. This will be rendered plainer, when we treat of the two curvatures of surfaces which are most common.

The card and pasteboard maker, who constructs a great number of objects with sheets of paper, or pasteboard pasted or glued together, produces a great variety of developable surfaces, both in relation to their position and form.

When the coachmaker has constructed the frame-work of a carriage, or put together the pieces of wood and iron which form the angular and solid parts of the coach—the frames of the doors, windows, &c.—he must close up the spaces marked by these frames, and by the wood-work which forms the body of the coach. He does this, in general, by thin flexible pieces of wood, bent according to some developable surface, which will extend to the frame-work at every point, and fill up all the intermediate spaces. The coachmaker ought, therefore, also to be acquainted with the solution of the problem explained by figures 2 and 3.

The brazier, the stove-maker, the tinsmith, ought also to be acquainted with the solution of this problem. In the construction of stoves, for example, and of many of the kettles and coppers used in the kitchen, in order to adjust the upper part of the stove or cauldron to the pipe, the workman must frequently make a developable surface, which will pass through a lower base, ABCD, of some certain form, fig. 5, pl. 10, and through an upper part *abcd*, having the form of a circular pipe. He must in this case, know exactly what form to give the metal plate, or system of metal plates, being planes, which being suitably bent, will form a developable surface, adapted both to ABCD and *abcd*. The solution of this problem will be given when treating of tangents, in the Fourteenth Lesson.

In place of covering surfaces by small sheets, as in fig. 4, it is sometimes more convenient, and therefore preferable, to cover them with long bands, which are developable surfaces.

When soldiers wore armour, the greater number of the pieces which covered their bodies and their limbs had developable surfaces; generally they were cylindrical or conical bands, easily made out of metal plates, to which it was only necessary to give a degree of curvature. There were very few pieces, such as the helmet, which it was necessary to make curved in two directions, and sometimes, even in constructing them, they were made of developable surfaces.

In ship-building, we have an example of a very fine application of developable surfaces disposed in bands, or in the planking of ships.

When the timbers of a vessel are fitted into the keel and placed erect, she has the appearance of a sort of *carcase*, MNOPQ, fig. 6, the ribs or timbers being connected in pairs 1.1, 2.2, 3.3... Being elevated in vertical planes, they leave large void spaces betwixt them (xyz, fig. 8, represents the elevation of the timbers in middle of the vessel). To cover the bottom completely, the form of which is determined by the timbers, planks are employed, of a given thickness, and accurately fashioned, according to a given and proper mould. The flat sides of the planks are placed against the outer part of the timbers, and bent closely to them at every point, into developable surfaces, each plank being adjusted side by side, and end to end, to the contiguous planks, so as to cover, with great precision, every part of the bottom. A rigorous method of cutting each plank, so that the whole shall exactly cover the timbers, is supplied by geometry.

Let us suppose that the ship has already been planked, from the keel up to ABCD, fig. 6, and that it is required to fit the next plank to the space immediately above, included between the lines ABCD, *abcd*. From a point x, to another point y, both being conveniently situated between ABCD, and *abcd*, a cord is extended so that it touches every timber on the outward part. Let us suppose that the plank to be fixed is prepared and placed, and that the cord is suddenly glued or fastened along the surface of the plank next the timbers. Let us develope, that is to say, take down the plank, and restore it to its straight position. The cord, when fastened along the timbers, marked on them, or on the surface of the vessel's bottom, the shortest distance between the points x, y, and this cord, will continue to mark the shortest line, which can be traced between its extremities, on the developable and now developed surface; that is to say, on the plank. But the shortest line which can be traced on this plane, is a right line. Whence the cord *xy*, will be in a right line, fig. 6, *double*, if it retains on the plank the position which constituted it on the bottom of the vessel, the shortest distance between *x* and *y*.

When the cord is extended along the timbers, it is marked on its length at the points 1, 2, 3, &c. and through them, perpendicular to the cord, treenails or small pieces of wood are fixed, which touch with their lower end, the upper edge of the plank already fixed, ABCD, and at their upper end, the line to which the upper edge of the next plank is to extend, *abcd*, or they extend to the two lines to which the new plank is to be adapted.

The cord *xy*, is taken from the timber and applied to the plank GHKL, fig. 6 *double*, so that the treenails, I 1 *I*, II 2 *II*, III 3 *III*, IV 4 *IV*, are perpendicular to the cord. The polygons I, II, III, IV,...*I*, *II*, *III*, *IV*, are first drawn on the plank, and through them a continued curve is drawn. They represent exactly, the upper and lower limit longitudinally or edges of the plank.

It is not sufficient to have found these curves; we must also know at each point I, II, III, IV,...*I*, *II*, *III*, *IV*,...the angle which the plank, to be applied, ought to make with the bottom, in order that the edges may apply exactly to the edges of the contiguous planks. These are ascertained by placing one branch of the bevel in the direction of each treenail, and the other along the edge of the plank already fixed, and perpendicular to the edge of this plank which touches the timbers. These angles must be respectively transferred to I, II, III, IV,...*I*, *II*, *III*, *IV*,...when the plank is worked out by the shipwright, whether he employs an axe or an adze.

In order to prevent confusion, as soon as the shipwright has ascertained, by means of the bevel, the angle which the joining of the new plank makes at I, II, III, IV, with the contiguous plank already fixed, he places one side, *ts*, of the bevel, on the side of some thin plank NP, fig. 6, *treble*, and along the other side *sr*, of the bevel, he draws a right line. All these lines being placed in the same order as the treenails 1, 2, 3, 4,...which, correspond with the points I, II, III, IV, it is very easy for him to find what beveling his plank will require at the points I, II, III, IV,...in order to give the narrow side (cant) of the plank, a proper slope in relation to its other sides.

It is essential to remark, that this method, assuming no particular form for the ship's bottom, may be applied to every species of architecture, civil or military, as well as to building ships. It is one of the most favourable examples of the advantages derived from applying in the arts those geometrical properties we discover in surfaces.

Developable models and patterns.—In a great number of arts, when it is required to form surfaces terminated by certain lines, these surfaces are divided into parts, which may be supposed to be developable surfaces. Their shape is then imitated by a model or pattern of paper, or other flexible substance, which will form true developable surfaces, without wrinkles, and without tearing. By models and patterns of this description, tailors and mantua-makers fashion the pieces of the garments, which they adapt to the geometrically irregular surface of the human body.

A very useful application of geometry consists in showing us how we may cut out the different pieces of which clothes are made, so as to lose the smallest possible quantity of stuff. Though it is not customary to employ either rulers* or compasses to resolve the problem, we must not, therefore, suppose that the tailor or the mantua-maker does not perform a mathematical operation, and even a very complicated operation, which requires a correct eye, a process of reasoning, and much knowledge of the human form, as well as of the shape, which must be given to the separate pieces of which clothing is made.

Independent of economy, comfort, ease, gracefulness, and elegance, ought all to be studied in clothing; and they are regulated by principles, which, in many respects, approach the laws of geometry and mechanics. The remarks already made on the subject of draperies and hangings in relation to developable surfaces, susceptible of being shortened or lengthened in places, and which

* The tailor employs instead of a ruler, a flexible measure, which is a developable surface, divided into equal parts. Being applied round the body and the limbs, it enables the tailor to ascertain their dimensions, which being transferred to a piece of cloth constituting a plane, by developing the measure, gives the points through which the tailor first draws the outlines, and then cuts out his work.

constitutes their flexibility and elasticity, may be, in a great measure, applied to clothing. Materials of this description, having the property of adapting themselves to the human form, easily take that shape which is dictated by fashion. To use the terms of art, such stuffs make the most becoming dresses.

If, in addition to being very elastic, the materials of clothing are also very supple and very light; they may be formed into those numerous inflexions, those varied and numberless folds which accord with the laws of good taste. Instead of preserving a hard rigidity, and a geometrical inflexibility, light and flexible cloths, when once arranged, are sensible of the smallest impulsion, and take a waving appearance, which reminds us of the varied agitations and graces of life. The artists of antiquity appear to have had stuffs of this description for the models of those very elegant draperies with which some of their statues are adorned. In modern times, muslins and cachemeres are the materials which best fulfil these conditions.

Clothing, to be perfect, should allow all parts of the body and all the limbs to move with ease and facility; which requires a certain amplitude and lightness, as well as a precise form, adapted to the different parts. But mankind having, in general, connected solemnity, importance, and dignity, with slowness of motion, those persons, who by their office are obliged to display these qualities, must have a *costume* which is consistent only with such a motion. The processional robes of the priest, the toga of the senator, and the monarch's mantle, are made of ample dimensions, and of materials so little flexible, that they form developable surfaces with large folds, which cannot be put into hasty motion by the agitation of the air.

The military tunic, the light drapery of the public dance, the gay ball dress, on the contrary, should all be fashioned, so that each dimension should be as small as possible. We should also choose the lighest, most flexible, and most waving stuffs, in order to display the human form, and its various movements, with gracefulness and fidelity.

Under this point of view, the choice of the materials, and the form of the garments, ought to be regulated by considerations derived from the theory of the fine arts, and connected with the organization of society. Considered in relation to convenience, comfort, and health, they belong to the more positive interests of the social state. Finally, in relation to industry, we must look to mechanics and geometry to furnish us with a measure for the forms and the qualities of these products, and with the means of manufacturing them in the best manner. These sciences teach us, also, what shape and adaptation of them are best calculated to produce, by the flexion of the stuff, and the collecting of primitive plane surfaces, that variety of pleasing forms which we observe in the clothing and the drapery of a people, over all of whom the fine arts exercise their agreeable influence.

We shall return to the subject of developable surfaces, after we have explained the principles of intersections and of tangents, to point out further applications of them, not less important than those already adverted to. At present we must consider indefinite surfaces.

Irregular non-developable, or indefinite Surfaces.—Surfaces engendered by consecutive right lines, which cannot be regarded as forming a succession of very narrow plane faces, are called irregular non-developable, or indefinite surfaces.

To form an idea of these indefinite surfaces, imagine a ladder, fig. 9 and 10, pl. 10, of which the two sides are not in the same plane. Place the ladder on the ground, so that both the sides may be in a horizontal direction, but not in the same vertical plane. Figure 9, will represent the vertical projection, and fig. 10, the horizontal projection. The sides AB, CD, fig. 9, will cross in some point 4,IV. If from this point we draw a vertical line, it will pass, fig. 10, through the point 4, in CD, and through the point IV in AB. Let us now, setting out from 4 and IV, divide the two uprights, AB, CD, into equal parts, at the points 1, 2, 3, 4, 5, 6, 7...I, II, III, IV, V, VI, VII, and draw the right lines 1.I, 2.II, 3.III, 4.IV, 5.V, &c., and we shall form a crooked or lop-sided ladder. This represents an indefinite surface.

The arms of a windmill are ladders of this description, being formed of long diverging sides and cross bars perpendicular to one of them. The rope ladders or rigging of the top-gallant masts of a ship, form also such surfaces, but they want one side.

Indefinite surfaces may be considered as composed of very narrow irregular faces, analogous to the ladder just described. The sides which are the boundaries of these irregular faces, are called edges.

Application to ship-building.—To plank or cover a ship's bottom, developable surfaces are formed of the planks, as already explained, fig. 6, pl. 10. Some parts of the ship, towards the stem and stern, are so much curved, that it would be impossible to make the largest planks cover more than a small space, if the shipwright endeavoured to preserve the exact form of some developable surfaces. In the representation of planking, fig. 11, it will be seen that much wood would be wasted, were the rectangular plank to be cut into the curved form 1, 2, 3, 4, 5, 6, 7, VII, VI, V, IV, III, II, I. Let us suppose, however, that we give a gentle and regular curvature to the cord, *abcdefg*, fig. 11, we shall then obtain a line, which we can apply completely to a plank of much less dimensions than the form given in fig. 12.

But if we proceed to bend the plank, cut into the form of fig. 11, it will not exactly fill the space for which it is destined on the ship's bottom. Mechanical means must be employed to force it into the required position. Such an operation almost always changes the developable surface into an indefinite surface.

For those parts of the vessel where the curvature is very great, the planks cannot be sufficiently bent, without a risk of breaking them by the operation. In this case we proceed thus:—

Method of fitting the circular planking.—Let us suppose that it is required to adapt a plank called circular, on account of its great curvature, to a vessel's bottom, below the plank marked ABC, fig. 13, pl. 10, to a ship's timbers. A ruler, represented by the right line ED, is held fast, by the help of which an idea of a plane surface is conceived, which will mark on the timbers three points, *m, n, o*, of ABC. It is assumed here, that the short portion of the round piece ABC, has only a simple curve. If there were between *m, n, o*, points too far distant from the plane curve *mno*, these distances must be laid down by distinct points, which would be an additional operation. Neglecting this additional operation, we draw through the points *m, n, o*, the right lines *m* 1, *n* 2, *o* 3, perpendicular to ED, and measure their length. We then take a bevel, and place one branch in the direction *m* 1, the other branch is placed on the surface of the vessel's bottom, both branches remaining in a plane perpendicular to ED, *mno*. The same operation is performed at the other points *n, o*, of the curve *mno*. The succession of positions in which the second branch of the bevel is placed, will form a non-developable surface, which will be the inner side of the plank that is to be adapted. The outer side is worked by forming a second non-developable surface, at the same distance in every point from the first, in order that the plank may be every where of the same thickness. For the narrow side which comes in contact with ABC, we have again recourse to the bevel; and the angle is found, which the second branch placed successively in *m, n, o*, against the surface of the timbers, forms with the lower side of the plank ABC, already fixed. It will then only be necessary to work the plank according to these angles, taking care to refer them to their proper places.

When a vessel is to be built, the timbers, as already described, are first set up; they are fixed together in pairs, the pairs are placed in parallel vertical planes, fig. 14. These timbers are then connected temporarily, by means of strong ribands or narrow planks, bolted to each pair of timbers, and placed lengthwise along both sides of the vessel. The curves they are to form are determined beforehand in the mould loft. For such parts of the ship as have only a small degree of curvature in a longitudinal direction, it is sufficient to work the planks which are to form the riband into the form of long quadrangular prisms, with the necessary beveling. The pieces are bent so that their ends meet at the points indicated on the different timbers. If the small portion of the bottom occupied by each of these longitudinal ribands, is developable into a rectilineal surface, the narrow plank may be bent without difficulty through its whole length and breadth to the timbers. If, on the contrary, the portion of the timbers, covered by the riband, and which ought to be in contact with it, is a non-developable surface, their perfect contact does not take place, and great exertions are necessary to adapt the riband closely and exactly to the timbers in the direction determined by the builder. In the parts of the vessel which are very much curved, recourse is had to the following method.

ABC, fig. 14, forming part of the plane of the riband, is marked by two cords; one of which is nailed on the timbers in the direction ABC, and the other, DE, is held at some distance from the ship's bottom. The angle formed by this plane, and the surface of the timbers, is measured on the different timbers A, B, C, by the bevel. Then placing the mould of the curve ABC, on the piece of wood, fig. 15, which is to form the riband, ABC is marked off, and the piece cut into notches opposite to A, B, C, till the bevel shews that they are exactly of the same form as the angles measured on the vessel. The wood is then cut away between the notches, so as to form either a developable or non-developable surface. In the interior of this surface, the points a, b, c, are marked, every where equally distant from ABC, and then the points a', b', c', at a distance from a, b, c, equal to the breadth of the riband, are also marked. We thus obtain the *first* side $abcc'b'a'$, of the riband which is applied to the timbers. The upper and lower sides are worked square with abc $c'b'a'$, and these two sides are made at every part of the same breadth. The fourth side is then worked square with the third. The formation of this piece, and the whole method of measuring and applying the planks to a ship's bottom, may be made very clear by the help of models, to the students in maritime towns; in towns in the Interior, if it should be thought that the explanation cannot be easily given, it may be omitted.

In civil architecture, also, non-developable surfaces are employed for the arch stones of some arches, and for some staircases.

The steps of a staircase ought to be, as is well known, horizontal and level in the part where the foot rests on them, in ascending or descending. They are of the form represented in fig. 16, by ABCFE, DEFGH, the joints by means of which each step bears on that which is immediately below, and supports that which is next above it, being represented by BC, EF, GH. In a staircase with parallel steps, or a straight staircase, the joints BC, EF, GH, are all parallel to one another; they are level, and have the form of parallelograms.

But when the staircase takes a curvilinear direction, or is what is called a round or well staircase, the work of placing the steps is much more complicated. It is evident, fig. 17, that the steps have not the same breadth throughout, being narrower towards the centre of the staircase, than at the exterior part. Consequently, the inclination of the staircase, measured by the lower line GFC, fig. 16, is less in proportion as we recede from the axis of the staircase. The joint, EF, of the steps, which is every where at right angles to GFC, approaches to a horizontal direction as we proceed towards the outer part of the staircase, and to a vertical direction as we approach to the centre.

The series of perpendiculars, EF, to the sunk edge E, forms, therefore, a tortuous surface or ladder, like that of figs. 9 and 10. The joint EF, of two consecutive steps, is, therefore, an indefinite surface. When all the plane surfaces of a step have been formed, according to the rules of the most simple geometry, it only remains to form the face of the joint EF.

For this purpose, divide the length of each step into equal parts, and through the points of division 1, 2, 3, marked on the sunk edge, OE, fig. 17, draw the right lines 1.1, 2.2, 3.3, perpendicular to the edge, and terminating in the next sunk edge above OB.

Fig. 18 represents, on a large scale, the elevation of the step OEB, seen perpendicular to OE. E 1, E 2, E 3, represent 1.1, 2.2, 3.3, of fig. 17 If we draw fig. 18, E I, E II, E III, perpendicular to E 1, E 2, E 3, these lines will represent the direction of the face of the joint of the two steps which touch each other in EO.

for the corresponding points 1, 2, 3. It will be sufficient to lay off the angles AEI, AEII, AEIII, with the bevel to find the inclination of the face of the point EF, in each of the points 1, 2, 3, fig. 16, of the contiguous steps.

The principles of this construction may be made very clear, if recourse for explanation can be had to a model, either in wood or in plaster.

Staircases considered as continued surfaces, belong, at least for their lower surfaces, to spiral surfaces, which possess great interest for the arts; and fall to be treated of in the Twelfth Lesson.

ELEVENTH LESSON.

Surfaces of Revolution.

SURFACES of Revolution are, with the exception of plane surfaces, the most easy to construct, and the most frequently employed in the arts. Their properties are called perpetually into use in mechanics, and Nature in her varied phenomena continually reproduces them before our eyes.

If we conceive any curve whatever, ABC, fig. 1, pl. 11, which is made to turn round an axis AC, the surface formed by it is a surface of revolution. The motion given to the curve, is called a circular or rotatory motion, and when the rotation is complete, that is to say, when the curve has revolved through 360 degrees, it is called a revolution.

In this motion every point B, B′, B″, describes a circle. All these circles have their planes B*b*, B′*b*′, B″*b*″, parallel to one another, and at right angles to the axis AC, on which all their centres O, O′, O″, are situated. In the Sixth Lesson, these various properties have been demonstrated.

It is not necessary that the curve ABB′B″C, should be plane, in order to produce a surface of revolution by revolving round AC. If

we draw from each point of the curve, B, B′, B″, perpendiculars BO, B′O′, B″O″, to the axis AC, these perpendiculars will not vary either in length or distance, when all of them are brought into the same plane. Their extremities B, B′, B″, will in this case, form a plane curve, which, revolving round the axis, will produce the same surface of revolution as the proposed curve.

The simple curve which produces the surface of revolution, by turning round the axis AC, is called the meridian of this surface. The circles B*b*, B′*b*′, B″*b*″, the planes of which are perpendicular to the axis and parallel to one another, have been called, for this reason, the parallel circles, or simply the parallels.

We can describe as many different kinds of surfaces of revolution as we can form different kinds of figures, by means of right lines, circles and other curves, and by their various combinations. These kinds even may be divided into very distinct species, according to the situation of the axis in relation to the generating line. We shall examine successively, the most simple surfaces of revolution, and such as are of most importance in the arts.

Surfaces of revolution produced by the motion of a right line.—If the line is perpendicular to the axis, it describes a plane by its revolution. In the Sixth Lesson, the different methods employed in the arts to form plane surfaces, by means of this property of a right line, have been mentioned.

If the producing line is parallel to the axis OO, fig. 2, it will describe a circular cylinder, the properties of which, as well as its applications to the arts, have been explained in the Eighth Lesson.

If the producing line passes through one point of the axis OO, and is oblique in relation to the axis, it will describe a cone, fig. 3, the properties and applications of which have been explained in the Ninth Lesson.

When the producing right line is not parallel to the axis, and relative to it, is in the same position as one side of a non-developable ladder is to the other side, the right line will produce a surface of revolution, of which the two curves, fig. 4, are in opposite directions.

When a right line, AB, fig. 4, pl. 11, does not pass through the axis OO, we may conceive a second line *ab*, placed symmetrically in relation to the plane OO, which passes through this axis. The two right lines AB, *ab*, necessarily intersect each other in some point P, situated in the plane of symmetry. If we make them revolve with an equal motion around the axis, whether they equally approach towards, or recede from the plane OO*o*, it will always be their plane of symmetry, and they will always intersect each other in some point placed in this plane. Let us cause the plane of symmetry, and the right lines AB, *ab*, to revolve round the axis. The two right lines being so disposed as always to intersect each other in the plane OO*o*, the several points where they intersect each other will form a curve, which is the meridian of a surface of revolution, generated by the two right lines AB, *ab*. These two right lines, therefore, on revolving round OO, will generate the same surface. Fig. 4, represents the two series of right lines which form this surface. Students will comprehend, perfectly, these two series of lines, if they are instructed to make a model with two circles of card paper, united by an axis, and with threads equally oblique, but running in opposite directions.

Sheers.—M. Ferry, formerly examiner at the *Ecole Polytechnique*, has made an ingenious pair of sheers with rectilineal blades; one, AB, fig. 4, being fixed, and the other *ab*, fig. 4, turning round an axis OO. The latter always remains in contact with the former as it moves, and divides the bodies which are placed between them.

Reels.—*Winders.*—Some of these are formed of rods, AB, *ab*, fig. 4, which revolve round the axis OO. The thread is wound on the hollow part of the curve, and cannot fall off. When a skain is wound on the reel, to remove it, all the rods are brought towards the axis, by means of a simple piece of mechanism.

Of the Sphere.—To produce this surface, we have only to cause the circle AM, BN, fig. 5, to revolve round one of its diameters, AB. All the points of the circumference of the meridian circle, being at the same distance from the centre O, will always remain at the same distance, when this circle is made to revolve round the axis AOB. All the points, therefore, in the surface of a sphere, are equally distant from a centre O, which is the centre of the sphere.

Every point in the plane of the meridian AMBN, but situated outside of or within it, is farther from or nearer to the centre O, than the points of the circumference AMBN. Every point of space, therefore, which is situated in the plane of any other meridian, will be further from the centre of the sphere than the meridian of the sphere, if it be situated outside of it, and nearer, if situated within.

Thus, not only all the points on the surface of a sphere are situated at the same distance from the centre, but no other point is at the same distance.

Every plane which passes through the centre of a sphere, divides it in the direction of a curve, all the points of which are at a distance from the centre equal to the radius of the sphere. This curve is, therefore, a circle. If we make these different circles revolve each on its diameter, we shall produce spheres, all having the same diameter and the same radius. We shall, therefore, always produce the same sphere.

Every cord, *mn*, of the circle AMBN, fig. 5, is smaller than the diameter MN, and it is smaller in proportion as it is removed from the centre of the sphere. But when the circle turns round an axis, AOB, perpendicular to the cord MN, the semi-cord *om*, forms a plane, and its extremity describes a circumference, which is wholly on the sphere. Consequently, 1st, every section of a sphere *mn*, made by a plane, is a circle. 2nd. These circles are all smaller than those, the centre of which is the centre of the sphere, and which, for this reason, are called the great circles of the sphere. 3rd. The small circles are less in proportion as their centre is further from the centre of the sphere.

Means of forming a Sphere.—Fix on the axis of a turning lathe, AB, fig. 6, the body which it is proposed to cut into the form of a sphere. At a certain distance from this axis, fix the half circle *aTb*, the diameter of which, $ab = AB$, and is parallel to it. The point M, of a cutting instrument, which projects from TM, = to the distance from *ab* to AB, and made to glide along *aTb*, parallel to it, will describe the meridian circle AMB. If the turning lathe, therefore, is made to revolve, the meridian will describe a sphere. The cutting instrument may be so placed, that its foot may glide along a circle *aTb*, having for its centre, the centre of the meridian circle, and always directed towards the centre O, of the two circles AMB, *aTb*. It is evident that TM, *tm*, representing the difference of the radii of the two circles, as T traverses the circle *aTb*, M must necessarily be always on the meridian circle. The cutting part of the instrument will, therefore, rest continually on the surface of the sphere.

In this manner, spherical mill-stones are made, as well as cannon balls, which are solid spheres. To make bomb and howitzer shells, which are hollow spheres, it is necessary to construct a mould, having the form of the parts distinguished by lines, fig. 8, pl. 11, and consisting of two spheres, one solid, A, and the other hollow, BBB. Between these two spheres, the shell is cast. In this case, the correctness of the operation depends on several circumstances: 1st. The two parts A and BBB, ought to be perfectly spherical; 2nd. Their centres ought to be in the same point. If these conditions are not exactly fulfilled, the motion of the shell when discharged will not be correct.

In the circle A*m*, B*m*, fig. 9, let us draw the cord *mm'*, and the radius O*o*A, perpendicular to the cord. By making the figure A*m*O, revolve round the axis AOB. 1st. The arc of the circle A*m*, will engender the spherical cap (*calotte*); 2nd. The segment of the circle, *m*A*m'*, will produce the spherical segment; 3rd. The sector of the circle, O*m*, A*m'*, will engender the spherical sector.

We must now resolve such problems as frequently occur in the arts.

What is the surface of the spherical cap *m*A*m'*, fig. 9, and of the complete sphere? What is the volume of a segment of a sphere, of a sector of a sphere, and of a complete sphere?

To ascertain the surface of the spherical cap, mAm', fig. 9, let us suppose that in place of the arc mAm', of the meridian circle of the sphere, we substitute a polygon with a very great number of sides, mn, np. Let us make this polygon revolve round the axis AOB, of the cap every part of the right line mn, np will form a truncated cone, of which AOB will be the axis. The whole surface of these truncated cones will differ less from the surface of the spherical cap mAm', in proportion as the sides of the polygon $mnpAp'n'm'$ are numerous. But the surface of a right-lined truncated cone mm' nn', equals the sum of the circumference of the two bases, multiplied by the half of the edge mn. Thus,

Surface of the truncated cone $mm'n'n$ = (circum. mm' + circum. nn') $\frac{1}{2}$ mn.

Surface of the truncated cone $nn'p'p$ = (circum. nn' + circum. pp') $\frac{1}{2}$ np.

and so on successively.

If we draw nh parallel to the axis, the rectangular triangle mnh, is similar to the rectangular triangle Oig, formed by Oi, perpendicular to the cord mn, by ig, perpendicular to the axis AO, also perpendicular, therefore, to nh, and by Og, perpendicular to mh.

The two triangles are therefore similar, and we shall have nh : nm : : ig : iO : : the circumference having ig for its radius, or ii' for its diameter, to the circumference having iO for its radius, or AB for its diameter, supposing that the number of sides of the polygon is so great, that there is no assignable difference between Oi and Om = OA, the radius of the sphere.

Whence, mn × circumference ii' = nh × circumference AB.

But $ii' = \frac{1}{2}$ (mm' + nn') wherefore

$mn \times \frac{1}{2}$ (circum. mm' + circum. nn') = nh × circum. AB.

The first term of this equation, is the surface of the truncated cone $mm'n'n$, the second term is the circumference of the meridian circle, multiplied by nh, the height of the truncated cone.

When the polygon mnp, therefore, is formed of a great number of extremely small sides, the surface which it engenders, equals the meridian circumference of the sphere, multiplied by the sum of the heights nh, ph'....of the truncated cones, produced by the rotation of the sides of the polygon.

I. The surface, therefore, of the spherical cap mAm', is equal to the circumference of a great circle of the sphere, multiplied by the sine Ao of the cap.

II. The surface of the sphere is equal to the circumference of a great circle multiplied by its diameter.

But the surface of a great circle $AmBm'$, equals the circumference, multiplied by the half of the radius, or the quarter of the diameter. The surface, therefore, of the sphere is equal to four times the surface of its great or meridian circle.

If we know that to cover a circle, AmBm'A, fig. 9, equally at every point it requires a certain weight, or quantity of sheet lead, or copper, or iron, or of paint, we may conclude that it will require four times this quantity or weight of metallic plates or paint to cover a whole sphere, having this circle for its meridian; and that with twice this weight or quantity, we can cover an arch or a dome in the form of a hemisphere, having the same circle for its base.

Measure of the volume of the sphere.—In considering the surface of the sphere, as composed of a great number of very small faces, we may regard each of them as the base of a pyramid, having its summit at the centre of the sphere. The volume of the whole of these pyramids will be the volume of the sphere. The volume of each pyramid is equal to the surface of its base, multiplied by the third of its height, which is, in this case, the third part of the radius. The whole volume of the sphere, therefore, will be equal to the sum of all the small faces, which have been substituted for its surface, multiplied by the third part of its radius. Thus, the volume of the sphere is measured by its surface, multiplied by the third part of its radius, or four times the surface of its great circle, multiplied by the third part of its radius.

We shall see, in like manner, that the volume of the sector of a sphere, OmAm'O, fig. 9, is equal to the surface of the cap *mAm'*, multiplied by the third part of the radius of the sphere.

If from this volume, we subtract the volume of the cone *mOm'*, we shall have the volume of the spherical segment, *mAm'* = $\frac{2}{3}$ circumference, A*m*B*m'* × A*o* × A*o* — ' circumference *mm'* × O*o* × *mo*.

The process employed to calculate the surface of a sphere, furnishes us with a method of constructing a sphere, which is often employed in the arts. If it be proposed to cover a spherical vault or arch with metallic plates, or any other substance, we divide the arch, by means of a succession of parallel planes, into zones or circular bands, $mm'n'n$, $nn'p'p$, &c. fig. 9, which are supposed to be conical, and, consequently, developable. The truncated cone $mm'n'n$, supposed to be developed, may be drawn in this manner.

Produce mn, $m'n'$, fig. 9, till they meet in the point s, the summit of the cone, of which the truncated cone $mm'n'n$, is a part. If we develope the cone, all the points of each base mm', nn', being equally distant from the summit s, fig. 9, will develope themselves in the form of the two arcs of the circle, MM′, NN′, fig. 9 *double*, both having the same centre, S.

We shall have, fig. 9, and fig. 9 *double*, the circle mm' = to the arc MKM′, and the circle nn' = arc NLN′. Let us now ascertain the value of the angle MSM′. The arc MKM′, equals the circumference, of which the radius is mo. But this circumference is to that of which SM is the radius : : mo : SM. The circle, therefore, having mo for its radius = MKM′ = the circle having SM for its radius $\times \frac{mo}{SM}$. Thus the arc MKM′, represents $\frac{mo}{SM} \times 360°$ of the circumference, having SM for radius.

It is only necessary to perform a multiplication and a division, to have the number of degrees of the angle MSM′, and, consequently, to know this angle itself. When this has been ascertained, with SM = sm, and SN = sn, as radii, draw the two arcs MKM′, and NLN′, fig. 9 *double*, we shall then have the zone MKM′N′LN, which being bent naturally, by joining the two ends MN, M′N′, will produce the truncated cone $mm'n'n$, fig. 9.

Tinmen and card or paper-box makers, by means of sheets of metal or paper cut into circular bands, and afterwards soldered or pasted, have very often occasion to construct surfaces which approach the form of a sphere, in proportion as the bands are narrow and numerous. The method just described, will be of use to them, and very often, also, to architects and carpenters.

Having explained the manner of forming a spherical surface with cones, we must now proceed to point out the means of forming it with cylinders.

Let us suppose that we make a great number of meridians, being planes, pass through the axis AOB, of a sphere, fig. 10, in such a manner as to divide the space around the axis into very small plane angles. Let us moreover suppose a succession of planes, perpendicular to the axis of the sphere, and, consequently, parallel to one another: 1st, they will divide the sphere into a succession of parallel circles; 2nd, they will divide the meridian circles into a succession of points, at equal distances from one another, on these circles. These points will be the summits of regular similar polygons, of which the corresponding sides will be parallels. All the sides parallel to any given direction, will form a cylinder, of which the edges will pass, at the same time, through two consecutive meridian circles. By this manner, we shall obtain a succession of cylindrical bands, similar in form to the sides of a melon; and the more these bands are multiplied, the more the surface thus produced will approach the form of a sphere.

Applications.—Oiled silk, skins, paper, gause, silk, &c, when employed to construct balloons, tennis balls, terrestrial and celestial globes for the study of geography or astronomy, umbrellas, parasols, hemispherical shades for lamps,* &c. &c. are united by cylindrical sides. In umbrellas, parasols, &c., the direction of the meridians is marked by pieces of whalebone, or of iron wire.

In order that the cylindrical sides shall, when united, form a surface, of which the joints or seams shall be meridians of the same sphere, they must be made of the following form:—

The breadths *mm'* = MM', *nn'* = NN', of one side are proportionals to the radii OM, ON, of the parallel circles, by reason of the similar triangles OMM', ONN'. OM, ON, being the radii of the parallel circles which correspond to *mm'* and *nn'*, we shall consequently have OM : ON : : MM' : NN' : : *mm'* : *nn'*. We may easily know, therefore, the breadths which correspond to the different points of each piece, and, consequently, its form.

* In France, it is a general practice to make the hemispherical shades put over lamps, which in this country are almost always made of glass, of fine cotton. The text alludes to these. Tr.

Application to Geography and Astronomy.—A very important use has been made of the properties of the sphere in both these sciences.

The form of the earth is evidently that of a surface of revolution, which differs but very little from the sphere. Centuries passed away before mankind could imagine and believe that the earth was spherical in every direction,—that it was what we call a globe, and had the form of a sphere. Before astronomers could ascertain that the earth is not precisely a sphere,—that it is flattened in one direction and swelled out in the perpendicular direction,—it was necessary that geometry and mechanical science should have made a simultaneous progress. When geographers had ascertained that the surface of the earth was spherical, they divided this surface in the following manner:—

The right line around which the earth appeared to them to make a complete revolution in twenty-four hours, they called the earth's axis. The points where this axis intersects the surface of the earth, they called the poles of the earth. They called all the planes which pass through the poles, meridians, and the lines which these planes form by intersecting the surface of the earth, meridian circles. All the circles drawn on the surface of the earth by planes parallel to one another and perpendicular to the axis, they called parallels.

Regarding the earth as a surface of revolution, two parallels are every where at the same distance from each other, and the meridians measure the distance which separates the parallels on the surface. The parallel which passes through the centre of the earth is the largest and is called the *equator*, because it divides the earth into two equal parts, called *hemi* or *half* spheres.

The northern hemisphere is that which contains the north pole, and, consequently, France, England, Europe, &c. are situated in the northern hemisphere. The other hemisphere contains the south pole, and is called the southern hemisphere.

If we conceive 360 plane meridians at an equal distance from one another, they will divide every parallel, including the equator, into 360 equal parts or degrees, which are degrees of longitude. If we divide the space between two of these 360 meridians, into 60 equal parts by new meridians, they will divide that space, or a degree of longitude, into 60 equal parts or minutes.

If the parallels are at equal distances from one another, and amount to 180°, they will divide the meridians into 360 equal parts, which are degrees of latitude. Intermediate parallels would subdivide these degrees into minutes, seconds, thirds, &c.

Division of the surface of the earth into spherical squares, in order to describe objects.—In the same manner as we divide the surface of a plane into squares, by lines parallel to one another, and at right angles to another series of lines, in order to mark out the position of figures traced on the plane, so the surface of the globe is divided into spherical squares, by circles parallel and at right angles to one another, in order to point out with precision, on this surface, the situation of all the remarkable lines and points; such as the site of cities, the course of rivers, the direction of chains of mountains, the form of the sea-coast, &c. The position of any point is fully indicated, when we know on what meridian and on what parallel, of either hemisphere, it is placed. The parallels are reckoned from the equator 0°, through the degrees of latitude 1°, 2°, 3°, to 90°, to the north pole on the one hand or to the south pole on the other. The meridians are reckoned in degrees of longitude 1°, 2°, 3°, 4°, to 180°, setting out from some assumed point. In France the meridian of the Observatory at Paris, in England the meridian of the Observatory at Greenwich, is this assumed point. Other nations use the meridian which passes through the Peak of the island of Teneriffe. The degrees are counted to the east and to the west; and at 180°, we are on the meridian circle of that place from which the degrees are reckoned.

When we know in which hemisphere any part of the globe is situated, if we know, 1st. the number of degrees which mark its latitude, and, 2nd. the number of degrees which mark its longitude, we know its precise position, which cannot be confounded with any other.

Geography, astronomy, and navigation, have all been much indebted to the various persons who have ascertained the exact number of degrees and minutes, both of latitude and longitude, which mark the position of the chief towns on the globe, and of its most conspicuous points. It is done, as has been stated, by expressing with two numbers, the position of a point on a sphere; a method perfectly analogous to that which we employ to indicate by two numbers, the position of any point on a plane.

This analogy has given occasion to represent the spherical surface of the earth, by a map, which is a plane divided into squares by right lines.

The parallel right lines, at equal distances, I.I, II.II, III.III,...fig. 2, pl. 5, represent the meridians laid down in their natural length. The parallel right lines 1.1, 2.2, 3.3,...represent parallels of latitude, not merely laid down, but elongated; for 1.1 = 2.2 = 3.3, &c.. though the parallels in fact diminish as they are removed from the equator.

Let us suppose for the present, that the divisions 1.2, 2.3, 3.4, &c. are enlarged in proportion to the corresponding parallels 1.1, 2.2, 3.3, &c. If we suppose the squares to be very small, we may regard each of those traced on the sphere as a plane. Its length and breadth will be proportional to the length and breadth of the square, enlarged proportionably in both directions on the plane chart.

Thus, in what are called reduced charts, every figure drawn on the sphere will be transferred to similar squares; each of the parts, consequently, of these figures, will, be similar:—their lines making the same angles with one another, as well as with the parallels and meridians. Sea charts, in particular, are all constructed on these principles.

Application to directing a ship's course.—If we suppose that a ship's course is always directed so as to form, at every point, the same angle with the meridian, her voyage will be represented by a straight line, drawn from the point of departure to the place of her arrival. The same line, therefore, will inform the navigator of the angle which the ship's course ought to make with the meridian, when he is navigating on a spherical ocean, or on a sea of which the surface is a surface of revolution.

In asserting that the figure of the earth is spherical, geographers only intended to say, that the globe, notwithstanding the inequalities of every species which its surface offers to our observation, deviates very little from the form of a sphere in proportion to its size. In fact, the height of the highest mountains, is not equal to the thousandth part of the diameter of a sphere bearing the nearest possible resemblance in form and size to the earth. The small protuberances on the rind of an orange, project more, in relation to its volume, than the highest mountains in relation to the volume of the earth.

To measure these inequalities with great precision, it is supposed, setting out from some point, such as the surface of the sea or a lake, that we draw the surface of a sphere having the same centre as the earth, and on which meridians and parallels are marked, corresponding to those of the earth.

To settle the position of any point whatever of the earth's surface, its altitude above this sphere is ascertained, as well as the degree of latitude and longitude which inform us of the parallel and the meridian which pass by a perpendicular, drawn from the point in question to the surface of the sphere.

When the equilibrium of fluids has been explained to the student, he will see in what manner, by means of the barometer, we can measure the height of the different points of the globe, and transfer them to the surface of a sphere, assumed as the term of comparison. Such measurements are not merely objects of curiosity; they are necessary to the engineer, that he may ascertain exactly the height of the ascents and descents to be passed, in carrying a road or canal from one point to another: they are of use also, for distinguishing the globe into regions, the heights of which influence, if they do not determine, the nature of every climate, as well as many other natural phenomena.

Independent of the numberless inequalities which form the undulations more or less extensive, and more or less marked, of the superficies of the earth, the globe also differs in its general outline from the form of a sphere. It is flattened towards the two poles, and, consequently, swelled out towards the equator. We are, therefore, nearer the centre of the earth when we are at either pole, than when we are in the temperate zones, and much nearer than when we are on the equator.

For many arts it is essential to know and calculate the quantity which the earth is flattened at the poles, for it makes the degrees of latitude comparatively long towards the pole and short towards the equator.* It influences the effects of gravity, to which all bodies are subject, making it greater at the poles than at the equator; so that a pendulum, as it is transported from the pole to the equator, vibrates slower and slower. Other things being equal, the column of air at the pole is heavier than the column of air at the equator; a circumstance which influences the motion of hydraulic machines, steam-engines, &c.

In the second and third volumes, in treating of machines and prime movers, we shall explain the law, according to which the gravity of bodies, the vibration of the pendulum, and the weight of the atmosphere, vary at different parts of the earth; and point out the consequences of this variation in different arts.

* The student may form an idea of this fact, by examining fig. 36, pl. 4. In the flattened curve, BDEFG, he may consider BG as the equator, and E the pole. The arc DF, having a radius DP, greater than BO and GQ, the degrees of the arc DEF, are larger than those of BD and FG.

Celestial sphere.—The sphere, divided into squares by parallels and meridians, is also made use of to distinguish in the heavens the position of the stars, as it is to distinguish on earth the situation of places. It is assumed, 1st that the heavens form a sphere, having the same centre and the same axis as the earth; and, 2nd. that all the stars are situated on the surface of this sphere.

A great number of the heavenly bodies, viz. all those called fixed stars, always remain at the same distance from one another on the celestial sphere, and their relative position does not alter.

If there were any one star placed on the celestial sphere, precisely in the direction of the axis, or on the pole, it would remain immovable, while all the others would revolve round it. The star, now called the pole star, is in fact, situated very near the celestial pole of our hemisphere, and, consequently, describes only a very small circle.

All the heavenly bodies vary their position in relation to our globe. Astronomers measure the number of degrees in latitude and longitude, which mark this position, at certain hours of the day; and on every day. When they have thus distinguished in the heavens a succession of isolated points, indicating sufficiently the path followed by the star, they draw through these points a continued curve, which thus marks out the path traversed by the star in its apparent motion on the surface of the celestial sphere.

From the study of the curves, traced on the celestial sphere by the motion of the heavenly bodies, astronomers have perceived that they are plane curves, and may be drawn on a right lined circular cone or conical surface of revolution, constituting that important study conic sections. The planets describe ellipses by their movements, comets describe parabolas, and the sun is situated in one of the focii of these curves. (See Lesson Thirteenth.)

The application of geometry to describe the course of the stars, is of so much importance, that but for it we should never have discovered the great law of attraction, which explains the different forces and motions of our whole planetary system, and gives to the modern science of astronomy, an immense superiority over the astronomy of the ancients. Thus, from the boy's top,—from the labours of the common tinman, who makes a tunnel in the form of a right lined circular cone, and who cuts it slanting, if he wishes to adapt it to a vessel in an oblique position, to the exalted studies of the astronomer who calculates the course of the celestial bodies, and ascertains the form of these visual cones, of which the bases are the curves, described by the centre of the stars, we find the same geometry, the same surfaces, the same sections, and the same curves. The principles of geometry, therefore, are equally useful in the most common arts of life, and in the most sublime sciences.

My principal object in showing the connection which exists between such opposite subjects, is to render those notions familiar, at the mention of which the people are apt to be terrified,—supposing them to be surrounded with insurmountable difficulties,—but which may be easily comprehended, when we perceive their analogy with conceptions, which appear to us some of the most vulgar, because they are connected with common labour, and are executed every day before our eyes or by our own hands. This, I will venture to assert, is the true philosophy of geometry, whether it be applied to the sciences or to the arts.

When we observe with attention the appearance of the heavens during a beautiful night, we perceive that the radiant bodies which glitter in the celestial canopy, do not remain fixed in relation to us. Like the sun, they rise in the east, star after star, and constellation after constellation; they mount towards the zenith, and again sink towards the west, disappearing till the ensuing night.

In this motion, each of the heavenly bodies describes a circle, and all these circles have the same axis, the axis of the earth. Thus the heavens appear to us as if they revolved round the axis of the earth. During a long succession of ages, the human race seem to have believed that all the heavenly bodies moved round our globe, which, in the vulgar creed, was supposed to remain immovable in the centre of the universe. Geometry has taught us the secret of this illusion. The mathematical knowledge gathered in several successive generations, seeming to prove that enlarged experience is more than equivalent to an additional sense, has unfolded to our wondering and revering minds, the one simple and grand cause of the eternal and harmonious (apparent) movement of the celestial bodies.

We are at such a vast distance from the stars, that visual rays projecting from the same star to parts of the globe most distant from each other, appear all parallel. The appearance of the heavens will, therefore, be the same, whether we are on the surface or at the centre of the earth. Let us suppose ourselves placed at the centre. If the heavens revolve completely round the earth in twenty-four hours, the earth does not revolve. If, on the contrary, the sun is immovable, it will be necessary that the earth should revolve round its own axis. In this motion, the only points in the heavens which would appear fixed would be the celestial poles, corresponding to the poles of our globe. The distances of the heavenly bodies from these poles being variable, every star, though appearing to rise and sink, in relation to the horizon at the different parts of the earth, would be always, on a visual ray, forming the same angle with that directed towards the pole, and which represents the axis of the world. Every star, therefore, would appear to us to move on the same cone of visual rays, and all the stars in advancing on their respective cones, would always appear to us to remain at their respective distances. The appearance of the heavens, therefore, would be precisely the same, supposing the earth to revolve on its own axis, as if we suppose the earth fixed, and the heavens to revolve round it.

It is thus by a property, a very simple one also, of the revolution of planes and points round a fixed axis, that we are enabled to ascertain the exact identity in the appearance of the heavens to us, whether the earth be fixed, and the celestial canopy be carried round its axis; or whether, on the contrary, the celestial vault remain immovable, and the earth revolve on itself. When the student is acquainted with the laws of circular motion, he will see the reasons why geometers have decided in favour of the latter system.

The sphere is not the only surface of revolution which can be generated by making a circle revolve round a right line. If we suppose that the axis of the surface does not pass through the centre of the circle, we shall form one of that species of surfaces which are called *annular;* because *rings*, such as are made use of in the arts, are a particular example of this species of surfaces. It is evident that all the meridians will divide the ring into equal circles, fig. 12, pl. 11, and also that all the parallels will intersect the surface in circles.

The rings worn on the finger are generally annular surfaces.

In the arts we find rings employed, under the name of ring-bolts, for the purpose of making ropes fast to them, hooking *tackles* on to them, and a variety of similar purposes. They are made of a ring, ABC, fig. 13, welded through the eye of a bolt, EDH, and fastened into the ground, into a wall, or into a ship's deck.

The figure of a ring, or a portion of it, is also employed in the decorations of architecture.

The *doucine*, AA, the *quarter of a circle*, QQ, fig. 14, in the capitals and bases of columns, are the fourth parts of an annular surface, formed by the revolution of a circle round the axis of the column; the *torus*, BB, is the half of an annular surface, formed by the revolution of a circle round the axis of a column or of an arch.

Architects also employ annular surfaces to construct arches. In that fine building, the *Halle au Bled* of

Paris, we see a large semicircular arch, ABC, fig. 15, around which is an annular surface, having for its meridian the two semicircles ADE, CFG.

Circular vases of an antique form, such as are represented in fig. 16, are composed of the cylindrical parts AB, CD, EF, GH, and of the annular parts *mn*, *pq*, *rs*, *tu*, *xy*.

When a joiner is making mouldings for a full curved doorway, the circular parts of his rebating plane describe annular surfaces.

A bell, ABCD, which in our houses is such a domestic convenience, which in our manufactories summons the workman to his labour, and in our churches calls men to worship their Creator with one common accord, is a surface of revolution, which is composed equally of conical and annular parts.

Sailors make use of a sort of half circular ring, called a *thimble*. Its exterior surface is made hollow, to receive a rope, which encircles the thimble, and is fastened round it, so that it cannot get lose; while another rope passed through the ring, moves backwards and forwards at pleasure.

For a long time astronomers were unable to explain the appearances of the planet Saturn, accompanied by its ring, which assumes at different times the different appearances represented by I, II, III, fig. 11. When they had become better geometricians, they easily ascertained that the *ring*, which varies in its aspects, I, II, III, appearing sometimes to surround and sometimes to cross the planet, is in fact, permanent and invariable in form and size. The plain and easy method of projection has been found sufficient to explain all the difficulty.

The pullies or sheeves which form the principal part of *blocks* are cylinders, very flat in proportion to their circumference. The edge is hollowed into an annular surface, having the arc of a circle for its generator.

A *wheel* is an annular surface, which is very much used in the arts.

The *fellies* of carriage-wheels, *m*, *m*, *m*, fig. 18, form also an annular surface of revolution. At the centre of the wheel is a solid piece, called the *nave*, ABCD. The spokes, or radii, at equal distances from one another, connect the nave, which is also a surface of revolution, with the ring formed by the fellies. The latter, consisting of portions equal to one another, are covered by the *tire*, a flat band or hoop of iron, which extends over the whole circumference of the wheel, and covering the joinings of the fellies, is nailed to them.

Some wheels have all the spokes in the same plane, *r*RR*r*, fig. 18, and on them the tire is at right angles to this plane, and forms a cylinder.

In other wheels the spokes, S*s*, S*s*, are placed in the direction of so many edges of a right lined circular cone. The tire being in them at right angles to the direction of the spokes, forms a conical surface. These are conical wheels.

When we come to examine the mechanical properties of wheels, we shall compare the advantages and disadvantages of these two species of surfaces of revolution for moving burdens from place to place.

Casks may be considered as surfaces of revolution remarkable for the simplicity of their structure. They are made of thin planks (staves), narrower towards the ends than at the middle, and joined at their sides, in such a manner that being forcibly bent, and kept bent by means of parallel hoops, AB, *ab*, *cd*, CD, fig. 19, a surface of revolution is produced, of which these circles are the parallels, and the joinings of the staves the meridians.

To close these surfaces of revolution, a circular plane is formed of other narrow and thin planks (the heading), the edge or circumference of which is fashioned into a conical shape, so as to fit into a circular notch or groove, made across the whole breadth of the inner face of each stave.

After the staves are reduced to a convenient thickness, the cooper makes their ends narrow, by pushing them over a large fixed plane, called a *jointer;* and completing this operation without any other guide than the eye, very often makes his casks offensively irregular.

To work staves according to a form perfectly regular, some persons have thought of employing geometrical means. Let us suppose that each stave is bent between three or any greater number of fixed points, ABC, fig. 20, pl. 11, and that *Oo*, representing the axis of the cask of which ABC is one of the staves, we place a jointer so that its iron shall be in a meridian plane, or, in other words, pass through the axis; and let us further suppose, that the iron could both turn round this axis, and move backwards and forwards in the plane of the meridian. The jointer being brought to a convenient distance from the stave, ABC, the upper part of the small face is first worked and afterwards turning the stave, the lower part is worked, according to the form which is proper for the meridian or profile of the cask. Staves thus fashioned would form a very exact surface of revolution.

On this principle a large manufactory of casks was established near Glasgow, but it no longer exists. In France there is one which promises to succeed very well.

When all the staves are brought together, their two ends are sawed in a direction perpendicular to the axis; the groove for the heading is made by an instrument called a *croze,* having one side flat, which rests on the circumference formed by the ends of the staves. A straight and projecting iron is fixed on a vertical stock, at a proper distance below the flat side, to cut out the groove. The two headings being placed together, are fashioned into a circle, the radius of which is equal to the radius of the groove. Temporary hoops are employed in the first instance till all the separate parts are made to fit; they are then removed, and the cask closed permanently at every part with iron or wooden hoops.

Of all the vessels constructed of pieces of wood for containing liquids, casks are the most advantageous; but the wood must be of a good kind, and the casks ought to be correctly made.

In the arrangement of the objects forming the cargo of ships, it is very often required to stow a great number of casks in tiers one above another, AB, CD, EF, fig. 21. The separate tiers, when not above three, are called the ground tier, middle tier, and upper tier; when above three they are numbered, the lower one, or No. 1, being, however, still called the ground tier. It is of great importance to know beforehand the height of each tier, in order to ascertain what portion of the hold of a ship will be occupied by the casks of wine, water, and spirits which she is to take on board, and, of course, what space will remain for the rest of the cargo.*

Though the combination, fig. 21, produces a gain of 27 hundredths of the radius of the casks, there is a considerable space lost, which is avoided by the modern method of keeping the water on board ships, in iron tanks of a cubical form, adapted to every part, even the most irregular, of a ship's hold. The metal, moreover, is better for preserving the water sweet than the wooden staves.

In arsenals, whether for the service of the army or navy, cannon balls, howitzer and bomb-shells, grenades

* It should be observed, that the three barrels *m*, *n*, *p*, supposed to be equal in size, all touch each other in the lines joining their centres; therefore, the centres are at a distance from each other equal to the greatest diameter of the barrel. Let the right line *nh*, in the triangle *mnp*, be drawn from the summit *n*, perpendicular to *mp*, making $mh = hp = 1$, we shall have $mn = 2$; and because the square of the hypothenuse is equal to the sum of the squares of the other sides of the triangle, we shall have $nh^2 = mn^2 - mh^2 = 4 - 1 = 3$.

From this we see that *nh* is nearly equal to 1,73. But the centres *m*, *p*, are at a distance from their support equal to the radius of the cask = 1, consequently, the height of the centre *n*, above the support, is 2,73.

If the cask *n*, was placed directly over the cask *p*, the height of the centre *n*, above the support, would be three times the radius; consequently, the sinking of each range of casks into the hollows between the casks of the range below, causes a gain of nearly twenty-seven hundredths of the radius.

of the same diameter or calibre, are piled up regularly by horizontal planes, fig. 22. In general, the base of each pile is a rectangle; and each forms a rectangular prism, with symmetrically truncated sides.

To determine the number of balls contained in a prismatic pile, truncated symmetrically, similar to that of fig. 22, count the number of balls in one of the triangular faces ABC, which, taking r as the number of rows of balls, will be,

$$1 + 2 + 3 \dots + r.$$

Multiply this total by the sum of the balls contained in the three extreme rows $Aa + Bb + Cc$, which represent the edges of the truncated symmetrical prism $ABCabc$.

Let n be the number of balls in Aa, each of the ranges Bc, Cc, will reckon $r - 1$ balls more than Aa. Thus $Aa + Bb + Cc = 3n + 2r - 2$.

Whence the total number of balls in the pile is $(1 + 2 + 3 \dots + r) \times (3n + 2r - 2)$, a product easy to calculate.

When there is only one ball in the row Aa, the prism becomes a quadrangular pyramid, of which the number of balls is,

$$(1 + 2 + 3 + \dots + r) \times \tfrac{1}{3}(3 + 2r - 2)$$

Or, $$(1 + 2 + 3 \dots + r) \times \tfrac{1}{3}(2r + 1).$$

When the pile is triangular, $Aa = 1$, $Bb = 1$, $Cc = r$, whence $Aa + Bb + Cc = r + 2$.

For a triangular pile, therefore, r being the number of rows, we shall have,

$$(1 + 2 + 3 \dots + r) + \tfrac{1}{3}(r + 2.)$$

TWELFTH LESSON.

Spiral Surfaces.

Before explaining the properties of spiral surfaces, and the applications of them in the arts, it is necessary to examine the curves by which such surfaces are produced.

Having drawn the rectangle OH*ka*, fig. 1, pl. 12, let us divide it into bands of an equal breadth, by means of the parallel right lines A*b*, B*c*, C*d*, &c. Draw the oblique lines A*a*, B*b*, C*c*, D*d*, which will be parallel to each other, for they intercept equal portions of the other parallels, AB = *ab*, BC = *bc*, CD = *cd*, &c.

Let us suppose that the rectangle is bent into any cylindrical form, having OH for one of its edges. Shut up the cylinder, so that *ak* will coincide exactly with OH. The point *a* will fall on the point O, *b* will fall on A, *d* on C, &c. The edges being all parallel to OA and *ak*, will be represented in the rectangle OH*ka*, by the right lines PQ, RS, TU, &c., parallels to the sides OH*ak*. But in the rectangle, all these parallel right lines intersect the obliques A*a*, B*b*, C*c*, under the same angle, for all these obliques are parallels. If we bend the rectangle on the cylinder, fig. 3, none of the angles formed by the obliques

Aa, Bb, Cc, fig. 1, and by the edges PQ, RS, TU, will be changed. It will, therefore, be a property of the obliques Aa, Bb, Cc, that in rejoining each other at the points A and *b*, B and *c*, C and *d*, &c., of fig. 1, they will form a curve, having the same angle at every point with the edges of the cylinder. This curve is called a *helix* or cylindrical spiral.

If we fold up the rectangle, so as to form a cylinder the base of which is a circle, we obtain the *helix* most frequently employed in the arts.

Let us now suppose that two points, setting out from H, advance at the same time; one following the direction of the side H*k* of the rectangle, fig. 1, the other following the oblique line H*h*. Let us suppose that the two points pass in the same time, 1st, to PQ; 2nd, to RS; 3rd, to TU....According to the properties of proportional lines, we shall have,

HQ : Q*q* : : HS : S*s* : : HU : U*u*, &c.

The point, therefore, which moves in the oblique direction H*h*, separates from the base H*k*, by the quantities Q*q*, S*s*, U*u*.... proportionals to the distance of the edge OH, from the edges PQ, RS, TU...

If we make, therefore, one of the edges, HO, of the cylinder turn round it, while a point is advancing along this edge, in such a manner that the spaces passed through by the point and the edge are proportionals, the point will describe a *helix* or spiral, as represented, fig. 3.

The spiral, therefore, is produced by the revolution of a point round an axis, it being at the same time moved forward in a direction parallel to the axis, in proportion to the quantity it revolves round the same axis.

A turner, consequently, can describe a spiral on a cylinder, with a cutting instrument which advances parallel to the axis, and proportionally to the quantity which the cylinder revolves round the axis. At each turn of the cylinder, also, the turner's instrument ought to advance an equal length, which length being every where the same, is the *pace* of the spiral or helix. The distance of the different revolutions of the helix or spiral, therefore, measured on each edge, is always the same.

Let us suppose that we take off an impression, fig. 2, by printing or any other means, of fig. 1; that is to say, that we form a second figure symmetrical with the first, and that it is bent on a cylinder, fig. 4, equal to the cylinder fig. 3, we shall then form a spiral turning in a contrary direction to that of fig. 3.

The spiral, fig. 3, is said to be turned to the right; that of fig. 4, to be turned to the left. When the cylinders, figs. 3 and 4, are equal, and the motion of revolution is equal, the spiral turned to the right is symmetrical to that turned to the left.

Spiral figure of screws.—In place of making a single point revolve round an axis, we may substitute any plane figure whatever, such as a triangle, fig. 5, or a square, fig 6, &c. We shall then generate surfaces which may either project from or be sunk into cylinders, which, in like manner, may be either solid or hollow. The spiral parts, whether raised or sunk, are called threads. They are formed round the cylinder by a triangle or square, either raised or sunk, advancing along the helix, always preserving the generating figure in the same position, in relation to the outline of the helix and to the direction of the axis of the cylinder.

The cylinder ABCD, fig. 5 and 6, which has the thread on its convex surface, is called the male screw; the hollow cylinder having a spiral thread formed in its concave surface, is called the female screw.

Let us suppose that we have two cylinders of the same diameter, on the surface of which we have formed the same spiral, and by it we cut two threads, one projecting, the other sunk; they will form a male and female screw, of the same thread and the same pace: the male screw may be introduced into the female screw, by making it advance, and turning it as it advances, without leaving any vacuum between them, and without the male screw being diminished in size at any point.

Let us insert the end of the projecting thread on the male screw, into the commencement of the sunk thread on the female screw, the arbors of both screws being so adjusted that their axes may be in a right line. This being done, one of the cylinders remaining fixed, let us turn round the other, so that each point of its thread may advance parallel to the axis, and proportional to the quantity which it turns, according to the relation indicated by the curvature of the helix by which the threads of the screw have been formed,—the profile of the surface of the projecting screw will describe the surface itself of the sunk screw, and the projecting thread will thus adapt itself at every point without compression, and leaving no vacuum to the sunk thread. This is the movement of a screw in a nut. Figures of both triangular and flat screws have been constructed geometrically, and with the greatest care, figs. 5 and 6, in order that the student may imitate the projection of screws on a large scale They will find it one of the best geometrical exercises which can be suggested to them.

As there are two species of spirals, one turned to the right the other to the left, so there are two species of screws and nuts, one turned to the right and the other to the left. It is plain that a screw turned to the right, will not enter a nut the thread of which is turned to the left, and *vice versa*.

Screws are continually used in the arts. They are sometimes employed to change a circular movement into a movement in a right line; and sometimes to produce a change in the contrary direction. This subject, however, will be treated of in the second volume, on machines.

We must remark fig. 1, that the pace OA = AB,.... of a screw, may be very small in relation to the length, H*k*, of the cylinder; moreover, the triangle H*kh*, is a scale composed of parts, Q*q*, S*s*, U*u*,...which are to one another : : 1 : 2 : 3 : ... This scale is similar to that formed, fig. 5, Lesson 5. If the base is equally divided into the parts HQ, QS, SU, a very sensible error in these lengths will be much less considerable in the heights Q*q*, S*s*, U*u*....

In the arts this geometrical property is already employed, to divide right lines very exactly into equal parts, by means of a screw.

Let it be proposed to divide the ruler AB, fig. 7, into equal parts. Suppose that the pace of the screw MN, the axis of which is parallel to AB, is the 10th part of the circumference of the cylinder on which the screw is cut; and that this base has for its radius, the tenth part of the radius of a circular plate, PQ, of which the circumference is divided into equal parts. Suppose further, that the errors of the divisions of PQ, may amount to the 1000th part, which would be inexcusable in any instrument intended for precise and correct operations, the circumference of PQ, is one hundred times greater than the pace of the screw, and each turn of PQ, only causes the index YX, which is moved by the screw, to advance or recede one pace. The error in the space traversed by the index, cannot, therefore, be more than one hundredth part of the errors in the divisions of the circle PQ. When the errors of PQ do not exceed one thousandth part, they cannot exceed one hundred thousandth part in AB; that is to say, a distance much less than can be appreciated by the sharpest and most attentive sight.

Let us turn round the circle PQ, so that a fixed index, Z, corresponds successively with the somewhat close divisions 1, 2, 3,...of the circle; we shall divide the right line AB, into very minute parts, the inequalities of which cannot be recognised by our senses.

The machines for cutting screws are made in proportion to the relations which must be established between the longitudinal divisions A, B, and the divisions of a circle PQ. It will be necessary, perhaps, to explain these machines to the students by exhibiting them.

Screws differ very much, according to the form of the thread, the section of which, perpendicular to the directing spiral, is sometimes an equilateral triangle, and sometimes a square; the former produces a screw with a triangular thread, fig. 5, the latter a screw with a flat thread, fig. 6.

Screws are employed to move rulers or parallel cylinders, nearer to or further from one another, without changing their parallelism. Let us conceive, for example, two equal screws placed each at one end of a pair of cylinders, in such manner, that in turning the screws, they will force the axes of the cylinders to approach to or recede from one another. When the two screws are moved an equal quantity, we shall cause the cylinders to approach or recede equally; but we may have an index fixed at each screw, so that the space traversed by it shall be one, two, or three hundred times greater than the distance longitudinally between each turn of the thread; and, therefore, a movement of the screws, which makes the index pass through one division, will not affect the cylinders more than the one, two, or three hundredth part of the motion of the index. By this means we can regulate their movements and their distance from one another with extreme precision, which in many operations is of very great importance.

Screws may also be applied in many other similar modes; such as that of measuring or moving minute bodies through minute points of space with a precision, far beyond what we could obtain by our senses alone; of which we have almost numberless examples in the employment of adjusting screws, in optical and astronomical instruments.

When it is required to fix an instrument, which stands on three or four legs, at a very correct level, to each of the legs an adjusting screw is adapted, which is turned a little round, as either of the legs of the instrument must be raised or lowered. In this manner the instrument is brought to the true position by almost insensible degrees, and the motion can be stopped at the precise point. Adjusting screws are used in reflecting telescopes, for the purpose of placing the mirrors in a proper position; and in other instruments to separate some of their parts, or bring them closer together.

Nature has set before man in her works, particularly in the vegetable world, numerous examples of spirals. Creeping plants wind and mount upwards round a vertical cylinder, such as the trunk of a tree, or bush, or pole, describing a spiral. In some instances, the plant sends forth long shoots, which it suspends to some other object by filaments that wind round it in a spiral form. Several of the fibres or vessels in the interior of plants, are coiled up like spirals. There are many vegetables also, of which the separate leaves, branches, or fruit, form, as they grow out of the tree, a spiral round the cylindrical trunk which bears them.

Man has imitated the spiral forms of the vegetable world, either to unite bodies together, or to penetrate into them.

When surgeons bind up a limb, having the form of a cylinder, such as a finger, a leg, or an arm, for example, they roll their bandages round it in a spiral direction, in order to cover a space, gradually and with an equal pressure throughout, which is much larger than the width of the bandage; which may thus be easily fastened at the end.

In explaining the properties of the wedge and of the screw, in the second volume, we shall describe, more in detail, gimblets, augers, corkscrews, gunworms, &c. &c.

Twisted columns.—We sometimes see trunks of trees around which a branch of ivy has grown in a spiral direction, and has so compressed the trunk, that it could only grow between the turns of the ivy; and it thus assumes the form of a screw with a circular thread. A tree of this description was probably the original model of twisted columns, fig. 8, which having neither the simplicity nor the strength of straight columns, can only please perverted imaginations.

A more graceful ornament, and one more worthy of the fine arts, is that of garlands of flowers, twisted spirally round regular columns; or, what is perhaps still more graceful, round the light dresses of young maidens, when they are adorned to share in the dance, threading, perhaps, a twisted maze, which almost approaches a spiral; or to mingle with happy companions in those sportful games, in which they pursue each other round a circle of these same companions, presenting the form of a spiral wound round a ring. But let us return to more useful applications.

The worm of a still, fig. 9, resembles a corkscrew in its form, but it is hollow. It is generated by the motion of a circle, the centre of which moves along a helix, to which the plane of the circle is always perpendicular. When a fluid, converted into the state of vapour by heat, passes into the worm, which is placed within a cask filled with cold water, the vapour becomes condensed, and reaches the bottom of the worm in the state of a fluid. In spiral tubes, therefore, brandy and other spirits, the produce of distillation, are condensed into fluids.

The straw-hat maker forms cylinders, fig. 10, pl. 12, out of narrow and level straw-plait, which being every where of the same thickness, represents the bands $AabB$, $BbcC$, &c. fig. 1. Bent into the form of a cylinder, and sewed together at the edges, they form a cylindrical surface. By a similar method, taking care to extend or contract one of the edges of the plait, we can construct a plane, a cone, or a sphere.

In proportion as the straw-plait is narrow, and as there is little necessity to expand or contract one of its edges, so the workman is enabled to make his commodity approach that precise geometrical form it is intended to possess. The beauty of Leghorn straw-hats, is derived from the perfect regularity and equality of the plait, it being very narrow, and the straw very fine and uniform in its colour and appearance.

Spiral springs, such as are used for coaches, are frequently employed in the arts, but we must postpone the explanation of their properties till we treat of elasticity.

The hair of some persons curls naturally in a spiral direction, and others curl their hair in this manner, by twisting it round a hot cylinder of small diameter, or simply by rolling it up, in a spiral form, round a paper, called a *papillotte*; sometimes pressing it with hot iron pincers, and sometimes doing nothing more to it, but keeping it thus rolled up for several hours. The heat of the pincers dissipates the moisture with which the hair is impregnated, and which has a tendency to relax it, and make it fall down straight from the head; the compression gives it a spiral curve, which it preserves for a longer or shorter time, according to the nature of the hair and the state of the atmosphere.

The art of the hair-dresser, and of the painter who wishes to represent a fine head of hair, consists in grouping the spiral curls, thus formed in masses, so as to harmonise with the general dress and physiognomy of the person represented by one, and adorned by the other. In this, perhaps trifling art, the Greeks and Romans have supplied us, as in more important arts, with some elegant models, which may be referred to as examples, in which the combinations of these spiral forms are managed in the most happy manner.

We must now, however, refer to a species of spirals much more important than most of those hitherto alluded to, viz. thread, rope, and every species of twisted cordage.

To make cloth and rope, yarn more or less fine, is first spun of hemp. of flax, of the *phormium tenax,* of the bark of some trees, &c.; cotton, wool, hair, and hides, cut into strips, are also used for the same purpose. Before the yarn is spun, the fibres of the raw material are drawn out parallel, either by carding or hackling, and are divided into very fine filaments, as equal as possible in length and thickness.

Spinning hemp and flax.—In early times, these materials were spun by the spindle, or bobbin, about which it was wound as fast as it was twisted. In this operation, the spinner, by means of the finger and thumb of her right hand, turns the point of the spindle as fast round as she can, which gives a sufficient twist to the thread not rolled on the spindle. At the same time she gradually draws, with her left hand, the material off the distaff, as the spindle revolves and descends towards the earth. When she has thus spun a length, she winds it on the spindle, and fastens it to the end by a hitch or knot, that can be easily made. The same process is then repeated, and the filaments, which were loosely thrown in a parallel direction round the distaff, assume a spiral form, and constitute a thread.*

This mode of spinning is the slowest and the most ancient. It was superseded in the first instance by a very simple wheel, fig. 11, which is put in motion by the hand or the foot of the spinner. As the thread is twisted, it is gradually rolled on the bobbin, which is only a mechanical spindle. The wheel twists the filaments into a thread. The spinner, therefore, has only to give motion to the wheel, and draw the flax from the distaff, forming it between her fingers, as the wheel turns, into a thread which is every where of equal thickness.

The thread is wound on the bobbin of the wheel by means of wings, fig. 12, provided with hooks. The wings are fixed on the axle *mn*, which passes through the bobbin, or wooden cylinder *rs*, on which the thread is to be rolled. The motion given to the cylinder is more rapid than that given to the wings, so that the bobbin pulls the thread, which is thus gradually rolled around it.

Let us suppose, to fix our ideas on this part of the process, that the cylinder makes five complete revolutions, while the wings make only four: the thread must be consequently wound once round the cylinder when it has made five revolutions, and the wings only four. These different rotatory movements are given by means of the large wheel, OAB, fig. 11, *mn*, *pq*, being two other small wheels, of which the respective diameters are to one another : : 4 : 5. It is evident that the cords A*mn*B, A*pq*B, extended over the grooves of the small and the large wheel, pass over the same space of the large wheel,

* This ancient method of spinning is still practised in several parts of France, particularly by women, who have at the same time some other occupation; in country places, both of Italy and Spain, and in some parts of the Highlands of Scotland and the Shetland Islands. Tr.

AB, when it is turned, and, consequently, that the pinion *mn* must make five complete revolutions, while *pq* only makes four. This is the relation between the wheels which it is required to establish.

The advantages of the wheel, compared to the spindle, are very great; ages were required before it was invented, and much, very much, has it been surpassed by modern inventions.

In spinning cotton or wool, the material is first formed, by the process of carding, into large flakes, being every where of the same breadth and the same thickness; and it is afterwards drawn out, so as to form narrow ribands. Being slightly twisted, these are converted into slivers or rovings, which are again twisted, and gradually drawn out one by the side of another, either by the hand or by a machine. As the process continues, they are turned round themselves that they may be equally twisted; that is to say, that both the quantity of materials may be equal at every point, and also that this quantity may every where be equally twisted, so that the thread may be throughout of the same size. In this process, each filament forms a spiral, having for its axis the axis of the fine cylinder represented by the thread.

The machine for spinning cotton, consists of a large wheel, OAB, fig. 13, of a spindle provided with a small wheel CD, and of an endless cord ABCD. The spindle receives the thread in the same manner as the common spindle already described, —the thread extending itself into a *sliver* or *roving* in the part not twisted. The workman takes hold of this with one hand, at a convenient distance from the spindle, drawing it out as the spindle revolves, and with the other hand turns the large wheel; as it revolves, the spindle revolving more rapidly, communicates its revolutions to the cotton and forms it into a thread, the elements of which are curved spirally. The twisting of these spirals depends, 1st, on the rapidity with which AOB revolves, and 2nd, on the rate at which the cotton is drawn out from the spindle. When a portion of the material is sufficiently twisted, so as to form a thread of the required thickness, the spinner turns back the wheel that she may undo the spiral which the thread forms at the point of the spindle, and then holding it perpendicularly to the axis of the spindle, gives the wheel a turn in the contrary direction, when the thread, in place of twisting, is wound on the spindle, forming there also a succession of spirals. By this machine, therefore, the same operations are performed as by the original spindle and distaff, when the thread is twisted by the finger and thumb.

One of the most original and most remarkable parts of the modern spinning machines, is that which is substituted for the fingers of the spinner. The light flakes of cotton, as they come from the carding machine, are made to pass between three pairs of cylinders or flatteners, so disposed, that the *first* pair turns round less quickly than the *second*, and the second less than the *third;* by the revolutions of which the cotton is stretched out and narrowed. It afterwards passes through a second series of three pairs of cylinders, in which it is twisted a little, and is then wound on bobbins. A certain number of these are then placed on vertical axes, and ranged in order, on a machine which performs all the operations formerly executed by the spinner; it draws out the cotton, twists it, and winds it when twisted, on other bobbins. The thread is in this case also drawn out by means of three pairs of cylinders, each pair revolving with different degrees of velocity.* The thread is then wound on a bobbin,

* Although, as stated by the author, three pairs of rollers are used in spinning cotton, yet two only are efficient in forming the thread, the middle pair having the velocity of the first in the series, and, therefore, serving merely to support and conduct the skein. The first pair of rollers, or those which first receive the skein, are called the holding rollers, and the third pair, or those placed at the front of the machine, moving with a greater velocity than the first, are called the drawing rollers The thickness of the skein or roving being given, the size of the thread depends upon the relative velocity of the first and third pair of rollers. Where the fibre or staple of the

provided with wings, like those of the common spinning-wheel. This mode is called the continuous method of spinning, because the operation is carried on without suspending or altering the different species of motion.

In the machines called mules, mentioned in the Second Lesson, the cotton is drawn out, not merely by the different degrees of velocity in the different pairs of revolving cylinders, but also by making all the bobbins on which the cotton is wound, alternately approach to and recede from the cylinders. When the bobbins recede, the thread is drawn out; when they advance, the thread is wound on them. The thread is twisted when the bobbins have reached the end of their course.

A frame for spinning coarse cotton carries 108 bobbins, one for spinning fine cotton carries 216; it is managed by one spinner, and two *knotters*.* On an average, it spins in eleven hours about 90lbs. of the cotton thread. No. 30. Thus, three persons can manage a number of threads, which would require 216 spinners with the wheel or spindle, and each thread is spun much quicker than by the fingers.

Thus, the means supplied by geometry of forming vegetable filaments, twisted spirally, into cylindrical threads having always the same diameter, has multiplied the resources of man, in this single instance, at least seventy fold. The student will be more instructed by

material to be spun is tortuous and tenacious, as is the case with the wool of sheep, another intermediate pair of rollers has occasionally been used, to keep the skein flat, or to preserve the fibres from starting; and in spinning flax by machinery, on account of the length of its fibres, and consequent necessary distance of the holding and drawing rollers, several pairs have been introduced, as conductors or supporters of the skein In the adjustment of the rollers, in regard to distance and relative velocity, consists the great practical difficulty of the art of forming threads by machinery.

* In the manufacturing parts of England, they are called *piecers*, as the art consists simply in applying the broken end of the thread to the skein, when it leaves the front rollers; and the revolution of the spindle, by twisting the fibres spirally, repairs it completely.

these remarks, if he can, at the same time, have an opportunity of inspecting models of wheels and spinning frames.

Silk, as it is produced by the silk-worm, is rolled up in the form of a spiral, on a surface of revolution called a *cocoon.* The first operation consists in developing the thread of the cocoon and winding it on a bobbin; it is then slightly twisted by being wound on a second bobbin. The threads thus prepared, are so twisted in the first instance, that all the points which were before in a right line on their cylindrical surface, form a spiral. These threads are united two, three, and even four together, by twisting them in a contrary or reverse direction. Of course, this second operation untwists each thread a little, but it unites all the threads spirally together in their whole lengths. In this state, the silk thread is called *organzine.*

This operation is similar to that of making rope with hemp. By the effect of both the twistings, the parts of each thread have a tendency to return to their straight position, by untwisting in one direction; while the united threads have a tendency to untwist in a different direction. The equilibrium established by the two twistings, prevents the threads from untwisting, which they would do, were they not constrained by some force extrinsic to themselves. On this subject, however, which belongs to the doctrine of *forces*, and extends beyond the bounds of geometry, I cannot now enter into more ample details.

Hemp is at first twisted into single threads called rope-yarns, the twisting being all in one direction; several of these yarns are then twisted together in an opposite direction, to form what are called *strands;* three or four of which, as the rope is to be three or four *stranded*, are twisted together in an opposite direction from the strand, and in the same direction as the yarn to form a common rope. To form a *cable-laid* rope, three or four lengths of this common rope, called in this case strands, are twisted up in a contrary direction, that is to say, as the original strands are twisted, and these may be again twisted together to form large cables.

The cables, shrouds, and standing rigging, of large ships, are generally *laid up*, as it is technically called, or twisted in this latter mode; their common ropes, braces, haulyards, &c. &c. are twisted in the former mode.

In England, ropes are now made by means of some very ingenious machinery, the regular geometrical movements of which have been attended with the best effects. With one third less material, or even a greater diminution, according to the size of the rope, we now obtain, in consequence of this improved method of manufacturing, equal strength; and this may be quoted as one of the best examples known of substituting scientific combinations of mechanic powers, for operations that almost may be called merely manual. Further details will be given of these machines in the Second Volume.

We have yet to speak of a species of non-developable or indefinite surfaces, frequently employed both in civil and naval architecture, as well as in constructing machines; viz. spiral surfaces, generated by the movement of a right line, or of the arc of a circle.

Spiral surfaces of stairs.—Among the various non-developable surfaces examined in the Tenth Lesson, those formed by winding staircases are spiral surfaces.

The spiral surface of the winding staircase is formed by the motion of a horizontal right line, one end of which is applied to the axis of the tower in which the staircase is built, the other being applied to a spiral traced in the direction of the interior circumference of the tower.

If we give the same height to each of the steps, they will all be of the same breadth, at equal distances from the centre. Consequently, if ABC, fig. 14, is the circle which represents the base of the cylinder forming the tower, in which the staircase is to be built, every other circle drawn from the same centre as the first will be divided into equal parts by the horizontal projection of the steps.

SCREW OF ARCHIMEDES.—The spiral surface of a staircase in a round tower, is a perfect representation of the screw of Archimedes, so called from this great geometrician, who invented it. The application of this screw to raise water, will be described, when we speak of hydraulic machines, in the third volume.

Having had occasion to construct the screw of Archimedes in wood, I proceeded in the following manner:

I began by dividing the circumference ABCD, fig. 19, pl. 12, into as many equal parts as I wished to employ pieces of wood to form a complete revolution of the spiral. I had the prisms squared, of which the base ODC was the sector, representing one of the equal divisions thus formed on the cylindrical face, having DC for the horizontal projection; and I drew a right line, inclined according to the direction of the helix, which the spiral surface traces on the cylinder ABCD.

I divided the radii OD, OC, into equal parts, D*d*, *dd'*,... C*c*, *cc'*,... then I caused the pieces of wood, already squared, to be sawed, with a saw always kept at an equal distance from the two points D, C, so that, *first*, on the upper base of the piece of wood the course of the saw always terminated at D, while on the lower base it always terminated at C; secondly, on the upper base, a cut ended in *d* and *d'*, when, on the lower base, the same cut ended in *c* and *c'*. Each of these cuts of the saw is the side of a polygon, representing the outline of a spiral curve, placed on the spiral surface to be produced.

With a very small plane, the iron of which was circular, always held in a horizontal position, and only stopping at the saw cut in CD, and at the vertical in O, the superfluous wood was gradually cut away to obtain the upper spiral surface of the screw of Archimedes.

When this was done, by means of a square, the faces of the joint in OD and OC, were made to square with the upper face. Finally, drawing on the faces of the joint, and on the surface CD, right lines equal and similar to those which limit the upper face of the screw, we were able to work the lower face by the means just described for working the upper face.

Let us here remark, that a ruler bent without any effort on the cylindrical outline ABCD, so as to pass by the two points C, D, indicates by its outline a perfect arc of the spiral or helix, which may produce great exactness, by means of the approximation just mentioned, giving many of the horizontal divisions by the saw, which stop at the axis O, on one part, and on the other at the helix, drawn by means of the bending rule.

It is not uninteresting to remark, that the joints which are squared with the spiral surface, are themselves the elements of a spiral surface of the same description. The latter surfaces tracing, on cylinders with circular bases, helixes, which even intersect, at right angles, the helixes drawn on the former surfaces.

If it be required that the upper part of the pieces which compose the spiral balustrade shall have the form of the staircase, the upper face OCD, must be allowed to retain its plane horizontal form, and the right exterior face OD its plane vertical form, working the surfaces of the joint and the lower surface of the staircase, by the means pointed out in the Tenth Lesson.

Often, in place of constructing a circular staircase, the steps of which extend entirely to the solid pillar O, fig. 14, pl. 12, the steps reach no farther than the circle $a'b'c'$, fig. 15, which represents horizontally a ledge in stone or wood, projecting a little above and below each step. Such staircases are called open winding stairs; several of which may be seen in the most elegant coffee-houses of Paris, admirably constructed, and which, apparently without support, surprise the beholder by their lightness and boldness.

There are open staircases, fig. 16, the outer part of which is not circular.

Whatever may be the base, $ABCD^h$, fig. 16, of the cylinder which represents the wall, inclosing the staircase, we always trace on it a spiral or helix, which advances in the direction of the outline $ABCD^h$, proportionably to the quantity it is vertically elevated. From each point of this curve, we then draw a horizontal line *Aa*, *Bb*, *Cc*,...at right angles to the cylinder, having $ABCD^h$, for its base. *Aa* is made equal to *Bb*, to *Cc*, &c.; and *abcd*, which is also a spiral, is to be drawn; this is the interior outline of the open spiral which forms the staircase. The working of the different parts of the spiral surface or of the staircase, is not more difficult than the working of that of figs. 14 and 15.

When it is required to make a staircase very strong, in place of generating the lower surface, by means of a horizontal right line, applied at the same time to the axis of the tower, and to a spiral traced on its inner surface, the lower surface is often terminated by the arc of a circle, fig. 17, having this horizontal line for its diameter, and situated in the vertical plane. In this manner a sort of spiral surface is formed, presenting everywhere the same section.

It is necessary in some arts to make spiral surfaces in steps upon a cone. Thus, watchmakers, with the cylinder or box which contains the main spring, combine a cone having on its surface a spiral path, fig. 18. A very fine chain, made in a masterly manner, winds, from one end, round the cylinder, in the form of a helix, and from the other end on the conical spiral. The variable relation of the diameter of the cone at different heights, to the diameter of the cylinder, compensates exactly for the diminution of the power of the spring, as it expands; and it consequently acts with an invariable force. This, also, is one of the numerous subjects with which the student will be made better acquainted, when the principles of machines are explained in the second volume.

THIRTEENTH LESSON.

Intersection of Surfaces.

WHEN two surfaces cut each other, the succession of points, common to them both, is called their intersection; it is a right line or curve, according to the form and position of the two surfaces.

Bodies terminated by portions of surfaces, distinct in their form and direction, offer at the boundaries of such surfaces projecting or retreating lines, which are the intersections of these surfaces. The rectilinear edges, for example, which separate the different faces of the prism and pyramid, are the intersections of the surfaces represented by these faces.

When one body crosses another, or is implanted in it, the surface of the former is partly hidden by the latter; the hidden part is separated from the visible part, by a line which is the intersection of the surface of the former body with the surface of the second.

Thus, in fig. 1, pl. 13, the line of intersection for the two prisms ABCD*abcd*, MNPQ*m'n'p'q'*, the first of which penetrates the second, is the periphery *mnpq* which separates the hidden from the visible part of the latter. To determine the horizontal and vertical projection of the intersection of surfaces, descriptive geometry supplies us with an easy method, which it will be very useful for every student to practise at length, by drawing the intersections of a great number of surfaces. On this point, we must content ourselves with pointing out the general course of proceeding, and we shall begin with the intersection of planes.

To represent the intersection of two planes of projection, the one vertical, the other horizontal, the paper is divided into two parts by a horizontal line, AB, fig. 2. The part of the paper above the line represents the vertical plane of projection; the part below the line represents the horizontal plane of projection. The latter is in general the ground plan; and then the intersection, AB, of the two planes, is commonly called the ground line.

In order that the representation may be perfect, we must double the paper up square; AB marking the direction of the doubling, the lower part of the paper remaining horizontal, and the upper part becoming vertical. At least, we must do this in thought; and it is done naturally in imagination, when on two planes we represent objects, the position of which is known to us. Thus we see, below the ground line, the plan of an edifice, and above it, the elevation of the edifice, with doors, windows, &c. &c. Even when the paper on which the plan and elevation are drawn is placed on a horizontal table, we restore, in our thoughts, the elevation of the edifice, and imagine it vertical: on the contrary, if the paper were placed vertically, as against the wall, the plan would, nevertheless, appear horizontal, if it represented such objects as a garden, a landscape, &c. It is necessary, therefore, for the student, to imagine the horizontal or vertical projection of volumes, of surfaces, or merely of lines, in its true position, as represented above or below the ground line.

To indicate the position of a point situated out of the two planes of projection, two right lines are drawn from it, one at right angles to the vertical plane, the other at right angles to the horizontal plane, and the position of the ends of these two right lines is marked on both planes of projection.

In order to abridge and facilitate the understanding of this mode of representation, supposing P the point situated in space to be projected, I will distinguish by P_v, fig. 2, its vertical projection, and by P_h, its horizontal projection. Thus the letters *v* and *h*, placed below one or more letters, indicate the vertical or horizontal projection of points, lines, surfaces, or volumes situated in space.

Through the point P, fig. 2, and fig. 2 *double*, situated in space, let us make a plane pass at right angles to the ground-line AB, it will, on this account, form right angles with both planes of projection, whence it will contain the perpendiculars let fall from the point P, one to the vertical, and the other to the horizontal plane of projection. Constructing a rectangle, fig. 2 *double*, having for its sides the two perpendiculars PP_v, PP_h, being the intersections of the plane which contains them, with the horizontal and vertical planes of projection, we shall have $MP_v = PP_h$, $MP_h = PP_v$. If we turn the horizontal plane of projection round, to bring it to the paper which contains the vertical plane of projection, during this movement, MP_v and MP_h, will not cease to be at right angles with the intersection AMB of the two planes of projection. In order, therefore, that two points P_v, P_h, fig. 2. shall be respectively the vertical, and the horizontal projection of the same point P, the right line P_vP_h, must be at right angles with the ground line AB.

The part MP_v of this line, is the distance of the point P, from the horizontal plane, and the part MP_h, is the distance of the point P, from the vertical plane.

Projections of the right line.—When a succession of points form a right line PQ, fig. 3, all the perpendiculars let fall from these points on a plane, will form a third plane, intersecting both the other planes in a right line. If we have, therefore, only the projections P_v, P_h; Q_v, Q_h, fig. 3, of the two extremities of the right line PQ, joining the points P_v and Q_v, P_h and Q_h, by a right line, we shall have two projections of the right line PQ. It is by the intersection of planes that we procure these projections.

To represent a plane according to the method of projections, we must employ another method; as follows:

The plane to be represented intersects each plane of projection, according to a right line: it intersects these two planes, at the same time, in the point M, fig. 4, situated on the ground line. The intersections PM, MQ, with the two planes of projection, are called the *traces* of the plane PMQ.

The position of a plane is fully determined by that of two right lines which it contains; and, consequently, two *traces* of a plane are sufficient to make us acquainted with its position.

Let us now suppose that it is required to find the vertical projection p_v, fig. 4, of a point p, placed on the plane PMQ, and that we know the horizontal projection p_h, of this same point. The two projections p_v, p_h, of the point p, are necessarily on a line perpendicular to the ground line: let us draw this perpendicular. Through the point p, draw on the plane PMQ, a horizontal line; it will be parallel to the horizontal line PM; whence its projection p_h, m_h, will be parallel to PM. But the point m_h, which is on the ground line AMB, can only belong to a point m_v, situated on the vertical plane of projection. Therefore, $m_h m_v$, perpendicular to AB, contains the point m_v, of which m_h is the horizontal projection. This point is also on the trace MQ, wherefore it is at m_v. If we draw $m_v p_v$, parallel to AMB, this line will represent, on the vertical plane, the projection of mp; the vertical projection, therefore, of the point p, is situated at the same time on $m_v p_v$, and on $p_h p_v$; and, consequently, it is at the point p_v, the intersection of these two right lines. Whence p_v is the vertical projection of the point, which has p_h for its horizontal projection.

Let the traces MP and MQ, SR and ST, fig. 5, of two planes be given, and let it be required to find the intersection of these two planes. 1st, the point D, being on both the vertical traces, belongs to this intersection, and as it is on the vertical plane of projection, it is projected in D_h, on the ground line AB. 2nd, the point EE$_h$, being on both the horizontal traces, belongs to the intersection of the two planes, and as it is on the horizontal plane, its vertical projection EE$_v$ is on the ground line. We have now got, therefore, two points of the right line, in which the two planes intersect each other; viz. the first point D_v, D_h, the second E_v, E_h. The right line to which these two points belong, has for it projection therefore the two right lines D_vE_v, D_hE_h; which is the intersection sought.

Projections of a Polygon.—Any polygon whatever, ABCDE, fig. 6, terminated by right lines, has for its projections two polygons of the same number of sides, $A_vB_vC_vD_vE_v$, $A_hB_hC_hD_hE_h$, the corresponding summits of which are on the same verticals A_vA_h, B_vB_h, &c.

As the intersection of two planes is always a right line, of which the projections are also right lines, it follows that a body terminated by plane faces, is also bounded by rectilinear edges, which are the intersections of these faces. Such a body is represented by drawing the right lines, which are the projections of each edge. The summits which terminate each edge, are placed on the same vertical in both planes of projection.

In fig. 7, a pyramid SABC, is represented both horizontally and vertically, by the projection of its edges, and the corresponding summits are projected in S_v and S_h; A_v and A_h; B_v and B_h; C_v and C_h, on the right lines S_vS_h, A_vA_h, B_vB_h, and C_vC_h, perpendiculars to the ground line MN.

By the intersections of planes and of right lines, descriptive geometry teaches us how to determine the length of a right line, the two projections of which are known;

and the superficies of a plane figure, given by the twc projections of its outline; the angle formed by two right lines, of which the projections are known; the angle formed by two planes, of which the horizontal and vertical traces are known; the shortest distance between two right lines given by their projections; the angle made by a right line given by its projections, with a plane given by its traces, &c. The solution of all these problems, however, can only be fully shown in a course of instruction on linear design; but when known, almost numberless applications may be made of them in the most important arts; in architecture, in stone-masonry, in carpentry, in ship-building, in constructing machines, making implements, &c.

Workmen will not only be able to draw the horizontal plans and vertical projections of buildings, of machines, of ships, &c. but they may easily make a section of these objects by any plane whatever. The plane of this section, meeting right lines, given by their horizontal and vertical projection, will produce points and angles which they will be able to determine. The different planes given by their *traces* will have a right line for their *intersection* with the plane of the section; the students will ascertain these right lines, and will produce a faithful and complete representation of every part of the building which is not curvilinear.

The carpenter, for example, may represent exactly all the pieces of a floor or a plane roof. By sections and projections he may obtain the form of every beam, rafter, king-post, joist, purline, &c. These various pieces are terminated by plane faces and by rectilinear edges; he may draw the projections of these edges; these pieces abut against each other, and the linee which mark the places of these abutments are the intersections of the plane faces of the pieces of wood in contact; and he may determine these intersections by the easy methods just pointed out. All the pieces of wood are not square or rectangular; he may measure the angles made by the different sides of the same piece, and by the adjacent sides of contiguous pieces; and he may in like manner find the direction, length, and breadth of each side of every piece.

By following this method, without, in fact, being sensible that he is using geometry, a good practical carpenter succeeds in ascertaining, by projection and sections, every rectilinear part of the carpentry of a building. Thus, a skilful carpenter, who draws with precision and correctness the plans of his different pieces of work, possesses, in fact, a very extensive knowledge of geometry. It is of little consequence that he does not give to his lines, surfaces, and solids, the names employed by the professors of the science, and consecrated by long usage in books. The things are, in principle, the same, though the names be different. Science neither loses in value nor utility, by being clothed in common language, and taught without didactic display.

These observations may be extended to masons as well as to carpenters. The stone-mason is obliged to prepare every one of the principal stones of which any regular building is constructed, according to some precise form, and so that the small stones placed by the side, or on the top of one another, in some regular order, required by a regard to solidity and durability, shall produce that precise form laid down by the architect in his plans of the edifice. The stone-mason, on the principle of horizontal and vertical projection, generally divides the walls into a succession of intersecting planes, and then the form of each stone is determined, 1st, by the exterior and interior face of the walls, and 2nd, by the intersecting planes, called planes of *joining*, because it is according to these planes that the adjacent stones are joined together.

The shape of stones in common upright walls is easily drawn, because they are all parallelopipeds, of which all the contiguous faces are at right angles to each other, and all the opposite edges are parallels. But when the walls incline in any direction, forming together other than right angles, each stone must be cut by a less easy rule: the angles made by the inclined faces with the horizontal faces,—the angles made by the direction of the edges of one wall with the direction of the edges of the contiguous wall, &c. must be determined. The upper part of doors and windows, though a plane, is very often formed of several pieces placed alongside of each other, and larger above than below, in order that they may not fall by their own weight. In this case also, the angles made by the edges and sides, or faces of these stones, the dimensions of each, so that the whole shall occupy only a predetermined space, must be ascertained before the workman begins to cut any one stone. All these problems may be solved by the method of intersections.

Young men, who are destined to be architects, builders, and leading hands among stone-masons, are taught to work, by making in plaster, of proper dimensions, models of arches, door-ways, window-frames, staircases, &c. &c. To each piece they give its proper form, determining geometrically, its angles, edges, and joints. Such instruction and such exercise cannot be too much recommended; and it is desirable, that in giving this instruction, the parts to be cut should be arranged in the order adopted in this work of plane, cylindrical, conical, developable, undefinable surfaces, and surfaces of revolution. It would also be desirable that carpenters, cabinet-makers, joiners, &c. should be taught to make complex pieces of carpentry, joinery, &c. by models, and according to geometrical principles. Their utility as workmen would not be lessened either to themselves or others; their education would be more rapid and more beneficial, and uniting the dignity of science with their daily labours, they would look on them with pride, and conceive them to be honourable.

Intersection of right lines, and planes with curved surfaces.—We shall treat of the intersections of the right line, and of the planes with these surfaces, according to the order in which we have examined them.

Projections of the Cylinder.—To obtain them, we describe on one of the planes of projection, on the horizontal plane, for example, *the trace*, or the intersection of the cylinder with the plane. All the edges of the cylinder being parallel, the projections of them are also parallel. As soon, therefore, as we have determined the direction $C_h c_h$, $C_v c_v$, fig. 9, pl. 13, the two projections of one edge, we shall have the direction of all the other edges. Generally, it is thought sufficient to mark, in horizontal and vertical projection, the extreme edges, $A_v a_v$ and $E_v e_v$; $B_h b_h$ and $D_h d_h$.

Intersection of the cylinder with a plane.—We have seen how to ascertain the intersection of a right line with a plane, when the traces of the plane and the projections of the line are known. If we perform this operation for the different edges of the cylinder, each edge will give a point of intersection, which may be projected horizontally and vertically. The whole of these points will form a horizontal and a vertical curve, which are the two projections of the intersection sought.

In practice, the intersections are very often traced on the surfaces by bringing them into contact. Let us suppose, for example, fig. 10, that the cylinder be the funnel of a stove, and that the plane be a sheet of iron through which the funnel is to pass; it is placed in the direction according to which it is to be fixed, but so as not to press against the sheet of iron. A ruler may now be applied to the cylinder, in the direction of the edges of this surface, one end of it touching the iron plate. If it be applied to each edge of the cylinder, or to as many parts of its surface as may be necessary, and the point where the ruler touches the iron plate be marked at each position, the whole of these points will, when united, form the curve of intersection of the two surfaces.

Let us suppose that any convenient length, continually preserved, be marked on the ruler, from the end which touches the iron plate, and that at this length we mark a succession of points on the cylinder or funnel, they will form a curve, which is the intersection of the cylinder with a plane. If we move either the plate or the cylinder parallel to its own direction, in consequence of the equality of parallels, comprised between parallels, the two curves traced, one on the plane and the other on the cylinder, will touch each other in every point, and coalesce into one. Having drawn these two curves, either the cylinder or the plane is, or both surfaces are, worked out in the direction of these curves, according to the purpose for which the funnel and the plate are to serve.

This method is correct, whatever may be the form of the cylinder, and even if the plate, instead of being a plane, should be curved in any direction.

Shipwrights employ this method to trace, at the spot, the curve of intersection between the surface of the ship's bow and of the decks, with the bowsprit and masts, and to cut holes and steps for the latter.

Application of the intersection of cylinders to transmitted shadows.—When a surface, terminated by prominent edges, intercepts the rays of the sun, if we draw a line parallel to the sun's rays, through each point of the intercepting surface, these parallels will form a cylinder, which will separate, beyond the surface, the part in the light from the part in the shade. If there is an object behind the cylinder, altogether in the shade, the sun is completely hidden or eclipsed by the surface which transmits the shade. If the object is only partially in the shade, and we determine the intersection of the surface of this body with the cylinder, the curve thus determined will separate on the object the part in the shade from the part in the light. We thus obtain a line of separation between the light and shade on the opaque body, by the curve of intersection of the surface of this body with the cylinder, which marks, in space, the limits of the sun's rays, intercepted by the opaque surface.

Let us take a ruler, and hold it parallel to the sun's rays, placing one end of it against the surface which causes the shade, and the other on the object, partially illuminated. Each position of the ruler thus held will mark a point on the object, and the whole of the points thus marked will be the line of separation between the light and shade.

Painters, engravers, and draftsmen, must form precise ideas of the cylinders projected by the shadows of bodies. To them it will be very useful to ascertain, rigorously, by the method of projection and the intersections of surfaces, the figure of the shadows produced by many different bodies, varying in position and form, on numerous other bodies also varying in their position and form. By this means they may acquire a species of demonstrative experience as to the effect of the sun's rays in producing shadows, and they will be saved from committing those gross errors, of which, from wanting some little knowledge of geometry applied to their art, they are occasionally guilty.

In architectural designs, the precision of transmitted shadows is essentially necessary, because all the objects represented, walls, columns, arches, &c. are rigorous geometrical forms. The architect who desires to shade his plans, and thus to be enabled to judge what will be the effects of light and shade on the appearance of his buildings, must accustom himself to determine the figure of transmitted shadows with scrupulous fidelity.

In architectural plans, and in drawing designs of machinery, it is assumed that the sun's rays come, descending at an angle of 45°, from left to right. When the plan of objects is drawn by the pen, without being washed in, the outlines which are towards the parts placed in shade are distinguished by being wider than the others, or than the lines separating the parts placed in the light. This explanation will be sufficient to give an idea of *relief* in drawings, which might otherwise be supposed, when seen, to be mere designs or plans.

In fig. 11, for example, I can immediately distinguish, on merely looking at the sides in relief and in shade, that ABCD is a frame in relief, and *abcd* a sunk frame. Students who have to draw the plans of buildings and machines, must accustom themselves to make strong and fine lines with great skill; for if their lines cannot be distinguished, an object may be supposed to be in relief which is below the surface, and an object below the surface may be supposed to be in relief.

Application to perspective.—When it is necessary to place a shaded architectural design in perspective, the point to which all the parallel rays converge must be determined by the general method explained in the Ninth Lesson. Whenever the perspective of any one point is found,* if we unite, on the picture, this point with the converging point of the sun's rays, we shall have the perspective of the ray which passes through the given point; or if the point is *opaque*, the perspective of the shadow produced or transmitted by it. Any curve placed in perspective will have for its shadow a succession of right lines, all meeting at the point of concourse, like the edges of a cone.

Intersections of the cone and the plane.—These intersections, which are specially denominated Conic Sections, when a circular right-lined or oblique cone is considered, are of the greatest importance both in science and art. The study of them alone, like the study of triangles, constitutes a separate and considerable branch of geometry; it is, so to speak, an intermediate step between elementary geometry, and the highest branches of the science. On the present occasion, I can only point out, in a few words, the essential forms of conic sections, and their principal applications.

The horizontal and vertical projections of the intersection of the cone with a plane, are determined in the same

* In general the plane of the picture PMQ, fig. 8, is supposed to be vertical; and according to this, $X_v S_v$, $X^h S_h$, being the two projections of a right line, and x_h, the intersection of $X^h S_h$, with the horizontal trace of the plane, it will be sufficient to draw the vertical line $x_h x^v$; ax_v will be the height of the point mentioned above, on a vertical elevated from x, on the picture. This method will enable us to determine the intersection of any ray drawn from the point of view, and, consequently, the perspective of all the points in any given figure.

manner as the intersections of the cylinder; that is to say, by determining the horizontal and vertical projection of the intersection of this plane with each edge of the cone. The curve which passes through the points thus determined on each plane of projection, is the projection required.

Let us take a right-lined circular cone, fig. 12, the most simple and regular of all cones; every section of this cone made by a plane parallel to the base, like the base, is a circle, and the properties of the circle and its circumference have been explained in the Third Lesson.

I. The Ellipsis.—If we divide the cone by a plane PQ, fig. 12, pl. 13, oblique to the axis, and this plane meet all the edges of the cone, the section of the cone thus produced, is an ellipsis; a curve closed on all sides, of which the principal properties are the following:—

It has a centre O, fig. 13, and two axes AB, CD, which intersect each other at right angles. Every line, SOT, drawn through the centre O, and terminated at the circumference of the ellipsis, is divided by the centre into two equal parts; it is a diameter which divides the ellipsis into two parts, one of which would exactly cover the other by turning this diameter end for end.

Each of the two axes divides the ellipsis into two symmetrical parts. Thus, every line, MPN, perpendicular to one of the axes AB, is divided by this axis into two equal parts, PM, PN. Consequently, if we turn the semi-ellipsis ACB, on AB, as a hinge, all the points of the outline ACB will fall exactly on the points of the outline ADB.

If the centre of the ellipsis is also the centre of a circle, having the axis AB for its diameter, by producing OD and PN to d and n on the circle, we shall always have the proportion OD : Od : : PN : Pn, which applies to all the right lines PNn, parallel to the axis COD. Thus, the ellipsis may be considered, in one direction, as a circle flattened proportionably in all its parts.

On the contrary, if we draw the circle CbD, fig. 13, *double*, on the small axis CD, as a diameter, we shall have the following proportion for every right line FgG, perpendicular to the axis CD, terminated at *g* in the circle, and at G in the ellipsis. O*b* : OB : : F*g* : FG.

Thus, the ellipsis may be considered in another direction, as a circle elongated proportionably in all its parts.

A circle being drawn on an inclined plane, represented by the right line AB, fig. 14, it is required to find its projection on a horizontal plane.

Let *ab* be the projection of the diameter AB, the most inclined of all the lines in the circle; *o* being the projection of the centre O, if we draw *cod* perpendicular to *ab*, and make *oc* = OC = the radius of the circle, the curve *abcd*, will be the projection of this circle; it will be an ellipsis. Let us draw MN perpendicular to the diameter AB, of the circle, drawn on the plane AB; the horizontal line MN, will be on the plane of the circle, and, consequently, equal to its projection *mn*. Thus, the perpendiculars *mn*, will be only so much nearer the great axis *cod*, as the perpendiculars MN, are nearer to the radius CO, in the relation of OM to *om*. The projection of the circle, therefore, is nothing more than this circle flattened proportionably in all its parts; it is an ellipsis.

It is a general principle, therefore, that whenever a circle is projected on a plane, which is not perpendicular to it, the projection is an ellipsis, and the great axis of this ellipsis equals the diameter of the circle.

I cannot treat of a multitude of properties belonging to the ellipsis, but there is one to which I must call your attention, in consequence of the numerous and important applications of which it is susceptible.

If we mark two fixed points F and *f*, fig. 15, by two stakes or poles, to which a cord is fastened, longer than the distance between F and *f*, and if we then, holding the cord extended, and advancing sometimes towards F, and sometime to *f*, draw the line described by the end of the cord, we shall describe a curve which is an ellipsis; and called the gardener's ellipsis, because they use this method to trace ellipses in their flower gardens.

A very remarkable property of the ellipsis is that at each of its points C, the two rectilineal portions of the cord FC and *f*C, make the same angle in C, with the curve, or with its tangent *t*CT.

To demonstrate this, produce FC, and take in it C*g* = C*f*, and draw *fg*. Draw also the right line TC*t*, perpendicular to *fg*. The oblique lines C*f* and C*g*, being equals, the angle *f*C*t* = *g*CT = FC*t*. Moreover, for any point whatever *t*, of CT*t*, the sum of the distances of the broken line F*t* + *ft* = F*t* + *tg*, is greater than the right line FC*g* = FC + *f*C. The point *t*, therefore, is out of the ellipsis. Thus, the right line TC*t*, cannot touch the ellipsis, except at C; it is a tangent. The tangent to the ellipsis at C, therefore, forms the same angle with the two vector radii. The same species of demonstration may be applied to the properties of the parabola and hyperbola hereafter mentioned.

Application to Optics.—We learn from experience, that a ray of light, FC, fig. 15, which impinges on a curved or other surface ACB, takes another direction C*f*, or it is said to be reflected in the direction C*f*; so that the two rays FC and C*f*, make the same angle with the curve or surface. If an ellipsis, therefore, be made to reflect the light like a plane mirror, every luminous ray FC, emanating from the point F, must, when reflected, take the direction C*f*, which passes through *f*.

The two points F and *f*, are called the foci. All the rays of light, therefore, emanating from one focus, and reflected by the surface of the ellipsis, will pass through the other focus.

Application to Acoustics.—Sound, like light, is propagated in right lines; it is reflected also in right lines, with an angle of reflection equal to the angle of incidence. If the circumference of the ellipsis, therefore, is constructed of materials which reflect sound, all the sounds emanating from the focus F, will be reflected through the focus *f*, which will be an echo of F.

Halls have been built in the form of an ellipsis, fig. 15, and have justified theory by experience. If we speak at the focus F, in a low voice, so as to be heard only at a small distance, at O, for example, the effects of the echo will render the words pronounced at F, distinctly audible at the other focus f.

A cruel use has been made of this echoing property of ellipses. Prisons have been constructed, in which the prisoners were chained near the focus F, and could not pronounce the least word without being heard at the other focus of an elliptical arch, separated from F by a partition, which hindered the prisoner from seeing the gaoler, who was appointed to listen in secret to every word he uttered.

The curves described by the course of the planets round the sun are ellipses, having the centre of the sun in one of their foci. Thirty centuries, passed in studying astronomy and geometry, were necessary to discover this truth, now confirmed by experience ; which prepared the way for the great and beautiful discoveries in modern astronomy.

If we make the ellipsis revolve round the large axis AFfB, which passes through both foci, we shall form a surface of revolution possessed of the following property ; every luminous ray, or every vibration of sound, FC, which emanates from the focus F, will be reflected in the direction of a right line which passes through the second focus f.

In the same manner as all ellipses are constructed by the circle, elongated or flattened proportionably at every point, so with the ellipsoid of revolution, constructed by making an ellipsis revolve on one of its axes, we may form every ellipsoidal surface, whether elongated or flattened. It is sufficient to point this out, without entering into farther details.

There is a mode of tracing an ellipsis by a continued motion, which is sometimes employed by artists. AOB, COD, fig. 16, being the two axes, if we draw a right line, MN = OA, and take in it, produced, if necessary, N = OC, the point M always remaining on the small, and the point N on the large axis, and making the right line advance or recede in every possible position, its extremity P, will trace the ellipsis ABCD.

On this principle, instruments have been constructed for drawing an ellipsis by a continued motion, which are true elliptical compasses.

In a paper published in the *Journal de l'Ecole Polytechnique*, I have shown how this species of description, by a continued motion, may be applied to draw any ellipsoidal surface, using a right line, of which three ascertained points always remain in three fixed planes, while a fourth, made to advance or recede in every direction, describes an ellipsoidal surface. This method may be applied in those drawings or operations which are required in constructing elliptical arches.

II. THE PARABOLA, fig. 17, is formed from the cone ABO*ba*, by a plane QR, parallel to one of the edges of the cone. It is a curve *mnp*, closed on one side, open on the other, and extending into infinity, its two branches *nm*, *np*, separating more and more.

The parabola MNP, fig. 18, has only one *axis* NL, in relation to which its two branches, NM, MP, are symmetrical. It has one focus F.

Produce the axis by a quantity, NG = NF, the distance of the focus from the summit of the parabola, and draw through the point G, the right line XY, perpendicular to the axis. If we produce the reflected radius IK to H, on the line XY, the point I, of the parabola, will be equally distant from the focus and from XY; therefore FI = HI. Take a square, with a cord fastened at F, and placed along XY, having a second cord directed along the square; if we hold the two cords in I, so that FI = IH, and allow both the cords to be equally unrolled, as the square is removed from the axis, the point I will describe a parabola.

If we suppose the ellipsis gradually lengthened, the two foci will gradually separate from each other. If we remain at one of the foci, that part of the ellipsis which extends rounds this focus, will come gradually to resemble a parabola, and at length will be of the same figure as this curve.

The orbits described by comets seem to be parabolas, the sun being at the focus; they are ellipses very much elongated.

In the elongation of the ellipsis, the vector radii drawn from the focus, removed to a distance, towards the other focus, approach more and more to parallel lines, as the foci are separated; and they become parallels, if we suppose the two foci at an infinite distance from each other. The ellipsis is then, rigorously speaking, a parabola, and the radii proceeding from the focus where the spectator is situated, are so reflected by this curve, as not to meet the axis, except in infinite space, where the other focus is supposed to be. In the parabola, therefore, the rays emanating from the focus are reflected by the curve, parallel to the axis.

The parabola is made use of accordingly, to receive the light proceeding from one focus, and to reflect it in one beam parallel to the axis, in place of allowing it to disperse itself towards all the points of space.

Light-houses on this principle, are erected at various places on the sea-coast, at the entrance of harbours, at the mouths of rivers, over or near dangerous shoals, and in various other situations, in which it is necessary to point out to the mariner a safe path, or warn him against danger. Both fires and lamps are employed for this purpose. It is necessary they should be seen as far off as possible, they are placed, therefore, in the focus of a piece of plated copper, having the form of a parabola, which revolves on its axis, fig. 18, pl. 13. It is the *paraboloid* of revolution. All the rays reflected from the surface, called a paraboloidal reflector, form a beam of parallel rays, having for their base, the parallel circle ABCD, which forms also the base of the surface of the reflector ABCDN.

Sometimes the *paraboloid* is fixed, and in that case the light can only be seen, at a great distance, at the moment when we cross the axis of the paraboloid. Sometimes the paraboloid turns round a vertical axis; then it sends, alternately to every point of the horizon, the light which it reflects, and the sailor knows, by the regular appearance and disappearance of the light, that it is not placed there by chance, and is intended as a guide. The difference of duration in the intervals of light and darkness, at different light-houses, occasioned by making the paraboloid revolve with more or less rapidity, serves as a mark to distinguish one light from another on the same coast.

III. An Hyperbola is the section *mnp*, *m'n'p'*, fig. 19, pl. 13, made in a cone by a plane, which divides the two parts, AOB, *aOb*. It has two separate portions, each of which has two branches, like the parabola, but with this difference, that the branches of the hyperbola tend much more rapidly to become straight than those of the parabola; so that the two branches of the closest hyperbola, having the same axis and summit as the parabola, always end by projecting beyond the branches of the parabola.

The hyperbola ABC, *abc*, fig. 20, has two axes; it has two foci, F, *f*, like the ellipsis; but, in place of the sum of the radii vector being *constant*, as in the ellipsis, in the hyperbola *their difference* is constant. The two radii FM, *f*M, form the same angle also with the curve; but the curve, in place of containing the radii vector, like the ellipsis, passes between them. There exist also, two right lines, XO*x*, ZO*z*, which make the same angle with the large axis FO*f*, and which, without there being a possibility that they should ever meet the two branches of the hyperbola, approach towards them in proportion as the distance is increased from the centre O, through which they pass. They are called the *asymptotes* of the curve.

Intersection of the cone with curved surfaces.—To determine this, it is sufficient to make a succession of planes pass through the summit of the cone. They will divide the cone in the direction of its rectilinear edges, and they will divide the curved surfaces in the direction of other lines, the intersection of which with these edges will be the points of the curve desired.

Application to Optics.—Objects are made visible to us, as was explained in the Ninth Lesson, by means of luminous rays transmitted from every point of them to the centre of our eye. Each line projected by these luminous rays becomes the base of a cone; and if we trace the intersection of this cone with the surface employed as a picture, we shall obtain the perspective of the illuminated line.

In general pictures are plane surfaces, but sometimes they have the form of cylinders or hemispheres.

Panoramas are cylindrical pictures, the point of view being placed on the axis of the cylinder. By this means the artist is able to represent on the surface of the cylinder, all the objects of nature which can be seen round a given point as far as the horizon. The name panorama, given to such pictures, signifies *universal view*, because it represents all the objects which can be discovered from a single point. The *trace* of panoramas is, therefore, nothing else but the intersection of the cylindrical surface forming the picture with one or several conical surfaces, having their summits at the point of view, and for their bases all the lines in nature which the artist proposes to represent.

In order to simplify the work, in painting this species of perspective, the horizon is divided into a great number of equal parts; into twenty, for example. The objects to be represented, which fall within each twentieth part of the horizon, are first drawn in perspective on common plain sheets of paper. On a canvas representing the developement of the cylindrical surface forming the picture, the twenty vertical and parallel bands embracing the whole horizon, or rather the objects contained in each of them, are painted side by side; and finally, the canvas is extended against the cylindrical wall of the rotunda that constitutes the panoramic building.

The truth of this species of representation, when well executed, is so striking, that it frequently produces a belief that the spectator is actually looking at nature itself. No other mode of representation makes us better acquainted with the general aspect of any country round a given point,—an advantage not possessed even by a plan in relief, and which the representation of a part of the horizon on a plane surface can never give.

Magic mirrors.—The analogy of the geometrical conceptions employed in both, makes me mention, in conjunction with panoramas, a remarkable mechanical illusion. Its object is to trace figures on a plane, so that when reflected by cylindrical or conical mirrors, they represent to the eye of the spectator regular objects and natural forms. To draw such figures on a plane, we must conceive, *first*, all the edges of the cones which place each object in perspective on the mirror; *second*, the reflected rays, considering these edges as rays of incidence. Each reflected ray, by its intersection with the plane, gives one point, and the whole of the points thus determined is the figure which must be drawn. The pleasure derived from such a picture arises from the surprise at seeing the most irregular, and even sometimes the most hideous figures, suddenly transformed, by the reflection of the mirror, into beautiful or elegant forms, which gratify our love of beauty, and our sense of propriety.

Pictures painted on domes.—In large buildings, such as temples and palaces, the archways, vaults, and domes, are very often painted in perspective, the outlines of the pictures being formed by the intersection of conical surfaces with the surfaces of these archways or domes. To execute them well, it is necessary that the artist should have studied profoundly the laws of perspective; for he has to draw figures, which, when seen from a short distance, may differ very much in form and position from nature, and when seen at the proper point of view, must appear in their proper and natural form and position.

Conical shadows.—When a luminous point, a torch, a candle, or a beam of light, passing through a small opening, falls on opaque objects, their shadow is projected in such a manner, that a conical surface separates in space the light from the shadow. If we desire to trace the shadow which one body illuminated by a single point produces on another, we must determine the intersection of the conical surface, formed by the body which transmits the shadow, with the object to which the shadow is transmitted.

On this subject, as on that of shadows transmitted by parallel rays, youthful artists may be reminded, that they will find a great advantage in ascertaining beforehand, by geometrical means, the exact form of many of the transmitted shadows of this species, in order to accustom themselves to the forms and figures which constitute the results, and enable themselves more accurately to appreciate the effects of light as to the form of shadows; which will add much to the truth of their productions.

By following a method analogous to that just pointed out, the student may find, *first*, the intersections of developable or indefinite surfaces with other surfaces which determine the points, where the latter are met by each of the right lines constituting the edges of the former; *second*, the intersections of surfaces of revolution with other surfaces, by finding out the points where the latter are intersected by parallel circles drawn on the former, &c. In all these operations, the talent of the artist consists in properly selecting his planes of projection, in order to have, for the projections of the generating lines of each surface, simple curves, which may be easily drawn.

FOURTEENTH LESSON.

On tangents, and on planes tangents to curves and surfaces.

In order to facilitate geometrical conception and demonstration, we sometimes, for a curve ABCDEFG, fig. 1, pl. 14, substitute a polygon, of which the small sides AB, BC, CD, DE,...approach to the nature of that part of the curve which lies between these different points.

If we draw the right line XABY, through the two points A, B, marked, we will suppose, on the curve as near one another as possible, it will, so to speak, coalesce with the curve, in the short space which separates the two points A and B. It will mark the direction of this small part of the curve ABCDEFGH, and we then say that the right line XABY is a tangent to the curve in the small part AB.

This mode of finding the tangents to a curve, is only, it must be remembered, an approximation; let us, therefore, try by a simple method to form a rigorous geometrical conception of a true tangent.

In the circle ABC, fig. 2, draw the radius OA, and at the extremity A draw XAY perpendicular to the radius. In the Third Lesson, it has been demonstrated that every point in XAY, except A, is outside of the circle. The right line XAY, which touches the circle in a single point, is the tangent to the circle. We cannot draw through the point A, either to the right or to the left, a right line between the tangent and the circle XAY. Let us draw any line AZ, through the point A, and ON perpendicular to OZ; ON, the perpendicular, will necessarily be shorter than the oblique OA, whence AZ will pass into the circle, and consequently will not pass, proceeding from the point A, between the circle and the tangent XAY.

As a very small portion of the circle proceeding from the tangent follows the direction of this tangent, we may regard a point very near A, on the circle, as being also on the tangent, which is sufficient to point out its direction; and it will be less incorrect, in proportion as the second point is nearer to the former.

The radius OA, perpendicular to the tangent XAY, is also perpendicular to that part of the curve which, proceeding from A, follows the direction of the tangent. This perpendicular is said to be *normal* to the tangent, and thus the *radius* of the circle is *normal to its circumference*. In the arts, continual use is made of the properties of tangents and normals, to give a settled and determined form to the direction of lines and surfaces.

Let us first see how to draw regular polygons, by means of tangents to the circle.

Let *abcdef*,...fig. 3, be a regular polygon; O being its centre, we shall have O*a* = O*b* = O*c* = O*d*..., and *ab* = *bc* = *cd*.... The triangles *a*O*b*, *b*O*c*, *c*O*d*..., are therefore equal to one another; as are, consequently, also the perpendiculars OA, OB, OC..., drawn from O, to *ab*, *bc*, *cd*....A circle, therefore, described from the point O, as a centre, with the radius OA = OB = OC = OD..., has for its tangents all the sides of the regular polygon *abcde*.

The polygon *abcde*..., is said to be circumscribed about the circle ABCD...; every regular polygon, therefore, may be circumscribed about a circle.

It is easy to see, 1st, that the circumference of a circle is greater than the perimeter of every polygon, ABCD, inscribed within it, and smaller than that of every polygon *abcd* circumscribed about it ; 2nd, that the surface of the circle is larger than that of every inscribed, and smaller than that of every circumscribed polygon.

By multiplying the sides of polygons, inscribed within and circumscribed about the circle, its radius being unity, we may calculate two outlines, differing less from each other than any length measurable by our instruments; and yet such that one of these outlines shall be larger, and the other smaller, than the circumference of the circle.

Regular polygons have in the same manner been found, of which the surface of one was larger, and the surface of the other smaller than that of the circle, and yet differing from each other less than any dimensions given beforehand. Thus, by numbers of close approximation, we may express the circumference and the surface of a circle having unity for its radius.

The same method may be employed to determine the perimeter and surface of any other portion of space terminated by any other species of curve.

This remarkable method is called by geometricians, the method of limits. It gives a rigorous demonstration of a great number of mathematical calculations and principles, which we have been accustomed to give as nearly rigorous, or as not differing sensibly from the truth.

If we wish to cut a surface, such as a piece of sheet iron or of pasteboard, of a circular figure ABCD, fig. 3, we begin by tracing with tangents a polygon circumscribed about a circle, and with a file, a plane, a pair of scissors, or other suitable instrument, we cut away the angles *a*, *b*, *c*, *d*.... Then we form a polygon with double the number of sides, and reduce the figure much nearer to the form of the circumference of a circle. By continuing the same process, we at length form a polygon, of which the sides are so numerous and so small, that the summits and the angles become imperceptible, and the circle appears perfectly regular.

In the construction of doors, windows, archways, either semicircular or gothic, the uprights AM, CN, figs. 4 and 5, are vertical, and at right angles to the horizontal radius AO = OC, fig. 4, and to AC, fig. 5; consequently the uprights are tangents to the arch in A and in C.

In the elliptical arch ABCD, fig. 6, formed like the handle of a basket, there are arcs of a circle, AB, BC, CD, the centres of which, *m*, O, *n*, are thus placed:—

1st. O, *m*, and the point B, where the arcs AB and BC meet, are all in one right line. 2nd. O, *n*, and the point C, where the arcs BC and CD meet, are also all in one right line. If XBY, therefore, is at right angles to O*m*B, and if ZCT is at right angles to O*n*C, the two lines will be at the same time tangents—the former to the arcs AB and BC in B, and the latter to the arcs BC and CD in C. As the arcs of circles thus drawn have the same tangent, there is no angle, or sharp turning, or abruptness, at the point of their intersection.

Whenever it is required to substitute for a continued curve the arcs of circles which are as near as possible of the same form, and from which no interruption to continuity arises, the circles should meet, so that at the point of meeting they have the same tangent.

Planes, tangents to surfaces.—Parallel to a given plane, let us make, in the surface AGB, fig. 7, a succession of plane sections, AB, CD, EF; they will gradually diminish as they approach the limits of the surface, and we shall at length arrive at a point G, which alone will be on a plane MN, parallel to all the sections.

Let us draw on the surface various curves, AGB, *a*G*b*, passing through the point G; through this point, let tangents to these curves be drawn. As no right line can pass between the tangents and the curves, all the tangents must be placed on the plane MN.

Thus every plane, forming in G a tangent to the surface AGB, contains all the right lines, tangents in G to all the curves drawn through this point on the same surface. We must except, however, *singular points*, such as the summit of the cone, &c.; but these points are always exceptions on surfaces.

Let us take the sphere, for an instance of the general rule. The parallel sections AB, CD, EF, fig. 8, are circles, the centres of which o, o', o'', are placed on a right line $oo'o''$,... G, perpendicular to the plane of all the circles, and passing through the centre of the sphere. If we draw a plane MN, through the extremity G, parallel to that of the sections, and consequently at right angles to oG, it will be the tangent to the sphere.

Every other point, in fact, of the plane, will be further from the centre of the sphere than G, and consequently out of the sphere; wherefore the plane will only touch the sphere in the point G. Every plane drawn through *go*G will divide the sphere, according to a circle of which *go*G will be the diameter, and of which the tangent in G will be perpendicular to *go*G. Now all the lines in G, perpendicular to the right line *go*G, are in the plane perpendicular to this right line, and pass through G; the tangent plane MN, therefore, contains all the tangents to the meridian circles having *go*G for a diameter. It might be demonstrated with equal facility, that every small circle drawn on the sphere through the point G, has its tangent in G, placed on the plane MN.

The line *go*G, fig. 8, perpendicular at G to the tangent plane, is called normal for surfaces, as well as for lines.

Let us apply these elementary principles to the different species of surfaces we have already examined in the preceding Lessons.

Planes, tangents to cylinders.—Suppose the cylinder AB*Cabc*, fig. 9, terminated by two bases, situated in parallel planes, and having all their corresponding lines parallel. If B*b* is an edge, the tangents MBN, and *mbn*, to the two curves at B and *b*, will be parallel. The same will be true of every other tangent *m'b'n'*, to the curve *a'b'c'*, parallel to the bases, *b'* being on the edge B*b*. The series of parallel tangents MBN, *m'b'n'*...*mbn*, passing through the edge B*b'b*, which is a right line, will form a plane. This plane is a tangent to the cylinder through the whole length of the edge.

Formation of planes by tangents to the cylinders.—The baker who rolls his rolling pin parallel to itself, forms of the paste a plane, which is by turns a tangent to each edge of the cylindrical surface of the roller.

The gardener obtains the same result for the paths and greensward of the garden, over which he drags a rolling cylinder. As the earth is gradually levelled, it becomes a tangent to the cylinder, through the whole length of the different edges of this surface.

The coachmaker suspends some carriages by a leathern brace on each side, fig. 11. This brace is adapted to the lower and cylindrical form of the body of the carriage, and is prolonged, so that its upper surface, a plane, forms a tangent to the body of the carriage. When the body is balanced before and behind, it advances or recedes on this tangent plane, which being the same on both sides, prevents the transversal shaking, which is the most harsh and disagreeable in carriages that are not suspended.*

Construction of a cylinder by tangent planes.—In the Lesson which treats of cylinders, the method of working a solid body, so that its surface shall be cylindrical, has been mentioned, and to it we shall now again refer. The bases are marked out on the two ends of the piece of wood, stone, or whatever is to be made into a cylinder; and then two polygons are circumscribed about these bases, having their corresponding sides equal and parallel: with a saw, a plane, or some instrument proper for making surfaces, small planes are worked by the parallel sides of these polygons. By this means, we obtain a polygonal prism circumscribed about the cylinder; because its various faces will be everywhere tangents to the surface of the cylinder. As we cut away the edges of the prism to form other planes, tangents to the cylinder, we bring the former to resemble the latter; and the more these planes are multiplied, or the more frequently the edges

* This method of suspending carriages, is, we believe, no longer practised in England, though still common on the Continent.

Tr.

are cut away, the more will the prism approach to the rigorous geometrical form of a cylinder.

Planes, tangents to the cone.—If we draw an edge SABC on the cone, fig. 12, all the tangents in A, B, C, to the parallel sections A*a*, B*b*, C*c*, are parallel to one another. The whole of these tangents form the plane PQMN, tangent to the cone through the whole length of the edge SABC.

This property of the cone permits us, by circumscribing its base with a polygon, to construct a pyramid, the faces of which shall be tangents to the cone through their whole length. By cutting away successively with any proper instrument the edges of this pyramid, we form new tangent planes, multiply more and more the edges, and thus form a surface, which represents a cone with any required degree of precision. (See Tenth Lesson.)

Planes, tangents to developable surfaces.—The property which the same tangent plane has of touching the cylinder or cone through the whole length of an edge, belongs also to other species of developable surfaces. Such surfaces may at all times be considered as formed of a great number of extremely small conical faces, having, like those of the cone, the same plane a tangent through the whole length of each edge.

We can make a developable surface pass through two given curves, by circumscribing polygons about these curves, so that the same plane will pass at the same time through one side of each polygon; this plane will be the tangent to the developable surface. Cutting away the edges formed by the intersection of these planes, we may multiply the sides of the two circumscribing polygons, and, of course, the small planes which are tangents to the developable surfaces required to be produeed.

Cylinders, tangents to each other in the direction of one of their edges.—Placing two right lined circular cylinders ABCD, BCEF, fig. 10, close to each other, so that their axes shall be parallel, and distant from each other by a quantity equal to the sum of the radii of the bases, we shall find that the two cylinders will touch each other in the whole length of the edge BC. The two surfaces will have, therefore, through the whole length of this edge, the same plane, tangent to both. Let us now suppose, that we have before and behind these two cylinders, a horizontal table, of which the upper part shall lie in the direction of this plane. If we place a plate of metal on one of the two tables, and force it through between the cylinders, equally distant from each other, we shall flatten this plate so that its two parallel faces will become tangent planes, the upper face to the upper cylinder, and the lower face to the lower cylinder. The flattening of metallic plates, therefore, by means of revolving cylindrical rollers, is founded on the property of planes being tangents to cylindrical surfaces.

Cones and Cylinders, tangents in one direction.—When a cylinder ABCD, and a cone ADE, fig. 13, pl. 14, have the same edge AD, and at D, the same tangent MQ, the plane drawn through MQ, and through the edge AD, is a tangent both to the cone and cylinder, through the whole extent of AD. The cone and the cylinder are, therefore, through this extent tangents to each other.

Blacksmiths, tinsmiths, coppersmiths, &c., apply this property in bending sheets of iron, tin, copper, &c., into a cylindrical form. They place the plate of metal so that the direction of the edges of the cylinder may lie also in the direction of the conical point of an anvil, represented by ADE. Then with a hammer, the head of which is hollowed cylindrically, they gradually bend the metallic plate along the whole length of the right line, according to which the cone touches the plate. They are thus sure to form a cylindrical surface. In the same manner they form a conical, or other developable surface, by augmenting or diminishing gradually, the curve of the metallic plate, according as the hammer falls on the edge of contact AD, nearer to or farther from the summit A.

Cylinders, tangents to other surfaces, and enveloping them.—If we suppose that a right line, proceeds forward, constantly parallel to its primitive direction, always remaining a tangent to any given surface, it will form a cylinder, which will be a tangent to the proposed surface, through the whole succession of the points of contact of the edges of the cylinder and of this surface.

Cylinders enveloping the sphere.—Let us suppose, for example, that we have a sphere *abcd*, fig. 14, and that a right line, always remaining a tangent to the sphere, moves parallel to an axis drawn through the centre of the sphere, we shall, in this manner, form a right lined circular cylinder, which will touch the sphere through the whole extent of a great circle *amcn*. We may make the sphere advance or recede in the cylinder without its ceasing to touch the cylinder in the direction of a circle, parallel to *amcn*, and perpendicular to the axis of the cylinder.

In the arts, frequent use is made of this property. Whenever a sphere is to receive a direction parallel to a rectilinear axis XOY, fig. 14, it is made to move in a cylinder which envelopes and touches it at every point.

Such is the principle on which all fire-arms, guns, muskets, pistols, mortars, howitzers, &c. are constructed. The interior surface of these instruments is a right lined circular cylinder, and the balls, bullets, or bombs, which they discharge towards a given object, are spheres, to to which a precise direction is given by their being compelled to move in the line of the axis of the cylinder.

To ascertain, 1st, that the diameter of the balls is not too large, which would prevent them entering the fire-arm destined to receive them; 2d, that it is not too small, which might prevent the direction given to the ball being that of the axis of the cylinder, or would destroy the accuracy of the aim, moulds are employed, fig. 15, which are themselves right lined circular cylinders, having very short edges. The gunner taking hold of the handle AB*ab*, trys the ball in every direction, to see if it will pass through the mould without leaving too great a space. This is called ascertaining the calibre of balls.

Application to Shadows.—Nature presents examples to us, at every moment, of cylindrical surfaces formed by right lines parallel to one another, all being tangents to the same surface. When any opaque object, terminated by a curved surface, is illuminated by the sun, it throws a shadow behind it. The rays which separate this shadow from the part illuminated by the sun, are necessarily the rays which touch the body without being intercepted by it. They are, consequently, tangents to its surface. The whole of the points, therefore, which limit, in space, the shadow created by the object, form a cylinder, the whole of the edges of which are tangents to the object. All the points of contact between the surface of the object and the cylinder which limits the shadow transmitted by the object, taken together, form a curve, which is the line of separation between the light and shadow on the surface of the illuminated body.

When we require to determine with precision the shadows transmitted by objects to a plane surface, we must construct cylinders formed in this manner by tangents to the surface of the objects parallel to the supposed direction of the sun's rays, and then we must determine the intersection of this cylindrical surface with the surface of the plane, to which the shadow is transmitted. This is an important study for the draftsman and the architect.

If we make the illuminated object advance or recede, parallel to itself, in the direction indicated by the sun's rays, each of the points in it will describe a right line parallel to the rays; thus, all the points of the object which are on the cylinder,—the limit to the shadow transmitted by the object,—will follow the direction of the rays, which will always continue to be tangents to the surface of the object; and the same cylinder will always be the limit of the shadow transmitted by the object. This cylinder, which constantly surrounds the object in all its positions, is called, in relation to it, an enveloping surface.

Thus the right lined cylinder is the enveloping surface of the sphere which moves in a right line, and always preserves the same radius. The interior of the barrel of the musket or cannon, and the cylinder it would form, if produced, is the enveloping surface of the space traversed by the ball.

We can form a cylindrical surface, which shall be the envelope of a sphere of a constant radius, the centre of the sphere moving in a right line. In fact, this takes place when we fire a ball through a soft and non-frangible substance.

Conversely, we can construct a sphere, by making a cylinder revolve round a right line perpendicular to its axis, and passing through it. In each position of the cylinder it will touch the sphere in the direction of a meridian circle, and the whole of these meridians will form the sphere itself; supposing these meridians drawn very close to each other, we can substitute for the tangent cylinders, the cylindrical sides comprised between two consecutive meridians. We shall then, however, return to the method by approximation given in the Eleventh Lesson.

By the same means we can construct, 1st, surfaces of any form whatever by other surfaces which they touch in every point, and which are made to move in a direction parallel to the edges of the cylinder; 2d, any surface whatever employing a system of cylinders which touch it at each of their edges.

Application in Joinery.—When the joiner or cabinet-maker has to form mouldings for an object of a curvilinear figure, he uses a plane, the iron of which represents the profile or transverse section of the mouldings; the wood of the instrument being of a cylindrical form, having this profile for its base. He guides his plane in working, so that it always forms a tangent to the shape which the moulding is to have. In this motion, the cylindrical surface of the plane becomes successively the tangent to the moulding, through the whole extent of the profile given by the plane-iron; and the moulding is the enveloping surface of the cylinder which constitutes the plane itself.

Conical surfaces may be considered in the same manner, and supply us with analogous results.

Let us suppose that we draw through a given point S, fig. 16, all the tangents SA, SB, SC,...which can be drawn to the sphere O, we shall form a right lined cone, tangent to the sphere through the whole extent of the circle ABCD, serving as a base to the cone. In fact, if we make the great circle ABE, revolve round the axis SO, drawn through S and through O, the centre of the sphere; 1st, the circle will generate the sphere; 2nd, SA, SB, the tangents to this great circle, will generate the cone.

If we suppose that the centre O, moves on the axis SO, augmenting or diminishing the radius of the sphere in proportion to its distance from the point S, in consequence of the properties of similar figures, the sphere will always have for tangents all the edges SA, SB, SC,...of the cone SABCD. This cone is, therefore, the envelope of the space passed through by the sphere, of which the centre moves in a right line, and the radius of which augments or diminishes, in proportion to the distance of the centre from a fixed point in the right line.

By substituting for the sphere any other curved surface whatever, we can, from any point at a distance from the surface, draw all the right lines which form the edges of a cone, touching it in each of these edges. If the point taken for the summit of the cone is luminous, the cone thus formed will mark, behind the object, the limit of the shadow transmitted by this object to any surface whatever. If we desire to draw in a rigorous manner the limit of the shadow transmitted by the object on any surface, we must determine the intersection of this surface with the cone which is the limit of the shadow transmitted by the illuminated object.

Explanation of Eclipses.—By applying these principles to astronomy, the cultivators of that science have succeeded in determining the form and duration of eclipses. Let us suppose that the moon must pass almost in a right line between the earth and the sun. Regarding the sun and the moon as two spheres, we may conceive a right lined circular cone, which will be the envelope of both, and which would mark in the heavens the limits of the shadow transmitted by the moon. As long as the earth is completely beyond the limits of this cone of shadow, the sun will not be eclipsed, but whenever a portion of the earth comes within the cone, this portion will be deprived of the light of the sun; on it the sun will be eclipsed by the moon, and there will be, as it is called, an *eclipse of the sun*. If we ascertain the respective positions of the three bodies, and the intersection of the surface of the earth with the cone,—the envelope of the sun and moon,—at every instant the eclipse may continue, this intersection will mark a certain space on the earth. At those places only, that are situated within this space, there will be at the given time a total eclipse. If we trace also all the intersections, during the time the same eclipse continues, the points, which may be beyond these intersections, will not be totally eclipsed, and those points, within the intersections, will be eclipsed for a longer or shorter period. Thus, by geometry, all the circumstances connected with an eclipse of the sun are ascertained. With equal facility also, it ascertains all the circumstances of an eclipse of the moon.

If we suppose a right lined circular cone, which envelopes at the same time the surface of the earth and the surface of the sun, when the moon enters into the conical shadow transmitted by the earth, there will be an eclipse of the moon. If the whole of the moon is included within the cone, the eclipse will be total; and the eclipse is partial, when only a part of the moon enters within the cone. In the latter case, we can ascertain for any given moment the form and extent of the eclipse, by ascertaining the intersections of the cones,—the envelopes of the sun and the earth,—with the surface of the moon.

When we look at any object whatever, and extend to it, as we have just done to the sun, the visual rays which are tangents to it, they determine on this object the limit of the points visible to us, which is called the apparent limit or visible outline of the object we are considering.

In painting, we draw on the surface of the picture the apparent outlines of an object; they are the intersections of this surface with that of a cone, all the edges of which are tangents to the object, and the summit of which is at the centre of our eye. A knowledge of the cones, therefore, which are the envelopes of objects, is indispensable to place those objects in proper perspective, which are not terminated exclusively by right lines.

When a luminous sphere *oab*, fig. 19, pl. 14, illuminates an opaque sphere OAB, we may conceive, first, a cone S*a*AB*b*, which envelopes both spheres at the same time, and which marks on the sphere OAB, the absolute separation of the shadow and the light; we may conceive afterwards, a second cone *mn*TNM, situated between the two spheres. In the space IMN, comprised within this cone on the illuminated sphere, the whole of the luminous sphere may be seen. But from any point within the space AMNB, only a portion of the illuminated sphere can be seen; there is a partial shadow, therefore, which is called the *penumbra*. When it is required to shade objects with great precision, both the shadows and the penumbras must be designated with care, which may be done by following methods, analogous to those just pointed out.

If the two surfaces *aob*, AOB, had no analogy, the same cone could not envelope them, and be at the same time tangents to both. It would then be a developable surface, which may be constructed by supposing that a plane remains a tangent to both surfaces, and presents successively all the positions compatible with this condition. In each position, let us join by a right line the two points in which the plane is a tangent to the two surfaces. The whole of these right lines will form a developable surface, which will separate light from shade, shadows from penumbras, according as it may be beyond the luminous object and the object illuminated, or as it, may pass between the two objects. I regret that the narrow limits of this course, and the exposition of elementary principles to which I must confine myself, do not allow of my doing more than merely indicating all these admirable properties of developable surfaces.

When a place is fortified, it is necessary that it should not be possible to direct any projectile, from within gun-shot, directly at the level parts of the works where their defenders are to be stationed. A developable surface is formed in imagination, which is at the same time a tangent to the upper part of the fortification, and to the summits of the ground within gun-shot around the place. This developable surface must at no point meet the plane where the defenders are stationed, nor even a surface raised above this plane the height of an ordinary man. When this condition is fulfilled, the interior of the place is said to be *defiladed.* The geometrical methods employed to obtain this result are called *methods of defilading.*

In the arts, frequent use is made of enveloping cones, to give a particular form to objects. The *sabotier*, or wooden shoemaker, (whose art is scarcely known in England, and the mention of it only preserved, as a pleasing illustration,) employs a rectilinear cutting instrument to form *sabots*, or wooden shoes; one end of the instrument is fixed, and the other is provided with a handle, which he seizes with his right hand; with his left he holds in a fixed position the piece of wood that is to be made into a shoe, and he shapes it with the instrument. Each cut produces a conical surface, which is a tangent to the *sabot*, in the whole length of a certain curve. The whole of the curves thus formed, produce at length the surface of the shoe, which is the enveloping surface of all the cones described by the instrument.

When the turner has to make any object of the figure of a surface of revolution, he takes at first a very narrow chisel or gouge, and cuts notches almost down to the outline of the required surface. He then takes a broad chisel, which he holds in a direction that is a tangent to the figure the object is to have. In each position, the edge of the chisel describes a cone, and the whole of the cones thus formed, making the chisel deviate every time, a little from its position and direction, present a succession of conical bands every where tangents to the surface of revolution, which is enveloped at every point, and in the end produced by these cones.

The hoops of casks and of masts composed of several pieces are cones, tangents to the surfaces of revolution of the masts and the casks.

Among the various methods of making surfaces, some ensure the artist more or less continuity in different directions, which renders them more or less advantageous, according to the wants which the products are intended to accommodate.

Let us now examine the enveloping surfaces which may be formed by the flexion of certain lines to which the enveloped surfaces are attached.

Let us suppose that a fixed thread represents the axis of a cylinder or a circular cone, or any other surface of revolution, and that we attach to this thread the centre of every sphere enveloped tangentially by the cylinder, by the cone, or by any other surface of revolution. Let us now bend the thread in the direction of any curve, the enveloping surface of all the spheres will be no longer a cylinder, a cone, or any other surface of revolution, but a surface composed of a series of circles, each of which will be common to one of the spheres and to the enveloping surface.

When we bend the axis of the cylinder, the enveloping surface is formed of a series of circles, all equal to the great circles of the equal spheres which were originally enveloped by the cylinder. All these circles have their plane at right angles to the curve formed by the bended axis; and their centre is on this axis.

The worm of a still is an enveloping surface of this species, formed, 1st, by bending the axis of the cylinder in the direction of a cylindrical spiral; 2nd, by taking the envelope of all the equal spheres which have their centre on this axis.

In the winding arched-over staircase the circular arch, is in the same manner the envelope of the equal spheres, having their centre on the outline of a spiral, of which the pace is equal to that of the steps of the staircase.

Each turn of a common rope of three strands is in like manner the envelope of the space which would be passed through by a sphere, the centre of which should move in the direction of the spiral drawn through the middle of the strand.

There are some caterpillars and other reptiles which are formed of short cylindrical rings, the articulations of which can be lengthened or shortened at the will of the animal. When these animals wind and turn, their skin forms a surface which continually varies in shape, but which never ceases to have the figure of those geometrical surfaces just described.

When the axis of a right-lined circular cylinder is bent in the direction of a circle, it is again transformed into a surface of revolution, or into that annular surface examined in the Eleventh Lesson; the mode of generating which, and its projections, were then described.

If the enveloping surfaces of a sphere, having a constant radius, are intersected by a plane at right angles to the curve described by the centres of these spheres, 1st, the plane will be at every point at right angles to the envelope; 2nd, the section is a constant quantity, for it is the great circle of equal spheres.

We may, without committing an error appreciable by our senses, suppose that the curve on which the centres of the spheres is placed is a polygon of infinitely small sides. The envelope will then be composed of a succession of zones or cylindrical bands, which touch, in the direction of the circles, the enveloped spheres. The whole of the circles of contact form the enveloped surface, whatever may be the number of the sides of the polygon, and even if this polygon becomes a continued curve.

When it is necessary to make a certain quantity of water pass through a conduit or pipe of a circular section, the section of the pipe must be equal at every point, if the water flow with the same velocity at every point, and be not anywhere impeded. The surface of the pipe must, in this case, be the envelope of a sphere of a constant radius.

Every species of canal and pipe for the distribution of water, must, in like manner, have for its section a curve or a polygon, the superficies of which is constant. For the sake of regularity and facility of execution, the same figure is given to the section generally through the whole length, except in cases where some insurmountable difficulty does not allow it.

In treating of *the centre of gravity*, in the Second Volume on MACHINES, I shall give an easy and simple method of determining the volume of bodies, and of those portions of space which are terminated by the *pipe* or *canal surfaces* just described, and which will admit of a multitude of applications.

The smith, the plumber, the glass-maker, the potter, the copper-smith, all make a number of articles which have the form of canal-surfaces. They first form solid or hollow prisms, which they bend to a certain extent; and their principal art consists in preserving, at every point of the objects they bend, the constant form which the transverse sections ought to have.

Ringlets, rings, necklaces, collars, corkscrews, spiral springs, pipes bent in the form of a curve, syphons, barometer tubes, the veins of the human body, are all examples of the surfaces under consideration.

In speaking of the intersection of surfaces, it was stated that double curved surfaces might be represented by a species of rings, or cylindrical or conical drums, like the trunks of columns; the inconvenience of this method for the surfaces of pipes, is, that there is a want of continuity in the longitudinal direction, and the transverse sections are not constant.

There are some towns in which the tin and coppersmiths have a particular art in working metallic plates, so as to give them a double curve, and preserve a regular and constant section at every point. The workmen of Lyons, in this respect, for example, are much superior even to those of Paris.

The engineer, in laying down the curved part of canals, employs a particular geometrical method, the objects of which are to preserve at every point a constant form in the section, and to preserve the position of his plane every where at right angles to the surface of the canal.

In place of supposing that a surface of constant dimensions traverses a certain space, of which the envelope is sought, let us suppose that a surface changes its size but not its form.

The most simple case of this description is that already examined of a sphere, fig. 16, which alters its radius, while its centre moves in a right line. Its envelope is a surface of revolution. Every sphere is touched, or enveloped, in the direction of a parallel circle, by this surface of revolution; and the whole of the parallel circles form the surface of revolution.

Let us suppose that with the axis of the surface of revolution the centres of these spheres are connected, and let us bend this axis according to any curve. The new envelope of all the spheres will vary in size with the spheres themselves; but it will touch, it will envelope, every sphere in the form of a circle.

In nature we find a great number of surfaces of this description. The serpent, when he holds himself straight, has the form of a surface of revolution nearly approaching to that of an elongated cone. He bends and twists himself in a great many directions; the surface of his skin changes its form at every instant, but it continually forms the envelope of a series of spheres, which may be supposed to be enveloped tangentially by the surface of its skin.

The figure of the serpent, when coiled or only curved in that particular form which is called serpentine, is imitated by the arts in such things as the musical instrument which bears its name, fig. 17; as the trumpet, fig. 18; the hunting horn, fig. 21; corkscrews, gimblets, &c.

If we suppose the serpent winds itself up in a spiral direction, having its tail at the centre, fig. 20, it forms a surface very much resembling the figure of a great number of shells.

The horns of most animals have at their extremity the form of the surfaces we are now treating of, fig. 22. A great number of musical instruments are made of the same form, such as bugle and French horns.

To make wind instruments which shall give a correct and soft sound, their curved surfaces ought to be very continuous; consequently, the means selected to make them ought to be such as will preserve this continuity in the longitudinal direction, along which the air is forced, as well as in the transverse direction, the section of which must always be circular.

The various methods already pointed out of constructing different surfaces, will enable the student to judge which of them is the best for manufacturing wind instruments, and may often enable him to substitute a more exact method than some of those now adopted.

It is not sufficient that we can give to objects a preciseness of form more or less pleasing by these ingenious methods; we must also, for the satisfaction of our sight, give to the surface of numberless commodities, the produce of art and industry, such a polish as seems by its regularity and brilliancy to confer on them an additional value. On this account, there is, in a multitude of arts, a final operation, which consists in polishing, furbishing, planishing, &c. &c.; and which is generally effected by the polishing instruments describing, by their movements, surfaces which are tangents to the objects to be polished; so that the latter form the definitive envelope of the spaces passed through by the former.

Let us take, for an example, the barrel of a musket. A flat piece of wood, smoothly planed, is held tangentially to the trunk of the cone formed by the exterior of the barrel, and moved backwards and forwards on one edge of the cone; the space thus traversed is a plane, the tangent to the cone. By repeating the same operation for all the edges of the cone, we have at length, by the time the barrel is polished, the cone itself as the envelope of all the tangent planes.

To polish a sphere, it may be made to advance or recede in a cylinder, presenting all the parts of the sphere alternately to the action of the cylinder. It may also be placed on a lathe, the axis of which passes through the centre of the sphere; and the sphere is then made to revolve under a polishing iron, which is fixed in various positions, being all tangents to this surface. Thus the sphere is polished by means of cones, of which it is the envelope.

Mirrors are polished by rubbing them with surfaces, of which the tangent plane, in all their positions, is the very plane which is required for the polished surface of the mirror. The same may be said of lenses, both plane and spherical, which are used by opticians.

When the shipwright with his adze, dubs the planks of the ship's side, at each stroke of his instrument he takes away the superfluous wood, according to the figure of a surface of revolution which is a tangent to the dubbed surface. He may be said to polish or smooth the ship. The surface of the vessel, when the operation is completed, is the envelope of the surfaces of revolution formed by the motion of the adze.

The explanation which I have now given, though much too brief in my opinion, will perhaps be sufficient to convince workmen how very much the study of geometrical forms, which are distinguished by lines and surfaces, is fruitful in direct, varied, and important applications in most arts. It is from not having reflected on the forms created by nature, or given to the products of art, that we do not see in them geometrical figures, or the geometrical properties they possess, or the means which these characteristic properties supply of accurately describing or correctly forming any object.

Whenever the workman and the artist shall become sensible of the utility of examining the form of different objects, he will make it a continual and almost involuntary study. He will look at the products of industry as a naturalist looks at nature,—with an attentive and scrutinizing eye, recognising in every new object some analogy with those great natural families and genera with which he is already familiar, and observing differences which serve to distinguish species, varieties, and individuals. Such a study will not end, though even this is not unimportant, in gratifying curiosity; but it will have for the improvement of the arts important consequences which we scarcely dare to predict.

But we shall not attain to great perfection, nor even carry our improvements to a considerable extent, unless we adopt and practise continually the rigorous methods of geometrical design. If artists study descriptive geometry with care, they will find that it supplies them, not only with the means of making accurate drawings, but also complete demonstrations of those useful properties, which in this brief Work are merely mentioned. Let us always remember, that manufacturing industry will remain imperfect, and be little more than mere mechanical drudgery, till a knowledge of the principles of the arts, of linear design, and of descriptive geometry, is universally spread through our workshops and manufactories, and made the basis of manual operations. To labour without thought is characteristic of the instinctive toils of animals. To man is given the high faculty of knowing and explaining what his hands perform: and perhaps we should all be more ready to note the principles of our operations, if we continually recollected that those which appear to us from long habit, the most mechanical and easy, had their origin in a close observation of nature; and that many of the forms and objects now produced by our most common workmen, were at one time highly esteemed as the offspring of inventive or philosophic genius.

FIFTEENTH LESSON.

Curvature of Lines and Surfaces.

LET us suppose that we move on a curve, while we continue to look, at every point of our progress, in the direction of the tangent to this curve. We must in this case not only move forward, but turn at every moment towards the bending part of the line that we follow. The curvature of this line is proportional to the quantity which we thus turn, divided by every small portion of space through which we move.

If the curve be a circle, to traverse equal arcs we must turn equal quantities; the curvature of the circle is the same, therefore, at all its points.

If we move successively on two unequal circles, fig. 1, pl. 15, having R and r for their radii respectively, then 3.14.... ×2R will be the circumference of the large circle, and 31.4.... ×$2r$ will be the circumference of the small circle. But when we move round a whole circle, and always on the circumference, we always turn equal to 360°, wherefore the curvatures of the two circles C and c, are to one another as,

$$\frac{360}{3.\ 1415 \times 2R} : \frac{360}{3.\ 1415 \times 2r} \text{ or} : : \frac{1}{R} : \frac{1}{r}.$$

Thus, in fig. 1, the circumference of the small circle is more curved than that of the large circle, in the inverse ratio of the lesser to the greater radius. The curvature of circles is, therefore, in an inverse ratio to the length of their radii. When a radius, consequently, is very large, the curvature of the circle appears to be almost nothing.

Application to the curvature of the Earth.—The radius of the earth, for example, being more than six million *metres*, its great circle is a million times less curved than a circle with a radius of six *metres*, and eight million times less curved than such a circle as the wheel of a carriage.* Its curvature, therefore, is not apparent to us in short spaces, and only becomes perceptible when at sea, or on vast plains.

A knowledge of the curvature of the earth enables us to measure, by approximation, the height of mountains and of the sea-coast, when we know also the distance of these from the point where we are at the moment.

Let AB, fig. 2, be the radius of the earth, and CD the mountain, the summit of which, D, begins to disappear from the eye of a voyager who has proceeded from D and arrived at B. If we know the distance BC, drawing the radius ACD, we can immediately, measure CD. When the angle BAC is very small, the arc BC is very nearly equal to a perpendicular let fall from B on AD. We shall then have, very nearly,

$$AB : BC :: BC : CD;$$

that is to say, that the radius of the earth is to the distance BC of the mountain from the spot where the voyager is, as this distance is to CD, the height of the mountain; consequently, $CD = \frac{BC^2}{AB.}$

Seamen, when they know the height CD of the masts, the hull, or any other part of a vessel, judge of the distance she is from them by an inverse method, which, during war, is of very great importance.

We have just seen that the radius of different circles

* The measures of the original have been preserved in the translation, on account of the proportion involved in them. The actual length of the radius of the earth is 7,290,901 yards. See page 51.
Tr.

gives us a measure of their curvature, and we shall now see that it also enables us to measure the curvature of all other curves.

One of the happiest conceptions in geometry, is that of measuring the curvature of curve lines by right lines. Such a conception simplifies extremely all our operations relative to this curvature.

Let AA′A″Z, fig. 3, be any curve, the curvature of which we desire to know. We take three by three the points of it near one another; and through three consecutive points A, A′, A″, we describe a circle ABC, which will have the same curvature as the curve AZ, in the very small arc AA′A″. We can do the same for every other point, and thus determine the circles which have the same curvature as the curve at its different points, and of course we can ascertain the radii of these circles.

The circle ABC, which in one point A has the same curvature as any curve AZ, is called the *osculator* circle of this curve: the radius AO, of the circle, is the radius of curvature, as the centre of the circle is the centre of curvature.

The radius being at right angles to the circumference of the circle in A, and the circumference of the circle in A, A′, and A″, being the same both for the circle and the curve, it follows that the radius of the curvature is at right angles, or *normal* to the curve, the curvature of which it measures.

Let us suppose that from the different points AA′A″, fig. 4, situated close to one another, perpendiculars or normals to the curve AZ have been drawn, and that we have ascertained the length AO of the radius of the curvature at A; the length AO′ of the radius of the curvature at A′, the length AO″ of the radius of the curvature at A″, &c.; we shall then have, the points A, A′, being on the arc of the circle of which O is the centre, OA = OA′, and for the same reason, O′OA = O′A″, O″O′A = O″A‴,...

Fasten, at the point A, the end of a thread of invariable length, and extend it in the direction AO, and in the direction marked by the points O, O′, O″,... which are the centres of the curvature of AZ. Now make the point A advance, keeping the thread extended, without allowing it to glide along OO′O″: the part AO of the thread will describe a small arc of the circle AA′, which will be entirely on the curve AZ, for its centre is the centre of the curvature O of AZ, proceeding from the point A.

When we reach A′, the thread will be extended in a right line from A′ to O′. When the point A advances to pass from A′ to A″, the thread, extended in a right line from O′, will describe the arc of a circle A′A″, of which O′ will be the centre. In the same manner, the point A, in passing from A″ to A‴, will describe an arc A″, A‴, the centre of which is at O″, &c.

When we know, therefore, a succession of points O, O′, O″,... very near one another, being the centres of curvature of the line AZ, we may, by means of a flexible thread, invariable in its length, very easily draw the curve AZ. This method will be more rigorously exact, in proportion as the centres O, O′, O″,...are at the least possible distance from one another. It will be perfectly exact if these points follow one another without any intervening space, and form a continued curve.

Though this method be only used as one of approximation, the curve AZ, will be represented with much more exactness and continuity, than by substituting for it a polygon, formed by the chords or the tangents to the curve. With the new curve, all the arcs of the circle substituted for the curve AZ, agree longitudinally; there are no longer any angles, as at the summits of polygons, nor straight sides supplying the place of portions of the curve.

We must, therefore, employ this latter method to produce a form approaching to curves, which cannot be exactly executed, whenever the continuity of the curvature is a matter of great importance.

We have seen that the thread AOO′O″,...will keep extended while the point A, its extremity, describes the curve AZ. If we examine the curve OPQ....X which is described by the point originally marked on the thread, we shall see that XO‴ equals the whole length of the portion of thread originally bent in the direction OO′O″ .... O‴.

The curve OPQX, which serves to develope the curve OO′O″....O‴, is said to be *developing*, and the latter curve is *developed*, so that its length is every where equal to the radius of curvature OO′, PO″, QO‴, .. XO‴, of the curve OPQX.

In the arts, the developing curves, and particularly the developing curve of the circle, fig. 5, are of great utility. The engineer employs the latter to form *cams* in a proper manner.

Let us suppose that a pile, AB, figs. 6, 7, 8, pl. 15, is so placed in a groove, that it can only move upwards and downwards in some determined vertical direction; this motion upwards and downwards is what the engineer has to produce.

For this purpose, a horizontal cylindrical beam C, is so placed, that it cuts tangentially, on a short projecting *chin* or plate DE, the under part of which is situated in a right line with the centre of the beam C, when the pile-driver has descended to the lowest point, fig. 6.

On the circumference of the beam, an arc of a circle is fixed, OPQR, being the developing curve of the circumference OO′O″O‴, of the circle, which serves as the base of the beam.

When this beam revolves, the point O′ first reaches the position occupied by O, and then the tangent O′P of the circle, becomes vertical, fig. 7. The projecting piece ED, carrying with it the pile-driver, must, therefore, at this point, have moved upwards one height = O′P. The beam continuing to revolve, O″ will reach the original position of O, and then the projecting chin and the pile-driver, will be raised from O to Q. Finally, the beam continuing to revolve, O‴ will reach the original position of O, fig. 8, and O‴R will be vertical. The projecting piece D, being no longer supported, does not prevent the descent of the pile-driver, which accordingly falls freely by its own weight, and remains at rest, till the *cam* having finished its revolution on the beam, returns to raise the pile-driver again.

This movement has the great advantage of being unattended by jerks, and, as will be explained in treating of mechanics, is executed without any loss of power.

In the Thirteenth Lesson, we have examined the ellipsis, which is a very important curve. This curve ABC, fig. 9, being symmetrical in relation to its axes, its *develope* DEF, is, in relation to them, also symmetrical. The greatest curvature of the ellipsis is at the extremity of its longer axis, and its least curvature is at the extremity of its shorter axis.

If we had occasion to construct a large ellipsis, fig. 9, possessing great continuity, we might draw the *develope* DEF, and describe the curve ABC, by means of a thread and a cord bent in the direction, sometimes of DE, and sometimes of EF.

It is of consequence to remark, that even if we take a polygon for DEF, that is to say, a succession of lines forming angles, the curve ABC, will not be rectilineal in any one part, nor will it have any angles. It will, therefore, have two elements of continuity, which are not possessed by DEF. The curve of which ABC shall be the developing curve, will have still more continuity; for its radii of curvature, augment or diminish by insensible degrees, even when the radii of the curvature ABC, follow each other without continuity, as in the construction of the curve called basket handle (*anse de panier*). See Lesson Fourth, fig. 36, pl. 4.

Now you are aware that there are different species of continuity, of which it is essential that you should distinctly classify the gradations in your minds.

1st. We can represent a curve by a succession of isolated points very close to one another, fig. 10, pl. 15. Such are dotted lines in drawing; such are the lines marked by a row of trees planted at a greater or less distance from one another, either in a curve or in a right line, which the eye easily imagines when these curves have any continuity, though in these cases the continuity is only indicated by isolated points. The same is the case in those plans, such for example as the plan of a ship's bottom, which designate, by numerals, the position of a certain number of points for each curve of the bottom.

2d. We can represent a curve by a series of right lines which are chords to it; AA′, A′A″, A″A‴,... fig. 11, or which are tangents to it, AA′A″,... fig. 12. By this latter method there is a continuity in the succession of points, but not in the direction from one another; for at each summit A′A″A‴,... of the polygon, the direction is suddenly changed.

3d. We can substitute for the curve, a series of arcs of circles AA′, A A″, A″A‴,... fig. 4, having very nearly the same radius of curvature as the line which they represent; there is then continuity in the succession of points and in their direction. If the arcs are very small, we have continuity both in the direction and in the curvature of the curve. In this manner, architects, as already mentioned, draw the profile of elliptical arches, and engineers draw the arches of bridges which are not circular.

In the arts there is a necessity, according to the importance of the operation proposed, and degree of exactness required for its success, to use these different degrees of continuity, both in constructing objects and in giving motion; but it is for those who direct these operations, to decide which of the above methods unites most advantages for them; which is the most simple of execution, while it is sufficiently exact.

Shipwrights make use of a mechanical method, when they desire to give great continuity of direction and of curvature to the lines, by the help of which they determine the form of a ship's bottom and afterwards construct it, which we will explain. They first mark the isolated points through which the curve is to pass; on each side of these points they drive nails, so that a thin ruler or batten, bent into the form prescribed by the situation of the nails, may be held between them. A curve, drawn by means of chalk or pencil, along the bent batten, passes of course through all the points A, A′, A″,... fig. 13. A great degree of nicety is necessary in performing this operation, in order that the curvature of the line may be made from one end to the other, by insensible deviations, and possess that degree of continuity which contributes to lessen the resistance of the water when the ship moves through it. Shipwrights will find, in this respect, a great advantage from studying geometrical forms. Such study will enable them both to judge of the beauty of curves with more promptness and certainty, and form them better.

The means proper for drawing large curves, cannot be used for small designs drawn on paper; and in place of using long wooden battens, small slips of whalebone are employed. Some of them, of an equal thickness throughout, serve to draw curves, the curvature of which only varies a small quantity; others, gradually thinned towards one or both ends, serve to draw parts of a curve, or any curve, diminishing gradually like them from one extremity to the other. The slips of whalebone are bent, so that they pass through the points indicated on the plane as belonging to the curve to be made, which is drawn by a pencil resting against the curved whalebone. Pieces of lead P, P′, P″,...fig. 14, pl. 15, covered with paper or cloth, and having the form of a triangle, that they may be used with greater ease, supply on the paper the place of the nails used in larger plans, such as those of the draftsmen in the mould loft.

In order to make the curves pass through the given points, draftsmen very often employ an instrument which they call a *pistolet*, fig. 15. As its curves are much varied, it may generally be so placed as to draw by degrees a figure which offers no angle, and of which the curves are at no point sharply interrupted.

Hitherto we have only spoken of curved lines drawn on a plane surface; or lines, as they are called, of a single curvature.

But some lines cannot be drawn on a plane, because they are curved in two directions; such are spirals traced on cones, on cylinders, &c.

For lines of double curvature, as well as for those of single curvature, we can always take the points immediately consecutive of which they consist, three by three, and through these three points draw a circle, which will be the osculator circle of the curve, in the extent of the small portion of the curve comprised within the three points. The plane of the osculator circle, is called the *osculator plane* of the curve. No other, proceeding from the portion marked by the three points, can approach nearer to the curve of double curvature. By means of osculator planes and circles, employing also a series of arcs, which agree with each other tangentially, we may practically and by approximation, but with great continuity, draw every species of curve with doublè curvature.

Relative to such curves, there are many interesting considerations which might be developed; but they are not sufficiently elementary, nor are they susceptible of such immediate and frequent applications in the ordinary business of life, that they can be noticed here in the manner they deserve. The curvature of surfaces is, on the contrary, perpetually a subject of consideration, and indispensable in many operations.

Curvature of the Sphere.—The sphere is the surface, of which the curvature is the most easy to measure and determine. Let us take any point A, fig. 16, of the sphere, and draw from the centre O, the radius AO. This radius will measure the curvature in A, of all the sections made in the sphere, by a plane which contains the radius AO. It will also measure the curvature of the sphere, which is the same in all directions, and at every point of the surface. The radius of the sphere is, therefore, at all times the radius of its curvature, and that of all the sections made by a plane in which this radius is situated.

The right lined circular cylinder, considered in the direction of its base, has for its radius of curvature the radius of the sphere which it envelopes, and which it touches, according to the circumference of its base. But considered in the direction of the edge AB, fig. 17, it has no curvature, and if it were required to find the length of the radius of the osculator circle of the cylinder in the direction of its length, we should see that the radius must be infinite.

The same holds good of the right lined circular cone. In the direction of its base, its radius of curvature is the radius of the sphere which it envelopes; in the direction of its edge, the curvature of the line is *nil*.

Other species of cylinders and cones, and in general all developable surfaces, have no curvature in the direction of their rectilinear edges, while in the direction *of their base*, or perpendicular to the edges, they have a curvature more or less marked.

In the cylinder and cone, the sections made according to a radius AO, of the base, figs. 17 and 18, always have their centre of curvature within the surface. Thus, throughout the whole extent of the same edge AA′A″,... B of the conical and cylindrical surfaces, the radii of curvature AO, A′O′, A″O″, lie in the same direction, and are parallel.

The same circumstance does not hold good for non-developable surfaces. If, for example, we attend to the non-developable surface of a staircase, we shall always find that in one direction, the curvature turns its concavity downwards, while in the direction perpendicular to this, it turns it upwards.

The hollow part or *throat* of a shiver in which the rope lies, fig. 19, pl. 15, has its least curvature in a direction at right angles to the axis of the shiver, and its centre of least curvature placed on the same axis; while in the direction parallel to the axis, the *throat* has its centre of greatest curvature in some point *n*, equi-distant from *m* and *p*, the edges of the throat.

Here then are three classes of surfaces very distinct from one another, when regarded in relation to their curvature.

In the first class, the curvature of the lines which can be traced on each surface, all lie in the same direction; this class includes the sphere, the ellipsoids, the surface of an egg, of a chesnut, of the cocoon of the silk-worm, &c.

In the second class there is only one direction in which the curvatures are marked, and in the other, they are nothing. This class only includes developable surfaces, cylinders, cones, &c.

In the third class, one part of the curvature lies in one direction, and another part in the opposite direction; so that if we draw, from any given point of the surface, the normal to it, one part of the centres of curvature of the sections is placed on this normal, on one side of the surface, and the other part is placed on the other side.

The varied surface of the human body presents all the three classes of surfaces. To the first class belong the projecting forms of the extremities; the heel, the knee-pan, the knee, the shoulder, the ends of the fingers; all the double curvatures of which lie in one direction.

A part of the thighs, of the legs and arms, has no curvature in one direction, and they belong to the second class of surfaces.

In all the joints of the arms, of the fingers, in the connection of the head and the body, with the neck, &c. we see surfaces of the third description; that is to say, having two curvatures in opposite directions.

Amidst these general forms, the practised eye of the sculptor and of the painter, discovers a multitude of shadowings into one another, in the succession and gradation of the curvatures of the different parts of the body. As he represents these nicer connexions correctly or not, or as he reproduces them with more or less fidelity, he forms those *chefs-d'œuvre*, the truth of which is the admiration of enlightened judges; or he shapes out those rude sketches, the gross negligence of which are offensive to the eye of a man of taste.

The curvature of the various parts of the surface of our body, depends on the form of the bones and of the muscles, to which the skin is a smooth, polished, and continuous covering. The intelligent artist should always take care, therefore, that his representations reveal those hidden forms beneath the skin, which shadow out themselves on its surface.

The works of some artists are very faulty, by making some parts of the body appear too projecting, too much curved, or too much swelled, in order to designate more distinctly these hidden anatomical forms, even when they ought not to attract the notice of the eye. This affectation is only a species of charlatanism, quite unworthy of the great masters.

The surface of our face is endowed with a most valuable property of changing its form, in close connexion with the changes in our momentary or permanent passions or affections; such changes being intended to reveal to the eye of others, the feelings that are passing within us. Permanent affections, or long and frequently indulged passions, give some particular curvature to the flexible parts, and even to the aspect of the fixed parts, of which attentive observation teaches us to recognise the slightest difference. On this is founded the science of physiognomy.

Even our momentary passions produce changes of form more or less distinct, more or less fleeting, and the study of them also is of great importance for the cultivation of the fine arts; they offer an infinite variety, among which the man of genius selects with unerring precision, those forms which accord best with the graceful, severe, profound, or terrible character of his works.

There is one study recently introduced, of which I have yet to speak. I mean the relation which is thought to exist between the exterior form of the head and the capacities of the mind. Besides a general regularity observable in the two principal curves of the cranium, inflections and local variations, more or less remarkable, are observed in different individuals.

The parts more or less curved, more or less prominent, have been called organs or *bumps*, and are considered to be external signs of our faculties and our affections, which are thought to be strong and intense, or weak and diffusive, as the organs are more or less conspicuously developed.

It is easy to throw an air of ridicule and contempt on such doctrines, particularly when they are promulgated with a pedantic display of a new scientific nomenclature; but the careful observer of the laws of nature, is never in haste to scatter censure or bestow profuse praise, when important doctrines or novel principles are in question. Even if it should be true that the desire of explaining every thing, should have caused a too extensive enumeration of supposed indications of our affections and passions, it would be sufficient that a small number of intellectual powers were indicated, with more or less certainty, by the form of the cranium, in order to make the varieties of its curves, highly worthy of occupying the meditation of the wise.

The various parts which compose the bodies of animals, are formed either straight or curved, and have such a volume as adapts them to certain motions and certain modes of life. Comparative anatomy is the name of the somewhat recent science, which has for its object to observe the connexion between the forms of different animals, and their habits; and it will receive salutary precision, as well as give accuracy to its results, by referring to geometrical measurements, not only the principal dimensions of each part of the bony skeleton, but also of the size and the direction of the curvature of each element of the skeleton, particularly the parts in contact, or the joints.

To carry geometrical precision and description into comparative anatomy might not only benefit that science, but might also lead to results useful to man. Animals in order to satisfy their wants, execute, with singular perfection, many operations similar to some practised in the arts, though we have not yet carried them beyond mediocrity; and in the means provided by nature for the actions of these animals, we might find many varied and ingenious models.

Herbivorous animals have their teeth perfectly suited for crushing vegetable food, and notwithstanding they wear away by continual use, they always preserve the same form: our mill-stones, however, which perform a similar office, are speedily altered, and they must be continually re-cut, in order to grind well. In this particular case, therefore, art is much inferior to nature. Under the influence of this idea, M. Molard, a member of the Institute, has endeavoured to construct machines for grinding, after the model of the grinding teeth of horses, in which there is no longer any necessity to re-cut the stones, in order that they may continue to perform the grinding properly.

Industry, therefore, is interested, that anatomists, geometricians, and mechanicians, should ascertain in concert, the curvatures and the functions of the various parts of animals.

We must now, however, pass from these general considerations, on the importance of the study of curved surfaces, both to the arts and natural history, to consider those geometrical characters which are adapted to give by some simple means, the elements and the varieties of these curvatures.

For surfaces of the first class we can always trace an ellipsis, projected parallel to its plane ABCD, fig. 20, which ellipsis will represent, proceeding from any point P, the form of a trench of the surface, made parallel to the plane *mn*, tangent to the surface in P, and very close to MN: PO being the distance of the point P, from the intersecting plane MN, if we draw a series of circles through P, and through the circumference of the ellipsis, having their centres on the normal or perpendicular PO, we shall have all the osculator circles of the sections made in the surface by the planes of these circles.

The smallest of these circles will pass through the summits B, D, of the small axis of the ellipsis; the largest will pass through the summits A, C, of the great axis of the ellipsis. Fig. 20 *double*, pl. 15, represents all the circles reduced to the same plane, which pass through the normal PO*p*, of fig. 20.

In the surfaces of the first class, therefore, all of which have their curvature in the same direction, that of the largest curvature AB, is perpendicular to the direction of the least curvature CD.

For all surfaces, therefore, of which the double curvatures are in the same direction proceeding from each point, the direction of the largest curvature is perpendicular to the direction of the least.

The outline of the ellipsis being symmetrical in relation to its two axes, the osculator circles which pass through this outline, and through the perpendicular or normal PO*p*, will also be symmetrical in relation to the axes AC, BD, that is to say, in relation to the two directions of the largest and least curvature.

Thus, the intermediate curvatures of the sections perpendicular to the surface,—curvatures which proceed by a continued gradation from the least to the largest, are disposed symmetrically, in relation to the direction of the largest and least curvature, proceeding from each point of the surface.

A plane which divides the surfaces of the third class, infinitely near to the tangent plane, gives a section, of which the form is that of an hyperbola. The direction of the axes of this hyperbola, gives the direction of the two axes of largest and least curvature. The intermediate curves are symmetrically disposed in relation to these axes. In fig. 21, pl. 15, the sections made in the hollow rim of a shiver, of which the two curves lie in contrary directions, by two planes very near the plane MN, tangent to the gorge of the shiver in P, are represented. These sections possess the form of two *indicating hyperbolas.* If the teacher is provided with this figure to exhibit, so much the better.

Surfaces of the second class may be considered as the common limit of the two other classes. They share the properties belonging to the other surfaces,—having their largest and least curvature perpendicular to each other, with all the intermediate curves symmetrically disposed in relation to the principal curves.

I have given the name of *indicators* to those curves which possess the property of *indicating* the nature and the relations of the curvature of surfaces, and I have given an account of all the means of using them, in order to discover the essential properties of the curvature of surfaces. I can only refer, for further information, to my investigations on this subject, given in my *Developpemens de Géométrie,* and to the applications I have made of them, (*Applications de Géométrie*), to the stability of floating bodies, to the construction of vessels, to levelling roads, and to the optical phenomena produced by the reflection of pencils of light on any kind of curved mirrors.

Let us now suppose, that proceeding from any point of a surface, we always advance in the direction of its greatest curvature, we shall trace a line; and all the lines thus traced, will cover the whole surface, forming *the system* of lines of greatest curvature.

On the other hand, if we advance, proceeding from any point in the direction of the least curve, we shall form another line, and all the lines thus drawn, will also cover the whole surface. They form the system of lines of least curvature.

Finally, the lines of the greatest curvature will be at every point perpendicular to the lines of least curvature.

The lines of curvature have one property valuable for the arts, which I shall content myself with merely indicating. If we draw from every point of the same line of curvature, a line perpendicular to the surface, all these perpendiculars together will form a surface, which will necessarily be developable.*

In the cylinder, fig. 22, the lines of least curvature are the rectilinear edges, of which the curvature is *zero*. The lines of greatest curvature are the sections, made in the planes at right angles to the axis, and the outline of these sections is evidently at right angles with each edge. In the cylinder, therefore, the lines of greatest, and those of least curvature, are at right angles.

In the cone, fig. 23, in which also the edges are the lines of least curvature, the lines of the greatest curvature are found as follows. The point of one leg of a pair of compasses is placed on the summit of the cone, and various curves are drawn with the other point, at different openings of the compasses, but all at right angles with the edges; for in developing the cone these curves become circles, of which the edges of the cone will be the radii.

In surfaces of revolution the meridians are lines of one curvature, and the parallels are lines of the other curvature, the former being, as we know, every where at right angles with the parallels.

The celebrated Monge applied in a very beautiful manner these properties in stone-masonry.

When arches, having curved surfaces, are to be constructed, they are divided into such small compartments that each of them may be formed of one stone.

* The demonstration of this principle, and of all the properties relative to the curvature of lines and surfaces, is contained in my *Developpemens de Géométrie*.

After having prepared that part of the stone which is to occupy one of these compartments, and having given it the shape proper for the surface of the arch, (the *intrados*) the sides or faces called the joints of the arch stones, which are to meet one another, are then made of the proper form. To adapt these stones in the best manner possible to all the purposes required of them, 1st, the faces of the joints should have a simple form easily made; 2d, the whole of the stones when united, should have the greatest possible solidity. The latter circumstance requires that the faces of the joints should be every where at right angles to the curve of the arch; and it is easy to see for what reason. If the face of any one stone made an obtuse angle with the curve of the arch, the face of the adjacent stone would make an acute angle with it; and when any pressure or weight came on the arch, the stone terminated by the obtuse-angled edge, would break the one terminated by the acute angle, or would make it start out. The former circumstance, giving facility of execution, requires that the faces should be planes, or at least developable surfaces. When this form is adopted, a piece of paper, pasteboard, or other flexible substance, may be readily cut into the shape which the joint must have; and it will enable the artisan, merely by bending it properly, to see if it is every where adjusted to the joint, which may be made at right angles with the curve of the arch by means of the common square.

But these conditions require that we should find at every point, some developable surfaces at right angles with the arch and with one another; they also require that we should select for the faces of the joinings on the surface of the arch, the lines of its curvature.

When we construct cylindrical surfaces, therefore, fig. 24, we select for the joints in one direction, their parallel edges at equal distances; they are the lines of least curvature: in the other direction, we select the curves at right angles with these edges; they are the lines of the greatest curvature. The surfaces of the joints formed by the *normals* to the surface, according to these edges and these curves, are planes, which intersect one another at right angles. In this case the labour of the stone-mason becomes as simple as possible.

When conical surfaces are to be constructed, fig. 25, such as those for arched doors and windows, wider at one side of the wall than at the other, and such as those for embrasures, in casemates, the edges of the cone, and the curves perpendicular to them, are in like manner chosen for the lines of the joints.

When an arch is to be made, having the form of a surface of revolution, fig. 26, a dome for example, regular compartments, formed by meridians and parallels, are drawn on the surface. The lines at right angles with the arch, in the direction of the meridians, form planes, which are the vertical joinings of the arch stones. The lines at right angles with the arch, in the direction of the parallels, form cones, which are the horizontal joinings of the stones, and these joints are developable, because they correspond to the lines of curvature. Finally, the conical joints are intersected at right angles by the plane joints, which, for cones, are meridian planes.

I shall not extend farther this magnificent application, so simple, so general, so fruitful in its principle and in its consequences. It is well adapted, however, to convince us of the importance to the arts, of the study of curved surfaces, and their chief properties. The fine arts may also derive great advantage from the study of them.

By the varied effects of light and shade, we form correct notions, at a single glance, not only of the illuminated or projecting points, and of the well-defined edges of objects, but also of those apparent outlines, which give a distinct character and figure to every object. Delicate minglings of light and shade, more or less strong, enable us, even in those parts where neither point nor line is particularly remarkable, to distinguish the form and species of objects, and the degree of curvature which belongs to each part of their surface.

The study of this part of the science is useful to men of all professions, as well as to artists; it enables them rapidly to form just and complete ideas of the true form of those objects which they consider, whether it be for business or amusement.

Let us examine the process by which the eye conveys to us the idea of curved surfaces; and for this purpose, let us suppose that a sphere ABC, fig. 27, is illuminated in a certain direction by the sun's rays.

I begin by drawing the line of separation between the light and shade, LLL, according to the principles laid down in the Fourteenth Lesson. I mark with dark strokes the part in the shade, and there remains illuminated only the part LLLBC, fig. 27. Thus the moon appears to us in its various phases, from the smallest illuminated crescent, fig. 29, pl. 15, to the first quarter, fig. 28, when one half of the moon is illuminated, and the other is in the shade; and afterwards, as it approaches towards the full moon, in the form seen at fig. 27, it being, when full, wholly illuminated; waning again towards the close, when, as in a total eclipse, no part of it is illuminated for the spectator. If I consider only the illuminated part LLLB, without the shadowings, there is nothing to inform me that this part belongs to a sphere, rather than to a surface depressed or projecting in the direction of the visual ray. Let us see how we judge of this difference.

There is on the surface, supposed to be as brilliant as a mirror, a certain point, O, fig. 27, whence the spectator would perceive the image of the sun or the luminous body. This is the point, where the light reflected by the surface is most brilliant, and it is called the *brilliant* point. We must determine its position.

This may be easily done, if we can draw the normal in O to the surface of the object. For, 1st, the two rays, the incident and the reflected, are in the same plane as this normal; 2d, they make the same angle with it. According to these conditions, descriptive geometry teaches us the means of finding the *brilliant* points of various surfaces, for any given position of the eye, and for the general direction of the rays of light. In proportion as these rays reach the surface and are also reflected under a more oblique angle, there is more light lost, and the surface appears less luminous. Around the point O, it may be conceived that we can trace a series of lines, through the whole length of which the object would appear to the spectator equally illuminated; such lines are said to be of equal *tints*. When they are traced, it will be sufficient to apply a series of *tints*, stronger or weaker according to the degree of light which corresponds to each line, and we shall paint, with great exactness, the gradual diminution of the light on the portion of the illuminated surface.

These lines, by their form and position, will enable us to distinguish perfectly the nature of the surface and its curvatures. They will have for the cylinder, for the cone, and in general for all developable surfaces, a peculiar character easily distinguished; for the sphere, for surfaces of revolution, and for annular surfaces, they will have another character; and they will have a different character for spiral surfaces, and for non-developable surfaces.

Though the lines to which we have just alluded are not visible on objects, and though nature produces deviations in tints by insensible and infinite gradations, the eye is not the less accustomed to study them, to perceive their differences, and to recognise those general and characteristic forms possessed by the infinite varieties of light and shade, which belong to different species of surfaces.

In this respect, however, a great difference is observed among different classes of men, as their professions lead them to pay attention to particular species of surfaces. Thus, the coppersmith, the tinsmith, and the cooper, distinguish with great facility, if surfaces, or the fractions of surfaces, are cylindrical, or conical, or in general developable; but they are not so capable of distinguishing other forms.

The turner, whether he work in wood or metal, and the potter, as well as all those persons who continually make surfaces of revolution, will distinguish by a single glance, and without touching an object, if a surface, or any portion of a surface, is a surface of revolution, or if it is flattened or elongated in any part; but they are less skilful in ascertaining other forms.

The architect will form a correct opinion of the varied form of cylinders and cones, similar to those which occur in buildings, as well as of surfaces of revolution, such as domes and columns; but he will not be so capable of distinguishing accurately the form of such surfaces as he seldom meets with in his peculiar pursuits.

It is of some consequence that all classes of the people should be accustomed to ascertain, by a single look, the chateristics of surfaces, and to decide whether or not they are made correctly. Such a study would be one means of promoting the progress of industry and of the fine arts. This subject, however, will be treated in detail, when an explanation is given in the Third Volume, On Motive Powers, of the studies and observations which may add to the acuteness of our senses, and multiply and enlarge the means which they supply for directing our labours.

Sculptors ought to accustom themselves to distinguish, by sight alone, whether the two curves at every part of a surface which they have to reproduce, lie in the same or in opposite directions,—to distinguish which is the direction of the greatest, and which of the least curvature,—to follow the direction of both these curvatures on surfaces, in order to reproduce the general character of those surfaces which they conceive or copy, which alone can give the character of truth to their productions.

The painter, who is to represent, by colours and tints on a surface which has only two dimensions, the *relief* of objects of three dimensions, ought to study profoundly the manner in which each species of surface influences the various tints, in order to reproduce similar effects with his pencil.

The engineer and draftsman must engage in the same studies, to attain the same fidelity of imitation, and the same truth of conception.

INDEX

THE END.

LONDON:
PRINTED BY S. AND R. BENTLEY, DORSET STREET.

SCENES and SKETCHES of a SOLDIER'S LIFE in IRELAND. By the Author of the "Eventful Life," &c. 12mo, 5s.

This little work is one of uncommon interest. It is really the production of a Soldier: one who like the Soldier of the 71st Regiment, whose Journal was so very favourably received, can select with judgment the most remarkable scenes and occurrences that happened to fall within his observation, in the course of a varied and adventurous life, and paint them naturally and vividly. The character and feelings of the British Soldiers, and of the Irish Peasantry, both Catholics and Protestants, are in this little narrative pourtrayed to the life, by an intelligent and candid observer.

LECTURES ON THE PHILOSOPHY OF THE HUMAN MIND. By the late THOMAS BROWN, M.D., Professor of Moral Philosophy in the University of Edinburgh. Second Edition, corrected. In four vols. 8vo, 2*l*. 12*s*. 6*d*.

WORKS of PROFESSOR LESLIE, viz.:—ELEMENTS OF NATURAL PHILOSOPHY. Vol. I., including Mechanics and Hydrostatics. In 8vo, with Engravings, 14s.

GEOMETRICAL ANALYSIS, AND THE GEOMETRY OF CURVE LINES. In 8vo, with 24 Engravings, and numerous Cuts, 16s.

THE ELEMENTS OF GEOMETRY AND PLANE TRIGONOMETRY. Fourth Edition. 8vo, 10s. 6d.

THE PHILOSOPHY OF ARITHMETIC; exhibiting a progressive View of the Theory and Practice of Calculation. Second Edition, improved and enlarged. 8vo. 9s.

EULER'S LETTERS to a GERMAN PRINCESS, on different subjects in Natural Philosophy. A new Edition, with Life and Notes, by DAVID BREWSTER, LL.D., Sec. R.S.E. &c. &c. In two vols. 12mo. with Plates, 16s.

In this Edition the translation has received very essential improvements. The plates have been re-engraved, and much improved; and a Life of the Author has been added, together with various Notes, which the Editor trusts will be both interesting and useful to the reader.—*Preface.*

SUPPLEMENT to THE ETYMOLOGICAL DICTIONARY of the SCOTTISH LANGUAGE. By JOHN JAMIESON, D.D., Fellow of the Royal Society of Edinburgh,—of the Society of the Antiquaries of Scotland—of the American Antiquarian Society,—and Associate of the First Class on the Royal Foundation of the Royal Society of Literature. In two vols. 4to, 5*l*. 5*s*.

WATT'S BIBLIOTHECA BRITANNICA, or General Index to British and Foreign Literature. In 2 parts—Authors and Subjects. In 4 large and very closely-printed vols. 4to.

*** This is unquestionably one of the most intrinsically valuable books that have been published in this country for many years. It is both a COMPLETE CATALOGUE OF THE WORKS OF EVERY AUTHOR, and a KEY TO ALL THAT HAS BEEN WRITTEN ON EVERY SUBJECT.

The work was originally published in Eleven Parts, at 1*l*. 1*s*. each, and was completed only two years ago. The remainder of the impression having been purchased by W. TAIT, Edinburgh, and C. TAIT, London, Copies may be had of them for a limited time, at 6*l*. 6*s*. Above 900 copies have already been sold.

Mr. Dibdin, in speaking of Watt's book, says, "the uses and advantage of that wonderful work are manifest and indispensable; and it should never fail to be a library companion in all collections of extent or importance."

A COMPARATIVE VIEW OF CHRISTIANITY, with all the other FORMS of RELIGION which have existed, particularly in regard to their moral tendency. By WILLIAM LAURENCE BROWN, D.D. Principal of Marischal College, Aberdeen, &c. &c. 2 vols. 8vo. 18s.

*** For a most favourable review of this book, see the *Evangelical Magazine* for September.

STORER'S VIEWS IN EDINBURGH; comprising above ONE HUNDRED VIEWS of the Public Buildings, Principal Streets, and Picturesque Antiquities of Edinburgh, and of the romantic Scenery in the neighbourhood, with descriptive Letter-press and a History of the City. In 2 vols. 8vo. 1*l*. 5*s*. originally published at 3*l*. 2*s*. 6*d*. Copies on LARGE PAPER, with PROOF Impressions, published at 5*l*. may be had for 2*l*. 2*s*.; and TWELVE COPIES, with PROOFS, on INDIA PAPER, at 4*l*. 4*s*., instead of 10*l*. the original price.

Storer's Views of Edinburgh are not more desirable for their beautiful execution than for their uncommon fidelity of represent[illegible]