29 November 2018

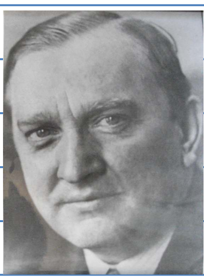
Review $(X, \|\cdot\|)$ a normed space.

Cauchy

Def. $x_n \to x$ if $\forall \varepsilon > 0, \exists N(\varepsilon) < \infty$ such that $n \geq N \Rightarrow \|x_n - x\| < \varepsilon$.

Def. (x_n) is Cauchy if $\forall \varepsilon > 0, \exists N(\varepsilon) < \infty$ such that $m, n \geq N \Rightarrow \|x_n - x_m\| < \varepsilon$.

Important $x_n \to x \Rightarrow (x_n)$ Cauchy, but in general the Converse is False!



Banach

Def. (a) $(X, \|\cdot\|)$ is complete if (x_n) Cauchy $\Rightarrow \exists\, x \in X$ s.t. $x_n \to x$.

(b) More generally, $S \subset X$ is complete if (x_n) Cauchy, $x_n \in S\ \forall n \geq 1$, $\Rightarrow \exists\, x \in S$ such that $x_n \to x$.

Need to Know

Proposition $(X, \|\cdot\|)$ a normed space

(a) If $S \subset X$ is complete, then S is closed.

(b) If $(X, \|\cdot\|)$ is complete and $S \subset X$ is closed, then S is complete.

(c) All finite-dimensional subspaces of X are complete.

$(C[a,b], \|\cdot\|_1)$ is not complete.

$(C[a,b], \|\cdot\|_\infty)$ is complete.

Fact (Not on Final Exam) Every normed space $(X, \|\cdot\|_X)$ has a "completion", that is a complete normed space $(Y, \|\cdot\|_Y)$ such that

a) $X \subset Y$, $\forall x \in X$, $\|x\|_Y = \|x\|_X$

b) $\overline{X} = Y$ (closure of X in Y)

c) $Y = X + \{$limit points of Cauchy seq.'s in $X\}$

Also not on Final Exam: How to prove that all finite-dimensional $(X, \|\cdot\|)$ are complete?

(a) Learn enough about the real numbers $\mathbb{R}$ to prove that $(\mathbb{R}, |\cdot|)$ is complete. ("Dedekind cuts")

(b) Let $\{v^1, \ldots, v^k\}$ be a basis for X, and let (x_n) be a Cauchy sequence. $\forall n \geq 1$, write $x_n = \alpha_n^1 v^1 + \alpha_n^2 v^2 + \cdots + \alpha_n^k v^k$.

Claim (x_n) is Cauchy in $(X, \|\cdot\|)$ if, and only if, $\forall\, 1 \leq i \leq k$, (α_n^i) is a Cauchy sequence in $(\mathbb{R}, |\cdot|)$. $\square$

(x_n) Cauchy $\iff$ its representation is Cauchy

$$X \xleftrightarrow[x = \alpha^1 v^1 + \cdots + \alpha^k v^k]{[x]_{\{v\}} = \begin{bmatrix} \alpha^1 \\ \alpha^2 \\ \vdots \\ \alpha^k \end{bmatrix}} \mathbb{R}^n$$

Part (a) is covered in Math 451 and (b) is covered in EECS 600.

Rob 501 Handout: Grizzle
A Useful Cauchy Sequence in $(\mathbb{R}, |\cdot|)$

Proposition Let $0 \le c < 1$ and let (a_n) be a sequence of real numbers satisfying, $\forall\ n \ge 1$,

$$|a_{n+1} - a_n| \le c|a_n - a_{n-1}|.$$

Then (a_n) is Cauchy in $(\mathbb{R}, |\cdot|)$.

Proof:

Claim 1: $\forall\ n \ge 1$, $|a_{n+1} - a_n| \le c^n |a_1 - a_0|$.

Proof: First observe that

$$|a_3 - a_2| \le c|a_2 - a_1| \le c^2 |a_1 - a_0|.$$

Then complete the proof is by induction.

Claim 2: $\forall\ n \ge 1, k \ge 1$, $|a_{n+k} - a_n| \le \frac{c^n}{1-c}|a_1 - a_0|$.

Proof:

$$\begin{aligned}
|a_{n+k} - a_n| &\le |a_{n+k} - a_{n+k-1} + a_{n+k-1} - a_{n+k-2} + \cdots a_{n+1} - a_n| \\
&\le |a_{n+k} - a_{n+k-1}| + |a_{n+k-1} - a_{n+k-2}| + \cdots |a_{n+1} - a_n| \\
&\le c^{n+k-1}|a_1 - a_0| + c^{n+k-2}|a_1 - a_0| + \cdots + c^n |a_1 - a_0| \\
&\le c^n \Big(\sum_{i=0}^{k-1} c^i \Big)\ |a_1 - a_0| \\
&\le c^n \Big(\sum_{i=0}^{\infty} c^i \Big)\ |a_1 - a_0| \\
&\le c^n \Big(\frac{1}{1-c} \Big)\ |a_1 - a_0| \\
&\le \frac{c^n}{1-c}\ |a_1 - a_0|
\end{aligned}$$

Claim 3: (a_n) is Cauchy

Proof: Consider m and n. WLOG, suppose $m \geq n$. If $m = n$, then $|a_m - a_n| = 0$. Thus assume $m = n + k$ for some $k \geq 1$. Then

$$|a_m - a_n| = |a_{n+k} - a_n| \leq \frac{c^n}{1-c}|a_1 - a_0| \xrightarrow[n\to\infty, m\to\infty]{0},$$

and thus it is Cauchy.

Remark: Because WLOG we could assume $m \geq n$, from $n \to \infty$, we have both $n \to \infty$ and $m \to \infty$.

Let $(X, \|\cdot\|)$ be a normed space.

<u>Def.</u> (a) Let $S \subset X$ be a subset and $T: S \to S$ a mapping (function). T is a <u>contraction mapping</u> if $\exists$ $0 \le c < 1$ such that, $\forall x, y \in S$,

$$\|T(x) - T(y)\| \le c\|x - y\|.$$

(b) $x^* \in X$ is a <u>fixed point</u> of T if

$$T(x^*) = x^*.$$

Contraction Mapping Theorem

<u>If</u> $T: S \to S$ is a contraction mapping on a complete subset S, then $\exists$ a unique $x^* \in X$ such that $T(x^*) = x^*$. Moreover, $\forall x_0 \in S$, the sequence $x_{k+1} = T(x_k)$, $k \ge 0$ is Cauchy and converges to x^*.

<u>Proof.</u> Let $x_0 \in S$, define $\forall n \geq 0$,

$x_{n+1} = T(x_n)$.

<u>Claim 1</u> (x_n) is Cauchy.

$$\begin{aligned} \|x_{n+1} - x_n\| &= \|T(x_n) - T(x_{n-1})\| \\ &\leq c\,\|x_n - x_{n-1}\| \end{aligned}$$

$\therefore$ From our proof that $|a_{n+1} - a_n| \leq c\,|a_n - a_{n-1}|$ is Cauchy, we deduce that (x_n) is Cauchy as well. $\square$

Because S is complete, $\exists\, x^* \in S$ such that $x_n \longrightarrow x^*$. Is it true that $T(x^*) = x^*$?

$$\begin{aligned} \|T(x^*) - x^*\| &= \|T(x^*) - x_n + x_n - x^*\| \\ &= \|T(x^*) - T(x_{n-1}) + x_n - x^*\| \\ &\leq \|T(x^*) - T(x_{n-1})\| + \|x_n - x^*\| \\ &\leq c\,\underbrace{\|x^* - x_{n-1}\|}_{\xrightarrow[n\to\infty]{} 0} + \underbrace{\|x_n - x^*\|}_{\xrightarrow[n\to\infty]{} 0} \end{aligned}$$

$\leq 0.$

$\therefore \; T(x^*) = x^*$.

Is it unique? Suppose $y^* \in S$ such that $T(y^*) = y^*$. Do we have $x^* = y^*$?

$$\|x^* - y^*\| = \|T(x^*) - T(y^*)\|$$
$$\leq c\,\|x^* - y^*\| \qquad 0 \leq c < 1$$

$\therefore \; \|x^* - y^*\| = 0 \qquad \therefore \; x^* = y^*$! $\square$

<u>Remarks</u> Claim 1 of (a_n) redone for (x_n)

$$\|x_{n+1} - x_n\| = \|T(x_n) - T(x_{n-1})\|$$
$$\leq c\,\|x_n - x_{n-1}\|$$
$$\leq c^2\,\|x_{n-1} - x_{n-2}\|$$
$$\vdots$$
$$\leq c^n\,\|x_1 - x_0\| .$$

Next Big Topic: How to guarantee that functions have a max or a min?

$f: S \to \mathbb{R}$, can we guarantee $\exists\, x_*$ and x^* such that

$$f(x_*) = \inf_{x \in S} f(x) \text{ and } f(x^*) = \sup_{x \in S} f(x) \text{ ??}$$

This will take us to

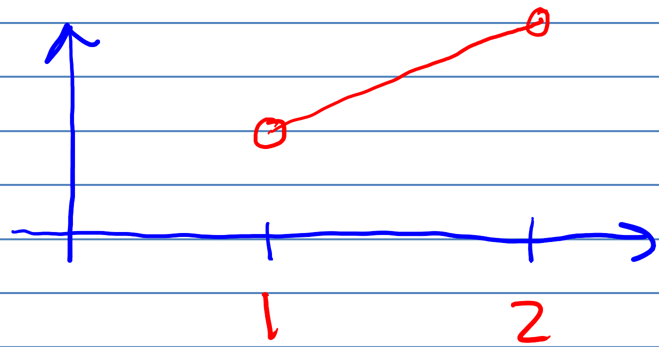
(a) continuous functions

(b) "compact sets", which in $\mathbb{R}^n$, are sets that are both closed and bounded.

Let's see why?

$(X, \|\cdot\|) = (\mathbb{R}, |\cdot|)$

$S = \cancel{(0,1)}\ (1,2)$, $f: S \to \mathbb{R}$ given by $f(x) = x$



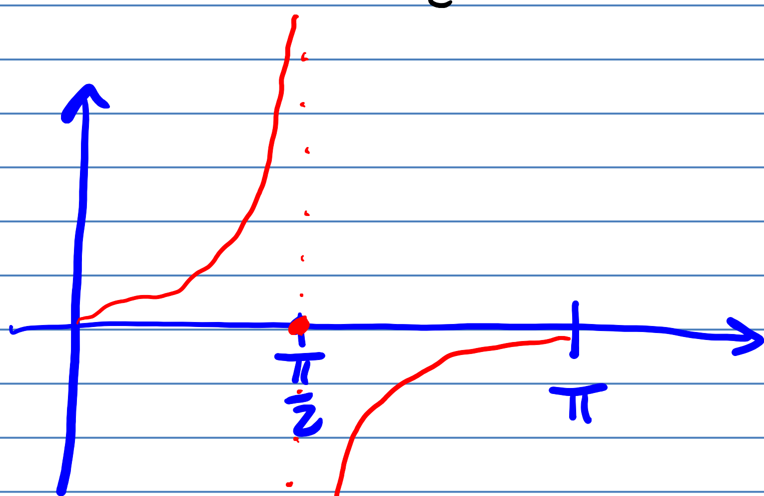
f has neither a max nor a min.

$S = [0, \infty)$, $f(x) = e^{-x}$ does not have a minimum.

$S = [0, \pi]$ $f: S \to \mathbb{R}$ by

$$f(x) = \begin{cases} \tan(x) & x \neq \pi/2 \\ 0 & x = \pi/2 \end{cases}$$

$f(x)$ has no max & no min



Def. Let $(X, \|\cdot\|)$ and $(Y, |||\cdot|||)$ be two normed spaces. A function $f: X \to Y$ is continuous at $x_0 \in X$ if, $\forall \varepsilon > 0$, $\exists\, \delta(\varepsilon, x_0) > 0$ such that $\|x - x_0\| < \delta \Rightarrow |||f(x) - f(x_0)||| < \varepsilon$.

- f is continuous on $S \subset X$ if f is continuous at $\forall x_0 \in S$.

Negate the definition of continuous at x_0 !!!