A Rudimentary treatise on the Elements of plane Trigonometry.

THE ELEMENTS

OF

PLANE TRIGONOMETRY.

BY JAMES HANN,

MATHEMATICAL MASTER OF KING'S COLLEGE SCHOOL, LONDON,

HONORARY MEMBER OF THE PHILOSOPHICAL SOCIETY OF NEWCASTLE UPON TYNE,
JOINT AUTHOR OF MECHANICS FOR PRACTICAL MEN,
AUTHOR OF A TREATISE ON THE STEAM ENGINE,
AND OF THEORETICAL AND PRACTICAL MECHANICS &c.

SECOND EDITION CORRECTED.

LONDON:
JOHN WEALE, 59, HIGH HOLBORN.
1854.

Cambridge:
Printed at the University Press.

ADVERTISEMENT TO THE SECOND EDITION.

MR. HANN'S Rudimentary Treatise on Plane Trigonometry having been carefully revised and corrected, the publisher submits this new edition to the judgment of Mathematical Teachers and Students with an increased amount of confidence. He would also respectfully invite their attention to his cheap collection of Mathematical Tables, forming Vols. 94 and 95 of his Series of Scientific Treatises; they will be found amply sufficient for all the practical purposes of Trigonometry, and will therefore furnish every necessary aid in computations connected with the present subject. The following brief extract, from the Author's Preface to the former edition, is perhaps sufficiently descriptive of the character of the work:

"I have given, to illustrate the principles, a great number of examples fully worked out, and which I hope will be of service to those who have not the aid of a teacher.

"In compiling the work, the best authors, whether French or English, have been consulted. I may refer to the excellent Works of Bonnycastle, Cape, De Morgan, Gaskin, Hall, Hind, Hymers, Snowball, Woodhouse, and Gregory; and to Davies's edition of Hutton's Course.

"The problems have been principally taken from the Ladies and Gentleman's Diaries, the Cambridge Problems, and Leybourn's Repository.

"The demonstration of Demoivre's Theorem is taken from an able French work on Trigonometry, by Lefebure De Fourcy."

J. HANN.

CONTENTS.

CHAPTER I.

CHAPTER II.

CHAPTER III.

CHAPTER IV.

CHAPTER V.

TRIGONOMETRY.

CHAPTER I.

1. TRIGONOMETRY was originally considered to be the doctrine of triangles, but in its present improved state it has a much more extensive signification, which we shall hereafter shew even in this rudimentary treatise.

2. In estimating angular measures, we suppose the right angle to be the primary one, and to be divided into 90 equal parts, each of which is called a degree; each degree is supposed to be divided into 60 equal parts, each of which is called a minute; each minute is supposed to be divided into 60 equal parts, each of which is called a second, and so on to thirds, fourths, &c. Here one degree is considered as the angular unit.

3. Modern French writers, instead of using the sexagesimal division, use the centesimal; and it is to be regretted that the latter is not universally used, from the great ease with which all calculations are made in that division.

We shall, however, shew how to reduce French into English measures, and *vice versâ*.

If E and F represent the number of English degrees and French grades in the same angle,

$$\frac{E}{90} = \frac{F}{100} \text{ or } \frac{E}{9} = \frac{F}{10};$$

$$\therefore\ E = \frac{9F}{10} = F - \frac{F}{10}; \text{ and } F = \frac{10E}{9} = E + \frac{E}{9}.$$

4. The circumference of a circle is known to be about 3·14159 times its diameter, or, in other words, the ratio of the circumference to the diameter is represented by 3·14159; for this number writers generally put the Greek letter π.

B

$\therefore$ circumference $= \pi D$; where D is the diameter, or
$2\pi r$, where r is the radius of the circle

Hence the length of the arc of a quadrant is $\frac{\pi r}{2}$; of a semicircle, or 180^0, is πr; and of 270^0, or three quadrants, is $\frac{3\pi r}{2}$.

Now if any arc a subtend an angle of A^0, then since $\frac{\pi r}{2}$ subtends 90^0, and that by Euclid VI. 33, angles are proportional to the arcs which subtend them,

$$A^0 : 90^0 :: a : \frac{\pi r}{2};$$

$$\therefore\ A^0 = \frac{180^0}{\pi} \cdot \frac{a}{r} \ldots\ldots\ldots\ldots \quad (1).$$

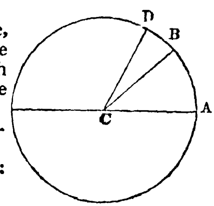
From this expression any one of the quantities may be found when the others are given.

Ex. 1. Find the length of an arc of 45^0 of a circle whose radius is 10 feet;

$$45^0 = \frac{180^0}{\pi} \cdot \frac{a}{10};$$

$$\therefore\ a = \frac{45^0}{180^0} \times 10 \times 3{\cdot}14159 = 7{\cdot}8539 \text{ feet.}$$

5. Most modern writers on Trigonometry take also for the unit of angular measure the number of degrees in an angle, subtended by an arc equal to the radius*. If U^0 represent that angle, then by equation (1),



* If ACB be an angle at the centre of a circle, subtended by an arc equal to the radius of the circle, then, since by the 33rd Proposition of the 6th Book of Euclid, the angles at the centre of a circle are to each other as the arcs on which they stand,

Angle ACB : four right angles :: arc AB : circumference, but AB is an arc equal to the radius,

$\therefore$ Angle ACB : four right angles :: $r : 2\pi r$ :: $1 : 2\pi$,

$\therefore\ ACB = \frac{\text{four right angles}}{2\pi}$, which, being independent of r, is constant for any circle; it may therefore be used to measure other angles.

$$U^\circ = \frac{180^\circ}{\pi} \cdot \frac{r}{r} = \frac{180^\circ}{\pi} = \frac{180^\circ}{3 \cdot 14159} = 57^\circ \cdot 29578.$$

Hence, $A^\circ = 57^\circ \cdot 29578 \left(\frac{a}{r}\right)$ or $A^\circ = U^\circ \left(\frac{a}{r}\right) \ldots\ldots (2).$

And since $U^\circ = 57^\circ \cdot 29578$ is constant,

$$A^\circ \text{ varies as } \frac{a}{r}, \text{ i.e. as } \frac{\text{arc}}{\text{radius}};$$

and taking U° as the unit, we have

$A^\circ = \frac{a}{r}$ which is called the circular measure of the angle.

From equation (2) we see that the measuring unit, U°, must be multiplied by the fraction $\frac{a}{r}$ to find the angle; thus if the circular measure of an angle be $\frac{5}{10}$, then

$$A^\circ \dot{=} \frac{5}{10} (57^\circ \cdot 29578) = 28^\circ \, 64789.$$

If the circular measure be $\frac{6}{5}$ then

$$A^\circ = \frac{6}{5} (57^\circ \cdot 29578) = \frac{12}{10} (57^\circ \cdot 29578) = 68^\circ \cdot 754936.$$

Now, suppose we take an angle of $22^\circ \, 27' \, 39''$, then this is put into decimals at once by the centesimal division, without putting down any work on paper, it being $22^\circ \cdot 2739$; whereas, by the sexagesimal, we must proceed in the following manner:

$$\begin{array}{r|l} 60 & 39 \\ \hline 60 & 27 \cdot 65 \\ \hline & 22 \cdot 4608 \end{array}$$

If we wish to find how many grades and minutes are contained in this angle, here

$$E = 22 \cdot 4608$$

$$\frac{E}{9} = 2 \cdot 4956$$

$E + \frac{E}{9} = 24 \cdot 9564$, which at sight is $24^g \, 95' \, 64''$.

Find the number of degrees and minutes in $46^g\ 56'\ 36''$.

$$F = 46{\cdot}5636$$
$$\frac{F}{10} = 4{\cdot}65636$$
$$E = F - \frac{F}{10} = 41{\cdot}90724$$
$$60$$
$$54{\cdot}4344$$
$$60$$
$$26{\cdot}064$$
$$E^0 = 41^0\ 54'\ 26''.$$

(1) If F' and F'', E' and E'' represent the magnitude of a French and English minute and second respectively, shew that

$$\frac{F'}{E'} = \frac{3\cdot3^2}{2\cdot5^2} : \frac{F''}{E''} = \frac{3\cdot3^3}{2\cdot5^3}.$$

$F' = 1$ French min.;

$\therefore\ F' \times 100 \times 100 =$ a quadrant,

$E' = 1$ English min.;

$\therefore\ E' \times 60 \times 90 =$ a quadrant,

hence, $F' \times 10000 = E' \times 5400$,

$$F' \times 50 = E' \times 27,$$
$$\frac{F'}{E'} = \frac{27}{50} = \frac{3\cdot3^2}{2\cdot5^2}.$$

Also, $F'' = 1$ French second;

$\therefore\ F'' \times 100 \times 100 \times 100 =$ a quadrant,

$E'' = 1$ English second;

$\therefore\ E'' \times 60 \times 60 \times 90 =$ a quadrant,

hence, $F'' \times 1000000 = E'' \times 324000$,

$$F'' \times 250 = E'' \times 81;$$
$$\frac{F''}{E''} = \frac{81}{250} = \frac{3\cdot3^3}{2\cdot5^3}.$$

(2) Compare the interior angles of a regular octagon and dodecagon:

In a polygon of n sides, $A^0 = 180^0 - \dfrac{360^0}{n}$.

In octagon $= 180^0 - \dfrac{360^0}{8} = 180^0 - 45^0 = 135^0$.

$$\text{In dodecagon} = 180^\circ - \frac{360^\circ}{12} = 180^\circ - 30^\circ = 150^\circ,$$

$$135 : 150 :: 9 : 10.$$

The ratio is therefore 9 to 10.

(3) The earth being supposed a sphere, of which the diameter is 7980 miles, find the length of an arc of 1°.

$$A^\circ = \frac{180^\circ}{\pi} \times \frac{a}{r}; \quad \therefore \ 1^\circ = \frac{180^\circ}{\pi} \times \frac{a}{\frac{7980}{2}} = \frac{180^\circ}{\pi} \times \frac{a}{3990},$$

$$\text{but } \frac{180^\circ}{\pi} = 57^\circ{\cdot}29578;$$

$$\therefore \ 1^\circ = 57^\circ{\cdot}29578 \times \frac{a}{3990},$$

$$a = \frac{3990}{57{\cdot}29578} = 69{\cdot}6 \text{ miles.}$$

(4) Find the diameter of a globe when an arc of 25° of the meridian measures 4 feet.

$$A^\circ = \frac{180^\circ}{\pi} \times \frac{a}{r}; \quad \therefore \ 25^\circ = \frac{180^\circ}{\pi} \times \frac{4}{r} = \frac{180^\circ}{\pi} \times \frac{8}{2r};$$

$$\therefore \ 2r = \frac{180}{\pi} \times \frac{8}{25} = 57{\cdot}29578 \times \frac{8}{25}$$

$$= 18{\cdot}3346 \text{ feet.} \quad \text{Answer.}$$

(5) Find the number of degrees in a circular arc 30 feet in length, of which the radius is 25 feet.

$$A^\circ = \frac{180^\circ}{\pi} \times \frac{a}{r} = \frac{180^\circ}{\pi} \times \frac{30}{25} = \frac{180^\circ}{\pi} \times \frac{6}{5}$$

$$= 57^\circ{\cdot}29578 \times \frac{6}{5} = 68^\circ{\cdot}75493.$$

(6) Find the number of degrees in an angle of which the circular measure is ·7854, the value of π being 3·1416.

$$A^\circ = \frac{180^\circ}{\pi} \times \frac{a}{r} = \frac{180^\circ}{3{\cdot}1416} \times {\cdot}7854 = \frac{180^\circ}{4} = 45^\circ.$$

(7) The interior angles of a rectilineal figure are in arithmetical progression, the least angle is 120°, and the common difference 5°; required the number of sides.

Sum of angles $= n\pi - 2\pi$ by Euc. (I. 32), or

$120^0 + 125^0 + 130^0$, &c., to n terms $= 180^0 n - 360^0$,

$$\{240 + (n-1)5\}\frac{n}{2} = 180n - 360,$$

$$120n + \frac{5n^2 - 5n}{2} = 180n - 360,$$

$$240n + 5n^2 - 5n = 360n - 720,$$

$$5n^2 - 125n = -720,$$

$$n^2 - 25n + \frac{625}{4} = \frac{625}{4} - 144 = \frac{625 - 576}{4} = \frac{49}{4},$$

$$n - \frac{25}{2} = \pm\frac{7}{2},$$

$$n = \frac{25 \pm 7}{2} = 16 \text{ or } 9.$$

The last is the congruent value of n, since no angle can be so great as 180^0; $\therefore$ the figure has nine sides.

(8) One regular polygon has two sides more than another, and each of its angles exceeds each angle of the other polygon by 15^0; find the number of sides in each.

$$nA = n\pi - 2\pi, \text{ equation (1)},$$

$$(n-2)(A-15) = (n - 2 \times 2)\pi,$$

$$nA - 2A - 15n + 30 = n\pi - 4\pi, \text{ equation (2)}.$$

Subtracting equation (2) from (1),

$$2A + 15n - 30 = 2\pi,$$

$$2A = 2\pi - 15n + 30,$$

$$A = \frac{2\pi - 15n + 30}{2}, \text{ and from equation (1)}$$

$$A = \frac{n\pi - 2\pi}{n} = \frac{180n - 360}{n};$$

$$\therefore \frac{2\pi - 15n + 30}{2} = \frac{180n - 360}{n},$$

$$390n - 15n^2 = 360n - 720,$$

$$15n^2 - 30n = 720,$$

$$n^2 - 2n = 48,$$

$$n^2 - 2n + 1 = 49,$$

$$n-1=\pm 7,$$
$$n=1\pm 7=8 \text{ or } -6,$$
$$\text{and } n-2=6.$$

An octagon and a hexagon.

(9) The angles in one regular polygon are twice as many as in another polygon; and an angle of the former is to an angle of the latter as 3 : 2; find the number of sides.

Let n = number of sides in 1st, A = each angle,

$2n$ = number of sides in 2nd, B = each angle,

$$A=\left(\frac{n-2}{n}\right)\pi; \quad B=\left(\frac{2n-2}{2n}\right)\pi=\frac{n-1}{n}\pi,$$
$$\frac{B}{A}=\frac{n-1}{n-2} \text{ also } =\frac{3}{2};$$
$$\therefore\ 2n-2=3n-6,$$
$$n=4,$$
$$2n=8.$$

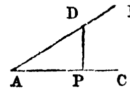
(10) The angles of a quadrilateral are in increasing geometrical progression, and the difference between the third angle and the fourth part of the first is 90°; find the angles.

Let A, Ar, Ar^2, Ar^3, be the angles;

$$\therefore\ A(1+r+r^2+r^3)=360^\circ,$$
$$\text{and } A\left(r^2-\frac{1}{4}\right)=90^\circ;$$
$$\therefore\ (1+r+r^2+r^3)=(4r^2-1);$$
$$\therefore\ r^3-3r^2+r+2=0;$$
$$\therefore\ r^4-3r^3+r^2+2r=0,$$
$$(r^4-3r^3+\frac{9}{4}r^2)-(r^2-\frac{3}{2}r)+\frac{1}{4}=\frac{r^2}{4}-\frac{r}{2}+\frac{1}{4};$$
$$\therefore\ r^2-\frac{3}{2}r-\frac{1}{2}=\frac{r}{2}-\frac{1}{2};$$
$$\therefore\ r^2=2r; \quad \therefore\ r=2.$$
$$\text{But } A\left(r^2-\frac{1}{4}\right)=90^\circ; \quad \therefore\ \frac{15A}{4}=90^\circ$$
$$\therefore\ 15A=360^\circ; \quad \therefore\ A=24^\circ.$$

And the angles are 24°, 48°, 96°, 192°.

6. Trigonometrical Ratios or Definitions.



Let BAC be any angle, and from any point D in AB let fall the perpendicular DP on AC, then if we represent the angle BAC by A,

$\frac{DP}{AD}$ is the sine of A; $\frac{AP}{AD}$ is the cosine of A;

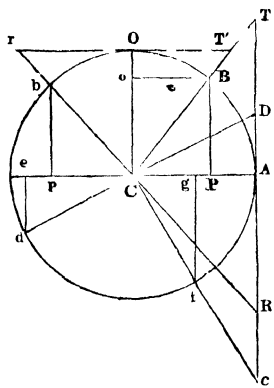
$\frac{DP}{AP}$ is the tangent of A; $\frac{AD}{DP}$ is the cosecant of A;

$\frac{AD}{AP}$ is the secant of A; $\frac{AP}{DP}$ is the cotangent of A.

For the sake of abbreviation the above quantities are generally put $\sin A$, $\cos A$, $\tan A$, $\sec A$, $\operatorname{cosec} A$, $\cot A$. $1-\cos A$ is defined to be the versed sine of A, or vers A*.

By the 47th Prop of the 1st Book of Euclid,

$$AP^2+DP^2=AD^2 \ldots\ldots\ldots\ldots\ldots (1)$$



* It may be perhaps necessary to remark that the following definitions have been given by most English writers till within the last few years. In the annexed figure, take any arc AB, draw BP and AT each perpendicular to the diameter AE, and produce CB to meet AT in T, then BP is called the sine of the angle ACB to the radius CB; CP is called the cosine; AT the tangent; CT the secant; AP the versed sine; OT' the cotangent; CT' the cosecant.

If we take the arc Ab greater than one quadrant and less than two, then bp is called the sine; Cp the cosine; AR the tangent; CR the secant; Ap the versed sine.

If the arc Abd be greater than two quadrants but less than three, then de is called the sine; Ce the cosine; AD the tangent; CD the secant; Ae the versed sine.

If the arc $Abdf$ be greater than three quadrants but less than four, then fg is called the sine; Cg the cosine; Ac the tangent; Cc the secant; and Ag the versed sine.

Let the angle $ACB=A$, $r=$ radius CB, then since $CP^2+PB^2=CB^2$ by 47th Prop. 1st book of Euclid, we have $\cos^2 A+\sin^2 A=r^2$ ....(1)

Also $AC^2+AT^2=CT^2$, or

$$r^2+\tan^2 A=\sec^2 A \ldots\ldots\ldots\ldots\ldots (2)$$

Divide both sides of this equation by AD^2, and we have

$$\frac{AP^2}{AD^2} + \frac{DP^2}{AD^2} = \frac{AD^2}{AD^2};$$

$$i.e.\ \left(\frac{AP}{AD}\right)^2 + \left(\frac{DP}{AD}\right)^2 = 1:$$

but $\frac{AP}{AD}$ is the cosine of A, and $\frac{DP}{AD}$ is the sine of A, by the above definitions;

$$\therefore\ \cos^2 A + \sin^2 A = 1 \ldots\ldots\ldots\ldots\ldots(2).$$

From this equation we have,

$$\cos^2 A = 1 - \sin^2 A\ ;$$

$$\therefore\ \cos A = \sqrt{(1 - \sin^2 A)},$$

$$\text{and } \sin^2 A = 1 - \cos^2 A,$$

$$\sin A = \sqrt{(1 - \cos^2 A)}\,;$$

and since by the definitions $\frac{DP}{AD} = \sin A$, and $\frac{AP}{AD} = \cos A$, we have $DP = AD \sin A$, and $AP = AD \cos A$.

$$\text{Now } \tan A = \frac{DP}{AP} = \frac{AD \sin A}{AD \cos A} = \frac{\sin A}{\cos A},$$

From (1) $\cos^2 A = r^2 - \sin^2 A$, or $\cos A = \sqrt{(r^2 - \sin^2 A)}$

and $\sin^2 A = r^2 - \cos^2 A$, or $\sin A = \sqrt{(r^2 - \sin^2 A)}$

From (2) $\tan^2 A = \sec^2 A - r^2$.

The triangles BCP, TCA, OCT', and oCB, are all similar,

$$\therefore\ \frac{AT}{AC} = \frac{BP}{CP} \text{ or } \frac{\tan A}{r} = \frac{\sin A}{\cos A}$$

$$\text{or } \tan A = \frac{r \sin A}{\cos A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

$$\frac{\cot A}{r} = \frac{\cos A}{\sin A}$$

$$\text{or } \cot A = \frac{r \cos A}{\sin A} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (4)$$

$$\frac{\sec A}{r} = \frac{r}{\cos A} \text{ or } \sec A = \frac{r^2}{\cos A} \ldots\ldots\ldots\ldots\ldots (5)$$

$$\frac{\text{cosec } A}{r} = \frac{r}{\sin A} \text{ or cosec } A = \frac{r^2}{\sin A} \ldots\ldots\ldots\ldots\ldots (6)$$

$$\text{versin } A = r - \cos A \ldots\ldots\ldots\ldots\ldots\ldots\ldots (7).$$

$$\cot A = \frac{AP}{DP} = \frac{AD \cos A}{AD \sin A} = \frac{\cos A}{\sin A},$$

$$\sec A = \frac{AD}{AP} = \frac{AD}{AD \cos A} = \frac{1}{\cos A},$$

$$\operatorname{cosec} A = \frac{AD}{DP} = \frac{AD}{AD \sin A} = \frac{1}{\sin A},$$

$$\sec^2 A = \frac{AD^2}{AP^2} = \frac{AP^2 + DP^2}{AP^2} = 1 + \left(\frac{DP}{AP}\right)^2 = 1 + \tan^2 A;$$

or by dividing equation (1) by AP^2,

$$\left(\frac{AP}{AP}\right)^2 + \left(\frac{DP}{AP}\right)^2 = \left(\frac{AD}{AP}\right)^2,$$

$$\text{or } 1 + \tan^2 A = \sec^2 A;$$

$$\therefore \sec A = \surd(1 + \tan^2 A).$$

Also dividing equation (1) by DP^2, we have

$$\left(\frac{AP}{DP}\right)^2 + \left(\frac{DP}{DP}\right)^2 = \left(\frac{AD}{DP}\right)^2 \text{ that is}$$

$$\cot^2 A + 1 = \operatorname{cosec}^2 A;$$

$$\therefore \operatorname{cosec} A = \surd(1 + \cot^2 A).$$

(1) The sine of an angle is equal to the cosine of its complement.

$$\text{Since } \angle ADP = 90^\circ - A,$$

$$\text{and } \sin ADP = \frac{AP}{AD} = \cos A,$$

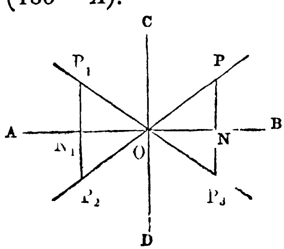
$$\cos ADP = \frac{DP}{AD} = \sin A;$$

$$\text{that is, } \sin(90^\circ - A) = \cos A,$$

$$\text{and } \cos(90^\circ - A) = \sin A.$$

(2) To shew that $\sin A = \sin(180^\circ - A)$.

Let BOP_1 be greater than a right angle; draw P_1N_1 perpendicular to AB; make the angle BOP equal to the angle AOP_1, and $OP = OP_1$, and let fall the perpendicular PN, then the triangles P_1ON_1 and PON are evidently equal.



$$\sin BOP_1 = \frac{P_1N_1}{OP_1}; \ \sin BOP = \sin(180^\circ - BOP_1) = \frac{PN}{OP};$$

$$\text{but } P_1N_1 = PN, \text{ and } OP_1 = OP;$$

$$\therefore \sin BOP_1 = \sin(180^\circ - BOP_1):$$

that is, the sine of an angle is equal to the sine of the supplement of that angle:

$$\cos BOP_1 = \frac{ON_1}{OP_1} = -\frac{ON}{OP},$$

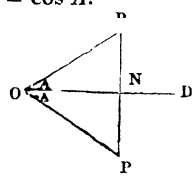
$$\cos BOP = \cos(180^\circ - BOP_1) = \frac{ON}{OP};$$

$$\therefore \cos BOP_1 = -\cos(180^\circ - BOP_1);$$

or the cosine of an angle is equal to minus the cosine of its supplement.

(3) Sin $(-A) = -\sin A$, and $\cos(-A) = \cos A$.

If the line OP revolve round the point O from OD upwards till it describes an angle $NOP = A$, then if it revolve downwards from OD till it describes the angle *NOP*, then $NOP_1 = -A$.



$$\sin A = \frac{PN}{OP}, \text{ and } \sin NOP_1 = \sin(-A) = \frac{P_1N}{OP_1};$$

$$\text{but } \frac{P_1N}{OP_1} = -\frac{PN}{OP};$$

$$\therefore \sin(-A) = -\sin A;$$

$$\cos A = \frac{ON}{OP}, \text{ and } \cos NOP_1 = \cos(-A) = \frac{ON}{OP_1} = \frac{ON}{OP};$$

$$\therefore \cos A = \cos(-A).$$

From these we readily see that

$$\tan A = -\tan(-A),$$

$$\cot A = -\cot(-A),$$

$$\sec A = \sec(-A),$$

$$\text{cosec } A = -\text{cosec}(-A).$$

After one revolution is completed, the sines, cosines, &c., take the same values as before; therefore the sine of any angle is the same as the sine of 360° + that angle, or

$$\sin A = \sin(2\pi + A);$$

In the same way,

$$\sin A = \sin(4\pi + A),$$

and for the general form, where n is a whole number, we have

$$\sin A = \sin(2n\pi + A).$$

Similarly,

$$\sin(\pi - A) = \sin(3\pi - A) = \sin(5\pi - A);$$

or, generally,

$$\sin(\pi - A) = \sin\{(2n + 1)\pi - A\};$$

$$\text{but } \sin A = \sin(\pi - A);$$

$$\therefore \sin(2n\pi + A) = \sin\{(2n + 1)\pi - A\}.$$

$$\text{Also } \sin A = -\sin\{(2n + 1)\pi + A\}$$
$$= -\sin(2n\pi - A),$$

$$\cos A = \cos(2n\pi + A) = -\cos\{(2n + 1)\pi - A\},$$
$$\tan A = \tan(2n\pi + A) = -\tan\{(2n + 1)\pi - A\},$$
$$\sec A = \sec(2n\pi + A) = -\sec\{(2n + 1)\pi - A\}.$$

(4) In the figure at page 10, it is clear that if we suppose the line OP, originally coinciding with OB, to revolve round O as a centre, in each of the quadrants the various trigonometrical quantities will have precisely the same value; for the sake of distinction these quantities are affected with different signs. Thus, suppose $ON = ON_1 = x$, then ON, measured to the right of O, is called $+x$, and ON_1 measured to the left of O, is called $-x$. Also, any line PN above the line AB is considered positive, and any line P_3N below the same line is considered negative; thus if $PN = +y$, then P_3N is $-y$. This is purely conventional, for we might have taken lines to the left to be positive, and those to the right to be negative, and so on; but when we once fix on the positive direction, the negative direction must necessarily be opposite to it.

If we suppose the line OP to revolve upwards, the angle PON is considered positive; but, if downwards, the angle P_3ON is considered negative: thus, the angle PON is $+A$, and the angle P_3ON is $-A$.

(5) We may now proceed to trace the values of the sine, cosine, &c., throughout the four quadrants. In the first quadrant $\frac{PN}{OP}$, which is the sine of A, is positive; and $\frac{ON}{OP}$, which is the cosine of A, is also positive; it is clear that as

the sine increases, the cosine decreases, and when OP coincides with OC, then $PN = OP$, and $ON = 0$; hence the sin $90^0 = 1$ and cos $90^0 = 0$. In the second quadrant the sine is also positive, but the cosine or $\frac{ON_1}{OP}$ is negative, since ON_1 is measured in the opposite direction to ON; as OP revolves from OC towards OA the sine decreases, and the cosine increases negatively; so that when OP coincides with OA the sine is 0 and the cosine is -1, that is, sin $180^0 = 0$ and cos $180^0 = -1$.

In the third quadrant both the sine and cosine are negative, and when OP coincides with OD, then the sin $270^0 = -1$, and the cos $270^0 = 0$. In the fourth quadrant the sine is negative but the cosine positive, and when the line OP has completed a whole revolution by coinciding again with OB, then we have the

$$\text{sine } 360^0 = 0, \text{ and cosine } 360^0 = 1.$$

Since $\tan A = \frac{\sin A}{\cos A}$; in the first quadrant, both sin A and cos A being positive, tan A is also positive, and when $A = 90^0$ we have $\tan 90^0 = \frac{\sin 90^0}{0} = \frac{1}{0} = \text{infinity}$; that is, the tangent of 90^0 is infinite.

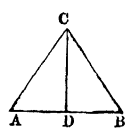
In the second quadrant $\tan A = \frac{+\sin A}{-\cos A}$, and is therefore negative, and $\tan 180^0 = \frac{\sin 180^0}{\cos 180^0} = \frac{0}{-1} = 0$.

In the third quadrant $\tan A = \frac{-\sin A}{-\cos A}$, and is therefore positive, and $\tan 270^0 = \frac{\sin 270^0}{\cos 270^0} = \frac{-1}{0} = -\text{infinity}$.

In the fourth quadrant $\tan A = \frac{-\sin A}{+\cos A}$, and consequently negative, and $\tan 360^0 = \frac{\sin 360^0}{\cos 360^0} = \frac{0}{1} = 0$.

(6) To find sine, cosine, &c., of 30^0, 60^0, 45^0.

Let the angle be 60^0.



In the equilateral triangle ACB, let fall the perpendicular CD, which bisects both the angle ACB and the base AB.

Now, $\dfrac{AD}{AC} = \sin ACD = \sin 30^0$, but $AD = \dfrac{1}{2} AC$;

$$\therefore \sin 30^0 = \frac{\frac{1}{2} AC}{AC} = \frac{1}{2};$$

$$\therefore \cos 30^0 = \sqrt{(1 - \sin^2 30^0)} = \sqrt{\left(1 - \frac{1}{4}\right)} = \frac{\sqrt{3}}{2}, \text{ but}$$

$$\cos 30^0 = \sin 60^0; \ \therefore \sin 60^0 = \frac{\sqrt{3}}{2};$$

$$\cos 60^0 = \sin 30^0; \ \therefore \cos 60^0 = \frac{1}{2};$$

$$\sin 30^0 = \frac{1}{2},$$

$$\cos 30^0 = \frac{\sqrt{3}}{2},$$

$$\tan 30^0 = \frac{\sin 30^0}{\cos 30^0} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}},$$

$$\cot 30^0 = \frac{\cos 30^0}{\sin 30^0} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3},$$

$$\sec 30^0 = \frac{1}{\cos 30^0} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}},$$

$$\text{cosec } 30^0 = \frac{1}{\sin 30^0} = \frac{1}{\frac{1}{2}} = 2;$$

$$\sin 60^0 = \frac{\sqrt{3}}{2},$$

$$\cos 60^0 = \frac{1}{2},$$

$$\tan 60^0 = \frac{\sin 60^0}{\cos 60^0} = \frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = \sqrt{3}$$

$$\cot 60^0 = \frac{\cos 60^0}{\sin 60^0} = \frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{1}{\sqrt{3}}$$

$$\sec 60^0 = \frac{1}{\cos 60^0} = \frac{1}{\frac{1}{2}} = 2,$$

$$\text{cosec } 60^0 = \frac{1}{\sin 60^0} = \frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}}$$

When the angle is 45^0,

$$\sin 45^0 = \cos (90^0 - 45^0) = \cos 45^0.$$

Now, since $\cos^2 45^0 + \sin^2 45^0 = 1$, and $\cos 45^0 = \sin 45^0$, we have $2\cos^2 45^\circ = 1$,

$$\cos^2 45^\circ = \frac{1}{2},$$

or, $\cos 45^\circ = \dfrac{1}{\sqrt{2}}$;

and therefore, $\sin 45^\circ = \dfrac{1}{\sqrt{2}}$;

and since $\sin 45^\circ = \cos 45^\circ$,

$$\tan 45^\circ = \frac{\sin 45^\circ}{\cos 45^\circ} = 1,$$

$$\cot 45^\circ = \frac{\cos 45^\circ}{\sin 45^\circ} = 1,$$

$$\sec 45^\circ = \frac{1}{\cos 45^\circ} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2},$$

$$\operatorname{cosec} 45^\circ = \frac{1}{\sin 45^\circ} = \frac{1}{\frac{1}{\sqrt{2}}} = \sqrt{2}.$$

The following results the student should commit to memory:

$$\cos^2 A + \sin^2 A = 1,$$

$$\cos A = \sqrt{(1 - \sin^2 A)};\ \sin A = \sqrt{(1 - \cos^2 A)},$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{1}{\cot A} = \sqrt{(\sec^2 A - 1)},$$

$$\cot A = \frac{\cos A}{\sin A} = \frac{1}{\tan A} = \sqrt{(\operatorname{cosec}^2 A - 1)},$$

$$\sec A = \frac{1}{\cos A} = \sqrt{(1 + \tan^2 A)},$$

$$\operatorname{cosec} A = \frac{1}{\sin A} = \sqrt{(1 + \cot^2 A)},$$

$$\operatorname{vers} A = 1 - \cos A.$$

Examples.

Ex. 1. If $\tan A = \dfrac{4}{3}$, compute the sine and versed sine.

$$\frac{\sin A}{\cos A} = \frac{4}{3},$$

$$3 \sin A = 4\sqrt{(1 - \sin^2 A)},$$

$$9 \sin^2 A = 16 - 16 \sin^2 A,$$

$$25 \sin^2 A = 16,$$

$$5 \sin A = 4,$$

$$\sin A = \frac{4}{5},$$

$$\cos A = \sqrt{(1 - \sin^2 A)} = \sqrt{\left(1 - \frac{16}{25}\right)} = \sqrt{\frac{9}{25}} = \frac{3}{5},$$

$$\text{vers } A = 1 - \cos A = 1 - \frac{3}{5} = \frac{2}{5}.$$

Ex. 2. $6 (\sin A)^2 = 5 \cos A$, find sine, cosine, tangent.

$$6 - 6 \cos^2 A = 5 \cos A,$$

$$6 \cos^2 A + 5 \cos A = 6,$$

$$\cos^2 A + \frac{5}{6} \cos A = 1,$$

$$\cos^2 A + \frac{5}{6} \cos A + \frac{25}{144} = 1 + \frac{25}{144} = \frac{169}{144},$$

$$\cos A + \frac{5}{12} = \pm \frac{13}{12},$$

$$\cos A = \frac{2}{3}, \text{ or } -\frac{3}{2},$$

$$\sin A = \sqrt{(1 - \cos^2 A)} = \sqrt{\left(1 - \frac{4}{9}\right)} = \sqrt{\frac{5}{9}} = \frac{\sqrt{5}}{3},$$

$$\tan A = \frac{\sin A}{\cos A} = \frac{\frac{\sqrt{5}}{3}}{\frac{2}{3}} = \frac{\sqrt{5}}{2}.$$

Ex. 3. $\sin A \cos A = \frac{\sqrt{3}}{4}$. Find sin, cos, and arc.

$$\sin A \sqrt{(1 - \sin^2 A)} = \frac{\sqrt{3}}{4},$$

$$\sin^2 A\,(1-\sin^2 A)=\frac{3}{16},$$

$$\sin^4 A-\sin^2 A=-\frac{3}{16},$$

$$\sin^4 A-\sin^2 A+\frac{1}{4}=\frac{1}{16},$$

$$\sin^2 A-\frac{1}{2}=\pm\frac{1}{4},$$

$$\sin^2 A=\frac{3}{4},\text{ or }\frac{1}{4},$$

$$\sin A=\frac{\sqrt{3}}{2},\text{ or }\frac{1}{2};$$

$$\therefore\ A=60^0,\text{ or }30^0.$$

$$\cos A=\frac{1}{2},\text{ or }\frac{\sqrt{3}}{2}.$$

Ex. 4. $\tan A+\cot A=\dfrac{4}{\sqrt{3}}$, find $\tan A$, and $\cot A$.

$$\tan A+\frac{1}{\tan A}=\frac{4}{\sqrt{3}},$$

$$\tan^2 A-\frac{4}{\sqrt{3}}\tan A=-1,$$

$$\tan^2 A-\frac{4}{\sqrt{3}}\tan A+\frac{4}{3}=\frac{1}{3},$$

$$\tan A-\frac{2}{\sqrt{3}}=\pm\frac{1}{\sqrt{3}},$$

$$\tan A=\sqrt{3},\text{ or }\frac{1}{\sqrt{3}},$$

$$\cot A=\frac{1}{\sqrt{3}},\text{ or }\sqrt{3}.$$

Ex. 5. If $3\sin A+5\sqrt{3}.\cos A=9$, find A.

$$5\sqrt{3}.\sqrt{(1-\sin^2 A)}=3\,(3-\sin A),$$

$$25.3.(1-\sin^2 A)=9\,(9-6\sin A+\sin^2 A),$$

$$25 - 25\sin^2 A = 27 - 18\sin A + 3\sin^2 A,$$

$$28\sin^2 A - 18\sin A = -2,$$

$$\sin^2 A - \frac{9}{14}\sin A = -\frac{1}{14},$$

$$\sin^2 A - \frac{9}{14}\sin A + \frac{81}{784} = \frac{81-56}{784} = \frac{25}{784},$$

$$\sin A - \frac{9}{28} = \frac{\pm 5}{28},$$

$$\sin A = \frac{\pm 5 + 9}{28} = \frac{1}{2}, \text{ or } \frac{1}{7};$$

$$\therefore\ A = 30^\circ.$$

Ex. 6. $\left.\begin{aligned}\sin A + \sin B &= \frac{1+\sqrt{3}}{2},\\ \sin A \sin B &= \frac{\sqrt{3}}{4}.\end{aligned}\right\}$ Find A and B;

$$\sin^2 A + 2\sin A\sin B + \sin^2 B = \frac{1 + 2\sqrt{3} + 3}{4},$$

$$4\sin A\sin B = \frac{4\sqrt{3}}{4},$$

$$\sin^2 A - 2\sin A\sin B + \sin^2 B = \frac{1 - 2\sqrt{3} + 3}{4},$$

$$\sin A - \sin B = \frac{\sqrt{3}-1}{2},$$

$$\sin A + \sin B = \frac{1+\sqrt{3}}{2},$$

$$2\sin A = 1; \qquad \therefore\ \sin A = \frac{1}{2}; \qquad \therefore\ A = 30^\circ,$$

$$2\sin B = \sqrt{3}; \qquad \therefore\ \sin B = \frac{\sqrt{3}}{2}; \qquad \therefore\ B = 60^\circ.$$

Ex 7. Shew that $\sec^2 A \operatorname{cosec}^2 A = \sec^2 A + \operatorname{cosec}^2 A$.

$$\sec^2 A \operatorname{cosec}^2 A = \sec^2 A\,(1 + \cot^2 A) = \sec^2 A + \sec^2 A\cot^2 A,$$

$$= \sec^2 A + \frac{1}{\cos^2 A}\cdot\frac{\cos^2 A}{\sin^2 A} = \sec^2 A + \frac{1}{\sin^2 A}$$

$$= \sec^2 A + \operatorname{cosec}^2 A.$$

Ex. 8. If $\tan^2 A + 4\sin^2 A = 6$, prove that $A = 60^\circ$.

$$\frac{\sin^2 A}{\cos^2 A} + 4\sin^2 A = 6, \text{ or, } \frac{\sin^2 A}{1-\sin^2 A} + 4\sin^2 A = 6,$$

$$\sin^2 A + 4\sin^2 A - 4\sin^4 A = 6 - 6\sin^2 A,$$

$$4\sin^4 A - 11\sin^2 A = -6,$$

$$\sin^4 A - \frac{11}{4}\sin^2 A + \frac{121}{64} = \frac{121}{64} - \frac{96}{64} = \frac{25}{64},$$

$$\sin^2 A - \frac{11}{8} = \pm\frac{5}{8}; \quad \therefore \sin^2 A = \frac{11}{8} \pm \frac{5}{8} = 2, \text{ or } \frac{3}{4};$$

$$\therefore \sin A = \frac{\sqrt{3}}{2} = \sin 60^\circ; \quad \therefore A = 60^\circ.$$

The $\sqrt{2}$ being greater than unity is inadmissible.

Ex. 9. If $\sin x \cos x + a\sin^2 x = b$, find x.

$$\sin x\sqrt{(1-\sin^2 x)} = b - a\sin^2 x,$$

$$\sin^2 x - \sin^4 x = b^2 - 2ab\sin^2 x + a^2\sin^4 x,$$

$$(a^2+1)\sin^4 x - (2ab+1)\sin^2 x = -b^2,$$

$$\sin^4 x - \left(\frac{2ab+1}{a^2+1}\right)\sin^2 x = -\frac{b^2}{a^2+1},$$

$$\sin^4 x - \left(\frac{2ab+1}{a^2+1}\right)\sin^2 x + \left\{\frac{2ab+1}{2(a^2+1)}\right\}^2$$

$$= \frac{4a^2b^2 + 4ab + 1}{4(a^2+1)^2} - \frac{b^2}{a^2+1}$$

$$= \frac{4ab + 1 - 4b^2}{4(a^2+1)^2},$$

$$\sin^2 x - \frac{2ab+1}{2(a^2+1)} = \pm\frac{\sqrt{(4ab+1-4b^2)}}{2(a^2+1)},$$

$$\sin^2 x = \frac{2ab + 1 \pm \sqrt{(1+4ab-4b^2)}}{2(a^2-1)},$$

$$\sin x = \left\{\frac{2ab+1 \pm \sqrt{(1+4ab-4b^2)}}{2(a^2+1)}\right\}^{\frac{1}{2}}.$$

Ex. 10. $25\sin A(\sin A - \cos A) = 4$. Prove that $\sin A = \frac{4}{5}$.

$$\sin^2 A - \sin A\cos A = \frac{4}{25},$$

$$\sin^2 A - \frac{4}{25} = \sin A \cos A$$

$$= \sin A \sqrt{(1 - \sin^2 A)},$$

$$\sin^4 A - \frac{8}{25} \sin^2 A + \frac{16}{625} = \sin^2 A - \sin^4 A,$$

$$2 \sin^4 A - \frac{33}{25} \sin^2 A = - \frac{16}{625},$$

$$\sin^4 A - \frac{33}{50} \sin^2 A = - \frac{8}{625},$$

$$\sin^4 A - \frac{33}{50} \sin^2 A + \left(\frac{33}{100}\right)^2 = \frac{1089}{10000} - \frac{8}{625} = \frac{961}{10000},$$

$$\sin^2 A - \frac{33}{100} = \pm \frac{31}{100},$$

$$\sin^2 A = \frac{33}{100} \pm \frac{31}{100} = \frac{64}{100}, \text{ taking the upper sign;}$$

$$\therefore \sin A = \frac{8}{10} = \frac{4}{5}.$$

Ex. 11. $6 \tan A + 12 \cot A = 5 \sqrt{3} . \sec A$. Find $\tan A$.

$$6 \tan A + \frac{12}{\tan A} = 5 \sqrt{3} . \sqrt{(1 + \tan^2 A)},$$

$$36 \tan^2 A + 144 + \frac{144}{\tan^2 A} = 75 + 75 \tan^2 A,$$

$$39 \tan^4 A - 69 \tan^2 A = 144,$$

$$13 \tan^4 A - 23 \tan^2 A = 48,$$

$$\tan^4 A - \frac{23}{13} \tan^2 A = \frac{48}{13},$$

$$\tan^4 A - \frac{23}{13} \tan^2 A + \left(\frac{23}{26}\right)^2 = \frac{529}{676} + \frac{2496}{676} = \frac{3025}{676},$$

$$\tan^2 A - \frac{23}{26} = \pm \frac{55}{26},$$

$$\tan^2 A = \frac{23}{26} \pm \frac{55}{26} = \frac{78}{26} = 3;$$

$$\therefore \tan A = \sqrt{3}; \quad \therefore A = 60^\circ \text{ (p. 14).}$$

Ex. 12. If $m = \tan A + \sin A$, and $n = \tan A - \sin A$, find an equation involving only m and n.

$$m = \tan A + \sin A,$$
$$n = \tan A - \sin A,$$
$$\tan A = \frac{m+n}{2}, \quad \sin A = \frac{m-n}{2},$$
$$\sin A = \frac{\tan A}{\sqrt{(1 + \tan^2 A)}};$$
$$\frac{m-n}{2} = \frac{\frac{m+n}{2}}{\sqrt{\left\{1 + \left(\frac{m+n}{2}\right)^2\right\}}};$$
$$\frac{(m-n)^2}{4} = \frac{\frac{(m+n)^2}{4}}{1 + \frac{(m+n)^2}{4}} = \frac{(m+n)^2}{4 + (m+n)^2},$$
$$4(m-n)^2 + (m-n)^2(m+n)^2 = 4(m+n)^2$$
$$(m^2 - n^2)^2 = 4(m+n)^2 - 4(m-n)^2 = 16\,mn,$$
$$m^2 - n^2 = \sqrt{(16mn)} = 4\sqrt{(mn)}.$$

Ex. 13. If $a(\sin\theta)^2 + b(\cos\theta)^2 = m$ .... (1),

$b(\sin\phi)^2 + a(\cos\phi)^2 = n$ .... (2),

and $a \tan\theta = b \tan\phi$ .... (3),

then $\frac{1}{a} + \frac{1}{b} = \frac{1}{m} + \frac{1}{n}$.

$$a(1 - \cos^2\theta) + b\cos^2\theta = m,$$
$$a - a\cos^2\theta + b\cos^2\theta = m,$$
$$(a-b)\cos^2\theta = a - m,$$
$$\cos^2\theta = \frac{a-m}{a-b}; \quad \therefore \sec^2\theta = \frac{a-b}{a-m} \text{ or } 1 + \tan^2\theta = \frac{a-b}{a-m};$$
$$\therefore \tan^2\theta = \frac{a-b}{a-m} - 1 = \frac{m-b}{a-m}.$$

In the same manner,

$$\tan^2\phi = \frac{n-a}{b-n};$$

$$\therefore \frac{\tan^2\theta}{\tan^2\phi} = \frac{m-b}{a-m} \cdot \frac{b-n}{n-a},$$

but by equation 3,

$$\frac{\tan\theta}{\tan\phi} = \frac{b}{a}; \quad \frac{\tan^2\theta}{\tan^2\phi} = \frac{b^2}{a^2};$$

$$\therefore \frac{b^2}{a^2} = \frac{m-b}{a-m} \cdot \frac{b-n}{n-a} = \frac{bm - b^2 - mn + bn}{an - mn - a^2 + am},$$

$$ab^2n - b^2mn - a^2b^2 + ab^2m = a^2bm - a^2b^2 - a^2mn + a^2bn;$$

$$\therefore (a^2 - b^2)\,mn = a^2bm - ab^2m + a^2bn - ab^2n$$
$$= ab\,(am - bm + an - bn)$$
$$= ab\,\{(a-b)\,m + (a-b)\,n\};$$

$$\therefore (a+b)\,mn = ab\,(m+n),$$

$$\therefore \frac{a+b}{ab} = \frac{m+n}{mn};$$

$$\therefore \frac{1}{a} + \frac{1}{b} = \frac{1}{m} + \frac{1}{n}.$$

(1) Prove that

$$\sin A = \frac{\tan A}{\sec A} = \frac{\cos A}{\cot A} = \frac{\tan A \cot A}{\operatorname{cosec} A},$$

$$\cos A = \frac{\sin A}{\tan A} = \frac{\cot A}{\operatorname{cosec} A} = \frac{\sin A \operatorname{cosec} A}{\sec A},$$

$$\tan A = \frac{\cos A}{\sin A \cot^2 A} = \frac{\cos A \sec A}{\cot A} = \frac{\sin A \operatorname{cosec} A}{\cot A},$$

$$\cot A = \frac{\cos A \sec A}{\tan A} = \frac{\sin A \operatorname{cosec} A}{\tan A} = \frac{\sin A}{\cos A \tan^2 A},$$

$$\sec A = \frac{\tan A}{\sin A} = \frac{\sin A \operatorname{cosec} A}{\cos A} = \frac{\tan A \cot A}{\cos A},$$

$$\operatorname{cosec} A = \frac{\sec A}{\tan A} = \frac{\cos A \sec A}{\sin A} = \frac{\tan A \cot A}{\sin A}.$$

(2) $\cos^2 A \cot^2 A = \cot^2 A - \cos^2 A.$

(3) $\dfrac{\cos^2 A - \cos^2 B}{\cos^2 A \cos^2 B} = \tan^2 B - \tan^2 A.$

(4) If $m = \operatorname{cosec} A - \sin A$; and $n = \sec A - \cos A$;

then $m^{\frac{4}{3}} n^{\frac{2}{3}} + m^{\frac{2}{3}} n^{\frac{4}{3}} = 1.$

(5) Given $\frac{\sin A}{\cos A} = 1$, find A. $A = 45^\circ$.

(6) $\sin A \cos A = \frac{\sqrt{3}}{4}$ find A. $A = 30^\circ$.

(7) $\tan A + \cot A = 2$, find A. $A = 45^\circ$.

(8) Write down the sum of all the exterior and also of all the interior angles of a polygon of n sides; and thence deduce the value of each of the vertical angles of the triangles made by producing both ways all the sides of a regular polygon, and verify the result in the cases where $n = 3$ and $n = 4$.

(9) $\frac{\tan A}{\sqrt{\sec A}} + \sqrt{\left(1 - \frac{1}{\sec A}\right)} = \sec A$, find $\sec A$:

$$\sec A = \frac{1 + \sqrt{5}}{2}.$$

(10) $\sqrt{(\tan^4 A - 1)} + \sqrt{\left(1 - \frac{1}{\tan^4 A}\right)} + 1 = \sec^2 A$, find $\tan A$;

$$\tan A = \left(\frac{1 + \sqrt{2}}{2}\right)^{\frac{1}{2}}.$$

(11) $\sin^3 A + \frac{1}{2} \cos^2 A = 1$, find A.

$$A = 90^\circ.$$

(12) $\sin^4 A - 2\sin^2 A - 1 = 2\sin A - \cos^2 A$, find A.

$$A = 270^\circ.$$

(13) $\tan^4 A - \tan^2 A + 1 = 4\tan A + \sec^2 A$, find $\tan A$.

(14) If $\sin A + \cos B = a$,

$\sin B + \cos A = b$,

then $\sin B = \frac{b}{2} \pm \frac{a}{2} \sqrt{\left(\frac{4}{a^2 + b^2} - 1\right)}$.

(15) Given $\tan x = \frac{3}{\tan x} + \frac{4}{\sqrt{\tan x}} + 3$, find $\tan x$:

$$\tan x = \frac{7 \pm \sqrt{13}}{2}.$$

(16) Given
$$\left.\begin{array}{r} \tan x \tan y = \tan^2 x - \tan^2 y \\ 2\tan^2 x = \dfrac{\tan^2 x + \tan^2 y}{\tan x - \tan y} \end{array}\right\}.$$

Find $\tan x$ and $\tan y$, without quadratics,

$$\tan x = \frac{\sqrt{5}+5}{4};\ \tan y = \frac{\sqrt{5}}{2}.$$

(17)
$$\left.\begin{array}{l} \tan x - \tan y = 2 \\ \dfrac{1}{12}(\tan x + \tan y) = \dfrac{1}{\tan x \tan y} \end{array}\right\} \begin{array}{l} \tan x = 3, \\ \tan y = 1. \end{array}$$

(18) $\dfrac{1}{4}\tan^2 x = \dfrac{1}{\tan x - 1}$; find $\tan x$:

$$\tan x = 2.$$

(19) $\tan^2 x + 48 \cot^2 x = 4 \cot x + 18$,

$$\tan x = 4 \text{ and } -2.$$

(20) $\dfrac{4\sec^2 x}{3} = 65 \cot x - 39$; find $\tan x$:

$$\tan x = \frac{3}{2}.$$

(21)
$$\left.\begin{array}{r} \sin^2 x + \sin y = \dfrac{3}{2} \\ \sin x \sin y = \dfrac{1}{2} \end{array}\right\} \begin{array}{l} \text{find } x \text{ and } y: \\ x = 90^\circ \text{ and } y = 30^\circ. \end{array}$$

(22) $\dfrac{1}{7}\tan^3 x = 3\dfrac{1}{7} - \tan x$; find $\tan x$:

$$\tan x = 2.$$

(23) If $\tan^2 A - \cot A = \dfrac{1}{8}(\cot A - 1)$, $A = 45^\circ$.

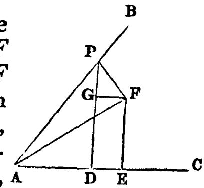
(24) $(\tan A - 1)^2 - 8 \cot A (1 - 2\cot A) = 8$: find $\tan A$.

(25) Prove that

$$\cfrac{1}{\tan A + \cfrac{1}{\tan A + \text{\&c., } ad\ inf.}} = \frac{1}{2}\{\sqrt{(\sec^2 A + 3)} - \tan A\}.$$

CHAPTER II.

7. To find the sine and cosine of the sum and difference of two angles.



Let the angle $FAC = A$, and the angle $FAP = B$, then $PAC = A + B$, draw PF perpendicular to AF, and from P and F let fall the perpendiculars PD and FE on AC, and draw FG parallel to AC. Now, because GF is parallel to AC, the alternate angles, AFG and FAC are equal, but $FAC = A$, therefore $AFG = A$; and since AFP is a right angle, the angle PFG is the complement of AFG; but FGP is a right angle, therefore FPG is the complement of PFG, that is, the angles FPG and AFG are the complements of the same angle (PFG), they are therefore equal; but AFG has been proved equal to A, hence $FPG = A$.

$$\sin(A+B) = \frac{DP}{AP} = \frac{DG + GP}{AP} = \frac{EF + PG}{AP}$$

$$= \frac{EF}{AP} + \frac{PG}{AP}$$

$$= \frac{EF}{AF} \cdot \frac{AF}{AP} + \frac{PG}{PF} \cdot \frac{PF}{AP};$$

but by the definitions, $\dfrac{EF}{AF} = \sin A$,

and $\dfrac{AF}{AP} = \cos B$; also $\dfrac{PG}{PF} = \cos FPG = \cos A$,

and $\dfrac{PF}{AP} = \sin B$, hence

$$\therefore \sin(A+B) = \sin A \cos B + \cos A \sin B.$$

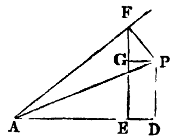
$$\cos(A+B) = \frac{AD}{AP} = \frac{AE - DE}{AP} = \frac{AE}{AP} - \frac{GF}{AP},$$

(because $GF = DE$)

$$= \frac{AE}{AF} \cdot \frac{AF}{AP} - \frac{GF}{PF} \cdot \frac{PF}{AP}$$

$$= \cos A \cos B - \sin A \sin B.$$

To find the sine and cosine of $A - B$.



Let the angle $DAF = A$, and the angle $PAF = B$, then $\angle\, PAD = A - B$: from any point P draw the perpendiculars PF and PD on AF and AD, also draw PG parallel to AD, then $\angle\, PFG =$ the complement of AFE, and therefore $= A$, because AFG is the complement of A; and because GD is a rectangle, $ED = PG$, and $PD = GE$.

$$\sin(A - B) = \frac{PD}{AP} = \frac{FE - FG}{AP} = \frac{FE}{AP} - \frac{FG}{AP}$$

$$= \frac{FE}{AF} \cdot \frac{AF}{AP} - \frac{FG}{FP} \cdot \frac{FP}{AP}$$

$$= \sin A \cos B - \cos A \sin B,$$

$$\cos(A - B) = \frac{AD}{AP} = \frac{AE + ED}{AP} = \frac{AE}{AP} + \frac{PG}{AP}$$

$$= \frac{AE}{AF} \cdot \frac{AF}{AP} + \frac{PG}{PF} \cdot \frac{PF}{AP}$$

$$= \cos A \cos B + \sin A \sin B.$$

8. Now, since $\sin(A + B) = \sin A \cos B + \cos A \sin B$, if we make $B = A$, then

$\sin(A + A) = \sin 2A = \sin A \cos A + \cos A \sin A = 2 \sin A \cos A$; also,

since $\cos(A + B) = \cos A \cos B - \sin A \sin B$; make $B = A$; then $\cos(A + A) = \cos 2A = \cos A \cos A - \sin A \sin A$

$$= \cos^2 A - \sin^2 A;$$

but since $\cos^2 A = 1 - \sin^2 A$, we have also

$$\cos 2A = 1 - \sin^2 A - \sin^2 A = 1 - 2\sin^2 A;$$

$$\text{also } \sin^2 A = 1 - \cos^2 A;$$

$$\therefore \cos 2A = \cos^2 A - (1 - \cos^2 A) = 2\cos^2 A - 1.$$

Since A may be any angle, we have the sine of any angle equal to twice the sine of half that angle multipled by the cosine of half that angle; and the cosine of any angle is equal to the square of the cosine of half that angle minus the sine square of half that angle. For the sines we have

$$\sin A = 2 \sin \frac{A}{2} \cos \frac{A}{2},$$

$$\sin \frac{1}{2} A = 2 \sin \frac{1}{4} A \cos \frac{1}{4} A,$$

$$\sin \frac{1}{3} A = 2 \sin \frac{1}{6} A \cos \frac{1}{6} A,$$

$$\sin \frac{1}{n} A = 2 \sin \frac{1}{2n} A \cos \frac{1}{2n} A.$$

In the same manner for the cosines,

$$\cos 2A = 1 - 2 \sin^2 A,$$

$$\cos A = 1 - 2 \sin^2 \frac{1}{2} A,$$

$$\cos \frac{1}{2} A = 1 - 2 \sin^2 \frac{1}{4} A,$$

$$\cos \frac{1}{3} A = 1 - 2 \sin^2 \frac{1}{6} A,$$

$$\cos \frac{1}{n} A = 1 - 2 \sin^2 \frac{1}{2n} A.$$

Also $\cos 2A = 2 \cos^2 A - 1,$

$$\cos A = 2 \cos^2 \frac{1}{2} A - 1,$$

$$\cos \frac{1}{2} A = 2 \cos^2 \frac{1}{4} A - 1,$$

$$\cos \frac{1}{3} A = 2 \cos^2 \frac{1}{6} A - 1,$$

$$\cos \frac{1}{n} A = 2 \cos^2 \frac{1}{2n} A - 1.$$

9. To find $\sin 3A$.

$$\sin 3A = \sin (2A + A).$$

By putting $2A$ for A, and A for B in the $\sin (A + B)$, we have

$$\sin (2A + A) = \sin 2A \cos A + \cos 2A \sin A;$$

but $\sin 2A = 2\sin A\cos A$, and $\cos 2A = 1 - 2\sin^2 A$

$$\begin{aligned}\sin(2A + A) &= 2\sin A\cos A\cos A + (1 - 2\sin^2 A)\sin A,\\ &= 2\sin A\cos^2 A + \sin A - 2\sin^3 A,\\ &= 2\sin A(1 - \sin^2 A) + \sin A - 2\sin^3 A,\\ &= 2\sin A - 2\sin^3 A + \sin A - 2\sin^3 A,\\ &= 3\sin A - 4\sin^3 A;\end{aligned}$$

that is, $\sin 3A = 3\sin A - 4\sin^3 A$.

To find $\cos 3A$.

$$\begin{aligned}\cos 3A &= \cos(2A + A) = \cos 2A\cos A - \sin 2A\sin A,\\ &= (2\cos^2 A - 1)\cos A - 2\sin A\cos A\sin A,\\ &= 2\cos^3 A - \cos A - 2\cos A\sin^2 A,\\ &= 2\cos^3 A - \cos A - 2\cos A(1 - \cos^2 A),\\ &= 2\cos^3 A - \cos A - 2\cos A + 2\cos^3 A,\\ &= 4\cos^3 A - 3\cos A.\end{aligned}$$

In the same way we may find the $\sin 4A$, &c., but it will be better to express them in general terms, as in the next Article.

10. Now, since

$$\sin(A + B) = \sin A\cos B + \cos A\sin B,$$
$$\sin(A - B) = \sin A\cos B - \cos A\sin B,$$

by addition we have

$$\sin(A + B) + \sin(A - B) = 2\sin A\cos B \quad\ldots\ldots\ldots(1).$$

By subtraction,

$$\sin(A + B) - \sin(A - B) = 2\cos A\sin B \quad\ldots\ldots\ldots(2).$$

By multiplication,

$$\begin{aligned}\sin(A + B)\sin(A - B) &= \sin^2 A\cos^2 B - \cos^2 A\sin^2 B\\ &= \sin^2 A(1 - \sin^2 B) - \sin^2 B(1 - \sin^2 A)\\ &= \sin^2 A - \sin^2 B = (\sin A + \sin B)(\sin A - \sin B)\ldots(3).\end{aligned}$$

Also, $\cos(A + B) = \cos A\cos B - \sin A\sin B$,

$\cos(A - B) = \cos A\cos B + \sin A\sin B$.

By addition,

$$\cos(A + B) + \cos(A - B) = 2\cos A\cos B \quad\ldots\ldots\ldots\ldots(4).$$

By subtraction,

$$\cos(A - B) - \cos(A + B) = 2\sin A\sin B \quad\ldots\ldots\ldots\ldots(5).$$

By multiplication,

$$\cos(A+B)\cos(A-B) = \cos^2 A \cos^2 B - \sin^2 A \sin^2 B$$
$$= \cos^2 A(1-\sin^2 B) - \sin^2 B(1-\cos^2 A)$$
$$= \cos^2 A - \sin^2 B$$
$$= (\cos A + \sin B)(\cos A - \sin B) \quad \ldots\ldots\ldots\ldots\ldots (6).$$

Now by equations (1) and (4):

$$\sin(A+B) + \sin(A-B) = 2\sin A\cos B,$$
$$\cos(A+B) + \cos(A-B) = 2\cos A\cos B;$$

for B put a, and for A put na; then,

$$A+B = na + a = (n+1)a, \text{ and } A-B = na - a = (n-1)a,$$
$$\sin(n+1)a + \sin(n-1)a = 2\sin na\cos a,$$
$$\cos(n+1)a + \cos(n-1)a = 2\cos na\cos a.$$

By giving to n different values, the sine or cosine of any multiple arc may be found.

11. Now, since half the sum of any two quantities added to half their difference gives the greater, and half the difference subtracted from half the sum gives the less, we have

$$A = \tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B),$$
$$B = \tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B).$$

Putting $\frac{1}{2}(A+B)$ for A, and $\frac{1}{2}(A-B)$ for B, in $\sin(A+B)$ and $\cos(A+B)$;

$$\sin A = \sin\{\tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B)\}$$
$$= \sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) + \cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \ldots (1).$$

In the same manner,

$$\sin B = \sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) - \cos\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \ldots (2).$$

By adding (1) and (2),

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \quad \ldots\ldots\ldots\ldots\ldots (3).$$

By subtracting (2) from (1),

$$\sin A - \sin B = 2\sin\tfrac{1}{2}(A-B)\cos\tfrac{1}{2}(A+B) \quad \ldots\ldots\ldots\ldots\ldots (4).$$

$$\cos A = \cos\{\tfrac{1}{2}(A+B) + \tfrac{1}{2}(A-B)\}$$
$$= \cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) - \sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \ldots (5).$$

$$\cos B = \cos\{\tfrac{1}{2}(A+B) - \tfrac{1}{2}(A-B)\}$$
$$= \cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) + \sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \ldots (6).$$

Add (5) and (6):

$$\cos A + \cos B = 2\cos\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B) \quad\ldots\ldots\ldots\ldots(7).$$

Subtract (5) from (6):

$$\cos B - \cos A = 2\sin\tfrac{1}{2}(A+B)\sin\tfrac{1}{2}(A-B) \quad\ldots\ldots\ldots\ldots(8).$$

By dividing (3) by (7):

$$\frac{\sin A + \sin B}{\cos A + \cos B} = \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} = \frac{\sin\frac{1}{2}(A+B)}{\cos\frac{1}{2}(A+B)}$$
$$= \tan\tfrac{1}{2}(A+B)\ldots\ldots\ldots(9).$$

Divide (4) by (7):

$$\frac{\sin A - \sin B}{\cos A + \cos B} = \frac{2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)}{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)} = \tan\tfrac{1}{2}(A-B).(10).$$

Divide (3) by (4):

$$\frac{\sin A + \sin B}{\sin A - \sin B} = \frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)}$$
$$= \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)} \quad\ldots\ldots\ldots(11).$$

Divide (7) by (8):

$$\frac{\cos B + \cos A}{\cos B - \cos A} = \frac{2\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}$$
$$= \cot\tfrac{1}{2}(A+B)\cot\tfrac{1}{2}(A-B) \quad\ldots\ldots\ldots(12).$$

Divide (4) by (8):

$$\frac{\sin A - \sin B}{\cos B - \cos A} = \frac{2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)}{2\sin\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}$$
$$= \cot\tfrac{1}{2}(A+B) \quad\ldots\ldots\ldots\ldots\ldots\ldots\ldots(13).$$

12. To express the $\tan(A \pm B)$ in terms of $\tan A$ and $\tan B$.

$$\tan(A+B) = \frac{\sin(A+B)}{\cos(A+B)} = \frac{\sin A\cos B + \cos A\sin B}{\cos A\cos B - \sin A\sin B}.$$

Divide the numerator and denominator of this fraction by $\cos A\cos B$; then

$$\tan(A+B) = \frac{\dfrac{\sin A\cos B}{\cos A\cos B} + \dfrac{\cos A\sin B}{\cos A\cos B}}{1 - \dfrac{\sin A\sin B}{\cos A\cos B}} = \frac{\tan A + \tan B}{1 - \tan A\tan B},$$

$$\tan(A-B)=\frac{\sin(A-B)}{\cos(A-B)}=\frac{\sin A\cos B-\cos A\sin B}{\cos A\cos B+\sin A\sin B}$$

$$=\frac{\dfrac{\sin A\cos B}{\cos A\cos B}-\dfrac{\cos A\sin B}{\cos A\cos B}}{1+\dfrac{\sin A\sin B}{\cos A\cos B}}=\frac{\tan A-\tan B}{1+\tan A\tan B}.$$

13. To express the $\cot(A\pm B)$ in terms of $\cot A$ and $\cot B$.

$$\cot(A+B)=\frac{\cos(A+B)}{\sin(A+B)}=\frac{\cos A\cos B-\sin A\sin B}{\sin A\cos B+\cos A\sin B}.$$

Dividing numerator and denominator by $\sin A\sin B$,

$$\cot(A+B)=\frac{\dfrac{\cos A\cos B}{\sin A\sin B}-1}{\dfrac{\sin A\cos B}{\sin A\sin B}+\dfrac{\cos A\sin B}{\sin A\sin B}}=\frac{\cot A\cot B-1}{\cot B+\cot A};$$

making $B=A$, we have

$$\cot 2A=\frac{\cot^2 A-1}{2\cot A}.$$

In precisely the same manner we obtain

$$\cot(A-B)=\frac{\cot A\cot B+1}{\cot B-\cot A}.$$

If in $\tan(A+B)$ we make $B=A$,

$$\text{then}\ \tan(A+A)=\frac{\tan A+\tan A}{1-\tan A\tan A};$$

$$\text{that is,}\ \tan 2A=\frac{2\tan A}{1-\tan^2 A}.$$

If for $2A$ we put A, and therefore $\frac{A}{2}$ for A, we have

$$\tan A=\frac{2\tan\frac{A}{2}}{1-\tan^2\frac{A}{2}}.$$

In the same manner

$$\tan\frac{A}{2}=\frac{2\tan\frac{A}{4}}{1-\tan^2\frac{A}{4}},$$

$$\tan\frac{A}{3} = \frac{2\tan\frac{A}{6}}{1-\tan^2\frac{A}{6}},$$

$$\tan\frac{A}{n} = \frac{2\tan\frac{A}{2n}}{1-\tan^2\frac{A}{2n}}.$$

14. Inverse trigonometrical functions.

The quantities $\sin^{-1}A$, $\cos^{-1}A$, &c., are called inverse functions, and have the following signification:—

$\sin^{-1}A$ is put for an angle whose sine is A,

$\cos^{-1}A$ is put for an angle whose cosine is A, &c., &c.

Let $\sin\alpha = a$, and $\sin\beta = b$,

$$\text{then } \alpha = \sin^{-1}a, \text{ and } \beta = \sin^{-1}b,$$

$$\cos\alpha = \sqrt{(1-a^2)}, \text{ and } \cos\beta = \sqrt{(1-b^2)},$$

$$\sin(\alpha+\beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta$$
$$= a\sqrt{(1-b^2)} + b\sqrt{(1-a^2)};$$

$$\therefore \alpha+\beta = \sin^{-1}\{a\sqrt{(1-b^2)} + b\sqrt{(1-a^2)}\},$$

$$\text{but } \alpha = \sin^{-1}a, \text{ and } \beta = \sin^{-1}b;$$

$$\therefore \sin^{-1}a + \sin^{-1}b = \sin^{-1}\{a\sqrt{(1-b^2)} + b\sqrt{(1-a^2)}\}.$$

Also in the same manner,

$$\sin^{-1}a - \sin^{-1}b = \sin^{-1}\{a\sqrt{(1-b^2)} - b\sqrt{(1-a^2)}\}.$$

By proceeding in the same manner for the cosine,

$$\cos^{-1}a + \cos^{-1}b = \cos^{-1}\{ab - \sqrt{(1-a^2)}\sqrt{(1-b^2)}\},$$

$$\cos^{-1}a - \cos^{-1}b = \cos^{-1}\{ab + \sqrt{(1-a^2)}\sqrt{(1-b^2)}\}.$$

Proceeding in the same way for the tangents,

$$\text{if } \tan\alpha = a, \ \tan\beta = b,$$

$$\text{then } \alpha = \tan^{-1}a, \text{ and } \beta = \tan^{-1}b,$$

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1-\tan\alpha\tan\beta} = \frac{a+b}{1-ab};$$

$$\therefore \alpha+\beta = \tan^{-1}\left(\frac{a+b}{1-ab}\right),$$

$$\text{but } \alpha = \tan^{-1}a, \text{ and } \beta = \tan^{-1}b,$$

$$\therefore \tan^{-1} a + \tan^{-1} b = \tan^{-1}\left(\frac{a+b}{1-ab}\right),$$

and since $\tan(\alpha - \beta) = \dfrac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta} = \dfrac{a-b}{1+ab}$;

$$\therefore \tan^{-1} a - \tan^{-1} b = \tan^{-1}\left(\frac{a-b}{1+ab}\right).$$

In the same way,

$$\cot^{-1} a \pm \cot^{-1} b = \cot^{-1}\left(\frac{ab \mp 1}{b \pm a}\right).$$

15. The following formulæ, which are sometimes called the formulæ of verification, are very useful:—

$$\text{since } \cos^2 A + \sin^2 A = 1,$$
$$\text{and } 2\sin A\cos A = \sin 2A.$$

By adding and subtracting we have

$$\cos^2 A + 2\sin A\cos A + \sin^2 A = 1 + \sin 2A,$$
$$\text{and } \cos^2 A - 2\sin A\cos A + \sin^2 A = 1 - \sin 2A.$$

By extracting the square root of each of these,

$$\cos A + \sin A = \sqrt{(1 + \sin 2A)},$$
$$\cos A - \sin A = \sqrt{(1 - \sin 2A)}.$$

When A is less than 45°, $\sin A$ is less than $\cos A$. By adding,

$$2\cos A = \sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)},$$
$$\cos A = \frac{1}{2}\sqrt{(1+\sin 2A)} + \frac{1}{2}\sqrt{(1-\sin 2A)}.$$

Subtracting,

$$2\sin A = \sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)},$$
$$\sin A = \frac{1}{2}\sqrt{(1+\sin 2A)} - \frac{1}{2}\sqrt{(1-\sin 2A)}.$$

When A is greater than 45°, then $\sin A$ is greater than $\cos A$, and the above will become

$$\cos A = \frac{1}{2}\{\sqrt{(1+\sin 2A)} - \sqrt{(1-\sin 2A)}\},$$
$$\sin A = \frac{1}{2}\{\sqrt{(1+\sin 2A)} + \sqrt{(1-\sin 2A)}\}.$$

To find the $\sin 18^\circ$.

Since the cosine of any angle is equal to the sine of its complement,

$$\cos 54^0 = \sin 36^0,$$

but $\cos 54^0 = \cos(36^0 + 18^0) = \cos 36^0 \cos 18^0 - \sin 36^0 \sin 18^0$;

$$\therefore\ \cos 36^0 \cos 18^0 - \sin 36^0 \sin 18^0 = \sin 36^0,$$

but $\cos 36^0 = 1 - 2\sin^2 18^0$, and $\sin 36^0 = 2 \sin 18^0 \cos 18^0$;

$\therefore\ \cos 18^0 (1 - 2\sin^2 18^0) - 2\sin^2 18^0 \cos 18^0 = 2 \sin 18^0 \cos 18^0$.

Dividing by $\cos 18^0$,

$$1 - 2\sin^2 18^0 - 2\sin^2 18^0 = 2\sin 18^0;$$

$$\therefore\ 4\sin^2 18^0 + 2\sin^2 18^0 = 1,$$

$$\sin^2 18^0 + \frac{1}{2}\sin 18^0 = \frac{1}{4},$$

$$\sin^2 18^0 + \frac{1}{2}\sin 18^0 + \left(\frac{1}{4}\right)^2 = \frac{1}{4} + \frac{1}{16} = \frac{5}{16},$$

$$\sin 18^0 + \frac{1}{4} = \frac{\sqrt{5}}{4},$$

$$\sin 18^0 = \frac{\sqrt{5} - 1}{4}.$$

The same result may be obtained by using the formula for the $\sin 3A$; or by the 10th proposition of the 4th book of Euclid.

$$\cos 18^0 = \sqrt{(1 - \sin^2 18^0)}$$

$$= \sqrt{\left\{1 - \left(\frac{\sqrt{5}-1}{4}\right)^2\right\}} = \sqrt{\frac{16 - 5 + 2\sqrt{5} - 1}{16}}$$

$$= \sqrt{\frac{5 + \sqrt{5}}{8}} = \frac{\sqrt{(5 + \sqrt{5})}}{2\sqrt{2}},$$

$$\sin 36^0 = 2 \sin 18^0 \cos 18^0$$

$$= \frac{2(\sqrt{5} - 1)}{4} \cdot \frac{\sqrt{(5 + \sqrt{5})}}{2\sqrt{2}} = \frac{(\sqrt{5} - 1)\sqrt{(5 + \sqrt{5})}}{4\sqrt{2}}$$

$$= \frac{\sqrt{\{(5 - 2\sqrt{5} + 1)(5 + \sqrt{5})\}}}{4\sqrt{2}} = \frac{\sqrt{(20 - 4\sqrt{5})}}{4\sqrt{2}}$$

$$= \frac{1}{2\sqrt{2}} \cdot \sqrt{\frac{20 - 4\sqrt{5}}{4}} = \frac{1}{2\sqrt{2}} \cdot \sqrt{(5 - \sqrt{5})}.$$

Now the $\sin 36^0 = \cos 54^0$,

$$\therefore \cos 54^0 = \frac{1}{2\sqrt{2}} \cdot \sqrt{(5 - \sqrt{5})},$$

$$\sin 54^0 = \sqrt{(1 - \cos^2 54^0)} = \sqrt{\left(1 - \frac{5 - \sqrt{5}}{8}\right)}\quad .$$

$$= \sqrt{\frac{8 - 5 + \sqrt{5}}{8}} = \sqrt{\frac{3 + \sqrt{5}}{8}} = \sqrt{\frac{6 + 2\sqrt{5}}{16}}$$

$$= \frac{\sqrt{(5 + 2\sqrt{5} + 1)}}{4} = \frac{\sqrt{5} + 1}{4}.$$

Or thus, $\sin 54^0 = \cos 36^0 = \cos 2\,(18^0) = \cos^2 18^0 - \sin^2 18^0$

$$= \frac{5 + \sqrt{5}}{8} - \frac{6 - 2\sqrt{5}}{16} = \frac{10 + 2\sqrt{5} - 6 + 2\sqrt{5}}{16}$$

$$= \frac{4 + 4\sqrt{5}}{16} = \frac{1 + \sqrt{5}}{4}.$$

Now since $\sin 18^0 = \cos 72^0$, and $\cos 18^0 = \sin 72^0$, we have the following results:

$$\sin 18^0 = \frac{\sqrt{5} - 1}{4}; \qquad \cos 18^0 = \frac{\sqrt{(5 + \sqrt{5})}}{2\sqrt{2}};$$

$$\sin 36^0 = \frac{\sqrt{(5 - \sqrt{5})}}{2\sqrt{2}}; \quad \cos 36^0 = \frac{\sqrt{5} + 1}{4};$$

$$\sin 54^0 = \frac{\sqrt{5} + 1}{4}; \qquad \cos 54^0 = \frac{\sqrt{(5 - \sqrt{5})}}{2\sqrt{2}};$$

$$\sin 72^0 = \frac{\sqrt{(5 + \sqrt{5})}}{2\sqrt{2}}; \quad \cos 72^0 = \frac{\sqrt{5} - 1}{4}.$$

By the formula, page 34, we may find the $\sin 15^0$.

$$\sin 15^0 = \frac{1}{2}\{\sqrt{(1 + \sin 30^0)} - \sqrt{(1 - \sin 30^0)}\}$$

$$= \frac{1}{2}\left\{\sqrt{\left(1 + \frac{1}{2}\right)} - \left\{\sqrt{\left(1 - \frac{1}{2}\right)}\right\}\right.$$

$$= \frac{1}{2}\left(\frac{\sqrt{3}}{\sqrt{2}} - \frac{1}{\sqrt{2}}\right) = \frac{\sqrt{3} - 1}{2\sqrt{2}}.$$

This might have been found by putting

$$\sin 15^0 = \sin (45^0 - 30^0).$$

Examples.

(1) Shew that

$$\cos(60^\circ + A) + \cos(60^\circ - A) = \cos A.$$

$$\cos(60^\circ + A) = \cos 60^\circ \cos A - \sin 60^\circ \sin A,$$
$$\cos(60^\circ - A) = \cos 60^\circ \cos A + \sin 60^\circ \sin A.$$

By addition,

$$\cos(60^\circ + A) + \cos(60^\circ - A) = 2\cos 60^\circ \cos A = 2 \times \frac{1}{2}\cos A$$
$$= \cos A \left(\text{because } \cos 60^\circ = \frac{1}{2}\right).$$

(2) $\sin(30^\circ + A) + \sin(30^\circ - A) = \cos A.$

$$\sin(30^\circ + A) = \sin 30^\circ \cos A + \cos 30^\circ \sin A,$$
$$\sin(30^\circ - A) = \sin 30^\circ \cos A - \cos 30^\circ \sin A.$$

By addition,

$$\sin(30^\circ + A) + \sin(30^\circ - A) = 2\sin 30^\circ \cos A = \cos A.$$

(3) $\sin A = \dfrac{2\tan \frac{1}{2}A}{1 + \tan^2 \frac{1}{2}A}$;

$$\sin A = \frac{\sin A}{1} = \frac{2\sin\frac{1}{2}A \cos\frac{1}{2}A}{1} = \frac{\dfrac{2\sin\frac{1}{2}A\cos\frac{1}{2}A}{\cos^2\frac{1}{2}A}}{\dfrac{1}{\cos^2\frac{1}{2}A}},$$

(by dividing numerator and denominator by $\cos^2 \frac{1}{2}A$)

$$= \frac{2\tan\frac{1}{2}A}{\sec^2\frac{1}{2}A} = \frac{2\tan\frac{1}{2}A}{1+\tan^2\frac{1}{2}A}.$$

(4) $2\cos(45^\circ + \frac{1}{2}A)\cos(45^\circ - \frac{1}{2}A) = \cos A.$

$$\cos(45^\circ + \tfrac{1}{2}A) = \cos 45^\circ \cos\tfrac{1}{2}A - \sin 45^\circ \sin\tfrac{1}{2}A$$
$$= \frac{1}{\sqrt{2}}(\cos\tfrac{1}{2}A - \sin\tfrac{1}{2}A);$$
$$\cos(45^\circ - \tfrac{1}{2}A) = \cos 45^\circ \cos\tfrac{1}{2}A + \sin 45^\circ \sin\tfrac{1}{2}A$$
$$= \frac{1}{\sqrt{2}}(\cos\tfrac{1}{2}A + \sin\tfrac{1}{2}A).$$

By multiplication,

$$2\cos(45^\circ + \tfrac{1}{2}A)\cos(45^\circ - \tfrac{1}{2}A)$$
$$= 2 \times \frac{1}{\sqrt{2}} \times \frac{1}{\sqrt{2}}(\cos^2\tfrac{1}{2}A - \sin^2\tfrac{1}{2}A)$$
$$= \cos^2\tfrac{1}{2}A - \sin^2\tfrac{1}{2}A = \cos A.$$

(5) Given

$\sin(x+a)+\cos(x+a)=\sin(x-a)+\cos(x-a)$, find x.

$$\sin(x+a)-\sin(x-a)=\cos(x-a)-\cos(x+a),$$
$$\sin(x+a)=\sin x\cos a+\sin a\cos x,$$
$$\sin(x-a)=\sin x\cos a-\sin a\cos x;$$
$$\therefore\ \sin(x+a)-\sin(x-a)=2\sin a\cos x,$$
$$\cos(x-a)=\cos x\cos a+\sin x\sin a,$$
$$\cos(x+a)=\cos x\cos a-\sin x\sin a;$$
$$\therefore\ \cos(x-a)-\cos(x+a)=2\sin x\sin a;$$
hence, $2\sin a\cos x=2\sin a\sin x$;
$$\therefore\ \cos x=\sin x;$$
$$\therefore\ x=45^0.$$

(6) If $\sin x+\cos 2x=\sqrt{\frac{5}{4}}$; prove that $x=18^0$.

Since $\cos 2x=1-2\sin^2 x$, we have

$$\sin x+1-2\sin^2 x=\frac{\sqrt{5}}{2},$$
$$\sin^2 x-\frac{1}{2}\sin x=\frac{2-\sqrt{5}}{4};$$
$$\sin^2 x-\frac{1}{2}\sin x+\frac{1}{16}=\frac{2-\sqrt{5}}{4}+\frac{1}{16}=\frac{8-4\sqrt{5}}{16}+\frac{1}{16}$$
$$=\frac{9-4\sqrt{5}}{16}=\frac{(\sqrt{5}-2)^2}{16},$$
$$\sin x-\frac{1}{4}=\frac{\sqrt{5}-2}{4},$$
$$\sin x=\frac{\sqrt{5}-2}{4}+\frac{1}{4}=\frac{\sqrt{5}-1}{4}=\sin 18^0;$$
$$\therefore\ x=18^0.$$

(7) Find the tangent of 15^0.

$$\tan 15^0=\tan(45^0-30^0)=\frac{\tan 45^0-\tan 30^0}{1+\tan 45^0\tan 30^0}$$
$$=\frac{1-\frac{1}{\sqrt{3}}}{1+\frac{1}{\sqrt{3}}}=\frac{\sqrt{3}-1}{\sqrt{3}+1}=\frac{(\sqrt{3}-1)^2}{2}=2-\sqrt{3}.$$

(8) If $x = \sin^{-1}\frac{3}{5}$, and $y = \sin^{-1}\frac{4}{5}$, then, $x + y = 90^\circ$.

$$\text{Since } \sin x = \frac{3}{5}; \quad \cos x = \sqrt{\left\{1 - \left(\frac{3}{5}\right)^2\right\}} = \frac{4}{5},$$

$$\text{and } \sin y = \frac{4}{5}; \quad \cos y = \sqrt{\left\{1 - \left(\frac{4}{5}\right)^2\right\}} = \frac{3}{5},$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y$$

$$= \frac{3}{5} \times \frac{3}{5} + \frac{4}{5} \times \frac{4}{5} = \frac{9}{25} + \frac{16}{25} = \frac{25}{25} = 1,$$

$$\text{but } \sin 90^\circ = 1; \quad \therefore x + y = 90^\circ,$$

or we may put it in the following form,

$$\sin^{-1}\frac{3}{5} + \sin^{-1}\frac{4}{5} = \sin^{-1}\left(\frac{3}{5} \times \frac{3}{5} + \frac{4}{5} \times \frac{4}{5}\right) = \sin^{-1} 1.$$

(9) Shew that $4 \sin 9^\circ = \sqrt{(3 + \sqrt{5})} - \sqrt{(5 - \sqrt{5})}$.

By Art. 21, $\sin A = \frac{1}{2}\{\sqrt{(1 + \sin 2A)} - \sqrt{(1 - \sin 2A)}\}$,

we use the under sign as the angle is less than 45°.

Now, in the above formula let $A = 9^\circ$, then, $2A = 18^\circ$,

$$\sin 9^\circ = \frac{1}{2}\{\sqrt{(1 + \sin 18^\circ)} - \sqrt{(1 - \sin 18^\circ)}\}$$

$$= \frac{1}{2}\left\{\sqrt{\left(1 + \frac{\sqrt{5} - 1}{4}\right)} - \sqrt{\left(1 - \frac{\sqrt{5} - 1}{4}\right)}\right\}$$

$$= \frac{1}{4}\{\sqrt{(3 + \sqrt{5})} - \sqrt{(5 - \sqrt{5})}\};$$

$$\therefore 4 \sin 9^\circ = \sqrt{(3 + \sqrt{5})} - \sqrt{(5 - \sqrt{5})}.$$

(10) Shew that $2 \cos 11^\circ 15' = \sqrt{\{2 + \sqrt{(2 + \sqrt{2})}\}}$.

$$\text{Since } 2\cos^2 \tfrac{1}{2}A - 1 = \cos A,$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{1 + \cos A}{2}},$$

$$\text{let } A = 45^\circ, \text{ then } \tfrac{1}{2}A = 22^\circ 30',$$

$$\cos 22^\circ 30' = \sqrt{\frac{1 + \cos 45^\circ}{2}} = \frac{\sqrt{1 + \frac{1}{\sqrt{2}}}}{2}$$

$$= \sqrt{\frac{\sqrt{2} + 1}{2\sqrt{2}}} = \sqrt{\frac{2 + \sqrt{2}}{4}} = \frac{\sqrt{(2 + \sqrt{2})}}{2},$$

$$\cos 11^\circ 15' = \sqrt{\frac{1+\cos 22^\circ 30'}{2}} = \frac{\sqrt{1+\frac{\sqrt{(2+\sqrt{2})}}{2}}}{2}$$

$$= \sqrt{\frac{2+\sqrt{(2+\sqrt{2})}}{4}} = \frac{\sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}}{2};$$

$$\therefore\ 2\cos 11^\circ 15' = \sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}.$$

Or it might have been put in this form:

$$2\cos^2\frac{45^\circ}{2} - 1 = \cos 45^\circ = \frac{1}{\sqrt{2}},$$

$$2\left(2\cos^2\frac{45^\circ}{2} - 1\right) = \frac{2}{\sqrt{2}} = \sqrt{2},$$

$$4\cos^2\frac{45^\circ}{2} = 2+\sqrt{2},$$

$$2\cos\frac{45^\circ}{2} = \sqrt{(2+\sqrt{2})},$$

$$2\left(2\cos^2\frac{45^\circ}{4} - 1\right) = \sqrt{(2+\sqrt{2})};$$

$$\therefore\ 2\cos\frac{45^\circ}{4} = \sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}.$$

This is the same as $2\cos\frac{45^\circ}{2^2} = \sqrt{\{2+\sqrt{(2+\sqrt{2})}\}}$, and by continuing the operation to $\frac{45^\circ}{2^n}$, we have

$$2\cos\frac{45^\circ}{2^n} = \sqrt{\{2+\sqrt{(2+ \text{to } n+1 \text{ terms})}\}}.$$

(11) If A, B, and C be in arithmetical progression,

$$\sin A - \sin B = 2\sin(A-B)\cos B.$$

Since B is an arithmetical mean between A and C, we have

$$\tfrac{1}{2}(A+C) = B \ \ldots\ldots\ (1),\quad \tfrac{1}{2}(A-C) = B - C \ \ldots\ldots\ (2),$$

$$A - B = B - C \ \ldots\ldots\ldots\ (3).$$

By page 29,

$$\sin A - \sin C = 2\sin\tfrac{1}{2}(A-C)\cos\tfrac{1}{2}(A+C)$$

$$= 2\sin(B-C)\cos B;\ \text{by (2) and (1)}$$

$$= 2\sin(A-B)\cos B;\ \text{by (3)}.$$

(12) **Show that**

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=45^0.$$

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}$$

$$=\left(\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}\right)+\left(\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}\right),$$

$$\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}=\tan^{-1}\left\{\frac{\frac{1}{3}+\frac{1}{5}}{1-\frac{1}{3}\cdot\frac{1}{5}}\right\}=\tan^{-1}\frac{4}{7},$$

$$\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{8}}{1-\frac{1}{7}\cdot\frac{1}{8}}\right\}=\tan^{-1}\frac{3}{11};$$

$$\therefore\ \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=\tan^{-1}\frac{4}{7}+\tan^{-1}\frac{3}{11},$$

$$\text{but }\tan^{-1}\frac{4}{7}+\tan^{-1}\frac{3}{11}=\tan^{-1}\left\{\frac{\frac{4}{7}+\frac{3}{11}}{1-\frac{4}{7}\cdot\frac{3}{11}}\right\}=\tan^{-1}(1)=45^0;$$

$$\therefore\ \tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{8}=45^0.$$

(13) Given $\text{vers}^{-1}\frac{x}{a}-\text{vers}^{-1}(1-b)=\text{vers}^{-1}\frac{bx}{a}$; find $\frac{x}{a}$.

$$\cos^{-1}\left(1-\frac{x}{a}\right)-\cos^{-1}\{1-(1-b)\}=\cos^{-1}\left(1-\frac{bx}{a}\right),$$

$$\therefore\ \text{p. }32,\ \left\{1-\frac{x}{a}\right\}b+\sqrt{(1-b^2)}\sqrt{\left\{1-\left(1-\frac{x}{a}\right)^2\right\}}=1-\frac{bx}{a};$$

$$\therefore\ \frac{1-b}{\sqrt{(1-b^2)}}=\sqrt{\frac{1-b}{1+b}}=\sqrt{\left\{1-\left(1-\frac{x}{a}\right)^2\right\}};$$

$$\therefore\ \left(1-\frac{x}{a}\right)^2=1-\frac{1-b}{1+b}=\frac{2b}{1+b},$$

$$1-\frac{x}{a}=\pm\sqrt{\frac{2b}{1+b}};$$

$$\therefore \frac{x}{a} = 1 \mp \sqrt{\frac{2b}{1+b}}.$$

(14) If $\sin x = \frac{1}{\sqrt{3}}$ and $\cos y = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3}}$, then $x - y = 30^0$.

$$\cos y = \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3}} = \frac{\sqrt{6}+1}{2\sqrt{3}}$$

$$\sin y = \sqrt{\left\{1 - \left(\frac{\sqrt{6}+1}{2\sqrt{3}}\right)^2\right\}} = \frac{\sqrt{(5-2\sqrt{6})}}{2\sqrt{3}}$$

$$\cos x = \sqrt{\left(1 - \frac{1}{3}\right)} = \sqrt{\frac{2}{3}}$$

$$\sin(x-y) = \sin x \cos y - \cos x \sin y$$

$$= \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{6}+1}{2\sqrt{3}} - \frac{\sqrt{2}}{\sqrt{3}} \cdot \frac{\sqrt{(5-2\sqrt{6})}}{2\sqrt{3}}$$

$$= \frac{\sqrt{6}+1}{6} - \frac{\sqrt{2} \cdot \sqrt{(5-2\sqrt{6})}}{6}$$

$$= \frac{\sqrt{6}+1-\sqrt{(10-4\sqrt{6})}}{6} = \frac{\sqrt{6}+1-\sqrt{(4-4\sqrt{6}+6)}}{6}$$

$$= \frac{\sqrt{6}+1-\sqrt{(\sqrt{6}-2)^2}}{6} = \frac{\sqrt{6}+1-(\sqrt{6}-2)}{6} = \frac{3}{6} = \frac{1}{2} = \sin 30^0.$$

Hence $x - y = 30^0$.

(15) If $4 \sin x \sin 3x = 1$, find x.

$$4 \sin x(3 \sin x - 4 \sin^3 x) = 1,$$

$$12 \sin^2 x - 16 \sin^4 x = 1,$$

$$\sin^4 x - \frac{3}{4} \sin^2 x = -\frac{1}{16},$$

$$\sin^4 x - \frac{3}{4} \sin^2 x + \frac{9}{64} = \frac{9}{64} - \frac{1}{16} = \frac{5}{64},$$

$$\sin^2 x - \frac{3}{8} = \pm \frac{\sqrt{5}}{8}$$

$$\sin^2 x = \frac{3 \pm \sqrt{5}}{8} = \frac{6 \pm 2\sqrt{5}}{16} = \frac{(\sqrt{5} \pm 1)^2}{16},$$

$\sin x = \frac{\sqrt{5} \pm 1}{4}$ taking the upper sign, $x = 54^0$, and taking the under sign, $x = 18^0$.

(16) If $\cos x = \sqrt{\frac{2}{3}}$ and $\cos y = \frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}$, show that $x + y = 60^\circ$.

$$\sin x = \sqrt{\left(1 - \frac{2}{3}\right)} = \sqrt{\frac{1}{3}} = \frac{1}{\sqrt{3}},$$

$$\sin y = \sqrt{\left\{1 - \left(\frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}}\right)^2\right\}} = \frac{\sqrt{(7 - 2\sqrt{6})}}{2\sqrt{3}}$$

$$= \frac{\sqrt{(\sqrt{6}-1)^2}}{2\sqrt{3}} = \frac{\sqrt{6}-1}{2\sqrt{3}},$$

$$\cos(x+y) = \cos x \cos y - \sin x \sin y$$

$$= \sqrt{\frac{2}{3}} \cdot \frac{\sqrt{3}+\sqrt{2}}{2\sqrt{3}} - \frac{1}{\sqrt{3}} \cdot \frac{\sqrt{6}-1}{2\sqrt{3}}$$

$$\frac{\sqrt{6}+2}{6} - \frac{\sqrt{6}-1}{6} = \frac{\sqrt{6}+2-\sqrt{6}+1}{6} = \frac{3}{2} - \frac{1}{2} = \cos 60^\circ.$$

(17) If $\tan 2x = \frac{2 \tan x}{1 - 4\tan^2 x}$, show that the radius of the arc x is $\frac{1}{2}$.

$$\text{Since } \tan 2x = \frac{2\tan x}{1 - \tan^2 x}$$

$$\tan 2x = \frac{2r^2 \tan x}{r^2 - \tan^2 x} \text{ to radius } r;$$

$$\therefore \frac{2r^2 \tan x}{r^2 - \tan^2 x} = \frac{2\tan x}{1 - 4\tan^2 x}$$

$$\frac{r^2}{r^2 - \tan^2 x} = \frac{1}{1 - 4\tan^2 x}$$

$$r^2 - 4r^2 \tan^2 x = r^2 - \tan^2 x$$

$$4r^2 = 1$$

$$r^2 = \frac{1}{4}; \therefore r = \frac{1}{2}.$$

(18) If $\sin(\alpha+\theta)\sin\theta = \cos^2\frac{\alpha}{2}$, show that $\theta = 90^\circ - \frac{\alpha}{2}$.

$$\text{Since } \cos^2\frac{\alpha}{2} = \frac{1+\cos\alpha}{2},$$

$$(\sin\alpha\cos\theta + \cos\alpha\sin\theta)\sin\theta = \frac{1+\cos\alpha}{2},$$

$$\sin\alpha\sin\theta\cos\theta + \cos\alpha\sin^2\theta = \frac{1+\cos\alpha}{2},$$

$$\frac{1}{2}\sin\alpha\sin 2\theta + \frac{1-\cos 2\theta}{2}\cos\alpha = \frac{1+\cos\alpha}{2},$$

$$\sin\alpha\sin 2\theta + (1-\cos 2\theta)\cos\alpha = 1+\cos\alpha,$$

$$\sin\alpha\sin 2\theta - \cos\alpha\cos 2\theta = 1;$$

$$\text{or, } \cos\alpha\cos 2\theta - \sin\alpha\sin 2\theta = -1;$$

$$\text{i. e., } \cos(\alpha+2\theta) = -1;$$

$$\therefore\ \alpha + 2\theta = 180^\circ;$$

$$\therefore\ \theta = \frac{180^\circ - \alpha}{2} = 90^\circ - \frac{\alpha}{2}.$$

(19) Cos $n\theta + \cos(n-2)\theta = \cos\theta$, then $\theta = \dfrac{60^\circ}{n-1}$.

$$\cos\{(n-1)+1\}\theta + \cos\{(n-1)-1\}\theta = \cos\theta,$$

$$\cos(n-1)\theta\cos\theta - \sin(n-1)\theta\sin\theta + \cos(n-1)\theta\cos\theta$$
$$+\sin(n-1)\theta\sin\theta = \cos\theta;$$

$$\therefore\ 2\cos(n-1)\theta = 1,$$

$$\cos(n-1)\theta = \frac{1}{2},$$

$$(n-1)\theta = 60^\circ;$$

$$\therefore\ \theta = \frac{60^\circ}{n-1}.$$

(20) Show that

$$\sin^{-1}\frac{1}{\sqrt{5}} + \cot^{-1}3 = 45^\circ.$$

Let $z = \cot^{-1}3$; $\therefore\ \cot z = 3$,

$$\frac{\cos^2 z}{\sin^2 z} = 9 \text{ or } \frac{1-\sin^2 z}{\sin^2 z} = 9;\ \therefore\ \sin^2 z = \frac{1}{10} \text{ and } \sin z = \frac{1}{\sqrt{10}};$$

the above becomes $\sin^{-1}\dfrac{1}{\sqrt{5}} + \sin^{-1}\dfrac{1}{\sqrt{10}} = 45^\circ$,

$$\sin^{-1}\frac{1}{\sqrt{5}} + \sin^{-1}\frac{1}{\sqrt{10}} = \sin^{-1}\left(\frac{1}{\sqrt{5}}\cdot\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}}\cdot\frac{2}{\sqrt{5}}\right)$$

$$= \sin^{-1}\left(\frac{3}{\sqrt{50}} + \frac{2}{\sqrt{50}}\right) = \sin^{-1}\frac{5}{\sqrt{50}} = \sin^{-1}\frac{1}{\sqrt{2}} = 45^\circ.$$

(21) Show that

$$\tan^{-1}\frac{1}{7} + 2\tan^{-1}\frac{1}{3} = 45^\circ.$$

This is the same as $\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}+\tan^{-1}\frac{1}{3}=45^\circ$,

$$\tan^{-1}\frac{1}{7}+\tan^{-1}\frac{1}{3}=\tan^{-1}\left\{\frac{\frac{1}{7}+\frac{1}{3}}{1-\frac{1}{7}\cdot\frac{1}{3}}\right\}=\tan^{-1}\left\{\frac{\frac{10}{21}}{\frac{20}{21}}\right\}=\tan^{-1}\frac{1}{2},$$

$$\text{then }\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\left\{\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}\right\}=\tan^{-1}\left\{\frac{\frac{5}{6}}{\frac{5}{6}}\right\}$$
$$=\tan^{-1}1=45^\circ.$$

(22) Find $\sin A$ from the equation

$$\tan\tfrac{1}{2}A=\operatorname{cosec}A-\sin A.$$

$$\sin A=\operatorname{cosec}A-\tan\tfrac{1}{2}A=\frac{1}{\sin A}-\tan\tfrac{1}{2}A,$$

$$=\frac{1}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}-\frac{\sin\frac{1}{2}A}{\cos\frac{1}{2}A}=\frac{1}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}-\frac{2\sin^2\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}$$
$$=\frac{1-2\sin^2\frac{1}{2}A}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}=\frac{\cos A}{\sin A};$$

$$\therefore\ \sin^2A=\cos A=\sqrt{(1-\sin^2A)},$$
$$\sin^4A=1-\sin^2A,$$
$$\sin^4A+\sin^2A=1,$$
$$\sin^4A+\sin^2A+\frac{1}{4}=1+\frac{1}{4}=\frac{5}{4},$$
$$\sin^2A+\frac{1}{2}=\frac{\sqrt{5}}{2},$$
$$\sin^2A=-\frac{1}{2}\pm\frac{\sqrt{5}}{2},$$
$$\sin A=\sqrt{\frac{-1\pm\sqrt{5}}{2}}.$$

(23) $\cot^2(45^\circ+\frac{1}{2}A)=\dfrac{2\operatorname{cosec}2A-\sec A}{2\operatorname{cosec}2A+\sec A}$.

$$\cot^2(45^\circ+\tfrac{1}{2}A)=\frac{(\cos\frac{1}{2}A-\sin\frac{1}{2}A)^2}{(\cos\frac{1}{2}A+\sin\frac{1}{2}A)^2}$$
$$=\frac{\cos^2\frac{1}{2}A-2\sin\frac{1}{2}A\cos\frac{1}{2}A+\sin^2\frac{1}{2}A}{\cos^2\frac{1}{2}A+2\sin\frac{1}{2}A\cos\frac{1}{2}A+\sin^2\frac{1}{2}A}$$

$$=\frac{1-2\sin\frac{1}{2}A\cos\frac{1}{2}A}{1+2\sin\frac{1}{2}A\cos\frac{1}{2}A}=\frac{1-\sin A}{1+\sin A}$$

$$=\frac{1-\dfrac{1}{\operatorname{cosec} A}}{1+\dfrac{1}{\operatorname{cosec} A}}=\frac{\operatorname{cosec} A-1}{\operatorname{cosec} A+1}$$

$$=\frac{\sec A\operatorname{cosec} A-\sec A}{\sec A\operatorname{cosec} A+\sec A}$$

$$=\frac{2\operatorname{cosec} 2A-\sec A}{2\operatorname{cosec} 2A+\sec A}.$$

(24) Prove $\dfrac{2\sin A-\sin 2A}{2\sin A+\sin 2A}=\tan^2\frac{1}{2}A.$

$$\frac{2\sin A-\sin 2A}{2\sin A+\sin 2A}=\frac{2\sin A-2\sin A\cos A}{2\sin A+2\sin A\cos A}=\frac{1-\cos A}{1+\cos A}$$

$$=\frac{1-(1-2\sin^2\frac{1}{2}A)}{1+\cos^2\frac{1}{2}A-1}=\frac{2\sin^2\frac{1}{2}A}{2\cos^2\frac{1}{2}A}=\frac{\sin^2\frac{1}{2}A}{\cos^2\frac{1}{2}A}=\tan^2\frac{1}{2}A.$$

(25) Prove that $\dfrac{\sin A+\sin 3A+\sin 5A}{\cos A+\cos 3A+\cos 5A}=\tan 3A.$

By formula, page 29,

$$\sin A+\sin 5A=2\sin 3A\cos 2A,$$

$$\cos A+\cos 5A=2\cos 3A\cos 2A,$$

$$\sin A+\sin 3A+\sin 5A=\sin 3A+2\sin 3A\cos 2A,$$
$$=\sin 3A\,(1+2\cos 2A),$$

$$\cos A+\cos 3A+\cos 5A=\cos 3A+2\cos 3A\cos 2A$$
$$=\cos 3A\,(1+2\cos 2A);$$

$$\therefore\ \frac{\sin A+\sin 3A+\sin 5A}{\cos A+\cos 3A+\cos 5A}=\frac{\sin 3A\,(1+2\cos 2A)}{\cos 3A\,(1+2\cos 2A)}$$
$$=\tan 3A.$$

(26) If $A+B+C=90°$, prove that

$$\cot A+\cot B+\cot C=\cot A\cot B\cot C.$$

By Art. 19, $\cot(A+B)=\dfrac{\cot A\cot B-1}{\cot B+\cot A}.$

Put $A+B$ for A, and C for B, and we have

$$\cot(A+B+C)=\frac{\cot(A+B)\cot C-1}{\cot C+\cot(A+B)}=0,$$

because the $\cot 90^0=0$;

$$\therefore\ \cot(A+B)\cot C-1=0,$$

$$\cot(A+B)\cot C=1,$$

$$\text{or}\quad \frac{\cot A\cot B-1}{\cot B+\cot A}\cot C=1,$$

$$\cot A\cot B\cot C-\cot C=\cot A+\cot B;$$

$$\therefore\ \cot A\cot B\cot C=\cot A+\cot B+\cot C.$$

(27) If $A+B+C=180^0$, prove that

$$\cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1,$$

$$\cos A=-\cos(B+C)=-\cos B\cos C+\sin B\sin C,$$

$$\cos^2 A=-\cos A\cos B\cos C+\sin B\sin C\cos A,$$

$$\cos^2 B=-\cos A\cos B\cos C+\sin A\sin C\cos B,$$

$$\cos^2 A+\cos^2 B$$
$$=-2\cos A\cos B\cos C+\sin C(\sin B\cos A+\sin A\cos B),$$

$$\cos^2 A+\cos^2 B+2\cos A\cos B\cos C=\sin C\sin(A+B)$$
$$=\sin C\sin C=\sin^2 C$$
$$=1-\cos^2 C;$$

$$\therefore\ \cos^2 A+\cos^2 B+\cos^2 C+2\cos A\cos B\cos C=1.$$

On the same supposition,

$$\sin 2A+\sin 2B+\sin 2C=4\sin A\sin B\sin C,$$

$$\sin 2A+\sin 2B+\sin 2C$$
$$=2\sin A\cos A+2\sin B\cos B+2\sin C\cos C,$$

but $\cos A=-\cos(B+C)$, $\cos C=-\cos(A+B)$,
and $\sin B=\sin(A+C)$;

$$\therefore\ \sin 2A+\sin 2B+\sin 2C=-2\sin A\cos(B+C)$$
$$+2\cos B\sin(A+C)-2\sin C\cos(A+B)$$

= (by expanding)

$$-2\sin A\cos B\cos C+2\sin A\sin B\sin C,$$
$$+2\sin A\cos B\cos C+2\sin A\sin B\sin C,$$
$$-2\cos A\cos B\sin C,$$
$$+2\cos A\cos B\sin C,$$
$$=4\sin A\sin B\sin C.$$

Also, $\sin A + \sin B + \sin C = 4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C$:

$$\sin C = \sin(A+B),$$

$$\sin A + \sin B = 2\sin\tfrac{1}{2}(A+B)\cos\tfrac{1}{2}(A-B),$$

$$\sin C = \sin(A+B) = 2\sin\tfrac{1}{2}(A+B)\cos(A+B),$$

$$\tfrac{1}{2}(A+B) = 90^0 - \tfrac{1}{2}C,$$

$$\sin\tfrac{1}{2}(A+B) = \sin(90^0 - \tfrac{1}{2}C) = \cos\tfrac{1}{2}C,$$

$$\sin A + \sin B + \sin C = 2\cos\tfrac{1}{2}C\{\cos\tfrac{1}{2}(A+B) + \cos\tfrac{1}{2}(A-B)\},$$

but $\cos\frac{1}{2}(A+B) + \cos\frac{1}{2}(A-B) = 2\cos\frac{1}{2}A\cos\frac{1}{2}B$,

$$\therefore \sin A + \sin B + \sin C = 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$$

Or thus; since $A + B + C = 180^0$, then $\frac{1}{2}A + \frac{1}{2}B + \frac{1}{2}C = 90^0$,

$\sin A + \sin B + \sin C$

$$= 2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A + 2\sin\tfrac{1}{2}B\cos\tfrac{1}{2}B + 2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C$$
$$= 2\cos\tfrac{1}{2}A\cos(\tfrac{1}{2}B + \tfrac{1}{2}C) + 2\sin\tfrac{1}{2}B\sin(\tfrac{1}{2}A + \tfrac{1}{2}C)$$
$$+ 2\cos\tfrac{1}{2}C\cos(\tfrac{1}{2}A + \tfrac{1}{2}B),$$

= (by expanding)

$$2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C - 2\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\cos\tfrac{1}{2}A$$
$$+ 2\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C + 2\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C\cos\tfrac{1}{2}A$$
$$+ 2\sin\tfrac{1}{2}B\sin\tfrac{1}{2}A\cos\tfrac{1}{2}C$$
$$- 2\sin\tfrac{1}{2}B\sin\tfrac{1}{2}A\cos\tfrac{1}{2}C$$

$$= 4\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C.$$

(28) If $\sin 2x = (\sin 3x)^2$, show that $x = 15^0$.

Subtracting each side from unity, $1 - \sin 2x = 1 - \sin^2 3x$,

$$\cos^2 x - 2\sin x\cos x + \sin^2 x = \cos^2 3x,$$

$$\cos x - \sin x = \cos 3x = 4\cos^3 x - 3\cos x,$$

$$\sin x = 4\cos x(1 - \cos^2 x) = 4\cos x\sin^2 x;$$

$$\therefore \sin x = 0. \quad \therefore x = 0, \text{ or } \pi,$$

$$\text{and } 2\sin x\cos x = \frac{1}{2},$$

$$\text{or } \sin 2x = \frac{1}{2} = \sin 30^0;$$

$$\therefore x = 15^0.$$

(29) $\left.\begin{array}{l} \cos x + \cos y = \dfrac{1}{2}(\sqrt{2} + \sqrt{3}) \\ \cos 3x + \cos 3y = -\dfrac{1}{\sqrt{2}} \end{array}\right\}$, find x and y.

The second equation is the same as

$$4(\cos^3 x + \cos^3 y) - 3(\cos x + \cos y) = -\frac{1}{\sqrt{2}}; \text{ (page 28.)}$$

$$\therefore \cos^5 x + \cos^3 y = \frac{1}{4}\left\{\frac{3}{\sqrt{2}} + \frac{3\sqrt{3}}{2} - \frac{1}{\sqrt{2}}\right\},$$

$$\text{or } (\cos x + \cos y)(\cos^2 x - \cos x \cos y + \cos^2 y)$$

$$= \frac{1}{4}\left(\sqrt{2} + \frac{3\sqrt{3}}{2}\right);$$

$$\therefore \cos^2 x - \cos x \cos y + \cos^2 y = \frac{1}{4}\left(\frac{2\sqrt{2} + 3\sqrt{3}}{\sqrt{2} + \sqrt{3}}\right)$$

$$= \frac{2\sqrt{6} + 9 - 4 - 3\sqrt{6}}{4}, \text{ by multiplying by } (\sqrt{3} - \sqrt{2});$$

$$\therefore \cos^2 x - \cos x \cos y + \cos^2 y = \frac{1}{4}(5 - \sqrt{6}),$$

$$\text{but } \cos^2 x + 2\cos x \cos y + \cos^2 y = \frac{1}{4}(5 + 2\sqrt{6});$$

$$\therefore 3\cos x \cos y = \frac{1}{4}.3\sqrt{6};$$

$$\therefore \cos x \cos y = \frac{1}{4}\sqrt{6};$$

$$\therefore \cos^2 x - 2\cos x \cos y + \cos^2 y = \frac{1}{4}(5 - 2\sqrt{6});$$

$$\therefore \cos x - \cos y = \pm\frac{1}{2}(\sqrt{3} - \sqrt{2});$$

$$\text{but } \cos x + \cos y = \frac{1}{2}(\sqrt{3} + \sqrt{2}).$$

By adding these equations, we have

$$\cos x = \frac{\sqrt{3}}{2}, \text{ or } \frac{1}{\sqrt{2}}; \therefore x = 30^\circ, \text{ or } 45^\circ.$$

By subtracting them

$$\cos y = \frac{1}{\sqrt{2}}, \text{ or } \frac{\sqrt{3}}{2}; \therefore y = 45^\circ, \text{ or } 30^\circ.$$

(1) Prove the following theorems,

$$\sin A = \frac{2\cos\frac{1}{2}A}{\operatorname{cosec}\frac{1}{2}A} = \frac{2\tan\frac{1}{2}A}{\sec^2\frac{1}{2}A} = \frac{2}{\tan\frac{1}{2}A + \cot\frac{1}{2}A}$$

$$= \frac{1}{\cot\frac{1}{2}A - \cot A};$$

$$\cos A = \frac{\sin 2A}{2\sin A} = \frac{\operatorname{cosec}\frac{1}{2}A - 2\sin\frac{1}{2}A}{\operatorname{cosec}\frac{1}{2}A} = \frac{\cot\frac{1}{2}A - \tan\frac{1}{2}A}{\cot\frac{1}{2}A + \tan\frac{1}{2}A}$$

$$= \frac{1}{1 + \tan A\tan\frac{1}{2}A};$$

$$\tan A = \frac{2}{\cot\frac{1}{2}A - \tan\frac{1}{2}A} = \frac{2\cot\frac{1}{2}A}{\cot^2\frac{1}{2}A - 1} = \frac{1 + \sec A}{\cot\frac{1}{2}A};$$

$$\sec A = \frac{\sec^2\frac{1}{2}A}{2 - \sec^2\frac{1}{2}A} = \frac{\operatorname{cosec}^2\frac{1}{2}A}{\operatorname{cosec}^2\frac{1}{2}A - 2} = \frac{\cot\frac{1}{2}A + \tan\frac{1}{2}A}{\cot\frac{1}{2}A - \tan\frac{1}{2}A};$$

$$\cot A = \frac{\cot^2\frac{1}{2}A - 1}{2\cot\frac{1}{2}A} = \frac{2 - \sec^2\frac{1}{2}A}{2\tan\frac{1}{2}A} = \frac{\operatorname{cosec}^2\frac{1}{2}A - 2}{2\cot\frac{1}{2}A};$$

$$\operatorname{cosec} A = \frac{1 + \tan^2\frac{1}{2}A}{2\tan\frac{1}{2}A} = \frac{\cot\frac{1}{2}A + \tan\frac{1}{2}A}{2} = \frac{\sec\frac{1}{2}A\operatorname{cosec}\frac{1}{2}A}{2}$$

(2) $\cos^4 A - \sin^4 A = \sin 2A.$

(3) $\tan(45^\circ + A) + \tan(45^\circ - A) = 2\sec 2A.$

(4) $\frac{1}{2}\tan(45^\circ + A) - \frac{1}{2}\tan(45^\circ - A) = \tan 2A.$

(5) $\dfrac{\cos 2A}{1 + \sin 2A} = \dfrac{1}{\tan 2A + \sec 2A}.$

(6) $\dfrac{\sin 2A}{\cot A} + \cos 2A = 1.$

(7) $4\sin^2(30^\circ + A) = 2\sin^2 A + \sqrt{3}.\sin 2A + 1.$

(8) $\cos 2A + \cos 2B = 2\cos(A + B)\cos(A - B).$

(9) $\cos\frac{1}{2}A = \dfrac{\tan A}{\sec A\sqrt{\operatorname{vers} A}}.$

(10) $8\cot 2A\operatorname{cosec}^2 A = \cot A\operatorname{cosec}^2 A - \tan A\sec^2 A.$

(11) $(\cos A + \sqrt{-1}\sin A - 1)(\cos A - \sqrt{-1} - 1)$

$$= 4\sin^2\tfrac{1}{2}A.$$

(12) Prove that $\dfrac{\sin\theta}{1+2\cos\theta} = \dfrac{1}{4}\cot\dfrac{\theta}{2} - \dfrac{3}{4}\cot\dfrac{3\theta}{2}$.

(13) Shew that $\tan 9^\circ = 1 + \sqrt{5} - \sqrt{(5+2\sqrt{5})}$.

(14) Prove that $\tan(45^\circ - x)\tan(45^\circ - 3x)$

$$= \frac{1-2\sin 2x}{1+2\sin 2x}.$$

(15) Eliminate θ from the equations

$$\cos^3\theta + a\cos\theta = b\ ;\ \sin^3\theta + a\sin\theta = c.$$

(16) If $\tan\theta = \tan^3\dfrac{\phi}{2}$, and $\cos^2\phi = \dfrac{m^2-1}{3}$,

shew that $m = \dfrac{2}{\{(\cos\theta)^{\frac{2}{3}} + (\sin\theta)^{\frac{2}{3}}\}^{\frac{1}{2}}}$.

(17) $\tan(a+x)\tan(a-x) = \dfrac{1-2\cos 2a}{1+2\cos 2a}$, find $\sin x$,

$x = 30^\circ$.

(18) If $\dfrac{n\tan x}{\cos^2(a-x)} = \dfrac{m\tan(a-x)}{\cos^2 x}$, shew that

$$\tan(a-2x) = \frac{n-m}{n+m}\tan a.$$

(19) If $\left.\begin{array}{l}4x = 3a\cos\theta + a\cos 3\theta \\ 4y = 3a\sin\theta - a\sin 3\theta\end{array}\right\}$ shew that $(a^2 - x^2 - y^2)^3$

$= 27a^2x^2y^2$.

(20) If $\sin(\beta + x) + \sin(2\alpha - \beta + x)$
$= (\cos^2\beta - \cos^2\alpha)\sin(\alpha + 2x) - \sin\alpha(\sin^2\beta - \sin^2\alpha)$,
shew that $\sec x = \sin(\alpha + \beta)\tan(\alpha - \beta)$.

(21) If $2\tan^{-1}x = \sin^{-1}2y$, then $y = \dfrac{x}{1+x^2}$.

(22) Given $\left\{\begin{array}{l}\cos\phi + \cos\theta = a \\ \cos 5\phi + \cos 5\theta = b\end{array}\right\}$ find $\cos\phi$ and $\cos\theta$.

(23) Given $\text{vers}^{-1}(1+x) - \text{vers}^{-1}(1-x)$

$= \tan^{-1} 2\sqrt{(1-x^2)}$, to find x,

$x = \pm 1$, $\dfrac{1}{2}$ and -1.

(24) $\tan^{-1}\left(\frac{1}{x-1}\right) - \tan^{-1}\left(\frac{1}{x+1}\right) = \frac{\pi}{2}$, find x,

$$x = \pm \frac{2}{\sqrt{3}-1}.$$

(25) $\sec^{-1} a - \sec^{-1} b = \sec^{-1} \frac{x}{b} - \sec^{-1} \frac{x}{a}$, find x,

$$x = \pm ab.$$

(26) $\tan^{-1} x - \tan^{-1} y = \tan^{-1} \frac{1}{2y} - \tan^{-1} \frac{1}{2x} = \frac{\pi}{12}$,

find x and y.

(27) Prove that $A + \tan^{-1}(\cot 2A) = \tan^{-1}(\cot A)$.

(28) Find the sum of n terms of the series,

$$\tan^{-1} a + \tan^{-1} \frac{a}{1 + 1.2a^2} + \tan^{-1} \frac{a}{1 + 2.3a^2} + \&c.$$

(29) If $A + B + C = 180^\circ$, then

(1) $\sin^2 \frac{1}{2} A + \sin^2 \frac{1}{2} B + \sin^2 \frac{1}{2} C + 2 \sin \frac{1}{2} A \sin \frac{1}{2} B \sin \frac{1}{2} C = 1$.

(2) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$.

(3) $\cos 2A + \cos 2B + \cos 2C = 1 - 4 \sin A \sin B \sin C$.

(30) If $A + B + C = 90^\circ$, then

(1) $\cos 2A + \cos 2B + \cos 2C = 1 + 4 \sin A \sin B \sin C$.

(2) $\tan A \tan B + \tan A \tan C + \tan B \tan C = 1$.

(3) $\tan A + \tan B + \tan C = \tan A \tan B \tan C$
$+ \sec A \sec B \sec C$.

CHAPTER III.

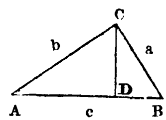
16. There are six parts in every plane triangle, viz., the three sides and the three angles, any three of which being given, the others can be found, except the case where the three angles are given; for it is clear that if you draw an indefinite number of lines parallel to the base AB of the triangle ABC, all the triangles so formed will have their

angles equal, but the triangles may have their sides of any length whatever (see fig.) The angles of a triangle are generally designated by A, B, C, and the sides opposite to them by the letters a, b, c.

The first proposition which we set out to prove is the following:

The sines of the angles of a plane triangle are proportional to the sides which subtend them; that is,

$$\sin A : \sin B :: a : b.$$



From any angle C of the triangle ABC, let fall the perpendicular CD, then, by the definitions,

$$\frac{CD}{AC} = \sin A, \text{ and } \frac{CD}{CB} = \sin B,$$

dividing the former by the latter, we have

$$\frac{\sin A}{\sin B} = \frac{CD}{AC} \cdot \frac{CB}{CD} = \frac{CB}{AC};$$

$$i.\ e.\ \sin A : \sin B :: CB : AC :: a : b.$$

In a similar way, by letting fall perpendiculars from A on BC, and B on AC, we have

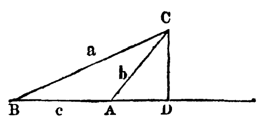
$$\sin B : \sin C :: b : c,$$

$$\sin A : \sin C :: a : c.$$

We now proceed to find the cosine of an angle in terms of the sides.

When the triangle is obtuse angled.

By Euclid, Book II. Prop. 12.



$$BC^2 = AB^2 + AC^2 + 2AB.AD \ldots (1).$$

Now $\frac{AD}{AC} = \cos CAD = -\cos BAC$,

because the cosine of an angle is equal to minus the cosine of its supplement;

$$\therefore AD = -AC \cos BAC,$$

this substituted in (1) we have

$$BC^2 = AB^2 + AC^2 - 2AB.AC \cos BAC,$$

$$a^2 = c^2 + b^2 - 2bc\cos A;$$

$$\therefore \cos A = \frac{c^2 + b^2 - a^2}{2bc}.$$

When the triangle is acute angled, as in the first figure, we have, by Euclid, Book II. Prop. 13,

$$AC^2 = AB^2 + BC^2 - 2AB.DB \dots (2).$$

$$\frac{DB}{BC} = \cos B, \ \therefore DB = BC\cos B;$$

this substituted in (2) gives

$$AC^2 = Ab^2 + BC^2 - 2AB.BC\cos B,$$

$$\text{or } b^2 = c^2 + a^2 - 2ac.\cos B;$$

$$\therefore \cos B = \frac{a^2 + c^2 - b^2}{2ac}.$$

Now, by putting c for a, and a for c, and A for B, we have

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}.$$

In the same manner

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}.$$

17. By p. 27, $\cos A = 1 - 2\sin^2 \frac{1}{2}A$,

$$\therefore 2\sin^2 \tfrac{1}{2}A = 1 - \cos A = 1 - \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{2bc - b^2 - c^2 + a^2}{2bc}$$

$$= \frac{a^2 - (b^2 - 2bc + c^2)}{2bc} = \frac{a^2 - (b - c)^2}{2bc}.$$

Now, since the difference of the squares of any two quantities is equal to the product of the sum and difference of the same quantities, we have

$$2\sin^2 \tfrac{1}{2}A = \frac{(a + b - c)(a - b + c)}{2bc} \dots\dots (1).$$

Also, by p. 27, $\cos A = 2\cos^2 A - 1$;

$$\therefore 2\cos^2 \tfrac{1}{2}A = 1 + \cos A = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc + b^2 + c^2 - a^2}{2bc}$$

$$= \frac{(b + c)^2 - a^2}{2bc} = \frac{(b + c + a)(b + c - a)}{2bc} \dots\dots (2).$$

Multiply (1) by (2),

$$4\sin^2\tfrac{1}{2}A\cos^2\tfrac{1}{2}A=\frac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{4b^2c^2}.$$

Extracting the square root,

$$2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A=\sqrt{\frac{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)}{4b^2c^2}},$$

but $2\sin\frac{1}{2}A\cos\frac{1}{2}A=\sin A$;

$$\therefore\ \sin A=\frac{\sqrt{\{(a+b+c)(a+b-c)(a-b+c)(-a+b+c)\}}}{2bc}\ldots(3).$$

This is generally put under a different form by putting the perimeter

$$a+b+c=2S.$$

From this equation subtract $2c$,

$$a+b-c=2S-2c=2(S-c).$$

From the same equation subtract $2b$,

$$a-b+c=2S-2b=2(S-b).$$

Also from the same subtract $2a$,

$$-a+b+c=2S-2a=2(S-a).$$

Substituting these values in equation (3),

$$\sin A=\frac{\sqrt{\{2S\,2(S-c)\,2(S-b)\,2(S-a)\}}}{2bc}$$

$$=\frac{\sqrt{\{16(S-a)(S-b)(S-c)\}}}{2bc}$$

$$=2\frac{\sqrt{\{S(S-a)(S-b)(S-c)\}}}{bc}\ldots\ldots(4).$$

This is sometimes done in the following manner:—

$$\sin^2 A=(1-\cos A)(1+\cos A)=\left(1-\frac{b^2+c^2-a^2}{2bc}\right)\left(1+\frac{b^2+c^2-a^2}{2bc}\right)$$

$$=\frac{(a+c-b)(a+b-c)(b+c+a)(b+c-a)}{4b^2c^2},$$

which is the same as equation (3).

From equation (1),

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(a+b-c)(a-b+c)}{4bc}} = \sqrt{\frac{(S-b)(S-c)}{bc}}.$$

From (2),

$$\cos \tfrac{1}{2}A = \sqrt{\frac{(b+c+a)(b+c-a)}{4bc}} = \sqrt{\frac{S(S-a)}{bc}};$$

$$\therefore \tan \tfrac{1}{2}A = \frac{\sin \frac{1}{2}A}{\cos \frac{1}{2}A} = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}}.$$

These values being of great importance, we shall collect them for easy reference.

$$\sin A = \frac{2}{bc}\sqrt{\{S(S-a)(S-b)(S-c)\}} \ldots\ldots (a),$$

$$\sin \tfrac{1}{2}A = \sqrt{\frac{(S-b)(S-c)}{bc}} \ldots\ldots\ldots\ldots\ldots (b),$$

$$\cos \tfrac{1}{2}A = \sqrt{\frac{S(S-a)}{bc}} \ldots\ldots\ldots\ldots\ldots\ldots\ldots (c),$$

$$\tan \tfrac{1}{2}A = \sqrt{\frac{(S-b)(S-c)}{S(S-a)}} \ldots\ldots\ldots\ldots\ldots (d).$$

By the same method as above, $\sin B$ and $\sin C$ may be determined; but they may be more easily found from equation (4) thus, put B for A; a for b and b for a, and we have

$$\sin B = \frac{2}{ac}\sqrt{\{S(S-a)(S-b)(S-c)\}} \ldots\ldots (e).$$

Also putting c for A, a for c, and c for a, we have

$$\sin C = \frac{2}{ab}\sqrt{\{S(S-a)(S-b)(S-c)\}} \ldots\ldots (f).$$

PRACTICAL OBSERVATIONS.

Equation (a) may be used in all cases where A does not approach near to 90°.

Equation (b) may be used in all cases where A approaches near to 90°.

Equation (c) may also be used in all cases where A approaches near to 90°.

Equation (d) may be used in all cases where A does not approach near to 180°.

18. The area of the triangle $ACB = \frac{AB.CD}{2}$,

$$\text{but } \frac{CD}{AC} = \sin A; \ \therefore \ CD = AC \sin A;$$

$$\therefore \ \Delta = \frac{AB.AC \sin A}{2} \ldots\ldots\ldots\ldots(5).$$

Where Δ represents the area of the triangle. We may also express the area of a triangle in terms of the sides from the above.

By equation (4),

$$\sin A = \frac{2}{bc} \sqrt{\{S(S-a)(S-b)(S-c)\}}.$$

This substituted in equation (5) gives

$$\Delta = \frac{1}{2} AB.AC.\frac{2}{bc}\sqrt{\{S(S-a)(S-b)(S-c)\}}$$

$$= \frac{1}{2} bc.\frac{2}{bc}\sqrt{\{S(S-a)(S-b)(S-c)\}}$$

$$= \sqrt{\{S(S-a)(S-b)(S-c)\}}.$$

19. When two sides and their included angle are given,

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

Add unity to each side of this equation,

$$\frac{a}{b} + 1 = \frac{\sin A}{\sin B} + 1; \ \therefore \ \frac{a+b}{b} = \frac{\sin A + \sin B}{\sin B} \ldots(1).$$

Subtract unity from each side of the same equation,

$$\frac{a}{b} - 1 = \frac{\sin A}{\sin B} - 1; \ \therefore \ \frac{a-b}{b} = \frac{\sin A - \sin B}{\sin B} \ldots(2).$$

Divide (1) by (2) and we have

$$\frac{a+b}{a-b} = \frac{\sin A + \sin B}{\sin A - \sin B}.$$

But by Art. 17, $\sin A + \sin B = 2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)$,
and $\sin A - \sin B = 2 \sin \frac{1}{2}(A-B) \cos \frac{1}{2}(A+B)$;

$$\therefore \frac{a+b}{a-b}=\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\sin\frac{1}{2}(A-B)\cos\frac{1}{2}(A+B)}=\tan\tfrac{1}{2}(A+B)\cot\tfrac{1}{2}(A-B)$$

$$=\frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}=\frac{\cot\frac{1}{2}C}{\tan\frac{1}{2}(A-B)},$$

since $\tan\frac{1}{2}(A+B)=\cot\frac{1}{2}C$.

In logarithms,

$$\log\tan\tfrac{1}{2}(A-B)=\log(a-b)-\log(a+b)+\log\cot\tfrac{1}{2}C.$$

From $\frac{c}{a}=\frac{\sin C}{\sin A}$, we have

$$\log c=\log a+\log\sin C-\log\sin A;$$

c may also be found from the formula,

$\cos C=\frac{a^2+b^2-c^2}{2ab}$; which may be adapted to logarithms in the following manner:

$$c^2=a^2+b^2-2ab\cos C;$$

but $\cos C=1-2\sin^2\frac{1}{2}C$;

$$\therefore c^2=a^2+b^2-2ab(1-2\sin^2\tfrac{1}{2}C)$$

$$=a^2+b^2-2ab+4ab\sin^2\tfrac{1}{2}C$$

$$=(a-b)^2+4ab\sin^2\tfrac{1}{2}C$$

$$=(a-b)^2\left\{1+\frac{4ab}{(a-b)^2}\sin^2\tfrac{1}{2}C\right\};$$

now since $\frac{4ab}{(a-b)^2}\sin^2\frac{1}{2}C=\left\{\frac{2a^{\frac{1}{2}}b^{\frac{1}{2}}\sin\frac{1}{2}C}{a-b}\right\}^2$,

which, being a square, is necessarily a positive quantity, and as this may be of any magnitude whatever, we may put

$$\tan^2\theta=\frac{4ab}{(a-b)^2}\sin^2\tfrac{1}{2}C;$$

then $c^2=(a-b)^2(1+\tan^2\theta)=(a-b)^2\sec^2\theta$;

$$\therefore c=(a-b)\sec\theta,$$

which is adapted for logarithmic computation, thus:

$$\log c=\log(a-b)+\log\sec\theta-10.$$

The following is useful in cases where a and b are nearly equal:

$$c^2=a^2+b^2-2ab\cos C;$$

but $\cos C=2\cos^2\frac{1}{2}C-1$;

$$\therefore\ c^2 = a^2 + b^2 - 2ab\,(2\cos^2 \tfrac{1}{2} C - 1)$$
$$= a^2 + b^2 + 2ab - 4ab\ \cos^2 \tfrac{1}{2} C$$
$$= (a+b)^2 - 4ab\cos^2 \tfrac{1}{2} C = (a+b)^2 \left\{1 - \frac{4ab\cos^2 \frac{1}{2} C}{(a+b)^2}\right\}.$$

Now $\dfrac{4ab\cos^2 \frac{1}{2} C}{(a+b)^2}$ is always less than unity,

for $\sqrt{ab}$ is less than $\frac{1}{2}(a+b)$*, or $(a+b)^2$ is greater than $4ab$, and since the cosine cannot exceed unity, it is evident that the above is a proper fraction, and we may put

$$\sin^2\theta = \frac{4\,ab\cos^2 \frac{1}{2} C}{(a+b)^2};$$

$$\therefore\ c^2 = (a+b)^2\,(1 - \sin^2\theta) = (a+b)^2 \cos^2\theta,$$

or $c = (a+b)\cos\theta$,

$$\log c = \log\,(a+b) + \log\cos\theta - 10;$$

$$\text{since } \tan\theta = \frac{2\,a^{\frac{1}{2}}\,b^{\frac{1}{2}}}{a-b}\sin \tfrac{1}{2} C,$$

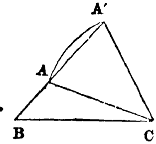
$$\log\tan\theta = \log 2 + \frac{1}{2}\log a + \frac{1}{2}\log b + \log\sin \tfrac{1}{2} C - \log\,(a-b).$$

$$\text{Also, since } \sin\theta = \frac{2\,a^{\frac{1}{2}}\,b^{\frac{1}{2}}}{a+b}\ \cos \tfrac{1}{2} C,$$

$$\log\sin\theta = \log 2 + \frac{1}{2}\log a + \frac{1}{2}\log b + \log\cos \tfrac{1}{2} C - \log\,(a+b).$$

ON THE AMBIGUOUS CASE.

20. When two sides of a triangle and the angle opposite one of them are given, there is evidently an ambiguity; for if CA be taken less than BC, but greater than a perpendicular from C on BA or BA produced, and if with C as centre and CA as radius, a circle be

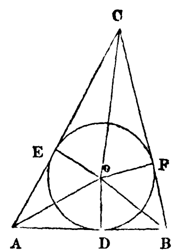


* $\dfrac{a+b}{2} > \sqrt{ab}$ or $a + b > 2\sqrt{ab}$ or $a^2 + 2ab + b^2 > 4ab$,
or $a^2 - 2ab + b^2 > 0$, or $(a-b)^2 > 0$;
for any even power of a quantity is positive.

described, it must necessarily cut the line BAA' in two points, while in all other cases it will either fall short of it or touch it only in one point. When AC is less than $a \sin B$, the triangle is impossible: but when $AC = a \sin B$, the triangle is right angled, and there is no ambiguity. And when AC is greater than BC, there can be only one triangle, and, if it fall between these limits, there is an ambiguity, for there will be two triangles having the same data; thus when the least side AC is opposite the given angle B, it is evident that either ABC or $A'BC$ may be the triangle required.

21. To find the area of a triangle in the terms of the radius of the inscribed circle, and sides of the triangle.

The triangle ACB is composed of the triangles AOB, AOC, and BOC; O being the centre of the inscribed circle.



Now the area of the triangle

$$AOB = \frac{AB \,.\, OD}{2} = \frac{cr}{2},$$

$$AOC = \frac{AC \,.\, OE}{2} = \frac{br}{2},$$

$$BOC = \frac{BC \,.\, OF}{2} = \frac{ar}{2}.$$

Hence the area of the whole triangle

$$ACB = \frac{ar}{2} + \frac{br}{2} + \frac{cr}{2} = \frac{a+b+c}{2} \,.\, r.$$

Since $\frac{a+b+c}{2} = S$, the semi-perimeter, we may express the radius in terms of the sides; putting, as before, Δ for the area of the triangle,

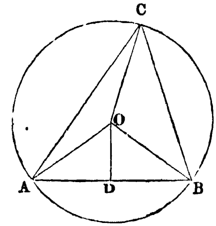
$$\Delta = rS; \quad \therefore \; r = \frac{\Delta}{S} \;\ldots\ldots\ldots\ldots\ldots\ldots\ldots (1);$$

but $\Delta = \sqrt{\{S(S-a)(S-b)(S-c)\}}$ by p. 56;

$$\therefore \; r = \frac{\sqrt{\{S(S-a)(S-b)(S-c)\}}}{S}$$

$$= \sqrt{\frac{(S-a)(S-b)(S-c)}{S}} \;\ldots\ldots\ldots (2).$$

22. To find the area of the triangle in terms of the radius of the circumscribing triangle and the sides.



By the 20th prop. of the third book of Euclid, the angle AOB is double the angle ACB.

Draw OD perpendicular to AB, which will bisect the angle AOB and the side AB.

Now by p. 56,

$$\Delta = \frac{AC.CB \sin ACB}{2} = \frac{AC.CB \sin AOD}{2}$$

$$= \frac{ba}{2} \cdot \frac{AD}{OA} = \frac{ba}{2} \cdot \frac{1}{2} \cdot \frac{AB}{OA} = \frac{abc}{4R}.$$

We may now find the radius of the circumscribing circle in terms of the sides; for since

$$\Delta = \frac{abc}{4R}, \therefore R = \frac{abc}{4\Delta} = \frac{abc}{4\sqrt{\{S(S-a)(S-b)(S-c)\}}} \ldots (3).$$

Multiplying (2) by (3),

$$Rr = \frac{abc}{4S} = \frac{abc}{2(a+b+c)}.$$

By dividing (3) by (2),

$$\frac{R}{r} = \frac{abc}{4(S-a)(S-b)(S-c)}.$$

Examples.

(1) In a plane triangle if a, b, c be the sides, and A, B, C the angles, prove that

$$c = b \cos A \pm \sqrt{(a^2 - b^2 \sin^2 A)}.$$

$$\text{Since } \cos A = \frac{b^2 + c^2 - a^2}{2bc},$$

$$2bc \cos A = b^2 + c^2 - a^2,$$

$$c^2 - 2bc \cos A = a^2 - b^2,$$

$$c^2 - 2bc \cos A + b^2 \cos^2 A = a^2 - b^2 + b^2 \cos^2 A$$

$$= a^2 - (1 - \cos^2 A) b^2,$$

$$c^2 - 2bc \cos A + b^2 \cos^2 A = a^2 - b^2 \sin^2 A,$$

$$c - b \cos A \quad \sqrt{(a^2 - b^2 \sin^2 A)},$$

$$c = b \cos A \pm \sqrt{(a^2 - b^2 \sin^2 A)}.$$

(2) Find the area of a square when the difference between the diagonal and the side $= m$ feet.

Let $x =$ side of the square, then $x + m =$ diagonal,

$$x^2 + x^2 = (x + m)^2, \text{ or } 2x^2 = (x + m)^2,$$

$$x\sqrt{2} = x + m,$$

$$x\sqrt{2} - x = m,$$

$$x(\sqrt{2} - 1) = m,$$

$$x = \frac{m}{\sqrt{2} - 1} = (\sqrt{2} + 1)m,$$

$$x^2 = m^2(\sqrt{2} + 1)^2.$$

(3) Given the area, angle C, and $a + b$; find the sides of the triangle

$$ab \sin C = 2\Delta,$$

$$ab = \frac{2\Delta}{\sin C}.$$

Let $a + b = m$,

$$a^2 + 2ab + b^2 = m^2,$$

$$4ab = \frac{8\Delta}{\sin C};$$

$$\therefore \; a^2 - 2ab + b^2 = m^2 - \frac{8\Delta}{\sin C},$$

$$a - b = \sqrt{\left(m^2 - \frac{8\Delta}{\sin C}\right)},$$

but $a + b = m$;

$$\therefore \; a = \frac{m + \sqrt{\left(m^2 - \frac{8\Delta}{\sin C}\right)}}{2},$$

$$b = \frac{m - \sqrt{\left(m^2 - \frac{8\Delta}{\sin C}\right)}}{2},$$

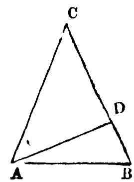
$$\cos C = \frac{a^2 + b^2 - c^2}{2ab},$$

whence c may be found; or c may be found by the formula

$$c^2 = (a+b)^2 - 4ab\cos^2\frac{C}{2}. \quad \text{(p. 58).}$$

(4) Given c, the base of an isosceles triangle, and p the perpendicular from one of the equal angles upon the opposite side: shew that the area $= \dfrac{pc^2}{4\sqrt{(c^2-p^2)}}$.

By the 13th Prop. of the second book of Euclid



$$AC^2 = c^2 + BC^2 - 2BC.BD,$$

$$\text{but } AC = BC;$$

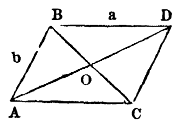
$$\therefore\ BC^2 = c^2 + BC^2 - 2BC.BD;$$

$$\therefore\ BC = \frac{c^2}{2BD} = \frac{c^2}{2\sqrt{(c^2-p^2)}},$$

$$\text{area} = \frac{BC.AD}{2} = \frac{pc^2}{4\sqrt{(c^2-p^2)}}.$$

(5) If θ be the angle between the diagonals of a parallelogram whose sides a and b are inclined to each other at an angle α,

$$\text{then } \tan\theta = \frac{2ab\sin\alpha}{a^2-b^2}.$$



Let $DB = a$, and $AB = b$,

then $a^2 = OB^2 + OD^2 + 2BO.OD\cos\theta$,

$$b^2 = OB^2 + AO^2 - 2AO.OB\cos\theta,$$

$$a^2 - b^2 = 2AO.OB\cos\theta + 2BO.OD\cos\theta$$

$$= 2BO(AO+OD)\cos\theta = 2BO.AD\cos\theta,$$

area of parallelogram $= ab\sin\alpha$; but $AD.BO\sin\theta$ is also equal the area, hence by division

$$\frac{ab\sin\alpha}{a^2-b^2} = \frac{AD.BO\sin\theta}{2BO.AD\cos\theta} = \frac{1}{2}\tan\theta,$$

$$\therefore\ \tan\theta = \frac{2ab\sin\alpha}{a^2-b^2}.$$

(6) In a triangle ABC prove that

$$\frac{\sin(A-B)}{\sin C} = \frac{a^2-b^2}{c^2},$$

$$\frac{\sin(A-B)}{\sin C} = \frac{\sin A \cos B - \cos A \sin B}{\sin C}$$

$$= \frac{\sin A}{\sin C}\cos B - \frac{\sin B}{\sin C}\cos A$$

$$= \frac{a}{c} \cdot \frac{a^2 + c^2 - b^2}{2ac} - \frac{b}{c} \cdot \frac{b^2 + c^2 - a^2}{2bc}$$

$$= \frac{a^2 + c^2 - b^2}{2c^2} - \frac{b^2 + c^2 - a^2}{2c^2}$$

$$= \frac{a^2 - b^2}{c^2}.$$

(7) Prove that the area of a triangle ABC

$$= \frac{a^2 + b^2 + c^2}{4(\cot A + \cot B + \cot C)}.$$

Since $\cos A = \dfrac{b^2 + c^2 - a^2}{2bc}$, and by p. 56, $\sin A = \dfrac{2\Delta}{bc}$

$$\cos B = \frac{a^2 + c^2 - b^2}{2ac}, \qquad \sin B = \frac{2\Delta}{ac};$$

$$\cos C = \frac{a^2 + b^2 - c^2}{2ab}, \qquad \sin C = \frac{2\Delta}{ab}.$$

By dividing each in the first group by the corresponding one in the second, we have

$$\cot A = \frac{b^2 + c^2 - a^2}{4\Delta}, \quad \cot B = \frac{a^2 + c^2 - b^2}{4\Delta}, \quad \cot C = \frac{a^2 + b^2 - c^2}{4\Delta};$$

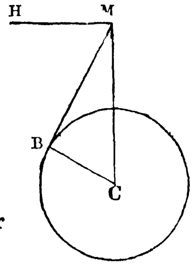
$\therefore$ by addition,

$$\cot A + \cot B + \cot C = \frac{b^2 + c^2 - a^2 + a^2 + c^2 - b^2 + a^2 + b^2 - c^2}{4\Delta}$$

$$= \frac{a^2 + b^2 + c^2}{4\Delta};$$

$$\therefore \Delta = \frac{a^2 + b^2 + c^2}{4(\cot A + \cot B + \cot C)}.$$

(8) From the top of a mountain a miles high, the visible horizon appeared depressed A°; required the diameter $(2r)$ of the earth, and the distance (d) of the horizon.



In the annexed figure $CM = r + a$, and since $HMB = A$, BCM is evidently $= A$,

$$CM \cos A = BC \text{ or } (r+a)\cos A = r,$$

$$\therefore\ r = \frac{a \cos A}{1 - \cos A} = \frac{\dfrac{a \cot A}{1 - \cos A}}{\sin A};$$

(Dividing numerator and denominator by sin A)

$$= \frac{a \cot A \sin A}{2 \sin^2 \frac{1}{2} A} = \frac{a \cot A \,.\, 2 \sin \frac{1}{2} A \cos \frac{1}{2} A}{2 \sin^2 \frac{1}{2} A}$$

$$= a \cot A \cot \tfrac{1}{2} A;$$

$$\frac{BC}{BM} = \cot A, \text{ or } \frac{r}{d} = \cot A; \ \therefore\ d = \frac{r}{\cot A} = \frac{a \cot A \cot \frac{1}{2} A}{\cot A}$$

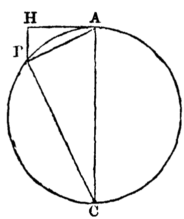
$$= a \cot \tfrac{1}{2} A.$$

(9) If L = length in miles of an arc of a great circle of the earth, and D the depression in feet of one extremity of it below a tangent to the other, then $D = \frac{2}{3} L^2$ nearly.

$AC = 7920$ miles nearly, $AB = L$, $HB = D$ feet $= \frac{D}{5280}$ miles,

$\because$ the triangles AHB and ACB are similar;

$$\therefore\ HB : L :: L : AC,$$



$$\frac{D}{5280} : L :: L : 7920,$$

$$D \cdot \frac{7920}{5280} = L^2.$$

$$D \cdot \frac{3}{2} = L^2,$$

$$\therefore\ D = \frac{2L^2}{3} \text{ nearly.}$$

(10) Given the perimeter, and the area of the triangle, and the angle A; find a.

By p. 55, $\cos \frac{1}{2} A = \sqrt{\frac{S(S-a)}{bc}}$,

$$bc\sin A = 2\Delta, \quad \therefore\ bc = \frac{2\Delta}{\sin A},$$

$$\cos^2 \tfrac{1}{2}A = \frac{S(S-a)}{2\Delta}.\sin A = \frac{S(S-a)}{2\Delta}.2\sin\tfrac{1}{2}A\cos\tfrac{1}{2}A.$$

Divide by $\sin\frac{1}{2}A\cos\frac{1}{2}A$, and

$$\cot\tfrac{1}{2}A = \frac{S^2 - Sa}{\Delta}, \text{ or } Sa = S^2 - \Delta\cot\tfrac{1}{2}A;$$

$$\therefore\ a = \frac{S^2 - \Delta\cot\frac{1}{2}a}{S}.$$

(11) Given the area, side c, and $A+B$; find $A-B$.

$$\text{Since } \cos C = \frac{a^2+b^2-c^2}{2ab},$$

$$a^2+b^2-c^2 = 2ab\cos C,$$

$$a^2+b^2 = c^2+2ab\cos C,$$

$$2ab = \frac{4\Delta}{\sin C},$$

$$a^2+2ab+b^2 = c^2+2ab\cos C+\frac{4\Delta}{\sin C}$$

$$= c^2 + \frac{4\Delta}{\sin C}\cos C + \frac{4\Delta}{\sin C}$$

$$= c^2 + 4\Delta\{\cot C + \sqrt{(1+\cot^2 C)}\},$$

$$a+b = \sqrt{[c^2+4\Delta\{\cot C+\sqrt{(1+\cot^2 C)}\}]}.$$

In the same way

$$a-b = \sqrt{[c^2+4\Delta\{\cot C-\sqrt{(1+\cot^2 C)}\}]},$$

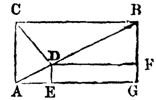
$$\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}A+B}{\tan\frac{1}{2}A-B} = \sqrt{\frac{c^2+4\Delta\{\cot C+\sqrt{(1+\cot^2 C)}\}}{c^2+4\Delta\{\cot C-\sqrt{(1+\cot^2 C)}\}}},$$

$$\therefore \tan\tfrac{1}{2}(A-B) = \tan\tfrac{1}{2}(A+B)\sqrt{\frac{c^2+4\Delta\{\cot C-\sqrt{(1+\cot^2 C)}\}}{c^2+4\Delta\{\cot C+\sqrt{(1+\cot^2 C)}\}}}.$$

(12) If from one of the angles of a rectangle a perpendicular be drawn to its diagonal, and from their intersection, lines be drawn perpendicular to the sides containing the opposite angle, then putting P and p for these last perpendiculars, and D for the diagonal, $P^{\frac{2}{3}} + p^{\frac{2}{3}} = D^{\frac{2}{3}}$.

Let the angle $BAG = \theta$, then $AD\sin\theta = DE = P$;

but by (Euc. VI. 8th Cor.) $AB \,.\, AD = AC^2$, $\therefore AD = \dfrac{AC^2}{AB}$;



but $AC = AB \sin\theta$;

$$\therefore AD = \frac{AB^2 \sin^2\theta}{AB} = AB \sin^2\theta;$$

and since $P = AD \sin\theta$, we have

$$P = AB \sin^3\theta = D \sin^3\theta.$$

In the same way we have

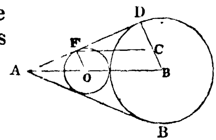
$$p = D\cos^3\theta,$$

$$\therefore\ P^{\frac{2}{3}} = D^{\frac{2}{3}} \sin^2\theta, \text{ and } p^{\frac{2}{3}} = D^{\frac{2}{3}}\cos^2\theta.$$

By addition,

$$P^{\frac{2}{3}} + p^{\frac{2}{3}} = D^{\frac{2}{3}}(\sin^2\theta + \cos^2\theta) = D^{\frac{2}{3}}.$$

(13) If two circles, whose radii are a and b, touch each other externally, and if θ be the angle contained by the two common tangents to these circles, shew that



$$\sin\theta = \frac{4(a-b)\sqrt{ab}}{(a+b)^2}.$$

Let $AB = x$, $OB = EC = a + b$, $CD = BD - EO = a - b$;

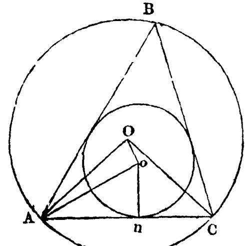
$$\sin\tfrac{1}{2}\theta = \frac{BD}{AB} = \frac{DC}{EC} = \frac{a-b}{a+b},$$

$$\cos\tfrac{1}{2}\theta = \sqrt{(1-\sin^2\tfrac{1}{2}\theta)} = \sqrt{\left\{1 - \left(\frac{a-b}{a+b}\right)^2\right\}} = \frac{2\sqrt{ab}}{a+b},$$

$$\sin\theta = 2\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta = \frac{4(a-b)}{(a+b)^2}\sqrt{ab}.$$

(14) To find the distance between the centres of the inscribed and circumscribed circles, the figure being the triangle ABC.

Let O and o be the centres of circumscribed and inscribed circles,



$$Oo = \delta,\quad OA = R,\quad on = r,$$

$$\therefore\ Ao = \frac{r}{\sin\frac{1}{2}A},$$

$$\begin{aligned}\angle\, OAo &= OAC - oAc \\ &= \tfrac{1}{2}(180^\circ - AOC) - \tfrac{1}{2}A \\ &= 90^\circ - B - \tfrac{1}{2}A \\ &= 90^\circ - B - \{90^\circ - \tfrac{1}{2}(B+C)\} \\ &= \tfrac{1}{2}(C-B);\end{aligned}$$

$$\delta^2 = Ao^2 + Ao'^2 - 2\,AoAo\,.\cos OAo$$

$$= R^2 + \frac{r^2}{\sin^2\frac{1}{2}A} - \frac{2Rr}{\sin\frac{1}{2}A}\cos\tfrac{1}{2}(C-B);$$

$$\therefore\ (\delta^2 - R^2)\sin^2\tfrac{1}{2}A = r^2 - 2Rr\,.\sin\tfrac{1}{2}A\cos\tfrac{1}{2}(C-B)$$

$$= r^2 - 2Rr\cos\tfrac{1}{2}(B+C)\cos\tfrac{1}{2}(C-B);$$

$$\therefore\ (\delta^2 - R^2)(1-\cos A) = 2r^2 - 2Rr(\cos B + \cos C).$$

Similarly,

$$(\delta^2 - R^2)(1-\cos C) = 2r^2 - 2Rr(\cos B + \cos A).$$

Subtract

$$(\delta^2 - R^2)(\cos C - \cos A) = -2Rr\,.(\cos C - \cos A),$$

$$\therefore\ \delta^2 - R^2 = -2Rr;$$

$$\therefore\ \delta^2 = R^2 - 2Rr; \qquad \therefore\ \delta = \sqrt{(R^2 - 2Rr)}.$$

(15) If the angles of a triangle be in geometrical proportion, ratio $\frac{1}{2}$, the greatest side

$$= 2\text{ perimeter} \times \sin 12^\circ 51' 25''\tfrac{5}{7}.$$

Let $P = a + b + c$, a = greatest side,

$$A + \tfrac{1}{2}A + \tfrac{1}{4}A = 180^\circ.$$

$$\tfrac{1}{2}A = B,\ \tfrac{1}{4}A = C,$$

$$A = 102^\circ 51' 25''\tfrac{5}{7},$$

$$B = \ \ 51^\circ 25' 42''\tfrac{6}{7},$$

$$C = \ \ 25^\circ 42' 51''\tfrac{3}{7};$$

$$P = a + b + c = a\left\{1 + \frac{b}{a} + \frac{c}{a}\right\} = a\left\{1 + \frac{\sin B}{\sin A} + \frac{\sin C}{\sin A}\right\}$$

$$= a\,\frac{\sin A + \sin B + \sin C}{\sin A};$$

$$\therefore\ a = \frac{P\sin A}{\sin A + \sin B + \sin C}$$

$$= \frac{2P\sin\frac{1}{2}A\cos\frac{1}{2}A}{4\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C} = \frac{P\sin\frac{1}{2}A}{2\cos\frac{1}{2}B\cos\frac{1}{2}C}.$$

But $\frac{1}{2}A = B$, $\frac{1}{2}B = C$,

$$a = \frac{P \sin B}{2 \cos C \cos \frac{1}{2}C} = \frac{2P \sin \frac{1}{2}B \cos \frac{1}{2}B}{2 \cos C \cos \frac{1}{2}C} = \frac{P \sin C \cos C}{\cos C \cos \frac{1}{2}C}$$

$$= \frac{P \sin C}{\cos \frac{1}{2}C} = \frac{2P \sin \frac{1}{2}C \cos \frac{1}{2}C}{\cos \frac{1}{2}C} = 2P \sin \frac{1}{2}C,$$

$= 2$ perimeter $\times \sin 12^{\circ}\,51'\,25''\,\frac{5}{7}$. Answer.

(16) The sides of a triangle are in arithmetical proportion, its area $= \frac{3}{5}$ of the area of an equilateral triangle, having the same perimeter, shew that the greatest angle $= 120^{\circ}$, and that the sides are as $3 : 5 : 7$.

Let $a - x$, a, $a + x =$ the sides, then $3a =$ perimeter;

$\therefore$ $a =$ side of equilateral Δ, and $\frac{a}{2} =$ half its base,

$$\sqrt{\left(a^2 - \frac{a^2}{4}\right)} = \sqrt{\frac{4a^2 - a^2}{4}} = \frac{a\sqrt{3}}{2} = \text{the altitude} = \frac{a}{2}\sqrt{3}.$$

Now $\because$ both Δ's may have the common base a; $\therefore$ area equilateral Δ : area of the other :: altitude of equilateral Δ : altitude of the other;

$$\text{or, } 1 \quad \frac{3}{5} :: \frac{a}{2}\sqrt{3} : \frac{3a}{10}\sqrt{3} = AD,$$

$$\sqrt{\left\{(a+x)^2 - \frac{27a^2}{100}\right\}} - \sqrt{\left\{(a-x)^2 - \frac{27}{100}a^2\right\}} = a,$$

$$\sqrt{\left\{(a+x)^2 - \frac{27}{100}a^2\right\}} = a + \sqrt{\left\{(a-x)^2 - \frac{27}{100}a^2\right\}};$$

$$\therefore\ (a+x)^2 - \frac{27}{100}a^2$$

$$= a^2 + 2a\sqrt{\left\{(a-x)^2 - \frac{27}{100}a^2\right\}} + (a-x)^2 - \frac{27}{100}a^2;$$

$$\text{or, } 2ax = a^2 + 2a\sqrt{\left\{(a-x)^2 - \frac{27}{100}a^2\right\}} - 2ax,$$

$$4ax = a^2 + 2a\sqrt{\left\{(a-x)^2 - \frac{27}{100}a^2\right\}},$$

$$4x - a = 2\sqrt{\left\{(a-x)^2 - \frac{27}{100}a^2\right\}};$$

$$16x^2 - 8ax + a^2 = 4(a-x)^2 - \frac{27}{25}a^2$$

$$= 4a^2 - 8ax + 4x^2 - \frac{27}{25}a^2,$$

$$12x^2 = 3a^2 - \frac{27}{25}a^2,$$

$$4x^2 = a^2 - \frac{9}{25}a^2,$$

$$4x^2 = \frac{16a^2}{25},$$

$$x^2 = \frac{4a^2}{25}; \quad \therefore \; x = \frac{2}{5}a,$$

$$a - x = \frac{3}{5}a,$$

$$a + x = \frac{7}{5}a,$$

$$AC = 3, \quad BC = 5, \quad AB = 7,$$

$\therefore \; \frac{3a}{5}, \; a, \; \frac{7a}{5}$ are the three sides, which are as $3a, \; 5a, \; 7a$, or as $3, 5, 7$;

$$\cos C = \frac{AC^2 + BC^2 - AB^2}{2AC.BC} = \frac{3^2 + 5^2 - 7^2}{2 \times 3 \times 5} = -\frac{15}{30} = -\frac{1}{2},$$

but $\cos 120^\circ = -\frac{1}{2}$, therefore $C = 120^\circ$.

(17) The angles of a triangle are as the numbers $1, 2, 3$; and the perpendicular from the greatest angle upon the opposite side is p. Shew that the area $= \frac{2p^2}{\sqrt{3}}$,

$$A + 2A + 3A = 180^\circ;$$

$$\therefore \; A = 30^\circ; \quad B = 60^\circ; \quad C = 90^\circ.$$

$\sin A : \sin ACD :: p : AD$, (fig. p. 52)

$$AD = p\frac{\sin ACD}{\sin A} = p\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}} = p\sqrt{3};$$

$$\sin B : \sin BCD :: p : DB,$$

$$DB = p\frac{\sin BCD}{\sin B} = p\frac{\frac{1}{2}}{\frac{\sqrt{3}}{2}} = \frac{p}{\sqrt{3}},$$

$$AD + DB = AB = p\sqrt{3} + \frac{p}{\sqrt{3}} = \frac{3p+p}{\sqrt{3}} = \frac{4p}{\sqrt{3}},$$

$$\text{but area} = \frac{AB \times p}{2} = \frac{4p}{\sqrt{3}} \times \frac{p}{2} = \frac{2p^2}{\sqrt{3}}.$$

(18) The angles of a triangle are as the numbers 2, 3, 5, and the side opposite the greatest angle = 100 feet; find the remaining sides and the angles:

$$2A + 3A + 5A = 180^\circ;$$

$$\therefore \left.\begin{aligned} A &= 36^\circ \\ B &= 54^\circ \\ C &= 90^\circ \end{aligned}\right\} \text{the angles,}$$

$$\sin C : \sin B :: c : b,$$

$$b = c\frac{\sin B}{\sin C},$$

$$\log b = \log c + \log \sin B - \log \sin C,$$

$$\begin{aligned} \log c &= \log 100 &&= 2{\cdot}00000 \\ \log \sin B &= \log \sin 54^\circ &&= 9{\cdot}90796 \\ & && \overline{11{\cdot}90796} \\ \log \sin C &= \log \sin 90^\circ &&= 10{\cdot}00000 \\ & && \overline{1{\cdot}90796} = \log 80{\cdot}9. \end{aligned}$$

$$\sin C : \sin A :: c : a,$$

$$a = c\frac{\sin A}{\sin C}.$$

$$\log a = \log c + \log \sin A - \log \sin C$$

$$\log c = \log 100 = \quad 2{\cdot}00000$$

$$\log \sin A = \log \sin 36^0 = \underline{9{\cdot}76922}$$

$$11{\cdot}76922$$

$$\log \sin C = \log \sin 90^0 = \underline{10{\cdot}00000}$$

$$1{\cdot}76922 = \log 58{\cdot}78$$

$$\left.\begin{array}{l} a = 58{\cdot}78 \\ b = 80{\cdot}9 \end{array}\right\} \text{sides.}$$

(19) Find a point within a triangle at which the three sides shall subtend equal angles:

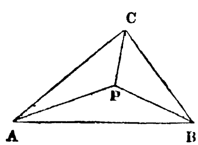
$$\left.\begin{array}{l} BC = a \\ AC = b \\ AB = c \end{array}\right\} \qquad \left.\begin{array}{l} AP = x \\ BP = y \\ PC = z \end{array}\right\}.$$

Then $\angle\, APB = BPC = APC = 120^0$;

$\therefore$ the cosine of each $= -\frac{1}{2}$, and the sine of each $= \frac{\sqrt{3}}{2}$;

$$a^2 = BP^2 + PC^2 - 2\,BP\,.\,PC \cos 120^0$$

$$= BP^2 + PC^2 - 2\,BP\,.\,PC\left(-\frac{1}{2}\right)$$



$$= BP^2 + PC^2 + BP\,.\,PC,$$

$$b^2 = AP^2 + PC^2 + AP\,.\,PC,$$

$$c^2 = AP^2 + BP^2 + AP\,.\,PB$$

$$a^2 + b^2 + c^2 = 2x^2 + 2y^2 + 2z^2 + xy + yz + xz.$$

Let $\triangle\, ABC = m^2$,

$$m^2 = \triangle BPC + \triangle APB + \triangle APC;$$

$$\therefore\ m^2 = \frac{yz \sin 120^0}{2} + \frac{xy \sin 120^0}{2} + \frac{xz \sin 120^0}{2}$$

$$= \frac{\sqrt{3}}{2}\ (xy + yz + xz),$$

$$xy + yz + xz = \frac{4\,m^2}{\sqrt{3}},$$

$$3\,xy + 3\,yz + 3\,xz = 4m^2\sqrt{3}.$$

But $2\,(x^2 + y^2 + z^2) + (xy + yz + xz) = a^2 + b^2 + c^2$;

$$\therefore\ 2\,(x^2 + y^2 + z^2) + 4\,(xy + yz + xz) = a^2 + b^2 + c^2 + 4m^2\sqrt{3},$$

$$x^2 + y^2 + z^2 + 2(xy + yz + xz) = \left(\frac{a^2 + b^2 + c^2}{2} + 2m^2\sqrt{3}\right),$$

$$x + y + z = \sqrt{\left(\frac{a^2 + b^2 + c^2}{2} + 2m^2\sqrt{3}\right)} = \beta,$$

$$x + y + z = \beta,$$

$$x + y = \beta - z,$$

$$x^2 + 2xy + y^2 = (\beta - z)^2, \text{ but } x^2 + y^2 + xy = c^2;$$

$$\therefore\ xy = (\beta - z)^2 - c^2.$$

Also, since $xy + xz + yz = \dfrac{4m^2}{\sqrt{3}}$,

$$xy = \frac{4m^2}{\sqrt{3}} - z(y + x); \text{ but } x + y + z = \beta;$$

$$\therefore\ xy = \frac{4m^2}{\sqrt{3}} - z(\beta - z);$$

$$\therefore\ (\beta - z)^2 - c^2 + z(\beta - z) = \frac{4m^2}{\sqrt{3}};$$

$$\text{or } (\beta - z + z)(\beta - z) - c^2 = \frac{4m^2}{\sqrt{3}};$$

$$\therefore\ \beta^2 - \beta z - c^2 = \frac{4m^2}{\sqrt{3}},$$

$$\beta z = \beta^2 - c^2 - \frac{4m^2}{\sqrt{3}} = \frac{a^2 + b^2 + c^2}{2} + 2m^2\sqrt{3} - c^2 - \frac{4m^2}{\sqrt{3}}$$

$$= \frac{a^2 + b^2 - c^2}{2} + \frac{2m^2}{\sqrt{3}};$$

$$\therefore\ z = \frac{1}{\beta}\left(\frac{a^2 + b^2 - c^2}{2} + \frac{2m^2}{\sqrt{3}}\right).$$

$$\text{Also } x = \frac{1}{\beta}\left(\frac{b^2 + c^2 - a^2}{2} + \frac{2m^2}{\sqrt{3}}\right),$$

$$\text{and } y = \frac{1}{\beta}\left(\frac{a^2 + c^2 - b^2}{2} + \frac{2m^2}{\sqrt{3}}\right).$$

(20) An object 6 feet high, placed on the top of a tower, subtends an angle $= \tan^{-1} \cdot 015$, at a place whose horizontal distance from the foot of the tower is 100 feet, determine the tower's height.

Let $DAB = \alpha$, $AD = x$, $AB = y$,

$$\text{then } \tan \alpha = \cdot 015, \quad \therefore \alpha = 51' \, 33'';$$

$$\therefore \sin \alpha = \cdot 01513, \quad \cos \alpha = \cdot 99989 \text{ nearly.}$$

$$\text{Now } 2 \,\triangle\, DAB = xy \sin \alpha = \cdot 01513 xy.$$

$$\text{Also } 2 \triangle ADB = AC \times DB = 100 \times 6 = 600;$$

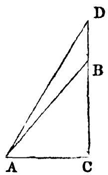
$$\therefore \cdot 015 xy = 600,$$

$$xy = 4000$$

$$\cos \alpha = \frac{x^2 + y^2 - 6^2}{2xy}, \quad \text{or } \cdot 9999 = \frac{x^2 + y^2 - 6^2}{2xy},$$

$$\cdot 9999 = \frac{x^2 + y^2 - 36}{80000} \quad \therefore x^2 + y^2 = 80028$$

$$2xy = 80000$$



$$x^2 + 2xy + y^2 = 160028$$

$$x^2 - 2xy + y^2 = 28$$

$$x + y = 400 \cdot 035$$

$$x - y = 5 \cdot 291$$

$$2x = 405 \cdot 326$$

$$x = 202 \cdot 663$$

$$2y = 394 \cdot 744$$

$$y = 197 \cdot 372,$$

$$BC = \sqrt{(y^2 - AC^2)}$$

$$= \sqrt{(197 \cdot 372^2 - 100^2)}$$

$$= \sqrt{28955 \cdot 76384}$$

$$= 170 \cdot 23188 \text{ feet.}$$

(21) From the top of a mountain 3 miles high, the visible horizon appeared depressed $2^\circ \, 13' \, 27''$; required the diameter of the earth, and the distance (d) of the horizon.

Referring to the figure and formulæ at page 64, we have

$$r = a \cot A \cot \tfrac{1}{2} A,$$

$$\frac{r}{d} = \cot A, \quad r = d \cot A;$$

$$\therefore a \cot A \cot \tfrac{1}{2} A = d \cot A$$

$$d = a \cot \tfrac{1}{2} A.$$

Here $a = 3$ miles, and $A = 2^\circ\, 13'\, 27''$;

$$\therefore\ d = 3 \cot 1^\circ\, 6'\, 43\tfrac{1}{2}''$$

$$\begin{aligned} \log 3 &= 0{\cdot}47712 \\ \log \cot 1^\circ\, 6'\, 43\tfrac{1}{2}'' &= 11{\cdot}71194 \\ & \overline{12{\cdot}18906} \\ & 10 \\ & \overline{2{\cdot}18906} = \log \text{ of } 154{\cdot}54; \end{aligned}$$

$$\therefore\ d = 154{\cdot}54,$$

$$r = a \cot A \cot \tfrac{1}{2} A$$

$$= a \cot 2^\circ\, 13'\, 27'' \cot 1^\circ\, 6'\, 43\tfrac{1}{2}'',$$

$$\log r = \log a + \log \cot 2^\circ\, 13'\, 27'' + \log 1^\circ\, 6'\, 43\tfrac{1}{2}'' - 20$$

$$\begin{aligned} \log 3 &= {\cdot}47712 \\ \log \cot 2^\circ\, 13'\, 27'' &= 11{\cdot}41075 \\ \log \cot 1^\circ\, 6'\, 43\tfrac{1}{2}'' &= 11{\cdot}71194 \\ & \overline{23{\cdot}59981} \\ & 20 \\ & \overline{3{\cdot}59981} = \log 3979 \end{aligned}$$

$$\therefore\ r = 3979;\quad \therefore\ 2r = 7958.$$

(22) A person, at a known distance from two towers, observes that their apparent altitudes are the same; he then walks a given distance towards them, till the angle of elevation of one is double that of the other; find the heights of the towers.

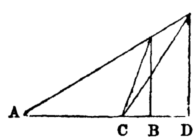
Let x and y = the heights,

$AB = a$, and α, β, the angles,

$AD = b$,

$CB = c$,

$CD = d$,



$$\tan \alpha = \frac{x}{a} = \frac{y}{b}, \quad \tan \beta = \frac{y}{d}, \quad \tan 2\beta = \frac{x}{c};$$

$$\text{but } \tan 2\beta = \frac{2 \tan \beta}{1 - \tan^2 \beta} = \frac{2\dfrac{y}{d}}{1 - \dfrac{y^2}{d^2}} = \frac{2dy}{d^2 - y^2} = \frac{x}{c};$$

$$\text{and } \because x = \frac{ay}{b},$$

$$\frac{2dy}{d^2 - y^2} = \frac{\frac{ay}{b}}{c} = \frac{ay}{bc}; \qquad \therefore \frac{2d}{d^2 - y^2} = \frac{a}{bc},$$

$$(d^2 - y^2)\, a = 2dcb; \qquad \therefore d^2 - y^2 = \frac{2dcb}{a},$$

$$y^2 = d^2 - \frac{2dcb}{a},$$

$$y = \sqrt{\left(d^2 - \frac{2dcb}{a}\right)},$$

$$x = \frac{ay}{b} = \frac{a}{b}\sqrt{\left(d^2 - \frac{2dcb}{a}\right)},$$

$$x = \frac{ad}{b}\sqrt{\left(1 - \frac{2cb}{ad}\right)}.$$

(23) In the arc AB of a circle, centre O, AC is taken $=\sin AB$; then will sector $COB =$ segment ACB.

The sector COB + sector COA

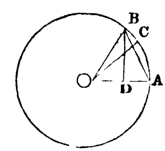
$= \Delta BOA$ + segment BCA,

or sector $COB + AC\frac{r}{2}$

$=$ segment $ACB + BD\frac{r}{2}$,

but $BD =$ arc AC;

$\therefore$ sector $COB =$ segment ACB.



PRACTICAL REMARKS ON THE SOLUTION OF TRIANGLES.

23. In working out the various problems in the solution of triangles by the tables, when one or more logarithms have to be subtracted, it is better to write down their complements, or what each of them wants of 10, instead of the logarithms themselves, and then add them altogether, subtracting as many tens from the index of the sum as there are logarithms to be subtracted.

Thus, if the logarithm to be subtracted be 3·6947362, it is the same thing as to add its complement 6·3052638, and then to subtract 10 from the sum; the easiest way to perform the subtraction is to begin at the left hand, and subtract each figure from 9 except the last figure on the right, which must be subtracted from 10.

If the index of the logarithm be greater than 10, write down what it wants of 19, and take the rest of the figures from 9 as before; observing, that after the addition is made 20 must be taken from the index. Thus, the complement of the logarithm of 13·6874563 is 6·3125437. If the logarithm of a decimal which has a negative index is to be subtracted, add the index, considered as positive, to 9, and then find the complement of the rest of the figures as usual, and subtract 10 from the index of the sum. Thus, the complement of the logarithm of $\bar{2}$·3674586 is 11·6325414.

24. Before proceeding to the next examples we shall shew how the trigonometrical tables are constructed. (See Lefebure de Fourcy's Trigonométrie, p. 35.)

Let us at first find the sin 10″; we have before stated that if the diameter of a circle be unity the circumference $= 3{\cdot}14159265358979 3 = \pi$, and when the radius is unity the semicircumference $= \pi$, and since there are 64800 seconds in 180^0, we shall have in parts of the radius

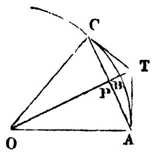
$$\text{arc } 10'' = \frac{\pi}{64800} = {\cdot}000048481368110.$$

Now in very small arcs the sine and the arc are very nearly equal, hence the above may be considered as a good approximate value of the sin 10″.

25. We proceed now to shew that in the first quadrant, the arc is greater than the sine and less than the tangent.

Let AP be the sine of the arc AB, and AT the tangent, and suppose this figure to turn round OP till the point A falls on C, we shall have the arc $AC >$ the chord AC, and, consequently, the arc $AB > AP$; hence, the arc is greater than the sine. We have, also, the arc $AC < (AT + CT)$, and therefore $AB < AT$.

Hence it follows that if $\frac{\tan a}{\sin a}$ differs little from 1, the ratio $\frac{a}{\sin a}$ differs still less.



Also, as the arc decreases the ratio of that arc to its sine becomes as near to unity as we please; that is, this ratio has unity for its limit.

And since $\tan a = \frac{\sin a}{\cos a}$; $\therefore \frac{\tan a}{\sin a} = \frac{1}{\cos a}$.

Now, diminishing the arc (which is supposed less than 90^0) the cosine increases and approaches unity as nearly as we please, that is, the ratio $\frac{1}{\cos a}$ or its equal $\frac{\tan a}{\sin a}$ continually diminishes, and has unity for its limit.

The arc being greater than the sine and less than the tangent, the ratio $\frac{a}{\sin a}$ can never be less than 1 nor greater than $\frac{\tan a}{\sin a}$, and since this last ratio can approach unity as nearly as we please, the former one will also approach unity as nearly as we please, and it is on this account that we take the arc of $10''$ for the sin $10''$.

26. We must now proceed to determine the degree of approximation

$$\sin a = 2 \sin \frac{1}{2} a \cos \frac{1}{2} a, \text{ and } \tan \frac{1}{2} a > \frac{1}{2};$$

$$\text{or } \frac{\sin \frac{1}{2} a}{\cos \frac{1}{2} a} > \frac{1}{2} a; \therefore \sin \frac{1}{2} a > \frac{1}{2} a \cos \frac{1}{2} a;$$

$$\text{or } 2 \sin \frac{1}{2} a > a \cos \frac{1}{2} a;$$

$$\text{and } 2 \sin \frac{1}{2} a \cos \frac{1}{2} a > a \cos^2 \frac{1}{2} a, \text{ or } \sin a > a \cos^2 \frac{1}{2} a;$$

$$\text{but } \cos^2 \frac{1}{2} a = 1 - \sin^2 \frac{1}{2} a;$$

$$\therefore \sin a > a \left(1 - \sin^2 \frac{1}{2} a\right)$$

$$\sin a > a \left\{1 - \left(\frac{1}{2} a\right)^2\right\} > a - \frac{a^3}{4}.$$

Applying this result to the arc of 10″, taking the nearest value to five places of decimals, we shall have the arc of

$$10'' < \cdot 00005,\ \text{then}\ \frac{1}{4}(\text{arc}\ 10'')^3 < \cdot 000000000000032;$$

hence we have

$$\sin 10'' > \{\cdot 000048481368110 - \cdot 000000000000032\};$$

$$\text{or}\ \sin 10'' > \cdot 000048481368078,$$

we see that this sine only begins to differ from the arc of 10″ at the 13th place of decimals, and taking the nearest decimal it is the same; hence it follows that we may take

$$\sin 10'' = \cdot 0000484813681,$$

and we may be assured that the error will be less than unity at the 13th place of decimals,

$$\text{and since}\ \cos 10'' = \sqrt{1 - \sin^2 10''},\ \text{we have}$$

$$\cos 10'' = \cdot 9999999988248.$$

27. We can now successively find the sines and cosines of 20″, 30″,... and so on up to 45°, by means of the formulæ (1) and (4), given at page 28,

$$\sin(a+b) = 2\cos b \sin a - \sin(a-b),$$

$$\cos(a+b) = 2\cos b \cos a - \cos(a-b);$$

we can consider the arcs $a-b$, a, $a+b$, as three consecutive terms of an arithmetical progression whose common difference is b, if we put t, t', t'' for these three terms, we have

$$\sin t'' = 2\cos b \sin t' - \sin t,$$

$$\cos t'' = 2\cos b \cos t' - \cos t;$$

consequently, to obtain the sine and cosine for every 10″ we must make $b = 10''$, and putting α and β for the known values of the sine and cosine of 10″, we have

$\sin 0 = 0,$	$\cos 0 = 1,$
$\sin 10'' = \alpha,$	$\cos 10'' = \beta,$
$\sin 20'' = 2\beta \sin 10'',$	$\cos 20'' = 2\beta \cos 10'' - 1,$
$\sin 30'' = 2\beta \sin 20'' - \sin 10'',$	$\cos 30'' = 2\beta \cos 20'' - \cos 10'',$
$\sin 40'' = 2\beta \sin 30'' - \sin 20'',$	$\cos 40'' = 2\beta \cos 30'' - \cos 20'',$
&c. &c.	&c. &c.

But as β differs very little from unity, 2β is nearly $= 2$, and the calculation can be still more abridged; put k for the difference $2 - 2\beta$, we shall have $k = \cdot 0000000023504$, and $2\beta = 2 - k$, consequently the value of sin t'' becomes

$$\sin t'' = 2 \sin t' - k \sin t' - \sin t;$$

$$\therefore \sin t'' - \sin t' = (\sin t' - \sin t) - k \sin t'.$$

In such a long series of calculation the errors multiply considerably, and we therefore cannot be sure of the decimals being correct to the end, but, by using the formulæ given in Art. 15, we may verify the results, for the number of decimals common to both processes may be relied on as exact.

We may remark that when the radius is unity the sines and cosines will be fractional, and, consequently, their logarithms will be negative, in order to render them positive, the radius is taken $= 10^{10}$.

EXAMPLES.

In the right-angled triangle ACB, given the hypothenuse $AB = 1785{\cdot}395$ yards and the angle $ABC = 59^{0}\,37'\,42''$, to find the other two sides.

To find AC or b.	To find BC.
$b = c \sin B,$	$a = c \cos B,$
or $\log b = \log c + \log \sin B - 10$	$\log a = \log c + \log \cos B - 10$
$\log 1785{\cdot}395$ ... $3{\cdot}2517343$	$\log 1785{\cdot}395$ ......$3{\cdot}2517343$
$\log 59^{0}\,37'42''$... $9{\cdot}9358919$	$\log \cos 59^{0}\,37'42''$...$9{\cdot}7038132$
$\log b$ $3{\cdot}1876262$	$\log a$ $2{\cdot}9555475$
$\therefore\ b = 1540{\cdot}0374.$	$\therefore\ a = 902.708.$

In the triangle ABC (see fig. p. 52), given $a = 9459{\cdot}31$ yards, $b = 8032{\cdot}29$ yards, $c = 8242{\cdot}58$ yards, find A;

$$\sin \tfrac{1}{2} A = \sqrt{\frac{(S-b)\,(S-c)}{bc}},$$

$2S = 25734{\cdot}18$, $S = 12867{\cdot}09$, $S - b = 4834{\cdot}8$, $S - c = 4624{\cdot}51$,

$$\log \sin \tfrac{1}{2} A = 10 + \frac{1}{2}\{\log (S-b) + \log (S-c) - \log b - \log c\},$$

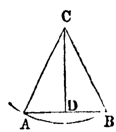
$$2 \log \sin \tfrac{1}{2} A = 20 + \log (S-b) + \log (S-c) - \log b - \log c,$$

$$\begin{aligned}
\log(S-b) &= 3{\cdot}6843785\\
\log(S-c) &= 3{\cdot}6650657\\
\text{Comp. } \log b &= 6{\cdot}0951606\\
\text{Comp. } \log c &= 6{\cdot}0839368\\
\hline
2\log\sin\tfrac{1}{2}A &= 19{\cdot}5285416,\\
\log\sin\tfrac{1}{2}A &= 9{\cdot}7642708,\\
\tfrac{1}{2}A &= 35^\circ 31' 47'',\\
A &= 71^\circ 3' 34''.
\end{aligned}$$

CHAPTER IV.

ON POLYGONS.

28. To find the radius of a circle described about a regular polygon of n sides.



Let $R = AC =$ radius, AB one of the sides of the polygon $= a$, ACB the angle subtending one of the sides of the polygon, then n times the angle ACB will be the sum of all the angles, or 360°;

$$\therefore\ \angle ACB = \frac{360^\circ}{n};\quad \therefore\ \angle ACD = \frac{180^\circ}{n},$$

since CD bisects the angle ACB,

$$R\sin ACD = AD, \text{ or } 2R\sin ACD = AB;$$

$$\therefore\ R = \frac{AB}{2\sin ACD} = \frac{AB}{2\sin\dfrac{180^\circ}{n}} = \frac{1}{2}AB\operatorname{cosec}\frac{180^\circ}{n}$$

$$= \frac{1}{2}a\operatorname{cosec}\frac{180^\circ}{n}.$$

29. If now we suppose AB to be the side of a regular polygon described about a circle, then CD will be the radius of that circle, which call r,

$$\text{then } \frac{CD}{AD} = \cot ACD = \cot\frac{180^\circ}{n};$$

$$\therefore\ CD = r = AD \cot\frac{180^0}{n} = \frac{1}{2}AB\cot\frac{180^0}{n} = \frac{1}{2}a\cot\frac{180^0}{n};$$

$$\therefore\ R + r = \frac{1}{2}a\left(\cot\frac{180^0}{n} + \operatorname{cosec}\frac{180^0}{n}\right)$$

$$= \frac{1}{2}a\left\{\frac{\cos\frac{180^0}{n}}{\sin\frac{180^0}{n}} + \frac{1}{\sin\frac{180^0}{n}}\right\}$$

$$= \frac{1}{2}a\left\{\frac{1+\cos\frac{180^0}{n}}{\sin\frac{180^0}{n}}\right\} = \frac{\frac{a}{2}\left(2\cos^2\frac{90^0}{n}\right)}{2\sin\frac{90^0}{n}\cos\frac{90^0}{n}} = \frac{a}{2}\cot\frac{90^0}{n},$$

$$R.r = \frac{1}{4}a^2\operatorname{cosec}\frac{180^0}{n}\cot\frac{180^0}{n} = \frac{1}{4}a^2\frac{\operatorname{cosec}^2\frac{180^0}{n}}{\sec\frac{180^0}{n}},$$

$$\frac{R}{r} = \frac{\frac{1}{2}a\operatorname{cosec}\frac{180^0}{n}}{\frac{1}{2}a\cot\frac{180^0}{n}} = \frac{\operatorname{cosec}\frac{180^0}{n}}{\cos\frac{180^0}{n}\operatorname{cosec}\frac{180^0}{n}} = \frac{1}{\cos\frac{180^0}{n}}.$$

30. To find the area of a regular polygon of n sides inscribed in a circle.

The polygon is composed of as many triangles as there are sides of the figure:

area of the triangle $ACB = AD.CD = r\sin\frac{180^0}{n}\,r\cos\frac{180^0}{n}$

$$= r^2\sin\frac{180^0}{n}\cos\frac{180^0}{n} = \frac{1}{2}r^2\sin\frac{360^0}{n}.$$

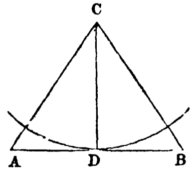
Hence the area of the inscribed polygon $= n$ times the triangle $ACB = \frac{n}{2}r^2\sin\frac{360^0}{n}$.

Or thus, the area of the triangle $ACB = \frac{AC.CB\sin ACB}{2}$

$$= \frac{r^2\sin\frac{360^0}{n}}{2};$$

$$\therefore \text{ area of polygon } = \frac{n}{2} r^2 \sin \frac{360^\circ}{n}.$$

31. For the circumscribing polygon.



Let $CD = r$, $\frac{AD}{CD} = \tan ACD = \tan \frac{180^\circ}{n}$;

$$\therefore AD = CD \tan \frac{180^\circ}{n} = r \tan \frac{180^\circ}{n},$$

area of $\triangle ACB = AD . CD = r^2 \tan \frac{180^\circ}{n}$,

area of polygon $= n$ times the triangle $ACB = nr^2 \tan \frac{180^\circ}{n}$,

$$\frac{\text{area of the inscribed polygon}}{\text{area of the circumscribed polygon}} = \frac{\frac{n}{2} r^2 \sin \frac{360^\circ}{n}}{nr^2 \tan \frac{180^\circ}{n}} = \frac{\frac{1}{2} \sin \frac{360^\circ}{n}}{\tan \frac{180^\circ}{n}}$$

$$= \frac{\frac{1}{2} . 2 \sin \frac{180^\circ}{n} \cos \frac{180^\circ}{n}}{\frac{\sin \frac{180^\circ}{n}}{\cos \frac{180^\circ}{n}}} = \cos^2 \frac{180^\circ}{n}.$$

The area of a circle whose radius is $r = \pi r^2$.

From the above the area of a polygon inscribed in a circle, is

$$\frac{n}{2} r^2 \sin \frac{360^\circ}{n} = \frac{n}{2} r^2 \sin \frac{2\pi}{n} = \pi r^2 \frac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}}.$$

As we increase the number of sides, the area of the polygon approaches nearer and nearer to that of the circle, and when n is infinitely great it becomes equal to it, for then

$\frac{2\pi}{n}$ is an infinitely small angle, therefore $\dfrac{\sin \frac{2\pi}{n}}{\frac{2\pi}{n}} = 1$;

$\therefore$ area of the circle $= \pi r^2$.

Examples on the two preceding Chapters.

(1) Given the perimeter of a right-angled triangle, and the radius of the inscribed circle; determine the sides,

$$S = \frac{a+b+c}{2}, \qquad \frac{ab}{2} = \Delta = Sr,$$

$$ab = 2Sr, \qquad a+b+c = 2S, \qquad a+b = 2S - c,$$

$$a+b = 2S - c = 2S - (S-r) = S + r,$$

$$a + b = S + r,$$

$$a^2 + 2ab + b^2 = S^2 + 2Sr + r^2,$$

$$4ab \quad = \quad 8Sr;$$

$$a^2 - 2ab + b^2 = S^2 - 6Sr + r^2,$$

$$a - b = \sqrt{(S^2 - 6Sr + r^2)}$$

$$a + b = 2S - c,$$

$$2a = 2S - c + \sqrt{(S^2 - 6Sr + r^2)},$$

$$a = \frac{2S - c + \sqrt{(S^2 - 6Sr + r^2)}}{2},$$

$$2b = 2S - c - \sqrt{(S^2 - 6Sr + r^2)},$$

$$b = \frac{2S - c - \sqrt{(S^2 - 6Sr + r^2)}}{2},$$

$$c = S - r$$

$$a^2 + 2ab + b^2 = 4S^2 - 4Sc + c^2$$

$$a^2 \qquad + b^2 = \qquad\qquad c^2$$

$$2ab = 4S^2 - 4Sc, \qquad \text{but } ab = 2Sr,$$

$$\therefore\ 4Sr = 4S^2 - 4Sc$$

$$4Sc = 4S^2 - 4Sr$$

$$4c = 4S - 4r$$

$$c = S - r.$$

(2) Given the perimeter of the same triangle, and the radius of the circumscribed circle; find the sides,

$$\frac{a+b+c}{2} = S, \ \frac{ab}{2} = \Delta = \frac{abc}{4R},$$

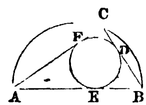
$$a + b + c = 2S \qquad 2R = c,$$

$$a + b = 2S - 2R,$$

$$a^2 + 2ab + b^2 = 4S^2 - 8SR + 4R^2,$$
$$a^2 \qquad + b^2 = \qquad 4R^2;$$
$$\therefore\ 2ab \qquad = 4S^2 - 8SR;$$
$$\therefore\ a^2 - 2ab + b^2 = 4R^2 + 8SR - 4S^2;$$
$$\therefore\ a - b = \sqrt{(4R^2 + 8SR - 4S^2)},$$
$$a + b = 2S - 2R,$$
$$2a = 2S - 2R + \sqrt{(4R^2 + 8SR - 4S^2)},$$
$$a = \frac{2S - 2R + \sqrt{(4R^2 + 8SR - 4S^2)}}{2},$$
$$2b = 2S - 2R - \sqrt{(4R^2 + 8SR - 4S^2)},$$
$$b = \frac{2S - 2R - \sqrt{(4R^2 + 8SR - 4S^2)}}{2}.$$

(3) If (D) and (d) be the diameters of the two circles, and (a), (b) the sides which include the right angle; shew that $D + d = a + b$.

$$AF + FC + CD + DB = b + a,$$
$$AF = AE \text{ and } BE = BD,$$
$$FC = r \text{ and } DC = r;$$
$$\because\ AE + BE + 2r = a + b,$$



but $AE + EB = D$.

Hence $a + b = D + 2r = D + d$.

(4) The sides of a triangle are as 3, 5, 6; shew that

$$R : r :: 45 : 16,$$

$$Rr = \frac{abc}{2(a+b+c)} = \frac{3.5.6}{2(3+5+6)} = \frac{3.5.3}{3+5+6} = \frac{45}{14},$$

$$r = \sqrt{\frac{(S-a)(S-b)(S-c)}{S}} = \sqrt{\frac{(7-3)(7-5)(7-6)}{7}} = \sqrt{\frac{8}{7}},$$

$$R = \frac{45}{14} \div r = \frac{45}{14} \div \sqrt{\frac{8}{7}} = \frac{45}{14} \times \sqrt{\frac{7}{8}};$$

$$R : r :: \frac{45}{14} \times \sqrt{\frac{7}{8}} : \sqrt{\frac{8}{7}},$$

$$45 \times 7 : 14 \times 8;$$

$$R : r :: 45 : 16.$$

(5) If the side of a pentagon inscribed in a circle be 1, the radius $= \frac{\sqrt{(5+\sqrt{5})}}{\sqrt{10}}$.

$$\text{Here } n = 5, \ a = 1,$$

$$\text{perimeter} = 5 = 2nr \sin \frac{\pi}{n}$$

$$= 2 \times 5r \sin \frac{180^\circ}{5},$$

$$\therefore \ 1 = 2r \sin 36^\circ$$

$$= 2r \frac{\sqrt{(5-\sqrt{5})}}{2\sqrt{2}}$$

$$= r \frac{\sqrt{(5-\sqrt{5})}}{\sqrt{2}},$$

$$\therefore \ r = \frac{1}{\frac{\sqrt{(5-\sqrt{5})}}{\sqrt{2}}}$$

$$= \frac{\sqrt{2}}{\sqrt{(5-\sqrt{5})}}$$

$$= \frac{\sqrt{2}\sqrt{(5+\sqrt{5})}}{\sqrt{(25-5)}},$$

$$= \frac{\sqrt{2}\sqrt{(5+\sqrt{5})}}{\sqrt{20}},$$

$$= \frac{\sqrt{(5+\sqrt{5})}}{\sqrt{10}}.$$

(6) Find the actual value of a side of a twenty-four sided regular figure inscribed in a given circle.

Here $n = 24$. Put $A = \frac{15^\circ}{2}$, $2A = 15^\circ$,

$$\sin 15^\circ = \frac{\sqrt{3}-1}{2\sqrt{2}}.$$

$$\text{Perimeter} = 2nr \sin \frac{\pi}{n}$$

$$= 2 \times 24r \sin \frac{180^\circ}{24}$$

$$= 2 \times 24r \sin \frac{15^\circ}{2},$$

$$\text{but } \sin A = \frac{1}{2}\{\sqrt{(1 + \sin 2A)} - \sqrt{(1 - \sin 2A)}\}$$

$$= \frac{1}{2}\left\{\sqrt{\left(1 + \frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \sqrt{\left(1 - \frac{\sqrt{3}-1}{2\sqrt{2}}\right)}\right\};$$

$$\therefore \text{perimeter} = 2 \times 24r \frac{1}{2}\left\{\sqrt{\left(1 + \frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \sqrt{\left(1 - \frac{\sqrt{3}-1}{2\sqrt{2}}\right)}\right\}$$

$$= 2 \times 12r\left\{\sqrt{\left(1 + \frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \sqrt{\left(1 - \frac{\sqrt{3}-1}{2\sqrt{2}}\right)}\right\}$$

$$= 24r\left\{\sqrt{\left(1 + \frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \sqrt{\left(1 - \frac{\sqrt{3}-1}{2\sqrt{2}}\right)}\right\},$$

$$\text{but } a = \frac{\text{perimeter}}{24};$$

$$\therefore a = 24r \frac{\sqrt{\left(1 + \frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \sqrt{\left(1 - \frac{\sqrt{3}-1}{2\sqrt{2}}\right)}}{24},$$

$$= r\left\{\sqrt{\left(1 + \frac{\sqrt{3}-1}{2\sqrt{2}}\right)} - \sqrt{\left(1 - \frac{\sqrt{3}-1}{2\sqrt{2}}\right)}\right\}.$$

(7) If r be the radius of the inscribed circle,

$$\text{area} = r^2 (\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C).$$

By Euc. Book IV. Prop. 4,

$$\angle OBA = \tfrac{1}{2}B, \quad \angle OCA = \tfrac{1}{2}C.$$

$$\text{Now } Ba = Oa \cot \tfrac{1}{2}B, \quad Ca = Oa \cot \tfrac{1}{2}C;$$

$$\therefore a = Oa \cot \tfrac{1}{2}B + Oa \cot \tfrac{1}{2}C,$$

$$a = r(\cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C),$$

$$b = r(\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}C),$$

$$c = r(\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B),$$

$$a + b + c = r(\cot \tfrac{1}{2}B + \cot \tfrac{1}{2}C)$$
$$+ r(\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}C) + r(\cot \tfrac{1}{2}A + \cot \tfrac{1}{2}B),$$

$$S = \frac{r(\cot\frac{1}{2}B + \cot\frac{1}{2}C) + r(\cot\frac{1}{2}A + \cot\frac{1}{2}C) + r(\cot\frac{1}{2}A + \cot\frac{1}{2}B)}{2}$$

$$= r\,\frac{(2\cot\frac{1}{2}A + 2\cot\frac{1}{2}B + 2\cot\frac{1}{2}C)}{2}$$

$$= 2r\,\frac{(\cot\frac{1}{2}A + \cot\frac{1}{2}B + \cot\frac{1}{2}C)}{2},$$

$$S = r(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C).$$

But area $= S.r$,

$$\text{area} = rr(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C);$$

$$\therefore \text{area} = r^2(\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C).$$

(8) The perimeters of an equilateral triangle, a square, and a hexagon, each including the same area, are as $\sqrt[4]{27}$, $\sqrt[4]{16}$, $\sqrt[4]{12}$.

Let each side of the $\triangle ABC$ at p. 82 $= a$, $3a =$ perimeter,

$$CD = \sqrt{a^2 - \left(\frac{a}{2}\right)^2} = \sqrt{\frac{3a^2}{4}}, \text{ and } \frac{CD.a}{2} = \text{area } \triangle;$$

$$\therefore \text{ side of equal square} = \sqrt{\frac{CD.a}{2}}$$

$$= \sqrt{CD}.\sqrt{\frac{a}{2}} = \sqrt[4]{\frac{3a^2}{4}}.\sqrt[4]{\frac{a^2}{4}} = \sqrt[4]{\frac{3a^4}{16}},$$

$$4.\sqrt[4]{\frac{3a^4}{16}} = \text{perimeter of square.}$$

$$\text{Area of } \triangle = CD.\frac{a}{2} = \sqrt{\frac{3a^2}{4}}.\sqrt{\frac{a^2}{4}} = \sqrt{\frac{3a^4}{16}}$$

= area of square = area of hexagon. Let $x =$ side of hexagon.

$$\text{Area of hexagon} = \frac{nx^2}{4}.\cot\frac{\pi}{n} = \frac{6x^2}{4}.\cot\frac{180^0}{6} = \frac{6x^2}{4}\cot 30^0$$

$$= \frac{3}{2}x^2.\sqrt{3} = \sqrt{\frac{3a^4}{16}}, \quad \frac{27x^4}{4} = \frac{3a^4}{16}, \quad 36x^4 = a^4,$$

$$6x^2 = a^2, \quad x\sqrt{6} = a; \quad \therefore x = \frac{a}{\sqrt{6}} = \text{side of hexagon:}$$

$$6\frac{a}{\sqrt{2}} = a\sqrt{6} = a\sqrt{3}.\sqrt{2} = \text{perimeter of hexagon:}$$

$$3a : 2a\sqrt[4]{3} : a\sqrt{2}.\sqrt{3} = \sqrt[4]{27} : \sqrt[4]{16} : \sqrt[4]{12}.$$

(9) If O be the centre of the circle inscribed in the triangle ABC,

$$OA^2 + OB^2 + OC^2 = ab + ac + bc - 12Rr.$$

See fig. page 59.

$$\left.\begin{aligned} AO^2 &= r^2 + AE^2 = r^2 + r^2 \cot^2 \tfrac{1}{2}A, \\ OB^2 &= r^2 + BF^2 = r^2 + r^2 \cot^2 \tfrac{1}{2}B, \\ OC^2 &= r^2 + EC^2 = r^2 + r^2 \cot^2 \tfrac{1}{2}C, \end{aligned}\right\}$$

$$AO^2 + OB^2 + OC^2 = 3r^2 + r^2\left(\cot^2\frac{A}{2} + \cot^2\frac{B}{2} + \cot^2\frac{C}{2}\right),$$

by Art. 21, and p. 55,

$$= 3\,\frac{(s-a)(s-b)(s-c)}{s} + \frac{(s-a)(s-b)(s-c)}{s}$$

$$\left\{\frac{s(s-a)}{(s-b)(s-c)} + \frac{s(s-b)}{(s-a)(s-c)} + \frac{s(s-c)}{(s-b)(s-a)}\right\}$$

$$= 3\,\frac{(s-a)(s-b)(s-c)}{s} + (s-a)^2 + (s-b)^2 + (s-c)^2$$

$$= 3s^2 - 3sa - 3sb + 3ab - 3sc + 3ac + 3bc - \frac{3abc}{s}$$

$$+ s^2 - 2sa + a^2 + s^2 - 2sb + b^2 + s^2 - 2sc + c^2$$

$$= 6s^2 - 5s(a+b+c) + a^2 + b^2 + c^2 + 2ab + 2ac$$

$$+ 2bc + ab + ac + bc - \frac{3abc}{s}$$

$$= 6s^2 - 5s\,.\,2s + (a+b+c)^2 + ab + ac + bc - 12Rr$$

$$= 6x^2 - 10s^2 + 4s^2 + ab + ac + bc - 12Rr$$

$$= ab + ac + bc - 12Rr;$$

$$\therefore\ AO^2 + OB^2 + OC^2 = ab + ac + bc - 12Rr.$$

(10) The area of a regular hexagon inscribed in a circle is a mean proportional between the areas of the inscribed and circumscribed equilateral triangles.

$$\text{Area of hexagon} = \frac{nr^2}{2}\sin\frac{2\pi}{n}$$

$$= \frac{6r^2}{2}\sin\frac{360^\circ}{6}$$

$$= 3r^2 \sin 60^\circ$$

$$= 3r^2\,\frac{\sqrt{3}}{2}.$$

$$\text{Area of inscribed triangle} = \frac{nr^2}{2} \sin \frac{2\pi}{n}$$
$$= \frac{3r^2}{2} \sin \frac{360^\circ}{3}$$
$$= \frac{3r^2}{2} \sin 120^\circ$$
$$= \frac{3r^2}{2} \frac{\sqrt{3}}{2}$$
$$= \frac{3r^2\sqrt{3}}{4}.$$

$$\text{Area of circumscribed } \Delta = nr^2 \tan \frac{\pi}{2}$$
$$= 3r^2 \tan 60^\circ$$
$$= 3r^2\sqrt{3};$$

$$\therefore \text{ as } \frac{3r^2\sqrt{3}}{4} : \frac{3r^2\sqrt{3}}{2} :: \frac{3r^2\sqrt{3}}{2} : 3r^2\sqrt{3}.$$

(11) The square of the side of a pentagon inscribed in a circle is equal to the sum of the squares of the sides of a regular hexagon and decagon inscribed in the same circle.

$$\text{Pentagon, perimeter} = n \times AB = 2nr \sin \frac{\pi}{n};$$

$$AB = 2r \sin \frac{180^\circ}{5} = 2r \sin 36^\circ$$
$$= 2r \frac{\sqrt{(5-\sqrt{5})}}{2\sqrt{2}} = r \frac{\sqrt{(5-\sqrt{5})}}{\sqrt{2}},$$
$$AB^2 = \frac{(5-\sqrt{5})\, r^2}{2}.$$

$$\text{Hexagon, perimeter} = n\, AB_{\prime} = 2nr \sin \frac{\pi}{n},$$

$$AB_{\prime} = 2r \sin \frac{180^\circ}{6} = 2r \sin 30^\circ$$
$$= 2r \frac{1}{2} = r; \quad \therefore AB_{\prime}^2 = r^2.$$

Decagon, perimeter $= n\,AB_{\prime\prime} = 2\,nr\sin\dfrac{180^\circ}{n}$,

$$AB_{\prime\prime} = 2r\sin\frac{180^\circ}{10} = 2r\sin 18^\circ$$

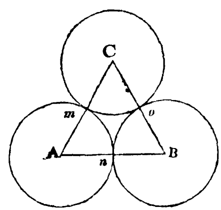
$$= 2r\frac{\sqrt{5}-1}{4} = r\frac{\sqrt{5}-1}{2};$$

$$\therefore\ AB_{\prime\prime}^2 = \frac{(3-\sqrt{5})\,r^2}{2},$$

$$AB_{\prime}^2 + AB_{\prime\prime}^2 = \frac{(3-\sqrt{5})\,r^2}{2} + r^2 = \frac{(3-\sqrt{5}+2)\,r^2}{2}$$

$$= \frac{(5-\sqrt{5})\,r^2}{2} = AB^2.$$

(12) Three equal circles touch each other; shew that the space between them is nearly equal to the square described upon a fifth part of the diameter; find the area when the circles are unequal, and the radii as the numbers 1, 2, 3.



Let rad. $= \dfrac{1}{2}$ diam. $= 1$, $\pi = 3{\cdot}1416$,

$$\text{arc } mn = \frac{2\,(3{\cdot}1416)}{6} = {\cdot}5236 \times 2$$

$$= 1{\cdot}0472.$$

Sector $Amn = \dfrac{1}{2}\,r \times \text{arc } mn = {\cdot}5236$;

$\therefore$ the area of the three sectors $= 1{\cdot}5708$,

$$Cn^2 = AC^2 - An^2; \qquad \therefore\ Cn = \sqrt{(AC^2 - An^2)}$$

$$= \sqrt{(AC^2 - 1)} = \sqrt{(4-1)} = \sqrt{3};$$

$$\Delta ABC = \frac{AB \times Cn}{2} = 1{\cdot}73205.$$

Now the space $mno = \Delta ABC - 3$ sector Amn

$$= 1{\cdot}73205 - 1{\cdot}5708 = {\cdot}161.$$

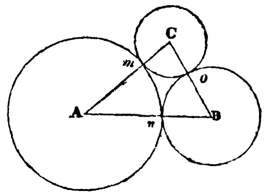
Now the square described on $\dfrac{1}{5}$ of the diam. $= \dfrac{4}{25} = {\cdot}16$.

Hence space $mno = \left(\dfrac{\text{diam.}}{5}\right)^2$ nearly.

(13) Let rad. $Co = 1$, $Bo = 2$, $Am = 3$, then the diameters are 2, 4, 6; and the sides of the $\triangle ABC$ are

$$BC = 3,\ AC = 4,\ AB = 5,\ \therefore\ AB^2 = AC^2 + BC^2;$$

$$\therefore\ \angle C = 90^\circ,\ \sin a = \frac{BC}{AB},$$



$$\log \sin A = \log BC - \log AB + 10,$$

$$\log 3 = \cdot 47712$$
$$\log 5 = \cdot 69897$$
$$10$$

$$\log \sin 36^\circ 52' = 9{\cdot}77815;$$

$$\therefore\ A = 36^\circ{\cdot}8666,$$
$$B = 53^\circ{\cdot}1333,$$

$$360^\circ : 36^\circ{\cdot}866 :: 3{\cdot}1416 \times 6 : 1{\cdot}9305 = \text{arc } mn,$$

$$\text{sector } Amn = \frac{1{\cdot}9305 \times 3}{2} = 2{\cdot}8958,$$

$$360^\circ : 53^\circ{\cdot}133 :: 3{\cdot}1416 \times 4 : 1{\cdot}8546 = \text{arc } no,$$

$$\text{sector } Bno = \frac{1{\cdot}8546 \times 2}{2} = 1{\cdot}8546,$$

$$\frac{\text{circumference of} \odot moC}{4} = \frac{3{\cdot}1416 \times 2}{4} = 1{\cdot}5708 = \text{arc } mo,$$

$$\text{sector } Com = \frac{1{\cdot}5708 \times 1}{2} = \cdot 7854,$$

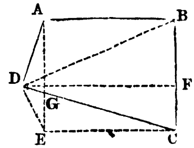
$$2{\cdot}8958$$
$$1{\cdot}8546$$
$$\cdot 7854$$

$$\text{sum of sectors} = 5{\cdot}5358$$

$$\triangle ABC = \frac{AC \times BC}{2} = \frac{4 \times 3}{2} = 6;$$

$$\therefore\ \text{space } nmo = \triangle ABC - \text{sum of sectors} = \cdot 4642.$$

(14) Given $AB = a$, $BC = b$, $CD = c$, three sides of a quadrilateral, and $\angle ABC = \theta$, $\angle BCD = \phi$; shew that twice the area $= ab \sin \theta + cb \sin \phi - ac\ (\theta + \phi)$.



Let $ABCD$ be the quadrilateral.

From the point A draw $AE \parallel BC$, and from the point D draw $DF \parallel$ to AB or EC,

$$\text{then } cb \sin\phi = 2\Delta BCD,$$

$$\begin{aligned} \text{and } ab \sin\theta &= \text{the figure } ABCE \\ &= \square\, ABFG + GFCE, \\ &= 2\Delta ABD + \square\, GFCE; \end{aligned}$$

$$\begin{aligned} \therefore\ ab \sin\theta + bc \sin\phi &= 2\Delta(ABD + BCD) + \square\, GFCE \\ &= 2\,\text{quadri.} + \square\, GFCE \\ &= 2\,\text{quadri.} + 2\Delta DEC \\ &= 2\,\text{quadri.} + ac\sin(\phi + \theta); \end{aligned}$$

$$\therefore\ ab\sin\theta + bc\sin\phi - ac\sin(\phi+\theta) = 2\text{ quadrilateral.}$$

(15) If R and r be the radii of the circumscribed and inscribed circles of a triangle:

$$r = 4R \sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C.$$

See Ex. p. 86.

$$\begin{aligned} \tfrac{1}{2}(a+b+c) &= r\{\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C\}, \\ \text{area} &= r^2\{\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C\}, \\ \text{but } \tfrac{1}{2}A &+ \tfrac{1}{2}B + \tfrac{1}{2}C = 90^\circ; \end{aligned}$$

$\therefore$ by Ex. 26, p. 45,

$$\begin{aligned} \text{area} &= r^2\{\cot\tfrac{1}{2}A + \cot\tfrac{1}{2}B + \cot\tfrac{1}{2}C\} \\ &= r^2 \cot\tfrac{1}{2}A \cot\tfrac{1}{2}B \cot\tfrac{1}{2}C. \end{aligned}$$

And by Art. 22, Chap. III., and formulæ, p. 55,

$$\begin{aligned} \text{area} &= 2R^2 \sin A \sin B \sin C \\ &= 4R^2 \sin\tfrac{1}{2}A \cos\tfrac{1}{2}A\ 2\sin\tfrac{1}{2}B\cos\tfrac{1}{2}B\ 2\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C \\ &= 16R^2 \sin\tfrac{1}{2}A\cos\tfrac{1}{2}A\sin\tfrac{1}{2}B\cos\tfrac{1}{2}B\sin\tfrac{1}{2}C\cos\tfrac{1}{2}C \\ &= r^2 \cot\tfrac{1}{2}A \cot\tfrac{1}{2}B \cot\tfrac{1}{2}C \\ &= r^2 \frac{\cos\tfrac{1}{2}A\cos\tfrac{1}{2}B\cos\tfrac{1}{2}C}{\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C}; \end{aligned}$$

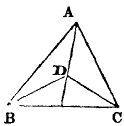
$$\therefore\ r^2 \frac{1}{\sin\frac{1}{2}A}\,\frac{1}{\sin\frac{1}{2}B}\,\frac{1}{\sin\frac{1}{2}C} = 16R^2 \sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C,$$

$$r^2 \frac{1}{\sin\frac{1}{2}A \sin\frac{1}{2}B \sin\frac{1}{2}C} = 16R^2 \sin\tfrac{1}{2}A \sin\tfrac{1}{2}B \sin\tfrac{1}{2}C;$$

$$\therefore\ r^2 = 16R^2 (\sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C)^2;$$

$$\therefore\ r = 4R \sin \tfrac{1}{2}A \sin \tfrac{1}{2}B \sin \tfrac{1}{2}C.$$

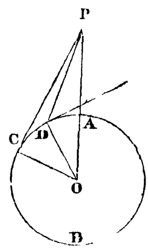
(16) In the triangle ABC take a point D, join DA, DB, DC; given AB, AC, the $\angle ABD$, $\angle ACD$, and $\angle BDC$, find BC.



$AB = a$, $\angle ABD = \alpha$,
$AC = b$, $\angle ACD = \beta$,
$\angle BDC = \gamma$,

$$\angle DBC + \angle DCB = \pi - \gamma,$$
$$\angle ABC + \angle ACB = (\pi - \gamma + \alpha + \beta),$$
$$\angle BAC = \pi - (\pi - \gamma + \alpha + \beta),$$
$$\angle BAC = \gamma - \alpha - \beta,$$
$$BC = \sqrt{(AB^2 + AC^2 - 2AB.AC \cos BAC)},$$
$$BC = \sqrt{\{a^2 + b^2 - 2ab \cos (\gamma - \alpha - \beta)\}}.$$

(17) $ABCD$ is the circumference of a circle, O its centre, from C draw a tangent to the circle meeting the radius OA produced, in P, join PD; then if $CP = a$, $DP = b$, θ = angle formed by DP and a tangent at D, prove that $r = \dfrac{a^2 - b^2}{2b}$ cosec θ, r being the radius of the circle.



COP is a right-angled triangle;

$$\therefore\ OP^2 = a^2 + r^2,$$

and in $\triangle ODP$, the sides OD, DP, and $\angle ODP$ are known.

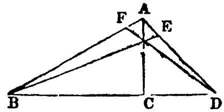
$$\therefore\ OP^2 = b^2 + r^2 - 2br \cos\left(\frac{\pi}{2} + \theta\right);$$

$$\therefore\ a^2 + r^2 = b^2 + r^2 - 2br \cos\left(\frac{\pi}{2} + \theta\right);$$

$$\therefore\ a^2 - b^2 = 2br \sin\theta;$$

$$\therefore\ r = \frac{a^2 - b^2}{2b \sin\theta} = \frac{a^2 - b^2}{2b} \text{ cosec } \theta.$$

(18) Given the three perpendiculars from the angles on the opposite sides; find the sides.



Let $AB = x$, $AD = y$, $BD = z$,
$AC = a$, $BE = b$, $DF = c$,
$2\Delta = cx = by = az$,
$2\Delta = 2\sqrt{\{S(S-x)(S-y)(S-z)\}}$;

$$\therefore\ cx = 2\sqrt{\{S(S-x)(S-y)(S-z)\}}$$

$$= 2\sqrt{\left(\frac{x+y+z}{2}\cdot\frac{y+z-x}{2}\cdot\frac{x+z-y}{2}\cdot\frac{x+y-z}{2}\right)}$$

$$= \frac{1}{2}\sqrt{\left\{\left(x+\frac{cx}{b}+\frac{cx}{a}\right)\left(\frac{cx}{b}+\frac{cx}{a}-x\right)\left(x+\frac{cx}{a}-\frac{cx}{b}\right)\left(x+\frac{cx}{b}-\frac{cx}{a}\right)\right\}}$$

$$= \frac{1}{2}x^2\sqrt{\left\{\left(1+\frac{c}{b}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{c}{a}-1\right)\left(1+\frac{c}{a}-\frac{c}{b}\right)\left(1+\frac{c}{b}-\frac{c}{a}\right)\right\}},$$

$$2c = x\sqrt{\left\{\left(1+\frac{c}{b}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{c}{a}-1\right)\left(1+\frac{c}{a}-\frac{c}{b}\right)\left(1+\frac{c}{b}-\frac{c}{a}\right)\right\}};$$

$$\therefore\ x = \frac{2c}{\sqrt{\left\{\left(1+\frac{c}{b}+\frac{c}{a}\right)\left(\frac{c}{b}+\frac{c}{a}-1\right)\left(1+\frac{c}{a}-\frac{c}{b}\right)\left(1+\frac{c}{b}-\frac{c}{a}\right)\right\}}}.$$

Similarly,

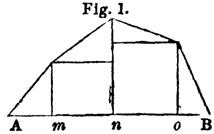
$$y = \frac{2b}{\sqrt{\left\{\left(1+\frac{b}{a}+\frac{b}{c}\right)\left(\frac{b}{a}+\frac{b}{c}-1\right)\left(1+\frac{b}{c}-\frac{b}{a}\right)\left(1+\frac{b}{a}-\frac{b}{c}\right)\right\}}},$$

$$z = \frac{2a}{\sqrt{\left\{\left(1+\frac{a}{b}+\frac{a}{c}\right)\left(\frac{a}{b}+\frac{a}{c}-1\right)\left(1+\frac{a}{c}-\frac{a}{b}\right)\left(1+\frac{a}{b}-\frac{a}{c}\right)\right\}}}.$$

(19) If l = the base of a polygon, abc, &c., the sides beginning from the base, $\alpha\beta\gamma$, &c., the angles made by abc, &c., with the base, then

$$l = a\cos\alpha + b\cos\beta + c\cos\gamma + \&c.$$

Fig. 1.



Let $AB = l$, from the upper angles draw lines perpendicular and parallel to the base. Then (fig. 1),

$Am = a\cos\alpha$, $mn = b\cos\beta$,
$no = c\cos\gamma$, $oB = d\cos\delta$,

$$AB = Am + mn + no + oB = a\cos\alpha + b\cos\beta + c\cos\gamma + d\cos\delta;$$

$$\therefore\ l = a\cos\alpha + b\cos\beta + c\cos\gamma + \&c.$$

Fig. 2.

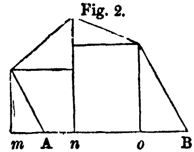


Fig. 2. $\frac{Am}{a} = \cos(\alpha - \pi) = -\cos\alpha$

$-Am = a\cos\alpha,\ mn = Am + An = b\cos\beta,$

$no = c\cos\gamma,\ oB = d\cos\delta,$

$AB = -Am + Am + An + no + oB$

$= a\cos\alpha + b\cos\beta + c\cos\beta + \&c.;$

$\therefore\ l = a\cos\alpha + b\cos\beta + c\cos\gamma + \&c.$

32. The two following rules may be found useful to practical men; the first is when two sides and the included angle are given, and the second when the three sides are given.

By Art. 19, $\frac{a+b}{a-b} = \frac{\tan\frac{1}{2}(A+B)}{\tan\frac{1}{2}(A-B)}.$

Hence the following rule:

As the sum of any two sides of a plane triangle is to their difference, so is the tangent of half the sum of their opposite angles to the tangent of half their difference.

By Art. 16, $\cos A = \frac{b^2+c^2-a^2}{2bc};$

$$\therefore\ a^2 - b^2 = c^2 - 2bc\cos A$$
$$= C(c^2 - 2b\cos A),$$
$$(a+b)(a-b) = C(c - 2b\cos A) = c(AD \sim DB),$$
$$c : a+b :: a-b : AD \sim DB.$$

From this we have the following rule:

Take the longest side for the base; then as the base is to the sum of the other two sides, so is their difference to the difference of the segments of the base.

Half the sum of the segments added to half their difference gives the greater segment, and half their difference subtracted from half their sum gives the least.

The following logarithmic forms for right-angled triangles may also be useful, although they are only the definitions at page 8 put into logarithms.

Referring to the figure, page 52, considering C to be a right angle, we have by the definitions,

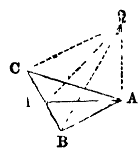
$$\sin A = \frac{BC}{AB} = \frac{a}{c}, \quad \cos A = \frac{AC}{AB} = \frac{b}{c}, \quad \tan A = \frac{BC}{AC} = \frac{a}{b}:$$

in logarithms,

$$\log a = \log c + \log \sin A - 10 = \log b + \log \tan A - 10,$$
$$\log b = \log c + \log \cos A - 10 = \log a - \log \tan A - 10,$$
$$\log c = \log a - \log \sin A + 10 = 10 + \log a - \log \sin A.$$

HEIGHTS AND DISTANCES, ETC.

(1) In the year 1784, a base BC being measured on Blackheath, of a mile in length, the angles of elevation of Lunardi's balloon were taken, at the same time, by observers placed at its two extremities and in the middle; the one at B being $46^\circ 10'$, that at P $55^\circ 8'$, and that at C $54^\circ 30'$: required the height OA of the balloon.—Bonnycastle's *Trigonometry*.



Investigation.

By right-angled triangles,

$AB = OA \cot OBA$; $AP = OA \cot OPA$; $AC = OA \cot OCA$.

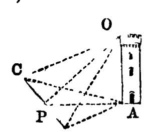
Now $AB^2 + AC^2 = 2AP^2 + 2BP^2$,

$$\text{or } AB^2 + AC^2 - 2AP^2 = 2BP^2,$$

hence by substitution we have

$$OA^2 \cot^2 OBA + OA^2 \cot^2 OCA - 2OA^2 \cot^2 OPA = 2BP^2,$$
$$OA^2 (\cot^2 OBA + \cot^2 OCA - 2 \cot^2 OPA) = 2BP^2;$$
$$\therefore \; OA^2 = \frac{2BP^2}{\cot^2 OBA + \cot^2 OCA - 2 \cot^2 OPA},$$
$$OA = \frac{BP\sqrt{2}}{\sqrt{\{\cot^2 OBA + \cot^2 OCA - 2 \cot^2 OPA\}}}.$$

(2) Observing an object OA at a distance, I took its angle of elevation OBA at the place where I stood, and found it to be $50^\circ 23'$; I then measured a distance PB of 60 yards, in the most convenient direction the ground afforded, and at this station found its elevation OPA to be $40^\circ 33'$; after which I measured on, in the



same line, 50 yards further to C, and at this place found its elevation OCA to be $30^\circ 49'$; from which it is required to determine the height of the object, and its distance from each of the three stations B, P, C.

Investigation.

Since $AB = OA \cot OBA$, $AP = OA \cot OPA$, $AC = OA \cot OCA$,

$\cos BPA = \dfrac{BP^2 + AP^2 - AB^2}{2AP.BP}$, and $\cos CAP$, which is equal to

$$-\cos BPA = -\frac{CP^2 + AP^2 - AC^2}{2AP.CP};$$

$$\therefore \frac{BP^2 + AP^2 - AB^2}{BP} = -\frac{CP^2 + AP^2 - AC^2}{CP},$$

$$CP.AB^2 + BP.AC^2 - (CP+BP)AP^2 = (CP+BP).CP.BP;$$

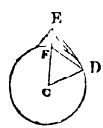
$\therefore$ by substitution we have

$$CP.OA^2 \cot^2 OBA + BP.OA^2 \cot^2 OCA - (CP+BP) \times$$

$OA^2 \cot^2 OPA = (CP + BP).CP.BP$, this reduced gives

$$OA = \sqrt{\frac{CP \cot^2 OBA + BP \cot^2 OCA - (CP+BP) \cot^2 OPA}{(CP+BP).CP.BP}}.$$

(3) If the height of the mountain called the Peak of Teneriffe be $2\frac{1}{3}$ miles, as it is very nearly, and the angle taken at the top of it, as formed between a plumb-line and a line conceived to touch the earth in the horizon, or farthest visible point, be $88^\circ 2'$; it is required from these measures to determine the magnitude of the whole earth, and the utmost distance that can be seen on its surface from the top of the mountain, supposing the form of the earth to be perfectly globular?



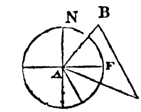
Let EF represent the height of the mountain, ED the tangent $\angle E = 88^\circ 2'$, DC a semidiameter of the earth.

Since EDC is a right angle, and $\angle C = 1^\circ 58'$, $\angle CDF = \angle CFD$, each of which $= 89^\circ 1'$, and $\angle EFD = 90^\circ 59'$, consequently

Then, as sin ∠ EDF 0° 59′ .... log ar. co. 9·765443
: sin ∠ EFD 90° 59′ .... log 9·999936
:: EF $2\frac{1}{3}$ miles .... log 0·367977
: ED 135·94 .... log 2·133356

And (Euc. III. 36) $\frac{(135{\cdot}94)^2}{2{\cdot}333\dot{3}} - 2{\cdot}333\dot{3} = 7916$ miles.

(4) A point of land was observed, by a ship at sea, to bear E. by S.; and after sailing N.E. 12 miles, it was found to bear S.E. by E. It is required to determine the place of that headland, and the ship's distance from it at the last observation.



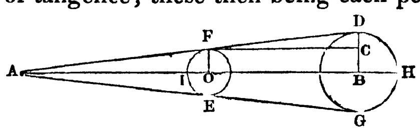
Let C be the point of land; A the former and B the latter place of observation. In the triangle ABC are given the angle A 56° 15′, the angle B 101° 15′, and the side AB 12 miles; consequently the angle C is 22° 30′.

Then, as sin 22°30′ ∠ C .... log. ar. co. 10·417160
: sin 56°15′ ∠ A .... log 9·919846
:: 12 miles AB .... log 1·079181

: 26·0728 miles BC, distance required, log 1·416187

(5) A string passes round the circumferences of two given circles, in the same plane, and at a given distance from each other: required the length of the string.

Let $GEIFDH$ represent the string touching the two circles, draw from the centres B and O the radii BD, OF to the points of tangence; these then being each perpendicular



to FD are parallel to one another; through F draw FC parallel to OB, then $FC = OB$ and $FO = BC$;

Let $OB = 14$ in., $OF = 1$ in., and $BD = 8\frac{1}{2}$ in.

Now $FD = \sqrt{FC^2 - DC^2} = \sqrt{OB^2 - (BD - OF)^2}$

$= \sqrt{(14^2 - 7{\cdot}5^2)} = 11{\cdot}82159$;

and $\sin FCD = \frac{FD}{FC} = \frac{FD}{OB} = \frac{11{\cdot}82159}{14} = {\cdot}844399$;

$$\therefore \angle FCD = \angle OBC = \angle IOF = \angle 57^\circ 36' 25''.$$

$$\text{Hence } \angle DBH = 180^\circ - \angle DBO = 180^\circ - 57^\circ 36' 25'' = 122^\circ 23' 35''.$$

Then, as $360^\circ : 57^\circ 36' 25'' :: 6{\cdot}2832$ (circumference of the less circle) : $1{\cdot}00543$ = the arc FI. Also $360^\circ : 122^\circ 23' 35'' :: 53{\cdot}4072$ (larger circum.) : $18{\cdot}15742$ = arc DH.

Hence $IF + FD + DH = 30{\cdot}98444$, twice which, or $61{\cdot}96888$ inches = 5ft. 2in. nearly, is the length of the band.

(6) The side of a hill forms an inclined plane whose angle of inclination is known; required the direction in which a rail or other road must run along the side of the hill, in order that it may have an ascent of 1 in every n feet.

(7) Wanting to know the height of an inaccessible tower; at the least distance from it, on the same horizontal plane, I took its angle of elevation equal to 58°; then going 300 feet directly from it, found the angle there to be only 32°; required its height, and my distance from it at the first station? Height 307·53, distance 192·15 ft.

(8) Being on a horizontal plane, and wanting to know the height of a tower placed on the top of an inaccessible hill: I took the angle of elevation of the top of the hill 40°, and of the top of the tower 51°; then measuring in a line directly from it to the distance of 200 feet farther, I found the angle at the top of the tower to be $33^\circ 45'$. What is the height of the tower? 93·33148 feet.

(9) From a window near the bottom of a house, which seemed to be on a level with the bottom of a steeple, I took the angle of elevation of the top of the steeple equal 40°; then from another window, 18 feet directly above the former, the like angle was $37^\circ 30'$: required the height and distance of the steeple? Height 210·44, distance 250·79 ft.

(10) Wanting to know the height of, and my distance from, an object on the other side of a river, which appeared to be on a level with the place where I stood, close by the side of the river; and not having room to measure backward, in the same line, because of the immediate rise of the bank, I placed a mark where I stood, and measured in a direction from the object, up the ascending ground, to the distance of 264 feet, where it was evident that I was above the level of

the top of the object; there the angles of depression were found to be, viz., of the mark left at the river's side 42°, of the bottom of the object 27°, and of its top 19°. Required the height of the object, and the distance of the mark from its bottom? Height 57·26 ft., distance 150·50 ft.

(11) Wanting to know my distance from an inaccessible object O, on the other side of a river, and having no instrument for taking angles, but only a chain or cord for measuring distances, from each of two stations, A and B, which were taken at 500 yards asunder, I measured in a direct line from the object O, 100 yards, viz., AC and BD each equal to 100 yards; also the diagonal AD measured 550 yards, and the diagonal BC, 560. What was the distance of the object from each station A and B? AO 536·81 yds., BO 500·47 yds.

(12) The elevation of a tower is observed. At a station (a) feet nearer, the elevation is the complement of the former; (b) feet nearer still, it is double the first elevation. Shew that the height is

$$\sqrt{\left\{(a+b)^2 - \frac{a^2}{4}\right\}}.$$

(13) If the angles A, B, C of a triangle ABC be as the numbers 2, 3, 4 respectively; then

$$2 \cos \frac{A}{2} = \frac{a+c}{b}.$$

(14) If in the three edges which meet at one angle of a cube, three points A, B, C be taken at distances a, b, c from the angle respectively; the area of the triangle ABC formed by joining the three points with each other

$$= \frac{1}{2} \sqrt{(a^2b^2 + a^2c^2 + b^2c^2)}.$$

(15) If the tangents of the angles of a plane triangle be in a geometric progression, whose ratio is n, shew that

$$\sin 2C = n \sin 2A.$$

(16) If the tangents of the semiangles of a plane triangle be in arithmetical progression, the cosines of the whole angles will also be in arithmetical progression. If R and r be the radii of the circles described about and inscribed in the triangle ABC, then the area of the triangle

$$\Delta = Rr (\sin A + \sin B + \sin C).$$

(17) In a right-angled triangle, CD, CE are drawn from the right angle C, making angles α, β with the hypothenuse; the area of the triangle

$$CDE = \frac{a^2b^2}{2c^2}(\cot \alpha \sim \cot \beta).$$

(18) If α, β, γ denote the lines bisecting the opposite sides a, b, c of any plane triangle, shew that

$$\alpha^4 + \beta^4 + \gamma^4 = \frac{9}{16}(a^4 + b^4 + c^4);$$

$$\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = \frac{9}{16}(a^2b^2 + b^2c^2 + c^2a^2).$$

(19) Let a, b, c be the sides of a plane triangle, and α, β, γ the lines bisecting the angles and terminating in the opposite sides; then will

$$\frac{\alpha\beta\gamma}{abc} = \frac{4(a+b+c)}{(a+b)(b+c)(c+a)} \times \text{area of the triangle.}$$

(20) If lines be drawn from the angles of a plane triangle to a point within it, so as to make equal angles with each other, their sum will be represented by

$$\sqrt{\{a^2 - 2ab\cos(C + 60^\circ) + b^2\}}.$$

(21) The squares of the sides of a plane triangle added to the squares of the radii of the four circles respectively touching these sides is equal to four times the square of the circumscribing diameter; required a demonstration.

(22) If a, b, c be the sides of a triangle, and α, β, γ the perpendiculars upon them from the opposite angles, shew that

$$\frac{\alpha^2}{\beta\gamma} + \frac{\beta^2}{\alpha\gamma} + \frac{\gamma^2}{\alpha\beta} = \frac{bc}{a^2} + \frac{ac}{b^2} + \frac{ab}{c^2}.$$

(23) If R be the radius of the circle circumscribed about a given triangle ABC, and r the radius of the inscribed circle, shew that

$$\frac{1}{4}\cdot\frac{r}{R} = \frac{1}{2}\left\{1 - \left(\sin^2\frac{A}{2} + \sin^2\frac{B}{2} + \sin^2\frac{C}{2}\right)\right\}.$$

(24) In any plane triangle, the square of the distance

between the centre of the inscribed circle and the intersection of the perpendiculars

$$= 4R^2 - 2Rr - \frac{a^3 + b^3 + c^3}{a + b + c}.$$

(25) ABC is a plane triangle, P a point within it from which the sides subtend equal angles, it is required to find any number of such triangles having the sides, and the distances AP, BP, CP all expressed in whole numbers. (See fig. page 71).

(26) If $T(1)$ denote the sum of the angles of any polygon, $T(3)$ the sum of all the products that can be made by taking them by threes, $T(5)$ the sum of all the products that can be made by taking them by fives, and so on, then supposing radius to be unity

$$T(1) - T(3) + T(5) - T(7) + \&c. = 0.$$

(27) The sum of the diameters of the inscribed and circumscribing circles of any plane triangle is equal to

$$a \cot A + b \cot B + c \cot C.$$

(28) If a, b, c denote the sides and S the semiperimeter of any plane triangle, the square of the line drawn from the vertex to the lower extremity of the vertical diameter of the circumscribing circle is

$$\frac{bc\,(b + c)^2}{4S\,(S - a)}.$$

(29) In a plane triangle let r be the radius of the inscribed circle, R, R' and R'' the radius of the three circles touching the sides of the triangle externally; then will

$$\frac{1}{r} = \frac{1}{R} + \frac{1}{R'} + \frac{1}{R''},$$

required a demonstration.

(30) Let a, b, c be the three sides of a triangle, and r, r', r'', r''' the radii of the four circles which touch them, then

$$ab + ac + bc = rr' + rr'' + rr''' + r'r'' + r'r''' + r''r''',$$

that is, the sum of all the rectangles made by the binary combinations of the sides is equal to the sum of all the rectangles made by the binary combinations of the four radii of the circles of contact.

(31) If from one of the angles of a regular polygon of n sides, lines be drawn to all the angles, shew that their sum $= a \operatorname{cosec}^2 \frac{90^0}{n}$, $2a$ being the length of a side of the polygon.

(32) Having given three lines drawn from any point within a square to three of its angular points, determine a side of the square.

(33) The shadows of two vertical walls, which are at right angles to each other, and are a and a_1 feet in height, are observed when the sun is due south, to be b and b_1 feet in breadth; shew that if α be the sun's altitude, and β the inclination of the first wall to the meridian,

$$\cot \alpha = \sqrt{\left(\frac{b^2}{a^2} + \frac{b_1^{\,2}}{a_1^{\,2}}\right)}, \quad \cot \beta = \frac{ab_1}{a_1 b}.$$

(34) The distance between the centres of two wheels $= a$, and the sum of their radii $= c$; shew that the length of a string which crosses between the two wheels and just wraps round them

$$= 2\left\{\sqrt{(a^2 - c^2)} + c \cos^{-1}\left(-\frac{c}{a}\right)\right\}.$$

(35) If an equilateral triangle have its angular points in three parallel straight lines of which the middle one is distant from the outside ones by a and b, its side will be expressed by the following formula, viz.

$$\text{side} = 2\sqrt{\frac{a^2 + ab + b^2}{3}},$$

$$\text{or } 2\sqrt{\frac{(a+b)^2 - ab}{3}}:$$

required the proof.

(36) The hypothenuse (c) of a right-angled triangle ABC is trisected in the points D, E: prove that if CD, CE be joined, the sum of the squares of the sides of the triangle $CDE = \frac{2c^2}{3}$.

(37) Four places situated at unequal but given distances in the same straight line, appear to a spectator in the same plane with them to be at equal distances from each other; find the position of the spectator.

(38) The area of a quadrilateral inscribed in a circle

$$=\sqrt{\{(S-a)(S-b)(S-c)(S-d)\}}.$$

(39) If D and D' be the diagonals of a quadrilateral figure and ϕ their angle of intersection, then the

$$\text{area} = \frac{1}{2} D . D' \sin \phi.$$

CHAPTER V.

DEMOIVRE'S THEOREM.

33. This theorem, to which we give the name of the French geometer that discovered it, is as follows:

$$(\cos \phi + \sqrt{-1} \sin \phi)^n = \cos n\phi + \sqrt{-1} \sin n\phi \;\ldots\ldots\; (A).$$

It expresses that, to raise the binomial

$$\cos \phi + \sqrt{-1} \sin \phi,$$

to any power whatever, it is sufficient to multiply the arc ϕ by the index of the power, we may either put the sign

$$+ \text{ or } - \text{ before } \sqrt{-1},$$

for that is the same as changing ϕ into $-\phi$.

The case where the index is a positive whole number is the first we shall proceed to consider.

By multiplication we find

$$(\cos \phi + \sqrt{-1} \sin \phi)(\cos \psi + \sqrt{-1} \sin \psi)$$

$$= \cos \phi \cos \psi - \sin \phi \sin \psi + \sqrt{-1} (\sin \phi \cos \psi + \cos \phi \sin \psi).$$

Now from the formulæ Art. 13, the real part of this product is equal to $\cos(\phi + \psi)$, and the imaginary part is equal to $\sqrt{-1} \sin(\phi + \psi)$; then

$$(\cos \phi + \sqrt{-1} \sin \phi)(\cos \psi + \sqrt{-1} \sin \psi)$$

$$= \cos(\phi + \psi) + \sqrt{-1} \sin(\phi + \psi).$$

That is to say, by multiplying together the two expressions of the form $\cos\phi + \sqrt{-1}\,\sin\phi$, we obtain a similar expression, which contains the sum of the two arcs.

In order to multiply the product by a new factor of the same form, it is sufficient to add the new arc to the other two, and thus we may continue the operation to any number of factors. Then, if we suppose that there are n factors, each equal to $\cos\phi + \sqrt{-1}\,\sin\phi$, it will become

$$(\cos\phi + \sqrt{-1}\,\sin\phi)^n = \cos n\phi + \sqrt{-1}\,\sin n\phi \;\ldots\ldots\; (1).$$

Let us consider the case where the exponent is fractional.

In replacing ϕ by $\frac{\phi}{n}$, the formula (1) becomes

$$\left(\cos\frac{\phi}{n} + \sqrt{-1}\,\sin\frac{\phi}{n}\right)^n = \cos\phi + \sqrt{-1}\,\sin\phi,$$

and, by extracting the nth root, and putting a fractional index in the place of the radical sign, the formula (A) will be demonstrated for the index $\frac{1}{n}$; for we have

$$(\cos\phi + \sqrt{-1}\,\sin\phi)^{\frac{1}{n}} = \cos\frac{\phi}{n} + \sqrt{-1}\,\sin\frac{\phi}{n} \;\ldots\ldots\ldots\; (2).$$

Generally, the expression $A^{\frac{m}{n}}$ signifies that we ought to raise A to the mth power, and afterwards take the nth root of the result. Consequently, if we raise $\cos\phi + \sqrt{-1}\,\sin\phi$ to the power m by the formula (1), and if afterwards we extract the nth root by the formula (2), it will become

$$(\cos\phi + \sqrt{-1}\,\sin\phi)^{\frac{m}{n}} = \cos\frac{m\phi}{n} + \sqrt{-1}\,\sin\frac{m\phi}{n} \;\ldots\ldots\; (3).$$

This is the formula (A) in which n is changed into any positive fraction $\frac{m}{n}$.

Finally, when the index is negative, we observe that

$$(\cos n\phi + \sqrt{-1}\,\sin n\phi)(\cos n\phi - \sqrt{-1}\,\sin n\phi)$$
$$= \cos^2 n\phi + \sin^2 n\phi = 1;$$

and from this we find,

$$\frac{1}{\cos n\phi + \sqrt{-1}\sin n\phi} = \cos n\phi - \sqrt{-1}\sin n\phi\,;$$

or which is the same thing, (by formula 1),

$$(\cos\phi + \sqrt{-1}\sin\phi)^{-n} = \cos(-n\phi) + \sqrt{-1}\sin(-n\phi) \;\ldots\ldots\ldots\; (4),$$

since the cosine of a positive arc is the same as the cosine of an equal arc taken negatively.

Thus, the formula (A) is true, when we take for n any positive or negative number whatever.

We have not used irrational indices seeing that they are not of any utility, as we cannot replace them by commensurable numbers, which however may differ therefrom as little as we please. And as to imaginary indices, they are not susceptible of any interpretation.

The formula (A), which is so simple and elegant, appears to have a serious defect when the index is fractional. In fact, the first member, being then equivalent to a radical expression, ought to have many values, whilst the second member presents only one. The explanations which follow have for their object to correct this imperfection.

Let us return to formula (2) in which n is a positive whole number. From the principles of Algebra, the first member, which is equivalent to $\sqrt[n]{(\cos\phi + \sqrt{-1}\sin\phi)}$, ought then to have n different values; and in order that the second may give all of them, we shall shew that it will suffice to replace ϕ by all the arcs which have the same sines and cosines as ϕ itself. The general expression of these arcs is $\phi + kC$, C designating the whole circumference, and k any whole positive or negative integer number. By putting $\phi + kC$ in the place of ϕ, the second member of formula (2) becomes,

$$\cos\frac{\phi + kC}{n} + \sqrt{-1}\sin\frac{\phi + kC}{n} \;\ldots\ldots\ldots\ldots\ldots\; (5).$$

And in this state, we say that it has precisely the same values as the first member.

And first, since n is an integer, it is clear, from formula (1), that by raising this second member to the power n, we arrive again at the form $\cos\phi + \sqrt{-1}\sin\phi$.

In the second place, if we make successively $k=0$, $k=1$, $k=2$ ...... $k=n-1$, we obtain n different values.

In fact, let there be any two values,

$$\cos\frac{\phi+\alpha C}{n}+\sqrt{-1}\sin\frac{\phi+\alpha C}{n},$$

$$\text{and }\cos\frac{\phi+\beta C}{n}+\sqrt{-1}\sin\frac{\phi+\beta C}{n};$$

in which α and β are integer numbers $<n$. In order that they may be equal, it is necessary to have separately, an equality between the real and the imaginary parts; then the difference of the two arcs $\frac{\phi+\alpha C}{n}$ and $\frac{\phi+\beta C}{n}$ ought to be equal to one or more circumferences: but that difference, which is $\frac{(\alpha-\beta)C}{n}$, is *less* than C, seeing that α and β are $<n$: consequently the two values are unequal.

In the third place, if we take for k other numbers than 0, 1, 2 ...... $n-1$, we shall find new values. In fact, all the other positive or negative integer numbers may be represented by the formula $n\gamma+n'$, γ being any integer number, positive or negative, and n' a positive integer $<n$; moreover by making $k=n\gamma+n'$ the expression (5) becomes

$$\cos\left(\gamma C+\frac{\phi+n'C}{n}\right)+\sqrt{-1}\sin\left(\gamma C+\frac{\phi+n'C}{n}\right),$$

or rather, suppressing the useless circumferences,

$$\cos\frac{\phi+n'C}{n}+\sqrt{-1}\sin\left(\frac{\phi+n'C}{n}\right);$$

and as n' is a positive number $<n$, that value is comprehended under those we have obtained by putting

$$k=0,\ 1,\ 2\ \ldots\ldots\ n-1.$$

Thus the second member of the formula (2) will acquire the generality, which it ought to have, if we take care to have for the arc ϕ not only the arc ϕ itself, but also the arcs

$$\phi+C,\ \phi+2C,\ \ldots\ldots\ \phi+(n-1)C.$$

Formula (3), ought also to be interpreted in an analogous manner, as we find by raising $\cos\phi+\sqrt{-1}\sin\phi$ to the

mth power, and extracting the nth root of the result. The formula (1), relating to the case of a whole positive index, gives

$$(\cos\phi + \sqrt{-1}\sin\phi)^m = \cos m\phi + \sqrt{-1}\sin m\phi,$$

and afterwards, for the nth root, the formula (2) gives

$$(\cos\phi + \sqrt{-1}\sin\phi)^{\frac{m}{n}} = \cos\frac{m\phi}{n} + \sqrt{-1}\sin\frac{m\phi}{n}.$$

But in order that the second member may have the same extension as the first, we must, after what we have already explained, put $m\phi + kC$ in the place of $m\phi$; or, which is the same thing, replace $m\phi$ successively by

$$m\phi,\ m\phi + C,\ \ldots\ldots\ m\phi + (n-1)\,C.$$

34. Formulæ for $\sin n\phi$, $\cos n\phi$, and $\tan n\phi$. Referring back to Demoivre's formulæ,

$$\cos n\phi + \sqrt{-1}\sin n\phi = (\cos\phi + \sqrt{-1}\sin\phi)^n \ \ldots\ldots\ldots\ (1),$$

in which we suppose n to be a positive integer. Changing ϕ into $-\phi$ it becomes

$$\cos n\phi - \sqrt{-1}\sin n\phi = (\cos\phi - \sqrt{-1}\sin\phi)^n \ldots\ldots\ (2).$$

Adding and subtracting (1) and (2),

$$\cos n\phi = \frac{(\cos\phi + \sqrt{-1}\sin\phi)^n + (\cos\phi - \sqrt{-1}\sin\phi)^n}{2} \ldots (3),$$

$$\sin n\phi = \frac{(\cos\phi + \sqrt{-1}\sin\phi)^n - (\cos\phi - \sqrt{-1}\sin\phi)^n}{2} \ldots (4).$$

Expanding by the binomial theorem, and suppressing the terms which destroy each other, we obtain

$$\cos n\phi = (\cos\phi)^n - \frac{n(n-1)}{1.2}(\cos\phi)^{n-2}(\sin\phi)^2$$
$$+ \frac{n(n-1)(n-2)(n-3)}{1.2.3.4}(\cos\phi)^{n-4}(\sin\phi)^4 - \&c.\ldots(5),$$

$$\sin n\phi = n(\cos\phi)^{n-1}\sin\phi - \frac{n(n-1)(n-2)}{1.2.3}(\cos\phi)^{n-3}(\sin\phi)^3$$
$$+ \frac{n(n-1)(n-2)(n-3)(n-4)}{1.2.3.4.5}(\cos\phi)^{n-5}(\sin\phi)^5 - \&c.\ldots(6).$$

These formulæ express the sine and cosine of the multiple $n\phi$ in terms of the sine and cosine of the simple arc ϕ: the law of the terms is evident, and in the same manner as the binomial formulæ from which they are derived, the terms must be continued until we find a term equal to nothing.

To find $\tan n\phi$, we must divide (6) by (5),

$$\frac{\sin n\phi}{\cos n\phi} = \tan n\phi = \frac{n \tan \phi - \frac{n(n-1)(n-2)}{1.2.3} \tan^3 \phi - \&c.}{1 - \frac{n(n-1)}{1.2} \tan^2 \phi + \&c.}.$$

35. Development of the sine and cosine in a series.

From equations (5) and (6), Euler has deduced the series which expresses the sine and cosine of an arc in terms of the arc itself.

Retaining the supposition that n is a whole number, we can dispose of ϕ in such a manner that $n\phi$ may be equal to any arc x; putting then $n\phi = x$, we have $n = \frac{x}{\phi}$, and consequently the equations (5) and (6) will become

$$\cos x = (\cos \phi)^n - \frac{x(x-\phi)}{1.2} (\cos \phi)^{n-2} \left(\frac{\sin \phi}{\phi}\right)^2 + \frac{x(x-\phi)(x-2\phi)(x-3\phi)}{1.2.3.4} (\cos \phi)^{n-4} \left(\frac{\sin \phi}{\phi}\right)^4 - \&c. \ldots (1),$$

$$\sin x = x(\cos \phi)^{n-1} \left(\frac{\sin \phi}{\phi}\right) - \frac{x(x-\phi)(x-2\phi)}{1.2.3} (\cos \phi)^{n-3} \left(\frac{\sin \phi}{\phi}\right)^3 + \frac{x(x-\phi)(x-2\phi)(x-3\phi)(x-4\phi)}{1.2.3.4.5} (\cos \phi)^{n-5} \left(\frac{\sin \phi}{\phi}\right)^5 - \&c. \ldots (2).$$

Let us now conceive ϕ to diminish to zero and n to increase to infinity, then the above formulæ will preserve no trace either of ϕ or of n, and will contain the arc x only.

When ϕ becomes zero, we have

$$\cos \phi = 1 \text{ and also } \frac{\sin \phi}{\phi} = 1.$$

Admitting that the powers of $\cos \phi$ and of $\frac{\sin \phi}{\phi}$ be also equal to unity, however great the indices may be, the above formulæ become

$$\cos x = 1 - \frac{x^2}{1.2} + \frac{x^4}{1.2.3.4} - \frac{x^6}{1.2.3.4.5.6} + \&c.\ldots(3),$$

$$\sin x = x - \frac{x^3}{1.2.3} + \frac{x^5}{1.2.3.4.5} - \frac{x^7}{1.2.3.4.5.6.7} + \&c.\ldots(4).$$

The following very sensible observations on these series are made by a correspondent of the *Mechanics' Magazine.* (See No. 1256, pages 231, 232).

"Now to many students, who are not altogether without thinking what they are about, and yet have not attained to clear notions on the distinction between *things themselves* and the *numerical measures* of those things—the above series are a sad stumbling-block, and well they may be. The reader who has these misty notions, naturally asks—'How can a sine or cosine, which I know to be a number or ratio, be expressed in terms of *an angle*, which I conceive to be an *opening?*' How indeed! About the *ne plus ultra* of all conceivable absurdities would be the following equation; sine of a certain opening

$$= \text{that opening} - \frac{1}{6}\,(\text{cube of that opening}) + \&c.$$

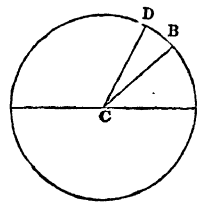
"The absurdity arises from considering the symbol (ϕ) as standing for the Euclidian angle or opening, instead of for a certain numerical fraction or ratio which *measures* that angle, and which bears the same proportion to *other fractions* (the measures of other angles), which the Euclidian angles themselves do. Cos ϕ is really an abridgment for 'cosine of that Euclidian angle the length of whose arc bears the same proportion to the length of the radius, as the number ϕ does to unity.'

"Again, the *use* of these series is seldom seen, and because the series on the left hand side of the equation is infinite in the number of its terms, the student is apt sometimes to conceive that an immense number of those terms must be taken to get anything like a tolerably near value for cos ϕ or sin ϕ. To obviate this fancy, and at the same time to shew clearly the real nature of the equation, let us take a numerical example and work it out. Suppose, then, we endeavour to find the numerical value of the cosine of 30 degrees by means of the series. Here (ϕ) means

$$\frac{\text{length of arc subtending the angle of } 30^0}{\text{length of radius}},$$

and this is the first thing to find.

"Now, we know that the length of the circumference of a circle bears the proportion to its diameter of (π) to unity: π being nearly equal to 3·1416. We have then



$$\frac{\text{arc } AB}{\frac{1}{2}\pi} = \frac{30^0}{90^0} = \frac{1}{3};$$

$\therefore$ arc $AB = \frac{1}{3} \times$ a quarter of the circumference, $= \frac{\text{semi-circumference}}{6}$;

$$\therefore \frac{\text{arc } AB}{\text{radius}} = \frac{1}{6} \cdot \frac{\text{semi-circumference}}{\text{radius}} = \frac{1}{6}\pi . = \frac{3{\cdot}1416}{6},$$

so that here (ϕ) is equal to ·5236.

"Take only the first three terms of the series for cos ϕ, so that

$$\cos 30^0 = 1 - \frac{1}{2}(\cdot 5236)^2 + \frac{1}{24}(\cdot 5236)^4.$$

Keep only four places of decimals in multiplying, and the second side becomes

$$= 1 - 0{\cdot}137078 + 0{\cdot}003130 = \cdot 866052.$$

"Now turn to a table of natural sines, &c., and you will find the cosine of 30^0 to be ·8660254, which agrees with that just found from the series up to the fifth place of decimals, and differs less from it than the cosines of $30^0 1'$, or $29^0 59'$, differ from the cosine of 30^0."

36. In the applications of the higher branches of mathematics we often find it necessary to express $(\sin\phi)^n$ and $(\cos\phi)^n$ in terms of the sine and cosine of the simple arc. We proceed in the following manner:

$$\text{Let } \cos\phi + \sqrt{-1}\sin\phi = x,$$

$$\text{then } \cos\phi - \sqrt{-1}\sin\phi = \frac{1}{x}.$$

$$\text{By addition, } 2\cos\phi = x + \frac{1}{x},$$

$$\text{and by subtraction, } 2\sqrt{-1}\sin\phi = x - \frac{1}{x};$$

$$\therefore\ 2^n(\cos\phi)^n = \left(x + \frac{1}{x}\right)^n$$

$$= x^n + nx^{n-2} + n\left(\frac{n-1}{2}\right)x^{n-4} + \ldots \frac{n}{x^{n-2}} + \frac{1}{x^n}$$

= (collecting into pairs the terms equidistant from each extremity of the series)

$$\left(x^n + \frac{1}{x^n}\right) + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + n\left(\frac{n-1}{2}\right)\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \&c.$$

$$\text{but } \{\cos\phi + \sqrt{-1}\sin\phi\}^n = \cos n\phi + \sqrt{-1}\sin n\phi = x^n,$$

$$\{\cos\phi - \sqrt{-1}\sin\phi\}^n = \cos n\phi - \sqrt{-1}\sin n\phi = \frac{1}{x^n},$$

by adding and subtracting, we have

$$2\cos n\phi = x^n + \frac{1}{x^n}, \text{ and } 2\sqrt{-1}\sin n\phi = x^n - \frac{1}{x^n};$$

$$\therefore\ \left(x^n + \frac{1}{x^n}\right) + n\left(x^{n-2} + \frac{1}{x^{n-2}}\right) + n\frac{n-1}{2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) + \&c.$$

$$= 2\cos n\phi + 2n.\cos(n-2)\phi + 2n\frac{n-1}{2}\cos(n-4)\phi, \&c.;$$

$$\therefore\ 2^{n-1}(\cos\phi)^n = \cos n\phi + n\cos(n-2)\phi$$

$$+ n\frac{n-1}{2}\cos(n-4)\phi, \&c.;$$

the last term of the series being

$$\frac{1}{2}n\left(\frac{n-1}{2}\right)\ldots\frac{n-\frac{1}{2}n+1}{\frac{1}{2}n}, \text{ when } n \text{ is even,}$$

$$\text{and } n\left(\frac{n-1}{2}\right)\ldots\frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)}\cos\phi, \text{ when } n \text{ is odd.}$$

From what has been given above, we have for the sine

$$2\sqrt{-1}\sin\phi = x - \frac{1}{x}, \text{ and } 2\sqrt{-1}\sin n\phi = x^n - \frac{1}{x^n},$$

$$(2\sqrt{-1}\ \sin\phi)^n = \left(x - \frac{1}{x}\right)^n = \left\{x + \left(-\frac{1}{x}\right)\right\}^n$$

$$= x^n - nx^{n-2} + \frac{n(n-1)}{2}x^{n-4}\ldots\frac{(-1)^n}{x^n}\ \ldots\ldots\ldots\ldots\ (a)$$

$$= \left\{x^n + \left(\frac{-1}{x}\right)^n\right\} - n\left\{x^{n-2} + \left(\frac{-1}{x}\right)^{n-2}\right\}$$

$$+ \frac{n(n-1)}{2}\left\{x^{n-4} + \left(\frac{-1}{x}\right)^{n-4}\right\}\ \ldots\ldots\ldots\ldots\ (b),$$

when n is even the series (b) becomes

$$(-1)^{\frac{n}{2}}2^n\sin^n\phi = \left(x^n + \frac{1}{x^n}\right) - n\left(x^{n-2} + \frac{1}{x^{n-2}}\right)$$

$$+ n\frac{n-1}{2}\left(x^{n-4} + \frac{1}{x^{n-4}}\right) - \ldots$$

$$+ (-1)^{\frac{n}{2}}n\frac{n-1}{2}\ldots\frac{n-\frac{1}{2}n+1}{\frac{1}{2}n}$$

$$= 2\cos n\phi - n\ .\ 2\cos(n-2)\phi + n\frac{n-1}{2}2\cos(n-4)\phi -$$

$$\ldots\ldots + (-1)^{\frac{n}{2}}n\frac{n-1}{2}\ldots\frac{n-\frac{1}{2}n+1}{\frac{1}{2}n};$$

$$\therefore\ (-1)^{\frac{n}{2}}2^{n-1}\sin^n\phi = \cos n\phi - n\cos(n-2)\phi$$

$$+ n\frac{n-1}{2}\cos(n-4)\phi - \ldots$$

$$\ldots\ldots + (-1)^{\frac{n}{2}}\tfrac{1}{2}n\frac{n-1}{2}\ldots\frac{n-\frac{1}{2}n+1}{\frac{1}{2}n}.$$

If n be odd, then $(\sqrt{-1})^n = \sqrt{-1}(\sqrt{-1})^{n-1} = (-1)^{\frac{n-1}{2}}\sqrt{-1}$; and the last term of (b) is

$$(-1)^{\frac{1}{2}(n-1)}n\frac{n-1}{2}\ldots\frac{n-\frac{1}{2}(n-1)+1}{\frac{1}{2}(n-1)}\left(x - \frac{1}{x}\right).$$

Hence, when n is an odd number (b) becomes

$$(-1)^{\frac{n-1}{2}} 2^n . \sqrt{-1} \sin^n \phi = \left(x^n - \frac{1}{x^n}\right) - n\left(x^{n-2} - \frac{1}{x^{n-2}}\right)$$

$$+ n\frac{n-1}{2}\left(x^{n-4} - \frac{1}{x^{n-4}}\right) - \dots$$

$$\dots\dots (-1)^{\frac{1}{2}(n-1)} n\frac{n-1}{2} \dots \frac{n - \frac{1}{2}(n-1) + 1}{\frac{1}{2}(n-1)}\left(x - \frac{1}{x}\right),$$

$$= 2\sqrt{-1} \sin n\phi - n \,.\, 2\sqrt{-1} \sin (n-2)\phi + \dots$$

$$\dots\dots + (-1)^{\frac{n-1}{2}} n\frac{n-1}{2} \dots \frac{n - \frac{1}{2}(n-1) + 1}{\frac{1}{2}(n-1)} 2\sqrt{-1} \sin \phi;$$

$$\therefore (-1)^{\frac{n-1}{2}} 2^{n-1} \sin^n \phi = \sin n\phi - n \sin (n-2)\phi$$

$$+ n\frac{n-1}{2} \sin (n-4)\phi - \dots$$

$$\dots\dots + (-1)^{\frac{n-1}{2}} n\frac{n-1}{2} \dots \frac{n - \frac{1}{2}(n-1) + 1}{\frac{1}{2}(n-1)} \sin \phi.$$

(1) Sum the series

$$\cos \phi + \cos 2\phi + \cos 3\phi + \&c. + \cos n\phi.$$

The sum is equal to the two following series,

$$\frac{1}{2}(x + x^2 + x^3 + \&c. + x^n),$$

$$\frac{1}{2}\left(\frac{1}{x} + \frac{1}{x^2} + \frac{1}{x^3} + \&c. + \frac{1}{x^n}\right);$$

$$\therefore = \frac{1}{2}\left(\frac{x^{n+1} - x}{x-1} + \frac{(x^n - 1)}{x^n(x-1)}\right)$$

$$= \frac{1}{2}\frac{(x^n - 1)(x^{n+1} + 1)}{x^n(x-1)} = \frac{1}{2}\frac{\left(x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}\right)\left(x^{\frac{n+1}{2}} + \frac{1}{x^{\frac{n+1}{2}}}\right)}{x^{\frac{1}{2}} - \frac{1}{x^{\frac{1}{2}}}}.$$

$$\text{Now, } \frac{x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}}{\sqrt{x} - \frac{1}{\sqrt{x}}} = \sqrt{\left\{\frac{x^{\frac{n}{2}} - \frac{1}{x^{\frac{n}{2}}}}{\sqrt{x} - \frac{1}{\sqrt{x}}}\right\}^2}$$

$$= \sqrt{\frac{\left(x^{\frac{n}{2}} + \frac{1}{x^{\frac{n}{2}}}\right)^2 - 4}{\left(\sqrt{x} + \frac{1}{\sqrt{x}}\right)^2 - 4}} = \sqrt{\frac{\left(2 \cos \frac{n\phi}{2}\right)^2 - 4}{\left(2 \cos \frac{\phi}{2}\right)^2 - 4}}$$

$$= \sqrt{\frac{1 - \cos^2 \frac{n\phi}{2}}{1 - \cos^2 \frac{n\phi}{2}}} = \frac{\sin \frac{n\phi}{2}}{\sin \frac{n\phi}{2}} :$$

consequently the sum of the series is equal to

$$\frac{\sin \frac{n\phi}{2}}{\sin \frac{n\phi}{2}} \times \cos \left(\frac{n+1}{2}\right)\phi.$$

(2) Sum the series $\operatorname{cosec} \phi + \operatorname{cosec} 2\phi \operatorname{cosec} 2^2\phi$ to n terms.

$$\operatorname{cosec} \phi = \cot \frac{\phi}{2} - \cot \phi,$$

$$\operatorname{cosec} 2\phi = \cot \phi - \cot 2\phi,$$

&c.

$$\operatorname{cosec} 2^{n-1} \phi = \cot 2^{n-2} \phi - \cot 2^{n-1} \phi ;$$

hence, by addition,

$$\operatorname{cosec} \phi + \operatorname{cosec} 2\phi + \text{\&c. } \operatorname{cosec} 2^{n-1} \phi = \cot \frac{\phi}{2} \cot 2^{n-1} \phi.$$

(3) Sum the series

$$\frac{1}{2} \tan \frac{\phi}{2} + \frac{1}{2^2} \tan \frac{\phi}{2^2} + \frac{1}{2^3} \tan \frac{\phi}{2^3} + \text{\&c. to } n \text{ terms.}$$

$$\text{Since } \tan \frac{\phi}{2} = \cot \frac{\phi}{2} - 2 \cot \phi,$$

we have $\frac{1}{2}\tan\frac{\phi}{2} = \frac{1}{2}\cot\frac{\phi}{2} - \cot\phi$,

$$\frac{1}{2^2}\tan\frac{\phi}{2^2} = \frac{1}{2^2}\cot\frac{\phi}{2^2} - \frac{1}{2}\cot\frac{\phi}{2},$$

&c....................

$$\frac{1}{2^n}\tan\frac{\phi}{2^n} = \frac{1}{2^n}\cot\frac{\phi}{2^n} - \frac{1}{2^{n-1}}\cot\frac{\phi}{2^{n-1}}.$$

By addition we have the sum

$$= \frac{1}{2^n}\cot\frac{\phi}{2^n} - \cot\phi.$$

(4) Sum the series $\sin\phi + \sin 2\phi + \sin 3\phi + \&c.$

MISCELLANEOUS EXAMPLES.

(1) The square of the distance between the centres of two of the escribed circles of a triangle exceeds the square of the sum of their radii, by square of the opposite side of the triangle.

(2) Let a, b, c be the three distances of a point from three successive angles of a square field, then if S be the side of the square, and Δ the area of the triangle, whose sides are a, $b\sqrt{2}$, c, it is required to prove that

$$S^2 = \frac{1}{2}(a^2 + c^2) \pm 2\Delta.$$

(3) The continued product of the radii of any three circles mutually touching each other, divided by the radius of the circle passing through their points of contact, gives the area of the triangle formed by joining their centres.

(4) Shew that if there be inscribed in a circle a regular figure, each of whose sides is an m^{th} part of the radius, the secant of the angle at the centre subtending each side

$$= \frac{2m^2}{2m^2 - 1}.$$

(5) Two regular polygons of the same number of sides being described, the one within, and the other without, the

same circle, what will be the number of sides, when the whole space intercepted *between* the two polygonal boundaries is an assigned part of either polygon? Ex. What is the figure when the exterior area is $\frac{1}{3}$ of the interior polygon?

(6) In a right-angled triangle, if the hypothenuse (c) be divided into segments (x), (y) by the line which bisects the right angle, and $t =$ the tangent of half the difference of the acute angles,

$$x : y :: 1 + t : 1 - t.$$

(7) To divide a given angle into two others, whose sines shall be in the ratio of $m : n$.

(8) A circle is inscribed in an equilateral triangle, an equilateral triangle in the circle, a circle again in the latter triangle, and so on; if r, r_1, r_2, r_3, &c., be the radii of the circles, shew that

$$r = r_1 + r_2 + r_3 + \text{\&c.}$$

(9) If a, b, c be the sides of a triangle, and p, q, r perpendiculars from a point within the triangle bisecting the sides, prove that

$$4\left(\frac{a}{p} + \frac{b}{q} + \frac{c}{r}\right) = \frac{abc}{pqr}.$$

(10) It is required to determine the continued product of n terms of the series

$$(\sin\phi \cos\frac{1}{2}\phi)^{\frac{1}{2}} (\sin\frac{1}{2}\phi \cos\frac{1}{4}\phi)^{\frac{1}{4}} (\sin\frac{1}{4}\phi \cos\frac{1}{8}\phi)^{\frac{1}{8}}$$

$$\ldots\ldots (\sin\frac{1}{2^{n-1}}\phi \cos\frac{1}{2^n}\phi)^{\frac{1}{2^n}}.$$

(11) The circumference of the inner of two concentric circles, of which R and r are the radii, is divided into n equal parts; shew that the sum of the squares of the lines drawn from the points of division to any point in the circumference of the outer circle $= n(R^2 + r^2)$.

(12) The ratio between the area of an equilateral and equiangular decagon described about a circle, and that of another within the same circle, is equal to

$$\frac{8}{7+\cfrac{1}{4+\cfrac{1}{4+\ldots\ldots}}}$$

(13) In a plane triangle ABC, having given the sum of the sides AC, CB, the perpendicular from the vertex C upon the base AB; and the difference of the segments of the base made by the perpendicular; find the sides of the triangle.

(14) The length of a road in which the ascent is 1 foot in 5, from the foot of a hill to the top is a mile and two-thirds. What will be the length of a zigzag road, in which the ascent is 1 foot in 12?

(15) At each end of a horizontal base measured in a known direction from the place of an observer, the angle which the distance of the other end and a certain object subtends is observed, as also the angle of elevation of the object at one end of the base. Find its height and bearing.

(16) A staff one foot long stands on the top of a tower 200 feet high. Shew that the angle which it subtends at a point in the horizontal plane 100 feet from the base of the tower is nearly $6'\cdot51''$.

(17) Find the degrees, minutes, and seconds, in the angle whose circular measure is $\cdot1$; also the values of $\cos m\pi$ and $\tan^{-1}(-1)^m$ (m an integer).

(18) Two circles have a common radius (r) and a circle is described touching this radius and the two circles: prove that the radius of the circle which touches the three $= \dfrac{3r}{56}$.

(19) At noon a column in the E.S.E. cast upon the ground a shadow, the extremity of which was in the direction N.E.: the angle of elevation of the column being a^0, and the distance of the extremity of the shadow from the column (a) feet, determine the height of the column.

(20) $\mathrm{Tan}^{-1}\left(\dfrac{\sqrt{2}+1}{\sqrt{2}-1}\right) - \tan^{-1}\left(\dfrac{1}{\sqrt{2}}\right) = \dfrac{\pi}{4}$.

(21) If A, B, C be the angles of a triangle,

$$\cos nA + \cos nB + \cos nC = 1 \pm 4 \sin \frac{nA}{2} \sin \frac{nB}{2} \sin \frac{nC}{2},$$

$$\text{or } = -1 \pm 4 \cos \frac{nA}{2} \cos \frac{nB}{2} \cos \frac{nC}{2},$$

according as n is odd or even.

(22) In a regular polygon of n sides inscribed in a circle whose radius is (r), if a be the distance of any point from the centre, and perpendiculars be drawn from this point upon all the sides of the polygon, the sum of the squares of the lines joining the feet of the adjacent perpendiculars is

$$n \sin^2 \frac{2\pi}{n} (r^2 + a^2).$$

(23) Four objects situated at unequal but given distances in the same straight line, appear to a spectator in the same plane with them to be at equal distances from each other, it is required to determine his position.

(24) If a', b' be the segments of the hypothenuse made by a line bisecting the right angle, then

$$\frac{a'b'}{ab} = \frac{a^2 + b^2}{(a + b)^2}.$$

(25) A boy flying a kite at noon, when the wind was blowing α^0 from the south, and the angular distance of the kite's shadow from the north was β^0, the wind suddenly changed to $\alpha_1{}^0$ from the south, and the shadow to $\beta_1{}^0$ from the north, and the kite was raised as much above 45^0 as it had before been below that elevation. Shew that θ^0 being the angular elevation of the sun, and $45^0 - \phi^0$ that of the kite at first,

$$\tan^2 \theta^0 = \frac{\sin \beta \sin \beta}{\sin (\alpha - \beta) \sin (\alpha_1 - \beta_1)},$$

$$\tan^2 (45^0 - \phi^0) = \frac{\sin (\alpha - \beta) \sin \beta_1}{\sin (\alpha_1 - \beta_1) \sin \beta}.$$

FINIS.

ERRATA.

Page 42, line 8, for $\frac{3}{2}$ read $\frac{3}{6}$.

... 64 ... 4, the thick and thin line to be transposed.

... 96 Supply *B* in fig. 2, as in fig. 1.