# ELEMENTS

OF

# Analytic Geometry

BY


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# PREFACE

This book has been written not to expound anything new but to provide a convenient and useful class manual. It seeks to give first discipline and second useful knowledge.

This discipline should consist first in training to think, and second in training to express thought.

This training to think will consist mainly in training to form clear, precise, and correct mathematical conceptions and to form trains of deductive reasoning constructed with perfect logical soundness and cogency. Hence the student should be frequently required to give clear, precise, and correct definitions of the terms he is using, to show every point, line and angle discussed in a neat clear and correct figure and to translate his final equation into ordinary language until at last each letter and symbol in it shall at once, and clearly, express to him the magnitude or the operation for which it stands. He should also be required to state clearly and correctly each step in the argument, to state it in its logical order and to fortify each step by quoting the proof for it; so that when he reaches his final equation he shall feel that it must necessarily be true. Hence an attempt has been made in this book to cast the demonstrations as far as possible in the form of those given in elementary geometry.

In explaining his work at the blackboard, the student should be required first to state his theorem in the best language he can command, then to illustrate its meaning by applying it to the figure, then to give his demonstration with all logical rigor and clearness of language, and finally to draw his conclusion correctly; doing all this in the spirit of one who feels that he has something important which he wishes to persuade his hearers is true.

Originality and skill in the art of manipulating the symbols and in applying the methods of Analytic Geometry will come from the solution of a large number of well-selected examples.

In the making of this manual, Mr. James G. Hardy has helped me very much by judicious and scholarly criticism and suggestion.

JOSEPH J. HARDY.

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## ERRATA

Page 16, line 8, for "of a" read "of the."

Page 18, Fig. 9, YOY′ should be drawn through the left hand intersection of the curve and X′X.

Page 25, line 2, for "curves" read "loci."

Page 113, lines 16, 17, 19, for "=" read "≡."

Page 117, line 11, for "opposite" read "same."

Page 117, line 12, for "from" read "as."

Page 122, line 2, for "$+a^2 \sin^2 \theta' + b^2 \cos \theta'$)" read "$+(a^2 \sin^2 \theta' + b^2 \cos^2 \theta')$."

Page 238, line 2, for "§ 15" read "[15]."

Page 261, line 15, for "$2B^2\left(x+\frac{E}{B}\right)$" read "$2B^3\left(x+\frac{E}{B}\right)$."

Page 339, lines 16, 17, for "=" read "≡."

Page 345, line 10, for "$\frac{a''}{D'}$" read "$\frac{a''}{D'}z^2$."

Page 359, line 22, for "ADEA′GH" read "ADEA′GH, Fig. 163."

NOTICE.—The references to geometry and trigonometry will be found in the appendix, page 361.

# ANALYTIC GEOMETRY

## CHAPTER I

### Constants and Variables

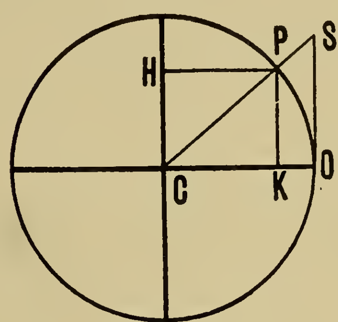



Fig. 1

1. In Fig. 1 let the point P move continually along the circumference of the circle. Then the sine PK, the cosine PH, and the tangent SO will change their values continually and may be made to take an infinite number of different values in consequence of this change in the position of the point P.

The radius CP, however, will always retain the same value throughout the operation of this change in the position of the point P.

Hence we call the sine PK, the cosine PH, and the tangent SO *variables*, but we call the radius CP a *constant*.

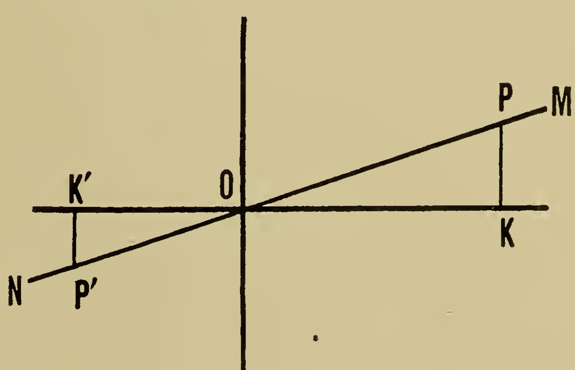



Fig. 2

2. In Fig. 2 the tangent of the angle POK is $\frac{PK}{OK}$, which we will represent by *t*.

Now let the point P move continually along the line MN. Then PK and OK will change their values continually, and may be made to take an infinite number of different values in consequence of this change in the position of the point P. The tangent *t*, however, will always retain the same value during the operation of this change in the position of the point P, because its angle POK does not change.

Hence we call PK and OK *variables*, but we call *t* a *constant*.

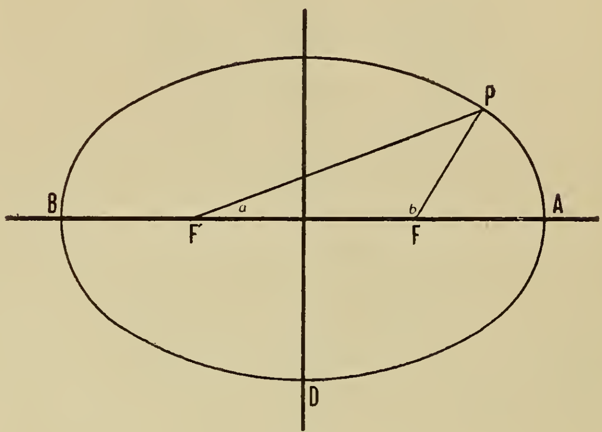



Fig. 3

3. In Fig. 3 let F′ and F represent two fixed pins, to which the ends of an inelastic thread are fastened. At P let a pencil be pressed against the thread. Let the point of the pencil be moved continually so as always to keep the thread stretched. The pencil will then trace out a curved line like APBDA.

Let the length of the thread $F'P + PF$ be represented by *l*, the angle PF′F by *a*, and the angle PFF′ by *b*.

Then F′P, FP and the angles *a* and *b* will change their values continually, and may be made to take an infinite number of different values in consequence of this change in the position of the point P, but *l*, the length of the broken line F′PF, and the length of F′F always retain the same value during the operation of this change in the position of the point P.

We call F′P, FP, *a* and *b* *variables*, but we call *l* and F′F *constants*.

4. **A Constant.**—A *constant* quantity is one that always retains the same value throughout the operation of a given change.

5. **A Variable.**—A *variable* quantity is one that may be made to take an infinite number of different values in consequence of the operation of a given change.

6. **The Change.**—In analytic geometry the *change* which affects the values of the quantities investigated is generally the motion of a point along a given line or surface.

# CHAPTER II

## Location of Points in a Plane

7. The position of a point in a plane may be indicated by means of its distances from any two fixed intersecting straight lines in the plane, these distances being measured parallel to the fixed lines.

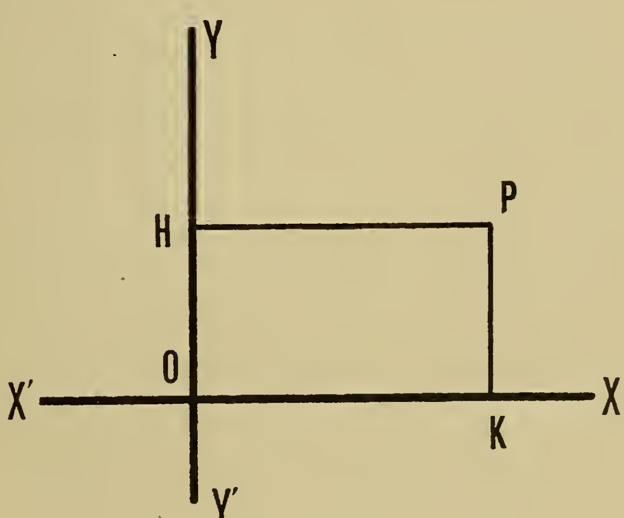



Fig. 4

Thus, if YY′ and XX′ are any two fixed intersecting straight lines in the plane YOX, we may indicate the position of the point P in that plane by giving its distance HP from the line YY′, measured parallel to XX′, and its distance KP from the line XX′, measured parallel to YY′.

For convenience we will let the line XX′ be horizontal.

**8. The Coordinate Axes.**—The fixed intersecting lines YY′ and XX′ are called the *axes of coordinates*.

**9. The Axes of Abscissas and Ordinates.**—The horizontal axis is called the X *axis*, or the *axis of abscissas*; the other axis is called the Y *axis*, or the *axis of ordinates*.

**10. The Origin.**—The point where the axes intersect each other is called the *origin*.

**11. Names of the Angles.**—The four angles which the axes make with each other are called the *first*, *second*, *third* and *fourth* angles.

The *first* angle is the one above the X axis and to the right of the Y axis.

The *second* angle is the one above the X axis and to the left of the Y axis.

The *third* angle is the one below the X axis and to the left of the Y axis.

The *fourth* angle is the one below the X axis and to the right of the Y axis.

**12. The Coordinates.**—The distances of any point from the axes, measured parallel to the axes, are called the *coordinates* of the point.

**13. The Abscissa.**—That coordinate of a point which is parallel to the X axis is called the *abscissa* of the point.

**14. The Ordinate.**—That coordinate of a point which is parallel to the Y axis is called the *ordinate* of the point.

Thus in Fig. 4, PH is the abscissa of the point P.
Since PH = OK, OK is often called the abscissa of P.
PK is the ordinate of the point P.

15. An *abscissa* is considered *positive* when it extends from the Y axis towards the right, and *negative* when it extends from the Y axis towards the left.

16. An *ordinate* is considered *positive* when it extends from the X axis upwards, and *negative* when it extends from the X axis downwards.

17. In giving the coordinates of a point we name the abscissa of the point first.

Thus when we say that the coordinates of a point are —3, 5, we mean that its abscissa is —3, and its ordinate is 5.

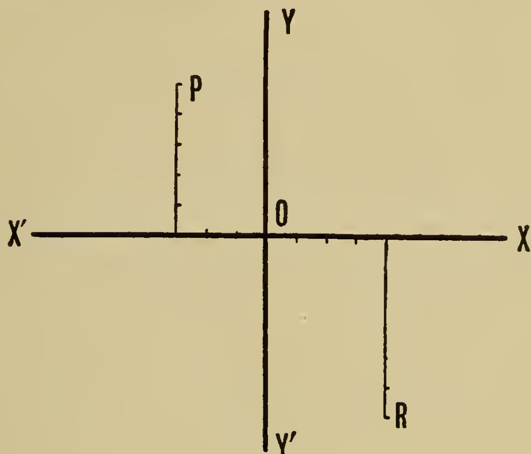



Fig. 5

18. To locate this point in the plane YOX, we take any convenient length, as a quarter of an inch, for our unit of length, and measure off, as in Fig. 5, 3 such units along the X axis from the origin towards the left, since the abscissa is —3, and from the point thus reached measure off 5 units from the X axis parallel to the Y axis and upwards, since the ordinate is 5. Hence P is the point whose coordinates are —3, 5.

19. Again, to locate the point whose coordinates are 4, —6, we measure off 4 units along the X axis from the origin to the right, since the abscissa is positive, and from the point thus reached measure off 6 units parallel to the X axis and downward, since the ordinate is negative. Hence R is the point whose coordinates are 4, —6.

20. Similarly locate in the plane YOX the following points :

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| 1. | The | point | whose | coordinates | are | 4, | 6. |
| 2. | " | " | " | " | " | 3, | —5. |
| 3. | " | " | " | " | " | —5, | 2. |
| 4. | " | " | " | " | " | —7, | —4. |
| 5. | " | " | " | " | " | 0, | 5. |
| 6. | " | " | " | " | " | 4, | 0. |
| 7. | " | " | " | " | " | 0, | 0. |

What are the signs of the coordinates of a point in the third angle? of one in the first angle? of one in the second angle? of one in the fourth angle?

# CHAPTER III

## The Construction of Loci

21. By means of an equation we can often locate one or more series of points which together form a geometrical line or figure.

EXAMPLE I

Take the equation $x^2 + y^2 = 25$.

| | | |
|---|---|---|
| [1] | | $x^2 + y^2 = 25,$ |
| [2] | Then | $y^2 = 25 - x^2$ |
| [3] | And | $y = \pm\sqrt{25 - x^2}.$ |

Now in [3] let $x = 0$, 1, 2, 3, etc., successively, and find the corresponding values of $y$. Then let $x = -1, -2, -3,$ etc., and find the corresponding values of $y$.

We will get

| | | |
|---|---|---|
| [1] | $y = \pm 5$ | when $x = 0,$ |
| [2] | $y = \pm 4.8$ | " $x = 1,$ |
| [3] | $y = \pm 4.5$ | " $x = 2,$ |
| [4] | $y = \pm 4$ | " $x = 3,$ |
| [5] | $y = \pm 3$ | " $x = 4,$ |
| [6] | $y = 0$ | " $x = 5,$ |
| [7] | $y = \pm\sqrt{-11}$ | " $x = 6.$ |

We see from [7] that $y$ is imaginary when $x = 6$. It is also imaginary for all values of $x$ greater than 6.

Now substituting the negative values of $x$ in [3] we get

| | | | |
|---|---|---|---|
| [8] | $y = \pm 4.8$ | when | $x = -1$, |
| [9] | $y = \pm 4.5$ | " | $x = -2$, |
| [10] | $y = \pm 4$ | " | $x = -3$, |
| [11] | $y = \pm 3$ | " | $x = -4$, |
| [12] | $y = \pm 0$ | " | $x = -5$, |
| [13] | $y = \pm \sqrt{-11}$ | " | $x = -6$. |

We see from [13] that $y$ is imaginary when $x = -6$, and that it is also imaginary for all values of $x$ which are negative and whose absolute values are greater than 6.

Now taking the first set of values of $x$ and $y$, namely, $x=0$ and $y = \pm 5$, we can locate, by the method given in Sections 18 and 19, two points on the axis of ordinates, one 5 units above the axis of abscissas, and the other 5 units below it. Thus we get the two points A and B of Fig. 6.

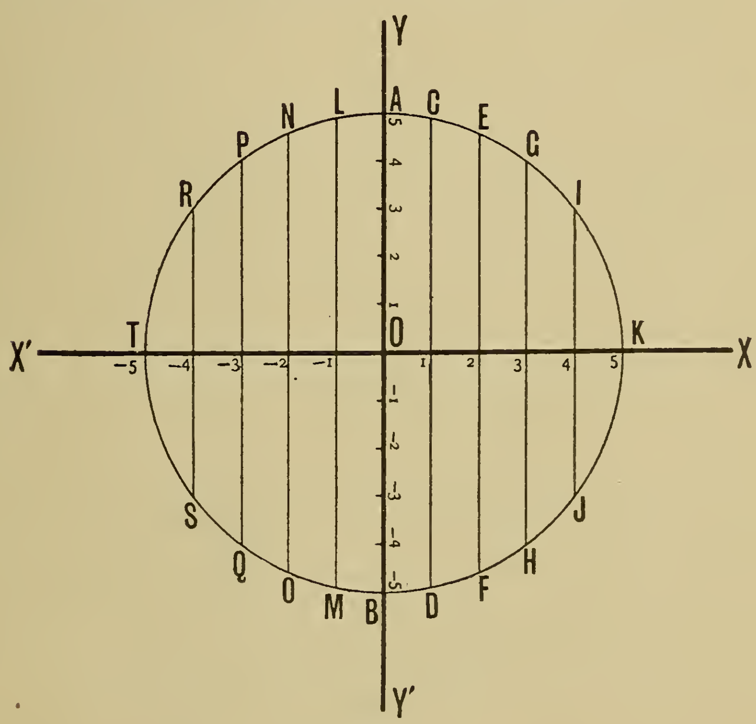



Fig. 6

Taking the second set of values, namely, $x = 1$ and $y = \pm 4.8$, we can locate two more points C and D of Fig. 6.

Taking the third set of values, namely, $x = 2$ and $y = \pm 4.5$, we can locate two more points E and F of Fig. 6.

Similarly,

| | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|
| the 4th | set | of | values | gives | us | the | points | G and H, |
| " 5th | " | " | " | " | " | " | " | I and J, |
| " 6th | " | " | " | " | " | " | point | K. |

The 7th set of values, namely, $x=6$ and $y=\pm\sqrt{-11}$, shows that it is impossible to locate, by means of this equation, any point whose abscissa shall be —6.

Moreover, since $y$ is imaginary for all values of $x$ greater than 6, it is impossible, by this equation, to locate any real point to the right of K.

Again,

| | | | | | | | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| by | the | 8th | set | of | values | we | can | locate | the points | L and M, |
| " | " | 9th | " | " | " | " | " | " | " | N and O, |
| " | " | 10th | " | " | " | " | " | " | " | P and Q, |
| " | " | 11th | " | " | " | " | " | " | " | R and S, |
| " | " | 12th | " | " | " | " | " | " | " point | T. |

The 13th set, namely, $x=-6$ and $y=\pm\sqrt{-11}$, shows that it is impossible, by means of this equation, to locate any point whose abscissa is —6.

Since $y$ is imaginary for all negative values of $x$ whose absolute values are greater than 6, it is impossible, by means of this equation, to locate any point to the left of T.

By joining all the points located in this way we get the circle AKBT.

It is obvious, that if, in addition to the values of $x$ used above, we take fractional values between them as 0.1, 0.2, 0.3, etc.; 1.1, 1.2, 1.3, etc.; 2.1, 2.2, 2.3, etc.; points may be located as near to each other as we please, and the line AKBT can be made as nearly continuous as we please.

EXAMPLE 2

Take the equation $xy=10$.

[1] $$xy=10,$$

[2] $$y=\frac{10}{x}.$$

If in [2] we let $x = 0, 1, 2, 3$, etc., and find the corresponding values of $y$, we get

| | | | |
|---|---|---|---|
| [1] | $y = \infty$ | when | $x = 0$, |
| [2] | $y = 10$ | " | $x = 1$, |
| [3] | $y = 5$ | " | $x = 2$, |
| [4] | $y = 3\frac{1}{3}$ | " | $x = 3$, |
| [5] | $y = 2\frac{1}{2}$ | " | $x = 4$, |
| [6] | $y = 2$ | " | $x = 5$, |
| [7] | $y = 1\frac{2}{3}$ | " | $x = 6$, |
| [8] | $y = 1\frac{3}{7}$ | " | $x = 7$, |
| [9] | $y = 1\frac{1}{4}$ | " | $x = 8$, |
| [10] | $y = 1\frac{1}{9}$ | " | $x = 9$, |
| [11] | $y = 1$ | " | $x = 10$. |

If we let $x = -1, -2, -3$, etc., we get

| | | | |
|---|---|---|---|
| [12] | $y = -10$ | when | $x = -1$, |
| [13] | $y = -5$ | " | $x = -2$, |
| [14] | $y = -3\frac{1}{3}$ | " | $x = -3$, |
| [15] | $y = -2\frac{1}{2}$ | " | $x = -4$, |
| [16] | $y = -2$ | " | $x = -5$, |
| [17] | $y = -1\frac{2}{3}$ | " | $x = -6$, |
| [18] | $y = -1\frac{3}{7}$ | " | $x = -7$, |
| [19] | $y = -1\frac{1}{4}$ | " | $x = -8$, |
| [20] | $y = -1\frac{1}{9}$ | " | $x = -9$, |
| [21] | $y = -1$ | " | $x = -10$. |

Now, from the second set of values, namely, $x = 1$, and $y = 10$, by the method given in Sections 18 and 19, we locate the point A in Fig. 7.

By the third set, $x = 2$ and $y = 5$, we locate the point B.

By the fourth set, $x = 3$ and $y = 3\frac{1}{3}$, we locate the point C.

By the fifth set we locate the point D.

By the sixth set we locate the point E.

Similarly, the remaining sets of positive values enable us

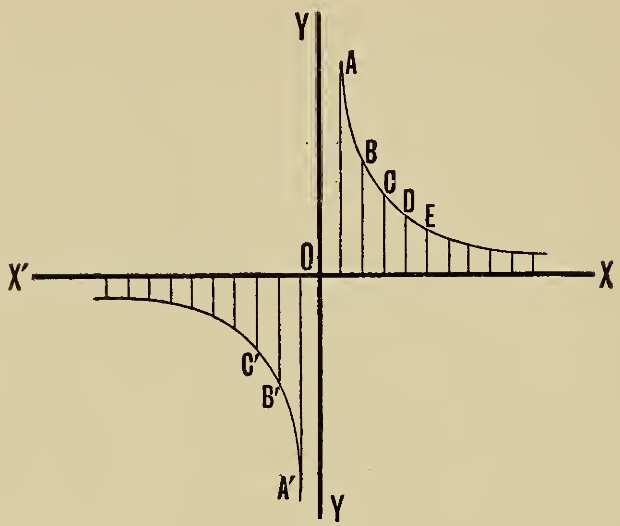



Fig. 7

to locate points to the right of the axis of ordinates and above the axis of abscissas.

By joining these points we get the line ABCDE.

In the same way the 12th, 13th, 14th, etc. sets of values enable us to locate points to the left of the axis of abscissas and below the axis of ordinates.

Joining these points we get the line A′B′C′.

It is obvious, that if, in addition to the values of $x$ used above, we take fractional values between them as 0.1, 0.2, 0.3, etc., and 1.1, 1.2, 1.3, etc., we can locate points as near to each other as we please, and so make each of the lines ABCDE and A′B′C′ as nearly continuous as we please.

The first set of values indicates that the point whose coordinates are $x = 0$, and $y = \infty$, is on the axis of ordinates at an infinite distance above the axis of abscissas. Hence the line EDCBA continually approaches the Y axis, can be made to approach as near to it as we please, but can never touch it.

If in [2] we make $x = \infty$, we get $y = 0$. This set of values, indicates that the point whose coordinates are $x = \infty$ and $y = 0$ is on the axis of abscissas at an infinite distance to the right of the axis of ordinates. Hence the line EDCBA continually approaches the X axis in the same way as it approaches the Y axis.

Show that A'B'C' approaches the axes in the same way.

EXAMPLE 3

Take the equation $16y^2 - 9x^2 = -144$.

[1] $16y^2 - 9x^2 = -144,$

[2] $16y^2 = 9x^2 - 144,$

[3] $y^2 = \frac{9}{16}x^2 - 9,$

[4] $y = \pm\sqrt{\frac{9}{16}x^2 - 9}.$

Now, in [4], if we let $x = 0, 1, 2, 3$, etc., and find the corresponding values of $y$, we get

| | | |
|---|---|---|
| [1] | $y = \pm\sqrt{-9}$ | when $x = 0$, |
| [2] | $y = \pm\sqrt{-8\frac{7}{16}}$ | " $x = 1$, |
| [3] | $y = \pm\sqrt{-6\frac{6}{8}}$ | " $x = 2$, |
| [4] | $y = \pm\sqrt{-3\frac{15}{16}}$ | " $x = 3$, |
| [5] | $y = 0$ | " $x = 4$, |
| [6] | $y = \pm 2\frac{1}{4}$ | " $x = 5$, |
| [7] | $y = \pm 3.3$ | " $x = 6$, |
| [8] | $y = \pm 4.3$ | " $x = 7$, |
| [9] | $y = \pm 6$ | " $x = 8$. |

If in [4] we let $x = -1, -2, -3$, etc., and find the corresponding values of $y$, we get

| | | |
|---|---|---|
| [10] | $y = \pm\sqrt{-8\frac{7}{16}}$ | when $x = -1$, |
| [11] | $y = \pm\sqrt{-6\frac{6}{8}}$ | " $x = -2$, |
| [12] | $y = \pm\sqrt{-3\frac{15}{16}}$ | " $x = -3$, |
| [13] | $y = 0$ | " $x = -4$, |
| [14] | $y = \pm 2\frac{1}{4}$ | " $x = -5$, |
| [15] | $y = \pm 3.3$ | " $x = -6$, |
| [16] | $y = \pm 4.3$ | " $x = -7$, |
| [17] | $y = \pm 5.2$ | " $x = -8$. |

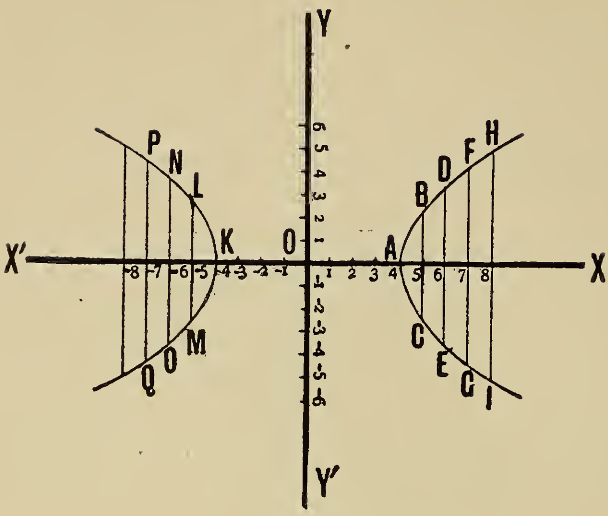



Fig. 8

The first four sets of values show that for this equation $y$ is imaginary when $x = 0$, 1, 2, or 3, and hence it is impossible, by means of this equation, to locate any real point whose abscissa shall be 0, 1, 2, or 3.

The fifth set of values, namely $x = 4$ and $y = 0$, enables us to indicate the point A on the axis of abscissas to the right of the axis of ordinates.

| | | | | | | | |
|---|---|---|---|---|---|---|---|
| The | 6th | set | of values | locates | the | points | B and C, |
| " | 7th | " | " | " | " | " | D and E, |
| " | 8th | " | " | " | " | " | F and G, |
| " | 9th | " | " | " | " | " | H and I. |

It is obvious that every set of values of $x$ greater than 8 will give two values of $y$, which will be equal, but which will have opposite signs, and hence that we may locate points farther and farther to the right.

Joining the points located we get the line HDAEG.

The 10th, 11th, and 12th sets of value show that for this equation $y$ is imaginary when $x = -1$, $-2$, or $-3$, and hence that it is impossible, by means of this equation, to locate any real points whose abscissas are $-1$, $-2$, or $-3$.

The 13th set locates the point K on the axis of abscissas to the left of the axis of ordinates.

The 14th set locates the points L and M.

Similarly each of the other sets enables us to locate two points to the left of the axis of ordinates, and at equal distances above and below the axis of abscissas.

It is obvious that every negative value of $x$ whose absolute value is greater than 8, will give us two values of $y$, which will be equal, but which will have opposite signs, and hence that we may locate points farther and farther to the left.

Joining the points located we get the line PLKOQ.

It is obvious, that if, in addition to the values of $y$ used above, we use the fractional values between them, we can locate points as near to each other as we please, and so make each of the lines HDAEG and PLKOQ as nearly continuous as we please.

Draw the lines represented by the following equations:

[4] $3y - 4x = 0,$

[5] $3y + 4x = 0,$

[6] $4y^2 = 3x^3,$

[7] $10y = x^2 - x - 20,$

[8] $y = x^2,$

[9] $y = x - \frac{1}{9}x^3,$

[10] $y^2 = 10x,$

[11] $16y^2 + 9x^2 - 144 = 0,$

[12] $16y^2 - 9x^2 + 144 = 0.$

In the next three examples substitute the values of $x$ before expanding the right hand members.

[13] $y^2 = (x-2)\ (x-7)^2,$

[14] $y^2 = (x-3)\ (x-7)\ (x-11),$

[15] $y^2 = (x-3)\ (x-7)\ (x-11)\ (x-13).$

**22.** It is obvious from the method by which the points were located in the line obtained by means of the equation of Example 1, that all points on the line AGKHBQTP, in Fig. 6, are fixed in position by one and the same law, namely, that the square of the abscissa plus the square of the ordinate is always equal to 25. For if in each of the sets of values of $x$ and $y$ taken to locate the points, we square the value of $x$ and also of $y$, and add them, we will always get 25.

23. The line AGKHBQTP is called the *locus* of the equation $x^2 + y^2 = 25$.

24. It is also obvious from the method by which the points were located in the lines ABC and A′B′C′ of Fig. 7, obtained by means of the equation of Example 2, that all the points on those lines are fixed in position by one and the same law, namely, that the abscissa, multiplied by the ordinate, is always equal to 10. For if in each of the sets of values of $x$ and $y$, taken to locate those points, we multiply the value of $x$ by that of $y$, we get 10.

25. The lines ABC and A′B′C′ together are called the *locus* of the equation $xy = 10$.

26. Similarly it may be shown that all the points on the lines HAI and PKQ of Fig. 8, obtained by means of the equation of Example 3, are fixed in position by one and the same law, namely, that 16 times the square of the ordinate minus 9 times the square of the abscissa is always equal to $-144$.

27. The lines HAI and PKQ together are called the *locus* of the equation $16y^2 - 9x^2 = -144$.

### EXERCISE

28. Give the law according to which the points are fixed in position in each of the lines obtained by means of each of the equations given above, namely, equations 4, 5, 6, 7, etc.

Hence we may define a locus as follows:

29. **A Locus.**—*A locus is the whole assemblage of points, each of which is fixed in position by one and the same law.*

30. It is obvious, that if in Example 1, page 6, in addition to the values of $x$ and $y$ there used, we consider the fractional values between them, then there are an infinite number of sets of values of $x$ and $y$, satisfying the equation corresponding to an infinite number of points on the locus. Hence if the $x$ and

$y$ of the equation represent a point P on the locus KGANR, etc., then by moving the point P along the locus, we may make the $x$ and $y$ of the equation take an infinite number of different values.

31. Hence the $x$ and $y$ of the equation $x^2 + y^2 = 25$ are variables. by § 5.

32. The equation $x^2 + y^2 = 25$ is called *the equation of the locus* KGPSJK (Fig. 6).

33. It is also obvious, that if in Example 2, page 8, in addition to the values of $x$ and $y$ there taken, we consider the fractional values between them, then there are an infinite number of sets of values of $x$ and $y$ satisfying the equation corresponding to an infinite number of points on the locus. Hence if the $x$ and $y$ of the equation represent the coordinates of a point P on the locus, then by moving the point P along the locus we may make the $x$ and $y$ of the equation take an infinite number of different values.

34. Hence the $x$ and $y$ of the equation $xy = 10$ are variables. by § 5.

35. The equation $xy = 10$ is called *the equation of the locus* ABCA′B′C′ (Fig. 7).

36. In the same way we may show that by moving the point P along the locus of the equation of Example 3, the $x$ and $y$ of that equation may be made to take an infinite number of different values. Therefore the $x$ and $y$ of the equation $16y^2 - 9x^2 = -144$ are variables. by § 5.

37. The equation $16y^2 - 9x^2 = -144$ is called *the equation of the locus* HBAIPKQ (Fig. 8).

38. It is obvious, from the method by which each of the loci in Examples 1, 2, and 3 was drawn, that the values of the $x$ and $y$ in each of the sets of values which satisfy the equations of the loci, are so related to each other that

1st, the $x$ and $y$ of each set of values of the variables which satisfy the equation stand for the coordinates of a particular point on the locus, and

2d, the values of the coordinates of each point on the locus satisfy the equation of the locus.

Hence we may define the equation of a locus as follows :

39. **The Equation of a Locus.**—The *equation of a locus* is one in which the variables stand for the coordinates of every point on the locus.

40. *Corollary 1.—The values of the coordinates of every point on a locus must satisfy the equation of a locus.*

41. *Corollary 2.—If the values of the coordinates of a point satisfy the equation of a locus, that point must be on the locus.*

by § 38, 1st

## Measurement of Arcs

42. An arc of any circle whose length is equal to the length of the radius of that circle is often taken as the unit for measuring arcs of that circle.

[1] The circumference $= \pi D = 2\pi R$ by Geom. 29.

Now, when the length of the radius is taken as the unit for measuring the lengths of arcs, [1] becomes

[2] The circumference $= 2\pi$,

and

[3] The semi-circumference $= \pi = 3.1416$.

That is, the length of the semi-circumference of any circle is equivalent to 3.1416 units when each unit is as long as the radius of the circle.

Hence, when the length of the radius is taken as the unit for measuring arcs

[4] $180^\circ = 3.1416$,

[5] $0^\circ = 0$,

[6] $10^\circ = 0.17$,

[7] $20^\circ = 0.35$,

[8] $30^\circ = 0.52$,

[9] $40^\circ = 0.70,$
[10] $50^\circ = 0.87,$
[11] $60^\circ = 1.05,$
[12] $70^\circ = 1.22,$
[13] $80^\circ = 1.40,$
[14] $90^\circ = 1.57,$
[15] $180^\circ = 3.14,$
[16] $190^\circ = 3.31,$
[17] $200^\circ = 3.49,$
[18] $210^\circ = 3.66,$
[19] $220^\circ = 3.84,$
[20] $230^\circ = 4.01,$
[21] $240^\circ = 4.19,$
[22] $250^\circ = 4.36,$
[23] $260^\circ = 4.54,$
[24] $270^\circ = 4.71.$

*Corollary.*—Since by equation 2

[1] $2\pi = 360^\circ,$
[2] Then $\pi = 180^\circ,$
[3] and $\frac{\pi}{2} = 90^\circ.$

EXAMPLES

Draw the curves of sines whose equation is

[1] $y = \sin x.$

For the values of the arc $x$ in this equation take the series of values given in the second column of the table in § 42.

For the values of $y$ take from the Trigonometrical Tables the natural sines of the number of degrees corresponding to each of these values of $x$. In this way we will get the following sets of values of $x$ and $y$:

| | |
|---|---|
| When $x = 0$, | $y = 0$, |
| $x = 0.17$, | $y = 0.17$, |
| $x = 0.35$, | $y = 0.34$, |
| $x = 0.52$, | $y = 0.50$, |
| $x = 0.70$, | $y = 0.64$, |
| $x = 0.87$, | $y = 0.77$, |
| $x = 1.05$, | $y = 0.87$, |
| $x = 1.22$, | $y = 0.94$, |
| $x = 1.40$, | $y = 0.98$, |
| $x = 1.57$, | $y = 1.00$, |
| $x = 3.14$, | $y = 0.00$, |
| $x = 3.31$, | $y = -0.17$, |
| $x = 3.49$, | $y = -0.34$, |
| $x = 3.66$, | $y = -0.50$, |
| $x = 3.84$, | $y = -0.64$, |
| $x = 4.01$, | $y = -0.77$, |
| $x = 4.19$, | $y = -0.87$, |
| $x = 4.36$, | $y = -0.94$, |
| $x = 4.54$, | $y = -0.98$, |
| $x = 4.71$, | $y = -1.00$. |

If with these values of $x$ and $y$ we locate a series of points by the method given in §§ 18 and 19, we will get the following locus:

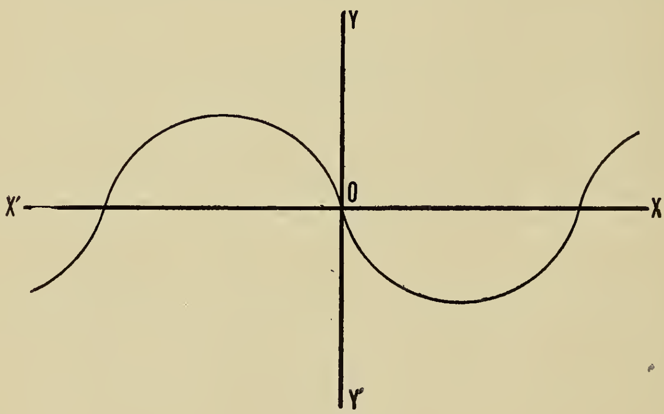



Fig. 9

2. Draw the curve of tangents whose equation is

$$y = \tan x.$$

For the values of the arc $x$ in this equation take the series of values given in the second column of the table in § 42.

For the values of $y$ take from the Trigonometrical Tables the natural tangents of the number of degrees corresponding to each of these values of $x$. In this way we get the following sets of values for $x$ and $y$:

When $x = 0.$ $\qquad y = 0,$

$x = \pm 0.17, \qquad y = \pm 0.18,$

$x = \pm 0.35, \qquad y = \pm 0.36,$

$x = \pm 0.52, \qquad y = \pm 0.58,$

$x = \pm 0.70, \qquad y = \pm 0.84,$

$^*x = \pm 0.79, \qquad y = \pm 1.00,$

$x = \pm 0.87, \qquad y = \pm 1.19,$

$x = \pm 1.05, \qquad y = \pm 1.73,$

$x = \pm 1.22, \qquad y = \pm 2.75,$

$x = \pm 1.40, \qquad y = \pm 5.67,$

$x = \pm 1.57, \qquad y = \infty.$

$^*x = \pm 0.79$ is the value of $45°$ obtained as in § 42.

If with these values of $x$ and $y$ we locate a series of points by the method given in §§ 18 and 19, we will get the following locus.

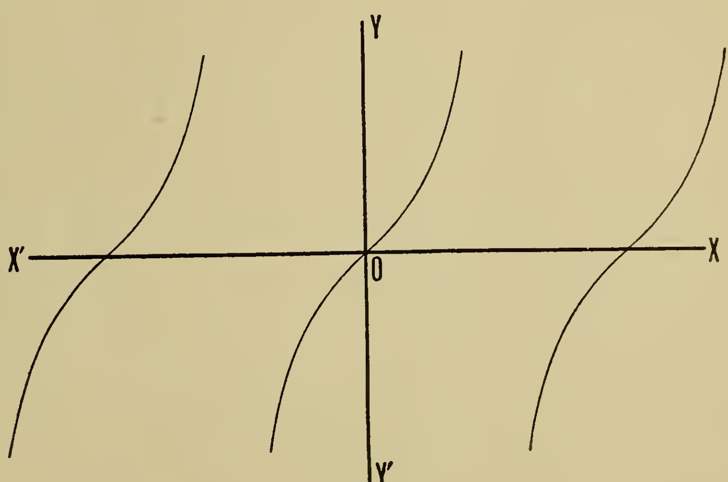



Fig. 10

3. Draw the curve of secants whose equation is

$$y = \sec x.$$

# CHAPTER IV

## The Intersection of Loci

### EXAMPLES

43. Where does the locus of $16y^2 + 9x^2 - 144 = 0$ cut the Y axis? Where does it cut the X axis?

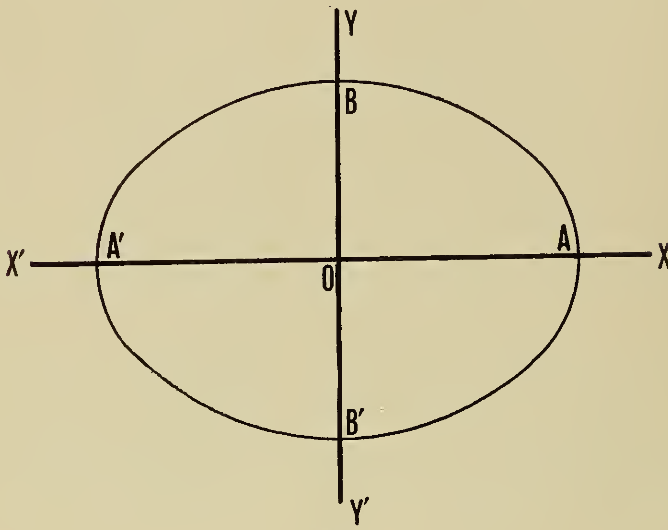



Fig. 11

From Example 11, p. 13, we learn that A'BAB' is the locus of

[1] $$16y^2 + 9x^2 - 144 = 0.$$

Since the point B is on the locus, its coordinates must satisfy the equation of the locus. by § 40.

But the coordinates of B are $x = 0$ and $y = \text{OB}$.

Substituting these values for the $x$ and $y$ of [1] we get

[2] $$16.\overline{\text{OB}}^2 - 144 = 0,$$

[3] or $$16.\overline{\text{OB}}^2 = 144,$$

[4] $$\overline{\text{OB}}^2 = 9,$$

[5] $$\text{OB} = \pm 3.$$

Hence $16y^2 + 9x^2 - 144 = 0$ cuts the $y$ axis in two points, one above and the other below the origin, and both at a distance of 3 units from it.

Since the point A is on the locus, its coordinates must satisfy the equation of the locus. by § 40.

But the coordinates of A are $x = \text{OA}$ and $y = 0$.

Substituting these values of $x$ and $y$ into [1] we get

[6] $$9 \cdot \overline{\text{OA}}^2 = 144,$$

[7] $$\overline{\text{OA}}^2 = 16,$$

[8] $$\text{OA} = \pm 4.$$

Hence $16y^2 + 9x^2 - 144 = 0$ cuts the X axis in two points, one to the right and the other to the left of the origin, and both at a distance of 4 units from it.

**44.** Where does the locus of $16y^2 - 9x^2 + 144 = 0$ cut the X axis? Where does it cut the Y axis?

Let A, Fig. 12, represent the point in which the locus of this equation cuts the X axis.

Then since A is on the locus, its coordinates must satisfy the equation of the locus. by § 40.

But the coordinates of A are $x = \text{OA}$ and $y = 0$.

Substituting these for the $x$ and $y$ of equation

[1] $$16y^2 - 9x^2 + 144 = 0,$$

we get

[2] $$-9 \cdot \overline{\text{OA}}^2 = -144,$$

[3] $$\overline{\text{OA}}^2 = 16,$$

[4] $$\text{OA} = \pm 4.$$

Hence this locus cuts the X axis in two points on opposite sides of the origin, and both at a distance of 4 units from it.

Let B, Fig. 12, represent the point where the locus is supposed to cut the Y axis.

Then if B is on the locus, its coordinates must satisfy the equation of the locus. by § 40.

But the coordinates of B are $x = 0$ and $y = OB$.
Substituting these values for the $x$ and $y$ of [1] we get

[5] $$16.\overline{OB}^2 = -144,$$

[6] $$\overline{OB}^2 = -9,$$

[7] $$OB = \pm\sqrt{-9}.$$

Equation [7] shows that it is impossible for this locus to cut the Y axis.

From Example 3, p. 11, we learn that the locus of [1] is LAKMA'N in Fig. 12.

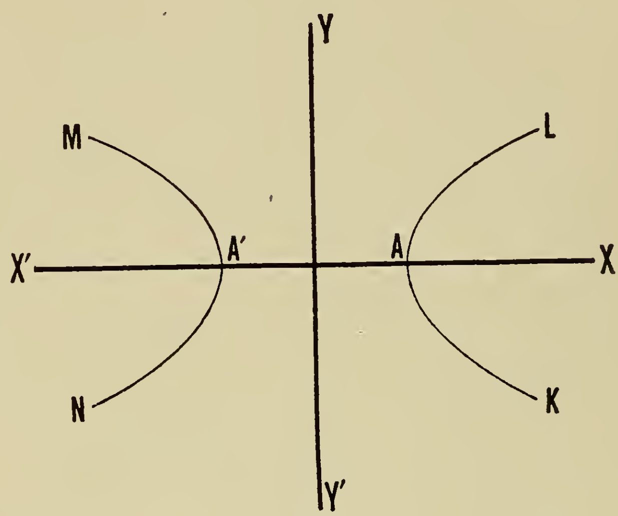


Fig. 12

Hence we may devise the following

45. **Rule.**—To find where any locus cuts the X axis, make $y = 0$ in the equation of the locus, and the values of $x$ obtained from the equation will be the abscissas of the cutting points.

For, the values of the coordinates of the cutting point must be $y = 0$, and $x =$ the abscissa of the cutting point.

But since the cutting point is on the locus, the values of these coordinates must satisfy the equation of the locus.

by § 40.

Hence, if we substitute these into the equation of the locus the terms involving $y$ will disappear and leave an equation whose only unknown quantity is $x$, the abscissa of the cutting point.

Similarly, we may devise the following

46. **Rule.**—To find the point where any locus cuts the Y axis, make $x = 0$ in the equation of the locus, and the values of $y$ obtained from the equation will be the ordinates of the cutting points.

EXAMPLES

Where do the loci of the following equations cut the axes?

1. $x^2 + y^2 = 9$. Ans. $x = \pm 3, y = 0$.
   $y = \pm 3, x = 0$.
2. $25y^2 + 16x^2 = 400$. Ans. $x = \pm 5, y = 0$.
   $y = \pm 4, x = 0$.
3. $25y^2 - 16x^2 = 400$. Ans. $x = \pm 5, y = 0$.
4. $y^2 = 10x$. Ans. $x = 0, \quad y = 0$.
5. $y^2 = (x - 2)(x - 7)^2$. Ans. $x = 2$ and $7$.
6. $y^2 = (x-3)(x-7)(x-11)(x-13)$. Ans. $x = 3, 7, 11, 13$.
7. $y^2 = (x - a)(x - b)(x - c)$. Ans. $x = a$, $b$ and $c$.

EXAMPLES

47. Where does $x^2 + y^2 = 25$ cut $xy = 10$?

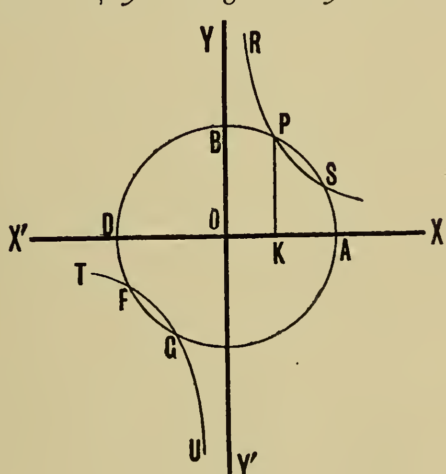



Fig. 13

From Example 1, p. 6, we see that ABD is the locus of

[1] $$x^2 + y^2 = 25.$$

From Example 2, p. 8, we see that RPSTU is the locus of

$$[2] \qquad xy = 10.$$

Let $x'$ and $y'$ be the coordinates of the point P.

Then, since the cutting point P is on the locus ABD, its coordinates must satisfy the equation of this locus. by § 40.

Hence, substituting them into [1] we get

$$[3] \qquad x'^2 + y'^2 = 25.$$

Since P is also on the locus RPSTU, its coordinates must satisfy the equation of that locus. by § 40.

Hence, substituting them into [2] we get

$$[4] \qquad x'y' = 10.$$

Now, since $x' =$ OK and $y' =$ PK in both [3], and [4] these equations are simultaneous and may be solved by algebra.

From [4] we get

$$[5] \qquad y' = \frac{10}{x'}.$$

Substituting this value of $y'$ into [3] we get

$$[6] \qquad x'^2 + \frac{100}{x'^2} = 25,$$

$$[7] \qquad x'^4 - 25x'^2 = -100,$$

$$[8] \qquad x'^4 - 25x'^2 + \frac{625}{4} = \frac{225}{4},$$

$$[9] \qquad x'^2 = 20 \text{ or } 5,$$

$$[10] \qquad x' = \pm 4.5 \text{ or } \pm 2.2.$$

Substituting these values of $x'$ *separately* into [3] we get

$$[11] \qquad y' = 2.2 \text{ when } x' = 4.5,$$

$$[12] \qquad y' = -2.2 \quad \text{“} \quad x' = -4.5,$$

$$[13] \qquad y' = 4.5 \quad \text{“} \quad x' = 2.2,$$

$$[14] \qquad y' = -4.5 \quad \text{“} \quad x' = -2.2.$$

Equations [11], [12], [13], and [14] show that the curves cut in four points, P, S, F, and G.

Hence we may derive the following

48. **Rule.**—To find where two curves cut each other, treat their equations as simultaneous and solve them. The values $x$ and $y$ thus found will be the coordinates of the cutting points.

For, since as in Fig. 13 the cutting point P is on both curves, its coordinates must satisfy the equation of both curves.

by § 40.

Hence we may let the abscissa of the cutting point be represented by $x'$, and the ordinate of the cutting point be represented by $y'$ in the equation of both curves. Then the equations will be simultaneous and may be solved by algebra.

The values of $x'$ and $y'$, obtained by solving these equations, will be the coordinates of the cutting points.

EXAMPLES

Solve each of the following examples and illustrate it by drawing its locus.

1. Where does $y = x + 5$ cut $y = x^2$?
   Ans. $x = 2.8, y = 7.8.$
   $x = 3.2, y = -1.8.$
2. Where does $y = 2x$ cut $xy = 18$? Ans. $x = 3, y = 6.$
   $x = -3, y = -6.$
3. Where does $x^2 + y^2 = 25$ cut $y = x^2$?
   Ans. $x = \pm 1.5, y = 2.2.$
4. Where does $y^2 = 4x$ cut $x^2 + y^2 = 25$?
   Ans. $x = 3.4, y = \pm 3.7.$
5. Where does $y = x^2$ cut $4y^2 = 3x^3$?
   Ans. $x = \frac{3}{4}, y = \pm \frac{9}{16}.$
   $x = 0, y = 0.$
6. Where does $x^2 + y^2 = 16$ cut $36y^2 + 9x^2 = 324$?
   Ans.
7. Where does $100y^2 + 64x^2 = 6400$ cut $64y^2 - 36x^2 = -2304$?
   Ans.

# CHAPTER V

## The Straight Line

49. **The Intercept.**—The *intercept* of any line on either axis is the distance from the origin to that line measured along that axis.

50. Every line which cuts the X axis makes four angles with it. Starting on the axis in the positive direction from the common vertex of these angles and moving counter clockwise, the angles are named first, second, third, and fourth.

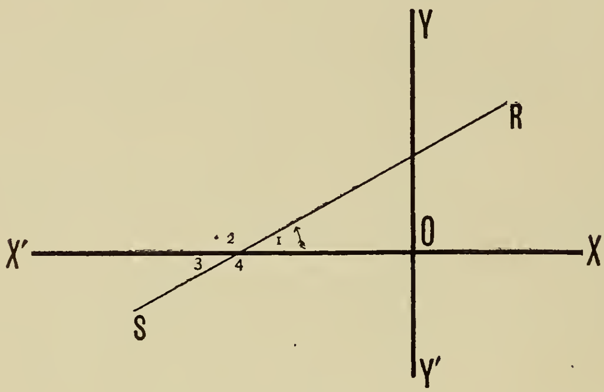



Fig 14

51. **The Inclination.**—The *inclination* of a line is the first of the four angles which it makes with the X axis.

52. **The Slope.**—The *slope* of a line is the tangent of its inclination.

PROPOSITION I

53. *The equation of a straight line in terms of its slope and Y intercept is*

$$y = sx + b,$$

*in which s is its slope and b its Y intercept.*

Let YY′ be the axis of ordinates and XX′ the axis of abscissas, and let MN be any straight line in their plane.

Let $s \equiv \tan$ BRH, the slope of MN ; and $b \equiv$ OB, its Y intercept.

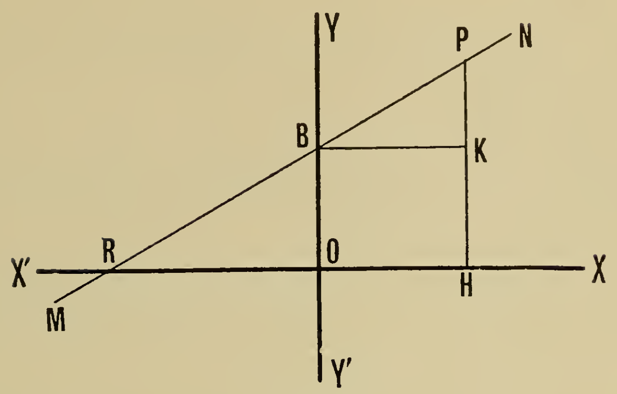



Fig. 15

Let P be *any* point on the line and draw its coordinates PH and OH.

Let $x \equiv \text{OH}$ and $y \equiv \text{PH}$.

We are to prove that

$$y = sx + b$$

is the equation of MN.

Draw BK ∥ OH.

| | | | |
|---|---|---|---|
| [1] | | $\angle \text{PBK} = \angle \text{PRH}$, | by Geom. 8. |
| [2] | hence | $\tan \text{PBK} = \tan \text{PRH} = s$, | |
| [3] | | $\tan \text{PBK} = \dfrac{\text{PK}}{\text{BK}}$, | by Trig. 3. |
| [4] | hence | $\dfrac{\text{PK}}{\text{BK}} = s$, | |
| [5] | and | $\text{PK} = s.\text{BK}$. | |
| [6] | But | $\text{BK} = \text{OH}$, | by Geom. 17. |
| [7] | hence | $\text{PK} = s.\text{OH}$. | |
| [8] | | $\text{KH} = \text{BO}$. | by Geom. 17. |

By adding [7] and [8] we get

[9] $\text{PK} + \text{KH} = s.\text{OH} + \text{BO}$,

[10] or $y = sx + b$.

Now the $x$ and $y$ of [10] stand for the coordinates of the point P. But, since P is *any* point on the line MN, we may move it either way as far as we please along that line.

Hence the $x$ and $y$ of [10] stand for the coordinates of *every* point on the line MN.

By moving the point P along the line, we may make the $x$ and $y$ of [10] take an *infinite* number of different values. They are therefore variables. by § 5.

But $s$ and $b$ always retain the same values as the point P moves along the line MN. They are therefore constants.
by § 4.

Therefore, since the variables in [10] stand for the coordinates of *every* point on the line MN, [10] must be the equation of the line MN. by § 39.

Q. E. D.

54. *Corollary 1.—Any equation of the form*

$$y = constant.x + constant,$$

*is the equation of some straight line.*

For, since MN may be any straight line, its intercept $b$ may have any value from 0 to $+\infty$, or to $-\infty$.

Hence $b$ may be any constant whatever.

Also, since MN may be any straight line, its inclination may have any value from 0° to 180°, and therefore its slope may have any value from 0 to $+\infty$, from 0 to $-\infty$.

Hence $s$ may be any constant whatever.

Therefore, $y = sx + b$ includes every equation of the form $y =$ const.$x +$ const.

Hence every equation of that form must be the equation of some straight line.

55. *Corollary 2.—The equation of a straight line passing through the origin is*

$$y = sx.$$

For, for every such line $b = 0$, and [10] becomes

$$y = sx.$$

56. *Corollary 3.—The equation of a straight line parallel to the X axis is*

$$y = b.$$

For, for every such line the inclination must be 0° or 180°, and hence the slope must be 0, by Trig. 19, and [10] must become

$$y = b.$$

Similarly, it may be shown, that if $c$ be any constant, then

$$x = c,$$

is the equation of a line parallel to the Y axis.

EXAMPLES

1. What is the inclination and slope of the line $y = \frac{1}{2}x + 5$? What is its Y intercept?

Its slope $= \frac{1}{2}$, by § 53.

or the tangent of its inclination $= \frac{1}{2}$, by § 52.

hence its inclination $= 26° \; 34'$.

2. What is the inclination of each of the following lines:

$y = \frac{6}{10}x + 2.$ Ans. 30° 58′.

$y = x + 3.$ Ans. 45°.

$y = 3x + 1.$ Ans. 71° 34′.

3. Which of the four angles made by the axes does the line $y = -2x + 5$ cross?

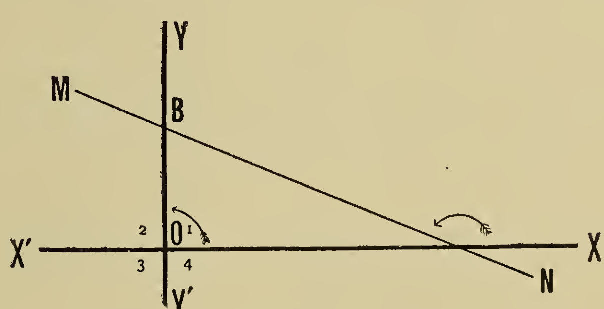



Fig. 16

The Y intercept of this line is 5. by § 53.

Hence the line cuts the Y axis 5 units above the origin, at B.

The slope of the line is $-2$. by § 53.

Hence the inclination is an obtuse angle. by Trig. 16.

Therefore, $y = -2x + 5$ must lie across the first of the four angles made by the axes, as MN in Fig. 16.

4. Show which of the four angles each of the following lines crosses:

$$y = -2x - 5,$$
$$y = 2x - 5,$$
$$y = 2x + 5.$$

5. Show how each of the following lines lies:

$$y = 2x,$$
$$y = -5,$$
$$y = 6.$$

6. What are the intercepts of $y = 5x + 10$ on the two axes?

7. What is the area of the triangle between $y = 5x + 10$ and the two axes?

8. What is the area of the triangle between $y = -5x - 10$ and the two axes?

9. What is the area of the riangle between $y = -10x + 5$ and the two axes?

PROPOSITION II

57. *The equation of a straight line passing through a fixed point is*

$$y' - y = s(x' - x),$$

*in which $x'$ and $y'$ are the coordinates of the fixed point, $x$ and $y$ the coordinates of any point on the line and $s$ is the slope of the line.*

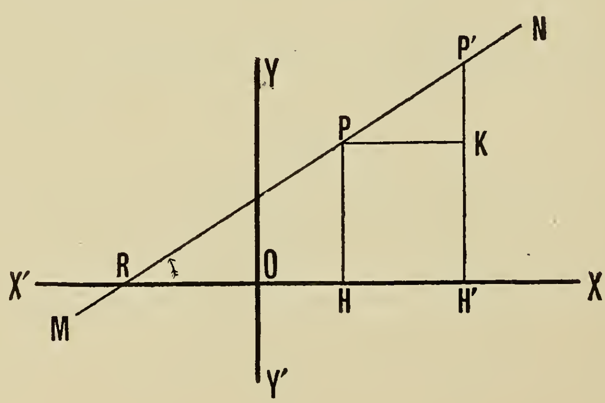



Fig. 17

Let YY′ be the axis of ordinates and XX′ the axis of abscissas.

Let P′ be the fixed point and MN any straight line passing through that point.

Let P be *any* point on the line MN and draw the coordinates of P and P′.

Let $x \equiv$ OH and $y \equiv$ PH,
$x' \equiv$ OH′ " $y' \equiv$ P′H′.
Let $s \equiv$ tan PRH, the slope of MN.

We are to prove that

$$y' - y = s\,(x' - x)$$

is the equation of MN.

[1] Draw PK ∥ XX′.

[2] $\angle$ P′PK $= \angle$ PRH, by Geom. 8.

[3] hence tan P′PK = tan PRH = $s$.

[4] But tan P′PK $= \dfrac{\text{P'K}}{\text{PK}}$, by Trig. 3.

hence $\dfrac{\text{P'K}}{\text{PK}} = s$,

[5] and P′K = $s$.PK.

[6] P′K = P′H′ − PH = $y' - y$, by Geom. 17.

[7] PK = OH′ − OH = $x' - x$, by Geom. 17.

Substituting these values of P′K and PK into [5] we get

[8] $y' - y = s\,(x' - x)$.

Now the $x$ and $y$ of [8] stand for the coordinates of the point P. But since P is *any* point on the line MN, we may move it either way as far as we please along that line.

Hence the $x$ and $y$ of [8] stand for the coordinates of *every* point on the line MN.

By moving the point P along the line MN, we may make the $x$ and $y$ of [8] take an infinite number of different values. They are therefore variables. by § 5.

But $s$, $x'$, and $y'$ always retain the same value as the point P moves along the line MN. They are therefore constants. by § 4.

Therefore, since the variables in [8] stand for the coordinates of every point on the line MN, [8] must be the equation of that line. by § 39.

Q. E. D.

EXAMPLES

1. What is the equation of the line whose inclination is 45° and which passes through the point whose coordinates are —5, 10?

Since this is a straight line passing through a given point, its equation must be of the form

[1] $$y' - y = s(x' - x),$$ by § 57.

in which $s = \tan 45° = 1,$ by Trig. 26.

$$x' = -5 \text{ and } y' = 10.$$

Substituting these values into [1] we get

[2] $$y - 10 = 1\,(x + 5).$$

Simplifying this equation we get

[3] $$y = x + 15,$$

which is the equation of the line passing through the point — 5, 10, and whose inclination = 45°.

2. What is the equation of the line whose inclination is 45°, and which passes through the point whose coordinates are — 5, — 10?

What intercepts does it cut from the axes?

Ans. $y = x - 5$.
Y intercept = — 5,
X intercept = 5.

3. What is the area of the triangle between the axes and the line whose inclination is 135°, and which passes through the point whose coordinates are — 2, — 5?

Ans. $y = -x - 7$.
Area of the triangle = $24\frac{1}{2}$.

4. Where does the line whose slope is 10, and which passes through the point whose coordinates are 1, 5, cut the line

whose slope is $-2$, and which passes through the point whose coordinates are $-1$, $-5$?

PROPOSITION III

58. *The equation of a straight line passing through two fixed points is*

$$y' - y = \frac{y'' - y'}{x'' - x'}(x' - x),$$

*in which $x'$ and $y'$ are the coordinates of one of the fixed points, $x''$ and $y''$ are the coordinates of the other, and $x$ and $y$ the coordinates of any point on the line.*

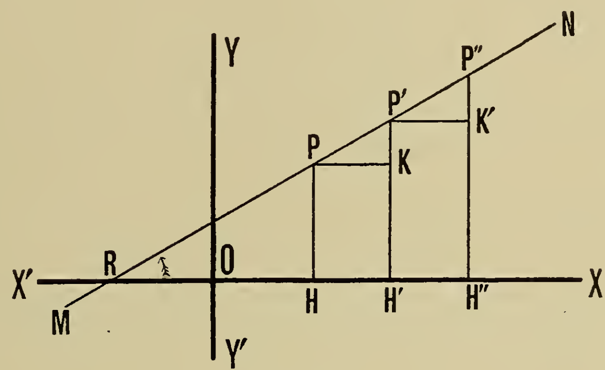



Fig. 18

Let YY' be the axis of ordinates and XX' the axis of abscissas.

Let P' and P'' be the two fixed points, and MN a straight line passing through them.

Let P be any other point on the line MN, and draw the coordinates of P, P' and P''.

$$\begin{aligned} \text{Let } x &\equiv OH \quad \text{and } y \equiv PH, \\ x' &\equiv OH' \quad \text{"} \quad y' \equiv P'H', \\ x'' &\equiv OH'' \quad \text{"} \quad y'' \equiv P''H''. \end{aligned}$$

We are to prove that

$$y' - y = \frac{y'' - y'}{x'' - x'}(x' - x),$$

is the equation of MN.

Draw PK and $P'K' \parallel XX'$.

[1] Now $\angle P''P'K' = \angle P'PK$. by Geom. 8.

[2] Hence $\tan P''P'K' = \tan P'PK$.

[3] $\tan P''P'K' = \frac{P''K'}{P'K'}$, by Trig. 3.

[4] $\tan P'PK = \frac{P'K}{PK}$, by Trig. 3.

[5] hence $\frac{P''K'}{P'K'} = \frac{P'K}{PK}$, by [2].

[6] and $P'K = \frac{P''K'}{P'K'} PK$.

[7] But $P'K = y' - y$,

[8] $P''K' = y'' - y'$,

[9] $P'K' = x'' - x'$,

[10] $PK = x' - x$,

[11] Hence $y' - y = \frac{y'' - y'}{x'' - x'}(x' - x)$. by [6].

Now the $x$ and $y$ of [11] stand for the coordinates of the point P. Since P is any point on the line MN, we may move it either way as far as we please along that line. Hence the $x$ and $y$ of [11] stand for the coordinates of every point on the line MN.

By moving the point P along the line MN, we may make the $x$ and $y$ of [11] take an infinite number of different values. They are therefore variables. by § 5.

But $x'$, $x''$, $y'$ and $y''$ always retain the same values as the point P is moved along the line MN. They are therefore constants. by § 4.

Therefore, since the variables in [11] stand for the coordinates of every point on the line MN, [11] must be the equation of MN. by § 39.

Q. E. D.

59. *Corollary 1.—The fraction* $\frac{y'' - y'}{x'' - x'}$ *is the slope of the line which passes through the two fixed points whose coordinates are* $x'$, $y'$, *and* $x''$, $y''$.

For

[1] $\tan P''P'K' = \dfrac{P''K'}{P'K'}$, by [3], §58.

[2] but $\tan P''P'K' = \tan PRH$, by Geom. 8.

[3] hence $\tan PRH = \dfrac{P''K'}{P'K'}$.

[4] But $\dfrac{P''K'}{P'K'} = \dfrac{y''-y'}{x''-x'}$,

[5] hence $\dfrac{y''-y'}{x''-x'} = \tan PRH =$ the slope. by § 52.

60. *Corollary 2.—The length of the line joining any two given points is*

$$\sqrt{(y''-y')^2 + (x''-x')^2},$$

*in which $x''$, $y''$ and $x'$, $y'$ are the coordinates of the two given points.*

For, in Fig. 18

[1] $\overline{P'P''}^2 = \overline{P''K'}^2 + \overline{P'K'}^2$, by Geom. 26.

[2] hence $\overline{P'P''}^2 = (y''-y')^2 + (x''-x')^2$,

[3] or $P'P'' = \sqrt{(y''-y')^2 + (x''-x')^2}$.

EXAMPLES

1. What is the equation of the line which passes through the two points 3, 5, and $-2$, $-7$?

The equation of this line must be of the form

[1] $y' - y = \dfrac{y''-y'}{x''-x'}(x'-x)$, by § 58.

in which

$$x' = 3 \text{ and } y' = 5,$$
$$x'' = -2 \text{ and } y'' = -7.$$

Substituting these values into [1] we get

[2] $y - 5 = \dfrac{-7-5}{-2-3}(x-3)$.

Simplifying this equation we get

[3] $$y = \frac{12}{5}x - \frac{11}{5},$$

which is the equation of the line passing through the two points 3, 5 and $-2$, $-7$.

2. What is the equation of the line which passes through the two points $-5$, 8 and 3, $-10$?

Ans. $y = -2\frac{1}{4}x - 3\frac{1}{4}$.

3. Where does the line which passes through the points $-3$, $-5$, and 5, 8 cut the axes?

Ans. At the point $\frac{1}{13}$, $-\frac{1}{8}$.

4. Where does the line which passes through $-5$, 3 and 3, $-5$ cut the line which passes through 5, 5 and the origin?

Ans. At the point $-1$, $-1$.

PROPOSITION IV

61. *The tangent of the angle between two straight lines is given by the equation*

$$\tan \varphi = \frac{s' - s}{1 + s's},$$

*in which $\varphi$ is the angle between the two lines, $s'$ is the slope of one of the lines and $s$ is the slope of the other.*

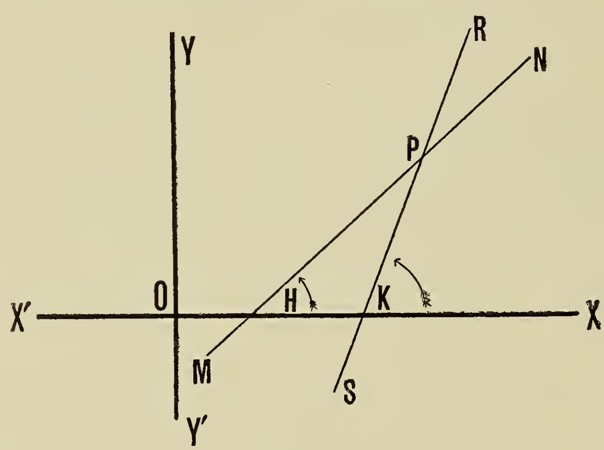



Fig. 19

Let MN and RS be any two straight lines.

[1] Let $\varphi \equiv \angle$ HPK.

Let $y = sx + b$ be the equation of the line MN.

[2] Then $s = \tan$ PHX. by § 53.

Let $y = s'x + b'$ be the equation of the line RS.

[3] Then $s' = \tan PKX.$ by § 53.

We are to prove that

$$\tan \varphi = \frac{s' - s}{1 + s's}.$$

[4] $\angle PKX = \angle PHK + \angle HPK,$ by Geom. 13.

[5] hence $\angle HPK = \angle PKX - \angle PHK,$

[6] and $\tan HPK = \tan (PKX - PHK),$

[7] but $\tan (A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B},$ by Trig. 17.

Since the A and B of [7] may be any angles whatever, let $A = PKX$ and $B = PHK$.

Substituting these values of A and B into [7] we get

[8] $\tan (PKX - PHK) = \frac{\tan PKX - \tan PHK}{1 + \tan PKX \tan PHK}.$

Comparing [6] and [8] we get

[9] $\tan HPK = \frac{\tan PKX - \tan PHK}{1 + \tan PKX \tan PHK}.$

Substituting the values of tan HPK, tan PKX, and tan PHK found in [1], [2] and [3] into [9] we get

[10] $\tan \varphi = \frac{s' - s}{1 + s's}.$

Q. E. D.

62. *Corollary 1.—Whenever two straight lines are perpendicular to each other*

$$1 + s's = 0,$$

*in which s' is the slope of one of the two lines and s is the slope of the other.*

For when RS ⊥ MN

[1] $\angle HPK = 90°,$

[2] and $\tan \varphi = \tan 90° = \infty,$ by Trig. 19.

[3] hence $\frac{s' - s}{1 + s's} = \infty,$ by [10.]

But when the value of the fraction $\frac{s'-s}{1+s's}$ is infinitely large its denominator is infinitely small.

[4] Hence $1+s's=0.$

63. *Whenever two straight lines are parallel to each other*

$$s'-s=0,$$

*in which s' is the slope of one of the two lines and s is the slope of the other.*

For when RS ‖ MN

[1] $s'=s,$ by Geom. 8.

[2] Hence $s'-s=0.$

EXAMPLES

1. What is the angle between the two lines $y=3x+1$ and $y=\frac{1}{2}x+2$?

Let $s'=3$, the slope of the first line, and $s=\frac{1}{2}$, the slope of the second line.

Substituting the values of $s$ and $s'$ into [10], § 61 we get

$$\tan\varphi=\frac{3-\frac{1}{2}}{1+\frac{3}{2}}=1.$$

Hence the angle between the lines $=45°$.

2. What is the angle between the lines $y=3x+1$, and $y=x-5$? Ans. $26° 34'$.

3. What is the angle between the lines $y=x-5$, and $y=-x+2$? Ans. $90°$.

4. What is the angle between $y=3x+2$, and $y=\frac{1}{2}x$? Ans. $45°$

5. What is the angle between $y=\frac{1}{10}x$, and $y=\frac{4}{5}x$? Ans. $32° 56' 36''$.

PROPOSITION V

64. *The equation of a straight line in terms of its intercepts on both axes is*

$$\frac{x}{a} + \frac{y}{b} = 1,$$

*in which a is its X intercept and b is its Y intercept.*

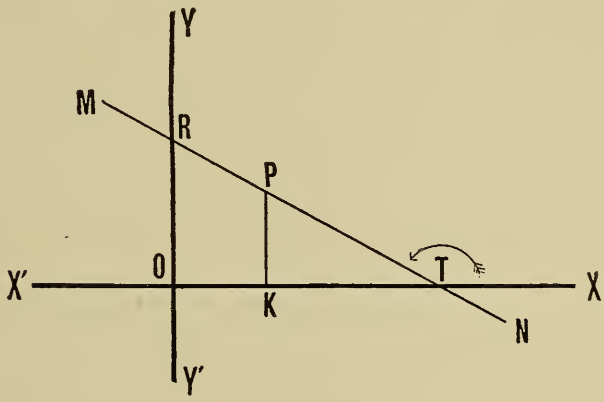



Fig. 20

Let MN be any straight line and P any point on it.

Let $a \equiv$ OT and $b \equiv$ OR,
$x \equiv$ OK " $y \equiv$ PK.

We are to prove that the equation of MN is

$$\frac{x}{a} + \frac{y}{b} = 1.$$

[1] KT : PK :: OT : OR, by Geom. 25 and 31.

[2] or $a - x : y :: a : b.$

[3] Hence $ab - bx = ay,$

[4] or $bx + ay = ab,$

[5] hence $\frac{x}{a} + \frac{y}{b} = 1.$

As in Propositions I and II we can show that the $x$ and $y$ in [5] are variables and stand for the coordinates of every point on the line MN. Therefore [5] is the equation of MN. by § 39.

EXAMPLES

1. Which of the four angles made by the axes is crossed by the line

$$\frac{x}{2} - \frac{y}{3} = -1?$$

Multiplying the equation through by $-1$ we get

$$-\frac{x}{2} + \frac{y}{3} = 1.$$

Changing two of the signs of the first fraction we get

$$\frac{x}{-2} + \frac{y}{3} = 1.$$

This is the form of [5]. Hence

$a = -2 =$ the X intercept of the line,

$b = \phantom{-}3 =$ the Y intercept of the line.

Hence the line $\frac{x}{2} - \frac{y}{3} = -1$ crosses the second angle.

Which of the four angles made by the axes does each of the following lines cross?

2. $$-\frac{x}{2} + \frac{y}{3} = -1.$$

3. $$\frac{x}{-5} - \frac{y}{2} = 1.$$

4. $$\frac{x}{3} - \frac{y}{-5} = -1.$$

5. $$\frac{x}{2} - \frac{y}{3} = 2.$$

PROPOSITION VI

65. *The equation of a straight line in terms of the perpendicular drawn to it from the origin, and the inclination of this perpendicular is*

$$x \cos \alpha + y \sin \alpha - p = 0$$

*in which $p$ is the perpendicular and $\alpha$ the angle which it makes with the X axis.*

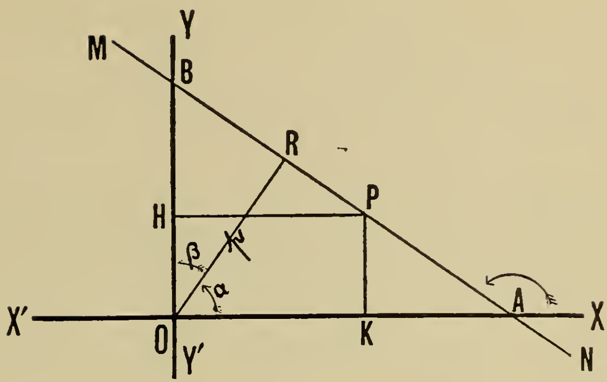



Fig. 21

Let MN be any straight line. Let OR be a straight line drawn from the origin perpendicular to MN.

Let $p \equiv OR$ and $\alpha \equiv \angle ROA$.

Let P be any point on the line MN and draw its coordinates PK and OK.

Let $x \equiv OK$ and $y \equiv PK$.

We are to prove that the equation of MN is

$$x \cos \alpha + y \sin \alpha - p = 0.$$

Draw PH ‖ OX.

The triangles PBH, ORB and ORA are similar.

by Geom. 54.

[1] Hence $\angle OBR = \angle ROA = \alpha$ by Geom. 31.

[2] and $p : OB :: PH : PB$, by Geom. 31.

[3] hence $p.PB = OB.PH = (BH + y)\,x = BH.x + xy.$

[4] Now $PB = \dfrac{BH}{\cos OBR}$, by Trig. 2.

[5] hence by [1] $PB = \dfrac{BH}{\cos \alpha}$,

[6] and $x = PH = BH.\tan \alpha.$ by Trig. 3

Substituting the values of PB and $x$ found in [5] and [6] into [3] we get

[7] $p \dfrac{BH}{\cos \alpha} = BH.x + y.BH.\tan \alpha,$

[8] or $$\frac{p}{\cos\alpha} = x + y\frac{\sin\alpha}{\cos\alpha},$$ by Trig. 6.

[9] hence $$x\cos\alpha + y\sin\alpha - p = 0.$$

Q. E. D.

66. *Corollary 1.—The equation of a straight line in terms of the perpendicular drawn to it from the origin, and the angles which this perpendicular makes with both axes is*

$$x\cos\alpha + y\cos\beta - p = 0.$$

For let $\beta \equiv \angle$ BOR.

[1] Then $\cos\beta = \sin\alpha.$ by Trig. 20.

Substituting this value of $\sin\alpha$ into [9] we get

[2] $$x\cos\alpha + y\cos\beta - p = 0$$

EXAMPLES

1. $\frac{1}{2}\sqrt{3}.x + \frac{1}{2}y = 5$ is the equation of a line in terms of the perpendicular drawn to it from the origin and the inclination of the perpendicular.

What angles does the perpendicular make with the axes? What is the length of the perpendicular? Ans. $\alpha = 30°$
$\beta = 60°$
$p = 5$

2. A line drawn from the origin perpendicular to a second line is 3 inches long, and its inclination is 45°. What is the equation of the second line in terms of the perpendicular to it from the origin, and the inclination of the perpendicular?

3. A line whose inclination is 150° cuts $y = 3x$ five inches from the origin. What is its equation in terms of the perpendicular to it from the origin and the inclination of the perpendicular?

PROBLEM

To prove that

[1] $2 - \frac{2x - y}{3} - \frac{3}{4}y = \frac{2}{3}(3x + y) - 5$ is the equation of a straight line.

Clearing the equation of fractions we get

[2] $$24 - 8x + 4y - 9y = 24x + 8y - 60.$$

Transposing we get

[3] $$-13y = 32x - 84.$$

Dividing by $-13$ we get

[4] $$y = -\tfrac{32}{13}x + \tfrac{84}{13}.$$

Now [4] is of the form $y =$ const. $x +$ const. Hence it is the equation of a straight line. by. § 54.

PROPOSITION VII

67. *Every equation of the first degree containing two variables only is the equation of a straight line.*

For, as in the preceding problem, by clearing of fractions, transposing, etc., every such equation can be reduced to the form

[1] $$Ay = Bx + C,$$

in which A, B and C are constants.

Dividing [1] through by A we get

[2] $$y = \frac{B}{A}x + \frac{C}{A},$$

in which $\frac{B}{A}$ and $\frac{C}{A}$ are constants.

Equation [2] is of the form $y =$ const.$x +$ const.

But every equation of that form is the equation of a straight line. by § 54.

Therefore every equation of the first degree containing two variables only is the equation of a straight line.

Q. E. D.

EXAMPLES

1. What are the slope and the intercepts of the line

$$3\frac{x+2}{4} + \frac{y}{3} - 1 = \frac{x}{2} - 4\left(y - \tfrac{3}{4}\right) + 2?$$

Clearing the equation of fractions, transposing, etc., we get

[1] $$3x + 52y = 66,$$

[2] $$y = -\tfrac{3}{52}x + \tfrac{66}{52},$$

Hence $-\frac{3}{52} =$ the slope, by § 53.

$\frac{66}{52} =$ the Y intercept, by § 53.

$22 =$ the X intercept, by § 45.

What are the slope and the intercepts of each of the following lines? Draw the lines.

2. $$\frac{x}{3} - \frac{2(y-3)}{2} = 4y - \frac{2x-5}{2} + 1.$$

3. $$2(x+y) - \frac{x}{5} = y - 3 + \frac{2x}{3}.$$

4. $$\tfrac{2}{3} - \left(\frac{x}{2} - 3y\right) = \tfrac{4}{5}\left(\frac{y}{3} - x\right) + 2.$$

PROPOSITION VIII

68. *If the equation of a straight line be given in the form*

$$Ax + By + C = 0,$$

*then by dividing this equation through by* $-\sqrt{A^2 + B^2}$, *we will get the equation of the same line in the form*

$$x \cos \alpha + y \sin \alpha - p = 0.$$

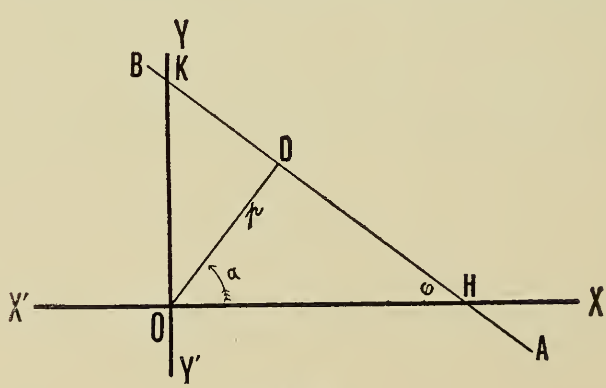



Fig. 22

In Fig. 22 let

[1] $$Ax + By + C = 0,$$

be the equation of the line AB. From O draw OD $\perp$ AB, and let

$$p \equiv OD.$$

We are to prove that if we divide [1] through by $-\sqrt{A^2+B^2}$ we will get the equation of AB in the form

$$x \cos \alpha + y \sin \alpha - p = 0.$$

Dividing [1] through by $-\sqrt{A^2+B^2}$ we get

[2] $$-\frac{A}{\sqrt{A^2+B^2}}.x - \frac{B}{\sqrt{A^2+B^2}}y - \frac{C}{\sqrt{A^2+B^2}} = 0.$$

[3] Now $$-\frac{A}{\sqrt{A^2+B^2}} = -\frac{\frac{C}{AB}.A}{\frac{C}{AB}\sqrt{A^2+B^2}} =$$

$$\frac{-\frac{C}{B}}{\sqrt{\frac{A^2C^2+B^2C^2}{A^2B^2}}} = \frac{-\frac{C}{B}}{\sqrt{\left(\frac{C}{B}\right)^2+\left(\frac{C}{A}\right)^2}}.$$

From [1] we get

[4] $$y = -\frac{A}{B}.x - \frac{C}{B}.$$

Making $x = 0$ in [4] we get

[5] $$-\frac{C}{B} = OK.$$ by § 46.

Making $y = 0$ in [4] we get

[6] $$-\frac{C}{A} = OH.$$ by § 45.

Substituting these values of $-\frac{C}{B}$ and $-\frac{C}{A}$ into the last member of [3] we get

[7] $$-\frac{A}{\sqrt{A^2+B^2}} = \frac{OK}{\sqrt{OK^2+OH^2}} = \frac{OK}{KH}.$$ by Geom. 26.

[8] But $\frac{OK}{KH} = \sin \varphi = \cos \alpha$, by Trig. 1 and 20.

[9] hence $-\frac{A}{\sqrt{A^2+B^2}} = \cos \alpha$.

[10] Similarly $-\frac{B}{\sqrt{A^2+B^2}} = -\frac{\frac{C}{AB}.B}{\frac{C}{AB}\sqrt{A^2+B^2}}$

$$= \frac{-\frac{C}{A}}{\sqrt{\left(\frac{C}{B}\right)^2 + \left(\frac{C}{A}\right)^2}} = \frac{OH}{KH}.$$

[11] But $\frac{OH}{KH} = \cos \varphi = \sin \alpha$.

[12] Hence $-\frac{B}{\sqrt{A^2+B^2}} = \sin \alpha$.

[13] $-\frac{C}{\sqrt{A^2+B^2}} = -\frac{C}{B} \times \frac{B}{\sqrt{A^2+B^2}} = +\frac{C}{B} \sin \alpha$.

by [12].

But by [5]

[14] $-\frac{C}{B} = OK$.

Hence [13] becomes

[15] $\frac{C}{\sqrt{A^2+B^2}} = OK \sin \alpha$.

But by Trig. 1

[16] $OD = p = OK \sin OKD = OK \sin \alpha$.

by Geom. 54 and 31.

Hence [15] becomes

[17] $\frac{C}{\sqrt{A^2+B^2}} = p$.

Hence by [9], [12] and [17], [2] becomes

[18] $$x \cos \alpha + y \sin \alpha - p = 0.$$

Q. E. D.

69. *Scholium.*—Let us agree to call the direction from the origin to the line AB the positive direction. Then $p$ will always be positive. Let us agree also to use the positive value of $\sqrt{A^2 + B^2}$. Then from [17] we see that if, in any equation of the form

$$Ax + By + C = 0,$$

the C be negative, we must change the signs of all the terms in the equation, so as to make C positive, before computing the value of $p$.

EXAMPLES

[1] Given the equation of a line in the form

$$3x + 4y + 15 = 0,$$

to find the equation of the same line in the form [18].

What are the values of cos $\alpha$, sin $\alpha$, and $p$? Draw the line.

Ans. $-\frac{3}{5}x - \frac{4}{5}y - 3 = 0.$
$\cos \alpha = -\frac{3}{5}.$
$\sin \alpha = -\frac{4}{5}.$
$p = 3.$

[2] Given the equation $4x - 3y - 15 = 0$, to change it into the form [18]. Ans. $\frac{4}{5}x - \frac{3}{5}y - 3 = 0.$

[3] Given the line $6y - 8x = 5$, to change it into the form [18]. Ans. $-\frac{8}{10}x + \frac{6}{10}y - \frac{1}{2} = 0.$

[4] Find the length of the perpendicular drawn from the origin to the line $12y - 5x = 26$, and the inclination of this perpendicular. Ans. $p = 2.$
Inclination $= 22° \, 37'.$

PROPOSITION IX

70. *The distance from any point whose coordinates are $x'$ and $y'$ to the line $x \cos \alpha + y \sin \alpha - p = 0$, is*

$$x' \cos \alpha + y' \sin \alpha - p.$$

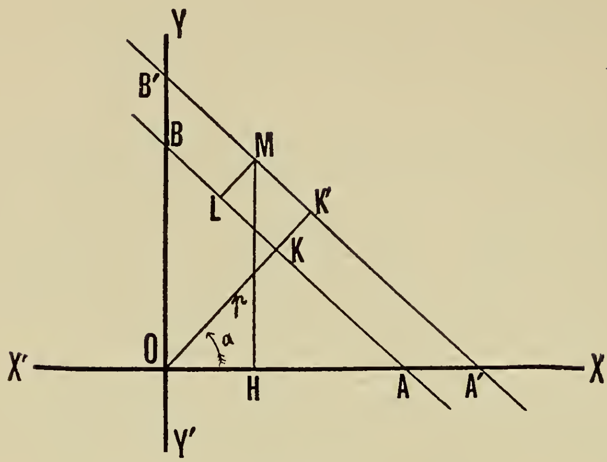



Fig. 23

Let M be any point, and AB any straight line.

Let $x \cos \alpha + y \sin \alpha - p = 0$ be the equation of AB.

Draw $ML \perp AB$. Let $D \equiv ML$.

Let OH and MH be the coordinates of the point M, and let $x' \equiv OH$, $y' \equiv MH$.

We are to prove that

[1] $$D = x' \cos \alpha + y' \sin \alpha - p.$$

Draw $OK \perp AB$.

Then since $x \cos \alpha + y \sin \alpha - p = 0$ is the equation of AB,

[2] $\alpha \equiv \angle KOA$, by § 65.

[3] and $p \equiv OK$. by § 65.

Through M draw $A'B' \parallel AB$.

Then $A'B' \perp OK$. by Geom. 6.

Let $p' \equiv OK'$.

Then the equation of $A'B'$ is

[4] $x \cos \alpha + y \sin \alpha - p' = 0$. by § 65.

[5] Hence $p' - p = KK'$.

[6] But $KK' = LM = D$, by Geom. 17.

[7] hence $D = p' - p$.

[8] Hence $p' = p + D$.

Substituting this value of $p'$ into [4] we get

[9] $$x \cos \alpha + y \sin \alpha - p - D = 0,$$

for the equation of A'B'.

Now, since the point M is on the line A'B', its coordinates $x'$ and $y'$ must satisfy the equation of A'B'.

Hence substituting $x'$ and $y'$ into [9] we get

[10] $$x' \cos \alpha + y' \sin \alpha - p - D = 0.$$

[11] Hence $D = x' \cos \alpha + y' \sin \alpha - p$.

Q. E. D.

Now by §69 the direction from O to K is the positive direction. Hence from [5] and [6] we see that D is positive. Hence we have the following corollary:

71. *Corollary 1.—When D is positive, the point from which the perpendicular is drawn is on the opposite side of AB from the origin; when D is negative, the point from which the perpendicular is drawn is on the same side of AB as the origin.*

72. *Corollary 2.—Or, when D is positive, the direction from the line AB to the point from which the perpendicular is drawn, is the same as the direction from the origin to the line AB; and when D is negative, the direction from AB to the point from which the perpendicular is drawn is opposite to the direction from the origin to AB.*

73. *Corollary 3.—The distance from the point whose coordinates are $x'$, $y'$, to the line $Ax + By + C = 0$ is*

$$D = \pm \frac{Ax' + By' + C}{\sqrt{A^2 + B^2}}.$$

For by [9], [12] and [17] of § 68

$$\cos \alpha = -\frac{A}{\sqrt{A^2 + B^2}}.$$

$$\sin \alpha = -\frac{B}{\sqrt{A^2 + B^2}}.$$

$$p = \frac{C}{\sqrt{A^2 + B^2}}.$$

Substituting these values of $\cos\alpha$, $\sin\alpha$, $p$ into [11] we get

$$[12]\quad D = -\left(\frac{Ax'}{\sqrt{A^2+B^2}} + \frac{By'}{\sqrt{A^2+B^2}} + \frac{C}{\sqrt{A^2+B^2}}\right).$$

$$[13]\quad D = -\frac{Ax' + By' + C}{\sqrt{A^2+B^2}}.$$

But the direction in which D is drawn from the line AB may be the same as the direction of $p$, or the opposite. Hence D may be positive or negative. Therefore [13] may be written

$$D = \pm\frac{Ax' + By' + C}{\sqrt{A^2+B^2}}.$$

EXAMPLES

1. Find the distance from the point 3, 2 to the line

$$3x + 4y - 5 = 0.$$

Draw a figure showing the line, point, and distance.

Ans. $D = 2\frac{2}{5}$.

2. Find the distance from the point $-2$, 5 to the line

$$4x - 3y + 8 = 0.$$

Draw a figure showing the line, point, and distance.

Ans. $D = -3$.

3. Find the distance from the point $-3$, $-3$ to the line

$$3y + 11 = -4x.$$

Draw a figure showing the line, point, and distance.

Ans. $D = -2$.

4. Find the distance from the point 1, $-1$ to the line

$$3x - 12 = 4y.$$

Draw a figure showing the line, point, and distance.

Ans. $D = -1$.

# CHAPTER VI

## Oblique Axes

74. Hitherto we have drawn the axes of coordinates perpendicular to each other. It is often more convenient, however, to take for the axes two lines which are oblique to each other.

When the axes are perpendicular to each other they are called Rectangular Axes, and the coordinates of a point are called Rectangular Coordinates.

When the axes are oblique to each other they are called Oblique Axes, and the coordinates of a point are called Oblique Coordinates.

In both cases the axes are called Rectilinear Axes and the coordinates of a point are called Rectilinear Coordinates.

### PROPOSITION X

75. *When the axes are oblique the equation of a straight line is*

$$y = \frac{\sin I}{\sin(\omega - I)} x + b,$$

*in which b is the Y intercept, I the inclination of the line, and ω the angle between the axes.*

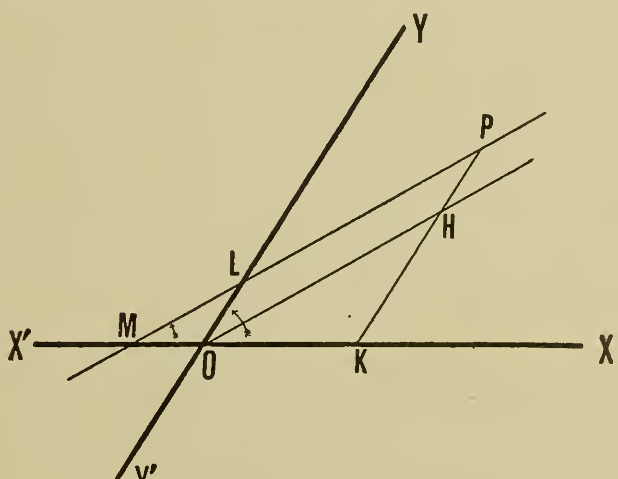



Fig. 24

Let YY′ be the Y axis, and XX′ the X axis,

and $\omega \equiv \angle YOX$.

Let PM be any line and let P be any point on that line.

Draw PK ∥ YY′.

Let $x \equiv OK$, $y \equiv PK$, $b \equiv OL$, and $I \equiv \angle LMO$.

We are to prove that the equation of PM is

$$y = \frac{\sin I}{\sin(\omega - I)} x + b.$$

Draw OH parallel to PM.

[1] Then $\frac{HK}{OK} = \frac{\sin HOK}{\sin OHK}$. by Trig. 14.

[2] But $\angle OHK = \angle LOH$, by Geom. 7.

[3] and $\angle LOH = \angle LOK - \angle HOK$.

[4] Hence $\angle OHK = \angle LOK - \angle HOK$.

Substituting this value of $\angle OHK$ into [1] we get

[5] $\frac{HK}{OK} = \frac{\sin HOK}{\sin (LOK - HOK)}$.

[6] $\angle HOK = \angle LMO \equiv I$, by Geom. 8.

[7] and $\angle LOK - \angle HOK = \omega - I$.

[8] Hence $\frac{HK}{OK} = \frac{\sin I}{\sin(\omega - I)}$,

[9] and $HK = \frac{\sin I}{\sin(\omega - I)} OK$.

[10] Now $PH = OL$. by Geom. 17.

Adding [9] and [10] we get

[11] $PK = \frac{\sin I}{\sin(\omega - I)} OK + OL$,

[12] or $y = \frac{\sin I}{\sin(\omega - I)} x + b$.

Q. E. D.

76. *Corollary 1.— The equation of a straight line through the origin is* $y = \frac{\sin I}{\sin(\omega - I)} x$.

76*a*. *Corollary 2.— The equation of a line parallel to the Y axis is* $x = c =$ *any const.*

77. *Scholium.*—If the axes be made rectangular, then $\omega = 90°$ and

$$\frac{\sin I}{\sin(\omega - I)} = \frac{\sin I}{\sin(90° - I)} = \frac{\sin I}{\cos I} = \tan I.$$

But by § 52 $\tan I$ is the slope of the line PM, which, as in § 53, we represent by $s$. Hence when the axes are rectangular

$$\frac{\sin I}{\sin(\omega - I)} = s,$$

and [12] becomes

$$y = sx + b,$$

as in § 53.

PROPOSITION XI

78. *When the axes are oblique the equation of a straight line passing through a fixed point is*

$$y' - y = \frac{\sin I}{\sin(\omega - I)}(x' - x),$$

*in which* $x'$ *and* $y'$ *are the coordinates of the fixed point,* $I$ *the inclination of the line and* $\omega$ *the angle between the axes.*

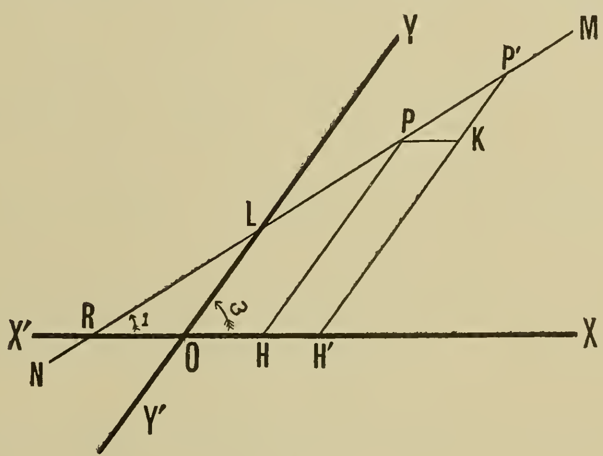



Fig. 25

Let P′ be any fixed point and MN any straight line passing through that point.

Let P be any point on MN and draw PH and P'H' ‖ YY' and PK ‖ XX'.

Let $x \equiv$ OH and $y \equiv$ PH,
$x' \equiv$ OH' " $y' \equiv$ P'H',
and $I \equiv$ PRH " $\omega \equiv$ YOX.

We are to prove that the equation of MN is

$$y' - y = \frac{\sin I}{\sin(\omega - I)}(x' - x).$$

[1] $\angle$P'PK $= \angle$PRH $= I$. by Geom. 8.

[2] $\angle$RLO $= \angle$YOX $- \angle$LRO. by Geom. 13.

[3] But $\angle$RLO $= \angle$PP'K. by Geom. 8.

[4] Hence $\angle$PP'K $= \angle$YOX $- \angle$LRO $= \omega - I$.

[5] $\dfrac{\text{P'K}}{\text{PK}} = \dfrac{\sin \text{P'PK}}{\sin \text{PP'K}} = \dfrac{\sin I}{\sin(\omega - I)}$, by Trig. 14.

[6] hence $\text{P'K} = \dfrac{\sin I}{\sin(\omega - I)}\text{PK}$.

[7] But P'K $=$ P'H' $-$ PH $= y' - y$,

[8] and PK $=$ OH' $-$ OH $= x' - x$. by Geom. 17.

[9] Hence $y' - y = \dfrac{\sin I}{\sin(\omega - I)}(x' - x)$. by [6].

Q. E. D.

79. *Scholium.*—If the axes be made rectangular, then $\omega = 90°$ and

$$\frac{\sin I}{\sin(\omega - I)} = \frac{\sin I}{\sin(90° - I)} = \frac{\sin I}{\cos I} = \tan I.$$

But $\tan I$ is the slope of the line MN, which, as in § 53, we will represent by $s$. Hence when the axes are rectangular

$$\frac{\sin I}{\sin(\omega - I)} = s,$$

and [9] becomes

$$y' - y = s(x' - x),$$

as in § 57.

## PROPOSITION XII

80. *When the axes are oblique the equation of a straight line passing through two given points is*

$$y' - y = \frac{y'' - y'}{x'' - x'}(x' - x),$$

*in which $x'$ and $y'$ are the coordinates of one of the given points and $x''$ and $y''$ are the coordinates of the other.*

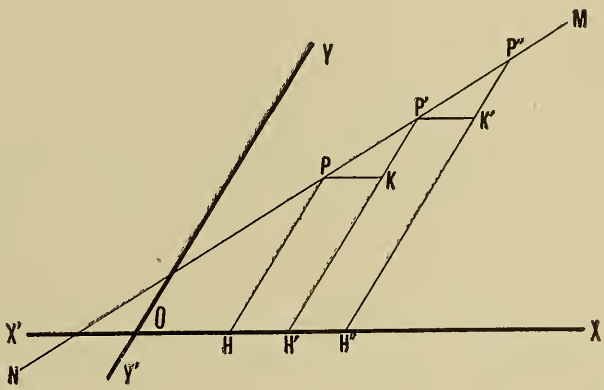



Fig. 26

Let P′ and P″ be the two given points, and MN a line passing through them.

Let P be any other point on the line MN and draw the coordinates of P, P′ and P″.

$$\text{Let } x \equiv OH \text{ and } y \equiv PH,$$
$$x' \equiv OH' \text{ '' } y' \equiv P'H',$$
$$x'' \equiv OH'' \text{ '' } y'' \equiv P''H''.$$

Draw PK and P′K′ ∥ XX′.

The demonstration is the same as in § 58.

### EXERCISES

1. Prove that the lines $y = x + 1$, $y = 2x + 2$, $y = 3x + 3$ intersect in the point $(-1, 0)$.

2. Find the angle which the lines $3x - y = 5$, $2y - x = 8$ make with each other. Ans. 45°.

3. Find the equations of the lines passing through the point $(3, -1)$ and making angles of $45°$ with $2y + 3x = 6$.

Ans. $5y + x + 2 = 0$.
$y - 5x + 16 = 0$.

4. Find the equation of the line passing through $(-3, 5)$ perpendicular to $3y - 2x - 2 = 0$. Ans. $2y + 3x - 1 = 0$.

5. The vertices of a triangle are $(2, 3)$, $(4, -5)$, $(-3, -6)$. What are the equations of its sides?

Ans. $x - 7y = 39$.
$9x - 5y = 3$.
$4x + y = 11$.

6. Let the sides of a triangle be given by $y = x + 1$, $x = 4$, $y = -x - 1$. What are the equations of the sides of the triangle formed by lines joining the middle points of the sides of the given triangle? Ans. $y = -x + 4$.
$y = x - 4$.
$2x = 3$.

7. Required the equation of a straight line passing through the origin and the intersection of the lines $3x - 2y + 4 = 0$ and $3x + 4y = 5$. Also find the distance between the two points. Ans. $9x + 2y = 0$.
$\frac{1}{6}\sqrt{85}$.

8. A parallelogram is formed by the lines

$$x = a;\quad x = b;\quad y = c;\quad y = d;$$

what are the equations of its diagonals?

Ans. $(d - c)x - (b - a)y = ad - bc$.
$(d - c)x + (b - a)y = bd - ac$.

9. Find the value of $s$ provided the line $y = sx$ passes through $(1, 4)$. Ans. $s = 4$.

10. Required the equation of a straight line perpendicular to $Ax + By + C = 0$ and making an intercept $b$ on the Y axis.

Ans. $Bx = A(y - b)$.

11. What is the equation of the line perpendicular to $\frac{x}{a} + \frac{y}{b} = 1$ and passing through $(a, b)$?

Ans. $ax - by = a^2 - b^2$.

12. Required the equations of the perpendiculars erected at the middle points of the sides of the triangle whose vertices are $(5, -7)$, $(1, 11)$, $(-4, 13)$. Prove that these perpendiculars meet in a point.

13. The points $(1, 2)$, $(2, 3)$ being equidistant from the point $(x, y)$, write the equation which expresses that fact.

Ans. $(x-1)^2 + (y-2)^2 = (x-2)^2 + (y-3)^2$,

or $x + y = 4$.

14. Find the distance between the points $(x'', y'')$, $(x', y')$, the axes making the angle $\varphi$ with each other.

Ans. $D = \sqrt{(x''-x')^2 + (y''-y')^2 + 2(x''-x')(y''-y')\cos\varphi}$.

15. What is the length of the perpendicular from the origin to the line $\frac{x}{3} + \frac{y}{4} = 1$?

16. Find the point equidistant from the three lines

$$4x + 3y - 7 = 0, \quad 5x + 12y - 20 = 0, \quad 3x + 4y - 8 = 0,$$

and its distance from each. Ans. $(2, 3)$; $2$.

17. Two lines are given, one passing through the points $(1, 2)$, $(-4, -3)$, the other through $(1, -3)$, and making an angle of $45°$ with the first; what are the equations of the lines? Ans. $y = x + 1$ and $y = -3$.

# CHAPTER VII

## Transformation to New Axes

### PROPOSITION XIII

81. *If we have given the equation of a locus referred to any pair of axes, we can find the equation of the same locus referred to any pair of axes parallel to the first by putting*

$$m + x' \text{ for } x,$$
$$\text{and} \quad n + y' \text{ for } y,$$

*in the original equation; m and n being the coordinates of the new origin, and x' and y' the coordinates of any point on the locus referred to the new axes.*

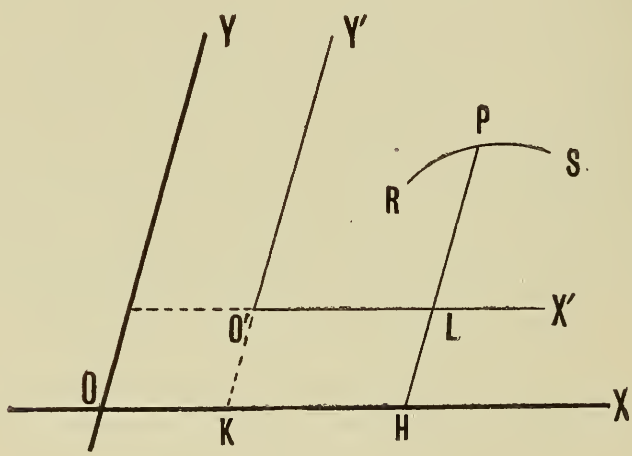



Fig 27

In Fig. 27 let OY and OX be the original axes, and O′Y′ and O′X′ be the new axes.

Let RS be any locus and P any point on it.

Draw PH parallel to OY. Then OH and PH will be the coordinates of the point P referred to the original axes, and O′L and PL will be its coordinates referred to the new axes.

$$\text{Let } m \equiv OK \text{ and } n \equiv O'K,$$
$$x' \equiv O'L \quad \text{“} \quad y' \equiv PL.$$

[1] Let $f(x, y) = 0,$

be the equation of RS referred to the original axes. Then

$$x = \text{OH and } y = \text{PH}.$$

We are to prove that we can find the equation of RS referred to O′X′ and O′Y′ by putting

$m + x'$ for $x$

and $n + y'$ for $y$

into [1].

[2] Now $\text{OH} = \text{OK} + \text{O'L},$ by Geom. 17.

[3] or $x = m + x',$

[4] and $\text{PH} = \text{PL} + \text{O'K}.$ by Geom. 17.

[5] Hence $y = n + y'.$

If we substitute these values of $x$ and $y$ into [1] we get

[6] $f(m + x', n + y') = 0.$

In this equation $m$ and $n$ are constants, but $x'$ and $y'$ are the coordinates of any point on RS. Hence [6] is the equation of RS referred to the new axes. by § 39.

Q. E. D.

## PROPOSITION XIV

82. *If we have given the equation of any locus referred to any pair of axes, we can find the equation of the same locus referred to any other pair of axes having the same origin by putting*

$$\frac{x' \sin(\theta - \alpha) + y' \sin(\theta - \beta)}{\sin \theta} \quad \textit{for} \quad x,$$

*and*

$$\frac{x' \sin \alpha + y' \sin \beta}{\sin \theta} \quad \textit{for} \quad y,$$

*into the equation of the locus; $x'$ and $y'$ being the coordinates of any point on the locus referred to the new axes, $\theta$ the angle between the original axes, $\alpha$ the inclination of the new axis of $X$, and $\beta$ the inclination of the new axis of $Y$.*

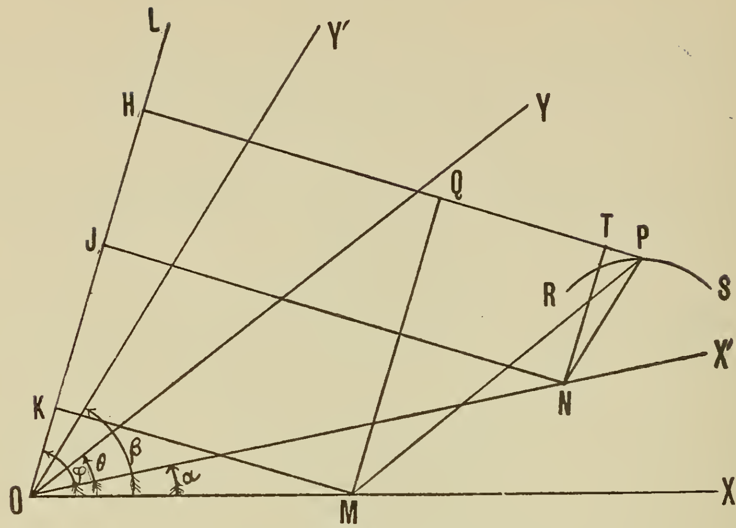



Fig. 28

In Fig. 28 let RS be any curve and P any point on it.

Let OY and OX be the original axes and OY′ and OX′ the new axes.

Draw PM ∥ OY and PN ∥ OY′. Then OM and PM are the coordinates of P referred to the original axes, and ON and PN its coordinates referred to the new axes.

$$\text{Let } x' \equiv \text{ON and } y' \equiv \text{PN}.$$

[1] Let $$f(x, y) = 0,$$

be the equation of RS referred to the original axes. Then

$$x \equiv \text{OM and } y \equiv \text{PM}.$$

$$\text{Let } \alpha \equiv \text{X}'\text{OX},$$
$$\beta \equiv \text{Y}'\text{OX},$$
$$\theta \equiv \text{YOX}.$$

We are to prove that we can find the equation of RS referred to OX′ and OY′ by putting

$$\frac{x' \sin(\theta - \alpha) + y' \sin(\theta - \beta)}{\sin \theta} \quad \text{for } x,$$

and $$\frac{x' \sin \alpha + y' \sin \beta}{\sin \theta} \quad \text{for } y.$$

into [1].

Draw any line OL through the origin and let

$$\varphi \equiv \angle \text{LOX}.$$

Draw PH, NJ and MK $\perp$ OL.

Draw NT and MQ $\parallel$ OL.

[2] Now $\text{OK} = \text{OM} \cos \varphi$. by Trig. 2.

[3] Hence $\text{OK} = x \cos \varphi$;

[4] also $\text{MQ} = \text{PM} \cos \text{PMQ}$, by Trig. 2.

[5] but $\text{MQ} = \text{KH}$, by Geom. 17.

[6] and $\angle\text{PMQ} = \angle\text{YOH} = \varphi - \theta$. by Geom. 11.

Substituting these values of MQ and PMQ into [4] we get

[7] $\text{KH} = y \cos (\varphi - \theta)$.

Adding [3] and [7], member to member, we get

[8] $\text{OK} + \text{KH} = x \cos \varphi + y \cos (\varphi - \theta)$,

[9] or $\text{OH} = x \cos \varphi + y \cos (\varphi - \theta)$.

[10] Similarly $\text{OJ} = \text{ON} \cos (\varphi - \alpha) = x' \cos (\varphi - \alpha)$,

[11] and $\text{JH} = \text{NT} = \text{PN} \cos \text{PNT} = y' \cos (\varphi - \beta)$.

Adding [10] and [11], member to member, we get

[12] $\text{OH} = x' \cos (\varphi - \alpha) + y' \cos (\varphi - \beta)$.

Hence from [9] and [12] we get

[13]

$$x \cos \varphi + y \cos (\varphi - \theta) = x' \cos (\varphi - \alpha) + y' \cos (\varphi - \beta).$$

Since OL is any line drawn through O, [13] must be true whatever be the angle $\varphi$. Hence we may draw OL so that

[14] $\varphi = \dfrac{\pi}{2} + \theta$, see § 42, Cor.

[15] then $\varphi - \theta = \dfrac{\pi}{2}$.

[16] Hence $\cos \varphi = -\sin \theta$, by Trig. 21.

[17] and $\cos(\varphi - \theta) = \cos\frac{\pi}{2} = 0.$ by Trig. 19.

Again by [14]

[18] $\cos(\varphi - \alpha) = \cos(\frac{\pi}{2} + \overline{\theta - \alpha}) = -\sin(\theta - \alpha).$

by Trig. 21.

[18′] Similarly $\cos(\varphi - \beta) = -\sin(\theta - \beta).$

Hence by [16], [17], [18] and [18′], [13] becomes

$$-x \sin\theta = -x' \sin(\theta - \alpha) - y' \sin(\theta - \beta).$$

[19] $$x = \frac{x' \sin(\theta - \alpha) + y' \sin(\theta - \beta)}{\sin\theta}.$$

Again, since [13] must be true whatever be the angle $\varphi$, we may draw OL so that

$$\varphi = \frac{\pi}{2}.$$ see § 42, Cor.

Then [13] becomes

$$y \sin\theta = x' \sin\alpha + y' \sin\beta,$$ by Trig. 19 and 20.

[20] and $$y = \frac{x' \sin\alpha + y' \sin\beta}{\sin\theta}.$$

Substituting the values of $x$ and $y$ found in [19] and [20] into [1] we get

[21]
$$f\left(\frac{x' \sin(\theta - \alpha) + y' \sin(\theta - \beta)}{\sin\theta}, \frac{x' \sin\alpha + y' \sin\beta}{\sin\theta}\right) = 0.$$

Now in [21] $\alpha$, $\beta$ and $\theta$ are constants, but $x'$ and $y'$ are the coordinates of any point on RS. Hence [21] is the equation of RS referred to the new axes. by § 39.

Q. E. D.

83. *Cor.—If we have given the equation of any locus referred to any pair of rectangular axes, we can find the equation of the same locus referred to any other pair of rectangular axes having the same origin by putting*

$$x' \cos\alpha + y' \sin\alpha \quad \textit{for} \quad x,$$

*and* $\qquad x' \sin \alpha + y' \cos \alpha$ *for* $y$,

*into the equation of the locus.*

For, if the axes are rectangular, then in Fig. 28

[22] $$\theta = \frac{\pi}{2}.$$

[23] Hence $$\frac{\pi}{2} - \angle YOX' = \alpha.$$

[24] Also $$\angle Y'OX' = \frac{\pi}{2}.$$

[25] Hence $$\frac{\pi}{2} - \angle YOX' = \angle Y'OY,$$

[26] and by [23] $$\angle Y'OY = \alpha.$$

Now by Fig. 28

$$\beta = \theta + \angle Y'OY.$$

[27] Hence by [22] and [26] $$\beta = \frac{\pi}{2} + \alpha.$$

Hence since $$\theta = \frac{\pi}{2},$$

and $$\beta = \frac{\pi}{2} + \alpha,$$

equations [19] and [20] become

[28] $$x = x' \cos \alpha - y' \sin \alpha,$$

[29] $$y = x' \sin \alpha + y' \cos \alpha.$$

Substituting these values of $x$ and $y$ into [1], we get

[30] $$f(x' \cos \alpha - y' \sin \alpha,\ x' \sin \alpha + y' \cos \alpha) = 0.$$

Since $x'$ and $y'$ are the coordinates of any point on RS, [30] must be the equation of RS referred to the new axes.

Q. E. D.

PROPOSITION XV

84. *If we have given the equation of a curve referred to any pair of axes, we may find the equation of the same curve referred to any other pair of axes by putting*

$$m + \frac{x'' \sin(\varphi - \alpha) + y'' \sin(\varphi - \beta)}{\sin \varphi} \quad \textit{for } x,$$

*and*

$$n + \frac{x'' \sin \alpha + y'' \sin \beta}{\sin \varphi} \quad \textit{for } y,$$

*into the equation of the curve; m and n being the coordinates of the new origin, and x'' and y'' the coordinates of any point on the curve referred to the new axes, α and β the inclinations of the new axis of X and axis of Y respectively to the original axis of X, and φ the angle between the original axes.*

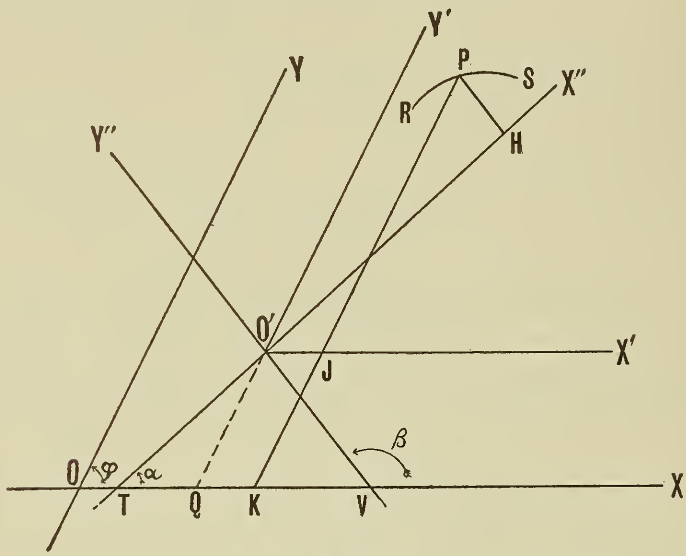



Fig. 29

In Fig. 29 let OX and OY be the original axes, and let O'X'' and O'Y'' be the new axes.

Let OK and PK be the coordinates of P referred to the original axes.

[1] Let $f(x, y) = 0$

be the equation of any curve RS referred to the original axes, and P be any point on that curve.

Then $x \equiv OK$ and $y \equiv PK$.

Let O'H and PH be the coordinates of P referred to the new axes, and let $x'' \equiv O'H$ and $y'' \equiv PH$

The coordinates of the new origin are OQ and O'Q.

Let $m \equiv$ OQ and $n \equiv$ O'Q.

" $\alpha \equiv \angle$ X''TX,

" $\beta \equiv \angle$ Y''VX,

" $\varphi \equiv \angle$ YOX.

We are to prove that the equation of RS referred to the new axes is

$$f\left(m + \frac{x'' \sin(\varphi - \alpha) + y'' \sin(\varphi - \beta)}{\sin \varphi},\right.$$

$$\left. n + \frac{x'' \sin \alpha + y'' \sin \beta}{\sin \varphi}\right) = 0.$$

Through O' draw O'X' ‖ OX and O'Y' ‖ OY. The coordinates of P referred to O'X' and O'Y' will be

$$x' = \text{O'J and } y' = \text{PJ}.$$

[2] Then $x = m + x'$, by § 81.

[3] and $y = n + y'$. by § 81.

[4] But $x' = \dfrac{x'' \sin(\varphi - \alpha) + y'' \sin(\varphi - \beta)}{\sin \varphi}$, by § 82.

[5] and $y' = \dfrac{x'' \sin \alpha + y'' \sin \beta)}{\sin \varphi}$. by § 82.

Substituting these values of $x'$ and $y'$ into [2] and [3] we get

[6] $x = m + \dfrac{x'' \sin(\varphi - \alpha) + y'' \sin(\varphi - \beta)}{\sin \varphi}$.

[7] $y = n + \dfrac{x'' \sin \alpha + y'' \sin \beta}{\sin \varphi.}$.

Substituting these values of $x$ and $y$ into [1] we get

[8] $$f\left(m + \frac{x'' \sin(\varphi - \alpha) + y'' \sin(\varphi - \beta)}{\sin \varphi},\right.$$

$$\left. n + \frac{x'' \sin \alpha + y'' \sin \beta}{\sin \varphi}\right) = 0.$$

Now in [8] the $x''$ and $y''$ are the coordinates of any point on the curve RS referred to the new axes, but all the other quantities in the equation are constants. Hence [8] is the equation of RS referred to the new axes. by Trig. 39.

EXAMPLES

1. Required the equation of the line $y = 3x + 1$ when we remove the origin to (2, 3), the axes remaining parallel to themselves. Ans. $y = 3x + 4$.

2. Given the equation $y^2 + x^2 - 4x + 4y - 8 = 0$; required its form when the origin is at (2, — 2) Ans. $x^2 + y^2 = 16$.

3. If we turn the axes through an angle $= 45°$, what does the equation $x^2 + y^2 = a^2$ become?

4. What does the equation $x^2 + y^2 - 4x - 6y = 18$ become when the origin is at the point (2, 3), the axes still being rectangular?

5. What do the following equations become when the origin is changed to the point given?

$x + y + 2 = 0$; the new origin (— 2, 0). Ans. $x + y = 0$.

$3x^2 + 4xy + y^2 - 5x - 6y - 3 = 0$; $(\frac{7}{2}, -4)$.

Ans. $12x^2 + 16xy + 4y^2 = 1$.

$x^2 - 6xy + y^2 - 6x + 2y + 1 = 0$; (0, 1).

6. Show that the degree of an equation is not altered by a transformation of coordinates.

# CHAPTER XVIII

## Polar Coordinates

85. The position of a point may be indicated by giving its distance and direction from a given fixed point in a given fixed straight line.

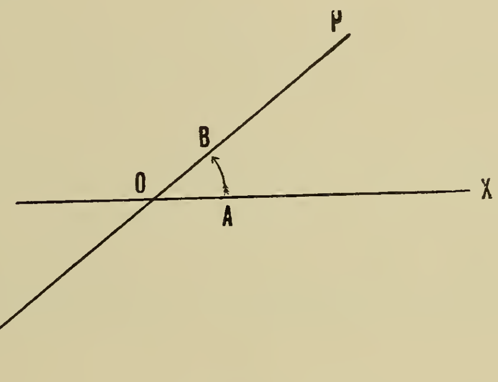



Fig. 30

For instance, if OX be a given fixed straight line, and O a given fixed point in it, then we may indicate the position of any point P by giving the angle, which a line drawn from O through P makes with OX, and also the distance along this line from O to P.

86. **The Initial Line.**—The given fixed straight line is called the *initial line.*

87. **The Pole.**—The given fixed point is called the *pole.*

88. **The Radius Vector.**—A straight line drawn from the *pole* to any point is called the *radius vector* of that point.

OP is the *radius vector* of the point P.

89. **The Vectorial Angle.**—The angle between the *initial line* and the *radius vector* is called the *vectorial angle.*

POX is the *vectorial angle.*

90. **The Vectorial Arc.**—The *vectorial angle* is measured by an arc whose center is the *pole*, and which lies between

the *initial line* and the *radius vector.* This is called the *vectorial arc.*

AB is the *vectorial arc.*

91. The *vectorial arc* is considered positive when it is generated by a point revolving about the pole, from the *initial line*, counter clockwise.

92. The *vectorial angle* is considered negative when it is generated by a point revolving about the pole, from the *initial line*, clockwise.

93. The *vectorial angle* is said to be positive when the *vectorial arc* is positive, and negative when the *vectorial arc* is negative.

94. The *radius vector* is positive when it is drawn from the *pole* towards the terminus of the *vectorial arc.*

95. The *radius vector* is negative when it is drawn from the *pole* in the direction opposite to that of the terminus of the *vectorial arc.*

In Fig. 30 OP is positive.

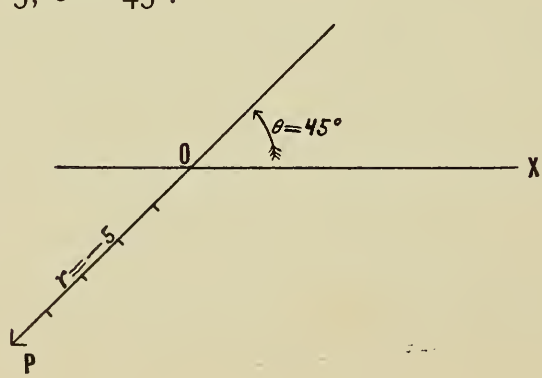
96. We may indicate the position of the point P′ by giving the angle BOA and the distance OP′. In that case OP′ is negative.

The *radius vector* is denoted by $r$.
The *vectorial angle* is denoted by $\theta$.

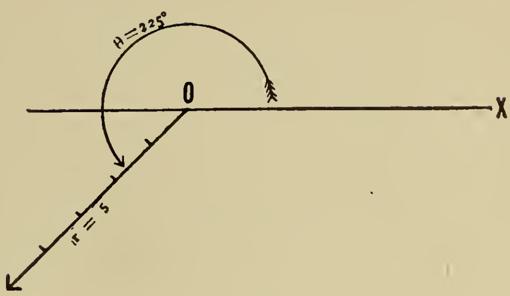
EXAMPLES

Locate the points whose coordinates are given in the following examples:

1. $r = -5$, $\theta = 45°$.



2. $r = 5$, $\theta = 225°$.




3. $r = 10$, $\theta = 135°$.

4. $r = -10$, $\theta = 135°$.

What is the position of the pole with respect to the two points in Examples 3 and 4 ?

5. $r = 10$, $\theta = 315°$.

What is the position of this point with respect to that in Example 4 ?

## PROPOSITION XVI

97. *The polar equation of a straight line is*

$$\rho = \frac{p}{\cos(\theta - \alpha)},$$

*in which $p$ represents the perpendicular from the pole to the line, $\alpha$ the inclination of this perpendicular, $\rho$ the radius vector of any point on the line, and $\theta$ the vectorial angle.*

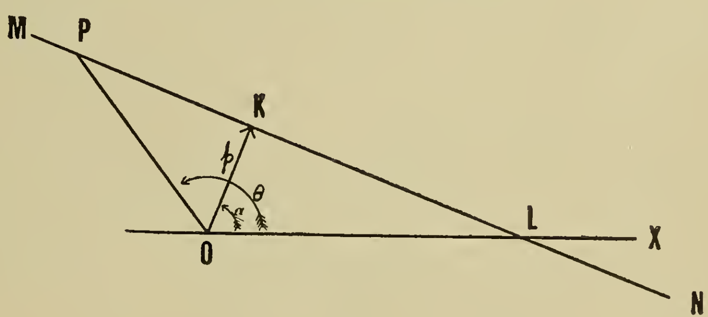



Fig. 31

Let OX be the initial line and O the pole.

Let MN be any straight line, OK the perpendicular drawn from the pole to this line, and P any point on the line MN.

Let $\alpha \equiv$ KOL and $p \equiv$ OK,
$\theta \equiv$ POX " $\rho \equiv$ OP.

We are to prove that

$$\rho = \frac{p}{\cos(\theta - \alpha)}.$$

[1] $$\cos POK = \frac{OK}{OP},$$ by Trig. 2.

[2] or $$\cos(\theta - \alpha) = \frac{p}{\rho}.$$

[3] Hence $$\rho = \frac{p}{\cos(\theta - \alpha)}.$$

Q. E. D.

98. *Corollary 1.—The equation of a line perpendicular to the initial line is*

$$\rho = \frac{p}{\cos \theta}.$$

For if MN be ⊥ to OX, OK must coincide with OX. by Geom. 5.

[1] Hence $\alpha = 0$,

and [3] will become

[2] $$\rho = \frac{p}{\cos \theta}.$$

99. *Corollary 2.—To find where a straight line cuts the initial line make $\theta = 0$, and then $\rho$ will be the distance from the pole to the cutting point.*

For, since in Fig. 31 P represents any point on the line MN, it may represent the point L.

Then $\theta = 0$, and by [3], § 97

[1] $$\rho = \frac{p}{\cos(-\alpha)} = \frac{p}{\cos \alpha}.$$ by Trig. 12.

[2] But $$OL = \frac{OK}{\cos KOL} = \frac{p}{\cos \alpha}.$$ by Trig. 2.

[3] Hence $\rho = OL$.

EXAMPLES

1. The perpendicular from the pole to a straight line is 5 inches long and makes an angle of 60° with the initial line. What is the equation of the line? Where does it cut the initial line?

$$\text{Here } \alpha = 60° \text{ and } p = 5.$$

Substituting these values into [3] of § 97, we get

$$\rho = \frac{5}{\cos(\theta - 60°)},$$

which is the equation of the line.

To find where the line cuts the initial line:

$$\text{Let } \theta = 0. \qquad \text{by § 99.}$$

$$\text{Then} \qquad \rho = \frac{5}{\cos(-60°)} = \frac{5}{\frac{1}{2}} = 10. \qquad \text{by Trig. 12.}$$

2. The perpendicular from the pole upon a straight line is 10 inches long, and makes an angle of 240° with the initial line. Where does it cut the initial line?

$$\text{Here } \alpha = 240° \text{ and } p = 10.$$

Substituting these values into [3] of § 97, we get

$$\rho = \frac{10}{\cos(\theta - 240°)},$$

which is the equation of the line.

To find where it cuts the initial line:

$$\text{Let } \theta = 0.$$

$$\text{Then} \qquad \rho = \frac{10}{\cos(-240°)} = \frac{10}{\cos 240°},$$

$$\text{and} \qquad \rho = \frac{10}{-\frac{1}{2}} = -20.$$

3. Show that the equation of transformation from a rectangular to a polar system of coordinates, the origin, and pole being non-coincident, are

$$x = a + r\cos(\theta + \varphi),$$
$$y = b + r\sin(\theta + \varphi),$$

where the origin is $(a, b)$, the $\angle \varphi$ is the $\angle$ of the initial line with the X axis, and $\theta$ is the vectorial angle.

# CHAPTER IX

## The Ellipse

100. The Ellipse.—An *Ellipse* is the locus of a point moving in a plane in such a way that the sum of its distances from two fixed points in the plane is constant.

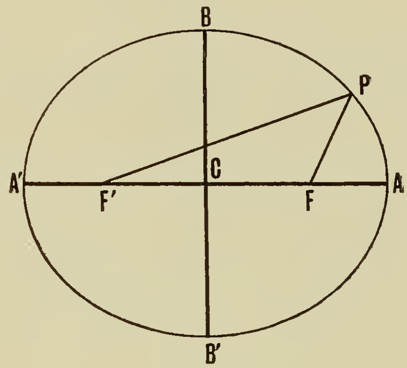



Fig. 32

Let F and F' be the two fixed points in the plane.

Let P be a point moving in this plane in such a way that PF + PF' is constant.

Then the locus ABA'B' traced out by P is an ellipse.

PROBLEM

101. To draw an ellipse.

Let F and F' in Fig. 32 represent two fixed pins. Take an inelastic thread longer than the distance FF'. Fasten an end of it to each pin. Press the point of a pencil against the thread so as to stretch it. Then if, in the figure, P represent the point of a pencil, F'PF will represent the thread.

Now move the pencil so as always to keep the thread stretched. Then since the thread is inelastic and is always stretched, the sum of the distances PF and PF', from the pencil point to the fixed pins, is constant. Hence the locus traced out by the pencil point must be an ellipse by § 100.

Q. E. D.

102. *Corollary.—It is obvious that the ellipse is a closed curve, and that it will cut the straight line drawn through the two fixed points F and F' in two points as A and A'.*

103. **The Foci.**—The two fixed points F and F′ are called the *foci.*

104. **The Focal Radii.**—The distances from the foci to any point on the ellipse are called the *focal radii* of that point.

FP and F′P are the *focal radii* of the point P.

105. **The Vertices.**—The points in which the ellipse cuts the straight line which passes through the foci are called the *vertices* of the ellipse.

A and A′ are *vertices.*

106. **The Transverse Axis.**—The line which joins the vertices is called the *transverse axis* of the ellipse.

AA′ is the *transverse axis.*

107. **The Center.**—The middle of the transverse axis is called the *center* of the ellipse.

108. **The Conjugate Axis.**—The *conjugate axis* is a straight line drawn through the center perpendicular to the transverse axis and terminated both ways by the ellipse.

BB′ is the *conjugate axis.*

PROPOSITION I

109. *The sum of the focal radii of any point on an ellipse is equal to the transverse axes.*

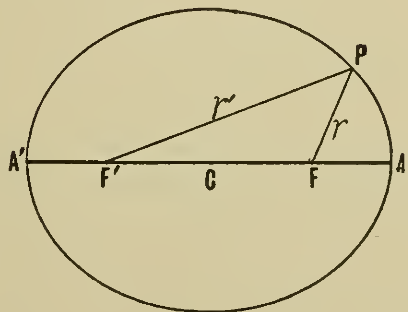



Fig. 33

Let P be any point on the ellipse, $r$ and $r'$ its focal radii, and AA′ its transverse axis.

We are to prove that

$$r' + r = AA'.$$

When the moving point P reaches A

[1] $r' + r = F'A + AF$. by § 101.

When the moving point reaches A′

[2] $r' + r = F'A' + FA'$.

[3] Hence $F'A + AF = FA' + A'F'$,

[4] or $FF' + AF + AF = FF' + A'F' + F'A'$.

[5] $FF' = FF'$.

By subtracting we get

[6] $2AF = 2A'F'$.

[7] Hence $AF = A'F'$.

Substituting A′F′ for AF in [1] we get

[8] $r' + r = F'A + A'F' = AA'$.

Q. E. D.

110. *Corollary.—The foci of an ellipse are equally distant from the center.*

For

[1] $CA = CA'$, by § 107.

[2] and $AF = A'F'$, by § 109, [7].

[3] hence $CA - AF = CA' - A'F'$,

[4] or $CF = CF'$.

PROPOSITION II

111. *The equation of the ellipse is*

$$a^2y^2 + b^2x^2 = a^2b^2,$$

*in which $a$ represents the semi-transverse axis, $b$ the semi-conjugate axis, and $x$ and $y$ represent the coordinates of any point on the ellipse.*

Let ABA′B′ be an ellipse, AA′ its transverse axis, BB′ its conjugate axis, and C its center.

Let $a \equiv CA$ and $b \equiv CB$.

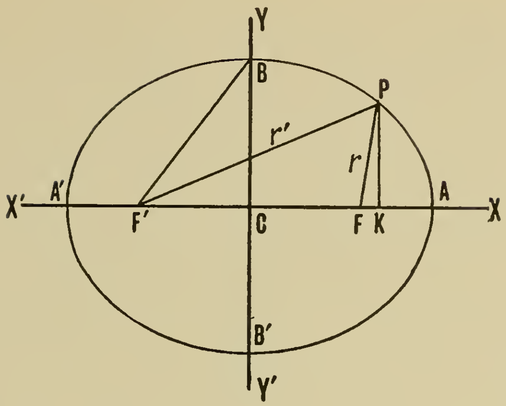



Fig. 34

Let P be any point on the ellipse and draw its ordinate PK.

Let $x \equiv \mathrm{CK}$ and $y \equiv \mathrm{PK}$.

We are to prove that

$$a^2y^2 + b^2x^2 = a^2b^2$$

is the equation of the ellipse.

Let F and F′ be the foci, and draw the focal radii FP and F′P.

Let $r \equiv \mathrm{FP}$ and $r' \equiv \mathrm{F'P}$.

Let $c \equiv \mathrm{CF}$.

[1] Then $\mathrm{CF'} = c,$ by § 110.

[2] $r^2 = \overline{\mathrm{FK}}^2 + \overline{\mathrm{PK}}^2,$ by Geom. 26.

[3] but $\mathrm{FK} = x - c,$

[4] hence $r^2 = (x - c)^2 + y^2,$

[5] and $r = \sqrt{(x - c)^2 + y^2}.$

[6] Again $r'^2 = \overline{\mathrm{F'K}}^2 + \overline{\mathrm{PK}}^2,$ by Geom. 26.

[7] but $\mathrm{F'K} = x + c.$

[8] Hence $r'^2 = (x + c)^2 + y^2,$

[9] and $r' = \sqrt{(x + c)^2 + y^2}.$

[10] Now $r' + r = \mathrm{AA'} = 2a,$ by § 109.

[11] hence $\sqrt{(x - c)^2 + y^2} + \sqrt{(x + c)^2 + y^2} = 2a.$

[12] $\sqrt{(x-c)^2+y^2} = 2a - \sqrt{(x+c)^2+y^2}$.

[13] $(x-c)^2+y^2 = 4a^2 - 4a\sqrt{(x+c)^2+y^2} + (x+c)^2+y^2$.

[14] $x^2-2cx+c^2 = 4a^2-4a\sqrt{(x+c)^2+y^2}+x^2+2cx+c^2$.

[15] $4a\sqrt{(x+c)^2+y^2} = 4a^2+4cx$.

[16] $a\sqrt{x+c)^2+y^2} = a^2+cx$.

[17] $a^2[(x+c)^2+y^2] = a^4+2a^2cx+c^2x^2$.

[18] $a^2x^2+2a^2cx+a^2c^2+a^2y^2 = a^4+2a^2cx+c^2x^2$.

[19] $a^2y^2+(a^2-c^2)x^2 = a^2(a^2-c^2)$.

In [19] $x$ and $y$ stand for the coordinates of any point on ABA′B′. Hence they may stand for the coordinates of the point B, which are

$$x = 0 \text{ and } y = b.$$

Substituting these values of $x$ and $y$ into [19], we get

[20] $a^2b^2 = a^2(a^2-c^2)$,

[21] $b^2 = a^2-c^2$.

Substituting this value of $(a^2-c^2)$ into [19], we get

[22] $a^2y^2+b^2x^2 = a^2b^2$.

Since in [22] the $x$ and $y$ stand for the coordinates of any point on the ellipse, that equation must be the equation of the ellipse. by § 39.

Q. E. D.

112. *Corollary.—The semi-transverse axis is equal to the distance from the focus to the extremity of the conjugate axis.*

[1] $b^2 = a^2-c^2$. by § 111, [21].

[2] Hence $a^2 = b^2+c^2$,

[3] but $\overline{\text{F}'\text{B}}^2 = b^2+c^2$. by Geom. 26.

[4] Hence $a^2 = \overline{\text{F}'\text{B}}^2$,

[5] and $a = \text{F}'\text{B}$.

**The Circle.**—The *circle* is an ellipse in which the conjugate and transverse axes are equal to each other.

PROPOSITION III

113. *The equation of the circle is*

$$x^2 + y^2 = r^2,$$

*in which r is the radius of the circle.*

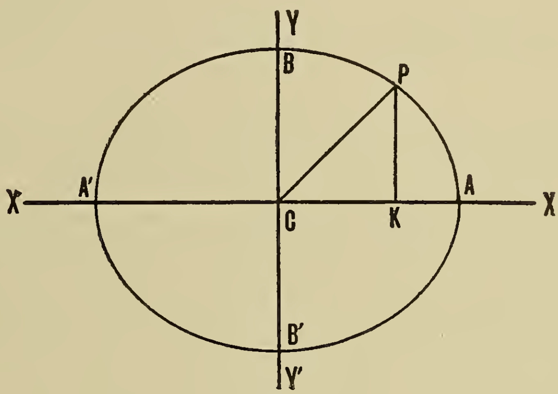



Fig. 35

Let P be any point on the ellipse and PK its ordinate.

$x \equiv$ CK and $y \equiv$ PK.

$a \equiv$ CA and $b \equiv$ CB.

The equation of the ellipse is

[1] $a^2y^2 + b^2x^2 = a^2b^2.$ by § 111.

Let $b = a,$

[2] then $b^2 = a^2.$

Substituting $a^2$ for $b^2$ in [1], we get

[3] $a^2y^2 + a^2x^2 = a^4.$

[4] Hence $y^2 + x^2 = a^2,$

[5] or $\overline{PK}^2 + \overline{CK}^2 = \overline{CA}^2,$

[6] but $\overline{PK}^2 + \overline{CK}^2 = \overline{CP}^2.$ by Geom. 26.

[7] Hence $\overline{CP}^2 = \overline{CA}^2,$

[8] hence CP = CA.

Now, if the point P be moved along the ellipse, CP will always retain the same value, since it is always equal to CA. CP is therefore constant.

Since P is any point on the ellipse, and CP is constant, the ellipse must be a circle. by Geom. 18.

Since in [4] the $x$ and $y$ stand for the coordinates of any point on this circle, that equation must be the equation of the circle. by § 39.

Substituting $r$ for the $a$ in [4], we get

[9] $$x^2+y^2=r^2,$$

which is the equation of the circle when the center is at the origin.

114. *Corollary 1.—The circle is called the equilateral ellipse.*

115. *Corollary 2.—When the center is not the origin, the equation of the circle is*

$$(x-m)^2+(y-n)^2=r^2.$$

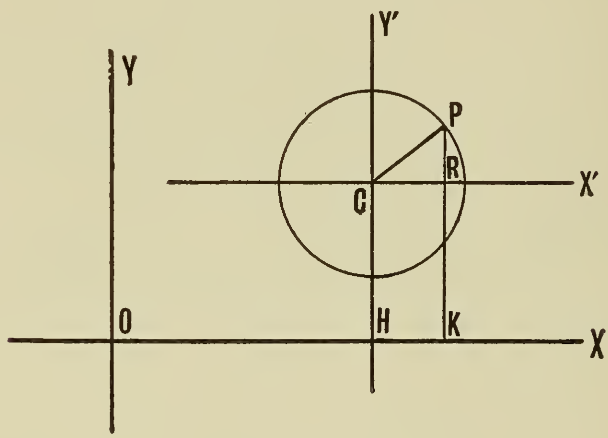



Fig. 36

Let the new origin O be at any other point than C.

Let Y'C and X'C be the old axes and YO and XO be the new axes.

Let P be any point on the circumference and let

$x \equiv$ OK and $y \equiv$ PK,
$x' \equiv$ CR " $y' \equiv$ PR,
$m \equiv$ OH " $n \equiv$ CH.
and $r \equiv$ CP.

[1] Then $$x'^2 + y'^2 = r^2,$$

is the equation of the circle referred to the old axes. by § 113.

[2] But $x' = x - m$ and $y' = y - n$. by Geom. 17.

Substituting these values of $x'$ and $y'$ into [1], we get

[3] $$(x - m)^2 + (y - n)^2 = r^2.$$

Since $x$ and $y$ are the coordinates of any point on the circumference of the circle, [3] must be the equation of the circle referred to the new axes. by § 39.

116. **The Circumscribed Circle.**—If from the center of an ellipse as a center with a radius equal to the semi-transverse axis of the ellipse, a circle be drawn, the circle is said to be *circumscribed* about the ellipse.

117. **Corresponding Ordinates.**—Ordinates drawn from the ellipse and from the circle on the same side of the transverse axis, and meeting the transverse axis at the same point, are called *corresponding ordinates*, and the points on the curves from which they are drawn, *corresponding points*.

PROPOSITION IV

118. *If a circle be circumscribed about an ellipse, any ordinate of the ellipse is to the corresponding ordinate of the circle as the semi-conjugate axis of the ellipse is to its semi-transverse axis.*

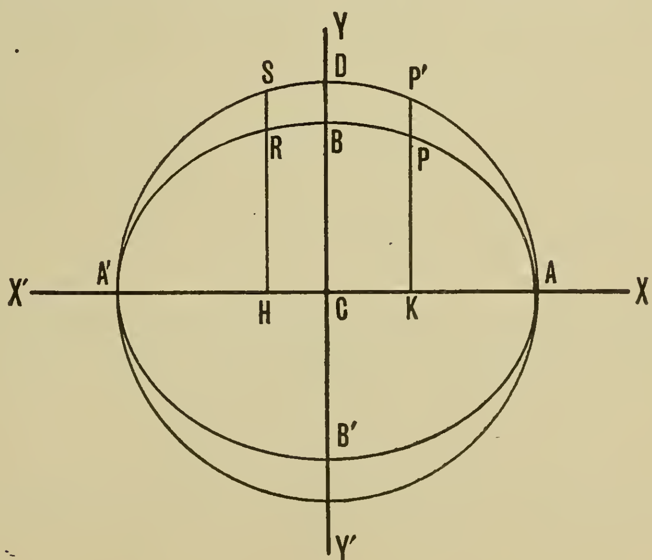



Fig. 37

Let P be any point on the ellipse and draw its ordinate PK. Produce PK till it meets the circle at some point P′.

Let $x \equiv$ CK, $y \equiv$ PK and $y' \equiv$ P′K.

Let $a \equiv$ CA and $b \equiv$ CB.

We are to prove that

$$y : y' :: b : a.$$

Since the point P is on the ellipse, its coordinates $x$ and $y$ must satisfy the equation of the ellipse. by § 40.

Hence, letting the $x$ and $y$ of that equation stand for the coordinates of the point P, we have

[1] $$a^2y^2 + b^2x^2 = a^2b^2,$$ by § 111.

[2] hence $$y^2 = \frac{b^2}{a^2}(a^2 - x^2).$$

Since the point P′ is on the circle, its coordinates $x$ and $y'$ must satisfy the equation of the circle. by § 40.

Hence substituting $x$ and $y'$ for the $x$ and $y$ of that equation, we get

[3] $$x^2 + y'^2 = r^2.$$ by § 113.

[4] Hence $$y'^2 = r^2 - x^2.$$

But since, by § 116, $r = a$, this equation becomes

[5] $$y'^2 = a^2 - x^2.$$

Now, dividing the members of [2] by the corresponding members of [5], we get

[6] $$\frac{y^2}{y'^2} = \frac{b^2}{a^2}.$$

[7] Hence $$y : y' :: b : a.$$

Q. E. D.

119. *Corollary 1.—Any two ordinates of an ellipse are to each other as the corresponding ordinates of the circumscribed circle. That is*

$$PK : RH :: P'K : SH.$$

120. *Corollary 2.—If from the center of the ellipse a circle be drawn having the conjugate axis as a diameter, the abscissa of*

*any point on the ellipse will be to the abscissa of the corresponding point on the circle as the semi-transverse axis is to the semi-conjugate axis.*

121. *Corollary 3.—To draw an ellipse when its axes are given.*

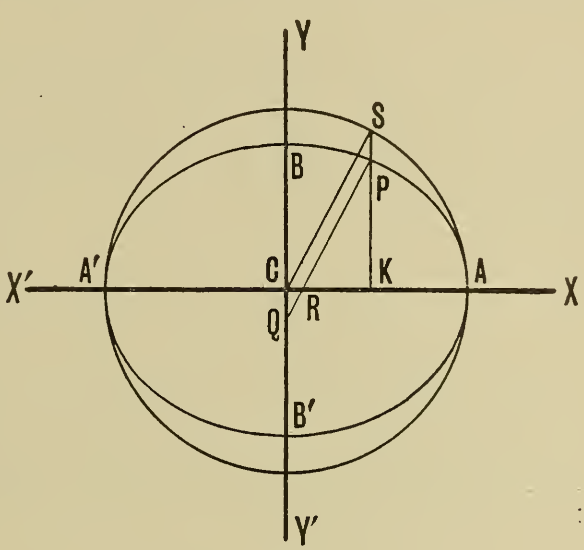


Fig. 38

Let XX' be the X axis, and YY' the Y axis.

Let $a \equiv CA$ and $b \equiv CB$.

On a stiff ruler mark off a length $PQ = a$, and a length $PR = b$. Then place Q on the Y axis, as at Q in Fig. 38, and place R upon the X axis, as at R in the figure. Now slide R and Q along the axes, and P will trace out an ellipse.

For, circumscribe a circle about the ellipse.

Through any point P on the ellipse, draw an ordinate PK and produce it till it meets the circle at S.

Let $x \equiv CK$ and $y \equiv PK$.

[1] Then $PK : SK :: b : a$. by § 118.

[2] Hence $PK : \sqrt{\overline{CS}^2 - \overline{CK}^2} :: b : a$, by Geom. 27.

[3] or $y : \sqrt{a^2 - x^2} :: b : a$.

[4] Hence $ay = b\sqrt{a^2 - x^2}$, by Geom. 21.

[5] hence $a^2y^2 + b^2x^2 = a^2b^2,$

which is the equation of the ellipse. by § 111.

Therefore, as the points R and Q move along the axes, P traces out an ellipse. by § 41.

PROPOSITION V

122. *The squares of the ordinates of any two points on an ellipse are to each other as the products of the segments which they make on the transverse axis.*

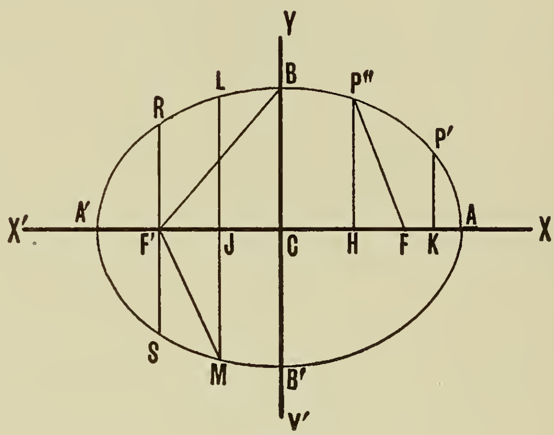



Fig 39

Let P' and P'' be any two points on an ellipse, and let P'K and P''H be their ordinates.

$$\text{Let } x' \equiv CK \text{ and } y' \equiv P'K.$$
$$x'' \equiv CH \quad " \quad y'' \equiv P''H.$$

We are to prove that

$$y'^2 : y''^2 :: A'K.KA : A'H.HA.$$

$$\text{Let } a \equiv AC = A'C.$$

[1] Then $A'K = a + x'$ and $KA = a - x'$,

[2] also $A'H = a + x''$ " $HA = a - x''$.

Since the point P is on the ellipse, its coordinates $x'$ and $y'$ must satisfy the equation of the ellipse. by § 40.

Substituting $x'$ and $y'$ for $x$ and $y$ in that equation, we get

[3] $a^2y'^2 + b^2x'^2 = a^2b^2.$ by § 111.

Since the point P$''$ is on the ellipse, its coordinates must also satisfy the equation of the ellipse. by § 40.

Substituting $x''$ and $y''$ for the $x$ and $y$ of that equation, we get

[4] $$a^2y''^2 + b^2x''^2 = a^2b^2.$$

By transposing and factoring, [3] becomes

[5] $$a^2y'^2 = b^2(a^2 - x'^2),$$

and [4] becomes

[6] $$a^2y''^2 = b^2(a^2 - x''^2).$$

Dividing the members of [5] by the corresponding members of [6], we get

[7] $$\frac{y'^2}{y''^2} = \frac{a^2 - x'^2}{a^2 - x''^2} = \frac{(a + x')(a - x')}{(a + x'')(a - x'')}.$$

[8] Hence $y'^2 : y''^2 :: (a + x')(a - x') : (a + x'')(a - x'')$,

[9] or $y'^2 : y''^2 ::$ A$'$K.KA : A$'$H.HA.

Q. E. D.

123. *Corollary 1.—Ordinates at equal distances from the center are equal.*

Let CJ $\equiv x'''$ and LJ $\equiv y'''$.

Let CJ = CH.

[1] Then $y''^2 : y'''^2 ::$ AH.HA$'$ : AJ.JA$'$, by § 122.

[2] CJ = CH,

[3] AC = A$'$C. by § 107.

Adding [2] and [3] we get

[4] AC + CJ = A$'$C + CH,

[5] that is AJ = HA$'$.

[6] Again A$'$C = AC,

[7] and CJ = CH.

Subtracting [7] from [6] we get

[8] A$'$C — CJ = AC — CH,

[9] that is JA$'$ = AH.

Multiplying the members of [5] by the corresponding members of [9], we get

[10] $AJ.JA' = AH.HA'$

Hence [1] becomes

[11] $y''^2 : y'''^2 :: AH.HA' : AH.HA'$.

[12] Hence $\frac{y''^2}{y'''^2} = 1$.

[13] Hence $y''^2 = y'''^2$,

[14] and $y'' = y'''$.

Q. E. D.

124. **The Parameter.**—The *parameter* of an ellipse is the double ordinate which passes through the focus.

Thus RS is the parameter.

The parameter is sometimes called the *latus rectum.*

125. *Corollary 2.—The parameter is a third proportional to the transverse and conjugate axes.*

For, since the point R is on the ellipse, its coordinates CF′ and RF′ must satisfy the equation of the ellipse. Hence substituting CF′ for $x$ and RF′ for $y$ in that equation, we get

[1] $a^2\overline{RF'}^2 + b^2\overline{CF'}^2 = a^2b^2$. by § 111.

[2] But $\overline{CF'}^2 = \overline{F'B}^2 - b^2$, by Geom. 27.

[3] and $F'B = a$. by § 112.

[4] Hence $\overline{CF'}^2 = a^2 - b^2$.

[5] Hence $a^2\overline{RF'}^2 + b^2(a^2 - b^2) = a^2b^2$, by [1].

[6] or $a^2\overline{RF'}^2 + a^2b^2 - b^4 = a^2b^2$,

[7] and $a^2\overline{RF'}^2 = b^4$.

[8] $aRF' = b^2$.

[9] Hence $2a2RF' = 4b^2$,

[10] and $2RF : 2b :: 2b : 2a$. by Geom. 56.

[11] $2RF' = RS$ by § 123.

[12] hence $RS : BB' :: BB' : AA'$.

126. *Corollary 3.—The ellipse is symmetrical with respect to both axes.*

127. *Corollary 4.—If the ordinates of any two points on an ellipse be equally distant from the center, the points will be equally distant from the adjacent foci. That is*

$$F'M = FP''.$$

128. **The Eccentricity.**—The *eccentricity* of an ellipse is the quotient of the distance from the center to the focus by the semi-transverse axis.

Let $e \equiv$ the eccentricity.

In Fig. 39 let $c \equiv \text{CF}$, and $a \equiv \text{CA}$.

[1] Then $$e = \frac{\text{CF}}{\text{CA}} = \frac{c}{a},$$

[2] and $$c = ae.$$

[3] But $$a^2 - b^2 = c^2.$$ by § 111, [21].

[4] Hence $$\frac{a^2 - b^2}{a^2} = \frac{c^2}{a^2} = \left(\frac{c}{a}\right)^2 = e^2,$$

[5] and $$\frac{b^2}{a^2} = 1 - e^2.$$

129. *Corollary 5.—The eccentricity of an ellipse is less than 1.*

For in [1] CF is less than CA.

130. *Corollary 6.—The eccentricity of a circle is 0.*

For when the ellipse takes the form which we call the circle, $b = a$ and [4] becomes

$$e^2 = \frac{a^2 - b^2}{a^2} = \frac{0}{a^2} = 0.$$

EXAMPLES

1. The semi-transverse axis of the earth's orbit is about 93,000,000 miles, and its eccentricity is $\frac{1}{60}$.

Find the length of the conjugate axis, the distance from the focus to the center, and the greatest and least distance of the earth from the sun, which is at one of the foci.

2. The semi-transverse axis of Jupiter's orbit is 483,000,000 miles, and its semi-conjugate axis is 478,000,000 miles.

Find the eccentricity of the orbit, the distance from the focus to the end of the conjugate axis, and the greatest and least distance of Jupiter from the sun, which is at one of the foci.

3. Find the semi-axes and eccentricity of each of the following ellipses:

$$25y^2 + 16x^2 = 400.$$

$$\frac{x^2}{3} + \frac{y^2}{2} = 1.$$

$$cy^2 + x^2 = d.$$

4. The equation of an ellipse is $16y^2 + 9x^2 = 144$. What is the distance of its foci from the center, and the distance of each focus from the vertices?

5. The distance from the focus to the end of the conjugate axis of an ellipse is 5, and its eccentricity is $\frac{3}{5}$. What is the equation of the ellipse and the distance of the focus from the vertices?

6. The semi-conjugate axis of an ellipse is 8, and the distance from its focus to its center is 6. What is the equation of the ellipse? Where does the ellipse cut the circle whose radius is 9?

PROPOSITION VI

131. *If $r'$ be the longer and $r$ be the shorter focal radius of any point on an ellipse, then*

$$r' = a + ex,$$

*and*

$$r = a - ex,$$

*in which $x$ is the abscissa of the point, $e$ the eccentricity, and $a$ the semi-transverse axis of the ellipse.*

Let $r' \equiv F'P$ and $r \equiv FP$,
$x \equiv CK$ " $y \equiv PK$,
$c \equiv CF$ " $a \equiv CA$,
and $e \equiv$ the eccentricity.

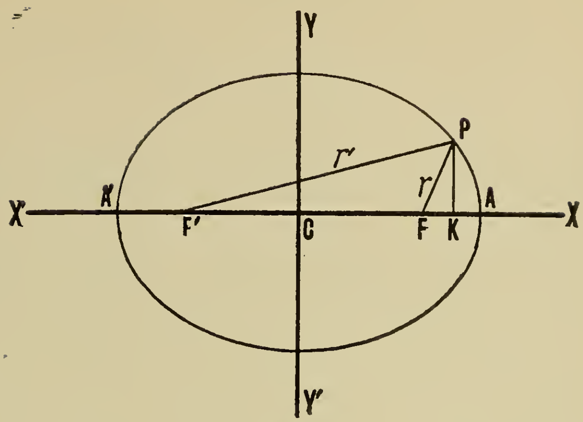



Fig. 40

[1] Then $r'^2 = \overline{PK}^2 + \overline{F'K}^2$, by Geom. 26.

[2] but $F'K = CK + CF' = x + c$.

[3] Hence $r'^2 = y^2 + (x + c)^2$.

[4] $r^2 = \overline{PK}^2 + \overline{FK}^2$. by Geom. 26.

[5] But $FK = CK - CF = x - c$.

[6] Hence $r^2 = y^2 + (x - c)^2$.

By subtracting the members of [6] from the corresponding members of [3], we get

[7] $r'^2 - r^2 = (x + c)^2 - (x - c)^2 = 4cx$.

[8] Hence $(r' + r)(r' - r) = 4cx$.

[9] But $r' + r = 2a$. by § 109.

[10] Hence $2a(r' - r) = 4\,cx$.

[11] But $e = \dfrac{c}{a}$. by § 128.

[12] Hence $c = ae$.

[13] Hence $2a(r' - r) = 4aex$,

[14] and $r' - r = 2ex$.

[15] $r' + r = 2a$. by § 109.

By adding [14] and [15]

[16] $2r' = 2a + 2ex$,

[17] and $r' = a + ex$.

By subtracting [14] from [15]

[18] $$2r = 2a - 2ex.$$

[19] $$r = a - ex.$$

132. **The Secant.**—A *secant* is a straight line cutting a curve in two points.

If one of the two cutting points remains fixed and we make the other move along the curve till the moving point coincides with the fixed point, the secant will revolve about the fixed point as a pivot.

133. **The Tangent.**—When the two points in which the secant cuts the curve coincide, the secant is called a *tangent*.

PROPOSITION VII

134. *The equation of the tangent to an ellipse is*

$$y' - y = -\frac{b^2 x'}{a^2 y'}(x' - x),$$

*in which $x'$ and $y'$ are the coordinates of the point of tangency and $a$ and $b$ are the semi-axes.*

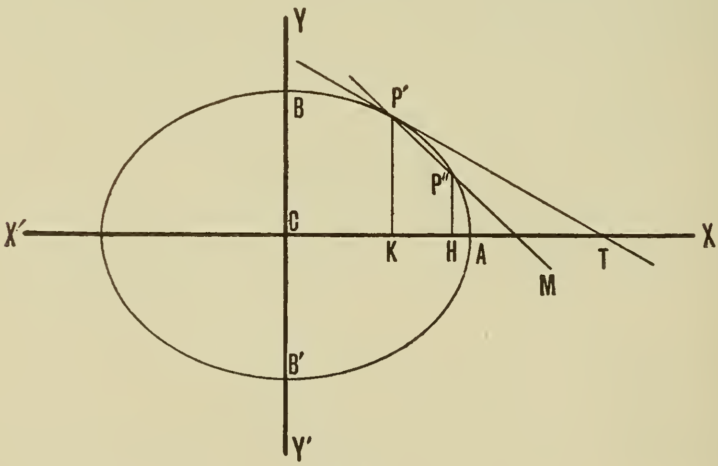



Fig. 41

Let P′T be a tangent to the ellipse at the point P′.

Let $x' \equiv$ CK and $y' \equiv$ P′K,

Let $a \equiv$ CA " $b \equiv$ CB.

We are to prove that

$$y'-y=-\frac{b^2x'}{a^2y'}(x'-x)$$

is the equation of the tangent to the ellipse.

Let P′M be a secant cutting the ellipse at the two points P′ and P″.

Let $x''\equiv$ CH and $y''\equiv$ P″H.

Since the secant is a straight line passing through the two points P′ and P″, its equation must be of the form

[1] $$y'-y=\frac{y''-y'}{x''-x'}(x'-x).$$ by § 58.

Since P′ is on the ellipse, its coordinates $x'$ and $y'$ must satisfy the equation of the ellipse. by § 40.

Substituting $x'$ and $y'$ for the $x$ and $y$ in that equation, we get

[2] $$a^2y'^2+b^2x'^2=a^2b^2.$$ by § 111.

Similarly, since P″ is on the ellipse, its coordinates $x''$ and $y''$ must satisfy the equation of the ellipse.

Substituting $x''$ and $y''$ for the $x$ and $y$ in that equation, we get

[3] $$a^2y''^2+b^2x''^2=a^2b^2.$$

Subtracting the members of [2] from the corresponding members of [3], we get

[4] $$a^2(y''^2-y'^2)+b^2(x''^2-x'^2)=0.$$

[5] $$b^2(x''^2-x'^2)=-a^2(y''^2-y'^2).$$

[6] $$b^2(x'+x'')(x''-x')=-a^2(y'+y'')(y''-y').$$

[7] $$\frac{y''-y'}{x''-x'}=-\frac{b^2}{a^2}\frac{(x'+x'')}{(y'+y'')}.$$

Substituting this value of $\frac{y''-y'}{x''-x'}$ into [1], we get

[8] $$y'-y=-\frac{b^2(x'+x'')}{a^2(y'+y'')}(x'-x).$$

Now let the point P″ move along the ellipse towards P′.

Then the secant P′M will revolve about the point P′ and will continually approach the tangent P′T, and when P″ reaches P′ it will coincide with the tangent. But when P″ reaches P′ we will have

[9] $$x'' = x' \text{ and } y'' = y',$$

and the fraction in [8] will become

[10] $$-\frac{b^2 2x'}{a^2 2y'} = -\frac{b^2 x'}{a^2 y'}.$$

Substituting this value of the fraction into [8], we get

[11] $$y' - y = -\frac{b^2 x'}{a^2 y'}(x' - x).$$

Now, the $x$ and $y$ of [8] stand for the coordinates of every point on the secant in every position which it takes as it revolves about P′. Hence they stand for the coordinates of every point on it when it coincides with the tangent.

But when the secant coincides with the tangent, [8] takes the form of [11]. Hence the $x$ and $y$ of [11] stand for the coordinates of every point on the tangent.

Therefore [11] is the equation of the tangent. by § 39.

Q. E. D.

135. *Corollary 1.—The fraction* $-\frac{b^2 x'}{a^2 y'}$ *is the slope of the tangent.*

For the fraction $\frac{y'' - y'}{x'' - x'}$ is the slope of P′M, by § 59, and $-\frac{b^2 x'}{a^2 y'}$ is the form that $\frac{y'' - y'}{x'' - x'}$ takes when the secant coincides with the tangent.

136. *Corollary 2.—The equation of the tangent to a circle is*

$$y' - y = -\frac{x'}{y'}(x' - x).$$

For when the ellipse takes the form which we call the circle $b = a$ and [11] becomes

$$y' - y = -\frac{x'}{y'}(x' - x).$$

137. **The Subtangent.**—The *subtangent* is the distance measured along the X axis, from the ordinate of the point of tangency to the tangent.

138. *Corollary 3.—The length of the subtangent of an ellipse is* $\frac{a^2 - x'^2}{x'}$.

For, since the point T is on the tangent P′T, its coordinates $x = \text{CT}$ and $y = 0$, must satisfy the equation of the tangent. by § 40.

Substituting CT and 0 for the $x$ and $y$ of that equation, we get

[1] $$y' = -\frac{b^2 x'}{a^2 y'}(x' - \text{CT}),$$

[2] $$a^2y'^2 = -b^2x'^2 + b^2x'\text{CT},$$

[3] $$a^2y'^2 + b^2x'^2 = b^2x'\text{CT}.$$

Since the point P′ is on the ellipse, its coordinates $x'$ and $y'$ must satisfy the equation of the ellipse. by § 40.

Substituting $x'$ and $y'$ for the $x$ and $y$ of that equation, we get

[4] $$a^2y'^2 + b^2x'^2 = a^2b^2.$$

[5] Hence $$b^2x'\text{CT} = a^2b^2,$$ by [3].

[6] $$\text{CT} = \frac{a^2}{x'},$$

[7] and $$\text{KT} = \text{CT} - \text{CK} = \frac{a^2}{x'} - x' = \frac{a^2 - x'^2}{x'}.$$

139. *Corollary 4.—When the ellipse takes the form which we call the circle,*

$$\textit{the subtangent} = \frac{y'^2}{x'}.$$

For then

[1] $$a = r,$$

[2] and $$\frac{a^2 - x'^2}{x'} = \frac{r^2 - x'^2}{x'} = \frac{y'^2}{x'}.$$ by § 138.

Draw a figure showing all the lines referred to in this corollary.

140. *Corollary 5.—If different ellipses have the same transverse axis, and ordinates be drawn to each from the same point on the transverse axis and tangents be drawn at the extremities of these ordinates, then all the subtangents will be equal to each other.*

For in the case of each ellipse

$$\text{the subtangent} = \frac{a^2 - x'^2}{x'}. \qquad \text{by § 138.}$$

Draw a figure showing all the lines referred to in this corollary.

141. *Corollary 6.—To draw a tangent to an ellipse from any given point on it.*

Circumscribe a circle about the ellipse, as in § 116. Draw an ordinate, cutting both ellipse and circle. Draw a tangent to the circle at the point where the ordinate cuts it. Join the point where this tangent cuts the transverse axis produced to the point where the ordinate cuts the ellipse. This last line drawn will be the tangent required.

Draw a figure and give the proof.

EXAMPLES

Required the equation of the tangent to and find the subtangent of each of the following ellipses :

1. $2x^2 + 4y^2 = 38$, when $(1, 3)$ is the point of tangency.
2. $x^2 + 4y^2 = 20$, " $(2, 2)$ " " " " "
3. $\frac{x^2}{m^2} + \frac{y^2}{n^2} = 1$, " $(m, n)$ " " " " "

142. **The Normal.**—The *normal* to a curve is a straight line perpendicular to the tangent at the point of tangency.

PROPOSITION VIII

143. *The equation of the normal to an ellipse is*

$$y' - y = \frac{a^2}{b^2}\frac{y'}{x'}(x' - x),$$

*in which $x'$ and $y'$ are the coordinates of the point of tangency and a and b the semi-axes.*

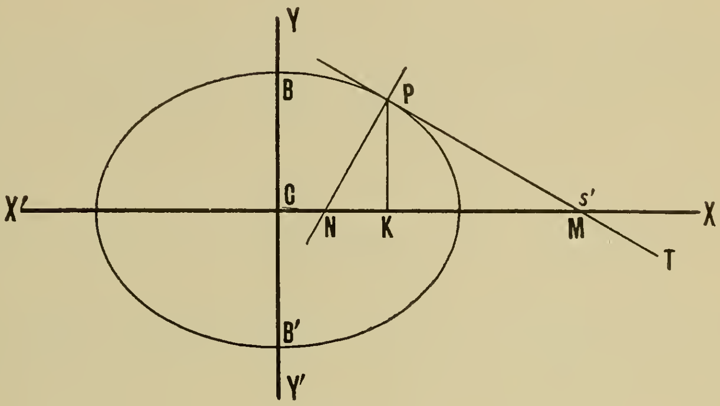



Fig. 42

Let PN be the normal and PT the tangent to the ellipse at the point P. Draw the ordinate PK.

Let $x' \equiv$ CK and $y' \equiv$ PK.

Let $s' \equiv$ the slope of the tangent PT.

The normal is a straight line passing through a fixed point, namely, the point of tangency. by § 142.

Hence its equation must be of the form

[1] $$y' - y = s(x' - x),$$ by § 57.

in which $x'$ and $y'$ are the coordinates of the fixed point, here the point of tangency, $s$ is the slope of the line, that is, the slope of the normal, and $x$ and $y$ the coordinates of any point on the line, here the coordinates of any point on the normal PN.

But the normal is perpendicular to the tangent. by § 142.

[2] Hence $$1 + ss' = 0,$$ by § 62.

[3] but $$s' = -\frac{b^2 x'}{a^2 y'}.$$ by § 135.

[4] Hence $$1 + s\left(-\frac{b^2 x'}{a^2 y'}\right) = 0.$$

[5] Hence $$s = \frac{a^2 y'}{b^2 x'}.$$

Substituting this value of $s$ into [1], we get

[6] $$y'-y=\frac{a^2y'}{b^2x'}(x'-x).$$

Now, in [6], $x$ and $y$ are the coordinates of any point on the normal PN. Hence [6] is the equation of the normal.

by § 39.

Q. E. D.

144. *Corollary 1.—The fraction* $\frac{a^2y'}{b^2x'}$ *is the slope of the normal.*

145. **The Subnormal.**—The *subnormal* is the distance measured along the transverse axis from the ordinate of the point of tangency to the normal.

146. *Corollary 2.—The length of the subnormal is* $\frac{b^2}{a^2}x'$.

For in Fig. 42 the point N is on the normal, hence its coordinates $y=0$ and $x=\text{CN}$ must satisfy the equation of the normal.

Substituting these values of $x$ and $y$ into the equation of the normal, we get

[1] $$y'-0=\frac{a^2y'}{b^2x'}(x'-\text{CN}). \qquad \text{by § 143.}$$

[2] Hence $$\text{CN}=\frac{a^2-b^2}{a^2}x'.$$

[3] But $$\text{NK}=\text{CK}-\text{CN}=x'-\frac{a^2-b^2}{a^2}x'.$$

[4] Hence $$\text{NK}=\frac{b^2}{a^2}x'.$$

146*a*. *Corollary 3.—The intercept of the normal on the X axis is equal to* $e^2x'$.

For

$$\frac{a^2-b^2}{a^2}=e^2. \qquad \text{by § 128, [4].}$$

Hence [2] becomes

$$\text{CN}=e^2x'.$$

147. *Corollary 4.—The equation of the normal to the circle is*

$$y' - y = \frac{y'}{x'}(x' - x).$$

For when the ellipse takes the form which we call the circle, $b = a$ and the equation of the normal becomes

$$y' - y = \frac{y'}{x'}(x' - x).$$

EXAMPLES

1. A straight line touches the ellipse whose semi-axes are 4 and 3 at the point whose coordinates are $-3$ and 1.9. What is its slope?

2. A line whose inclination is 45° touches $25y^2 + 16x^2 = 400$. What are the coordinates of the point of tangency?

Ans. $x = -3.9$.
$y = 2.5$.

3. Find the equation of the tangent drawn to $16y^2 + 9x^2 = 144$ at the extremity of the parameter.

Ans. $y = -0.67x + 4.04$.

4. Find the equation of the tangent to the ellipse $a^2y^2 + b^2x^2 = a^2b^2$ at the extremity of the parameter.

5. A straight line touches $25y^2 + 16x^2 = 400$ at the point whose coordinates are $-3$, 3.2. What are the angles which it makes with the axes? What is the area of the triangle between it and the axes?

6. A tangent to $a^2y^2 + b^2x^2 = a^2b^2$ makes equal angles with the axes. What are the coordinates of the point of tangency?

Ans. $x = \frac{a^2}{\sqrt{a^2 + b^2}}$,

$y = \frac{b^2}{\sqrt{a^2 + b^2}}$.

7. A normal is drawn to $25y^2 + 16x^2 = 400$ at the point whose coordinates are $-3$ and 3.2. What are the angles which it makes with the axes? What are its intercepts? Find the distances measured along the normal from the point of tangency to each of the axes.

Find the equation of the normal and the value of the subnormal in each of the following ellipses :

8. $3y^2 + 4x^2 = 39$, when (3, 1) is the point of tangency.

9. $x^2 + 4y^2 = 20$, " (2, 2) " " " " "

10. $a^2y^2 + b^2x^2 = a^2b^2$ " $(a, 0)$ " " " " "

PROPOSITION IX

148. *In an ellipse the normal bisects the interior, and the tangent the exterior angle between the focal radii of the point of tangency.*

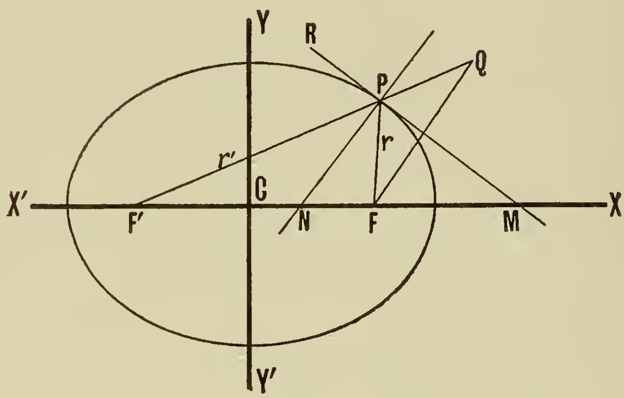



Fig. 43

Let PN be the normal and PM the tangent to the ellipse at the point P.

Let FP and F′P be the focal radii of the point P.

We are to prove that the normal PN bisects the angle FPF′ and that the tangent PM bisects the angle FPQ.

[1] $CN = e^2x'$. § 146*a*.

[2] $NF = CF - CN = c - e^2x' = ae - e^2x'$. by § 128, [2].

[3] Hence $NF = e(a - ex')$.

[4] Similarly $NF' = CF' + CN = e(a + ex')$.

[5] Hence $\frac{NF}{NF'} = \frac{a - ex'}{a + ex'}$,

[6] and $\frac{NF}{NF'} = \frac{r}{r'}$, by § 131.

[7] or $NF : NF' :: r : r'$.

Take $QP = PF = r$ and draw QF.

Substituting this value of $r$ into [7], we get

[8] $NF : NF' :: QP : r'$.

Hence is QF ∥ to PN. by Geom. 24.

[9] Hence $\angle F'PN = \angle PQF$, by Geom. 8.

[10] and $\angle NPF = \angle PFQ$. by Geom. 7.

[11] But $\angle PFQ = \angle PQF$, by Geom. 16.

[12] hence $\angle F'PN = \angle NPF$.

Q. E. D.

Again

[13] $\angle F'PR + \angle F'PN = \angle FPN + \angle FPM$, by § 142.

[14] and $\angle F'PN = \angle FPN$, by [12].

[15] hence $\angle F'PR = \angle FPM$.

[16] But $\angle F'PR = \angle QPM$, by Geom. 4.

[17] hence $\angle QPM = \angle FPM$.

Q. E. D.

149. *Corollary 1.—To draw a tangent to an ellipse at a given point on it.*

Draw focal radii to the given point. Produce either of these focal radii and bisect the exterior angle between them. The bisector is the tangent. Draw a figure and give the proof.

EXAMPLES

1. Focal radii are drawn to the point 3, 3.4 on the ellipse $36y^2 + 16x^2 = 576$. Find the length of the focal radii and the angle between them.

2. Focal radii are drawn to the point 3, 3.2, on the ellipse $25y^2 + 16x^2 = 400$. Find the angle which they make with a tangent to the ellipse at the same point and the length of the perpendicular drawn from the focus to the tangent.

150. **A Chord.**—A *chord* of an ellipse is a straight line terminated both ways by the ellipse.

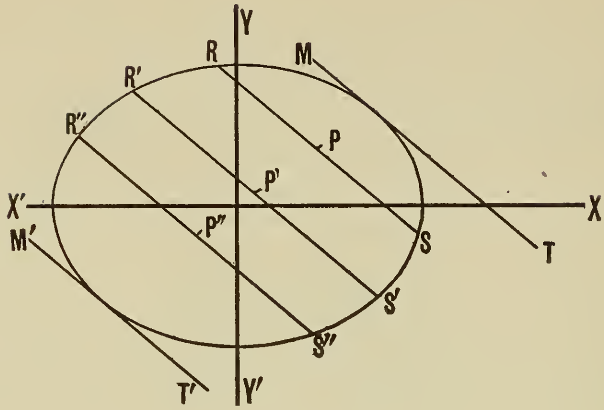



Fig 44

Let RS, R′S′, and R″S″ be parallel chords, and P, P′, and P″ their middle points.

Let MT and M′T′ be two tangents parallel to these chords.

Conceive the number of parallel chords between MT and M′T′ to be increased continually until they touch each other side by side. Then their middle points will also touch each other and form a continuous line. Such a system is called a complete system of parallel chords.

151. **The Bisector of a Complete System of Parallel Chords.**—The *bisector of a complete system of parallel chords* is the line which contains all the middle points of those chords.

152. **The Diameter.**—The *diameter* of an ellipse is that part of the bisector of a complete system of parallel chords which is terminated both ways by the ellipse.

PROPOSITION X

153. *The equation of the diameter of an ellipse is*

$$y = \left(-\frac{b^2}{a^2} \cot \varphi\right)x,$$

*in which $\varphi$ is the inclination of the system of chords bisected by the diameter, and a and b are the semi-axes.*

Let RS represent any one of a complete system of parallel chords, $\varphi$ its inclination, and P its middle point.

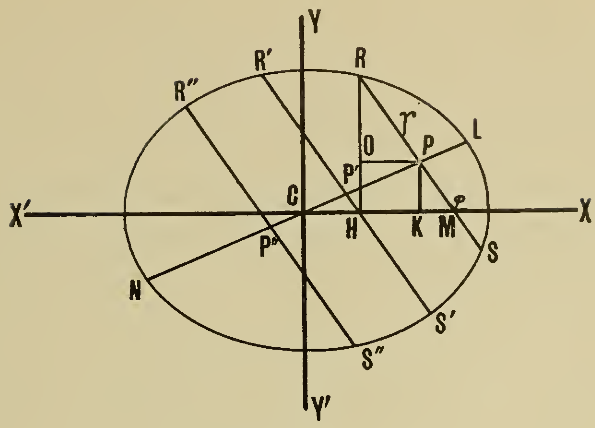



Fig. 45

Let $x \equiv CK$ and $y \equiv PK$,
$x' \equiv CH$ " $y' \equiv RH$,
and $r \equiv PR = PS$.

Draw $PO \parallel XX'$.

[1] $\angle RPO = \angle PMK$. by Geom. 8.

[2] But $\angle PMK = 180 - \varphi$.

[3] Hence $\angle RPO = 180 - \varphi$,

[4] and $\cos RPO = - \cos \varphi$. by Trig. 23.

[5] $PO = r \cos RPO$. by Trig. 2.

[6] Hence by [4] $PO = - r \cos \varphi$,

[7] and $x' = CK - PO = x + r \cos \varphi$.

[8] $RO = r \sin RPO$. by Trig. 1.

[9] Hence $RO = r \sin \varphi$, by Trig. 22.

[10] and $y' = PK + RO = y + r \sin \varphi$.

Now the point R is on the ellipse, and hence its coordinates $x'$ and $y'$ must satisfy the equation of the ellipse. by § 40.

Substituting $x'$ and $y'$ for the $x$ and $y$ in the equation of the ellipse we get

[11] $a^2 y'^2 + b^2 x'^2 = a^2 b^2$. by § 111.

Substituting the values of $x'$ and $y'$ found in [7] and [10] into [11], we get

[12] $a^2 (y + r \sin \varphi)^2 + b^2 (x + r \cos \varphi)^2 = a^2 b^2$.

Squaring the binomials and factoring with respect to $r^2$ and $r$, we get

$$[13]\quad (a^2 \sin^2 \varphi + b^2 \cos^2 \varphi)r^2 + 2(a^2y \sin \varphi + b^2x \cos \varphi)r = a^2b^2 - a^2y^2 - b^2x^2.$$

Now, since P is the middle point of RS, the two values of $r$ in [13] must be equal to each other, and hence by the theory of quadratic equations

$$[14]\qquad 2(a^2y \sin \varphi + b^2x \cos \varphi) = 0.$$

$$[15]\qquad a^2y \sin \varphi = - b^2x \cos \varphi.$$

$$[16]\qquad y = \left(-\frac{b^2}{a^2}\,\frac{\cos \varphi}{\sin \varphi}\right)x,$$

$$[17]\quad \text{and} \qquad y = \left(-\frac{b^2}{a^2} \cot \varphi\right)x. \qquad \text{by Trig. 9 and 6.}$$

Now the $x$ and $y$ of [17] stand for the coordinates of the point P. But since RS represents any one of the complete system of parallel chords, P is any point on their bisector, and since the diameter LN is a part of the bisector, P is any point on that diameter.

Hence the $x$ and $y$ of [17] stand for the coordinates of any point on the diameter which bisects the system of chords represented by RS, and therefore [17] is the equation of that diameter. by § 39.

Q. E. D.

154. *Corollary 1.—The diameter of an ellipse is a straight line passing through the center.*

Let the chord RS move across the ellipse, always remaining parallel to itself. Then its middle point P will trace out the diameter which bisects the system of chords represented by RS.

Now as P moves along the diameter, $a$, $b$, and $\varphi$ always retain the same value and therefore are constants. by § 5.

Hence the expression $-\frac{b^2}{a^2} \cot \varphi$ is a constant. Now if we represent this constant by $s$, [17], which is the equation of the diameter, may be written

[18] $$y = sx.$$

But [18] is the equation of a straight line passing through the origin. by § 55.

Hence, since the origin is at the center of the ellipse, the diameter is a straight line passing through the center.

155. *Corollary 2.—If $\theta$ is the inclination of any diameter, and $\varphi$ the inclination of its system of bisected chords, then*

$$\tan\theta \tan\varphi = -\frac{b^2}{a^2}.$$

Let $\theta$ be the inclination of any diameter. Then $\tan\theta$ is its slope. by § 52.

Since by Corollary 1, the diameter is a straight line, the coefficient of $x$ in its equation must be the slope of the diameter. by § 53.

[1] Hence $$\tan\theta = -\frac{b^2}{a^2}\cot\varphi.$$ by § 153.

[2] But $$\cot\varphi = \frac{1}{\tan\varphi}.$$ by Trig 9.

[3] Hence $$\tan\theta\tan\varphi = -\frac{b^2}{a^2}.$$

Q. E. D.

PROPOSITION XI

156. *If any diameter bisect a system of chords which are parallel to a second diameter, then that second diameter will bisect a system parallel to the first diameter.*

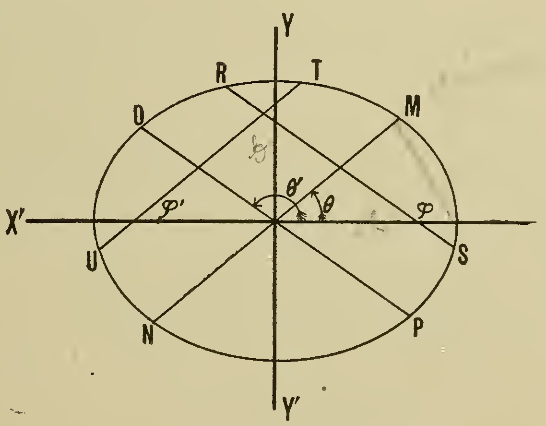



Fig. 46

Let the diameter MN bisect a system of parallel chords represented by RS, and let the diameter OP bisect the system represented by TU, and let RS be parallel to OP.

We are to prove that TU ∥ MN.

Let $\theta$ be the inclination of MN.
" $\varphi$ " " " " RS.
" $\theta'$ " " " " OP.
" $\varphi'$ " " " " TU.

[1] Then $\tan\theta\tan\varphi = -\frac{b^2}{a^2}$, by § 155.

[2] and $\tan\theta'\tan\varphi' = -\frac{b^2}{a^2}$.

[3] Hence $\tan\theta\tan\varphi = \tan\theta'\tan\varphi'$.

But RS is parallel to OP. by hypoth.

[4] Hence $\theta' = \varphi$, by Geom. 8.

[5] and $\tan\theta' = \tan\varphi$.

Substituting this value of $\tan\varphi$ into [3], we get

[6] $\tan\theta\tan\theta' = \tan\theta'\tan\varphi'$.

[7] Hence $\tan\theta = \tan\varphi'$,

[8] and $\theta = \varphi'$.

Hence TU ∥ MN, by Geom. 9.

and the system of chords parallel to TU will be parallel to MN. by Geom. 10.

Therefore OP bisects a system of chords which are parallel to MN.

Q. E. D.

157. **Conjugate Diameters.**—Two diameters are said to be *conjugate* to each other when each bisects a system of chords which are parallel to the other.

158. *Corollary 1.—If $\theta$ be the inclination of any diameter, and $\theta'$ the inclination of its conjugate, then*

$$\tan\theta\tan\theta' = -\frac{b^2}{a^2}.$$

[1] For $\tan\theta\tan\varphi = -\frac{b^2}{a^2}$. by § 155.

[2] But $\tan\varphi = \tan\theta'$. by § 157.

[3] Hence $\tan\theta\tan\theta' = -\frac{b^2}{a^2}$.

PROPOSITION XII

159. *The tangent to an ellipse at the extremity of any diameter is parallel to the conjugate of that diameter.*

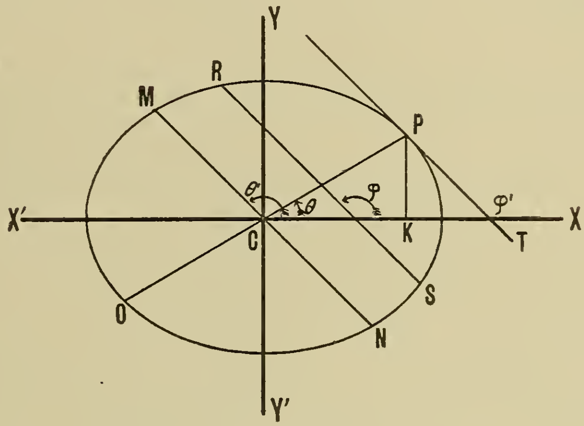



Fig. 47

Let OP be any diameter and MN its conjugate.

Let PT be the tangent at the extremity of OP.

We are to prove that PT ∥ MN.

Let RS be any one of the system of chords bisected by OP.

Let $\theta$ be the inclination of OP,
" $\theta'$ " " " " MN,
" $\varphi$ " " " " RS,
and $\varphi'$ " " " " PT.

Let $x' = \text{CK}$ and $y' = \text{PK}$.

RS will be parallel to MN. by § 157.

[1] Hence $\varphi = \theta'$, by Geom. 8.

[2] and $\tan\varphi = \tan\theta'$.

Since the point P is on the diameter OP, its coordinates $x'$ and $y'$ must satisfy the equation of that diameter. by § 40.

The equation of the diameter OP is

[3] $$y = (-\frac{b^2}{a^2}\cot\varphi)x.$$ by § 153.

Substituting $x'$ and $y'$ for the $x$ and $y$ of this equation, we get

[4] $$y' = (-\frac{b^2}{a^2}\cot\varphi)x',$$

[5] hence $$\cot\varphi = -\frac{a^2y'}{b^2x'}.$$

[6] But $$\cot\varphi = \frac{1}{\tan\varphi}.$$ by Trig. 9.

[7] Hence $$\frac{1}{\tan\varphi} = -\frac{a^2y'}{b^2x'},$$

[8] and $$\tan\varphi = -\frac{b^2x'}{a^2y'}.$$

[9] Hence $$\tan\theta' = -\frac{b^2x'}{a^2y'}.$$ by [2].

[10] But $$\tan\varphi' = -\frac{b^2x'}{a^2y'}.$$ by § 135.

[11] Hence $$\tan\theta' = \tan\varphi',$$ by [9] and [10].

[12] and $$\theta' = \varphi'.$$

Therefore PT ‖ MN by Geom. 9.

Q. E. D.

160. *Corollary 1.—The two tangents at the extremities of a diameter are parallel to each other.*

161. *Corollary 2.—The four tangents at the extremities of two conjugate diameters form a parallelogram circumscribed about the ellipse.*

PROPOSITION XIII

162. *Given the coordinates of one extremity of any diameter of an ellipse to find the coordinates of the extremities of its conjugate.*

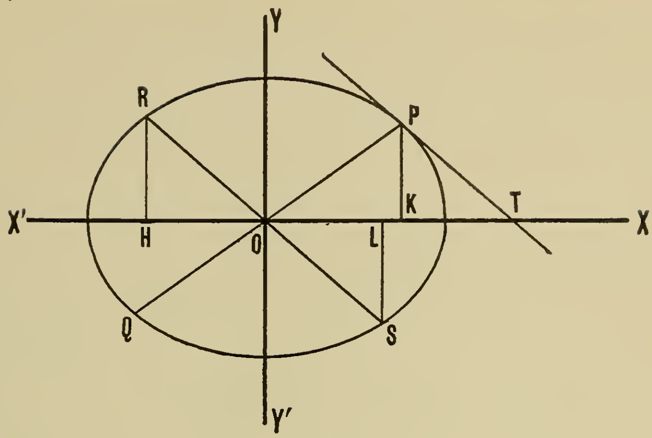



Fig 48

Let PQ be any diameter and RS its conjugate.

Let OK and PK be given.

We are to find OH and RH, and OL and LS.

$$\text{Let } x' \equiv \text{OK and } y' \equiv \text{PK.}$$
$$\text{" } x'' \equiv \text{OH and } y'' \equiv \text{RH.}$$

Draw PT tangent to the ellipse at the point P.

Since RS is a straight line passing through the origin, its equation must be of the form

[1] $$y = sx.$$ by § 55.

But RS ∥ PT. by § 159.

[2] Hence $\angle \text{ROX} = \angle \text{PTX}$, by Geom. 8.

[3] and $\tan \text{ROX} = \tan \text{PTX}$.

[4] But $\tan \text{ROX} = s$, by [1] and § 53.

[5] and $\tan \text{PTX} = -\dfrac{b^2 x'}{a^2 y'}$, by § 135.

[6] hence $$s = -\frac{b^2 x'}{a^2 y'}.$$

If we substitute this value of $s$ into [1], the equation of RS becomes

[7] $$y = -\frac{b^2 x'}{a^2 y'} x.$$

Now since the point R is on the diameter RS, its coordinates $x''$ and $y''$ must satisfy the equation of RS. by § 40.

Hence substituting $x''$ and $y''$ for the $x$ and $y$ of [7], we get

[8] $$y'' = -\frac{b^2x'}{a^2y'}x''.$$

The equation of the ellipse is

[9] $$a^2y^2 + b^2x^2 = a^2b^2,$$ by § 111.

and since the point R is on the ellipse, its coordinates $x''$ and $y''$ must satisfy the equation of the ellipse. by § 40.

Hence substituting $x''$ and $y''$ for the $x$ and $y$ of [9], we get

[10] $$a^2y''^2 + b^2x''^2 = a^2b^2.$$

Now since in both [8] and [10], $x''$ stands for OH and $y''$ stands for RH, these equations are simultaneous, and therefore can be solved by algebra.

Squaring both members of [8], we get

[11] $$y''^2 = \frac{b^4x'^2}{a^4y'^2}x''^2.$$

Substituting this value of $y'$, into [10], we get

[12] $$a^2\frac{b^4x'^2}{a^4y'^2}x''^2 + b^2x''^2 = a^2b^2.$$

[13] $$\frac{b^2x'^2}{a^2y'^2}x''^2 + x''^2 = a^2.$$

[14] $$b^2x'^2x''^2 + a^2y'^2x''^2 = a^4y'^2.$$

[15] $$(a^2y'^2 + b^2x'^2)x''^2 = a^4y'^2.$$

Now since the point P is on the ellipse, its coordinates $x'$ and $y'$ must satisfy the equation of the ellipse.

Substituting $x'$ and $y'$ for the $x$ and $y$ of [9], we get

[16] $$a^2y'^2 + b^2x'^2 = a^2b^2.$$

Substituting the value of the left hand member of this equation into [15], we get

[17] $$a^2b^2x''^2 = a^4y'^2.$$

[18] Hence $$x'' = \pm\frac{a}{b}y'.$$

That is, $OH = -\frac{a}{b}y'$ and $OL = \frac{a}{b}y'$.

Substituting $+\frac{a}{b}y'$ for $x''$ in [8], we get

[19] $$y'' = -\frac{b^2x'}{a^2y'}\frac{a}{b}y' = -\frac{b}{a}x' = LS.$$

Substituting $-\frac{a}{b}y'$ for $x''$ in [8], we get

[20] $$y'' = \frac{b^2x'}{a^2y'}\frac{a}{b}y' = \frac{b}{a}x' = RH.$$

163. *Corollary.—All diameters are bisected by the center.*

| | | | |
|---|---|---|---|
| [1] | For | OL = OH, | by [18]. |
| [2] | and | LS = RH. | by [19] and [20]. |
| [3] | Hence | OS = OR. | by Geom. 15. |

EXAMPLES

1. A straight line touches $36y^2 + 16x^2 = 576$ at the point whose coordinates are 3 and 3.4. What is the equation of the diameter conjugate to the one which passes through the point 3, 3.4?

2. The inclination of a diameter of $25y^2 + 16x^2 = 400$ is $45°$. What is the equation of the tangent at the vertex?

Ans. $y = -\frac{16}{25}x + 32$.

3. A straight line touches $16y^2 + 9x^2 = 144$ at 2, 2.6. Where does the diameter which is parallel to this line cut the curve?

Ans. At $-3.1$, 1.5, and at 3.1, $-1.5$.

4. A diameter cuts $25y^2 + 16x^2 = 400$ at $-3$, 3.2. What is the equation of the tangent parallel to that diameter?

PROPOSITION XIV

164. *The sum of the squares of any pair of conjugate diameters is equal to the sum of the squares of the axes.*

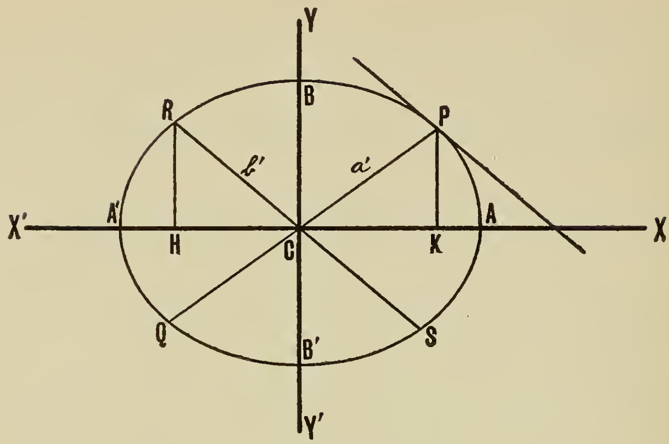



Fig. 49

Let PQ and RS be any pair of conjugate diameters, and AA′ and BB′ the axes.

We are to prove that

$$\overline{PQ}^2 + \overline{RS}^2 = \overline{AA'}^2 + \overline{BB'}^2.$$

Let $x' \equiv CK$ and $y' \equiv PK$.
$x'' \equiv CH$ " $y'' \equiv RH$.
$a \equiv AC$ " $b \equiv BC$.
$a' \equiv CP$ " $b' \equiv CR$.

[1] $a'^2 = x'^2 + y'^2$. by Geom. 26.

[2] $b'^2 = x''^2 + y''^2$. by Geom. 26.

[3] But $x''^2 = \frac{a^2}{b^2} y'^2$, by § 162, [18].

[4] and $y''^2 = \frac{b^2}{a^2} x'^2$. by § 162, [20].

[5] Hence $b'^2 = \frac{a^2}{b^2} y'^2 + \frac{b^2}{a^2} x'^2$. by [2].

Adding [1] and [5], we get

[6] $a'^2 + b'^2 = x'^2 + \frac{b^2}{a^2} x'^2 + \frac{a^2}{b^2} y'^2 + y'^2$,

[7] and $a'^2 + b'^2 = (a^2 + b^2) \frac{x'^2}{a^2} + (a^2 + b^2) \frac{y'^2}{b^2}$,

[8] $$a'^2 + b'^2 = (a^2 + b^2)\left(\frac{x'^2}{a^2} + \frac{y'^2}{b^2}\right).$$

Now since the point P is on the ellipse, its coordinates $x'$ and $y'$ must satisfy the equation of the ellipse. by § 40.

Substituting $x'$ and $y'$ for the $x$ and $y$ in that equation, we get

[9] $$a^2y'^2 + b^2x'^2 = a^2b^2,$$ by § 111.

[10] hence $$\frac{y'^2}{b^2} + \frac{x'^2}{a^2} = 1.$$

Substituting the value of the left hand member of this equation into [8], we get

[11] $$a'^2 + b'^2 = a^2 + b^2.$$

[12] Hence $$4a'^2 + 4b'^2 = 4a^2 + 4b^2.$$

[13] But $a' = \frac{1}{2}PQ$, $b' = \frac{1}{2}RS$, $a = \frac{1}{2}AA'$ and $b = \frac{1}{2}BB'$. by § 163.

[14] Hence $$\overline{PQ}^2 + \overline{RS}^2 = \overline{AA'}^2 + \overline{BB'}^2.$$ by Geom. 32.

Q. E. D.

## PROPOSITION XV

165. *The parallelogram formed by tangents to an ellipse at the extremities of any pair of conjugate diameters is equivalent to the rectangle whose sides are equal to the axes of the ellipse.*

Let PQ and RS be any pair of conjugate diameters, and let LMNO be the parallelogram formed by tangents at their extremities.

Let DEFG be the rectangle whose sides are equal to the axes AA′ and BB′.

We are to prove that

LMNO = DEFG.

Let $a \equiv CA$ and $b \equiv CB$,

$a' \equiv CP$ " $b' \equiv CR$,

$x' \equiv CH$ " $y' \equiv PH$,

$x'' \equiv CK$ " $y'' \equiv RK$,

$\theta \equiv \angle PCH$ " $\varphi \equiv \angle RCH$.

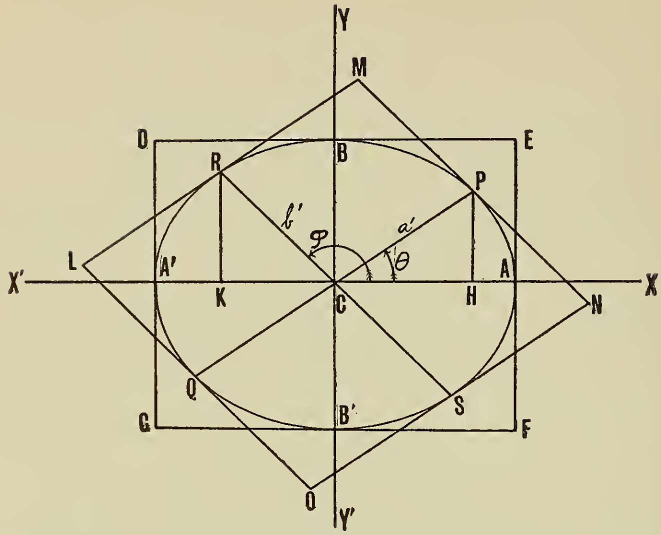



Fig. 50

[1] $\text{LMNO} = \text{LO} \times \text{NO} \sin \text{LON}.$ by Trig. 15.

$\text{LO} \parallel \text{RS},$ by § 159.

and $\text{NO} \parallel \text{PQ}.$ by § 159.

[2] Hence $\angle \text{LON} = \angle \text{RCP},$ by Geom. 11.

[3] and $\sin \text{LON} = \sin \text{RCP} = \sin(\varphi - \theta).$

[4] $\text{LO} = \text{RS}.$ by Geom. 17.

[5] $\text{RS} = 2b'.$ by § 163.

[6] Hence $\text{LO} = 2b'.$

[7] Similarly $\text{NO} = 2a'.$

Substituting these values of sin LON, LO and NO into [1], we get

[8] $\text{LMNO} = 4a'b' \sin(\varphi - \theta).$

[9] $\sin(\varphi - \theta) = \sin\varphi \cos\theta - \cos\varphi \sin\theta.$ by Trig. 13.

[10] $\sin\varphi = \sin \text{RCK}.$ by Trig. 22.

[11] But $\sin \text{RCK} = \dfrac{\text{RK}}{\text{RC}} = \dfrac{y''}{b'}.$ by Trig. 1.

[12] Hence $\sin\varphi = \dfrac{y''}{b'}.$

[13] $\cos\theta = \dfrac{\text{CH}}{\text{CP}} = \dfrac{x'}{a'}.$ by Trig. 2.

[14] $\cos\varphi = -\cos \text{RCK}.$ by Trig. 23.

[15] $\cos \text{RCK} = \frac{\text{CK}}{\text{CR}} = \frac{x''}{b'},$

[16] hence $\cos\varphi = -\frac{x''}{b'}.$

[17] $\sin\theta = \frac{\text{PH}}{\text{CP}} = \frac{y'}{a'}.$

Substituting these values of $\sin\varphi$, $\cos\theta$, $\cos\varphi$ and $\sin\theta$ into [9], we get

[18] $\sin(\varphi - \theta) = \frac{y''}{b'}\frac{x'}{a'} + \frac{x''}{b'}\frac{y'}{a'} = \frac{y''x' + x''y'}{a'b'}.$

[19] But $x'' = \frac{a}{b}y',$ by § 162, [18].

[20] and $y'' = \frac{b}{a}x'.$ by § 162, [20].

[21] Hence $\sin(\varphi - \theta) = \frac{\frac{b}{a}x'^2 + \frac{a}{b}y'^2}{a'b'} = \frac{b^2x'^2 + a^2y'^2}{a'b'\,ab}.$

Since the point P is on the ellipse, its coordinates $x'$ and $y'$ must satisfy the equation of the ellipse. by § 40.

Substituting $x'$ and $y'$ for the $x$ and $y$ in that equation, we get

[22] $a^2y'^2 + b^2x'^2 = a^2b^2.$

Substituting the value of the left hand member of this equation into [21], we get

[23] $\sin(\varphi - \theta) = \frac{a^2b^2}{a'b'ab} = \frac{ab}{a'b'}.$

Substituting this value of $\sin(\varphi - \theta)$ into [8], we get

[24] $\text{LMNO} = 4a'b'\frac{ab}{a'b'} = 4ab.$

[25] But $4ab = 2a2b = \text{AA}' \times \text{BB}',$

[26] and $\text{AA}' = \text{GF}.$ by Hypoth.

[27] Hence $4ab = \text{GF} \times \text{BB}' = \text{DEFG}.$ by Geom. 28.

[28] Therefore $\text{LMNO} = \text{DEFG}$. by [24].

Q. E. D.

166. *Corollary.—The sides of the rectangle circumscribed about an ellipse are equal to the axes.*

PROPOSITION XVI

167. *The area of an ellipse is $\pi$ times the product of the semi-axes.*

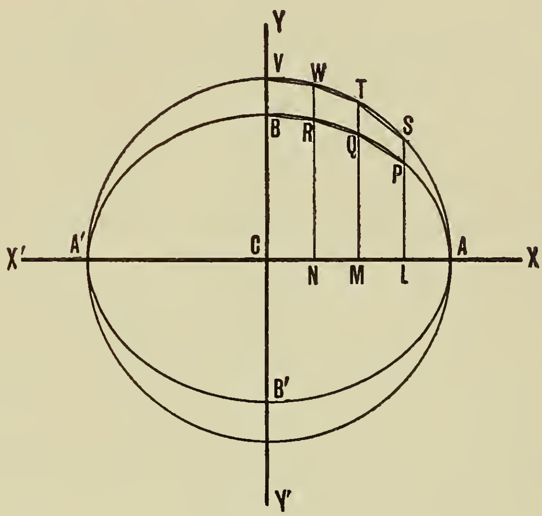



Fig. 51

Let $E \equiv$ the area of the ellipse.

" $a \equiv \text{CA}$ and $b \equiv \text{CB}$.

We are to prove that

$$E = \pi ab.$$

Circumscribe a circle about the ellipse.

Draw ordinates to the circle, cutting the ellipse in the points P, Q, and R.

Join the points P and Q, Q and R, R and B, S and T, T and W, W and V.

[1] $\text{PQML} = (\text{PL} + \text{QM}) \dfrac{\text{ML}}{2}$, by Geom. 61.

[2] and $\text{STML} = (\text{SL} + \text{TM}) \dfrac{\text{ML}}{2}$. by Geom. 61.

[3] Hence $\frac{\text{PQML}}{\text{STML}} = \frac{\text{PL} + \text{QM}}{\text{SL} + \text{TM}}$.

[4] PL : QM : : SL : TM. by § 119.

[5] Hence PL : SL : : QM : TM, by alternation.

[6] and PL + QM : SL + TM : : PL : SL. by Geom. 22.

[7] But PL : SL : : $b : a$. by § 118.

[8] Hence PL + QM : SL + TM : : $b : a$,

[9] or $\frac{\text{PL} + \text{QM}}{\text{SL} + \text{TM}} = \frac{b}{a}$.

Substituting the right hand member of this equation into [3], we get

[10] $\frac{\text{PQML}}{\text{STML}} = \frac{b}{a}$.

[11] Similarly $\frac{\text{QRNM}}{\text{TWNM}} = \frac{b}{a}$,

[12] and $\frac{\text{RBCN}}{\text{WVCN}} = \frac{b}{a}$.

[13] Hence $\frac{\text{PQML}}{\text{STML}} = \frac{\text{QRNM}}{\text{TWNM}} = \frac{\text{RBCN}}{\text{WVCN}}$,

[14] or PQML + QRNM + RBCN : STML + TWNM + WVCN : : PQML : STML. by Geom. 22.

Let $C =$ the area of the circumscribing circle.

$\Sigma T_c =$ the sum of the trapezoids in the quarter of the circle and

$\Sigma T_e =$ the sum of the trapezoids in the quarter of the ellipse.

Then [14] will become

[15] $\Sigma T_e : \Sigma T_c$ : : PQML : STML,

[16] or $\frac{\Sigma T_e}{\Sigma T_c} = \frac{\text{PQML}}{\text{STML}} = \frac{b}{a}$. by [10].

[17] Hence $a \Sigma T_e = b \Sigma T_c$.

Now let the number of trapezoids be increased continually. Then the two members of [17] will be variables. by § 5.

[18] Hence limit $a\Sigma T_e =$ limit $b\Sigma T_c$. by Geom. 20.

But while the number of trapezoids increases, the $a$ and $b$ of [17] always retain the same values. They are, therefore, constants. by § 4.

[19] Hence $a$ limit $\Sigma T_e = b$ limit $\Sigma T_c$.

[20] Limit $\Sigma T_e = \frac{E}{4}$. by Geom. 19.

[21] Limit $\Sigma T_c = \frac{C}{4}$. by Geom. 19.

[22] Hence $a\frac{E}{4} = b\frac{C}{4}$, by [19].

[23] and $aE = bC$.

[24] But $C = \pi a^2$. by Geom. 30.

[25] Therefore $E = \pi ab$.

Q. E. D.

PROPOSITION XVII

168. *If the inclinations of two diameters be supplementary angles, the diameters must be equal to each other.*

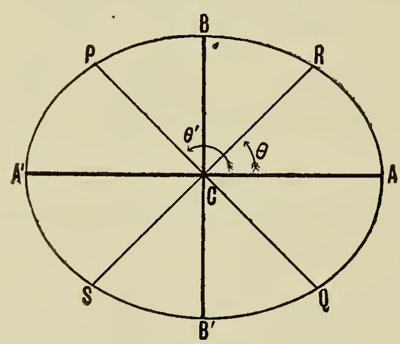



Fig. 52

Let $\theta$ be the inclination of the diameter RS, and $\theta'$ be the inclination of the diameter PQ.

Let $\theta' = 180° - \theta$.

We are to prove that

$$RS = PQ.$$

[1] $\theta' = 180° - \theta.$ by Hypoth.

[2] Also $\theta' = 180° - \angle PCA'.$ by Geom. 3.

[3] Hence $180° - \theta = 180° - \angle PCA',$

[4] and $\theta = \angle PCA'.$

Now about BB′ as an axis revolve BAB′ till it comes into the plane of BA′B′.

Then since

$$\angle BCA = \angle BCA',$$

CA will take the direction of CA′.

And since

$$\theta = \angle PCA',$$

CR will take the direction of CP.

Now ARB and A′PB are symmetrical with respect to BB′. by § 126.

Hence R will fall upon P, and therefore CR will coincide with CP.

Therefore $RS = PQ.$ by § 163.

Q. E. D.

PROPOSITION XVIII

169. *The two conjugate diameters whose inclinations are supplementary angles are, when produced, the diagonals of the rectangle formed on the axes.*

Let RS and PQ be two conjugate diameters whose inclinations $\theta$ and $\theta'$ are supplementary angles.

Let LMNO be the rectangle on the axes.

We are to prove that RS and PQ produced are the diagonals of the rectangle LMNO.

Since RS and PQ are conjugates

[1] $\tan\theta \tan\theta' = -\dfrac{b^2}{a^2}.$ by § 158.

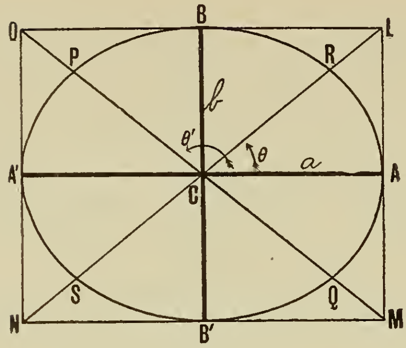



Fig. 53

But since $\theta$ and $\theta'$ are supplementary

[2] $$-\tan\theta = \tan\theta'.$$ by Trig. 24.

Substituting this value of $\tan\theta'$ into [1], we get

[3] $$-\tan^2\theta = -\frac{b^2}{a^2}.$$

[4] Hence $$\tan\theta = \pm\frac{b}{a},$$

[5] or $$\tan \mathrm{RCA} = \frac{b}{a}, \text{ and } \tan \mathrm{RCA'} = -\frac{b}{a}.$$

[6] But $$\tan \mathrm{LCA} = \frac{\mathrm{LA}}{\mathrm{CA}},$$ by Trig. 3.

[7] or $$\tan \mathrm{LCA} = \frac{b}{a}.$$

Hence by [5] and [7] we get

[8] $$\tan \mathrm{RCA} = \tan \mathrm{LCA}.$$

[9] Hence $$\angle \mathrm{RCA} = \angle \mathrm{LCA}.$$

Therefore the diameter RS coincides with the diagonal LN.

Similarly it may be shown that the diameter PQ coincides with the diagonal OM.

Q. E. D.

170. *Corollary 1.—The conjugate diameters whose inclinations are supplementary angles are equal to each other.* by § 168.

171. *Corollary 2.—There are only two conjugate diameters whose inclinations are supplementary angles.*

For there are only two diagonals of the rectangle formed on the axes. But the conjugate diameters whose inclinations are supplementary coincide with these diagonals. by § 169.

Hence there can only be two such conjugate diameters.

172. **Equi-Conjugate Diameters.**—The two conjugate diameters which, when produced, are the diagonals of the rectangle formed on the axes are called the *equi-conjugate diameters* of the ellipse.

## The Directrix

173. **The Directrix.**—The *directrix* of an ellipse is a straight line drawn perpendicular to the X axis on the opposite side of the focus from the vertex, and at such a distance from the vertex that the distance from the focus to the vertex divided by the distance from the vertex to the perpendicular is equal to the eccentricity of the ellipse.

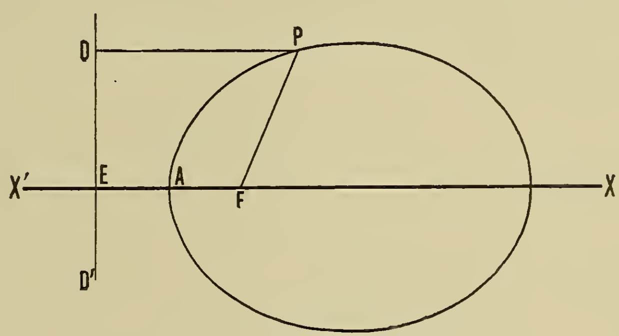



Fig. 54

In Fig. 54 if $\frac{FA}{AE} = e$, then DD′ is the directrix.

174. **Focal Radius.**—The distance from any point on the ellipse to the focus is called the *focal radius* of that point.

175. **The Directral Distance.**—The distance from any point on the ellipse to the directrix is called the *directral distance* of that point.

PD is the directral distance of the point P.

PROPOSITION XIX

176. *The ratio between the focal radius and the directral distance of any point on an ellipse is constant and is equal to the eccentricity of the ellipse.*

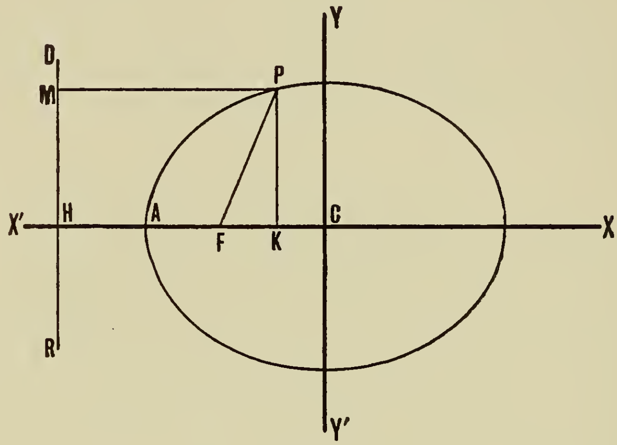



Fig. 55

Let P be any point on the ellipse, and let DR be the directrix and F the focus. Join P to F and draw PM perpendicular to DR.

Let $e \equiv$ the eccentricity.

We are to prove that

$$\frac{PF}{PM} = e.$$

Draw the ordinate PK.

Let $x \equiv CK$ and $y \equiv PK$.

$m \equiv HK$,

$a \equiv CA$,

$c \equiv CF$,

and $p \equiv FH$.

[1] $p = FH = FA + AH.$

[2] $\frac{FA}{AH} = e.$ by § 173.

[3] Hence $AH = \frac{FA}{e} = \frac{a - c}{e} = \frac{a - ae}{e}.$ by § 128, [2].

[4] Hence

$$p = \text{FA} + \frac{a - ae}{e} = a - c + \frac{a - ae}{e} = a - ae + \frac{a - ae}{e},$$

[5] or $p = \dfrac{ae - ae^2 + a - ae}{e} = \dfrac{a(1 - e^2)}{e}$.

[6] Also $x = \text{CH} - \text{HK} = c + p - m = c - (m - p)$.

The equation of the ellipse is

[7] $a^2y^2 + b^2x^2 = a^2b^2$. by § 111

Since the point P is on the ellipse, the $x$ and $y$ of the equation of the ellipse may stand for the coordinates of that point. Hence substituting the value of $x$ given in [6] into [7], we get

[8] $a^2y^2 + b^2[c - (m - p)]^2 = a^2b^2$,

[9] or $a^2y^2 + b^2[c^2 - 2c(m - p) + (m - p)^2] = a^2b^2$.

[10] Hence $y^2 + \dfrac{b^2c^2}{a^2} - 2\dfrac{b^2c}{a^2}(m - p) + \dfrac{b^2}{a^2}(m - p)^2 = b^2$.

[11] But $\dfrac{b^2}{a^2} = 1 - e^2$, by § 128, [5].

[12] and $c^2 = a^2e^2$. by § 128, [2].

Substituting these values into [10], we get

[13] $y^2 + b^2e^2 - 2\dfrac{b^2e}{a}(m - p) + (1 - e^2)(m - p)^2 = b^2$.

[14]

$$y^2 + b^2e^2 - 2\frac{b^2e}{a}(m - p) + (m - p)^2 - e^2(m - p)^2 = b^2.$$

[15]

$$y^2 + (m - p)^2 = -b^2e^2 + 2\frac{b^2e}{a}(m - p) + e^2(m - p)^2 + b^2.$$

From [11] we get

[16] $b^2 = a^2(1 - e^2)$.

Substituting this value of $b^2$ and the value of $p$ given in [5] into [15], we get

[17] $y^2 + (m - p)^2 = e^2m^2$,

[18] or $\overline{PK}^2 + \overline{FK}^2 = e^2\overline{HK}^2.$

[19] Hence $\overline{PF}^2 = e^2\overline{HK}^2.$ by Geom. 26.

[20] But $\overline{HK}^2 = \overline{PM}^2.$ by Geom. 17.

[21] Hence $\overline{PF}^2 = e^2\overline{PM}^2,$

[22] and $\frac{PF}{PM} = e.$

Q. E. D.

177. *Corollary 1.—In an ellipse the focal distance is less than the directral distance.*

For $e$ is less than 1. by § 129.

Hence PF is less than PM.

178. *Corollary 2.—The distance from the centre of an ellipse to the directrix is equal to* $\frac{a}{e}$.

For in Fig. 55

[1] $CF = c = ae,$ by § 128, [2].

[2] and $FH = p = \frac{a - ae^2}{e}.$ by § 176, [5].

[3] Hence $CH = CF + FH = ae + \frac{a - ae^2}{e} = \frac{a}{e}.$

PROPOSITION XX

179. *The equation of the ellipse when any pair of conjugate diameters are taken as the axes is*

$$a'^2y^2 + b'^2x^2 = a'^2b'^2,$$

*in which $a'$ and $b'$ are the semi-conjugate diameters.*

Let P be any point on the ellipse, and LM and NO be any two conjugate diameters.

Let LM be the new axis of abscissas and NO the new axis of ordinates.

Draw PS ∥ YY′ and PK ∥ NO.

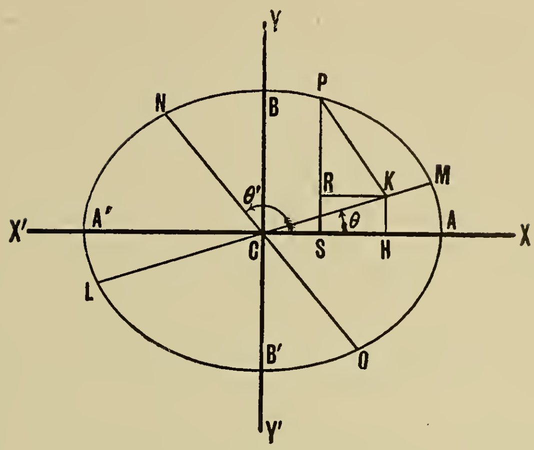



Fig. 56

Let $x \equiv$ CS and $y \equiv$ PS.
$x' \equiv$ CK " $y' \equiv$ PK.
$a' \equiv$ CM " $b' \equiv$ CN.
$\theta \equiv$ MCA " $\theta' \equiv$ NCA.

We are to prove that

$$a'^2y^2 + b'^2x^2 = a'^2b'^2$$

is the equation of the ellipse referred to the diameters LM and NO.

[1] $\text{PR} = \text{PK}.\sin \text{PKR} = y' \sin \theta'$. by Trig. 1 and 22.
[2] $\text{RS} = \text{KH} = \text{CK}.\ \sin \theta = x' \sin \theta$. by Trig. 1.
[3] Hence $y = \text{PR} + \text{RS} = y' \sin \theta' + x' \sin \theta$.
[4] $\text{RK} = \text{PK}.\ \cos \text{PKR} = -y' \cos \theta'$. by Trig. 2 and 23.
[5] $\text{CH} = \text{CK}.\ \cos \theta = x' \cos \theta$.
[6] Hence $x = \text{CH} - \text{RK} = x' \cos \theta + y' \cos \theta'$.

When XX' and YY' are taken as the axes the equation of the ellipse is

[7] $a^2y^2 + b^2x^2 = a^2b^2$. by § 111.

Substituting for the $x$ and $y$ of this equation their values given in [3] and [6], we get

[8] $a^2[y'^2 \sin^2 \theta' + 2x'y' \sin \theta \sin \theta' + x'^2 \sin^2 \theta]$
$+ b^2[y'^2 \cos^2 \theta' + 2x'y' \cos \theta \cos \theta' + x'^2 \cos^2 \theta] = a^2b^2$.

[9] Hence $(a^2 \sin^2 \theta + b^2 \cos^2 \theta)x'^2$
$+ a^2 \sin^2 \theta' + b^2 \cos \theta')y'^2$
$+ 2(a^2 \sin \theta \sin \theta' + b^2 \cos \theta \cos \theta')x'y' = a^2b^2.$

[10] But $\tan \theta \tan \theta' = -\dfrac{b^2}{a^2}.$ by § 158.

[11] Hence $a^2 \tan \theta = -\dfrac{b^2}{\tan \theta'},$

[12] or $a^2 \dfrac{\sin \theta}{\cos \theta} = -b^2 \dfrac{\cos \theta'}{\sin \theta'}.$ by Trig. 6.

[13] Hence $a^2 \sin \theta \sin \theta' = -b^2 \cos \theta \cos \theta'.$

[14] Hence $a^2 \sin \theta \sin \theta' + b^2 \cos \theta \cos \theta' = 0.$

Substituting 0 for this binomial in [9] we get

[15] $(a^2 \sin^2 \theta + b^2 \cos^2 \theta)x'^2 + (a^2 \sin^2 \theta' + b^2 \cos^2 \theta')y'^2 = a^2b^2.$

Since the $x'$ and $y'$ of this equation stand for the coordinates of any point on the ellipse when LM and NO are taken as the axes, this equation is the equation of the ellipse referred to LM and NO as axes. by § 39.

Since the point M is on the ellipse, its coordinates $a'$ and 0 must satisfy the equation of the ellipse. by § 40.

Substituting these values for the $x'$ and $y'$ of [15], we get

[16] $(a^2 \sin^2 \theta + b^2 \cos^2 \theta)a'^2 = a^2b^2.$

[17] Hence $a^2 \sin^2 \theta + b^2 \cos^2 \theta = \dfrac{a^2 b^2}{a'^2}.$

Since the point N is on the ellipse, its coordinates 0 and $b'$ must also satisfy the equation of the ellipse.

Substituting these values for the $x'$ and $y'$ of [15], we get

[18] $(a \sin^2 \theta' + b^2 \cos^2 \theta')b'^2 = a^2b^2.$

[19] Hence $a^2 \sin^2 \theta' + b^2 \cos^2 \theta' = \dfrac{a^2 b^2}{b'^2}.$

Substituting the right hand sides of [17] and [19] into [15] we get

[20] $\dfrac{a^2 b^2}{a'^2} x'^2 + \dfrac{a^2 b^2}{b'^2} y'^2 = a^2b^2.$

[21] Hence $a'^2y'^2 + b'^2x'^2 = a'^2b'^2.$

Since the old axes are no longer to be used, we may drop the accents over the $x$ and $y$ and write the equation

[22] $$a'^2y^2 + b'^2x^2 = a'^2b'^2.$$

Q. E. D.

PROPOSITION XXI

180. *The equation of the tangent to an ellipse when any pair of conjugate diameters is taken as the axes is*

$$y' - y = -\frac{b'^2x'}{a'^2y'}(x' - x),$$

*in which $x'$ and $y'$ are the coordinates of the point of tangency and $a'$ and $b'$ are the semi-conjugate diameters.*

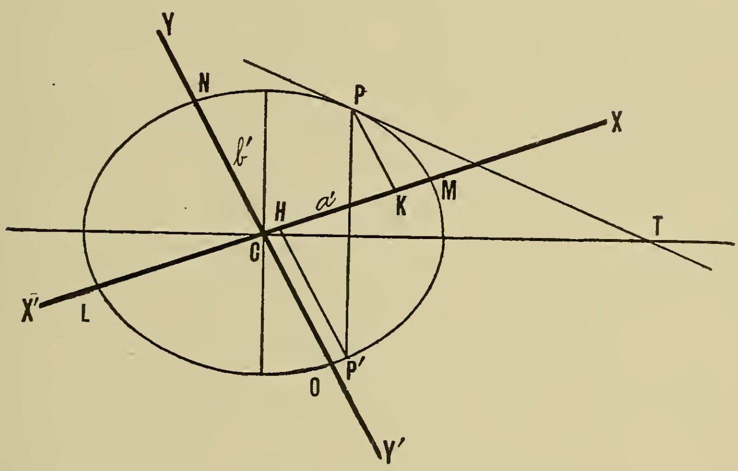



Fig. 57

Let LM and NO be two conjugate diameters.

Let LM be taken as the X axis and NO as the Y axis.

Let PT be tangent to the ellipse at P.

Draw the ordinates PK and P'H.

Let $x' \equiv$ CK and $y' \equiv$ PK.

$x'' \equiv$ CH " $y'' \equiv$ P'H.

$a' \equiv$ CM " $b' \equiv$ CN.

We are to prove that

$$y' - y = -\frac{b'^2 x'}{a'^2 y'} (x' - x)$$

is the equation of the tangent PT.

The secant PP′ is a straight line cutting the curve in two points, therefore its equation must be of the form

[1] $$y' - y = \frac{y'' - y'}{x'' - x'}(x' - x).$$ by § 58.

Since the point P is on the ellipse, its coordinates $x'$ and $y'$ must satisfy the equation of the ellipse.

Substituting them into the equation of the ellipse referred to conjugate diameters, we get

[2] $$a'^2 y'^2 + b'^2 x'^2 = a'^2 b'^2.$$ by § 179.

Since the point P′ is on the ellipse, its coordinates $x''$ and $y''$ must satisfy the equation of the ellipse.

Substituting them into the equation of the ellipse referred to conjugate diameters, we get

[3] $$a'^2 y''^2 + b'^2 x''^2 = a'^2 b'^2.$$ by § 179.

Proceeding as in § 134, we get

$$y' - y = -\frac{b'^2 x'}{a'^2 y'}(x' - x).$$

Q. E. D.

## PROPOSITION XXII

181. *When any pair of conjugate diameters is taken as the axes, the equation of the chord which joins the points of tangency of two tangents drawn to an ellipse from the same point without it, is*

$$a'^2 yy' + b'^2 xx' = a'^2 b'^2,$$

*in which $x'$ and $y'$ are the coordinates of the point from which the two tangents are drawn, and $a'$ and $b'$ are the semi-conjugate diameters.*

Let PT and P′T be two tangents drawn to the ellipse from the same point T.

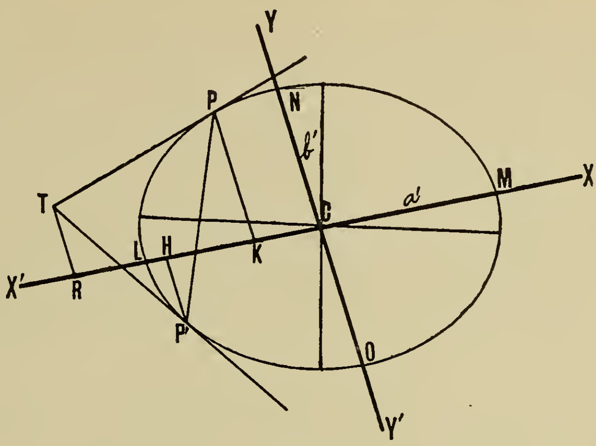



Fig 58

Let PP′ be the chord joining their points of tangency.

Let any two conjugate diameters LM and NO be taken as the axes, and draw the ordinates PK, P′H and TR.

Let $x' \equiv$ CR and $y' \equiv$ TR.
$a' \equiv$ CM " $b' \equiv$ CN.

We are to prove that

$$a'^2yy' + b'^2xx' = a'^2b'^2,$$

is the equation of the chord PP′.

Let $x'' \equiv$ CK and $y'' \equiv$ PK.
$x''' \equiv$ CH " $y''' \equiv$ P′H.

The equation of PT is

[1] $$y'' - y = -\frac{b'^2x''}{a'^2y''}(x'' - x).$$ by § 180.

[2] Hence $$a'^2yy'' - a'^2y''^2 = -b'^2xx'' + b'^2x''^2,$$

[3] and $$a'^2yy'' + b'^2xx'' = a'^2y''^2 + b'^2x''^2.$$

Since the point P is on the ellipse, its coordinates, $x''$ and $y''$, must satisfy the equation of the ellipse.

Substituting them into the equation of the ellipse referred to conjugate diameters, we get

[4] $$a'^2y''^2 + b'^2x''^2 = a'^2b'^2.$$ by § 179.

Hence substituting $a'^2b'^2$ for the second member of [3], we get

[5] $$a'^2yy'' + b'^2xx'' = a'^2b'^2,$$

which is the equation of PT.

Similarly we may show that the equation of P′T is

[6] $$a'^2yy''' + b'^2xx''' = a'^2b'^2.$$

Since the point T is on the tangent PT, its coordinates $x'$ and $y'$, must satisfy the equation of PT.

Hence substituting them for the $x$ and $y$ in [5], we get

[7] $$a'^2y'y'' + b'^2x'x'' = a'^2b'^2.$$

Similarly, since the point T is on the tangent P′T, by substituting the coordinates of T into [6], we get

[8] $$a'^2y'y''' + b'^2x'x''' = a'^2b'^2.$$

[9] Now $$a'^2yy' + b'^2xx' = a'^2b'^2,$$

is the equation of a straight line. by § 67.

But the coordinates $x''$, $y''$ of the point P will satisfy this equation, for if they are substituted for the $x$ and $y$ in it, we get a true equation, *viz.*, [7].

Hence the straight line represented by [9] must pass through the point P.

The coordinates $x'''$, $y'''$ of the point P′ will also satisfy [9], for if they are substituted for the $x$ and $y$ in it, we get a true equation, *viz.*, [8].

Hence the straight line represented by [9] must also pass through the point P′.

Therefore since the line represented by [9] passes through both points P and P′, it must be the chord PP′.

Therefore [9] must be the equation of the chord PP′.

Q. E. D.

182. *Corollary.—When the transverse axis is taken as the X axis and the conjugate axis as the Y axis, the equation of the chord becomes*

$$a^2yy' + b^2xx' = a^2b^2.$$

EXAMPLES

1. From the point 10, 5, two tangents are drawn to

$16y^2 + 9x^2 = 144$. Find the slope of the chord which joins the two points of tangency. Ans. Slope $= -\frac{9}{8}$.

2. Two tangents are drawn to $16y^2 + 9x^2 = 144$ at the extremities of the chord $y = -\frac{3}{4}x + 3$. Where do the tangents meet? Ans. $x = 4, y = 3$.

Construct a figure showing the tangents and the chord in each example.

## PROPOSITION XXIII

183. *The two tangents at the extremities of any chord of an ellipse meet on the diameter which bisects that chord.*

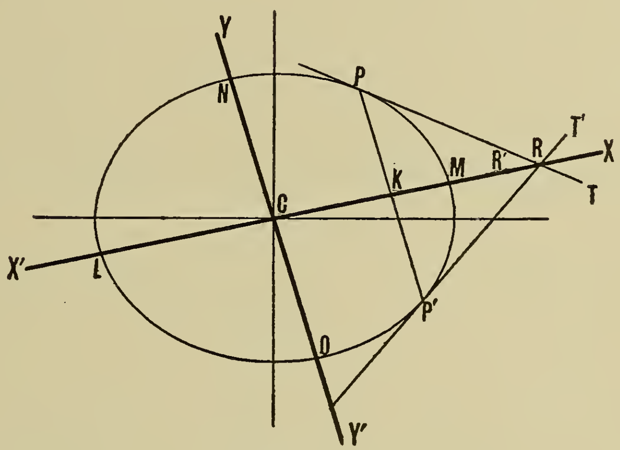



Fig. 59

Let PT and P′T′ be two tangents to the ellipse at the extremities of the chord PP′, and let LM be the diameter which bisects that chord.

We are to prove that PT and P′T′ meet on the diameter LM.

Let NO be the diameter which is conjugate to LM.

Let LM be taken as the X axis and NO as the Y axis.

$$\text{Let } a' \equiv \text{CM and } b' \equiv \text{CN}.$$
$$x' \equiv \text{CK}.$$

Now if R be the point where PT cuts the axis LM, then by § 180 and § 45

[1] $CR = \frac{a'^2}{x'}$.

Similarly, if R′ be the point where P′T′ cuts the axis LM,

[2] then $CR' = \frac{a'^2}{x'}$.

[3] Hence $CR = CR'$. by Geom. 1.

Therefore the points R and R′ coincide with each other and the two tangents meet the diameter at the same point.

Q. E. D.

## PROPOSITION XXIV

183. *If two tangents be drawn at the extremities of any focal chord,*

(*1*) *the two tangents will meet on the directrix, and*

(*2*) *the line joining the intersection of the two tangents to the focus will be perpendicular to the focal chord.*

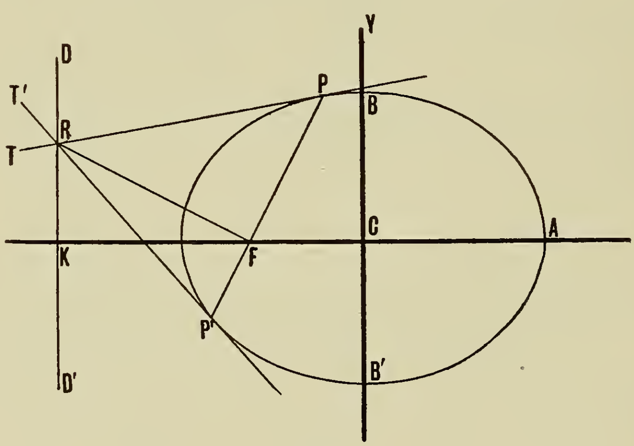



Fig. 60

Let PT and P′T′ be two tangents drawn to the ellipse at the extremities of the focal chord PP′. Let R be the intersection of the two tangents, and DD′ the directrix.

We are to prove

(1) that R will be on the directrix DD′.

(2) that RF will be the perpendicular to PP′.

Let $x' \equiv CK$ and $y' \equiv RK$.
$a \equiv CA$ " $b \equiv CB$.

Let $x$ and $y$ be the coordinates of any point on the chord PP′.

The equation of the chord PP′ is

[1] $$a^2yy' + b^2xx' = a^2b^2,$$ by § 182.

in which the $x'$ and $y'$ are the coordinates of the point R.

[2] $$CF = ae.$$ by § 128; [2].

Since the point F is on the chord PP′, its coordinates $x = ae$ and $y = 0$ must satisfy the equation of that chord.

Substituting these values of $x$ and $y$ into [1], we get

[3] $$b^2aex' = a^2b^2.$$

[4] Hence $$x' = \frac{a}{e}.$$

But $\frac{a}{e}$ is the distance from the center to the directrix.

by § 178.

Hence R, the intersection of the two tangents, must be on the directrix.

Q. E. D.

Again, since RF is a straight line passing through the two fixed points R and F, its equation must be of the form

[5] $$y' - y = \frac{y'' - y'}{x'' - x'}(x' - x).$$ by § 58.

In [5] let $x''$ and $y''$ stand for the coordinates of the point F, and $x'$ and $y'$ stand for the coordinates of the point R.

Then $x'' = ae$, by § 128, [2].
and $y'' = 0$.

Substituting these values of $x''$ and $y''$ into [5], we get

[6] $$y' - y = \frac{-y'}{ae - x'}(x' - x).$$

But $$x' = \frac{a}{e}$$ by [4].

Substituting this value of $x'$ into [6], we get

[7] $$y'-y=\frac{ey'}{a-ae^2}(x'-x).$$

[8] Hence $$y'-y=\frac{aey'}{a^2(1-e^2)}(x'-x),$$

which is the equation of RF.

[9] By [4] $$x'=\frac{a}{e}.$$

Substituting this value of $x'$ into [1], we get

[10] $$a^2yy'=-b^2\frac{a}{e}x+a^2b^2,$$

[11] or $$a^2ey'y=-b^2ax+a^2b^2e.$$

[12] Hence $$y=-\frac{b^2}{aey'}(x-ae).$$

[13] But $$b^2=a^2(1-e^2).$$ by § 128, [5].

[14] Hence $$y=-\frac{a^2(1-e^2)}{aey'}(x-ae),$$

which is the equation of the chord PP′.

Let $s\equiv$ the slope of the line RF,
and $s'\equiv$ " " " " " PP′.

From [8] we get

[15] $$s=\frac{aey'}{a^2(1-e^2)}$$ by § 53.

From [14] we get

[16] $$s'=-\frac{a^2(1-e^2)}{aey'}$$ by § 53.

[17] Hence $$ss'=\frac{aey'}{a^2(1-e^2)}\times-\frac{a^2(1-e^2)}{aey'}=-1,$$

[18] and $$1+ss'=0.$$

Therefore RF and PP′ are perpendicular to each other.
by § 62.

Q. E. D.

PROPOSITION XXV

184. *The locus of the intersection of two tangents to an ellipse which are perpendicular to each other, is a circle whose center is the center of the ellipse.*

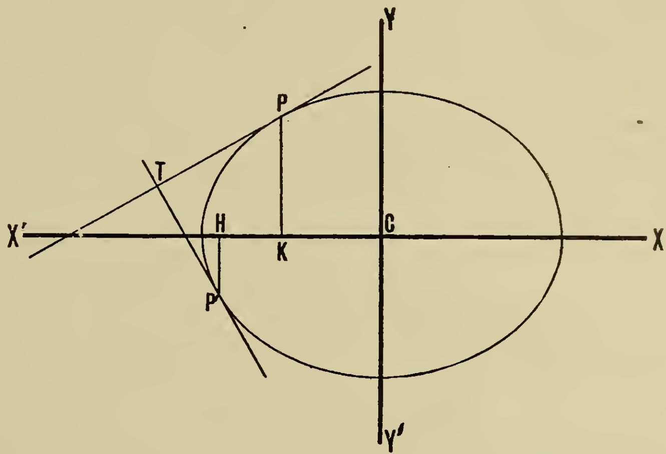



Fig. 61

Let PT and P′T be the two tangents to an ellipse at the points P and P′. Let them be perpendicular to each other at the point T.

Let P and P′ move along the ellipse, but so that PTP′ shall always be a right angle.

We are to prove that the locus of the point T will be a circle whose center is at the origin.

Let $x' \equiv$ CK and $y' \equiv$ PK.

The equation of the tangent PT is

[1] $$y' - y = -\frac{b^2 x'}{a^2 y'}(x' - x).$$ by § 134.

[2] Hence $a^2 y' y = a^2 y'^2 + b^2 x'^2 - b^2 x' x.$

Now since the point P is on the ellipse, its coordinates must satisfy the equation of the ellipse. by § 39.

[3] Hence $a^2 y'^2 + b^2 x'^2 = a^2 b^2.$ by § 111.

Substituting this value of $a^2 y'^2 + b^2 x'^2$ into [2], we get

[4] $$a^2 y' y = a^2 b^2 - b^2 x' x,$$

[5] hence $$y = \frac{b^2}{y'} - \frac{b^2 x'}{a^2 y'} x.$$

[6] But $$\frac{b^2}{y'} = \sqrt{\frac{b^4}{y'^2}} = \sqrt{b^2 \frac{a^2 b^2}{a^2 y'^2}}$$ which by [3]

$$= \sqrt{b^2 \frac{a^2 y'^2 + b^2 x'^2}{a^2 y'^2}} = \sqrt{\frac{a^2 b^2 y'^2}{a^2 y'^2} + \frac{b^4 x'^2}{a^2 y'^2}}.$$

[7] Hence $$\frac{b^2}{y'} = \sqrt{b^2 + a^2 \frac{b^4 x'^2}{a^4 y'^2}} = \sqrt{b^2 + a^2 \left(\frac{b^2 x'}{a^2 y'}\right)^2}.$$

Now substituting this value of $\frac{b^2}{y'}$ into [5], we get

[8] $$y = -\frac{b^2 x'}{a^2 y'} x + \sqrt{b^2 + a^2 \left(\frac{b^2 x'}{a^2 y'}\right)^2}.$$

Let $s \equiv -\frac{b^2 x'}{a^2 y'}$,

then [8] becomes

[9] $$y = sx + \sqrt{b^2 + a^2 s^2},$$

which is the equation of any tangent PT to an ellipse.

[10] Let $$y = s'x + \sqrt{b^2 + a^2 s'^2},$$

be the equation of the tangent P′T.

Then since PT and P′T are ⊥ to each other

[11] $$1 + ss' = 0.$$ by § 62.

[12] Hence $$s' = -\frac{1}{s}.$$

Substituting this value of $s'$ into [10], we get

[13] $$y = -\frac{x}{s} + \sqrt{b^2 + \frac{a^2}{s^2}},$$

which is the equation of P′T.

By transposition [9] and [13] become

[14] $$y - sx = \sqrt{b^2 + a^2 s^2}.$$

[15] $$y + \frac{x}{s} = \sqrt{b^2 + \frac{a^2}{s^2}}.$$

From [14] we get by squaring

[16] $y^2 - 2syx + s^2x^2 = b^2 + a^2s^2$, the equation of PT.

[17] From [15] $y^2 + 2\frac{xy}{s} + \frac{x^2}{s^2} = b^2 + \frac{a^2}{s^2}$.

Clearing [17] of fractions, we get

[18] $s^2y^2 + 2sxy + x^2 = b^2s^2 + a^2$, the equation of P′T.

Since the point T is on both tangents its coordinates must satisfy [16] and [18]. Hence we will let the $x$ and $y$ of [16] and [18] be the coordinates of T. Then these equations are simultaneous and may be combined.

Adding [16] and [18], we get

[19] $(1 + s^2)y^2 + (1 + s^2)x^2 = a^2(1 + s^2) + b^2(1 + s^2)$.

[20] Hence $y^2 + x^2 = a^2 + b^2$,

in which $x$ and $y$ are the coordinates of the point T.

Hence [20] is the equation of the locus traced out by the point T when the points P and P′ move along the ellipse.

by § 39.

Let $a^2 + b^2 = r^2$.

[21] Then [20] becomes $x^2 + y^2 = r^2$,

which is the equation of a circle whose center is at the origin. by § 113.

Therefore the locus traced out by the point T is a circle whose center is the center of the ellipse.

Q. E. D.

PROPOSITION XXVI

185. *If any chord of an ellipse pass through a fixed point and tangents be drawn at its extremities, and if the chord be made to revolve about the fixed point as a pivot, then the locus of the intersection of the two tangents will be a straight line whose equation is*

$$a'^2yy' + b'^2xx' = a'^2b'^2,$$

*in which $x'$ and $y'$ are the coordinates of the fixed point about which the chord revolves, and $a'$ and $b'$ are the semi-conjugate diameters which are taken as axes.*

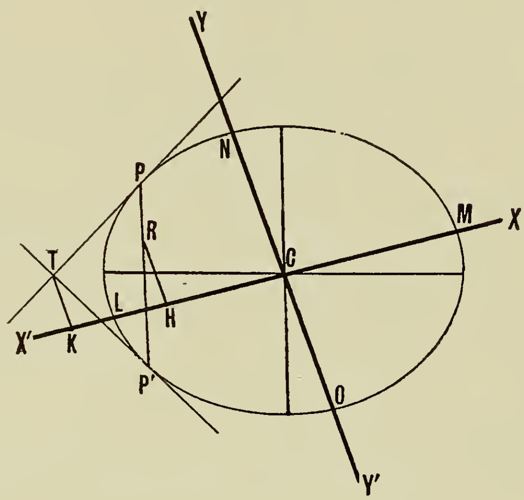



Fig. 62

Let PP′ be the chord passing through the fixed point R and let PT and P′T be the two tangents drawn to the ellipse at its extremities.

Let LM and NO be two conjugate diameters taken as axes.

$$\text{Let } x' \equiv CH \text{ and } y' \equiv RH,$$

$$a' \equiv CM \quad \text{"} \quad b' \equiv CN.$$

Let PP, revolve about R as a pivot.

We are to prove that the locus traced out by T is a straight line and that its equation is

$$a'^2yy' + b'^2xx'' = a'^2b'^2.$$

$$\text{Let } x'' \equiv CK \text{ and } y'' \equiv TK.$$

The equation of the chord PP′ is

[1] $$a'^2yy'' + b'^2xx'' = a'^2b'^2.$$ by § 181.

Since the point R is on this chord, its coordinates must satisfy the equation of this chord. by § 40.

Hence substituting $x'$ and $y'$ for the $x$ and $y$ in [1], we get

[2] $$a'^2y'y'' + b'^2x'x'' = a'^2b'^2.$$

Now as PP′ revolves about R, T will trace out a locus. Moreover [2] will be true for the coordinates of the point T wherever it may be as it traces out this locus. Hence the $x''$ and $y''$ of [2] stand for the coordinates of every point on the locus traced out by T. Therefore [2] must be the equation of that locus. by § 39.

This locus must also be a straight line. by § 67.

Since T is any point on the locus traced out by the intersection of the tangents, we may drop the accent marks from its coordinates $x''$ and $y''$ and write them $x$ and $y$. Then [2] may be written

[3] $$a'^2yy' + b'^2xx' = a'^2b'^2.$$

Therefore the locus of T is a straight line and its equation is [3].

Q. E. D.

186. *Corollary.—When the axes of the ellipse are taken as the axes of coordinates, the equation of the locus of the intersection of the tangents becomes*

$$a^2yy' + b^2xx' = a^2b^2.$$

187. **Supplemental Chords.**—Two chords drawn from the same point on an ellipse to the extremities of any diameter are called *supplemental chords.*

PROPOSITION XXVII

188. *If a chord be parallel to any diameter of an ellipse, the supplemental chord will be parallel to the conjugate of that diameter.*

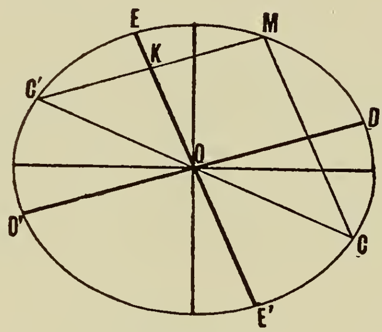



Fig. 63

Let MC and MC′ be two supplemental chords drawn to the extremities of the diameter CC′.

Let DD′ and EE′ be two conjugate diameters and let

$$MC \parallel EE'.$$

We are to prove that

$$MC' \parallel DD'.$$

[1] $\frac{C'O}{C'C} = \frac{C'K}{C'M}$. by Geom. 23.

[2] But $\frac{C'O}{C'C} = \frac{1}{2}$. by § 163.

[3] Hence $\frac{C'K}{C'M} = \frac{1}{2}$.

[4] Hence $C'K = \frac{1}{2}C'M$.

That is, the diameter EE′ ∥ MC bisects a system of chords ∥ MC′. But by hypothesis EE′ and DD′ are conjugate, and therefore EE′ bisects a system of chords parallel to DD′. by § 157.

Therefore $MC' \parallel DD'$. by Geom. 10.

Q. E. D.

## Polar Equations of the Ellipse

### PROPOSITION XXVIII

189. *When the right hand focus is taken as the pole, the polar equation of the ellipse is*

$$r = \frac{a(1-e^2)}{1+e\cos\theta},$$

*in which a is the semi-transverse axis of the ellipse, e is its eccentricity, r the radius vector of any point on the ellipse, and θ the vectorial angle.*

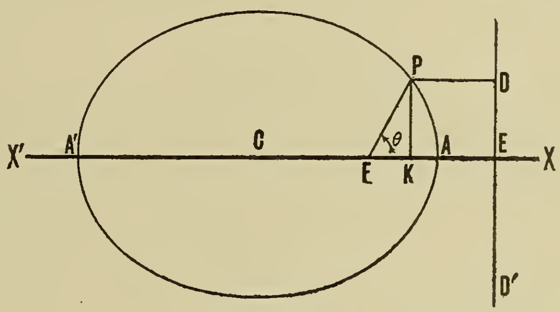



Fig. 64

Let F be the pole, XX′ the initial line and DD′ the directrix.

Let P be any point on the ellipse. Draw PD ⊥ the directrix and PK ⊥ XX′.

Let $\theta \equiv \angle PFX$, $r \equiv FP$, and $e \equiv$ the eccentricity.

We are to prove that

$$r = \frac{a(1-e^2)}{1+e\cos\theta}$$

is the polar equation of the ellipse.

| | | | |
|---|---|---|---|
| [1] | | $PF = ePD$. | by § 176. |
| [2] | Hence | $PF = e(EF - FK)$, | |
| [3] | or | $PF = eEF - eFK$. | |
| [4] | But | $EF = \frac{a(1-e^2)}{e}$, | by § 176, [5]. |

[5] and $FK = r \cos \theta$. by Trig. 2.

[6] Hence by [3] $PF = r = a(1 - e^2) - er \cos \theta$,

[7] and $r(1 + e \cos \theta) = a(1 - e^2)$.

[8] Hence $$r = \frac{a(1 - e^2)}{1 + e \cos \theta}.$$

Q. E. D.

190. *Corollary 1.—The polar equation of the ellipse will become*

$$r = \frac{a(1 - e^2)}{1 - e \cos \theta}$$

*when the left hand focus is the pole.*

191. *Corollary 2.—The polar equation of the ellipse may be written*

$$r = \frac{p}{2(1 + e \cos \theta)},$$

*in which $p$ is the parameter.*

[1] For $(1 - e^2) = \frac{b^2}{a^2}$. by § 128, [5].

Substituting into [8], § 189, we get

[2] $$r = \frac{b^2}{a} \frac{1}{1 + e \cos \theta}.$$

Let $p$ be the parameter.

[3] Then $\frac{p}{2} : b :: b : a$. by § 125.

[4] Hence $\frac{p}{2} = \frac{b^2}{a}$.

Substituting into [2] we get

[5] $$r = \frac{p}{2(1 + e \cos \theta)}.$$

Q. E. D.

EXAMPLES

1. What is the value of $r$ when $\theta = 0$? What line in Fig. 64 does $r$ then represent? Ans. $r = a - ae$.
$r = FA$.

2. What is the value of $r$ when $\theta = 180°$? What line in Fig. 64 does $r$ then represent. Ans. $r = a + ae$.
$r = FA'$.

3. What is the value of $\theta$ when $r$ is drawn to the extremity of the conjugate axis?
Ans. $\theta = \cos^{-1}(-e)$.

4. The eccentricity of an ellipse is $\frac{1}{2}$. What is the value of $\theta$ when $r = \frac{1}{2}a$? Ans. $\theta = 0$.

## PROPOSITION XXIX

192. *When the pole is at the center the polar equation of the ellipse is*

$$r^2 = \frac{b^2}{1 - e^2 \cos^2 \theta},$$

*in which b is the semi-conjugate axis of the ellipse, e its eccentricity, r the radius vector of any point on the ellipse, and θ the vectorial angle.*

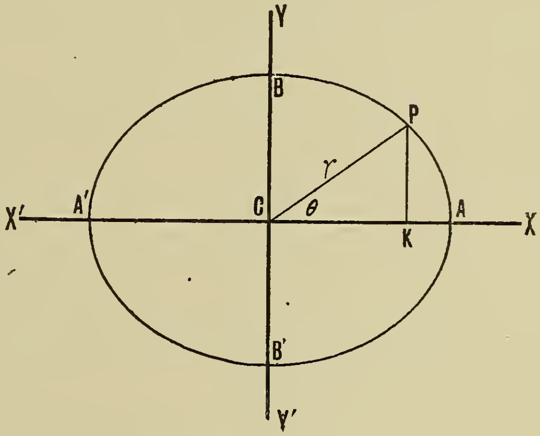



Fig. 65

Let P be any point on the ellipse and draw its ordinate PK.

$\theta \equiv \angle PCK$ and $r \equiv CP$,
$a \equiv CA$ " $b \equiv CB$.

We are to prove that

$$r^2 = \frac{b^2}{1 - e^2 \cos^2 \theta}.$$

Let $x \equiv$ CK and $y \equiv$ PK.

[1] $x = r \cos \theta$. by Trig. 2.

[2] $y = r \sin \theta$. by Trig. 1.

When YY′ and XX′ are taken as the axes of coordinates, the equation of the ellipse is

[3] $a^2y^2 + b^2x^2 = a^2b^2$. by § 111.

Substituting the values of $x$ and $y$ given in [1] and [2], into [3], we get

[4] $a^2r^2 \sin^2 \theta + b^2r^2 \cos^2 \theta = a^2b^2$.

[5] Hence $r^2 = \dfrac{a^2b^2}{a^2 \sin^2 \theta + b^2 \cos^2 \theta}$.

[6] But $\sin^2 \theta = 1 - \cos^2 \theta$. by Trig. 5.

[7] Hence $r^2 = \dfrac{a^2b^2}{a^2 - (a^2 - b^2) \cos^2 \theta}$,

[8] and $r^2 = \dfrac{b^2}{1 - \dfrac{a^2 - b^2}{a^2} \cos^2 \theta}$.

[9] But $\dfrac{a^2 - b^2}{a^2} = e^2$. by § 128, [4].

[10] Hence $r^2 = \dfrac{b^2}{1 - e^2 \cos^2 \theta}$.

Q. E. D.

193. *Corollary.— When the pole is at the center the polar equation of the ellipse may be written*

$$r^2 = \frac{a^2(1 - e^2)}{1 - e^2 \cos^2 \theta}.$$

[1] For $\dfrac{b^2}{a^2} = 1 - e^2$. by § 128, [5].

[2] Hence $b^2 = a^2(1 - e^2)$.

Substituting this value of $b^2$ into [10] § 192, we get

[3] $r^2 = \dfrac{a^2(1 - e^2)}{1 - e^2 \cos^2 \theta}$.

EXAMPLES

1. The center being the pole, what is the value of $r$ when $\theta = 0$? Ans. $r = a$.

2. What is the value of $r$ when $\theta = 90°$? Ans. $r = b$.

3. In an ellipse whose eccentricity is $\frac{1}{2}$, what is the value of $r$ when $\theta = 60°$? Ans. $r = 0.89a$.

4. What is the value of $r$ in the equilateral ellipse? Ans. $r = b$.

5. Two tangents are drawn from a point to a circle; required the equation of the chord of contact in the following:

(1) From (4, 2) to $x^2 + y^2 = 9$. Ans. $4x + 2y = 9$.

(2) From $(a, b)$ to $x^2 + y^2 = c^2$. Ans. $ax + by = c^2$.

6. Find the equation of a circle through (4, 0), (0, 4), (6, 4). Ans. $x^2 + y^2 - 6x - 6y + 8 = 0$.

*Suggestion.*—Join (4, 0) to (0, 4) and to (6, 4) by straight lines; then erect perpendiculars at the middle points of these two lines; their intersection will be the center of the circle, and the distance from this center to any one of the points will be the radius. Then make use of § 115.

7. Find the equation of a circle through (0, 0), $(-8a, 0)$, $(0, 6a)$. Ans. $x^2 + y^2 + 8ax - 6ay = 0$.

8. Find the equation of a circle through (10, 4), (17, —3), and radius = 13.

9. Find the equation of a circle touching each axis at a distance of 4 units from the origin.
Ans. $x^2 + y^2 - 8x - 8y + 16 = 0$.

10. Find the equation of a circle through (5, 6), and having its center at the intersection of $y = 7x - 3$; $4y - 3x = 13$.
Ans. $(x - 1)^2 + (y - 4)^2 = 20$.

11. What must be the value of $s$ in order that the line $y = sx - 4$ may touch the circle $x^2 + y^2 = 2$?
Ans. $s = \pm\sqrt{7}$.

12. Required the equation of a tangent to the ellipse $\frac{x^2}{5} + \frac{y^2}{6} = 1$, whose inclination to the X axis is 45°.

13. If $3y = 5x$ is a diameter of $\frac{x^2}{4} + \frac{y^2}{9} = 1$, what is the equation of the conjugate diameter? Ans. $20y + 29x = 0$.

14. Required the area of $\frac{x^2}{4} + \frac{y^2}{10} = 1$. Ans. $2\pi\sqrt{10}$.

15. The extremities of a line of constant length slide along the coordinate axes. Required the locus traced by any point of the line.

16. If from the extremity of any diameter straight lines be drawn to the foci, prove that their product is equal to the square of half the conjugate diameter.

17. Find the equation of a diameter parallel to the normal drawn to the ellipse at $(x, y)$, the semi-axes being $a$ and $b$.

18. Write the equations of diameters conjugate to the lines $x - y = 0$; $x + y = 0$; $ax = by$; $ay = bx$.

Ans. $b^2x + a^2y = 0$; $b^2x - a^2y = 0$;
$a^3y + b^3x = 0$; $bx + ay = 0$.

19. The center of an ellipse is at (4, 7), the major and minor axes are 14 and 8. Required its equation, the axes being parallel to the axes of coordinates.

20. Prove that the length of a line drawn from the center to a tangent and parallel to either focal radius of the point of contact is equal to the semi-major axis.

21. The minor axis = 12, the double focal ordinate = 5, required the equation of the ellipse, the origin being at the left hand vertex.

Ans. $\frac{25x^2}{144} + y^2 = 5x$.

# CHAPTER X

## The Hyperbola

194. **The Hyperbola.**—The *hyperbola* is the locus of a point moving in a plane in such a way that the difference between its distances from two fixed points in the plane is constant.

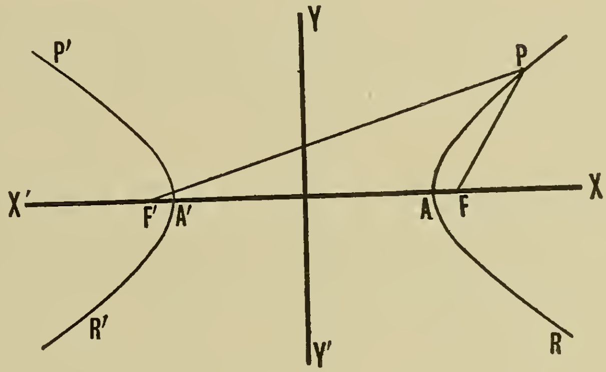



Fig. 66

Let F and F' be the two fixed points in the plane. Let P be a point moving in this plane in such a way that PF' — PF is constant.

Then the line PAR traced out by P is one branch of an hyperbola.

The line P'A'R' traced out by a point P' moving in the same way as P is another branch of the hyperbola.

The whole locus PARP'A'R' is an hyperbola.

PROBLEM

195. To draw an hyperbola.

Let F'H (Fig. 67) represent a ruler which may be moved about the point F' as a pivot.

Let $l \equiv$ the length of the ruler and $a \equiv$ any constant.

Take an inelastic thread whose length is $l - 2a$ and fasten one end of it at F and the other at H'. Place a pencil point

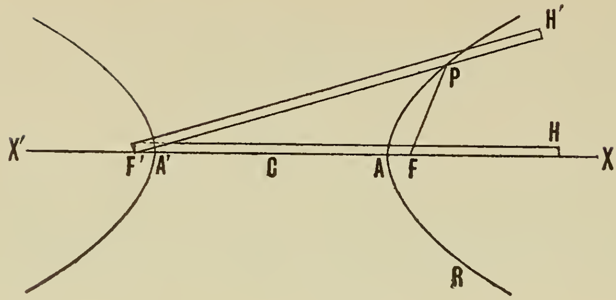



Fig. 67

against the thread so as to form a loop at some point P on the edge of the ruler. Now, keeping the thread stretched and the pencil point against the edge of the ruler, turn the ruler about the point F'.

As the point P moves F'P and FP are both increased or both decreased by the same amount. Hence F'P — PF remains constant, and therefore the locus traced out by P will be one branch of an hyperbola.

The other branch of the hyperbola is drawn by taking the point P as the pivot about which the ruler is turned.

It will be shown hereafter that in Fig. 67

$$CA = a.$$

196. *Corollary.—It is obvious that the locus drawn in this way must cut the line F'F in two points A and A'.*

197. **The Foci.**—The two fixed points are called the *foci*.

198. **The Focal Radii.**—The distances from the foci to any point on the hyperbola are called the *focal radii* of that point.

199. **The Vertices.**—The points in which the hyperbola cuts the straight line passing through the foci are called the *vertices* of the hyperbola.

200. **The Transverse Axis.**—The line which joins the vertices is called the *transverse axis*.

201. **The Center.**—The middle of the transverse axis is called the *center* of the hyperbola.

**202. The Conjugate Axis.**—A straight line drawn through the center perpendicular to the transverse axis, bisected by it and equal to twice the square root of the difference between the square of the distance from the focus to the center and the square of the semi-transverse axis is called the *conjugate axis*.

PROPOSITION I

203. *The difference between the focal radii of any point on the hyperbola is equal to the transverse axis.*

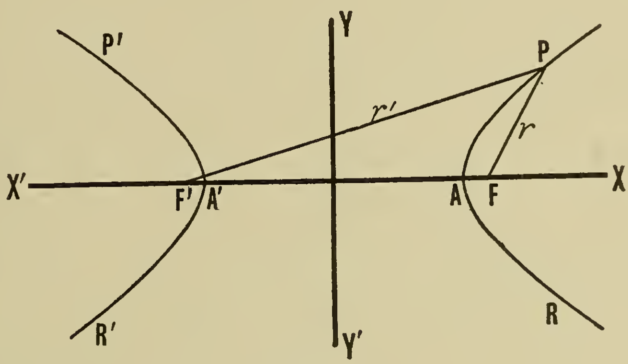



Fig. 68

Let P be any point on the hyperbola, $r$ and $r'$ its focal radii, and AA′ its transverse axis.

We are to prove that

$$r' - r = AA'.$$

Let the point P move along the branch PAR of the hyperbola. When it reaches A we have

[1] $r' - r = F'A - FA.$ by § 195.

Now let P′ move along the branch P′A′R′ until it reaches A′.

[2] Then $r' - r = FA' - F'A'.$

From [1] and [2] we get

[3] $F'A - FA = FA' - F'A'.$

[4] Now $F'A = F'A' + A'A,$

[5] and $FA' = FA + AA'.$

Substituting these values into [3] we get

[6] $F'A' + A'A - FA = FA + A'A - F'A'$;

[7] hence $2F'A' = 2FA$,

[8] and $F'A' = FA$.

Substituting the right hand side of [4] into [1], we get

[9] $r' - r = F'A' + A'A - FA$.

But from [8] we get

[10] $F'A' - FA = 0$.

Therefore [9] becomes

[11] $r' - r = A'A$.

Q. E. D.

PROPOSITION II

204. *The equation of an hyperbola is*

$$a^2y^2 - b^2x^2 = -a^2b^2,$$

*in which a is the semi-transverse axis, b the semi-conjugate axis, and x and y the coordinates of any point on the hyperbola.*

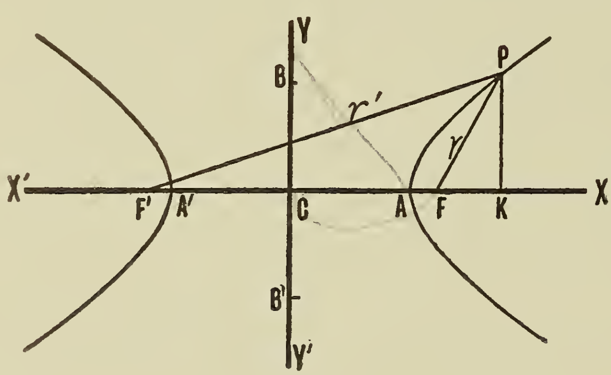


Fig. 69

Let BB′ be the conjugate and AA′ the transverse axis of the hyperbola.

Let AA′ and BB′ produced be the axes of coordinates.

Draw PK ∥ YY′.

Let $x \equiv CK$ and $y \equiv PK$,
$a \equiv CA$ " $b \equiv CB$.

We are to prove that

[1] $a^2y^2 - b^2x^2 = -a^2b^2$

is the equation of the hyperbola.

Let $r \equiv PF$, $r' \equiv PF'$, and $c \equiv CF$.

[2] $r'^2 = \overline{F'K}^2 + \overline{PK}^2$. by Geom. 26.

[3] But $F'K = x + c$.

[4] Hence $r'^2 = (x + c)^2 + y^2$,

[5] or $r' = \sqrt{(x + c)^2 + y^2}$.

[6] $r^2 = \overline{FK}^2 + \overline{PK}^2$. by Geom. 26.

[7] But $FK = x - c$.

[8] Hence $r^2 = (x - c)^2 + y^2$,

[9] or $r = \sqrt{(x - c)^2 + y^2}$.

[10] But $r' - r = 2a$. by § 203.

[11] Hence $\sqrt{(x + c)^2 + y^2} - \sqrt{(x - c)^2 + y^2} = 2a$.

Clearing this equation of radicals as in § 111, [11], we get

[12] $a^2y^2 - (c^2 - a^2)x^2 = -a^2(c^2 - a^2)$.

[13] But $b = \sqrt{c^2 - a^2}$. by § 202.

[14] Hence $b^2 = c^2 - a^2$.

Substituting this value of $c^2 - a^2$ into [12], we get

[15] $a^2y^2 - b^2x^2 = -a^2b^2$.

Since in [15] the $x$ and $y$ stand for the coordinates of any point on the hyperbola, that equation must be the equation of the hyperbola.

Q. E. D.

205. *Corollary 1.—Since in the ellipse $b^2$ stands for $a^2 - c^2$ but in the hyperbola it stands for $c^2 - a^2 = -(a^2 - c^2)$, any equation of the ellipse may be changed into the corresponding equation of the hyperbola by substituting $-b^2$ for $b^2$.*

206. *Corollary 2.—The distance from the center to the focus is equal to the distance from the vertex to the extremity of the conjugate axis.*

For

[1] $b^2 = c^2 - a^2$. by § 202.

[2] Hence $c^2 = a^2 + b^2$.

[3] But $a^2 + b^2 = \overline{AB}^2$. by Geom. 26.

[4] Hence $c^2 = \overline{AB}^2$,

[5] and $CF = AB$.

207. **The Equilateral Hyperbola.**—The *equilateral hyperbola* is that hyperbola whose axes are equal to each other.

208. *Corollary.—The equation of the equilateral hyperbola is*

$$y^2 - x^2 = -a^2,$$

*in which a is the semi-transverse axis.*

The equation of any hyperbola is

[1] $a^2y^2 - b^2x^2 = -a^2b^2$. by § 204.

But in the equilateral hyperbola

[2] $a^2 = b^2$. by § 207.

Hence dividing [1] by [2] member by member, we get

[3] $y^2 - x^2 = -a^2$.

Compare with § 113.

## PROPOSITION III

209. *If to an hyperbola and the equilateral hyperbola which has the same transverse axis, ordinates be drawn to the same point on the transverse axis, then the ordinate of the first hyperbola will be to the ordinate of the equilateral hyyerbola as the conjugate axis is to the transverse axis.*

Let PAM be any hyperbola, AA′ its transverse axis, and BB′ its conjugate axis.

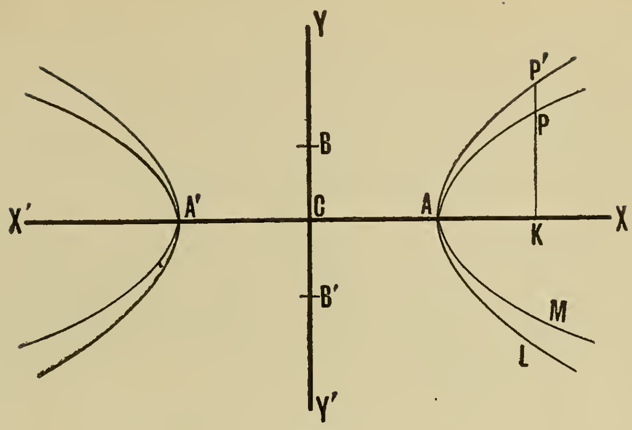



Fig. 70

Let P′AL be the equilateral hyperbola having the same transverse axis.

Draw the ordinates P′K and PK to the same point K.

Let $x \equiv$ CK, $y \equiv$ PK and $y' \equiv$ P′K.
Let $a \equiv$ CA and $b \equiv$ CB.

We are to prove that

PK : P′K :: BB′ : AA′.

Since P is on the hyperbola PAM, its coordinates $x$ and $y$ must satisfy the equation of that hyperbola. by § 40.

Hence, letting $x$ and $y$ of the equation of the hyperbola stand for the coordinates of the point P, we get

[1] $$a^2y^2 - b^2x^2 = -a^2b^2,$$ by § 204.

[2] or $$y^2 = \frac{b^2}{a^2}(x^2 - a^2).$$

Since the point P′ is on the equilateral hyperbola P′AL, its coordinates $x$ and $y'$ must satisfy the equation of the equilateral hyperbola. by § 40.

Hence substituting $x$ and $y'$ for the $x$ and $y$ of that equation, we get

[3] $$y'^2 = x^2 - a^2.$$ by § 208.

Dividing the members of [2] by the corresponding members of [3], we get

[4] $$\frac{y^2}{y'^2} = \frac{b^2}{a^2}.$$

[5] Hence $$\frac{y}{y'} = \frac{b}{a} = \frac{BB'}{AA'},$$

[6] or $$y : y' :: BB' : AA'.$$

Q. E. D.

210. **Conjugate Hyperbolas.**—Two *hyperbolas* are *conjugate* to each other when the transverse axis of each is the conjugate axis of the other.

PROPOSITION IV

211. *The equation of the conjugate to any hyperbola is*

$$a^2y^2 - b^2x^2 = a^2b^2,$$

*in which b is the semi-transverse and a the semi-conjugate axis of the conjugate hyperbola.*

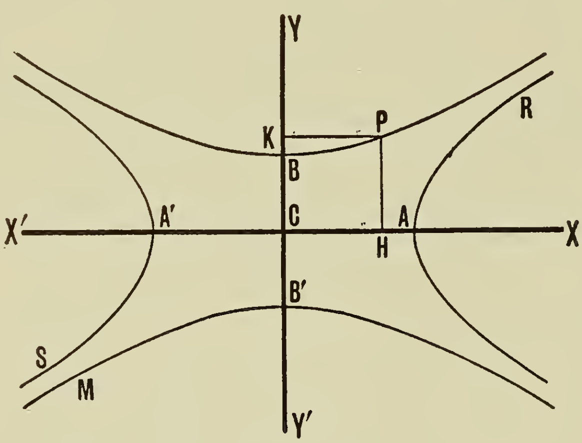



Fig 71

Let PBB'M be conjugate to the hyperbola RAA'S.

Let $a \equiv$ CA and $b \equiv$ CB.

Let P be any point on the conjugate hyperbola PBB'M.

We are to prove that the equation of the hyperbola PBB'M is

$$a^2y^2 - b^2x^2 = a^2b^2.$$

If for the conjugate hyperbola PBB'M we take YY' for the X axis, and XX' for the Y axis, then since the point P is on the hyperbola PBB'M, its coordinates $x =$ PH and $y =$ PK must satisfy the equation of the hyperbola. by § 40.

But for the conjugate hyperbola, $b$ is the semi-transverse axis, and $a$ is the semi-conjugate axis. by § 210.

Hence substituting PH for $x$, PK for $y$, $b$ for $a$ and $a$ for $b$ into the equation of the hyperbola, we get

[1] $$b^2\overline{PK}^2 - a^2\overline{PH}^2 = - a^2b^2.$$ by § 204.

But for the hyperbola RAA'S and in all other theorems, we have taken XX' for the X axis and YY' for the Y axis. Hence for the sake of uniformity, we will do the same for the present theorem.

Then PK = CH will be represented by $x$ and PH will be represented by $y$.

Substituting $x$ for PK and $y$ for PH in [1], we get

[2] $$b^2x^2 - a^2y^2 = - a^2b^2,$$

[3] Hence $$a^2y^2 - b^2x^2 = a^2b^2.$$

Now since in [3] $x$ and $y$ are the coordinates of P, and P is any point on the hyperbola PBB'M, [3] must be the equation of that hyperbola. by § 39.

Q. E. D.

*Corollary.—The equations of any hyperbola and its conjugate differ only in the signs of their absolute terms.*

### PROPOSITION V

**212.** *The squares of the ordinates of any two points on an hyperbola are to each other as the products of the segments which they make on the transverse axis.*

Let P and P' be any two points on an hyperbola, and let PK and P'H be their ordinates.

Let $x' \equiv$ CK and $y' \equiv$ PK.

$x'' \equiv$ CH " $y'' \equiv$ P'H.

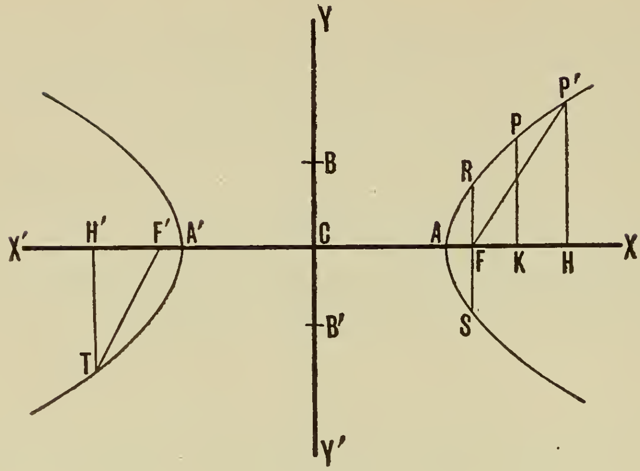



Fig. 72

We are to prove that

$$y'^2 : y''^2 :: A'K.KA : A'H\ HA.$$

$$\text{Let } a \equiv CA = CA'.$$

[1] $A'K = a + x'$ and $KA = x' - a.$

[2] $A'H = a + x''$ and $HA = x'' - a.$

Since the point P is on the hyperbola, its coordinates $x'$ and $y'$ must satisfy the equation of the hyperbola. by § 40.

Substituting $x'$ and $y'$ for the $x$ and $y$ of that equation, we get

[3] $a^2y'^2 - b^2x'^2 = - a^2b^2.$ by § 204.

Similarly, since the point P′ is also on the hyperbola, we get

[4] $a^2y''^2 - b^2x''^2 = - a^2b^2.$

By transposing and factoring, [3] becomes

[5] $a^2y'^2 = b^2(x'^2 - a^2),$

and [4] becomes

[6] $a^2y''^2 = b^2(x''^2 - a^2).$

Dividing the members of [5] by the corresponding members of [6], we get

[7] $\dfrac{y'^2}{y''^2} = \dfrac{x'^2 - a^2}{x''^2 - a^2}.$

[8] Hence $y'^2 : y''^2 :: (x' + a)(x' - a) : (x'' + a)(x'' - a)$,

[9] or $y'^2 : y''^2 ::$ A'K.KA : A'H.HA.

Q. E. D.

213. *Corollary 1.—Ordinates at equal distances from the center are equal.*

214. *Corollary 2.—The hyperbola is symmetrical with respect to both axes.*

215. *Corollary 3.—If the ordinates of any two points on an hyperbola be at equal distances from the center, the points will be equally distant from the adjacent foci.*

That is if CH = CH′,

then will F′T = FP′.

216. **The Parameter.**—The *parameter* of an hyperbola is the double ordinate which passes through the focus.

217. *Corollary.—The parameter is a third proportional to the transverse and conjugate axis.*

Since the point R is on the hyperbola, its coordinates CF and RF must satisfy the equation of the hyperbola. by § 40.

Hence substituting CF for $x$ and RF for $y$ in that equation, we get

[1] $a^2.\overline{RF}^2 - b^2.\overline{CF}^2 = -a^2b^2$. by § 204.

[2] But $\overline{CF}^2 = a^2 + b^2$. by § 202.

[3] Hence $a^2.\overline{RF}^2 - b^2(a^2 + b^2) = -a^2b^2$,

[4] or $a^2.\overline{RF}^2 - a^2b^2 - b^4 = -a^2b^2$.

[5] Hence $a^2.\overline{RF}^2 = b^4$.

[6] $a.RF = b^2$.

[7] $2a.2RF = 4b^2$.

[8] $2RF : 2b :: 2b : 2a$. by Geom. 56.

[9] or RS : BB′ :: BB′ : AA′. by § 213.

**218. The Eccentricity.**—The *eccentricity* of an hyperbola is the quotient of the distance from the focus to the center by the semi-transverse axis.

Let $e \equiv$ the eccentricity.

In Fig. 72 let $c \equiv$ CF and $a \equiv$ CA.

[1] Then $$e = \frac{\text{CF}}{\text{CA}} = \frac{c}{a},$$

[2] and $$c = ae.$$

[3] But $$a^2 + b^2 = c^2.$$ by § 202.

[4] Hence $$\frac{a^2 + b^2}{a^2} = e^2,$$

[5] and $$\frac{b^2}{a^2} = e^2 - 1.$$

219. *Corollary.—The eccentricity of an hyperbola is greater than 1.*

For in [1], § 218, CF is greater than CA.

EXAMPLES

1. What are the semi-axes and the eccentricities of the following hyperbolas?

$$25y^2 - 16x^2 = -400,$$

$$3y^2 - 2x^2 = 12,$$

and $$\frac{x^4}{4} - y^2 = m.$$

2. The equation of an hyperbola is $16y^2 - 9x^2 = -144$. What is the distance of the focus from the center, and the distance of the focus from each of the vertices?

3. The distance from the vertex of an hyperbola to the end of the conjugate axis is 5, and its semi-transverse axis is 4. What is the eccentricity and the equation of the hyperbola?

4. Find the eccentricity of an equilateral hyperbola.

## PROPOSITION VI

**220.** *If $r'$ be the longer and $r$ the shorter focal radius of any point on the hyperbola, then*

$$r' = ex + a,$$

$$\text{and} \qquad r = ex - a,$$

*in which $e$ is the eccentricity and $a$ the semi-transverse axis of the hyperbola.*

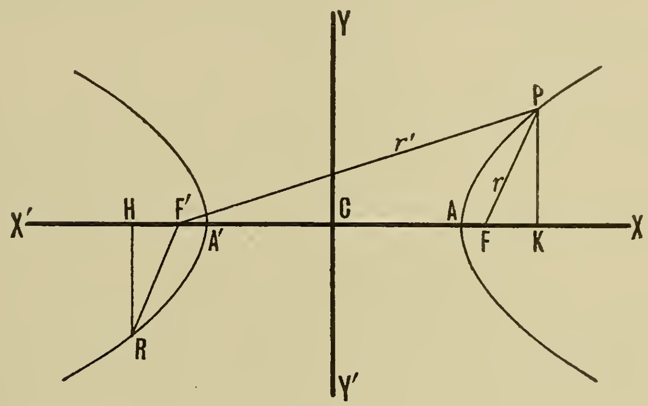



Fig. 73

In Fig. 73

let $r' \equiv F'P$ and $r \equiv FP$,

$x \equiv CK$ " $y \equiv PK$,

and $c \equiv CF$ " $a \equiv CA$.

We are to prove that

$$r' = ex + a,$$

$$\text{and } r = ex - a.$$

[1] $r'^2 = y^2 + (x + c)^2$. by Geom. 26.

[2] Hence $r'^2 = y^2 + x^2 + 2cx + c^2$.

[3] But $a^2y^2 - b^2x^2 = -a^2b^2$. by § 204.

[4] Hence $y^2 = \dfrac{b^2}{a^2}(x^2 - a^2)$.

Substituting this value of $y^2$ into [2], we get

[5] $r'^2 = \dfrac{b^2}{a^2}(x^2 - a^2) + x^2 + 2cx + c^2$.

[6] But $c^2 = b^2 + a^2$. by § 202.

[7] Hence $r'^2 = \frac{b^2}{a^2}(x^2 - a^2) + x^2 + 2cx + a^2 + b^2$,

[8] or $r'^2 = \frac{b^2 x^2}{a^2} - b^2 + x^2 + 2cx + a^2 + b^2$.

[9] Now $c = ae$. by § 218, [2].

[10] Hence $r'^2 = \frac{b^2 + a^2}{a^2} x^2 + 2aex + a^2$,

[11] or $r'^2 = e^2x^2 + 2aex + a^2 = (ex + a)^2$. by § 218, [4].

[12] Hence $r' = ex + a$.

Similarly it may be shown that

[13] $r = ex - a$.

Q. E. D.

## PROPOSITION VII

221. *The equation of the tangent to an hyperbola is*

$$y' - y = \frac{b^2x'}{a^2y'}(x' - x),$$

*in which $x'$ and $y'$ are the coordinates of the point of tangency, and $a$ and $b$ are the semi-axes.*

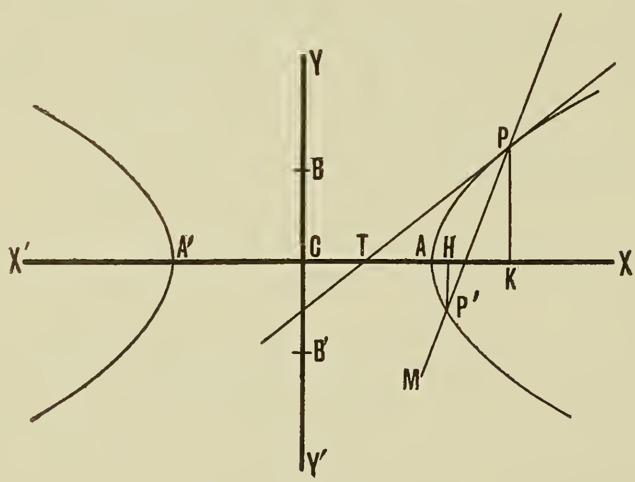



Fig. 74

Let PT be a tangent to the hyperbola at the point P.

Let $x' \equiv$ CK and $y' \equiv$ PK.

$a \equiv$ CA " $b \equiv$ CB.

We are to prove that

$$y' - y = \frac{b^2 x'}{a^2 y'}(x' - x).$$

Let PM be a secant cutting the hyperbola at the two points P and P'.

Let $x'' \equiv \text{CH}$ and $y'' \equiv \text{P'H}$.

Since the secant is a straight line passing through two fixed points, its equation must be of the form

[1] $$y' - y = \frac{y'' - y'}{x'' - x'}(x' - x).$$ by § 58.

Since P is on the hyperbola, its coordinates $x'$ and $y'$ must satisfy the equation of the hyperbola. by § 40.

Hence substituting $x'$ and $y'$ for the $x$ and $y$ of that equation, we get

[2] $$a^2 y'^2 - b^2 x'^2 = -a^2 b^2.$$ by § 204.

Similarly, since the point P' is on the hyperbola, its coordinates $x''$ and $y''$ must also satisfy the equation of the hyperbola. by § 40.

Hence substituting $x''$ and $y''$ for the $x$ and $y$ of that equation, we get

[3] $$a^2 y''^2 - b^2 x''^2 = -a^2 b^2.$$ by § 204.

Subtracting the members of [2] from the corresponding members of [3], we get

[4] $$a^2(y''^2 - y'^2) - b^2(x''^2 - x'^2) = 0,$$

[5] or $$a^2(y''^2 - y'^2) = b^2(x''^2 - x'^2).$$

[6] Hence $$a^2(y'' - y')(y' + y'') = b^2(x'' - x')(x' + x''),$$

[7] and $$\frac{y'' - y'}{x'' - x'} = \frac{b^2}{a^2}\,\frac{x' + x''}{y' + y''}.$$

Substituting this value of $\frac{y'' - y'}{x'' - x'}$ into [1], we get

[8] $$y' - y = \frac{b^2}{a^2}\,\frac{x' + x''}{y' + y''}(x' - x).$$

Now let the point P' move along the hyperbola towards P.

Then the secant will revolve about the point P as a pivot and will continually approach the tangent PT, and when P′ reaches P it will coincide with the tangent.

When P′ reaches the point P we will have

[9] $$x'' = x' \text{ and } y'' = y',$$

and the fraction in [8] becomes

[10] $$\frac{b^2}{a^2}\,\frac{2x'}{2y'} = \frac{b^2 x'}{a^2 y'},$$

and [8] becomes

[11] $$y' - y = \frac{b^2 x'}{a^2 y'}\,(x' - x).$$

Now the $x$ and $y$ of [8] stand for the coordinates of every point on the secant in every position which it takes as it revolves about P. Hence they stand for the coordinates of every point on it when it coincides with the tangent.

But when the secant coincides with the tangent, [8] takes the form of [11]. Hence the $x$ and $y$ of [11] stand for the coordinates of every point on the tangent.

Therefore [11] is the equation of the tangent. by § 39.

Q. E. D.

222. *Corollary.—The fraction* $\frac{b^2 x'}{a^2 y'}$, *is the slope of the tangent.*

For proof compare § 135.

223. **The Subtangent.**—The *subtangent* is the distance measured along the X axis from the ordinate of the point of tangency to the tangent.

*Corollary 1.—The length of the subtangent is* $\frac{x'^2 - a^2}{x'}$.

For proof compare § 138.

224. *Corollary 2.—If different hyperbolas have the same transverse axis and ordinates be drawn to each from the same point on the transverse axis, then all the subtangents will be equal to each other.*

Compare § 140.

Draw a figure showing all the lines referred to in this corollary.

225. **The Normal.**—The *normal* to an hyperbola is a straight line perpendicular to the tangent at the point of tangency.

PROPOSITION VIII

226. *The equation of the normal to an hyperbola is*

$$y' - y = -\frac{a^2 y'}{b^2 x'}(x' - x),$$

*in which $x'$ and $y'$ are the coordinates of the point of tangency, and $a$ and $b$ are the semi-axes.*

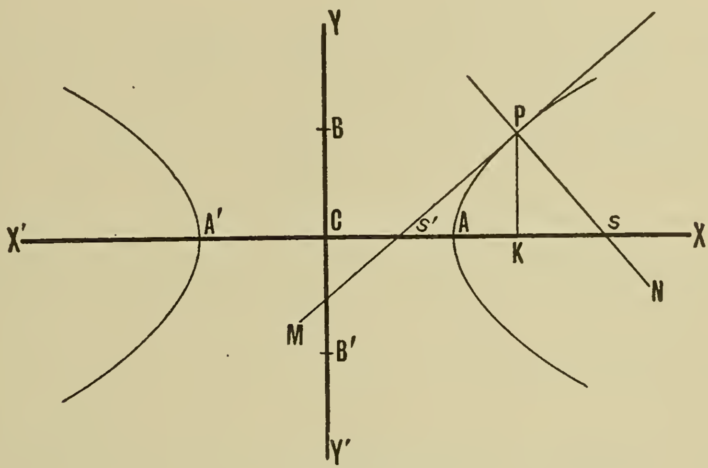



Fig. 75

Let PN be the normal and PM the tangent to the hyperbola at the point P.

Let $x' \equiv$ CK and $y' \equiv$ PK,

and $s' \equiv$ the slope of the tangent PM.

We are to prove that the equation of PN is

$$y' - y = -\frac{a^2 y'}{b^2 x'}(x' - x).$$

The normal is a straight line passing through a fixed point, namely the point of tangency. by § 225.

Hence its equation must be of the form

[1] $$y' - y = s(x' - x),$$ by § 57.

in which $x'$ and $y'$ are the coordinates of the fixed point, here the point of tangency; $s$ is the slope of the line, here the slope of the normal; and $x$ and $y$ the coordinates of any point on the line, here the coordinates of any point on the normal PN.

The normal is perpendicular to the tangent. by § 225.

[2] Hence $$1 + ss' = 0.$$ by § 62.

[3] But $$s' = \frac{b^2 x'}{a^2 y'}.$$ by § 222.

[4] Hence $$1 + s\frac{b^2 x'}{a^2 y'} = 0,$$

[5] and $$s = -\frac{a^2 y'}{b^2 x'}.$$

Substituting this value of $s$ into [1], we get

[6] $$y' - y = -\frac{a^2 y'}{b^2 x'}(x' - x).$$

Now in [6] the $x$ and $y$ are the coordinates of any point on the normal PN. Hence [6] is the equation of the normal.

by § 39.

Q. E. D.

227. *Corollary.—The fraction* $-\frac{a^2 y'}{b^2 x'}$ *is the slope of the normal.*

228. **The Subnormal.**—The *subnormal* is the distance measured along the transverse axis from the ordinate of the point of tangency to the normal.

229. *Corollary.—The length of the subnormal is* $\frac{b^2}{a^2}x'$.

For the proof compare § 146.

EXAMPLES

1. Required the equation of the normal and the value of the subnormal in the following hyperbolas:

$9y^2 - 4x^2 = -36$, the point of tangency being (4, ord. +)

$\frac{x^2}{a} - \frac{y^2}{b} = 1$, the point of tangency being $(\sqrt{a}, 0)$.

PROPOSITION IX

230. *The tangent to an hyperbola bisects the interior, and the normal the exterior angle between the focal radii of the point of tangency.*

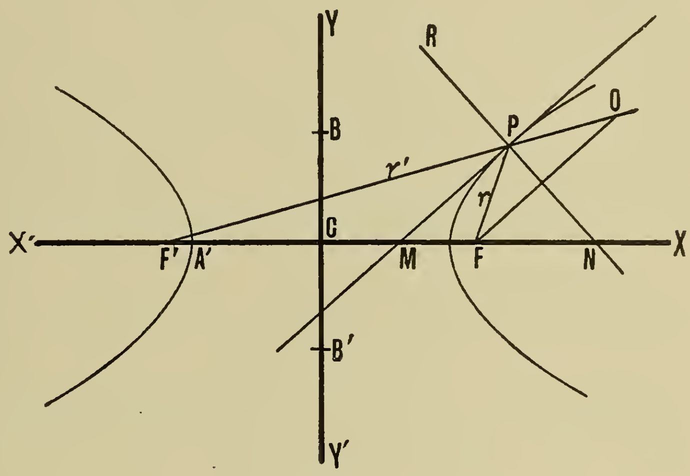



Fig. 76

Let PM be the tangent and PN the normal to the hyperbola at the point P.

Let FP and F′P be the focal radii at the point P.

We are to prove that

PM bisects $\angle$ F′PF,

and that

PN bisects $\angle$ FPO.

Since the point M is on the tangent, its coordinates $x =$ CM and $y = 0$ must satisfy the equation of the tangent. by § 40.

Hence substituting these values for the $x$ and $y$ of that equation, we get

[1] $$y' = \frac{b^2 x'^2}{a^2 y'} - \frac{b^2 x'}{a^2 y'} \text{CM}.$$

Hence $a^2y'^2 - b^2x'^2 = -\text{CM}b^2x'$.

But $a^2y'^2 - b^2x'^2 = -a^2b^2$. by § 221, [2].

[2] Then $a^2b^2 = \text{CM}b^2x'$,

[3] or $\text{CM} = \dfrac{a^2}{x'}$.

[4] Now $\text{MF} = \text{CF} - \text{CM}$,

or $\text{MF} = ae - \dfrac{a^2}{x'}$. by § 218, [2].

[5] hence $\text{MF} = \dfrac{a}{x'}(ex' - a)$.

[6] Also $\text{MF}' = \text{CF}' + \text{CM}$,

or $\text{MF}' = ae + \dfrac{a^2}{x'}$.

[7] $\text{MF}' = \dfrac{a}{x'}(ex' + a)$.

[8] Then from [5] and [7] $\dfrac{\text{MF}}{\text{MF}'} = \dfrac{ex' - a}{ex' + a}$.

[9] Hence $\dfrac{\text{MF}}{\text{MF}'} = \dfrac{r}{r'}$, by § 220, [12] and [13].

[10] or $\text{MF} : \text{MF}' :: r : r'$.

Take $\text{PO} = \text{PF} \equiv r$,

and draw OF.

Substituting this value of PF or $r$ into [10], we get

[11] $\text{MF} : \text{MF}' :: \text{OP} : r'$.

Hence $\text{OF} \parallel \text{PM}$, by Geom. 24.

[12] and $\angle \text{F}'\text{PM} = \angle \text{POF}$, by Geom. 8.

[13] and $\angle \text{FPM} = \angle \text{PFO}$. by Geom. 7.

[14] But $\angle \text{PFO} = \angle \text{POF}$. by Geom. 16.

[15] Hence $\angle \text{F}'\text{PM} = \angle \text{FPM}$.

Q. E. D.

Again

[16] $\angle \text{F}'\text{PR} + \angle \text{F}'\text{PM} = \angle \text{FPM} + \angle \text{FPN}$, by § 225:

[17] and $\angle \text{F}'\text{PM} = \angle \text{FPM}$. by [15].

[18] Hence $\angle F'PR = \angle FPN.$

[19] But $\angle F'PR = \angle OPN.$ by Geom. 4.

[20] Hence $\angle FPN = \angle OPN.$

Q. E. D.

231. *Corollary.—To draw a tangent to an hyperbola.*

Compare § 149.

232. **A Chord.**—A *chord* of an hyperbola is a straight line terminated both ways by the hyperbola.

233. **The Bisector of a Complete System of Parallel Chords.**—The *bisector of a complete system of parallel chords* is the line which contains all the middle points of those chords.

234. **The Diameter.**—A *diameter* of an hyperbola is that part of the bisector of a complete system of parallel chords which is bounded both ways by the hyperbola.

PROPOSITION X

235. *The equation of a diameter of an hyperbola is*

$$y = \left(\frac{b^2}{a^2} \cot \varphi\right) x,$$

*in which $\varphi$ is the inclination of the system of chords bisected by diameter, and a and b are the semi-axes.*

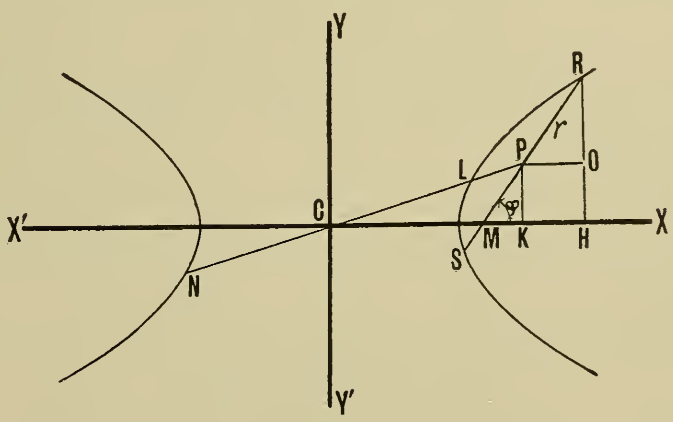



Fig. 77

Let RS represent any one of a complete system of parallel chords, $\varphi$ its inclination, and P its middle point. Let LN be the diameter which, when produced, bisects the system of chords represented by RS.

We are to prove that the equation of LN is

$$y = \left(\frac{b^2}{a^2}\cot\varphi\right)x.$$

$$\begin{aligned} \text{Let } x &\equiv \text{CK and } y \equiv \text{PK}, \\ x' &\equiv \text{CH} \quad \text{"} \quad y' \equiv \text{RH}, \\ \text{and } r &\equiv \text{PR} = \text{PS}. \end{aligned}$$

[1] $\angle\text{RPO} = \angle\text{PMK}.$ by Geom. 8.

[2] $\cos\text{RPO} = \cos\text{PMK} = \cos\varphi.$

[3] $\text{PO} = r.\cos\text{RPO}.$ by Trig. 2.

[4] $\text{PO} = r.\cos\varphi.$

[5] $x' = \text{CK} + \text{PO} = x + r.\cos\varphi.$

[6] $\text{RO} = r.\sin\text{RPO}.$ by Trig. 1.

[7] $\text{RO} = r.\sin\varphi.$

[8] $y' = \text{PK} + \text{RO} = y + r.\sin\varphi.$

Now the point R is on the hyperbola, and hence its coordinates $x'$ and $y'$ must satisfy the equation of the hyperbola. by § 40.

Substituting $x'$ and $y'$ for the $x$ and $y$ of that equation, we get

[9] $a^2y'^2 - b^2x'^2 = -a^2b^2.$ by § 204.

Substituting the values of $x'$ and $y'$ found in [5] and [8] into [9], we get

[10] $a^2(y + r.\sin\varphi)^2 - b^2(x + r.\cos\varphi)^2 = -a^2b^2.$

Squaring the binomials and factoring with respect to $r^2$ and $r$, we get

[11] $(a^2\sin^2\varphi - b^2\cos^2\varphi)\,r^2 + 2(a^2y\sin\varphi - b^2x\cos\varphi)r = -(a^2y^2 - b^2x^2 + a^2b^2).$

Now since P is the middle point of RS, the two values of $r$ in [11] must be equal to each other, and hence by the theory of quadratic equations

[12] $2(a^2y \sin \varphi - b^2x \cos \varphi) = 0.$

[13] Hence $a^2y \sin \varphi = b^2x \cos \varphi.$

[14] Hence $y = \frac{b^2}{a^2} \frac{\cos \varphi}{\sin \varphi} x.$

[15] Hence $y = \left(\frac{b^2}{a^2} \cot \varphi\right) x.$ by Trig. 6 and 9.

Now the $x$ and $y$ of [15] stand for the coordinates of the point P. But, since RS represents any one of the system of parallel chords, P may be any point on their bisector PN, and since the diameter LN is a part of the bisector, P may be any point on that diameter.

Hence the $x$ and $y$ of [15] stand for the coordinates of any point on the diameter LN, which bisects the system of chords represented by RS, and therefore [15] is the equation of that diameter. by § 39.

Q. E. D.

236. *Corollary 1.—The diameter of an hyperbola is a straight line passing through the center.*

For proof compare § 154.

237. *Corollary 2.—If $\theta$ be the inclination of any diameter, and $\varphi$ the inclination of its system of bisected chords, then*

$$\tan \theta \tan \varphi = \frac{b^2}{a^2}.$$

For proof compare § 155.

PROPOSITION XI

238. *If any diameter bisect a system of chords which are parallel to a second diameter, then that second diameter will bisect a system which are parallel to the first diameter.*

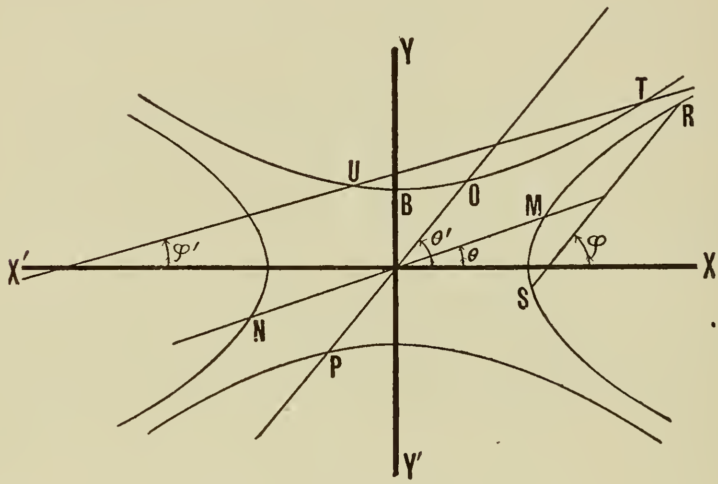



Fig. 78

Let MN bisect a system of chords represented by RS, and let RS be parallel to OP. Let OP bisect a system of parallel chords represented by TU.

We are to prove that

TU ∥ MN.

Let $\theta$ be the inclination of MN,
$\varphi$ " " " " RS,
$\theta'$ " " " " OP,
and $\varphi'$ " " " " TU.

[1] Then $\tan\theta\tan\varphi = \frac{b^2}{a^2}$, by § 237.

[2] and $\tan\theta'\tan\varphi' = \frac{b^2}{a^2}$. by § 237.

[3] Hence $\tan\theta\tan\varphi = \tan\theta'\tan\varphi'$.

But RS ∥ OP by Hypoth.

[4] Hence $\theta' = \varphi$,

[5] and $\tan\theta' = \tan\varphi$.

Substituting this value of tan $\varphi$ into [3], we get

[6] $$\tan\theta = \tan\varphi',$$

[7] and $$\theta = \varphi'.$$

Hence TU ∥ MN. by Geom. 9.

and the system of chords parallel to TU will be parallel to MN. by Geom. 10.

Therefore OP bisects a system of chords which are parallel to MN.

Q. E. D.

239. **Conjugate Diameters.**—Two *diameters* are said to be conjugate to each other when each bisects a system of chords which are parallel to the other.

240. *Corollary.—If $\theta$ be the inclination of any diameter, and $\theta'$ the inclination of its conjugate, then*

$$\tan\theta \tan\theta' = \frac{b^2}{a^2}.$$

For proof compare § 158.

PROPOSITION XII

241. *The tangent to an hyperbola at the extremity of any diameter is parallel to the conjugate of that diameter.*

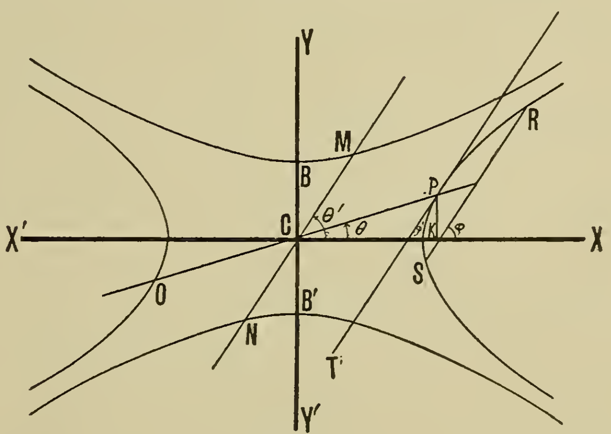



Fig. 79

Let OP be any diameter and MN its conjugate. Let PT be a tangent at the extremity of OP.

We are to prove that

$$PT \parallel MN.$$

Let RS be any one of the system of chords bisected by OP.

Let $\theta$ be the inclination of OP,
$\theta'$ " " " " MN,
$\varphi$ " " " " RS,
and $\varphi'$ " " " " PT.

Let $x' \equiv CK$ and $y' \equiv PK$.

We have $RS \parallel MN$. by § 239.

[1] Hence $\varphi = \theta'$, by Geom. 8.

[2] and $\tan\varphi = \tan\theta'$.

Since the point P is on the diameter OP, its coordinates $x'$ and $y'$ must satisfy the equation of that diameter which is

[3] $$y = \left(\frac{b^2}{a^2}\cot\varphi\right)x.$$ by § 235.

Substituting $x'$ and $y'$ for the $x$ and $y$ of this equation, we get

[4] $$y' = \left(\frac{b^2}{a^2}\cot\varphi\right)x'.$$

[5] Hence $\cot\varphi = \frac{a^2 y'}{b^2 x'}$.

[6] But $\cot\varphi = \frac{1}{\tan\varphi}$. by Trig. 9.

[7] Hence $\frac{1}{\tan\varphi} = \frac{a^2 y'}{b^2 x'}$,

[8] and $\tan\varphi = \frac{b^2 x'}{a^2 y'}$.

[9] Hence by [2] $\tan\theta' = \frac{b^2 x'}{a^2 y'}$.

[10] But $\tan\varphi' = \frac{b^2 x'}{a^2 y'}$. by § 222.

[11] Hence $\tan\theta' = \tan\varphi'$. by [9] and [10].

Therefore $PT \parallel MN$. by Geom. 9.

Q. E. D.

242. *Corollary 1.—The two tangents at the extremities of any diameter are parallel to each other.*

243. *Corollary 2.—The four tangents at the extremities of any pair of conjugate diameters form a parallelogram inscribed within the two conjugate hyperbolas.*

PROPOSITION XIII

244. *Given the coordinates of the extremity of any diameter to find the coordinates of the extremities of its conjugate.*

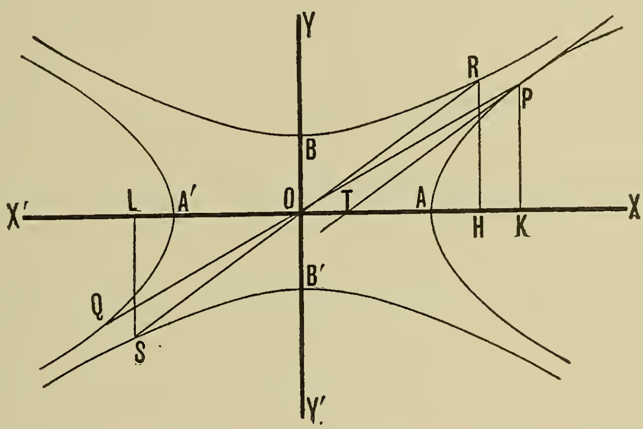



Fig. 80

Let PQ and RS be any two conjugate diameters. Let OK and PK be given.

We are to find OH and RH, OL and LS.

Let $x' \equiv$ OK and $y' \equiv$ PK,
and $x'' \equiv$ OH " $y'' \equiv$ RH.

Draw the tangent PT.

Since RS is a straight line passing through the origin, its equation must be of the form

[1] $y = sx.$ by § 55.

RS ∥ PT. by § 241.

[2] Hence $\angle$ ROX $= \angle$ PTX, by Geom. 8.

[3] and tan ROX $=$ tan PTX.

[4] But by [1] $\tan \text{ROX} = s$, by § 53.

[5] and $\tan \text{PTX} = \frac{b^2 x'}{a^2 y'}$. by § 222.

[6] Hence $s = \frac{b^2 x'}{a^2 y'}$.

If we substitute this value of $s$ into [1], the equation of RS becomes

[7] $$y = \frac{b^2 x'}{a^2 y'} x.$$

Now, since the point R is on the diameter RS, its coordinates $x''$ and $y''$ must satisfy the equation of RS. by § 40.

Hence substituting $x''$ and $y''$ for the $x$ and $y$ of [7], we get

[8] $$y'' = \frac{b^2 x'}{a^2 y'} x''.$$

Since R is also on the conjugate hyperbola RBB'S, its coordinates $x''$ nnd $y''$ must satisfy the equation of that hyperbola, which is

[9] $$a^2y^2 - b^2x^2 = a^2b^2.$$ by § 211.

Hence substituting $x''$ and $y''$ for the $x$ and $y$ of that equation, we get

[10] $$a^2y''^2 - b^2x''^2 = a^2b^2.$$

Now, since in both [8] and [10] $x''$ stands for OH and $y''$ stands for RH, these equations are simultaneous and therefore can be solved by algebra.

Squaring both members of [8], we get

[11] $$y''^2 = \frac{b^4x'^2}{a^4y'^2} x''^2.$$

Substituting this value of $y''^2$ into [10], we get

[12] $$a^2 \frac{b^4x'^2}{a^4y'^2} x''^2 - b^2x''^2 = a^2b^2,$$

[13] $$\frac{b^2x'^2}{a^2y'^2} x''^2 - x''^2 = a^2,$$

[14] $$b^2x'^2x''^2 - a^2y'^2x''^2 = a^4y'^2,$$

[15] and $$(a^2y'^2 - b^2x'^2)x''^2 = - a^4y'^2.$$

Now, since the point P is on the hyperbola PAA′Q, its coordinates $x'$ and $y'$ must satisfy the equation of that hyperbola. by § 40.

Hence substituting $x'$ and $y'$ for the $x$ and $y$ of that equation, we get

[16] $$a^2y'^2 - b^2x'^2 = -a^2b^2.$$ by § 204.

Substituting the value of the left hand member of this equation into [15], we get

[17] $$a^2b^2x''^2 = a^4y'^2.$$

[18] Hence $$x'' = \pm \frac{a}{b} y' = \text{OH or OL}.$$

Substituting $+\frac{a}{b}y'$ for the $x''$ in [8], we get

[19] $$y'' = \frac{b^2 x'}{a^2 y'} \frac{a}{b} y' = \frac{b}{a} x' = \text{RH}.$$

Substituting $-\frac{a}{b}y'$ for $x''$ in [8], we get

[20] $$y'' = -\frac{b}{a} x' = \text{SL}.$$

245. *Corollary.—All diameters are bisected by the center.*

For proof compare § 163.

PROPOSITION XIV

246. *The difference of the squares of any two conjugate diameters of an hyperbola is equal to the difference of the squares of its axes.*

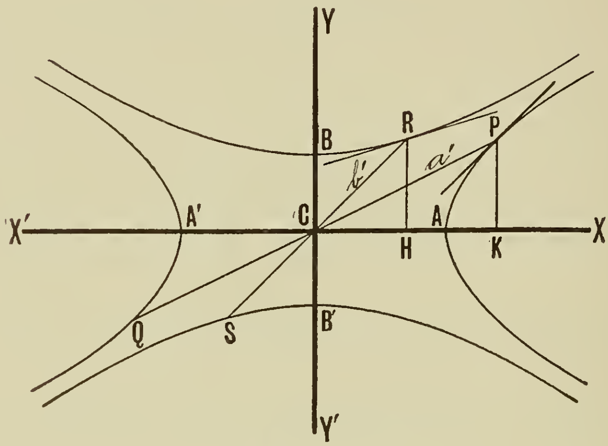



Fig. 81

Let PQ and RS be any two conjugate diameters, and AA′ and BB′ the axes.

We are to prove that

$$\overline{PQ}^2 - \overline{RS}^2 = \overline{AA'}^2 - \overline{BB'}^2.$$

Let $x' \equiv CK$ and $y' \equiv PK$,
$x'' \equiv CH$ " $y'' \equiv RH$,
$a \equiv CA$ " $b \equiv CB$,
and $a' \equiv CP$ " $b' \equiv CR$.

[1] $a'^2 = x'^2 + y'^2$, by Geom. 26.

[2] and $b'^2 = x''^2 + y''^2$. by Geom. 26.

[3] But $x''^2 = \frac{a^2}{b^2} y'^2$, by § 244.

[4] and $y''^2 = \frac{b^2}{a^2} x'^2$. by § 244.

[5] Hence by [2] $b'^2 = \frac{a^2}{b^2} y'^2 + \frac{b^2}{a^2} x'^2$.

Subtracting the members of [5] from the corresponding members of [1], we get

[6] $$a'^2 - b'^2 = x'^2 + y'^2 - \frac{a^2}{b^2}y'^2 - \frac{b^2}{a^2}x'^2.$$

[7] $$= x'^2\left(1 - \frac{b^2}{a^2}\right) + y'^2\left(1 - \frac{a^2}{b^2}\right).$$

[8] $$= \frac{a^2 - b^2}{a^2}x'^2 + \frac{b^2 - a^2}{b^2}y'^2.$$

[9] $$= \frac{a^2 - b^2}{a^2}x'^2 - \frac{a^2 - b^2}{b^2}y'^2.$$

[10] $$= (a^2 - b^2)\left(\frac{x'^2}{a^2} - \frac{y'^2}{b^2}\right).$$

[11] $$a'^2 - b'^2 = (a^2 - b^2)\frac{b^2x'^2 - a^2y'^2}{a^2b^2}.$$

Since the point P is on the hyperbola, its coordinates $x'$ and $y'$ must satisfy the equation of the hyperbola. by § 40.

Substituting $x'$ and $y'$ for the $x$ and $y$ of that equation, we get

[12] $$a^2y'^2 - b^2x'^2 = -a^2b^2.$$

[13] $$b^2x'^2 - a^2y'^2 = a^2b^2.$$

Substituting the value of the left hand member of this equation into [11], we get

[14] $$a'^2 - b'^2 = a^2 - b^2.$$

[15] Hence $\overline{\text{PQ}}^2 - \overline{\text{RS}}^2 = \overline{\text{AA}'}^2 - \overline{\text{BB}'}^2$. by § 245.

Q. E. D.

PROPOSITION XV

247. *The parallelogram formed by tangents to two conjugate hyperbolas at the extremities of any pair of conjugate diameters is equal to the rectangle whose sides are equal to the axes of the hyperbolas.*

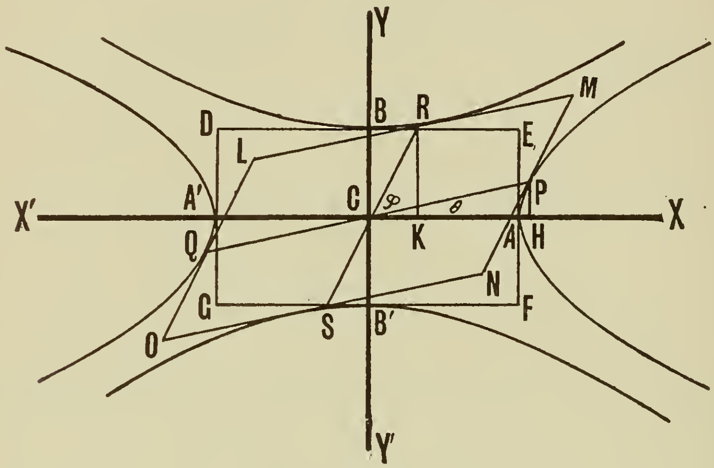



Fig. 82

Let PQ and RS be any pair of conjugate diameters, and LMNO the parallelogram formed by tangents to the hyperbolas at the extremities of these diameters.

Let DEFG be the rectangle whose sides are equal to the axes AA′ and BB′.

We are to prove that

$$\text{LMNO} = \text{DEFG}.$$

$$\text{Let } a \equiv \text{CA and } b \equiv \text{CB},$$
$$a' \equiv \text{CP} \quad \text{''} \quad b' \equiv \text{CR},$$
$$x' \equiv \text{CH} \quad \text{''} \quad y' \equiv \text{PH},$$
$$x'' \equiv \text{CK} \quad \text{''} \quad y'' \equiv \text{RK},$$
$$\theta \equiv \angle \text{PCH and } \varphi \equiv \angle \text{RCK}.$$

[1] $\text{LMNO} = \text{LO} \times \text{NO} \sin \text{LON}$. by Trig. 15.

[2] Hence $\text{LMNO} = 4a'b' \sin \text{RCP}$. by § 245.

[3] or $\text{LMNO} = 4a'b' \sin (\varphi - \theta)$.

[4] $\sin(\varphi - \theta) = \sin\varphi\cos\theta - \cos\varphi\sin\theta$. by Trig. 13.

[5] $$\sin\varphi = \frac{RK}{CR} = \frac{y''}{b'}.$$

[6] $$\cos\theta = \frac{CH}{CP} = \frac{x'}{a'}.$$

[7] $$\cos\varphi = \frac{CK}{CR} = \frac{x''}{b'}.$$

[8] $$\sin\theta = \frac{PH}{CP} = \frac{y'}{a'}.$$

[9] Hence $\sin(\varphi - \theta) = \frac{y''}{b'}\frac{x'}{a'} - \frac{x''}{b'}\frac{y'}{a'} = \frac{y''x' - x''y'}{a'b'}$.

[10] But $x'' = \frac{a}{b}y'$, by § 244.

[11] and $y'' = \frac{b}{a}x'$. by § 244.

Substituting these values of $x''$ and $y''$ into [9], we get

[12] $$\sin(\varphi - \theta) = \frac{\frac{b}{a}x'^2 - \frac{a}{b}y'^2}{a'b'} = \frac{b^2x'^2 - a^2y'^2}{a'b'\,ab}.$$

Now since the point P is on the hyperbola PAA′Q, its coordinates $x'$ and $y'$ must satisfy the equation of that hyperbola. by § 40.

Substituting $x'$ and $y'$ for the $x$ and $y$ in that equation, we get

[13] $a^2y'^2 - b^2x'^2 = -a^2b^2$. by § 202.

[14] Hence $b^2x'^2 - a^2y'^2 = a^2b^2$.

Substituting $a^2b^2$ for $b^2x'^2 - a^2y'^2$ in [12], we get

[15] $$\sin(\varphi - \theta) = \frac{a^2b^2}{a'b'ab} = \frac{ab}{a'b'}.$$

Substituting this value of $\sin(\varphi - \theta)$ into [3], we get

[16] $$LMNO = 4a'b'\frac{ab}{a'b'} = 4ab.$$

[17] But $4ab = 2a2b = AA' \times BB'$, by § 245.

[18] and $AA' = GF$. by Geom. 17.

[19] Hence $4ab = GF \times BB' = DEFG$. by Geom. 28.

Therefore from [16] and [19], we get

[20] $$LMNO = DEFG.$$

Q. E. D.

## The Directrix

248. **The Directrix.**—The *directrix* of an hyperbola is a straight line drawn perpendicular to the X axis on the opposite side of the vertex from the focus, and at such a distance from the vertex that the distance from the focus to the vertex divided by the distance from the vertex to the perpendicular is equal to the eccentricity of the hyperbola.

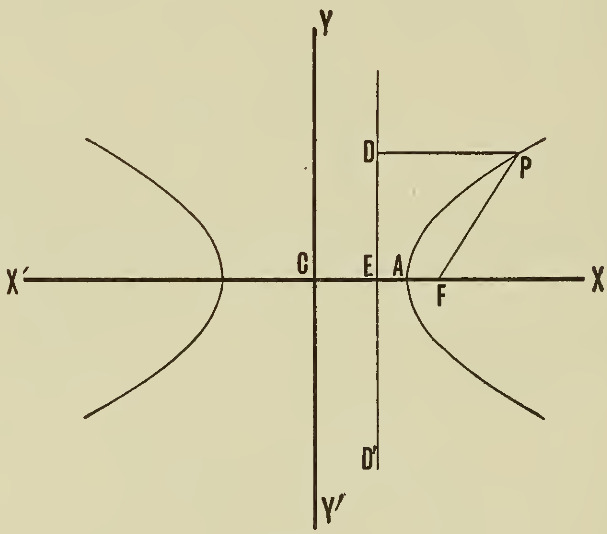



Fig. 83

In Fig. 83, if $\frac{FA}{AE} = e$, then DD′ is the directrix.

249. **The Focal Distance.**—The distance from any point on an hyperbola to the focus is called the *focal distance* of that point.

FP is the focal distance of the point P.

250. **The Directral Distance.**—The distance from any

point on an hyperbola to the directrix is called the *directral distance* of that point.

PD is the directral distance of the point P.

PROPOSITION XVI

251. *The ratio between the focal and directral distances of any point on an hyperbola is constant and is equal to the eccentricity of the hyperbola.*

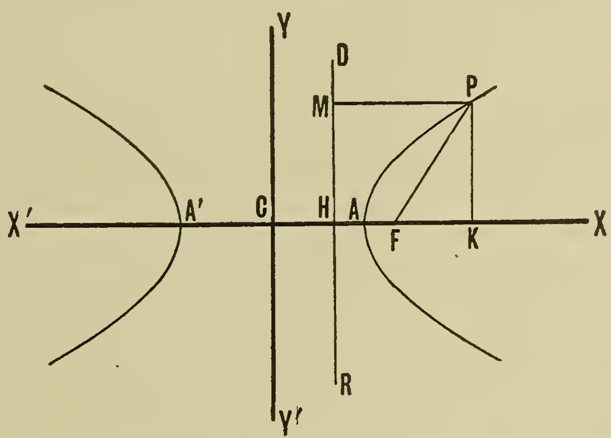



Fig. 84

Let P be any point on the hyperbola, and let DR be the directrix.

Join P to F and draw PM $\perp$ DR.

We are to prove that

$$\frac{PF}{PM} = e.$$

Draw the ordinate PK.

Let $x \equiv CK$, $y \equiv PK$, $x' \equiv HK$,
$a \equiv CA$, $c \equiv CF$, and $p \equiv FH$.

[1] $$p \equiv FH = FA + AH.$$

[2] $$\frac{FA}{AH} = e.$$ by § 248.

[3] Hence $AH = \frac{FA}{e} = \frac{c - a}{e} = \frac{ae - a}{e}$. by § 218, [2].

[4] Hence by [1]

$$p = FA + \frac{ae - a}{e} = c - a + \frac{ae - a}{e} = ae - a + \frac{ae - a}{e} =$$

$$\frac{ae^2 - ae + ae - a}{e}.$$

[5] Hence $p = \frac{ae^2 - a}{e} = \frac{a(e^2 - 1)}{e}$.

[6] $x = CH + HK = c - p + x' = c + (x' - p)$.

The equation of the hyperbola is

[7] $a^2y^2 - b^2x^2 = -a^2b^2$. by § 204.

Since the point P is on the hyperbola, the $x$ and $y$ of [7] may stand for the coordinates of that point.

Substituting the value of $x$ given in [6] into [7], we get

[8] $a^2y^2 - b^2[c + (x' - p)]^2 = -a^2b^2$.

[9] Hence $a^2y^2 - b^2[c^2 + 2c(x' - p) + (x' - p)^2] = -a^2b^2$.

[10] $y^2 - \frac{b^2c^2}{a^2} - \frac{2b^2c}{a^2}(x' - p) - \frac{b^2}{a^2}(x' - p)^2 = -b^2$.

[11] But $\frac{b^2}{a^2} = e^2 - 1$, by § 218, [5].

[12] and $c^2 = a^2e^2$ by § 218, [2].

[13] Hence

$$y^2 - b^2e^2 - 2\frac{b^2e}{a}(x' - p) - (e^2 - 1)(x' - p)^2 = -b^2.$$

[14] $y^2 - b^2e^2 - 2\frac{b^2e}{a}(x' - p) - e^2(x' - p)^2 + (x' - p)^2 = -b^2$.

[15] $y^2 + (x' - p)^2 = b^2e^2 + 2\frac{b^2e}{a}(x' - p) + e^2(x' - p)^2 - b^2$.

From [11] we get

[16] $b^2 = a^2(e^2 - 1)$.

Substituting this value of $b^2$ and the value of $p$ given in [5] into [15], we get

[17] $y^2 + (x' - p)^2 = e^2 x'^2$,

[18] or $\overline{PK}^2 + \overline{FK}^2 = e^2 \overline{HK}^2$.

[19] Hence $\overline{PF}^2 = e^2 \overline{HK}^2$, by Geom. 26.

[20] and $PF = eHK = ePM$. by Geom. 17.

[21] Hence $\frac{PF}{PM} = e$.

Q. E. D.

252. *Corollary 1.—In an hyperbola the focal distance is greater than the directral distance.*

For $e$ is greater than 1. by § 219.

Hence in § 251, [21], PF must be greater than PM.

253. *Corollary 2.—The distance from the center of an hyperbola to the directrix is equal to* $\frac{a}{e}$.

For in Fig. 84

[1] $CF = c = ae$, by § 218, [2].

[2] and $FH = \frac{ae^2 - a}{e}$ by § 251, [5].

[3] Hence $CH = CF - FH = ae - \frac{ae^2 - a}{e} = \frac{a}{e}$.

PROPOSITION XVII

254. *The equation of the hyperbola when any pair of conjugate diameters are taken as the axes is*

$$a'^2y^2 - b'^2x^2 = -a'^2b'^2,$$

*in which $a'$ and $b'$ are the semi-conjugate diameters.*

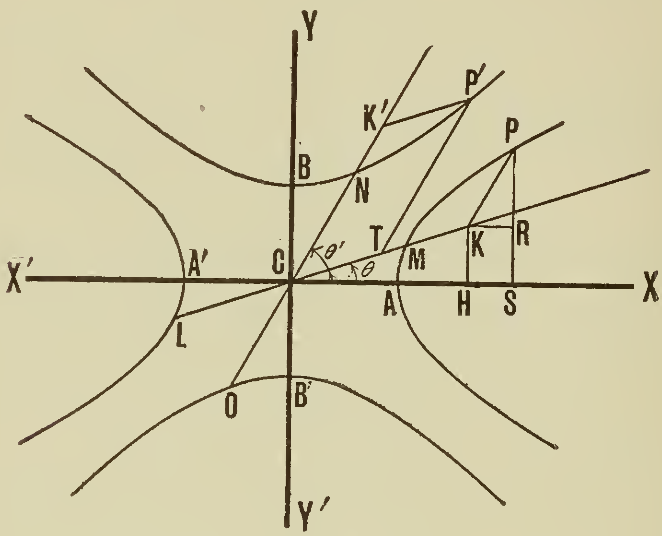



Fig. 85

Let P be any point on the hyperbola.

Let LM and NO be two conjugate diameters.

Let LM be the new axis of abscissas, and NO the new axis of ordinates.

Draw PS ∥ YY′ and PK ∥ NO.

Let $x \equiv$ CS and $y \equiv$ PS.

$x' \equiv$ CK " $y' \equiv$ PK,

$a' \equiv$ CM " $b' \equiv$ CN.

$\theta \equiv \angle$KCX, $\theta' \equiv$ NCX.

We are to prove that

$$a'^2y^2 - b'^2x^2 = -a'^2b'^2,$$

is the equation of the hyperbola referred to the diameters LM and NO.

[1] $PR = PK \sin PKR = y' \sin \theta'$. by Trig. 1. and Geom. 11.

[2] $RS = KH = CK.\sin \theta = x' \sin \theta$. by Trig. 1.

[3] Hence $y = PR + RS = y' \sin \theta' + x' \sin \theta$.

[4] Again $RK = PK.\cos PKR = y' \cos \theta'$, by Trig. 2. and Geom. 11.

[5] and $CH = CK.\cos \theta = x' \cos \theta$. by Trig. 2.

[6] Hence $x = CH + RK = x' \cos \theta + y' \cos \theta'$.

When XX′ and YY′ are taken as the axes, the equation of the hyperbola is

[7] $a^2y^2 - b^2x^2 = - a^2b^2$. by § 204.

Substituting for the $x$ and $y$ of this equation their values given in [3] and [6], we get

[8] $$a^2[y'^2 \sin^2 \theta' + 2x'y' \sin \theta \sin \theta' + x'^2 \sin^2 \theta] - b^2[y'^2 \cos^2 \theta' + 2x'y' \cos \theta \cos \theta' + x'^2 \cos^2 \theta] = -a^2b^2.$$

[9] Or $$(a^2 \sin^2 \theta - b^2 \cos^2 \theta)x'^2 + (a^2 \sin^2 \theta' - b^2 \cos^2 \theta')y'^2 + 2(a^2 \sin \theta \sin \theta' - b^2 \cos \theta \cos \theta')x'y' = - a^2b^2.$$

[10] But $\tan \theta \tan \theta' = \dfrac{b^2}{a^2}$. by § 240.

[11] Hence $a^2 \tan \theta = \dfrac{b^2}{\tan \theta'}$,

[12] or $a^2 \dfrac{\sin \theta}{\cos \theta} = b^2 \dfrac{\cos \theta'}{\sin \theta'}$. by Trig. 6.

[13] Hence $a^2 \sin \theta \sin \theta' = b^2 \cos \theta \cos \theta'$,

[14] and $a^2 \sin \theta \sin \theta' - b^2 \cos \theta \cos \theta' = 0$.

[15] Substituting 0 for this binomial in [9], we get

[16]
$$(a^2 \sin^2 \theta - b^2 \cos^2 \theta)x'^2 + (a^2 \sin^2 \theta' - b^2 \cos^2 \theta')y'^2 = - a^2b^2,$$
which is the equation of the hyperbola when LM and NO are taken as the axes of coordinates.

Since the point M is on the hyperbola, its coordinates $a'$ and 0 must satisfy the equation of the hyperbola. by § 40.

Substituting these values for the $x'$ and $y'$ of [16], we get

[17] $$(a^2 \sin^2 \theta - b^2 \cos^2 \theta) a'^2 = -a^2 b^2.$$

[18] Hence $$a^2 \sin^2 \theta - b^2 \cos^2 \theta = -\frac{a^2 b^2}{a'^2}.$$

Now, if instead of taking the point P, we had taken any point P$'$ on the conjugate hyperbola, and had taken YY$'$ for the rectangular X axis and XX$'$ for the rectangular Y axis; and had taken NO for the oblique X axis and ML for the oblique Y axis; and also had taken BCN for $\theta$ and BCM for $\theta'$, then in the same way that we obtained [16] we would have obtained

[19] $$(\overline{CB}^2 \sin^2 BCN - \overline{CA}^2 \cos^2 BCN)\overline{CK'}^2 +$$
$$(\overline{CB}^2 \sin^2 BCM - \overline{CA}^2 \cos^2 BCM)\,\overline{P'K'}^2 = -a^2 b^2.$$

If we take ML for the X axis and NO for the Y axis, then $P'K' = CT$ may be represented by $x''$ and $CK' = P'T$ by $y''$, and [19] will become

[20]
$$(b^2 \cos^2 \theta' - a^2 \sin^2 \theta') y''^2 + (b^2 \cos^2 \theta - a^2 \sin^2 \theta) x''^2 = -a^2 b^2.$$

In [20] $x''$ and $y''$ stands for the coordinates of any point on the conjugate hyperbola when LM is taken as the X axis and NO is taken as the Y axis.

Since the point N is on the conjugate hyperbola, its coordinates 0 and $b'$ must satisfy [20].

Substituting these values for the $x''$ and $y''$ of [20], we get

[21] $$(b^2 \cos^2 \theta' - a^2 \sin^2 \theta') b'^2 = -a^2 b^2.$$

[22] $$a^2 \sin^2 \theta' - b^2 \cos^2 \theta' = \frac{a^2 b^2}{b'^2}.$$

Substituting the right hand members of [18] and [22] into [16], we get

[23] $$-\frac{a^2 b^2}{a'^2} x'^2 + \frac{a^2 b^2}{b'^2} y'^2 = -a^2 b^2.$$

[24] Hence $$a'^2 y'^2 - b'^2 x'^2 = -a'^2 b'^2.$$

In [24] the $x'$ and $y'$ stand for the coordinates of any point on the hyperbola PML, and hence [24] is the equation of that hyperbola.

Now since the oblique axes only are to be used, there is no further need of the accents over the $x$ and $y$, and hence the equation of the hyperbola may be written

[25] $$a'^2y^2 - b'^2x^2 = -a'^2b'^2.$$

Q. E. D.

PROPOSITION XVIII

255. *When any pair of conjugate diameters are taken as the axes of coordinates, the equation of the tangent to an hyperbola is*

$$y' - y = \frac{b'^2x'}{a'^2y'}(x' - x).$$

*in which $x'$ and $y'$ are the coordinates of the point of tangency, and $a'$ and $b'$ are the semi-conjugate diameters.*

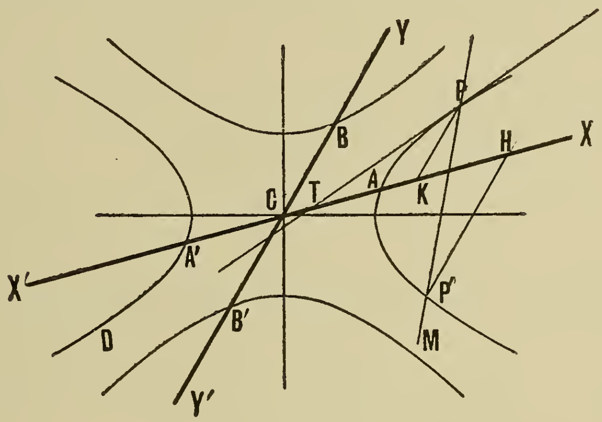



Fig. 86

Let AA′ and BB′ be any two conjugate diameters.

Let AA′ be the X axis and YY′ the Y axis.

Let PT be tangent to the hyperbola PAA′D at the point P.

Let $x' \equiv$ CK and $y' \equiv$ PK,
$a' \equiv$ CA " $b' \equiv$ CB.

We are to prove that

$$y' - y = \frac{b'^2x'}{a'^2y'}(x' - x),$$

is the equation of PT.

Let PM be a secant cutting the hyperbola at P and P′.

Let $x'' \equiv$ CH and $y'' \equiv$ P′H.

Since the secant PM is a straight line passing through two fixed points, its equation must be of the form

[1] $$y'-y=\frac{y''-y'}{x''-x'}(x'-x).$$ by § 58.

Since P is on the hyperbola, its coordinates $x'$ and $y'$ must satisfy the equation of the hyperbola when AA′ and BB′ are taken as the axes.

Substituting $x'$ and $y'$ for the $x$ and $y$ of that equation, we get

[2] $$a'^2y'^2-b'^2x'^2=-a'^2b'^2.$$ by § 254.

Similarly, since the point P′ is also on the hyperbola, we may also get

[3] $$a'^2y''^2-b'^2x''^2=-a'^2b'^2.$$

Now proceeding as in § 134, we get

[4] $$y'-y=\frac{b'^2x'}{a'^2y'}(x'-x).$$

Q. E. D.

255*a*. *Corollary.—The equation of the tangent may also be written*

[5] $$a'^2y'y-b'^2x'x=-a'^2b'^2.$$

PROPOSITION XIX

256. *When any pair of conjugate diameters are taken as the axes, the equation of the chord which joins the points of tangency of two tangents drawn to an hyperbola from the same point without it is*

$$a'^2yy'-b'^2xx'=-a'^2b'^2.$$

*in which $x'$ and $y'$ are the coordinates of the point from which the two tangents are drawn, and $a'$ and $b'$ are the semi-conjugate diameters.*

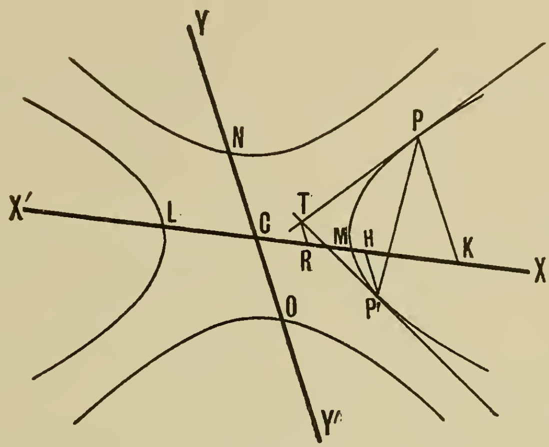



Fig. 87

Let PT and P′T be two tangents drawn to an hyperbola from the same point T.

Let PP′ be the chord joining the points of tangency.

Let any two conjugate diameters LM and NO be taken as the axes.

Let $x' \equiv$ CR and $y' \equiv$ TR,

$a' \equiv$ CM " $b' \equiv$ CN.

We are to prove that

$$a'^2yy' - b'^2xx' = -a'^2b'^2,$$

is the equation of the chord PP′.

Let $x'' \equiv$ CK and $y'' \equiv$ PK.

$x''' \equiv$ CH " $y''' \equiv$ P′H.

The equation of PT is

[1] $$y'' - y = \frac{b'^2x''}{a'^2y''}(x'' - x). \qquad \text{by § 255.}$$

[2] Hence $a'^2yy'' - a'^2y''^2 = b'^2xx'' - b'^2x''^2,$

[3] and $a'^2yy'' - b'^2xx'' = a'^2y''^2 - b'^2x''^2.$

Since the point P is on the hyperbola, its coordinates $x''$ and $y''$ must satisfy the equation of the hyperbola.

Hence substituting $x''$ and $y''$ for the $x$ and $y$ of the equation of the hyperbola, we get

[4] $$a'^2y''^2 - b'^2x''^2 = -a'^2b'^2.$$

Now substituting $-a'^2b'^2$ for the right hand member of [3], we get

[5] $$a'^2yy'' - b'^2xx'' = -a'^2b'^2,$$

for the equation of PT.

Similarly we may show that the equation of P′T is

[6] $$a'^2yy''' - b'^2xx''' = -a'^2b'^2.$$

Since the point T is on the tangent PT, its coordinates $x'$ and $y'$ must satisfy [5], the equation of PT.

Hence substituting $x'$ and $y'$ for the $x$ and $y$ of [5], we get

[7] $$a'^2y'y'' - b'^2x'x'' = -a'^2b'^2.$$

Similarly since the point T is also on the tangent P′T, we get

[8] $$a'^2y'y''' - b'^2x'x''' = -a'^2b'^2.$$

Now equation

[9] $$a'^2y'y - b'^2x'x = -a'^2b'^2$$

is the equation of a straight line. by § 67.

But the coordinates $x''$ and $y''$ of the point P will satisfy this equation, for if they are substituted for the $x$ and $y$ in it we get [7].

Hence the straight line represented by [9] must pass through the point P. by § 41.

The coordinates $x'''$ and $y'''$ of the point P′ will also satisfy [9], for if they are substituted for the $x$ and $y$ in it we get [8].

Hence the straight line represented by [9] must also pass through the point P′. by § 41.

Hence, since the straight line represented by [9] passes through both the points P and P′, it must be the chord PP′.

Therefore [9] must be the equation of the chord PP′.

Q. E. D.

257. *When the transverse axis of the hyperbola is taken as the X axis and the conjugate axis as the Y axis, the equation of the chord becomes*

$$a^2yy' - b^2xx' = -a^2b^2.$$

PROPOSITION XX

258. *The two tangents at the extremities of any chord of an hyperbola meet on the diameter which bisects that chord.*

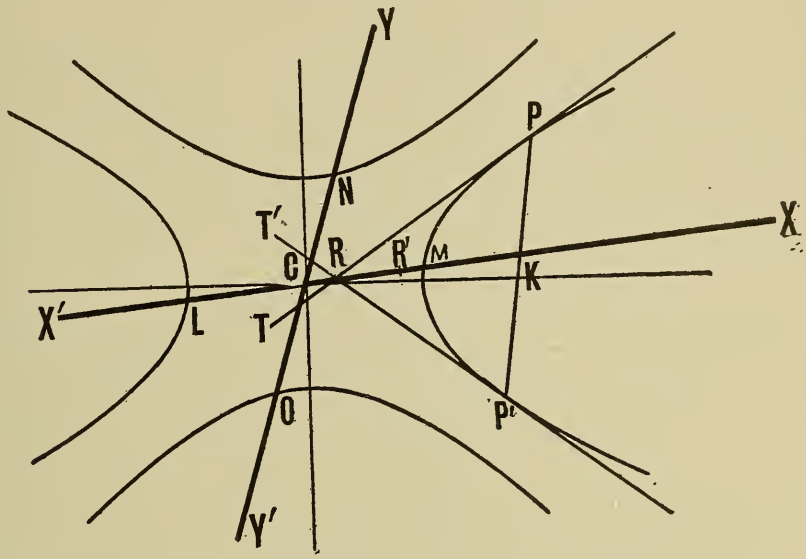



Fig. 87*a*

Let PT and P′T′ be tangents to the hyperbola at the extremities of the chord PP′.

Let LM be the diameter which bisects that chord.

We are to prove that that PT and P′T′ meet on the diameter LM.

Let ON be the diameter which is conjugate to LM.

Let LM be taken as the X axis and NO as the Y axis.

Let $a' \equiv$ CM, $b' \equiv$ CN, and $x' \equiv$ CK.

If R be the point where PT cuts LM, the X axis, then as in [3] of § 230, we get

[1] $$CR = \frac{a'^2}{x'}.$$ by § 45.

Similarly, if R′ be the point where P′T′ cuts LM, the X axis, then

[2] $$CR' = \frac{a'^2}{x'}.$$

[3] Hence $CR = CR'$,

which shows that both tangents meet the diameter LM at the same point.

Q. E. D.

PROPOSITION XXI

259. *If two tangents be drawn through the extremities of any focal chord of an hyperbola,*

(*1*) *the two tangents will meet on the directrix ;*

(*2*) *the line joining the intersection of the two tangents to the focus will be perpendicular to the focal chord.*

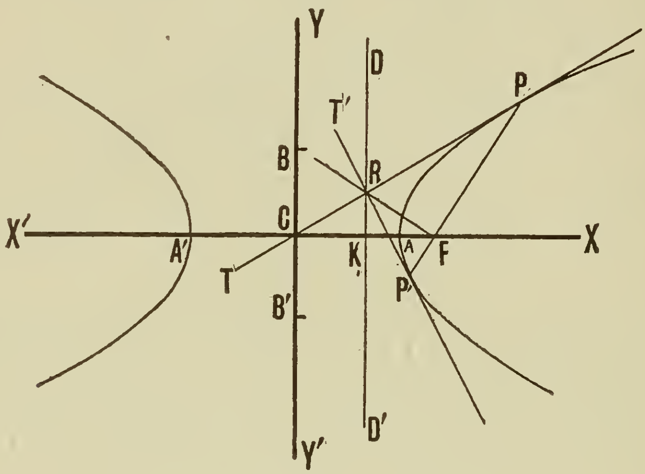



Fig. 88

Let PT and P′T′ be two tangents drawn to the hyperbola at the extremities of the focal chord PP′. Let R be the intersection of the two tangents.

Let DD′ be the directrix.

We are to prove that R will be on the directrix DD′.

$$\text{Let } x' \equiv \text{CK and } y' \equiv \text{RK},$$
$$a \equiv \text{CA} \quad \text{"} \quad b \equiv \text{CB}.$$

Let $x$ and $y$ be the coordinates of any point on the chord PP′.

The equation of PP′ is

[1] $$a^2yy' - b^2xx' = -a^2b^2,$$ by § 257.

in which $x'$ and $y'$ are the coordinates of the point R.

Since the point F is on the chord PP′, its coordinates $x = ae$ and $y = 0$, must satisfy the equation of that chord. by § 40.

Substituting these values for the $x$ and $y$ of [1], we get

[2] $$-b^2aex' = -a^2b^2.$$

[3] Hence $$x' = \frac{a}{e}.$$

But since by § 253, $\frac{a}{e}$ is the distance from the center to the directrix, R the intersection of the two tangents, must be on the directrix.

Q. E. D.

Again, since RF is a straight line passing through the two fixed points R and F, its equation must be of the form

[4] $$y' - y = \frac{y'' - y'}{x'' - x'}(x' - x).$$ by § 58.

In [4] let $x''$ and $y''$ stand for the coordinates of the point F, and $x'$ and $y'$ stand for the coordinates of the point R.

Then $x'' = ae$ and $y'' = 0$.

Substituting these values of $x''$ and $y''$ into [4], we get

[5] $$y' - y = \frac{-y'}{ae - x'}(x' - x).$$

But $$x' = \frac{a}{e}.$$ by [3].

Substituting this value of $x'$ into [5], we get

[6] $$y' - y = \frac{ey'}{a - ae^2}(x' - x).$$

[7] Hence $$y' - y = \frac{aey'}{a^2(1 - e^2)}(x' - x),$$

which is the equation of the line RF.

[8] By [3] $$x' = \frac{a}{e}.$$

Substituting this value of $x'$ into [1], we get

[9] $$a^2y'y = b^2\frac{a}{e}x - a^2b^2.$$

[11] Hence $a^2ey'y = b^2ax - a^2b^2e$,

[12] and $y = \frac{b^2}{aey'}(x - ae)$.

[13] But $b^2 = a^2(e^2 - 1) = -a^2(1 - e^2)$. by § 218, [5].

[14] Hence $y = -\frac{a^2(1 - e^2)}{aey'}(x - ae)$,

which is the equation of the chord PP′.

Let $s \equiv$ the slope of the line RF,
and $s' \equiv$ " " " " chord PP′.

From [7] we get

[15] $s = \frac{aey'}{a^2(1 - e^2)}$. by § 53.

From [14] we get

[16] $s' = -\frac{a^2(1 - e^2)}{aey'}$. by § 53.

[17] Hence $ss' = \frac{aey'}{a^2(1 - e^2)} \times -\frac{a^2(1 - e^2)}{aey'} = -1$,

[18] or $1 + ss' = 0$.

Therefore RF and PP′ are perpendicular to each other.
by § 62.

Q. E. D.

PROPOSITION XXII

260. *The locus of the intersection of two tangents to an hyperbola which are perpendicular to each other, is a circle whose center is at the origin.*

Let PT and P′T be two tangents to an hyperbola at the points P and P′. Let them be perpendicular to each other at the point T.

Let $x' \equiv$ CK and $y' \equiv$ PK.

Let P and P′ move along the hyperbola in such a way that $\angle$PTP′ shall always be a right angle.

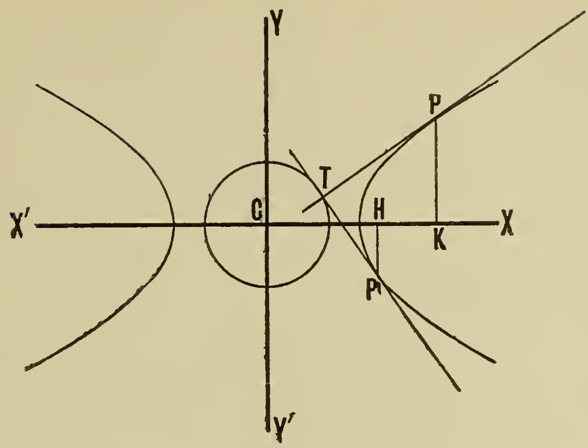



Fig. 89

We are to prove that the locus of the point T will be a circle whose center is at the origin.

The equation of the tangent PT is

[1] $$y'-y=\frac{b^2x'}{a^2y'}(x'-x).$$ by § 221.

[2] Hence $a^2y'y-a^2y'^2=b^2x'x-b^2x'^2$,

[3] and $a^2y'y=b^2x'x+a^2y'^2-b^2x'^2$.

Since the point P is on the hyperbola, its coordinates $x'$ and $y'$ must satisfy the equation of the hyperbola.

[4] Hence $a^2y'^2-b^2x'^2=-a^2b^2$. by § 204.

Substituting this value of $a^2y'^2-b^2x'^2$ into [3], we get

[5] $$a^2y'y=b^2x'x-a^2b^2.$$

[6] Hence $$y=\frac{b^2x'}{a^2y'}x-\frac{b^2}{y'}.$$

[7] Now $\frac{b^2}{y'}=\sqrt{\frac{b^4}{y'^2}}=\sqrt{b^2\frac{a^2b^2}{a^2y'^2}}$, which by [4]

$$=\sqrt{b^2\frac{b^2x'^2-a^2y'^2}{a^2y'^2}}=\sqrt{\frac{b^4x'^2}{a^2y'^2}-\frac{a^2b^2y'^2}{a^2y'^2}}.$$

[8] Hence $$\frac{b^2}{y'}=\sqrt{a^2\frac{b^4x'^2}{a^4y'^2}-b^2}=\sqrt{a^2\left(\frac{b^2x'}{a^2y'}\right)^2-b^2}.$$

Substituting this value of $\frac{b^2}{y'}$ into [6], we get

[9] $$y = \frac{b^2 x'}{a^2 y'} x - \sqrt{a^2 \left(\frac{b^2 x'}{a^2 y'}\right)^2 - b^2}.$$

Now let $$s = \frac{b^2 x'}{a^2 y'}.$$

Then [9] becomes

[10] $$y = sx - \sqrt{a^2 s^2 - b^2},$$

which is the equation of any tangent PT to an hyperbola.

[11] Let $$y = s'x - \sqrt{a^2 s'^2 - b^2},$$

be the equation of the tangent P′T, which is perpendicular to PT.

Then

[12] $$1 + ss' = 0.$$ by § 62.

[13] Hence $$s' = -\frac{1}{s}.$$

Substituting this value of $s'$ into [11], we get

[14] $$y = -\frac{x}{s} - \sqrt{\frac{a^2}{s^2} - b^2},$$

for the equation of P′T.

By transposition [10] and [14] become

[15] $$y - sx = -\sqrt{a^2 s^2 - b^2},$$

[16] and $$y + \frac{x}{s} = -\sqrt{\frac{a^2}{s^2} - b^2}.$$

By squaring [15] and [16] become

[17] $$y^2 - 2sxy + s^2 x^2 = a^2 s^2 - b^2,$$

[18] and $$y^2 + 2\frac{xy}{s} + \frac{x^2}{s^2} = \frac{a^2}{s^2} - b^2.$$

Clearing [18] of fractions, we get

[19] $$s^2 y^2 + 2sxy + x^2 = a^2 - s^2 b^2.$$

Adding [17] and [19], we get

[20] $(1+s^2)y^2+(1+s^2)x^2=a^2(1+s^2)-(1+s^2)b^2$.

[21] Hence $y^2+x^2=a^2-b^2$,

in which $x$ and $y$ are the coordinates of the point T. by § 48.

Hence [21] is the equation of the locus traced out by T. by § 39.

Now let $a^2-b^2=r^2$.

[22] Then by [21] $x^2+y^2=r^2$,

which is the equation of a circle whose center is at the origin. by § 113.

Therefore the locus traced out by T is a circle whose center is at the origin.

Q. E. D.

PROPOSITION XXIII

261. *If any chord of an hyperbola pass through a fixed point and tangents be drawn at its extremities, and if the chord be made to revolve about the fixed point as a pivot, then the locus of the intersection of the two tangents will be a straight line whose equation is*

$$a'^2yy'-b'^2xx'=-a'^2b'^2,$$

*in which $x'$ and $y'$ are the coordinates of the fixed point about which the chord revolves, and $a'$ and $b'$ of the semi-conjugate diameters which are taken as axes.*

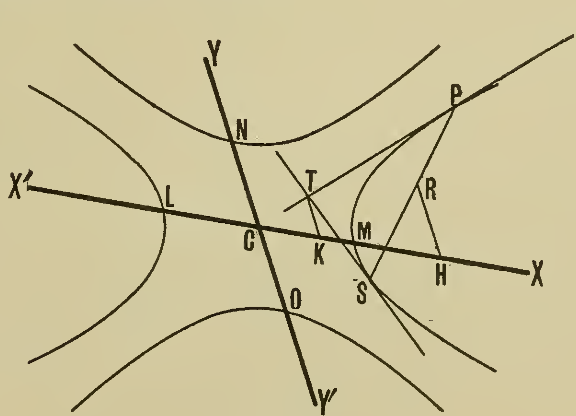



Fig. 90

Let PS be a chord passing through the fixed point R, and let PT and ST be tangents drawn at its extremities.

Let LM and NO be two conjugate diameters taken as axes.

$$\text{Let } x' \equiv \text{CH} \quad \text{and} \quad y' \equiv \text{RH},$$
$$a' \equiv \text{CM} \quad \text{“} \quad b' \equiv \text{CN}.$$

Let PS revolve about R as a pivot.

We are to prove that the locus traced out by T is a straight line and that its equation is

$$a'^2yy' - b'^2xx' = -a'^2b'^2.$$

$$\text{Let } x'' = \text{CK} \quad \text{and} \quad y'' = \text{TK}.$$

The equation of the chord PS is

[1] $$a'^2yy'' - b'^2xx'' = -a'^2b'^2.$$ by § 256.

Since the point R is on this chord, its coordinates $x'$ and $y'$ must satisfy the equation of this chord. by § 40.

Substituting $x'$ and $y'$ for the $x$ and $y$ of [1], we get

[2] $$a'^2y'y'' - b'^2x'x'' = -a'^2b'^2.$$

Now as PS revolves about R, T will trace out a locus. Moreover [2] will be satisfied by the coordinates of the point T wherever it may be as it traces out this locus.

Hence the $x''$ and $y''$ of [2] stand for the coordinates of every point on the locus traced out by T.

Therefore [2] must be the equation of that locus. by § 39.

This locus must be a straight line. by § 67.

But since T is *any* point on this straight line traced out by the intersection of the tangents, we may drop the accent marks from its coordinates and write them $x$ and $y$, hence [2] may be written

[3] $$a'^2yy' - b'^2xx' = -a'^2b'^2.$$

Therefore the locus traced out by T is a straight line and its equation is

[4] $$a'^2yy' - b'^2xx' = -a'^2b'^2.$$

262. **Supplemental Chords.**—Two chords drawn from the same point on an hyperbola to the extremities of any diameter are called *supplemental chords.*

PROPOSITION XXIV

263. *If a chord be parallel to any diameter of an hyperbola the supplemental chord will be parallel to the conjugate diameter.*

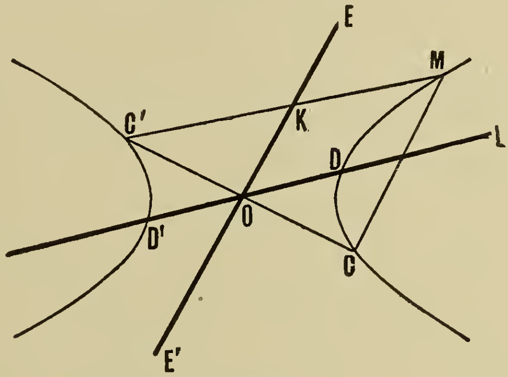



Fig 91

Let MC and MC′ be two supplemental chords drawn to the extremities of the diameter CC′.

Let DD′ and EE′ be two conjugate diameters.

Let MC ∥ EE′.

We are to prove MC′ ∥ DD′.

[1] $\frac{C'O}{C'C} = \frac{C'K}{C'M}$. by Geom. 23.

[2] But $\frac{C'O}{C'C} = \frac{1}{2}$. by § 245.

[3] Hence $\frac{C'K}{C'M} = \frac{1}{2}$,

[4] and $C'K = \frac{1}{2}C'M$.

Hence the diameter EE′ ∥ MC bisects a system of chords ∥ MC′. But by hypothesis the diameters EE′ and DD′ are conjugate; and therefore EE′ bisects a system of chords which are ∥ DD′.

Therefore MC′ ∥ DD′. by Geom. 10.

Q. E. D.

## Polar Equation of the Hyperbola

PROPOSITION XXV

264. *When the right hand focus is taken as the pole, the polar equation of the hyperbola is*

$$r = \frac{a(e^2 - 1)}{1 - e \cos \theta},$$

*in which a is the semi-transverse axis of the hyperbola, e its eccentricity, r the radius vector of any point on it, and θ the vectorial angle of that point.*

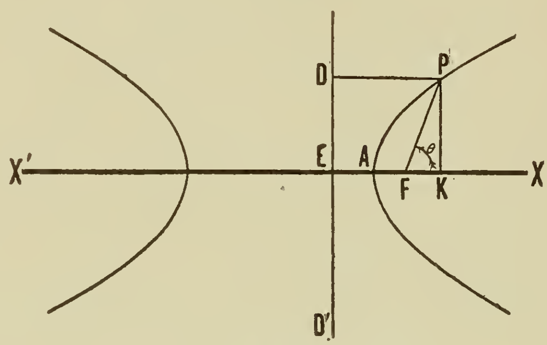



Fig. 92

Let F be the pole, XX′ the initial line, and DD′ the directrix.

Let P be any point on the hyperbola.

Let $\theta \equiv \angle$ PFX, $r \equiv$ FP, and $e \equiv$ the eccentricity.

We are to prove that

$$r = \frac{a(e^2 - 1)}{1 - e \cos \theta},$$

is the polar equation of the hyperbola.

Draw PD $\perp$ DD′ and PK $\perp$ XX′.

[1] PF $= e$PD. by § 251.

[2] Hence PF $= e$(EF + FK), by Geom. 17.

[3] or PF $= e$.EF + $e$.FK.

[4] But $EF = \frac{a(e^2-1)}{e}$, by § 251, [5].

[5] and $FK = r \cos\theta$. by Trig. 2.

Hence from [3], [4] and [5], we get

[6] $r = a(e^2-1) + er\cos\theta$.

[7] Hence $r(1 - e\cos\theta) = a(e^2-1)$,

[8] and $r = \frac{a(e^2-1)}{1-e\cos\theta}$.

Q. E. D.

265. *Corollary.—When the left hand focus is taken as the pole, the polar equation of the hyperbola becomes*

$$r = \frac{a(e^2-1)}{e\cos\theta - 1}.$$

EXAMPLES

1. What is the equation of the hyperbola conjugate to $25y^2 - 16x^2 = -400$?

2. What is the eccentricity of the hyperbola conjugate to $16y^2 - 9x^2 = -144$?

3. Find the equations of the two tangents to $4y^2 - 12x^2 = -48$, at the upper and at the lower extremities of the parameter. Ans. $y = 2x - 2$.
$y = -2x + 2$.

4. How far is it from the intersection of these tangents to the directrix?

5. What is the angle between the two tangents drawn to $16y^2 - 9x^2 = -144$ at the extremities of the parameter?

6. Find the equations of the two tangents drawn to $a^2y^2 - b^2x^2 = -a^2b^2$ at the extremities of the parameter.
Ans. $y = ex - a$.
$y = -ex + a$.

7. Find the angle between these two tangents and the distance of their intersection from the directrix.

Ans. $\tan^{-1}\frac{2e}{1-e^2}$.
Distance $= 0$.

8. Where does the tangent whose inclination is 45° touch $25y^2 - 16x^2 = -400$? Ans. At $x' = 8\frac{1}{3}$.
$y' = 5\frac{1}{3}$.

9. Where does the tangent whose inclination is 45° touch $a^2y^2 - b^2x^2 = -a^2b^2$?

Ans. At $x' = \dfrac{a^2}{\sqrt{a^2 - b^2}}$

$y' = \dfrac{b^2}{\sqrt{a^2 - b^2}}$

10. Where does the tangent at the vertex of its conjugate cut $16y^2 - 9x^2 = -144$? Ans. $x = \pm 5.6$.
$y = \pm 3$.

11. Where does the tangent at the vertex of its conjugate cut $a^2y^2 - b^2x^2 = -a^2b^2$? Ans $x = a\sqrt{2}$.
$y = \pm b$.

12. Where does the tangent drawn from the upper vertex of its conjugate touch $16y^2 - 9x^2 = -144$?

Ans. At $x = 4\sqrt{2}$.
$y = -3$.

13. Where does the tangent drawn from the upper vertex of its conjugate touch $a^2y^2 - b^2x^2 = -a^2b^2$.

Ans. At $x = a$.
$y = -b$.

14. What are the coordinates of the upper end of the parameter of $a^2y^2 - b^2x^2 = -a^2b^2$?

15. The equation of the tangent drawn through the upper extremity of the parameter of an hyperbola is $y = \frac{5}{4}x - 4$. What is the equation of the hyperbola?

Ans. $16y^2 - 9x^2 = -144$.

16. The tangent drawn through the upper end of the parameter of an hyperbola whose semi-transverse axis is 5, passes through the vertex of the conjugate hyperbola. What is the equation of the first hyperbola and what is its eccentricity? Ans. $25y^2 - 25x^2 = -625$.
$e = \sqrt{2}$.

17. From what point must two tangents to $9y^2 - 16x^2 = -144$ be drawn that they may touch the hyperbola at the extremities of a chord which passes through the vertex of the hyperbola and the upper end of the parameter?

Ans. From $x = 3$.
$y = 2$.

18. About what point must a chord of $36y^2 - 16x^2 = -576$ revolve in order that the locus of the intersection of the two tangents drawn through the extremities of this chord may bisect the positive halves of the axes of the hyperbola?

Ans. $x' = 12$.
$y' = -8$.

19. About what point must a chord of $36y^2 - 16x^2 = -576$ revolve in order that the locus of the intersection of the two tangents drawn through the extremities of this chord may bisect the positive half of the transverse axis and the negative half of the conjugate axis?

Ans. $x' = 12$.
$y' = 8$.

20. About what point must a chord of $a^2y^2 - b^2x^2 = -a^2b^2$ revolve that the locus of the intersection of the two tangents drawn through the extremities of the chord may bisect the positive half of the transverse axis and the negative half of the conjugate axis?

Ans. $x' = 2a$.
$y' = 2b$.

21. About what point must a chord of $a^2y^2 - b^2x^2 = -a^2b^2$ revolve in order that the locus of the intersection of the two tangents drawn through the extremities of this chord may bisect the positive halves of the axes of the hyperbola?

Ans. About $x = 2a$.
$y = -2b$.

# CHAPTER XI

## Asymptotes

**266. The Asymptotes.**—The diagonals of the rectangle constructed upon the axes of an hyperbola are called the *asymptotes* of the hyperbola.

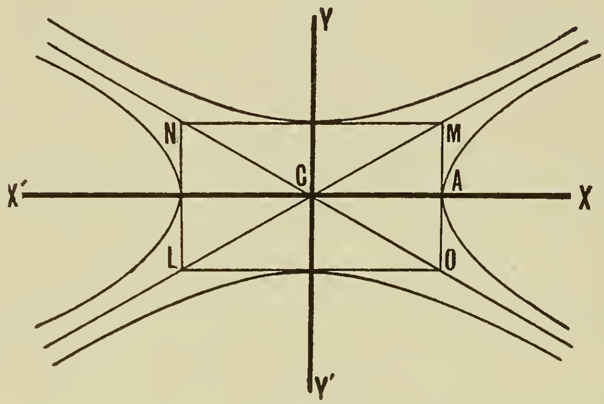



Fig. 93

In Fig. 93 LM and NO are the asymptotes.

PROPOSITION XXVI

267. *The equations of the asymptotes of an hyperbola are*

$$y = \frac{b}{a} x,$$

$$\textit{and} \quad y = -\frac{b}{a} x,$$

*in which a and b are the semi-axes of the hyperbola.*

For in Fig. 93 the asymptote LM is a straight line passing through the origin. by § 266.

Hence its equation must be of the form

[1] $y = sx$. by § 55.

[2] Hence $s = \tan \text{MCA}$. by § 53.

[3] But $\tan \text{MCA} = \dfrac{\text{MA}}{\text{CA}} = \dfrac{b}{a}$. by Trig. 3.

[4] Hence $s = \dfrac{b}{a}$.

Substituting this value of $s$ into [1], we get

[5] $$y = \frac{b}{a}x,$$

for the equation of the asymptote LM.

Similarly it may be shown that the equation of the asymptote NO is

[6] $$y = -\frac{b}{a}x.$$

PROPOSITION XXVII

268. *As the asymptote extends outward from the center of the hyperbola, the distance between it and the hyperbola continually approaches 0, and if the asymptote be made long enough this distance may be made less than any assignable quantity, but the asymptote can never touch the hyperbola.*

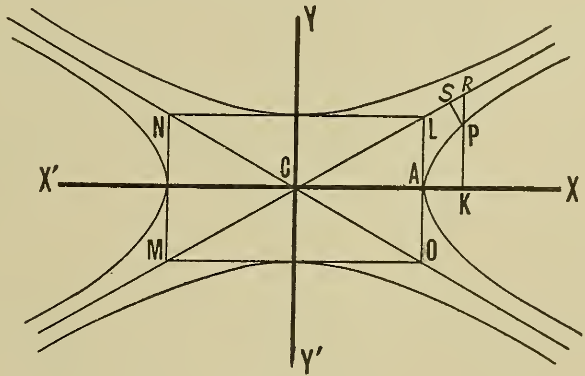



Fig. 94

Let LM be one of the asymptotes of the hyperbola.

Let R be any point on it and draw RK ⊥ XX′ and cutting the hyperbola at the point P.

Through P draw PS ⊥ LM.

Let $x \equiv$ CK and $y \equiv$ PK.

We are to prove that as the point S is taken further and further from the center of the hyperbola, PS continually approaches 0, can be made less than any assignable quantity, but can never become 0.

The equation of LM is

[1] $$y = \frac{b}{a} x,$$ by § 267.

[2] or $$RK = \frac{b}{a} CK.$$

The equation of the hyperbola is

[3] $$a^2y^2 - b^2x^2 = - a^2b^2.$$ by § 204.

[4] Hence $$y = \frac{b}{a}\sqrt{x^2 - a^2},$$

[5] or $$PK = \frac{b}{a}\sqrt{\overline{CK}^2 - a^2}.$$

[6] Hence $$RP = RK - PK = \frac{b}{a}\left(CK - \sqrt{\overline{CK}^2 - a^2}\right).$$

[7] Hence

$$RP = \frac{b}{a}\,\frac{\overline{CK}^2 - \overline{CK}^2 + a^2}{CK + \sqrt{\overline{CK}^2 - a^2}} = \frac{ab}{CK + \sqrt{\overline{CK}^2 - a^2}}.$$

[8] Now $PS = RP.\cos RPS$, by Trig. 2.

[9] and $\angle RPS = \angle LCA.$ by Geom. 12.

[10] Hence $PS = RP \cos LCA.$

Substituting the value of RP given in [7] into [10], we get

[11] $$PS = \frac{ab}{CK + \sqrt{\overline{CK}^2 - a^2}} \cos LCA.$$

Now let the point R move along the asymptote away from the center.

The quantities *ab* and cos LCA do not change their values in consequence of this motion of the point R. The only part of the right hand member of [11] that does change its value is the denominator of the fraction, which increases continually.

Since the numerator of this fraction is constant, and its denominator is increasing, the value of the fraction is continually approaching 0.

Since the denominator can be made as large as we please, the value of the fraction can be made as small as we please.

Since the numerator is constant, the value of the fraction can never become 0.

Therefore, as the asymptote extends outward from the center the distance PS between it and the hyperbola continually approaches 0, can be made less than any assignable quantity, but can never become 0. Q. E. D.

269. *Corollary.—As the asymptote extends outwards from the center, the distance from it to the conjugate hyperbola continually approaches* 0, *but it can never touch the conjugate hyperbola.*

It is obvious that any other straight line passing through the center must meet either the hyperbola or its conjugate. Hence we may define the asymptote as follows:

270. **The Asymptote.**—The *asymptote* of an hyperbola is a straight line passing through the center, whose distance from the hyperbola can be made less than any assignable quantity by taking a point on the hyperbola far enough from its center, but can never be made 0.

## PROPOSITION XXVIII

271. *The tangent to an hyperbola can be made to coincide as nearly as we please with the asymptote by moving the point of tangency far enough from the center.*

For, the equation of the tangent is

[1] $$y'-y=\frac{b^2x'}{a^2y'}(x'-x).$$ by § 221.

[2] Hence $$y=\frac{b^2x'x}{a^2y'}-\frac{b^2}{y'}.$$ as in § 260, [6].

Since the point of tangency is on the hyperbola, its coordinates $x'$ and $y'$ must satisfy the equation of the hyperbola.

by § 40.

[3] Hence $a^2y'^2 - b^2x'^2 = -a^2b^2$, by § 204.

[4] and $y' = \frac{b}{a}\sqrt{x'^2 - a^2}$.

Substituting this value of $y'$ into [2], we get

[5] $$y = \frac{b^2}{a^2}x\frac{x'}{\frac{b}{a}\sqrt{x'^2 - a^2}} - \frac{b^2}{y'}.$$

[6] Hence $$y = \frac{b}{a}x\frac{1}{\sqrt{1 - \frac{a^2}{x'^2}}} - \frac{b^2}{y'}.$$

Now let the point $(x, y)$ remain fixed and the point of tangency move away from the center continually.

[7] Then limit $y$ = limit $\left(\frac{b}{a}x\frac{1}{\sqrt{1 - \frac{a^2}{x'^2}}} - \frac{b^2}{y'}\right)$.

The value of $\sqrt{1 - \frac{a^2}{x'^2}}$ will continually approach 1, and the value of $\frac{b^2}{y'}$ will approach 0.

Thus as $x'$ and $y'$ increase indefinitely, the limit of $\sqrt{1 - \frac{a^2}{x'^2}}$ is 1, and the limit of $\frac{b^2}{y'} = 0$. by Geom. 19.

When these limits are substituted into [7], it becomes

[8] $$y = \frac{b}{a}x,$$

which is the equation of the asymptote.

Hence by moving the point of tangency far enough from the center we can make [6] coincide as nearly as we please with [8].

Q. E. D.

PROPOSITION XXIX

272. *When any pair of conjugate diameters are taken as the axes of coordinates, the equations of the asymptotes are*

$$y = \frac{b'}{a'}x$$

$$\textit{and} \quad y = -\frac{b'}{a'}x,$$

*in which a' and b' are the semi-conjugate axes.*

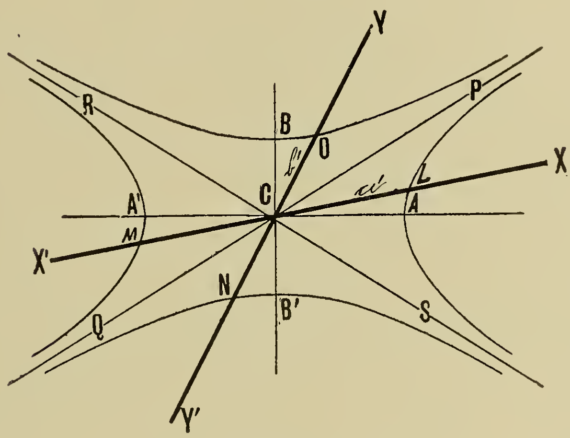



Fig. 95

Let PQ and RS be the asymptotes.

Let LM and NO be any two conjugate diameters.

Let LM be the X axis and NO the Y axis.

Let $a' \equiv$ CL and $b' \equiv$ CO.

We are to prove that

[1] $$y = \frac{b'}{a'}x$$

is the equation of PQ,

[2] and $$y = -\frac{b'}{a'}x$$

is the equation of RS.

When LM and NO are taken as the axes of coordinates, the equation of the hyperbola is

$$a'^2y^2 - b'^2x^2 = -a'^2b'^2. \qquad \text{by § 254.}$$

Since this equation is of exactly the same form as [3], § 268, and since [1] is of exactly the same form as [1] of the same section, then by the same method as was used in that section we may prove that the distance from $y = \frac{b'}{a'}x$ to the hyperbola is continually approaching 0, can be made less than any assignable quantity, but that it can never become 0.

Hence

$$y = \frac{b'}{a'}x$$

is the equation of the asymptote. by § 270.

Similarly it may be shown that

$$y = -\frac{b'}{a'}x$$

is the equation of the other asymptote.

Q. E. D.

PROPOSITION XXX

273. *The asymptote is the diagonal of every parallelogram whose sides are tangents parallel to two conjugate diameters.*

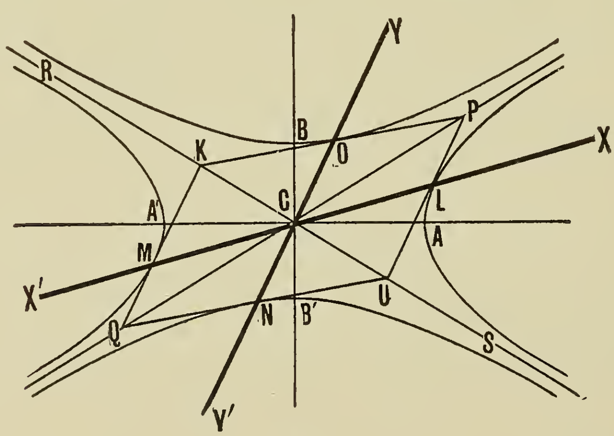



Fig. 96

Let LM and NO be two conjugate diameters.

Let LM be the X axis and NO the Y axis.

Let $a' \equiv$ CL and $b' \equiv$ CO.

Let PQ and RS be the asymptotes.

Draw the tangents LP and OP parallel to the conjugate diameters LM and NO.

We are to prove that the diagonal PC of the parallelogram PLCO coincides with PQ, and the diagonal KC of the parallelogram KOCM coincides with RS.

Let $I \equiv \angle$PCL,

and $\omega \equiv \angle$OCL.

The equation of the diagonal PC will be

[1] $$y = \frac{\sin I}{\sin(\omega - I)}x.$$ by § 76.

[2] But $\omega - I = \angle$OCP $= \angle$CPL. by Geom. 7.

[3] Hence $$y = \frac{\sin \text{PCL}}{\sin \text{CPL}}x.$$

[4] PL : CL : : sin PCL : sin CPL. by Trig. 14.

[5] Hence $b' : a'$ :: sin PCL : sin CPL,

[6] or $$\frac{b'}{a'} = \frac{\sin \text{PCL}}{\sin \text{CPL}}.$$

Substituting the left hand member of this equation into [3], we get

[7] $$y = \frac{b'}{a'}x.$$

But this is the equation of PQ. by § 272.

Therefore the diagonal PC coincides with the asymptote PQ.

Similarly it may be shown that the diagonal CK coincides with the asymptote RS.

Q. E. D.

PROPOSITION XXXI

274. *When the asymptotes are taken as the axes of coordinates, the equation of the hyperbola is*

$$xy = \frac{a^2 + b^2}{4}$$

*in which a and b are the semi-axes of the hyperbola.*

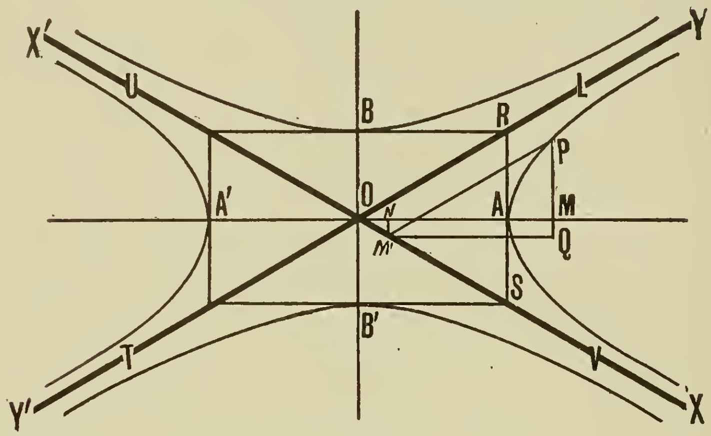



Fig. 97

Let LT and UV be the asymptotes of the hyperbola.

Let LT be taken as the Y axis and UV as the X axis.

Let P be any point on the hyperbola, and draw PM $\perp$ AA' produced and PM' $\parallel$ LT.

Let $x \equiv$ OM and $y \equiv$ PM,
$x' \equiv$ OM' " $y' \equiv$ PM',
$a \equiv$ OA " $b \equiv$ OB.

We are to prove that

$$xy = \frac{a^2 + b^2}{4}$$

is the equation of the hyperbola.

[1] Now tan ROA $= \frac{b}{a}$, by Trig. 3.

[2] and tan SOA $= \frac{b}{a}$. by Trig. 3.

[3] Hence $\tan \text{ROA} = \tan \text{SOA}$,

[4] and $\angle \text{ROA} = \angle \text{SOA}$.

Let $\beta \equiv \angle \text{ROA} = \angle \text{SOA}$.

[5] $x = \text{ON} + \text{M}'\text{Q} = \text{OM}' \cos \text{SOA} + \text{PM}' \cos \text{PM}'\text{Q}$, by Trig. 2.

[6] or $x = x' \cos \beta + y' \cos \beta = (x' + y') \cos \beta$.

[7] Again $y = \text{PQ} - \text{M}'\text{N} =$
$\text{PM}' \sin \text{PM}'\text{Q} - \text{OM}' \sin \text{SOA}$.
by Trig. 1.

[8] or $y = y' \sin \beta - x' \sin \beta = (y' - x') \sin \beta$.

Now when the axes of the hyperbola are taken as the axes of coordinates, the equation of the hyperbola is

[9] $$a^2y^2 - b^2x^2 = - a^2b^2.$$ by § 204.

Substituting the values of $x$ and $y$ given in [6] and [8] into this equation, we get

[10] $a^2(y' - x')^2 \sin^2 \beta - b^2(x' + y')^2 \cos^2 \beta = - a^2b^2$.

[11] Hence $(b^2 \cos^2 \beta - a^2 \sin^2 \beta)(x'^2 + y'^2) +$
$2(b^2 \cos^2 \beta + a^2 \sin^2 \beta)x'y' = a^2b^2$.

[12] But $\tan \beta = \dfrac{\text{RA}}{\text{OA}} = \dfrac{b}{a}$, by [1].

[13] or $\dfrac{\sin \beta}{\cos \beta} = \dfrac{b}{a}$. by Trig. 6.

[14] Hence $a \sin \beta = b \cos \beta$.

[15] and $a^2 \sin^2 \beta = b^2 \cos^2 \beta$.

Substituting these values into [11], we get

[16] $$4a^2 x'y' \sin^2 \beta = a^2b^2.$$

[17] $$4 x'y' \sin^2 \beta = b^2.$$

[18] But $\sin \beta = \dfrac{\text{RA}}{\text{OR}} = \dfrac{b}{\sqrt{a^2 + b^2}}$.

Substituting this value of $\sin \beta$ into [17], we get

[19] $$4x'y' = a^2 + b^2,$$

[20] and $$x'y' = \frac{a^2 + b^2}{4}.$$

Now since $x'$ and $y'$ are the coordinates of any point on the hyperbola, this must be the equation of the hyperbola.

by § 39.

If the asymptotes are the only axes used in any discussion, we may drop the accent marks and write the equation

[21] $$xy = \frac{a^2 + b^2}{4}.$$

Q. E. D.

275. *Corollary.*—Since $a$ and $b$ may be any constants whatever $\frac{a^2 + b^2}{4}$ may be any constant.

Let this constant be represented by $c$.

*Then the equation of the hyperbola may be written*

$$xy = c.$$

PROPOSITION XXXII

276. *When the asymptotes are taken as the axes of coordinates the equation of the tangent to the hyperbola is*

$$\frac{x}{x'} + \frac{y}{y'} = 2,$$

*in which $x'$ and $y'$ are the coordinates of the point of tangency.*

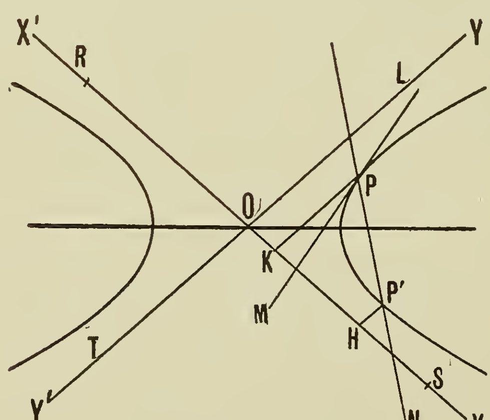



Fig. 98

Let LT and RS be the asymptotes of the hyperbola.

Let LT be the Y axis and RS the X axis.

Let PM be a tangent to the hyperbola at any point P.

Let $x' \equiv$ OK and $y' \equiv$ PK.

We are to prove that

$$\frac{x}{x'} + \frac{y}{y'} = 2$$

is the equation of the tangent PM.

Let PN be a secant cutting the hyperbola at the two points P and P′.

Let $x'' \equiv$ OH and $y'' \equiv$ P′H.

Since the secant PN is a straight line passing through the two points P and P′, its equation must be of the form

[1] $$y' - y = \frac{y'' - y'}{x'' - x'}(x' - x). \qquad \text{by § 58.}$$

Since the point P is on the hyperbola, its coordinates $x'$ and $y'$ must satisfy the equation of the hyperbola. by § 40.

[2] Hence $$x'y' = \frac{a^2 + b^2}{4} \qquad \text{by § 274.}$$

For the same reason $x''$ and $y''$, the coordinates of the point P′, must satisfy the equation of the hyperbola.

[3] Hence $$x''y'' = \frac{a^2 + b^2}{4}. \qquad \text{by § 274.}$$

[4] Hence $$x''y'' = x'y',$$

[5] and $$y'' = \frac{x'y'}{x''}.$$

Substituting this value of $y''$ into [1], we get

[6] $$y' - y = -\frac{y'x'' - x'y'}{x''^2 - x'x''}(x' - x)$$

[7] and $$y' - y = -\frac{y'}{x''}(x' - x).$$

Now let the point P′ move along the curve towards P. The secant will continually approach the tangent, and when P′ reaches P the secant will coincide with the tangent and

[8] $$x'' = x'.$$

Substituting this value of $x''$ into [7], we get

[9] $$y' - y = -\frac{y'}{x'}(x' - x).$$

[10] Hence $$1 - \frac{y}{y'} = -1 + \frac{x}{x'}$$

[11] or $$\frac{x}{x'} + \frac{y}{y'} = 2.$$

Since the $x$ and $y$ of this equation are the coordinates of any point on the tangent PM, it must be the equation of the tangent. by § 39.

Q. E. D.

PROPOSITION XXXIII

277. *The segment of any tangent to an hyperbola lying between the asymptotes is bisected by the point of tangency.*

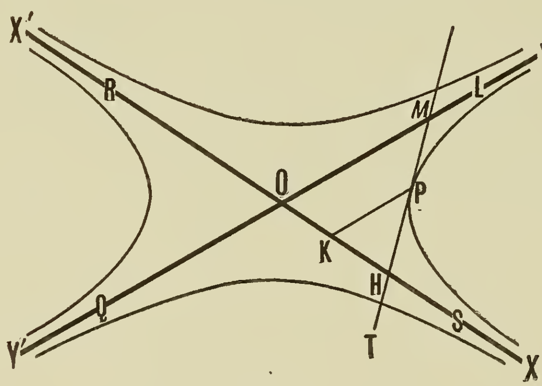



Fig. 99

Let LQ and RS be the asymptotes.

Let LQ be the Y axis and RS the X axis.

Let MT be any tangent to the hyperbola at the point P.

We are to prove that

$$PM = PH.$$

Draw PK ∥ YY′.

Let $x' \equiv$ OK and $y' \equiv$ PK.

The equation of MT is

[1] $\frac{x}{x'} + \frac{y}{y'} = 2.$ by § 276.

[2] Hence OH = $2x'$ = 2OK. by § 45.

Since PK ∥ OM,

[3] OK : OH :: PM : MH. by Geom. 23.

[4] And OK : 2OK :: PM : MH, by [2].

[5] or $\frac{1}{2} = \frac{PM}{MH}$.

[6] Hence MH = 2PM,

[7] and PM = PH.

Q. E. D.

278. *Corollary.—The product of the segments of the asymptotes between the center and any tangent is equal to the sum of the squares of the semi-axes.*

For

[1] OM = $2y'$, by § 276 and § 46.

[2] and OH = $2x'$. by § 45.

[3] Hence OM.OH = $4x'y'$,

[4] and OM.OH = $a^2 + b^2$. by § 274.

## PROPOSITION XXXIV

279. *The area of the triangle formed by any tangent to an hyperbola, and the segments of the asymptotes between this tangent and the center is equal to the rectangle of the semi-axes.*

Let LT and RS be the asymptotes.

Let LT be the Y axis and RS the X axis.

Let MH be any tangent to the hyperbola.

Let $a \equiv$ OA and $b \equiv$ OB.

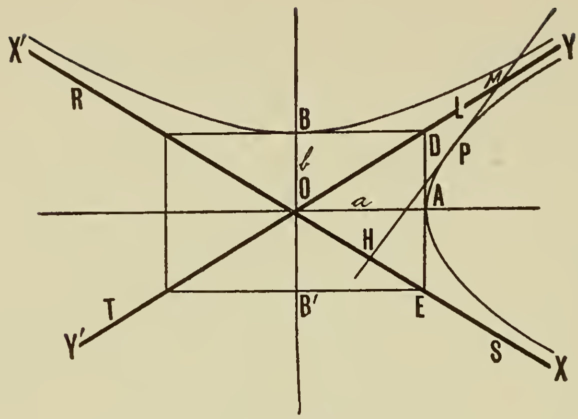



Fig. 100

We are to prove that

the area of MOH $= ab$.

Let $\omega \equiv \angle$ DOA.

[1] Then $\tan\omega = \dfrac{\text{DA}}{\text{OA}} = \dfrac{b}{a}$, by Trig. 3.

[2] or $\dfrac{\sin\omega}{\cos\omega} = \dfrac{b}{a}$. by Trig. 6.

[3] Hence $a\sin\omega = b\cos\omega$,

[4] and $a^2\sin^2\omega = b^2\cos^2\omega = b^2 - b^2\sin^2\omega$. by Trig. 5.

[5] Hence $\sin^2\omega = \dfrac{b^2}{a^2+b^2}$.

[6] Therefore $1 - \cos 2\omega = \dfrac{2b^2}{a^2+b^2}$. by Trig. 18.

[7] Hence $\cos 2\omega = 1 - \dfrac{2b^2}{a^2+b^2} = \dfrac{a^2-b^2}{a^2+b^2}$,

[8] and $\cos^2 2\omega = \dfrac{(a^2-b^2)^2}{(a^2+b^2)^2}$.

[9] Hence $\sin^2 2\omega = 1 - \dfrac{(a^2-b^2)^2}{(a^2+b^2)^2} = \dfrac{4a^2b^2}{(a^2+b^2)^2}$,

[10] and $\sin 2\omega = \dfrac{2ab}{a^2+b^2}$.

[11] But $2\omega = \text{DOE}$. by Geom. 15.

Hence from [10] we get

[12] $$\sin \text{DOE} = \frac{2ab}{a^2 + b^2}.$$

[13] Now the area of MOH $= \frac{1}{2}$OM.OH sin DOE, by Trig. 31.

[14] and $\text{OM.OH} = a^2 + b^2$. by § 278.

Hence substituting the right hand members of [12] and [14] into [13], we get

[15] the area of MOH $= \frac{1}{2}(a^2 + b^2)\, \dfrac{2ab}{a^2 + b^2} = ab.$

Q. E. D.

# CHAPTER XII

## The Parabola

280. **The Parabola.**—The *parabola* is the locus of a point moving in a plane in such a way that its distance from a fixed point in the plane and its distance from a fixed line in the plane are always equal to each other.

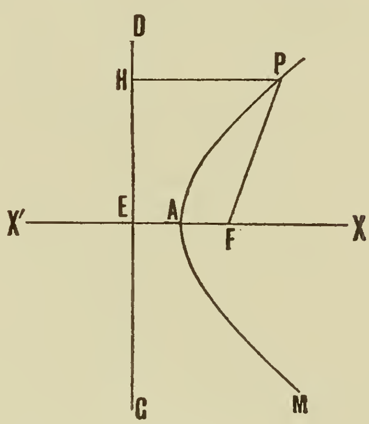



Fig. 101

Let F be a fixed point and DG a fixed line, and let P be a point moving in the plane DFG in such a way that PF and PH shall always be equal to each other.

Then the line PAM traced out by P is called a parabola.

281. **The Focus.**—The fixed point is called the *focus*.

282. **The Directrix.**—The fixed line is called the *directrix*.

283. **The Focal Radius.**—The distance from the focus to the moving point is called the *focal radius* of that point.

284. **The Axis.**—The *axis* of the parabola is a straight line drawn through the focus perpendicular to the directrix.

285. **The Vertex.**—The point where the parabola cuts the axis is called the *vertex* of the parabola.

PROPOSITION I

286. *To draw a parabola which shall have a given point for its focus and a given line for its directrix.*

FIRST METHOD; BY THREAD AND RULER.

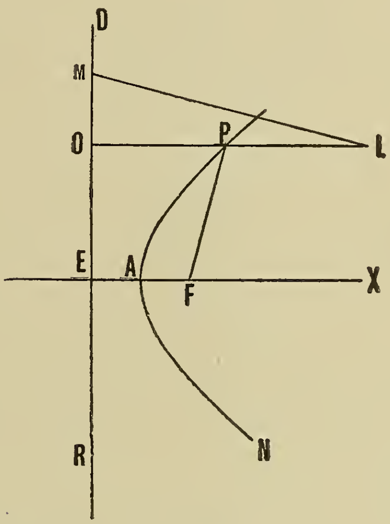



Fig. 102

Let F be the given focus and DR the given directrix.

Let LMO be a ruler in the form of a right triangle.

Take an inelastic thread whose length is equal to LO. Fasten one end of it at L and the other end at F. Press the thread against the ruler at P by the point of a pencil so as always to keep it stretched.

Now move the ruler so that its side MO shall slide along DR.

The line PAN traced out by the pencil point will be a parabola.

For at every point P on this line we shall have

[1] $$LP + PO = LP + PF$$

[2] hence $$PF = PO.$$

Therefore PAN is a parabola. by § 280.

SECOND METHOD; BY POINTS.

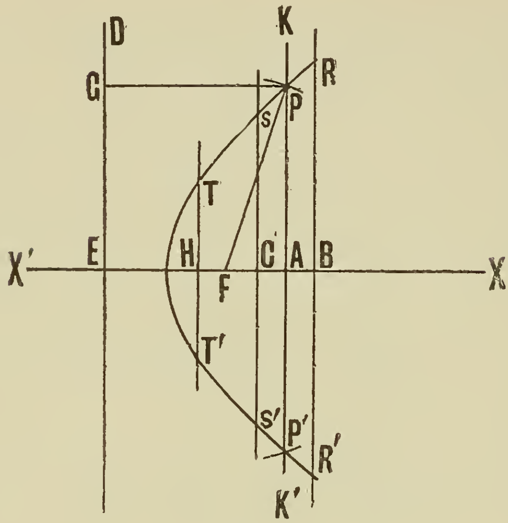



Fig. 103

Let F be the given focus, and DE the given directrix.

Let XX′ be the axis. Take any point A on the axis and draw the perpendicular KAK′.

With AE as a radius and F as a center, draw a circle cutting KK′ in the two points P and P′.

P and P′ will be on the parabola required.

For draw PG ⊥ DE and join F with P.

| | | | |
|---|---|---|---|
| [1] | Then | PG = EA. | by Geom. 17. |
| [2] | But | EA = FP. | by construction. |
| [3] | Hence | PG = FP. | |

Therefore P is on the parabola. by § 280.

Similarly it may be shown that P′ is also on the parabola.

By drawing perpendiculars through B, C and H we can determine other points R and R′, S and S′, T and T′, by the same method by which we determined P and P′, and can show that they are also on the parabola.

By taking a sufficient number of perpendiculars we can determine as many points on the parabola and points as near to each other as we please. By joining these points we get the parabola required.

PROPOSITION II

287. *The equation of the parabola is*

$$y^2 = 2px,$$

*in which $p$ represents the distance from the focus to the directrix.*

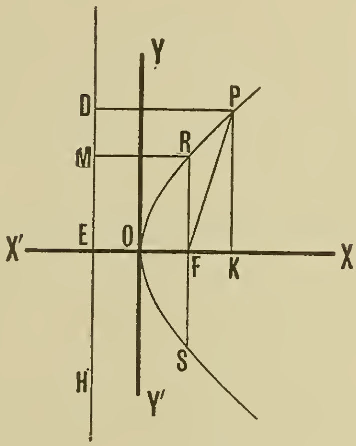



Fig. 104

Let F be the focus, DE the directrix, and P any point on the parabola.

Let $x \equiv OK$, $y \equiv PK$, and $p \equiv EF$.

We are to prove that

$$y^2 = 2px$$

is the equation of the parabola.

[1] $\overline{PF}^2 = \overline{FK}^2 + \overline{PK}^2$. by Geom. 26.

[2] $PF = PD = EK$. by § 280, and Geom. 17.

[3] Hence $PF = EK = p + FK$.

Substituting this value of PF into [1], we get

[4] $(p + FK)^2 = \overline{FK}^2 + \overline{PK}^2$.

[5] Hence $p^2 + 2pFK + \overline{FK}^2 = \overline{FK}^2 + \overline{PK}^2$,

[6] and $p^2 + 2pFK = y^2$.

[7] But $FK = OK - OF = x - OF$,

[8] and $OF = OE = \frac{1}{2}p$. by § 280.

[9] Hence $FK = x - \frac{1}{2}p$.

Substituting this value of FK into [6], we get

[10] $p^2 + 2p(x - \frac{1}{2}p) = y^2$.

[11] Hence $p^2 + 2px - p^2 = y^2$,

[12] or $y^2 = 2px$.

Q. E. D.

288. *Corollary 1.—The squares of the ordinates of any two points on a parabola are to each other as their abscissas.*

289. *Corollary 2.—Ordinates at equal distances from the vertex are equal to each other.*

290. **The Principal Parameter.**—The *principal parameter* of a parabola is the double ordinate that passes through the focus, the axis of the parabola and the tangent at the vertex being the axes of coordinates.

291. *Corollary 1.—The principal parameter of a parabola is equal to twice the distance from the focus to the directrix.*

In Fig. 104 let RS be the parameter.

[1] RF = RM. by § 280.

[2] RM = EF = $p$. by Geom. 17.

[3] Hence RF = $p$.

[4] FS = RF. by § 289.

[5] Hence FS = $p$,

[6] and RS = RF + FS = $2p$.

292. *Corollary 2.—The principal parameter of a parabola is a third proportional to any abscissa and its corresponding ordinate.*

Since in Fig. 104 the point P is on the parabola, its coordinates must satisfy the equation of the parabola. by § 40.

Hence letting the $x$ and $y$ of that equation stand for the coordinates of the point P, we get

[1] $y^2 = 2px$. by § 287.

[2] Hence $x : y :: y : 2p$. by Geom. 56.

[3] But $2p$ = the parameter. by § 291.

[4] Hence $x : y :: y$ : Parameter.

Q. E. D.

## PROPOSITION III

293. *The equation of the tangent to the parabola is*

$$y' - y = \frac{p}{y'}(x' - x),$$

*in which $x'$ and $y'$ are the coordinates of the point of tangency and $p$ is half the principal parameter.*

For proof compare § 134.

294. *Corollary 1.—The fraction $\frac{p}{y'}$ is the slope of the tangent.*

295. *Corollary 2.—The vertex of a parabola bisects the subtangent.*

## PROPOSITION IV

296. *The equation of the normal to a parabola is*

$$y' - y = -\frac{y'}{p}(x' - x),$$

*in which $x'$ and $y'$ are the coordinates of the point of tangency, and $p$ is half the principal parameter.*

For proof compare § 143.

297. *Corollary 1.—The fraction $-\frac{y'}{p}$ is the slope of the normal.*

298. *Corollary 2.—The subnormal of a parabola is constant and is equal to half the parameter.*

299. **A Diameter.**—A *diameter* of a parabola is a straight line drawn from any point on it in the positive direction parallel to the axis of the parabola.

PROPOSITION V

300. (*a*) *The tangent to a parabola bisects the angle between the focal radius of the point of tangency and a diameter produced through the point of tangency.*

(*b*) *The normal bisects the angle between the focal radius of the point of tangency and a diameter drawn from the point of tangency.*

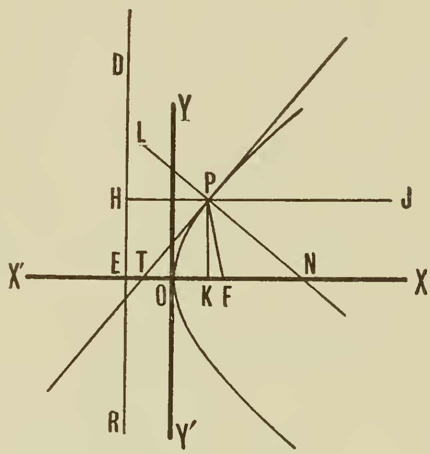


Fig. 105

Let PT be a tangent to the parabola at P, JH a diameter produced through P, and FP the focal radius of the point P.

We are to prove that

$$\angle FPT = \angle HPT.$$

Let $x' \equiv OK$ and $y' \equiv PK$.

Let $p \equiv EF$ and DR be the directrix.

| | | | |
|---|---|---|---|
| [1] | | $FP = PH.$ | by § 280. |
| [2] | But | $PH = EK.$ | by Geom. 17. |
| [3] | Hence | $FP = EK = EO + OK.$ | |
| [4] | But | $EO = FO = \frac{1}{2}p.$ | by § 280. |
| [5] | Hence | $FP = \frac{1}{2}p + x'.$ | |
| [6] | | $FT = FO + OT.$ | |
| [7] | But | $OT = OK = x'.$ | by § 295. |

Hence by [4] and [7], [6] becomes

[8] $FT = \frac{1}{2}p + x'.$

Hence from [5] and [8], we get

[9] $FP = FT.$

[10] Hence $\angle FPT = \angle FTP.$ by Geom. 16.

[11] But $\angle FTP = \angle HPT.$ by Geom. 7.

[12] Hence $\angle FPT = \angle HPT.$

Q. E. D.

Again let PN be the normal to the parabola at the point P, and PJ a diameter drawn from the point of tangency.

We are to prove that

$$\angle FPN = \angle NPJ.$$

[13] $\angle TPL = \angle TPN.$ by § 142.

[14] $\angle HPT = \angle FPT.$ by [12].

[15] Hence $\angle HPL = \angle FPN.$

[16] But $\angle HPL = \angle NPJ.$ by Geom. 4.

[17] Hence $\angle FPN = \angle NPJ.$ by Geom. 1.

Q. E. D.

PROPOSITION VI

301. *When any diameter and the tangent at its extremity are taken as the axes, the equation of the parabola is*

$$y^2 = 2p'x,$$

*in which $p'$ is the principal parameter divided by the square of the sine of the inclination of the tangent.*

Let LN be any diameter and TR the tangent at its extremity.

Let LN be taken as the new X axis and TR as the new Y axis.

Let P be any point on the parabola, and draw PM ∥ YY′.

$$\text{Let } x \equiv LM \text{ and } y \equiv PM.$$

$$\text{Let } p' \equiv \frac{p}{\sin^2 LRA}.$$

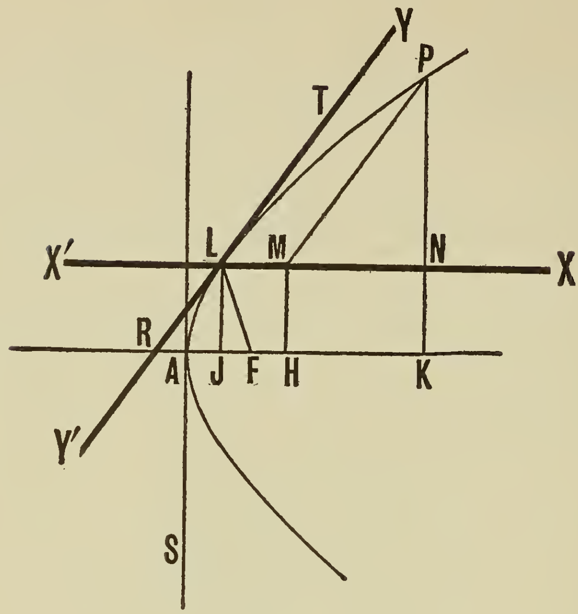



Fig. 106

We are to prove that

$$y^2 = 2p'x$$

is the equation of the parabola.

Let $x' \equiv AJ$ and $y' \equiv LJ$,

$x'' \equiv AK$ " $y'' \equiv PK$.

[1] $x'' = AK = AJ + LM + MN = x' + x + y \cos PMN$. by Trig. 2.

[2] But $\angle PMN = \angle LRA$. by Geom. 11.

[3] Hence $x'' = x' + x + y \cos LRA$.

[4]

$y'' = PK = LJ + PN = y' + y \sin PMN = y' + y \sin LRA$. by Trig. 1 and Geom. 11.

Since $x''$ and $y''$ are the coordinates of any point on the parabola, when AK and AS are taken as the axes of coordinates, the equation of the parabola will be

[5] $$y''^2 = 2px''.$$ by § 287.

Now substituting the values of $x''$ and $y''$ found in [3] and [4] into [5], we get

[6] $(y' + y \sin \angle RA)^2 = 2p(x' + x + y \cos \angle RA.)$

[7] Hence

$$y^2 \sin^2 \angle RA + 2y(y' \sin \angle RA - p \cos \angle RA) + y'^2 - 2px' = 2px.$$

[8] $\tan \angle RA = \frac{p}{y'}$, by § 294.

[9] or $\frac{\sin \angle RA}{\cos \angle RA} = \frac{p}{y'}$. by Trig. 6.

[10] Hence $y' \sin \angle RA - p \cos \angle RA = 0$.

Since the point L is on the parabola, its coordinates $x'$ and $y'$ must satisfy the equation of the parabola. by § 40.

Hence substituting them for the $x$ and $y$ in the equation of the parabola, we get

[11] $y'^2 = 2px'$.

[12] Hence $y'^2 - 2px' = 0$.

Substituting the right hand members of [10] and [12] into [7], we get

[13] $y^2 \sin^2 \angle RA = 2px$.

[14] Hence $y^2 = \frac{2p}{\sin^2 \angle RA} x$.

Now since

$$p' \equiv \frac{p}{\sin^2 \angle RA}$$

[14] becomes

[15] $y^2 = 2p'x$.

Q. E. D.

302. *Corollary 1.—Ordinates drawn to the same point on any diameter are equal.*

For since

[1] $y^2 = 2p'x$, by § 301.

[2] then $y = \pm \sqrt{2p'x}$.

Now [2] shows that for every positive value of $x$ there are two values of $y$ numerically equal but with opposite signs.

303. *Corollary 2.—Every diameter bisects a system of chords parallel to the tangent at its extremity.*

Compare § 159.

304. **The Parameter of any Diameter.**—The *parameter of any diameter* is a third proportional to any abscissa on that diameter and its corresponding ordinate.

305. *Corollary 1.—The parameter of any diameter is equal to the principal parameter divided by the square of the sine of the inclination of the tangent at the vertex of the diameter.*

For by [14] of § 301

[1] $$y^2 = \frac{2p}{\sin^2 \text{LRA}} x.$$

[2] Hence $$x : y :: y : \frac{2p}{\sin^2 \text{LRA}}.$$ by Geom. 56.

Therefore $\frac{2p}{\sin^2 \text{LRA}}$ = the parameter of the diameter LM. by § 304.

306. *Corollary 2.—The parameter of any diameter is four times the distance from the focus to its extremity.*

For in Fig. 106

[1] $\text{FL} = \text{FR}$. by § 300, [9].

[2] But $\text{FR} = \text{AF} + \text{AR}$,

[3] and $\text{AR} = \text{AJ}$. by § 295.

[4] Hence $\text{FR} = \text{AF} + \text{AJ}$.

[5] Hence by [1] $\text{FL} = \text{AF} + \text{AJ} = \frac{1}{2}p + x'$. by § 280.

[6] Again $y'^2 = 2px'$. by § 287.

[7] Hence $$x' = \frac{y'^2}{2p}.$$

[8] $$\tan \text{LRA} = \frac{p}{y'},$$ by § 294.

[9] hence $$y' = \frac{p}{\tan \text{LRA}} = p \cot \text{LRA},$$ by Trig. 9.

[10] and $y'^2 = p^2 \cot^2 \text{LRA}$.

Substituting this value of $y'^2$ into [7], we get

[11] $$x' = \frac{1}{2}p \cot^2 LRA.$$

Substituting this value of $x'$ into [5], we get

[12] $$FL = \frac{1}{2}p + \frac{1}{2}p \cot^2 LRA,$$

[13] or $$FL = \frac{1}{2}p(1 + \cot^2 LRA).$$

[14] hence $$FL = \frac{1}{2}p \operatorname{cosec}^2 LRA.$$ by Trig. 10.

[15] Hence $$FL = \frac{p}{2 \sin^2 LRA}$$ by Trig. 7.

[16] Therefore $$\text{Parameter} = \frac{2p}{\sin^2 LRA} = 4FL.$$ by § 305.

Q. E. D.

PROPOSITION VII

307. *The squares of ordinates to any diameter of a parabola are to each other as the corresponding abscissas.*

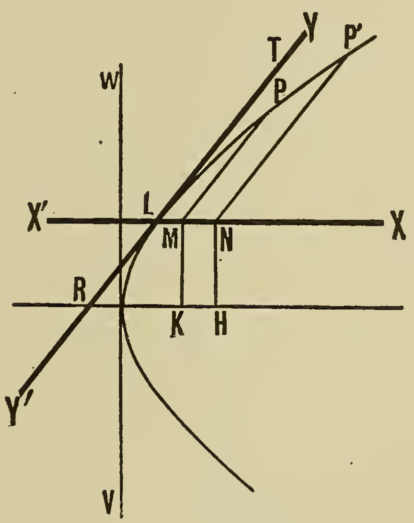



Fig. 107

Let LN be any diameter and TR a tangent at its extremity.

Let LN be taken as the X axis and TR as the Y axis.

Let PM and P'N be any two ordinates drawn to the diameter LN.

$$\text{Let } x \equiv LM \quad \text{and} \quad y \equiv PM.$$
$$x' \equiv LN \quad \text{"} \quad y' \equiv P'N.$$

We are to prove that

$$y^2 : y'^2 :: x : x'.$$

Since the point P is on the parabola, its coordinates must satisfy the equation of the parabola.

[1] Hence $y^2 = 2p'x$. by § 301.

[2] Similarly $y'^2 = 2p'x'$. by § 301.

[3] Hence $\frac{y^2}{y'^2} = \frac{x}{x'}$,

[4] or $y^2 : y'^2 :: x : x'$.

Q. E. D.

PROPOSITION VIII

308. *When any diameter and the tangent at its extremity are taken as the axes of coordinates, the equation of any tangent to the parabola is*

$$y' - y = \frac{p'}{y'}(x' - x),$$

*in which $x'$ and $y'$ are the coordinates of the point of tangency of the latter tangent, and $p'$ is equal to half the principal parameter divided by the square of the sine of the inclination of the tangent taken as the Y axis.*

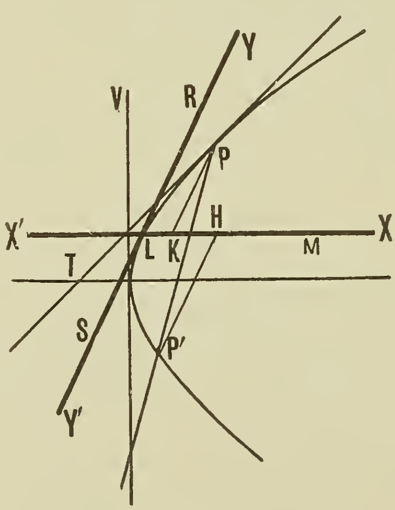



Fig. 108

Let LM be any diameter and RS the tangent at its extremity.

Let LM be the X axis and RS the Y axis.

Let PT be a tangent to the parabola at any point P.

$$\text{Let } x' \equiv \text{LK} \quad \text{and} \quad y' \equiv \text{PK},$$

$$\text{and let } p' \equiv \frac{p}{\sin^2 \text{RLX}}.$$

We are to prove that

$$y' - y = \frac{p'}{y'}(x' - x)$$

is the equation of the tangent to the parabola.

Let PP′ be a secant cutting the parabola at the two points P and P′.

$$\text{Let } x'' \equiv \text{LH} \text{ and } y'' \equiv \text{P'H}.$$

For the method of demonstration see § 221.

PROPOSITION IX

309. *The equation of the chord which joins the points of tangency of two tangents drawn to a parabola from any point without it is*

$$yy' = p(x + x'),$$

*in which $x'$ and $y'$ are the coordinates of the point from which the tangents are drawn, and $p$ is half the principal parameter.*

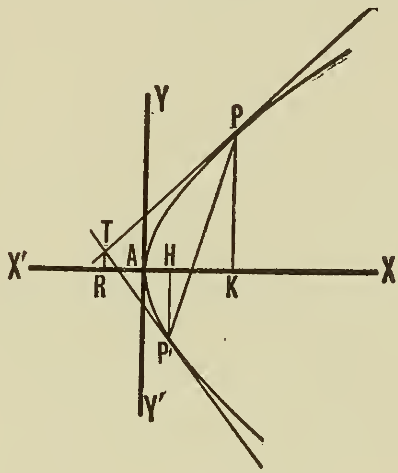



Fig. 109

Let PT and P′T be any two tangents drawn to the parabola from the same point T.

Let PP′ be the chord joining their points of tangency.

Let $x' \equiv$ AR and $y' \equiv$ TR.

We are to prove that

$$yy' = p(x + x')$$

is the equation of PP′.

Let $x'' \equiv$ AK and $y'' \equiv$ PK,
$x''' \equiv$ AH " $y''' \equiv$ P′H.

The equation of PT is

[1] $$y'' - y = \frac{p}{y''}(x'' - x).$$ by § 293.

[2] Hence $$y''^2 - yy'' = px'' - px,$$

[3] and $$yy'' = px + y''^2 - px''.$$

Since the point P is on the parabola, its coordinates $x''$ and $y''$ must satisfy the equation of the parabola.

Hence substituting $x''$ and $y''$ for the $x$ and $y$ of the equation of the parabola, we get

[4] $$y''^2 = 2px''.$$ by § 287.

Substituting this value of $y''^2$ into [3], we get

[5] $$yy'' = p(x + x'')$$

for the equation of PT.

Similarly we may show that the equation of P′T is

[6] $$yy''' = p(x + x''').$$

Now since the point T is on the tangent PT, its coordinates $x'$ and $y'$ must satisfy the equation of PT.

Substituting $x'$ and $y'$ for the $x$ and $y$ of that equation, we get

[7] $$y'y'' = p(x' + x'').$$

Similarly, since the point T is on the tangent P′T, we get

[8] $$y'y''' = p(x' + x''').$$

Now equation

[9] $$yy' = p(x + x')$$

is the equation of a straight line. by § 67.

But the coordinates $x''$ and $y''$ of the point P will satisfy this equation, for if they are substituted for the $x$ and $y$ in it, we get [7].

Hence the straight line represented by [9] must pass through P. by § 41.

The coordinates $x'''$ and $y'''$ of the point P′ will also satisfy [9], for if they are substituted for the $x$ and $y$ in it, we get [8].

Hence the straight line represented by [9] must also pass through P′. by § 41.

Therefore since the line represented by [9] passes through both the point P and the point P′, that equation must be the equation of PP′. Q. E. D.

PROPOSITION X

310. *If any chord of a parabola pass through a fixed point and tangents be drawn at its extremities, and if the chord be made to revolve about the fixed point, then the locus of the intersection of the two tangents will be a straight line whose equation is*

$$yy' = p(x + x'),$$

*in which $x'$ and $y'$ are the coordinates of the fixed point, and $p$ is half the principal parameter.*

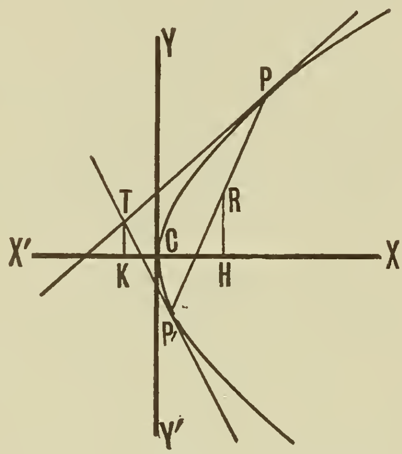



Fig. 110

Let PP′ be a chord passing through the fixed point R, and let PT and P′T be tangents drawn at its extremities and intersecting at the point T.

$$\text{Let } x' \equiv \text{CH} \quad \text{and} \quad y' \equiv \text{RH}.$$

Let PP′ revolve about R as a pivot.

We are to prove that the locus traced out by T is a straight line, and that its equation is

$$yy' = p(x + x').$$

$$\text{Let } x'' \equiv \text{CK} \quad \text{and} \quad y'' \equiv \text{TK}.$$

The equation of the chord PP′ is

[1] $$yy'' = p(x + x'').$$ by § 309.

Since the point R is on this chord, its coordinates $x'$ and $y'$ must satisfy the equation of that chord. by § 40.

Substituting $x'$ and $y'$ for the $x$ and $y$ of [1], we get

[2] $$y'y'' = p(x' + x'').$$

Now let PP′ revolve about R as a pivot. Then T will trace out a locus. Moreover [2] will be satisfied by the coordinates of the point T wherever it may be, as it traces out this locus.

Hence $x''$ and $y''$ of [2] stand for the coordinates of every point on the locus traced out by T.

Therefore [2] must be the equation of that locus. by § 39.

This locus is a straight line. by § 67.

Since T stands for any point on this straight line traced out by the intersection of the tangents, we may drop the accent marks from its coordinates and write them $x$ and $y$. Hence [2] may be written

$$yy' = p(x + x').$$

Therefore the locus is a straight line and its equation is

$$yy' = p(x + x').$$

Q. E. D.

PROPOSITION XI

311. *If two tangents be drawn at the extremities of any focal chord of a parabola*

(*1*) *the tangents will meet on the directrix;*

(*2*) *the line joining the intersection of the two tangents to the focus will be perpendicular to the focal chord.*

Let PT and P′T′ (Fig. 111) be the two tangents drawn to the parabola at the extremities of the focal chord PP′, and let R be the intersection of the two tangents.

Let DD′ be the directrix.

We are to prove

(1), that R will be on the directrix DD′, and

(2), that RF will be perpendicular to PP′.

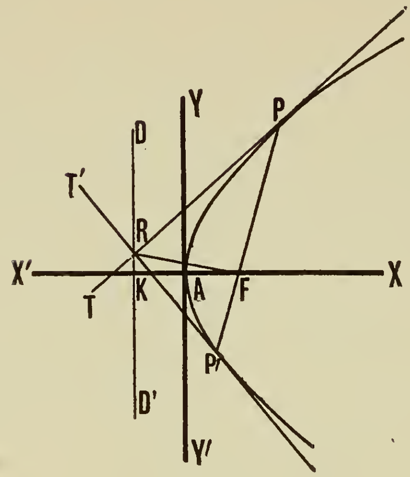



Fig. 111

Let $x' \equiv AK$, $y' \equiv RK$, and $p \equiv FK$.

The equation of PP′ is

[1] $$yy' = p(x' + x),$$ by § 309.

in which $x'$ and $y'$ are the coordinates of the point R and $x$ and $y$ are the coordinates of any point on the chord PP′.

Since the point F is on the chord PP′, its coordinates $x = \frac{1}{2}p$ and $y = 0$ must satisfy the equation of the chord. by § 40.

Substituting these values for the $x$ and $y$ of [1], we get

[2] $$0 = p(x' + \tfrac{1}{2}p).$$

[3] Hence $$x' = AK = -\tfrac{1}{2}p.$$

But $-\frac{1}{2}p$ is the distance from the vertex of the parabola to the directrix. by § 280.

Therefore the point R is on the directrix. Q. E. D.

Since RF is a straight line passing through the two fixed points R and F, its equation must be of the form

[4] $$y' - y = \frac{y'' - y'}{x'' - x'}(x' - x).$$ by § 58.

Now in [4] let $x''$ and $y''$ stand for the coordinates of the point F, and $x'$ and $y'$ the coordinates of the point R.

Then $x'' = \frac{1}{2}p$ and $y'' = 0$.

Substituting these values into [4], we get

[5] $$y'-y=\frac{-y'}{\frac{1}{2}p-x'}(x'-x).$$

[6] But $x'=-\frac{1}{2}p.$ by [3].

[7] Hence $$y'-y=-\frac{y'}{p}(-\tfrac{1}{2}p-x).$$

[8] $$-y=-\frac{y'}{p}(-\tfrac{1}{2}p-x)-y'.$$

[9] $$y=\frac{y'}{p}(-\tfrac{1}{2}p-x)+y'.$$

[10] $$y=-\frac{y'}{p}x+\frac{y'}{2},$$

which is the equation of the line RF. by § 39.

From [3] we get

[11] $$x'=\text{AK}=-\tfrac{1}{2}p.$$

Substituting this value of $x'$ into [1], we get

[12] $$yy'=p(-\tfrac{1}{2}p+x).$$

[13] $$y=-\frac{p}{y'}(\tfrac{1}{2}p-x),$$

[14] $$y=\frac{p}{y'}x-\frac{p^2}{2y'},$$

which is the equation of the chord PP′. by § 39.

Now let $s\equiv$ the slope of RF,
and $s'\equiv$ the slope of PP′.

Then from [10] we get

[15] $$s=-\frac{y'}{p},$$

and from [14] we get

[16] $$s'=\frac{p}{y'}.$$

[17] Hence $$ss'=-\frac{y'}{p}\,\frac{p}{y'}=-1.$$

[18] Hence $1+ss'=0.$

Therefore RF is perpendicular to PP′. by § 62.

Q. E. D.

PROPOSITION XII

312. *The two tangents drawn at the extremities of any focal chord are perpendicular to each other.*

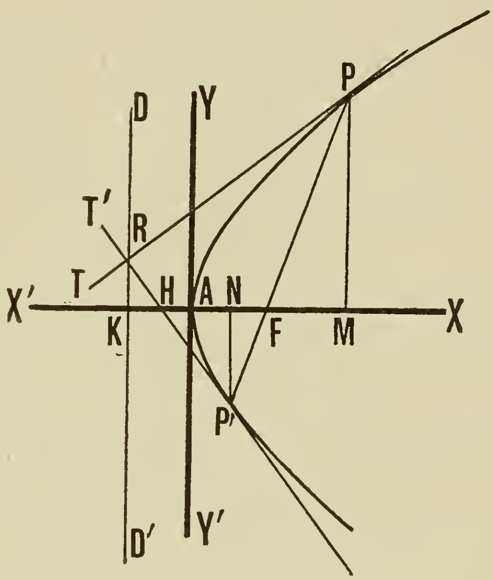



Fig. 112

Let PT and P′T′ be the two tangents drawn at the extremities of the focal chord PP′.

Let R be the point on the directrix at which the two tangents meet.

We are to prove that PT is perpendicular to P′T′.

Let $x' \equiv AK$ and $y' \equiv RK$.
$x'' \equiv AM$ " $y'' \equiv PM$,
$x''' \equiv AN$ " $y''' \equiv P'N$.

Since PT is a straight line passing through the two given points P and R, its equation must be of the form

[1] $$y' - y = \frac{y'' - y'}{x'' - x'}(x' - x).$$ by § 58.

in which

[2] $$\frac{y'' - y'}{x'' - x'} = \text{the slope of PT}.$$ by § 59.

[3] But $\frac{p}{y''} =$ the slope of PT. by § 294.

[4] Hence $\frac{y'' - y'}{x'' - x'} = \frac{p}{y''}$.

Since the point P is on the parabola,

[5] $y''^2 = 2px''$. by § 287 and § 40.

[6] Hence $x'' = \frac{y''^2}{2p}$.

[7] Also $x' = -\frac{1}{2}p$. by § 311, [3].

Substituting the right hand members of [6] and [7] into [4] we get

[8] $$\frac{y'' - y'}{\frac{y''^2}{2p} + \frac{p}{2}} = \frac{p}{y''}.$$

[9] Hence $y''^2 - y'y'' = \frac{y''^2}{2} + \frac{p^2}{2}$,

[10] $y''^2 - 2y'y'' = p^2$,

[11] and $y'' = \sqrt{p^2 + y'^2} + y'$.

[12] Hence $\frac{p}{y''} = \frac{p}{\sqrt{p^2 + y'^2} + y'} =$ the slope of PT. by [3].

Since P′T′ is also a straight line passing through the two given points P′ and R, its equation must be of the form

[13] $y' - y = \frac{-y''' - y'}{x''' - x'}(x' - x)$, by § 58.

in which

[14] $\frac{-y''' - y'}{x''' - x'} =$ the slope of P′T′. by § 59.

[15] But $-\frac{p}{y'''} =$ the slope of P′T′. by § 294.

[16] Hence $\frac{-y''' - y'}{x''' - x'} = -\frac{p}{y'''}$.

Solving this equation as we did [4], we get

[17] $y''' = \sqrt{p^2 + y'^2} - y'$.

[18] Hence $-\frac{p}{y'''} = -\frac{p}{\sqrt{p^2+y'^2}-y^2} =$ the slope of P′T′. by § 15.

Now let $s \equiv$ the slope of PT,

and $s' \equiv$ the slope of P′T′.

[19] Then $ss' = -\frac{p}{\sqrt{p^2+y'^2}-y'} \times \frac{p}{\sqrt{p^2+y'^2}+y'} =$

$$-\frac{p^2}{p^2+y'^2-y'^2} = -1.$$

[20] Hence $1 + s's = 0.$

Therefore PT and P′T′ are perpendicular to each other.

by § 62.

Q. E. D.

PROPOSITION XIII

313. *When the focus is taken as the pole, the polar equation of the parabola is*

$$r = \frac{p}{1 - \cos\theta},$$

*in which p is half the principal parameter, r is the radius vector of any point on the parabola, and θ is its vectorial angle.*

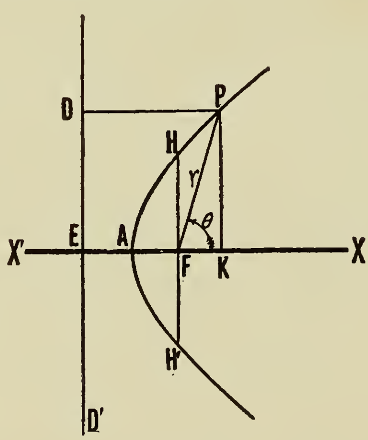



Fig. 113

Let the focus F be the pole.

Let P be any point on the parabola, $r$ its radius vector, and $\theta$ its vectorial angle.

Let $p \equiv$ one half the principal parameter HH′.

We are to prove that

$$r = \frac{p}{1 - \cos\theta},$$

is the equation of the parabola.

Let DD′ be the directrix.

Draw PD ⊥ DD′ and PK ⊥ XX′.

[1] FP = PD. by § 280.

[2] But PD = EK. by Geom. 17.

[3] Hence FP = EK = EF + FK,

[4] also EF = $p$, by § 291.

[5] and FK = $r \cos\theta$. by Trig. 2.

[6] Hence by [3] FP = $p + r\cos\theta$,

[7] or $r = p + r\cos\theta$.

[8] Hence $r(1 - \cos\theta) = p$.

[6] Therefore $r = \dfrac{p}{1 - \cos\theta}$.

Q. E. D.

EXAMPLES

1. What is the value of $r$ when $\theta = 0$? Ans. $r = \infty$.

2. What is the value of $r$ when $\theta = 90°$?
Ans. $r = p =$ FH.

3. What is the value of $r$ when $\theta = 180°$?
Ans. $r = \dfrac{p}{2} =$ AF.

4. What is the value of $r$ when $\theta = 60°$?

5. What is the value of $r$ when $\theta = 270°$?

# CHAPTER XIII

## The Conic

314. **The Conic.**—A *conic* is the locus of a point moving in a plane in such a way that its distance from a fixed point in the plane, and its distance from a fixed line in the plane always have the same ratio to each other.

315. **The Focus.**—The fixed point is called the *focus* of the conic.

316. **The Directrix.**—The fixed line is called the *directrix* of the conic.

PROPOSITION I

317. *The ellipse is a conic.*

For by § 176 the ellipse is only a particular case of the locus defined in § 314.

318. *Corollary.—The ellipse is a conic in which the distance of any point on the conic from the focus is less than its distance from the directrix.*

Compare § 176 and § 129.

PROPOSITION II

319. *The hyperbola is a conic.*

For by § 251 the hyperbola is only a particular case of the locus defined in § 314.

320. *Corollary.—The hyperbola is a conic in which the distance of any point on the conic from the focus is greater than its distance from the directrix.* by § 252.

PROPOSITION III

321. *The parabola is a conic.*

For by § 280 the parabola is only a particular case of the locus defined in § 314.

322. *Corollary.—The parabola is a conic in which the distance of any point on the conic from the focus is equal to its distance from the directrix.* by § 280.

## Poles and Polars

323. Equations [5], § 255*a*, [9], § 256, and [4], § 261, and the similar equations of the other conics show that the coordinates of every point on the tangent and the coordinates of the point of tangency; the coordinates of every point on the chord of contact and the coordinates of the intersection of the corresponding tangents; and the coordinates of every point on the locus of the intersection of two tangents, and the coordinates of the fixed point about which the corresponding chord of contact revolves, are connected in the same way.

That is, each of these three lines is connected with a fixed point in the same way in which each of the other lines is connected with a fixed point.

324. **The Polar.**—If the coordinates of every point on any line have the same relation to the coordinates of a certain fixed point that the coordinates of every point on a tangent have to the coordinates of the point of tangency, that line is called the *polar* of the fixed point.

325. **A Pole.**—The fixed point is called the *pole* of the line.

When the *pole* is without the conic the *polar* is the *chord of contact* of the two tangents drawn from the pole.

When the *pole* is on the conic the *polar* is a *tangent* to the conic.

When the *pole* is within the conic the *polar* is the *locus of the intersection* of the two tangents drawn at the extremities of the chord of contact passing through the pole.

PROPOSITION IV

326. *In an hyperbola or an ellipse the polar of a point on any diameter or on any diameter produced is parallel to the conjugate diameter.*

I. FOR THE HYPERBOLA.

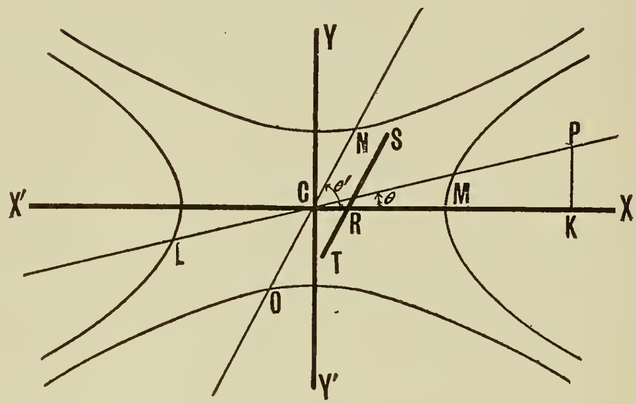



Fig. 114

Let P be any fixed point on the diameter LM produced, and let ST be the polar of that point.

Let NO be conjugate to LM.

We are to prove that ST is parallel to NO.

$$\text{Let} \quad x' \equiv \text{CK} \quad \text{and} \quad y' \equiv \text{PK},$$
$$\theta \equiv \angle\text{MCK} \quad \text{"} \quad \theta' \equiv \angle\text{NCK}.$$

Since LM is a straight line passing through the two points C and P, its equation must be of the form

[1] $$y' - y = \frac{y'' - y'}{x'' - x'}(x' - x), \qquad \text{by § 58.}$$

in which $x''$ and $y''$ are the coordinates of the point C, and $x'$ and $y'$ are the coordinates of the point P.

Since C is the origin,

$$x'' = 0 \quad \text{and} \quad y'' = 0.$$

Substituting these values into [1], we get

[2] $$y' - y = \frac{y'}{x'}(x' - x).$$

[3] Hence $$y'x' - yx' = y'x' - y'x,$$

[4] and $y = \frac{y'}{x'} x,$

which is the equation of LM.

[5] Hence $\frac{y'}{x'} = \tan \theta.$ by § 55.

Since LM and NO are conjugate to each other,

[6] $\tan \theta \tan \theta' = \frac{b^2}{a^2}.$ by § 240.

[7] Hence $\frac{y'}{x'} \tan \theta' = \frac{b^2}{a^2}.$

[8] Hence $\tan \theta' = \frac{b^2 x'}{a^2 y'}.$

The equation of the polar ST is

[9] $a^2yy' - b^2xx' = - a^2b^2.$ by § 324 and § 221.

[10] Hence $y = \frac{b^2 x'}{a^2 y'} x - \frac{b^2}{y'}.$

[11] Hence $\tan \text{SRK} = \frac{b^2 x'}{a^2 y'}.$ by § 53.

By comparing [8] and [11], we get

[12] $\tan \theta' = \tan \text{SRK}.$

Therefore ST is parallel to NO. by Geom. 9.

Q. E. D.

2. FOR THE ELLIPSE.

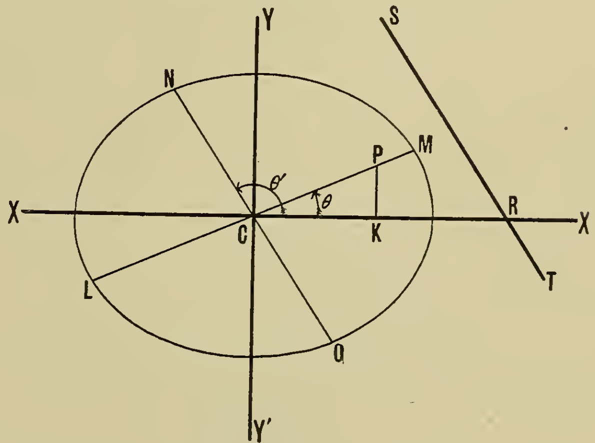



Fig. 115

For demonstration compare that for the hyperbola.

PROPOSITION V

327. *For a parabola the polar of a point on any diameter is parallel to the ordinates of that diameter.*

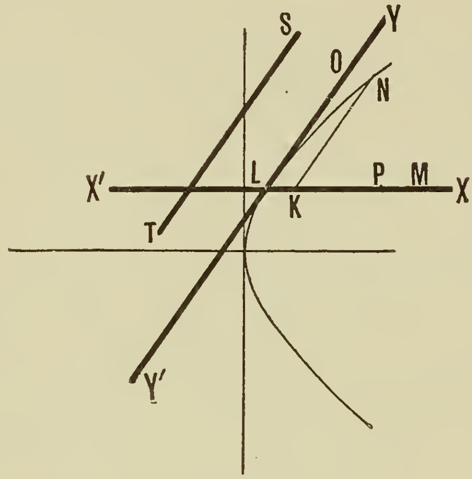



Fig. 116

Let P be any fixed point on the diameter LM, and let ST be the polar of that point.

Let NK be an ordinate to the diameter LM.

We are to prove ST ∥ NK.

Let LO be a tangent to the parabola at L.

Let LM be taken as the X axis and LO as the Y axis.

Then the equation of ST will be

[1] $$y'-y=\frac{p}{y'}(x'-x),$$ by §324 and §293.

[2] Hence $$y'y=p(x'+x).$$

Now since the point P is on the X axis, $y'=0$ and [2] becomes

[3] $$x=-x'.$$

But this is the equation of a line parallel to the Y axis. by § 76*a*.

Hence ST ∥ LO.

But NK ∥ LO. by § 14.

Therefore the polar ST ∥ the ordinate NK.

Q. E. D.

328. *Corollary.—The polar of any point on an axis of a conic is perpendicular to that axis.*

## PROPOSITION VI

329. *In an hyperbola and in an ellipse the distance from the center measured along any diameter to the polar of any point on that diameter or diameter produced, is a third proportional to the semi-diameter and the distance of the point from the center.*

I. FOR THE HYPERBOLA.

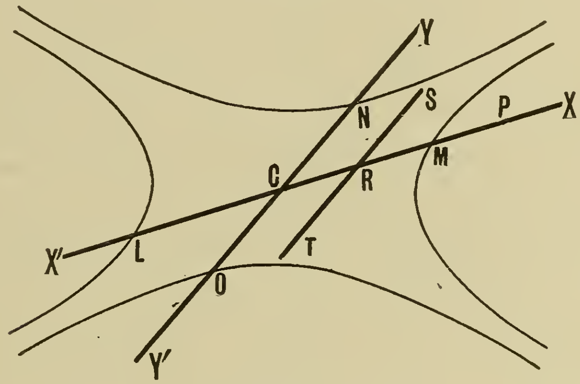



Fig. 117

Let ST be the polar of the point P on the diameter LM produced, and let R be the point where this polar cuts LM.

$$\text{Let } a' \equiv CM \quad \text{and} \quad x' \equiv CP.$$

We are to prove that

$$CR : a' :: a' : CP.$$

Let NO be conjugate to LM.

Let LM be taken as the X axis and NO as the Y axis.

$$\text{Let } b' \equiv CN.$$

Then the equation of the polar ST will be

[1] $a'^2yy' - b'^2xx' = -a'^2b'^2$. by § 324 and 255*a*.

Now the polar ST cuts the X axis LM where $y = 0$ and $x = CR$. by § 45.

Substituting these values into [1], we get

[2] $$-b'^2x'\text{CR} = -a'^2b'^2,$$

[3] and $$\text{CR} = \frac{a'^2}{x'}.$$

[4] Hence $$\text{CR} : a' :: a' : x',$$

[5] or $$\text{CR} : a' :: a' : \text{CP}.$$

Q. E. D.

2. FOR THE ELLIPSE.

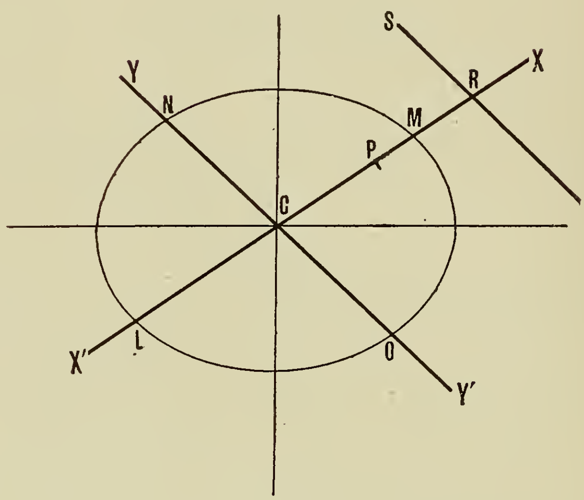



Fig. 118

Let SR be the polar of the point P on the diameter LM, and let R be the point where this polar cuts LM produced.

Let $a' \equiv \text{CM}$ and $x' \equiv \text{CP}$.

We are to prove

$$\text{CR} : a' :: a' : \text{CP}.$$

For demonstration compare that of the hyperbola.

## PROPOSITION VII

330. *The distance from the extremity of any diameter of a parabola measured along that diameter to the polar of any point on it is equal to the distance from the vertex to the point, but is measured in the opposite direction.*

For in § 327 we have proved that

$$x = -x',$$

in which $x$ is the abscissa of any point on the polar, and $x'$ is the abscissa of its pole.

Q. E. D.

PROPOSITION VIII

331. *In any conic the directrix is the polar of the adjacent focus.*

I. FOR THE HYPERBOLA

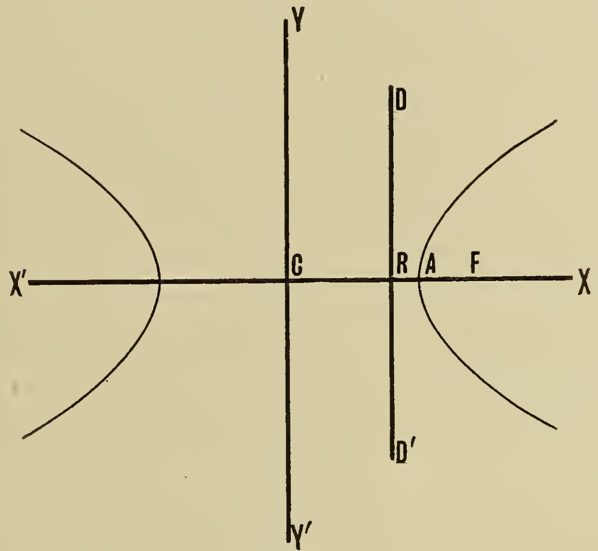



Fig. 119

Let DD′ be the directrix and F the adjacent focus.

We are to prove that DD′ is the polar of the focus F.

Let $a \equiv CA$ and $p \equiv RF$.

[1] Then $CR = CF - RF.$

[2] But $CF = c = ae,$ by § 218, [2].

[3] and $RF = p = \dfrac{a(e^2 - 1)}{e}$ by § 251, [5].

[4] Hence by [1] $CR = ae - \dfrac{a(e^2 - 1)}{e} = \dfrac{a}{e}.$

The directrix ⊥ the X axis. by § 248.

The equation of the polar of the focus is

[5] $$y' - y = \frac{b^2x'}{a^2y'}(x' - x).$$ by § 324 and § 221.

[6] Or $$a^2yy' - b^2xx' = -a^2b^2,$$

in which $x' = ae$ and $y' = 0$.

Substituting these values into [6], we get

[7] $$-b^2xae = -a^2b^2,$$

[8] and $$x = \frac{a}{e},$$

which is the equation of a line parallel to the Y axis, and at a distance from the origin equal to $\frac{a}{e}$. by § 56.

The polar of the focus is therefore perpendicular to the X axis. by Geom. 6.

Since the origin is at the center, the distance of the polar of the focus from the center is $\frac{a}{e}$.

Hence from [4] and [8] it follows that the directrix and the polar of the focus are at the same distance from the center and both perpendicular to the X axis.

Therefore the directrix coincides with the polar of the focus. by Geom. 2.

Q. E. D.

2. FOR THE ELLIPSE

For the demonstration of the ellipse compare that of the hyperbola.

3. FOR THE PARABOLA

Let DD′ (Fig. 120) be the directrix and F the focus.

We are to prove that DD′ is the polar of the focus F.

[1] $$AR = AF.$$ by § 280.

The distance of the polar of the focus from A is equal to AF but measured in the opposite direction from A. by § 330.

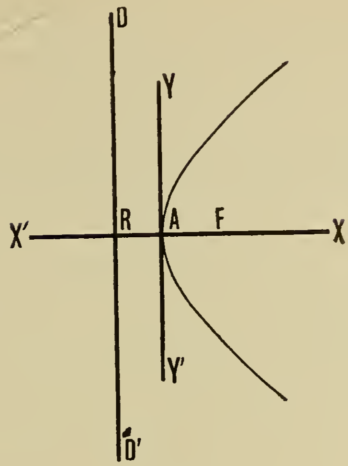



Fig. 120

Therefore the directrix and the polar of the focus pass through the same point R.

The directrix is perpendicular to the X axis. by § 284.

The polar of the focus is perpendicular to the X axis. by § 327 and § 14.

Therefore the directrix coincides with the polar of the focus. by Geom. 2.

Q. E. D.

EXAMPLES.

1. The slope of a tangent to the parabola $y^2 = 6x$ is 3. What is the equation of the tangent? Ans. $y = 3x + \frac{1}{2}$.

2. Given the parabola $y^2 = 8x$. What is the parameter of the diameter $y - 16 = 0$? Ans. 136.

3. Required the chord of contact of tangents drawn from $(-2, 5)$ to $y^2 = 8x$. Ans. $5y - 4x + 8 = 0$.

4. A tangent to $y^2 = 4x$ makes an angle of $45°$ with the X axis. What is the point of tangency?

5. Given the parabola $y^2 = 4x$; required the equation of the chord which is bisected by the point (2, 1).

6. Required the equation of the right line passing through the vertex of any parabola and the extremity of the focal ordinate.

7. The headlights on locomotives contain parabolic reflectors. Why?

8. The equation of a parabola is $y^2 = 10x$. Through the point (7, ord. +) we draw a tangent and a normal. Required the lengths of the tangent, normal, subtangent and subnormal.

9. The points of contact of two tangents are $(x_1, y_1)$, $(x_2, y_2)$. Find their intersection.

10. Find the equation of a straight line touching the parabola $y^2 = 16x$ and passing through $(-4, 8)$.

# CHAPTER XIV

## The General Equation of the Second Degree

### PROPOSITION I

331. *To determine the forms of all the loci which are represented by equations of the second degree containing two variables only.*

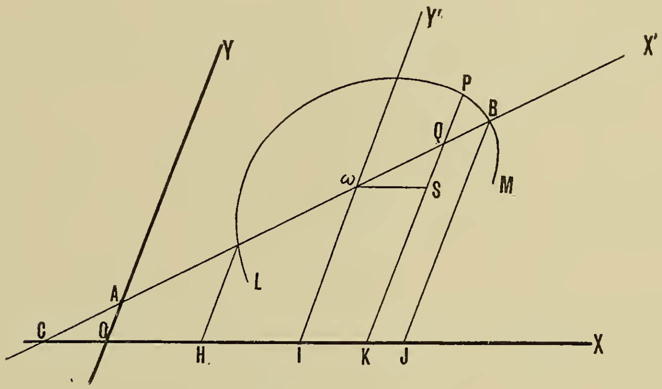



Fig. 121

The general equation of the second degree may be written

[1] $Ax^2 + 2Bxy + Cy^2 + 2Dx + 2Ey + F = 0.$

If the $x$ and $y$ in this equation be variables and represent the coordinates of a point moving in a plane, then whatever real values its coefficients may have, the equation will represent some locus, real or imaginary.

Let LM of Fig. 121 represent that locus. We are to find the forms of LM for all real values of $A$, $B$, $C$, $D$, $E$, $F$.

It will be convenient to divide the investigation into two parts.

1st. To find the forms of LM when $C \neq$* 0.

2nd. To find the forms of LM when $C = 0$.

* Read $\neq$ "is not equal to."

FIRST PART

$$C \neq 0.$$

Equation [1] may then be written

[2] $Cy^2 + 2(Bx + E)y + Ax^2 + 2Dx + F = 0.$

Finding the values of $y$ from this, we get

[3] $y = -\dfrac{Bx + E}{C} \pm \dfrac{1}{C}\sqrt{(Bx+E)^2 - C(Ax^2+2Dx+F)}.$

[4] Let $R \equiv (Bx + E)^2 - C(Ax^2 + 2Dx + F).$

[5] Then

$$R \equiv B^2x^2 + 2BEx + E^2 - ACx^2 - 2CDx - CF,$$

[6] or $R \equiv (B^2 - AC)x^2 + 2(BE - CD)x + E^2 - CF.$

[7] Let $\Delta \equiv + \begin{vmatrix} A & B & D \\ B & C & E \\ D & E & F \end{vmatrix}.$

Wentworth's College Alg., § 399.

The coefficients of [6] may easily be remembered by noticing that they are the minors corresponding to $F$, $D$ and $A$ in $\Delta$.

$$B^2 - AC = - \begin{vmatrix} A & B \\ B & C \end{vmatrix}.$$

$$BE - CD = \begin{vmatrix} B & C \\ D & E \end{vmatrix},$$

and $$E^2 - CF = - \begin{vmatrix} C & E \\ E & F \end{vmatrix}.$$

Multiplying both sides of [7] by $-C$, we get

[8] $-C\Delta = -AC^2F + ACE^2 + C^2D^2 + CFB^2 - 2CBDE.$

Adding $B^2E^2 - B^2E^2$ to the right hand member of [8], we get

[9] $-C\Delta =$

$$-AC^2F + ACE^2 - B^2E^2 + CFB^2 + B^2E^2 - 2CBDE + C^2D^2$$
$$= AC(E^2 - CF) - B^2(E^2 - CF) + (BE - CD)^2$$
$$= (AC - B^2)(E^2 - CF) + (BE - CD)^2.$$

[10] or $-C\Delta = (BE - CD)^2 - (B^2 - AC)(E^2 - CF)$.

Let
$$a \equiv B^2 - AC,$$
$$b \equiv BE - CD,$$
$$c \equiv E^2 - CF.$$

Then equation [6] becomes

[11] $$R = ax^2 + 2bx + c,$$

and [10] becomes

[12] $$b^2 - ac = -C\Delta.$$

Now it will be found that the form of LM depends largely upon the coefficient of $x^2$ in [6].

This coefficient may take three different values.

1st. $B^2 - AC < 0$.

2nd. $B^2 - AC > 0$.

3rd. $B^2 - AC = 0$.

*First Case*

$$B^2 - AC < 0.$$

In this case we may have

1st. $C\Delta < 0$.

2nd. $C\Delta > 0$.

3rd. $C\Delta = 0$.

*First* when $C\Delta < 0$.

[13] Put $ax^2 + 2bx + c = 0$.

Now since $C\Delta < 0$,
$$-C\Delta > 0;$$

hence by [12]
$$b^2 - ac > 0.$$

Hence the roots of [13] are real and unequal.

Wentworth's College Algebra, § 141.

Let these roots be $x'$ and $x''$ and let $x' < x''$,

[14] then $R = ax^2 + 2bx + c = a(x - x')(x - x'')$.

Wentworth's College Algebra, § 153.

But since in this first case $B^2 - AC < 0$, $a$ is negative, and [14] may be written

[15] $$R = -a(x - x')(x - x'').$$

Substituting $R$ for the quantity under the $\sqrt{\phantom{x}}$ in [3], we get

[16] $$y = -\frac{Bx + E}{C} \pm \frac{1}{C}\sqrt{R}.$$

[17] Let $$Y \equiv \frac{1}{C}\sqrt{R},$$

[18] then $$y = -\frac{Bx + E}{C} \pm Y,$$

which is the equation of LM. by § 39.

If in Fig. 121 P be any point on LM, then in [18]

[19] $$y \equiv \text{PK} \quad \text{and} \quad x \equiv \text{OK}.$$

Let AB be the line whose equation is

[20] $$y = -\frac{Bx + E}{C}.$$

Let the coordinates of the point Q on this line be

$$x \equiv \text{OK} \quad \text{and} \quad y''' \equiv \text{QK}.$$

[21] Then $$y''' = -\frac{Bx + E}{C}.$$ by § 40.

Now if we subtract [21 from [18], member for member, we get

[22] $$y - y''' = \pm Y,$$

[23] or $$\text{PK} - \text{QK} = \pm Y,$$

[24] or $$\text{PQ} = \pm Y.$$

Take HI = IJ,

and draw IY′ ∥ OY.

If now we take $\omega Y'$ and $\omega X'$ as our axes of coordinates, then $\omega$Q and PQ will be the coordinates of the point P.

[24*a*] Let $\omega Q \equiv X$.

[24*b*] $PQ = Y$. by [24].

In Fig. 121 let O be so taken that

$$x' \equiv OH \quad \text{and} \quad x'' \equiv OJ.$$

Then $x'' = OH + 2HI$,

hence $x' + x'' = 2(OH + HI)$,

[25] and $\dfrac{x' + x''}{2} = OI$.

Hence by [19] and [25], we get

[26] $OK = x = \dfrac{x' + x''}{2} + \omega S$.

Now by Trig. 14 and Geom. 11 we get

$$\frac{\omega S}{\omega Q} = \frac{\sin \omega QS}{\sin \omega SQ} = \frac{\sin CAO}{\sin COA}.$$

But $\dfrac{\sin CAO}{\sin COA}$ is constant. Representing this constant by $k$, we get

[27] $\dfrac{\omega S}{\omega Q} = k$.

[28] $\omega S = k\omega Q = kX$. by [24*a*].

Substituting into [26], we get

[29] $x = \dfrac{x' + x''}{2} + kX$.

Substituting this value of $x$ into [15], we get

[30] $R = -a\left(\dfrac{x' + x''}{2} + kX - x'\right)\left(\dfrac{x' + x''}{2} + kX - x''\right)$,

[31] whence $R = -a\left(kX + \dfrac{x'' - x'}{2}\right)\left(kX - \dfrac{x'' - x'}{2}\right)$.

Now let $h \equiv \dfrac{x'' - x'}{2}$,

[32] then $R = -a(k^2X^2 - h^2)$.

Substituting this value of $R$ into [17], we get

[33] $$Y = \frac{1}{C}\sqrt{-a(k^2X^2 - h^2)}.$$

[34] $$C^2Y^2 = -ak^2X^2 + ah^2.$$

[35] $$\frac{C^2}{ah^2}Y^2 = -\frac{k^2}{h^2}X^2 + 1.$$

[36] Let $$\frac{C^2}{ah^2} \equiv \frac{1}{b'^2},$$

[37] and $$\frac{k^2}{h^2} \equiv \frac{1}{a'^2}.$$

Substituting into [35], we get

[38] $$\frac{Y^2}{b'^2} = -\frac{X^2}{a'^2} + 1.$$

[39] Whence $a'^2Y^2 + b'^2X^2 = a'^2b'^2$.

Now since by [24*a*] and [24*b*] $X = \omega Q$ and $Y = PQ$, [39] is the equation of the ellipse.

*Second* when $C\Delta > 0$.

In this case, by [12], $b^2 - ac < 0$. Hence the roots of

$$ax^2 + 2bx + c = 0$$

are imaginary. Wentworth's College Algebra, § 141.

Hence $ax^2 + 2bx + c$ is negative when $a$ is negative.
Wentworth's College Algebra, § 180.

But in this first case

$$B^2 - AC < 0,$$

[40] or $$a < 0,$$

that is, $a$ is negative. Hence $ax^2 + 2bx + c$ is negative.

Hence by [11] $R$ is negative, and therefore $\sqrt{R}$ is imaginary.

Hence by [16] $y$ is imaginary.

Therefore when $C\Delta > 0$, [1] cannot represent any real locus.

*Third* when $C\Delta = 0$.

Since in this first part of the investigation $C \neq 0$, then $\Delta = 0$.

Hence by [12]

[41] $$b^2 - ac = 0.$$

Therefore the roots $x'$ and $x''$ of [13] are real and equal.

Wentworth's College Algebra, § 141.

[42] Hence $$R = a(x - x')(x - x') = a(x - x')^2,$$

and $$\frac{1}{C}\sqrt{R} = \frac{1}{C}(x - x')\sqrt{a}.$$

Hence [16] becomes

[43] $$y = -\frac{Bx + E}{C} \pm \frac{x - x'}{C}\sqrt{a}.$$

Now since by [40] $a$ is negative, the last term of this equation is imaginary, and the equation itself represents two imaginary straight lines.

Therefore when $B^2 - AC < 0$, and

1st. $C\Delta < 0$, [1] represents an ellipse;

2nd. $C\Delta > 0$, [1] represents an imaginary ellipse;

3rd. $C\Delta = 0$, [1] represents two imaginary straight lines.

*Second Case*

$$B^2 - AC > 0.$$

In this case again we may have

| | |
|---|---|
| 1st | $C\Delta < 0$. |
| 2nd. | $C\Delta > 0$. |
| 3rd. | $C\Delta = 0$. |
| *First* let | $C\Delta < 0$. |

By [12] we see that when $C\Delta < 0$, $b^2 - ac > 0$. Hence again $x'$ and $x''$ are both real and unequal. Also since $B^2 - AC > 0$, $a$ is positive. Therefore [33] will become

[44] $$Y = \frac{1}{C}\sqrt{a(k^2X^2 - h^2)}.$$

[45] $$C^2Y^2 = ak^2X^2 - ah^2.$$

[46] $$\frac{C^2Y^2}{ah^2} = \frac{k^2}{h^2}X^2 - 1.$$

Let $$\frac{C^2}{ak^2} = \frac{1}{b'^2},$$

and $$\frac{k^2}{h^2} = \frac{1}{a'^2},$$

then [46] will become

[47] $$\frac{Y^2}{b'^2} = \frac{X^2}{a'^2} - 1.$$

[48] and $$a'^2Y^2 - b'^2X^2 = -a'^2b'^2,$$

which is the equation of the hyperbola.

*Second* let $C\Delta > 0$. Then by [12] $b^2 - ac < 0$, and the roots of [13] are imaginary.

Now since in this second case $a$ is positive, as in [31], we get

$$R = a\left(kX + \frac{x'' - x'}{2}\right)\left(kX - \frac{x'' - x'}{2}\right),$$

but since $x'$ and $x''$ are imaginary, we may write this

$$R = a(kX + h\sqrt{-1})(kX - h\sqrt{-1})$$
$$= a(k^2X^2 + h^2).$$

Therefore by [17] $Y = \frac{1}{C}\sqrt{a(k^2X^2 + h^2)}$.

$$C^2Y^2 = ak^2X^2 + ah^2.$$
$$C^2Y^2 - ak^2X^2 = ah^2.$$
$$\frac{C^2}{ah^2}Y^2 - \frac{k^2}{h^2}X^2 = 1.$$

Put $$\frac{C^2}{ah^2} \equiv \frac{1}{b'^2} \text{ and } \frac{k^2}{h^2} \equiv \frac{1}{a'^2},$$

then $$\frac{Y^2}{b'^2} - \frac{X^2}{a'^2} = 1,$$

or $$a'^2Y^2 - b'^2X^2 = a'^2b'^2,$$

which is the equation of the hyperbola conjugate to [48].

*Third* let $C\Delta = 0$.

Again by [12], when $C\Delta = 0$, $b^2 - ac = 0$, and the roots of [13] will be real and equal; and again as in [42]

[49] $$R = a(x - x')^2,$$

and as in [43]

[50] $$y = -\frac{Bx + E}{C} \pm \frac{x - x'}{C}\sqrt{a}.$$

Now since $a$ is positive, this equation represents two real straight lines.

Therefore when $B^2 - AC > 0$, and

1st. $C\Delta < 0$, [1] represents an hyperbola;

2nd. $C\Delta > 0$, [1] represents the conjugate hyperbola;

3rd. $C\Delta = 0$, [1] represents two straight lines.

*Third Case*

$$B^2 - AC = 0.$$

Since $a \equiv B^2 - AC,$

[51] then $a = 0,$

and by [6] we get

$$R = 2(BE - CD)x + E^2 - CF,$$

[52] or by [11] $R = 2bx + c.$

Substituting this value of $R$ into [3], we get

[53] $$y = -\frac{Bx + E}{C} \pm \frac{1}{C}\sqrt{2bx + c}$$

[54] $$y = -\frac{Bx + E}{C} \pm \frac{1}{C}\sqrt{2b\left(x + \frac{c}{2b}\right)}.$$

Now the values of $y$ given by this equation will be real so long as

$$x \geqq \frac{c}{2b}.$$

Since by [51]

$$a = 0,$$

[55] then $(BE - CD)^2 = -C\Delta.$ by [10].

[56] Hence $b^2 = -C\Delta,$

so that when $b \neq$ to 0, $C\Delta$ is negative,

[57] and when $b = 0$, $C\Delta$ is $0$.

Hence in this case we can only have

1st. $C\Delta < 0$.

2nd. $C\Delta = 0$.

*First* let $C\Delta < 0$.

Then by [17] and [52] we get

[58] $Y = \frac{1}{C}\sqrt{2bx + c} = \frac{1}{C}\sqrt{2b\left(x + \frac{c}{2b}\right)}$.

In Fig. 121 let

$$x_1 = OI = -\frac{c}{2b},$$

then by [58] $Y = \frac{1}{C}\sqrt{2b(x - x_1)}$.

Now as in [29] we may show that

$$x = x_1 + kX.$$

[59] Hence $Y = \frac{1}{C}\sqrt{2b(x_1 + kX - x_1)}$,

[60] and $Y = \frac{1}{C}\sqrt{2bkX}$,

[61] and $Y^2 = 2\frac{bk}{C^2}X$.

Let $p \equiv \frac{bk}{C^2}$,

[62] then $Y^2 = 2pX$,

which is the equation of the parabola.

*Second* let $C\Delta = 0$,

Then by [57]

$$b = 0,$$

and [53] becomes

[63] $$y = -\frac{Bx + E}{C} \pm \frac{\sqrt{c}}{C},$$

which is the equation of two parallel straight lines.

Therefore when $B^2 - AC = 0$, and

1st. $C\Delta < 0$, [1] represents a parabola.

2nd. $C\Delta = 0$, [1] represents two parallel straight lines.

### SECOND PART

$$C = 0.$$

Equation [1] may then be written

[64] $$[2(Bx+E)]y + Ax^2 + 2Dx + F = 0.$$

[65] $$y = -\frac{Ax^2 + 2Dx + F}{2(Bx+E)}.$$

[66] $$y = -\frac{A}{2B}x - \frac{D}{B} + \frac{AE}{2B^2} + \frac{\left(-\frac{AE^2}{B^2} + \frac{2DE}{B} - F\right)}{2(Bx+E)}.$$

[67] $$y = -\frac{A}{2B}x + \frac{AE}{2B^2} - \frac{D}{B} + \left(\frac{-AE^2 + 2BDE - FB^2}{2B^2(Bx+E)}\right)$$

But when $C = 0$,

$$\Delta = -AE^2 + 2BDE - FB^2$$

Hence [67] becomes

[68] $$y = -\frac{A}{2B}x + \frac{AE}{2B^2} - \frac{D}{B} + \frac{\Delta}{2B^2(Bx+E)}.$$

[69] $$y = -\frac{A}{2B}x + \frac{AE}{2B^2} - \frac{D}{B} + \frac{\Delta}{2B^2\left(x + \frac{E}{B}\right)}.$$

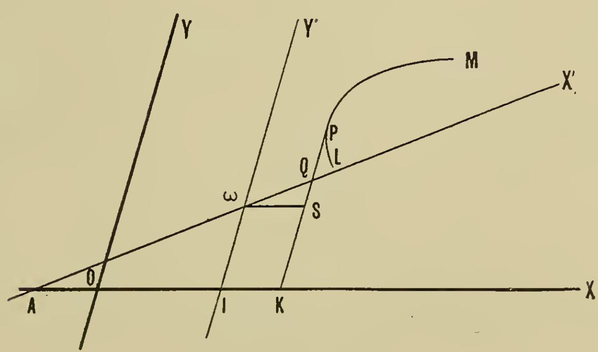



Fig. 122

In Fig. 122 let LM be the locus of [69], and let P be any point on LM.

Then $x \equiv OK$ and $y \equiv PK$.

Let AX′ be the line whose equation is

[70] $$y = -\frac{A}{2B}x + \frac{AE}{2B^2} - \frac{D}{B},$$

and let $y''' \equiv QK$,

[71] then $$y''' = -\frac{A}{2B}x + \frac{AE}{2B^2} - \frac{D}{B}.$$ by § 40.

Then by subtracting [71] from [69], we get

[72] $$PQ = y - y''' = \frac{\Delta}{2B^3\left(x + \frac{E}{B}\right)},$$

[73] Let $$Y \equiv PQ = \frac{\Delta}{2B^3\left(x + \frac{E}{B}\right)},$$

let $$OI = -\frac{E}{B},$$

and let IY′ be the line whose equation is

[74] $$x' = -\frac{E}{B}.$$

[75] Then $$\omega S = x - x' = x + \frac{E}{B}.$$

Then as in [28] of the first part

[76] $$\omega Q = k'\left(x + \frac{E}{B}\right).$$

Now if we take AX′ for a new axis of X and IY′ for the new axis of Y, then the coordinates of P will be

[77] $X \equiv \omega Q$ and $Y \equiv PQ$,

[78] and $$X = k'\left(x + \frac{E}{B}\right),$$

[79] hence $$x + \frac{E}{B} = \frac{X}{k'}.$$

Then [73] becomes

[80] $$Y = \frac{\Delta}{2B^3 \dfrac{X}{k'}} = \frac{k'\Delta}{2B^3X}.$$

[81] $$XY = \frac{k'\Delta}{2B^3}.$$

Since $\dfrac{k'\Delta}{2B^3}$ is constant, [81] is the equation of an hyperbola referred to its asymptotes. by § 275.

Therefore when $C = 0$, [1] represents an hyperbola referred to its asymptotes.

SUMMARY

$$C \neq 0.$$

Then when $B^2 - AC < 0$, and

1st. $C\Delta < 0$, [1] represents an ellipse;
2nd. $C\Delta > 0$, [1] represents an imaginary ellipse;
3rd. $C\Delta = 0$, [1] represents two imaginary straight lines.

When $B^2 - AC > 0$, and

1st. $C\Delta < 0$, [1] represents an hyperbola;
2nd. $C\Delta > 0$, [1] represents the conjugate hyperbola;
3rd. $C\Delta = 0$, [1] represents two straight lines.

When $B^2 - AC = 0$, and

1st. $C\Delta < 0$, [1] represents a parabola;
2nd. $C\Delta = 0$, [1] reprasents two parallel straight lines.

$$C = 0.$$

Then [1] represents an hyperbola.

331*a*. *Corollary.—Every equation of the second degrce containing two variables only is the equation of a conic.*

*Remark.*—The lines represented by [43], [49] and [61] of § 330 are limiting cases of the corresponding conics.

# NON-CONICS

332. **A Non-Conic.**—Any plain locus which is not a conic we will call a *non-conic*. Among the non-conics are

Higher Plane Loci,
Spirals,
The Logarithmic Curve,
Trigonometric Loci.

## HIGHER PLANE LOCI

333. **A Higher Plane Locus.**—A *higher plane locus* is a locus whose equation is of a higher degree than the second.

### The Lemniscate

334. **The Lemniscate.**—If from the center of an equilateral hyperbola a perpendicular be drawn to a tangent to the hyperbola, and the point of tangency be made to move along the hyperbola, the locus traced out by the intersection of the perpendicular and tangent is called the *lemniscate*.

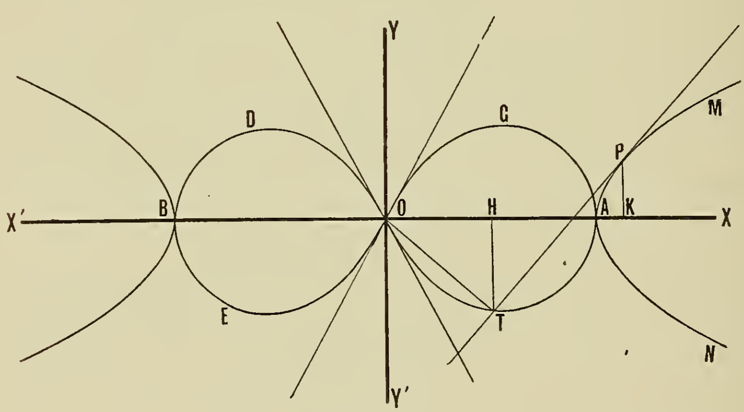



Fig. 123

Let MAN be an equilateral hyperbola; OT a perpendicular drawn from the center to the tangent which touches the hyperbola at the point P, and let P move along the hyperbola.

Then the locus traced out by T, the intersection of the perpendicular and the tangent, is the lemniscate.

PROPOSITION

335. *The equation of the lemniscate is*

$$(x^2+y^2)^2=a^2(x^2-y^2).$$

Let $x \equiv OH$ and $y \equiv TH$,
$x' \equiv OK$ " $y' \equiv PK$,
$a \equiv OA$.

The equation of MAN is

[1] $$x^2-y^2=a^2,$$ by § 208.

and since P is on the hyperbola

[2] $$x'^2-y'^2=a^2.$$ by § 40.

Hence the equation of the tangent to the equilateral hyperbola may be written

[3] $$y=\frac{x'}{y'}x-\frac{a^2}{y'}.$$ by §§ 221 and 207.

Therefore the equation of the perpendicular OT is

[4] $$y=-\frac{y'}{x'}x.$$ by §§ 55 and 62.

Since the point T is on both the perpendicular and the tangent, we may let the $x$ and $y$ in [3] and [4] stand for the coordinates of the point T. by § 40.

Then [3] and [4] become simultaneous. Solving them we get

[5] $$x'=\frac{a^2x}{x^2+y^2} \text{ and } y'=-\frac{a^2y}{x^2+y^2}.$$

Substituting these values into [2] and reducing, we get

[6] $$(x^2+y^2)^2=a^2(x^2-y^2).$$

Q. E. D.

336. *Corollary.—The polar equation of the lemniscate is*

$$r^2 = a^2 \cos 2\theta,$$

*in which* $r \equiv OT$ *and* $\theta = \angle AOT$ *reckoned counter clockwise.*

This equation may be found by transforming [6] to polar coordinates.

*Scholium.*—Let $\theta = 0$, then $r = \pm a$.

Let $0 < \theta < 45°$,

then $1 > \cos 2\theta > 0$.

Hence $a^2 \cos 2\theta < a^2$,

and $r^2 < a^2$.

Therefore $r$ has two values numerically equal, less than $a$, and of opposite signs.

Let $\theta = 45°$, then $r = 0$.
Let $45° < \theta < 135°$, then $r$ is imaginary.
Let $\theta = 135°$, then $r = 0$.
Let $\theta = 180°$, then $r = \pm a$.

The curve therefore consists of two loops, one to the right and the other to the left of the origin, symmetrical with respect to the X axis, and reaching to the distance $a$ from the origin.

## The Cissoid

337. **The Cissoid.**—If on opposite sides of the center of a circle and at equal distances from it two ordinates be drawn to any diameter; and if through the upper extremity of either ordinate and the extremity of the diameter farthest from it a line be drawn, and this line be made to revolve about the extremity of the diameter, then the locus of the point where this line cuts the second ordinate is the *cissoid.*

Let QR and ST be two ordinates drawn to the diameter OA at equal distances from C and on opposite sides of it, and let OL be a line through S, the upper extremity of the ordinate ST, and O the extremity farthest from ST of the diameter OA.

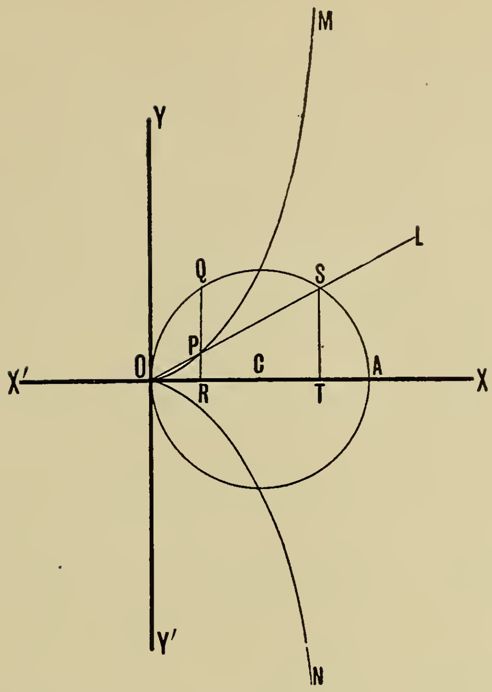



Fig. 124

Let P be the point where OL cuts the second ordinate QR. Now let OL revolve about the point O.

Then the locus MPON traced out by P is the cissoid.

PROPOSITION

338. *The equation of the cissoid is*

$$y^2 = \frac{x^3}{2a - x}.$$

Let $x \equiv$ OR, $y \equiv$ PR and $a \equiv$ OC.

[1] CR = CT, by construction.

[2] hence OR = TA.

[3] Again $\overline{ST}^2 = OT.TA$, by Geom. 60.

[4] or $\overline{ST}^2 = (2a - TA)TA$.

Hence by [2]

[5] $$\overline{ST}^2 = (2a - OR)OR,$$

[6] or $$ST = \sqrt{x(2a - x)}.$$

Now OPR and OST are similar, by Geom. 51.

[7] hence OR : PR :: OT : ST, by Geom. 31.

[8] or $x : y :: 2a - x : ST$.

Hence by [6]

[9] $$x : y :: 2a - x : \sqrt{x(2a - x)},$$

[10] and $$y = \frac{x\sqrt{x(2a - x)}}{2a - x} = \pm\sqrt{\frac{x^3}{2a - x}}.$$

[11] Therefore $$y^2 = \frac{x^3}{2a - x}.$$

Q. E. D.

*Scholium.*—From [10] we see

(1) that the curve is symmetrical with respect to the X axis,

(2) it passes through the origin,

(3) it has the line $x = 2a$ for an asymptote.

339. *Corollary.—The polar equation of the cissoid is*

$$r = 2a \tan\theta \sin\theta.$$

## The Witch

340. **The Witch.**—If to any diameter of a circle an ordinate be drawn and this ordinate be produced until the produced ordinate is to the ordinate itself as the whole diameter is to either segment of the diameter, and then the ordinate be moved continually in the direction of this segment; the locus of the extremity of the produced diameter is the *witch.*

In the circle OLM let LK be an ordinate to the diameter OM. Let LK be produced until

[1] PK : LK :: OM : KM,

and let PK move continually in the direction of the arrow.

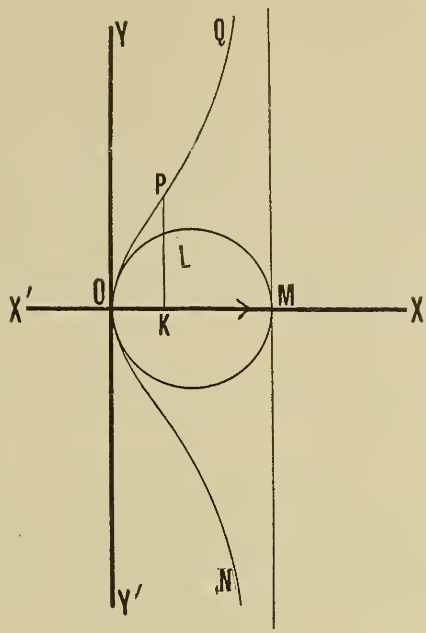



Fig. 125

Then NOPQ, the locus of the point P, is the witch.

Let $x \equiv OK$, $y \equiv PK$ and $2a \equiv OM$.

PROPOSITION

341. *The equation of the witch is*

$$y^2 = \frac{4a^2x}{2a - x}.$$

[2] $OK : LK :: LK : KM.$ by Geom. 60.

[3] Hence $LK = \sqrt{OK.KM} = \sqrt{x(2a - x)}.$

Therefore from [1] and [3] we get

[4] $y : \sqrt{x(2a - x)} :: 2a : 2a - x.$

[5] Hence $y^2 = \frac{4a^2x}{2a - x}.$

Q. E. D.

342. *Corollary.*—

(*1*) *The witch is symmetrical with respect to the X axis ;*

(2) *It lies wholly to the right of the Y axis;*

(3) *The line $x = 2a$ is an asymptote to it.*

## The Conchoid

343. **The Conchoid.**—If while the center of a circle moves along a fixed straight line, one of its diameters always passes through a fixed point, then the locus of the extremities of the diameter is the *conchoid.*

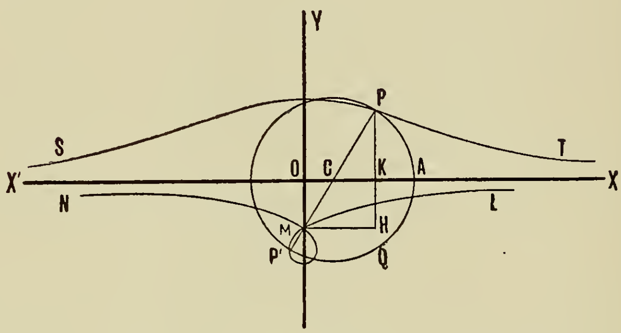



Fig. 126

Let the center C of the circle PQP′ move along the fixed line X′X, while the diameter PP′ passes through the fixed point M; then the locus SPT–LMN of the extremities P and P′ of the diameter is the conchoid.

Let P be any point on the locus and draw its ordinate PK.

$$\text{Let } x \equiv OK \quad \text{and} \quad y \equiv PK,$$
$$a \equiv OM \quad \text{"} \quad b \equiv CP.$$

PROPOSITION

344. *The equation of the conchoid is*

$$x^2y^2 = (b^2 - y^2)(a + y)^2.$$

Draw MH ∥ CK.

Produce PK until it meets MH at H.

Since the triangles MPH and CPK are similar, by Geom. 25.

[1] PK : CK :: PH : MH. by Geom. 31.

[2] then $PK : \sqrt{\overline{CP}^2 - \overline{PK}^2} :: PK + OM : OK$,
by Geom. 27 and 17.

[3] or $y : \sqrt{b^2 - y^2} :: y + a : x$,

[4] hence $x^2y^2 = (b^2 - y^2)(a + y)^2$.

Q. E. D.

From [4] we get the equation of the conchoid in another form.

[5] $$x = \pm \frac{a + y}{y} \sqrt{b^2 - y^2}.$$

*Scholium.*—When $x = 0$, [5] becomes

$$y = \pm b.$$

Hence the curve cuts the Y axis above and below the origin and $b$ units from it.

When $y$ is numerically less than $b$, [5] shows that every real value of $y$, either positive or negative, gives two values of $x$ numerically equal, but with opposite signs.

Hence there is one branch of the curve above the X axis and another below it, and the curve is symmetrical with respect to the Y axis.

When $y = 0$, we find $x = \pm \infty$.

Hence the branches extend an indefinite distance to the right and left of the origin.

Let $b > a$.

Then when $y = -a$, we have $x = 0$,

and when $y = -b$, we have $x = 0$.

Hence the curve cuts the Y axis below the origin at the distance $-a$ and $-b$ from it.

When $a < y < b$, $x$ has two values numerically equal, but with opposite signs.

Hence there is a loop below the origin.

If we take M for the pole and MY for the initial line and let

$$r \equiv MP \quad \text{and} \quad \theta \equiv YMP,$$

we get the following :

345. *Corollary.—The polar equation of the conchoid is*

$$r = a.\sec\theta \pm b.$$

## The Limacon

346. **The Limacon of Pascal.**—If one of the two points in which a secant cuts a circle remains fixed while the other moves along the circumference of the circle, and if the length of the exterior segment of the secant produced through this latter point remains constant, then the locus of the extremity of this secant is the *limaçon of Pascal.*

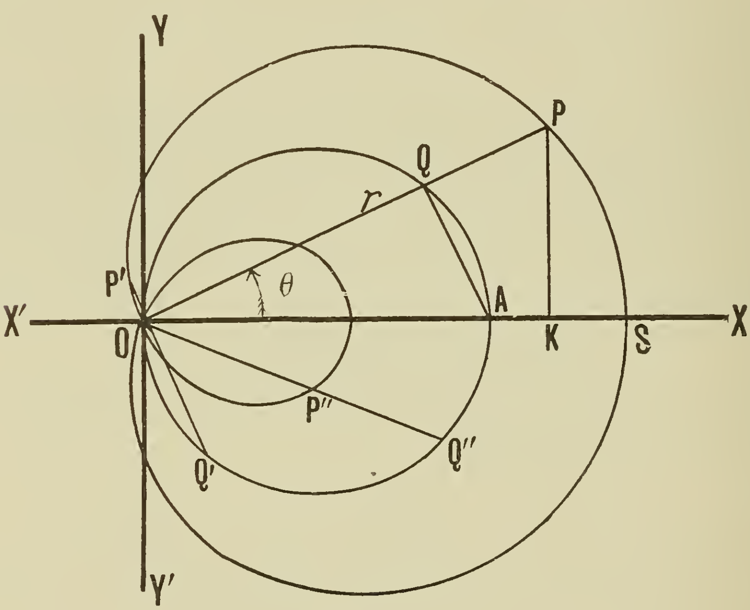



Fig. 127

In Fig. 127 let the point O remain fixed while Q moves along the circumference of the circle OQQ′. Let the length of the exterior segment PQ remain constant.

Then the locus of P is the limaçon.

Let $x \equiv OK$ and $y \equiv PK$,
$d \equiv OA$ " $s \equiv PQ$.

PROPOSITION

347. *The equation of the limaçon is*

$$(x^2+y^2-dx)^2-s^2(x^2+y^2)=0.$$

[1] For $OP=s+OQ=\sqrt{x^2+y^2}$, by Geom. 26.

[2] hence $s+d.\cos AOQ=\sqrt{x^2+y^2}$.
by Geom. 55, Trig. 2.

Now from the triangle OPK we get

$$\cos AOQ = \frac{x}{\sqrt{x^2+y^2}}.$$

[3] Hence by [2]

$$s+d\frac{x}{\sqrt{x^2+y^2}}=\sqrt{x^2+y^2},$$

[4] or $s\sqrt{x^2+y^2}+dx=x^2+y^2$.

[5] Hence $(x^2+y^2-dx)^2=s^2(x^2+y^2)$.

Q. E. D.

By expanding [5] we get

[6] $y^4+y^2(2x^2-2dx-s^2)+x^2[(x-d)^2-s^2]=0$.

[7] Let $v\equiv y^2$, $b\equiv 2x^2-2dx-s^2$,
and $c\equiv x^2[(x-d)^2-s^2]$.

Then [6] becomes

[8] $$v^2+bv+c=0.$$

By the theory of quadratics we know that $v$ will be real when

[9] $$b^2-4c \geqq 0,$$

[10] that is, if $(2x^2-2dx-s^2)^2-4x^2[(x-d)^2-s^2] \geqq 0$,

[11] hence if $$x \geqq \frac{s^2}{4d}.$$

When $x>d-s$ and $x<d+s$,

[12] $$c = x^2[(x-d)^2 - s^2] < 0.$$

For let $$x = d - s + r,$$

then by the second inequality above

$$d - s + r < d + s.$$

Hence $$r^2 < 2\,rs.$$

Again let $$x = d + s - r,$$

then also $$r^2 < 2rs.$$

But $$x - d = s - r,$$

hence $$(x-d)^2 - s^2 = r^2 - 2rs < 0,$$

and $$x^2[(x-d)^2 - s^2] < 0.$$

That is, $c$ is negative, and therefore one of the values of $v$ in [8], say $v_1$, is positive, and the other, say $v_2$, is negative.

Hence from [7] we get

[13] $$y^2 = v_1,$$

[14] and $$y^2 = -v_2.$$

From [14] we see that two of the values of $y$ in [8] will be imaginary, and from [13] that the other two will be real, equal and will have opposite signs.

From [5] we see that

1st, when $x = 0$ and $y = 0$, the equation of the locus is satisfied.

[15] Hence the limaçon passes through the origin.

2nd, when $y = 0$, then $x = d \pm s$.

[16] Hence the limaçon meets the X axis at two points.

$$x = d + s \quad \text{and} \quad x = d - s.$$

FIRST CASE

Let $s < d$.

Then when $$x < d - s,$$

$$c < 0,$$

and $$2x^2 - 2dx - s^2 < -2ds + s^2.$$

But $s < d,$

hence $2x^2 - dx - s^2 < 0,$

[17] or $b < 0.$

Consequently in this case both the values of $v$ in [8] are positive, and hence all four values of $y$ in [6] are real and equal two and two.

Hence we see by [11], [15], [16], that from

$$x = -\frac{s^2}{4d} \text{ to } x = 0,$$

and from $x = 0$ to $x = d - s,$

the limaçon has two branches, both symmetrical with respect to the X axis.

From [13] and [14] we see that from

$$x = d - s \text{ to } x = d + s,$$

the limaçon has but one branch, and it is symmetrical with respect to the X axis.

Therefore when $s < d$, the limacon will be of the form given in Fig. 127.

SECOND CASE

Let $s = d.$

In this case $-\frac{s^2}{4d} = -\frac{d}{4},$

and $d - s = 0.$

Hence again when $x < d - s = 0.$

$b < 0.$

Therefore from $-\frac{d}{4}$ to 0, $y$ will have four real values equal two and two; the point $d - s$ will be at the origin; the loop will become the point O. From 0 to $2d$, $y$ will have two values equal and of opposite signs, and the limaçon will take the form shown in the following figure.

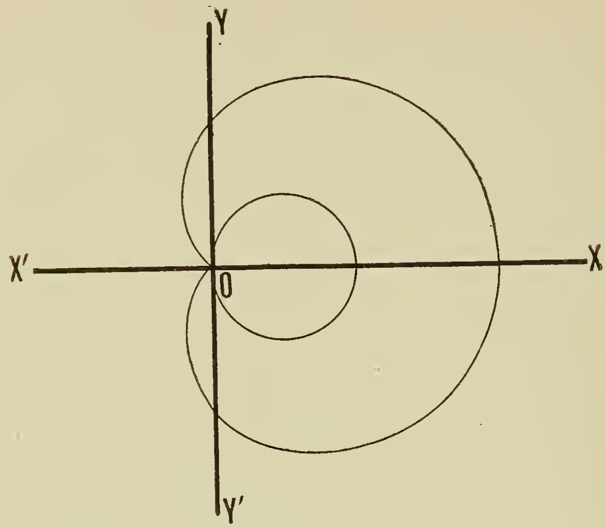



Fig. 128

The limaçon is then called the cardioid.

Since in this case $s = d$, [5] becomes

[17] $$(x^2 + y^2 - dx)^2 = d^2(x^2 + y^2),$$

which is the equation of the cardioid.

THIRD CASE

Let $d < s < 2d$.

Then $d - s$ is negative.

Also when $x < d - s$,

we have $b < 0$.

Therefore from $-\frac{s^2}{4d}$ to $d - s$ the $y$ in [5] has four real values equal two and two with opposite signs. From $d - s$ to $d + s$ it has two real and equal values with opposite signs.

Since $x = 0$ and $y = 0$ satisfy [5], the origin is an isolated point on the limaçon. Hence in this case the limaçon takes the form shown in the following figure.

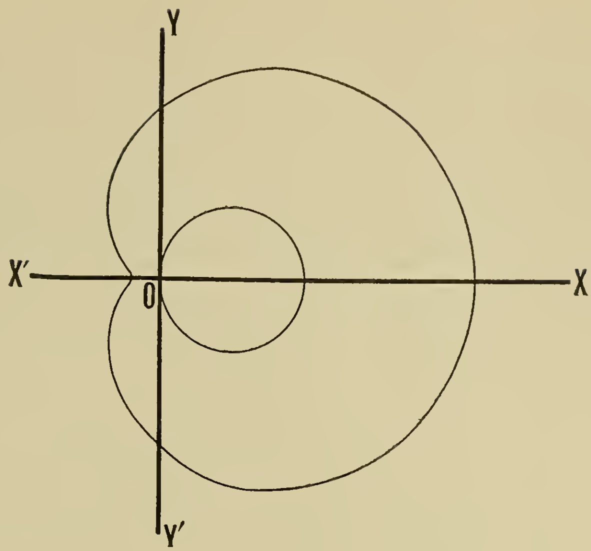



Fig. 129

FOURTH CASE

Let $s = 2d$.

Then $$-\frac{s^2}{4d} = -d \text{ and } d - s = -d.$$

Now when $x = -d$ in [8], $b = 0$ and $c = 0$ and the four values of $y$ in [6] become 0.

From $-d$ to 3d, $y$ has two real and equal values with opposite signs. Therefore the limaçon takes the form of Fig. 130.

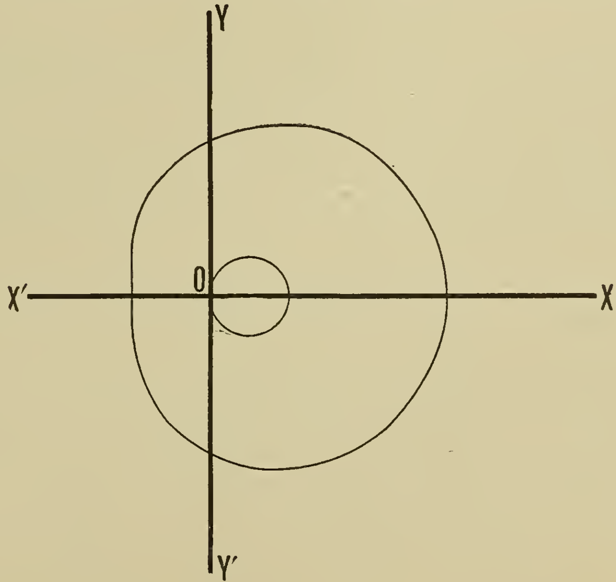



Fig. 130

FIFTH CASE

Let $s > 2d$.

Then the values of $y$ in [6] will be imaginary from $-\frac{s^2}{4d}$ to $d - s$, but from $d - s$ to $d + s$, $y$ will have two real and equal values with opposite signs. Therefore the limaçon will take the form shown in Fig. 131.

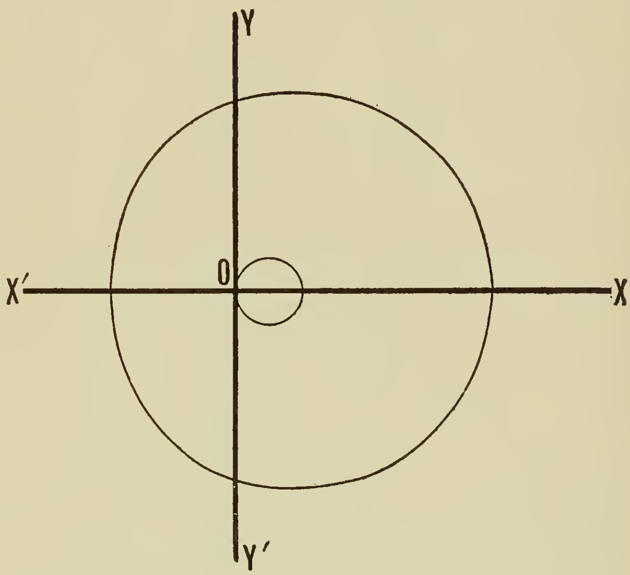



Fig. 131

336. *Corollary.—The polar equation of the limaçon is*

$$r = s + d \cos \theta.$$

In Fig. 127 let O be the pole and POX the vectorial angle.

Let $\theta \equiv$ POX and $r \equiv$ OP.

| | | | |
|---|---|---|---|
| | Now | AQO is a right angle, | by Geom. 55. |
| [1] | hence | $OQ = OA \cos \theta = d \cos \theta$. | by Trig. 2. |
| [2] | But | $r = PQ + OQ$, | |
| [3] | hence | $r = s + d \cos \theta$. | |

Q. E. D.

*Scholium.*—If we take the point P′, then

[4] $OQ' = OA. \cos AOQ' = -OA \cos AOP'$, by Trig. 23.

[5] or $OQ' = -d \cos \theta$.

[6] But $r \equiv OP'$ and $s \equiv Q'P'$,

[7] hence $r = s - OQ'$.

Substituting the value of OQ′ given in [5] into [7], we get

[8] $r = s + d \cos \theta$.

If we take the point P″, then

[9] $-OP'' = -OQ'' + Q''P''$,

[10] or $OP'' = OQ'' - Q''P''$.

[11] Hence $r = s + d \cos \theta$.

## SPIRALS

346. A Spiral.—A *spiral* is the locus traced out by a point revolving in a plane about a fixed point and receding from it according to some fixed law.

### Logarithmic Spiral

347. The Logarithmic Spiral.—The *logarithmic spiral* is the locus of a point revolving about a fixed point in such a way that the logarithm of its radius vector is always equal to a constant multiplied by the number of radians in its vectorial angle.

PROPOSITION

348. *The equation of the logarithmic spiral is*

$$r = a^{m\theta}.$$

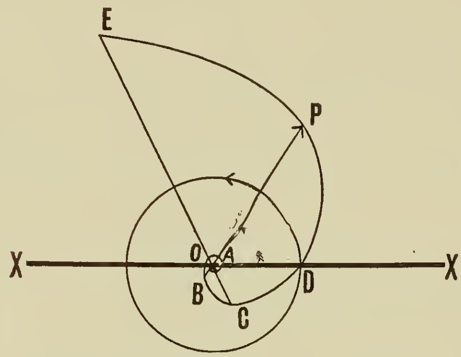



Fig. 132

In Fig. 132 let P move about the fixed point O in such a way that the logarithm of OP is always equal to a constant multiplied by the number of radians in ∠POD. Then will ABCDE, the locus traced out by P, be the logarithmic spiral.

Let O be the pole and OX the initial line.

Let $r \equiv$ OP,

$\theta \equiv$ the number of radians in ∠POD,

and $m \equiv$ any constant.

By the definition of the spiral we have

[1] $$\log r = m\theta.$$

Then by the definition of a logarithm we have

[2] $$r = a^{m\theta}.$$

Q. E. D.

PROBLEM

349. To construct the logarithmic spiral whose equation is

[1] $$r = 2\theta.$$

The number of degrees in one radian is 57°.3. by Trig. 27.

Therefore by [1] we may make the following table.

TABLE

| Number of degrees in $\theta$. | $\theta$. | Length of $r$. |
|---|---|---|
| 0 | 0 | 1 |
| $57°.3$ | 1 | 2 |
| $114°.6$ | 2 | 4 |
| $-57.3$ | $-1$ | .5 |
| $-114.6$ | $-2$ | .25 |
| $-\infty$ | $-3$ | 0 |

Since by the table $r = 1$ when $\theta = 0$, the point whose coordinates are $r = 1$, $\theta = 0$, is the point D in Fig. 132 at the distance of a unit from the pole.

Since by the table when the number of degrees in $\theta = 57°.3$, $r = 2$, we may locate a second point P on the spiral by laying off $\angle POX = 57°.3$ and making $OP = 2$. Similarly we may locate the point E.

Since by the table $r = 0.5$ when $\theta = -57°.3$, we may locate the point C in the figure by laying off an angle of $57°.3$ measured from OX clockwise, and making $OC = 0.5$. Similarly we may locate the point B.

In this way we may locate any number of points and draw the spiral.

350. *Corollary 1.—Since when $\theta = 0$, $r = 1$, we see that any logarithmic spiral cuts the initial line at a unit's distance from the pole.*

*Corollary 2.—Since when $\theta = -\infty$, $r = 0$, the spiral makes an infinite number of revolutions within the circle whose radius is 1.*

*Corollary 3.—Since when $\theta = \infty$, $r = \infty$, the spiral makes an infinite number of revolutions outside of the circle whose radius is 1.*

## The Spiral of Archimedes

351. The Spiral of Archimedes.—The *spiral of archimedes* is the locus traced out by a point moving about a fixed point in such a way that the ratio of its radius vector to its vectorial angle is constant.

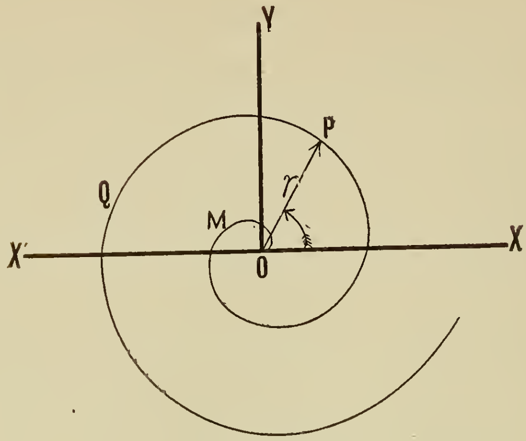



Fig. 133

Let the point P revolve about the fixed point O in such a way that $\frac{\text{OP}}{\text{POX}}$ is constant; then the locus OMPQ is the spiral of archimedes.

PROPOSITION

352. *The equation of the spiral of archimedes is*

$$r = c.\theta.$$

Let $r \equiv$ OP and $\theta \equiv$ the number of radians in POX and c $\equiv$ a constant.

Since by definition

[1] $$\frac{r}{\theta} = c,$$

[2] $$r = c.\theta.$$

Q. E. D.

353. *Corollary.*—

(*1*) *Since when* $\theta = o$, $r = o$, *the spiral passes through the pole ;*

(*2*) *Since when* $\theta = \infty$, $r = \infty$, *the spiral makes an infinite number of turns about the pole.*

## The Hyperbolic Spiral

354. **The Hyperbolic Spiral**, or the reciprocal of the spiral of Archimedes.—The *hyperbolic spiral* is the locus traced out

by a point revolving in a plane about a fixed point in such a way that the product of its radius vector and the number of radians in its vectorial angle is constant.

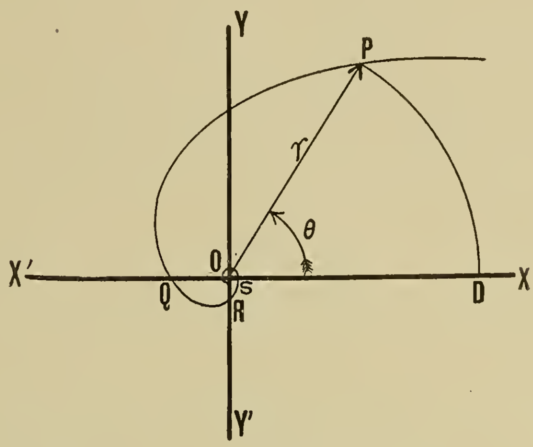



Fig. 134

In Fig. 134 let O be a fixed point and OX a fixed straight line. Let P revolve about O in such a way that OP multiplied by the number of radians in ∠POD is constant; then will PQRS, the locus traced out by P, be the hyperbolic spiral.

PROPOSITION

355. *The equation of the hyperbolic spiral is*

$$r\theta = c.$$

Let $r \equiv$ OP, $\theta \equiv$ the number of radians in POX, and $c \equiv$ a constant.

Then by definition we get

[1] $$r\theta = c.$$

356. *Corollary.—*

(*1*) *The spiral makes an infinite number of revolutions about the pole before reaching it;*

(*2*) *Since* $r = \frac{c}{\theta}$, *there is no point on the spiral whose vectorial angle is zero.*

357. *Corollary 2.—The constant c is the circumference of a circle whose radius is equal to the length of the radius vector at the end of the first revolution.*

For at the end of the first revolution

[2] $$\theta = 2\pi.$$

Let $r' \equiv r$ at the end of the first revolution.

Then by [1]

[3] $$2\pi r' = c,$$

and $2\pi r' =$ the circumference of a circle whose radius $= r'$.
by Geom. 29.

358. *Corollary 3.—The arc of a circle between any point on the spiral and the initial line is equal to the circumference of the circle whose radius is the length of r at the end of the first revolution.*

[4] For $$\frac{\text{arc PD}}{r} = \theta,$$ by Trig. 28.

[5] or $$\text{arc PD} = r\theta.$$

Hence by [1]

[6] $$\text{arc PD} = c.$$

Therefore by § 357 the arc PD is equal to the circumference of the circle whose radius is equal to the length of $r$ at the end of the first revolution.

## The Parabolic Spiral

359. The Parabolic Spiral.—The *parabolic spiral* is the locus traced out by a point revolving in a plane about a fixed point in such a way that the ratio of the square of its radius vector to the number of radians in its vectorial angle is constant.

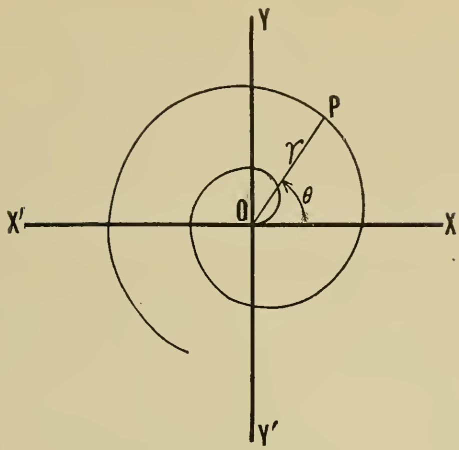



Fig. 135

PROPOSITION

360. *The equation of the parabolic spiral is*

$$\frac{r^2}{\theta} = c.$$

For the equation follows at once from the definition.

## The Lituus

361. **The Lituus.**—The *lituus* is the locus traced out by a point revolving in a plane about a fixed point in such a way that the product of the square of its radius vector and the number of radians in its vectorial angle is constant.

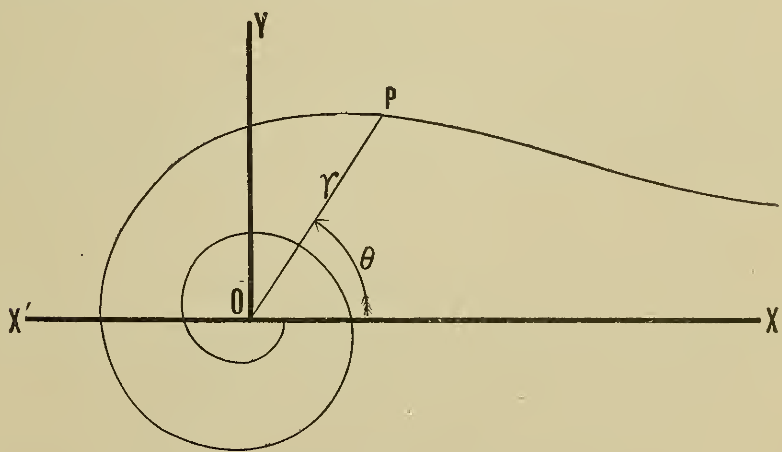



Fig. 136

PROPOSITION

362. *The equation of the lituus is*

$$r^2\theta = c.$$

The equation follows at once from the definition.

363. *Corollary.—The initial line is an asymptote to the lituus.*

## THE LOGARITHMIC CURVE

364. **The Logarithmic Curve.**—The *logarithmic curve* is the locus of a point moving in a plane in such a way that its abscissa is always equal to the logarithm of its ordinate.

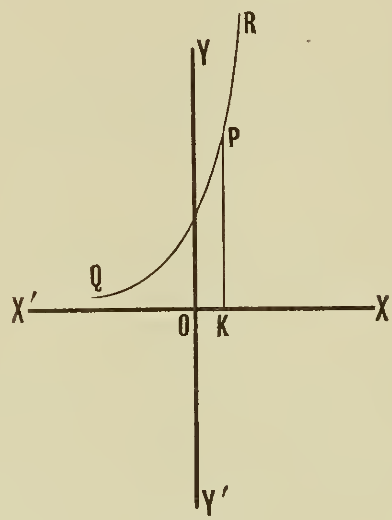



Fig. 137.

Let P be any point in the plane YOX.

Let P move in the plane YOX so that OK is always equal to the logarithm of PK. Then RPQ will be the logarithmic curve.

PROPOSITION

365. *The equation of the logarithmic curve is*

$$y = a^x.$$

Let $x \equiv$ OK and $y \equiv$ PK.

Then by the definition of the curve

[1] $$x = \log y,$$

hence, by the definition of a logarithm,

[2] $$y = a^x.$$ Q. E. D.

366. *Corollary 1.—The whole of the logarithmic curve lies above the X axis.*

For since negative numbers have no logarithms, $y$ can never be negative in [2].

367. *Corollary 2.—Every logarithmic curve must cut the Y axis at a point one unit above the origin.*

For when $$x = 0,$$

[3] $$y = a^0 = 1,$$

whatever may be the value of the base $a$.

368. *Corollary 3.—The X axis is an asymptote to the curve.*

For when $$x = -\infty.$$

$$y = a^{-\infty} = \frac{1}{a^\infty} = 0.$$

## TRIGONOMETRICAL LOCI

369. **A Trigonometrical Locus.**—A *trigonometrical locus* is a locus, one of whose rectangular coordinates is a *trigonometrical function*.

### The Cycloid

370. **The Cycloid.**—If a circle roll upon a fixed straight line, the locus traced out by a given point on the circumference of the circle is called a *cycloid*.

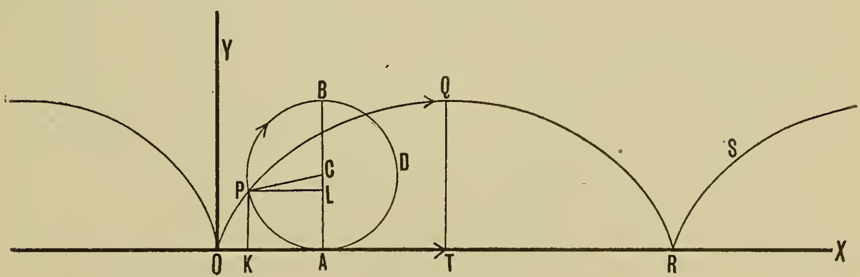



Fig. 138

Let X′X be the fixed straight line, and APBD a circle rolling upon that straight line in the direction of OT. Let P be a given point on the circumference of the circle.

Then the locus OPQRS, traced out by the point P, is a cycloid.

PROPOSITION

371. *The equation of the cycloid is*

$$x = r \text{ vers}^{-1} \frac{y}{r} - \sqrt{2ry - y^2}.$$

In Fig. 138 let

$x \equiv$ OK and $y \equiv$ PK,
$r \equiv$ CP " $\theta \equiv \angle$PCL.

[1] Now OK = OA — KA,

and, since the circle rolls in the direction OT,

[2] OA = the arc AP.

[3] But the arc AP = $r\theta$, by Trig. 28.

[4] hence OA = $r\theta$.

[5] Again KA = PL, by Geom. 17.

[6] but PL = $r \sin \theta$, by Trig. 1.

[7] Hence KA = $r \sin \theta$.

Substituting the values of OA and KA found in [4] and [7] into [1], we get

[8] OK = $r\theta - r \sin \theta$.

[9] Therefore $x = r\theta - r \sin \theta$.

[10] Again PK = AL = CA — CL. by Geom. 17.

[11] Now CL = $r \cos \theta$, by Trig. 2.

[12] hence PK = CA — $r \cos \theta$.

[13] PK = $y$ and CA = $r$. by Geom. 18.

Substituting the values of CA and PK found in [13] into [12], we get

[14] $$y = r - r\cos\theta.$$

Multiplying both sides of [14] by $2r$, we get

[15] $$2ry = 2r^2 - 2r^2\cos\theta.$$

Squaring both sides of [14], we get

[16] $$y^2 = r^2 - 2r^2\cos\theta + r^2\cos^2\theta.$$

Subtracting [16] from [15], we get

[17] $$2ry - y^2 = r^2 - r^2\cos^2\theta = r^2(1 - \cos^2\theta).$$

Now since $1 - \cos^2\theta = \sin^2\theta$, by Trig. 5.
equation [17] becomes

[18] $$2ry - y^2 = r^2\sin^2\theta,$$

[19] and $$\sqrt{2ry - y^2} = r\sin\theta.$$

Hence [9] becomes

[20] $$x = r\theta - \sqrt{2ry - y^2}.$$

[21] Now $$\cos\theta = \frac{\text{CL}}{r},$$ by Trig. 2.

[22] hence $$\cos\theta = \frac{r - y}{r},$$

[23] or $$\theta = \cos^{-1}\frac{r - y}{r}.$$ by Trig. 30.

Hence [20] becomes

[24] $$x = r\cos^{-1}\frac{r - y}{r} - \sqrt{2ry - y^2}.$$

[25] $$\cos^{-1}\frac{r - y}{r} = \text{vers}^{-1}\left(1 - \frac{r - y}{r}\right) = \text{vers}^{-1}\frac{y}{r}.$$
by Trig. 29.

Hence [24] may be written

[26] $$x = r\,\text{vers}^{-1}\frac{y}{r} - \sqrt{2ry - y^2}.$$

Q. E. D.

372. *Corollary.—When the highest point of the rolling circle is taken as the origin, and the diameter perpendicular to the line upon which the circle rolls as the X axis, the equation of the cycloid is*

$$y = r \text{ vers}^{-1} \frac{x}{r} + \sqrt{2rx - x^2}.$$

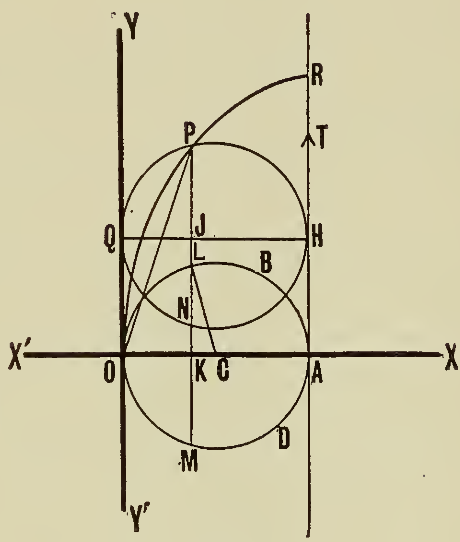



Fig. 139

In Fig. 139 let AT be the fixed straight line upon which the circle ABOD rolls in the direction of the arrow. Let P be any given point on the circle.

Let $x \equiv$ OK and $y \equiv$ PK,

$r \equiv$ CL " $\theta \equiv \angle$LCK.

[1] Then PK = PJ + JK.

[2] Now PN = LM, by Geom. 57.

[3] hence PJ = LK. by Geom. 59.

[4] Again JK = HA, by Geom. 17.

hence [1] becomes

[5] PK = HA + LK.

[6] Again AR = the arc ALO,

[7] and HR = the arc HP,

[8] hence HA = the arc PQ.

[9] But PQN = LOM, by Geom. 58.

[10] hence the arc PQ = the arc LO, by Geom. 59.

[11] hence by [8] HA = the arc LO.

[12] LK = $r \sin \theta$.

Hence by [11] and [12], [5] becomes

[13] PK = the arc LO + $r \sin \theta$.

[14] Hence $y = r\theta + r \sin \theta$. by Trig. 28.

[15] Also $x \equiv$ OK = CO — CK = $r - r \cos \theta$. by Trig. 2.

Now from [14] and [15], as in the demonstration of the proposition, we may get

[16] $$y = r \operatorname{vers}^{-1}\frac{x}{r} + \sqrt{2rx - x^2}.$$

373. Other trigonometrical loci are the following:

The curve of sines, see page 17.

The curve of tangents, see page 18.

The curve of secants, see page 19.

# SOLID ANALYTIC GEOMETRY

## CHAPTER I

### Points and Directions in Space

374. *The position of a point in space may be indicated by means of its distances from each of three well known, fixed, intersecting planes which are perpendicular to each other.*

Thus, in Fig. 140, let AB, CD and EF be three well known, fixed, intersecting planes perpendicular to each other. Let P be any point in space, and draw

PS $\perp$ to the plane AB,
PQ $\perp$ " " " EF,
PM $\perp$ " " " CD.

Then the position of the point P is indicated by giving the lines PS, PQ, and PM.

375. **The Coordinate Planes.**—The three well known, fixed, intersecting planes are called the *coordinate planes.*

The plane AB is called the ZX plane.
" " CD " " " ZY "
" " EF " " " XY "

376. **The Origin.**—The point in which the coordinate planes intersect each other is called the *origin.*

377. **The Coordinate Axes.**—The lines in which the coordinate planes intersect each other are called the *coordinate axes.*

The line in which the plane ZX intersects the plane XY is called the X axis.

The line in which the plane ZY intersects the plane XY is called the Y axis.

The line in which the plane ZX intersects the plane ZY is called the Z axis.

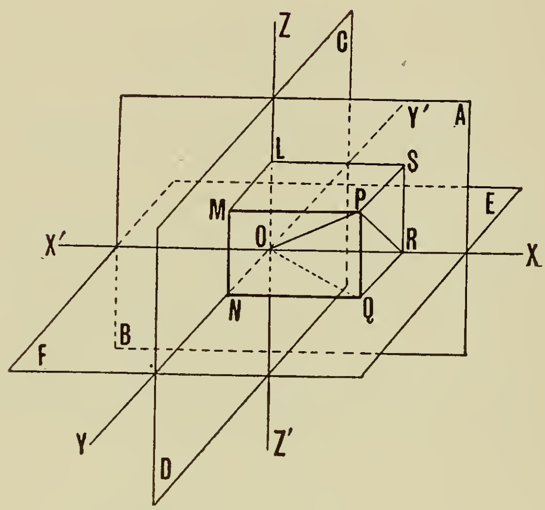



Fig. 140

378. *Corollary.—The axes are perpendicular to each other.*

For the plane ZX ⊥ the plane ZY, by construction.
and " " XY ⊥ " " ZY, by construction.
Hence the line XX' ⊥ " " ZY, by Geom. 44.

Hence XX' must be ⊥ to YY' and ZZ'. by Geom. 33.

Similarly it may be proved that ZZ' is ⊥ to YY' and XX', and that YY' is ⊥ to ZZ' and XX'.

379. **The Coordinates of a Point.**—The three distances of a point from the coordinate planes are called the *coordinates of the point.*

The coordinates of P are PM, PS and PQ, and they are respectively parallel to the axes XX', YY' and ZZ'.

For PM is ⊥ the plane ZY, by construction.

and it was shown in § 378 that XX' is ⊥ the plane ZY.

Hence PM $\parallel$ XX′. by Geom. 35.

Similarly it may be proved that

PS $\parallel$ YY′ and PQ $\parallel$ ZZ′.

The coordinate $\parallel$ to the X axis is called the $x$ coordinate.
" " $\parallel$ " " Y " " " " $y$ "
" " $\parallel$ " " Z " " " " $z$ "

Let $x \equiv$ the $x$ coordinate.
$y \equiv$ " $y$ "
$z \equiv$ " $z$ "

Pass a plane through PS and PQ; another through PS and PM, and another through PM and PQ.

PS is $\perp$ the ZX plane. by construction.
Hence plane PQRS " $\perp$ " ZX " by Geom. 43.
Also PQ " $\perp$ " XY " by construction.
Hence " PQRS " $\perp$ " XY " by Geom. 43.
Hence since the plane ZX is $\perp$ to the plane PQRS,
and " " " XY " $\perp$ " " " PQRS,
their intersection XX′ " $\perp$ " " " PQRS.
by Geom. 44.
Hence OR $\perp$ SR. by Geom. 33.

Now OL $\perp$ OR, by § 378.
and we have just proved that

SR $\perp$ OR.

Hence OL $\parallel$ SR. by Geom. 46.

Similarly it may be proved that

SL $\parallel$ OR.

Hence OLSR is a parallelogram.

It may also be proved that OLMN, MNQP, PQRS, ONQR and MLSP are all parallelograms.

Hence PM = QN = OR $\equiv x$,
PS = ML = ON $\equiv y$,
and PQ = MN = OL $\equiv z$. by Geom. 47.

*Remark.*—The coordinate planes need not be taken perpendicular to each other as above. When they are so taken the system of coordinates is then called a rectangular system, otherwise it is called an oblique system.

PROPOSITION I

380. *If $\rho$ be the distance from the origin to any point and $x$, $y$, $z$ be the coordinates of that point, then*

$$\rho^2 = x^2 + y^2 + z^2.$$

In Fig. 140 let P be any point in space.

Let $\rho \equiv OP$ be its distance from the origin.

$$x \equiv OR, \quad y \equiv RQ \quad \text{and } z \equiv PQ.$$

| | | | |
|---|---|---|---|
| | Now | $PQ \perp OQ.$ | by Geom. 33. |
| [1] | Hence | $\rho^2 = \overline{OQ}^2 + z^2.$ | by Geom. 26. |
| [2] | But | $\overline{OQ}^2 = x^2 + y^2,$ | by Geom. 26. |
| [3] | Hence | $\rho^2 = x^2 + y^2 + z^2.$ | |

Q. E. D.

381. **The Direction Cosines** of a line drawn through the origin.—The cosines of the three angles which any line passing through the origin makes with the axes are called the *direction cosines* of the line.

In Fig. 140 let $\alpha \equiv \angle POX,$

$$\beta \equiv \angle POY,$$

$$\gamma \equiv \angle POZ.$$

Then $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines of the line OP.

PROPOSITION II

382. *If $x$, $y$, $z$ be the coordinates of any point in space, $\rho$ the length of the line drawn from the origin to that point, and $\cos\alpha$, $\cos\beta$, $\cos\gamma$ the direction cosines of this line,*

*then* $x = \rho \cos\alpha,$

$$y = \rho \cos\beta,$$

$$z = \rho \cos\gamma.$$

For in Fig. 140 OR $\perp$ PR. by Geom. 33.

[1] Hence OR = OP cos POX, by Trig. 2.

[2] or $x = \rho \cos \alpha$.

Similarly for $y$ and $z$.

PROPOSITION III

383. *If cos α, cos β, cos γ be the direction cosines of any line drawn through the origin, then*

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1.$$

For $x^2 = \rho^2 \cos^2 \alpha$,

$y^2 = \rho^2 \cos^2 \beta$,

$z^2 = \rho^2 \cos^2 \gamma$. by § 382.

[1] Hence $x^2 + y^2 + z^2 = \rho^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)$.

[2] But $x^2 + y^2 + z^2 = \rho^2$. by § 380.

[3] Hence $\rho^2 = \rho^2(\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma)$,

[4] and $\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1$.

Q. E. D.

PROPOSITION IV

384. *If x′, y′, z′ be the coordinates of any point in space; x″, y″, z″ the coordinates of any other point in space; and D the length of the line joining these points, then*

$$D^2 = (x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2.$$

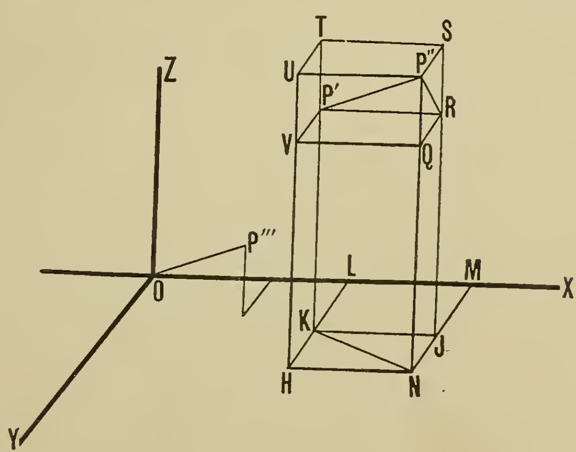



Fig. 141

In Fig. 141 let P′ and P″ be the two points.

$$D \equiv \text{P}'\text{P}'',$$

$$x' \equiv \text{OL}, \quad y' \equiv \text{LK}, \quad z' \equiv \text{P}'\text{K},$$

$$x'' \equiv \text{OM}, \quad y'' \equiv \text{MN}, \quad z'' \equiv \text{P}''\text{N}.$$

We are to prove that

$$D^2 = (x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2.$$

As in § 380, it may be shown that

[1] $\overline{\text{P}'\text{P}''}^2 = \overline{\text{P}'\text{R}}^2 + \overline{\text{RQ}}^2 + \overline{\text{P}''\text{Q}}^2.$

[2] But $\text{P}'\text{R} = \text{KJ} = \text{LM} = x'' - x'.$ by Geom. 17.

[3] $\text{RQ} = \text{JN} = \text{KH} = y'' - y',$ by Geom. 17.

[4] and $\text{P}''\text{Q} = z'' - z'.$

Substituting these values into [1], we get

[5] $D^2 = (x'' - x')^2 + (y'' - y')^2 + (z'' - z')^2.$

Q. E. D.

385. **The Angle Between Two Lines in Space.**—By *the angle between two lines in space*, which do not intersect, we mean the angle between either of them, and a line drawn through any point on it parallel to the other.

386. **The Direction Angles of Any Line in Space.**—By *the direction angles of any line in space* we mean the angles which a parallel to the given line drawn through the origin makes with the axes.

387. **The Direction Cosines of Any Line in Space.**—By *the direction cosines of any line in space* we mean the direction cosines of a parallel to this line drawn through the origin.

PROPOSITION V

388. *If $x'$, $y'$, $z'$, and $x''$, $y''$, $z''$ be the coordinates of any two points in space, $\rho$ the length of the line joining them, and $\cos \alpha$, $\cos \beta$, $\cos \gamma$ the direction cosines of that line, then*

$$x'' - x' = \rho \cos \alpha,$$

$$y'' - y' = \rho \cos \beta,$$

$$z'' - z' = \rho \cos \gamma.$$

In Fig. 141 let OP‴ be drawn through the origin ∥ to P′P″.

Let $\alpha \equiv P'''OX$, $\beta \equiv P'''OY$, $\gamma \equiv P'''OZ$.

[1] Then $P'R = P'P'' \cos P''P'R$, by Trig. 2.

[2] or $x'' - x' = \rho \cos P''P'R$.

[3] Similarly $y'' - y' = \rho \cos P''P'V$,

[4] and $z'' - z' = \rho \cos P''P'T$.

[5] But $P''P'R = P'''OX = \alpha$,

$P''P'V = P'''OY = \beta$,

and $P''P'T = P'''OZ = \gamma$. by Geom. 11.

Hence from [2], [3] and [4], we get

$$x'' - x' = \rho \cos \alpha,$$
$$y'' - y' = \rho \cos \beta,$$
$$z'' - z' = \rho \cos \gamma.$$

Q. E. D.

PROPOSITION VI

389. *If* $\cos \alpha$, $\cos \beta$, $\cos \gamma$ *be the direction cosines of any line in space, and* $l$, $m$, $n$ *be any three quantities proportional to them, then*

$$\cos \alpha = \frac{l}{\sqrt{l^2 + m^2 + n^2}},$$

$$\cos \beta = \frac{m}{\sqrt{l^2 + m^2 + n^2}},$$

$$\cos \gamma = \frac{n}{\sqrt{l^2 + m^2 + n^2}}.$$

[1] For $\frac{l}{\cos \alpha} = \frac{m}{\cos \beta} = \frac{n}{\cos \gamma}$. by hypothesis.

[2] Now let $r \equiv \frac{l}{\cos \alpha}$.

Then from [1] we also get

[3] $r = \frac{m}{\cos \beta}$ and $r = \frac{n}{\cos \gamma}$.

From [2] and [3] we get

[4] $l = r \cos\alpha,\quad m = r\cos\beta,\quad n = r\cos\gamma.$

Then by squaring and adding the corresponding members of [4], we get

[5] $l^2 + m^2 + n^2 = r^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma).$

[6] But $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$ by § 383.

[7] hence $l^2 + m^2 + n^2 = r^2,$

[8] and $r = \sqrt{l^2 + m^2 + n^2}.$

From [4] we get

[9] $$\cos\alpha = \frac{l}{r}.$$

[10] $$\cos\beta = \frac{m}{r}.$$

[11] $$\cos\gamma = \frac{n}{r}.$$

Now substituting the value of $r$ found in [8] into [9], [10] and [11], we get

[12] $$\cos\alpha = \frac{l}{\sqrt{l^2 + m^2 + n^2}}.$$

[13] $$\cos\beta = \frac{m}{\sqrt{l^2 + m^2 + n^2}}.$$

[14] $$\cos\gamma = \frac{n}{\sqrt{l^2 + m^2 + n^2}}.$$

Q. E. D.

390. *Corollary.—We can determine the direction of any line by means of any three numbers proportional to the direction cosines of that line.*

391. **Directors of a Line.**—Any three numbers proportional to the direction cosines of a line are called *directors of that line.*

392. *Corollary.—The direction cosines of a line are directors of that line.*

For in [4], § 389, $r$ may be any number whatever. It may therefore be 1. In that case we have

$$l = \cos\alpha, \quad m = \cos\beta, \quad n = \cos\gamma.$$

PROPOSITION VII

393. *If the directors of any line are proportional to the directors of a second line, the two lines are parallel.*

For let $l$, $m$ and $n$ be the directors of the first line, and let $l'$, $m'$ and $n'$ be the directors of the second line.

[1] Then $\dfrac{l'}{l} = \dfrac{m'}{m} = \dfrac{n'}{n}$. by hypothesis.

[2] Let $r \equiv \dfrac{l'}{l}$.

Then from [1] we get

[3] $r = \dfrac{m'}{m}$ and $r = \dfrac{n'}{n}$.

Hence from [2] and [3], we get

[4] $l' = rl, \quad m' = rm, \quad$ and $\quad n' = rn.$

Now let $\cos\alpha$, $\cos\beta$, $\cos\gamma$ be the direction cosines of the first line, and let $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$ be the direction cosines of the second line. Then by § 389, [12], [13] and [14] and by [4] above, we get

[5] $$\cos\alpha' = \frac{l'}{\sqrt{l'^2 + m'^2 + n'^2}} = \frac{rl}{\sqrt{r^2l^2 + r^2m^2 + r^2n^2}} = \frac{l}{\sqrt{l^2 + m^2 + n^2}} = \cos\alpha.$$

[6] Similarly $\cos\beta' = \cos\beta$,

[7] and $\cos\gamma' = \cos\gamma$.

Now since the direction cosines of the two lines are equal, the lines are parallel.

Q. E. D.

PROPOSITION VIII

394. *If* $\alpha$, $\beta$, $\gamma$ *are the direction angles of any line;* $\alpha'$, $\beta'$, $\gamma'$ *the direction angles of any other line, and* $V$ *is the angle between the two lines, then*

$$V = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'.$$

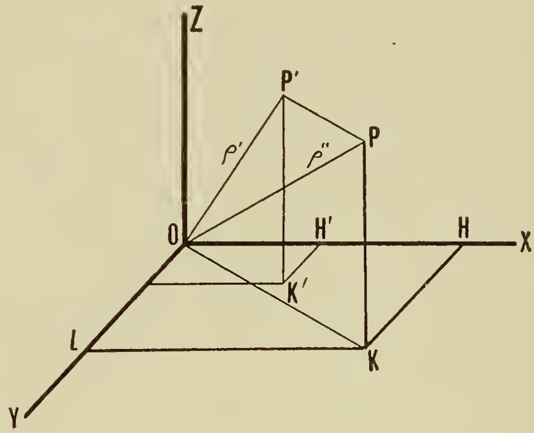



Fig. 142

Let $x'' \equiv OH$, $y'' \equiv HK$, $z'' \equiv PK$,
$x' \equiv OH'$ $y' \equiv H'K'$, $z' \equiv P'K'$,
$\alpha \equiv POX$, $\beta \equiv POY$, $\gamma \equiv POZ$,
$\alpha' \equiv P'OX$, $\beta' \equiv P'OY$, $\gamma' \equiv P'OZ$,
$\rho'' \equiv OP$ and $\rho' \equiv OP'$.

[1] Then
$x'' = \rho''\cos\alpha$, $y'' = \rho''\cos\beta$, and $z'' = \rho''\cos\gamma$, by § 382.

[2] and
$x' = \rho'\cos\alpha'$, $y' = \rho'\cos\beta'$, and $z' = \rho'\cos\gamma'$, by § 382.

[3] and $D^2 = (x''-x')^2 + (y''-y')^2 + (z''-z')^2$.
by § 384.

Hence by [1] and [2] we get

[4] $D^2 = (\rho''\cos\alpha - \rho'\cos\alpha')^2 + (\rho''\cos\beta - \rho'\cos\beta')^2 + (\rho''\cos\gamma - \rho'\cos\gamma')^2$.

[5] Hence $D^2 = \rho''^2(\cos^2\alpha + \cos^2\beta + \cos^2\gamma) + \rho'^2(\cos^2\alpha' + \cos^2\beta' + \cos^2\gamma') - 2\rho''\rho'(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma')$.

[6] Hence

$D^2 = \rho''^2 + \rho'^2 - 2\rho'\rho''(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma')$.

[7] But $D^2 = \rho''^2 - 2\rho'\rho''\cos V + \rho'^2$. by Trig. 25.

[8] Hence $\rho''^2 + \rho'^2 - 2\rho'\rho''\cos V =$

$\rho''^2 + \rho'^2 - 2\rho'\rho''(\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma')$.

[9] Hence $\cos V = \cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma'$.

Q. E. D.

395. *Corollary 1.—If $\cos\alpha$, $\cos\beta$, $\cos\gamma$, and $\cos\alpha'$, $\cos\beta'$, $\cos\gamma'$ be the direction cosines of two lines, then in order that the two lines be perpendicular to each other we must have*

[10] $\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = 0$.

For in order that the two lines shall be perpendicular to each other, we must have in [9], § 394

$\cos V = 0$. by Trig. 19.

396. *Corollary 2.—If $l$, $m$, $n$, and $l'$, $m'$, $n'$ be directors of two lines, then in order that the two lines shall be perpendicular to each other, we must have*

$$ll' + mm' + nn' = 0.$$

For as in § 389, [9], [10] and [11]

$$\frac{l}{r} = \cos\alpha,\ \frac{m}{r} = \cos\beta, \text{ and } \frac{n}{r} = \cos\gamma;$$

and $$\frac{l'}{r'} = \cos\alpha',\ \frac{m'}{r'} = \cos\beta', \text{ and } \frac{n'}{r'} = \cos\gamma'.$$

Substituting these values of the direction cosines into [10], § 395, we get

[11] $$\frac{ll' + mm' + nn'}{rr'} = 0.$$

[12] Hence $ll' + mm' + nn' = 0$.

PROPOSITION IX

397. *If a, b, c, and a', b', c' be the direction cosines of any two lines, and V be the angle between them, then*

$$\sin^2 V = (ab' - a'b)^2 + (bc' - b'c)^2 + (ca' - c'a)^2.$$

[1] For $\sin^2 V = 1 - \cos^2 V.$ by Trig. 5.

[2] But $1 = a^2 + b^2 + c^2.$ by § 383.

[3] Hence $\sin^2 V = a^2 + b^2 + c^2 - \cos^2 V.$

But by § 394

[4] $\cos^2 V =$
$$a^2a'^2 + b^2b'^2 + c^2c'^2 + 2aa'bb' + 2bb'cc' + 2cc'aa'.$$

[5] Hence $\sin^2 V =$
$$a^2 + b^2 + c^2 - a^2a'^2 - b^2b'^2 - c^2c'^2 - 2aa'bb' - 2bb'cc' - 2cc'aa'.$$

[6] Hence $\sin^2 V =$
$$a^2(1 - a'^2) + b^2(1 - b'^2) + c^2(1 - c'^2) - 2aa'bb' - 2bb'cc' - 2cc'aa'.$$

But by § 383

[7] $1 - a'^2 = b'^2 + c'^2,\quad 1 - b'^2 = a'^2 + c'^2$
and $1 - c'^2 = a'^2 + b'^2.$

[8] Hence $\sin^2 V =$
$$a^2(b'^2 + c'^2) + b^2(a'^2 + c'^2) + c^2(a'^2 + b'^2)$$
$$- 2aa'bb' - 2bb'cc' - 2cc'aa',$$

[9] or
$$\sin^2 V = a^2b'^2 - 2aa'bb' + a'^2b^2$$
$$+ b^2c'^2 - 2bb'cc' + b'^2c^2$$
$$+ a^2c'^2 - 2aa'cc' + a'^2c^2.$$

[10] Hence $\sin^2 V =$
$$(ab' - a'b)^2 + (bc' - b'c)^2 + (ca' - c'a)^2.$$

Q. E. D.

# CHAPTER II

## Projections

398. To project any line upon a second line, draw from each extreme of the first line a perpendicular to the second. The segment of the second line between these perpendiculars is called the projection of the first line upon the second.

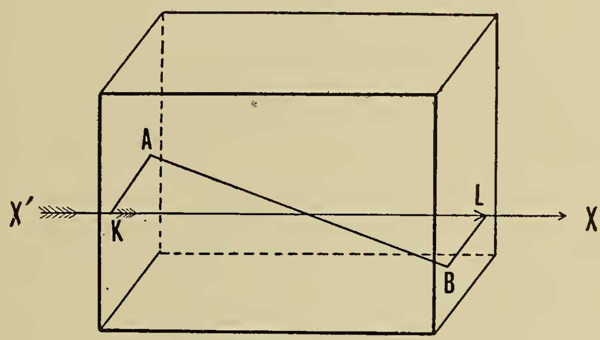



Fig. 143

To project the line extending from A to B upon the line X′X, we draw a perpendicular from A and another from B to the line X′X. Suppose these perpendiculars meet X′X at the points K and L. Then the line extending from K to L is the projection of AB on X′X.

399. If we take the direction X′X as the positive direction, then KL will be positive, and LK will be negative.

400. Let the projection of any line AB upon the X axis be represented by $\overline{AB}_x$, its projection upon the Y axis by $\overline{AB}_y$, and its projection upon the Z axis by $\overline{AB}_z$.

401. **The Projection of a Broken Line.**—The *projection of a broken line* is the algebraic sum of the projections of the segments of that line.

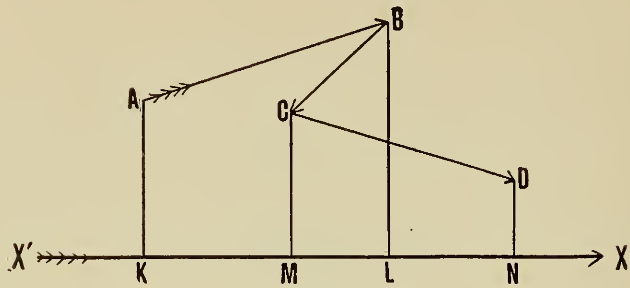


Fig. 144

Let the direction X′X be the positive direction.

The projection of ABCD on X′X is

$$KL - ML + MN = KN.$$

PROPOSITION I

402. *The projection of any closed contour upon a straight line is zero.*

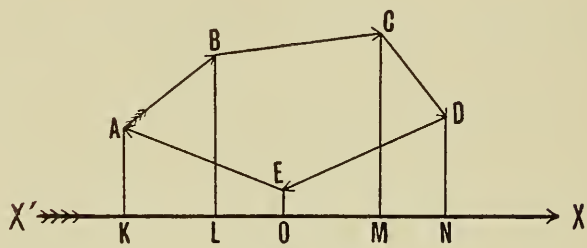


Fig. 145

Let X′X be any straight line. Let ABCDEA represent any closed contour.

We are to prove that

$$\overline{ABCDEA}_x = 0.$$

According to § 400 we will let

$$\overline{AB}_x \equiv KL,$$
$$\overline{BC}_x \equiv LM,$$
$$\overline{CD}_x \equiv MN,$$
$$\overline{DE}_x \equiv NO,$$
$$\overline{EA}_x \equiv OK.$$

[1] $\overline{ABCD}_x = KL + LM + MN = KN$. by § 401.

[2] $\overline{DEA}_x = NO + OK = NK = -KN$.

by § 401 and § 399.

Then by adding [1] and [2], we get

$$\overline{ABCDEA}_x = 0.$$

Q. E. D.

PROPOSITION II

403. *When any two broken lines or a straight line and a broken line have the same extremities their projections on the same line are equal.*

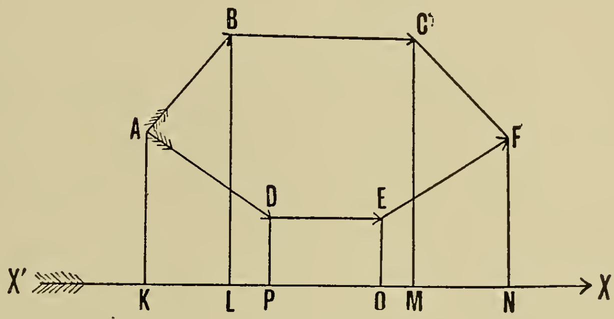



Fig. 146

Let ABC and ADEFC be two broken lines having the same extremities A and C.

Let X'X be any straight line upon which ABC and ADEFC are projected.

We are to prove that

$$\overline{ABC}_x = \overline{ADEFC}_x.$$

Now since the two broken lines form a closed contour, we have

[1] $\overline{KL + LM} + \overline{MN + NO + OP + PK} = 0$. by § 402.

[2] Hence $KL + LM = -MN - NO - OP - PK$,

[3] and $KL + LM = KP + PO + ON + NM$.

by § 399.

[4] or $\overline{ABC}_x = \overline{ADEFC}_x$. by § 401.

Q. E. D

PROPOSITION III

404. *The projection of any straight line upon a second straight line is equal to the length of the first multiplied by the cosine of the angle between them.*

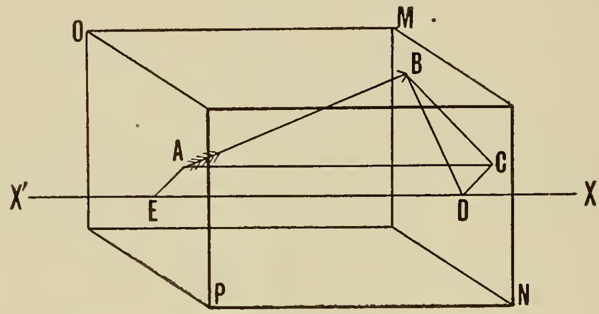



Fig. 147

Let AB be any straight line and X′X the line upon which we wish to project it.

We are to prove that $\overline{AB}_x$ is equal to AB times the cosine of the angle between AB and X′X.

Through A and B pass planes ⊥ to X′X at the points E and D.

Draw AC ∥ X′X and piercing the plane MN at C.

Draw BD and BC.

Since by construction ED is ⊥ to the planes, MN and OP, it must be ⊥ to BD and AE. by Geom. 33.

Hence ED is the projection of AB upon X′X. by § 398.

The plane MN is ∥ to the plane OP. by Geom. 38.

[1] Hence AC = ED. by Geom. 40.

AC ∥ X′X. by construction.

Hence AC ⊥ the plane MN, by Geom. 36.

and AC ⊥ BC. by Geom. 33.

[2] Then AC = AB cos BAC. by Trig. 2.

Hence by [1] we get

[3] ED = AB cos BAC.

Now AC ∥ X′X, by construction.

hence BAC is the ∠ between AB and X′X. by § 385.

Therefore by [3], ED, the projection of AB on X′X, is equal to AB multiplied by the cosine of the angle between AB and X′X.

Q. E. D.

PROPOSITION IV

405. *If two straight lines be parallel, their projections upon the same plane are parallel.*

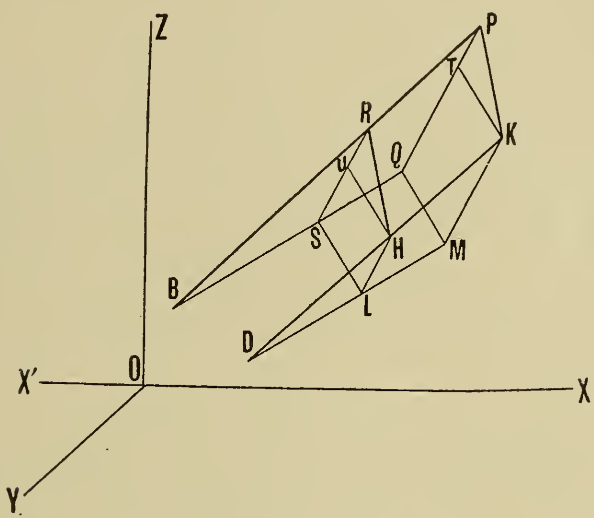



Fig. 148

Let PB and KD be two parallel straight lines.

Let BQ and DM be their projections upon the plane ZX.

We are to prove that

BQ ‖ DM.

Since Q is the projection of the point P on the plane ZX,

PQ is ⊥ the plane ZX. by Geom. 34.

Hence the plane PBQ is ⊥ the plane ZX. by Geom. 43.

Similarly the plane KDM is ⊥ the plane ZX.

Let R be any point on the line PB and draw

RH and PK ⊥ DK.

[1] Then RH ‖ PK. by Geom. 46.

Let S be the projection of the point R upon the plane ZX.

[2] Then RS ∥ PQ. by Geom. 35.

Now since

[3] RS ∥ PQ,

[4] and RH ∥ PK, by [1]

[5] the plane RSLH is ∥ to the plane PQMK. by Geom. 41.

[6] Hence SL ∥ QM. by Geom. 39.

[7] Draw HU ⊥ RS and KT ⊥ PQ.

[8] Now RS ⊥ SL " PQ ⊥ QM. by Geom. 33.

[9] Hence HU ∥ SL " KT ∥ MQ. by Geom. 46.

[10] Hence by [6] HU ∥ KT. by Geom. 37.

[11] But RS ∥ PQ ∥ KM ∥ HL. by Geom. 35.

[12] Hence HU = LS and KT = MQ. by Geom. 17.

Now since

[13] RH ∥ PK, by [1].

[14] and HU ∥ KT. by [10].

[15] Then ∠RHU = ∠PKT. by Geom. 41.

[16] Now HU = RH cos RHU, by Trig. 2.

[17] and KT = PK cos PKT. by Trig. 2.

[18] But RH = PK, by Geom. 17.

[19] and cos RHU = cos PKT. by [15].

Hence by [16], [17], [18] and [19]

[20] HU = KT.

Hence by [12]

[21] LS = QM.

Now since by [6] and [21], LS and MQ are both equal and parallel, SQML is a parallelogram. by Geom. 48.

Hence BQ ∥ DM.

Q. E. D.

# CHAPTER III

## Transformation of Coordinates

### PROPOSITION I

406. *If we have given the coordinates of a point referred to any system of rectangular axes we can find the coordinates of the same point referred to any other system of axes parallel to the first by putting*

$$x = m + x',$$
$$y = n + y',$$
$$z = o + z',$$

*in which* $x$, $y$, $z$ *are the coordinates of the point referred to the original axes, and* $m$, $n$, $o$ *the coordinates of the new origin referred to the original axes, and* $x'$, $y'$, $z'$ *the coordinates of the point referred to the new axes.*

In Fig. 149 let OX, OY and OZ be the old axes, and O′X′, O′Y′ and O′Z′ be the new axes.

Let P be any point in space and draw its coordinates with respect to each system of axes.

$$\begin{array}{llll} \text{Let } x \equiv \text{OH}, & y \equiv \text{HK} & \text{and} & z \equiv \text{PK}, \\ x' \equiv \text{O'R}, & y' \equiv \text{RQ} & \text{"} & z' \equiv \text{PQ}, \\ m \equiv \text{OL}, & n \equiv \text{LM} & \text{"} & o \equiv \text{MO'}. \end{array}$$

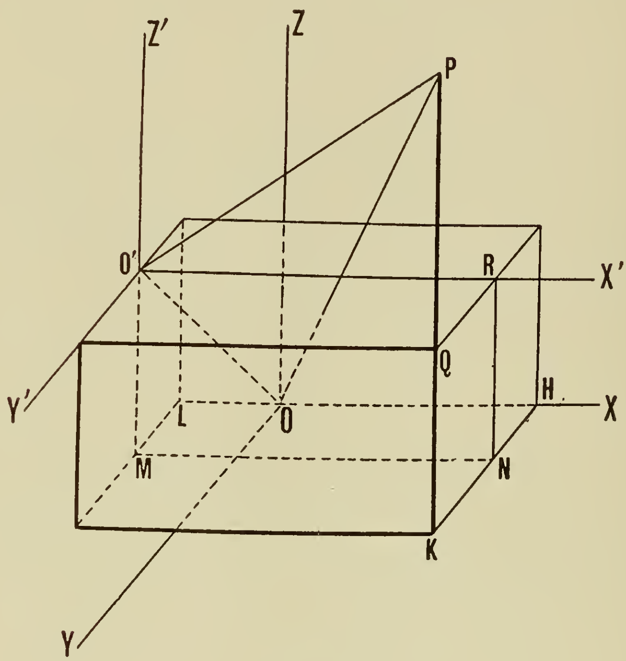



Fig. 149

Now using the notation of § 400, we have

[1] $\overline{\text{OP}}_x = \overline{\text{OO'}}_x + \overline{\text{O'P}}_x$. by § 403.

[2] But $\overline{\text{OO'}}_x = \overline{\text{OL}}_x + \overline{\text{LM}}_x + \overline{\text{MO'}}_x$, by § 403.

[3] and $\overline{\text{O'P}}_x = \overline{\text{O'R}}_x + \overline{\text{RQ}}_x + \overline{\text{PQ}}_x$. by § 403.

Hence

[4] $\overline{\text{OP}}_x = \overline{\text{OL}}_x + \overline{\text{LM}}_x + \overline{\text{MO'}}_x + \overline{\text{O'R}}_x + \overline{\text{RQ}}_x + \overline{\text{PQ}}_x$ .

[5] But $\overline{\text{OP}}_x = \overline{\text{OH}}_x + \overline{\text{HK}}_x + \overline{\text{PK}}_x$. by § 403.

[6] Hence $\overline{\text{OH}}_x + \overline{\text{HK}}_x + \overline{\text{PK}}_x =$

$\overline{\text{OL}}_x + \overline{\text{LM}}_x + \overline{\text{MO'}}_x + \overline{\text{O'R}}_x + \overline{\text{RQ}}_x + \overline{\text{PQ}}_x$.

[7] Hence OH cos 0° + HK cos 90° + PK cos 90° = OL cos 0° + LM cos 90° + MO′ cos 90° + O′R cos 0° + RQ cos 90° + PQ cos 90°. by § 404.

[8] Hence $OH = OL + O'R$, by Trig. 19.

[9] or $x = m + x'$.

[10] Similarly $y = n + y'$,

[11] and $z = o + z'$. Q. E. D.

## PROPOSITION II

407. *If we have given the coordinates of a point referred to any rectangular system of axes, we can find the coordinates of the same point referred to any other rectangular system having the same origin by putting*

$$x = x' \cos \alpha + y' \cos \alpha' + z' \cos \alpha'',$$
$$y = x' \cos \beta + y' \cos \beta' + z' \cos \beta'',$$
$$z = x' \cos \gamma + y' \cos \gamma' + z' \cos \gamma'',$$

*in which* $x$, $y$, $z$ *are the coordinates of the point referred to the original axes;* $x'$ $y'$, $z'$ *its coordinates referred to the new axes; the* $\alpha$*'s are the angles which the new axes make with the original X axis; the* $\beta$*'s the angles which the new axes make with the original Y axis; and the* $\gamma$*'s the angles which the new axes make with the original Z axis.*

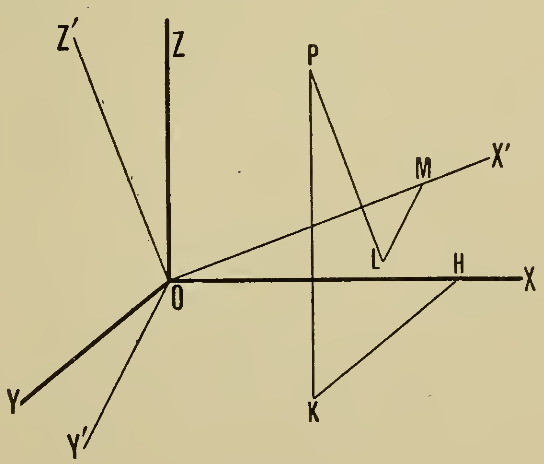



Fig. 150

In Fig. 150 let OX, OY and OZ be the original axes, and OX′, OY′ and OZ′ the new axes.

Let P be any point in space and draw its coordinates with respect to each system of axes.

Let $x \equiv OH$, $y \equiv HK$ and $z \equiv PK$,
$x' \equiv OM$, $y' \equiv ML$ " $z' \equiv PL$,

and in the following table let the angle between any two axes be in both the column and the horizontal line containing those axes.

| | OX | OY | OZ |
|---|---|---|---|
| OX′ | $\alpha$ | $\beta$ | $\gamma$ |
| OY′ | $\alpha'$ | $\beta'$ | $\gamma'$ |
| OZ′ | $\alpha''$ | $\beta''$ | $\gamma''$ |

Thus $\alpha$ is the angle between OX′ and OX,
and $\gamma''$ " " " " OZ′ " OZ, etc.

Now in Fig. 150

[1] $\overline{OH}_x + \overline{HK}_x + \overline{PK}_x = \overline{OM}_x + \overline{ML}_x + \overline{LP}_x$. by § 403.

[2] Hence $OH \cos 0^\circ + HK \cos 90^\circ + PK \cos 90^\circ =$
$OM \cos \alpha + ML \cos \alpha' + LP \cos \alpha''$. by § 404.

[3] Hence $x = x' \cos \alpha + y' \cos \alpha' + z' \cos \alpha''$. by Trig. 19.

Similarly by projecting upon the Y axis, we get

[4] $y = x' \cos \beta + y' \cos \beta' + z' \cos \beta''$.

And by projecting upon the Z axis, we get

[5] $z = x' \cos \gamma + y' \cos \gamma' + z' \cos \gamma''$.

Q. E. D.

408. *Corollary.—If we project the coordinates of the point upon the new axes, we get*

[6] $x' = x \cos \alpha + y \cos \beta + z \cos \gamma$,

[7] $y' = x \cos \alpha' + y \cos \beta' + z \cos \gamma'$,

[8] $z' = x \cos \alpha'' + y \cos \beta'' + z \cos \gamma''$.

Equations [3], [4], [5], [6], [7] and [8] may be conveniently indicated by means of the following

TABLE.

| | $x'$ | $y'$ | $z'$ |
|---|---|---|---|
| $x$ | $\cos\alpha$ | $\cos\alpha'$ | $\cos\alpha''$ |
| $y$ | $\cos\beta'$ | $\cos\beta'$ | $\cos\beta''$ |
| $z$ | $\cos\gamma$ | $\cos\gamma'$ | $\cos\gamma''$ |

Equations [3], [4], [5], [6], [7] and [8] may be readily obtained from the table above by means of the following rules:

409. *Rule 1.—The variable at the left on any horizontal line is equal to the sum of the products obtained by multiplying each cosine on that line by the variable at the top of the column containing the cosine.*

410. *Rule 2.—The variable at the top of any column is equal to the sum of the products obtained by multiplying each cosine in that column by the variable at the left on the horizontal line containing the cosine.*

411. *Corollary 1.*—$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1,$

$$\cos^2\alpha' + \cos^2\beta' + \cos^2\gamma' = 1,$$

$$\cos^2\alpha'' + \cos^2\beta'' + \cos^2\gamma'' = 1.$$ by § 383.

412. *Corollary 2.*—$\cos\alpha\cos\alpha' + \cos\beta\cos\beta' + \cos\gamma\cos\gamma' = 0.$

$$\cos\alpha\cos\alpha'' + \cos\beta\cos\beta'' + \cos\gamma\cos\gamma'' = 0.$$

$$\cos\alpha'\cos\alpha'' + \cos\beta'\cos\beta'' + \cos\gamma'\cos\gamma'' = 0.$$

by § 395.

# CHAPTER IV

## Spherical Coordinates of a Point in Space

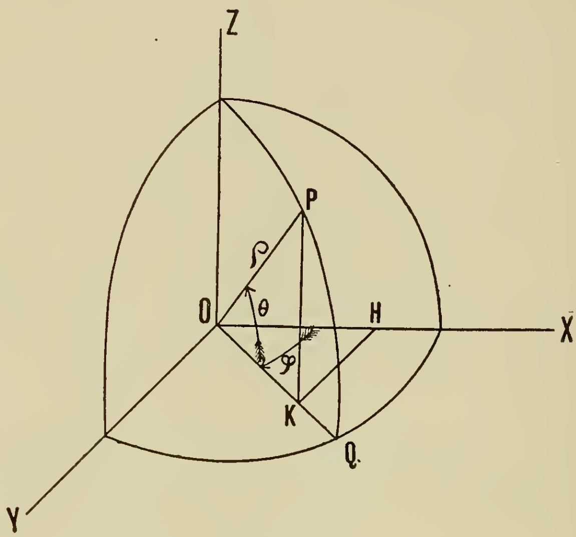



Fig. 151

In Fig. 151 let P be any point in space, and let

$$\varphi \equiv QOX, \quad \theta \equiv POQ \quad \text{and} \quad \rho \equiv OP.$$

Then if we know $\varphi$, $\theta$ and $\rho$, we know the position of the point P.

413. **The Radius Vector.**—The *radius vector* of a point in space is its distance from the origin.

414. **The Latitude of a Point.**—The *latitude of a point* in space is the angle between its radius vector and the projection of the radius vector upon the XY plane.

415. **The Longitude of a Point.**—The *longitude of a point* is the angle between the X axis and the projection of the radius vector upon the XY plane.

416. **The Spherical Coordinates of a Point.**—The radius vector, the latitude and the longitude of a point are called its *spherical coordinates*.

PROPOSITION I

417. *If we have given the rectangular coordinates of a point, we can find its spherical coordinates by putting*

$$x = \rho \cos \varphi \cos \theta,$$
$$y = \rho \cos \varphi \sin \theta,$$
$$z = \rho \sin \theta.$$

Let $x \equiv$ OH, $y \equiv$ HK and $z \equiv$ PK.

Then in Fig. 151, since the axes are rectangular, KHO is a right angle.

| | | | |
|---|---|---|---|
| [1] | Hence | OH = OK cos KOH, | by Trig. 2. |
| [2] | or | $x =$ OK cos $\varphi$. | |
| [3] | But | OK = $\rho \cos \theta$. | by Trig. 2. |
| [4] | Hence | $x = \rho \cos \varphi \cos \theta$. | |
| [5] | Again | KH = OK sin KOH, | by Trig. 1. |
| [6] | or | $y =$ OK sin $\varphi$. | |
| [7] | But | OK = $\rho \cos \theta$. | by Trig. 2. |
| [8] | Hence | $y = \rho \sin \varphi \cos \theta$. | |
| [9] | Again | PK = OP sin $\theta$, | by Trig. 1. |
| [10] | or | $z = \rho \sin \theta$. | |

418. *Corollary.—If* $\cos \alpha$, $\cos \beta$, $\cos \gamma$ *are the direction cosines of the radius vector of a point in space, then*

$$\cos \alpha = \cos \varphi \cos \theta,$$
$$\cos \beta = \cos \varphi \sin \theta,$$
$$\cos \gamma = \sin \theta.$$

For in Fig. 151

| | | | |
|---|---|---|---|
| [1] | | OH = OP cos POH, | by Trig. 2. |
| [2] | or | $x = \rho \cos \alpha$, | |
| [3] | and | $x = \rho \cos \varphi \cos \theta$. | by § 417, [4]. |
| [4] | Hence | $\cos \alpha = \cos \varphi \cos \theta$. | |
| [5] | Similarly | $y = \rho$ cos POH $= \rho \cos \beta$, | |
| [6] | and | $y = \rho \sin \varphi \cos \theta$. | by § 417, [8]. |
| [7] | Hence | $\cos \beta = \sin \varphi \cos \theta$. | |
| [8] | Also | $\cos \gamma = \sin \theta$. | |

# CHAPTER V

## The Plane

### PROPOSITION I

419. *The equation of a plane is*

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0,$$

*in which* $x$, $y$, $z$ *are the coordinates of any point in the plane,* $p$ *the length of the perpendicular drawn from the origin to the plane, and* $\cos \alpha$, $\cos \beta$, $\cos \gamma$ *the direction cosines of this perpendicular.*

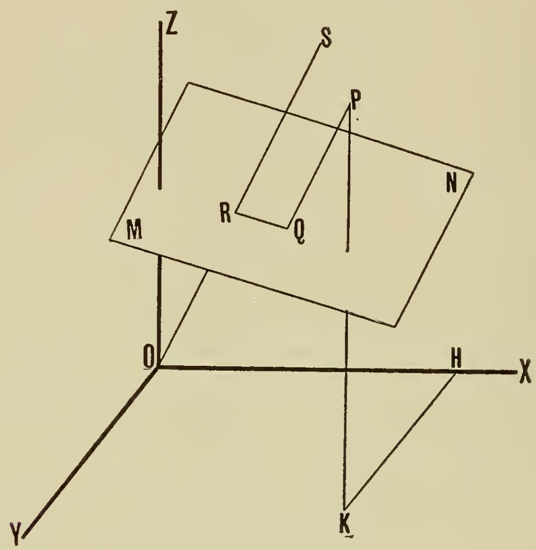



Fig. 152

Let MN be any plane, and OS a line drawn from the origin ⊥ to this plane at the point R.

Let P be any point in space and draw its coordinates PK, KH and OH.

Draw PQ ⊥ to the plane MN.

Let $\alpha$, $\beta$, $\gamma$, be the angles which OS makes with the axes.

Let $p \equiv$ the length of OR.

Let $\overline{PK}_s \equiv$ the length of the projection of PK upon OS.

Project the two broken lines ORQP and OHKP upon the line OS.

[1] Then $\overline{OH}_s + \overline{HK}_s + \overline{KP}_s = \overline{OR}_s + \overline{RQ}_s + \overline{QP}_s$. by § 403.

[2] Hence $x \cos \alpha + y \cos \beta + z \cos \gamma =$
$p \cos 0° + RQ \cos 90° + d \cos 0°$, by § 404.

[3] or $x \cos \alpha + y \cos \beta + z \cos \gamma = p + d$. by Trig. 19.

[4] Hence $x \cos \alpha + y \cos \beta + z \cos \gamma - p = d$.

If the point P is taken anywhere in the plane MN, then $d = 0$, and [4] becomes

[5] $$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0.$$

Now $\cos \alpha$, $\cos \beta$, $\cos \gamma$, and $p$ determine the point R.

And there can be but one plane drawn through R perpendicular to $p$. by Geom. 53.

Hence since there can be but one plane whose perpendicular from the origin has the length $p$ and whose direction cosines are $\cos \alpha$, $\cos \beta$, $\cos \alpha$, [5] determines the plane MN.

Since the $x$, $y$ and $z$ of [5] are the coordinates of any point on this plane, then [5] must be the equation of that plane.

Q. E. D.

420. *Corollary.*—

*The equation of a plane* ‖ *to the XY plane is* $z = p$.
" " " " " ‖ " " *ZX* " " $y = p$.
" " " " " ‖ " " *ZY* " " $x = p$.

PROPOSITION II

421. *Every equation of the first degree containing three variables only is the equation of a plane.*

[1] Let $Ax + By + Cz + D = 0$,

be any equation of the first degree containing only three variables. Then if this be the equation of a plane, it must be but another form of the equation

[2] $x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0.$ by § 419.

Therefore by multiplying [1] through by some constant as $\lambda$, we must be able to change [1] into the form of [2].

[3] That is $\lambda Ax + \lambda By + \lambda Cz + \lambda D = 0,$
must be only another form of [2].

[4] Hence
$$\lambda A = \cos \alpha,$$
$$\lambda B = \cos \beta,$$
$$\lambda C = \cos \gamma,$$
$$\lambda D = -p.$$

Now since $\lambda A$, $\lambda B$ and $\lambda C$ are the direction cosines of a line through the origin, then

[5] $\lambda^2 A^2 + \lambda^2 B^2 + \lambda^2 C^2 = 1,$ by § 383.

[6] or $\lambda^2 (A^2 + B^2 + C^2) = 1.$

[7] Hence $\lambda = \pm \dfrac{1}{\sqrt{A^2 + B^2 + C^2}}.$

Since $A$, $B$ and $C$ are real quantities, $\lambda$ must be real.

Substituting the positive value of $\lambda$ into [4], we get

[8]

$$\frac{A}{\sqrt{A^2 + B^2 + C^2}} = \cos \alpha; \qquad \frac{B}{\sqrt{A^2 + B^2 + C^2}} = \cos \beta;$$

$$\frac{C}{\sqrt{A^2 + B^2 + C^2}} = \cos \gamma; \qquad \frac{D}{\sqrt{A^2 + B^2 + C^2}} = -p.$$

Substituting the positive value of $\lambda$ into [3], we get

[9] $$\frac{A}{\sqrt{A^2 + B^2 + C^2}} x + \frac{B}{\sqrt{A^2 + B^2 + C^2}} y + \frac{C}{\sqrt{A^2 + B^2 + C^2}} z + \frac{D}{\sqrt{A^2 + B^2 + C^2}} = 0.$$

Hence from [8] and [9], we get

[10] $x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0.$

Therefore [1] is only another form of [2], and hence [1] is the equation of a plane.

Q. E. D.

From [8] we get the following

422. *Corollary.—If* $Ax + By + Cz + D = 0$ *be the equation of a plane*, $p$ *the* ⊥ *on it from the origin, and* $\alpha$, $\beta$, $\gamma$ *the direction angles of this* ⊥, *then*

$$p = \frac{-D}{\sqrt{A^2 + B^2 + C^2}}, \quad \cos \alpha = \frac{A}{\sqrt{A^2 + B^2 + C^2}},$$

$$\cos \beta = \frac{B}{\sqrt{A^2 + B^2 + C^2}} \text{ and } \cos \gamma = \frac{C}{\sqrt{A^2 + B^2 + C^2}}.$$

PROPOSITION III

423. *If* $x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0$

*be the equation of a plane, then*

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p \pm d = 0$$

*is the equation of a plane parallel to it.*

For since the direction cosines of the ⊥ drawn from the origin to the two planes are the same, by § 419. these ⊥'s must coincide with each other. Hence the two planes are ⊥ to the same straight line and are therefore ∥. by Geom. 38.

Q. E. D.

424. *Corollary.—The distance between the two planes is* $\pm d$.

For the distance from the origin to the first plane is $p$, and the distance from the origin to the second plane is $p \pm d$. Hence the distance between the planes must be $\pm d$. by § 419.

PROPOSITION IV

425. *The distance from any point $x'$, $y'$, $z'$ to the plane*

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = o$$

*is*

$$x' \cos \alpha + y' \cos \beta + z' \cos \gamma - p.$$

For let $x \cos \alpha + y \cos \beta + z \cos \gamma - p = o$ .
be the equation of any plane.

Let $x'$, $y'$ $z'$ be the coordinates of any point in space.

Let $d \equiv$ the distance from $x'$, $y'$, $z'$ to the plane.

Pass a plane through the point $x'$, $y'$, $z'$ ∥ to

$$x \cos \alpha + y \cos \beta + z \cos \gamma - p = 0.$$

Since these ∥ planes are everywhere equally distant, $d$ is the distance between them, and the equation of the second plane is

$$[1] \qquad x \cos \alpha + y \cos \beta + z \cos \gamma - p \pm d = 0.$$

Now since the point $x'$, $y'$, $z'$ is on the second plane, its coordinates must satisfy the equation of that plane.

Hence substituting $x'$, $y'$, $z'$ into [1], we get

$$[2] \qquad x' \cos \alpha + y' \cos \beta + z' \cos \gamma - p \pm d = 0.$$

$$[3] \quad \text{Hence} \quad \pm d = x' \cos \alpha + y' \cos \beta + z' \cos \gamma - p.$$

Q. E. D.

By §§ 422 and 425 we can easily prove the following

426. *Corollary.—If $d \equiv$ the distance from the point $x'$, $y'$, $z'$ to the plane*

$$Ax + By + Cz + D = o,$$

then
$$d = \frac{Ax' + By' + Cz' + D}{\sqrt{A^2 + B^2 + C^2}}.$$

427. **The Traces of a Plane.**—The *traces of a plane* are its intersections with the three coordinate planes.

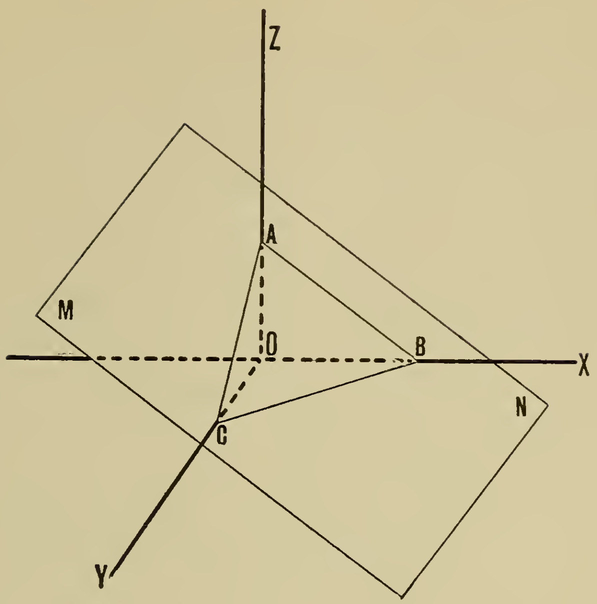



Fig. 153

In Fig. 153 AB is the trace of the plane MN on the plane ZX,

BC is its trace on the plane XY,

and AC is its trace on the plane ZY.

Let the equation of the plane MN be

[1] $$Ax + By + Cz + D = 0.$$

Now the plane MN cuts the plane XY where $z = 0$.

Hence making $z = 0$ in [1], we get

[2] $$Ax + By + D = 0,$$

for the equation of BC the trace of MN on XY.

Similarly making $y = 0$, we get

[3] $$Ax + Cz + D = 0,$$

for the equation of the trace on ZX.

[4] And making $x = 0$, we get

[5] $$By + Cz + D = 0,$$

for the equation of the trace on ZY.

428. *Scholium.*— The variables in the equation of a trace indicate the coordinate plane on which the trace lies.

429. **Intercepts of a Plane.**—The distances along the axes from the origin to any plane are called the *intercepts* of that plane.

In Fig. 153 the intercepts are OA, OB, OC.

The coordinates of the point B are $x = \text{OB}$, $y = 0$, and $z = 0$.

Since B is on the plane MN, its coordinates must satisfy the equation of that plane.

Hence substituting $x = \text{OB}$, $y = 0$, and $z = 0$ into the equation

$$Ax + By + Cz + D = 0,$$

we get

[1] $$A(\text{OB}) + D = 0.$$

[2] Hence $\text{OB} = -\dfrac{D}{A}$ = the intercept on the X axis.

[3] Similarly $\text{OC} = -\dfrac{D}{B}$ = the intercept on the Y axis.

[4] and $\text{OA} = -\dfrac{D}{C}$ = the intercept on the Z axis.

# CHAPTER VI

## Straight Lines

### PROPOSITION I

430. *The equations of a straight line in space are*

$$\frac{x - x'}{\cos \alpha} = \frac{y - y'}{\cos \beta} = \frac{z - z'}{\cos \gamma},$$

*in which $x'$, $y'$ $z'$ are the coordinates of a fixed point on the line; $x$, $y$, $z$ are the coordinates of any other point on it; and $\cos \alpha$, $\cos \beta$, $\cos \gamma$ are the direction cosines of the line.*

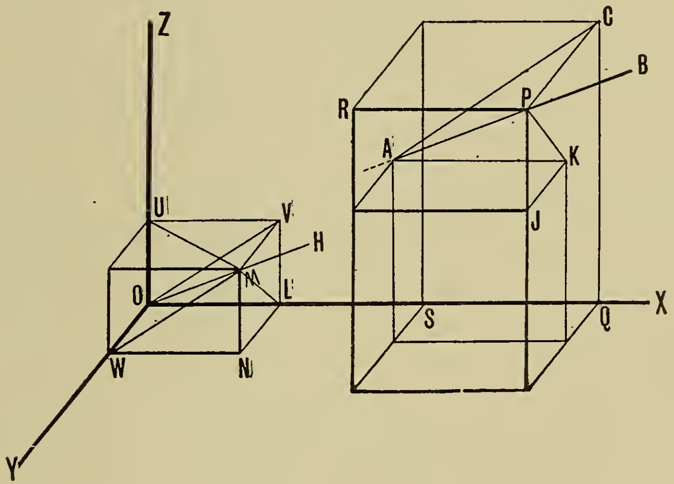



Fig. 154

Let AB be any line in space.

Let A be a *fixed* point on this line, and P be *any* point on it.

Let the coordinates of A be $x'$, $y'$ $z'$, and
" " " " P " $x$, $y$, $z$.

Through each of the points A and P pass a plane ⊥ to OX.

Then the plane RS is ∥ to the plane PQ. by Geom. 38.

Draw AK ∥ to OX.

Then AK is ⊥ to the plane PQ, by Geom. 36.

and AK ⊥ to PK, by Geom. 33.

and the ∠ AKP is a right angle.

Through the origin draw

OH ∥ AB.

[1] Then ∠LOH = ∠KAP. by Geom. 11.

Let M be any point on OH and through it pass planes ⊥ to the axes.

The coordinates of M will be OL, LN and MN.

Join M with U, L and W.

OL is ⊥ LM. by Geom. 33.

Hence ∠OLM = right ∠.

Therefore by [1] the two triangles OLM and AKP are similar. by Geom. 51.

[2] Hence $\frac{AK}{OL} = \frac{AP}{OM}$. by Geom. 31.

[3] But AK = SQ. by Geom. 40.

[4] Hence $\frac{SQ}{OL} = \frac{AP}{OM}$,

[5] or $\frac{x - x'}{OL} = \frac{AP}{OM}$.

It may easily be shown that PJK and MNL are similar. by Geom. 51.

[6] Hence $\frac{JK}{LN} = \frac{PK}{ML} = \frac{AP}{OM}$, by Geom. 31.

[7] or $\frac{y - y'}{LN} = \frac{AP}{OM}$.

[8] Similarly $\frac{z - z'}{MN} = \frac{AP}{OM}$.

[9] Hence $$\frac{x-x'}{OL}=\frac{y-y'}{LN}=\frac{z-z'}{MN}.$$

Now let OM be taken as the unit of length. Then OL, LN and MN are the direction cosines of the line, and [9] becomes

[10] $$\frac{x-x'}{\cos\alpha}=\frac{y-y'}{\cos\beta}=\frac{z-z'}{\cos\gamma}.$$

Since $x, y, z$ are the coordinates of any point on the line AB, and the other quantities in the equation are constants, [10] must be the equation of the line.

Q. E. D.

PROPOSITION II

431. *The equations of a straight line may be written*

$$x = x' + l\rho,$$
$$y = y' + m\rho,$$
$$z = z' + n\rho,$$

*in which $x', y'$ $z'$ are the coordinates of a fixed point on the line; $x, y, z$ the coordinates of any point on the line; $\rho$ the distance between these points; and $l, m, n$ the directors of the line.*

In Fig. 154

[1] $$OL = OM\cos\alpha,$$

[2] $$LN = OW = OM\cos\beta,$$

[3] $$MN = OU = OM\cos\gamma.$$ by Trig. 2.

Let $p$ be any constant whatever, then

[4] $$p.OL = p.OM\cos\alpha,$$

[5] $$p.LN = p.OM\cos\beta,$$

[6] $$p.MN = p.OM\cos\gamma.$$

Hence from [4], [5] and [6], we get

[7] $$\frac{p.OL}{\cos\alpha}=\frac{p.LN}{\cos\beta}=\frac{p.MN}{\cos\gamma}.$$

[8] Let $a \equiv p.\mathrm{OL}$,

[9] $b \equiv p.\mathrm{LN}$,

[10] $c \equiv p.\mathrm{MN}$,

[11] $d \equiv p.\mathrm{OM}$.

Then by [7], [8], [9] and [10], we get

[12] $$\frac{a}{\cos\alpha} = \frac{b}{\cos\beta} = \frac{c}{\cos\gamma}.$$

Let $\rho \equiv \mathrm{AP}$.

Then from § 430 [9] and the similar triangles KAP and LOM, we get

[13] $$\frac{x - x'}{\mathrm{OL}} = \frac{y - y'}{\mathrm{LM}} = \frac{z - z'}{\mathrm{MN}} = \frac{\rho}{\mathrm{OM}}.$$

Dividing this equation through by $p$, we get

[14] $$\frac{x - x'}{p.\mathrm{OL}} = \frac{y - y'}{p.\mathrm{LM}} = \frac{z - z'}{p.\mathrm{MN}} = \frac{\rho}{p.\mathrm{OM}}.$$

Hence by [8], [9], [10] and [11], [14] becomes

[15] $$\frac{x - x'}{a} = \frac{y - y'}{b} = \frac{z - z'}{c} = \frac{1}{d}\rho.$$

[16] Hence $$x = x' + \frac{a}{d}\rho.$$

[17] $$y = y' + \frac{b}{d}\rho.$$

[18] $$z = z' + \frac{c}{d}\rho.$$

Now let $l \equiv \frac{a}{d}$; $m \equiv \frac{b}{d}$; $n \equiv \frac{c}{d}$.

[19] Then $$\frac{l}{m} = \frac{\frac{a}{d}}{\frac{b}{d}} = \frac{a}{b} = \frac{\cos\alpha}{\cos\beta},$$ by [12].

[20] and $$\frac{m}{n} = \frac{\frac{b}{d}}{\frac{c}{d}} = \frac{b}{c} = \frac{\cos\beta}{\cos\gamma}.$$ by [12].

Then from [19] and [20], we get

$$[21] \qquad \frac{l}{\cos\alpha} = \frac{m}{\cos\beta} = \frac{n}{\cos\gamma}.$$

Hence $l$, $m$ and $n$ are the directors of the line AB. by § 391.

Substituting $l$, $m$ and $n$ into [16], [17] and [18], we get

$$[22] \qquad x = x' + l\rho,$$

$$[23] \qquad y = y' + m\rho,$$

$$[24] \qquad z = z' + n\rho.$$

Now in Fig. 154, $\rho$ is the distance along the line AB from the given point $x'$, $y'$, $z'$ to the point P. If then the point P be known, the value of $\rho$ is known. Substituting this value of $\rho$ into [22], [23] and [24] will give us one and only one set of values for $x$, $y$ and $z$.

On the other hand, if the values of $x$, $y$, $z$ are given, then [22], [23] and [24] will give us the length of $\rho$. Measuring off this length from the point $x'$, $y'$, $z'$ in the direction of the line AB, we get one and only one point on the line.

Hence for each point P on the line AB there is one and only one set of values of $x$, $y$, and $z$ that will satisfy [22], [23] and [24]; and for each set of values of $x, y$ and $z$ satisfying these equations there is one and only one point on the line AB.

Therefore

$$x = x' + l\rho,$$

$$y = y' + m\rho,$$

$$z = z' + n\rho,$$

determine the line AB and are called the equations of the line AB.

## PROPOSITION III

432. *The equations of a straight line may be written*

$$[1] \qquad x = sz + b,$$

$$[2] \quad \text{and} \qquad y = sz' + b'.$$

*Equation (1) being the equation of the projection of the line* *upon the ZX plane, s its slope upon the Z axis, and b its intercept on the X axis.*

*Equation (2) being the equation of the projection of the line on the ZY plane, s′ its slope on the Z axis, and b′ its intercept on the Y axis.*

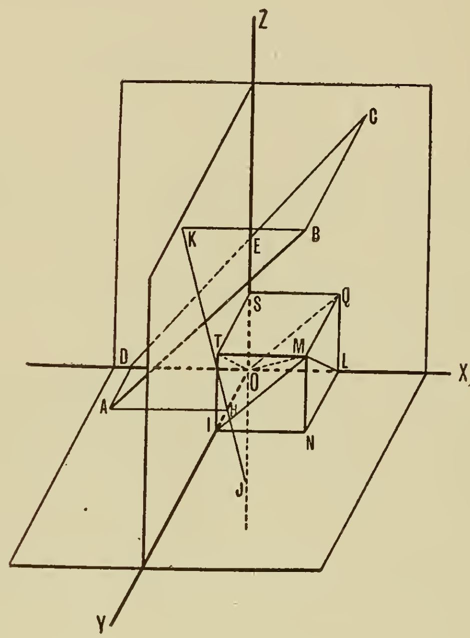



Fig. 155

Let AB be any line in space.

" DC be its projection on the plane ZX.

" KH " " " " " " ZY.

$$\text{Let } s \equiv \tan \text{ZEC},$$
$$b \equiv \text{DO},$$
$$s' \equiv \tan \text{HJO},$$
$$b' \equiv \text{HO}.$$

Through the origin draw OM ∥ to AB and a unit in length, and let its direction cosines be $\cos\alpha$, $\cos\beta$, $\cos\gamma$.

Let OQ be the projection of OM on the ZX plane.

Through M pass planes ⊥ to the axes.

[1] Then $\text{OL} = \cos\alpha$, $\text{LN} = \cos\beta$, $\text{MN} = \cos\gamma$.

[2] Now $\frac{OL}{MN} = \frac{IN}{MN} = \frac{SQ}{OS}$. by Geom. 47.

[3] But $\frac{SQ}{OS} = \tan QOS$.

[4] Hence $\frac{OL}{MN} = \tan QOS$.

Now OQ is ∥ DC. by § 405.

[5] Hence $\tan QOS = \tan ZEC = s$. by Geom. 11.

Hence by [4] and [5], we get

[6] $$\frac{OL}{MN} = s.$$

[7] Similarly $$\frac{LN}{MN} = s'.$$

Now by § 430, the equation of the line AB is

[8] $$\frac{x - x'}{\cos \alpha} = \frac{y - y'}{\cos \beta} = \frac{z - z'}{\cos \gamma}.$$

Hence by [1] and [8] we get

[9] $$\frac{x - x'}{OL} = \frac{y - y'}{LN} = \frac{z - z'}{MN}.$$

Now let $x'$, $y'$, $z'$ be the point A where AB pierces the XY plane.

[10] Then $x' = AH = DO = b$,

[11] $y' = AD = HO = b'$,

[12] and $z' = 0$.

Substituting these values of $x'$, $y'$, $z'$ into [9], we get

[13] $$\frac{x - b}{OL} = \frac{y - b'}{LN} = \frac{z}{MN}.$$

[14] hence $$x = \frac{OL}{MN} z + b.$$

Hence by [6] we get

[15] $$x = sz + b.$$

Now if we consider OZ as the axis of abscissas, and OX as the axis of ordinates, [15] will be the equation of the line DC. by § 53.

From [13] we also get

[16] $$y = \frac{LN}{MN} z + b'.$$

Hence by [7] we get

[17] $$y = s'z + b',$$

which is the equation of KH. by § 53.

Now one plane and only one can be drawn ⊥ to the plane ZX, which shall contain the projection DC. by Geom. 45.

This plane will contain the lines DA and CB. by Geom. 42.

Hence since it contains the points A and B, it must contain the line AB.

Similarly it may be shown that the only plane that can be passed through the projection KH ⊥ the plane ZY must also contain the line AB. Hence AB must be the intersection of these two planes.

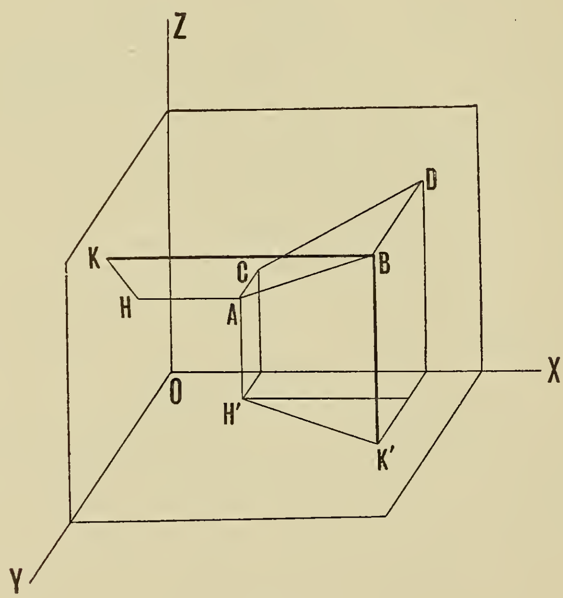



Fig. 156

Now since in Fig. 156 the only plane that can be passed through the projection DC perpendicular to the plane ZX must contain the line AB, and the only plane that can be passed through the projection KH perpendicular to the plane ZY must also contain the line AB, then the projections DC and KH determine the line AB. Therefore the equations of the projections DC and KH, namely,

$$x = sz + b,$$
$$y = s'z + b',$$

determine the line AB, and are called the equations of AB.

Q. E. D.

# CHAPTER VII

## Surfaces

### PROPOSITION I

433. *A surface may be represented by an equation of the form*

$$f(x, y, z) = 0.$$

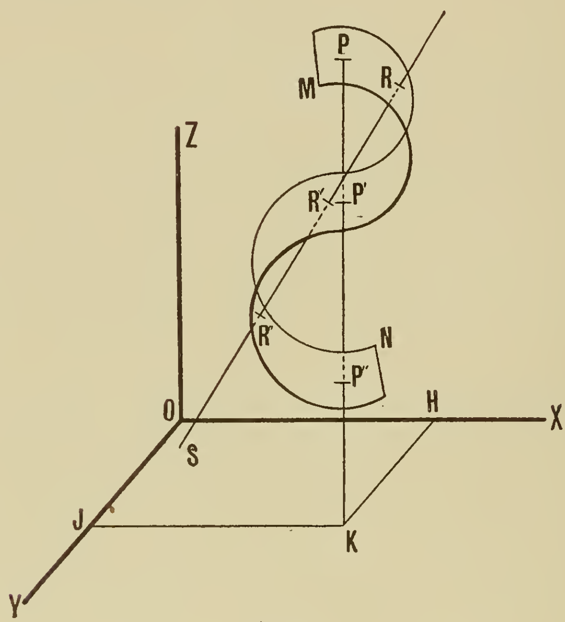



Fig. 157

In Fig. 157 let MN represent any surface, and let OX, OY and OZ be the axes of coordinates.

From any point P on the surface draw PK $\perp$ to the XY plane. It will generally cut the surface in other determinate points, as P′, P″, etc.

Draw $\qquad$ KH $\parallel$ OY,

and $\qquad$ KJ $\parallel$ OX.

$$\text{Let} \quad x \equiv OH,$$
$$y \equiv KH,$$
$$z \equiv PK,$$
$$z' \equiv P'K,$$
$$z'' \equiv P''K.$$

Now to the value of $z$ belonging to each point on the surface there corresponds one and only one value of $x$, and one and only one value of $y$. That is, the coordinates of every point on the surface have a definite relation to each other. This relation may be expressed by the equation

$$f(x, y, z) = 0,$$

which is therefore called the equation of the surface.

434. **The Equation of a Surface.**—The *equation of a surface* is one in which the variables represent the coordinates of every point on the surface.

### PROPOSITION II

435. *To find where a straight line cuts a surface, we must treat the equations of the straight line and of the surface as simultaneous and solve them. The values of x, y and z thus found are the coordinates of the cutting points.*

In Fig. 157 let MN represent any surface and RS any straight line cutting it at the points R, R′, R″.

[1] Let $f(x, y, z) = 0,$

be the equation of MN.

[2] and let $x = x' + l\rho,$

[3] $y = y' + m\rho,$

[4] $z = z' + n\rho,$

be the equations of the line RS. by § 431.

Let T be any point on the line RS, whose coordinates $x'$, $y'$, $z'$ we know.

Then by § 431, $x'$, $y'$, $z'$, $l$, $m$, $n$ in [2], [3] and [4] are known and $\rho = \text{TR}$.

Now since each of the cutting points, as R, is on the surface, the coordinates of R must satisfy [1].

Since the point R is also on the line RS, its coordinates must also satisfy [2], [3] and [4].

Therefore let $x$, $y$ and $z$ represent the coordinates of the point R. Then [1], [2], [3] and [4] are simultaneous and may be solved.

Hence substituting the values of $x, y, z$ given in [2], [3] and [4] into [1], we get an equation

$$f(x'+l\rho,\ y'+m\rho,\ z'+n\rho)=0,$$

whose only unknown quantity is $\rho$.

Solving the equation

$$[5] \qquad f(x'+l\rho,\ y'+m\rho,\ z'+n\rho)=0,$$

we get the values of $\rho$, that is TR, TR$'$, TR$''$, etc. The number of values of $\rho$ depends upon the degree of [5].

Substituting each of these values of $\rho$ successively into [2], [3] and [4], we get the coordinates of the points R, R$'$, R$''$, etc.

Q. E. D.

436. *Corollary.—To find where two surfaces cut each other we must treat the equations of the surfaces as simultaneous and solve them. The values of the variables thus found are the coordinates of the cutting points.*

# CHAPTER VIII

## Quadrics

437. **The Quadric.**—The locus of every equation of the second degree containing three variables only is called a *quadric.*

### PROPOSITION I

438. *A straight line intersects a quadric in two real, imaginary or coincident points.*

The general equation of the second degree containing three variables only is

$$[1] \quad Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'zx + 2B''xy + 2Cx + 2C'y + 2C''z + D = 0.$$

By § 437 every quadric can be represented by some form of this equation.

The equations of the straight line are

$$[2] \quad x = x' + l\rho,$$

$$[3] \quad y = y' + m\rho,$$

$$[4] \quad z = z' + n\rho. \qquad \text{by § 431.}$$

Now to find where the straight line cuts the quadric, we treat [1], [2], [3] and [4] as simultaneous and solve them. by § 435.

Hence substituting the values of $x, y$ and $z$ given in [2], [3] and [4] into [1], we get an equation of the form

$$[5] \quad A(x' + l\rho) + A'(y' + m\rho) + A''(z' + n\rho) + 2B(y' + m\rho)(z' + n\rho) + \text{etc.}$$

Since in [2], [3] and [4] $x'$, $y'$ $z'$, $l$, $m$ and $n$ are known quantities, $\rho$ will be the only unknown quantity in [5]. Again [5] contains the second and no higher powers of $\rho$. Hence when we solve it we get two and only two values for $\rho$.

Substituting these two values of $\rho$ successively into [2], [3] and [4], we will get two and only two sets of values of $x, y, z$. But each of these sets of values of $x$, $y$, $z$ locates a cutting point.

Hence the straight line cuts the quadric in two points and two only, which will be real or imaginary according as the values of $x$, $y$ and $z$ are real or imaginary, and will be coincident when the two sets of values are identical.

Q. E. D.

439. **A Chord.**—If a straight line cuts a quadric in two points, the part of the line joining these points is called a *chord* of the quadric.

PROPOSITION II

440. *Every section of a quadric made by a plane is a conic.*

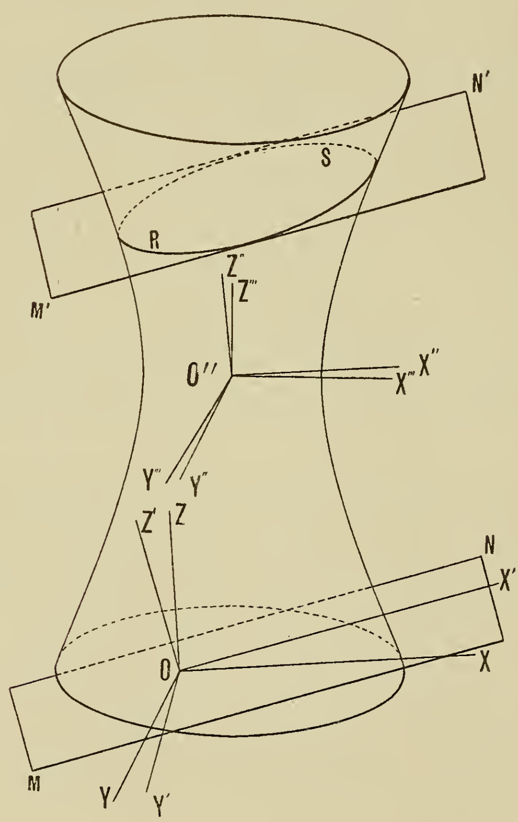



Fig. 158

For in Fig. 158 let the curved surface represent a quadric, and let

$$[1] \quad Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'zx + 2B''xy + 2Cx + 2C'y + 2C''z + D = 0,$$

be the equation of that quadric referred to the axes OX, OY and OZ. Then to get the equation of the quadric referred to any other system of axes OX$'$, OY$'$ and OZ$'$ having the same origin, we must put

$$[2] \quad x = x' \cos\alpha + y' \cos\alpha' + z' \cos\alpha'',$$

$$[3] \quad y = x' \cos\beta + y' \cos\beta' + z' \cos\beta'',$$

$$[4] \quad z = x' \cos\gamma + y' \cos\gamma' + z' \cos\gamma'',$$

into [1]. by § 407.

[5] Let $\alpha \equiv \cos\alpha,$

[6] $\beta \equiv \cos\beta,$

[7] $\gamma \equiv \cos\gamma,$

$\alpha' = \cos\alpha',$

etc.=etc.

in [2], [3] and [4] and substitute the results thus obtained into [1]. We will then get

$$[8] \quad A(\alpha x' + \alpha' y' + \alpha'' z')^2 + A'(\beta x' + \beta' y' + \beta'' z')^2 + A''(\ldots\ldots\ldots\ldots\ldots)^2 + 2B(\beta x' + \beta' y' + \beta'' z')(\gamma x' + \gamma' y' + \gamma'' z') + 2B'(\ldots\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots) + 2B''(\ldots\ldots\ldots\ldots\ldots)(\ldots\ldots\ldots\ldots\ldots) + 2C(\alpha x' + \alpha' y' + \alpha'' z') + 2C'(\ldots\ldots\ldots\ldots\ldots) + 2C''(\ldots\ldots\ldots\ldots\ldots) + D = 0.$$

[9] Now $(\alpha x' + \alpha' y' + \alpha'' z')^2 =$

$$\alpha^2 x'^2 + \alpha'^2 y'^2 + \alpha''^2 z'^2 + 2\alpha\alpha' x'y' + 2\alpha\alpha'' x'z' + 2\alpha'\alpha'' y'z',$$

[10] and $(\beta x' + \beta' y' + \beta'' z')(\gamma x' + \gamma' y' + \gamma'' z') =$

$$\beta\gamma x'^2 + \beta'\gamma' y'^2 + \beta''\gamma'' z'^2 + (\beta'\gamma + \beta\gamma')x'y' + (\beta''\gamma' + \beta'\gamma'')y'z' + (\beta''\gamma + \beta\gamma'')x'z'.$$

It will be found that when the terms in [8] are expanded as in [9] and [10] and the result is factored with respect to $x'^2$, $y'^2$, $z'^2$, $x'y'$, $x'z'$, $x'$, $y'$, and $z'$, we get an equation having exactly the same form as [1], which may therefore be written

$$[11] \quad ax'^2 + a'y'^2 + a'' z'^2 + 2by'z' + 2b'z'x' + 2b''x'y' \\ + 2cx' + 2c'y' + 2c''z' + D = 0.$$

The values of the coefficients $a$, $b$, $c$, etc., in this equation will depend upon the values of $\alpha$, $\beta$, $\gamma$, and $A$, $B$, $C$, etc. in [8].

Now in Fig. 158 let M′N′ be a plane ∥ to the plane X′Y′, that is, the plane MN.

The equation of M′N′ will be

$$[12] \quad z = p. \qquad \text{by § 420.}$$

Since OX′, OY′ and OZ′ may have any directions, the plane MN may slope in any direction.

Hence since M′N′ is parallel to MN, and $p$ may have any value, M′N′ may represent any plane.

To find where the plane M′N′ cuts the quadric, we must treat [11] and [12] as simultaneous and solve them. § 436.

Hence substituting the value of $z$ found in [12] into [11], we get

$$[13] \quad \alpha x'^2 + \alpha' y'^2 + \alpha'' p^2 + 2bpy' + 2b'px' + 2b''x'y' \\ + 2cx' + 2c'y' + 2c''p + D = 0.$$

Now since this is an equation of the second degree containing two variables only, it is the equation of a conic. by § 331*a*.

Hence in Fig. 158 RS, the common section of the plane and quadric, is a conic.

Q. E. D.

441. **The Center.**—The *center* of a quadric is the point which bisects every chord passing through it.

442. **A Diameter.**—A *diameter* of a quadric is any chord passing through the center.

PROPOSITION III

443. *The equation of a quadric referred to its center is*

$$Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'zx + 2B''xy + D' = 0.$$

For [1], § 440, is the equation of the quadric referred to the axes OX, OY and OZ of Fig. 158.

Now to get the equation of the quadric referred to the ∥ system O″X″, O″Y″ and O″Z″, we must put

[14] $$x = m + x',$$
[15] $$y = n + y',$$
[16] $$z = o + z',$$ by § 406.

into [1]. Making this substitution, we get

[17] $$Ax'^2 + A'y'^2 + A''z'^2 + 2By'z' + 2B'z'x' + 2B''x'y'$$
$$+ 2(Am + B''n + B'o + C)x'$$
$$+ 2(A'n + Bo + B''m + C')y'$$
$$+ 2(A''o + B'm + Bn + C'')z'$$
$$+ Am^2 + A'n^2 + A''o^2 + 2Bno + 2B'om + 2B''mn + 2Cm$$
$$+ 2C'n + 2C''o + D = 0.$$

Now suppose the new origin O″ in Fig. 158 to be such that when the quadric is referred to the axes O″X″, O″Y″, O″Z″, the coefficients of $x'$, $y'$ and $z'$ in its equation become 0.

Then from [17] we get

[18] $$Am + B''n + B'o + C = 0,$$
[19] $$A'n + Bo + B''m + C' = 0,$$
[20] $$A''o + B'm + Bn + C'' = 0,$$

or

[21] $$Am + B''n + B'o = -C,$$
[22] $$B''m + A'n + Bo = -C',$$
[23] $$B'm + Bn + A''o = -C''.$$

[24] Hence 
$$m = \frac{\begin{vmatrix} -C' & B'' & B \\ -C' & A' & B \\ -C'' & B & A'' \end{vmatrix}}{\begin{vmatrix} A & B'' & B' \\ B'' & A' & B \\ B' & B & A'' \end{vmatrix}};$$

$$n = \frac{\begin{vmatrix} A & -C & B' \\ B'' & -C' & B \\ B' & -C'' & A' \end{vmatrix}}{\begin{vmatrix} A & B'' & B' \\ B'' & A' & B \\ B' & B & A'' \end{vmatrix}}; \qquad o = \frac{\begin{vmatrix} A & B'' & -C \\ B'' & A' & -C' \\ B' & B & -C'' \end{vmatrix}}{\begin{vmatrix} A & B'' & B' \\ B'' & A' & B \\ B' & B & A'' \end{vmatrix}}.$$

Substituting these values of $m$, $n$ and $o$ into [17], we get by [18], [19] and [20],

[25] $$Ax'^2 + A'y'^2 + A''z'^2 + 2By'z' + 2B'z'x' + 2B''x'y' + D' = 0,$$

in which

[26] $$D' = Am^2 + A'n^2 + A''o^2 + 2Bno + 2B'mo + 2B''mn + 2Cm + 2C'n + 2C''o + D.$$

Since accents over the variables are no longer needed, we may drop them from [25], when we will get

[27] $$Ax^2 + A'y^2 + A''z^2 + 2Byz + 2B'xz + 2B''xy + D' = 0,$$

which is the equation of the quadric referred to the origin O''.

Now the form of [27] shows that if $x$, $y$, $z$ satisfy it, then $-x$, $-y$ and $-z$ must also satisfy it. That is, if any point $x, y, z$ be on the quadric, then the point $-x, -y, -z$ will also be on it. But these points are on opposite sides of the origin and equally distant from it.

Hence the origin O″ bisects every chord passing through it and must therefore be the center of the quadric. by § 441.

Hence [27] is the equation of the quadric referred to its center.

Q. E. D.

In [24] $m$, $n$ and $o$ are the coordinates of the origin O″. by § 406.

But we have just shown that O″ is the center of the quadric. Hence $m$, $n$ and $o$ are the coordinates of the center.

444. *Scholium.*—The right hand member of [26] is a function of $m$, $n$ and $o$ of exactly the same form as the left hand member of [1], § 440.

By [24] we see that the coordinates of the center will be real. That is, the quadric will have a center when the determinant which forms the denominators of the fractions is not 0, and that these coordinates will be infinite, that is, the quadric will have no center when this determinant is 0.

445. **Central Quadrics.**—Quadrics which have a center are called *central quadrics.*

446. **Non-Central Quadrics.**—Quadrics which have no center are called *non-central quadrics.*

# CHAPTER IX

## Central Quadrics

### PROPOSITION I

447. *The equation of the central quadric may be written*

$$\pm\frac{x^2}{a^2}\pm\frac{y^2}{b^2}\pm\frac{z^2}{c^2}=1.$$

For [27] is the equation of the quadric referred to the axes O″X″, O″Y″ and O″Z″ in Fig. 158.

To get the equation of the quadric referred to the axes O″X‴, O″Y‴, O″Z‴ we must, as in [8], put

$$x=\alpha x'+\alpha' y'+\alpha'' z',$$
$$y=\beta x'+\beta' y'+\beta'' z',$$
$$z=\gamma x'+\gamma' y'+\gamma'' z',$$

into [27].

Now it may be shown, as it was for [8], that this will give us another equation of exactly the same form as [27].

Hence this new equation may be written

[28] $ax'^2+a'y'^2+a''z'^2+by'z'+b'z'x'+b''x'y'+D'=0,$

By comparing this equation with [8], [9] and [10], it will be seen that the cofficients $a$, $a'$, $a''$, $b$, $b'$, $b''$ depend for their values upon $\alpha$, $\alpha'$, $\alpha''$, $\beta$, $\beta'$, $\beta''$, $\gamma$, $\gamma'$, $\gamma''$, and upon the coefficients $A$, $B$, $C$, etc., of [1], § 440. The latter are known and thefo rmer are direction cosines. by § 407.

Now let the axes O″X‴, O″Y‴, O″Z‴ be so chosen that in [28]

[29] $b=0,$

[30] $b'=0,$

[31] $b''=0.$

Equations [29], [30] and [31], and §§ 411 and 412 give us nine equations from which the nine direction cosines may be found.

When these are found and substituted into [28], that equation will become by [29], [30] and [31]

[32] $$ax'^2 + a'y'^2 + a''z'^2 + D' = 0.$$

The accents over the variables being no longer needed, this equation may be written

[33] $$ax^2 + a'y^2 + a''z^2 + D' = 0,$$

[34] or $$-\frac{a}{D'}x^2 - \frac{a'}{D'}y^2 - \frac{a''}{D'} = 1,$$

which is the equation of the quadric referred to the axes O″X‴, O″Y‴, O″Z‴.

Now since the nine direction cosines have been found, the coefficients of [34] will depend only upon the coefficients $A$, $B$, $C$, $A'$, etc., of [1], § 440, and will be positive or negative according as these latter are positive or negative.

Let us take the positive value of $\frac{D'}{a}$ and let $\sqrt{\frac{a}{D'}} \equiv \frac{1}{a_1}$.

[35] Then $$\frac{a}{D'} = \frac{1}{a_1^2}.$$

Now since the $a$ of [34] may be either positive or negative, the coefficient $-\frac{a}{D'}$ may be either positive or negative.

Hence by [35] we get

[36] $$-\frac{a}{D'}x^2 = \pm\frac{x^2}{a_1^2}.$$

Similarly we may get

[37] $$-\frac{a'}{D'}y^2 = \pm\frac{y^2}{b_1^2},$$

[38] and $$-\frac{a''}{D'}z^2 = \pm\frac{z^2}{c_1^2}.$$

Hence by [36], [37] and [38], equation [34] may be written

$$\pm\frac{x^2}{a_1^2} \pm \frac{y^2}{b_1^2} \pm \frac{z^2}{c_1^2} = 1,$$

or dropping subscripts,

[A] $$\pm \frac{x^2}{a^2} \pm \frac{y^2}{b^2} \pm \frac{z^2}{c^2} = 1,$$

which is the equation of the central quadric. Q. E. D.

448. Equation [A], § 447, the equation of the central quadric, may take four different forms:

1st. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

2nd. $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

3rd. $$-\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

4th. $$-\frac{x^2}{a^2} - \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

449. **The Ellipsoid.**—If every section of a quadric made by a plane parallel to one of the coordinate planes is an ellipse, the quadric is called an *ellipsoid*.

PROPOSITION II

450. *The equation of the ellipsoid is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

*in which a, b and c represent the semi-principal axes of the ellipsoid.*

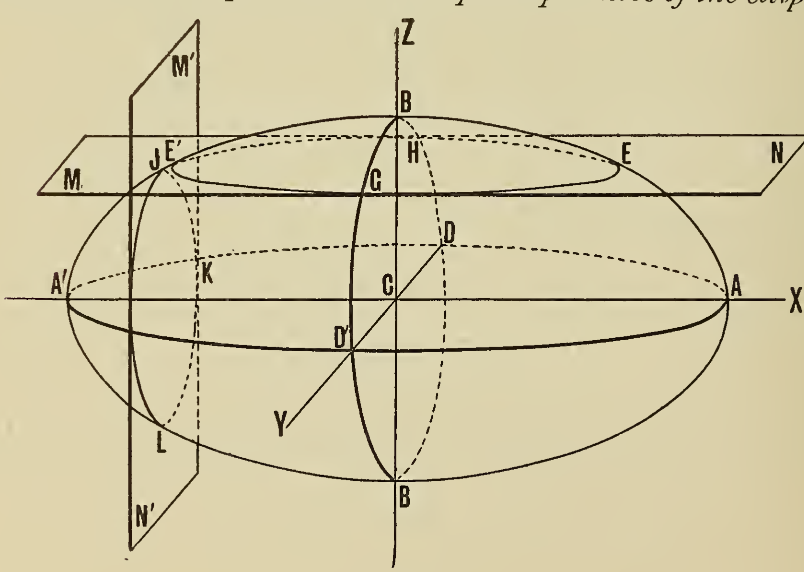



Fig. 156

To determine the quadric represented by

[1] $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1,$$

which is the first form of [A], § 447, we must determine the sections formed on this quadric by planes parallel to the coordinate planes.

Let ABD′B′DA′ be the quadric represented by [1].

Let MN be a plane ‖ to the XY plane.

The equation of the plane MN is

[2] $$z=p.$$ by § 420.

To determine the section formed on the quadric by the plane MN, we must treat [1] and [2] as simultaneous and solve them. by § 436.

There will be three cases

1st. when $p<c$.

2nd. $p=c$.

3rd. $p>c$.

FIRST CASE

$$p<c.$$

Substituting the value of $z$ found in [2] into [1], we get

[3] $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{p^2}{c^2}=1.$$

[4] Hence $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1-\frac{p^2}{c^2}.$$

Since in this case $p<c$, then $\frac{p^2}{c^2}<1$, hence $1-\frac{p^2}{c^2}$ is positive.

[5] Let $$1-\frac{p^2}{c^2}=q^2.$$

Then by [4] and [5] we get

[6] $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=q^2.$$

[7] Hence $$\frac{x^2}{(aq)^2}+\frac{y^2}{(bq)^2}=1,$$

which is the equation of the section EGE′H. by § 436.

Now [7] is the equation of an ellipse whose semi-axes are $aq$ and $bq$. by § 111.

[A] Hence EGE′H is an ellipse whose semi-axes are $aq$ and $bq$.

By [5] we see that

[8] $$q < 1.$$

[9] Hence $$aq < a,$$

[10] and $$bq < b.$$

SECOND CASE

$$p = c.$$

Since in this case $p = c$, then $\frac{p^2}{c^2} = 1$, hence by [5] we get

[11] $$q = 0,$$

and hence [6] becomes

[12] $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0,$$

[13] $$b^2x^2 + a^2y^2 = 0.$$

Since in [13] $a^2x^2$ and $b^2y^2$ are necessarily positive, this equation can only be satisfied when $x = 0$ and $y = 0$, that is, the ellipse EGE′H becomes a point.

[B] Therefore when $p = c$, the plane MN is tangent to the quadric.

THIRD CASE

$$p > c.$$

Since in this case $p > c$, then $\frac{p^2}{c^2} > 1$, hence by [5] $q^2$ will be negative and [6] becomes

[14] $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = -q^2,$$

[15] hence $$\frac{y^2}{b^2} = -\frac{x^2}{a^2} - q^2.$$

Now from [15] we see that for every value of $x$, $y$ is imaginary.

[C] Hence when $p > c$ the plane MN cannot cut the quadric.

Therefore from [A], [B] and [C] it follows that every section of the quadric represented by [1], made by a plane parallel to the XY plane is an ellipse.

Similarly it may be shown that every section of this quadric, made by a plane parallel to either of the other coordinate planes, is an ellipse.

Therefore the quadric represented by [1] is an ellipsoid.
by § 449.

**451. The Principal Axes.**—

When $p = 0$, [5] becomes

$$1 = q^2,$$

and [6] becomes

$$[16] \qquad \frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is the equation of the ellipse AD′A′D.

By [9] and [10] we see that the axes of this ellipse are greater than those of any of the other ellipses made by planes parallel to the XY plane.

Similarly we may show that the equation of the ellipse BD′B′D is

$$[17] \qquad \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1,$$

and that its axes $b$ and $c$ are greater than those of any of the other ellipses like JKL made by planes parallel to the ZY plane.

Hence $a$, $b$ and $c$ are called the *principal axes* of the quadric.

**452. The Ellipsoid of Revolution.**—The surface generated by revolving an ellipse about either of its axes is called an *ellipsoid of revolution*.

453. *Corollary.—If any two of the axes of an ellipsoid be equal to each other, it will be an ellipsoid of revolution.*

For if in [1] $b$ and $c$ be equal to each other, then every section of the ellipsoid made by a plane ‖ to the ZY plane will be a circle, and the ellipsoid in Fig. 159 may be generated by revolving the ellipse AB′A′B about the axis AA′.

If $a > b = c$, the ellipsoid is prolate.
" $a < b = c$, " " " oblate.
" $a = b = c$, " " " a sphere.

454. **The Hyperboloid of One Nappe.**—If a quadric be continuous and every section of it made by a plane ‖ to one of the coordinate planes be an ellipse and every section of it made by a plane ‖ to either of the other coordinate planes be an hyperbola, the quadric is called an *hyperboloid of one nappe.*

PROPOSITION III

455. *The equation of the hyperboloid of one nappe is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1.$$

To determine the quadric represented by

[1] $$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = 1,$$

which is the second form of [A], § 448, we must determine the sections formed on it by planes ‖ to the coordinate planes.

Let ABA′B′DD′ be the quadric represented by [1].

Let MN be a plane ‖ to the XY plane.

The equation of MN is

[2] $$z = p.$$ by § 420.

Substituting this value of $z$ into [1], we get

[3] $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 + \frac{p^2}{c^2},$$

which is the equation of the section EHE′G. by § 436.

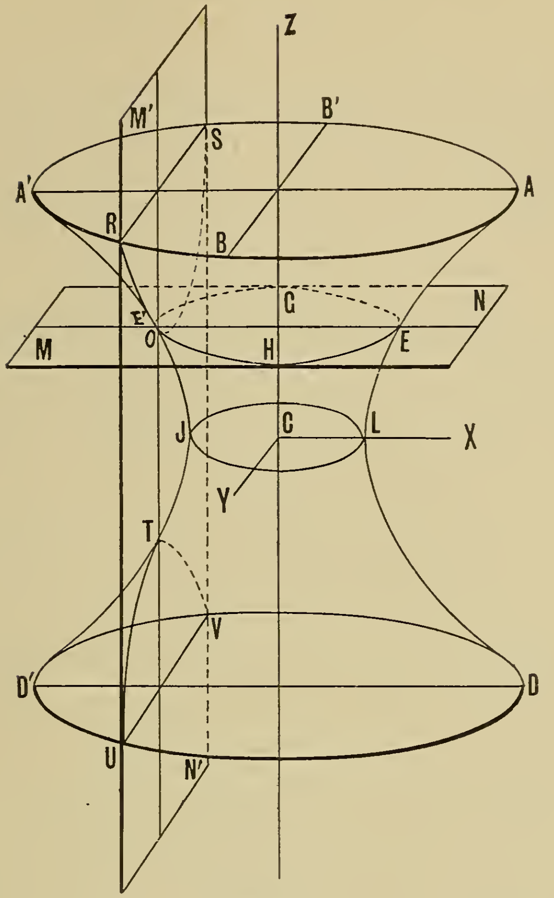



Fig. 160

[4] Now let $$1 + \frac{p^2}{c^2} = q^2.$$

[5] Then $$q^2 > 1,$$

and [3] becomes

[6] $$\frac{x^2}{(aq)^2} + \frac{y^2}{(bq)^2} = 1,$$

which is the equation of an ellipse whose semi-axes are $aq$ and $bq$. by § 111.

If the plane MN be passed through the center C then in [2]

[7] $$p = 0,$$

[8] and $$q = 1,$$

and [6] becomes

[9] $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

which is the equation of the ellipse JL, whose semi-axes are $a$ and $b$.

Now by [5]

[10] $$aq > a.$$

[11] $$bq > b.$$

Hence by [6] and [9] we see that the size of the ellipse increases continually as the plane MN is moved farther and farther from the center $C$, and that the quadric is continuous.

JL is called the *ellipse of the gorge*.

Let M′N′ be a plane ∥ to the ZY plane. Its equation will be

[12] $$x = p.$$ by § 420.

Substituting the value of $x$ found in [12] into [1], we get

[13] $$\frac{y^2}{b^2} - \frac{z^2}{c^2} = 1 - \frac{p^2}{c^2},$$

which is the equation of the hyperbola ORSTUV. by § 204.

Similarly it may be proved that the equation of every section of the quadric formed by a plane ∥ to the ZX plane is

[14] $$\frac{x^2}{a^2} - \frac{z^2}{c^2} = 1 - \frac{p^2}{c^2},$$

which is also the equation of an hyperbola.

Therefore by [6], [13] and [14] we see that the quadric represented by [1] is an hyperboloid of one nappe, § 454.

Q. E. D.

**456. The Hyperboloid of Revolution of One Nappe.**— The surface generated by revolving an hyperbola about its conjugate axis is called an *hyperboloid of revolution of one nappe*.

457. *Corollary.—If the two axes a and b of the hyperboloid of one nappe are equal to each other it is an hyperboloid of revolution of one nappe.*

458. **The Hyperboloid of Two Nappes.**—If a quadric is discontinuous and every section of it made by a plane ‖ to one of the coordinate planes is an ellipse, and every section of it made by a plane ‖ to either of the other coordinate planes is an hyperbola, the quadric is called an *hyperboloid of two nappes*.

PROPOSITION IV

459. *The equation of an hyperboloid of two nappes is*

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} - \frac{z^2}{c^2} = -1.$$

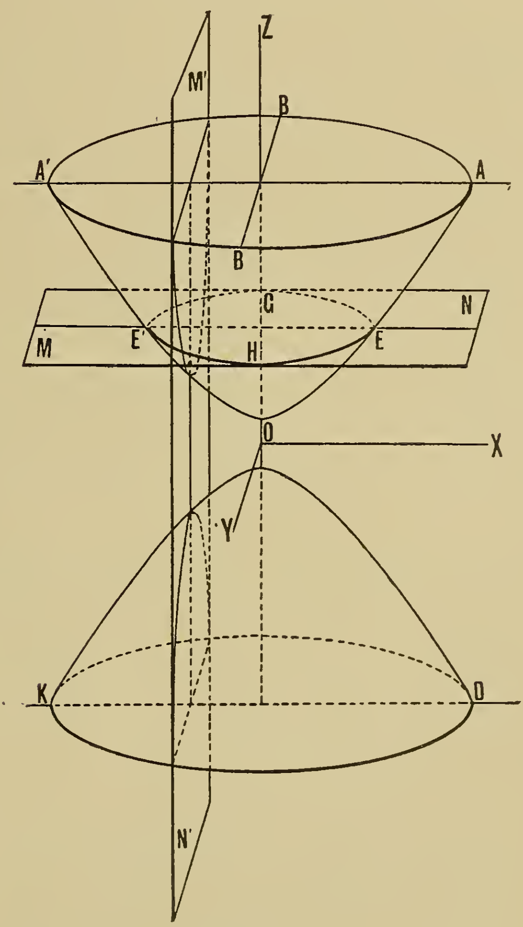



Fig. 161

The third form of equation [A] § 448 is

$$[1] \qquad -\frac{x^2}{a^2} - \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1.$$

Dividing this equation through by $-1$, we get

[2] $$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=-1.$$

Let ABA′B′DK be the quadric represented by [2].

Let MN be a plane ∥ to the XY plane.

Then, as in the preceding proposition, the equation of the section formed on the quadric by the plane MN is

[3] $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1+\frac{p^2}{c^2}.$$

[4] Let $$-1+\frac{p^2}{c^2}=q^2.$$

then [3] becomes

[5] $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=q^2,$$

which is the equation of an ellipse. by § 111.

As in § 450 there may be three cases:

1st. $p < c$.

2nd. $p = c$.

3rd. $p > c$.

FIRST CASE

$$p < c.$$

In this case since $p < c$, [4] shows that $q^2$ must be negative and [3] becomes

[6] $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-q^2.$$

[7] Hence $$\frac{y^2}{b^2}=-q^2-\frac{x^2}{a^2}.$$

[7] shows that for every real value of $x$, $y$ is imaginary, that is, the ellipse is imaginary.

Therefore the plane MN does not cut the quadric represented by [2] so long as $p < c$.

Hence the quadric represented by [2] is discontinuous.

SECOND CASE

$$p = c.$$

In this case [3] becomes

[8] $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 0,$$

and, as in § 450, [13], it can be shown that the section becomes a point.

Therefore the plane is tangent to the quadric when $p = c$.

THIRD CASE

$$p > c.$$

In this case since $p > c$, [4] shows that $q^2$ must be positive and [3] becomes

[9] $$\frac{x^2}{a^2} + \frac{y^2}{b^2} = q^2,$$

[10] hence $$\frac{x^2}{(aq)^2} + \frac{y^2}{(bq)^2} = 1,$$

which is the equation of an ellipse whose semi-axes are $aq$ and $bq$. by § 111.

Now by [4] we see that after $p$ becomes equal to $c$, $q$ increases continually as the plane MN is moved farther and farther from O. Hence the semi-axes $aq$ and $bq$ must also increase.

Therefore the ellipse EHE'G gets larger and larger as MN is moved farther and farther from the origin.

As in the preceding proposition, it may be shown that every section of the quadric made by any plane M'N' ‖ to the ZY plane is an hyperbola, and that every section of it made by any plane ‖ to the ZX plane is also an hyperbola.

Since the quadric represented by [2] is discontinuous and every section of it made by a plane ‖ to the XY plane is an ellipse, and every section of it made by a plane ‖ to either of the other coordinate planes is an hyperbola, therefore the quadric must be an hyperboloid of two nappes. by § 458.

Q. E. D.

**460. The Hyperboloid of Revolution of Two Nappes.**—The surface generated by revolving an hyperbola about its transverse axis is called an *hyperboloid of revolution of two nappes*.

461. *Corollary.—If the two imaginary axes a and b of the hyperboloid of two nappes are equal to each other, it is an hyperboloid of revolution of two nappes.*

462. *Scholium.*—The fourth form of [A], § 448, does not represent a real surface. For dividing it through by $-1$, we get

[1] $$\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=-1.$$

The equation of any plane ‖ to the XY plane is

[2] $$z=p.$$ by § 420.

Substituting this value of $z$ into [1] we get

[3] $$\frac{x^2}{a^2}+\frac{y^2}{b^2}=-1-\frac{p^2}{c^2},$$

which is the equation of the section formed on the surface represented by [1] by any plane ‖ to the XY plane. by § 436.

From [3] we get

[4] $$\frac{y^2}{b^2}=-1-\frac{p^2}{c^2}-\frac{x^2}{a^2}.$$

Since every term in the second member of this equation is necessarily negative, $y$ must be imaginary for every real value of $x$ in it.

That is, no plane ‖ to the XY plane and at a finite distance from the origin can cut the surface represented by [1].

Similarly it may be shown that no plane ‖ to either of the other coordinate planes can cut the surface represented by [1].

Therefore there can be no real surface represented by [1].

# CHAPTER X

## Non-Central Quadrics

463. On page 343, § 444, we have shown that the quadric will be non-central if

$$[1] \qquad \begin{vmatrix} A & B'' & B' \\ B'' & A' & B \\ B' & B & A'' \end{vmatrix} = 0.$$

In the more extensive treatises on this subject it is shown that when this determinant is 0, [1] § 440 can be reduced to the form

$$[B] \qquad A'y^2 + A''z^2 + 2Cx = 0.$$

Equation [B] may take two different forms according as the signs of $A'$ and $A''$ are alike or unlike.

1st. $A'y^2 + A''z^2 = 2Cx.$

2nd. $A'y^2 - A''z^2 = 2Cx.$

464. **The Elliptic Paraboloid.**—If every section of a non-central quadric made by a plane ‖ to one of the coordinate planes is an ellipse, and the sections formed on it by planes ‖ to the other two coordinate planes are parabolas, then the quadric is called an *elliptic paraboloid*.

PROPOSITION I

465. *The equation of an elliptic paraboloid is*

$$A'x^2 + A''y^2 = 2Cz.$$

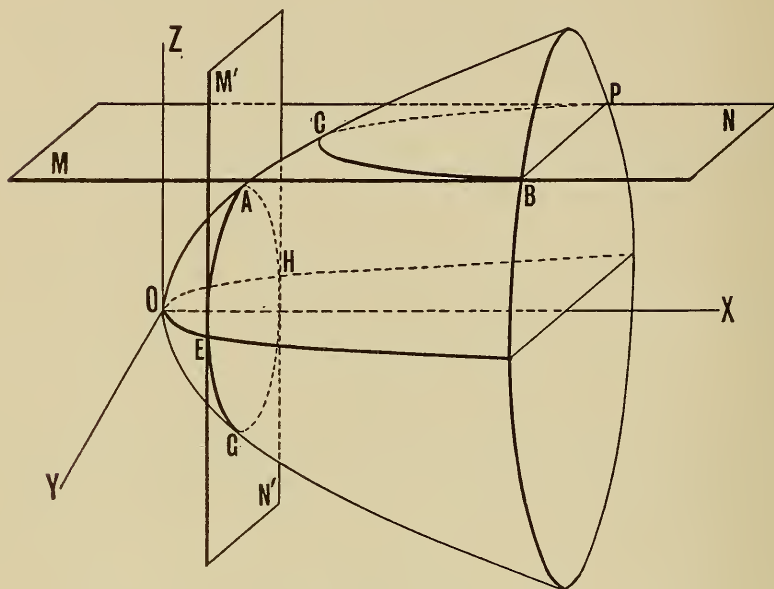



Fig. 162

The first form of [B], § 463, is

[1] $$A'y^2 + A''z^2 = 2Cx.$$

Let MN be a plane ‖ to the XY plane. Its equation is

[2] $$z = p.$$ by § 420.

The equation of the section CBP, made by the plane MN on the quadric represented by [1], is

[3] $$A'y^2 = 2Cx - A''p^2,$$ by § 436.

which is the equation of a parabola. by § 287.

Hence every section of the quadric made by a plane ‖ to the XY plane is a parabola.

The equation of a section made by a plane ‖ to the ZX plane is

[4] $$A''z^2 = 2Cx - A'p^2,$$ by § 436.

which is also the equation of a parabola. by § 287.

Hence every section of the quadric made by a plane ‖ to the ZX plane is a parabola.

The equation of the section AEGH, made by the plane M′N′ ‖ to the ZY plane is

[5] $$A'y^2 + A''z^2 = 2Cp,$$ by § 436.

which is the equation of an ellipse. by § 111.

Since every section of the quadric represented by [1] made by a plane ‖ to the ZY plane is an ellipse, and every section of it made by a plane ‖ to either of the other coordinate planes is a parabola the quadric is an *elliptic paraboloid.* by § 464.

466. **The Hyperbolic Paraboloid.**—If every section of a non-central quadric made by a plane ‖ to one of the coordinate planes is an hyperbola, and every section of it made by planes ‖ to the other coordinate planes is a parabola, the quadric is called an *hyperbolic paraboloid.*

467. *The equation of an hyperbolic paraboloid is*

$$A'y^2 - A''z^2 = 2Cx.$$

The second form of [B], § 463, is

[1] $$A'y^2 - A''z^2 = 2Cx.$$

Let MN be any plane ‖ to the ZY plane. Its equation is

[2] $$x = p.$$ by § 420.

The equation of the section ADEA′GH, made by the plane MN on the quadric represented by [1], is

[3] $$A'y^2 - A''z^2 = 2Cp,$$ by § 436.

which is the equation of an hyperbola. by § 204.

Hence every section of the quadric represented by [1], made by a plane ‖ to the ZY plane, is an hyperbola.

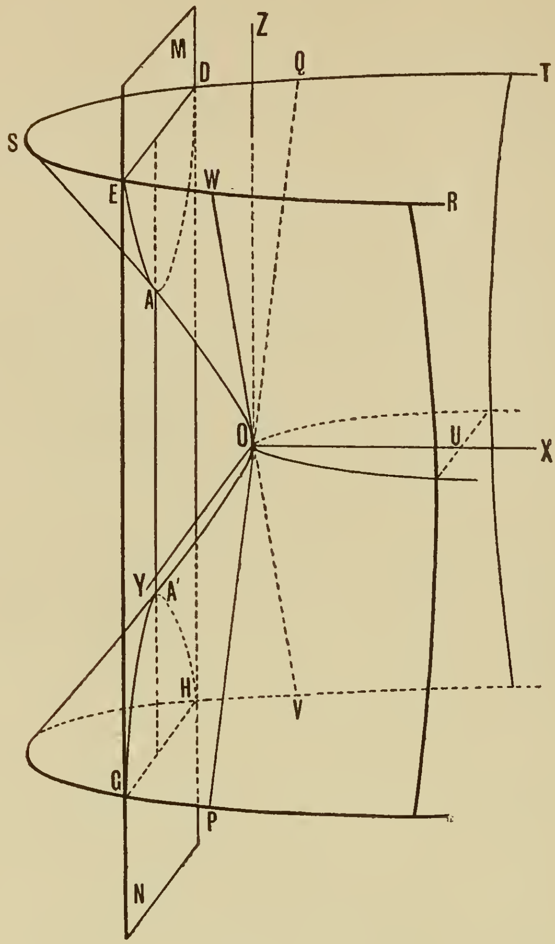



Fig. 163

Similarly every section of the quadric made by a plane ‖ to the ZX plane, is also an hyperbola.

The equation of the section RST, made by any plane ‖ to the XY plane, is

$$A'y^2 = 2Cx + A''p^2, \qquad \text{by § 436.}$$

which is the equation of a parabola. by § 287.

Hence every section of the quadric represented by [1], made by a plane ‖ to the XY plane, is a parabola.

Since every section of the quadric represented by [1], made by a plane ‖ to the XY plane, is a parabola, and every section of it made by a plane ‖ to either of the other coordinate planes, is an hyperbola, the quadric is an *hyperbolic paraboloid*. by § 466.

# APPENDIX

## PROPOSITIONS REFERRED TO IN THE TEXT

### GEOMETRY

1. Things which are equal to the same thing are equal to each other.

2. At any given point in a given straight line one perpendicular and only one can be erected.

3. If two adjacent angles have their exterior sides in a straight line, these angles are supplements of each other.

4. If one straight line intersects another straight line, the vertical angles are equal.

5. From a point without a straight line one perpendicular, and only one, can be drawn to this line.

6. If a straight line is perpendicular to one of two parallel lines, it is perpendicular to the other also.

7. If two parallel straight lines are cut by a third straight line, the alternate interior angles are equal.

8. If two parallel straight lines are cut by a third straight line, the exterior interior angles are equal.

9. When two straight lines are cut by a third straight line, so as to make the exterior interior angles equal, these two straight lines are parallel.

10. Two straight lines which are parallel to a third straight line, are parallel to each other.

11. Two angles whose sides are parallel, each to each, are either equal or supplementary.

12. Two angles whose sides are perpendicular, each to each, are either equal or supplementary.

13. The exterior angle of a triangle is equal to the sum of the two opposite interior angles.

14. Two right angles are equal if the hypotenuse and an acute angle of one are equal respectively to the hypotenuse and an acute angle of the other.

15. Two right triangles are equal if their legs are equal, each to each.

16. In an isosceles triangle the sides opposite the equal angles are equal.

17. Parallel lines comprehended between parallel lines are equal.

18. A circle is a portion of a plane bounded by a curved line called a circumference, all points of which are equally distant from a point within called the center.

19. The limit of a variable is

(1) a constant,
(2) towards which the variable continually approaches,
(3) and from which it may be made to differ by a quantity which is less than a given positive quantity however small this latter may be made.

20. If two variables are constantly equal and each approaches a limit, the limits are equal.

21. In every proportion the product of the extremes is equal to the product of the means.

22. In a series of equal ratios, the sum of the antecedents is to the sum of the consequents as any antecedent is to its consequent.

23. If a line be drawn through two sides of a triangle parallel to the third side, it divides those sides proportionally.

24. If a straight line divides two sides of a triangle proportionally, it is parallel to the third side.

25. If two triangles have their sides respectively parallel, or respectively perpendicular, they are similar.

26. The sum of the squares of the two legs of a right triangle is equal to the square of the hypotenuse.

27. The square of either leg of a right triangle is equal to the difference of the squares of the hypotenuse and the other leg.

28. The area of a rectangle is equal to the product of its base and altitude.

29. The ratio of the circumference of a circle to its diameter is constant. $C = 2\pi R$.

30. The area of a circle equals $\pi$ times the square of its radius.

31. Similar polygons are polygons having their homologous sides proportional and their homologous angles equal.

32. The square on any line is four times the square on half the line.

33. A straight line is perpendicular to a plane if it is perpendicular to every straight line of the plane drawn through its foot; that is, through the point where it meets the plane.

34. The projection of a point on a plane is the foot of the perpendicular from the point to the plane.

35. Two straight lines perpendicular to the same plane are parallel.

36. If one of two parallel lines is perpendicular to a plane, the other is also perpendicular to the plane.

37. If two straight lines are parallel to a third straight line they are parallel to each other.

38. Two planes perpendicular to the same straight line are parallel.

39. The intersections of two parallel planes with a third plane are parallel lines.

40. Parallel lines included between parallel planes are equal.

41. If two angles not in the same plane have their sides respectively parallel and lying in the same direction, they are equal, and their planes are parallel.

42. If two planes are perpendicular to each other, a perpendicular to one of them at any point of their intersection will lie in the other.

43. If a straight line is perpendicular to a plane, every plane passed through the line is perpendicular to the first plane.

44. If two intersecting planes are each perpendicular to a third plane, their intersection is also perpendicular to that plane.

45. Through a given straight line not perpendicular to a plane, one plane, and only one, can be passed perpendicular to the given plane.

46. Two straight lines in the same plane perpendicular to the same line are parallel.

47. In a parallelogram the opposite sides are equal and the opposite angles are equal.

48. If two sides of a quadrilateral are equal and parallel, then the other two sides are equal and parallel, and the figure is a parallelogram.

49. A straight line is inscribed in a circle if it is a chord.

50. Two triangles are similar if two angles of the one are respectively equal to two angles of the other.

51. Two right triangles are similar if an acute angle of the one is equal to an acute angle of the other.

53. At a given point in a straight line one plane perpendicular to the line can be drawn, and only one.

54. See 51.

55. An angle inscribed in a semicircle is a right angle.

56. If the product of any two factors be equal to the product of any other two factors, we may take the factors of either product as the means of a proportion if we take the factors of the other product as the extremes.

57. In the same circle or equal circles, chords equally distant from the center are equal.

58. Equal chords subtend equal arcs.

59. A radius perpendicular to a chord bisects the chord and also the arc which it subtends.

60. The perpendicular from any point in the circumference to the diameter of a circle is a mean proportional between the segments of the diameter.

61. The area of a trapezoid is equal to one-half the sum of the parallel sides multiplied by the altitude.

## TRIGONOMETRY.

1. $$\sin A = \frac{a}{c} = \frac{\text{opposite leg}}{\text{hypotenuse}}.$$

2. $$\cos A = \frac{b}{c} = \frac{\text{adjacent leg}}{\text{hypotenuse}}.$$

3. $$\tan A = \frac{a}{b} = \frac{\text{opposite leg}}{\text{adjacent leg}}.$$

4. $$\cot A = \frac{b}{a} = \frac{\text{adjacent leg}}{\text{opposite leg}}.$$

5. $$\sin^2 A + \cos^2 A = 1.$$

6. $$\tan A = \frac{\sin A}{\cos A}.$$

7. $$\sin A \times \operatorname{cosec} A = 1.$$

8. $$\cos A \times \sec A = 1.$$

9. $$\tan A \times \cot A = 1.$$

10. $$1 + \cot^2 A = \operatorname{cosec}^2 A.$$

11. $$\sin(-A) = -\sin A, \qquad \tan(-A) = -\tan A.$$

12. $$\cos(-A) = \cos A, \qquad \cot(-A) = -\cot A.$$

13. $$\sin(A - B) = \sin A \cos B - \cos A \sin B.$$

14. The sides of a triangle are proportional to the sines of the opposite angles.

15. The area of a parallelogram is equal to the product of any two adjacent sides by the sine of the included angle.

16. In quadrant 2 the sine and cosecant only are positive.

17. $$\tan(A - B) = \frac{\tan A - \tan B}{1 + \tan A \tan B}.$$

18. $\sin \frac{1}{2}A = \pm \sqrt{\frac{1 - \cos A}{2}}$; $\cos \frac{1}{2}A = \pm \sqrt{\frac{1 + \cos A}{2}}$.

19.

| | 0°. | 90°. | 180°. | 270°. | 360°. |
|---|---|---|---|---|---|
| cosine | 1 | $\pm 0$ | $-1$ | $\pm 0$ | 1 |
| tangent | $\pm 0$ | $\pm\infty$ | $\pm 0$ | $\pm\infty$ | $\pm 0$ |

20. $\sin A = \cos B$, $\cos A = \sin B$, when $A + B = 90$.

21. $$\cos(90 + A) = -\sin A.$$

22. $$\sin(180 - A) = \sin A.$$

23. $$\cos(180 - A) = -\cos A.$$

24. $$\tan(180 - A) = -\tan A.$$

25. The square of any side of a triangle is equal to the sum of the squares of the other two sides, diminished by twice their product into the cosine of the included angle.

26. $$\tan 45^\circ = 1.$$

27. A radian is the angle at the center of a circle subtended by an arc whose length is equal to the radius of the circle. It is equal to $57^\circ.3$

28. The number of radians in a given angle is equal to its arc divided by the radius of the circle.

29. $$\text{vers } A = 1 - \cos A.$$

30. $\cos^{-1}\frac{r-y}{r}$ is read thus: the arc whose cosine is $\frac{r-y}{r}$.

31. The area of a triangle is equal to half the product of two adjacent sides by the sine of the included angle.

quality and the increase in quantity of dairy products."—*H. W. Wiley in Journal of the American Chemical Society, February, 1897.*

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