perty of the angle of convergence, by which the most minute differences in the distances of objects and the slightest relief on their surfaces can be detected, and by which also in the abnormal conversion introduced in its action by the pseudoscope all our sensations are reversed. Therefore the pseudoscope is the great test of the phenomena of binocular vision; for by reversing certain sensations which by constant habit we may hardly notice, it renders them more conspicuous by the comparison of the abnormal state brought out by its action, and proves the theory of binocular vision in the most effective manner.

A truth is never better established than when it can be shown that the same principles are capable of producing contrary effects when they are applied in a contrary way.

Professor Wheatstone, by adding the pseudoscope to the stereoscope, has thus in the most scientific and ingenious manner completed his splendid discovery, and left very little (we might almost venture to say that he has left nothing) for further investigations in the physiology of binocular vision.

II. "On the Calculation of the Numerical Value of Euler's Constant, which Professor Price, of Oxford, calls E." By William Shanks, Esq., Houghton-le-Spring, Durham. Communicated by the Rev. B. Price, F.R.S. Received March 28, 1867.

In the year 1853 Dr. Rutherford, of the Royal Military Academy, Woolwich, sent a paper on the Computation of the value of π to the Royal Society, and the paper was published in the 'Proceedings' of that learned body*. The value of π is there given to 607 decimals, the first 440 being the joint production of Dr. Rutherford and the author of this paper, and the remaining 167 decimals having been calculated by the present writer, for the accuracy of which he alone is responsible. Subsequently, the Astronomer Royal, G. B. Airy, Esq., kindly presented the author's paper on the Calculation of the value of e, the base of Napier's logarithms, to upwards of 200 decimals; the aforesaid paper also contained the Napierian logarithms of 2, 3, and 5, as well as the modulus of the common system, all to upwards of 200 places of decimals. This paper was not, however, published, but deposited in the Archives of the Royal Society; but an abstract, containing the numerical results, was printed in the 'Proceedings'†. In a paper sent by the author to the Astronomer Royal, and forwarded by him to the Royal Society, will, the author believes, be found the reciprocal of the prime number 17389, consisting of a circulating period of no less than 17388 decimals, the largest on record. Some few remarks are also given touching circulates generally, and the easiest modes of obtaining them. The writer now desires to supplement what he then did, by giving the

* Vol. vi. p. 273. † Vol. vi. p. 397.

numerical value of Euler's constant, which is largely employed in "Infinitesimal Calculus," to a greater extent than has hitherto been found, and free from error.

In Crelle's Journal for 1860, vol. lx. p. 375, M. Oettinger has contributed an article on Euler's constant, and especially on "certain discrepancies" in the value given by former mathematical writers. Adopting the formula there employed, as being well adapted for the purpose, the writer of this paper has both corrected and extended what has been previously done; and as very great care has been bestowed upon the calculations, so as to exclude error, he confidently believes that his results are, as far as they go, absolutely correct. He may remark that, since the separate values of n in the formula (which, see below) produce identical results as far as they go, and the higher the value of n the more nearly we can approximate to the value of the constant, we thus have sufficient proof afforded of the correctness of the value found when n is 10, 20, 50, or 100. If the writer can command sufficient leisure, he may resume the calculation by and by, and, making n 1000, he may thus verify, as well as extend, the value of Euler's constant given in this paper. The numbers 10, 20, 50, 100, 200, and 1000, especially 10 and its integral powers, are more easily handled than others, particularly in those terms of the formula which contain Bernoulli's numbers. The harmonic progression is here "summed" much further than was requisite for finding E to 50 or 55 decimals; but this was of some importance in ensuring correctness in the decimal expression of each of the higher terms of S_{100} and S_{200}. It may be observed that the numbers of decimal places in E, obtained from n being 10, 20, 50, 100, and 200, are nearly proportional to $10^{\frac{1}{3}}$, $20^{\frac{1}{3}}$, $50^{\frac{1}{3}}$, $100^{\frac{1}{3}}$, and $200^{\frac{1}{3}}$—a rather curious coincidence.

The formula for Euler's constant, employed by M. Oettinger, as above stated, is—

$$\text{Constant} = Sn - \log_{\epsilon} n - \frac{1}{2n} + \frac{B_1}{2n^2} - \frac{B_2}{4n^4} + \frac{B_3}{6n^6} - \frac{B_4}{8n^8} + \dots \text{\&c., where}$$

$Sn = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \dots \frac{1}{n}$, and B_1, B_2, B_3, &c. are Bernoulli's numbers.

$1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{10}=2{\cdot}928\dot{9}6825\dot{3}$

$1+\frac{1}{2}+\frac{1}{3}+\dots\frac{1}{20}=3{\cdot}5977\dot{3}$ 96571 43681 91148 37690 68908 38779 38367
10245 37897 60291 30827 89243 77995 58542 59259
83201 52499 71906 31865 55446 61736 61220 10084
85838 20720 04363 02603 48526 $60\dot{2}$.

$1+\frac{1}{2}+\dots\dots\frac{1}{25}=3{\cdot}8159\dot{5}$ 81777 53506 86913 48136 76474 73449 06181
89635 55401 83780 86220 32609 11027 77063 20242
32778 03544 32662 95332 52228 09675 62970 52433
81377 45056 57669 24455 54624 85197 30818 82894
77830 46585 03877 77968 87905 92007 71409 68243
+ (circulating period consists of 1584 decimal places).

$1+\frac{1}{2}+\ldots\ldots\frac{1}{50}=4{\cdot}49920$ 53383 29425 05756 04717 92964 76909 19706
01823 96745 38296 58902 43217 68318 06557 86735
78157 82663 88434 12900 97472 82033 55398 82942
63084 20093 94581 03200 90115 13091 15572 24477
64889 95794 59834 14243 62248 23530 45299 19591
+ (circulating period consists of 1,275,120 decimal places).

$1+\frac{1}{2}+\ldots\ldots\frac{1}{100}=5{\cdot}18737$ 75176 39620 26080 51176 75658 25315 79089
72126 70845 16531 76533 95658 72195 57532 55049
66056 87768 92312 04135 49921 06986 97779 79182
73403 18617 00828 94825 42444 49096 57618 56974
16326 13467 07313 21114 47132 49733 09103 51089
+ (circulating period consists of 39,419,059,680 decimal places).

$1+\frac{1}{2}+\ldots\ldots\frac{1}{200}=5{\cdot}87803$ 09481 21444 47605 73863 97130 86163 68374
00246 53027 30844 64971 94472 28783 30029 84018
15499 64301 86679 89238 37326 83211 85439 05911
76542 77855 27568 86559 30203 06049 25715 75389
22254 75748 47845 75246 64079 54805 61627 08837
+ (circulating period consists of 2,498,236,128,143,832,017,541,600 decimal places).

The following value of Euler's constant has been found from the respective sums given above, of the 10, 20, 50, 100, and 200 first terms of the Harmonic Progression:—

E or Eul. const. $=\cdot57721$ 56649 01532 86060 6 $\left(\text{last term employed is}-\frac{B_{12}}{24.10^{24}}\right)$.

E $=\cdot57721$ 56649 01532 86060 65120 900 $\left(\text{last term is}-\frac{B_{12}}{24.20^{24}}\right)$.

E $=\cdot57721$ 56649 01532 86060 65120 90082 40243 1042 $\left(\text{last term is}+\frac{B_{13}}{26.50^{26}}\right)$.

E $=\cdot57721$ 56649 01532 86060 65120 90082 40243 10421 59335 9 $\left(\text{last term is}-\frac{B_{12}}{24.100^{24}}\right)$.

E $=\cdot57721$ 56649 01532 86060 65120 90082 40243 10421 59335 93995 35989 $\left(\text{last term is}+\frac{B_{13}}{26.200^{26}}\right)$.

Certainly 50 decimals are correct, and probably 55, in the value last given.

March 2, 1867.

Supplementary Paper to that of March 2, 1867, "On the Calculation of the Numerical Value of Euler's Constant." By WILLIAM SHANKS, Esq., Houghton-le-Spring, Durham. Received April 9, 1867.

When $n=500$, we have

$1+\frac{1}{2}+\frac{1}{3}\ldots\ldots\frac{1}{500}=6{\cdot}79282$ 34299 90524 60298 92871 45367 97369 48198
13814 39680 91166 43088 89685 43566 23790 55049
24576 49403 73586 56039+

$$E = \cdot 57721\ 56649\ 01532\ 86060\ 65120\ 90082\ 40243\ 10421\ 59335\ 93995\ 35988\ 05771\ 53865\ 48677 \left(\text{last term} - \frac{B_{14}}{28.500^{28}}\right).$$

When $n=1000$, we have

$$1+\tfrac{1}{2}+\tfrac{1}{3}+\ldots\tfrac{1}{1000} = 7\cdot 48547\ 08605\ 50344\ 91265\ 65182\ 04333\ 90017\ 65216\ 79169\ 70883\ 36657\ 73626\ 74995\ 76993\ 49165\ 20244\ 09599\ 34437\ 41184\ 50813+$$

$$E = \cdot 57721\ 56649\ 01532\ 86060\ 65120\ 90082\ 40243\ 10421\ 59335\ 93995\ 35988\ 05772\ 02455\ 61942\ 00508\ 15825 \left(\text{last term} + \frac{B_{15}}{30.1000^{30}}\right).$$

Hence we see that 54 decimal places are correct in the value of E (n being 200) last given in the paper dated March 2, 1867,—also that 59 decimals are correct in the value of E when $n=500$. When $n=1000$, probably 65 decimals in the value of E are correct.

When $n=1$, we readily find $E = \cdot 57 \left(\text{last term} - \frac{B_2}{4}\right)$.

„ $=2$, „ $E = \cdot 57721 \left(\text{last term} - \frac{B_4}{8.2^8}\right)$.

„ $=5$, „ $E = \cdot 57721\ 56649\ 015 \left(\text{last term} + \frac{B_{11}}{22.5^{22}}\right)$.

When $n=$	1,	E consists of	2 decimals.
„ $=$	10,	„	21 decimals.
„ $=$	100,	„	46 decimals.
„ $=$	1000,	„	65 decimals, probably.
„ $n=$	2,	„	5 decimals.
„	20,	„	28 decimals.
„	200,	„	54 decimals.
„ $n=$	5,	„	13 decimals.
„	50,	„	39 decimals.
„	500,	„	59 decimals.

From the above we may fairly infer that when n is increased in a *geometrical* ratio, the corresponding number of decimals obtained in the value of E increases only in something like an *arithmetical* one, and that probably from 50,000 to 100,000 terms in the Harmonic Progression would require to be summed in order to obtain 100 places of decimals in the value of E, Euler's constant.