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ADVERTISEMENT.

The Committee appointed by the *Royal Society* to direct the publication of the *Philosophical Transactions* take this opportunity to acquaint the public that it fully appears, as well from the Council-books and Journals of the Society as from repeated declarations which have been made in several former *Transactions*, that the printing of them was always, from time to time, the single act of the respective Secretaries till the Forty-seventh Volume; the Society, as a Body, never interesting themselves any further in their publication than by occasionally recommending the revival of them to some of their Secretaries, when, from the particular circumstances of their affairs, the *Transactions* had happened for any length of time to be intermitted. And this seems principally to have been done with a view to satisfy the public that their usual meetings were then continued, for the improvement of knowledge and benefit of mankind: the great ends of their first institution by the Royal Charters, and which they have ever since steadily pursued.

But the Society being of late years greatly enlarged, and their communications more numerous, it was thought advisable that a Committee of their members should be appointed to reconsider the papers read before them, and select out of them such as they should judge most proper for publication in the future *Transactions;* which was accordingly done upon the 26th of March, 1752. And the grounds of their choice are, and will continue to be, the importance and singularity of the subjects, or the advantageous manner of treating them; without pretending to answer for the certainty of the facts, or propriety of the reasonings contained in the several papers so published, which must still rest on the credit or judgment of their respective authors.

It is likewise necessary on this occasion to remark, that it is an established rule of the Society, to which they will always adhere, never to give their opinion, as a Body, upon any subject, either of Nature or Art, that comes before them. And therefore the thanks, which are frequently proposed from the Chair, to be given to the authors of such papers as are read at their accustomed meetings, or to the persons through whose hands they received them, are to be considered in no other light than as a matter of civility, in return for the respect shown to the Society by those communications. The like also is to be said with regard to the several projects, inventions, and curiosities of various kinds, which are often exhibited to the Society; the authors whereof, or those who exhibit them, frequently take the liberty to report, and even to certify in the public newspapers, that they have met with the highest applause and approbation. And therefore it is hoped that no regard will hereafter be paid to such reports and public notices; which in some instances have been too lightly credited, to the dishonour of the Society.

1900.

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  - *p.* Observatorio.

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- Melbourne.
  - *p.* Observatory.
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- AB. Royal Irish Academy.

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- AB. Società Italiana di Scienze Naturali.

Modena.
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Naples.
- *p.* Società di Naturalisti.
- AB. Società Reale, Accademia delle Scienze.
- B. Stazione Zoologica (Dr. DOHRN).

Padua.
- *p.* University.

Palermo.
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- AB. Consiglio di Perfezionamento (Società di Scienze Naturali ed Economiche).
- A. Reale Osservatorio.

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- *p.* Società Toscana di Scienze Naturali.

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- *p.* Rassegna delle Scienze Geologiche in Italia.

Italy (continued).

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- Luxembourg.
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- Odessa.
  - *p.* Société des Naturalistes de la Nouvelle-Russie.
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  - AB. Comité Géologique.
  - AB. Ministère de la Marine.
  - A. Observatoire Physique Central.

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  - AB. University.
- Edinburgh.
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  - *p.* Royal College of Physicians (Research Laboratory).

Scotland (continued).

Edinburgh (continued).
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- *p.* Royal Physical Society.
- *p.* Royal Scottish Society of Arts.
- AB Royal Society.

Glasgow.
- AB. Mitchell Free Library.
- *p.* Natural History Society.
- *p.* Philosophical Society.

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Belgrade.
- *p.* Académie Royale de Serbie.

Sicily. (See Italy.)

Spain.

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Madrid.
- *p.* Comisión del Mapa Geológico de España.
- AB. Real Academia de Ciencias.

Sweden.

Gottenburg.
- AB. Kongl. Vetenskaps och Vitterhets Samhälle.

Lund.
- AB. Universitet.

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- A. Acta Mathematica.
- AB. Kongliga Svenska Vetenskaps-Akademie.
- AB. Sveriges Geologiska Undersökning.

Upsala.
- AB. Universitet.

Switzerland.

Basel.
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Bern.
- AB. Allg. Schweizerische Gesellschaft.
- *p.* Naturforschende Gesellschaft.

Geneva.
- AB. Société de Physique et d'Histoire Naturelle.
- AB. Institut National Genevois.

Lausanne.
- *p.* Société Vaudoise des Sciences Naturelles.

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- *p.* Société des Sciences Naturelles.

Zürich.
- AB. Das Schweizerische Polytechnikum.
- *p.* Naturforschende Gesellschaft.
- *p.* Sternwarte.

Tasmania.

Hobart.
- *p.* Royal Society of Tasmania.

United States.

Albany.
- AB. New York State Library.

Annapolis.
- AB. Naval Academy.

Austin.
- *p.* Texas Academy of Sciences.

Baltimore.
- AB. Johns Hopkins University.

Berkeley.
- *p.* University of California.

Boston.
- AB. American Academy of Sciences.
- B. Boston Society of Natural History.
- A. Technological Institute.

Brooklyn.
- AB. Brooklyn Library.

Cambridge.
- AB. Harvard University.
- B. Museum of Comparative Zoology.

Chapel Hill (N.C.).
- *p.* Elisha Mitchell Scientific Society.

Charleston.
- *p.* Elliott Society of Science and Art of South Carolina.

Chicago.
- AB. Academy of Sciences.
- *p.* Astrophysical Journal.
- *p.* Field Columbian Museum.

Davenport (Iowa).
- *p.* Academy of Natural Sciences.

Granville (Ohio).
- *p.* Journal of Comparative Neurology.

Ithaca (N.Y.).
- A. Journal of Physical Chemistry.
- *p.* Physical Review (Cornell University).

Lawrence.
- *p.* Kansas University.

Madison.
- *p.* Wisconsin Academy of Sciences.

Mount Hamilton (California).
- A. Lick Observatory.

New Haven (Conn.).
- AB. American Journal of Science.
- AB. Connecticut Academy of Arts and Sciences.

New York.
- *p.* American Geographical Society.
- A. American Mathematical Society.
- *p.* American Museum of Natural History.
- AB. Columbia College Library.
- *p.* New York Academy of Sciences.
- *p.* New York Medical Journal.

United States (continued).

Philadelphia.
- AB. Academy of Natural Sciences.
- AB. American Philosophical Society.
- *p.* Franklin Institute.
- *p.* Wagner Free Institute of Science.

Rochester (N.Y.).
- *p.* Academy of Science.

St. Louis.
- *p.* Academy of Science.

Salem (Mass.).
- *p.* American Association for the Advancement of Science.
- AB. Essex Institute.

San Francisco.
- AB. California Academy of Sciences.

United States (continued).

Washington.
- AB. Patent Office.
- AB. Smithsonian Institution.
- AB. United States Coast Survey.
- B. United States Commission of Fish and Fisheries.
- AB. United States Geological Survey.
- AB. United States Naval Observatory.
- *p.* United States Department of Agriculture.
- A. United States Department of Agriculture (Weather Bureau).

West Point (N.Y.)
- AB. United States Military Academy.

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PHILOSOPHICAL TRANSACTIONS.

I. *The Kinetic Theory of Planetary Atmospheres.*

By G. H. Bryan, *Sc.D., F.R.S.*

Received March 15,—Read April 5, 1900.

1. The possibility of applying the Kinetic Theory to account for the presence or absence of different gases in the atmospheres surrounding the various members of our Solar System, and in particular to explain the absence of any visible atmosphere from the Moon, was first discussed by Waterston in 1846, in his memorable paper on "The Physics of Media," that so long remained unpublished in the archives of the Royal Society.* This application of the theory is distinctly mentioned in the abstract of Waterston's paper published in 1846,† which is reproduced by Lord Rayleigh as an appendix to the paper itself. Hence we may say that the kinetic theory of planetary atmospheres is as old as the kinetic theory of gases.

The present subject received the attention of Dr. Johnstone Stoney somewhere about the year 1867. It was brought under my notice by a note written by Sir Robert Ball in 1893,‡ and in that year I read a paper before the Nottingham meeting of the British Association on "The Moon's Atmosphere and the Kinetic Theory," in which numerical results were obtained sufficing to account for the absence of a visible atmosphere on the Moon and the existence of such gases as hydrogen in presence of the Sun. At that time, however, I did not see clearly how to take account of axial rotation, which evidently might play an important part in whirling off the atmospheres from certain planets, and thus the results given only represented the state of affairs at points along the polar axes of the bodies in question. Owing to this objection I did not deem it desirable to publish a more detailed paper than the abstract which appeared in the Nottingham Report.§

In the following year I published a note "On the Law of Molecular Distribution in the Atmosphere of a Rotating Planet," as an appendix to my Report on the Kinetic

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* 'Phil. Trans.,' A, 1892.

† 'Proc. Roy. Soc.,' vol. 5, p. 604.

‡ 'Science,' February 24, 1893.

§ 'British Association, Nottingham Report, 1893, p. 682.

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Theory;[*] but various circumstances have prevented me from following up the line of investigation thus started. In the interval, the new elements discovered by Professor RAMSAY have supplied fresh material for discussion, and in 1897 a paper "Of Atmospheres upon Planets and Satellites" was published by Dr. JOHNSTONE STONEY.[†]

The latter paper, however, does not in any way anticipate the methods that I had contemplated using, and there are so many points on which further investigation appeared desirable that a continuation of the present work seemed to me fully justified.

Since the present investigation was completed, I have seen Mr. S. R. COOK's paper,[‡] the method of which is closely analogous with that adopted in the calculations of my Nottingham paper.

2. WATERSTON'S *Method.*

WATERSTON'S investigations take no account of the fact that the molecular speeds of the molecules of the gas are distributed about a mean, according to such an arrangement as the BOLTZMANN-MAXWELL distribution. Hence in dealing with planetary atmospheres it is assumed that at any point all the molecules are moving with the same speed. On such an assumption the conditions for the existence of an atmosphere are very simple, especially if no account is taken of axial rotation. For supposing the celestial body under consideration to be a sphere of radius a, and that the acceleration of its attraction at distance r is μ/r^2, then v being the translational speed of a molecule at the surface of the body, it is evident that if

$$\tfrac{1}{2}v^2 > \frac{\mu}{a}$$

the molecules will describe hyperbolas, and will ultimately fly off to infinity, so that the body will lose its atmosphere; while if

$$\tfrac{1}{2}v^2 < \frac{\mu}{a}$$

the atmosphere will not extend beyond a concentric sphere whose radius r is given by

$$\tfrac{1}{2}v^2 = \mu\left(\frac{1}{a} - \frac{1}{r}\right).$$

If WATERSTON'S hypothesis were correct it would have the following advantages:—

(i.) It would enable an exact limit to be fixed for the temperature at which any gas could exist in the atmosphere of a planet.

(ii.) It would enable an exact limit to be fixed for the height of that atmosphere.

(iii.) It would account for the decrease of temperature of the atmosphere with increase of height.

WATERSTON concluded that the Moon was capable of retaining an atmosphere of air provided that its temperature was not greater than 2405° F.

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[*] 'British Association Report,' 1894, p. 100.

[†] 'Trans. Roy. Dublin Soc.,' vol. 6, p. 305.

[‡] 'The Astrophysical Journal,' vol. 11, 1.

3. *Application of* MAXWELL'S *Law.*

The results tabulated in my British Association paper of 1893 were based on the assumption that the speeds of the molecules were grouped according to MAXWELL'S Law, so that out of any N molecules at any point the number whose speed lies between c and $c + dc$ is

$$\frac{4\mathrm{N}h^{\frac{3}{2}}}{\sqrt{\pi}} e^{-hc^2} c^2 dc \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where $2/\sqrt{(\pi h)}$ is the mean speed, and $3/2h$ the mean square of the speeds of the molecules (WATSON, 'Kinetic Theory of Gases,' p. 6).

If v be the least velocity required to carry off a molecule to infinity, and given by the relation $\frac{1}{2}v^2 = \mu/a$, then the number of molecules having sufficient speed to fly off from the atmosphere will be

$$= \frac{4\mathrm{N}h^{\frac{3}{2}}}{\sqrt{\pi}} \int_v^\infty e^{-hc^2} c^2 dc,$$

which on putting $hc^2 = t^2$ gives

$$\frac{4\mathrm{N}}{\sqrt{\pi}} \int_{v\sqrt{h}}^\infty e^{-t^2} t^2 dt \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2).$$

To calculate the integral, we have on integration by parts

$$\int_t^\infty e^{-t^2} t^2 dt = \tfrac{1}{2} t e^{-t^2} + \tfrac{1}{2} \int_t^\infty e^{-t^2} dt$$
$$= \tfrac{1}{2} t e^{-t^2} + \tfrac{1}{2} \mathrm{Erfc}\, t \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3),$$

where Erfc t denotes the complement of the error function of t.

The values of $2\pi^{-\frac{1}{2}}$ Erf t are tabulated for small values of t in WOOLSEY JOHNSON'S 'Method of Least Squares,' p. 153, but in the present problem the values of t are necessarily large. Employing the series for Erfc t given in WOOLSEY JOHNSON, p. 46, Ex. 11, we deduce that the number of molecules moving with speed greater than the critical speed is

$$\frac{2\mathrm{N}}{\sqrt{\pi}} e^{-t^2} t \left\{ 1 + \frac{1}{2t^2} - \frac{1}{2^2 t^4} + \frac{1 \cdot 3}{2^3 t^6} - \frac{1 \cdot 3 \cdot 5}{2^4 t^8} + \ldots \right\} \quad . \quad . \quad . \quad . \quad (4),$$

where $t = v\sqrt{h}$.

For very large values of t, such as occur in many of the calculations, the first term of the series need alone be taken into account for purposes of rough approximation.

From this formula were calculated the results given in Table I. of my Nottingham paper, and which it may be convenient to reproduce here. This table shows the values of N when the number of molecules given by the above formula is taken to be unity; in other words, the number of molecules of oxygen or hydrogen at different temperatures out of which there is, on an average, *one* molecule moving with sufficient speed to overcome the attraction of the Earth, the Moon, Mars, or the Sun, as the case may be.

The wide difference in the numbers, due to the presence of the exponential factor, is sufficient to account for the practical permanence of the atmospheres of certain planets and the non-existence of certain gases in the atmospheres of others.

TABLE of Average Number of Molecules of Gas to every one whose Speed is sufficiently great to overcome the Attraction of the Corresponding Body.

Hydrogen at Temperature.		Oxygen at Temperature.		Surface of Moon.	Surface of Mars.	Surface of Earth.	Earth's atmosphere at height of 80 miles.	At Earth's distance from Sun.
Absolute.	Cent.	Absolute.	Cent.					
273°	0°	4368	4095°	3·6	3920	$6·0 \times 10^{19}$	$2·3 \times 10^{19}$	$2·7 \times 10^{307}$
68	− 205	1092	819	610·0	$5·0 \times 10^{15}$	$3·3 \times 10^{81}$	$7·6 \times 10^{79}$	$6·6 \times 10^{1233}$
17	− 246	273	0	$2·7 \times 10^{12}$	$1·0 \times 10^{65}$	$2·3 \times 10^{329}$	$5·7 \times 10^{322}$	$2·0 \times 10^{4740}$
$4\frac{1}{4}$	− 269	68	− 205	$6·9 \times 10^{51}$	$1·8 \times 10^{263}$	$4·5 \times 10^{1322}$	$1·5 \times 10^{1296}$	$1·7 \times 10^{19767}$

4. *Mr.* COOK'S *Method.*

Mr. COOK has employed a method almost identical with that explained above. He uses the formula (4), and proceeds to calculate the number of molecules crossing a unit of surface with velocity exceeding the critical velocity, which would just suffice to carry a molecule to infinity. For the Earth the number of molecules is computed under the following conditions:—

(1) For a spherical shell at the Earth's surface at a mean temperature of 5° C.

(2) For a spherical shell 200 kiloms. from the Earth's surface at a temperature of −66° C.

(3) For a spherical shell 20 kiloms. from the Earth's surface at a temperature of −66° C.

(4) For a spherical shell 50 kiloms. from the Earth's surface at a temperature of 180° C.

Apart from the fact that no account was taken of axial rotation in either my earlier calculations or those of Mr. COOK, the assumption of the BOLTZMANN-MAXWELL distribution prevents us from drawing any hard and fast line between gases which can exist, and gases which cannot exist, at any given temperature on any planet. It seems natural to think that, mathematically speaking, the condition of permanence would be satisfied if the number of molecules out of which one would attain the critical velocity were greater than the total number of molecules in the planet's atmosphere. But the loss of a good many cubic centimetres of air from our atmosphere in the course of a year might easily be taking place without producing any perceptible effects on its practical permanence.

5. *Dr.* STONEY'S *Method.*

If v be the critical velocity which would enable a particle to overcome the attraction of a planet, so that $\frac{1}{2}v^2 = \mu/a$, and u be the velocity of translation at the planet's equator due to axial rotation, it is clear that a particle moving in the proper direction with *relative* velocity $v - u$ would escape. If this relative velocity be a large multiple of the mean molecular velocity of the gas, then the escape of molecules will be of rare occurrence; if it be a comparatively small multiple, the gas cannot remain permanently on the planet. Dr. STONEY, in his paper of 1897, calculates the corresponding critical relative velocity $v' - u'$ at the top of the atmosphere of the planet; then, if w be the velocity of mean square in the gas, his condition of permanence is that the velocity-ratio $\frac{v' - u'}{w}$ must be great.

The limiting value of this ratio consistent with permanence might for convenience be called the critical velocity-ratio.

There being no well-defined theoretical limit to this velocity-ratio, Dr. STONEY has adopted the plan of judging the unknown from the known. Assuming that free hydrogen and helium could not exist in our atmosphere, while watery vapour does actually so exist, it is inferred that a velocity-ratio of 20 is consistent with permanence, while a velocity-ratio of 9·27 is incompatible with permanence. Applying these criteria to the case of other members of the Solar System, Dr. STONEY investigates the possibility of the existence of different gases in the atmospheres of the other members of the Solar System. Those for which the velocity-ratio falls between the two above limits are uncertain as constituents of the corresponding atmospheres, and any observations as to their actual existence or non-existence will enable closer limits to be fixed for the velocity-ratio. In making these calculations, Dr. STONEY assumes a temperature of $-66°$ C.

Now there are many circumstances which render a further investigation of the ocnditions of permanence of planetary atmospheres desirable. It is not obvious that inferences drawn from the value of the velocity-ratio in different planets are necessarily conclusive. Thus, for example, the relative velocity $v' - u'$ might be the same for two planets—one rotating very slowly and having the smaller gravitation potential, and the other rotating very rapidly and having the greater gravitation potential. The above velocity only determines the proportion of molecules *at the planet's equator moving in the direction of rotation* which would fly off. But it is clear that in the planet with the smaller potential and slower rotation a greater proportion of molecules at other points or moving in other directions would leave. For a non-rotating body, the velocity-ratio determines the proportion of *all* the molecules that would fly off, provided that they were moving away from the planet. A planet would get rid of its atmosphere much more quickly if the molecules flew off to infinity in all directions from all parts of its surface than if they were only whirled off near its equator.

Again, the molecules which at the *surface* of the planet happen to be moving with velocity greater than the critical velocity, so far from leaving its atmosphere, will only collide with other molecules, provided the atmosphere is of sufficient density to possess the attributes of ordinary gases, *i.e.*, is of density comparable with that of our atmosphere. It is only in media of extreme tenuity, possessing the properties ot high vacua, where the mean free path of the molecules is very great, that there is practically any chance of a molecule, which possesses sufficient speed, escaping without coming into collision with other molecules. In the case of the Earth this circumstance is taken into account by Dr. STONEY, where he calculates the critical velocity-ratio at a certain distance from the Earth's surface, this distance being assumed to represent the limit of height of the atmosphere. But this bring us to the question as to what is to be regarded as the limit of a planet's atmosphere, and we are thus brought to consider the problem of the law of stationary distribution of the molecules in the atmosphere of a rotating planet.

The original object of the present investigation was to clear up a number of obscure points such as those mentioned above, and at the same time to replace considerations of the critical velocity-ratio by results of a more statistical character. But the calculations have led to the result that certain gases believed by many physicists to escape from our atmosphere either do not so escape, or, if they do, that their escape takes place under different conditions as to temperature, &c., to those commonly assumed, or is a result of causes lying outside the principles of the Kinetic Theory of Gases.

6. *The Law of Molecular Distribution in the Atmosphere of a Rotating Planet.*

The modification of "MAXWELL'S Law" for the case of a gas in a field of external force has been discussed by several writers, the proofs given by WATSON being simple and neat.* The case of a rotating gas was discussed by MAXWELL, but his treatment is hardly lucid. We now proceed to investigate the modifications which must be made in the BOLTZMANN-MAXWELL distribution in order to take account of axial rotation. We assume the field of force due to the planet's attraction to be symmetrical about the axis of rotation.

Let F_1, F_2 be the functions determining the frequencies of given distributions of co-ordinates and momenta of two molecules. These satisfy the following conditions:—

(i.) In the absence of encounters, F_1 and F_2 are constant.

(ii.) When an encounter occurs, the values before and after the encounter are connected by the relation

$$F_1F_2 = F_1'F_2' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (5).$$

Condition (i.) is satisfied if F_1 and F_2 are functions of any of the integrals of the

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* 'Treatise on the Kinetic Theory of Gases,' § 11.

equations of motion of the molecules. In the case commonly considered the only known integral is the energy, and hence F_1 and F_2 are taken to be functions of the energies E_1 and E_2.

Condition (ii.), combined with the fact that the sum of the energies is unaltered by an encounter, then leads us to the well-known forms

$$F_1 \propto e^{-hE_1}, \qquad F_2 \propto e^{-hE_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6).$$

But if the field is symmetrical about the axis of z, and Γ_1, Γ_2 are the angular momenta of the molecules about this axis, we also know that—

(*a*) In the absence of an encounter Γ_1 and Γ_2 are constant;

(*b*) The total angular momentum is unaltered by encounters, so that the values before and after an encounter satisfy the relation $\Gamma_1 + \Gamma_2 = \Gamma_1' + \Gamma_2'$.

We thus see that a more general solution satisfying conditions (i.) and (ii.) now exists, of the form

$$F_1 = n_1 e^{-hE_1 + k\Gamma_1}, \qquad F_2 = n_2 e^{-hE_2 + k\Gamma_2} \quad . \quad . \quad . \quad . \quad . \quad . \quad (7),$$

the frequency functions F_1 and F_2 now containing exponential functions of the angular momentum as well as of the total energy. Take the case in which the molecules are regarded as material points of mass m, and suppose in the first instance that x, y, z, and u, v, w represent co-ordinates and velocities referred to *fixed* axes, the axis of z coinciding with the axis of rotation. Also let V be the gravitation potential of the field of force, and suppose $k = h\Omega$. Then the above expressions for F_1 and F_2 show that a permanent distribution can exist, in which the number of molecules whose co-ordinates and velocities lie within the limits $dx\,dy\,dz$ and $du\,dv\,dw$ respectively, is of the form

$$n \exp - hm\{\tfrac{1}{2}(u^2 + v^2 + w^2) - \Omega(vx - uy) - V\}\, dx\,dy\,dz\,du\,dv\,dw \quad . \quad (8),$$

where h, Ω, may be any constants up to this point, and n is proportional to the total number of molecules.

Now take axes of ξ, η, ζ, rotating about the axis of z or ζ with angular velocity Ω and instantaneously coinciding with the axes of x, y, z. Then the velocities of a molecule *relative* to these axes are ξ', η', ζ' respectively, where

$$\xi' = \frac{d\xi}{dt} = u + \Omega y, \quad \eta' = \frac{d\eta}{dt} = v - \Omega x, \quad \zeta' = \frac{d\zeta}{dt} = w,$$

and the expression determining the frequency of distribution becomes

$$n \exp - hm\,\{\tfrac{1}{2}(\xi'^2 + \eta'^2 + \zeta'^2) - V - \tfrac{1}{2}\,\Omega^2(\xi^2 + \eta^2)\}\, d\xi\,d\eta\,d\zeta\,d\xi'd\eta'd\zeta' \qquad (9).$$

The distribution of co-ordinates and relative velocities is thus the same as if the axes were fixed, and the potential of the field of force had the term $-\tfrac{1}{2}\Omega(\xi^2 + \eta^2)$ added to it. This term represents the potential of centrifugal force due to rotation with angular velocity Ω.

The above distribution, if it exist at any instant, will be permanent in the absence of encounters, and will be unaffected by encounters between the molecules; moreover, if the molecules form the atmosphere of a planet or other body rotating with angular velocity Ω, the distribution will clearly be unaffected by the molecules colliding with obstacles on the surface of the planet, while this could not be said of other distributions that might be assumed.

7. *Conditions for the above Law of Distribution.*

Let us now examine how far the conditions necessary for establishing this distribution are fulfilled in the atmosphere of a planet.

In the denser regions of the atmosphere where collisions are frequent, the effect of these collisions must be to distribute the velocities of the molecules, occurring per unit volume in the neighbourhood of a point, according to MAXWELL's law, the rate of progress towards the stationary state being calculable by the methods of BOLTZMANN, WATSON, BURBURY, and TAIT. The tendency to equalisation of the value of F in different regions of the atmosphere is determinable by the known methods of investigating diffusion, thermal conductivity, &c.

As we ascend in the planet's atmosphere, collisions between the molecules will become more and more infrequent, and at last we shall reach a region where practically all the molecules are describing free paths. These molecules will be those which are projected from the lower regions, and those which do not escape will frequently return to these lower regions. And since the proposed law of distribution remains permanent in the absence of intermolecular encounters, it follows that the molecules in question will remain distributed according to the same law, and this distribution will be brought about by the collisions which these molecules undergo in the less rarefied portions of the atmosphere.

At still greater distances from the planet we may have to deal with cases in which instead of there being a large number of molecules in a unit volume, the presence of a single molecule in, say, a cubic kilom., is an event of rare occurrence, owing to the smallness of $e^{+hm\mathrm{V}}$. But as the BOLTZMANN-MAXWELL distribution is generally accepted to be a theorem in probability, we may still apply the present distribution to determine the probability that a molecule may reach these regions. It has been objected that the error law cannot be applied to calculate the probability of events of exceptional occurrence; but it appears to be the generally accepted view that such cases should, if anything, be excluded. Practically, it is immaterial which course we take. If theory proves that the escape of a molecule is of such exceptional occurrence as will be found in some of the subsequent calculations, it will make no difference to the permanence of the atmosphere whether we assume the escaping molecule to exist or not.

The chief objection to the distribution is that it is an isothermal one, whereas in the lower regions of our atmosphere, in consequence of convection currents and winds, the adiabatic law more nearly represents the true distribution of density. The calculation of numerical data showing the effect of this correction must be deferred for future investigation, but a general discussion of the effects which may be expected to arise from this cause will be given later.

8. *Rate at which the Molecules are escaping across a Concentric Spherical Surface.*

Assuming the distribution given by (7) and supposing the planet to be spherical, so that the potential at distance r from the centre is M/r, let us calculate the number of molecules per unit time which are crossing a concentric spherical shell of radius r with sufficient velocity to carry them to infinity if they do not encounter other molecules in their subsequent paths.

For this purpose refer to polar co-ordinates, then the frequency of distribution (7) assumes the form

$$n \exp \left[- hm \left\{ \tfrac{1}{2} (u_1^2 + u_2^2 + u_3^2) - \Omega u_3 r \sin\theta - M/r \right\} \right] \\ r^2 \sin\theta \, dr \, d\theta \, d\phi \, du_1 \, du_2 \, du_3 \quad . \quad . \quad . \quad . \quad . \quad . \quad (10),$$

where u_1, u_2, u_3 are the velocity components relative to axes fixed in space coinciding with the directions of the line elements dr, $rd\theta$ and $r \sin r\theta\phi$ at the point (r, θ, ϕ). Thus u_1, u_2, u_3 are the velocities resolved along the vertical, the meridian and the parallel of latitude at the point.

To find the number of molecules crossing the spherical surface-element $r^2 \sin\theta \, d\theta \, d\phi$ with velocities within the limits of the multiple differential du_1, du_2, du_3, we write $u_1 \, dt$ for dr in the above expression, and if the number is required per unit time we divide by dt.

Of the molecules crossing the surface *outwards*, those which will escape from the planet's attraction if they do not encounter other molecules, have their resultant velocity greater than Q where $Q^2 = 2M/r$. Calling these the "escaping molecules," the number of escaping molecules across the element $r^2 \sin\theta \, d\theta \, d\phi$ per unit time is

$$r^2 \sin\theta \, d\theta \, d\phi \iiint n \exp \left[- hm \left\{ \tfrac{1}{2} (u_1^2 + u_2^2 + u_3^2) - \Omega u_3 r \sin\theta - M/r \right\} \right] u_1 \, du_1 \, du_2 \, du_3 \\ . \quad . \quad . \quad . \quad . \quad . \quad . \quad (11),$$

the limits of integration being defined by the relation $u_1^2 + u_2^2 + u_3^2 > Q^2$

Now transform the integral by putting

$$u_1 = q \sin\alpha \sin\beta, \; u_2 = q \cos\alpha, \; u_3 = q \sin\alpha \cos\beta,$$

and at the same time integrate with respect to θ and ϕ.

Then we obtain for the total number of escaping molecules the expression

$$\int_Q^\infty dq \int_0^\pi d\theta \int_0^{2\pi} d\phi \int_0^\pi d\alpha \int_0^\pi d\beta \,.\, n \exp\left[- hm \left\{\tfrac{1}{2}q^2 - \Omega rq \sin\alpha \cos\beta \sin\theta - \mathrm{M}/r\right\}\right]$$
$$q \sin\alpha \sin\beta \,.\, q^2 \sin\alpha \,.\, r^2 \sin\theta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (12).$$

Integrate first with respect to β and ϕ. We obtain

$$2\pi n \int_Q^\infty dq \int_0^\pi d\theta \int_0^\pi d\alpha \; e^{-hm\left\{\frac{1}{2}q^2 - \mathrm{M}/r\right\}} \left\{ e^{hm\Omega rq \sin\alpha \sin\theta} - e^{-hm\Omega rq \sin\alpha \sin\theta} \right\} \frac{rq^2 \sin\alpha}{hm\Omega}.$$

This expression cannot be integrated very simply with respect to α alone or θ alone, as we should obtain integrals of the forms $\int_0^\pi e^{k \sin x}\, dx$ and $\int_0^\pi e^{k \sin x} \sin x \, dx$.

It is therefore not possible to express the number of "escaping molecules" in the neighbourhood of a given parallel of latitude except by integrals of these forms.

But the double integration can be effected very simply by regarding α, θ as polar co-ordinates of a point on a sphere, and changing the axis of polar co-ordinates to one at right angles to the former axis.

Thus considering the integral

$$\int_0^\pi \int_0^\pi e^{k \sin\alpha \sin\theta} \sin\alpha \, d\alpha \, d\theta,$$

we choose two new variables, χ, ψ, defined by the relations

$$\left.\begin{aligned} \sin\alpha \sin\theta &= \cos\chi, \\ \sin\alpha \cos\theta &= \sin\chi \cos\psi, \\ \cos\alpha &= \sin\chi \sin\psi, \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (13),$$

and by spherical geometry or otherwise

$$\sin\alpha \, d\alpha \, d\theta = \sin\chi \, d\chi \, d\psi$$

and the integral transforms into

$$\int_0^{2\pi} d\psi \int_0^{\frac{1}{2}\pi} d\chi \; e^{k \cos\chi} \sin\chi \, d\chi = 2\pi \, \frac{e^k - 1}{k} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (14).$$

Substituting $k = hm\Omega rq$ and $k = -hm\Omega rq$ in turn in (12), we obtain

$$4\pi^2 n e^{h\mathrm{M}m/r} \int_Q^\infty \frac{e^{-\frac{1}{2}hmq^2}\left(e^{hm\Omega rq} + e^{-hm\Omega rq} - 2\right)}{h^2 m^2 \Omega^2} \, q\,dq \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (15).$$

The form most convenient for evaluating this integral must depend on the particular problem considered. If the effects of axial rotation on the rate of escape are small, then $2 \cosh hm\Omega rq$ may be expanded in powers of $hm\Omega rq$, and we thus obtain in the limiting case when $\Omega = 0$

$$4\pi^2 n e^{h\mathrm{M}m/r}\int_{\mathrm{Q}}^{\infty} e^{-\frac{1}{2}hmq^2}\, r^2 q^3 dq = 4\pi^2 n r^2 e^{h\mathrm{M}m/r}\,\frac{hm\mathrm{Q}^2+2}{h^2m^2}\,e^{-\frac{1}{2}hm\mathrm{Q}^2},$$

$$= 4\pi^2 n r^2\,\frac{1+hm\mathrm{M}/r}{h^2m^2} \quad . \quad . \quad . \quad . \quad (\text{since } \mathrm{Q}^2 = 2\mathrm{M}/r) \quad . \; (16),$$

a result more easily obtainable independently of the present method. If, however, the expansion of 2 cosh $hm\Omega rq$ in a series becomes inapplicable, we must express the integral by means of error functions. It becomes

$$\frac{4\pi^2 n e^{h\mathrm{M}m/r}}{h^2m^2\Omega^2}\left\{e^{\frac{1}{2}hm\Omega^2r^2}\int_{\mathrm{Q}}^{\infty} e^{-\frac{1}{2}hm(q-\Omega r)^2} q\,dq + e^{\frac{1}{2}hm\Omega^2r^2}\int_{\mathrm{Q}}^{\infty} e^{-\frac{1}{2}hm(q+\Omega r)^2} q\,dq - 2\int_{\mathrm{Q}}^{\infty} e^{-\frac{1}{2}hmq^2} q\,dq\right\}.$$

Now

$$\int_{\mathrm{Q}}^{\infty} e^{-\frac{1}{2}hmq^2} q\,dq = \frac{e^{-\frac{1}{2}hm\mathrm{Q}^2}}{hm}$$

$$\int_{\mathrm{Q}}^{\infty} e^{-\frac{1}{2}hm(q-\Omega r)^2}(q-\Omega r)\,dq = \frac{e^{-\frac{1}{2}hm(\mathrm{Q}-\Omega r)^2}}{hm}$$

$$\int_{\mathrm{Q}}^{\infty} e^{-\frac{1}{2}hm((q-\Omega r)^2}\,\Omega r\,dq = \frac{\Omega r}{\sqrt{(2hm)}}\,\mathrm{Erfc}\,[(\mathrm{Q}-\Omega r)\sqrt{(\tfrac{1}{2}hm)}],$$

and similarly for the reduction of the remaining integral. Hence we obtain finally

$$\frac{4\pi^2 n e^{h\mathrm{M}m/r}}{h^3m^3\Omega^2}\left\{e^{-\frac{1}{2}hm\mathrm{Q}^2}(e^{hm\Omega r\mathrm{Q}} + e^{-hm\Omega r\mathrm{Q}} - 2)\right.$$
$$\left. + \Omega r\sqrt{(\tfrac{1}{2}hm)}\Big(\mathrm{Erfc}\,[(\mathrm{Q}-\Omega r)\sqrt{(\tfrac{1}{2}hm)}] - \mathrm{Erfc}\,[(\mathrm{Q}+\Omega r)\sqrt{(\tfrac{1}{2}hm)}]\Big)\right\} \quad . \quad . \quad . \; (17),$$

and when the argument is large we may expand the error-function complements by means of the descending series referred to in § 3, equation (4). Remembering, too, that $\mathrm{Q}^2 = 2\mathrm{M}/r$, we obtain

$$\frac{4\pi^2 n}{h^3m^3\Omega}\left\{e^{hm\Omega r\mathrm{Q}}\left[1 + \frac{\Omega r}{4(\mathrm{Q}-\Omega r)}\left\{1 - \frac{1}{hm(\mathrm{Q}-\Omega r)^2} + \frac{1\,.\,3}{h^2m^2(\mathrm{Q}-\Omega r)^2} - \,.\,.\,.\right\}\right]\right.$$
$$\left. + e^{-hm\Omega r\mathrm{Q}}\left[1 - \frac{\Omega r}{4(\mathrm{Q}+\Omega r)}\left\{1 - \frac{1}{hm(\mathrm{Q}+\Omega r)^2} + \frac{1\,.\,3}{h^2m^2(\mathrm{Q}+\Omega r)^2} - \,.\,.\,.\right\}\right] - 2\right\}$$
$$. \quad . \quad . \quad (18).$$

This formula might be applied to form an estimate of the rate at which a planet is losing its atmosphere. We shall now, however, show that the law of distribution, and therefore the above formula, cease to hold good beyond a certain distance from the planet.

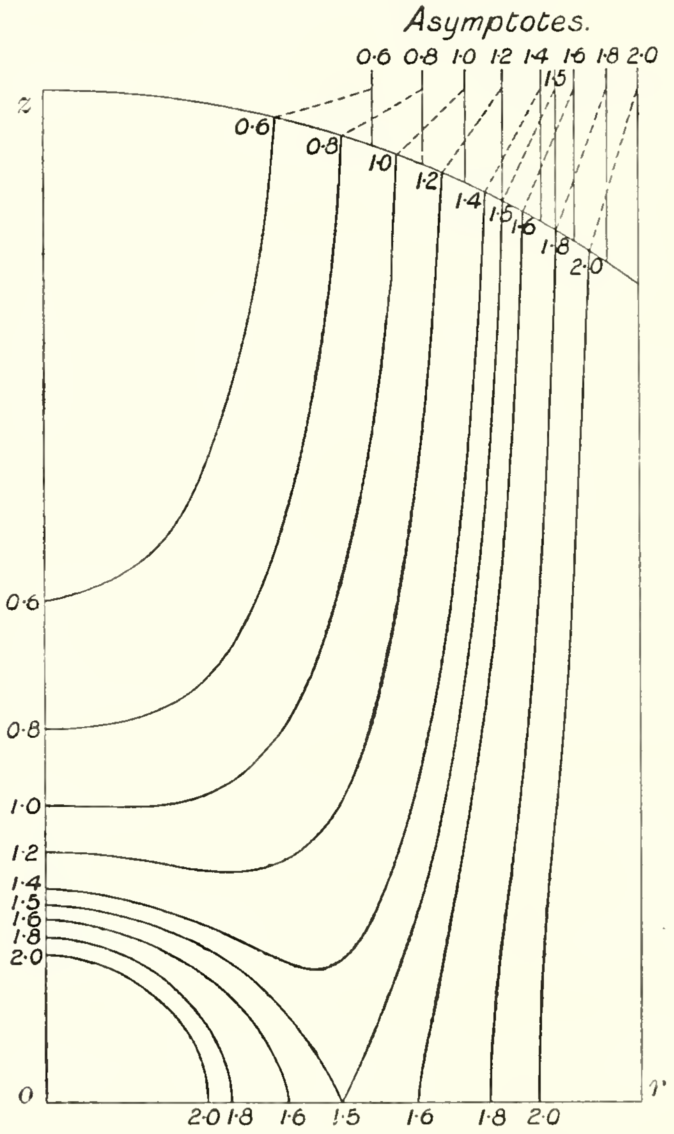
9. *Limit to the Height of a Planet's Atmosphere.*

It is obvious, on the *non*-kinetic view, that no matter could be retained in relative equilibrium beyond the distance at which a planet's attraction is just balanced by centrifugal force. On the kinetic theory, this limitation finds its interpretation in the fact that the density of the atmosphere according to the above law of distribution is proportional to

$$\exp - hm\{\mathrm{V} - \tfrac{1}{2}\Omega^2(\xi^2 + \eta^2)\} = \exp - hm\{\mathrm{V} - \tfrac{1}{2}\Omega^2 r^2\} \text{ where } r^2 = \xi^2 + \eta^2 \quad . \quad . (19).$$

If $d\mathrm{V}/dr > \Omega^2 r$, the density decreases as we proceed outwards from the axis of the planet, it becomes a minimum when $d\mathrm{V}/dr = \Omega^2 r$, and begins to increase again outwards when $d\mathrm{V}/dr < \Omega^2 r$. Hence the point at which centrifugal force is just balanced by the planet's attraction is the point of minimum density in the atmosphere according to the above law of *permanent* distribution. And since the atmosphere does not extend to infinity, we conclude that it cannot be permanent unless the density at the point of minimum density is infinitesimal, and practically zero.

This may be made clearer by drawing the meridian sections of the surfaces of equal density, and thus reproducing the results first obtained by EDOUARD ROCHE, of Montpellier. In the case of a spherical planet of mass M (astronomical units), these curves are given in cylindrical co-ordinates by the equation



$$\frac{M}{\sqrt{z^2 + r^2}} + \tfrac{1}{2}\Omega^2 r^2 = \text{const.}$$

This equation may be written

$$\left(\frac{r^2 + z^2}{a^2}\right)^{-\frac{1}{2}} + \tfrac{1}{2}\left(\frac{r}{a}\right)^2 = \text{C (a constant)} \quad . \quad . \quad . \quad . \quad . \quad . \quad (20),$$

where

$$a^3 = \frac{M}{\Omega^2}.$$

At the point of minimum density in the equatorial plane, $z = 0$, $r = a$, and the curve of equal density through this point is the one for which $C = 1{\cdot}5$. This curve has a double point at the point in question.

The curves for which $C > 1{\cdot}5$, and for which the density is therefore greater than the equatorial minimum, each consist of a closed oval and an infinite branch having the line $\frac{1}{2}(r/a)^2 = C$ as asymptote, while the curves for which $C < 1{\cdot}5$ consist of infinite branches only. It is clear then that if any gas of appreciable density were to pass by diffusion beyond the closed portion of the surface $C = 1{\cdot}5$, it would diffuse itself indefinitely over the unclosed surfaces of equal density, and its molecules would not be able, by their collisions, to maintain the equilibrium of the distribution of molecules within the surface.

Calling the surface $C = 1{\cdot}5$ the critical surface, one condition for permanence is that the molecules which reach this surface must be so few and far between, that collisions rarely take place between them.

Such molecules will then describe free trajectories under the planet's attraction. If their velocity be greater than that required to carry them to infinity, they will leave the planet, describing parabolic or hyperbolic orbits. If the velocity be less than that amount, they will describe ellipses, and return to the planet's atmosphere.

Now at the singular point of the critical surface, the velocity due to axial rotation is just sufficient to make a particle, if projected with it, describe a circular orbit. For a parabolic orbit the velocity is $\sqrt{2}$ times as great. Hence a molecule at the singular point moving tangentially in the direction of rotation, will just leave the planet if its relative translatory velocity, due to the temperature of the atmosphere which it has left, be $\sqrt{2} - 1$ times the velocity due to rotation. If it be moving in any other direction, or it be situated at any other point of the critical surface, its velocity will have to be correspondingly greater.

10. *The Critical Density-ratio.*

It thus becomes important before proceeding further to calculate and tabulate, for different gases at different temperatures, the ratio of their densities at the surfaces of different members of the Solar System to their densities at the corresponding critical surfaces, assuming the law of permanent distribution of § 6. This can easily be done from the formula, according to which the density at any point is proportional to

$$e^{-hm(V - \frac{1}{2}\Omega^2 r^2)}.$$

Calling the ratio in question the *critical density-ratio*, the condition for a permanent atmosphere requires this ratio to be very large. It must in fact be the ratio of a density of molecular distribution at the surface of the planet comparable with that of the gases in our atmosphere to one at the critical surface in which the molecules are very few and far between. For most purposes it will be sufficient to know the value of this ratio correct to the nearest power of 10, and this is shown quite as clearly by tabulating the logarithm of the ratio to base 10, for which purpose it is only necessary to multiply the difference of the potentials of gravitation and centrifugal force at the planet's surface and at the critical surface by $hm\mu$, where μ is the modulus of common logarithms. The tables are thus easy enough to calculate, but it is necessary to have them before one in order to arrive at any definite conclusions.

The logarithm to base 10 of the critical density-ratio, as thus defined, is equal to

$$\mu hm (V_0 + \tfrac{1}{2}u^2 - V_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (21),$$

where V_0 is the gravitation potential at the planet's surface,

u is the velocity at the planet's equator due to axial rotation,

V_1 is the combined potential of gravitation and centrifugal force at the critical surface. Since this is constant over the critical surface, we may take V_1 to be the potential, due to gravitation alone, at the extremity of the polar radius of the critical surface. This polar radius is $\frac{2}{3}$ of the equatorial radius, and the latter is obtained by equating the planet's attraction to centrifugal force.

hm is equal to $3/Q^2$, where Q^2 is the mean of the squares of the velocities of the molecules. This quantity is proportional to the molecular weight and inversely proportional to the absolute temperature.

Starting with the case of the Earth and hydrogen, I have used the data given in Dr. JOHNSTONE STONEY's paper, pp. 310, 312, viz.,

Earth's equatorial radius = 6378 kiloms.

Value of g at equator due to Earth's attraction at equator = 981·5 centims./sec.^{2}.

u the equatorial velocity due to rotation = 464 metre/sec.

$Q^2 = (11140)^2$ T, in C.G.S. units at absolute temperature T.

It will be convenient to assume a temperature of 100° absolute (−173° C.) in the first calculations, since the values for other temperatures can be more easily calculated by taking a round number to start with.

We thus obtain the following values for the various terms :—

$\mu hm V_0$ the term due to the gravitation potential at the Earth's surface = 65·718
Call this term A.

$\mu hm \,.\, \frac{1}{2}u^2$ the term due to centrifugal force at the Earth's surface $= {\cdot}1130$.
Call this term B.

$\mu hm\mathrm{V}_1$ the term due to the total potential at the critical surface $= 14{\cdot}880$.
Call this term C.

The logarithm of the critical density ratio is equal to $\mathrm{A} + \mathrm{B} - \mathrm{C}$.
Hence we obtain the following results :—

TABLE of Logarithms of the Critical Density-ratio for Hydrogen relative to the Earth.

Absolute temp.	Centigrade temp.	Log. crit. dens. ratio.
100°	−173°	50·951
200	− 73	25·475
300	+ 27	16·987

To make the meaning of these figures perfectly clear we notice, as remarked by Dr. STONEY, that at the bottom of our atmosphere the number of molecules in each cub. centim. of air is about 10^{21}. Supposing now that the earth was invested with an atmosphere of hydrogen containing 10^{21} molecules per cub. centim. at the surface of the earth. Then at a temperature of 27° C. there would be $10^{21-16{\cdot}987}$, or roughly 10^4 or 10,000 molecules in every cub. centim. at the critical surface; at −73° C. there would be on an average one molecule to every $10^{25{\cdot}175-21}$, or roughly $10^{4{\cdot}475}$ or 30,000 cub. centims. at the critical surface; at −173° there would only be one molecule per $10^{50{\cdot}951-21}$ or 10^{30} cub. centims. at the critical surface.

In the first case a considerable escape of gas would take place through molecules passing beyond the critical surface. In the second case it will be noticed that there would be about 33 molecules per cubic metre up at the critical surface, so that molecules would escape fairly frequently, though the rate of loss of the atmosphere would of course be very slow. In interpreting the third case it will be convenient to observe that the equatorial radius of the critical surface is about $4{\cdot}225 \times 10^9$ centims., and the volume of a sphere having this radius, and therefore enclosing the whole of the Earth's atmosphere, is about $3{\cdot}162 \times 10^{29}$ cub. centims. The calculated result for the frequency of distribution at the critical surface represents an average of about one-third of a molecule per volume equal to this vast enclosing sphere.

11. *Calculation for other Planets.*

In performing the calculations of the logarithm of the critical density-ratio for other planets, the data for the Earth may be taken as a starting-point, it being observed (as may be easily proved) that (i.) The term A representing the gravitation potential at the planet's surface.

$$\propto \frac{\text{mass of planet}}{\text{radius}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22).$$

(ii.) The term B representing the potential of centrifugal force at the surface of the equator,

$$\propto \left(\frac{\text{radius of planet}}{\text{time of axial rotation}}\right)^2 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (23).$$

(iii.) The term C representing the combined potential of gravitation and centrifugal force over the critical surface,

$$\propto \left(\frac{\text{mass of planet}}{\text{time of rotation}}\right)^{\frac{2}{3}} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (24).$$

(iv.) It will also be convenient to notice that the equatorial radius of the closed part of the critical surface (which we have denoted by a)

$$\propto (\text{mass})^{\frac{1}{3}} \times (\text{time of rotation})^{\frac{2}{3}} . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (25).$$

Employing the values for the radii, masses and times of rotation given in Dr. STONEY's paper, taking the Earth's as unity, I obtain the following results :—

TABLE of Logarithms of the Critical Density-ratio for Hydrogen for various Planets.

	100° absolute.	200° absolute.	300° absolute.
Venus . .	40·6360	20·3180	13·5453
Earth . .	50·951	25·475	16·987
Mars . .	10·4690	5·2345	3·4896
Jupiter . .	711·94	355·97	237·31
Saturn . .	165·98	82·99	55·33

The logarithm of the critical density-ratio being inversely proportional to the absolute temperature and directly proportional to the molecular weight of the gas, it becomes unnecessary to perform the calculations for other gases or other temperatures, as this is a mere matter of simple arithmetic.

The high values for the logarithm of the critical density-ratio on Jupiter and Saturn leave little doubt as to the possibility of the lightest gases, such as hydrogen, remaining practically permanent in the atmospheres of these planets. When we apply the methods to helium (assuming its molecular weight twice that of hydrogen) on the Earth and watery vapour (molecular weight nine times that of hydrogen) on Mars, we have the following results :—

	100° absolute.	200° absolute.	300° absolute.
Terrestrial helium . .	101·90	50·95	33·97
Water on Mars . . .	94·22	47·11	31·41

The values of the logarithm of the critical density-ratio at 200° absolute—a point sufficiently near the temperature −66° C. assumed in Dr. JOHNSTONE STONEY's paper—appear far too great to be consistent with any appreciable loss taking place from the planets at those temperatures. Even at 300° absolute or 27° C., the figures represent averages of 100 and 30,000 molecules per cub. kilom. at the critical surface for every 10^{21} molecules per cub. centim. at the surface of the planet. While the figures show that the conditions for water on Mars are less favourable than for terrestrial helium, they appear distinctly favourable to both these elements at the temperatures ordinarily assumed for planetary atmospheres. To examine this point more fully it is important to calculate limits for the rate at which a planet would lose its atmosphere for the supposed values of the critical density-ratio, and for the time in which this loss would become appreciable.

12. *Rate of Flow across Critical Surface.*

The total rate of effusion across the closed portion of the critical surface may be calculated from the formula

$$\tfrac{1}{4} q \rho \mathrm{S} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (26),$$

where q is the mean translational speed of the molecules, ρ the density, and S the area of the critical surface.

Without entering into the mathematical problem of the quadrature of the surface in question by actual integration, its area can be found sufficiently closely for our purpose by noticing that the closed portion in question consists of two caps cutting their common base at 60°, and whose heights are $\frac{2}{3}$ of the radius of the base. Now, if we take two *spherical* caps the radius of whose common base is a, and whose heights are each $\frac{2}{3}a$, the area of the closed surface formed by them $= \frac{26}{9}\pi a^2$; while if we take two spherical caps on the same base and cutting that base at an angle of 60° the area $= \frac{24}{9}\pi a^2$. As the closed part of the critical surface lies between the first and second pair of spherical caps, we know that its superficial area is $\frac{25}{9}\pi a^2$, correct to within ± 4 per cent., and this value we shall adopt.

$$\text{For the Earth } a = 6{\cdot}625 \text{ R, where R} = \text{Earth's radius;}$$
$$\therefore a = 4{\cdot}225 \times 10^9 \text{ centims.}$$

If we imagine all the molecules which cross the surface in question to leave the planet's atmosphere, and leave out of count all those which return after describing free trajectories, the formula gives a superior limit to the rate at which the planet is losing its atmosphere. To obtain an estimate of the time which must elapse before the loss in question would have an appreciable effect on the amount of the gas present in the atmosphere, let us calculate the time in which the loss would represent an amount of gas equivalent to the removal of a layer 1 centim. thick from the surface of the planet. If q be measured in centims. per sec., and if L represent the critical density-ratio, then the required time

$$= \frac{144}{25} \frac{a^2}{R^2} \frac{L}{q}.$$

Putting this equal to L/E years, we have

$$E = \frac{365 \cdot 25 \times 24 \times 60 \times 60 \times 25}{144} \frac{R^2 q}{a^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (27).$$

Calculating the value of log E from this formula for hydrogen at temperature 100° absolute, we have the following results :—

Hydrogen at $-173°$ C. $=$ 100° absolute.	Log E.
The Earth	14·40133
Venus.	14·35456
Mars	14·35149
Jupiter	13·47129
Saturn	13·27377

For other gases at other temperatures, the value of log E is easily deduced.

For E varies as q, which varies as the square root of the absolute temperature divided by the square root of the molecular weight. Thus for hydrogen at 200° absolute, the temperature is doubled, E is increased in the ratio $\sqrt{2} : 1$, and log E has to be increased by $\frac{1}{2} \log 2$. For oxygen, the molecular weight is 16 times as great as for hydrogen, E is decreased in the ratio of $4 : 1$, and log E has to be decreased by log 4 ; similarly for other cases. It will be seen, however, that log E is, roughly, somewhere about 14 for the majority of gases at ordinary temperatures relative to the Earth, Venus, or Mars, and rather less (namely, about 13) for the larger planets.

It follows that *if the logarithm of the critical density-ratio for a given gas at a given temperature relative to a given planet is equal to about* 14, *the total rate of effusion of that gas across the critical surface would be equivalent to the removal of the amount of that gas present in a layer* 1 *centim. thick over the surface of the planet in a period of time comparable with a year.*

If the logarithm of the critical density-ratio is 20, *the corresponding period of time would be comparable with a million years ; if* 26, *with a billion years, and so on.*

The rate of effusion is proportional to the quantity of gas present, so that in cases where a gas does not exist in appreciable quantities in the atmosphere of the planet, the figures enable us to calculate what *would* happen supposing the planet to be endowed with an atmosphere of any given gas at a given temperature.

We now obtain the results given in the following tables:—

NUMBER of Years in which the Efflux across the Critical Surface would equal the Removal of the Gas in a Superficial Layer 1 centim. thick.

I. EARTH'S ATMOSPHERE.

Hydrogen at absolute temp.	Helium at absolute temp.	Years.
100°	200°	$3 \cdot 54 \times 10^{36}$
150	300	$3 \cdot 06 \times 10^{19}$
200	400	$8 \cdot 40 \times 10^{10} = 84,000,000,000$
250	500	$6 \cdot 02 \times 10^{5} = 602,000$
300	600	$2 \cdot 22 \times 10^{2} = 222$

II. ATMOSPHERE OF MARS.

Vapour of water at absolute temperature.	Years.
200°	$1 \cdot 22 \times 10^{33}$
250	$3 \cdot 37 \times 10^{23}$
300	$1 \cdot 94 \times 10^{16}$
400	$2 \cdot 40 \times 10^{9} = 2,400,000,000$
500	$4 \cdot 28 \times 10^{4} = 42,800$
600	$1 \cdot 06 \times 10^{2} = 106$

The following table shows the corresponding absolute temperatures on the Earth and Mars at which the calculations for helium and water respectively would lead to approximately the same numerical results:—

Earth and helium . .	200°	300°	400°	500°	600°
Mars and water . . .	187	281	376	472	571

The removal of a layer of atmosphere 1 centim. thick from the surface of a planet would of course decrease the barometric pressures on that planet by an amount equal to the decrease for 1 centim. of altitude, and such a change would therefore be practically quite imperceptible. A secular diminution in atmospheric pressure would not make itself noticeable to any practical extent till it corresponded to an altitude comparable with, say, 100 metres. For such a change the numbers of years in the foregoing table would have to be multiplied by 10,000.

13. *Effect of Air Currents in the Lower Regions.*

As previously indicated, one effect of convection currents, or winds, is to produce a temperature-gradient in the lower regions of a planet's atmosphere, rendering the adiabatic law of distribution a closer approximation than the isothermal. While a discussion of the numerical results obtained with that law must be left for future investigation, there appear to be many reasons on general grounds why such a distribution should not lead to results differing widely from ours, and why these results should certainly not be more favourable to the escape of the lighter gases from the atmospheres of planets.

The correction required to take account of the temperature-gradient may be roughly estimated in various ways :—

(1) We may calculate the values of the critical density-ratio for the highest and lowest temperatures known to exist in the atmosphere, the actual state of affairs being intermediate between these extreme cases.

(2) We may apply a correction to the results above calculated by comparing the actual gradient of density in the strata of the atmosphere which have been explored with the gradient which would exist if the distribution were isothermal. To make this correction sufficient, it would be necessary to carry the investigation up to the height at which the temperature becomes sensibly constant.

Now if the density of the atmosphere at the assumed upper limit be appreciable, the factor by which the critical density-ratio must be multiplied in order to apply the correction will be finite. Such a correction, then, cannot possibly affect the permanence, for example, of helium in the Earth's atmosphere at ordinary temperatures, where, if the rate of escape were multiplied, say, a millionfold, it would still be inappreciable.

But the existence of these air currents has a further influence on the distribution of the atmosphere. For, according to the isothermal distribution of the kinetic theory, the density of the heavier constituents falls off much more rapidly with the altitude than that of the lighter ones, and we therefore believe that the lighter gases extend to altitudes at which the heavier ones have practically ceased to exist. The effect of air currents is to equalise the percentage composition of the atmosphere in the upper and lower regions ; in this way a greater percentage of the lighter constituents will be retained in the lower regions than would be the case if equilibrium were attained by diffusion. The conditions will be more favourable to the escape of the heavier constituents, or less favourable to the escape of the lighter ones, or both.

14. *Conclusions.*

Without entering into the debatable question of the "Age of the Earth," we may take Lord KELVIN'S estimate of 10^8 years as affording some indication of the order of magnitude of the times which have to be taken into account in tracing the past history of our planet under conditions more or less similar to those prevailing in present or geological times.

The times calculated in the previous paragraph range up to far higher orders of magnitude.

We may therefore safely draw the following conclusions :—

1. The Earth's attraction is capable, according to the kinetic theory, of retaining a gas of twice the weight of hydrogen in the form of a (practically) permanent atmosphere of uniform temperature, as high as any temperatures commonly existing in its present atmosphere.

2. The vapour of water is similarly capable, according to the kinetic theory, of existing on Mars in the form of a (practically) permanent atmosphere of uniform temperature, at any ordinary temperature. Hence it follows that *either*

(*a*) Helium does not escape from our atmosphere and water does not escape from that of Mars ;

Or (*b*) The escape takes place under far more favourable conditions, such as far higher temperatures than those assumed in previous investigations, or than we should be naturally led to assume from our knowledge of the conditions prevailing in those regions of our atmosphere that have been explored ;

Or (*c*) The escape is due to translational movements of molecules other than those investigated by the methods of the kinetic theory of gases.

[*Postscript added August*, 1900 ; *revised October*, 1900.

Since the above paper was read, Dr. JOHNSTONE STONEY has written several papers in which he maintains that helium does escape from our atmosphere. If this view be adopted, the present investigation must be regarded as a proof that the escape, instead of being attributable to the motions which the kinetic theory assigns to the molecules of a gas under ordinary conditions as to temperature, &c., must be due either (1) to the existence of other causes or conditions, or (2) to a divergence between the law of distribution which forms the basis of our commonly accepted kinetic theory and that occurring in an actual atmosphere.

Owing to the vastness of the subject, it appears only desirable for me in the present note to touch briefly on a few of the main points.

In connection with (1) Dr. STONEY points out that the actual distribution of molecules of a gas in any particular problem is dependent on two functions, π and δ, of which he supposes the first to determine the normal distribution, while the second represents the deviations due to disturbing causes. He is of opinion that the effect of the δ function is to increase the probability of a molecule acquiring a velocity considerably in excess of the mean velocity. In that case it must also correspondingly decrease the probability of the molecule possessing a velocity not considerably in excess of the mean. The mean itself may be increased, but as the present calculations are based on the hypothesis that the mean translational energy is proportional to the temperature, such an increase would be equivalent to an assumed increase of the temperature.

Now if the molecular distribution in the upper regions of the atmosphere were dependent entirely on encounters between the molecules taking place in those regions, the rare occurrence of such encounters would no doubt practically prevent their having any appreciable effect in bringing about the normal or "π" distribution; and the distribution, if it followed any law at all, would be determined by any disturbances whose aggregate effect was sufficiently marked to give rise to a "δ" function.

But it is my contention that the molecules in the upper regions in describing free paths frequently descend to the denser portions of the atmosphere where they collide with other molecules, and their place is supplied by molecules shot up from these lower regions. The causes tending to bring about the law of distribution investigated in this paper are therefore not of infrequent occurrence, and we are justified in assuming the effects of the "π" function to be considerable even in the higher strata of the atmosphere.

Going now to the disturbing causes, the following include the principal ones suggested by Dr. STONEY, viz. :—

(*a*) Tides in the atmosphere.
(*b*) Convection currents.
(*c*) Solar radiation.
(*d*) Atmospheric storms and the transferences of energy associated with them.
(*e*) Electrical disturbances giving rise to "prominences."

Of these causes, the effects of tide-generating forces still remain to be dealt with in a subsequent investigation. The same applies to convection currents, except so far as their general effects, introducing a temperature gradient, have been briefly mentioned. The effect of solar radiation is to increase the temperature of the gas on which it falls. But I freely admit that given a sufficiently high temperature helium will escape. Yet in his 1897 paper[*] Dr. STONEY assumes so low a temperature as $-66°$ C. or $207°$ absolute, while with considerably higher temperatures I find helium permanent. The question as to how the energy of solar radiation is absorbed by molecules and converted into kinetic energy of translation is one of great difficulty, and cannot be adequately discussed here, but if the view be accepted that the increase of translational kinetic energy takes place entirely at encounters, the effects of solar radiation in the upper layers of the atmosphere would seem to be small. Of the last two causes, if storms occasionally give rise to exceptionally high temperatures, it is clear again that our assumptions as to temperature conditions must be modified. Lastly, if the escape of gases be attributed to jets or prominences, the problem is

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[*] 'Scientific Transactions of the Royal Dublin Society,' vol. 6, p. 13.

removed from the field of study of the kinetic theory, and the velocities necessary for the escape of gases are largely dependent on the velocities of the jets as a whole.

But if disturbing causes of any kind have to be invoked in order to account for the escape of gases, the BOLTZMANN-MAXWELL doctrine being abandoned as insufficient, I do not see how the arguments used by Dr. STONEY, in his 1897 paper, can be regarded as conclusive. In that paper the condition for escape is made to depend on the ratio of the velocity necessary for escape to the velocity of mean square. This implies that the velocities of the molecules of a gas are distributed about the mean according to some definite fixed law, such as "MAXWELL'S Law," so that (as stated on pp. 310, 314 of his paper) a velocity of say nine times the velocity of mean square is sufficiently frequent to give rise to a marked escape of gas, while a velocity of 20 times the velocity of mean square occurs so seldom as to have no appreciable effect on the progress of events. But directly external disturbing causes are brought to bear on the question, there is no longer any necessary fixed relation between the velocities these are capable of producing, and the velocities of mean square of the molecules on which they act.

For example, if the disturbing causes take the form of jets or prominences, they will have the effect of impressing on all the molecules affected the common velocity of the jet. If several different gases occur in the same jet, the changes of velocity will bear no fixed relation to the velocities of mean square, but will be independent of the latter. Again, consider the effects of tide generating force. If on two different planets, one with a satellite and one without, the conditions were equally favourable to the permanence of a certain gas, the tide-generating force due to the satellite might remove the gas from one planet while it was retained on the other. Or again, a certain gas on one planet might, owing to the smallness of the disturbing causes, so rarely attain a velocity of 10 times the velocity of mean square that such occurrences had no appreciable influence. On another planet the disturbing causes might become so great as to frequently give the molecules a velocity of 20 times the velocity of mean square.

In connection with (2), when reading Dr. STONEY'S 1897 paper, I naturally imagined (as I believe others have done) that MAXWELL'S Law was tacitly assumed as the basis of his investigations, and my present calculations were undertaken in order to place the question on a statistical basis, in the expectation that the conclusions would confirm Dr. STONEY'S. I assumed that the object of the *à posteriori* method was to overcome a difficulty I had long felt, of drawing a hard and fast line between gases which do escape, and those which do not. The nearness of the values of the critical velocity-ratios found by Dr. STONEY for helium in our atmosphere and water on Mars, interpreted in the light of the kinetic theory, naturally justified Dr. STONEY'S inference that if the former gas escapes the latter will also escape. My calculations, however, show that on the assumed hypotheses neither gas escapes.

We are now told, however, that Dr. STONEY abandoned "MAXWELL'S Law" about thirty years ago. The *special* objections which at that time might have been raised against the law being applied to planetary atmospheres, and which arise from the necessity of taking account of the effects of gravitation, axial rotation, &c., appear to be met by the modified form of the BOLTZMANN-MAXWELL distribution used as the basis of the present investigations (§ 6. equation 9), and the arguments in § 7. There remains the more *general* objection now raised by Dr. STONEY that MAXWELL'S Law does not correctly represent the distribution in any actual gas, in support of which he remarks that the distribution according to MAXWELL'S Law is a function of one variable, while he thinks that the distribution in an actual gas may be represented by a far less simple law.

Hitherto it has been generally supposed, however, that the deviations of an actual gas from MAXWELL'S Law only become important in the case of dense gases, where the ratio of the volume of the molecules themselves to the total volume of the gas is no longer so small as to be negligible, where the time during which a molecule is encountering other molecules is not infinitely small compared with the time during which it is describing free paths, and where multiple encounters are not so exceptional that their effects may be neglected. So far as I am aware, the most successful attempt at dealing theoretically with such dense gases is Mr. BURBURY'S method, in which the view is advanced that the velocities of neighbouring molecules become correlated, and the distribution function involves two constants. It is precisely in the more rarefied portions of a planet's atmosphere that the conditions seem to me to approach most nearly to those assumed in the ordinary proofs of the permanency of MAXWELL'S Law, and of the generalised BOLTZMANN-MAXWELL distribution. Dr. STONEY'S objection appears to require either (a) that the gases in the upper atmosphere do conform much more nearly to MAXWELL'S Law than the denser gases near the Earth, and that the deviations in the latter are so great that experiments made with them lead to the velocity of mean square in the upper regions of the atmosphere being greatly under-estimated, or (b) that the Kinetic Theory of Gases must be abandoned in just those cases in which we have been accustomed to regard it as being least open to objection.

With regard to the contention that these deviations have been omitted from the present investigation, there do not as yet appear to be sufficient data available for including them in any calculation. The present methods may be utilised in the determination of such data. By calculations in which the unknown data are omitted, combined with experimental observations, it is possible to formulate an estimate of the extent to which the BOLTZMANN-MAXWELL distribution may fail to account for any experimentally observed phenomenon. The present investigation would then become an *à posteriori* determination indicating the extent to which the omitted causes must be invoked, and it would thus afford an estimate of their magnitude.]

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II. *Energy of Röntgen and Becquerel Rays, and the Energy required to produce an Ion in Gases.*

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The primary object of the investigations described in this paper was the determination of the amount of energy required to produce a gaseous ion when Röntgen rays pass through a gas, and to deduce from it the energy of the radiation emitted per second by uranium, thorium, and other radio-active substances. In order to determine the "ionic energy" (as it will be termed for brevity), it has been necessary to make a special investigation to measure accurately the heating effect of X rays when the rays are absorbed in metals, and also the absorption of the rays in gases.

The method employed to determine the ionic energy was briefly as follows:—The total energy of the rays emitted per second was determined by measuring the heating effect of a known proportion of the rays when absorbed in a metal. The total number of ions produced by complete absorption of the rays in the gas was deduced from measurements on the current produced by the ionization of a known volume of the gas and of the absorption of the rays in the gas, assuming the value of the ionic charge recently determined by J. J. Thomson.

On the assumption that all the energy of the X rays is absorbed in producing ions in the gas, the total energy of the rays, divided by the total number of ions produced, is a measure of the energy required to produce an ion.

In the course of the investigation the following subjects have been considered:—

(1.) Measurement of the heating effect of X rays and the total energy of the rays emitted per second.

(2.) Efficiency of a fluorescent screen excited by X rays as a source of light.

(3.) Absorption of X rays in gases at different pressures.

(4.) Energy required to produce an ion in gases, with deductions on—

(*a.*) Distance apart of the charges of ions in a molecule.

(*b.*) Minimum potential required to produce a spark in the gases.

(5.) Rate of emission of energy from the radio-active substances, uranium, thorium, radium, and polonium.

Heating Effect of X Rays.

Experiments on the heating effect of Röntgen radiation have been made by DORN.[*] The rays were partly absorbed in metal foil placed in one bulb of a differential air-thermometer. The heat absorbed by the metal was communicated to the gas and the resulting change of volume observed. In order to obtain a measure of the heat supplied, the heating effect due to a current in a wire placed inside the bulb was observed.

MOFFAT[†] has deduced the energy of X rays from photometric comparisons of a fluorescent screen with the Hefner amyl lamp, *assuming* the efficiency of a fluorescent screen excited by X rays as a source of light. Knowing the value of the energy of the visible light of the Hefner standard, the heating effect of the rays can be deduced.

In determining the heating effect of the rays, difficulties arise from which measurements of the heating effect of weak sources of visible light are free. In the first case, the inconstancy of an X-ray bulb as a source of radiation for measurements extending over long intervals is always a cause of trouble. In the second place, the X rays are only slightly absorbed in thin metal foil, while light rays are completely absorbed at the surface of thin metal coated with lampblack. Only a small portion of the energy of the rays is absorbed in passing through thin metal foil, and in consequence a bolometer like LANGLEY'S, where the change of resistance of a very thin metal sheet, due to heat supplied by the rays, is observed, is not very suitable for measurements on the energy of X rays. Ordinary thermopiles are open to grave objections, as will be explained later in this paper.

In order to measure the heating effect of the rays, a specially designed platinum bolometer was employed, and the heating effect was determined from the change of resistance of the platinum.

Description of Bolometer. (Fig. 1.)

A platinum strip, about 3 metres long, ·5 centim. wide, and ·003 centim. thick, was wound on an open mica frame made as light as was compatible with rigidity. The frame was 10 centims. square, and of a shape shown in fig. 1 (*a*). The platinum strip was wound round and round the frame, the strips on the front of the frame partly overlapping the corresponding ones at the back, but not touching them. The platinum strip (fifteen complete turns in all) was held in position by notches in the side of the mica frame, and the distance between each turn of the strip was 1 millim.

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[*] 'Wied. Annal.,' vol. 63, p. 150.

[†] 'Roy. Soc. Edin. Proc.,' 1898.

Two of these grids were constructed as similar as possible, and mounted in the same vertical plane on a wooden base.

Resistance of each grid = 4·2 ohms.

Area of platinum surface of grid = 92·2 sq. centims.

Fig. 1(*a*).

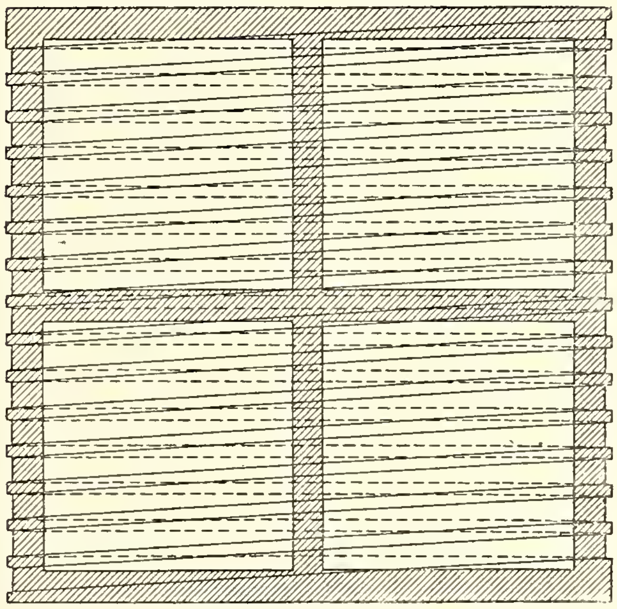
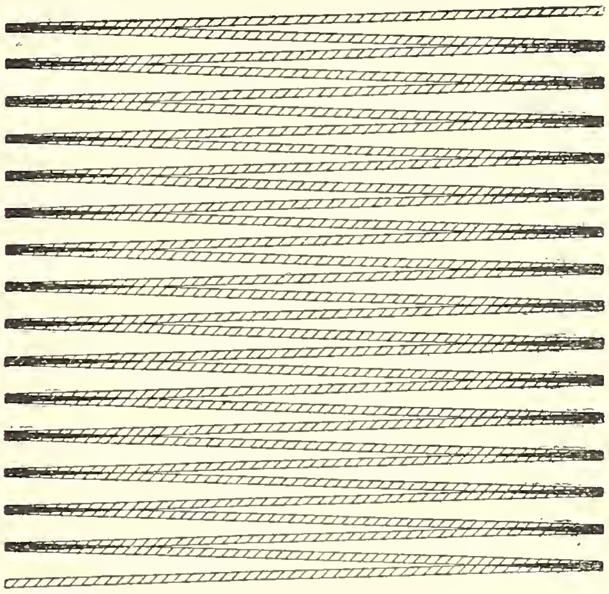


Fig. 1(*b*).



The X rays incident on the grid for the most part passed through two thicknesses of platinum, but, on account of the windings not completely overlapping, the rays in some portions passed through one thickness only. This was clearly shown in an X-ray photograph of the grid, which is sketched in fig. 1 (*b*), where the shaded portions are the areas where the rays only passed through one thickness of the platinum.

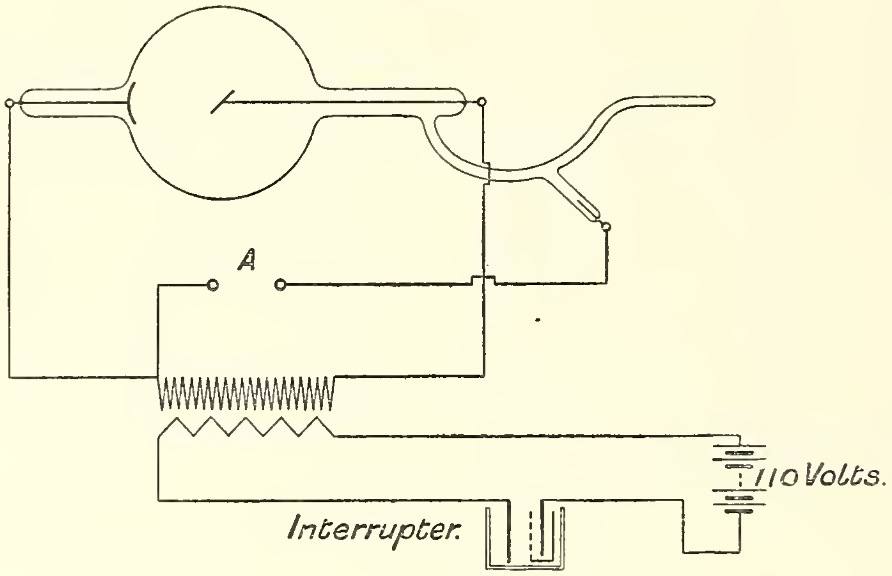
The absorption of the rays in the mica frame was very slight, and it was only on very careful inspection of the photograph that the outline of the frame could be observed. For the X rays employed, the intensity was cut down to ·45 of its value after passing through the grid.

Focus Tube.

The rays were excited in an automatic focus tube of the pattern shown in fig. 2, with a platinum anode and an aluminium cathode. The tube was excited by a large coil, using a Wehnelt interrupter on a 110-volt circuit. The alternative spark gap A was always kept the same length—about 5 inches. The bulb was a very hard one, and there was generally a fairly rapid succession of sparks across A during the working of the bulb in order to keep the vacuum constant. The constancy of the length of the spark A is of great importance in these experiments, in order to obtain rays of the same degree of penetration. A diminution of the spark length lowers the vacuum of the gas in the tube and produces rays of lower penetrating power.

After working the coil for 10 seconds, the platinum plate became red-hot and remained fairly constant during the next 30 seconds. It was generally found advisable to run the bulb at intervals for a quarter of an hour before beginning measurements, in order to get it into a steady state for the emission of rays of constant intensity. Under these conditions experiments could be made from day day with a maximum variation of intensity of 30 per cent., and generally with much less.

Fig. 2.



The bulb employed gave out rays of great intensity and great penetrating power. A fluorescent screen was brightly lighted at a distance of 20 feet from the bulb. With a "soft" tube and less intense rays, it would have been difficult to measure the heating effect with accuracy.

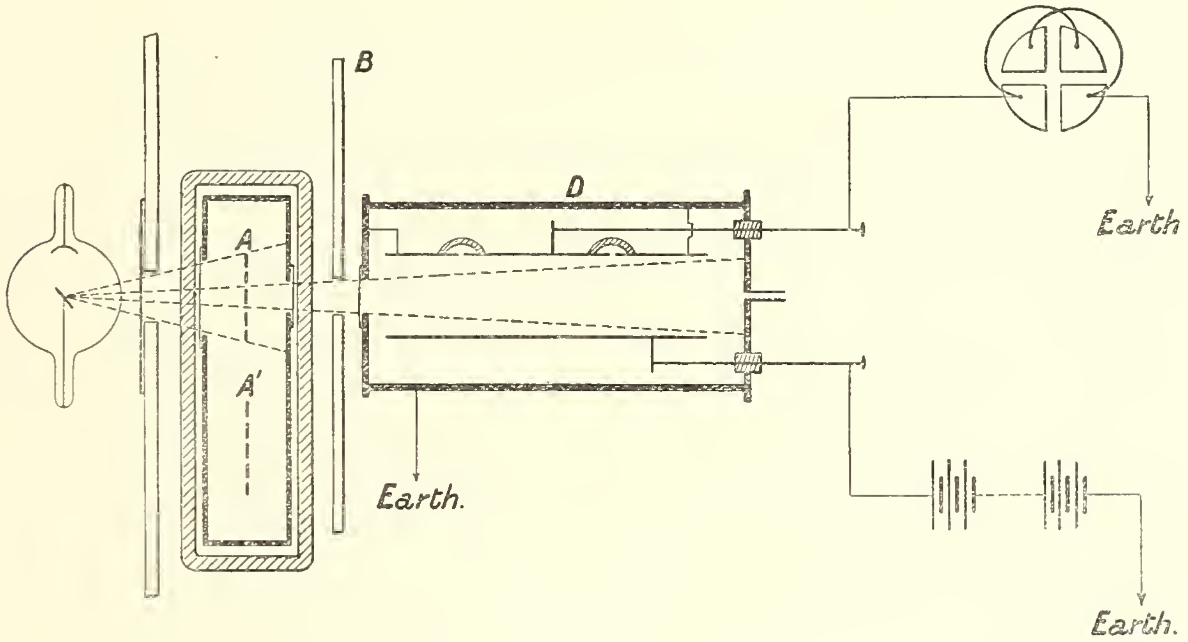
Wehnelt Interrupter.

The Wehnelt interrupter was of a simple pattern. One lead plate was placed inside a thick glass vessel (a Leclanché cell was used) with three holes about 1 millim. in diameter bored in one side. The glass vessel was placed inside an ebonite box with a lead electrode, and was filled with dilute sulphuric acid in the usual manner The acid was kept cool by having a water circulation through a coil of lead pipe in the ebonite box. By suitably tapering the holes in the glass the interrupter was made to work steadily at a slow speed and gave strong discharges in the coil. The E.M.F. employed was 110 volts and the current was about 15 amperes. The average number of breaks per second was 57. In the course of more than six months' work the glass vessel was only replaced once, on account of the gradual increase of the diameter of the holes.

Arrangement of Apparatus.

Fig. 3 shows the general arrangement of the experiment. The bulb and coil were completely enclosed in a small lead-covered room connected to earth. The rays passing through a circular hole in the lead, covered with aluminium, fell on one of the platinum grids A. A pencil of the rays, after traversing the grid, passed through a rectangular hole in the thick lead plate B, and made the air a partial conductor inside the discharge cylinder D. The vessel D merely served as a means of testing the constancy of the rays given out by the bulb by noting the current produced between the charged electrodes. The discharge apparatus will be described in detail later.

Fig. 3.



The two grids A and A′ formed two arms of a Wheatstone bridge (fig. 4). The other two arms were formed by a manganin cylinder potentiometer of 22 ohms, corresponding to a length of about 25 metres of wire. A sensitive low-resistance galvanometer was employed, and the deflection read with a telescope and scale.

The balance was first obtained for a momentary passage of the battery current. The rays were then turned on for a given time, generally either 30 or 45 seconds. The rays falling on the grid A were partly absorbed and heated the platinum to a slight extent, and the resistance consequently changed. The deflection from zero was noted immediately after the cessation of the rays.

In order to obtain a measure of the heating effect, a steady current was sent through the grid A for the same time as the rays acted. The magnitude of this current was adjusted until the deflection from zero was the same as for the rays.

When this is the case, the amount of heat, H, supplied per second by the rays is equal to the amount of heat generated by the current:

$$H = \cdot 24\, i^2 R \text{ gramme calories,}$$

where i = current through the grid,
and R = resistance of the grid.

The heating effect due to the rays was small and consequently care had to be taken to avoid disturbances of the balance due to outside causes. The grids were enclosed in a lead vessel with an aluminium window in front of the grid A. A thick covering of felt completely enclosed the lead vessel. Between the bulb and the grid there was one plate of aluminium 1 millim. thick and two sheets of thin aluminium, besides the felt covering. A lead screen in addition could be placed over the hole H. When the hole was covered thus with the lead plate, there was no disturbance of the zero, showing that the rays falling on the grid were responsible for the heating effect and the rays alone.

In practice it was found necessary to remove the sensitive astatic galvanometer employed a considerable distance away from the induction coil before the magnetic disturbances due to it were negligible. This necessitated additional leads, and in consequence more troublesome changes of the balance. For the most part, however, the changes of the balance point were gradual and, if necessary, could be accurately allowed for during the short time the grid was exposed to the rays.

Two observers were required, one to start and stop the rays and to note the electrometer deflections, and the other to observe the galvanometer deflections. With the aid of a simple system of signals, the experiments presented no serious difficulty.

In order that the rays should be, as far as possible, of constant intensity when falling on the grid, the hole in the lead plate between the bulb and the grid was covered with a lead screen, operated from a distance, during the first 15 seconds after the rays were turned on. By means of a cord the screen was suddenly removed and the rays stopped after a definite time by breaking the current.

The following table is an example of a succession of observations extending over several hours:—

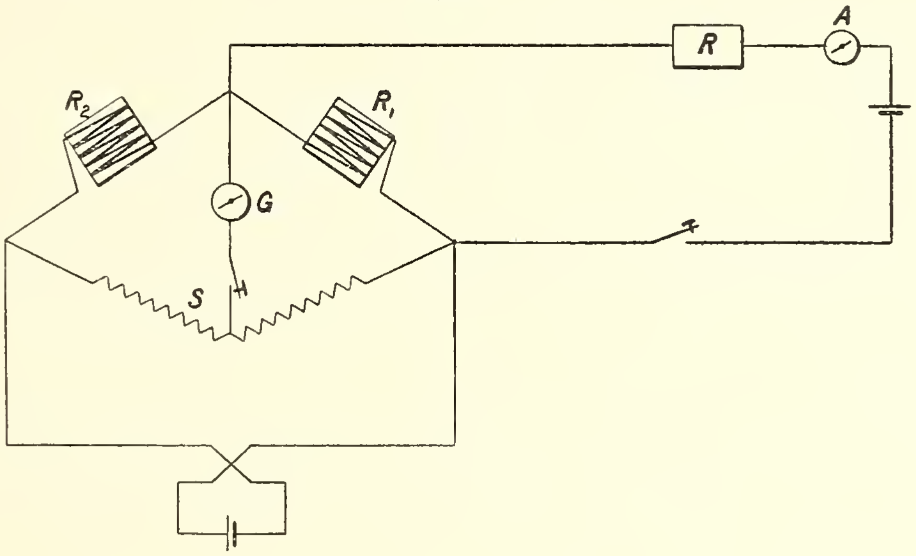
Time of exposure to the rays = 30 seconds.
Testing current through the grids = ·04 ampere.

Deflection of galvanometer in millims.	Deflection of electrometer in scale divisions per second.
17·0	5·17
18·6	5·53
18·4	5·33
17·8	5·27
Mean value = 17·9.	Mean deflection = 5·32

In order to measure the amount of heat corresponding to this deflection of the galvanometer, a steady current from a separate battery was passed through the grid previously exposed to the rays and for the same time.

Fig. 4 shows the connections. As it was necessary to determine the change of zero *immediately after* the passage of the current for a definite time, the arms of the bridge were undisturbed, and consequently a portion of the heating current passed through the grid R_2.

Fig. 4.



Resistance of grid R_1 exposed to rays = 4·2 ohms.

,, second grid R_2 = 4·28 ohms.

Total resistance of the other two arms and leads, S = 23·04 ohms.

If i be the current from the battery supplied for heating purposes then,

Current through R_1 = ·867 i.

,, ,, R_2 = ·133 i.

Heating effect on R_1 = ·752 i^2 gramme calories.

,, ,, R_2 = ·018 i^2 ,,

Difference in amount of heat supplied to R_1 and R_2 = ·734 i^2.

We thus see that most of the heating effect is confined to the grid R_1.

From a special series of experiments it was found that the deflection from zero of the galvanometer in a given time due to the heating by the current was very closely proportional to the square of the current. It was therefore not necessary to find experimentally the exact value of the current to give the same deflection as the rays, but from observations on one known current the results could be obtained by interpolation.

It was found that under the same conditions as the table given above, a current i = ·0200 ampere gave a mean deflection from zero in 30 seconds of 37·4 divisions. The mean deflection due to the rays was 17·9 divisions.

Thus the amount of heat supplied to the grid per second by the rays

$$= \frac{17{\cdot}9}{37{\cdot}4} \times ({\cdot}02)^2 \times {\cdot}734 = {\cdot}000141 \text{ gramme calorie.}$$

In the first stage of the investigation a null method was employed to measure the heat of the rays, but unexpected difficulties arose, and the method was abandoned. The battery current was kept steadily flowing through the grids, and the balance obtained. During the time the rays were on, a portion of the current through the grid was shunted through a resistance of known value. The value of the shunt resistance was adjusted until there was no change ot the balance immediately after the rays were stopped and the shunt circuit broken.

It was difficult, however, to obtain satisfactory results, partly on account of the inconstancy of the rays, but chiefly on account of the slight difference of heating effect of the two grids for equal currents. The strength of current through the grid was generally ·04 of an ampere, and with this current the inequality of the grids was immediately seen by a change of balance, when the current was applied for some time. The addition of a shunt to one grid caused a variation of current through both grids, and the change of temperature, due to inequality of the grids, introduced an error which was not negligible compared with the small heating effect of the rays. The method was not so rapid or certain a the one finally employed.

Measurement of Heating Effect by Thermopile.

Some experiments were made to see it a thermopile was suitable for a measure of the heating effect of X rays. The only thermopiles in the laboratory were of the ordinary solid type of 65 bismuth antimony couples. The thermopile was placed inside a metal tube covered with aluminium at one end and a rock-salt plate at the other. With a sensitive low-resistance galvanometer a deflection of 15 millims. could be obtained in 30 seconds. The rate of supply of heat was standardised by using a standard Hefner amyl-acetate lamp, the total radiation from which has been determined in absolute measure by TUMLIRZ.[*] It was observed that the thermopile, when exposed to the X rays, took up its final temperature very much more slowly than when exposed to the radiation from the lamp, and that the results obtained differed considerably from the bolometer method. The cause of the discrepancy lies in the unsuitability of a solid thermopile for measurements on X rays. The radiation from the lamp falling on the lampblack coating of the thermopile is absorbed at the surface of the metal, while the X rays penetrate a distance of the order of 1 millim. before much of the energy is absorbed. On account of this, the maximum rise of temperature near the surface of the junction on which the E.M.F. depends is less

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[*] 'Wied. Annal.,' vol. 38, p. 640.

with X rays than with light for equal intensities of radiation. The deflection of the galvanometer is thus less for X rays than for light-waves of equal energy.

From these considerations it is obvious that the solid type of thermopile is most unsuitable for such work; but a modified thermopile of thin plane sheets of metal, *e.g.*, iron and constantin, would probably give better results and be simpler to manipulate than the bolometer. With thin sheets the heat would be equally distributed over the cross-section due to diffusion, and no appreciable error would arise. The method, however, has the objection that the amount of heat must be standardised by a known lamp or source of radiant energy.

Total Energy of the Rays emitted per Second.

When X rays fall on a metal plate, the plate is heated, and the question at once arises whether we are justified in assuming that the energy of the rays stopped by the metal plate is transformed into heat in the plate. The experiments of PERRIN, SAGNAC, and J. J. THOMSON have clearly shown that when X rays strike a solid body, secondary rays are set up which ionize the gas and act on a photographic plate. These secondary rays are of a far less penetrating character than the rays that excited them; but on account of the ease with which they are absorbed in the gas, the amount of ionization per cub. centim. in the gas near the surface of the body may be greater than that due to the direct rays. The total number of ions produced by the scattered rays depends to a great extent on the density of the metal as well as on the intensity of the incident rays. The total number of ions produced by complete absorption of the scattered rays is generally only a small proportion of the number produced by complete absorption of the direct rays. Assuming that an ion in both cases requires the same expenditure of energy to produce it, the energy of the scattered rays is thus only a small proportion of the total energy of the incident rays.

The secondary rays are set up both at the points of incidence and emergence of the rays falling on the grid. The heating effect on the grid is thus less than the heat equivalent of the energy of the rays stopped by the grid by the portion of the energy used up in exciting secondary rays. The correction is probably small, and has been neglected in these experiments, but it is hoped in a future investigation to determine its value.

There is no evidence that the chemical energy of platinum is in any way altered by the passage of the rays through it, and, as far as our present knowledge goes, the energy of the rays stopped minus the energy of the scattered radiation, is transformed into heat within the platinum.

In a case where there is a chemical change, *e.g.*, when the rays fall on a photographic film, the heating effect would not be the equivalent of the energy absorbed.

It has been shown by RÖNTGEN and other observers that the intensity of the rays given out from the front surface of a platinum plate of a focus tube is approximately equal in all directions. In the experiment the rays fell normally on the centre of the grid, but on account of the size of the grid, the intensity of the rays could not be considered constant over its surface. The intensity of the rays diminishes, and the obliquity of the angle of incidence increases from the centre of the grid outwards. In consequence of this, a greater proportion of the incident radiation is absorbed at the edges than at the centre.

The intensity of the rays was cut down to about ·45 of its incident value in passing normally through the grid. It can be shown, by approximate integration over the surface of the grid, that for the distance of the grid from the source of the rays, namely, 26 centims., and the dimensions of the grid, the actual energy absorbed is about 2 per cent. less than if the rays had the same intensity over the surface of the grid as at the centre, and had fallen normally at all points of the grid.

For the special bulb employed, it was shown that the rate of supply of heat to the grid was equal to

$$\cdot 00014 \text{ gramme calorie per second.}$$

This corresponded to a maximum rise of temperature of about 1/200° C.

Distance of the centre of the grid from the source of the rays = 26 centims.

Area of grid = 92·2 sq. centims.

Now ·55 of the incident radiation was absorbed in the grid.

Total energy of the rays falling on the grid is approximately

$$= \cdot 00025 \text{ gramme calorie per second.}$$

Therefore the total heating effect due to all the rays emitted from the front of the plate (omitting absorption in the glass, air, and screens)

$$= \frac{2\pi \times (26)^2}{92\cdot 2} \times \cdot 98 \times \cdot 00025 = \cdot 011 \text{ gramme calorie per second,}$$

$$\text{or } \cdot 046 \text{ watt.}$$

Now the number of discharges per second in the bulb was 57, and TROUTON[*] has shown that the duration of the rays during each discharge of an induction coil is less than 10^{-3} second, and probably about 10^{-4} second. Assuming the average duration of the rays for each discharge is 10^{-4} second, the rate of emission of energy while it lasts

$$= 1\cdot 95 \text{ gramme calorie per second.}$$

The heating effect of the sun's rays falling normally on 1 sq. centim. surface is about

$$= \cdot 035 \text{ gramme calorie per second.}$$

The maximum rate of emission of energy as X rays from the bulb is thus about 56 times greater than the amount of energy per sq. centim. due to the sun's rays.

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[*] 'Brit. Assoc. Report,' 1896.

Efficiency of a Fluorescent Screen as a Source of Light.

Experiments were made to determine the efficiency of a fluorescent screen as a transformer of Röntgen radiation into visible light.

Photometric observations of the light emitted by a fluorescent screen, excited by X rays, have been made by A. MOFFAT,[*] who deduced the energy of the rays, by assuming that the coefficient of transformation of the energy into visible light was 4 per cent., the value found by E. WIEDEMANN[†] for the transformation of radiant energy into luminescence.

It was not the object of this investigation to make a complete photometric comparison, but to deduce an approximate coefficient of transformation for a definite experimental arrangement which could readily be reproduced in practice. For this purpose a piece of fluorescent screen was placed over one of the diffusive surfaces of a Lummer-Brodhun screen, and the diffused light of the screen compared with the diffused light of the amyl lamp in the usual manner. The ratio of the square of the distance of the screen from the bulb to the square of the distance of the lamp was taken as the ratio of the intensities of the light emitted by the X-ray bulb and lamp. In this case the amount of the light of the amyl lamp absorbed in the plaster of Paris surface was neglected.

Dr. E. SUMPNER[‡] has shown that a piece of blotting-paper reflected over 80 per cent. of the light incident upon it, and it was found experimentally that the plaster of Paris surface of the screen was a still better reflector.

The current in the discharge vessel was determined, during the measurements of the energy, in absolute measure, and also during the comparison of the screen with the amyl lamp, in order to correct for changes of intensity of the rays during the observations. In this way it was found, using a platino-barium cyanide screen, that

$$\frac{\text{Intensity of light from fluorescent screen}}{\text{Intensity of light from amyl lamp}} = \cdot 0206.$$

Now if the intensity, I, of the visible light from a Hefner amyl lamp is given by

$$I = K/r^2$$

from the experiment of TUMLIRZ[§] the value of K for the visible light is equal to

$$\cdot 00361 \text{ gramme calorie per second,}$$

and the total energy radiated by the lamp is 41·1 times the energy of the light radiation alone.

Now in the experiments with the bolometer the heating effect of the rays incident

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[*] 'Roy. Soc. Edinburgh Proc.,' 1898.

[†] 'Wied. Annal.,' vol. 37, p. 233.

[‡] 'Phil. Mag.,' February, 1893.

[§] 'Wied. Annal.,' vol. 38, p. 640.

on the grid, area 92·2 sq. centims., at a distance of 26 centims. from the source of rays, for the same strength of rays as those incident on the fluorescent screen, was ·00032 gramme calorie per second.

$I = K/r^2$, and the value of K, which represents the amount of energy due to the rays falling normally on a surface of 1 sq. centim. at a distance of 1 centim. from the source of rays

$$= \cdot 0023 \text{ gramme calorie.}$$

The intensity of the X rays was thus ·64 of the intensity of the visible light of the standard Hefner lamp.

Now the efficiency of transformation of X rays into light

$$= \frac{\text{energy radiated as light}}{\text{energy supplied by the rays}} = \frac{\cdot 0206 \times \cdot 00361}{\cdot 73 \times \cdot 0023} = \cdot 044,$$

since it was found electrically that ·73 of the rays were absorbed in the screen.

The efficiency of transformation is thus

4·4 per cent.

If we assume that 85 per cent. of the incident light is diffused from the surface of the Lummer-Brodhun screen, the efficiency of transformation is about 3·7 per cent.

A calcium tungstate screen, in which the absorption was ·36, gave almost the same efficiency of transformation.

The results we have obtained afford a simple means of expressing the intensity of X rays in absolute measure, assuming the coefficient of transformation of a fluorescent screen to be about 4 per cent.

Two experiments would be necessary—

(1.) The intensity of the light from the screen would be compared with a Hefner standard lamp.

(2.) The absorption of the rays by the screen would be measured electrically or photometrically by placing a portion of the screen to absorb the rays.

Let I_1 and I_2 be the intensities of X rays and a Hefner standard lamp in absolute measure, disregarding absorption of rays in glass, metal screens, air, &c. When there is equality of illumination let r_1 and r_2 be the distances of the source of rays and lamp from the Lummer-Brodhun screen.

Let r_1 and r_3 be the distances for equal illumination when the rays pass through a piece of the screen before falling on the Lummer-Brodhun screen.

Then ρ, the ratio of transmitted to incident rays for the fluorescent screen, is given by

$$\rho = r_2^2/r_3^2.$$

Ratio of incident energy absorbed $= 1 - \rho$.

Therefore it is readily seen that

$$\frac{I_1}{I_2} = \frac{r_1^2}{r_2^2(1-\rho)} \times \frac{100}{4{\cdot}4},$$

since efficiency is 4·4 per cent.

Thus, since $I_2 = {\cdot}00361$ gramme calorie,

$$\therefore \quad I_1 = {\cdot}082 \frac{r_1^2}{r_2^2(1-\rho)} \text{ gramme calorie.}$$

Thus two simple photometric comparisons would be required to express the energy of the radiation of any particular bulb in absolute measure.

For penetrating rays the absorption in the cardboard of the screen is negligible, but if necessary it can readily be allowed for.

The chief source of difficulty in the comparison is the difference in colour between the light from the Hefner lamp and a fluorescent screen. The fluorescent light appears a greenish-blue, and the amyl lamp a reddish-yellow, when seen side by side in the telescope of a Lummer-Brodhun screen.

Some experiments were made using a coloured glass to make the sources of light more nearly a match in colour. A greenish coloured glass was found to give a good colour match when interposed between the screen and amyl lamp. On determining by means of a thermopile the amount of the visible energy of the Hefner lamp which was allowed to go through the glass, it was found to be so small (less than 2 per cent.) that a special investigation was required to find the coefficient of transmission with accuracy. The experiments were not continued, owing to lack of time, but the evidence showed that by using a coloured screen in the path of the amyl lamp to give the same tint as the fluorescent screen, the efficiency of the transformation of the screen was much higher in that case than the 4 per cent. obtained by using no coloured glass or solution.

Energy required to produce an Ion.

The method employed to determine the energy required to produce an ion in gases exposed to Röntgen rays depended on the measurement of the heating effect of the rays, and of the total number of ions produced by the radiation in the gas. If H is the number of calories given out per second by the rays, and E the energy in ergs of the rays emitted per second, then

$$E = JH \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

If W is the average amount of energy required to produce an ion, then

$$nW = E \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

where n is the number of ions produced per second, supposing that all the energy of the rays, absorbed in the gas, is due alone to the production of ions.

In order to determine the number of ions, n, it is necessary to measure the maximum current that can be produced between two electrodes when all the ions produced by the rays in the gas reach the charged electrodes before there is any appreciable loss of their number due to recombination. If i is the maximum or *saturation* current through the gas, then

$$i = n\epsilon,$$

where ϵ is the charge on an ion.

The value of ϵ has been determined by J. J. THOMSON,[*] and is equal to $6\cdot5\ 10^{-10}$ electrostatic unit.

From (1) and (2)

$$n\mathrm{W} = \mathrm{JH},$$

therefore

$$\mathrm{W} = \frac{\mathrm{JH}\epsilon}{i}.$$

In order to determine W it is thus necessary to determine the value of H and i.

The considerations on which the method is based are :

(1.) When the X rays are absorbed by a solid substance, the greater proportion of the energy is given up to the substance in the form of heat.

(2.) The energy of the rays absorbed in passing through a given volume of the gas is used up in producing ions.

(1) has been considered earlier in the paper, and it has been shown that we are probably justified in assuming that a very large proportion of the energy due to rays absorbed in a substance like platinum is transformed into heat. A small proportion of the total energy is used up in setting up secondary rays at the point of incidence of the rays on a solid conductor and also at the point of emergence.

In regard to (2), one of the authors[†] has previously shown that the absorption of the rays in a gas is roughly proportional to the intensity of the ionization in the gas. Gases and vapours, which are made good conductors by the rays, also strongly absorb them. The absorption of the rays in the gas has no direct connection with the molecular weight or density of the gas. For example, in hydrochloric acid gas the rays are far more readily absorbed than in carbonic acid, a gas of greater density.

PERRIN has shown that the ionization of a gas is approximately proportional to the pressure. This result has been confirmed by us, and the authors have also found that the absorption of the rays varies directly as the pressure, *i.e.*, as the ionization of the gas. These results point to the conclusion that the absorption of the rays in a gas is closely connected with the number of ions produced. It is possible that there is a certain amount of scattering of the rays in passing through a gas, but if the apparent absorption of the rays were due in any great measure to scattering, we

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[*] 'Phil. Mag.,' Dec., 1898.

[†] RUTHERFORD, 'Phil. Mag.,' April, 1897.

should expect the absorption to depend chiefly on the density of the gas, and such is not the case. RÖNTGEN and others have observed that the gas itself which has been acted on by the rays gives out a radiation which is able to light up a fluorescent screen. This radiation may be due either to the scattering of the rays, or to the radiation caused by the recombination of the ions. In either case it is probable that the radiation is of a type similar to the secondary radiation set up at the surface of metals when X rays impinge upon them. This secondary radiation is far more readily absorbed in gases than the primary radiation, and would be absorbed in producing ions in the gas. The rate of discharge would be increased, and provided all the scattered, or secondary, radiation were used up in producing fresh ions between the electrodes, no correction for the amount of scattered radiation would be required. This of course proceeds on the assumption that the ions produced by the primary and secondary radiation are the same, and require the same amount of energy in each case to produce them.

It is not practicable to measure directly the total maximum current through the gas, due to the passage of all the ions produced between charged electrodes, as the rays could pass through several hundred metres of the gas before approximately complete absorption took place.

In practice the number of ions produced in a known small volume of the gas is determined, and also the coefficient of absorption of the rays by the gas. The total number of ions that would be produced, provided all the rays were absorbed, can be directly calculated.

An account will now be given of the experiments performed to measure the absorption of the rays in gases.

Absorption of the Rays in Gases.

The bulb employed gave out rays of great intensity and penetrating power, and the absorption of the rays in air was small. About 3 per cent. of the rays were absorbed in passing through a metre of air at atmospheric pressure and temperature. In order to measure the absorption, a delicate null method was employed. No direct method can be employed on account of the smallness of the absorption and the variation of the intensity of the rays during the experiments.

Fig. 5 shows the general arrangement of the apparatus. A similar method was employed by one of us* on a previous occasion to measure the absorption of the rays in gases.

Two long brass tubes, A, A′, 118 centims. long and 3·4 centims. in diameter, were placed horizontally at a slight angle to each other, and in such a position with regard to the bulb that the axes of the two tubes met at a point on the surface of the platinum plate of the anticathode. The ends of the tubes were covered with

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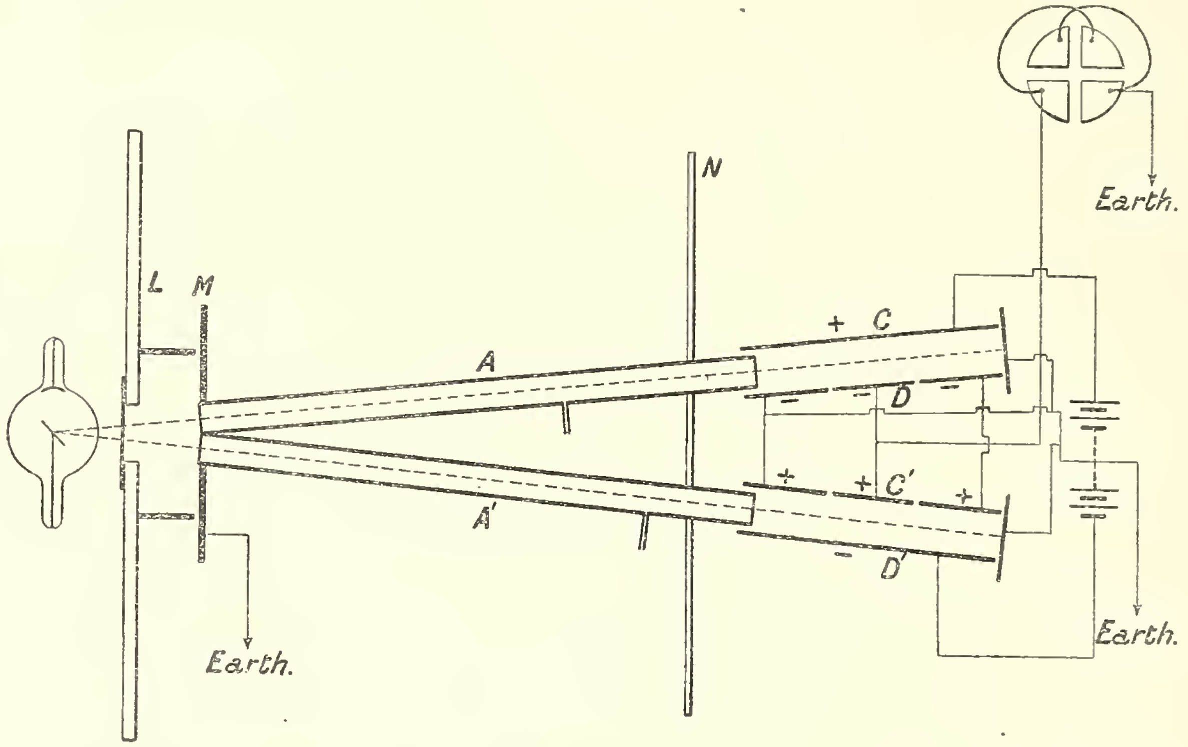
* E. RUTHERFORD, 'Phil. Mag.,' April, 1897.

aluminium caps 1 millim. in thickness, and were made air-tight and capable of standing a pressure of 3 atmospheres. The rates of discharge due to the rays after passing through the tubes, were taken between two sets of parallel plates, CD and C′ D′. The plates D and C′ were of the same size and cut into three portions, of which the centre plates were carefully insulated. The centre plates were thus surrounded by a guard ring, and the rates of discharge to the centre plates alone were measured. The centre plates, D and C′, were connected together, and to one pair of quadrants of the electrometer, the other pair of quadrants being connected to earth. The plates, C and D′, were connected to the terminals of a battery of small storage cells of 310 volts, *the middle point of which was to earth.* The electrometer will show no deflection if the intensity of the rays between C and D is exactly equal to the intensity between C′ and D′, since the current between C and D is equal and opposite to the current between C′ and D′.

Fig. 5.



Lead screens, L, M and N, were placed at the positions marked in the figure, in order to prevent any stray radiation from reaching the testing plates. The wires leading to the electrometer were enclosed in metal tubes, which were connected to earth in order to avoid any loss of charge due to stray radiation or disturbances by any electrostatic field. The electrometer was completely surrounded by a wire gauze The separation of the quadrants of the electrometer was operated from a distance by means of suitable keys. Such precautions are very necessary during the very dry Canadian winter, when the slightest movement causes frictional electrification. The table and the woodwork on which the apparatus was placed was covered over with metal, to prevent the collection of charges either from frictional electrification or the action of the rays near charged conductors.

The tubes were first adjusted so that the rays caused no movement of the electrometer needle. The tube, A, was then rapidly exhausted by means of a Fleuss pump. The intensity of the rays after emerging from the tube was thus greater than for the tube A′ on account of the less absorption, and the electrometer therefore showed a deflection. If the tube A′ were exhausted and the tube A filled with air, the electrometer gave a deflection in the opposite direction. If the end of one of the tubes was closed with a thick lead plate so that no rays could get through, then the rate of movement of the electrometer needle corresponded to the intensity, I, of the rays after emergence from the other tube.

If λ is the coefficient of absorption of the rays in the gas, then the intensity of the rays after passing through a distance d of the tube is $e^{-\lambda d}$ of its value if there had been no absorption.

Since the currents between C and D and between C′ and D′ are proportional to the intensities of the radiations between the plates, then

$$\frac{\text{Difference between currents}}{\text{Total current}} = \frac{\mathrm{I} - \mathrm{I}e^{-\lambda d}}{\mathrm{I}}$$

$$= 1 - e^{-\lambda d}$$

$$= \lambda d, \quad \text{if } \lambda d \text{ is small.}$$

In order to determine λd, we thus require the ratio of the number of divisions per second, given by the electrometer needle from the balance when one of them is exhausted, to the number per second when the end of the tube containing the gas is covered with a thick lead plate.

The following table gives the results for air at pressures in one tube ranging from ·5 of an atmosphere to 3 atmospheres, the other tube being exhausted :—

Difference of pressures in atmospheres.	Number of divisions in 5 secs. with one tube screened.	Deflection from balance in 20 secs.	λd.
·5	—	—	·0187
1	160	21·8	·034
2	169	48·0	·071
3	172	70·0	·102

The above results are the mean values of a series of measurements. The results for ·5 of an atmosphere were obtained at a different time from the others and with a different sensitiveness of the electrometer. The table shows that the absorption of the rays in the gas is approximately proportional to the pressure.

The value of d was 118 centims.

The value of λ for different pressures is thus given by the following table :—

Pressure in atmospheres.	Value of λ.
·5	·000158
1	·000288
2	·00060
3	·00086

The value of λ at atmospheric pressure and temperature obtained for the same bulb after daily use for two months was found to be ·000270. The value of λ at atmospheric pressure and temperature in the calculations is taken as the mean of these two values, and is thus given by

$$\lambda = {\cdot}000279.$$

It is probable that the rays were not homogeneous, and the value of λ must be considered as the mean value for the different kinds of rays. On account of the very small absorption of the rays it was difficult to determine with accuracy the absorption for pressures lower than half an atmosphere. The results, however, indicated that the absorption was, roughly, proportional to the pressure for still lower pressures.

The value of λ was determined for carbonic acid gas at normal pressure and temperature. The absorption was 1·59 times that of air, and the value of λ was found to be ·000457. The results were confirmed by varying the pressure of the air in one tube until there was no disturbance of the electrometer zero. The results agreed with the value obtained above, assuming the absorption in air is proportional to the pressure.

We see from the results given above that the radiation is reduced to half its value with no absorption after passing through a length of 24·7 metres of air at ordinary pressure and temperature.

The value of λ for uranium rays[*] is 1·6, or the absorption is 6000 times as great for uranium rays as for the X rays employed. The value of λ obtained some years ago for a much "softer" bulb was ·001, or about four times the absorption of the bulb employed in these experiments.

Measurement of the Current through the Gas.

In order to determine the amount of ionization in a known volume of gas, the apparatus shown in fig. 6 was employed.

The rays passed into a brass cylinder, 12 centims. in diameter and 30 centims. in length, through a rectangular orifice, O, at one end covered with an aluminium window, 1 millim. in thickness. Inside the cylinder two parallel rectangular plates,

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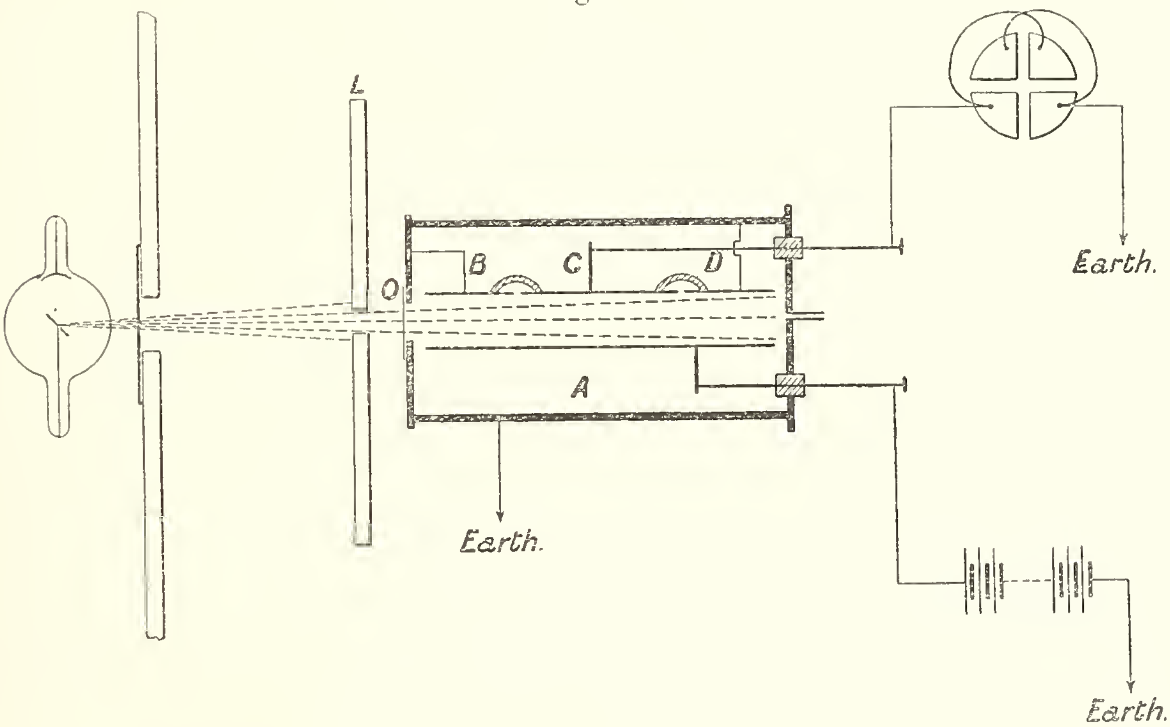
[*] E. Rutherford, 'Phil. Mag.,' January, 1899.

A and BCD, were fixed on a light wooden frame. The plate opposite to A was cut into three parts, B, C, D, and C was insulated from B and D. The plate A was connected to one pole of a battery of 310 volts, the other pole of which was connected to earth. The plate C was connected to one pair of quadrants of the electrometer, the other pair of which was connected to earth. The plates B and D were in connection with the cylinder, which was also connected to earth. The plates B and D thus corresponded to a partial guard ring for the plate C, and served two purposes. The electric field was rendered uniform from C to A, and most of the secondary radiation set up at the two ends of the cylinder was absorbed between A and B and between A and D, and thus did not produce appreciable ionization between A and C. A large lead plate, L, with a rectangular orifice, was so placed that the rays from the source passed into the cylinder, and did not fall on the parallel plates. This avoided the presence of secondary radiation. The two ends of the cylinders were covered inside with cardboard in order to make the amount of secondary radiation as small as possible. The amount of radiation set up at the surface of air and cardboard is very small. The amount of insulating material inside the cylinder was reduced as far as possible in order to avoid the collection of free charges on them, and consequent disturbance of the electric field. For this reason the plates were mounted on a wooden frame instead of an ebonite one. The wood was a sufficiently good conductor to quickly discharge any electrification that reached its surface.

Fig. 6.



The current between C and A is thus due only to the ions which were produced by the passage of the rays between them.

The length of the plate C was 12·06 centims., measuring from the centre of the air spaces. The distance between the plates was 4·16 centims. The rays, before entering the cylinder, passed through a rectangular orifice in a thick lead plate. Knowing the distance from the source of the rays and the area of the opening, the area of a section of the cone of rays at any point in the cylinder could be determined.

The results calculated in this way were compared with the area of the impression on a photographic plate at different distances from the orifice, and it was found that the correction to apply for the source of the rays, not being a point source, was practically negligible.

Let I_2 be the intensity of the rays at the beginning of the plate C, and S the area of the cross-section of the cone of rays at that point. The energy crossing the surface per second is I_2S.

If there was no absorption of energy in the gas, the energy crossing the cross-section of the cone of rays at the further end of the plate C would be the same as at the beginning. But in consequence of absorption the energy crossing the surface per second is

$$I_2Se^{-\lambda l},$$

where l = length of the plate C, and λ = the coefficient of absorption of the rays in the gas. The energy absorbed in the gas per second is equal to

$$I_2S(1 - e^{-\lambda l}) = I_2S\lambda l$$

if λl is small, as was the case in the experiments.

The current between the plates A and C was determined for a voltage sufficient to move all the ions to the electrodes before any appreciable recombination could take place.

Let n = total number of ions produced per second.
i = current between the plates through the gas.
e = charge on an ion.
W = average energy required to produce an ion.

Then
$$Wn = I_2S\lambda l \quad \text{and} \quad i = n\epsilon;$$

$$\therefore \quad W = \frac{I_2S\lambda l\epsilon}{i}.$$

The value of I_2 was determined from the heating effect of the rays, as explained in the earlier part of the paper.

Let I_1 = intensity of the rays at the surface of the bolometer.
ρ_1 = transmission ratio of the rays when passing through the platinum grid.

Then

$$\text{energy absorbed in the grid per second} = I_1A(1 - \rho_1),$$

where A = area of grid.

Assuming the value of the intensity uniform over the surface of the grid, and equal to the value at the centre, the total energy absorbed in the grid is slightly less than the above amount, and it has been shown earlier in the paper that very approximately the energy absorbed in the grid $= \cdot 98 \mathrm{I}_1 \mathrm{A}\ (1 - \rho_1)$.

In practice the current in the discharge cylinder was observed at the same time as the heating effect. The cylinder was placed behind the platinum grid in such a position that the rays entering the cylinder passed slightly to one side of the centre of the grid, thus avoiding the mica frame of the bolometer. The rays before entering the cylinder were cut down in intensity by their passage through the grid, by the enclosing envelope and the aluminium window in the discharge cylinder.

Let ρ_2 = transmission ratio of the rays through the platinum grid + the felt cover + the aluminium window, &c.

d_1 = distance of grid from source of rays.

Let d_2 = distance of the beginning of the plate C in the discharge cylinder from the source of the rays.

Then it can easily be seen that

$$\frac{\mathrm{I}_2}{\mathrm{I}_1} = \rho_2 \frac{d_1^{\,2}}{d_2^{\,2}} e^{-\lambda(d_1 - d_2)} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1).$$

The factor $e^{-\lambda(d_1 - d_2)}$ is nearly equal to unity, and is the correction for the absorption of the rays in the gas between the grid and the discharge cylinder.

If H is the number of heat units communicated to the bolometer per second, then

$$\cdot 98 \mathrm{A} \mathrm{I}_1 (1 - \rho_1) = \mathrm{JH} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

and

$$\mathrm{W} = \frac{\mathrm{I}_2 \mathrm{S} \lambda l \epsilon}{i} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3).$$

Dividing (3) by (2) and substituting the value of $\mathrm{I}_2 / \mathrm{I}_1$ from (1), we obtain

$$\mathrm{W} = \frac{\mathrm{JH}}{\cdot 98 \mathrm{A}(1 - \rho)} \cdot \frac{\mathrm{S} \lambda l \epsilon}{i} \cdot \rho_2 \cdot \frac{d_1^{\,2}}{d_2^{\,2}} e^{-\lambda(d_1 - d_2)}.$$

Determination of i.

The value of i was determined by an electrometer with an additional capacity of ·00248 of a microfarad in parallel. The heating effect on the bolometer and the quantity of electricity discharged between the plates of the cylinder were observed at the same time. A lead screen cut off the rays from the platinum grid and the discharge cylinder for 15 seconds after the bulb had started, for it was found that the rays gradually increased in intensity for the first 10 or 15 seconds. At the end of 15 seconds the lead screen was suddenly removed by a cord operated from a distance. After the passage of the rays for 30 or 45 seconds, the rays were stopped. The deflection from the zero of the bolometer was taken by one observer, while the deflection of the electrometer was taken by another. In the later experiments the capacity of the electrometer and connections was $\frac{1}{27}$ of the capacity added. and the total capacity of the circuit was ·00257 of a microfarad.

The following is an example of the determination of i in electrostatic units :—

1 Clark cell of 1·434 volts E.M.F. gave 57·0 divisions on the electrometer scale. The deflection of the electrometer due to the passage of the rays for 30 seconds was 160 divisions.

$$\begin{aligned} i &= \text{quantity of electricity per second} \\ &= \frac{\cdot 00257}{10^6} \times \frac{160}{30} \times \frac{1\cdot 434}{57} \times 3 \times 10^9 \\ &= 1\cdot 03 \text{ electrostatic units.} \end{aligned}$$

Determination of the Absorption of the Rays in the Bolometer, the Aluminium Window, &c.

The values of ρ_1 and ρ_2 were determined electrically by utilising the discharge cylinder of fig. 6. The rate of discharge in the cylinder was observed with the grid before the hole in the lead plate and then without the grid. The ratio of the currents in the two cases is proportional to the ratio of the intensities, since the ionization is proportional to the intensity of radiation. A mean value of the ratio for different portions of the grid was taken, and it was found that $\rho_1 = \cdot 453$. It is thus seen that the intensity of the radiation was cut down to a little more than half in passing through the platinum grid. The value of ρ_2 was determined in a similar manner. The intensity of the rays in this case was cut down more in consequence of passing through a thickness of felt, an aluminium window of ·1 centim. thickness, and a thin layer of aluminium, as well as the platinum grid. The value found for this ratio was $\rho_2 = \cdot 31$.

The absorption of the rays in the mica frame appears in the values of ρ_1 and ρ_2. The absorption, however, was small and practically negligible in any case.

Dimensions of the Apparatus and Values of Constants.

The area of the rectangular hole through which the rays passed into the discharge cylinder, and from which S was calculated, was 7·1 sq. centims. The length of the centre plate in the discharge cylinder was 12·06 centims. For most of the experiments, the distance d of the grid from the source of rays was 26·0 centims., and the distance of the hole in the lead plate from the source was 45·2 centims.

Area of platinum grid was 92·6 sq. centims.
Mean value of $\lambda = \cdot 000279$.
Value of $\epsilon = 6\cdot 5 \times 10^{-10}$ electrostatic unit.

The correction for the absorption in air between the grid and the discharge cylinder was negligible.

The following table gives the values of i, H, and W for different times of the exposure to the rays:—

Time of exposure to rays.	i in electrostatic units.	H in calories.	W in ergs.
45 secs.	·894	$1·49 \times 10^{-4}$	$2·22 \times 10^{-10}$
"	·976	$1·56 \times 10^{-4}$	$2·13 \times 10^{-10}$
"	1·045	$1·47 \times 10^{-4}$	$1·87 \times 10^{-10}$
"	1·115	$1·57 \times 10^{-4}$	$1·82 \times 10^{-10}$
"	·996	$1·38 \times 10^{-4}$	$1·84 \times 10^{-10}$
30 secs.	1·00	$1·34 \times 10^{-4}$	$1·79 \times 10^{-10}$
"	1·07	$1·47 \times 10^{-4}$	$1·83 \times 10^{-10}$
"	1·03	$1·45 \times 10^{-4}$	$1·87 \times 10^{-10}$
"	1·09	$1·41 \times 10^{-4}$	$1·72 \times 10^{-10}$

The mean value of W, the energy required to produce an ion in air at atmospheric pressure and temperature, is given by

$$W = 1·90 \times 10^{-10} \text{ erg.}$$

The energy required to produce a positive and a negative ion from a neutral molecule is twice this amount, and since one ion cannot be produced without the other, then $3·8 \times 10^{-10}$ erg is the smallest amount of energy that will produce ions in air.

The mean intensity of the rays in absolute measure at the surface of the bolometer is given by

$$JH = ·98AI_1(1 - \rho_1).$$

Taking the value of H as $1·5 \times 10^{-4}$ calorie, we find that

$$I_1 = 127 \text{ ergs.}$$

On account of the very short duration of the rays from each discharge, the maximum intensity of the radiation at any time is probably over a thousand times greater than the above value.

The energy absorbed per second in producing ions in the cylinder $= \frac{i}{\epsilon} W = ·29$ erg, taking $i = 1$ E.S. unit.

This absorption of energy is spread throughout a volume of over 100 cub. centims. of the gas, so that the absorption of energy per cub. centim. in the air is very small.

The value of W, the ionic energy, is seen to depend on the measurement of the current through the gas, the coefficient of absorption, and the heating effect of the rays. The absorption of the rays in the gas has to be determined separately from the current and heating effect, and an uncertainty consequently arises on account of the variation of the penetrating power of the rays as the bulb varies. The value of λ determined for the rays, under different conditions as regards the frequency of the Wehnelt interrupter, was found to be approximately the same, after the bulb had been in constant use for several months. It is probable that the type of rays does not on an average vary much from day to day, but the greatest source of error is probably due to the assumption that the rays are homogeneous in character. RÖNTGEN and others have shown, from experiments on the absorption of successive thicknesses of metal, that rays are not simple in character, but contain rays of widely different order of penetrating power, so that the value of λ is the mean value for the different types of rays.

The value of W also depends upon the value of ϵ, the charge on an ion, and if future investigations should assign a different value to ϵ, the value of W would be altered in a like ratio.

Energy required to produce an Ion in other Gases.

When the energy required to produce an ion in one gas is known, the energy required to produce an ion in another gas can be determined from the ratio of the absorptions of the rays and the intensity of ionization in the gases.

Let n_1 and n_2 be the number of ions produced per cub. centim. in two gases.

Let λ_1 and λ_2 be the coefficients of absorption.

Let W_1 and W_2 be the energies required to produce ions in the two gases.

Let i_1 and i_2 be the maximum currents through the gases.

Then for the same intensity of rays,

$$\frac{\text{absorption of energy in gas}_2}{\text{absorption of energy in gas}_1} = \frac{\lambda_2 I}{\lambda_1 I} = \frac{n_2 W_2}{n_1 W_1}$$

$$= \frac{i_2 W_2}{i_1 W_1}, \text{ since } i_1 = n_1\epsilon \text{ and } i_2 = n_2\epsilon, \text{ assuming charges on the ions are equal.}$$

Therefore
$$\frac{W_2}{W_1} = \frac{\lambda_2}{\lambda_1} \cdot \frac{i_1}{i_2}.$$

The ratios λ_2/λ_1 and i_1/i_2 can be readily determined, and if W_1 is known, then W_2 can be calculated without recourse to experiments on the heating effect of the rays in each case.

The value of λ_2/λ_1, the ratio of the absorption coefficient of carbonic acid gas to that of air, was found to be 1·59 for the rays employed. The ratio i_2/i_1 of the current in air and carbonic acid gas for a potential difference of 300 volts was found to be 1·43.

Therefore the energy required to produce an ion in carbon dioxide

$$= \frac{1·59}{1·43} W_1 = 1·11 W_1 = 2·11 \times 10^{-10} \text{ erg.}$$

This value is a little higher than in the case of air. The measured amount of i_2, the current in carbon dioxide, was somewhat less than the maximum, since the electromotive force applied was not sufficient to move all the ions to the plates before recombination. A correction for this would make the values for air and carbon dioxide more nearly equal.

Taking the value 1·53, found by J. J. THOMSON,* for the relative ionization in carbonic acid and air, the ionic energies are nearly the same.

The results for air and carbon dioxide show that the energy required to produce ions in the two gases is not very different. The results of a previous paper† showed that the absorption of X rays in gases was roughly proportional to the ionization produced. From this it follows that the energy required to produce ions in the gases examined was, roughly, the same.

The results obtained with uranium radiation‡ showed that the total number of ions produced by complete absorption of the radiation in air, oxygen, hydrogen, carbonic acid gas, hydrochloric acid gas, and ammonia were approximately the same. The results in that case were more readily obtained as the radiation was almost completely absorbed in a few centims. of the gas, and the maximum current through the gas was a measure of the total number of ions produced.

The recent results of McLENNAN§ also point strongly in the same direction. In his experiments, cathode rays were passed out of the discharge tube into another vessel, and the maximum current produced by the cathode rays was found for different gases. Using a constant supply of cathode rays, the current, *i.e.*, the total number of ions produced, was independent of the nature of the gas (provided the pressure of the gas was adjusted to give the same absorption of the rays in each case). The gases examined were air, hydrogen, oxygen, nitrogen, carbonic acid, nitrous oxide, and the total number of ions produced in them was nearly the same. Assuming that the same proportion of energy of the cathode rays was used up in producing ions in the gases, it follows that the energy required to produce an ion in all the gases is the same.

The results on the ionization of different gases by the agency of Röntgen, Becquerel, and cathode rays all strongly point to the conclusion that the same energy is required to produce an ion whatever the gas.

Variation of Ionic Energy with Pressure.

It has been shown earlier in the paper that from half an atmosphere to three atmospheres' pressure the absorption is proportional to the pressure. A special

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* J. J. THOMSON, 'Camb. Phil. Soc. Proc.,' vol. 10, Part I.

† E. RUTHERFORD, 'Phil. Mag.,' April, 1897.

‡ E. RUTHERFORD, 'Phil. Mag.,' January, 1899.

§ 'Roy. Soc. Proc.,' 1900; 'Phil. Trans.,' A, vol, 195.

investigation has shown that for the same range the intensity of the ionization is also approximately proportional to the pressure. This shows that for the pressures examined the ionic energy is independent of the pressure. The results on the variation of absorption with pressure for uranium* and thorium† radiation also point to the same conclusion.

In order to fully establish such a general law that the energy to produce an ion is independent of the gas and its pressure, a large number of careful experiments will be required. The results so far obtained can only be considered to show that such a law is approximately true. It is intended to continue these investigations on ionic energy for other gases besides air and carbonic acid.

Deductions from the Results.

If ions of the same kind are produced in a gas by different agencies, it is probable that the same amount of energy has been absorbed to produce the ions in the different cases. The only test we have at present for equality is to compare the velocity of the ions in the gas for a potential gradient of 1 volt per centim. J. J. THOMSON has shown that the charge on an ion produced by Röntgen rays is probably the same for the gases hydrogen, air, oxygen, and carbonic acid, and TOWNSEND‡ that it is equal to the charge on a hydrogen ion in the electrolysis of water. The velocity of the ions in a given electric field depends upon the ratio ϵ/m of the charge to the mass of the ion, and thus if the velocities of ions produced in the same gas by different agencies are the same, the masses must be the same, since the charges are equal.

It has recently§ been shown that the ions in the "electric wind" travel in air with the same velocity as the ions produced by rays. The energy used up in producing the ions can thus be immediately calculated.

Let i = the current through the gas due to the electric discharge from a wire or point.

Energy absorbed in producing ions $= n\,W = \frac{i}{\epsilon}\,W$,

where n is the number of ions produced per second and ϵ the charge.

Therefore, neglecting recombination of the ions,

$$\frac{\text{Energy required to produce ions}}{\text{Total loss of energy}} = \frac{\frac{i}{\epsilon}\,W}{Vi} = \frac{W}{\epsilon}\,\frac{1}{V},$$

where V is the potential of the discharging wire.

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* E. RUTHERFORD, 'Phil. Mag.,' January, 1899.

† R. B. OWENS, 'Phil. Mag.,' October, 1899.

‡ 'Phil. Trans.,' A, 1899.

§ CHATTOCK, 'Phil. Mag.,' October, 1899.

If $V = 6000$ volts $= 20$ electrostatic units, then the proportion of the total loss of energy used up in the production of ions

$$= \frac{1 \cdot 90 \times 10^{-10}}{6 \cdot 5 \times 10^{-10}} \cdot \frac{1}{20} = \frac{1}{69} \text{ approx.}$$

Thus quite an appreciable proportion of the total energy supplied is absorbed in producing ions. The proportion decreases with the increase of voltage.

Distance between the Ions in a Molecule.

If we suppose that most of the energy required to produce a positive and a negative ion from a neutral molecule is due to the work done in separating the ions from each other against the forces of electrical attraction, we can at once form an approximate estimate of the distance apart of the charges in the molecule.

The work done in separating a charge $+\epsilon$ from a charge $-\epsilon$, both charges supposed concentrated at points from a distance r to an infinite distance, is equal to ϵ^2/r.

If this is equal to the energy required to produce two ions, then

$$\frac{\epsilon^2}{r} = 3 \cdot 8 \times 10^{-10},$$

since $\epsilon = 6 \cdot 5 \times 10^{-10}$

$r = 1 \cdot 1 \times 10^{-9}$ centim. approximately.

The average diameter of an atom, calculated from various methods, is about 3×10^{-8} centim. This is a very much greater distance than the value found for the distance apart of the charges on the ions in a molecule. The results support the theory advanced by J. J. THOMSON, that ionization is produced by the removal of a negative ion from the molecule, and that the negative ion is only a small portion of the mass of the atom. The positive ion is supposed to remain attached to the rest of the molecule. It is to be expected from the theory that the distance of the charges from each other would be less than the diameter of an atom.

The energy required to produce an ion in air is very much greater than the energy required to produce an ion in the electrolysis of water. If V (1·46 volts) is the least E.M.F. required to dissociate water, the work done in moving a quantity of electricity ϵ is $V\epsilon$. The work done in producing an ion thus is $V\epsilon$ or $\frac{1 \cdot 46}{300} \times 6 \cdot 5 \times 10^{-10} = 3 \cdot 16 \times 10^{-12}$ erg, or about $\frac{1}{60}$ of the energy required to produce an ion in air by the agency of X rays, so that in water the ions are about two atoms apart

Least Potential required to produce a Spark in Air.

The ions in the "electric wind" in air have been shown to move with the same velocity as the ions produced by X rays. It is probable that the passage of a spark between two electrodes is heralded by the production of ions in the gas, and that these ions are of the same kind as the ions in air produced by X rays.

Let V be the difference of potential between two electrodes in air, one electrode being connected to earth. Suppose a pair of ions to be produced and to travel to the electrodes. A quantity of energy $3 \cdot 8 \times 10^{-10}$ erg is absorbed in their production, while the energy of the electric system is diminished by an amount $V\epsilon$. The energy required to produce the ions must be derived from the electric energy of the system. In order for an ion to be produced consistent with the conservation of energy, we must have V of such value that $V\epsilon$ is greater than the energy required to produce a pair of ions.

$$\therefore \quad V > \frac{3 \cdot 8 \times 10^{-10}}{6 \cdot 5 \times 10^{-10}} \text{ electrostatic unit.}$$

$$\therefore \quad V > 175 \text{ volts.}$$

Now PEACE (J. J. THOMSON, 'Recent Researches,' p. 89) has shown that it is impossible to produce a spark in air below about 300 volts, however close the electrodes are together. This is a somewhat greater value than the one found above, but is of the same order. STRUTT* has recently shown that the minimum potential difference for the passage of a spark in pure nitrogen is about 251 volts. As most of the ions in air are probably produced from the nitrogen molecules, this value makes the agreement still closer. The results obtained would indicate that it would be impossible to produce an ion, and therefore an electric spark, below 175 volts. If the energy required to produce an ion were the same at all pressures, the minimum sparking potential according to the above theory would be unaltered. This is borne out by PEACE'S results (*loc. cit.*, p. 86), where it is shown that the minimum potential difference for a spark between spherical electrodes ·001 centim. apart is approximately the same for pressures from 300 to 50 millims. of mercury. The minimum potential rises below 50 millims., indicating that the energy required to produce an ion may possibly increase below that pressure.

This theory would suggest that the minimum potential required to produce a spark conversely might be used as a means of determining the energy to produce an ion. The phenomenon, however, is more complex than this would indicate. The minimum sparking potential is to a small extent influenced by the metal used for the electrodes and also by the gas, and moreover it would leave unexplained the remarkable fact that when the electrodes are a small distance apart the spark does not follow the shortest path (*loc. cit.*) between them.

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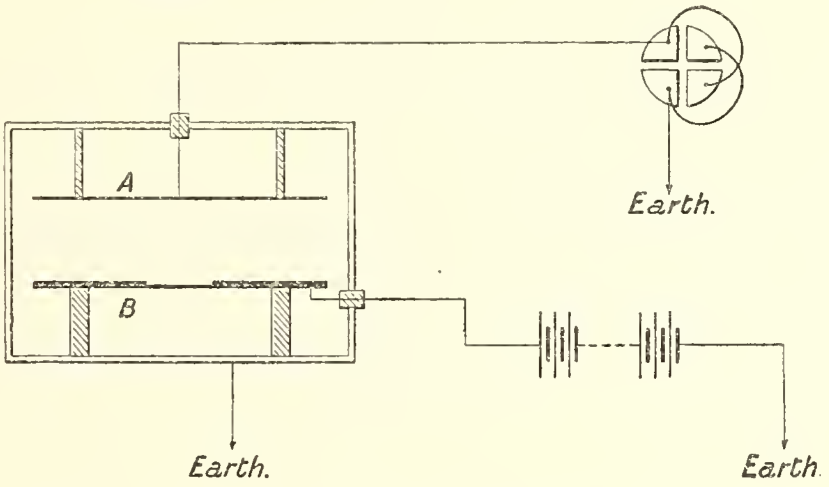
* 'Phil. Trans.,' A, 1900.

Energy of Radiation of Radio-active Substances.

In a previous paper it has been shown that the ions produced in air by uranium radiation have the same velocity as the ions produced by Röntgen rays. On the assumption that the same amount of energy is required to produce the ions, whether the agency is Röntgen or uranium rays, the energy of the radiation given out into the gas can be at once determined. Fig. 7 shows the arrangement of the apparatus to determine the current through air produced by the uranium rays.

Two large parallel plates, A and B, 4·1 centims. apart, were insulated from each other. A was connected to the electrometer in the usual manner, and B was connected to one pole of a battery of small cells of 310 volts. The uranium oxide employed was placed in a square shallow hole cut in a lead plate placed on the plate B. The current between A and B was determined for a potential difference of 310 volts, an amount sufficient to practically remove all the ions before recombination.

Fig. 7.



Since the area of the uranium surface is small compared with the area of the plates between which the uranium was placed, the total energy per second emitted by the surface S of uranium is approximately equal to IS, where I is the intensity of the radiation at the surface of the uranium.

If λ is the coefficient of absorption of the rays in air, and W is the energy required to produce an ion, the energy absorbed per second between the plates at a distance d apart is equal to

$$\mathrm{IS}\{1 - e^{-\lambda d}\} = n\mathrm{W} = \frac{i}{\epsilon}\mathrm{W},$$

where n is the number of ions produced per second, i is the current, and ϵ the charge on an ion.

The total energy emitted per second is equal to

$$\frac{i\mathrm{W}}{\epsilon(1 - e^{-\lambda d})}.$$

In the case of uranium it has been shown (*loc. cit.*) that apparently two types of radiation are emitted, one of which is readily absorbed in air. The ionization due to the more penetrating rays is in general a small part of the total, especially for thin layers of uranium; so that in the present calculation we will only consider the energy given out by uranium in the production of the more absorbed type of radiation.

The intensity of the radiation emitted from uranium falls to half its value after passing through 4·3 millims. (*loc. cit.*, p. 128). Therefore $\lambda = 1{\cdot}6$.

Since $d = 4{\cdot}1$, therefore $e^{-\lambda d}$ is small and may be neglected.

Thus the energy given out into the air is $\frac{i}{\epsilon}$ W. Now for a thick layer of uranium oxide (3·6 grammes spread over a surface of 38 sq. centims.) the current $i = {\cdot}0515$ electrostatic unit. Thus the energy emitted per unit area of uranium surface per second

$$= \frac{{\cdot}0515}{6{\cdot}5 \times 10^{-10} \times 38} \times 1{\cdot}90 \times 10^{-10} = {\cdot}0004 \text{ erg}$$

$$= 10^{-11} \text{ calorie per second, approximately.}$$

This amount of energy would suffice to raise 1 cub. centim. of water 1° C., assuming no radiation of heat, *in about* 3000 *years.*

It is a difficult matter to determine the total energy given off in the radiation by a given weight of uranium on account of the ease with which the radiation is probably absorbed by the heavy metal uranium itself in its passage through it. Some experiments were made on the current due to a given surface of uranium oxide when different depths of the active material were spread over it. The following are some of the results:—

SURFACE of Uranium Oxide = 38 sq. centims.

Weight of uranium oxide in grammes.	Current in E.S. units per second.
·138	·0201
·365	·0365
·718	·0471
1·33	·0515
3·63	·0560

The uranium oxide in the form of a fine powder was dusted on uniformly by means of a fine wire gauze. The results show that the current per gramme of uranium oxide is greater for small than for large thicknesses. Even with a very thin layer of uranium oxide in the form of powder it is probable that a large proportion of the energy emitted (supposed produced throughout the volume of the substance) is absorbed in the substance itself. An approximate determination of the total energy per second that can be radiated by 1 gramme of uranium could be determined by dissolving a few crystals, say, of uranium nitrate in water and pouring the solution over a large surface area. On evaporating the water a very thin film of the nitrate would be left on its surface. In such a case the rays produced throughout the volume of the film should reach the surface without much loss due to absorption, and the maximum current through the gas would be proportional to the total energy radiated. In this case half of the total energy would be absorbed in the substance on which the film was placed and only half would be efficient in producing ions.

In order to obtain an approximate value of the total energy of radiation, uranium oxide in the form of a very fine dust was spread over a surface of 38 sq. centims.

Weight of uranium oxide = ·138 gramme.

Current = ·02 electrostatic unit.

Total energy radiated into the gas per second = $1{\cdot}4 \times 10^{-10}$ calorie.

Energy per gramme of uranium oxide radiated into the air

$$= 10^{-9} \text{ calorie per second.}$$

$$= {\cdot}032 \text{ calorie per year.}$$

In our present state of knowledge it is uncertain whether the radiating power is confined to the surface of the uranium or is given out uniformly throughout the mass. In any case, the total energy radiated is probably greater than the value above on account of absorption of the radiation in the uranium itself, and also on account of the existence of a more penetrating type of radiation, the energy of which has been neglected in the above calculations.

Energy of Thorium Radiation.

The apparatus employed was the same as that for uranium. Thorium oxide was employed and the following results were obtained. Area of surface = 38 sq. centims. :—

Weight of thorium oxide.	Current.
gramme.	
·339	·0445 electrostatic unit per second.
·665	·0622 " " "

In previous papers by OWENS[*] and RUTHERFORD,[†] the behaviour of thorium oxide as a radio-active substance has been carefully examined. It has been shown that thorium compounds give out a material emission of some kind, which possesses temporary radio-active properties. This emanation is most apparent with thick layers of thorium oxide. In the present case the layer was not thick enough to give out much emanation, and the rate of discharge was due to the radiation alone. OWENS has shown that the radiation from thorium is approximately homogeneous.

The value of λ, the coefficient of absorption of thorium radiation in air, is ·69 and $d = 4{\cdot}1 \therefore 1 - e^{-\lambda d} = {\cdot}96$.

Thus for a weight of ·665 gramme the total energy radiated into the gas per unit area $= \frac{i\mathrm{W}}{{\cdot}96\epsilon\mathrm{A}}$ erg, and, on the same assumptions as for uranium, the energy radiated into the air per second $= 1{\cdot}2 \times 10^{-11}$ calorie, a somewhat greater value than for an equal weight of uranium oxide.

Excited Radio-activity due to Thorium.

Thorium compounds, in addition to the property of giving out a radio-active emanation, possess the power of exciting temporary radio-activity on all substances in their neighbourhood. The excited radiation is homogeneous in character, and is of a more penetrating type than the radiation from either uranium or thorium. The intensity of the excited radio-activity can be greatly increased by concentration on the negative electrode of small area by means of a strong electric field. On the assumption that the energy of the radiation excited on the electrode is dissipated in producing ions, an estimate can be formed of the energy stored up on the electrode.

In a particular experiment a fine platinum wire, ·018 centim. in diameter and 1 centim. long, caused the separation of about 10 coulombs of electricity before the radio-active power was lost. This corresponds to an emission of 2×10^{-5} calorie.

This by no means inconsiderable quantity of energy is in some way derived from the surface of a platinum wire ·056 sq. centim. in area, without the slightest appreciable change either in the weight or appearance of the wire.

Radium and Polonium.

The question of the equality of the velocity of the ions, produced by thorium radiation and the rays from the powerful radio-active substances radium and polonium, with the velocity of the ions produced by X rays, has not been specially investigated, but from the very close similarity of the types of these radiations, it seems very probable that the ions produced by all are the same.

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[*] 'Phil. Mag.,' October, 1899.

[†] 'Phil. Mag.,' January and February, 1900.

In one respect, however, some of [the radio-active substances, notably radium, differ in their type of radiation from X rays. Becquerel, Curie, and others, in a series of papers in the 'Comptes Rendus,' have shown that radium gives out a type of rays which are easily deflected by a magnet. This emission of rays similar in character to cathode rays of low velocity is very remarkable, but does not seem to be a necessary accompaniment of a radio-active substance. For example, Giesel found polonium gave out rays deflected by a magnet, while Becquerel could obtain no magnetic action for the same substance. The rays which are deflected by a magnet seem to be present or absent according to the mode of preparation of the substance, and depend possibly on the age of the specimens. Two impure and not very sensitive specimens of radium and polonium obtained from pitchblende have been tested by one of us, but no magnetic action has been observed. Becquerel has found no trace of magnetic action in uranium radiation, and one of the authors has tested both uranium and thorium radiations in a magnetic field at atmospheric pressure and obtained negative results.

The experiments of Curie and Becquerel have shown that, in radium, two types of rays are present, one of which is deflected by a magnetic field and the other is not. The non-deflected type is similar in character to secondary X rays, and the deflected ones similar to low velocity cathode rays.

We thus see that the phenomena exhibited by the radio-active substances are not simple, and that they differ from one to the other. It is still possible, however, to form an approximate estimate of the energy of the radiation whatever its kind, provided the energy is all completely absorbed in ionizing the gas, and produces ions of the same kind. It seems probable that the radium rays acted on by a magnetic field are a type of cathode rays, and that they ionize the gas in their passage through it. The results of McLennan* clearly show that the energy of the cathode rays is lost in its passage through the gas, due partly to the work done in ionizing the gas in its path. Provided the ions produced by the deflected and undeflected rays of the radio-active substances are the same, and absorb the same amount of energy in their production, the relative energies of the radiations emitted can be compared by noting the total maximum current produced by the rays when completely absorbed between the electrodes.

If $n =$ the ratio of the currents between parallel plates for equal areas and thicknesses of the test substances and uranium oxide when the plates are at a sufficient distance apart to approximately absorb all the rays, then

Energy radiated out by the test substances into the gas $= n \times$ the energy radiated by an equal area of uranium oxide.

This will probably apply roughly to the conductivity produced by the deflected and undeflected rays.

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* 'Roy. Soc. Proc.,' vol. 66, 1900; 'Phil. Trans.,' A, vol. 195.

In some experiments CURIE mentions using a specimen of radium 100,000 times more active than uranium. If this applies to measurement for equal weights and areas of radium and uranium, then the total energy radiated out into the gas by 1 gramme weight of radium is not less than 10^{-4} calorie per second, or 3200 calories per year.

It is evident that, unless energy is supplied from external sources, the substance cannot continue emitting energy at such a rate for many years, even supposing a considerable amount of energy may possibly be derived from rearrangements of the components of the molecule.

In the light of the results on the amount of energy given out by radio-active substances, it is of interest to consider some speculations as to the origin of the rays, and of the supply of energy required for a continuous emission of the radiation.

We will first briefly review the state of our present knowledge of the radio-active substances. Uranium, the first of the radio-active substances discovered, has been closely investigated. BECQUEREL has shown that it gives out radiation constantly from year to year, even when placed in the dark. The radio-activity is preserved in solution, and persists if the substance is recrystallised in the dark. The radiation given out is independent of the gas around it, and of the pressure of the gas, and is not much affected by considerable changes of temperature. The same radiation is given out by all the uranium compounds. The radio-activity appears to depend on the uranium molecule alone, and not what it is combined with. Pitchblende and other uranium minerals are active, and, as far as experiments have gone, continue radiating indefinitely.

In considering the question of the emission of energy per unit weight of uranium, an important point arises which it is difficult to decide satisfactorily by experiment, viz., whether the radio-activity is confined to the surface or possessed by the whole mass of the substance. At first sight the radio-activity appears to be superficial, since the intensity of the radiation does not increase with increase of thickness of uranium. Such an action, however, is to be expected, even though there is volume radio-activity, since the radiation can only penetrate to the surface from a very short depth below the surface. The increase of the intensity of the radiation with increase of thickness for thin layers and the action of solutions support, as far as they go, the supposition that the activity is throughout the volume. The energy given out in the interior of the substance would most probably be dissipated as heat in the material. If the radio-active power is possessed by the whole volume, it follows from the above supposition that the mineral pitchblende must have been radiating energy since its formation as a mineral. If we suppose the radiation has been going on constantly at its present rate in the course of 10,000,000 years, each gramme of uranium has radiated at least 300,000 calories.

It is difficult to suppose that such a quantity of energy can be derived from regrouping of the atoms or molecular recombinations on the ordinary chemical theory.

This difficulty is still further increased when we consider the emission of energy from radium, a substance 100,000 times more active than uranium. The emission of energy in that case is, at least, 3000 calories per year. If future experiments should show that radium, as well as uranium, gives out radiation at a constant rate from year to year, in order to account for such a rapid emission of energy, it would be necessary to suppose that the radio-active substance in some way acts as a transformer of energy. Such a supposition does not seem probable, and leads us into many difficulties.

On the view, however, advanced recently by J. J. Thomson, that an atom is not simple, but composed of a large number of positively and negatively charged electrons, the possible energy to be derived from the closer aggregation or regrouping of the components of a molecule is very much greater than on the atomic theory, as ordinarily understood. The energy required to completely dissociate a molecule into its component electrons would be many thousand times greater than the energy required to dissociate a molecule into its atoms. The energy that might be derived from a greater concentration or closeness of aggregation of the components of such a complex molecule would possibly be sufficient in the case of uranium to supply the energy for the emission of radiation for long periods of time. The sudden movements of electrons would set their charges in oscillation, and give rise to a series of electromagnetic pulses corresponding to X rays. The remarkable property of some of the radio-active substances in naturally emitting a kind of cathode rays shows that the present views of molecular actions require alteration or extension in order to explain such phenomena. The energy that might possibly be derived from regrouping of the constituents of the atom would not, however, suffice to keep up a constant emission of energy from a strong radio-active substance, like radium, for many years. It is of importance that experiments to test the constancy of the radiations of a powerful radio-active substance, like radium, should be carried out at definite intervals. If the radiation should keep constant from year to year, it would be strong evidence that the energy of the radiation was not derived at the expense of the chemical energy of the radio-active substance.

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III. *The Circulation of the Surface Waters of the North Atlantic Ocean.*

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[Plates 1—4.]

I.

The history of our knowledge of the currents of the North Atlantic Ocean up to the year 1870 has been written once for all by Petermann (1), who in that year published a memoir maintaining, contrary to the opinion of Findlay, Blunt, and Carpenter, that eastern and northern extensions of the Gulf Stream were the prime factors in the circulation. Petermann subjected practically the whole of the material in the way of observations then extant to an exhaustive critical examination, and came to conclusions which are worth quoting, in the summary, inasmuch as the observations of the twenty succeeding years did not seriously modify them :—

1. The hot source and core of the Gulf Stream extends from the Strait of Florida, along the North American coast at all times . . . . up to the 37th degree of northern latitude.

2. Under the 37th and 38th degree of latitude the hot core of the Gulf Stream turns away from the American coast toward the east beyond the meridian of Newfoundland and its banks to long. 40° W. . . . . From there it proceeds to the north-east, diffuses nearly across the entire Atlantic, and surrounds the whole of Europe, to the Arctic and the White Sea of Archangel, with a broad and permanent watercourse . . . . The south-west winds receive their high temperature from the Gulf Stream; and only through the Ocean, not by the winds, can warmth be carried into latitudes as high as those of the European coasts are.

3. The Gulf Stream, as a whole, is as yet but little explored; we only know its influence in some degree. How limited our knowledge of it is may be inferred, for instance, from the fact that there are most contradictory statements of its velocity and strength. A. G. Findlay, one of the principal authorities on the Gulf Stream,

11.3.1901

computed (1869) its velocity as requiring one to two years to reach Europe from Florida, while, according to my computation, two months would suffice. . . . .

4. To conclude from the soundings, obtained so far, the Gulf Stream must be, up to the Arctic Ocean, a deep and voluminous watercourse; if it should not be so, the Polar ice would reach the European coasts. . . . . The Gulf Stream, in its course, is more powerful and steady than all the winds; only the Polar ice and the Polar currents, in spring and summer, exercise a great influence over it. The Polar Stream presses at three places against it; first from the north-west, east of Newfoundland; then from the north, east of Iceland; at both these places the Polar Stream is buried and proceeds beneath the Gulf Stream, after having pushed it off literally to the south-east. But for the third time, at Bear Island, the Polar Stream comes directly against the Gulf Stream, from the north-east, splits it into two or three branches, and in places even presses it beneath its own waters, at least in July . . . .

5. These three conflicts with the Polar currents cause the summer (July) isothermal lines of the Gulf Stream to make deep cuts at the respective places, and to assume a certain concavity of form which will not be found in those of the winter (January). But even if the July curves, when compared with those of January, appear pressed back somewhat to the south, they show, nevertheless, on the whole a very high temperature for the entire Atlantic basin. . . . . A great depression of the temperature of the surface is caused by the Polar Stream descending east of Iceland, and, after its collision with the Gulf Stream, proceeding beneath the latter, principally when reaching the shallow German Sea. It is evident that this branch of the Polar Stream, and the winds blowing from it, are depressing also the summer temperature of a considerable part of Western Germany. . . . . It is pretty certain that a sub-surface Polar current reaches, in summer, from Iceland and Jan Mayen to the German coast, but there have been so far hardly any inquiries made about it. . . . .

6. In winter (January) the Gulf Stream is cut-into much less. The pressure of the Polar Stream at Newfoundland is hardly visible on the chart, the curves being simply parallel with the coasts; east of Iceland a Polar Stream proceeding to the south-east cannot be inferred at all from the observations of the temperature of the sea at Iceland, the Faeroe Islands, Scotland, and Norway, which bear toward each other quite different relations in January and in July. The relations in winter between Bear Island and Spitsbergen are yet unexplored, but we have known for a long time the grand effects of a relatively high-tempered sea up to Spitsbergen and Nova Zembla. . . . . The Polar streams, in conformity with the general laws of nature, are less powerful in the winter than in the summer; the Polar ice does not drift as far southward. . . . . The Gulf Stream is in winter more powerful than in summer; while the Polar streams, so to say, set at rest in some measure, withdraw their ice and concentrate it round the land. . . . .

7. The relations of the temperature of the Gulf Stream within themselves, are

about the same in January as in July. In the latter month the isothermal curve of 7°·5 C. runs from Newfoundland far toward the north, beyond the whole of Europe; it corresponds in general with the January curve of 2°·5 C.; the amplitude of the Gulf Stream, therefore, *i.e.*, the fluctuation between its maximum and minimum temperature (July to January, or August and February) would be, on the average, only about 5° C.

This general description was amplified in many directions by the additional data collected by special expeditions, and by the discussion of surface observations made on board merchant vessels, during the period 1870–90. Amongst the former are to be noted the contributions of H.M.S. "Challenger" (1873–76) (2), which made soundings in the western part of the Atlantic basin; of the Norwegian North Atlantic Expedition (1876), which explored the whole eastern part of the basin up to Spitsbergen; of H.M.SS. "Knight Errant" (1880) and "Triton" (1882), chiefly in the Faeroe-Shetland Channel; the different expeditions to the Polar Seas; the expeditions of the Danish ships "Fylla" and "Ingolf" (1877–79); of the United States Coast and Geodetic Survey (3), especially within the region of the Gulf Stream properly so called; and amongst the latter, the meteorological and hydrographical services of Denmark, Germany, Great Britain, and the United States.

Stated in the most general way, the effect of the increased information was to reduce the relative importance of the Gulf Stream current; it was recognised that while in the first instance most of the stream moving northward near the eastern seaboard of the United States was derived from the region outside the West Indies, and did not pass through the Strait of Florida, that stream did not continue as such much beyond the south-eastern extremity of the Newfoundland Banks (4). At the same time, observations at the higher latitudes, while defining more fully the general northward movement of the waters in the upper layers of the Eastern Atlantic, brought out the unexpected magnitude of the Polar streams moving southward both at the bottom and at the surface. Both results led, after much discussion, to increasing belief in the direct frictional action of wind as the prime factor in oceanic circulation, the gravitational influence of inequalities of specific gravity being relegated to second place, except in the greater depths: this doctrine was finally established by the observations of MURRAY (5) in Scottish lochs.

Up to this point little or no attempt was made to ascertain the limits within which the circulation in different parts of the ocean was liable to periodic or other variations, notwithstanding the emphasis with which PETERMANN had insisted on the need of it. The seasonal variations in the strength of the Gulf Stream, recognised by PETERMANN, and indeed the seasonal variations of all currents except in the monsoon regions, were practically ignored, either because their existence was disbelieved in, or because they were assumed to be so small as to be beyond investigation with the available observations, and therefore too small to prevent any observations being comparable. The only exception to this was the preparation by Meteorological Departments of charts showing the seasonal changes of mean surface temperature, changes which are in part affected, or effected, by variations in the currents, and a few "current charts," showing the mean result of numbers of surface current observations made at different seasons of the year (6). In these cases, however, it was of course necessary to ignore the possibility of irregular or long-period changes, trusting to a sufficiency of observations to give approximate seasonal values from the means.

It seems scarcely necessary to attempt to account for the gradual recognition, during the past decade, of the existence of extensive changes in the circulation of oceanic waters at different times. The discoveries of the "Challenger" Expedition with regard to the geographical distribution of marine organisms raised innumerable questions which demanded fuller knowledge of the physical and chemical conditions than could be obtained by merely extending and continuing observational work along the old lines. The development of a general mathematical theory of atmospheric circulation which agreed with the results of observation in the main outlines, but presented many local anomalies, required more detailed study of the physical conditions of parts of the earth's surface, and especially over the sea, for their elucidation. Lastly, the necessity for the regulation of some of the great fishing industries was becoming increasingly urgent, and it was more and more evident that any such regulation must be based on full scientific knowledge of the physical and chemical conditions upon which, directly or indirectly, the positions of the great fishing grounds had been shown to depend. All these influences worked in the same direction, and the economic importance of the fishery question strengthened the hands of the societies or government departments upon which the expense of further investigation must fall.

The first investigation carried out with the requisite detail and accuracy of method was that undertaken by Professor F. L. EKMAN, who directed the Swedish exploration of the Baltic in the year 1877 (7); and it is to the subsequent labours of Swedish hydrographers, and especially of Professor OTTO PETTERSSON, of Stockholm, that the development of the modern methods of research are very largely due. Put shortly, the outstanding feature may be said to be the application of the idea of the synoptic chart—the survey of the part of the ocean under investigation in such a manner that the physical or chemical conditions in its waters are known at successive instants of time, at intervals sufficiently short to allow of the changes being continuously traced. While allowing a full measure of credit to the Swedish hydrographers in this respect, it is necessary to notice that similar methods were developed independently by the officers of the United States Coast and Geodetic Survey and by Professor W. LIBBEY, Jun. (8), in their work on the Gulf Stream, and by MILL and MURRAY (9) in the Clyde Sea Area, and on various lakes and fjords in Scotland.

As the work of the Swedish hydrographers in the Baltic and Skagerak progressed, they were driven further and further seaward in their inquiries, until, in 1890, PETTERSSON and G. EKMAN (10) began a systematic investigation of the waters of the

North Sea. At the same time, the Swedish scientists, with the support of their government, made efforts to secure the co-operation of other nations interested in the regions under examination, which were so far successful that a preliminary reconnaissance consisting of four sets of observations, at three-monthly intervals, was made during 1893 and 1894 by expeditions sent out simultaneously by Sweden, Norway, Denmark, Germany, and Great Britain. These observations have been worked up and reported on by the directors of the several expeditions, and the results have been combined in a number of important papers by Professor PETTERSSON (11).

During 1891 and 1892 a Danish expedition made important investigations in the seas north of Iceland and east of Greenland, extending a line of soundings as far as Spitsbergen (12).

Although no further joint investigations have been carried on since those of 1893 and 1894, active work has been continued in nearly all the countries concerned. The Danish cruiser "Ingolf," under the command of Commodore C. F. WANDEL, made important investigations in the seas round Iceland and Greenland in 1895 and 1896 (13), and the Danish Hydrographic Office has continued the analysis and prompt publication of observations of surface temperature and of the position of the ice in the northern seas traversed by its merchant vessels (14). Investigations were begun by Norway, under the direction of Dr. JOHAN HJORT (15), in November, 1893, in time to take part in the later of the joint expeditions referred to above, and since then Dr. HJORT has continued his researches, especially with reference to fishery questions, with conspicuous success. Besides these we have the valuable observations made by the Nansen Expedition, and numerous observations have been made by yachts and other vessels cruising in high latitudes. Professor PETTERSSON and his colleagues have continued to follow the changes in the Skagerak and Kattegat. In 1896 some of the stations in the Faeroe-Shetland Channel visited by H.M.S. "Jackal" in the work of 1893–94, and some of the stations of the earlier expeditions near the Wyville-Thomson Ridge, were re-visited by H.M.S. "Research" (16).

Many of these observations, although not made on a preconcerted plan, can be discussed together, and they form a fairly effective continuation of the work of 1893–94, pending a systematic investigation of the whole area by international co-operation.

In tracing the movements of oceanic waters, five elements may be taken into account:—

1. Direct measurement, either by current meter, or by the drift of floating bodies. These measurements are practically restricted to surface movements, and they are attended by difficulties of two kinds; single observations of current from a vessel may be made in merely local and temporary streams, and observations of "drifters" may be affected by errors due to uncertainty of their path, the effect of local drifts in shifting them from one current to another, uncertainty as to the time at which they are found, and so on. The first method gives valuable results where very large

numbers of observations are dealt with, as in the admirable charts published by the Meteorological Office; the second must still be regarded as of doubtful value except where its indications can be very fully controlled by observations of another kind (17).

2. Observations of temperature; important at all depths, but especially at a considerable distance from the surface, where seasonal variations need not be taken into account, and where differences of salinity are extremely small. Temperature observations give as yet by far the most trustworthy information about the slow vertical or horizontal "creep" of waters in the greater depths. In shallow waters near land, temperature observations may be exceedingly difficult to interpret; the oceanic waters tend to have a small annual range of temperature which may vary greatly according to the degree of mixing of the surface waters by sea disturbance, tidal streams, &c., while the normally greater annual range of the land waters is affected by the source from which they are derived, *e.g.*, glaciers, &c.

3. Observations of salinity. In the cases just mentioned, and indeed in all surface waters, the amount of salt affords a surer guide as to movement than does the temperature, provided the determinations of salinity are made with sufficient exactness, chiefly because it is not liable to considerable local or seasonal variations, except by active mixing of waters from different sources. Thus in the open ocean, observations taken in a small area and within a few days of each other may show considerable irregularities of temperature, but the corresponding salinities will agree very closely; hence it is not necessary in the latter case to "generalise" the observations in showing the isohalines, and thereby introduce risk of error due to misinterpretation, to the same extent as in drawing the isothermals.

4. Dissolved Gases. The results of analysis of the gas contents have recently been employed with marked success by PETTERSSON (18) and KNUDSEN (19) in tracing the source of origin of different oceanic waters. It is unnecessary to refer to this in more detail here.

5. Plankton. Qualitative and quantitative examinations of the plankton contained in samples of sea-water have recently, in the opinion of CLEVE and others, proved a valuable assistance to the physical and chemical methods mentioned (20).

In attempting to investigate the movements of water within a given area by applying the synoptic method to the distribution of the elements just enumerated, the limitations imposed are determined by the extent of that area and the nature and rapidity of the changes from the observation of which the movements are to be inferred. Where the changes take place slowly, observations distributed over a considerable period may be regarded as having been made simultaneously in the middle of that period, and treated as in all respects comparable with each other; and further, if with a large number of observations distributed fairly evenly over the "period of observation," we find that the condition revealed by successive sets of observations can be shown to follow naturally from hypothetical continuous changes during the "intervals," we are justified in assuming that we have made a legitimate and adequate application of the method, and ultimately that the supposed changes have actually taken place. A more limited application of the method can be made where sets of observations are separated from others by intervals too long for the assumption of continuous change. Under this heading comes, for example, a comparison of the state of affairs at the same season in different years; which is valuable in that strictly seasonal variations are eliminated, and irregular or long-period changes can be studied by themselves.

The joint observations of 1893–94 were, for the most part, made within a period of a week or ten days, and except in certain localities, or under unusual weather conditions, it was found that, at least in so far as temperature and salinity were concerned, the observations of each set could be treated as simultaneous, and plotted together on curves or charts. Again, the interval of three months was, for the most part, found to be sufficiently short to allow of comparison of each set of observations with its successor, at least in the main outlines (21); and the general nature of the changes in temperature and salinity could be traced.

The net result of these comparisons was to establish, beyond all reasonable doubt, that the variations in the circulation already known to exist in more or less enclosed areas like the Baltic, occurred not only in the comparatively open North Sea, but even in the open channels connecting it with the Atlantic (22); that not only do seasonal variations of wide amplitude take place from month to month, but that irregular variations of probably equal magnitude render the type of circulation markedly different at the same period of different years. These latter variations, about which no chart of *average* conditions can give any information, certainly originate in the waters at or near the surface, and probably involve remote regions of the ocean. Although the existence of irregular variations in the surface conditions of the North Atlantic was recognised by PETERMANN, and probably accounts for the widely divergent views held about the general circulation by scientific men almost up to the present day, as well as for the persistent scepticism of a large number of efficient navigators, the large proportion which they bear to the whole mean movement in circulation has not been realised, and no systematic attempt has hitherto been made to ascertain their nature and extent. The reason for this is perhaps to be looked for in the tenacious hold which the idea of "rivers in the ocean" still retains in the minds of many with regard to currents.

The importance of a knowledge of the changes just referred to was strongly impressed upon me in the course of the work done under my direction on board H.M.S. "Jackal" in 1893 and 1894. Believing that, at least in so far as the open ocean is concerned, the greater part of the information required could be obtained from surface observations, I determined to see if the observations ordinarily recorded in the log-books of sea-going vessels would provide sufficient material for the construction of synoptic charts of temperature and salinity, when, as seemed necessary (partly because of the number of observations required and partly for the sake of comparison with existing charts of "mean" conditions and with meteorological charts) the "period of observation" was extended to one month and the "interval" restricted to the same period. Using data kindly furnished by the Danske Meteorologisk Institut, the British Meteorological Office, and other similar departments, I found it possible to construct, with a fair approximation to accuracy, charts showing the distribution of surface temperature over the North Atlantic during the months of May, August, and November, 1893, and February and May, 1894 (23). The usefulness of these charts, notwithstanding their being altogether of the nature of a first attempt, has been shown by PETTERSSON (24) and MEINARDUS (25). The attempt to construct similar charts of surface salinity was unsuccessful, the material, consisting of rough hydrometer determinations made on board ship, proving, naturally, perfectly worthless. (It was not uncommon to find salinities of 32 and 39 *pro mille* within a few days and miles of each other in the middle of the Atlantic.)

The preliminary experiment being so far successful, it seemed worth while to make a continuous series of monthly temperature charts extending over a considerable period, and if possible to obtain material for adding charts of salinity for the same time.

II. *Material used in Constructing the Charts.*

Having decided to attempt the preparation of charts showing the distribution of surface temperature and salinity for each month of the two years 1896 and 1897, a study of existing charts of mean surface movement and of my 1893 and 1894 charts satisfied me that the parallel of 40° N. lat. should be retained as the southern boundary. The Meteorological Council agreed to furnish me with extracts from logs of all temperature observations made north of that line during the years as they were received. The copying and arranging has been done throughout by Mr. W. G. JAMES, of the Meteorological Office, and the observations extend over 600 sheets, roughly 16,000 observations, forming the greater part of the temperature data utilised. In addition to the Meteorological Office observations the temperature values on the maps published monthly by the Danish Meteorological Department were inserted on the charts, along with occasional data, published and in manuscript, kindly sent to me by the United States Hydrographic Department, the Bureau Central Météorologique de France, and by Professor PETTERSSON. Many of the temperature observations received directly from the observers collecting samples of water were also sent to the Meteorological Office and came to me in duplicate, but a large number of additional observations was received and placed on the charts. In some cases verified thermometers were supplied to observers.

Temperatures were all reduced to the Centigrade scale before charting. Apart from the necessity for making the work comparable with that of workers in other countries, I found the difference of temperature represented by 1° C. better adapted to the work than the smaller Fahrenheit degree; the simpler numbers are also an advantage in charting large numbers of observations, and I found the occasional negative sign no trouble whatever. Taking into consideration the excellent quality of the instruments used, and the general nature of the observations, no attempt was made to correct the observations for either instrumental error or diurnal range.

With the view of obtaining material for constructing salinity charts the following arrangements were made. At my request the Meteorological Council authorised their secretary, Mr. R. H. SCOTT, F.R.S., to write to a number of the commanders and officers taking temperature observations, asking if they would be willing to undertake the additional trouble of preserving and sending to me samples of the water collected for their temperature and hydrometer observations, in bottles supplied, and to fill up a form with the necessary details as to ship's position, temperature, &c. The bottles, when full, were to be sent to me as soon as the ship reached its nearest point, and were to be replaced by a fresh supply as required.

Favourable replies were received from a considerable number of observers; and this number was added to by observers working for the Danish Hydrographical Department, the officials of which, under Commodore WANDEL, took a great deal of trouble in the matter. A number of volunteers were obtained in response to an appeal published in the 'Field' newspaper, amongst whom was the late Sir GEORGE BADEN POWELL. Mr. C. M. MUNDAHL, of Grimsby, added largely to the material obtained from the regions between Faeroe and Iceland, during 1896, by observations made on board his vessels, and I am indebted to friends for assistance from other observers.

The necessity for restricting the expense connected with the investigation, an important item of which was the cost of carriage of boxes to and from the vessels, made it necessary to keep the number of observers considerably below what could easily have been obtained, and even to diminish the number during the second year of the work. The detailed list of observers is given in Appendix I., and I have to express my most cordial thanks to them for the enthusiasm, skill, and accuracy, with which the work was carried out. In addition to the work of collecting and forwarding the samples, done with one exception altogether gratuitously, I received a number of letters containing valuable hints and suggestions bearing on the interpretation of the results.

The boxes used were each capable of holding thirty 6-ounce bottles, the bottom of the box being lined with felt, and the bottles kept in position by perforated frames, also lined with felt. The tops of the corks in the bottles were exactly level with the top of the box, so as to be in contact with the lid when closed, preventing all possibility of movement, and yet avoiding all need for special packing. The efficacy of the arrangement is apparent from the fact that not one of the 4100 samples was lost through a breakage in transit.

The bottles were made of ordinary blue glass, and were supplied by Messrs. BAIRD and TATLOCK.

A label was affixed to each, bearing a number (1 to 30), and the distinguishing number of its box: these labels were secured by a thick coating of shellac varnish. The corks, which were carefully selected, were soaked in very hot melted paraffin wax, and allowed to drain for a considerable time at a temperature above melting point. Except in cases where the samples had only been a short time in the bottles (as in, *e.g.*, the boxes supplied to R.M.S. "Teutonic"), the corks were stirred together in hot water and dried between each voyage, and all defective or dirty corks replaced. I have not found any difficulty with this method of the nature described by M. KNUDSEN (26), nor have I any reason to suppose that the paraffin could give rise to error in any of the analytical determinations. The trouble caused by particles getting into the pipettes is entirely avoided by taking care to remove superfluous paraffin in the manner described, and in the absence of paraffin particles of cork occasionally find their way into the samples.

As soon as the analyses of each box of samples were examined and booked for charting, the bottles were emptied, thoroughly washed in several changes of warm water, and allowed to drain for at least twelve hours. In view of Instruction *a* (Form A) it was thought unnecessary to attempt further drying.

The details of working were simple. One bottle was filled daily in accordance with the instructions given in Form A (Appendix II.), and the details as to position, &c., filled up in Form B in the line numbered to correspond with the number on the bottle. On the return of the ship to the home port the box was at once sent to Oxford, along with Form C, containing particulars as to forwarding of a fresh box.

III. *The Determinations of Salinity, &c.*

The estimations of salinity of the samples were made, in the first instance, from the amount of halogen contained. My own experience of this and other methods led me to agree entirely with the views so strongly expressed by Professor PETTERSSON as to its superiority; especially in dealing with large quantities of material. The mode of operation adopted was practically the same as that described in a paper published recently by Professor PETTERSSON (27), but a number of details throwing light on the accuracy of the work are of some importance. It is to be distinctly understood that no attempt has been made to obtain the highest possible degree of accuracy of which the methods are capable; the object was rather to work in such a way that the values obtained should be sufficiently close for the geographical purposes of the investigation, and that it should be possible to deal with the large quantities of material in the limited time at my disposal.

The chlorine determinations were made volumetrically with silver nitrate, using potassium chromate as index. The amount of sea-water used for each was 10 cub. centims., the titration being performed with a silver solution of about one-fifth normal strength. The pipette and burette used were made by GEISSLER, the former being furnished with a tap and finely drawn point, and with fiducial marks above and below the bulb, while the latter, also furnished with a very fine point, was divided to 0·1 cub. centim. and read to ·01 cub. centim. by means of a float. After a little practice the delivery of the burette could be easily regulated to ·02 cub. centim. In each determination, the pipette was washed out twice with the sample about to be titrated, and it was always filled direct from the bottle containing the sample.

The volume of the pipettes was ascertained by filling them repeatedly with carefully tested distilled water at different temperatures. The following set of weighings give some idea of the accuracy attained in filling. Pipettes marked ∴ and ⋮—

		Weight of distilled water.	
Pipette.	Temp.	(1).	(2).
∴	7°·3	10·0113	10·0080
⋮	7°·2	10·0118	10·0151
∴	8°·7	—	10·0117
⋮	8°·7	10·0152	10·0130
∴	12°·8	10·0068	10·0048
⋮	12°·8	10·0092	10·0093

These give for volume of pipette ⋮, which was used for the whole of the determinations—

At 7°·2 C. . . . . . 10·026 cub. centims.
8°·7 ,, ... 10·027 ,, ,,
12°·8 ,, ... 10·025 ,, ,,

showing that the volume of the pipette is constant within limits of measurement throughout the whole range of temperature ordinarily experienced in the laboratory, and that the difference of two fillings is not likely to exceed ·004 cub. centim., a degree of accuracy greater than that required by the burette. The amount of silver solution ordinarily required was something over 25 cub. centims., measured to ± ·02 cub. centim., whereas the pipette gives ± ·004 × 2·5 = ·01 cub. centim.

The determinations of the first 250 samples were made in duplicate, with results as shown by the following half-dozen, taken at random from the laboratory book:—

Sample.	Cub. centims. of silver solution. 1.	2.
11	27·51	27·51
12	25·91	25·91
13	25·59	25·58
14	24·88	24·87
15	28·71	28·72
16	27·56	27·55

These seemed so consistent, and as the work progressed the chance of a serious error failing to be caught on the charts seemed so small, that duplicate determinations were afterwards made of only every fourth or fifth sample, merely to make sure that one's eye was not "out" in deciding on the end-points.

The standardising of the silver solutions with sufficient accuracy gave some little trouble. The silver nitrate was obtained from Messrs. JOHNSON, MATTHEY & Co. as triple crystallised and fused, and notwithstanding very careful testing in various ways, no impurity of significant importance was detected, and in particular no free acid. The solution was made up in quantities of about 5 litres at a time, and each lot was stored in two Winchesters and treated, so far as standardising went, as two separate solutions. The strength of each solution was ascertained by titrating weighed quantities (about 10 grammes) of four different solutions, containing approximately the same amount of chlorine as an average sea-water; two of these solutions were prepared from carefully purified KCl, and two from similar NaCl, and it was arranged that the supply of more than one solution should not run down at the same time, so that each fresh solution should always control, and be controlled by, three old ones. The following specimens, selected at random, indicate the general accuracy obtained:—

Silver solution No. 24—	Gramme Cl in 1 cub. centim.
KCl solution A	·006923
" " B	·006936
NaCl " C	·006919
" " D	·006930
Mean	·006927

Silver solution No. 65—	
KCl solution A	·007267
" " B	·007259
NaCl " C	·007245
" " D	·007252
Mean	·007256

These values are consistent to within ·00002 gramme, again well within the limits required.

In the earlier part of the work, the comparison with some of the Sprengel tube determinations, to be described presently, raised some doubt in my mind as to the real accuracy of the method, and the reason which first suggested itself was the quite distinct difference in the appearance of the end reaction with KCl and NaCl solutions and with sea-water, the last being sharper and showing less tendency to "go back." This seemed to indicate an uncertainty due to a varying personal equation in judging the end-points with the sea-water and with the solutions ; but on repeated strength determinations of the same silver solution with the same NaCl solution, it was found that the uncertainty did not exceed 0·1 cub. centim. of the burette, giving as limits

Gramme Cl in 1 cub. centim.	·007747
,, ,, ,, ...	·007718
	·000029

This satisfied me as to the *consistency* of the strength determinations, but in order to estimate their *accuracy* I compared the mean given by the four solutions (·007724) with the mean of two gravimetric determinations obtained by weighing the AgCl precipitated from 30 cub. centims. of the silver solution, which was ·007722. A further comparison was made by mixing a number of samples together in two lots, A and B, and determining the chlorines by the Dittmar-Volhard method, which gives a better-defined reaction susceptible of "zig-zag" repetition, for comparison with the chromate method, with the following results—

		A.	B.
Dittmar-Volhard . .	1.	19·476	19·523
,, . .	2.	19·470	19·521
Chromate	1.	19·48	19·48
,, ...	2.	19·51	19·50

The following are eight duplicate determinations made with different silver solutions :—

	Cl. Silver solution.			
Sample.	No. 14.	No. 15.	No. 17.	Diff.
592	19·38	—	19·36	—·02
594	19·48	—	19·47	—·01
597	19·51	—	19·49	—·02
599	19·51	—	19·49	—·02
621	—	18·44	18·45	+·01
624	—	19·67	19·69	+·02
626	—	19·35	19·38	+·03
630	—	19·52	19·56	+·04

This gives ± ·02 as the probable error of any one chlorine estimation. Repeats of the twelve samples 1775—1779, 1787—1790, 1794, 1796, 1797, and 1800, with No. 40, No. 45, and No. 46 gave a probable error for each Cl = ± ·024.

These experiments, and the further experience gained in the course of the work, gave me perfect confidence in the chromate method for the purpose I had in view. The results seem to show conclusively that without spending the time necessary for the highest degree of accuracy, but merely taking ordinary care, the chlorines of large numbers of samples can be determined with comparatively little trouble, to within the admissible limit of error of ± ·03.

In working out the chlorine titrations and computing the corresponding salinities, I have employed tables of four-place logarithms, and have in no case gone beyond the second place of decimals in the results. The use of the third figure, which cannot be significant, is misleading, and its absence greatly simplifies the calculations.

Although I was unable to use the material collected for any special investigations into the properties of sea-water, I thought it desirable to determine the specific gravities of a certain number of each batch of samples, partly to get further evidence of the accuracy of the factors used in calculating the salinities, partly to act as a check on the chlorine work, and partly to give warning of any impurity in the samples. Six samples, usually the two saltest, the two freshest, and two others, were, as a rule, taken from each box, after the chlorine estimations were finished, and their specific gravities determined by means of Sprengel tubes containing about 50 cub. centims. The determination was made at 15° C., the tubes being set in a frame immersed in a tank, in which water was kept actively circulating by means of paddles, driven by a small motor. The thermometer used was a standard, made and tested by CASELLA, and repeatedly verified by myself in comparison with standards in use in the laboratory. The tubes were left in the tank for ten to twenty minutes after any change of volume was apparent, and set before the paddle was stopped—the whole time in the bath being thirty to fifty minutes. They were then carefully dried and placed in an open cardboard box beside the balance for some time before weighing.

The general accuracy of the pyknometer method, even when special precautions are not taken to get the best possible results, may be judged from the following, in which the tubes were filled with the same samples 1 and 2 :—

	Sample.		
Tube.	1.	2.	2 re-weighed next day.
1	1026·17	1026·22	1026·24
2	·12	·26	·21
3	·18	·19	·19
4	·13	·23	·19
5	—	·28	·29
6	—	·20	·20

In the case of the second sample the first weighing was made during the damp weather of a cyclone front, and the second under the drier conditions succeeding.

The following is a specimen of duplicate determinations taken in the ordinary course of routine work :—

Sample.	Tube.	$_4S_{15}$. 1.	2.	Diff.
1803	1	1005·12	1005·12	·00
1805	3	07·74	07·74	·00
1809	4	20·15	20·11	+·04
1817	5	20·06	20·05	+·01
1830	6	17·91	17·97	−·06

But these can scarcely be taken as representing the *certainty* of the method in dealing with large numbers of samples. On comparison with the chlorines a Sprengel determination was every now and then found to give a widely different salinity, and repetition almost invariably proved the latter to be wrong. Speaking generally, single Sprengel determinations are much less trustworthy than single chlorines, the possible sources of error in observation are more numerous, and the errors are much less easily detected, while the work is slower and more troublesome. It is worth while to note that thorough rinsing with distilled water before and after each observation, and washing through twice with each sample before filling, is not sufficient to keep the tubes thoroughly clean. A transparent, apparently gelatinous, film is slowly deposited on the walls of the tube, diminishing its volume and very slightly increasing its weight. It was found necessary to wash the tubes thoroughly at intervals with soda and strong nitric acid, drying them in a current of hot air. The following shows the effect of this treatment on a couple of tubes after being in use some time:—

	Tube. No. 3.	No. 4.
Washed with distilled water and dried in hot air: after being in use some time . .	19·4069	20·5916
Washed with soda and water, and dried. .	19·4051	20·5901

differences quite sufficient to affect results unfavourably, although the presence of the film causing it could not be detected by examination of the tube. The occurrence of this deposit was unfortunately not definitely recognised until the work had reached its later stages, and it is possible that its presence may slightly affect the ratios about to be discussed.

Grouping the pairs of values according to the chlorines, and averaging, we get the following table:—

Cl.	No. of samples.	Mean Cl.	Mean $_4S_{15}$.	$_4S_{15}$ from (1) below.
17·00 to 17·99	98	17·640	23·635	23·635
18·00 ,, 18·49	57	18·243	24·478	24·475
18·50 ,, 18·99	43	18·734	25·160	25·160
19·00 ,, 19·49	128	19·327	25·991	25·986
19·50 ,, 19·99	162	19·675	26·489	26·480
20·00 and over	41	20·220	27·239	27·245
	529			

These give the relation between ${}_4S_{15}$ and Cl

$$ {}_4S_{15} = 1\cdot399 \text{ Cl} - 1\cdot045 \quad (1), $$

a straight line which satisfies excellently the determinations of fifty-two samples in which Cl was less than 17·00 (28).

The samples used as standards in previous work (29) give for the same relation

$$ {}_4S_{15} = 1\cdot389 \text{ Cl} - 0\cdot805 \quad (2), $$

an equation which gives values agreeing fairly well with (1) for values of Cl met with in open water, but differing to a serious amount for very fresh waters, *e.g.*—

Cl.	${}_4S_{15}$ from (1).	${}_4S_{15}$ from (2).	Diff.
20	26·94	26·97	·03
17	22·74	22·80	·06
5	5·95	6·14	·19

The discrepancy indicates that in the waters of low salinity the chlorine forms a smaller percentage of the total salts in my samples than in those used as standards; the difference may possibly be accounted for by the fact that while in the low-salinity standards the salinity has been reduced by admixture of land water from the Baltic, my samples have been freshened by water from melting ice (30). It seems not impossible that the relative deficiency in chlorides known to exist in surface waters near melting ice may make itself apparent in this way in the North Atlantic, but the differences are so small as to require confirmation by other observers, and it would be quite impossible to deal with differences in variations quantitatively by any known methods. This point will be referred to again in discussing the salinities.

Partly with the object of checking the purity of the water-samples, and partly with a lingering hope that some variation might be found sufficiently large to be useful in studying the circulation, I made determinations of the total sulphates present in a number of the samples received during the first year. The method described by DITTMAR (31) was followed absolutely all through, about 40 cub. centims. (all that could be spared) being taken for each determination. The results are given in the last column of Table I. (p. 117, *et seq.*). The chief interest of these determinations is in the ratio of the total sulphate per kilogramme of water to the chlorine, usually expressed as $\frac{100 \text{ SO}_3}{\text{Cl}}$.

It may be doubted if, under ordinary conditions, the sulphate estimations in sea-water are really trustworthy to much less than 1 per cent. As the chlorines are taken as correct to ± ·03, we may neglect their errors, and treat the errors of observation in the fraction $\frac{100 \text{ SO}_3}{\text{Cl}}$ as wholly due to errors in the sulphate determinations; *i.e.*, we may take 1 per cent. as the probable error of a value of the fraction. To make this clearer, we may take hypothetical values of Cl and SO_3, such as might be obtained from the examination of one sample, as follows :—

Cl.	SO_3.	$\frac{100\ SO_3}{Cl.}$.
19·97	·00236	11·82
19·97	·00234	11·72
20·00	·00236	11·80
20·00	·00234	11·70
20·03	·00236	11·78
20·03	·00234	11·68

Thus we know the value of the fraction to approximately the first decimal place, even with a low standard of accuracy in the sulphate determinations. It would seem, however, that the margin allowed is unnecessarily large, involving, as it does, differences of about 4 milligrammes in weighing in an ordinary sea-water determination. The following duplicate series would indicate that an error of 2 milligrammes, or about $\frac{1}{2}$ per cent., covers most of the errors likely to be made in an analysis :—

Sample.	SO_3.		$\frac{100\ SO_3}{Cl.}$.	
	I.	II.	I.	II.
2314	·00238	·00237	11·88	11·84
2318	235	235	11·85	11·86
2326	235	235	11·84	11·86
2331	241	240	11·82	11·77
2337	242	242	11·89	11·89
2338	244	242	11·85	11·78

An examination of some of the values actually obtained confirms this :—

Cl.	No. of samples.	Mean of $\frac{100\ SO_3}{Cl.}$.	Probable error of one determination.
15 to 17·5	15	11·77	± ·047
17·5 to 20	20	11·75	± ·055
Over 20	14	11·74	± ·036

i.e., the probable error of any one determination of $\frac{100\ SO_3}{Cl}$ is ± ·05 approximately, or about $\frac{1}{2}$ per cent.

The values of $\frac{100\ SO_3}{Cl}$ obtained for the 322 samples examined were first arranged according to the latitude from which the samples were obtained, with the following result :—

Lat. N.	No. of samples.	Mean $\frac{100\ SO_3}{Cl}$.
Under 50°	139	11·74
50° to 60°	73	11·79
Over 60°	110	11·76

Arranged according to the chlorine contents of the samples, the mean values were :—

Cl.	No. of samples.	$\frac{100\ SO_3}{Cl}$.
0 to 5	2	11·49
5 ,, 10	2	11·64
10 ,, 15	3	11·69
15 ,, 17·5	15	11·77
17·5 ,, 20	286	11·75
Over 20	14	11·74

of which the four samples with chlorine less than 10 may be regarded as untrustworthy, on account of the small amount of the total sulphate weighed.

The net result is to show that so far as the surface waters of the North Atlantic are concerned, the proportion of sulphates to chlorides (× 100) is 11·754, and that no variation takes place in this quantity which can be determined by these methods. It may be interesting to note here that a mixture of samples collected in the Faeroe-Shetland Channel and North Sea on board H.M.S. "Jackal" n 1893, gave the following values in January, 1896 :—

$$Cl = 19{\cdot}473$$

$$SO_3 = {\cdot}00216 \text{ mean of 8 determinations}$$

$$\frac{100\ SO_3}{Cl} = 11{\cdot}10$$

showing a diminution in the proportion of sulphates far beyond possible limits of error, probably caused by reduction of the sulphates by organic matter. It seems therefore necessary that sulphate determinations pretending to high degrees of accuracy must be made from fresh samples.

In calculating salinities (p) from the values of Cl, the tables in the previous work (32) were extended by the formula and constants there employed, and used throughout. As the corresponding tables for obtaining salinity from given values of ${}_4S_{15}$ were computed from formulæ based on an extension of the same data through the chlorine values (33), the values of p obtained from the chlorines (p Cl) and from the Sprengel determination (p S) are not independent. As I fully expected when the work was begun that the ratio ${}_4S_{15}/Cl$ found for the standard samples (34) would hold good all through, the differences in the values of p Cl and p S seemed an easy method of checking the purity of the samples, and the accuracy of the work as it progressed. The following table gives the mean values of these differences :—

p.	No. of samples.	Mean diff. Δ p Cl − p S (*pro mille*).	r.	R.
0 to 5	5	+·17	±·06	±·03
5 ,, 10	9	+·17	·05	·02
10 ,, 15	7	+·13	·04	·01
15 ,, 20	4	+·15	·02	·01
20 ,, 25	4	+·12	·07	·03
25 ,, 30	12	+·13	·06	·02
30 ,, 35	183	+·03	—	—
Over 35	270	·00	·07	·00

The actual values of the differences correspond, of course, to the difference of density and chlorine ratios found, and indicate that while the values of p obtained from the chlorines are substantially correct within limits of error of observation for salinities above 30 *pro mille*, the values for fresher waters are too high by an amount rising to nearly 0·20 in waters below 5 *pro mille* salinity. So far as this inquiry is concerned, these errors are altogether negligible, as isohalines lower than 30 *pro mille* rarely appear on the charts, and when they do the number of observations is never more than sufficient to merely indicate their position roughly; indeed it seems very unlikely that in oceanic waters, sufficiently fresh to make it worth while to take the differences into account, the distribution of salinity would require so great refinements for its elucidation.

If we accept the mean values of Δ in the third column as correct, and allow for them accordingly, the second differences afford a measure of the combined errors of the chlorine and Sprengel determinations; the values of r give the probable error of one observation of the difference Δ, which amounts to about ± 0·05. The quantity R is the probable error of the values of Δ.

Table I. gives particulars as to the samples, and the results of the various determinations. The first column gives the "Laboratory Number" assigned to each sample on its receipt; the next the name of the vessel; the next two the date and hour of collection; the next two the latitude and longitude at time of collection; the next the surface temperature observed; the next the amount of chlorine estimated; the next the salinity calculated, by table (35), from the preceding; the next the ${}_4S_{15}$ from Sprengel tube. The last column gives the amount of SO_3 in grammes per kilo.

The working charts, upon which the temperatures and salinities were plotted, were drawn on MERCATOR's projection, to a scale of 1° long. = ·37 inch; each chart was divided into four sheets. After the plotting of the observations was completed the lines were drawn in in pencil and carefully revised three times, first independently, and then in comparison with the charts for the months preceding and following. The lines were then transferred, point by point, to blank maps, reprints of part of the Polar chart used in the "Challenger" Reports. These were then photographed and a set of prints coloured. From these Plates 1–4 have been prepared. In numbering the lines on the Salinity Maps the first figure and the decimal point have been omitted for all salinities over 30·00 *pro mille*—thus 34·5 appears as 45, and so on. Salinities below 30·0 are given in full. The colours used are as follows :—

Salinity	36·0 or over	dark blue.
,,	35·0 to 36·0	middle blue.
,,	34·0 ,, 35·0	light blue.
,,	33·0 ,, 34·0	dark green.
,,	32·0 ,, 33·0	light green.
,,	31·0 ,, 32·0	dark red.
,,	30·0 ,, 31·0	middle red.
,,	Under 30·0	light red.

The scale of temperature is :—

Over 25°	dark red.
20° to 25°	light red.
15° ,, 20°	yellow.
10° ,, 15°	green.
5° ,, 10°	light blue.
0° ,, 5°	middle blue.
Below 0°	dark blue.

In both the Temperature and Salinity Maps intermediate lines have been drawn where the observations seemed specially well able to define them, or when they seemed to throw additional light on the distribution (36).

IV. *The Distribution of Temperature and Salinity in the Surface Waters.*

January, 1896.—The observations for this month are limited to an area bounded by the 40th parallel and a line joining the Newfoundland Banks and the Faeroe Isles. The isothermal of 15° appears north of 40° N. lat. between the coast of Portugal and 35° W. long., and again between 60° and 65° W.—the mean position of the axis of the Gulf Stream (37). The 10° isothermal cuts the 40th parallel in 50° W., and again in about 60° W., further defining the head of the Gulf Stream ; the observations do not define the course of the 10° line between 40° and 50° W., but at 40° it reappears in 50° N. lat., following that parallel to about 22° W., where it swerves up to 55° N. to enclose an area of relatively warm water S.W. of the British Isles, turning south again some distance from their western coasts, and just touching the S.W. point of Ireland. Between the two areas outlined by the 10° isothermal an area of cold water extends S.E. from the Newfoundland Banks, the line of 5° runs to about lat. 48° N., long. 43° W., and then turns W., keeping well to the south of

Newfoundland and the Gulf of St. Lawrence, and approaching the coast between Nantucket Island and New York. Temperature falls quickly on the land side of the 5° line, especially to S.E. of Newfoundland. The isothermal of 7° fits into the loop of the 10° line when the latter follows the 50th parallel, *i.e.*, east of long. 40° W.; it indicates that the temperature gradient is turned southward rather than south-eastward.

On the eastern side, clear of the land, the distribution of temperature is extremely uniform. Temperature falls from 15° to 10° in 15° of latitude; the position of the 9° line is ill-defined. The lines of 7° and 8°, however, indicate that the fall becomes more rapid to north and west of the Faeroe Islands. The Faeroe-Shetland Channel is marked by an axis of over 7°, and a similar axis extends into the North Sea between Scotland and the Shetlands.

The distribution of salinity shows the same general features as that of temperature. On the western side the line of 35 *pro mille* closely agrees with the 10° isothermal in defining the northern border of the Gulf Stream, while the line of 36 *pro mille* agrees with that of 15°. On the eastern side the 36 *pro mille* line again agrees with the 15° isothermal, but the salinity remains about 35 over the whole area covered by the observations up to the Faeroe Islands. The head of the 10° line and the position of that of 7° in mid-ocean are reflected in the form of the line of 35·4 *pro mille* salinity, and again in the detached part of the 35 line running east from lat. 50° N. long. 40° W. The band of low temperature running S.E. from the Newfoundland Banks is represented, but in a much more marked degree, by water of low salinity; there is a steep gradient from 35·0 to 30·0 on both sides, and water of 34·0 extends down to lat. 40° N., and apparently spreads westward along the southern border of the Gulf Stream.

Thus we have in lat. 40° N. two surfaces of almost equally warm and salt water, one on each side of the Atlantic. One—the Gulf Stream water—stops off the land near the deep water line to the south of the Gulf of St. Lawrence; the other extends northwards along the coast of Europe, as far as the observations go, in a steadily narrowing tongue, and westwards to about long. 40° W. These two surfaces are entirely separated, down to 40° N., by a band of fresher colder water, stretching S.E. from the Newfoundland Banks.

February, 1896.—The observations for this month are increased by a line to Reykjavik.

In the lower latitudes the isothermal of 15° now follows the parallel of 40° N., except between about 12° and 22° W. long., when it bends slightly southward, and between 40° and 45° W., where there is a turn northward. The line of 5° has moved southward, but retains practically the same shape as in January. Temperature gradients have thus become much steeper west of 40° W. long. In mid-Atlantic the distribution of temperature shows little change, but in the eastern part of the ocean there is a tendency to equalisation; the water is slightly colder off the coast of Portugal, and apparently warmer between Faeroe and Iceland, the line of 8° having moved northward. Water of 8° or more now occupies the Faeroe-Shetland Channel, but the 8° line does not enter the North Sea. The two parts of the 10° isothermal have not altered their positions much, but they are now joined together by an almost straight line.

The salinity line shows similar changes of form. The line of 34·0 salinity occupies nearly the same position south of the Gulf of St. Lawrence and west of Newfoundland, but south-east of the Newfoundland Banks it does not extend so far south, the two parts of the line being now joined by a nearly straight portion. Water of 36 *pro mille* salinity appears north of 40° N. in patches, as far east as 40° W.; between 40° and 50° W. the fresh waters have entirely disappeared from the low latitudes. Little change is apparent in the eastern Atlantic; the 35·5 line is a little more to the north, and the 35·3 line stretches from Iceland to the Faeroe Islands, and forms a loop extending well into the North Sea. The low temperatures and salinities south-west of Iceland are to be noticed

The most important changes are thus, the advance of a warmer and salter area into the region north of lat. 40° N. and between long. 40° and 50° W., and a very slight northward extension of warmer and salter water along the whole of the European coast.

March, 1896.—The isothermal of 15° now appears north of 40° N. lat. only in mid-Atlantic; the position of its eastern portion remains almost unchanged, but it has retreated southward off the Gulf of St. Lawrence and the American coast. In this region the 5° and 10° lines are somewhat fuller to the south-east, and the gradient below 5° has become steeper, apparently on account of an extensive southward movement of ice-cold water from the Gulf of St. Lawrence and east of Newfoundland. In mid-Atlantic there is little or no change, and in the eastern part the conditions are the same as far north as 60° N., but north of this and east of 30° W. there is a marked fall of temperature, averaging about 1° between East Iceland and the Faeroe Islands. Temperature seems to have risen considerably to the west of Iceland.

Salinity observations are unfortunately not very well distributed for this month, the south-eastern part of the area being unrepresented. The most marked change is the advance of the 36 *pro mille* line to a point in about lat. 50° N. long. 10° W., off the south-west of Ireland, apparently due to a continuation north-eastwards of the movement of this line indicated by the comparison of the January and February maps; the angle formed by the 36 line is fairly acute; in all probability it may be supposed that it crosses long. 20° W. in about lat. 45° N. In the Western Atlantic the fresher waters appear further to the eastward; south-west of long. 50° the 36 line has disappeared. In the north-eastern region there is a slight but distinct freshening of the water; the line of 35·5 now scarcely goes north of lat. 50° N., and between Faeroe and Iceland the line of 35·2 replaces that of 35·3, and the latter does not now appear in the North Sea.

There is thus a marked lowering of temperature and salinity in the western region, a strong increase of salinity without any corresponding change of temperature in the south-eastern region, and a distinct lowering of temperature and salinity in the north-east.

April, 1896.—The area covered by the observations for this month is largely increased by the records of the whaler "Active," and vessels trading to Greenland.

In the south-west the isothermal of 15° reappears in the Gulf Stream region, and north of this the temperature gradient is exceedingly steep, for the 5° line retains the position of the previous month. Further east, in long. 50° W., temperature has fallen, the 15° line disappears, the 10° and 5° lines are packed close together, and the 0° isothermal comes down to lat. 45°. In mid-Atlantic the 15° line has moved northward, chiefly in the region of long. 25° to 30° W., where it appears in about lat. 45°, and it continues in lat. 40° to 42° to the Portuguese coast. There is a distinct rise of temperature west and north-west of the British Isles—the 10° isothermal touches the north-west of Ireland, and the 8° line has recovered its February position. The 5° line starts at the south-eastern corner of Iceland, runs to just north of the Faeroe Islands, turns north-east till it cuts the meridian of 0° in about lat. 64° N., and then goes north to lat. 70°, where it turns eastwards. The line of 0° follows a similar course somewhat to the north-west of the 5° line, but north of 72° it bends repeatedly, recrossing the meridian of 10° E. several times, the warmer water lying always to the eastward.

Temperature has apparently undergone little change to the west of Iceland, the 5° line starting from about the middle of the western coast. To the south and south-west of Greenland the isothermal forms a tongue pointing W. and N.W., temperature 0° to 5°; further up Davis Strait there are very low temperatures near the land.

In the eastern region the chief changes in salinity are a marked fall off the coast of the United States and a rise in about long. 58° W., making the lines run more S.W. and N.E., and packing them closer together. Between the 40th and 50th meridians all the lines have moved south-eastward, the 36 line a little, the 35 and 34 lines more, hence the gradients are steeper. In the south-eastern region the position of the 36 line last month is confirmed. and the 35·5 line remains unaltered.

A narrow belt running through the Faeroe-Shetland Channel, and apparently widening out beyond it, is enclosed by the 35·3 line. The 35 line skirts the coasts of the North Sea and the entrance to the Skagerak, and then sweeps westward to the meridian of 0° in about lat. 64° N. before finally turning N.E. Another branch of it runs close to the south of Iceland, and then apparently turns slowly to the north-east to form the north-western border of a wide belt which runs eastward to the north of Norway and spreads northwards and westwards to Spitsbergen and the meridian of 10° E.

It would seem that between Iceland and the Faeroe Islands the freshening of the surface water noticed last month continued; and that west of Iceland salter waters appeared farther north. The S-shaped form of the 35·0 and 35·3 lines in mid-Atlantic is one of the most important features of this chart, being a complete change, at least for the latter, since the previous month, and representing an increase of salinity over a wide belt just south of lat. 60° N. The westward-pointing tongue formed by these lines is continued into Davis Strait by the 34·5 and 34·0 lines.

Apart from the increased information afforded by this map, the most important features are—marked steepening of gradients, both of temperature and salinity, on the American side; extension of the cold and fresh area south-eastward, between long. 40° and 50° W.; absence of any change in the south-eastern region; slight freshening and steepening of temperature gradients east and south-east of Iceland; sudden extension of a salter area westwards just south of lat. 60° N.

May, 1896.—This month is marked by great rise of temperature, especially in the lower latitudes, with the advancing season. In the Gulf Stream region the isothermal of 20° reaches the 40th parallel between 55° and 65° W. long., and further east the 15° line runs up to lat. 50°. The 10° and 15° lines have again closed in south-east of Newfoundland, by a change like that between January and February. The 5° line has not moved much, but the water between it and the coast has become warmer, the line of 0° now only appearing off Cape Race. In mid-Atlantic the rise of temperature only appears south of 50° N. lat., the 12° and 15° isothermals have moved up, but the 10° line remains unchanged. East of long. 30° W. the distribution has become more complex, there is everywhere a rise of temperature—the 15° line comes up to 50° N. lat., and near to the entrance to the English Channel, the 12° line touches the north of Ireland, the 10° line runs up to 60° N. north-west of Scotland, and the 8° line nearly takes the place of the 5° line east of Iceland, although the latter has been very little displaced. But the *form* of the isothermals has altered considerably, the axis of maximum temperature in the Faeroe-Shetland Channel has become more strongly marked—note the two 9° lines, and the "head" of the 10° line—the 8° line is bent sharply round at the Faeroe Islands, and to south-east and south of Iceland the rise of temperature from last month is relatively small. West and south-west of Iceland the isothermals form a small wedge pointing northwards, while on both sides of the southern extremity of Greenland temperature seems to have fallen somewhat. On the west coast of Greenland temperatures are higher north of about 62° N.

Off the coasts of New England and Nova Scotia the fresher waters have retreated closer to the land, and the 36 line re-appears. There is little change off the Newfoundland Banks, but in the lat. 45° to 50° N. long. 40° to 45° W. area the salinity is greater. In the south-eastern region the 35·5 line retains its position, but east of long. 20° W. the great bend of the 36 line has disappeared, and the line runs straight on to the middle of the Bay of Biscay.

North of 50° N. lat. the 35 line has moved nearly 10° to the west, and it now runs nearly straight south along the meridian of 40° W. In the region between Norway and Iceland the lines have become much crowded together; a patch of water of over

35·2 north of the Shetlands protrudes into the North Sea, and north of the Faeroe Islands has narrow ridges running out N.E. and S.W. Salinity falls rapidly to below 33 *pro mille* between this patch and the Norwegian coast, and a tongue of water below 35 runs south-eastwards from the east of Iceland. There are indications of increased salinity to the west and north of Iceland and along the west coast of Greenland, but the April observations are insufficient to make this quite certain. It is probable that the 34 line reaches a much higher latitude in Davis Strait and keeps nearer the Greenland coast, and that the 35 line west of Spitsbergen has moved westward.

Thus the main features are—great increase of the seasonal element in the changes of temperature; closing in of warmer and salter water towards the land on the American side; equalising of salinity in the south-east; great freshening of water off the Norwegian coast and between Iceland and the Faeroes; probable increase of salinity west of Iceland and Greenland.

June, 1896.—The rise of temperature in the south-western region continues, but the distribution has now become very irregular—the isothermals probably representing only the general features of complex changes. The isothermal of 20° follows the parallel of 40° N. as far east as 35° W., the 10° line skirts the edge of the deeper waters, and the 5° line appears close to the south coast of Newfoundland, and again out to the eastward. Temperature has risen slightly in the south-eastern area, the line of 18° now appearing near the 40th parallel, but the 15° has scarcely moved, except near the land, where there is a marked rise.

In the north-east, the 10° line now follows the 60th parallel from long. 25° W. to long. 10° W., where it turns up into the Faeroe-Shetland Channel, defining a narrow belt of warm water west of the Shetlands. Temperature has risen a little between the Faeroes and Iceland, and east of Iceland, and the band of cold water is rather less clearly marked. West of Iceland a large rise has taken place, the 8° line taking the place of the 5° line and sweeping well out to sea, while to the north-west and north the 5° line replaces that of 0°.

South-east of Cape Farewell the 9° line retains nearly its last month's position, while the 5° bends far to the westward. West of Greenland, in lat. 60° N., temperature has risen 3°, further north there is little or no change along the coast. North of 70° N. temperatures are highest near the American coast, in the freshest waters.

South of Spitsbergen the temperature falls rapidly from 5° to 0° towards the east and north-east, slowly towards the west and north-west.

West of long. 50° W. salinity has diminished—the 36 line has disappeared, and the 33 line is crowded close down to the 35, which, however, has scarcely moved. In the lat. 40° to 50° N., long. 40° to 50° W. area, the increase of salinity noted last month continues, the 33 to 35 lines now running north and south, slightly to the east of long. 50° W. East of this area, salinity has increased, both the 36 and 35·5 lines being again convex northward, but this is partly due to a decrease of salinity in the Bay of

Biscay and off the Portuguese coast. Between 50° and 55° N. lat. the 35 line has moved eastward, almost to its old position, and it is closely followed by the line of 34·5.

Salter waters cover a large area to the eastward and northward, the 34 line touches the Norwegian coast, and the 35 line presses it closely. The lines of lower salinity (33 and 34) have retreated somewhat to the east of Iceland, and salter water extends up to the south coast of that island (line of 35·2). West of Iceland there is an immense increase of salinity; the 35 line, which has the same position as last month in lat. 60°, runs almost due north to the coast of Greenland.

Salinity has apparently increased considerably in the northern parts of Davis Straits. It seems likely that the 35 line should more closely follow that of the 34 west of Spitsbergen.

The general features are thus—lowering of salinity with irregular rise of temperature in the south-west; increase of salinity in the south (middle) and south-east; spreading of salter area in the Faeroe-Shetland region, and more especially towards the Norwegian coast: great increase of salinity west of Iceland: and lowering of salinity in mid-Atlantic just north of lat. 50° N.

July, 1896.—The isothermal of 25° now appears north of 40° N. lat. between about 52° and 64° W. long. The 20° line starts from New York and runs eastward very close to the 25° line, and between it and the land the distribution of temperature is very irregular, with warmer and colder patches, but rarely falling below 10°. East of Newfoundland there is a considerable area of water between 5° and 10°, and off the Labrador coast temperature seems to keep below 5° far to the eastward.

In the middle south region the surface is warmer generally, the line of 20° appears in about lat. 42° N. and the 15° line follows the 50th parallel. The 20° line runs eastward to the Portuguese coast, while the line of 15° goes north-east to the north of Ireland.

West and north-west of Scotland there is a further rise of about 1°, but the 10° and 12° lines continue the sharp bend towards S.E. shown by the 8° and 9° lines in the previous months. Except for a narrow band close to the land, the whole area between East Iceland and Norway is now about 5°, the 5° line cutting the 70th parallel in about long. 5° W., whence temperature falls to 0° off the Greenland coast. West of Iceland temperature rises to 10°, and the cold waters are apparently only met with close to the land. Temperatures are also considerably higher towards the eastern side of Davis Strait.

South of the American coast, as far east as long. 50° W., there is little change in the distribution of salinity, but to the south-east of Newfoundland the fresher waters extend much further eastward, the 35 line having moved nearly to long. 40° W. East of Newfoundland, in lat. 50°, the lines have opened out, the 34 line is further to the west and the 35 line further east. From long. 40° W. the 35 line takes another bend eastward to long. 30°, recurving in a wider loop than before to the same position as last month in lat. 57°. This change of form is closely followed by the line of 34·5.

In the south-eastern region the increase of salinity noticed last month is continued, the 36 line almost reaches the mouth of the Channel, and the 35·5 line takes a fuller northerly bend off the west of Ireland.

Observations are unfortunately not numerous in the Orkney and Shetland region. A central axis of 35 water probably extended far south into the North Sea during both June and July. East of Iceland the water is fresher, and a narrow strip of water of less than 34 *pro mille* salinity runs along the south coast of the island. The belt of 35 water to the west of Iceland is apparently narrower, but the observations do not make this quite clear.

The observations in the north of Davis Strait do not indicate any marked change.

The principal features are thus, continued rise of temperature everywhere, without much change in distribution, except to the south-east of Newfoundland, where the rise is slight; a slight increase of salinity in the south-eastern area; marked freshening east and south-east of Newfoundland and in mid-Atlantic between lat. 50° and 55°; freshening and narrowing of the low-salinity belt east of Iceland, and extension of that belt round the south side.

August, 1896.—In the south-western area, under the land, the distribution of temperature is much simpler. The position of the 20° and 25° isothermals is little changed, but instead of irregular patches of colder water there is a steady fall towards the north-east to 12° just off Cape Race. This means on the whole a considerable rise of surface temperature in the shallower waters. There is a great rise of temperature to the east of Newfoundland, but the area of relatively cold water retains its position, and its form is better defined by the greater number of observations—note the loops in the isothermals of 8°, 10°, 12°, and 15°.

Between the meridians of 30° and 50° W., the isothermal of 20° remains in its position near lat. 50° N., but the temperature has increased further south, the 20° line being now close to it. To the north of lat. 50° there is a very marked change of temperature, the 10° line has moved northward and westward and straightened out, closing in towards the coasts of Labrador and Greenland, while the 8° line, parallel to it, almost touches Cape Farewell.

In the south-eastern area temperature is practically unchanged, but to the west of Ireland it has fallen, causing the 15° isothermal to follow the 50th parallel closely for nearly 30 degrees. The line of 12°, which further west has moved somewhat to the north-west, following the 10° line, remains in the same position to the north and north-west of Scotland, but it has moved far to the eastward from the Faeroe-Shetland Channel, changing from a V to a U shape, and leaving an area with a temperature of 10° or less. This area probably extends far northward and eastward over the Norwegian Sea; the line of 9° reaches long. 30° E., and temperatures over 5° occur far along the north coast of Europe, and again south and west of Spitsbergen. North and east of Iceland the distribution of temperature has probably undergone little change.

Between Iceland and Greenland the cold-water area has increased, the line of 10° having moved to the north-east. Temperature seems somewhat higher in the middle of Davis Strait, but there may be more ice-cold water close to the coast of Greenland.

Salinity observations are unfortunately almost wanting for August along the parallel of 40° N. and 60° N. In the southern latitudes there seems to be little change. The line of 33 *pro mille* is in nearly the same position west of long. 50° W., but the 32 line is no closer to the land. The ends of the 34 and 35 lines in lat. 46° N. are in the same place as last month, and the part of the 36 line shown is also unchanged. In lat. 50° N. the 35 line is bent a little more to the south, and the 35·5 line more to the east, indicating a further lowering of salinity towards south and east. Salinity seems to have increased slightly off the mouth of the English Channel, and a tongue of 35 water protrudes further through the Straits of Dover into the North Sea. The axis of 35 extending into the North Sea, from the north, has withdrawn somewhat.

South of Iceland, the line of 35 salinity has moved closer inshore, and off the Faeroe Islands its position is also a little more to the north. The form of the line has, however, changed; it now runs out to the east, greatly narrowing the belt of water over 35, and the 34 line east of Iceland has moved southward and taken a similar form.

The line of 35 follows the coast of Norway, packing the 34 and 33 lines close in between it and the land, and forms a loop south of Spitsbergen in about long. 20° E.—which bends round westward and probably joins the other part of the line by following the meridian of 0°. An isolated patch of 35 water appears north of the White Sea.

A considerable freshening has apparently taken place to the north of Iceland, and there now seems to be no 35 water between Iceland and Greenland. It appears also from the form of the 35 and 34·5 lines that a considerable lowering of salinity has taken place in the area to the south-east of Cape Farewell.

The principal changes are therefore—small changes in the low latitudes: equalisation of temperatures and salinities west of long. 50° W.; slight rise of temperature in mid-Atlantic; the colder fresher waters form a wider belt to the east and south-east of Iceland, and they occupy more of the Denmark Strait and the area east of Cape Farewell; temperature has risen in mid-Atlantic between 50° and 60° N. lat., salinity has fallen a little in the southern part of that area, and more in the northern.

September, 1896.—In the south-west the 25° isothermal follows the 40th parallel from long. 50° W. to long. 70° W., and between these meridians temperature is unchanged northwards to the land, except for a very narrow strip of cold water (under 15°) close inshore. This strip extends westwards, south of Long Island to New York, bending the 20° line southward, and eastward and south-eastward past Cape Race, where it deflects the 15° and 20° lines southward.

Temperature is practically unchanged in the middle and eastern Atlantic between lat. 40° and 50° N.

Between lat. 48° and 55° N. the principal changes are in the west and north-west: lower temperatures prevail eastward from the coast of Labrador, and to the south-east off Newfoundland, and north-east the form of the isotherms has undergone a marked change, tongues of colder water pointing S.E. and E. respectively. Note especially the changes in the 10°, 9°, and 8° lines.

South of Greenland, temperature has risen: there are loops to the westward in the 8° and 9° lines. The rise of temperature extends north into Davis Strait: note the change of form and position of the isothermal of 5°.

To the west and south-west of Iceland temperature has fallen further, the 10° line does not now touch the west of the island at all. East of Iceland temperature has fallen considerably, and the cold area seems to have extended. A uniform general fall seems to have taken place in the Norwegian Sea up to high latitudes.

Comparing the salinities in the lower latitudes with both July and August; the salter waters appear further north; the 33 line has moved up towards the north-west of long. 55° W., and the gradients have become very steep, the 36 line pressing close up in long. 60° W. In long. 50° the isohalines bend southward, and there is a marked freshening of water south-east of Newfoundland; the 34 line now touching the 40th meridian.

In the south-eastern region the 36 line remains unmoved, but salinities of 36·5 appear near the Azores.

Along the whole length of a belt about 2° or 3° on each side of a line joining lat. 50° N., long. 40° W. with the west of Scotland there is everywhere a marked lowering of salinity: the 35 line is moved about 5° to the east, and the 35·5 line disappears from the area altogether. To the north of this there appears to be a slight increase of salinity across a strip which is widest south of Iceland, but extremely narrow further west. South-east of Greenland the area of fresher water seems to have extended, although the observations are insufficient to be certain of this. East of Iceland and south of Spitsbergen the 35 line has retreated southward.

The chief changes are thus—in the south-west a rise of temperature and salinity at the head of the Gulf Stream: and the appearance of a colder fresher band close to the land, which extends east and west; increase of salinity with small change of temperature in the south-east; slight fall of temperature and marked fall of salinity southwards and eastwards from the east of Newfoundland, and right across the middle belt of the Atlantic to the British Isles; fall of temperature and salinity east of Iceland, south-east of Greenland, and in the north round Spitsbergen.

October, 1896.—In this month the seasonal changes of temperature are again strongly marked. Temperature has fallen considerably near the 40th parallel west of long. 50° W., except in about long. 65° W., where the 25° line still appears. Near land the fall averages about 4°, and the 10° line now clears the south of Newfoundland.

East and south-east of Newfoundland there is a very great fall of temperature; the 20° line is cut through, and the 15° line bends south to lat. 42° N.; the curve of the 10° line has filled out, and moved a long way to the east.

In the lower latitudes the fall of temperature becomes less as we go eastward—till near the Portuguese coast. Near the Azores temperature is unchanged. The 15° line has moved irregularly southward except in mid-Atlantic, where it remains stationary. The 10° line has moved southward, and changed its form, since the greatest change has taken place west of the Faeroe Islands and of long. 30° W. Temperature has fallen considerably west of Iceland and in Davis Strait, and south of Greenland the distribution has become more uniform—note the form of the line of 7°.

Salinity does not appear to have changed much west of long. 50° W., except for a slight increase south of the Gulf of St. Lawrence. Between 50° and 40° W., however, the isohalines up to 35·5 have crowded together to the N.W. East of this the 36·0 line and the 35·5 lines have moved northward, and there is a marked increase of salinity east of 20° W. long. up to the south-east of Iceland. Note the position of 35·3. The 35 line is also pressed towards the coast of Norway, and there is apparently a large surface of 35 water in the North Sea. Salinity has also increased to the west of Iceland.

South of Greenland there is a freshening of the surface water, the bends of the 35 and 34·5 lines retain the form shown last month, but very much compressed, and driven southward and eastward. Salinity has probably diminished in Davis Strait—at least since May, the last month with adequate observations.

The chief points, besides the marked seasonal fall of temperature, are—great fall of temperature and salinity southward on the meridian of 50° W.; increase of salinity towards the north-west, and relatively small change of temperature east of that line; increase of salinity between Scotland and Iceland, in the North Sea, and west of Iceland; lowering of salinity and equalisation of temperature in the central region south of 60° N. lat.

November, 1896.—In this month the seasonal change becomes still more marked, and, as in the season of rising temperature (May), the distribution of temperature becomes very irregular. In mid-Atlantic, the fall of temperature in the lower latitudes is scarcely appreciable—note that the 15° line has hardly moved. On the 50th meridian, south-east of Newfoundland, the fall averages about 5°, but the isothermals retain almost the same form. West of this, the fall near land is 3° to 5°, but much less at the head of the Gulf Stream in lat. 40° N.

Off the European coast, the fall amounts to 2° or less, except in the north, in the Faeroe-Iceland region, where the 5° and 8° lines have scarcely moved. South of

Iceland the 10° line has retreated southward, but further west, *i.e.*, south of Greenland, the fall is inconsiderable till the American coast is approached. A marked fall has taken place east of Cape Farewell—note the change in the 8° line, and the change of places by the 7° and 5° lines.

There is practically no change of salinity from last month south of lat. 50° N. (the loop in the 36 line in mid-Atlantic is uncertain) except for an extension eastward of the 32 line from Cape Race. Between 50° N. and 53° N. the loops of the 35 and 35·5 lines have increased in size and now extend much further to the east, while to the north of 55° the 35·5 line now bends westward, and the 35 line shows a wider and fuller loop (compare September) to the west and north. The fresher waters east of Southern Greenland now extend to the south of Iceland—note the 34 line, and the whole line from the east of Iceland to the Faeroes, Shetlands, and Norway is probably occupied by water under 35. The observations show the 35 line between the Faeroes and Shetlands, but it must be regarded with suspicion.

South of lat. 50° N. there are therefore few variations beyond the ordinary seasonal changes of temperature; north of 50° N. the lines have moved westward; south and south-west of Iceland the water is fresher and colder.

December, 1896.—The distribution of temperature is again very irregular, and the seasonal change from November well marked. The 20° line north of 40° N. lat. has shrunk to a small arc between long. 55° and 60° W. The 10° line has, roughly, taken the place of the 15° line, and the 5° line of the 10°, as far east as long. 50° W. East and south-east of Newfoundland temperature has fallen to 0°; the 10° line has moved little, but beyond it temperature has fallen; the 15° line has retreated to the south-east.

In the south-eastern region a slight fall of temperature has taken place, but there is no marked change.

The 10° line has scarcely moved between long. 20° W. and the Irish coast, and north of this to between Iceland and the Shetlands there is little or no change (lines 8°, 7°, and 5°). Toward the Norwegian coast, however, the fall seems to have been considerable.

A considerable fall of temperature has evidently taken place in the northern central region—note the portion of the 8° line close to that of 10°, which has also moved southward.

Salinity has changed little in the fresher waters off the American coasts, the positions of the 32 and 33 isohalines being practically unaltered. Further seaward there is a marked increase of salinity northward and westward, indicated by the 34, 35, and 36 *pro mille* lines. Off Cape Race the belt of fresher water protrudes further to the south, while in the area bounded by 40° and 50° N. lat. and 40° and 50° W. long. there is a general equalisation—the 33 line has moved westward, and the 36 line southward. A similar tendency to more uniform distribution appears all over the Eastern Atlantic, the 36 line has moved southward, especially near the coast of Europe, while one part of the broken 35 line shown has moved northward, and the other runs along the south coast of Iceland.

The principal features are thus—marked seasonal fall of temperature off the American coast, with increased area of salter water round the head of the Gulf Stream. Slight fall of temperature in the south-eastern and eastern areas, with marked tendency to a more uniform distribution of both temperature and salinity.

January, 1897.—In the lower latitudes temperature has scarcely changed, except near the land—note the isothermals of 15° and 20°. The low temperature area off Cape Race has extended itself eastwards, but very slightly: the axis of the 0° line has not moved. In the Eastern Atlantic temperature has fallen in the Bay of Biscay, to the west of Ireland and Scotland, and in the North Sea, but there is no change in the Faeroe-Iceland region. In the northern region, between long. 30° and 40° W., temperature has fallen very considerably.

West of long. 52° W. the isohalines have straightened out; the 36 line has altered little, but lines of lower salinity have moved southward. East of Cape Race the fresher waters have moved considerably to southward and eastward (note the 30 line), while further south, along 50° W. long., the salter waters tend to close in, giving a similar straightening.

In the eastern and north-eastern part of the ocean there is an increase of salinity in all latitudes; the line of 36·5 reappears east of long. 30° W., and the isohalines 36, 35·5, 35·3, and 35 all trend more to the north. In the northern region (*i.e.*, south of Iceland) this increase seems to extend considerably to the westward.

Thus we have—small changes of temperature near 40° N. lat.; extension eastward of a cold freshwater area from Cape Race; in the eastern region a fall of temperature, decreasing northward, but a marked increase of salinity, which extends also northwestward.

Compared with the corresponding month of 1896, the area west of 50° W. long. is warmer and salter, the axis of greatest warmth and salinity being more to the east. The same differences hold good east of Cape Race, hence there is not the same southward protrusion of a cold and fresh area. There is again greater warmth and saltness in the south-eastern region. In the Faeroe-Iceland region temperatures appear to be lower, and probably also salinities, but it is impossible to be certain about the latter. In the northern central portion temperatures are certainly lower, probably also salinities.

February, 1897.—West of long. 50° W. a fairly uniform fall of about 5° has taken place; the observations are insufficient to specify details, but there does not appear to be much change in the general distribution. In the area between 40° and 50° N., and 50° and 60° W., the axis of low temperature has moved eastwards—a rise has taken place in the south-western part of the area—see line of 15°, and a fall in the north-eastern part—line of 10°.

East of long. 40° W., the line of 15° has moved southward, the 12° line remains stationary, and the lower isothermals have moved southward west of long. 10° W. Off the west of Scotland and in the Faeroe-Shetland Channel the water is warmer West of the Faeroes temperature has scarcely changed.

Salinity appears to have changed little in the western region, except for a fall in the neighbourhood of lat. 50° N. long. 40° W. East of long. 40° W. there is a further equalisation ; the 36 line has moved down to lat. 40° N., the 35·5 line has straightened out towards the N.W. of Ireland, and salinity has increased slightly in the Faeroe Channel and towards the south of Iceland. The 35 line has moved northward in the North Sea, but a tongue of 35 water protrudes through the Straits of Dover.

The changes in distribution during the month are therefore small, colder and fresher water has spread east of Newfoundland, the distribution of temperature and salinity has become more uniform in the south-east, north-east the surface water is slightly warmer and salter.

Compared with February, 1896, the distribution of temperature is more irregular in the south-west, the lines showing a double bend, while the 5° line is apparently further from the land. Salinity, on the other hand, shows more uniform distribution. Temperature and salinity are lower in the south central and south-eastern regions—note the lines of 15° and 36 *pro mille* ; and again south-east of Newfoundland. In lat. 50° N., long. 30° to 40° W., the conditions are little different from last year. Temperature and salinity are lower in the north-eastern regions and in the North Sea.

March, 1897.—South of Newfoundland and the Gulf of St. Lawrence the isothermals have crowded together—the 15° line has moved northward and straightened out between 50° and 60° W. long., and the 0° line appears nearly parallel and a long way from land. In the south central region the tongue formed by the 10° and 12° lines has disappeared, giving a rise of temperature in about lat. 46° N., long. 43° W. Further east little change is apparent south of lat. 50° N., except that the 15° line has reappeared on the Portuguese coast.

West of Ireland and Scotland, and south of Iceland, a distinct rise of temperature is apparent, and this extends along a narrow axis west of the Orkneys and Shetlands, and into the North Sea. East and south-east of Iceland, on the other hand, temperature seems to have fallen.

In the south-west the isohalines seem to be crowded together in a manner similar to the isothermals, but the February observations are hardly sufficient for comparison. Between 40° and 50° N. lat. there is a very distinct movement of the isohalines to the N.W.. off the Newfoundland Banks. East of this salinity has also increased, the 36 line having moved northward. In the north-east there appears to be little change, there is if anything a slight increase of saltness, except north-west of the Faeroe Islands.

Thus the principal features are—a rise of temperature and salinity at the head of the Gulf Stream, and a fall of temperature, and probably also of salinity, in the same region off the land ; a rise of temperature and salinity in all the south central and south-eastern regions; and a rise of temperature in the north-eastern, probably with slight increase of salinity, except between Iceland and the Faeroes.

Comparing 1897 with 1896, the characteristic difference in the south-west is the greater warmth and saltness along the line of 40° N. to the west of long. 50° W., and the crowding together of the lines immediately to the north of this. In the south-east and east, and probably in mid-Atlantic, the distribution of both temperature and salinity is much the same as last year, but in the Faeroe-Shetland Channel and North Sea temperature and salinity are lower. Temperature is very markedly lower to the west of Iceland.

April, 1897.—The distribution of temperature in the south-west has become very irregular: there is no very clear rise or fall, but the crowding together of last month has given rise to bending or "interdigitating" of the lines. Little change is apparent in the south centre, south-east, and east, except a slight seasonal rise, and in the south a tendency to equalisation of temperature shown by the straightening out of the 12° and 15° lines. North of the Faeroes temperature has fallen slightly—the 5° line has filled out southwards and eastwards. Temperature has risen to the west of Iceland, and west of Cape Farewell.

In the south-west there is distinct increase of salinity just north of lat. 40° N. and about long. 50° W.—note the appearance of the 36·5 line and the retreat of the 34 line, which now has a sharp bend. In the south centre there does not seem to be much change, but there is an increase of salinity in the east and north-east—note the position of the 35·5 line, and the replacing of the 35 line by that of 35·3. The increase of salinity does not, however, appear to extend north or west of the Faeroe Islands, to judge from the form of the 35·3 line.

Thus the changes are, on the whole, slight; the distribution of temperature has become irregular in the south-west, and in the south centre, south-east, and east up to the Faeroe Channel salinity has increased.

Compared with April, 1896, the temperature is much the same west of long. 50° W., but east of that meridian it is lower—the lines running more east and west—note the isothermals of 8°, 10°, 12°, 15°. Temperature is lower in the Faeroe-Shetland Channel, but while the 5° line is in nearly the same position, the water east and north-east of Iceland is warmer. East of Greenland temperature is lower—note the 5° line S.W. of Iceland and S. of lat. 60°, but to the west slightly warmer (3° line).

West of long. 50° W. salinity is on the whole greater, and between long. 40° W. and 50° W. there is a marked increase. The increase holds good all over the south-eastern and eastern area, up to the Faeroes, and across to the south of Iceland. Beyond this the form of the 35 line, so far as it can be depended on, supports the temperature observations in indicating more uniform conditions than last year.

West and south-west of Iceland the water is fresher, the 35 and 35·3 lines have retreated south-eastward.

May, 1897.—In the south-western region the seasonal rise of temperature is great near the land, and in lat. 40° N. the 20° line has reappeared. The 10° and the 15° lines have not altered their positions much. In the south central region temperature has risen, and there is again a marked rise near the European coast, but north and north-east of the Azores there is little or no change. The 8° line has filled out north-eastward between Iceland and the Faeroes, and off the Faeroes the 5° line has retreated northward, but east of Iceland itself there is little change. West of Iceland there is also little change ; temperature has risen considerably south-east of Greenland and again to the west of Greenland. The greater number of observations east and north-east of Newfoundland show a warm area extending westwards towards Labrador.

Salinity has fallen along the 40th parallel west of long. 50° W., the 36 line has moved southward and eastward, while the lines of lower salinity bend southward, along the coast, towards the west. East of Newfoundland the salter waters extend further to the north and west—note the 35 line.

In the south-eastern region the 36 line has retreated from the European coast and moved south-westward nearly to the Azores, while in the north-east the 35·5 line sends a tongue into the Faeroe Channel. The greatest change appears in the north central region, where the 35 line has moved westward and north-westward—bending up into Denmark and Davis Straits, and apparently running parallel to the Labrador coast. The 34·5 line runs well to the N.W. of Iceland. Note the position of the 35 line west and south of Spitsbergen and off the Norwegian coast.

The characteristic changes are therefore marked fall of salinity and absence of seasonal rise of temperature north and north-east of the Azores ; increase of salinity and rise of temperature in the whole of the northern area—in the Faeroe Channel, east of Greenland, west and south-west of Greenland, across to Labrador.

Comparing the distribution with 1896, the Gulf Stream "axis" in the south-west lies further to the eastward—compare temperature 20° and salinity 36 *pro mille*. In the central southern area temperature and salinity are both greater, the lower lines being crowded northwards and westwards towards Newfoundland and Labrador. In the south-eastern region temperature and salinity are lower.

In the north-eastern region temperature is very markedly lower—note the isothermals of 10°, 9°, 8°, and 5°. This difference holds good to a greater or less extent over the whole distance from south-eastern Iceland to the coast of Norway, into the North Sea, and along the south coast of Iceland. West of Spitsbergen temperature seems higher, east and south-east of Greenland rather lower, and west of Greenland markedly higher.

The salinity observations do not, unfortunately, show the distribution east of Iceland in 1897. It appears, however, that while salter water extended up the west side of the British Isles, the surface of the North Sea was fresher. The salter waters apparently spread farther to the west of Spitsbergen, but not so far north, and a similar difference occurs off the Labrador coast and in Davis Strait.

June, 1897.—The isothermal of 20° apparently follows the 40th parallel closely almost all the way across, representing a marked rise except at the head of the Gulf Stream. Temperature has also risen near the land, but has remained almost steady to the south-east of Newfoundland.

Temperature has risen markedly everywhere else, but on the whole uniformly; the general tendency is for the isothermals to become somewhat straighter, and to trend more S.W. to N.E.

In the south-western region, salinity has increased to the west of long. 60° W., but diminished east and south-east of Newfoundland. North and west of the Azores salinity has increased again, the 35·5 and 36 lines have moved northward, and the 36·5 line reappears. The 36 line fails, however, to reach the Portuguese coast, and to the west of the British Isles salinity has diminished slightly. In the north-western region the 35 line has retreated southward and eastward.

Thus the chief changes are:—the increased salinity in the central southern area; the small change of temperature and distinct fall of salinity on the north-west and off Newfoundland; and the slight fall of salinity west of the British Isles.

Comparing with 1896, the distribution of temperature is more regular. West of long. 50° W. the waters near land are warmer, and the gradients seawards not so steep. A lower temperature extends east from Cape Race, but to the south-east the lines are rather more crowded together.

In the eastern area there is little difference till we come to the Shetland-Iceland region, where temperature is distinctly lower—note the 10°, 8°, and 5° lines. To the west of Iceland and east and south-east of Greenland there is no marked difference.

The differences of salinity closely follow those of temperature in the south-west. The isohalines are not so crowded, the very fresh waters not extending so far southwards from the land; but on the other hand the fresher water shows farther eastward.

Observations in the eastern region are rather deficient for 1897, but so far as they go they indicate little or no difference. The 35 line is more to the westward between lat. 50° N. and 57° N., but below lat. 60° N. it is bent sharply eastward again, instead of continuing northward as in 1896.

July, 1897.—In the south-west the axis of highest temperature is now far to the west in about long. 65° W., where it has risen to 25°. In about long. 50° W. temperature has fallen a little, the 20° line is broken, and the 15° line is further south. The 10° line now forms a loop east of Newfoundland, extending to nearly 40° W., and recurving westward at least to long. 50° W. On the 40th meridian the isothermals have scarcely moved; temperature has risen towards the coast of Europe, but the lines have straightened out farther. West of Iceland there is a rise close to the land, but the direction of the isothermal has changed. Note also the changed form of the 10° line S.E. of Greenland. In the south-western region an immense freshening of the surface water has taken place. West of long. 50° W. the 35 line now encloses a wide area south and east of the land. East of this area, in long. 45° W., the lines up to 36 are crowded closely together, and beyond this little change is apparent from last month. In the north-eastern region there is again little change.

In mid-Atlantic the S-shape of the 35 line is more marked, and there is an eastward movement of the whole.

Thus the great feature is an expansion of a large cold fresh area east and south-east of Newfoundland, and the freshening and cooling of the water east of Labrador and east and south-east of Greenland.

Comparing with 1896: the cold fresh area extends further south and about the same distance east of Newfoundland; salinity and temperature are somewhat lower in the south-eastern area. In the Faeroe-Shetland Channel, temperature is lower; west of the Faeroes the 10° line takes a similar course to the south-east of Iceland; east of Iceland the course of the 5° line shows much lower temperatures. To the east and south-east of Greenland the surface water is colder and fresher, while in Davis Strait the general tendency is towards greater warmth and saltness.

August, 1897.—In this month the relatively cold area to the east of Newfoundland is less pronounced, but it becomes more marked towards the south; the lines close in eastward off the land, and westward in about long. 45° W. In the eastern region there is a general rise of temperature, which becomes very marked in the Faeroe-Shetland Channel—note the changed position of the 10° line. To the east, south-east, and west of Greenland the type of distribution remains the same. There is a great rise of temperature in the north of Davis Strait.

There is a marked fall of salinity west of Cape Race along the land. East of this the isohalines tend to run more north and south, but the higher lines are more to the eastward. In the north-eastern region the 35·3 line has opened out across the Faeroe-Shetland Channel, and the 35 line has shifted to the W. and N.W.

The most important changes are therefore the spreading of the fresh water area, which has become hotter near the land; and of the salter waters towards the north-east and north-west.

Temperature and salinity are higher off the United States coast than in 1896, but off Newfoundland the cold fresh area extends both farther east and farther south. In the south-east temperatures are much the same, salinities slightly lower; north-east temperatures are higher in the Faeroe-Shetland Channel, salinities apparently nearly the same. South of Spitsbergen the 35 water does not seem to cover such a large area, but (if we compare also July, 1897) it curves more to the west.

To the south-west of Iceland temperatures and salinities are both lower—note the 35 line and the 10° line—but this difference does not persist over any great area to the south or south-east of Greenland, at least in the case of the salinity.

September, 1897.—Temperature has fallen considerably in the south-western area, and especially to the east of Newfoundland, where the bend of the 12° line has filled out. In the south central and south-eastern region temperature is unchanged. East of Iceland, and in the Iceland-Scotland region, a considerable fall has taken place; the 10° line takes a wider bend, and the 12° line is turned round into the North Sea. To the east of Greenland the 10° line has retreated south-eastward, but in the area south of Greenland and east of Newfoundland and Labrador temperature has risen slightly, the 10° line and the 12° line being fuller. There appears to be little change in Davis Strait.

Salinity has changed little under the land west of Cape Race. To the east of Newfoundland the isohalines have become crowded together towards the land, and the 35 line has moved irregularly north-westward into the area north of 50° N. lat., and between long. 40° and 50° W. The 35·5 line appears close to the Greenland coast. West of the British Isles the 35·5 line has moved northward.

The principal change is therefore the extension of salter areas northward and north-westward.

Comparison with the corresponding month of 1896 is somewhat difficult in the south-western regions, as the observations for this year do not extend down to lat. 40°. Salinity is higher close to the land south and south-east of Newfoundland; temperature slightly lower—the 12° line has a wider bend. In the south centre and south-east temperature and salinity seem nearly the same in both years, but north of a line joining Cape Race and the north of Scotland salinity and temperature are nearly everywhere higher—note specially the 35 *pro mille* and 10° lines.

October, 1897.—A considerable fall of temperature appears all over the south-western region, but the fall is most marked round the south and east of Newfoundland, where a large area of relatively cold water runs S.W. and N.E. East of this area the fall of temperature is very slight, even along the west coast of Europe to north-west Scotland. East and south of Iceland a considerable fall has taken place—note the retreat of the 8° and 10° lines, and there is another marked fall south of Greenland, in about lat. 55°, and from there towards the Labrador coast.

The changes of salinity correspond to those of temperature. The cold area south-east of Newfoundland is an area of relatively fresh water: in the eastern and south-eastern regions there is little change: south of Greenland, in about lat. 55° N. a fresh water area extends eastwards, while between it and the Newfoundland area there is a narrow ridge of salter water pointing westward.

The principal change is thus the extension of a colder and fresher area south and east from the coast of Labrador.

The *type* of distribution of both temperature and salinity is the same as in 1896, but there are marked differences. The fresh cold area off Newfoundland was broader, and its axis turned more east and west, in 1897 than in 1896: in the eastern and south-eastern regions temperature was slightly higher in 1897, but salinity markedly lower. Salinity is, however, higher than in 1897 along the 55° N. lat. region west of long. 20° W., the area of relatively cold fresh water not extending so far eastward. Temperature and salinity seem both lower on the western side of Davis Strait

November, 1897.—In the south-western region the higher isothermals at the head of the Gulf Stream have scarcely moved, but temperature has fallen to the west along the land, and in the area south and east of Newfoundland. East of this area temperature remains unchanged till about long. 30° W., where there is a marked fall, the 15° line opening out southward: east of 25° W. there is again no change. In the north-eastern region the temperature has fallen very slightly, chiefly to the west of Scotland, where a large area of about 10° temperature appears. In the region north of 50° N. lat. and on each side of 40° W. long. a very marked fall has taken place—the 10° isothermal is bent sharply round in lat. 50° N. long. 47° W., and runs almost due east for about 8°.

West of Cape Race there is no significant change of salinity. The fresh water off Newfoundland extends further eastward, and east of long. 45° W. all the lines have spread out south-eastward and southward in the direction of the Azores. In the north-east all the higher isohalines have moved southward—the 35·5 line does not get beyond 54° N. lat., and the 35·3 line scarcely crosses lat. 60° N. Note, however, that the 35 line appears between Norway and Spitsbergen. In the central area between the 50th and 60th parallels there is little change in the north (35·0 and 35·3 lines south of Greenland); in the southern half there is the southward movement already referred to, and the 35 and 35·3 lines also bend farther eastward.

The chief changes are thus the increased area occupied by relatively cold and fresh water east of Newfoundland and Labrador, and the extension of this area southward towards the Azores.

At the head of the Gulf Stream (long. 55° to 60° W.) the temperature and salinity are much the same in the two years, but south of the Gulf of St. Lawrence and Newfoundland a much larger area is occupied in 1897 by water of temperature about 5° and salinity below 34 *pro mille*. The steep salinity gradients and complex distribution of temperature east of the intersection of the 50th parallel and 40th meridian in 1896 are not reproduced, for the colder fresher area extends more to the southward (*i.e.*, towards the Azores), and not so much to the eastward (towards Ireland). Note the different positions of the 10°, 12°, and 15° isothermals, and the 35·0, 35·5, and 36·0 isohalines.

In the north-eastern region and south of Iceland temperature and salinity are markedly higher in 1897: the isohaline of 35 runs south-west from Iceland right across the area occupied by water below 34 in 1896. The type of distribution is on the whole the same, the fresher and colder areas extending more *eastwards* in 1896 and *southwards* in 1897.

December, 1897.—West of Cape Race a considerable fall of temperature has taken place, the 10° line touching lat. 40° N. just off Cape Cod. South and east of Newfoundland temperature has fallen largely, an axis of ice-cold water stretching south-eastward over the region covered by the 15° line last month. North of the Azores temperature has risen a little, but elsewhere there is a general fall of about

2° in the south-eastern area. Temperature has not fallen much in the north-east, but south-west of Iceland the 10° line has moved a long way to the south-east. This line keeps its position in lat. 50° N. on the 40th meridian, but to the north temperature has fallen in this longitude—note the movement of the 5° isothermal.

The salinity observations are, unfortunately, not very numerous for this month, being the last. They are sufficient, however, to show an immense extension of the fresh-water area east and south-east of Newfoundland, and some increase of salinity in the south-eastern and eastern areas.

The most important facts are therefore the extension south-eastward of the cold and fresh area from Newfoundland, and the fall of temperature, and probably also of salinity, in the region south-east and south of Greenland.

Except near the United States coast, temperature and salinities are lower in 1897 than in 1896, and this difference is greatly exaggerated east and south-east of Newfoundland. In the east and south-east temperature and salinity are higher than in 1896. The line of 8° runs along the east side of the Faeroe-Shetland Channel instead of the west, but the line of 7° is rather more to the north, and the salinity seems practically the same.

V. *The Movements of the Surface Waters.*

The detailed description of the charts has shown that while considerable variations of an irregular type occur, there are nevertheless changes over the whole area covered which evidently represent a continuous sequence. The charts for each month differ from those for the months immediately preceding and following in such a way as to form a satisfactory intermediate step between the two: they tell, in fact, an intelligible story about the distribution of both temperature and salinity, showing progressive changes bearing certain relations to each other, having certain seasonal phases which occur in both years, and certain features in which the one year differs materially from the other. The important point is accordingly established, that the method employed is adequate to its purpose. Hence, without employing special ships or observers, a continuous survey of the surface changes in the North Atlantic could be kept up with comparatively little trouble or expense: a much larger number of observations than I was able to deal with could be obtained, worked up, and charted for about £300 a year.

In attempting to explain changes of temperature and salinity in the surface waters without a knowledge of those occurring in the layers below the surface, it is necessary to consider separately the probable effects on these elements of seasonal changes without movement of water, and of horizontal or vertical movements with or without seasonal changes superposed. The most important points appear to be these:—

1. The annual range of temperature increases with the latitude, and the normal temperature gradient northward is greater in winter than in summer. Temperature

rises and falls quickly in spring and autumn, and the changes of temperature are slow near the minimum and maximum. The waters in areas enclosed by land are abnormally warmed in summer, forming a *Sprungschicht*, and cold water tends to sink below the surface in winter on account of its greater specific gravity.

Salinity has practically no seasonal variation (38), except perhaps in regions of permanent winds, where the evaporation is great, as in the region of the Trades, and possibly also on the north side of the Atlantic anticyclone. Thus while changes of temperature of water are due both to actual warming or cooling and to admixture with other water, changes of salinity are almost wholly due to the latter.

2. In the North Atlantic surface water in low latitudes is normally warmer and salter than water to the north of it or below it; hence an intrusion of water from the north, or a mixture with waters from below, reduces both temperature and salinity at the surface; and a movement of this water northward is indicated by an increase of temperature and salinity. This holds good at all seasons, but it is specially true of temperature in summer.

3. Surface water in high latitudes and near land is normally relatively fresh, on account of the large admixture of water derived from the land or from ice in proportion to the amount of evaporation. During autumn, winter, and early spring the low temperature of this water is strongly marked, but in the hotter months of the year this is not so characteristic, especially where land influences are strong, either in the way of direct heating or addition of large quantities of warm land water. Similarly the surface layer is normally fresher than that underlying it, colder than it in winter, and warmer in summer.

A southward movement of this water is therefore indicated by the extension of relatively low salinity at all seasons, and this is accompanied by a fall of temperature, which is in general well marked in winter, but not in summer. It is to be noted that any considerable southward freshening is a certain indication of southward movement, for the freshening by mixture with underlying layers or by heavy rainfall is slight, even in low latitudes (39).

4. A mixture of the surface water with the underlying layer produces the same apparent effect, in (*a*) low latitudes as an intrusion of water from the north, and in (*b*) high latitudes as an intrusion of water from the south.

5. In the case of water moving in an easterly or westerly direction, inequalities of temperature tend to disappear, through prolonged exposure to uniform conditions, and such movements can frequently be traced on the salinity maps after they have ceased to appear on those of temperature. In spring and autumn, when the distribution of temperature on land and sea tends to great local irregularities of heating and cooling, the isothermals give no reliable information about such movements.

Applying these as general principles, and keeping especially in mind the constant danger of misinterpretation due to No. 4, the rate and amount of mixing of surface and under layers being practically an unknown quantity, the following seems a fair description of the movements of the surface waters so far as can be gathered from the charts :—

In January, 1896, two surfaces of warm and salt water, one off the American coast and the other extending eastward from mid-Atlantic to the coast of Europe, were entirely separated from each other at the surface by a band of cold fresh water running south-eastward from Newfoundland to the parallel of 40° N. This band is evidently an offshoot from a large area occupying the whole of the region off the Labrador coast, and another band extends due east from this area. There was probably a third similar band south-east of Greenland, and certainly one east of Iceland.

During the two following months there was a persistent movement eastward and slightly northward over nearly the whole distance between lat. 40° and 60° N. The result is the cutting off of the southern end of the cold fresh band from Newfoundland, and the banking up of warm salt water towards the eastern side in the lower latitudes. The greater part of this escapes northward, but to the north of lat. 50° N. it is overlaid by the colder, fresher water from the Labrador coast, which has also moved eastward, and takes part in the northerly drift movement as it nears the land (note the 8° isothermal and the 35·2 isohaline to the south of Iceland in February and March).

In the month of April there is the first marked indication that the general easterly movement is losing strength in the higher latitudes. The easterly drift from Labrador begins to retreat, or rather to be absorbed by mixing, and it shrinks rapidly all round its edge, giving the appearance of warm salt water, moving westward, to the south of Iceland. Farther south there is not the same weakening of the eastward movement, but there *is* evidence, both from temperature and salinity, that more water is making its escape south-eastward.

At the same time (April and May) the southward movement of the fresher waters along the land begins again. South-east of Newfoundland the higher isohalines do not give way, but the lower lines are crowded together by an increase in the streams from the land. The area covered by this water shows a great rise of temperature in May, and in June it expands southwards and contracts eastward, indicating that it is then largely due to the water from the Gulf of St. Lawrence, which rapidly becomes warmer.

All this time a strong current of cold fresh water runs south-eastward from the north and east of Iceland ; the north-easterly drift from the Atlantic comes to the surface only on the east side of the Faeroe-Shetland Channel, and the 35 water appears over a large part of the North Sea. This south-easterly current apparently covers a larger and larger area as the spring progresses ; the north-easterly drift gets narrower and narrower, and the fresher surface water extends westward along the south coast of Iceland, though this is, no doubt, partly due to increased outflow of water from the land.

To the west of Iceland the branch of the Irminger current going northward gains strength quickly in spring, and apparently reaches its greatest surface extension in June. During the same period salt water makes its way steadily northward along the west coast of Greenland; this is the westerly branch of the Irminger current discovered by the "Ingolf" expedition.

In July the drift of water eastward from the American coast attains immense proportions in the lower latitudes. The "banking-up" of salt water (35·5 *pro mille* and over) towards the European coast becomes more marked, and with it the tendency to spread northward. But the eastward movement is still apparent farther north, a tongue of land water makes its way east from Cape Race, and again there is a drift from the Labrador coast.

Along with this there is everywhere a large increase in the supplies of Polar water. Strong currents appear running southward close to the Labrador coast; the Irminger current is overwhelmed by a rush of water southward across the whole breadth of the Denmark Strait, which gradually spreads over a large area to the south-east of Greenland; the current to the east and north-east of Iceland is strengthened, though to a less degree; and to the north of Europe currents of relatively cold and fresh water extend southward to the coast, entirely covering the 35 *pro mille* water except in isolated patches.

These conditions continue for two months, with the general result that gradients of both salinity and temperature become steeper and steeper on the margins of the areas described. In September a drift, consisting partly of fresh water from near the Newfoundland Banks which has been delivered there by the Labrador current, extends across towards the British Isles; the northern area is largely covered by Polar water, and between the two is the only part of the western branch of the Irminger current which appears at the surface.

The autumn conditions following the culmination of the summer type in August and September are chiefly the result of a weakening of the easterly currents south of lat. 50° N., and a strengthening of them to the north. The Labrador current again makes its way to the southward round the Newfoundland Banks, the stream being not now turned eastward, hence there is an increase of salinity immediately to the eastward (due, according to the temperature observations, to mixture from below; it is interesting to note the rapidity with which the fresh water advances and retreats south of the Newfoundland Banks, where it evidently forms a very thin though widespread layer), while the weakening of the easterly movement also causes lower temperature and salinity in the south-east Atlantic.

North of lat. 50° N. there is a strong easterly drift from the coast of Labrador, and another from the east coast of Greenland. The current from the east coast of Iceland is also deflected more to the north-east, broadening the north-easterly current between that island and Scotland.

This brings us to a distribution of temperature and salinity in December very similar to that observed in the preceding January, the principal difference being that the salter and warmer waters press closer up in the south-east and south centre, the current south-east of Newfoundland being much restricted, while temperature and salinity are much lower between the Azores and Portugal.

The development of the conditions observed in the spring of 1896 accordingly occurs earlier in 1897, but with this difference, that the movement appears to be, on the whole, more from the south, and the easterly components are weaker. The result is that the salter warmer waters spread more uniformly northwards and westwards over the whole of the central area of the North Atlantic; they come closer to the south-east coast of Greenland and the coast of Labrador, and occupy a wider area in Davis Strait. At the same time, perhaps because there is less "banking-up" against the European coast, the stream northwards and through the Faeroe Channel appears to be weaker—this appears chiefly from the temperatures. Both branches of the Irminger current seem to have less penetrative power. East and north-east of Iceland the distribution is altogether more uniform, as if both warm and cold currents were weaker than last year.

This development evidently culminates in May, but in April the enlargement of the southern end of the Labrador current is already apparent. Another change is the increasing southward movement between the Azores and the coast of Portugal, which becomes more marked than in 1896.

The easterly drift from the lower latitudes becomes well defined in June and July, but it does not attain the same development eastwards as in 1896; off the Labrador coast the fresh waters do not extend so far to the east, while the higher isohalines retain their position near the Greenland side of the entrance to Davis Strait. The branch of the Irminger current west of Iceland is evidently weaker; Polar water spreads south-east from Greenland, but again to a less extent than in 1896. The current east of Iceland tends to spread eastward. but northward rather than southward.

In August the characteristic change is a large extension southward and eastward of the Labrador current, and the apparent retreat southward and eastward of the salter waters in the Azores region. At this season the easterly drift was more to the south-east than in 1896.

The increase of the Polar current in Denmark Strait is well marked, but not so well as in 1896, and the Polar waters are slower in making their way eastward. Salinity observations are unfortunately wanting to the east of Iceland, and it is dangerous to draw any conclusions from the temperatures on account of the higher temperatures everywhere near land. It would seem that the Polar water continued to move eastward from the east and north-east of Iceland, but there is no information as to how far it covered the north-easterly current from the Atlantic.

So far as the information goes, the north-east current between Scotland and Iceland was weaker during the whole of the early part of 1897 than in the corresponding period of 1896, and in the far north the whole circulation of both warm and cold streams at the surface seems to have been slower.

In September, 1897, the general direction of the drift loses its southerly component, and in October it becomes due east. Hence there is at first "banking-up" of water against the European coast and escape northward, causing the northward stream west of the British Isles to run stronger, and salter water to spread over the central area generally. After October the supply from the lower latitudes diminishes, but the drift eastward from the Labrador coast continues into November. It is to be remarked that the spreading of the Polar water eastward from the south-east of Greenland, so strongly marked in 1896, is scarcely noticeable in 1897.

Towards the end of the year the drift from the Labrador current, which has not contracted to its usual size after the expansion in August, expands southward again, and the "north and south" distribution becomes more marked than in 1896, partly because of this, and partly because of the belated strengthening of the northward streams on the eastern side.

It is a matter of some difficulty to ascertain with any detail how far the features common to the circulation in the two years described are reproduced every year. The most reliable means of comparison is probably the series of temperature and current charts published by the Meteorological Office (40), but these do not contain much information about the circulation north of 40° N. during the winter months. In the lower latitudes they show, however, what is important to our purpose, that the circulation round the Atlantic anticyclone is more active and definite in summer than in winter (January and August).

The current chart, of course, cannot define the source of the surface waters off the Newfoundland Banks, but in spring and early summer (April and June charts) the apparent direct continuation of the Gulf Stream becomes shorter, while there is increased eastward movement from the east of Newfoundland.

In August the easterly drift shows a tendency to divide near the south-west of Ireland, the greater part appearing to go southward, while a narrow stream moves to the north. Further north the general direction of movement is more to the south.

The charts for October and November show that the head of the Gulf Stream broadens and retreats, while in the north the easterly component becomes well marked; note the easterly direction of the arrows south-east of Greenland. The dividing line formed by a band of "no current" between Newfoundland and the British Isles is an important feature: it moves northward in winter, southward in summer.

It appears, therefore, that there are certain important seasonal changes in the surface circulation which occur in the two years 1896 and 1897, and which can, to some extent at least, be traced in the less definite outlines of the composite pictures obtained by the method of averages. These changes may be summarised as follows:—

The surface circulation in the North Atlantic between the parallels of 40° and

60° N., forms during winter part of a cyclonic movement, resulting in a southerly and south-easterly drift on the western side, and a northerly drift on the eastern side. In the lower latitudes, about 40° N., the easterly movement is comparatively weak. The northerly movement in the easterly half of the area is considerable, but it is hampered by the configuration of the land—the Faeroe Isles, Iceland, Greenland, &c., hence the water tends to spread widely over the surface northward and north-westward, but stream currents of any marked degree of energy are not developed. Hence there is a tendency at the end of winter towards uniform distribution of salinity and temperature, which is aided by a diminution in the supplies of Polar water.

In spring the north and south components become less marked, and the easterly movement becomes stronger everywhere south of lat. 50° N., the increase being most noticeable in the lowest latitudes. The greater angle now made by the drift with the European coast line causes increased "banking-up" of water against the land, and this water escapes by stream currents running northward and southward. The total discharge of these streams must depend on the rate and volume of the easterly drift, and the proportions discharged to north and south on the angle made by the drift with the coast. The direction of drift is probably always somewhat towards the north, but the resistance due to the configuration of the land and to the influence of the earth's rotation hinders the development of a northward stream sufficient to carry off the water as fast as it accumulates, and the banked-up water tends also to sink below the surface, causing the high temperatures observed at low levels in the Eastern Atlantic (41), and to accumulate further westward and north-westward, flowing out wherever opportunity presents itself, as in the branches of the Irminger current.

The northward discharge of Atlantic water from the lower latitudes is therefore greater during the months of spring and summer, when a stream current, independent of the local surface drift, sets northward between Scotland and Iceland, attaining its greatest strength in the Faeroe-Shetland Channel, and having marked inductive effects on the Polar bottom water at the Wyville-Thomson Ridge (42).

As the summer progresses, the drift circulation in the higher latitudes becomes weaker, but after midsummer the increased strength of the current from the south is replied to by an enormous delivery of Polar water—first and chiefly from the area between the east and north-east of Iceland and Jan Mayen, then later, as the ice breaks up, from the Polar current east of Greenland. The Labrador current also increases largely in volume, but a greater proportion of this increase is likely to be due to melting of ice by the warm air sent up by the cyclonic circulation developed over the continent of North America.

Except in the case of the Iceland—Jan Mayen Stream, which develops the features of a stream current of great energy, for reasons recently set forth by PETTERSSON (43), the Polar water forms large pools of fresh water to the east of the southern part of Greenland, and off the Labrador coast, where it seems to collect in a comparatively limited area, mixing little with the waters on which it lies, and being gradually warmed by the direct rays of the sun. Between Iceland and Jan Mayen, and in the Spitsbergen area, the quantity of water coming from the south is greater, the melting of the ice takes place more rapidly, and there is therefore a greater return of Polar water on the surface, forming streams which may cover over large portions of the northerly current, and penetrate southward into the Faeroe Channel and along the coasts of the British Isles.

This circulation reaches its greatest development in August or early in September, when the easterly movement again becomes marked in the higher latitudes—50° to 60° N.—and weakens in the lower. The northward discharge by a stream current in the eastern area accordingly diminishes, but the increased drift is shown by the spreading eastward of the Polar water from south-east of Greenland and from the Labrador coast. This can be distinctly traced in October and November, and it is followed by a transition into the partial cyclonic circulation first described, the *drift* nature of the circulation being characterised by the gradual lessening of gradients, both of temperature and salinity.

If it be admitted that the surface circulation undergoes the periodic changes described, it appears that they follow directly from the seasonal changes in the circulation of the atmosphere at the surface, modified by the position and form of the land.

During winter the prevailing winds on the east coast of Canada are north-westerly. Large cyclones make their way in continuous succession north-eastward across the Atlantic, the region of lowest average pressure forming a belt from the south-east of Greenland, round Iceland, and thence in a north-easterly direction; so that the prevailing winds to the right of that belt are west in the central area, south-west nearer Europe. Pressure is high, with anticyclonic circulation, over the Eurasian continent. The Atlantic anticyclone is during the winter months at its smallest and weakest, and the whole area down to 40° N. is therefore practically under the control of the equatorial side of the cyclonic circulation. Under the influence of the strong winds, which nowhere form a large angle with the coast-line, an immense system of surface drifts is developed, the water moving northward in the eastern half of the basin, and southward in the western half, while the main area of purely easterly drift is confined to the centre and to lower latitudes (44).

The characteristic feature of the winter circulation is therefore the purely drift nature of the surface currents: it is specially important to notice this in relation to the north-eastern part of the area, for the water moving northward between Scotland and Iceland is then a wide surface stream of small depth, consisting of mixed waters brought from different sources into the central Atlantic area during the preceding autumn.

The transition from winter to summer conditions in the atmospheric circulation consists of a gradual increase in size and strength of the Atlantic anticyclone, and diminution in the number and energy of the cyclones following the Iceland low-pressure belt. The effect is to weaken the drifts in the higher latitudes, and to strengthen them in the south, the latter being supported by the fall of pressure over the North American continent. The fall of pressure over Europe, and the tendency of the Atlantic anticyclone to project north-eastwards, causes steady westerly winds to prevail over a broad belt in the widest part of the North Atlantic, and the drifts accordingly set eastward against the land in much greater volume, and much further north, than in winter. Hence there is a great increase in the relief current moving northward: this current is known to extend down to 300 fathoms at the Wyville-Thomson Ridge, and to penetrate far into the Arctic regions (45).

The enormous quantities of warm Atlantic water sent north by this current are much more effective than the seasonal warming of the air in melting the ice of the Arctic seas, and the southward movement of the Polar water is apparent in July and August. The light variable winds prevalent at that season do not induce any marked drift of these waters.

During autumn the transition phases of spring are reversed; the coast currents again become weaker through the changes of wind force and direction due to the shrinkage of the Atlantic anticyclone, and the drift system is re-established. The first result is to spread the Polar waters eastward over the Atlantic area, where they are more or less rapidly absorbed by mixture with underlying water; but the mixed waters may partly or wholly cover over the weakened and retreating coast current so well marked in the summer season.

The additions made to our knowledge of the warm northerly currents in the higher latitudes of the Atlantic by recent expeditions have been fully summarised and discussed by PETTERSSON since this investigation was begun (46), and the conclusions arrived at with regard to them are fully supported by the extended surface observations in the lower latitudes. The great development of these streams is to be accounted for by the transfer of the warm salt waters, sent up along the American coast by the Gulf Stream, as surface drifts to the south-western coasts of Europe, where they are banked up, stored as it were in a vast reservoir, from which they escape northwards, southwards, and downwards, filling the whole basin of the eastern and north-eastern Atlantic, and overflowing as northward streams wherever the form of the basin makes it possible. These northward currents are permanent, but they suffer variations corresponding to the changes in the rate at which the drift-water is accumulated, and in their more remote branches they have a surface circulation superposed upon them—a *thermal* circulation in the late summer and early autumn, and a *drift* circulation in late autumn and winter.

The general circulation of the North Atlantic is therefore the result of a large number of factors, each of which is subject to wide variation. From a consideration of the mean result in its relation to the mean atmospheric circulation, it appears that the oceanic circulation is directly controlled by the winds: the form, position, and intensity of the whole of the Atlantic anticyclone, and of the cyclonic area to the north of it being taken into account. The movements of water set up directly by these systems are modified by, firstly and chiefly, the configuration of the land, and secondly by the effects of melting of ice.

The precise effects of variations in the atmospheric systems, which must in the first instance be regarded as the independent variables, can be ascertained to some extent by comparing the circulation in the two years 1896 and 1897. The changes in circulation required to account for the observed differences in the distribution of temperature and salinity have already been suggested, but it seems desirable to state them in a more general form before attempting to discuss their causes.

The principal point to be considered in the early part of 1897 is the weakness of the drift circulation compared to 1896. The south-easterly drift from the north-western area is weaker, and the surface waters generally are therefore warmer and salter. Again, the easterly drift towards Europe is weaker, there is less "banking up" of water on the European coast, and the outflow to the north-east and to the Irminger current is weaker; the main easterly drift appears, in fact, to be further south than last year, it consists more exclusively of Gulf Stream water, and its course is more towards the African coast.

The differences are of the same general type until August, when the large delivery southward in the Azores region becomes most strongly marked. The effect of the decreased strength of the northward streams during spring and early summer appears in the diminution in the supply of Polar water; the melting of ice has obviously gone on more slowly, and the increase in the fresh-water streams is smaller and occurs later. The difference is least marked in the case of the Labrador current, which depends least on the warm streams for its supply.

In the autumn the movement becomes more easterly and northerly, and the direction of the easterly drift is more towards the land in the south-west of Europe, causing more "banking-up" and consequently stronger northerly streams than at the corresponding period of 1896. The change, however, comes too late for these streams to produce the enormous melting of ice and consequent outflow of Polar water observed in the previous year, and the phase quickly gives way to the drift circulation of winter. The characteristic "north and south" feature becomes strongly developed, owing to the form of the autumn distribution just noted, and to the absence of Polar water spreading over the surface.

The construction of charts showing the distribution of atmospheric pressure and temperature during individual months is a matter of great difficulty. The discussion of material obtained from ships' logs is beyond the resources of the private investigator; the only means of getting at the information required is to utilise the monthly averages of observations made at coast stations surrounding the area, or on islands situated within it, and to eke out the information obtained with the excellent general summaries published in the 'U.S. Pilot Chart,' and the 'Bulletin mensuel du

Bureau météorologique de France.' Even this is a troublesome matter, as it is not easy to get all the corrections necessary for rendering the observations fully comparable, and many of the data are only published after long delays.

The simplest method of obtaining an approximate view of the atmospheric conditions prevalent during 1896 and 1897 seemed to be to use the *differences* of the monthly means from the long-period averages at a number of stations, thus avoiding all the troublesome reductions to sea-level, and to plot the differences on charts. The data for the two years were partly obtained from the publications of the meteorological services concerned, but through the kindness of the Director I was furnished with the as yet unpublished means for a large number of the stations of the Danske Meteorologisk Institut. The long-period means used were those in the "Challenger" Report on Atmospheric Circulation, and the differences are given in Table III.

The anomalies shown by the charts are not, of course, to be regarded as having sufficient local accuracy to be worth detailed quantitative discussion; even if they were it would not be possible to deal with them rigorously, for the relation between a drift current and the wind which causes it is still quite uncertain. It is necessary to look merely at the broader outlines, and to seek for differences which occur consistently over considerable areas and continue for successive months.

The most marked departure from the average distribution of pressure in 1896 is the excess in the lower latitudes during the first half of the year. With the single exception of the month of June there is continued high pressure from January to August, and the excess is greatest to the south and south-west of the British Isles. This indicates an unusual extension of the Atlantic anticyclone north-eastwards, and consequently stronger and steadier westerly and south-westerly winds, which would produce an unusually large easterly surface drift south of the 50th parallel, and excessive banking-up of water south-west of Great Britain. Hence we should expect all the branches of the northerly current to exceed their usual strength, and later in the year to find large supplies of Polar water making their appearance, the result of excessive melting of the Polar ice: this is precisely what the observations have shown.

In January 1896 the area of high barometric pressure extended over Iceland and Southern Greenland, the least excess being in the south-western area round Newfoundland; but during February, March, and April pressure was below the average in the north, and the deficiency increased eastwards to form a belt of specially low pressure lying along lat. 60° to the Norwegian coast in March. This would lead to abnormal easterly drifts from the Labrador-Greenland region during spring, which would ultimately join the northerly currents on the eastern side. The waters of the northerly streams are therefore chiefly derived in winter from the Gulf Stream area, while in spring there is an increasing admixture of water from the Labrador currents.

During the midsummer period, pressure was on the whole above the average; but the differences did not lead to any definite disturbance of the normal gradients, and so far as the winds are concerned the conditions are to be regarded as normal.

The months of September and October are characterised by deficient pressures in the low latitudes, due to the passage of cyclones from the south-west to the Bay of Biscay and the British Isles, and by relatively high pressures in Greenland and Iceland. This distribution would give an easterly tendency to the winds in the north, and the slowness of the easterly movement of the fresh surface waters derived from the ice is therefore probably abnormal. In November the cyclone track moved northward, and an anticyclone developed over South-western Europe, conditions which would increase the easterly movements in the higher latitudes in the Atlantic.

In December 1896 we find the beginning of a different distribution of pressure, which continues, somewhat irregularly, but with little interruption, till August 1897. The characteristic feature is pressure above the average in the north and west, and below it in the south and south-east, the region of deficient pressure being chiefly south and south-west of the British Isles. The Atlantic anticyclone does not therefore expand north-eastwards as far as it did in 1896, and the track of cyclones skirting it is further south and more directly eastward; it appears also that the cyclones were shallower or less numerous than usual.

Hence the main easterly drift is weaker on its northern margin, and the direction of movement is more to the southward, the chief region of banked-up water, the source of the northward-moving currents, is further south, and the currents receive less direct aid from the surface drifts. The relatively high pressure in the higher latitudes would give weaker westerly winds in the Atlantic, and therefore a weaker drift circulation, and less spreading of the Labrador stream water eastward. The drift delivery northward on the eastern side would be less, the winds being weaker and more westerly; but, on the other hand, the southward deflection of the main cyclone track would increase the easterly component of the winds between Iceland and Spitsbergen. The water sent northward by the current from the coast of Europe would therefore tend to mix less with the water underlying it, and on reaching the Spitsbergen region to drift westwards. We know, as a matter of fact, that an unusually large area west and south-west of Spitsbergen was open during 1897—probably the result of the enormous amount of warm water sent up in the preceding year, and that the open area was covered to an unusual extent by Atlantic water (47).

The supplies of Atlantic water being smaller, and the ice more remote, in 1897 than in 1896, the increase of the Polar streams in autumn is much less marked; hence an unusually large area is then occupied by the warmer and salter surface-waters, and this is maintained by the peculiar atmospheric changes which take place in the latter part of the year.

In September pressure is somewhat above the average south and south-west of the British Isles, but below it over Norway, a phase which would increase the strength of the westerly winds, but which is quickly modified by the spreading northward of the relatively high pressure over nearly the whole of Western Europe, while further west pressure keeps to the average, or falls a little below it. The gradients are thus stronger than usual for southerly winds, and the "north-and-south" form of the winter drift circulation becomes specially well marked.

The circulation of waters in the North Atlantic therefore not only follows the general seasonal changes in the atmospheric circulation, but the irregularities in the seasonal changes, which in these latitudes may amount to a large fraction of the whole, are accompanied by irregular variations in the oceanic streams, also amounting to large changes in the total movement; the oceanic changes bear similar relations to the atmospheric in both cases. The effect of changes in the direction and force of prevailing winds makes itself felt almost immediately on the "drift" circulation, while the relief currents produced by the banking-up of water are longer in responding, and "thermal" currents due to melting of ice by warm water below the surface take longer still. The difference in the time-interval arising in this way must lead to a smoothing out of the effects on the deeper movements of water, and it is probably only when unusual conditions persist for a long time, as in the case of the Atlantic anticyclone during 1896, that there is any considerable variation in them.

The principal conclusions may therefore be summed up as follows:—

1. The surface waters along the whole of the eastern seaboard of North America north of (about) lat. 30° N., consisting partly of water brought from the equatorial currents by the Gulf Stream and partly of water brought down by the Labrador current, are drifted eastward across the Atlantic towards south-western Europe, and banked up against the land outside the continental shelf (48). This continues all the year round, but it is strongest in summer, when the Atlantic anticyclone attains its greatest size and intensity; and the proportion of Gulf Stream water is greatest at that season.

2. The drifts in the northern part of the Atlantic area are under the control of the cyclones crossing it. The circulation set up accordingly reaches its maximum intensity in winter, and almost dies out in summer. In the winter the drifts tend to be south-eastward from the mouth of Davis Strait, eastward in mid-Atlantic, and north-eastward in the eastern region. In spring and autumn the movement is more easterly over the whole distance, and a larger quantity of water from the Labrador current is therefore carried eastward.

3. The water banked up in the manner described in (1) escapes partly downwards, partly southwards, and partly northwards. It occupies the whole of the eastern basin of the North Atlantic, and to the north it extends westward to Davis Strait, being confined below 300 fathoms depth by the ridges connecting Europe, the Faeroes, Iceland, and Greenland. Above that level it escapes northward by a strong

current through the Faeroe-Shetland Channel and between Faeroe and Iceland, and by the two branches of the Irminger current, one west of Iceland, the other west of Greenland. As it seems desirable that this northerly current should have a distinctive name, it might be well to call it the European stream, and its branches the Norwegian, Irminger, and Greenland streams respectively.

The strength and volume of the European stream is liable to considerable variation, according to the form and position of the Atlantic anticyclone, which causes the amount of banked-up water and the proportions escaping northward and southward to vary. It is also modified by the strength and direction of the surface drifts in its course. It is, however, always strongest in summer.

4. The Norwegian stream is by far the largest branch of the European, and it traverses the Norwegian Sea and enters the Arctic Ocean. The warm water thus sent northward melts enormous quantities of ice, and the fresh water derived from the ice moves southward in autumn, chiefly in a wide surface current, between Iceland and Jan Mayen, which may entirely cover over parts of the Norwegian stream. Part of the surface water also comes southward through the Denmark Strait, but the amount is much smaller, probably chiefly because the melting of the ice is slower, and the channel is longer blocked.

The Greenland branch of the European stream also causes melting of ice in Davis Strait, but the warm winds from the American continent and the large quantity of water received from the land are probably more effective in increasing the volume of the Labrador current.

5. The water from the melted ice is spread over the surface of the North Atlantic during late autumn and winter by the increasing drift circulation, and it is gradually absorbed by mixing with the underlying water.

6. The circulation described is liable to extensive variations corresponding to variations in the atmospheric circulation.

The meteorology of the North Atlantic area during the period under discussion, and the reaction of the oceanic upon the atmospheric circulations, really form part of a separate investigation, and will be discussed in another paper. Special interest attaches to the departures of the monthly temperatures from the mean and their relation to the pressure anomalies. One or two important points, however, may be touched on.

Quite recently, PETTERSSON and MEINARDUS (49) have shown that a relation exists between the mean barometric pressure over an oceanic area during the winter months, and the temperature of its surface waters; high temperatures tending to lower pressures, and low temperatures to higher pressures. This is probably seen on its largest scale in the southern hemisphere, where the areas west of the land masses (50), supplied with cold water by the Antarctic currents, coincide with the strongest developments of the southern high-pressure belt.

It has been shown that the expansion north-eastward of the Atlantic anticyclone during the early part of 1896 led to increased strength in the European stream, resulting in the delivery of unusually large quantities of warm water by the Norwegian stream, with subsequent excessive melting of Polar ice. Hence at the end of 1896 the northern seas were covered with water below the average temperature. But the characteristic of the first half of 1897 is the relatively high pressure persisting in this region, and the deflection southward of the main cyclone tracks, which is therefore probably the result of the low surface temperature in autumn, prolonged automatically by the weakness induced in the drift circulation. The presence of unusual quantities of warm water below the surface would, on the other hand, keep up the melting and retard the formation of ice, and temperature would be above the average in the higher latitudes, but below it in the regions usually free of ice because of the spreading of the ice-cold water. The influence of the warmer water would become gradually more apparent at the surface late in the winter, as the colder waters were absorbed by mixture.

The weakness of the Norwegian stream in 1897 and the comparatively open sea left in the preceding winter resulted in less melting of ice, and, consequently, a more limited distribution of Polar water; hence in the autumn the warmer water appeared more at the surface, and the result is relatively low pressure over the northern sea areas and a rapid development of the drift type of circulation.

The main result obtained by PETTERSSON is accordingly confirmed, but the problem is complicated by the varying influence of the high-pressure areas to the south and over the land. The key to the position seems to be the Atlantic anticyclone, which controls the low-pressure areas, both directly and indirectly, by its far-reaching effect on the oceanic circulation; and it seems scarcely likely that the causes modifying this system are confined to the Atlantic, even if they are to be found at the surface at all.

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28. The Laboratory Numbers of these samples (Table I.) are given in Appendix III.

29. Twelfth Report of the Fishery Board for Scotland, Part III., p. 379. The values of Cl and S for these samples are given in Table II., with the S calculated from the Cl by equation (2).
30. See PETTERSSON, On the Properties of Water and Ice, "Vega" Expedition Report. Stockholm, 1883; also J. Y. BUCHANAN, Proc. R.S.E., XIV., p. 129.
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33. *Loc. cit.*, p. 341, and Table XII.
34. Table II.
35. Notes 32 and 33.
36. In addition to my own data, salinity observations have been added to the charts from the Report of the "Ingolf" Expedition, Professor HJORT's papers, the Iagttagelser over Overfladevandets Temperatur, Saltholdighed og Plankton, paa Islandske og Grønlandske Skibsroute of Commodore WANDEL, PETTERSSON and EKMAN's papers on the North Sea and the Baltic, 1893–97 (Stockholm, 1897), and on Die hydrographischen Verhältnisse der oberen Wasserschichten des nördlichen Nordmeeres (Stockholm, 1898).
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TABLE I.

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1895.		N.	W.					
1	Teutonic . .	Dec. 19	midnight	51° 24′	13° 58′	10·6	19·79	35·75	—	—
2	,,	,, 20	noon . .	51 22	20 24	11·7	19·68	35·56	—	—
3	,,	,, 20	midnight	50 45	26 37	11·1	19·79	35·75	—	—
4	,,	,, 21	noon . .	50 24	31 40	5·6	19·64	35·48	—	—
5	,,	,, 21	midnight	49 18	37 28	6·1	19·67	35·54	26·45	—
6	,,	,, 22	noon . .	48 44	41 25	11·1	19·68	35·56	—	—
7	,,	,, 22	midnight	47 24	44 45	2·8	18·78	33·94	—	·00219
8	,,	,, 23	noon . .	45 44	50 3	2·8	17·96	32·48	—	·00209
9	,,	,, 23	midnight	44 23	55 17	6·1	18·07	32·66	24·23	—
10	,,	,, 24	noon . .	43 0	60 39	6·1	17·83	32·24	—	·00208
11	,,	,, 24	midnight	41 25	65 38	6·7	18·83	34·03	—	—
12	,,	,, 25	noon . .	40 22	70 8	8·9	18·90	34·16	—	·00217
		1896.								
13	,,	Jan. 1	midnight	40 22	68 37	2·8	19·22	34·72	—	—
14	,,	,, 2	noon . .	41 11	64 11	11·7	19·45	35·14	—	—
15	,,	,, 2	midnight	42 28	59 34	4·4	17·95	32·46	—	·00205
16	,,	,, 3	8 A.M. .	43 15	56 25	6·1	18·41	33·28	—	—
17	,,	,, 3	noon . .	43 38	54 45	5·0	18·14	32·79	24·32	—
18	,,	,, 3	midnight	45 5	49 58	5·6	18·06	32·64	—	·00208
19	,,	,, 4	8 A.M. .	46 3	46 42	8·3	18·47	33·39	—	—
20	,,	,, 4	noon . .	46 34	44 58	6·1	18·58	33·58	—	—
21	,,	,, 4	midnight	47 53	39 50	12·8	19·82	35·80	—	·00231
22	,,	,, 5	8 A.M. .	48 46	36 22	11·7	19·73	35·64	—	—
23	,,	,, 5	noon . .	49 19	34 41	11·7	19·67	35·54	—	—
24	,,	,, 5	midnight	50 16	28 59	9·4	19·81	35·79	26·69	—
25	,,	,, 6	8 A.M. .	50 37	25 12	8·9	19·70	35·59	—	—
26	,,	,, 6	noon . .	50 53	23 0	10·0	19·72	35·63	—	—
27	,,	,, 6	midnight	51 8	17 1	11·1	19·67	35·54	—	—
28	,,	,, 7	8 A.M. .	51 18	13 4	11·7	19·68	35·56	—	—
29	,,	,, 7	noon . .	51 33	10 44	7·8	19·67	35·54	26·54	—
30	,,	,, 7	midnight	52 32	5 56	8·3	19·22	34·72	—	·00224
		1895.								
31	Ethiopia . .	Dec. 22	noon . .	55 21	11 19	9·4	19·52	35·27	26·28	·00229
32	,,	,, 23	,,	55 1	19 27	10·0	19·59	35·40	26·34	—
33	,,	,, 24	,,	54 1	27 39	8·9	19·39	35·03	—	—
34	,,	,, 25	,,	52 31	35 19	7·2	19·31	34·89	—	—
35	,,	,, 26	,,	50 27	42 27	8·9	18·93	34·21	—	—
36	,,	,, 27	,,	47 54	49 30	1·1	17·87	32·31	—	—
37	,,	,, 28	,,	45 11	55 35	3·3	17·80	32·19	23·88	·00208
38	,,	,, 29	,,	42 40	60 59	5·0	17·80	32·19	23·82	—
39	,,	,, 30	,,	41 5	67 20	8·3	18·24	32·97	24·46	—
40	,,	,, 31	,,	40 30	72 50	6·7	18·06	32·64	—	·00211
		1896.								
41	,,	Jan. 12	,,	40 41	69 32	5·6	18·18	32·86	—	—
42	,,	,, 13	,,	41 49	63 54	6·7	18·05	32·63	24·23	—
43	,,	,, 14	,,	43 16	58 29	3·9	17·97	32·49	24·26	·00208
44	,,	,, 15	,,	44 54	52 51	3·9	17·92	32·40	—	—
45	,,	,, 16	,,	47 7	47 5	0·6	17·94	32·44	—	·00207
46	,,	,, 17	,,	49 24	41 32	11·1	19·39	35·03	26·05	—
47	,,	,, 18	,,	51 20	35 45	8·3	19·39	35·03	26·04	—
48	,,	,, 19	,,	52 58	29 10	7·8	19·32	34·91	—	·00226

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3
		1896.		N.	W.					
49	Ethiopia . .	Jan. 20	noon . .	54° 23′	21° 42′	10·6	19·58	35·38	26·33	—
50	,,	,, 21	,,	55 1	13 35	10·6	19·59	35·40	—	·00229
51	,,	,, 22	,,	55 16	6 26	7·2	19·32	34·91	25·97	·00225
52	Teutonic . .	,, 16	midnight	51 8	13 39	11·1	19·70	35·59	26·56	—
53	,,	,, 17	noon . .	50 39	19 29	11·1	19·66	35·52	—	·00231
54	,,	,, 17	midnight	49 57	23 41	11·1	19·60	35·41	—	—
55	,,	,, 18	noon . .	49 14	27 54	5·6	19·67	35·54	—	—
56	,,	,, 18	midnight	47 57	33 22	5·0	19·57	35·36	—	·00229
57	,,	,, 19	noon . .	46 40	38 51	7·2	19·69	35·58	—	—
58	,,	,, 19	midnight	44 45	42 53	16·1	19·81	35·79	26·66	—
59	,,	,, 20	noon . .	42 50	47 55	5·6	18·23	32·95	24·52	·00214
60	,,	,, 20	midnight	42 13	53 21	12·8	19·65	35·50	—	—
61	,,	,, 21	noon . .	41 35	58 48	11·1	19·88	35·91	—	·00234
62	,,	,, 21	midnight	41 23	63 44	10·0	19·71	35·61	—	—
63	,,	,, 22	noon . .	40 37	68 46	7·2	18·38	33·22	—	—
64	,,	,, 29	,,	Off Sandy Hook L. V.		0·0	17·76	32·11	23·91	·00207
65	,,	,, 29	midnight	40° 1′	68° 36′	6·1	18·48	33·41	—	—
66	,,	,, 30	noon . .	40 36	64 2	17·2	20·10	36·30	—	—
67	,,	,, 30	midnight	40 59	59 59	12·2	19·48	35·19	—	—
68	,,	,, 31	noon . .	41 23	53 15	6·1	18·87	34·10	—	·00221
69	,,	,, 31	midnight	42 38	49 9	6·1	18·51	33·46	24·75	—
70	,,	Feb. 1	noon . .	43 54	44 22	12·8	19·77	35·72	—	—
71	,,	,, 1	midnight	45 38	39 34	13·9	19·82	35·80	26·66	—
72	,,	,, 2	noon . .	47 20	34 46	11·1	19·64	35·48	—	·00231
73	,,	,, 2	midnight	48 33	29 15	10·0	19·76	35·70	—	—
74	,,	,, 3	noon . .	49 46	23 43	11·7	19·63	35·47	—	—
75	,,	,, 3	midnight	50 31	18 0	11·1	19·70	35·59	—	—
76	,,	,, 4	noon . .	51 17	12 .16	11·1	19·69	35·58	—	—
77	Laura . . .	Jan. 24	11 A.M. .	59 52	3 20	7·6	19·49	35·21	—	—
78	,,	,, 24	10 P.M. .	60 49	5 5	8·0	19·57	35·36	—	·00229
79	,,	,, 25	6 A.M. .	61 42	6 4	6·5	19·47	35·17	—	·00226
80	,,	,, 26	11 A.M. .	61 52	6 15	6·0	19·14	34·58	25·79	—
81	,,	Feb. 1	,,	62 8	6 20	6·2	19·39	35·03	—	—
82	,,	,, 3	2 P.M. .	62 28	7 2	6·2	19·46	35·15	—	—
83	,,	,, 3	10 P.M. .	62 36	9 30	8·0	19·54	35·30	—	—
84	,,	,, 4	11 A.M. .	62 39	12 55	8·0	19·54	35·30	26·39	—
85	,,	,, 4	10 P.M. .	62 42	15 7	8·2	19·60	35·41	—	—
86	,,	,, 5	11 A.M. .	62 54	17 45	8·0	19·56	35·34	—	—
87	,,	,, 5	10 P.M. .	63 19	19 46	8·0	19·55	35·32	—	—
88	,,	,, 6	11 A.M. .	63 17	20 18	7·0	19·48	35·19	—	—
89	,,	,, 6	4 P.M. .	63 35	22 0	6·0	19·36	34·98	—	·00224
90	,,	,, 6	10 P.M. .	64 0	23 2	5·5	18·18	32·86	24·41	·00212
91	,,	,, 7	1 A.M. .	64 18	22 45	4·5	19·30	34·87	—	·00225
92	,,	,, 11	8 A.M. .	64 17	22 42	4·2	18·91	34·17	—	·00220
93	,,	,, 11	noon . .	64 1	22 59	5·0	18·35	33·17	24·66	—
94	,,	,, 11	10 P.M. .	63 7	21 15	6·8	19·54	35·30	—	—
95	,,	,, 12	7 A.M. .	62 45	19 18	7·2	19·55	35·32	—	—
96	,,	,, 12	noon . .	62 40	18 10	7·8	19·50	35·23	—	—
97	,,	,, 12	10 P.M. .	62 32	15 23	8·1	19·55	35·32	—	·00228
98	,,	,, 13	7 A.M. .	62 26	12 31	8·0	19·54	35·30	—	·00229
99	,,	,, 13	noon . .	62 31	11 3	8·0	19·53	35·28	26·28	·00227
100	,,	,, 13	10 P.M. .	62 33	7. 55	7·2	19·53	35·28	—	—
101	,,	,, 14	7 A.M. .	62 35	6 58	7·0	19·50	35·23	—	—
102	,,	,, 15	,,	62 9	6 20	6·7	19·47	35·17	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
103	Laura . .	Feb. 15	10 P.M. .	61° 11′	5° 15′	7·0	19·49	35·21	—	—
104	,,	,, 16	7 A.M. .	60 8	3 31	8·3	19·61	35·43	26·48	—
105	,,	,, 16	noon . .	59 40	2 25	8·2	19·52	35·27	—	—
106	Teutonic . .	,, 13	midnight	—	—	8·9	19·77	35·72	—	—
107	,,	,, 14	noon . .	50 39	20 49	10·6	19·66	35·52	—	—
108	,,	,, 14	midnight	49 39	25 58	11·7	19·69	35·58	—	—
109	,,	,, 15	noon . .	48 40	31 7	12·2	19·71	35·61	—	—
110	,,	,, 15	midnight	47 18	35 37	9·4	19·71	35·61	—	·00231
111	,,	,, 16	noon . .	45 57	40 7	11·7	19·67	35·54	—	—
112	,,	,, 16	midnight	44 18	44 30	11·7	19·14	34·58	—	—
113	,,	,, 17	noon . .	42 40	48 53	8·9	18·69	33·78	25·08	·00219
114	,,	,, 17	midnight	42 6	53 41	5·0	19·08	34·47	—	—
115	,,	,, 18	noon . .	41 32	58 50	11·1	19·38	35·01	—	·00228
116	,,	,, 18	midnight	41 2	64 3	11·1	19·36	34·98	—	—
117	,,	,, 19	noon . .	40 32	69 36	3·9	18·26	33·00	—	·00214
118	,,	,, 19	midnight	off Fire	Island	-1·7	18·02	32·58	—	—
119	,,	,, 26	,,	—	—	-2·2	17·54	31·71	23·46	—
120	,,	,, 27	noon . .	40° 33′	64° 14′	15·6	20·19	36·46	27·21	·00237
121	,,	,, 27	midnight	40 56	59 21	10·0	19·40	35·05	26·06	—
122	,,	,, 28	noon . .	41 20	54 28	10·0	20·09	36·29	—	·00235
123	,,	,, 28	midnight	42 13	50 20	5·0	19·19	34·67	25·75	—
124	,,	,, 29	noon . .	43 6	46 12	15·6	19·98	36·09	26·91	—
125	,,	,, 29	midnight	44 51	41 43	15·6	19·92	35·98	—	—
126	,,	Mar. 1	noon . .	46 36	37 13	13·9	19·77	35·72	—	·00231
127	,,	,, 1	midnight	47 44	31 50	12·8	19·73	35·64	—	—
128	,,	,, 2	noon . .	49 13	26 27	12·2	19·71	35·61	—	—
129	,,	,, 2	midnight	50 6	21 0	11·1	19·71	35·61	—	·00233
130	,,	,, 3	noon . .	50 31	15 33	11·1	19·75	35·68	—	—
131	,,	,, 3	midnight	51 3	11 56	10·6	19·97	36·07	—	—
132	Ethiopia . .	Feb. 2	noon . .	55 6	14 55	10·0	19·65	35·50	26·43	—
133	,,	,, 3	,,	54 7	22 49	10·6	19·62	35·45	—	—
134	,,	,, 4	,,	53 22	28 46	8·3	19·36	34·98	—	—
135	,,	,, 5	,,	51 53	34 35	7·2	19·25	34·78	—	·00224
136	,,	,, 6	,,	50 8	39 48	11·1	19·62	35·45	—	—
137	,,	,, 7	,,	47 49	45 26	2·8	18·74	33·87	—	—
138	,,	,, 8	,,	45 26	50 28	2·8	17·92	32·40	24·10	·00211
139	,,	,, 9	,,	44 11	54 17	1·7	18·35	33·17	—	—
140	,,	,, 10	,,	42 58	60 14	4·4	18·37	33·20	—	—
141	,,	,, 11	,,	41 58	62 34	3·9	18·28	33·04	—	—
142	,,	,, 12	,,	41 12	64 49	5·0	18·40	33·26	24·74	—
143	,,	,, 13	,,	40 41	69 5	4·4	18·23	32·95	—	·00215
144	,,	,, 23	,,	40 34	68 13	4·4	18·15	32·80	—	—
145	,,	,, 24	,,	41 38	61 50	2·8	18·00	32·54	—	—
146	,,	,, 25	,,	42 31	55 37	3·9	18·28	33·04	—	·00216
147	,,	,, 26	,,	43 43	49 29	2·8	17·93	32·42	24·06	—
148	,,	,, 27	,,	46 29	44 4	3·9	18·68	33·76	25·08	—
149	,,	,, 28	,,	49 0	38 12	11·1	19·52	35·27	—	·00231
150	,,	,, 29	,,	51 9	31 32	9·4	19·35	34·96	—	—
151	,,	Mar. 1	,,	52 53	24 10	9·4	19·42	35·08	—	—
152	,,	,, 2	,,	54 15	16 26	10·0	19·57	35·36	26·39	—
153	,,	,, 3	,,	55 17	8 33	8·3	19·42	35·08	—	·00227
		1895.								
154	Loughrigg Holme	Dec. 29	,,	51 10	9 16	10·3	19·57	35·36	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1895.		N.	W.					
155	Loughrigg Holme	Dec. 30	noon	49° 41′	13° 11′	11·8	19·63	35·47	—	·00228
156	,,	,, 31	,,	48 12	16 57	11·8	19·63	35·47	26·40	—
		1896.								
157	,,	Jan. 1	,,	46 31	21 15	13·2	19·76	35·70	26·61	—
158	,,	,, 1	8 P.M.	45 51	22 41	12·9	19·71	35·61	—	—
159	,,	,, 2	4 A.M.	45 6	24 16	12·7	19·73	35·64	—	·00229
160	,,	,, 2	noon	44 20	25 53	13·8	19·80	35·77	—	—
161	,,	,, 2	8 P.M.	43 34	27 22	13·4	19·81	35·79	—	—
162	,,	,, 3	4 A.M.	42 49	28 48	13·9	19·89	35·93	—	—
163	,,	,, 3	noon	42 4	30 14	14·4	19·86	35·88	—	—
164	,,	,, 3	8 P.M.	41 19	31 35	14·6	19·87	35·89	—	—
165	,,	,, 4	4 A.M.	40 32	32 47	14·0	19·87	35·89	—	—
166	,,	,, 4	noon	39 47	33 59	15·8	20·02	36·16	26·96	—
167	,,	Feb. 15	,,	41 14	49 22	13·0	19·60	35·41	—	—
168	,,	,, 15	midnight	41 59	47 13	14·0	19·82	35·80	—	·00233
169	,,	,, 16	noon	42 41	45 10	13·9	19·75	35·68	—	—
170	,,	,, 16	midnight	43 26	43 6	15·6	19·96	36·05	26·82	—
171	,,	,, 17	noon	44 16	40 54	14·6	19·81	35·79	—	·00226
172	,,	,, 17	midnight	45 9	39 0	14·4	19·76	35·70	—	—
173	,,	,, 18	noon	45 53	36 45	13·1	19·75	35·68	—	—
174	,,	,, 18	midnight	46 25	34 13	12·2	19·75	35·68	—	—
175	,,	,, 19	noon	46 53	32 0	12·2	19·68	35·56	—	—
176	,,	,, 19	midnight	47 23	29 40	12·6	19·73	35·64	26·58	—
177	,,	,, 20	noon	47 59	27 9	12·1	19·66	35·52	—	·00229
178	,,	,, 20	midnight	48 17	24 40	11·8	19·74	35·66	—	—
179	,,	,, 21	noon	48 20	22 11	11·7	19·64	35·48	—	—
180	,,	,, 21	midnight	47 55	19 48	11·3	19·71	35·61	—	—
181	,,	,, 22	noon	47 53	17 28	11·7	19·69	35·58	—	—
182	,,	,, 22	midnight	48 25	15 20	11·6	19·68	35·56	—	—
183	,,	,, 23	noon	49 0	12 54	11·7	19·67	35·54	—	·00229
184	Laura	Mar. 7	6 A.M.	59 42	2 38	6·5	19·52	35·27	26·23	—
185	,,	,, 7	noon	60 1	3 13	8·0	19·70	35·59	—	—
186	,,	,, 7	10 P.M.	60 47	4 52	8·0	19·59	35·40	—	—
187	,,	,, 8	6 A.M.	61 37	6 33	6·3	19·49	35·21	—	—
188	,,	,, 12	10 P.M.	62 27	7 45	6·2	19·50	35·23	26·30	—
189	,,	,, 13	6 A.M.	62 40	10 25	7·2	19·50	35·23	—	—
190	,,	,, 13	noon	62 47	12 49	7·6	19·52	35·27	—	—
191	,,	,, 13	10 P.M.	63 4	16 27	7·4	19·50	35·23	—	—
192	,,	,, 14	6 A.M.	63 17	19 21	7·4	19·50	35·23	—	—
193	,,	,, 14	noon	63 34	21 10	6·2	19·48	35·19	26·22	·00229
194	,,	,, 14	10 P.M.	64 7	23 0	4·2	19·23	34·74	—	·00226
195	,,	,, 17	6 A.M.	64 12	22 10	2·5	18·81	34·00	—	·00222
196	,,	,, 17	noon	64 4	22 24	2·7	18·85	34·07	—	·00222
197	,,	,, 19	,,	63 46	22 41	5·4	19·26	34·80	—	·00227
198	,,	,, 19	10 P.M.	63 23	20 42	6·4	19·48	35·19	—	·00226
199	,,	,, 20	6 A.M.	63 1	19 41	7·5	19·51	35·25	—	—
200	,,	,, 20	noon	62 54	18 50	7·6	19·49	35·21	—	—
201	,,	,, 20	10 P.M.	62 46	17 24	7·6	19·51	35·25	26·25	—
202	,,	,, 21	6 A.M.	62 37	16 18	7·8	19·49	35·21	—	—
203	,,	,, 21	noon	62 13	14 27	7·9	19·52	35·27	—	—
204	,,	,, 21	10 P.M.	62 11	11 55	7·9	19·53	35·28	26·22	—
205	,,	,, 22	6 A.M.	62 15	9 0	7·7	19·53	35·28	—	—
206	,,	,, 22	noon	62 16	7 35	6·8	19·48	35·19	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
207	Laura . .	Mar. 23	noon . .	62° 10′	6° 15′	6·0	19·45	35·14	—	—
208	,,	,, 24	10 P.M. .	61 34	6 20	6·4	19·47	35·17	—	—
209	,,	,, 25	6 A.M. .	60 50	5 14	7·2	19·53	35·28	—	—
210	,,	,, 25	noon . .	60 10	3 46	8·2	19·57	35·36	26·34	—
211	Teutonic . .	,, 12	midnight	51 7	14 19	11·1	19·64	35·48	—	—
212	,,	,, 13	noon . .	50 42	20 48	11·1	19·58	35·38	—	·00231
213	,,	,, 13	midnight	49 35	26 36	10·0	19·60	35·41	—	—
214	,,	,, 14	noon . .	48 28	32 25	11·1	19·70	35·59	—	—
215	,,	,, 14	midnight	46 48	37 47	10·6	19·45	35·14	—	—
216	,,	,, 15	noon . .	45 7	42 10	6·7	19·92	35·98	26·89	—
217	,,	,, 15	midnight	43 42	46 29	1·1	18·64	33·69	25·03	—
218	,,	,, 16	noon . .	42 17	51 48	3·9	18·31	33·10	—	—
219	,,	,, 16	midnight	41 54	57 0	6·7	18·37	33·20	—	·00218
220	,,	,, 17	noon . .	41 20	62 12	6·7	18·58	33·58	—	—
221	,,	,, 17	midnight	40 57	66 59	4·4	19·10	34·51	—	—
222	,,	,, 18	noon . .	40 33	71 47	4·4	18·46	33·37	—	—
223	,,	,, 25	midnight	40 16	68 30	5·6	18·25	32·98	24·51	·00216
224	,,	,, 26	noon . .	40 34	64 12	8·9	18·90	34·16	—	·00223
225	,,	,, 26	midnight	40 55	59 34	13·3	19·17	34·63	—	—
226	,,	,, 27	noon . .	41 16	54 57	14·4	19·66	35·52	—	·00233
227	,,	,, 27	midnight	42 23	50 4	10·6	19·03	34·39	25·57	—
228	,,	,, 28	noon . .	43 30	45 12	15·0	19·85	35·86	26·70	—
229	,,	,, 28	midnight	45 14	40 22	13·3	19·50	35·23	—	·00233
230	,,	,, 29	noon . .	46 58	35 32	13·3	19·72	35·63	—	—
231	,,	,, 29	midnight	48 16	30 45	12·2	19·65	35·50	—	—
232	,,	,, 30	noon . .	49 34	24 48	11·7	19·71	35·61	—	—
233	,,	,, 30	midnight	50 26	18 54	11·7	19·72	35·63	—	—
234	,,	,, 31	noon . .	51 9	13 1	11·7	19·65	35·50	—	—
235	Ethiopia . .	,, 14	,,	54 58	14 17	10·0	19·51	35·25	—	—
236	,,	,, 15	,,	53 57	20 48	10·6	19·50	35·23	—	—
237	,,	,, 16	,,	52 54	25 56	8·9	19·20	34·69	—	—
238	,,	,, 17	,,	51 41	30 13	10·6	19·59	35·40	—	—
239	,,	,, 18	,,	50 16	35 17	11·1	19·48	35·19	—	·00230
240	,,	,, 19	,,	49 11	36 19	11·1	19·67	35·54	—	—
241	,,	,, 20	,,	48 14	39 8	11·1	19·68	35·56	—	—
242	,,	,, 21	,,	46 9	45 7	5·0	18·59	33·60	—	—
243	,,	,, 22	,,	44 22	50 40	2·8	17·81	32·21	23·90	·00210
244	,,	,, 23	,,	43 22	56 48	3·9	18·21	32·91	—	—
245	,,	,, 24	,,	42 2	63 36	2·8	18·02	32·58	—	—
246	,,	,, 25	,,	40 24	70 5	3·3	18·07	32·66	—	·00212
247	,,	,, 29	,,	40 28	69 0	3·9	18·26	33·00	—	—
248	,,	,, 30	,,	41 28	63 22	2·8	17·80	32·19	23·89	—
249	,,	,, 31	,,	42 20	58 17	7·8	18·75	33·89	25·18	—
250	,,	Apr. 1	,,	42 50	54 30	2·2	18·15	32·80	24·32	·00214
251	,,	,, 2	,,	44 28	49 0	0·0	18·17	32·84	26·43	—
252	,,	,, 3	,,	46 54	43 44	3·9	18·80	33·98	—	—
253	,,	,, 4	,,	49 17	37 46	13·3	19·62	35·45	—	·00230
254	,,	,, 5	,,	51 23	30 57	11·1	19·38	35·01	—	—
255	,,	,, 6	,,	53 8	23 34	11·1	19·48	35·19	—	—
256	,,	,, 7	,,	54 30	16 19	10·6	19·52	35·27	—	—
257	,,	,, 8	,,	55 19	8 5	9·4	19·22	34·72	—	·00221
258	Monarch . .	Feb. 13	,,	50 $21\frac{1}{2}$	2 33	8·3	19·55	35·32	—	—
259	,,	,, 14	,,	52 $12\frac{1}{2}$	5 19	9·4	19·45	35·14	—	—
260	,,	,, 15	,,	52 $56\frac{1}{2}$	4 $43\frac{3}{4}$	8·3	19·40	35·05	—	—
261	,,	,, 16	,,	52 $56\frac{3}{4}$	4 32	8·3	19·34	34·94	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
262	Monarch . .	Feb. 17	noon . .	52° 31½′	5° 42′	8·3	19·35	34·96	—	—
263	,,	,, 18	,,	52 13¾	6 18	8·9	19·33	34·93	—	—
264	,,	,, 19	,,	52 17	6 31	8·3	19·13	34·56	—	·00226
265	,,	,, 20	,,	52 13	6 14½	9·4	19·33	34·93	—	—
266	,,	,, 21	,,	Off	Passage	10·0	14·91	27·02	19·83	·00174
267	,,	,, 23	,,	,,	,,	8·9	11·47	20·82	15·06	·00134
268	,,	,, 24	,,	,,	,,	7·8	19·38	35·01	26·04	—
269	,,	Mar. 1	,,	56° 17′	5° 49¾′	7·8	18·89	34·14	—	—
270	,,	,, 2	,,	57 13	5 39½	7·8	18·63	33·67	—	—
271	,,	,, 3	,,	57 57½	5 45	7·2	18·95	34·24	—	—
272	,,	,, 4	,,	58 55	3 14½	7·2	19·14	34·58	—	—
273	,,	,, 5	,,	Cairston Rd., Stromness		6·7	18·84	34·05	—	—
274	,,	,, 6	,,	58° 43′	3° 0′	6·7	19·16	34·62	—	—
275	,,	,, 7	,,	Leith	Roads	6·1	17·80	32·19	—	—
276	,,	,, 8	,,	—	—	6·1	17·92	32·40	—	—
277	,,	,, 9	,,	—	—	5·6	17·85	32·28	—	—
278	,,	,, 10	,,	—	—	5·6	—	—	—	—
279	,,	,, 11	,,	Inch Keith Lt. Ho. N. 62½° E. (true) dist. 3·45′.		6·7	18·59	33·60	—	—
280	,,	,, 12	,,	Leith	Roads	5·6	18·60	33·62	—	—
					E.					
281	,,	,, 13	,,	53° 37½′	0° 40′	7·8	18·71	33·82	—	—
282	,,	,, 21	,,	Zandvoort 1′ E.		7·8	11·97	21·73	15·81	—
283	,,	,, 22	,,	52° 25′	4° 8′	6·7	18·55	33·53	—	—
284	,,	,, 24	,,	52 21¾	3 41	7·8	17·82	32·22	—	—
285	,,	Apr. 3	,,	50 42½	0 10	8·9	19·46	35·15	—	—
					W.					
286	,,	,, 4	,,	50 13	5 54½′	10·6	19·67	35·54	26·47	·00230
287	,,	,, 5	,,	51 23½	5 58	8·9	18·90	34·16	—	—
288	Frolic . .	,, 2	,,	59 30	6 10	7·8	19·44	35·12	—	·00230
289	,,	,, 2	6 P.M. .	60 0	7 56	8·9	15·57	35·36	—	—
290	,,	,, 3	6 A.M. .	60 50	10 53	9·4	19·48	35·19	—	—
291	,,	,, 3	noon . .	61 10	12 3	9·4	19·45	35·14	—	·00230
292	,,	,, 3	6 P.M. .	61 40	13 46	8·9	19·47	35·17	—	—
293	,,	,, 4	6 A.M. .	62 30	16 48	8·9	19·46	35·15	—	—
294	,,	,, 4	noon . .	62 40	17 28	8·9	19·45	35·14	—	—
295	,,	,, 4	6 P.M. .	63 0	18 42	8·3	19·41	35·07	26·05	—
296	,,	,, 5	6 A.M. .	63 10	19 18	7·8	19·33	34·93	—	—
297	,,	,, 5	noon . .	63 20	19 55	7·8	19·36	34·98	26·01	—
298	,,	,, 17	,,	63 30	20 16	8·9	19·42	35·08	26·14	—
299	,,	,, 17	6 P.M. .	63 10	19 9	8·9	19·43	35·10	—	·00230
300	,,	,, 18	6 A.M. .	62 20	16 8	8·9	19·43	35·10	26·11	—
301	,,	,, 18	noon . .	61 50	14 18	8·9	19·45	35·14	—	—
302	,,	,, 18	6 P.M. .	61 10	11 56	8·9	19·44	34·12	—	—
303	,,	,, 19	6 A.M. .	59 50	7 14	8·9	19·44	35·12	—	·00230
304	,,	,, 19	noon . .	59 30	6 5	10·0	19·50	35·23	26·23	—
305	,,	,, 19	6 P.M. .	59 0	4 25	9·4	19·03	34·39	—	—
306	,,	,, 20	6 A.M. .	57 40	—	8·3	19·17	34·63	—	—
307	,,	,, 20	noon . .	56 30	—	8·9	19·17	34·63	—	—
308	,,	,, 20	6 P.M. .	55 40	—	8·9	19·16	34·62	—	·00226
309	Teutonic . .	,, 10	noon . .	50 39	19 58	11·1	19·59	35·40	—	·00232
310	,,	,, 10	midnight	49 43	27 14	12·2	19·64	35·48	—	—
311	,,	,, 11	noon . .	48 47	34 31	12·2	19·58	35·38	—	—
312	,,	,, 11	midnight	47 17	37 1	13·3	19·83	35·82	26·63	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
313	Teutonic .	Apr. 12	noon . .	45° 47′	41° 32′	13·3	19·59	35·40	—	—
314	,,	,, 12	midnight	44 5	47 20	8·3	18·68	33·76	—	—
315	,,	,, 13	noon . .	42 24	51 8	8·3	18·31	33·10	—	—
316	,,	,, 13	midnight	41 52	56 38	10·0	20·10	36·30	—	—
317	,,	,, 14	noon . .	41 20	62 8	5·6	18·30	33·08	—	·00216
318	,,	,, 14	midnight	40 57	67 37	6·1	18·17	32·84	—	—
319	,,	,, 15	noon . .	40 34	73 7	7·8	17·72	32·04	23·71	—
320	,,	,, 22	midnight	40 9	69 47	6·7	18·36	33·19	—	·00215
321	,,	,, 23	noon . .	40 27	65 37	7·2	18·46	33·37	—	·00217
322	,,	,, 23	midnight	40 53	60 43	11·7	19·65	35·50	26·43	—
323	,,	,, 24	noon . .	41 19	55 48	16·7	20·25	36·57	27·17	·00238
324	,,	,, 24	midnight	42 10	50 56	6·1	18·47	33·39	24·93	—
325	,,	,, 25	noon . .	43 1	46 3	16·7	20·01	36·14	—	—
326	,,	,, 25	midnight	44 51	41 18	12·8	19·78	35·73	—	·00233
327	,,	,, 26	noon . .	46 41	36 33	13·3	19·77	35·72	—	—
328	,,	,, 26	midnight	48 0	31 11	12·2	19·71	35·61	—	—
329	,,	,, 27	noon . .	49 19	25 50	13·3	19·65	35·50	—	—
330	,,	,, 27	midnight	50 17	20 2	10·0	19·74	35·66	—	—
331	,,	,, 28	noon . .	51 16	14 14	12·2	19·63	35·47	—	—
332	Capricornus.	,, 14	1.30 P.M.	59 24	4 25	8·1	19·40	35·05	—	—
333	,,	,, 15	8 A.M. .	61 11	7 23	7·2	19·45	35·14	—	·00230
334	,,	,, 15	1.30 P.M.	61 42	8 55	7·5	19·47	35·17	—	—
335	,,	,, 15	7 P.M. .	62 15	10 23	7·5	19·45	35·14	—	—
336	,,	,, 16	8 A.M. .	63 15	14 4	7·8	19·43	35·10	—	—
337	,,	,, 16	2 P.M. .	63 46	15 42	6·7	19·39	35·03	—	—
338	,,	,, 16	9 P.M. .	63 37	17 0	7·2	19·40	35·05	26·08	—
339	,,	,, 17	8 A.M. .	63 28	17 53	5·8	18·65	33·71	25·00	—
340	,,	,, 18	,,	63 22	20 23	6·4	19·38	35·01	—	·00229
341	,,	,, 19	noon . .	63 22	20 23	6·1	19·29	34·86	—	—
342	,,	,, 20	,,	63 22	20 23	6·1	19·29	34·86	—	—
343	,,	,, 21	,,	63 22	20 23	6·4	19·30	34·87	—	—
344	,,	,, 22	,,	63 22	20 23	6·1	19·28	34·84	—	—
345	,,	,, 23	,,	63 22	20 23	6·4	19·29	34·86	—	—
346	,,	,, 24	,,	63 22	20 23	6·7	19·28	34·84	25·90	—
347	,,	,, 25	7 P.M. .	63 42	16 55	7·2	18·26	33·00	24·48	·00216
348	,,	,, 26	8 A.M. .	63 50	16 11	7·8	19·42	35·08	26·11	—
349	,,	,, 26	noon . .	63 50	16 11	8·1	19·42	35·08	—	·00228
350	,,	,, 26	6 P.M. .	63 50	16 11	8·1	19·42	35·08	—	—
351	,,	,, 27	8 A.M. .	62 39	11 6	8·1	19·41	35·07	—	—
352	,,	,, 27	noon . .	62 18	9 37	8·3	19·46	35·15	—	·00230
353	,,	,, 27	6 P.M. .	61 57	8 0	7·8	19·46	35·15	—	—
354	,,	,, 28	8 A.M. .	60 30	5 35	8·6	19·47	35·17	—	—
355	,,	,, 28	noon . .	60 0	4 45	9·7	19·52	35·27	—	·00229
356	,,	,, 28	6 P.M. .	59 24	3 54	8·6	19·47	35·17	—	—
357	Hercules . .	,, 24	7.30 A.M.	59 50	6 10	14·7	19·52	35·27	—	—
358	,,	,, 24	1 P.M. .	60 11	7 8	11·4	19·52	35·27	—	—
359	,,	,, 24	7.30 P.M.	60 38	8 25	8·6	19·52	35·27	—	—
360	,,	,, 25	8.30 A.M.	61 20	10 48	8·3	19·52	35·27	—	—
361	,,	,, 25	4 P.M. .	61 44	12 6	8·3	19·48	35·19	—	—
362	,,	,, 25	8 P.M. .	62 3	13 6	8·3	19·46	35·15	—	—
363	,,	,, 26	8.30 A.M.	62 49	15 6	8·3	19·47	35·17	—	—
364	,,	,, 26	noon . .	63 6	16 30	8·3	19·47	35·17	—	—
365	,,	,, 26	5 P.M. .	63 45	22 45	8·3	19·39	35·03	—	—
366	,,	,, 27	4 P.M. .	64 47	24 11	4·3	19·07	34·46	—	—
367	,,	,, 27	7 P.M. .	64 57	24 15	4·3	19·07	34·46	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
368	Hercules . .	Apr. 28	noon . .	64° 47′	24° 12′	4·3	19·09	34·49	—	—
369	,,	,, 28	6 P.M. .	64 57	24 15	4·3	19·07	34·46	—	—
370	,,	,, 29	10 A.M. .	64 57	24 15	4·3	19·08	34·47	—	—
371	,,	,, 29	5 P.M. .	64 47	24 12	4·3	19·06	34·44	—	—
372	,,	,, 30	2 P.M. .	64 57	24 15	4·3	19·07	34·46	—	—
373	,,	,, 30	8 P.M. .	64 47	24 12	4·3	19·07	34·46	—	—
374	,,	May 1	3 P.M. .	64 57	24 15	4·3	19·08	34·47	—	—
375	,,	,, 2	9 P.M. .	64 47	24 12	4·3	19·06	34·44	—	—
376	,,	,, 3	4 P.M. .	66 8	24 20	2·8	19·06	34·44	—	—
377	,,	,, 4	8 P.M. .	66 7	24 20	2·8	19·06	34·44	—	—
378	,,	,, 5	10 A.M. .	66 25	21 0	2·8	19·07	34·46	—	—
379	,,	,, 6	4 P.M. .	63 27	19 24	6·5	19·41	35·07	—	—
380	,,	,, 7	7 P.M. .	62 27	19 24	6·5	19·41	35·07	—	—
381	,,	,, 8	noon . .	62 45	16 50	8·3	19·42	35·08	—	—
382	,,	,, 8	4 P.M. .	62 26	15 27	8·9	19·45	35·14	—	—
383	,,	,, 8	9 P.M. .	62 8	14 18	8·9	19·41	35·07	—	—
384	,,	,, 9	9 A.M. .	61 38	11 45	10·0	19·51	35·25	—	—
385	,,	,, 9	5 P.M. .	60 52	9 38	10·0	19·50	35·23	—	—
386	,,	,, 9	9.30 P.M.	60 30	8 30	10·0	19·47	35·17	—	—
387	Ethiopia . .	Apr. 18	noon . .	54 46	13 29	10·6	19·54	35·30	—	—
388	,,	,, 19	,,	53 26	20 35	10·6	19·53	35·28	—	—
389	,,	,, 20	,,	51 43	27 25	11·1	19·53	35·28	—	—
390	,,	,, 21	,,	49 44	33 21	12·2	19·64	35·48	—	·00230
391	,,	,, 22	,,	47 57	38 6	11·7	19·59	35·40	—	—
392	,,	,, 23	,,	46 11	42 24	12·2	19·54	35·30	26·31	—
393	,,	,, 24	,,	44 17	46 50	4·4	18·43	33·32	—	·00214
394	,,	,, 25	,,	42 59	53 7	2·8	18·20	32·90	—	—
395	,,	,, 26	,,	42 35	59 57	3·9	18·10	32·72	—	—
396	,,	,, 26	,,	41 21	66 39	5·0	18·07	32·66	24·32	—
397	,,	,, 28	,,	40 25	73 32	7·2	17·66	31·93	—	·00208
398	,,	May 3	,,	40 30	68 18	6·7	18·10	32·72	—	—
399	,,	,, 4	,,	40 36	62 10	19·4	19·93	36·00	—	—
400	,,	,, 5	,,	41 14	55 38	18·9	20·05	36·21	27·06	·00233
401	,,	,, 6	,,	41 42	48 53	14·4	19·54	35·30	—	—
402	,,	,, 7	,,	44 43	43 29	13·9	19·66	35·52	26·49	—
403	,,	,, 8	,,	47 25	38 8	12·2	19·47	35·17	—	—
404	,,	,, 9	,,	49 39	32 6	11·1	19·35	34·96	26·11	—
405	,,	,, 10	,,	51 56	25 28	12·8	19·47	35·17	—	·00228
406	,,	,, 11	,,	53 42	17 54	12·2	19·53	35·28	—	—
407	,,	,, 12	,,	55 8	10 15	12·8	19·53	35·28	—	—
408	Frolic . .	Apr. 27	6 A.M. .	59 10	5 3	10·0	19·48	35·19	—	·00231
409	,,	,, 27	noon . .	59 40	6 44	10·0	19·50	35·23	—	—
410	,,	,, 27	6 P.M. .	60 10	8 30	9·4	19·53	35·28	—	—
411	,,	,, 28	6 A.M. .	61 10	12 0	8·9	19·45	35·14	—	—
412	,,	,, 28	noon . .	61 35	13 26	8·9	19·45	35·14	—	—
413	,,	,, 28	6 P.M. .	62 0	15 3	8·9	19·47	35·17	—	—
414	,,	,, 29	6 A.M. .	62 40	17 26	8·3	19·46	35·15	—	—
415	,,	,, 29	noon . .	63 0	18 41	8·3	19·40	35·05	26·07	—
416	,,	,, 29	6 P.M. .	63 10	19 18	8·9	19·15	34·60	25·74	·00226
417	,,	,, 30	6 A.M. .	63 25	19 54	8·3	19·40	35·05	—	—
418	,,	,, 30	noon . .	63 25	20 2	8·9	19·45	35·14	—	—
419	,,	,, 30	6 P.M. .	63 25	20 8	8·9	19·45	35·14	—	—
420	,,	May 4	6 A.M. .	64 30	23 8	7·2	19·38	35·01	—	—
421	,,	,, 4	noon . .	64 50	24 24	6·7	19·39	35·03	—	—
422	,,	,, 4	6 P.M. .	65 10	24 10	6·7	19·34	34·94	—	—

Table I. *(continued)*.

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
423	Frolic. . .	May 5	6 A.M. .	65° 10′	24° 20′	5·6	19·27	34·82	—	—
424	,,	,, 5	noon . .	65 30	24 56	6·7	19·28	34·84	—	—
425	,,	,, 8	,,	65 50	24 48	6·7	19·28	34·84	—	—
426	,,	,, 9	6 A.M. .	64 0	23 46	7·8	19·31	34·89	25·97	·00228
427	,,	,, 9	noon . .	63 40	23 30	7·8	19·09	34·49	25·57	—
428	,,	,, 9	6 P.M. .	63 0	20 16	9·4	19·47	35·17	26·18	—
429	,,	,, 10	6 A.M. .	62 20	16 28	10·0	19·42	35·08	—	—
430	,,	,, 10	noon . .	62 0	14 44	10·6	19·45	35·14	—	—
431	,,	,, 10	6 P.M. .	61 40	13 25	10·6	19·44	35·12	—	—
432	,,	,, 11	6 A.M. .	60 40	10 2	11·1	19·52	35·27	—	—
433	,,	,, 11	noon . .	60 0	8 26	11·7	19·52	35·28	—	—
434	,,	,, 11	6 P.M. .	59 20	5 46	12·8	19·54	35·30	—	·00229
435	Laura . . .	Apr. 27	6 A.M. .	60 4	3 26	9·2	19·53	35·28	—	—
436	,,	,, 27	noon . .	60 30	4 42	9·0	19·53	35·28	—	—
437	,,	,, 27	10 P.M. .	61 30	6 10	8·0	19·42	35·08	—	—
438	,,	,, 28	noon . .	61 54	7 26	7·0	19·44	35·12	—	—
439	,,	,, 28	10 P.M. .	62 17	11 4	7·7	19·48	35·19	—	—
440	,,	,, 29	6 A.M. .	62 32	13 47	7·6	19·49	35·21	—	—
441	,,	,, 29	noon . .	62 40	16 7	8·0	19·48	35·19	—	—
442	,,	,, 29	10 P.M. .	63 17	19 20	7·7	19·34	34·94	—	—
443	,,	,, 30	6 A.M. .	63 20	20 0	7·2	19·40	35·05	—	·00229
444	,,	,, 30	noon . .	63 44	22 14	7·2	18·95	34·24	25·50	—
445	,,	May 3	10 P.M. .	64 30	23 32	6·0	19·26	34·80	25·95	—
446	,,	,, 4	6 A.M. .	65 5	23 45	4·3	19·14	34·58	—	·00226
447	,,	,, 5	,,	65 16	24 0	4·7	18·95	34·24	—	—
448	,,	,, 8	,,	66 4	23 53	2·8	18·81	34·02	—	·00223
449	,,	,, 8	10 P.M. .	66 17	23 35	3·0	18·80	33·98	—	—
450	,,	,, 9	6 A.M. .	65 19	24 34	4·7	19·20	34·69	—	—
451	,,	,, 9	noon . .	64 34	23 24	6·8	19·26	34·80	—	—
452	,,	,, 13	6 A.M. .	63 47	22 48	8·0	18·07	32·66	24·24	·00214
453	,,	,, 13	noon . .	63 34	21 17	8·0	19·13	34·56	25·69	—
454	,,	,, 13	10 P.M. .	63 7	18 12	8·5	19·40	35·05	26·15	—
455	,,	,, 14	6 A.M. .	62 50	14 54	8·9	19·45	35·14	—	—
456	,,	,, 14	noon . .	62 38	12 37	8·7	19·44	35·12	—	—
457	,,	,, 14	10 P.M. .	62 27	8 40	8·6	19·45	35·14	—	—
458	,,	,, 15	6 A.M. .	62 26	7 0	7·5	19·39	35·03	—	—
459	,,	,, 16	10 P.M. .	61 8	5 45	9·0	19·45	35·14	—	·00231
460	,,	,, 17	6 A.M. .	60 25	4 17	9·7	19·54	35·30	—	—
461	,,	,, 17	noon . .	59 45	2 50	9·7	19·49	35·21	—	—
462	Loughrigg Holme	Mar. 21	,,	58 43	3 52	7·4	19·08	34·47	25·70	·00223
463	,,	,, 21	midnight	58 30	7 7	8·6	19·43	35·10	—	·00228
464	,,	,, 22	noon . .	58 18	10 8	9·9	19·48	35·19	—	·00230
465	,,	,, 22	midnight	58 1	12 21	9·1	19·49	35·21	—	—
466	,,	,, 23	noon . .	57 48	13 46	8·4	19·47	35·17	26·30	—
467	,,	,, 23	midnight	57 20	15 48	8·6	19·50	35·23	—	—
468	,,	,, 24	noon . .	56 33	18 48	9·5	19·44	35·12	—	—
469	,,	,, 24	midnight	55 40	21 32	9·4	19·47	35·17	—	—
470	,,	,, 25	noon . .	55 3	23 50	8·9	19·40	35·05	—	—
471	,,	,, 25	midnight	54 27	25 40	8·4	19·40	35·05	—	—
472	,,	,, 26	noon . .	53 47	27 49	8·0	19·34	34·94	—	·00225
473	,,	,, 26	midnight	53 1	29 39	7·8	19·29	34·86	—	—
474	,,	,, 27	noon . .	52 34	30 51	7·3	19·23	34·74	—	—
475	,,	,, 27	midnight	51 47	32 28	9·5	19·35	34·96	—	—
476	,,	,, 28	noon . .	50 54	34 14	8·1	19·25	34·78	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
477	Loughrigg Holme	Mar. 28	midnight	50° 14′	35° 44′	12·2	19·63	35·47	—	—
478	,,	,, 29	noon . .	49 32	37 0	11·8	19·52	35·27	—	—
479	,,	,, 29	midnight	48 54	38 30	12·8	19·60	35·41	—	—
480	,,	,, 30	noon . .	48 17	39 48	11·8	19·57	35·36	—	—
481	,,	,, 30	midnight	48 2	40 35	10·1	19·66	35·52	—	·00230
482	,,	,, 31	noon . .	47 53	41 8	11·6	19·53	35·28	—	—
483	,,	,, 31	midnight	47 21	42 11	12·3	19·71	35·61	26·63	—
484	,,	Apr. 1	noon . .	46 16	43 55	4·3	18·76	33·90	25·31	—
485	,,	,, 1	midnight	45 21	45 41	8·5	18·90	34·16	—	—
486	,,	,, 2	noon . .	44 39	47 4	4·6	18·46	33·37	—	—
487	,,	,, 2	midnight	43 43	48 36	2·8	18·43	33·32	—	·00213
488	,,	,, 3	noon . .	42 54	50 24	1·7	18·07	32·66	24·34	—
489	,,	,, 3	midnight	42 54	52 45	5·6	18·54	33·51	—	—
490	,,	,, 4	noon . .	43 3	54 49	3·7	18·25	32·98	—	—
491	,,	,, 4	midnight	43 2	57 10	3·0	18·22	32·93	—	—
492	,,	,, 5	noon . .	43 2	59 18	2·4	18·07	32·66	—	—
493	,,	,, 5	midnight	42 49	61 39	2·2	18·01	32·64	—	—
494	,,	,, 6	noon . .	42 36	64 1	1·3	17·48	31·63	23·52	·00206
495	,,	,, 6	midnight	42 28	66 35	3·9	18·11	32·73	—	—
496	,,	,, 7	noon . .	42 28	69 10	3·7	18·25	32·98	—	—
497	,,	,, 14	,,	43 2	66 34	4·1	17·87	32·31	—	—
498	,,	,, 14	midnight	42 58	65 8	2·2	17·60	31·83	—	—
499	,,	,, 15	noon . .	43 54	62 20	1·9	17·64	31·90	23·69	·00205
500	,,	May 12	,,	46 33	54 35	1·9	17·93	32·42	24·17	—
501	,,	,, 13	,,	46 51	51 17	1·8	17·95	32·46	—	·00211
502	,,	,, 13	midnight	47 14	48 44	0·8	18·10	32·72	—	—
503	,,	,, 14	noon . .	47 46	46 19	2·1	18·73	33·85	—	—
504	,,	,, 15	,,	48 47	41 13	11·5	19·55	35·32	—	—
505	,,	,, 15	midnight	49 11	38 33	8·3	18·67	33·75	—	—
506	,,	,, 16	noon . .	49 33	36 54	11·5	19·40	35·05	—	—
507	,,	,, 16	midnight	49 50	34 15	12·8	19·67	35·54	—	—
508	,,	,, 17	noon . .	50 7	30 36	12·9	19·55	35·32	—	·00227
509	,,	,, 17	midnight	50 15	27 55	13·0	19·54	35·30	—	—
510	,,	,, 18	noon . .	50 23	25 7	13·3	19·42	35·08	—	—
511	,,	,, 18	8 P.M. .	50 24	23 12	14·1	19·54	35·30	—	—
512	,,	,, 18	midnight	50 25	22 14	13·9	19·60	35·41	—	—
513	,,	,, 19	4 A.M. .	50 25	21 16	14·2	19·60	35·42	26·38	—
514	,,	,, 19	noon . .	50 25	19 17	14·7	19·62	35·45	—	—
515	,,	,, 19	4 P.M. .	50 22	18 22	14·8	19·67	35·54	—	—
516	,,	,, 19	midnight	50 14	16 33	14·1	19·63	35·47	—	·00231
517	,,	,, 20	8 A.M. .	50 8	14 45	14·9	19·62	35·45	—	—
518	,,	,, 20	noon . .	50 5	13 40	14·8	19·65	35·50	—	—
519	,,	,, 21	8 A.M. .	49 56	9 1	13·7	19·55	35·32	—	—
520	,,	,, 21	noon . .	49 55	8 5	14·1	19·68	35·56	26·58	—
521	,,	,, 21	6 P.M. .	49 49	6 37	12·8	19·67	35·54	—	—
522	Capricornus	,, 3	midnight	W. side of Pentland Firth.		8·1	19·16	34·62	—	—
523	,,	,, 4	8 A.M. .	59° 23′	5° 4′	9·7	19·49	35·21	—	—
524	,,	,, 4	noon . .	59 48	6 20	10·3	19·49	35·21	—	—
525	,,	,, 4	7 P.M. .	60 17	7 47	9·2	19·46	35·15	—	—
526	,,	,, 4	midnight	60 42	9 2	8·6	19·53	35·28	—	—
527	,,	,, 5	8 A.M. .	61 18	10 52	8·9	19·45	35·14	—	—
528	,,	,, 5	noon . .	61 38	11 54	9·2	19·50	35·23	—	—
529	,,	,, 5	7 P.M. .	62 13	13 44	8·6	19·45	35·14	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
530	Capricornus	May 5	midnight	62° 40′	15° 7′	8·3	19·45	35·14	—	—
531	,,	,, 6	8 A.M. .	63 3	16 22	8·9	19·45	35·14	—	—
532	,,	,, 6	noon . .	63 20	17 27	9·2	19·44	35·12	—	—
533	,,	,, 6	midnight	63 26	17 32	8·1	19·29	34·86	—	—
534	,,	,, 7	noon . .	63 50	16 24	8·3	19·30	34·87	—	—
535	,,	,, 7	midnight	63 50	16 24	7·8	19·33	34·93	—	—
536	,,	,, 8	8 A.M. .	63 3	19 46	8·3	19·42	35·08	—	—
537	,,	,, 8	noon . .	63 36	21 37	7·5	19·41	35·07	—	—
538	,,	,, 8	6 P.M. .	63 45	22 45	7·2	19·26	34·80	—	—
539	,,	,, 8	midnight	64 44	24 0	6·7	19·37	35·00	—	·00226
540	,,	,, 9	noon . .	65 50	24 10	3·9	19·00	34·33	25·53	—
541	,,	,, 9	midnight	66 36	22 47	3·3	19·12	34·54	25·66	·00224
542	,,	,, 10	noon . .	66 39	22 28	2·5	18·85	34·07	—	·00222
543	,,	,, 11	,,	66 39	22 28	2·2	18·79	33·96	25·26	·00221
544	,,	,, 17	midnight	63 44	16 0	8·1	19·38	35·01	—	·00228
545	,,	,, 18	noon . .	63 7	12 3	9·2	19·41	35·07	—	—
546	,,	,, 19	midnight	62 18	9 10	6·7	19·39	35·03	—	—
547	,,	,, 20	8 A.M. .	61 45	7 0	7·2	19·40	35·05	—	—
548	,,	,, 21	noon . .	61 9	6 26	8·9	19·49	35·21	26·12	—
549	,,	,, 21	,,	60 15	5 12	9·2	19·06	34·44	25·64	—
550	,,	,, 22	midnight	59 42	4 32	8·9	19·05	34·42	—	—
551	,,	,, 21	noon . .	59 0	3 36	9·7	19·06	34·44	—	—
552	Teutonic. .	,, 7	midnight	51 6	14 22	12·8	19·58	35·38	—	—
553	,,	,, 8	noon . .	50 39	20 46	13·3	19·58	35·38	—	·00229
554	,,	,, 8	midnight	49 28	27 1	13·9	19·63	35·47	—	—
555	,,	,, 9	noon . .	48 20	33 4	13·3	19·68	35·56	—	—
556	,,	,, 9	midnight	46 28	38 23	13·3	19·58	35·38	—	—
557	,,	,, 10	noon . .	44 50	43 10	14·4	19·78	35·73	—	—
558	,,	,, 10	midnight	43 18	47 42	8·9	18·43	33·32	—	—
559	,,	,, 11	noon . .	42 26	52 28	6·7	18·30	33·08	—	·00215
560	,,	,, 11	midnight	41 49	57 50	11·1	19·56	35·34	—	—
561	,,	,, 12	noon . .	41 12	63 31	14·4	19·67	35·54	—	·00231
562	,,	,, 12	midnight	40 43	68 43	7·2	18·21	32·91	—	·00213
563	,,	,, 20	,,	40 8	69 43	10·0	18·29	33·06	—	—
564	,,	,, 21	noon . .	40 31	64 58	16·7	19·78	35·73	—	·00232
565	,,	,, 21	midnight	40 50	60 6	17·8	20·13	36·35	—	·00235
566	,,	,, 22	noon . .	41 12	54 47	14·4	19·54	35·30	—	—
567	,,	,, 22	midnight	41 44	49 46	14·4	19·68	35·56	—	—
568	,,	,, 23	noon . .	43 21	45 0	17·8	20·14	36·37	—	—
569	,,	,, 23	midnight	45 15	40 15	16·1	19·67	35·54	—	—
570	,,	,, 24	noon . .	47 0	35 21	15·0	19·54	35·30	—	—
571	,,	,, 24	midnight	48 20	29 53	14·4	19·68	35·56	—	—
572	,,	,, 25	noon . .	49 32	24 16	15·6	19·68	35·56	—	—
573	,,	,, 25	midnight	50 32	18 26	15·0	19·67	35·54	—	—
574	,,	,, 26	noon . .	51 10	12 22	14·4	19·68	35·56	—	—
575	Frolic. . .	,, 17	6 A.M. .	59 10	5 4	10·0	19·49	35·21	—	—
576	,,	,, 17	noon . .	59 35	6 32	11·1	19·60	35·41	—	—
577	,,	,, 17	6 P.M. .	60 0	8 0	11·1	19·53	35·28	—	—
578	,,	,, 18	6 A.M. .	60 50	10 58	8·9	19·49	35·21	—	—
579	,,	,, 18	noon . .	61 10	12 17	9·4	19·47	35·17	—	—
580	,,	,, 18	6 P.M. .	61 35	13 47	9·4	19·50	35·23	—	—
581	,,	,, 19	6 A.M. .	62 20	16 37	8·9	19·50	35·23	—	—
582	,,	,, 19	noon . .	62 40	17 48	8·9	19·48	35·19	—	—
583	,,	,, 19	6 P.M. .	63 0	19 0	8·3	19·49	35·21	26·25	—
584	,,	,, 20	6 A.M. .	63 40	22 8	7·8	17·45	31·53	23·42	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
585	Frolic. . .	May 20	noon . .	64° 20′	23° 16′	7·2	19·34	34·94	25·99	—
586	,,	,, 20	6 P.M. .	65 0	24 35	6·7	19·23	34·74	—	—
587	,,	,, 21	noon . .	65 40	25 8	5·6	19·31	34·89	—	·00227
588	,,	,, 24	,,	66 20	24 38	5·0	19·02	34·37	—	·00223
589	,,	,, 26	,,	66 35	23 28	2·8	18·66	33·73	—	·00218
590	,,	,, 27	,,	66 50	23 15	1·1	18·30	33·08	24·55	·00216
591	,,	,, 30	,,	65 0	24 36	6·7	19·33	34·93	25·99	·00226
592	,,	,, 30	6 P.M. .	64 20	23 50	7·2	19·38	35·01	—	·00229
593	,,	,, 31	noon . .	63 20	22 46	7·8	18·50	33·44	—	—
594	,,	June 1	6 A.M. .	62 25	17 18	8·3	19·48	35·19	—	—
595	,,	,, 1	noon . .	62 5	16 10	8·9	19·56	35·34	—	—
596	,,	,, 1	6 P.M. .	61 40	14 38	8·9	19·49	35·21	—	—
597	,,	,, 2	6 A.M. .	60 30	10 36	9·4	19·51	35·25	—	—
598	,,	,, 2	noon . .	60 0	8 45	10·0	19·52	35·27	—	—
599	,,	,, 2	6 P.M. .	59 30	6 56	10·6	19·51	35·25	—	—
600	California .	Mar. 20	noon . .	31 6	47 36	—	20·20	36·48	—	·00238
601	,,	,, 20	midnight	31 6	49 56	—	20·08	36·27	—	·00238
602	,,	,, 21	noon . .	31 51	52 19	—	20·23	36·53	—	·00239
603	,,	,, 21	midnight	32 39	54 39	—	20·49	37·00	—	·00242
604	,,	,, 22	noon . .	33 35	56 34	—	20·06	36·23	—	—
605	,,	,, 22	midnight	34 16	58 27	—	20·20	36·48	—	—
606	,,	,, 23	noon . .	34 59	60 21	—	20·08	36·27	—	—
607	,,	,, 24	,,	36 43	64 51	—	20·21	36·49	—	·00237
608	,,	,, 24	midnight	37 17	66 43	—	20·24	36·55	—	—
609	,,	,, 25	noon . .	37 51	68 18	—	20·07	36·25	27·15	·00236
610	,,	,, 25	midnight	38 41	70 35	—	19·66	35·52	26·47	·00228
611	,,	,, 26	noon . .	39 37	71 50	—	19·40	35·05	26·18	·00229
612	,,	,, 26	midnight	40 27	73 53	—	18·12	32·75	—	·00213
613	,,	April 9	noon . .	40 25	70 22	—	18·30	33·08	—	·00215
614	,,	,, 10	,,	40 49	66 6	—	19·22	34·72	—	·00226
615	,,	,, 11	,,	41 27	61 34	—	18·11	32·73	—	·00211
616	,,	,, 11	midnight	41 49	59 14	—	19·78	35·73	—	—
617	,,	,, 12	noon . .	42 21	56 50	—	20·02	36·16	—	—
618	,,	,, 12	midnight	42 41	54 18	—	18·88	34·12	—	—
619	,,	,, 13	,,	45 15	49 9	—	18·15	32·80	—	—
620	,,	,, 14	noon . .	43 58	47 6	—	18·33	33·13	—	—
621	,,	,, 14	midnight	45 5	45 0	—	18·44	33·34	—	—
622	,,	,, 15	noon . .	46 19	42 52	—	18·27	33·02	—	—
623	,,	,, 15	midnight	47 22	40 39	—	18·60	33·62	—	—
624	,,	,, 16	noon . .	48 27	38 18	—	19·61	35·43	—	—
625	,,	,, 16	midnight	49 21	35 50	—	19·54	35·30	—	—
626	,,	,, 17	noon . .	50 20	35 10	—	19·35	34·96	—	—
627	,,	,, 17	midnight	51 4	30 18	—	19·45	35·14	—	—
628	,,	,, 18	noon . .	52 5	27 39	—	19·53	35·28	—	—
629	,,	,, 18	midnight	52 45	24 47	—	19·45	35·14	—	—
630	,,	,, 19	noon . .	53 25	21 52	—	19·52	35·27	—	—
631	,,	,, 19	midnight	53 55	18 56	—	19·61	35·43	—	—
632	,,	,, 20	noon . .	53 33	16 8	—	19·61	35·43	—	—
633	Corean . .	May 3	,,	off Queenstown		11·7	19·68	35·56	—	—
634	,,	,, 4	,,	51° 25′	15° 35′	12·2	19·72	35·63	26·68	—
635	,,	,, 5	,,	51 32	22 50	12·2	19·58	35·38	—	—
636	,,	,, 6	,,	51 44	30 15	11·1	19·55	35·32	—	—
637	,,	,, 7	,,	50 38	37 14	10·0	19·34	34·94	—	—
638	,,	,, 8	,,	48 44	43 35	7·8	18·74	33·87	—	·00221
639	,,	,, 9	,,	47 47	47 1	0·6	18·49	33·43	—	·00217

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
640	Corean . .	May 10	noon . .	47° 28′	50° 15′	1·1	18·07	32·66	24·24	—
641	,,	,, 11	,,	6′ off St. John's, Newfoundland		−2·2	18·14	32·79	—	—
642	,,	,, 15	,,	off C. Race		−1·1	17·75	32·09	—	—
643	,,	,, 16	,,	44° 50′	59° 20′	0·0	17·67	31·95	23·76	—
644	,,	,, 19	,,	41 45	66 0	7·8	18·32	33·11	—	—
645	,,	,, 20	,,	39 37	70 45	13·9	19·58	35·38	—	·00230
646	,,	,, 27	,,	39 32	70 47	14·4	18·94	34·23	—	—
647	,,	,, 28	,,	41 6	65 40	11·7	18·52	33·48	—	—
648	,,	,, 29	,,	42 48	60 23	5·6	18·09	32·70	—	—
649	,,	,, 30	,,	45 18	55 40	1·7	17·87	32·31	—	—
650	,,	,, 31	,,	46 12	53 45	2·2	17·80	32·19	—	—
651	,,	June 1	,,	off C. Race		1·7	17·69	31·99	—	—
652	,,	,, 3	,,	off St. John's		1·1	17·52	31·68	23·41	·00206
653	,,	,, 4	,,	48° 48′	48° 38′	0·0	17·77	32·13	—	·00209
654	,,	,, 5	,,	50 48	43 30	11·1	19·48	35·19	—	—
655	,,	,, 6	,,	52 30	37 20	8·9	19·20	34·69	—	—
656	,,	,, 7	,,	53 53	30 40	8·9	19·37	35·00	—	—
657	,,	,, 8	,,	54 55	23 50	12·8	19·55	35·32	—	—
658	,,	,, 9	,,	55 22	16 40	10·6	19·53	35·28	—	—
659	,,	,, 10	,,	55 17	9 30	13·3	19·51	35·25	26·26	—
					E.					
660	Otra . . .	Jan. 12	,,	57 38	1 47	7·2	19·49	35·21	—	—
661	,,	,, 28	,,	60 41	2 57	7·5	19·61	35·43	—	·00231
662	,,	,, 28	11 P.M. .	60 4	2 6	6·7	19·53	35·28	26·35	—
663	,,	,, 29	noon . .	58 54	0 37	6·9	19·51	35·25	—	—
664	,,	Feb. 5	,,	57 40	0 17	7·2	19·35	34·96	—	—
665	,,	,, 6	,,	60 4	4 57	5·8	18·68	33·76	25·19	·00221
666	,,	,, 19	,,	61 4	2 25	7·8	19·51	35·25	—	—
667	,,	,, 19	11 P.M. .	60 18	1 4	6·7	19·44	35·12	—	—
668	,,	Mar. 27	noon . .	59 22	1 41	6·4	19·47	35·17	—	·00230
669	,,	Apr. 13	7 P.M. .	62 22	5 10	5·6	18·45	33·36	24·85	·00219
670	,,	,, 14	noon . .	60 29	2 5	7·8	19·54	35·30	—	—
671	,,	,, 15	8 A.M. .	58 5	1 12	6·9	19·29	34·86	—	—
					W.					
672	,,	May 3	noon . .	60 19	7 51	8·3	19·44	35·12	—	—
673	,,	,, 3	midnight	61 8	10 54	8·3	19·50	35·23	—	—
674	,,	,, 4	noon . .	61 56	13 58	8·3	19·65	35·50	—	—
675	,,	,, 4	midnight	62 46	16 44	7·8	19·64	35·48	—	—
676	,,	,, 5	noon . .	63 35	20 40	7·2	19·62	35·45	—	—
677	,,	,, 5	midnight	64 5	23 15	7·5	19·20	24·69	—	—
678	,,	,, 10	,,	64 50	24 10	5·6	18·95	34·24	—	—
679	,,	,, 14	noon . .	66 10	19 50	0·8	18·58	33·58	—	—
680	,,	,, 15	2 A.M. .	66 15	18 55	0·6	18·12	32·75	—	—
681	,,	,, 17	noon . .	66 12	17 55	1·7	19·10	34·51	—	—
682	,,	,, 17	midnight	66 35	19 45	2·2	19·19	34·67	—	—
683	,,	,, 18	noon . .	66 18	23 50	2·2	19·29	34·86	—	—
684	,,	,, 22	midnight	64 30	13 50	2·5	19·08	34·47	—	—
685	,,	,, 24	1 A.M. .	65 25	13 40	2·8	18·09	32·70	—	·00213
686	,,	,, 24	10 P.M. .	65 45	14 5	2·8	18·26	33·00	—	—
687	,,	June 6	midnight	64 20	11 58	8·3	19·54	35·30	—	—
688	,,	,, 7	noon . .	63 5	10 5	8·3	19·61	35·43	26·30	—
689	,,	,, 8	1 A.M. .	61 35	7 45	7·8	19·60	35·41	—	·00231
690	Capricornus.	May 30	6 A.M. .	59 0	4 0	9·7	19·56	35·34	—	—
691	,,	,, 30	noon . .	59 33	5 38	10·0	19·56	35·34	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
692	Capricornus.	May 30	6 P.M. .	60° 4′	7° 5′	9·4	19·57	35·36	—	—
693	,,	,, 30	midnight	60 32	8 22	9·4	19·48	35·19	—	—
694	,,	,, 31	8 A.M. .	61 3	10 4	9·7	19·56	35·34	—	·00230
695	,,	,, 31	noon . .	61 22	11 3	9·4	19·55	35·32	26·38	·00230
696	,,	June 1	8 A.M. .	62 55	16 42	8·6	19·48	35·19	—	—
697	,,	,, 1	midnight	63 23	18 10	7·8	19·48	35·19	—	—
698	,,	,, 2	8 A.M. .	63 36	21 20	7·5	18·18	34·65	—	—
699	,,	,, 2	noon . .	63 45	22 42	7·5	19·40	35·05	—	—
700	,,	,, 3	,,	66 17	23 47	5·0	19·06	34·44	—	—
701	,,	,, 4	midnight	66 17	23 47	4·7	19·13	34·56	—	—
702	,,	,, 5	noon . .	66 27	24 15	5·6	19·17	34·63	—	—
703	,,	,, 7	,,	66 43	23 13	5·3	18·75	33·89	—	—
704	,,	,, 8	,,	66 43	23 13	5·0	18·89	34·30	—	—
705	,,	,, 11	,,	66 33	22 24	4·7	19·07	34·46	—	—
706	,,	,, 12	,,	66 33	22 24	5·0	19·02	34·37	—	—
707	,,	,, 13	,,	66 40	22 17	4·4	18·88	34·12	—	—
708	,,	,, 13	midnight	66 23	18 25	3·3	18·37	33·20	—	—
709	,,	,, 14	6 A.M. .	66 37	16 18	3·1	18·29	33·06	—	—
710	,,	,, 14	noon . .	66 22	14 27	2·8	18·34	33·15	—	—
711	,,	,, 14	6 P.M. .	65 40	13 27	3·9	18·24	32·97	—	·00216
712	,,	,, 14	midnight	65 0	12 24	3·1	18·68	33·76	25·19	·00222
713	,,	,, 15	noon . .	63 30	9 40	7·2	19·25	34·78	26·03	—
714	,,	,, 15	6 P.M. .	63 0	8 30	9·2	19·41	35·07	—	—
715	,,	,, 15	midnight	62 35	7 48	8·9	19·60	35·41	26·18	—
716	,,	,, 16	6 A.M. .	61 52	6 15	8·6	19·53	35·28	26·20	·00231
717	,,	,, 16	noon . .	61 2	5 25	10·6	19·67	35·54	—	·00231
718	,,	,, 16	6 P.M. .	60 10	4 37	10·6	19·56	35·34	—	—
719	,,	,, 16	midnight	59 26	4 6	11·9	19·23	34·74	—	—
720	Longhirst .	Jan. 28	noon . .	50 58	4 47	9·4	19·71	35·61	—	—
721	,,	,, 29	,,	48 22	7 29	12·2	19·73	35·64	—	—
722	,,	,, 30	,,	45 29	9 12	12·5	19·76	35·70	26·78	·00233
723	,,	,, 31	,,	42 27	10 35	14·4	19·97	36·07	26·86	·00235
724	,,	June 6	,,	48 13	61 56	7·2	16·80	30·40	22·54	·00198
725	,,	,, 7	,,	47 2	58 11	5·6	17·71	32·02	23·77	·00209
726	,,	,, 8	,,	46 25	54 10	5·0	17·87	32·31	—	—
727	,,	,, 9	,,	47 12	50 13	5·6	18·08	32·68	—	—
728	,,	,, 10	,,	48 32	46 14	5·0	18·42	33·30	—	·00217
729	,,	,, 11	,,	49 58	42 10	12·8	19·40	35·05	—	·00227
730	,,	,, 12	,,	51 28	38 29	11·1	19·13	34·56	—	—
731	,,	,, 13	,,	52 47	34 35	11·7	19·30	34·87	—	—
732	,,	,, 14	,,	53 37	29 55	11·4	19·40	35·05	—	—
733	,,	,, 15	,,	54 28	24 37	12·2	19·56	35·34	—	—
734	,,	,, 16	,,	54 58	19 20	13·3	19·68	35·56	26·38	—
735	,,	,, 17	,,	55 7	13 58	13·9	19·68	35·56	—	—
736	,,	,, 18	,,	55 18	8 17	15·0	19·59	35·40	—	—
737	,,	,, 19	,,	53 48	4 22	12·2	19·45	35·14	26·01	—
738	Teutonic. .	,, 4	midnight	51 5	14 25	13·9	19·61	35·43	—	—
739	,,	,, 5	noon . .	50 43	20 47	14·4	19·80	35·77	—	—
740	,,	,, 5	midnight	49 28	26 46	13·9	19·17	34·63	—	—
741	,,	,, 6	noon . .	48 20	32 56	13·3	19·71	35·61	—	·00233
742	,,	,, 6	midnight	46 48	38 28	15·0	19·78	35·73	—	—
743	,,	,, 7	noon . .	44 52	43 46	18·3	20·01	36·14	26·96	·00234
744	,,	,, 7	midnight	43 4	48 44	10·6	18·30	33·08	—	—
745	,,	,, 8	noon . .	42 2	53 50	15·6	18·21	32·91	—	—
746	,,	,, 8	midnight	41 37	58 18	8·9	18·38	33·22	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
747	Teutonic . .	June 9	noon . .	41° 2′	64° 46′	10·6	17·88	32·33	24·04	·00208
748	,,	,, 9	midnight	40 34	70 2	13·3	17·90	32·37	—	·00211
749	,,	,, 17	,,	40 20	69 59	20·6	18·14	32·79	—	—
750	,,	,, 18	noon . .	40 34	64 51	22·2	19·21	34·71	—	—
751	,,	,, 18	midnight	40 56	59 58	17·8	19·29	34·86	—	—
752	,,	,, 19	noon . .	41 19	54 41	16·7	19·22	34·72	—	—
753	,,	,, 19	midnight	42 3	49 34	12·2	18·30	33·08	—	—
754	,,	,, 20	noon . .	43 27	44 44	20·0	20·02	36·16	26·89	·00235
755	,,	,, 20	midnight	45 0	40 6	16·1	19·72	35·63	—	—
756	,,	,, 21	noon . .	47 28	34 53	15·0	19·62	35·45	—	·00231
757	,,	,, 21	midnight	48 41	29 1	14·4	19·51	35·25	—	—
758	,,	,, 22	noon . .	49 54	23 12	15·6	19·50	35·23	—	—
759	,,	,, 22	midnight	50 31	17 11	15·0	19·65	35·50	26·52	—
760	,,	,, 23	noon . .	51 16	11 5	15·6	19·68	35·56	—	—
761	Laura . .	,, 8	4 A.M. .	59 55	3 12	9·0	19·47	35·17	26·33	·00230
762	,,	,, 8	noon . .	60 56	4 54	9·2	19·53	35·28	26·31	·00231
763	,,	,, 8	8 P.M. .	61 45	6 20	8·0	19·39	35·03	—	—
764	,,	,, 10	4 A.M. .	62 8	6 26	7·0	19·40	35·05	—	—
765	,,	,, 11	,,	62 4	8 20	8·0	19·45	35·14	—	—
766	,,	,, 11	noon . .	62 16	11 28	9·0	19·45	35·14	—	—
767	,,	,, 11	8 P.M. .	62 32	14 25	9·2	19·48	35·19	—	—
768	,,	,, 12	4 A.M. .	62 48	17 30	9·0	19·40	35·05	—	—
769	,,	,, 12	noon . .	63 14	20 4	8·7	19·39	35·03	26·18	·00230
770	,,	,, 12	8 P.M. .	63 50	23 2	8·7	18·08	32·68	24·30	·00215
771	,,	,, 16	noon . .	64 35	23 23	8·5	19·24	34·76	—	·00228
772	,,	,, 16	8 P.M. .	64 59	24 25	8·1	19·09	34·49	—	·00225
773	,,	,, 17	4 A.M. .	66 4	23 59	6·0	18·87	34·10	—	—
774	,,	,, 18	,,	66 13	23 46	5·9	18·87	34·11	—	—
775	,,	,, 20	,,	65 22	24 26	7·6	18·65	33·71	—	—
776	,,	,, 22	,,	65 16	23 28	7·9	18·75	33·96	—	—
777	,,	,, 22	noon . .	64 28	23 26	9·2	18·93	34·21	—	—
778	,,	,, 25	4 A.M. .	64 5	22 55	9·8	18·73	33·85	—	—
779	,,	,, 25	noon . .	63 34	20 15	9·8	19·24	34·76	—	—
780	,,	,, 25	8 P.M. .	62 55	19 5	10·0	19·37	35·00	—	—
781	,,	,, 26	4 A.M. .	62 40	16 8	10·5	19·42	35·08	—	—
782	,,	,, 26	noon . .	62 30	13 7	10·5	19·47	35·17	—	—
783	,,	,, 26	8 P.M. .	62 25	9 56	10·0	19·41	35·07	—	—
784	,,	,, 27	4 A.M. .	62 26	7 4	8·7	19·39	35·03	—	—
785	,,	,, 28	8 P.M. .	61 22	6 20	9·7	19·46	35·15	—	—
786	,,	,, 29	4 A.M. .	60 38	4 38	10·7	19·47	35·17	26·32	—
787	,,	,, 29	noon . .	59 39	2 39	11·1	19·41	35·07	—	—
		1895.								
788	Moor . . .	Dec. 8	,,	48 6	5 35	12·8	19·61	35·43	26·53	·00232
789	,,	,, 8	midnight	45 41	7 45	13·3	19·71	35·61	—	—
790	,,	,, 9	noon . .	43 7	9 50	16·1	19·81	35·79	—	·00234
791	,,	,, 9	midnight	40 44	11 36	16·7	19·96	36·05	26·95	—
		1896.								
792	,,	Feb. 2	4 A.M. .	40 40	11 16	13·9	19·83	35·82	—	·00233
793	,,	,, 2	noon . .	42 4	10 10	13·6	19·86	35·88	—	—
794	,,	,, 2	midnight	44 22	8 30	12·8	19·79	35·75	—	·00234
795	,,	,, 3	noon . .	46 47	6 52	12·2	19·74	35·66	—	—
796	,,	Apr. 27	midnight	40 14	11 37	15·0	19·92	35·98	—	·00236
797	,,	,, 27	4 A.M. .	39 24	12 12	13·9	19·89	35·93	26·94	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
798	Moor . . .	June 18	4 P.M. .	39° 55′	11° 49′	19·4	19·92	35·98	—	·00236
799	,,	,, 18	midnight	41 41	10 39	16·7	19·89	35·93	—	—
800	,,	,, 19	noon . .	44 11	8 50	16·7	19·75	35·68	—	·00234
801	Corean . .	,, 17	,,	Off Ballycottin		16·7	19·41	35·07	—	—
802	,,	,, 18	,,	51° 7′	14° 23′	14·4	19·61	35·43	—	—
803	,,	,, 19	,,	51 40	20 33	14·4	19·73	35·64	26·56	·00231
804	,,	,, 20	,,	51 37	26 35	14·4	19·62	35·45	—	—
805	,,	,, 21	,,	51 26	34 0	12·8	19·10	34·51	25·75	—
806	,,	,, 22	,,	50 34	40 25	13·9	19·41	35·07	—	—
807	,,	,, 23	,,	48 55	46 38	8·3	19·17	34·63	—	—
808	,,	,, 24	,,	47 45	51 25	6·7	17·57	31·77	—	·00207
809	,,	,, 26	,,	45 34	57 44	9·4	17·67	31·95	—	—
810	,,	,, 27	,,	Off Halifax		12·8	17·19	31·09	—	·00202
811	,,	,, 29	,,	40° 38′	67° 10′	19·4	17·38	31·43	—	—
812	,,	,, 30	,,	39 2	73 50	21·7	19·54	35·30	26·35	·00230
813	,,	July 5	,,	40 6	69 40	20·0	18·38	33·22	—	—
814	,,	,, 6	,,	41 30	64 25	16·7	18·00	32·54	—	—
815	,,	,, 7	,,	43 37	59 20	11·1	17·56	31·75	—	·00207
816	,,	,, 8	,,	46 6	54 0	10·0	17·63	31·88	—	—
817	,,	,, 9	,,	Off Cape Race		8·9	17·68	31·97	—	—
818	,,	,, 10	,,	48° 58′	49° 3′	9·2	18·62	33·66	—	—
819	,,	,, 11	,,	51 17	43 0	12·2	19·48	35·19	—	·00229
820	,,	,, 12	,,	52 57	36 45	10·0	19·12	34·54	—	—
821	,,	,, 13	,,	54 7	29 45	13·3	19·31	34·89	—	—
822	,,	,, 14	,,	54 58	22 25	12·8	19·46	35·15	—	—
823	,,	,, 15	,,	55 29	14 30	13·9	19·53	35·28	26·38	—
824	,,	,, 16	,,	Off L. Foyle		14·4	19·33	34·93	—	—
825	Teutonic . .	,, 2	midnight	51° 6′	14° 25′	16·1	19·57	35·36	—	·00232
826	,,	,, 3	noon . .	50 37	20 35	15·6	19·73	35·64	—	—
827	,,	,, 3	midnight	49 39	26 26	15·0	19·72	35·63	—	—
828	,,	,, 4	noon . .	48 34	31 47	15·6	19·51	35·25	—	—
829	,,	,, 4	midnight	47 0	37 20	15·6	19·74	35·66	—	—
830	,,	,, 5	noon . .	45 19	42 40	17·8	19·61	35·43	—	·00231
831	,,	,, 5	midnight	43 23	47 29	14·4	18·35	33·17	—	—
832	,,	,, 6	noon . .	42 12	52 22	15·0	18·28	33·04	—	·00216
833	,,	,, 6	midnight	41 39	57 47	18·9	18·78	33·94	—	—
834	,,	,, 7	noon . .	41 8	62 58	23·3	19·61	35·43	26·57	—
835	,,	,, 7	midnight	40 44	68 23	13·3	18·20	32·90	24·37	—
836	,,	,, 15	,,	40 8	69 46	19·4	17·99	32·53	24·19	—
837	,,	,, 16	noon . .	40 31	64 53	19·4	18·45	33·36	—	—
838	,,	,, 16	midnight	40 56	59 38	23·3	19·34	34·94	—	—
839	,,	,, 17	noon . .	41 19	54 30	22·2	19·17	34·63	—	—
840	,,	,, 17	midnight	42 4	49 32	21·1	18·67	33·75	—	·00220
841	,,	,, 18	noon . .	43 43	44 45	18·9	18·73	33·85	—	—
842	,,	,, 18	midnight	45 30	39 57	17·8	19·60	35·41	—	—
843	,,	,, 19	noon . .	47 20	35 1	16·7	19·71	35·61	26·48	—
844	,,	,, 19	midnight	48 28	29 43	14·4	19·54	35·30	—	·00230
845	,,	,, 20	noon . .	49 34	24 40	16·7	19·77	35·72	—	—
846	,,	,, 20	midnight	50 22	19 0	15·6	19·83	35·82	—	·00232
847	,,	,, 21	noon . .	51 10	13 9	16·7	19·87	35·89	—	—
848	Ethiopia . .	June 27	,,	54 46	14 3	13·9	19·63	35·47	—	—
849	,,	,, 28	,,	53 34	21 25	13·3	19·60	35·41	—	—
850	,,	,, 29	,,	52 0	28 42	12·8	19·22	34·72	—	—
851	,,	,, 30	,,	49 50	35 27	15·0	19·48	35·19	—	·00231
852	,,	July 1	,,	47 30	40 44	15·6	19·52	35·27	26·49	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
853	Ethiopia . .	July 2	noon . .	44° 35′	45° 47′	13·9	18·17	32·84	24·54	—
854	,,	,, 3	,,	42 51	51 49	12·2	18·15	32·80	—	·00214
855	,,	,, 4	,,	42 31	58 23	13·3	18·14	32·79	—	—
856	,,	,, 5	,,	41 34	64 55	15·6	18·32	33·11	—	·00217
857	,,	,, 6	,,	40 36	71 3	17·8	17·86	32·30	24·07	—
858	,,	,, 12	,,	40 44	68 19	12·8	18·28	33·04	—	—
859	,,	,, 13	,,	41 56	62 27	17·2	17·96	32·48	—	·00211
860	,,	,, 14	,,	43 42	56 29	13·3	18·06	32·64	—	—
861	,,	,, 15	,,	45 30	50 32	10·0	17·81	32·21	—	—
862	,,	,, 16	,,	48 27	44 55	10·6	18·75	33·89	—	—
863	,,	,, 17	,,	50 41	38 50	13·9	19·08	34·47	25·67	—
864	,,	,, 18	,,	52 43	31 1	11·7	19·27	34·82	—	—
865	,,	,, 19	,,	54 7	22 38	13·9	19·69	35·58	26·50	—
866	,,	,, 20	,,	55 1	14 38	3·3	19·64	35·48	—	·00231
867	Aldgate . .	Mar. 20	,,	44 0	10 10	13·3	19·86	35·88	—	—
868	,,	,, 21	,,	41 3	11 48	14·4	19·96	36·05	—	—
869	,,	,, 22	,,	38 0	14 10	15·6	20·09	36·29	27·10	—
870	,,	June 25	,,	40 33	54 3	19·4	19·67	35·54	—	·00232
871	,,	,, 25	midnight	40 38	51 40	20·0	19·83	35·82	—	—
872	,,	,, 26	noon . .	41 17	48 56	20·6	20·03	36·18	—	—
873	,,	,, 26	midnight	42 4	46 45	19·4	20·12	36·33	—	—
874	,,	,, 27	noon . .	43 4	44 19	19·4	19·91	35·97	—	—
875	,,	,, 27	midnight	43 52	42 4	18·9	19·97	36·07	26·82	—
876	,,	,, 28	noon . .	44 43	39 58	17·8	19·84	35·84	—	—
877	,,	,, 28	midnight	45 27	37 38	16·7	19·66	35·52	—	—
878	,,	,, 29	noon . .	46 14	35 3	17·8	19·76	35·70	—	·00234
879	,,	,, 29	midnight	46 57	32 26	16·7	19·81	35·79	—	—
880	,,	,, 30	noon . .	47 43	30 4	16·1	19·93	36·00	—	—
881	,,	,, 30	midnight	48 9	26 55	16·1	19·62	35·45	—	—
882	,,	July 1	noon . .	48 38	24 35	17·2	20·06	36·23	—	—
883	,,	,, 1	midnight	48 55	21 56	16·7	19·69	35·58	—	·00232
884	,,	,, 2	noon . .	49 16	19 2	17·8	19·72	35·63	—	—
885	,,	,, 2	midnight	49 30	15 50	17·2	19·75	35·68	—	—
886	,,	,, 3	noon . .	49 48	13 20	17·8	19·68	35·56	—	—
887	,,	,, 3	midnight	49 47	10 10	17·2	19·72	35·63	—	—
888	,,	,, 4	noon . .	49 50	7 22	17·2	19·72	35·61	26·49	·00234
889	,,	,, 4	midnight	50 2	4 41	15·0	19·67	35·54	—	—
890	,,	,, 5	noon . .	50 29	1 49	17·2	19·53	35·28	—	—
891	,,	,, 5	midnight	Off Du	ngeness	15·0	19·61	35·43	—	—
					E.					
892	,,	,, 6	noon . .	52° 21′	3° 3′	15·6	19·22	34·72	—	—
893	,,	,, 6	midnight	53 38	5 27	15·6	18·63	33·67	—	—
894	,,	,, 20	noon . .	51 57	2 48	17·8	19·23	34·74	—	—
					W.					
895	,,	,, 21	,,	50 16	3 14	17·8	19·65	35·50	—	·00234
896	,,	,, 21	8 P.M. .	Off the	Lizard	16·7	19·71	35·61	26·50	—
897	Loughrigg Holme	June 18	noon . .	55° 25′	8° 9′	14·7	19·35	34·96	26·13	—
898	,,	,, 19	,,	55 18	15 56	13·7	19·58	35·38	—	·00232
899	,,	,, 20	,,	54 55	18 35	12·3	19·41	35·07	26·29	—
900	,,	,, 21	,,	54 28	24 31	11·9	19·49	35·21	—	—
901	,,	,, 22	,,	53 47	30 22	11·0	19·40	35·05	—	—
902	,,	,, 23	,,	53 8	35 16	9·7	19·25	34·78	—	—
903	,,	,, 24	,,	52 15	40 8	13·9	19·39	35·03	—	—
904	,,	,, 25	,,	50 43	44 8	12·7	19·48	35·19	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
905	Loughrigg Holme	June 26	noon . .	48° 38′	48° 15′	7·2	18·85	34·07	—	—
906	,,	,, 27	,,	46 35	52 43	5·3	17·92	32·40	—	·00210
907	,,	,, 28	,,	46 24	57 24	9·6	17·73	32·06	—	—
908	,,	July 22	noon . .	46 18	55 59	12·4	17·68	31·97	23·59	—
909	,,	,, 22	midnight	46 18	53 32	8·6	17·81	32·21	—	—
910	,,	,, 23	noon . .	46 43	51 24	11·7	17·81	32·21	—	—
911	,,	,, 23	midnight	47 25	49 5	9·8	17·75	32·09	—	·00208
912	,,	,, 24	noon . .	48 4	46 55	9·3	18·20	32·90	—	—
913	,,	,, 24	midnight	48 49	43 39	12·8	18·69	33·78	—	—
914	,,	,, 25	noon . .	49 27	41 0	16·1	19·34	34·94	—	·00227
915	,,	,, 25	midnight	49 44	37 58	14·4	19·59	35·40	—	—
916	,,	,, 26	noon . .	50 15	35 8	16·1	19·52	35·27	—	—
917	,,	,, 26	midnight	50 37	32 9	14·1	19·36	34·98	—	—
918	,,	,, 27	noon . .	50 57	29 25	15·0	19·64	35·48	—	·00231
919	,,	,, 27	midnight	51 4	26 37	15·0	19·65	35·50	26·27	—
920	,,	,, 28	noon . .	51 7	23 42	15·3	19·57	35·36	—	—
921	,,	,, 28	midnight	51 8	21 1	15·1	19·77	35·72	—	—
922	,,	,, 29	noon . .	51 9	18 19	15·6	19·70	35·59	—	—
923	,,	,, 29	midnight	51 15	15 25	15·4	19·74	35·66	26·47	—
924	,,	,, 30	noon . .	51 21	12 36	16·3	19·75	35·68	—	—
925	,,	,, 30	midnight	51 23	9 25	11·7	19·47	35·17	—	·00229
926	,,	,, 31	noon . .	52 2	6 40	15·0	19·46	35·15	—	—
927	Laura . . .	,, 19	8 P.M. .	59 36	2 37	11·7	19·35	34·97	—	—
928	,,	,, 20	4 A.M. .	60 28	4 26	11·9	19·50	35·23	26·28	—
929	,,	,, 20	noon . .	61 13	6 10	11·6	19·45	35·14	—	·00233
930	,,	,, 22	8 P.M. .	62 28	7 23	9·0	19·42	35·08	—	—
931	,,	,, 23	4 A.M. .	62 35	9 55	8·0	19·14	34·58	—	—
932	,,	,, 23	noon . .	62 53	12 50	11·0	19·48	35·19	—	·00231
933	,,	,, 23	8 P.M. .	63 1	15 47	11·0	19·90	35·95	—	—
934	,,	,, 24	4 A.M. .	63 10	18 48	10·7	19·24	34·76	—	·00228
935	,,	,, 24	noon . .	63 26	20 38	11·0	19·23	34·74	—	—
936	,,	,, 24	8 P.M. .	63 59	22 59	10·7	18·11	32·73	—	·00215
937	,,	Aug. 2	noon . .	64 6	22 56	10·7	18·73	33·85	—	—
938	,,	,, 2	8 P.M. .	63 30	20 34	11·5	18·80	33·98	25·29	—
939	,,	,, 3	4 A.M. .	62 54	18 6	11·3	18·08	32·68	24·30	—
940	,,	,, 3	noon .	62 48	15 11	11·9	19·36	34·98	—	·00230
941	,,	,, 3	8 P.M. .	62 39	12 20	11·5	19·45	35·14	—	—
942	,,	,, 4	4 A.M. .	62 31	9 27	10·5	19·46	35·15	—	—
943	,,	,, 4	noon . .	62 29	7 10	10·2	19·46	35·15	—	·00231
944	,,	,, 6	4 A.M. .	61 20	6 20	11·6	19·47	35·17	—	—
945	,,	,, 6	noon . .	60 29	4 25	12·2	19·52	35·27	26·36	—
946	,,	,, 6	8 P.M. .	59 50	2 34	11·7	19·40	35·05	—	—
947	Traveller .	Apr. 8	noon . .	59 29	4 38	8·6	19·63	35·47	—	—
948	,,	,, 9	,,	59 54	5 48	8·6	19·66	35·52	26·39	·00234
949	,,	,, 10	,,	60 41	5 50	8·1	19·64	35·48	—	—
950	,,	,, 11	,,	61 4	5 56	7·2	19·62	35·45	—	—
951	,,	,, 12	,,	60 30	6 52	7·2	19·64	35·48	—	·00233
952	,,	,, 13	,,	60 5	9 26	8·3	19·65	35·50	—	—
953	,,	,, 14	,,	60 5	12 38	8·2	19·63	35·47	—	—
954	,,	,, 15	,,	60 7	15 31	8·4	19·55	35·32	—	—
955	,,	,, 16	,,	60 32	17 24	8·3	—	—	—	—
956	,,	,, 17	,,	59 56	17 41	8·4	19·61	35·43	—	—
957	,,	,, 18	,,	59 46	18 38	8·6	19·57	35·36	—	·00231
958	,,	,, 19	,,	59 27	20 28	8·9	18·97	34·28	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
959	Traveller .	Apr. 20	noon . .	59° 20′	22° 42′	8·9	19·55	35·32	—	—
960	”	” 21	”	59 6	26 11	8·6	19·56	35·34	—	—
961	”	” 22	”	58 51	31 1	7·9	19.54	35·30	—	—
962	”	” 23	”	58 45	35 39	6·1	19·47	35·17	—	—
963	”	” 24	”	58 52	37 30	5·6	19·45	35·14	—	·00230
964	”	” 25	”	58 35	38 39	4·4	19·34	34·94	—	—
965	”	” 26	”	57 46	38 1	6·1	19·51	35·25	—	—
966	”	” 27	”	57 55	38 1	6·4	19·46	35·15	—	—
967	”	” 28	”	57 38	38 33	5·7	19·42	35·08	—	—
968	”	” 29	”	58 21	39 20	5·3	19·42	35·08	—	—
969	”	” 30	”	58 42	40 45	3·9	19·18	34·65	25·66	—
970	”	May 1	”	58 19	42 4	3·6	19·20	34·69	—	—
971	”	” 2	”	58 9	42 16	3·7	19·34	34·94	—	·00227
972	”	” 3	”	57 46	43 17	4·2	19·29	34·86	—	—
973	”	” 4	”	57 46	45 41	3·4	19·27	34·82	—	—
974	”	” 5	”	57 56	47 51	2·5	19·23	34·74	—	·00229
975	”	” 6	”	57 44	48 28	3·0	19·28	34·84	—	—
976	”	” 7	”	57 41	48 34	3·1	19·29	34·86	—	—
977	”	” 8	”	57 28	47 26	3·6	19·29	34·86	—	—
978	”	” 9	”	58 1	49 48	1·4	18·82	34·01	—	—
979	”	” 10	”	58 40	50 36	1·1	18·59	33·60	24·98	—
980	”	” 11	”	58 59	50 45	1·9	18·95	34·24	—	·00229
981	”	” 12	”	58 48	51 18	1·4	18·80	33·98	—	—
982	”	” 13	”	59 3	51 2	1·7	18·87	34·10	—	·00228
983	”	” 14	”	59 56	50 56	0·0	18·42	33·30	24·65	—
984	”	” 15	”	60 12	51 0	0·8	18·93	34·21	—	—
985	”	” 16	”	59 59	50 51	0·4	18·71	33·82	—	·00219
986	”	” 17	”	60 18	51 20	0·6	18·68	33·76	—	—
987	”	” 18	”	61 2	51 35	0·3	18·54	33·51	24·82	—
988	”	” 19	”	61 11	51 45	0·6	18·65	33·71	—	—
989	”	” 20	”	60 50	52 0	0·0	18·78	33·94	—	—
990	”	” 21	”	60 46	52 10	0·0	18·78	33·94	—	·00220
991	”	” 22	”	60 52	52 10	0·3	18·77	33·92	—	—
992	”	July 3	”	60 22	52 43	4·6	18·97	34·28	—	—
993	”	” 4	”	58 30	50 53	6·1	19·05	34·42	—	—
994	”	” 5	”	57 29	47 14	5·6	18·93	34·21	—	—
995	”	” 6	”	57 34	44 58	6·7	19·06	34·44	—	—
996	”	” 7	”	57 26	42 22	7·2	19·32	34·91	—	—
997	”	” 8	”	57 27	39 27	7·2	19·29	34·86	—	—
998	”	” 9	”	57 50	36 7	8·1	19·16	34·62	—	—
999	”	” 10	”	58 39	36 31	8·1	19·37	35·00	—	—
1000	”	” 11	”	58 42	35 52	8·3	19·50	35·23	—	—
1001	”	” 12	”	58 40	34 6	8·9	19·39	35·03	—	—
1002	”	” 15	”	58 12	22 3	11·9	19·56	35·34	26·25	—
1003	”	” 13	”	58 21	29 39	10·0	19·48	35·19	—	·00229
1004	”	” 14	”	58 21	26 7	11·7	19·52	35·27	—	—
1005	”	” 16	”	58 12	17 35	12·0	19·52	35·27	—	—
1006	”	” 17	”	58 0	14 43	12·8	19·52	35·27	—	—
1007	”	” 18	”	58 7	12 24	12·2	19·42	35·08	—	·00232
1008	”	” 19	”	58 18	8 0	13·3	19·51	35·25	—	·00231
1009	Para . . .	Apr. 10	”	46 48	14 51	12·5	20·10	36·30	—	—
1010	”	” 11	”	43 57	21 35	13·6	19·88	35·91	—	—
1011	”	” 12	”	40 40	27 32	15·0	19·97	36·07	—	·00232
1012	”	” 13	”	37 2	32 39	16·7	20·05	36·21	—	—
1013	”	May 22	”	37 48	38 5	20·0	20·24	36·55	27·09	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1014	Para . . .	May 23	noon . .	40° 36′	32° 0′	18·9	19·98	36·09	—	·00236
1015	,,	,, 24	,,	43 16	25 10	18·6	19·87	35·89	—	—
1016	,,	,, 25	,,	45 45	18 8	16·7	19·85	35·86	—	—
1017	,,	,, 26	,,	47 56	11 48	15·3	19·87	35·89	—	—
1018	,,	,, 27	,,	50 3	4 55	13·3	19·73	35·64	26·51	—
1019	,,	July 2	,,	48 47	6 58	16·1	19·93	36·00	—	—
1020	,,	,, 3	,,	45 52	14 7	18·9	20·01	36·14	—	—
1021	,,	,, 4	,,	42 52	20 55	20·6	19·92	35·98	—	·00234
1022	,,	,, 5	,,	39 56	27 25	21·7	19·63	35·47	27·37	—
1023	,,	Aug. 15	,,	41 26	29 28	25·0	19·72	35·63	26·76	—
1024	,,	,, 16	,,	44 27	22 39	20·6	20·04	36·19	—	—
1025	,,	,, 17	,,	47 3	15 8	19·4	20·09	36·29	—	·00236
1026	,,	,, 18	,,	49 15	6 57	17·5	19·86	35·88	—	—
1027	,,	,, 14	,,	37 22	34 55	25·6	—	—	—	—
1028	,,	,, 13	,,	33 21	40 18	29·4	—	—	—	—
1029	,,	,, 12	,,	29 15	45 11	32·8	—	—	—	—
1030	Wydale . .	Feb. 22	,,	49 12	6 24	11·1	19·65	35·50	—	—
1031	,,	,, 23	,,	47 3	8 20	12·3	19·78	35·73	—	—
1032	,,	,, 24	,,	44 18	10 24	12·8	19·78	35·73	—	·00235
1033	,,	,, 25	,,	41 28	12 16	13·9	19·85	35·86	—	—
1034	,,	Apr. 25	,,	40 6	57 10	18·3	20·00	36·12	—	—
1035	,,	,, 26	,,	40 48	53 50	14·4	19·78	35·73	—	·00234
1036	,,	,, 27	,,	41 44	50 4	12·8	19·64	35·48	—	—
1037	,,	,, 28	,,	43 3	47 20	8·9	19·08	34·47	—	—
1038	,,	,, 29	,,	44 12	43 20	14·2	19·83	35·82	—	—
1039	,,	,, 30	,,	45 13	39 18	14·4	19·82	35·80	—	—
1040	,,	May 1	,,	46 11	35 15	14·2	19·69	35·58	—	—
1041	,,	,, 2	,,	47 10	31 3	13·3	19·52	35·27	—	—
1042	,,	,, 3	,,	48 3	26 30	13·3	19·65	35·50	—	·00233
1043	,,	,, 4	,,	48 40	21 33	13·1	19·70	35·59	—	—
1044	,,	,, 5	,,	49 8	16 45	13·9	19·75	35·68	—	—
1045	,,	,, 6	,,	49 35	11 38	18·8	19·68	35·56	—	—
1046	,,	July 25	,,	40 51	64 0	22·2	18·01	32·56	—	·00214
1047	,,	,, 26	,,	42 22	60 17	18·3	17·92	32·40	24·10	—
1048	,,	,, 27	,,	43 23	56 31	16·7	17·67	31·95	23·71	—
1049	,,	,, 28	,,	44 26	52 19	15·6	18·02	32·58	—	—
1050	,,	,, 29	,,	45 24	48 9	10·6	17·47	31·60	23·37	—
1051	,,	,, 30	,,	46 22	44 0	12·8	18·35	33·17	—	—
1052	,,	,, 31	,,	47 21	40 45	18·9	19·88	35·91	—	—
1053	,,	Aug. 1	,,	48 20	37 20	17·2	19·61	35·43	—	—
1054	,,	,, 2	,,	48 48	33 25	16·1	19·57	35·36	—	—
1055	,,	,, 3	,,	49 20	29 3	15·6	19·23	34·74	—	—
1056	,,	,, 4	,,	49 47	23 51	17·2	19·65	35·50	—	—
1057	,,	,, 5	,,	49 54	19 14	17·2	19·74	35·66	—	—
1058	,,	,, 6	,,	49 51	14 21	16·7	19·81	35·79	26·59	—
1059	,,	,, 7	,,	49 50	10 24	17·2	19·85	35·86	—	·00234
1060	Teutonic .	July 30	midnight	51 24	14 8	15·0	19·64	35·48	—	—
1061	,,	,, 31	noon . .	51 24	20 27	15·6	19·69	35·58	—	—
1062	,,	,, 31	midnight	50 47	26 30	15·0	19·67	35·54	—	·00232
1063	,,	Aug. 1	noon . .	50 9	32 47	13·3	19·58	35·38	26·51	—
1064	,,	,, 1	midnight	48 47	38 45	16·1	19·61	35·43	26·26	—
1065	,,	,, 2	noon . .	47 28	44 33	11·7	18·37	33·20	—	—
1066	,,	,, 2	midnight	45 34	49 8	13·3	17·99	32·53	23·97	—
1067	,,	,, 3	noon . .	43 58	55 35	16·7	18·04	32·61	—	—
1068	,,	,, 3	midnight	43 1	60 39	16·7	17·98	32·51	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1069	Teutonic .	Aug. 4	noon . .	41° 38′	65° 45′	14·4	17·77	32·13	23·73	—
1070	,,	,, 4	midnight	40 16	71 44	18·9	18·20	32·90	—	—
1071	,,	,, 12	,,	40 9	69 52	23·3	18·21	32·91	—	—
1072	,,	,, 13	noon . .	41 3	65 27	20·0	18·23	32·95	—	—
1073	,,	,, 13	midnight	42 16	60 32	20·0	18·38	33·22	—	—
1074	,,	,, 14	noon . .	43 24	55 40	18·9	18·09	32·70	—	—
1075	,,	,, 14	midnight	44 52	50 48	15·6	17·86	32·30	—	·00210
1076	,,	,, 15	noon . .	46 24	45 40	15·0	18·14	32·79	24·30	—
1077	,,	,, 15	midnight	47 37	40 47	18·9	19·67	35·54	—	—
1078	,,	,, 16	noon . .	49 2	35 49	16·7	19·36	34·98	—	—
1079	,,	,, 16	midnight	49 55	30 16	16·1	19·45	35·14	—	—
1080	,,	,, 17	noon . .	50 50	24 31	16·1	19·64	35·48	—	·00230
1081	,,	,, 17	midnight	50 1	18 31	17·2	19·65	35·50	—	—
1082	,,	,, 18	noon . .	50 20	12 21	17·2	19·68	35·56	—	·00233
1083	Ethiopia . .	,, 1	,,	55 9	14 39	14·4	19·71	35·61	26·56	—
1084	,,	,, 2	,,	54 40	23 25	13·9	19·64	35·48	26·34	—
1085	,,	,, 3	,,	53 30	31 12	12·2	19·41	35·07	—	—
1086	,,	,, 4	,,	51 40	38 14	12·8	19·16	34·62	—	·00227
1087	,,	,, 5	,,	49 36	45 8	13·3	18·97	34·28	—	—
1088	,,	,, 6	,,	46 55	51 5	11·7	17·71	32·02	—	·00209
1089	,,	,, 7	,,	44 9	57 16	15·0	17·99	32·53	—	—
1090	,,	,, 8	,,	42 7	63 47	18·3	18·02	32·58	—	—
1091	,,	,, 9	,,	40 42	69 56	17·2	18·11	32·73	—	·00213
1092	,,	,, 16	,,	40 37	68 22	18·3	17·98	32·51	—	—
1093	,,	,, 17	,,	42 8	62 23	20·0	17·99	32·53	—	·00213
1094	,,	,, 18	,,	44 26	57 2	16·1	17·52	31·68	23·57	—
1095	,,	,, 19	,,	47 0	51 23	13·9	17·52	31·68	23·31	—
1096	,,	,, 20	,,	49 33	45 15	13·3	18·78	33·94	—	—
1097	,,	,, 21	,,	51 45	38 31	15·0	19·08	34·47	—	—
1098	,,	,, 22	,,	53 25	31 3	13·3	19·33	34·93	—	·00228
1099	,,	,, 23	,,	54 29	23 7	14·4	19·54	35·30	—	—
1100	,,	,, 24	,,	55 6	14 38	14·4	19·69	35·58	26·44	—
1101	Corean . .	July 26	,,	Galley	Head	17·2	19·45	35·14	—	·00229
1102	,,	,, 27	,,	51° 40′	16° 36′	16·7	19·79	35·75	26·71	—
1103	,,	,, 28	,,	51 41	22 48	15·6	19·63	35·47	—	—
1104	,,	,, 29	,,	51 52	27 28	13·9	19·29	34·86	—	—
1105	,,	,, 30	,,	51 38	34 15	14·4	19·24	34·76	—	·00226
1106	,,	,, 31	,,	50 31	41 14	13·3	18·91	34·17	25·41	—
1107	,,	Aug. 1	,,	48 46	48 2	11·1	18·77	33·92	—	—
1108	,,	,, 4	,,	45 46	55 49	13·3	17·58	31·79	—	—
1109	,,	,, 5	,,	44 31	62 10	14·4	16·94	30·64	22·58	—
1110	,,	,, 7	,,	40 48	67 25	16·7	16·88	30·54	—	·00201
1111	,,	,, 8	,,	39 3	73 26	24·4	17·95	32·46	—	—
1112	,,	,, 15	,,	39 34	71 22	25·6	17·61	31·85	—	—
1113	,,	,, 16	,,	41 5	66 29	16·1	17·87	32·31	—	·00210
1114	,,	,, 17	,,	42 47	61 48	18·9	17·90	32·37	23·91	—
1115	,,	,, 18	,,	44 34	56 36	17·8	17·81	32·21	—	—
1116	,,	,, 19	,,	46 34	52 54	15·6	17·93	32·42	—	—
1117	,,	,, 20	,,	48 12	51 5	13·9	17·43	31·52	—	—
1118	,,	,, 21	,,	50 19	45 25	12·2	18·77	33·92	—	—
1119	,,	,, 22	,,	52 27	39 3	14·4	18·92	34·19	—	—
1120	,,	,, 23	,,	53 58	31 55	13·9	19·30	34·87	—	—
1121	,,	,, 24	,,	54 48	24 19	16·7	19·51	35·25	—	—
1122	,,	,, 25	,,	55 7	16 25	13·9	19·59	35·40	—	—
1123	,,	,, 26	,,	55 10	8 58	13·9	19·53	35·28	26·27	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1124	Corean . .	Aug. 27	noon . .	off Greenock		—	—	—	—	—
1125	Longhirst .	July 7	,,	55° 28′	14° 20′	13·9	19·75	35·68	26·64	—
1126	,,	,, 8	,,	55 2	19 52	13·3	19·67	35·54	—	·00231
1127	,,	,, 9	,,	54 28	25 36	12·8	19·50	35·23	—	—
1128	,,	,, 10	,,	53 47	30 25	10·6	19·36	34·98	26·03	—
1129	,,	,, 11	,,	53 33	34 12	10·6	19·40	35·05	—	—
1130	,,	,, 12	,,	52 15	37 35	11·1	19·18	34·65	—	—
1131	,,	,, 13	,,	50 50	42 30	11·1	19·01	34·35	—	—
1132	,,	,, 14	,,	49 19	46 44	10·0	19·04	34·40	—	·00224
1133	,,	,, 15	,,	47 27	51 15	10·0	17·72	32·04	—	—
1134	,,	,, 16	,,	46 28	55 42	11·9	17·73	32·06	—	·00206
1135	,,	,, 21	,,	48 9	61 54	15·8	15·90	28·80	21·15	—
1136	,,	Aug. 15	,,	48 45	68 7	11·7	15·65	28·35	20·77	·00231
1137	,,	,, 16	,,	50 0	63 22	11·9	17·38	31·44	23·26	—
1138	,,	,, 17	,,	50 26	58 52	12·8	17·01	30·77	—	·00200
1139	,,	,, 18	,,	51 58	54 46	9·4	16·34	29·58	—	—
1140	,,	,, 19	,,	53 10	49 38	11·4	18·91	34·17	—	—
1141	,,	,, 20	,,	54 23	44 14	11·7	19·07	34·46	—	—
1142	,,	,, 21	,,	54 47	38 34	11·1	19·14	34·58	—	—
1143	,,	,, 22	,,	55 12	33 33	12·8	19·43	35·10	—	—
1144	,,	,, 23	,,	55 38	27 56	13·3	19·36	34·98	—	—
1145	,,	,, 24	,,	56 6	22 30	13·3	19·47	35·00	—	—
1146	,,	,, 25	,,	55 54	16 46	10·6	19·66	35·52	—	—
1147	,,	,, 26	,,	55 2	11 40	14·4	19·73	35·64	—	—
1148	,,	,, 27	,,	55 26	7 43	14·4	19·55	35·32	—	—
1149	,,	,, 28	,,	53 42	3 41	15·6	18·85	34·07	—	—
1150	Jamesia . .	,, 7	midnight	58 56	3 44	11·7	19·31	34·89	—	—
1151	,,	,, 8	noon . .	60 10	6 44	11·4	19·54	35·30	—	—
1152	,,	,, 8	11.30 P.M.	61 23	9 46	11·4	19·47	35·17	—	—
1153	,,	,, 9	noon . .	62 36	12 54	11·7	19·50	35·23	—	—
1154	,,	,, 9	midnight	63 40	14 58	11·4	19·15	34·60	—	—
1155	,,	,, 15	noon . .	63 23	16 34	10·8	18·97	34·28	25·61	—
1156	,,	,, 16	,,	63 28	17 34	10·3	18·99	34·32	25·48	—
1157	,,	,, 18	,,	64 0	15 14	10·6	19·43	35·10	—	—
1158	,,	,, 19	,,	63 48	13 49	11·4	19·42	35·08	—	—
1159	,,	,, 20	,,	62 34	10 10	11·7	19·49	35·21	—	—
1160	,,	,, 20	midnight	61 6	7 30	11·9	19·49	35·21	—	—
1161	,,	,, 21	noon . .	59 46	4 50	12·5	19·61	35·43	—	—
1162	,,	,, 21	10 P.M. .	59 16	4 26	11·9	19·46	35·15	—	—
1163	,,	,, 28	noon . .	59 56	4 30	10·3	19·50	35·23	—	—
1164	,,	,, 28	midnight	61 44	5 42	10·0	19·49	35·21	26·11	—
1165	,,	,, 29	noon . .	61 56	7 26	10·0	19·49	35·21	—	—
1166	,,	,, 29	midnight	62 16	8 16	9·4	19·50	35·23	—	—
1167	,,	,, 30	noon . .	62 16	8 20	9·4	19·50	35·23	—	—
1168	,,	,, 30	midnight	62 12	8 24	10·6	19·50	35·23	—	—
1169	,,	,, 31	noon . .	62 12	8 18	10·8	19·45	35·14	—	—
1170	,,	,, 31	midnight	62 12	8 24	10·3	19·52	35·27	—	—
1171	,,	Sept. 1	noon . .	62 20	8 20	10·6	19·49	35·21	—	—
1172	,,	,, 2	midnight	62 12	8 16	10·6	19·49	35·21	—	—
1173	,,	,, 3	noon . .	62 20	8 18	10·6	19·47	35·17	—	—
1174	,,	,, 3	midnight	62 12	8 20	10·3	19·45	35·14	—	—
1175	,,	,, 4	noon . .	62 22	8 16	10·0	19·48	35·19	—	—
1176	,,	,, 4	6 P.M. .	61 48	8 0	10·8	19·55	35·32	—	—
1177	,,	,, 4	midnight	60 58	6 40	11·1	19·33	34·93	26·03	—
1178	,,	,, 5	6 A.M. .	60 10	5 20	11·7	19·34	34·94	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
1179	Jamesia . .	1896. Sept. 5	noon . .	N. 59° 22′	W. 4° 6′	11·7	19·39	35·03	—	—
1180	Active . .	Apr. 16	„	62 33	E. 0 0	8·3	19·57	35·36	—	—
1181	„	„ 17	„	65 38	1 0	4·2	19·37	35·00	—	—
1182	„	„ 18	„	68 20	2 0	5·0	19·40	35·05	—	—
1183	„	„ 19	„	70 30	2 40	4·7	19·47	35·17	26·29	—
1184	„	„ 19	8 P.M. .	71 10	2 50	3·5	19·40	35·05	—	—
1185	„	„ 20	noon . .	71 52	7 0	−1·0	19·07	34·46	—	·00225
1186	„	„ 21	„	73 56	7 0	−0·3	19·07	34·46	—	—
1187	„	„ 22	„	74 51	6 0	−0·8	19·16	34·62	25·78	—
1188	„	„ 23	„	74 50	6 0	−1·3	18·99	34·32	—	—
1189	„	„ 24	„	76 0	6 0	−1·7	19·02	34·37	—	—
1190	„	„ 25	„	77 40	5 0	1·7	19·36	34·98	—	·00229
1191	„	„ 26	„	79 0	4 30	−0·6	19·26	34·80	25·84	—
1192	„	„ 27	„	80 15	4 0	0·6	19·16	34·62	—	—
1193	„	„ 28	„	79 50	3 30	0·0	19·10	34·51	—	·00227
1194	„	„ 29	4 P.M. .	79 40	3 30	−1·4	19·15	34·60	—	—
1195	„	„ 30	noon . .	79 13	3 0	−1·4	19·03	34·39	25·50	—
1196	„	May 1	4 P.M. .	78 50	2 0	−0·6	19·39	35·03	—	—
1197	„	„ 2	„	78 40	1 0	0·0	19·35	34·96	—	—
1198	„	„ 3	noon . .	79 0	1 30	−1·7	19·02	34·37	—	—
1199	„	„ 4	„	79 20	W. 0 45	−1·7	18·90	34·16	25·37	—
1200	„	„ 8	„	78 20	0 15	−0·6	19·18	34·65	—	—
1201	„	„ 10	4 P.M. .	79 30	E. 2 30	0·0	19·14	34·58	—	·00227
1202	„	„ 11	noon . .	79 50	2 30	0·8	19·20	34·69	—	—
1203	„	„ 12	„	79 45	2 30	−0·3	19·19	34·67	—	—
1204	„	„ 13	„	79 20	1 30	−0·8	18·92	34·19	—	—
1205	„	„ 14	„	79 15	1 0	−1·1	18·85	34·07	—	—
1206	„	„ 15	„	79 25	0 0	−0·6	18·92	34·19	—	·00222
1207	„	„ 16	„	79 50	2 0	−0·3	18·96	34·26	—	—
1208	„	„ 17	„	79 35	1 30	−1·1	18·90	34·16	—	—
1209	„	„ 18	„	79 40	1 30	−1·1	18·94	34·23	—	·00223
1210	„	„ 19	„	79 30	0 0	−1·1	18·81	34·00	—	—
1211	„	„ 20	„	79 10	W. 1 0	−1·4	18·77	33·92	—	—
1212	„	„ 21	„	79 0	2 0	−1·1	18·51	33·46	—	·00217
1213	„	„ 22	„	78 50	2 30	−1·4	18·17	32·84	—	—
1214	„	„ 23	„	78 55	3 0	−1·4	18·24	32·97	—	—
1215	„	„ 24	„	79 0	3 10	−1·2	18·25	32·98	—	—
1216	„	„ 25	„	78 40	3 30	−1·0	18·32	33·12	—	—
1217	„	„ 26	„	78 45	3 0	−1·3	18·17	32·84	—	—
1218	„	„ 27	„	79 0	4 30	−1·4	18·10	32·72	—	·00212
1219	„	„ 28	„	78 58	3 20	−1·4	18·10	32·72	—	·00211
1220	„	„ 29	„	78 20	5 0	−1·2	18·26	33·00	24·54	—
1221	„	„ 31	„	78 26	5 0	−1·3	18·25	32·98	—	·00213
1222	„	June 1	„	78 1	6 0	−1·6	18·28	33·04	—	—
1223	„	„ 2	4 P.M. .	77 50	7 30	−1·2	18·27	33·02	—	—
1224	„	„ 4	noon . .	78 5	4 0	−1·2	18·36	33·17	24·59	·00214
1225	„	„ 5	„	78 40	2 0	−0·7	19·06	34·44	25·56	—
1226	„	„ 6	„	79 0	1 30	−1·3	18·57	33·56	—	—
1227	„	„ 7	„	78 40	0 30	0·0	19·11	34·53	—	·00224
1228	„	„ 8	„	79 10	E. 1 20	−0·6	18·82	34·01	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	E.					
1229	Active . .	June 10	noon . .	79° 33′	1° 28′	0·3	—	—	—	—
1230	,,	,, 11	,,	79 28	1 40	0·9	18·89	34·14	—	·00224
1231										
1232	,,	,, 13	,,	79 37	2 0	0·8	18·78	33·94	—	—
1233	,,	,, 14	,,	79 30	3 0	1·4	19·03	34·39	—	·00221
1234	,,	,, 15	,,	79 25	2 0	1·3	—	—	—	—
1235	,,	,, 17	,,	79 35	2 0	1·4	18·76	33·90	—	—
1236	,,	,, 18	,,	78 56	0 40	−0·2	18·49	33·43	24·74	—
					W.					
1237	,,	,, 19	,,	78 40	1 0	−0·6	18·51	33·46	—	—
1238	,,	,, 20	,,	78 20	1 30	−0·1	18·62	33·66	—	—
1239	,,	,, 21	,,	77 20	2 30	0·4	18·95	34·24	—	—
1240	,,	,, 22	,,	76 30	3 30	0·4	19·14	34·58	—	·00226
1241	,,	,, 23	,,	75 40	8 40	−0·3	18·59	33·60	—	·00219
1242	,,	,, 24	,,	74 43	11 30	0·8	18·83	34·03	—	·00217
1243	,,	,, 25	,,	74 0	13 0	0·0	18·31	33·10	—	—
1244	,,	,, 26	,,	73 48	13 0	0·0	18·26	33·00	24·51	—
1245	,,	,, 27	,,	73 30	13 10	0·0	18·25	32·98	—	—
1246	,,	,, 28	,,	73 20	14 0	−0·8	17·98	32·51	—	·00211
1247	,,	,, 29	,,	73 0	16 0	−0·3	17·87	32·31	—	—
1248	,,	,, 30	,,	73 0	17 30	0·1	17·93	32·42	24·02	—
1249	,,	July 1	,,	72 50	17 40	0·5	17·86	32·30	—	·00210
1250	,,	,, 2	,,	72 50	18 0	0·3	17·90	32·37	—	—
1251	,,	,, 3	,,	73 4	19 20	0·1	17·89	32·35	—	—
1252	,,	,, 4	,,	72 45	18 40	0·8	17·76	32·11	23·78	—
1253	,,	,, 5	,,	72 40	18 30	0·8	17·65	31·92	—	—
1254	,,	,, 6	,,	72 50	18 20	0·8	17·56	31·75	—	—
1255	,,	,, 7	,,	72 45	18 0	0·4	17·73	32·06	—	—
1256	,,	,, 8	,,	72 20	17 40	0·0	17·64	31·90	—	—
1257	,,	,, 9	,,	72 20	17 40	0·5	17·57	31·77	—	—
1258	,,	,, 10	,,	72 18	17 40	1·1	15·21	27·55	—	—
1259	,,	,, 10	,,	72 18	17 40	—	—	—	—	—
1260	,,	,, 11	,,	72 25	18 0	1·1	17·17	31·06	23·01	—
1261	,,	,, 12	,,	72 30	18 30	0·8	17·63	31·88	—	—
1262	,,	,, 13	,,	72 30	19 0	0·6	17·53	31·70	—	·00207
1263	,,	,, 14	,,	72 35	19 30	0·8	17·19	31·10	—	—
1264	,,	,, 15	,,	72 30	19 0	0·6	17·45	31·56	—	—
1265	,,	,, 16	,,	72 35	18 20	0·6	17·46	31·57	—	—
1266	,,	,, 17	,,	72 41	18 0	0·4	17·37	31·41	—	—
1267	,,	,, 18	,,	72 27	17 40	0·0	17·60	31·83	—	—
1268	,,	,, 19	,,	72 22	17 40	0·2	17·50	31·65	—	—
1269	,,	,, 20	,,	72 20	17 40	0·0	17·56	31·75	—	·00207
1270	,,	,, 21	,,	72 13	18 0	0·7	17·57	31·77	—	·00207
1271	,,	,, 22	,,	72 36	18 30	2·2	17·45	31·56	—	·00202
1272	,,	,, 23	,,	72 10	18 0	1·0	17·24	31·18	—	·00203
1273	,,	,, 24	,,	72 12	19 10	1·7	16·16	29·25	—	—
1274	,,	,, 25	,,	72 35	17 50	0·9	17·22	31·14	—	—
1275	,,	,, 26	,,	72 35	17 40	−0·1	17·41	31·49	23·41	—
1276	,,	,, 27	,,	72 30	17 30	0·8	17·45	31·56	—	—
1277	,,	,, 28	,,	72 40	17 10	0·0	17·09	30·91	—	—
1278	,,	,, 29	,,	72 40	17 0	0·7	17·29	31·27	—	—
1279	,,	,, 30	,,	72 55	16 40	0·9	17·18	31·07	—	—
1280	,,	,, 31	,,	73 0	16 30	0·6	17·28	31·25	—	—
1281	,,	Aug. 1	,,	72 50	16 0	1·1	17·18	31·07	22·97	—
1282	,,	,, 2	,,	72 55	16 30	0·4	17·14	31·00	-	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1283	Active . .	Aug. 3	noon . .	73° 10′	16° 5′	0·9	16·92	30·60	22·56	·00199
1284	,,	,, 4	,,	73 5	16 55	1·1	16·43	29·74	—	—
1285	,,	,, 5	,,	73 10	17 50	1·4	17·12	30·97	—	—
1286	,,	,, 6	,,	73 10	17 50	1·2	17·22	31·14	—	—
1287	,,	,, 7	,,	72 50	19 0	1·6	16·77	30·34	—	·00198
1288	,,	,, 8	,,	72 24	18 40	0·8	16·89	30·55	—	·00198
1289	,,	,, 10	,,	72 0	18 0	1·1	16·77	30·34	—	—
1290	,,	,, 11	,,	71 30	18 20	1·1	16·88	30·54	—	—
1291	,,	,, 12	,,	71 10	20 0	1·9	17·07	30·87	—	—
1292	,,	,, 13	,,	70 20	21 25	1·9	15·35	27·81	—	·00181
1293	,,	,, 22	,,	69 0	19 30	0·6	16·99	30·73	22·69	—
1294	,,	,, 23	,,	68 10	17 0	3·6	17·96	32·48	—	—
1295	,,	,, 24	,,	66 54	13 30	5·6	18·53	33·49	—	—
1296	,,	,, 25	,,	65 15	10 30	6·9	18·42	33·30	—	—
1297	,,	,, 26	,,	62 10	7 40	9·9	19·49	35·21	—	·00231
1298	,,	,, 27	,,	60 40	4 30	11·1	19·52	35·27	—	—
1299	,,	,, 28	,,	59 30	1 45	11·7	19·46	35·15	—	—
					E.					
1300	Otaria . .	July 30	,,	71 8	27 20	9·5	19·08	34·47	—	—
1301	,,	Aug. 1	,,	71 40	38 10	6·2	19·29	34·86	—	—
1302	,,	,, 1	10 P.M. .	72 1	41 38	6·9	19·29	34·86	—	—
1303	,,	,, 2	noon . .	71 58	45 57	6·2	19·31	34·89	25·97	—
1304	,,	,, 2	10 P.M. .	72 7	47 55	5·9	19·08	34·47	—	·00222
1305	,,	,, 12	noon . .	72 22	51 28	5·1	17·24	31·18	23·02	·00203
1306	,,	,, 12	10 P.M. .	72 19	48 54	5·6	18·69	33·78	—	·00218
1307	,,	,, 13	noon . .	72 22	44 25	6·2	19·33	34·93	—	—
1308	,,	,, 13	10 P.M. .	72 16	41 40	6·7	19·35	34·96	—	—
1309	,,	,, 14	noon . .	72 24	38 7	5·3	19·33	34·93	—	—
1310	,,	,, 14	10 P.M. .	72 8	35 34	6·8	19·37	35·00	—	—
1311	,,	,, 15	noon . .	71 49	30 50	8·9	19·17	34·63	—	—
1312	,,	,, 15	10 P.M. .	71 22	28 26	9·2	19·21	34·71	—	—
1313	,,	,, 16	noon . .	71 5	27 10	8·8	18·84	34·05	—	—
1314	,,	,, 16	10 P.M. .	71 56	26 0	9·4	18·60	33·62	—	—
1315	,,	,, 26	,,	68 55	16 48	12·1	17·20	31·10	—	—
1316	,,	,, 27	noon . .	67 46	14 13	13·3	17·42	31·51	—	·00207
1317	,,	,, 28	,,	65 11	11 52	12·0	18·06	32·64	—	—
1318	,,	,, 29	,,	63 45	7 39	11·7	17·58	31·79	—	—
1319	,,	Sept. 1	,,	63 15	7 29	13·1	18·67	33·75	—	—
1320	,,	,, 2	,,	62 47	4 49	14·8	18·68	33·76	—	—
1321	,,	,, 2	10 P.M. .	62 18	4 36	14·0	18·27	33·02	24·42	·00213
1322	,,	,, 3	noon . .	61 31	2 9	13·2	19·55	35·32	—	·00229
1323	,,	,, 3	10 P.M. .	61 3	1 9	13·6	19·34	34·94	—	—
					W.					
1324	,,	,, 7	0.30 P.M.	60 1½	1 8⅓	12·3	19·51	35·25	—	—
1325	,,	,, 7	10 P.M. .	59 18	1 17	12·2	19·52	35·27	26·26	—
1326	,,	,, 8	noon . .	58 26	1 23	12·3	19·46	35·15	—	—
1327	,,	,, 8	10 P.M. .	57 48	1 34	12·2	19·38	35·01	—	—
1328	,,	,, 9	10 A.M. .	Near Bell Rock		13·1	18·87	34·10	—	—
1329	,,									
1330	Laura . .	Aug. 24	8 P.M. .	60° 8′	3° 37′	11·5	19·56	35·34	—	—
1331	,,	,, 25	4 A.M. .	60 53	5 20	11·0	19·56	35·34	—	—
1332	,,	,, 25	noon . .	61 26	6 24	10·7	19·07	34·46	—	—
1333	,,	,, 26	4 A.M. .	61 42	6 30	9·9	19·51	35·25	26·25	—
1334	,,	,, 27	noon . .	62 8	6 26	10·2	19·46	35·15	—	—
1335	,,	,, 28	4 A.M. .	62 27	7 8	10·0	19·46	35·15	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1336	Laura . .	Aug. 28	noon . .	62° 35′	9° 36′	10·5	19·51	25·25	—	—
1337	,,	,, 28	8 P.M. .	62 48	12 22	8·9	19·50	35·23	26·21	·00228
1338	,,	,, 29	4 A.M. .	62 52	15 38	10·0	19·44	35·12	—	—
1339	,,	,, 29	noon .	63 16	18 48	10·5	19·12	34·54	—	—
1340	,,	,, 29	8 P.M. .	63 30	20 50	9·5	19·17	34·63	—	—
1341	,,	,, 30	4 A.M. .	64 14	22 51	9·2	19·05	34·42	25·57	—
1342	,,	Sept. 4	8 P.M. .	64 10	22 48	10·1	18·31	33·10	—	—
1343	,,	,, 5	4 A.M. .	63 30	21 20	10·0	19·30	34·87	—	—
1344	,,	,, 5	noon .	63 6	18 56	10·0	19·10	34·51	—	—
1345	,,	,, 5	8 P.M. .	63 0	16 18	10·2	19·46	35·15	—	—
1346	,,	,, 6	4 A.M. .	62 52	12 34	10·1	19·51	35·25	—	—
1347	,,	,, 6	noon .	62 35	10 24	10·5	19·53	35·28	—	·00230
1348	,,	,, 6	8 P.M. .	62 29	7 33	9·4	19·54	35·30	—	·00229
1349	,,	,, 7	,,	61 54	6 30	9·5	19·46	35·15	26·18	—
1350	,,	,, 8	4 A.M. .	61 21	6 20	10·5	19·53	35·28	—	—
1351	,,	,, 8	noon .	60 41	5 4	11·5	19·48	35·19	—	·00230
1352	,,	,, 8	8 P.M. .	59 45	3 10	11·7	19·53	35·28	—	—
1353	Teutonic .	Aug. 27	midnight	51 24	14 26	14·4	19·73	35·64	26·64	·00231
1354	,,	,, 28	noon .	51 24	20 42	15·6	19·77	35·72	—	—
1355	,,	,, 28	midnight	50 46	26 36	16·1	19·49	35·21	—	—
1356	,,	,, 29	noon .	50 9	32 42	17·2	19·23	34·74	—	—
1357	,,	,, 29	midnight	48 56	38 38	17·8	19·40	35·05	26·15	·00229
1358	,,	,, 30	noon .	47 32	44 24	13·9	18·20	32·90	—	—
1359	,,	,, 30	midnight	45 56	49 56	11·1	17·83	32·24	—	—
1360	,,	,, 31	noon .	44 21	55 10	18·3	17·96	32·48	24·02	—
1361	,,	,, 31	midnight	43 5	60 10	16·7	17·70	32·01	23·75	·00208
1362	,,	Sept. 1	noon .	41 40	65 19	17·8	18·06	32·64	—	—
1363	,,	,, 1	midnight	40 26	69 54	15·6	18·04	32·61	—	—
1364	,,	,, 9	,,	40 11	70 2	17·8	18·66	33·73	25·19	—
1365	,,	,, 10	noon	40 57	66 12	23·3	18·30	33·08	—	—
1366	,,	,, 10	midnight	41 49	61 31	18·9	18·23	32·95	—	·00213
1367	,,	,, 11	noon .	43 4	56 39	21·7	18·62	33·66	—	—
1368	,,	,, 11	midnight	44 17	52 5	18·9	17·85	32·28	23·90	—
1369	,,	,, 12	noon .	45 45	47 32	14·4	17·59	31·81	23·54	—
1370	,,	,, 12	midnight	47 3	42 51	15·0	18·20	32·90	—	—
1371	,,	,, 13	noon .	48 35	37 43	17·8	19·53	35·28	—	—
1372	,,	,, 13	midnight	49 30	32 14	15·6	19·34	34·94	25·89	—
1373	,,	,, 14	noon .	50 33	26 35	16·7	19·51	35·25	—	—
1374	,,	,, 14	midnight	50 50	20 1	14·4	19·59	35·40	—	—
1375	,,	,, 15	noon .	51 22	15 2	16·1	19·71	35·61	26·55	—
1376	Loughrigg Holme .	Aug. 16	midnight	55 36	8 45	14·3	19·57	35·36	—	—
1377	,,	,, 17	noon .	55 57	12 4	14·7	19·55	35·32	—	—
1378	,,	,, 17	midnight	56 3	15 0	14·2	19·56	35·34	—	—
1379	,,	,, 18	noon . .	56 11	17 53	13·9	19·55	35·32	—	—
1380	,,	,, 18	midnight	56 14	20 48	13·9	19·58	35·38	—	—
1381	,,	,, 19	noon . .	56 16	23 44	13·7	19·52	35·27	—	—
1382	,,	,, 19	midnight	56 19	26 12	13·9	19·49	35·21	—	—
1383	,,	,, 20	noon . .	56 22	28 57	12·2	19·33	34·93	—	—
1384	,,	,, 20	midnight	56 11	32 10	12·1	19·35	34·96	—	—
1385	,,	,, 21	noon . .	56 7	35 22	12·1	19·26	34·80	—	—
1386	,,	,, 21	midnight	55 41	37 26	10·9	19·20	34·69	—	—
1387	,,	,, 22	noon . .	55 14	40 14	10·8	19·16	34·62	—	—
1388	,,	,, 22	midnight	54 44	43 23	10·5	19·16	34·62	—	—
1389	,,	,, 23	noon . .	54 13	46 39	11·4	19·03	34·39	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO$_3$.
		1896.		N.	W.					
1390	Loughrigg Holme	Aug. 23	midnight	53° 29′	49° 42′	11·7	19·02	34·37	—	—
1391	,,	,, 24	noon	52 46	52 28	7·2	17·62	31·86	—	—
1392	,,	,, 24	midnight	52 10	54 35	10·0	16·32	29·54	—	—
1393	,,	Sept. 12	,,	52 0	54 40	9·4	17·08	30·88	—	—
1394	,,	,, 13	noon	52 28	51 46	6·1	17·85	32·28	—	—
1395	,,	,, 13	midnight	52 57	49 5	10·6	19·06	34·44	—	—
1396	,,	,, 14	noon	53 12	46 17	11·6	19·14	34·58	—	—
1397	,,	,, 14	midnight	53 39	43 23	12·2	19·15	34·60	—	—
1398	,,	,, 15	noon	54 5	40 24	10·2	19·04	34·40	—	—
1399	,,	,, 15	midnight	54 42	37 33	11·0	19·20	34·69	—	—
1400	,,	,, 16	noon	55 13	34 37	10·7	19·40	35·05	—	—
1401	,,	,, 17	,,	55 42	28 1	12·3	19·30	34·87	—	—
1402	,,	,, 17	midnight	55 49	24 42	12·8	19·44	35·12	—	—
1403	,,	,, 18	noon	55 57	21 21	13·5	19·45	35·14	—	—
1404	,,	,, 18	midnight	55 53	18 4	12·8	19·59	35·40	—	—
1405	,,	,, 19	noon	55 48	14 48	13·1	19·52	35·27	—	—
1406	Ethiopia	,, 5	,,	55 16	14 21	15·0	19·20	34·69	—	—
1407	,,	,, 6	,,	54 27	22 59	15·6	19·46	35·15	—	—
1408	,,	,, 7	,,	53 18	31 1	13·3	19·28	34·84	—	—
1409	,,	,, 8	,,	51 39	38 19	15·6	19·10	34·51	—	—
1410	,,	,, 9	,,	49 42	43 54	15·0	18·95	34·24	—	—
1411	,,	,, 10	,,	47 23	50 24	14·4	17·51	31·66	23·52	—
1412	,,	,, 11	,,	44 51	56 13	17·8	17·80	32·19	23·87	—
1413	,,	,, 12	,,	42 27	61 42	18·3	17·44	31·54	—	—
1414	,,	,, 13	,,	40 55	67 46	16·7	17·97	32·49	—	—
1415	,,	,, 20	,,	40 45	68 28	16·1	17·99	32·53	—	—
1416	,,	,, 21	,,	42 23	62 44	16·1	17·73	32·06	—	—
1417	,,	,, 22	,,	44 25	56 54	16·1	17·65	31·92	—	—
1418	,,	,, 23	,,	46 59	51 16	14·4	17·41	31·49	23·30	—
1419	,,	,, 24	,,	49 29	45 29	12·2	18·61	33·64	—	—
1420	,,	,, 25	,,	51 22	38 53	13·3	19·07	34·46	—	—
1421	,,	,, 26	,,	53 3	31 59	11·7	19·20	34·69	25·87	—
1422	,,	,, 27	,,	54 11	24 26	12·8	19·33	34·93	—	—
1423	,,	,, 28	,,	54 55	16 27	12·8	19·56	35·34	—	—
1424	,,	,, 29	,,	55 35	8 29	13·3	19·51	35·25	—	—
1425	Corean	,, 6	,,	Off Fastnet		15·6	19·50	35·23	—	—
1426	,,	,, 7	,,	51° 43′	16° 42′	15·0	19·57	35·36	—	—
1427	,,	,, 8	,,	51 52	23 28	16·7	19·63	35·47	—	—
1428	,,	,, 9	,,	51 38	30 25	16·7	19·46	35·15	26·32	—
1429	,,	,, 10	,,	51 5	36 30	14·4	18·95	34·24	—	—
1430	,,	,, 11	,,	50 16	41 45	16·7	18·78	33·94	—	—
1431	,,	,, 12	,,	48 50	47 59	18·3	18·45	33·36	—	—
1432	,,	,, 13	,,	6′ E. of St. John's		13·9	17·19	31·09	23·08	—
1433	,,	,, 15	,,	45° 42′	57° 0′	16·1	17·55	31·73	—	—
1434	,,	,, 16	,,	8′ S. of Halifax		15·6	16·49	29·84	—	—
1435	,,	,, 17	,,	43° 42′	64° 12′	17·2	17·32	31·32	—	—
1436	,,	,, 18	,,	40 22	68 58	17·2	17·96	32·48	—	—
1437	,,	,, 19	,,	38 53	74 40	20·0	17·79	32·17	—	—
1438	,,	,, 26	,,	39 30	71 38	17·8	18·81	34·00	—	—
1439	,,	,, 27	,,	41 6	66 30	15·6	17·85	32·28	—	—
1440	,,	,, 28	,,	42 49	61 20	17·8	17·68	31·97	23·67	—
1441	,,	,, 29	,,	44 45	56 13	14·4	—	—	—	—
1442	,,	,, 30	,,	10′ E. of St. John's		11·1	17·20	31·11	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1443	Corean . .	Oct. 1	noon . .	49° 33′	47° 30′	11·7	18·78	33·94	—	—
1444	,,	,, 2	,,	51 40	40 55	14·4	19·04	34·40	25·55	—
1445	,,	,, 3	,,	53 2	33 54	11·7	19·27	34·82	—	—
1446	,,	,, 4	,,	53 35	27 5	12·8	19·31	34·89	—	—
1447	,,	,, 5	,,	54 49	20 6	12·2	19·66	35·52	—	—
1448	,,	,, 6	,,	56 24	12 40	11·7	19·66	35·52	26·46	—
1449	,,	,, 7	,,	Off Holy Isle		12·2	18·79	33·96	—	—
1450	Thorwaldsen	June 2	8 A.M. .	59° 55′	3° 4′	9·0	19·38	35·01	—	—
1451	,,	,, 2	noon . .	59 57	4 25	9·7	19·58	35·38	—	—
1452	,,	,, 2	4 P.M. .	59 56	5 41	10·2	19·58	35·38	—	—
1453	,,	,, 3	8 A.M. .	59 46	10 41	9·8	19·56	35·34	26·42	—
1454	,,	,, 3	noon . .	59 38	12 5	9·8	19·59	35·40	—	—
1455	,,	,, 3	4 P.M. .	59 28	13 15	10·0	19·59	35·40	—	—
1456	,,	,, 4	8 A.M. .	58 50	17 20	9·6	19·57	35·36	—	—
1457	,,	,, 4	noon . .	58 40	18 19	9·8	19·56	35·34	26·27	—
1458	,,	,, 4	4 P.M. .	58 36	18 58	10·0	19·56	35·34	—	—
1459	,,	,, 5	8 A.M. .	58 12	22 24	10·5	19·51	35·25	—	—
1460	,,	,, 5	noon . .	58 1	23 50	10·1	19·55	35·32	—	—
1461	,,	,, 5	4 P.M. .	57 56	24 57	10·5	19·50	35·23	—	—
1462	,,	,, 6	8 A.M. .	57 40	28 27	10·0	19·45	35·14	—	—
1463	,,	,, 6	noon . .	57 37	28 50	9·5	19·45	35·14	—	—
1464	,,	,, 6	4 P.M. .	57 32	29 0	10·6	19·40	35·05	—	—
1465	,,	,, 7	8 A.M. .	57 17	28 58	9·5	19·50	35·23	26·10	—
1466	,,	,, 7	noon . .	57 9	28 53	9·5	19·40	35·05	—	—
1467	,,	,, 7	4 P.M. .	57 10	29 7	9·8	19·44	35·12	—	—
1468	,,	,, 8	8 A.M. .	57 9	29 34	9·5	19·46	35·15	—	—
1469	,,	,, 8	noon . .	57 15	29 59	9·0	19·44	35·12	—	—
1470	,,	,, 8	4 P.M. .	57 15	29 59	9·5	19·46	35·15	26·05	—
1471	,,	,, 9	8 A.M. .	57 7	34 22	8·4	19·43	35·10	—	—
1472	,,	,, 9	noon . .	57 6	35 42	8·0	19·43	35·10	—	—
1473	,,	,, 9	4 P.M. .	57 7	36 46	7·6	19·37	35·00	—	—
1474	,,	,, 10	8 A.M. .	57 2	39 21	6·5	19·30	34·87	—	—
1475	,,	,, 10	noon . .	57 0	40 10	6·5	19·40	35·05	—	—
1476	,,	,, 10	4 P.M. .	57 1	40 15	6·2	19·34	34·94	—	—
1477	,,	,, 11	8 A.M. .	56 54	42 10	5·0	19·32	34·91	25·92	—
1478	,,	,, 11	noon . .	56 53	43 2	5·2	19·30	34·87	—	—
1479	,,	,, 11	4 P.M. .	56 56	43 31	5·2	19·18	34·65	—	—
1480	,,	Sept. 15	noon . .	56 42	45 29	8·0	19·11	34·53	—	—
1481	,,	,, 15	4 P.M. .	56 45	44 31	8·0	19·11	34·53	—	—
1482	,,	,, 16	noon . .	57 0	40 21	8·0	19·15	34·60	—	—
1483	,,	,, 16	4 P.M. .	57 3	39 59	9·2	19·21	34·71	25·88	—
1484	,,	,, 17	8 A.M. .	57 2	36 4	9·5	19·37	35·00	—	—
1485	,,	,, 17	noon . .	57 4	35 14	10·0	19·31	34·89	26·00	—
1486	,,	,, 17	4 P.M. .	57 4	34 19	10·5	19·40	35·05	—	—
1487	,,	,, 18	8 A.M. .	57 4	30 51	10·5	19·37	35·00	—	—
1488	,,	,, 18	noon . .	57 5	29 56	11·0	19·39	35·03	—	—
1489	,,	,, 18	4 P.M. .	57 9	29 1	11·5	19·46	35·15	—	—
1490	,,	,, 19	8 A.M. .	57 25	25 48	10·5	19·43	35·10	—	—
1491	,,	,, 19	noon . .	57 27	25 25	12·0	19·46	35·15	26·14	—
1492	,,	,, 19	4 P.M. .	57 29	24 56	13·0	19·49	35·21	—	—
1493	,,	,, 20	8 A.M. .	57 35	24 22	11·2	19·46	35·15	—	—
1494	,,	,, 20	noon . .	57 45	24 3	11·7	19·50	35·23	—	—
1495	,,	,, 20	4 P.M. .	57 54	23 37	12·2	19·53	35·28	—	—
1496	,,	,, 21	8 A.M. .	57 52	23 20	12·0	19·55	35·32	—	—
1497	,,	,, 21	noon . .	58 1	23 10	12·5	19·53	35·28	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO$_3$.
		1896.		N.	W.					
1498	Thorwaldsen	Sept. 22	noon . .	57° 51′	23° 15′	12·0	19·50	35·23	—	—
1499	,,	,, 23	8 A.M. .	58 10	20 26	11·5	19·54	35·30	26·22	—
1500	,,	,, 23	noon . .	58 15	19 30	12·0	19·50	35·23	—	—
1501	,,	,, 23	4 P.M. .	58 20	18 40	11·6	19·49	35·21	—	—
1502	,,	,, 24	8 A.M. .	58 43	15 21	11·5	19·53	35·28	—	—
1503	,,	,, 24	noon . .	58 47	14 38	10·8	19·55	35·32	—	—
1504	,,	,, 25	,,	59 5	11 26	12·0	19·58	35·38	—	—
1505	,,	,, 26	,,	59 37	9 18	11·5	19·52	35·27	—	—
1506	,,	,, 27	,,	59 37	7 7	11·5	19·55	35·32	—	—
1507	,,	June 12	4 P.M. .	58 13	47 14	5·2	18·97	34·28	25·50	—
1508	,,	,, 12	noon . .	57 54	46 9	5·5	19·05	34·42	—	—
1509	,,	,, 12	8 A.M. .	57 37	45 25	4·5	19·01	34·35	—	—
1510	Teutonic .	Sept. 25	noon . .	51 23	10 7	15·0	19·92	35·98	—	—
1511	,,	,, 25	midnight	51 23	15 51	14·4	19·68	35·56	—	—
1512	,,	,, 26	noon . .	51 23	20 50	15·0	19·64	35·48	—	—
1513	,,	,, 26	midnight	50 53	26 45	14·4	19·79	35·75	26·69	—
1514	,,	,, 27	noon . .	50 16	32 32	14·4	19·60	35·41	—	—
1515	,,	,, 27	midnight	48 55	38 12	14·4	19·20	34·69	—	—
1516	,,	,, 28	noon . .	47 29	43 55	14·4	18·51	33·46	—	—
1517	,,	,, 28	midnight	46 4	49 30	13·3	17·73	32·06	23·75	—
1518	,,	,, 29	noon . .	44 26	54 42	15·6	18·40	33·26	—	—
1519	,,	,, 29	midnight	43 34	58 21	16·1	17·96	32·48	—	—
1520	,,	,, 30	noon . .	41 30	65 43	16·7	17·75	32·09	—	—
1521	,,	,, 30	midnight	40 40	70 31	15·9	18·05	33·63	—	—
1522	,,	Oct. 7	,,	40 48	66 49	20·0	19·49	35·21	—	—
1523	,,	,, 8	noon . .	41 10	64 59	21·7	19·75	35·68	—	—
1524	,,	,, 8	midnight	42 7	60 58	17·2	17·97	32·49	—	—
1525	,,	,, 9	noon . .	43 23	56 9	16·7	18·47	33·39	24·79	—
1526	,,	,, 9	midnight	44 43	50 59	15·0	17·96	32·48	—	—
1527	,,	,, 10	noon . .	46 10	46 10	15.6	18·37	33·20	—	—
1528	,,	,, 10	midnight	47 39	41 9	16·7	19·70	35·59	—	—
1529	,,	,, 11	noon . .	49 7	36 7	16·7	19·68	35·56	26·37	—
1530	,,	,, 11	midnight	49 49	30 50	14·4	19·47	35·17	26·16	—
1531	,,	,, 12	noon . .	50 43	25 11	14·4	19·61	35·43	—	—
1532	,,	,, 12	midnight	50 57	19 22	13·9	19·74	35·66	—	—
1533	,,	,, 13	noon . .	51 23	13 31	13·3	19·91	35·97	—	—
1534	Granuaile .	June 14	,,	55 48	10 40	13·5	19·60	35·41	—	·00230
1535	,,	,, 14	4 P.M. .	56 17	11 25	12·8	19·57	35·36	—	—
1536	,,	,, 14	8 P.M. .	56 44	12 6	12·8	19·62	35·45	26·49	—
1537	,,	,, 14	midnight	57 10	12 50	11·7	19·60	35·41	—	·00230
1538	,,	,, 15	4 A.M. .	57 39	13 36	10·7	19·53	35·28	26·27	—
1539	,,	,, 15	11.30 A.M.	57 35	13 44	10·4	19·63	35·47	—	—
1540	,,	,, 15	4 P.M. .	57 23	13 27	11·1	19·58	35·38	—	—
1541	,,	,, 15	8 P.M. .	57 35	13 44	10·8	19·53	35·28	—	·00230
1542	,,	,, 15	midnight	57 32	13 48	10·8	19·63	35·47	—	—
1543	,,	,, 16	4 A.M. .	57 35	13 43	10·4	19·56	35·34	26·25	—
1544	,,	,, 16	8 A.M. .	57 36	13 40	10·6	19·59	35·40	—	—
1545	,,	,, 16	4.15 P.M.	57 36	12 0	11·5	19·60	35·41	—	·00230
1546	,,	,, 16	8.30 P.M.	57 36	10 50	11·7	19·63	35·47	—	—
1547	,,	,, 16	midnight	57 46	9 50	12·2	19·60	35·41	—	—
1548	,,	,, 17	4 A.M. .	57 48	8 42	12·1	19·54	35·30	—	—
1549	,,	,, 17	4 P.M. .	57 22	8 16	12·8	19·44	35·12	26·17	·00229
1550	China . .	Aug. 5	11 P.M. .	63 34	16 31	9·7	18·97	34·28	—	—
1551	,,	,, 6	noon . .	63 34	16 31	10·0	19·07	34·46	—	—
1552	,,	,, 6	midnight	63 37	15 12	11·7	19·22	34·72	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO$_3$.
		1896.		N.	W.					
1553	China . .	Aug. 7	noon . .	63° 37′	15° 12′	11·7	19·25	34·78	25·90	—
1554	,,	,, 8	,,	65 10	14 22	7·2	18·50	33·44	—	—
1555	,,	,, 8	midnight	66 23	14 34	8·3	18·21	32·91	—	—
1556	,,	,, 9	9 A.M. .	66 31	18 4	8·3	18·50	33·44	—	—
1557	,,	,, 9	11.30 P.M.	66 27	20 21	7·2	18·74	33·87	—	—
1558	,,	,, 10	noon . .	66 27	20 16	6·7	18·72	33·83	—	—
1559	,,	,, 11	midnight	66 27	20 16	6·7	18·56	33·55	—	—
1560	,,	,, 12	11 A.M. .	66 33	20 10	7·2	17·89	32·35	—	—
1561	,,	,, 13	9 P.M. .	63 35	18 10	7·2	18·48	33·41	24·81	—
1562	,,	,, 15	11 A.M. .	63 28	18 40	10·6	19·17	34·63	—	—
					E.					
1563	,,	,, 27	1 A.M. .	57 38	14 50	12·8	19·56	35·34	—	—
1564	,,	,, 27	2 P.M. .	57 38	14 50	12·2	19·53	35·28	—	—
1565	,,	Sept. 12	noon . .	56 56	14 20	13·3	19·51	35·25	26·24	—
1566	,,	,, 12	11 P.M. .	56 56	14 20	12·2	19·57	35·36	—	—
1567	,,	,, 11	,,	56 56	14 20	13·3	19·58	35·38	—	—
1568	,,	,, 28	,,	62 10	6 20	13·3	19·45	35·14	—	—
1569	,,	,, 29	noon . .	62 10	6 20	13·3	19·45	35·14	26·08	—
1570	,,	Oct. 14	11 P.M. .	62 12	6 15	5·1	19·46	35·15	—	—
1571	,,	,, 15	noon . .	62 12	6 14	6·1	19·50	35·23	—	—
1572	,,	,, 16	11 P.M. .	62 5	6 10	6·1	19·25	34·78	—	—
1573	,,	,, 17	noon . .	62 5	6 10	6·7	19·28	34·84	25·94	—
1574	,,	,, 18	11 P.M. .	62 3	6 12	6·7	19·25	34·78	—	—
1575	,,	,, 19	noon . .	62 3	6 12	7·2	19·27	34·82	—	—
1576	,,	,, 20	11 P.M. .	62 1	6 2	7·2	19·29	34·86	—	—
1577	,,	,, 21	noon . .	62 1	6 2	7·2	19·28	34·84	—	—
1578	,,	,, 22	11 P.M. .	62 15	6 15	6·1	19·28	34·84	—	—
1579	,,	,, 23	noon . .	62 16	6 16	6·1	19·28	34·84	—	—
					W.					
1580	Laura . .	Sept. 27	8 P.M. .	59 46	3 13	10·7	19·51	35·25	—	—
1581	,,	,, 28	4 A.M. .	60 45	5 2	10·5	19·57	35·36	—	—
1582	,,	,, 28	noon . .	61 5	5 47	9·5	19·50	35·23	—	—
1583	,,	,, 29	8 P.M. .	61 39	6 34	9·0	19·48	35·19	26·23	—
1584	,,	Oct. 2	,,	62 14	8 2	8·5	19·32	34·91	—	—
1585	,,	,, 3	4 A.M. .	62 24	10 24	9·0	19·46	35·15	—	—
1586	,,	,, 3	noon . .	62 30	13 11	9·5	19·50	35·23	—	—
1587	,,	,, 3	8 P.M. .	62 44	16 13	9·5	19·48	35·19	—	—
1588	,,	,, 4	4 A.M. .	62 51	18 2	8·5	19·45	35·14	—	—
1589	,,	,, 4	noon . .	63 29	21 11	7·8	18·77	33·92	25·25	—
1590	,,	,, 4	8 P.M. .	63 49	22 54	6·5	19·66	35·52	—	—
1591	,,	,, 5	4 A.M. .	64 12	22 20	5·2	19·09	34·49	—	—
1592	,,	,, 10	8 P.M. .	64 19	22 35	6·5	19·32	34·91	—	—
1593	,,	,, 11	4 A.M. .	65 5	24 16	6·5	19·33	34·93	—	—
1594	,,	,, 11	8 P.M. .	65 10	23 28	6·5	19·26	34·80	—	—
1595	,,	,, 15	8 A.M. .	66 9	23 50	5·2	19·14	34·58	25·56	—
1596	,,	,, 18	8 P.M. .	65 41	24 57	5·6	19·16	34·62	—	—
1597	,,	,, 19	4 A.M. .	64 40	24 5	6·0	19·35	34·96	—	—
1598	,,	,, 22	noon . .	63 50	22 59	6·1	19·20	34·69	—	—
1599	,,	,, 22	8 P.M. .	63 34	20 33	6·0	18·97	34·28	—	—
1600	,,	,, 23	4 A.M. .	63 24	20 0	6·2	19·31	34·89	—	—
1601	,,	,, 23	noon . .	62 55	17 23	8·7	19·49	35·21	—	—
1602	,,	,, 24	4 A.M. .	62 53	14 59	7·0	19·22	34·72	—	—
1603	,,	,, 24	noon . .	62 58	13 23	6·9	19·99	36·11	—	—
1604	,,	,, 24	8 P.M. .	62 14	13 0	7·0	19·64	35·48	—	—
1605	,,	,, 25	4 A.M. .	62 0	11 28	8·0	19·52	35·27	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1606	Laura . .	Oct. 25	noon . .	61° 45′	9° 29′	8·5	19·52	35·27	—	—
1607	,,	,, 25	8 P.M. .	61 40	8 4	8·2	19·53	35·28	—	—
1608	,,	,, 28	,,	60 38	4 8	8·0	19·57	35·36	—	—
1609	,,	,, 29	4 A.M. .	59 30	2 50	8·0	19·59	35·40	—	—
1610	Thyra . .	May 13	noon . .	60 7	5 3	10·0	19·60	35·41	—	—
1611	,,	,, 13	8 P.M. .	61 5	5 51	9·2	19·57	35·36	—	—
1612	,,	,, 14	noon . .	61 49	6 39	7·5	19·48	35·19	—	—
1613	,,	,, 16	4 A.M. .	63 45	10 3	8·0	19·50	35·23	26·26	—
1614	,,	,, 16	noon . .	64 31	11 48	2·0	18·94	34·23	—	—
1615	,,	,, 20	8 A.M. .	63 59	15 33	8·0	19·44	35·12	—	—
1616	,,	,, 20	noon . .	63 44	16 35	7·5	19·36	34·98	—	—
1617	,,	,, 20	8 P.M. .	63 21	18 51	7·0	19·38	35·01	26·04	—
1618	,,	,, 21	4 A.M. .	63 40	21 26	8·3	17·93	32·42	24·00	—
1619	,,	June 14	8 P.M. .	64 58	12 37	3·0	18·77	33·92	—	—
1620	,,	,, 15	4 A.M. .	64 11	10 33	5·0	19·11	34·53	—	—
1621	,,	,, 15	noon . .	63 22	8 39	8·5	19·46	35·15	—	—
1622	,,	,, 15	8 P.M. .	62 30	6 51	8·5	19·38	35·01	—	—
1623	,,	,, 17	noon . .	61 31	6 40	9·0	19·49	35·21	—	—
1624	,,	,, 17	8 P.M. .	60 43	4 50	10·5	19·51	35·25	—	—
1625	,,	,, 18	8 A.M. .	59 44	2 25	10·5	19·54	35·30	—	—
1626	,,	July 10	8 P.M. .	59 42	2 53	12·0	19·42	35·08	—	—
1627	,,	,, 11	8 A.M. .	60 52	4 57	12·6	19·57	35·36	—	—
1628	,,	,, 11	noon . .	61 21	6 2	12·0	19·52	35·27	—	—
1629	,,	,, 13	8 P.M. .	63 0	8 3	10·5	19·43	35·10	—	—
1630	,,	,, 14	4 A.M. .	63 56	9 56	9·8	19·38	35·01	—	—
1631	,,	Aug. 12	8 P.M. .	61 14	5 20	12·0	19·51	35·25	—	—
1632	,,	Sept. 21	noon . .	60 13	3 42	10·9	19·54	35·30	—	—
1633	,,	,, 21	8 P.M. .	61 5	5 3	9·5	19·44	35·12	—	—
1634	,,	,, 23	4 A.M. .	62 21	8 20	8·3	19·46	35·15	—	—
1635	,,	,, 23	noon . .	63 6	10 4	8·5	19·25	34·78	—	—
1636	,,	,, 23	8 P.M. .	63 57	11 43	6·5	19·01	34·35	—	—
1637	,,	,, 24	4 A.M. .	64 51	13 30	5·0	18·62	33·66	24·95	—
1638	,,	Oct. 31	noon . .	63 7	8 35	6·3	19·34	34·94	—	—
1639	,,	Nov. 2	,,	59 53	3 5	8·8	19·51	35·25	—	—
1640	America . .	Aug. 6	11.30 A.M.	—	—	12·9	19·49	35·21	—	—
1641	,,	,, 7	noon . .	—	—	12·9	19·40	35·05	—	—
1642	,,	,, 9	11.30 A.M.	—	—	13·9	19·97	36·07	—	—
1643	,,	,, 11	10.45 A.M.	57 40	8 20	12·8	19·51	35·25	26·33	—
1644	,,	,, 18	noon . .	56 30	14 0	14·4	19·97	36·07	—	—
1645	,,	,, 20	,,	56 30	14 10	13·9	19·52	35·27	—	—
1646	,,	,, 22	,,	57 0	11 45	13·9	19·54	35·30	—	—
1647	,,	Sept. 3	11.30 A.M.	58 10	8 56	12·8	19·46	35·15	26·15	—
1648	,,	,, 4	noon . .	57 30	11 5	12·8	19·55	35·32	—	—
1649	,,	,, 5	11.30 A.M.	56 36	14 22	13·3	19·60	35·41	—	—
1650	,,	,, 6	noon . .	56 40	14 27	13·9	19·54	35·30	—	—
1651	,,	,, 7	,,	57 0	14 20	13·9	19·49	35·21	26·21	—
1652	,,	,, 8	11.30 A.M.	56 30	14 42	12·8	19·52	35·27	—	—
1653	,,	,, 9	,,	56 30	14 36	13·3	19·54	35·30	—	—
1654	,,	,, 10	,,	57 0	14 25	13·3	19·32	34·91	—	—
1655	,,	,, 11	,,	58 0	8 30	12·8	19·29	34·86	—	—
1656	,,	,, 12	noon . .	59 0	6 25	12·8	19·25	34·78	—	—
1657	,,	,, 13	,,	56 0	9 5	12·8	19·29	34·86	—	—
1658	,,	,, 14	,,	55 30	9 10	12·8	19·31	34·89	—	—
1659	,,	,, 15	,,	56 39	8 52	12·8	19·52	35·25	—	—
1660	,,	,, 16	,,	56 30	8 48	12·8	19·47	35·17	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1661	America . .	Sept. 17	noon . .	56° 4′	8° 57′	13·3	19·46	35·15	—	—
1662	,,	,, 18	,,	57 0	8 54	13·9	19·55	35·32	—	—
1663	,,	Oct. 9	,,	62 0	9 20	11·7	19·54	35·30	26·22	—
1664	,,	,, 11	,,	62 25	11 7	12·8	19·54	35·30	—	—
1665	,,	,, 15	,,	63 10	12 52	12·8	19·54	35·30	—	—
1666	,,	,, 17	,,	63 0	14 45	12·2	19·53	35·28	—	—
1667	Teutonic . .	,, 22	midnight	51 24	13 47	12·2	19·83	35·82	—	—
1668	,,	,, 23	noon . .	51 24	19 57	13·3	19·59	35·40	—	—
1669	,,	,, 23	midnight	50 54	26 18	13·3	19·63	35·47	—	—
1670	,,	,, 24	noon . .	50 11	32 31	14·4	19·29	34·86	—	—
1671	,,	,, 24	midnight	48 54	38 24	16·7	19·44	35·12	—	—
1672	,,	,, 25	noon . .	47 38	44 32	13·3	18·41	33·28	—	—
1673	,,	,, 25	midnight	46 0	49 46	11·7	17·65	31·92	23·66	—
1674	,,	,, 26	noon . .	44 20	54 52	15·6	18·44	33·34	24·78	—
1675	,,	,, 26	midnight	43 3	60 1	14·4	18·61	33·64	—	—
1676	,,	,, 27	noon . .	41 39	65 23	12·2	18·12	32·75	—	—
1677	,,	,, 27	midnight	40 31	69 51	13·3	18·09	32·70	—	—
1678	,,	Nov. 4	,,	40 12	70 21	12·8	18·12	32·75	—	—
1679	,,	,, 5	noon . .	41 8	65 36	12·2	18·06	32·64	—	—
1680	,,	,, 5	midnight	42 16	60 55	18·9	18·82	34·01	—	—
1681	,,	,, 6	noon . .	43 21	56 15	14·4	18·83	34·03	—	—
1682	,,	,, 6	midnight	44 38	51 35	12·2	17·83	32·25	23·83	—
1683	,,	,, 7	noon . .	45 54	46 57	12·8	18·49	33·43	—	—
1684	,,	,, 7	midnight	47 2	42 16	13·3	18·66	33·73	—	—
1685	,,	,, 8	noon . .	48 30	37 8	17·2	19·76	35·70	—	—
1686	,,	,, 8	midnight	49 33	31 52	14·4	19·16	34·62	—	—
1687	,,	,, 9	noon . .	50 40	26 14	13·3	19·53	35·28	—	—
1688	,,	,, 9	midnight	50 59	20 22	12·8	19·62	35·45	—	—
1689	,,	,, 10	noon . .	51 10	14 9	12·2	19·63	35·47	—	—
1690	Eclipse . .	Apr. 9	,,	59 2	4 3	8·3	19·42	35·08	—	—
1691	,,	,, 10	,,	60 26	5 30	8·3	19·57	35·36	—	—
1692	,,	,, 11	,,	60 39	6 5	6·7	19·48	35·19	—	—
1693	,,	,, 12	,,	59 48	8 50	8·3	19·58	35·38	26·45	—
1694	,,	,, 13	,,	59 20	10 48	8·8	19·55	35·32	—	—
1695	,,	,, 14	,,	59 40	15 56	8·9	19·53	35·28	—	—
1696	,,	,, 15	,,	59 33	20 43	8·9	19·50	35·23	—	—
1697	,,	,, 16	,,	60 22	25 8	8·3	19·50	35·23	—	—
1698	,,	,, 17	,,	59 0	25 19	8·9	19·56	35·34	—	—
1699	,,	,, 18	,,	59 28	30 19	7·8	19·48	35·19	—	—
1700	,,	,, 19	,,	60 25	32 42	7·2	19·55	35·32	—	—
1701	,,	,, 20	,,	60 13	35 34	5·1	19·44	35·12	—	—
1702	,,	,, 21	,,	58 45	42 6	3·2	19·02	34·37	—	—
1703	,,	,, 22	,,	58 3	45 23	2·8	19·17	34·63	—	—
1704	,,	,, 23	,,	59 3	48 23	2·8	18·96	34·26	—	—
1705	,,	,, 24	,,	59 50	50 23	1·7	18·98	34·30	25·53	—
1706	,,	,, 25	,,	59 21	50 12	2·5	19·21	34·71	—	—
1707	,,	,, 26	,,	59 2	50 36	3·9	19·28	34·84	—	—
1708	,,	,, 27	,,	60 30	53 5	1·7	19·02	34·37	—	—
1709	,,	,, 28	,,	62 18	55 20	−0·6	18·78	33·94	25·17	—
1710	,,	,, 29	,,	63 54	52 32	−0·6	18·74	33·87	—	—
1711	,,	,, 30	,,	64 5	52 40	−1·1	18·54	33·51	24·79	—
1712	,,	May 1	,,	65 25	53 30	1·1	18·90	34·16	—	—
1713	,,	,, 2	,,	65 55	53 24	1·1	18·93	34·21	—	—
1714	,,	,, 3	,,	66 8	54 30	−0·6	18·95	34·24	—	—
1715	,,	,, 4	,,	68 25	55 30	−2·7	18·95	34·24	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1716	Eclipse .	May 5	noon . .	68° 40′	54° 23′	-1·7	18·60	33·62	—	—
1717	,,	,, 6	,,	68 53	55 19	-1·7	18·61	33·64	—	—
1718	,,	,, 7	,,	69 15	54 12	-1·1	18·58	33·58	—	—
1719	,,	,, 8	,,	69 0	55 21	-1·1	18·61	33·64	—	—
1720	,,	,, 9	,,	—	—	-1·1	18·62	33·66	—	—
1721	,,	,, 10	,,	68 45	55 13	-0·6	18·62	33·66	—	—
1722	,,	,, 11	,,	68 30	54 36	-1·1	18·54	33·51	—	—
1723	,,	,, 12	,,	68 12	55 42	-1·1	18·63	33·67	—	—
1724	,,	,, 13	,,	69 18	56 12	-1·1	18·66	33·73	—	—
1725	,,	,, 14	,,	69 9	53 25	-1·1	17·53	31·70	—	—
1726	,,	,, 15	,,	69 9	53 25	-1·0	17·53	31·70	—	—
1727	,,	,, 16	,,	69 9	53 25	-1·1	18·09	32·70	—	—
1728	,,	,, 17	,,	69 27	54 56	-1·1	18·67	33·75	—	—
1729	,,	,, 18	5 P.M. .	70 25	55 0	-1·6	18·51	33·46	—	—
1730	,,	,, 19	noon . .	70 25	55 0	-0·8	18·56	33·55	—	—
1731	,,	,, 20	,,	70 36	55 12	-1·0	18·60	33·62	—	—
1732	,,	,, 21	,,	70 20	55 0	-1·1	18·64	33·69	—	—
1733	,,	,, 22	,,	70 25	55 0	-1·1	18·55	33·53	—	—
1734	,,	,, 23	,,	70 25	55 0	-0·6	18·58	33·58	—	—
1735	,,	,, 24	,,	70 26	54 53	-0·6	18·60	33·62	—	—
1736	,,	,, 25	,,	70 29	54 53	0·6	18·61	33·64	—	—
1737	,,	,, 26	,,	70 40	54 30	-0·7	18·60	33·62	—	—
1738	,,	,, 27	,,	70 40	54 30	-1·0	18·54	33·51	—	—
1739	,,	,, 28	,,	72 46	56 10	-0·2	18·62	33·66	—	—
1740	,,	,, 29	,,	72 57	56 15	0·4	18·72	33·83	—	—
1741	,,	,, 30	,,	73 28	57 0	-1·6	18·20	32·90	—	—
1742	,,	June 1	,,	73 38	57 3	-0·9	18·49	33·43	—	—
1743	,,	,, 2	,,	73 55	58 0	-1·2	18·49	33·43	—	—
1744	,,	,, 3	,,	74 3	57 55	-0·7	18·57	33·56	—	—
1745	,,	,, 4	,,	74 13	58 20	-0·9	18·46	33·37	24·77	—
1746	,,	,, 5	,,	74 13	58 20	-1·1	18·48	33·41	—	—
1747	,,	,, 6	,,	74 13	58 20	-0·2	18·47	33·39	—	—
1748	,,	,, 7	,,	75 10	59 40	-0·2	18·48	33·41	—	—
1749	,,	,, 8	,,	75 10	59 40	-0·1	18·48	33·41	—	—
1750	,,	,, 9	,,	75 10	61 47	-0·3	18·52	33·48	—	—
1751	,,	,, 10	,,	75 10	61 47	-0·1	18·55	33·53	—	—
1752	,,	,, 11	,,	75 10	61 47	0·4	18·61	33·64	—	—
1753	,,	,, 12	,,	—	—	-0·4	18·53	33·49	—	—
1754	,,	,, 13	,,	—	—	-0·2	18·50	33·44	—	—
1755	,,	,, 14	,,	—	—	-0·4	18·52	33·48	—	—
1756	,,	,, 15	,,	—	—	-0·6	18·55	33·53	—	—
1757	,,	,, 16	,,	—	—	-0·2	18·55	33·53	24·89	—
1758	,,	,, 17	,,	—	—	0·1	18·36	33·19	—	—
1759	,,	,, 18	,,	—	—	0·3	18·28	33·04	—	—
1760	,,	,, 19	,,	—	—	-0·2	18·52	33·48	—	—
1761	,,	,, 20	,,	—	—	-0·2	18·51	33·46	—	—
1762	,,	,, 21	,,	75 29	67 18	-0·6	18·51	33·46	—	—
1763	,,	,, 22	,,	75 51	—	-0·8	18·53	33·49	—	—
1764	,,	,, 23	,,	76 15	68 46	0·0	18·24	32·97	—	—
1765	,,	,, 24	,,	76 28	70 4	-0·6	18·12	32·75	—	—
1766	,,	,, 25	,,	74 47	76 12	3·6	18·37	33·20	—	—
1767	,,	,, 26	,,	—	—	3·3	17·84	32·26	—	—
1768	,,	,, 27	,,	—	—	3·1	17·93	32·42	—	—
1769	,,	,, 28	,,	74 3	71 52	3·6	17·61	31·85	—	—
1770	,,	,, 29	,,	—	—	2·8	17·34	31·36	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1771	Eclipse . .	June 30	noon . .	73° 45′	72° 57′	6·4	17·40	31·47	—	—
1772	,,	July 1	,,	73 51	—	2·5	17·23	31·17	—	—
1773	,,	,, 2	,,	73 0	69 30	0·0	17·31	31·31	—	—
1774	,,	,, 3	,,	72 51	—	0·3	17·92	32·40	—	—
1775	,,	,, 4	,,	72 45	76 5	1·2	15·96	28·90	21·33	—
1776	,,	,, 5	,,	72 45	76 5	0·6	3·31	6·04	3·66	—
1777	,,	,, 6	,,	72 45	76 5	1·1	3·05	5·57	3·34	—
1778	,,	,, 7	,,	72 45	76 5	0·9	1·62	2·93	1·31	—
1779	,,	,, 8	,,	—	—	—	1·20	2·18	—	—
1780	,,	,, 9	,,	71 30	70 30	4·4	17·74	32·07	—	—
1781	,,	,, 10	,,	71 45	72 40	1·4	16·00	28·97	—	—
1782	,,	,, 11	,,	—	—	2·2	16·32	29·54	—	—
1783	,,	,, 12	,,	—	—	1·1	16·50	29·87	—	—
1784	,,	,, 13	,,	—	—	1·7	16·37	29·63	—	—
1785	,,	,, 14	,,	72 8	69 30	3·6	16·19	29·31	—	—
1786	,,	,, 15	,,	—	—	3·1	16·70	30·21	—	—
1787	,,	,, 16	,,	—	—	1·7	16·78	30·37	22·34	·00198
1788	,,	,, 17	,,	71 33	71 0	1·7	16·47	29·81	—	·00197
1789	,,	,, 18	,,	—	—	2·6	13·15	23·85	—	·00156
1790	,,	,, 19	,,	70 36	66 30	1·6	16·28	29·48	21·83	·00193
1791	,,	,, 20	,,	70 34	67 55	1·4	8·41	15·31	10·85	·00097
1792	,,	,, 21	,,	70 34	67 55	0·3	9·09	16·54	11·75	—
1793	,,	,, 22	,,	—	—	-0·1	5·36	9·78	6·50	—
1794	,,	,, 23	,,	—	—	0·1	5·52	10·06	6·79	·00065
1795	,,	,, 24	,,	—	—	0·3	15·48	28·04	20·60	·00183
1796	,,	,, 25	,,	69 45	67 30	0·2	5·76	10·49	7·06	·00069
1797	,,	,, 26	,,	69 45	67 30	0·5	4·26	7·78	4·99	·00051
1798	,,	,, 27	,,	69 45	67 30	1·1	6·86	12·50	8·86	·00082
1799	,,	,, 28	,,	69 45	67 30	0·4	4·92	8·97	5·94	·00058
1800	,,	,, 29	,,	69 45	67 30	0·5	5·49	10·01	6·74	·00065
1801	,,	,, 30	,,	69 45	67 30	0·6	8·74	15·90	11·18	·00104
1802	,,	Aug. 1	,,	69 45	67 30	0·8	8·32	15·14	—	·00097
1803	,,	,, 2	,,	69 45	67 30	0·3	4·28	7·82	5·12	·00049
1804	,,	,, 3	,,	69 45	67 30	0·5	6·04	11·01	—	·00070
1805	,,	,, 4	,,	69 45	67 30	0·4	6·22	11·34	7·74	—
1806	,,	,, 5	,,	69 45	67 30	1·1	4·36	7·96	—	·00050
1807	,,	,, 6	,,	—	—	—	6·29	11·46	—	—
1808	,,	,, 7	,,	69 43	67 20	0·4	16·59	30·02	—	—
1809	,,	,, 8	,,	70 18	68 8	1·1	15·09	27·34	20·13	—
1810	,,	,, 9	,,	70 18	68 8	0·2	12·88	23·36	—	—
1811	,,	,, 10	,,	70 17	69 20	3·9	12·95	23·49	—	—
1812	,,	,, 11	,,	70 17	69 20	4·4	13·21	23·96	—	—
1813	,,	,, 12	,,	70 17	69 20	4·3	13·32	24·15	—	—
1814	,,	,, 13	,,	70 17	69 20	4·3	—	—	—	—
1815	,,	,, 14	,,	70 10	69 12	4·6	13·31	24·14	—	—
1816	,,	,, 15	,,	—	—	4·4	13·82	25·05	—	—
1817	,,	,, 16	,,	70 15	69 5	1·1	15·15	27·45	20·06	·00179
1818	,,	,, 17	,,	70 28	68 45	1·7	15·00	27·18	—	—
1819	,,	,, 18	,,	70 28	68 45	1·7	14·14	25·63	—	—
1820	,,	,, 19	,,	70 28	68 45	1·6	15·22	27·57	—	—
1821	,,	,, 20	,,	70 28	68 45	2·2	14·88	26·96	—	·00174
1822	,,	,, 21	,,	70 28	68 45	1·8	14·38	26·07	—	—
1823	,,	,, 22	,,	—	—	2·9	13·52	24·52	—	—
1824	,,	,, 23	,,	—	—	4·4	13·79	25·00	—	—
1825	,,	,, 24	,,	—	—	2·2	14·48	26·25	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1826	Eclipse . .	Aug. 25	noon . .	70° 34′	68° 18′	1·4	15·03	27·24	—	—
1827	,,	,, 26	,,	70 57	70 34	1·7	14·15	25·65	—	—
1828	,,	,, 27	,,	70 57	70 34	0·6	13·22	23·98	—	—
1829	,,	,, 28	,,	70 57	70 34	1·7	13·22	23·98	—	—
1830	,,	,, ?	,,	—	—	3·1	13·51	24·50	17·97	—
1831	,,	,, ?	,,	70 57	70 34	2·5	13·48	24·44	—	—
1832	,,	Sept. 1	,,	70 54	70 0	2·8	2·60	4·74	2·74	—
1833	,,	,, 2	,,	70 40	69 50	2·1	16·45	29·76	21·93	—
1834	,,	,, 3	,,	70 54	70 0	2·9	6·98	12·72	8·66	—
1835	,,									
1836	,,									
1837	,,									
1838	,,	,, 7	,,	70 54	70 0	3·8	10·44	18·97	—	—
1839	,,	,, 8	,,	70 54	70 0	3·2	8·22	14·96	—	—
1840	,,	,, 9	,,	70 54	70 0	2·5	2·74	4·99	2·76	—
1841	,,	,, 10	,,	70 54	70 0	2·2	7·64	13·92	9·73	—
1842	,,	,, 11	,,	71 8	70 25	2·3	16·67	30·16	—	—
1843	,,	,, 12	,,	71 44	73 40	2·2	16·95	30·67	—	—
1844	,,	,, 13	,,	71 44	73 40	1·8	16·62	30·08	—	—
1845	,,									
1846	,,	,, 15	,,	71 50	73 10	1·7	17·68	31·97	—	—
1847	,,	,, 16	,,	72 2	74 40	1·4	17·20	31·12	23·03	—
1848	,,	,, 17	,,	72 2	74 40	1·1	17·20	31·11	—	—
1849	,,	,, 18	,,	72 15	74 35	1·4	17·32	31·32	—	—
1850	,,	,, 19	,,	72 45	76 10	0·8	17·90	32·37	—	—
1851	,,	,, 20	,,	72 35	76 12	1·0	17·37	31·40	23·24	—
1852	,,									
1853	,,	,, 22	,,	72 44	78 10	1·2	16·58	30·01	—	—
1854	,,	,, 23	,,	—	—	—	17·23	31·17	—	—
1855	,,	,, 24	,,	72 10	73 51	1·1	17·51	31·66	—	—
1856	,,	,, 25	,,	71 52	73 14	1·1	17·53	31·70	—	—
1857	,,	,, 26	,,	70 40	69 50	0·8	16·84	30·46	—	—
1858	,,	,, 27	,,	70 40	69 50	1·4	17·08	30·89	—	—
1859	,,	,, 28	,,	70 40	69 50	0·8	17·19	31·08	23·00	—
1860	,,	,, 29	,,	—	—	—	17·10	30·93	—	—
1861	,,	,, 30	,,	70 40	69 50	1·4	17·14	31·00	—	—
1862	,,	Oct. 1	,,	70 54	69 25	0·8	19·51	35·25	—	—
1863	,,	,, 2	,,	70 5	67 12	0·3	17·29	31·27	—	—
1864	,,	,, 3	,,	69 30	66 30	-0·6	17·44	31·54	—	—
1865	,,	,, 4	,,	—	—	0·0	16·37	29·63	—	—
1866	,,	,, 5	,,	69 30	66 30	0·0	17·24	31·18	—	—
1867	,,	,, 6	,,	69 30	66 30	0·3	17·31	31·31	—	—
1868	,,	,,	,,							
1869	,,	,, 8	,,	69 34	66 58	0·0	17·16	31·04	—	—
1870	,,	,, 9	,,	69 43	67 25	0·1	16·82	30·43	—	—
1871	,,	,, 10	,,	70 6	67 0	0·0	17·43	31·52	—	—
1872	,,	,, 11	,,	70 17	67 34	0·3	17·23	31·17	—	—
1873	,,	,, 12	,,	70 47	70 0	-0·1	17·29	31·27	—	—
1874	,,	,, 13	,,	70 47	70 0	-0·2	17·25	31·19	—	—
1875	,,	,, 14	,,	70 47	70 0	-0·3	17·37	31·42	—	—
1876	,,	,, 15	,,	70 49	69 20	0·0	17·43	31·52	—	—
1877	,,	,, 16	,,	70 17	68 10	0·1	17·23	31·17	—	—
1878	,,	,, 17	,,	70 4	67 25	-0·4	17·30	31·29	—	—
1879	,,	,, 18	,,	69 32	67 53	-0·8	17·32	31·32	—	—
1880	,,	,, 19	,,	69 32	67 53	-0·6	17·40	31·46	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1881	Eclipse . .	Oct. 20	noon . .	69° 13′	66° 30′	−0·6	17·56	31·75	—	—
1882	,,	,, 21	,,	67 52	63 40	−1·1	17·69	31·99	—	—
1883	,,	,, 22	,,	66 28	61 53	−1·4	17·97	32·49	—	—
1884	,,	,, 23	,,	66 12	60 30	−0·8	17·95	32·46	—	—
1885	,,	,, 24	,,	64 50	59 39	0·0	18·09	32·70	—	—
1886	,,	,, 25	,,	63 7	57 54	1·7	17·89	32·35	—	—
1887	,,	,, 26	,,	62 2	60 45	2·2	18·23	32·95	—	—
1888	,,	,, 27	,,	61 22	58 53	2·8	18·52	33·48	—	—
1889	,,	,, 28	,,	60 13	54 26	4·4	18·90	34·16	—	—
1890	,,	,, 29	,,	59 50	52 46	4·7	18·66	33·73	—	—
1891	,,	,, 30	,,	59 93	50 22	5·0	18·98	34·30	—	—
1892	,,	Nov. 1	,,	58 41	44 52	5·0	18·94	34·23	—	—
1893	,,	,, 2	,,	58 34	40 45	6·7	19·22	34·72	—	—
1894	,,	,, 3	,,	58 21	35 59	7·5	19·37	35·00	—	—
1895	,,	,, 4	,,	58 2	30 45	8·8	19·46	35·15	—	—
1896	,,	,, 5	,,	58 6	26 27	10·6	18·61	33·64	—	—
1897	,,	,, 6	,,	58 7	22 46	10·6	18·74	33·87	—	—
1898	,,	,, 7	,,	58 21	17 40	9·4	18·48	33·41	—	—
1899	,,	,, 8	,,	58 44	12 7	9·7	19·57	35·34	—	—
1900	,,	,, 9	,,	58 40	6 12	9·9	19·64	35·48	—	—
1901	Corean . .	Oct. 18	,,	Off Kinsale		10·0	19·67	35·54	—	—
1902	,,	,, 20	,,	52° 20′	16° 22′	11·1	19·75	35·68	—	—
1903	,,	,, 21	,,	52 12	22 35	12·2	19·58	35·38	—	—
1904	,,	,, 22	,,	51 52	29 45	12·8	19·62	35·45	26·29	—
1905	,,	,, 23	,,	50 58	37 10	12·2	19·17	34·63	—	—
1906	,,	,, 24	,,	50 3	43 50	14·4	19·23	34·74	—	—
1907	,,	,, 25	,,	48 3	49 50	8·3	17·22	31·14	—	—
1908	,,	,, 27	,,	46 5	55 24	10·0	17·60	31·83	23·57	—
1909	,,	,, 28	,,	44 48	61 51	12·2	17·12	30·97	22·85	—
1910	,,	,, 30	,,	42 0	66 10	11·7	17·79	32·17	—	—
1911	,,	,, 31	,,	39 30	71 15	10·6	18·94	34·23	—	—
1912	,,	Nov. 7	,,	39 28	71 55	10·6	19·14	34·58	—	—
1913	,,	,, 8	,,	41 5	67 12	10·0	18·03	32·59	—	—
1914	,,	,, 9	,,	42 39	61 46	13·3	18·29	33·06	—	—
1915	,,	,, 10	,,	44 6	57 5	12·8	18·48	33·41	—	—
1916	,,	,, 11	,,	25′ S. of St. John's.		7·8	17·42	31·52	—	—
1917	,,	,, 12	,,	48° 3′	51° 36′	6·7	17·53	31·70	—	—
1918	,,	,, 13	,,	50 1	45 55	10·6	19·15	34·60	—	—
1919	,,	,, 14	,,	51 54	39 45	11·7	19·16	34·62	—	—
1920	,,	,, 15	,,	53 23	33 3	10·6	19·22	34·72	—	—
1921	,,	,, 16	,,	54 52	26 10	11·1	19·45	35·14	—	—
1922	,,	,, 17	,,	55 24	18 50	11·1	19·69	35·58	—	—
1923	,,	,, 18	,,	55 24	11 16	10·0	19·67	35·54	—	—
1924	Loughrigg Holme	Oct. 9	,,	50 32	7 57	12·6	19·55	35·32	—	—
1925	,,	,, 9	midnight	51 3	10 48	11·7	19·55	35·32	—	—
1926	,,	,, 10	noon . .	51 33	13 17	13·1	19·66	35·52	—	—
1927	,,	,, 10	midnight	52 5	16 1	12·3	19·55	35·32	26·42	—
1928	,,	,, 11	noon . .	52 37	18 55	13·1	19·64	35·48	—	—
1929	,,	,, 11	midnight	52 52	21 45	12·8	19·47	35·17	—	—
1930	,,	,, 12	noon . .	53 28	25 4	11·8	19·39	35·03	—	—
1931	,,	,, 12	midnight	53 44	28 2	11·1	19·21	34·71	—	—
1932	,,	,, 13	noon . .	53 58	30 28	10·3	19·24	34·76	—	—
1933	,,	,, 13	midnight	54 5	33 50	10·1	19·34	34·94	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
1934	Loughrigg Holme	Oct. 14	noon . .	54° 11′	37° 0′	9·6	19·18	34·65	—	—
1935	,,	,, 14	midnight	53 58	40 7	9·4	19·04	34·40	—	—
1936	,,	,, 15	noon . .	53 45	43 4	10·2	19·04	34·40	—	—
1937	,,	,, 15	midnight	53 27	46 16	9·0	19·07	34·46	—	—
1938	,,	,, 16	noon . .	53 10	49 37	9·4	18·88	34·12	—	—
1939	,,	,, 16	midnight	52 29	52 26	5·6	17·94	32·44	24·01	—
1940	,,	,, 17	noon . .	51 48	55 14	3·9	17·41	31·49	—	—
1941	,,	Nov. 5	,,	46 10	57 18	10·1	17·63	31·88	—	—
1942	,,	,, 5	midnight	46 22	54 31	10·6	17·85	32·28	—	—
1943	,,	,, 6	noon . .	46 48	51 58	8·5	17·37	31·41	23·23	—
1944	,,	,, 6	midnight	47 25	49 30	9·8	17·75	32·09	—	—
1945	,,	,, 7	noon . .	48 1	47 3	7·3	17·38	31·43	—	—
1946	,,	,, 7	midnight	48 30	44 38	9·8	17·32	31·32	—	—
1947	,,	,, 8	noon . .	49 1	42 2	17·8	19·67	35·54	26·44	—
1948	,,	,, 8	midnight	49 19	39 26	15·9	19·52	35·27	—	—
1949	,,	,, 9	noon . .	49 37	36 42	15·4	19·46	35·15	—	—
1950	,,	,, 9	midnight	49 55	34 18	15·6	19·54	35·30	—	—
1951	,,	,, 10	noon . .	50 13	31 35	13·6	19·19	34·67	—	—
1952	,,	,, 10	midnight	50 26	28 51	13·8	19·39	35·03	—	—
1953	,,	,, 11	noon . .	50 38	25 57	12·7	19·36	54·98	—	—
1954										
1955										
1956										
1957										
1958										
1959										
					E.					
1960	California .	May 16	,,	37 16	3 2	17·8	20·46	36·94	—	—
1961	,,	,, 17	,,	37 47	7 29	17·8	20·43	36·89	—	—
1962	,,	,, 18	,,	36 51	12 18	17·2	20·54	37·08	—	—
1963	,,	,, 21	,,	43 8	15 31	16·7	21·27	38·39	28·77	—
					W.					
1964	,,	June 28	,,	37 13	10 29	18·3	20·08	36·27	—	—
1965	,,	,, 28	midnight	37 53	12 46	18·3	20·10	36·30	—	—
1966	,,	,, 29	noon . .	38 28	15 8	18·3	20·04	36·19	—	—
1967	,,	,, 29	midnight	39 6	17 41	18·3	19·97	36·07	26·78	—
1968	,,	,, 30	noon . .	39 45	20 26	18·3	19·99	36·11	—	—
1969	,,	,, 30	midnight	40 18	22 50	18·3	20·02	36·16	—	—
1970	,,	July 1	noon . .	40 50	26 0	18·3	19·97	36·07	—	—
1971	,,	,, 1	midnight	41 10	28 45	17·8	19·94	36·02	—	—
1972	,,	,, 2	noon . .	41 30	31 29	18·3	19·92	35·98	—	—
1973	,,	,, 2	midnight	41 38	34 6	20·6	19·97	36·07	—	—
1974	,,	,, 3	noon . .	41 46	36 48	21·1	19·97	56·07	—	—
1975	,,	,, 3	midnight	41 51	39 25	20·0	20·02	36·16	—	—
1976	,,	,, 4	noon . .	41 57	42 2	20·6	19·89	35·93	—	—
1977	,,	,, 4	midnight	42 5	44 37	20·6	20·07	36·25	—	—
1978	,,	,, 5	noon . .	42 12	47 12	21·1	19·98	36·09	—	—
1979	,,	,, 5	midnight	42 16	49 34	13·3	18·14	32·79	24·28	—
1980	,,	,, 6	noon . .	42 20	51 57	18·9	18·58	33·58	—	—
1981	,,	,, 6	midnight	42 14	54 20	15·0	18·52	33·48	—	—
1982	,,	,, 7	noon . .	42 8	56 43	14·4	18·22	32·93	—	—
1983	,,	,, 7	midnight	41 57	59 7	17·2	18·15	32·80	24·22	-
1984	,,	,, 8	noon . .	41 40	61 31	16·7	17·89	32·35	—	
1985	,,	,, 8	midnight	41 17	63 47	14·4	17·91	32·39	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO$_3$.
		1896.		N.	W.					
1986	California .	July 9	noon . .	40° 53′	66° 4′	16·1	19·61	35·43	—	—
1987	,,	,, 9	midnight	40 41	68 40	16·7	18·19	32·88	—	—
1988	,,	,, 10	noon . .	40 28	71 17	16·7	17·98	32·51	—	—
1989	,,	,, 19	,,	40 17	70 16	18·3	18·09	32·70	24·29	—
1990	,,	,, 19	midnight	40 22	67 48	18·9	19·06	34·44	—	—
1991	,,	,, 20	noon . .	40 27	65 20	21·1	18·88	34·12	—	—
1992	,,	,, 20	midnight	40 38	62 30	24·4	19·68	35·56	26·41	—
1993	,,	,, 21	noon . .	40 50	59 39	26·1	19·45	35·14	—	—
1994	,,	,, 21	midnight	40 56	56 51	25·6	19·32	34·91	—	—
1995	,,	,, 22	noon . .	41 3	54 2	25·6	19·26	34·80	—	—
1996	,,	,, 22	midnight	41 6	51 16	24·4	19·66	35·52	—	—
1997	,,	,, 23	noon . .	41 8	48 50	24·4	19·77	35·72	—	—
1998	,,	,, 23	midnight	41 19	46 26	24·4	19·68	35·56	—	—
1999	,,	,, 24	noon . .	41 29	44 3	24·4	19·75	35·68	—	—
2000	,,	,, 24	midnight	41 37	41 34	23·9	19·82	35·80	—	—
2001	,,	,, 25	noon . .	41 45	39 6	23·3	19·90	35·95	—	—
2002	,,	,, 25	midnight	41 42	36 24	22·8	19·97	36·07	—	—
2003	,,	,, 26	noon . .	41 39	33 43	22·8	19·95	36·04	—	—
2004	,,	,, 26	midnight	41 19	31 4	22·8	19·90	35·95	—	—
2005	,,	,, 27	noon . .	41 0	28 26	22·2	19·88	35·91	—	—
2006	,,	,, 27	midnight	40 32	25 56	22·2	20·03	36·18	—	—
2007	,,	,, 28	noon . .	40 4	23 26	21·7	20·03	36·18	—	—
2008	,,	,, 28	midnight	39 35	20 57	21·1	20·01	36·14	26·89	—
2009	,,	,, 29	noon . .	39 5	18 27	20·6	20·12	36·33	—	—
2010	,,	,, 29	midnight	38 32	16 8	20·6	20·11	36·31	—	—
2011	,,	,, 30	noon . .	38 0	13 50	20·0	20·15	36·39	—	—
2012	,,	,, 30	midnight	37 24	11 43	19·4	20·16	36·40	27·24	—
2013	,,	,, 31	noon . .	36 48	9 36	19·4	20·20	36·48	—	—
2014	,,	,, 31	midnight	36 20	7 15	20·0	20·15	36·39	—	—
2015	,,	Aug. 1	noon . .	36 10	4 44	20·0	20·12	36·33	—	—
2016	,,	,, 1	midnight	36 36	2 4	21·1	20·26	36·58	—	—
2017	,,	,, 2	noon . .	37 36	0 36	22·8	20·40	36·84	—	—
2018	,,	,, 2	midnight	38 37	E. 1 14	22·8	20·38	36·80	—	—
2019	,,	Sept. 3	,,	36 30	W. 7 29	21·1	20·22	36·51	—	—
2020	,,	,, 4	noon . .	37 6	9 46	21·1	20·10	36·30	—	—
2021	,,	,, 4	midnight	37 47	12 1	20·6	20·23	36·53	—	—
2022	,,	,, 5	noon . .	38 28	14 22	20·0	20·17	36·42	—	—
2023	,,	,, 5	midnight	39 5	16 36	20·0	20·11	36·31	27·21	—
2024	,,	,, 6	noon . .	39 43	18 51	20·0	20·19	36·46	—	—
2025	,,	,, 6	midnight	40 13	21 1	20·0	19·97	36·07	—	—
2026	,,	,, 7	noon . .	40 35	22 59	20·0	19·99	36·11	—	—
2027	,,	,, 7	midnight	41 2	25 13	20·0	20·10	36·30	—	—
2028	,,	,, 8	noon . .	41 26	27 34	20·0	20·01	36·14	—	—
2029	,,	,, 8	midnight	41 51	30 3	20·0	19·97	36·07	—	—
2030	,,	,, 9	noon . .	42 17	32 32	20·6	19·95	36·04	26·83	—
2031	,,	,, 9	midnight	42 28	34 30	21·1	19·83	35·82	—	—
2032	,,	,, 10	noon . .	42 39	36 28	21·7	19·84	35·84	—	—
2033	,,	,, 10	midnight	42 49	38 39	21·7	19·90	35·95	—	—
2034	,,	,, 11	noon . .	42 59	40 50	22·2	19·97	36·07	—	—
2035	,,	,, 11	midnight	43 16	43 3	22·8	19·94	36·02	—	—
2036	,,	,, 12	noon . .	43 32	45 16	23·3	19·75	35·68	—	—
2037	,,	,, 12	midnight	43 32	47 47	20·0	18·13	32·77	—	—
2038	,,	,, 13	noon . .	43 32	50 17	16·1	17·76	32·11	23·83	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
2039	California .	Sept. 13	midnight	43° 16′	53° 3′	20·0	17·99	32·53	—	—
2040	,,	,, 14	noon . .	43 1	55 50	21·1	17·97	32·49	—	—
2041	,,	,, 14	midnight	42 40	58 23	20·0	18·55	33·53	—	—
2042	,,	,, 15	noon . .	42 19	60 57	19·4	17·70	32·01	—	—
2043	,,	,, 15	midnight	41 50	63 30	17·8	17·54	31·71	—	—
2044	,,	,, 16	noon . .	41 22	66 6	17·8	17·65	31·92	23·64	—
2045	,,	,, 16	midnight	40 58	68 41	15·6	18·32	33·11	—	—
2046	,,	,, 17	noon . .	40 34	71 16	15·6	18·03	32·59	—	—
2047	,,	,, 26	midnight	40 24	72 37	18·9	17·86	32·30	—	—
2048	,,	,, 27	noon . .	40 23	70 12	16·7	17·87	32·31	—	—
2049	,,	,, 27	midnight	40 33	67 51	16·7	18·01	32·56	—	—
2050	,,	,, 28	noon . .	40 43	65 29	21·1	19·55	35·32	—	—
2051	,,	,, 28	midnight	40 54	63 9	22·2	19·48	35·19	—	—
2052	,,	,, 29	noon . .	41 6	60 49	22·2	19·91	35·97	—	—
2053	,,	,, 29	midnight	41 27	58 24	22·2	20·04	36·19	—	—
2054	,,	,, 30	noon . .	41 48	56 0	22.2	19·20	34·69	25·96	—
2055	,,	,, 30	midnight	42 6	53 39	21·7	19·67	35·54	—	—
2056	,,	Oct. 1	noon . .	42 24	51 18	20·0	17·92	32·40	24·00	—
2057	,,	,, 1	midnight	42 34	48 56	20·0	19·17	34·63	—	—
2058	,,	,, 2	noon . .	42 43	46 34	20·0	19·76	35·70	—	—
2059	,,	,, 2	midnight	42 45	44 10	23·3	19·86	35·88	—	—
2060	,,	,, 3	noon . .	42 47	41 47	23·3	19·87	35·89	—	—
2061	,,	,, 3	midnight	42 36	39 17	22·2	19·84	35·84	—	—
2062	,,	,, 4	noon . .	42 25	36 45	22·2	19·91	35·97	—	—
2063	,,	,, 4	midnight	42 10	34 11	22·2	19·92	35·98	—	—
2064	,,	,, 5	noon . .	41 56	31 36	22·2	19·91	35·97	—	—
2065	,,	,, 5	midnight	41 31	29 4	21·1	19·93	36·00	—	—
2066	,,	,, 6	noon . .	41 6	26 32	20·6	19·94	36·02	26·82	—
2067	,,	,, 6	midnight	40 41	24 2	21·1	19·94	36·02	—	—
2068	,,	,, 7	noon . .	40 16	21 33	21·1	20·14	36·37	—	—
2069	,,	,, 7	midnight	39 40	19 11	21·1	20·27	36·60	—	—
2070	,,	,, 8	noon . .	39 3	16 50	21·1	20·15	36·39	—	—
2071	,,	,, 8	midnight	38 23	14 33	20·6	20·26	36·58	27·26	—
2072	,,	,, 9	noon . .	37 44	12 16	20·6	20·14	36·37	—	—
2073	,,	,, 9	midnight	37 7	10 1	20·6	20·44	36·91	—	—
2074	,,	,, 10	noon . .	36 31	7 46	20·0	20·18	36·44	—	—
2075	,,	,, 10	midnight	36 0	5 50	20·0	20·12	36·33	—	—
2076	Anchoria .	Nov. 7	noon . .	55 18	14 9	10·3	19·64	35·48	—	—
2077	,,	,, 8	,,	54 39	23 5	11·4	19·70	35·59	—	—
2078	,,	,, 9	,,	53 20	31 29	11·4	19·17	34·63	—	—
2079	,,	,, 10	,,	51 47	38 46	13·9	19·42	35·08	—	—
2080	,,	,, 11	,,	49 47	44 27	14·4	19·19	34·67	25·90	—
2081	,,	,, 12	,,	47 22	50 46	8·6	17·55	31·73	—	—
2082	,,	,, 13	,,	44 41	56 28	11·9	18·21	32·91	—	—
2083	,,	,, 14	,,	42 13	63 27	11·7	18·25	32·98	24·50	—
2084	,,	,, 15	,,	40 36	69 40	11·1	17·92	32·40	—	—
2085	,,	,, 16	,,	40 33	71 3	12·2	18·12	32·75	—	—
2086	,,	,, 17	,,	40 33	71 3	12·2	18·02	32·58	—	—
2087	,,	,, 18	,,	40 34	72 34	12·2	18·02	32·58	—	—
2088	,,	,, 24	,,	40 48	68 40	11·7	17·93	32·42	—	—
2089	,,	,, 25	,,	42 8	62 11	11·7	19·17	34·63	—	—
2090	,,	,, 26	,,	44 20	56 31	10·3	18·43	33·32	—	—
2091	,,	,, 27	,,	47 6	51 11	0·0	17·63	31·88	23·57	—
2092	,,	,, 28	,,	49 16	45 16	5·8	18·78	33·94	—	—
2093	,,	,, 29	,,	51 11	39 17	10·8	19·22	34·72	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
2094	Anchoria .	Nov. 30	noon . .	52° 57′	32° 20′	9·7	19·23	34·74	—	—
2095	,,	Dec. 1	,,	54 2	25 34	12·2	19·34	34·94	25·91	—
2096	,,	,, 2	,,	55 1	17 30	10·3	19·54	35·30	—	—
2097	,,	,, 3	,,	55 22	9 24	9·7	19·56	35·34	—	—
2098	Teutonic. .	Nov. 19	midnight	51 24	13 37	11·1	19·60	35·41	—	—
2099	,,	,, 20	noon . .	51 18	18 59	12·2	19·63	35·47	—	—
2100	,,	,, 20	midnight	51 6	24 26	15·0	19·68	35·56	—	—
2101	,,	,, 21	noon . .	50 30	30 9	12·2	19·42	35·08	26·25	—
2102	,,	,, 21	midnight	49 26	35 41	10·0	19·52	35·27	—	—
2103	,,	,, 22	noon . .	48 19	41 30	11·1	19·94	36·02	—	—
2104	,,	,, 22	midnight	46 50	46 38	11·1	18·59	33·60	—	—
2105	,,	,, 23	noon . .	45 24	50 58	6·1	17·82	32·22	23·93	—
2106	,,	,, 23	midnight	44 10	56 2	5·6	18·16	32·82	—	—
2107	,,	,, 24	noon . .	42 39	61 17	12·2	18·57	33·56	—	—
2108	,,	,, 24	midnight	41 25	66 24	10·6	18·08	32·68	—	—
2109	,,	,, 25	noon . .	40 28	71 20	10·6	17·98	32·51	24·13	—
2110	,,	Dec. 2	midnight	40 10	70 7	8·9	18·09	32·70	—	—
2111	,,	,, 3	noon . .	41 1	65 43	8·9	18·03	32·59	—	—
2112	,,	,, 3	midnight	42 3	60 8	13·9	19·48	35·19	—	—
2113	,,	,, 4	noon . .	42 55	56 37	7·2	18·05	32·63	—	—
2114	,,	,, 4	midnight	44 11	51 56	6·1	18·69	33·78	—	—
2115	,,	,, 5	noon . .	45 48	47 6	4·4	18·67	33·75	—	—
2116	,,	,, 5	midnight	47 19	42 11	7·8	18·78	33·94	—	—
2117	,,	,, 6	noon . .	48 45	36 59	12·8	19·64	35·48	—	—
2118	,,	,, 6	midnight	49 39	31 37	12·2	19·46	35·15	—	—
2119	,,	,, 7	noon . .	50 22	26 15	11·1	19·57	35·36	—	—
2120	,,	,, 7	midnight	50 49	20 36	10·0	19·59	35·40	—	—
2121	,,	,, 8	noon . .	51 16	14 43	11·7	19·55	35·32	—	—
2122	Laura. . .	Nov. 18	,,	59 54	3 1	9·5	19·30	34·87	—	—
2123	,,	,, 18	8 P.M. .	60 40	4 38	8·5	19·32	34·91	—	—
2124	,,	,, 19	4 A.M. .	61 27	6 16	8·2	19·32	34·91	—	—
2125	,,	,, 24	8 P.M. .	62 28	8 12	7·5	19·29	34·86	—	—
2126	,,	,, 25	4 A.M. .	62 31	10 40	7·5	19·40	35·05	—	—
2127	,,	,, 25	noon . .	63 1	13 0	7·4	19·39	35·03	—	—
2128	,,	,, 25	8 P.M. .	63 8	15 50	7·2	19·29	34·86	—	—
2129	,,	,, 26	4 A.M. .	63 9	18 46	7·2	19·32	34·91	25·99	—
2130	,,	,, 26	noon . .	63 20	20 16	6·5	19·04	34·40	—	—
2131	,,	,, 26	8 P.M. .	63 48	22 50	7·2	19·33	34·93	—	—
2132	,,	,, 27	4 A.M. .	64 14	22 14	5·5	19·10	34·51	—	—
2133	,,	Dec. 3	noon . .	63 48	22 48	5·7	18·85	34·07	25·20	—
2134	,,	,, 3	8 P.M. .	62 58	21 0	7·0	19·41	35·07	—	—
2135	,,	,, 4	4 A.M. .	62 47	18 15	7·2	19·42	35·08	—	—
2136	,,	,, 4	noon . .	62 39	15 57	7·2	19·41	35·07	—	—
2137	,,	,, 4	8 P.M. .	62 32	13 8	7·5	19·49	35·21	—	—
2138	,,	,, 5	4 A.M. .	62 25	10 17	7·0	19·48	35·19	—	—
2139	,,	,, 5	noon . .	62 20	7 54	7·5	19·47	35·17	—	—
2140	,,	,, 9	,,	61 20	6 20	8·0	19·38	35·01	—	—
2141	,,	,, 9	8 P.M. .	60 28	4 27	9·0	19·58	35·38	26·18	—
2142	,,	,, 10	4 A.M. .	59 58	2 50	8·5	19·46	35·15	—	—
2143	Longhirst .	Nov. 19	noon . .	51 20	7 11	11·1	19·99	35·93	—	—
2144	,,	,, 20	,,	50 29	11 20	11·7	19·96	36·05	—	—
2145	,,	,, 21	,,	48 47	15 18	10·6	19·85	35·86	—	—
2146	,,	,, 22	,,	48 40	20 0	12·8	19·84	35·84	26·75	—
2147	,,	,, 23	,,	48 29	25 1	13·9	19·72	35·63	—	—
2148	,,	,, 24	,,	48 36	30 0	14·4	19·77	35·72	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
2149	Longhirst .	Nov. 25	noon . .	48° 34′	34° 52′	14·4	19·73	35·64	—	—
2150	,,	,, 26	,,	48 24	39 15	10·0	19·81	35·79	—	—
2151	,,	,, 27	,,	47 23	42 0	11·1	19·93	36·00	—	—
2152	,,	,, 28	,,	46 13	46 25	7·2	18·90	34·16	—	—
2153	,,	,, 29	,,	45 41	51 45	7·8	18·03	32·59	—	—
2154	,,	,, 30	,,	45 5	55 53	7·8	17·98	32·51	24·02	—
2155	,,	Dec. 1	,,	44 28	61 35	6·7	17·23	31·16	23·06	—
2156	,,	,, 2	,,	41 54	65 45	7·8	17·72	32·04	—	—
2157	,,	,, 3	,,	39 55	70 37	11·7	18·52	33·48	—	—
2158	,,	,, 11	,,	37 48	70 20	20·6	19·93	36·00	—	—
2159	,,	,, 12	,,	38 35	66 54	11·7	18·69	33·78	—	—
2160	,,	,, 13	,,	39 36	62 58	21·7	20·16	36·40	26·99	—
2161	,,	,, 14	,,	40 15	58 20	21·1	20·00	36·12	—	—
2162	,,	,, 15	,,	41 38	54 13	15·6	19·59	35·40	—	—
2163	,,	,, 16	,,	43 3	50 34	2·8	18·33	33·13	—	—
2164	,,	,, 17	,,	45 4	47 5	10·0	18·70	33·80	—	—
2165	,,	,, 18	,,	46 54	43 26	7·2	19·19	34·67	—	—
2166	,,	,, 19	,,	48 21	40 7	12·8	19·75	35·68	—	—
2167	,,	,, 20	,,	49 14	38 15	12·8	19·70	35·59	—	—
2168	,,	,, 21	,,	50 32	34 14	11·7	19·78	35·73	—	—
2169	,,	,, 22	,,	52 15	28 51	11·1	19·53	35·28	—	—
2170	,,	,, 23	,,	53 27	23 38	11·1	19·47	35·17	—	—
2171	,,	,, 24	,,	54 27	18 2	11·1	19·61	35·43	—	—
2172	,,	,, 25	,,	54 44	13 17	10·0	19·63	35·47	—	—
2173	California .	Nov. 20	,,	36 31	8 28	15·6	20·20	36·48	—	—
2174	,,	,, 20	midnight	37 20	10 55	15·6	20·14	36·37	—	—
2175	,,	,, 21	noon . .	38 9	13 22	15·6	20·22	36·51	—	—
2176	,,	,, 21	midnight	38 53	15 57	15·0	20·22	36·51	—	—
2177	,,	,, 22	noon . .	39 37	18 32	15·0	20·05	36·21	—	—
2178	,,	,, 22	midnight	40 19	21 8	15·6	20·13	36·35	—	—
2179	,,	,, 23	noon . .	41 2	23 45	15·6	20·01	36·14	—	—
2180	,,	,, 23	midnight	41 27	26 26	15·6	19·98	36·09	—	—
2181	,,	,, 24	noon . .	41 52	29 7	15·6	19·89	35·93	26·80	—
2182	,,	,, 24	midnight	42 13	31 46	15·6	19·89	35·93	—	—
2183	,,	,, 25	noon . .	42 18	32 39	15·6	19·95	36·04	—	—
2184	,,	,, 25	midnight	42 4	33 31	15·6	19·95	36·04	—	—
2185	,,	,, 27	noon . .	41 50	34 27	15·6	20·01	36·14	—	—
2186	,,	,, 27	midnight	41 48	36 14	16·1	19·99	36·11	—	—
2187	,,	,, 28	noon . .	41 45	38 1	16·1	19·97	36·07	—	—
2188	,,	,, 28	midnight	41 54	*40 25*	16·1	20·01	36·14	—	—
2189	,,	,, 29	noon . .	42 4	42 49	16·1	20·11	36·31	—	—
2190	,,	,, 29	midnight	42 33	44 52	16·1	20·15	36·39	27·05	—
2191	,,	,, 30	noon . .	42 53	46 52	16·1	19·47	35·17	—	—
2192	,,	,, 30	midnight	43 3	49 36	8·9	18·55	33·53	—	—
2193	,,	Dec. 1	noon . .	43 13	52 20	5·6	18·94	34·23	—	—
2194	,,	,, 1	midnight	42 59	54 53	12·2	19·05	34·42	—	—
2195	,,	,, 2	noon . .	42 44	57 26	12·2	18·65	33·71	24·98	—
2196	,,	,, 2	midnight	42 26	59 58	12·2	19·72	35·63	—	—
2197	,,	,, 3	noon . .	42 8	62 30	13·3	20·02	36·16	—	—
2198	,,	,, 3	midnight	41 40	65 13	8·9	18·16	32·82	—	—
2199	,,	,, 4	noon . .	41 4	68 6	7·8	18·01	32·56	24·07	—
2200	,,	,, 4	midnight	40 34	70 53	7·8	17·90	32·37	—	—
2201	,,	,, 5	noon . .	40 28	73 40	8·9	17·75	32·09	23·77	—
2202	Corean . .	Nov. 29	,,	off Queenstown		10·0	19·82	35·80	—	—
2203	,,	,, 30	,,	51° 48′	13° 51′	10·0	19·75	35·68	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO$_3$.
		1896.		N.	W.					
2204	Corean . .	Dec. 1	noon . .	51° 53′	20° 4′	12·2	19·68	35·56	—	—
2205	,,	,, 2	,,	51 42	26 16	14·4	19·61	35·43	—	—
2206	,,	,, 3	,,	51 32	31 6	10·0	19·60	35·41	—	—
2207	,,	,, 4	,,	51 18	35 8	11·1	19·68	35·56	—	—
2208	,,	,, 5	,,	50 50	39 18	8·3	19·26	34·80	—	—
2209	,,	,, 6	,,	50 15	41 45	12·2	19·90	35·95	26·91	—
2210	,,	,, 7	,,	49 56	43 28	7·2	19·30	34·87	—	—
2211	,,	,, 8	,,	48 15	48 56	0·6	18·73	33·85	—	—
2212	,,	,, 11	,,	46 0	56 10	3·3	17·77	32·13	—	—
2213	,,	,, 12	,,	44 33	62 0	4·4	17·30	31·28	23·03	—
2214	,,	,, 13	,,	43 48	64 4	5·0	17·18	31·07	—	—
2215	,,	,, 14	,,	40 33	68 15	8·3	18·02	32·58	—	—
2216	,,	,, 15	,,	39 4	73 15	10·6	18·58	33·58	—	—
2217	,,	,, 16	,,	38 56	73 27	11·1	18·76	33·90	—	—
2218	,,	,, 23	,,	39 30	71 12	8·3	18·49	33·43	—	—
2219	,,	,, 24	,,	39 58	66 54	12·2	19·49	35·21	—	—
2220	,,	,, 26	,,	43 57	57 52	3·3	17·85	32·28	—	—
2221	,,	,, 27	,,	off C.	Race	0·6	17·73	32·06	23·90	—
2222	,,	,, 28	,,	48° 7′	50° 50′	−2·2	18·07	32·66	—	—
2223	,,	,, 29	,,	50 4	45 24	10·0	19·61	35·43	—	—
2224	,,	,, 30	,,	51 50	39 6	5·6	19·27	34·82	—	—
2225	,,	,, 31	,,	53 25	32 10	5·6	19·42	35·08	26·03	—
		1897.								
2226	,,	Jan. 1	,,	54 58	25 22	9·4	19·58	35·38	—	—
2227	,,	,, 2	,,	56 0	19 35	8·9	19·56	35·34	—	—
2228	,,	,, 3	,,	55 45	12 20	8·9	19·68	35·56	—	—
		1896.								
2229	California .	Dec. 17	,,	40 5	71 36	7·8	18·08	32·68	—	—
2230	,,	,, 17	midnight	40 23	69 52	7·8	18·00	32·54	—	—
2231	,,	,, 18	noon . .	40 42	68 9	7·8	17·97	32·49	—	—
2232	,,	,, 18	midnight	41 2	65 54	7·8	18·33	33·13	—	—
2233	,,	,, 19	noon . .	41 23	63 40	10·0	18·41	33·28	—	—
2234	,,	,, 19	midnight	41 46	61 0	15·6	19·96	36·05	26·99	—
2235	,,	,, 20	noon . .	42 10	58 20	15·6	19·82	35·80	—	—
2236	,,	,, 20	midnight	42 36	55 37	13·9	19·25	34·78	—	—
2237	,,	,, 21	noon . .	42 52	53 5	13·3	18·91	34·17	—	—
2238	,,	,, 21	midnight	43 1	50 25	3·3	18·19	32·88	—	—
2239	,,	,, 22	noon . .	43 10	47 45	8·9	18·48	33·41	—	—
2240	,,	,, 22	midnight	43 10	45 1	14·4	18·45	33·39	24·77	—
2241	,,	,, 23	noon . .	43 10	42 17	14·4	19·68	35·56	26·53	—
2242	,,	,, 23	midnight	43 6	39 53	14·4	19·94	36·02	26·81	—
2243	,,	,, 24	noon . .	43 2	37 29	14·4	19·99	36·11	—	—
2244	,,	,, 24	midnight	42 45	34 50	15·6	19·84	35·84	—	—
2245	,,	,, 25	noon . .	42 28	32 11	15·6	19·51	35·25	—	—
2246	,,	,, 25	midnight	42 1	29 32	15·6	19·93	36·00	26·91	—
2247	,,	,, 26	noon . .	41 35	26 53	15·0	19·85	35·86	—	—
2248	,,	,, 26	midnight	41 4	24 11	15·0	19·79	35·75	—	—
2249	,,	,, 27	noon . .	40 33	21 29	15·0	19·89	35·93	—	—
2250	,,	,, 27	midnight	39 48	18 56	15·0	19·92	35·98	—	—
2251	,,	,, 28	noon . .	39 3	16 24	15·0	19·98	36·09	—	—
2252	,,	,, 28	midnight	38 18	14 2	15·0	19·99	36·11	—	—
2253	,,	,, 29	noon . .	37 33	11 40	15·0	20·11	36·31	—	—
2254	,,	,, 29	midnight	36 50	9 14	15·0	19·98	36·09	—	—
2255	,,	,, 30	noon . .	36 8	6 50	15·0	20·01	36·14	—	—
2256	,,	,, 30	midnight	off Tarifa, St. Gibraltar			20·01	36·14	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
2257	Teutonic. .	Dec. 17	midnight	51° 22′	13° 27′	10·0	19·69	35·58	—	—
2258	,,	,, 18	noon . .	51 26	19 23	11·7	19·68	35·56	—	—
2259	,,	,, 18	midnight	50 52	25 42	12·2	19·61	35·43	—	—
2260	,,	,, 19	noon . .	50 21	31 38	12·2	19·59	35·40	—	—
2261	,,	,, 19	midnight	49 6	36 57	11·7	19·73	35·64	—	—
2262	,,	,, 20	noon . .	48 2	42 1	12·8	19·86	35·88	26·77	—
2263	,,	,, 20	midnight	46 37	47 16	4·4	18·70	33·80	—	—
2264	,,	,, 21	noon . .	45 17	52 8	5·0	17·12	30·97	22·85	—
2265	,,	,, 21	midnight	43 56	57 10	6·1	18·11	32·73	—	—
2266	,,	,, 22	noon . .	42 29	62 40	5·6	17·80	32·19	—	—
2267	,,	,, 22	midnight	41 10	67 35	6·7	18·02	32·58	—	—
2268	,,	,, 23	noon . .	40 29	73 7	5·0	17·77	32·13	23·70	—
2269	,,	,, 30	midnight	40 9	70 7	8·3	18·04	32·61	—	—
2270	,,	,, 31	noon . .	41 8	65 8	10·0	18·61	33·64	—	—
2271	,,	,, 31	midnight	42 17	60 22	11·7	19·64	35·48	—	—
		1897.								
2272	,,	Jan. 1	noon . .	43 29	55 40	6·7	18·46	33·37	24·67	—
2273	,,	,, 1	midnight	44 52	50 53	3·3	17·95	32·46	—	—
2274	,,	,, 2	noon . .	46 6	46 16	-0·6	18·39	33·24	—	—
2275	,,	,, 2	midnight	47 29	41 16	6·7	18·83	34·03	—	—
2276	,,	,, 3	noon . .	48 44	36 35	12·2	19·74	35·66	26·49	—
2277	,,	,, 3	midnight	49 43	31 24	10·0	19·44	35·12	—	—
2278	,,	,, 4	noon . .	50 46	25 24	11·1	19·63	35·47	—	—
2279	,,	,, 4	midnight	51 6	19 28	11·1	19·66	35·52	—	—
2280	,,	,, 5	noon . .	51 35	13 45	11·1	19·73	35·64	—	—
		1896.			E.					
2281	Monarch. .	June 20	noon . .	53 16	1 0	14·4	19·01	34·35	—	—
					W.					
2282	,,	,, 21	,,	56 49	1 39	13·3	19·06	34·44	—	—
2283	,,	,, 22	,,	Sinclair	B. Wick	10·6	19·22	34·72	—	—
2284	,,	,, 23	,,	58° 36′	2° 43½′	11·1	19·38	35·01	—	—
2285	,,	,, 24	,,	Sinclair Bay		11·1	19·36	34·98	—	—
2286	,,	,, 26	,,	56° 5¾′	5° 41¾′	12·2	19·11	34·53	25·75	—
2287	,,	,, 27	,,	52 10	5 25	13·3	19·45	35·14	—	—
2288	,,	,, 28	,,	Morte B., Bristol Ch.		16·1	19·52	35·27	—	—
2289	,,	,, 29	,,	,,	,,	16·7	19·48	35·19	—	—
2290	,,	,, 30	,,	S. E. Bay, Lundy Id.		15·6	19·68	35·56	—	—
2291	,,	July 1	,,	Croyde	Bay	15·6	19·75	35·68	—	—
2292	,,	,, 2	,,	,,	,,	15·6	19·44	35·12	—	—
2293	,,	,, 3	,,	,,	,,	16·1	19·46	35·15	—	—
2294	,,	,, 4	,,	Ilfracombe		16·1	19·30	34·87	—	—
2295	,,	,, 5	,,	,,		16·1	12·50	35·23	—	—
2296	,,	,, 7	,,	Croyde	Bay	16·7	19·34	34·94	26·04	—
2297	,,	,, 8	,,	Lundy	Id.	16·1	19·67	35·54	—	—
2298	,,	,, 9	,,	50° 30′	2° 15′	16·7	19·66	35·52	—	—
					E.					
2299	,,	Sept. 13	,,	50 48	0 38	17·2	19·66	35·52	—	—
					W.					
2300	,,	,, 14	,,	50 29	2 24	17·2	19·62	35·45	—	—
2301	,,	,, 15	,,	51 21	5 19	16·1	19·75	35·68	—	—
2302	,,	,, 16	,,	Milford	Haven	15·6	19·20	34·69	25·81	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
2303	Monarch .	Sept. 30	noon .	51° 49′	5° 22′	15·0	19·56	35·34	—	—
2304	,,	Oct. 1	,,	51 50	5 15	13·9	19·65	35·50	26·55	—
2305	,,	,, 3	,,	51 48	5 24	14·4	19·64	35·48	—	—
2306	,,	,, 4	,,	55 40	5 59	12·8	19·13	34·56	—	—
2307	,,	,, 5	,,	Staosunaig		11·7	19·24	34·76	—	—
2308	,,	,, 6	,,	off Colonsay E. side.		11·7	19·21	34·71	—	—
2309	,,	,, 7	,,	N. end	Iona Sd.	12·2	19·25	34·78	—	—
2310	,,	,, 8	,,	Tobermory Hrb.		12·2	18·86	34·08	—	—
2311	Para . . .	Sept. 10	,,	48° 12′	6° 8′	17·2	19·83	35·82	—	—
2312	,,	,, 11	,,	45 31	11 29	17·8	20·00	36·12	—	—
2313	,,	,, 12	,,	40 9	22 40	22·2	20·21	36·49	27·33	—
2314	,,	Oct. 24	,,	38 39	33 10	19·7	20·06	36·23	27·00	·00237
2315	,,	,, 25	,,	41 20	28 4	18·9	20·07	36·25	—	—
2316	,,	,, 26	,,	43 57	21 20	17·2	20·06	36·23	—	—
2317	,,	,, 27	,,	46 32	14 20	13·3	20·01	36·14	—	—
2318	,,	Nov. 19	,,	48 19	6 16	11·1	19·80	35·77	—	·00235
2319	,,	,, 20	,,	45 13	12 56	12·8	20·00	36·12	—	—
2320	,,	,, 21	,,	41 39	19 34	16·7	20·23	36·53	—	—
2321	,,	,, 22	,,	38 8	25 23	18·3	20·25	36·57	—	—
		1897.								
2322	,,	Jan. 2	,,	39 5	32 24	17·2	20·28	36·62	27·23	—
2323	,,	,, 3	,,	42 14	26 18	13·9	20·01	36·14	—	—
2324	,,	,, 4	,,	44 42	19 10	12·2	19·89	35·93	—	—
2325	,,	,, 5	,,	47 37	11 30	11·7	19·85	35·86	—	—
2326	,,	,, 6	,,	Dodman Pt. N.W. 10′		10·6	19·84	35·84	—	·00235
					W.					
2327	,,	,, 1	,,	35° 20′	38° 2′	17·5	20·21	36·49	—	—
		1896.								
2328	,,	Dec. 31	,,	31 35	42 43	20·0	20·29	36·64	—	—
2329	,,	Nov. 23	,,	34 52	30 39	19·4	20·14	36·37	—	—
2330	,,	Oct. 23	,,	35 0	36 20	20·0	20·03	36·18	26·89	—
2331	,,	,, 22	,,	31 34	40 18	23·1	20·35	36·75	—	—
2332	,,	Sept. 23	,,	11 32	69 18	29·4	19·64	35·48	—	—
2333	,,	,, 22	,,	14 21	63 36	29·4	19·26	34·80	—	—
2334	,,	,, 20	,,	15 18	57 5	28·9	19·70	35·59	—	—
2335	,,	,, 19	,,	18 38	52 41	27·2	19·24	34·76	—	—
2336	,,	,, 18	,,	22 1	47 58	26·7	20·50	37·01	—	—
2337	,,	,, 17	,,	25 41	43 10	26·1	20·34	37·08	—	·00242
2338	,,	,, 16	,,	29 24	38 34	25·6	20·58	37·15	27·74	·00243
2339	,,	,, 15	,,	33 11	33 15	24·4	20·30	36·66	—	—
2340	,,	,, 14	,,	37 2	28 8	23·3	20·33	36·71	—	—
2341	Anchoria .	Dec. 12	,,	55 27	10 30	9·27	19·60	35·41	—	—
2342	,,	,, 13	,,	55 7	16 20	10·0	19·57	35·36	—	—
2343	,,	,, 14	,,	54 36	23 17	11·7	19·40	35·05	—	—
2344	,,	,, 15	,,	53 46	29 30	8·3	19·28	34·84	26·06	—
2345	,,	,, 16	,,	52 32	35 44	7·8	19·21	34·71	—	—
2346	,,	,, 17	,,	50 51	41 32	7·5	19·08	34·47	—	—
2347	,,	,, 18	,,	48 40	47 30	4·4	19·91	35·97	—	—
2348	,,	,, 19	,,	46 22	52 28	1·7	17·75	32·09	23·80	—
2349	,,	,, 20	,,	44 27	57 10	4·4	18·82	34·01	—	—
2350	,,	,, 21	,,	42 47	61 24	6·1	18·20	32·90	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1896.		N.	W.					
2351	Anchoria .	Dec. 22	noon . .	40° 48′	67° 42′	7·2	17·96	32·48	24·14	—
2352	,,	,, 27	,,	40 49	68 47	6·7	18·03	32·59	—	—
2353	,,	,, 28	,,	41 55	63 14	7·8	18·56	33·55	—	—
2354	,,	,, 29	,,	43 53	57 56	5·6	18·07	32·66	—	—
2355	,,	,, 30	,,	46 17	52 49	0·6	17·78	32·15	—	—
2356	,,	,, 31	,,	48 43	47 17	2·5	18·96	34·26	25·42	—
		1897.								
2357	,,	Jan. 1	,,	50 47	40 59	12·8	19·75	35·68	—	—
2358	,,	,, 2	,,	51 47	34 55	8·9	19·35	34·96	—	—
2359	,,	,, 3	,,	53 26	28 10	7·8	19·30	34·87	—	—
2360	,,	,, 4	,,	54 16	21 4	10·3	19·56	35·34	26·40	—
2361	,,	,, 5	,,	54 54	14 5	10·0	19·65	35·50	—	—
2362	,,	,, 6	,,	55 19	9 3	9·4	19·60	35·41	—	—
		1896.								
2363	Loughrigg Holme	Nov. 11	midnight	50 41	22 55	13·5	19·68	35·56	—	—
2364	,,	,, 12	noon . .	50 45	20 3	12·4	19·57	35·37	—	—
2365	,,	,, 12	midnight	50 43	18 4	11·7	19·64	35·48	—	—
2366	,,	,, 13	noon . .	50 41	14 5	11·9	19·64	35·48	—	—
2367	,,	,, 13	midnight	50 24	11 8	12·1	19·68	35·56	—	—
2368	,,	,, 14	noon . .	50 7	8 14	11·1	19·61	35·43	—	—
2369	,,	Dec. 3	,,	49 31	6 7	11·4	19·67	35·54	—	—
2370	,,	,, 3	midnight	48 6	6 52	11·8	19·74	35·66	—	—
2371	,,	,, 4	noon . .	47 38	7 5	11·9	19·79	35·75	—	—
2372	,,	,, 4	midnight	47 41	7 10	11·7	19·76	35·70	—	—
2373	,,	,, 5	noon . .	46 16	7 53	11·9	19·69	35·58	—	—
2374	,,	,, 5	midnight	45 24	7 27	12·2	19·75	35·68	—	—
2375	,,	,, 6	noon . .	45 18	8 42	12·2	19·76	35·70	—	—
2376	,,	,, 6	midnight	43 48	9 39	12·8	19·79	35·75	—	—
2377	,,	,, 7	noon . .	42 3	9 39	13·4	19·88	35·91	—	—
2378	,,	,, 7	midnight	40 18	9 39	14·6	19·95	36·04	26·94	—
2379	,,	,, 8	noon . .	39 26	9 40	14·8	19·94	36·02	—	—
		1897.								
2380	,,	Jan. 1	,,	40 20	9 32	13·2	19·55	35·32	—	—
2381	,,	,, 1	midnight	42 13	9 23	12·8	19·64	35·48	—	—
2382	,,	,, 2	noon . .	43 59	9 8	12·8	19·82	35·80	26·73	—
2383	,,	,, 2	midnight	45 50	8 26	11·4	19·68	35·56	—	—
2384	,,	,, 3	noon . .	47 41	7 45	11·7	19·82	35·80	26·77	—
2385	,,	,, 3	midnight	49 29	6 50	10·4	19·68	35·56	—	—
2386	,,	,, 4	noon . .	51 19	5 56	9·4	19·68	35·56	26·47	—
2387$_{(1)}$	Teutonic .	,, 14	midnight	51 22	13 5	10·0	19·71	35·61	—	—
2387$_{(2)}$	,,	,, 15	noon . .	51 24	20 30	11·1	—	—	—	—
2387$_{(3)}$	,,	,, 15	midnight	50 37	26 37	10·0	—	—	—	—
2388	,,	,, 16	noon . .	50 8	32 43	10·0	19·65	35·50	—	—
2389	,,	,, 16	midnight	48 52	38 52	13·3	19·88	35·91	26·79	—
2390	,,	,, 17	noon . .	47 44	43 20	7·8	18·02	32·58	24·12	—
2391$_{(1)}$	,,	,, 17	midnight	46 19	48 42	–0·6	16·50	29·87	—	—
2391$_{(2)}$	,,	,, 18	noon . .	44 35	54 2	5·6	—	—	—	—
2391$_{(3)}$	,,	,, 18	midnight	43 17	59 14	8·3	—	—	—	—
2392	,,	,, 19	noon . .	41 50	63 42	0·0	18·22	32·93	—	—
2393	,,	,, 19	midnight	41 1	67 24	–2·2	18·24	32·97	—	—
2394	,,	,, 26	,,	40 8	69 40	5·0	18·20	32·90	24·34	—

Table I (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
2395	Teutonic . .	Jan. 27	noon . .	40° 30′	64° 43′	15·6	19·98	36·09	—	—
2396	,,	,, 27	midnight	40 54	59 44	14·4	19·96	36·05	—	—
2397	,,	,, 28	noon . .	41 17	54 46	15·0	19·95	36·04	—	—
2398	,,	,, 28	midnight	41 56	50 33	15·0	19·43	35·10	26·03	—
2399	,,	,, 29	noon . .	43 12	45 52	15·6	20·09	36·29	—	—
2400	,,	,, 29	midnight	44 57	40 59	15·0	19·71	35·61	—	—
2401	,,	,, 30	noon . .	46 45	36 37	13·9	19·70	35·59	—	—
2402	,,	,, 30	midnight	47 53	31 21	11·7	19·77	35·72	—	—
2403	,,	,, 31	noon . .	49 22	25 56	12·2	19·65	35·50	—	—
2404	,,	,, 31	midnight	50 16	20 18	11·1	19·65	35·50	—	—
2405	,,	Feb. 1	noon . .	50 54	14 11	10·6	19·71	35·61	—	—
2406	Laura . .	Jan. 22	8 P.M. .	60 25	4 5	7·0	19·60	35·41	26·36	—
2407	,,	,, 23	4 A.M. .	60 45	4 54	7·0	19·45	35·14	—	—
2408	,,	,, 23	noon . .	61 30	6 42	6·6	19·45	35·14	—	—
2409	,,	,, 26	,,	62 7	6 30	5·7	19·40	35·05	—	—
2410	,,	,, 29	,,	62 29	7 58	7·0	19·51	35·25	—	—
2411	,,	,, 29	8 P.M. .	62 47	10 49	7·0	19·52	35·27	—	—
2412	,,	,, 30	4 A.M. .	63 0	13 38	7·0	19·49	35·21	—	—
2413	,,	,, 30	noon . .	63 9	16 58	7·3	19·53	35·28	—	—
2414	,, .	,, 30	8 P.M. .	63 26	20 5	5·5	19·29	34·86	—	—
2415	,,	,, 31	4 A.M. .	63 44	21 48	5·0	19·44	35·12	—	—
2416	,,	Feb. 6	noon . .	64 5	22 57	4·0	19·27	34·82	25·80	—
2417	,,	,, 6	8 P.M. .	63 8	21 35	6·2	19·50	35·23	—	—
2418	,,	,, 7	4 A.M. .	62 41	19 4	6·5	19·46	35·15	—	—
2419	,,	,, 7	noon . .	62 42	16 9	6·5	19·46	35·15	—	—
2420	,,	,, 7	8 P.M. .	62 35	13 7	7·2	19·45	35·14	—	—
2421	,,	,, 8	4 A.M. .	62 30	10 2	7·2	19·55	35·32	26·22	—
2422	,,	,, 8	noon . .	62 23	7 10	5·5	19·49	35·21	—	—
2423	,,	,, 9	,,	62 8	6 30	5·4	19·45	35·14	26·16	—
2424	,,	,, 10	4 A.M. .	61 57	6 36	5·6	19·43	35·10	—	—
2425	,,	,, 10	noon . .	61 27	5 36	6·0	19·45	35·12	—	—
2426	,,	,, 10	8 P.M. .	60 28	3 40	6·5	19·54	35·30	—	—
2427	Anchoria .	Jan. 16	noon . .	54 43	15 25	9·4	19·57	35·36	—	—
2428	,,	,, 17	,,	53 9	22 56	10·6	19·55	35·33	26·40	—
2429	,,	,, 18	,,	52 0	28 23	9·4	19·55	35·32	—	—
2430	,,	,, 19	,,	50 53	31 56	10·8	19·65	35·51	26·44	—
2431	,,	,, 20	,,	48 33	38 41	12·2	19·67	35·54	—	—
2432	,,	,, 21	,,	45 59	43 32	3·9	18·81	34·00	—	—
2433	,,	,, 22	,,	43 48	48 14	6·7	18·82	34·01	—	—
2434	,,	,, 23	,,	43 12	54 37	6·7	18·63	33·67	—	—
2435	,,	,, 24	,,	42 32	60 5	3·9	18·34	33·15	—	—
2436	,,	,, 25	,,	41 21	64 56	2·2	18·02	32·58	—	—
2437	,,	,, 26	,,	40 49	67 17	4·4	18·22	32·93	—	—
2438	,,	,, 27	,,	40 31	71 19	4·7	18·02	32·58	—	—
2439	,,	Feb. 1	,,	40 20	67 25	5·6	18·28	33·04	—	—
2440	,,	,, 2	,,	40 59	61 1	15·6	19·95	36·04	26·74	—
2441	,,	,, 3	,,	41 35	54 31	13·9	19·90	35·95	—	—
2442	,,	,, 4	,,	42 46	48 36	8·9	19·10	34·51	—	—
2443	,,	,, 5	,,	45 3	44 23	11·7	19·44	35·12	—	—
2444	,,	,, 6	,,	46 49	40 43	8·3	19·07	34·46	25·62	—
2445	,,	,, 7	,,	47 36	39 3	12·8	19·83	35·82	—	—
2446	,,	,, 8	,,	47 54	38 14	12·8	19·76	35·70	—	—
2447	,,	,, 9	,,	47 9	37 48	13·3	19·75	35·68	—	—
2448	,,	,, 10	,,	49 11	36 0	12·2	19·68	35·56	—	—
2449	,,	,, 11	,,	50 56	31 1	10·3	19·47	35·17	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
2450	Anchoria. .	Feb. 12	noon . .	52° 32′	25° 10′	11·4	19·57	35·36	—	—
2451	,,	,, 13	,,	53 47	18 58	10·6	19·63	35·47	—	—
2452	,,	,, 14	,,	54 52	12 3	9·4	19·68	35·56	—	—
2453	Laura. . .	Mar. 8	4 A.M. .	59 53	3 23	7·2	19·47	35·17	—	—
2454	,,	,, 8	noon . .	60 27	4 19	7·5	19·35	34·96	—	—
2455	,,	,, 8	8 P.M. .	61 22	6 12	6·5	19·36	34·98	—	—
2456	,,	,, 9	4 A.M. .	61 35	6 44	6·2	19·45	35·14	—	—
2457	,,	,, 10	,,	61 38	6 40	5·5	19·32	34·91	—	—
2458	,,	,, 13	noon . .	62 8	6 30	5·8	19·37	35·00	—	—
2459	,,	,, 14	4 A.M. .	62 30	6 49	5·5	19·11	34·53	—	—
2460	,,	,, 14	noon . .	62 32	8 10	7·0	19·47	35·17	—	—
2461	,,	,, 14	8 P.M. .	62 39	11 8	7·0	19·48	35·19	—	—
2462	,,	,, 15	4 A.M. .	62 52	14 6	7·0	19·49	35·21	—	—
2463	,,	,, 15	noon . .	62 58	17 39	7·5	19·48	35·19	—	—
2464	,,	,, 15	8 P.M. .	63 19	20 10	6·0	19·43	35·10	26·27	—
2465	,,	,, 16	4 A.M. .	63 41	22 40	5·0	19·34	34·94	—	—
2466	,,	,, 20	8 P.M. .	63 38	22 23	5·5	19·45	35·14	—	—
2467	,,	,, 21	4 A.M. .	63 32	21 22	5·7	19·48	35·19	—	—
2468	,,	,, 21	noon . .	63 29	20 24	5·7	19·43	35·10	—	—
2469	,,	,, 22	4 A.M. .	63 10	19 50	5·5	19·50	35·23	—	—
2470	,,	,, 22	noon . .	62 56	18 4	7·0	19·48	35·19	—	—
2471	,,	,, 22	8 P.M. .	62 48	15 38	7·7	19·56	35·34	—	—
2472	,,	,, 23	4 A.M. .	62 38	13 2	7·5	19·49	35·21	—	—
2473	,,	,, 23	noon . .	62 39	10 22	7·7	19·46	35·15	—	—
2474	,,	,, 23	8 P.M. .	62 28	7 38	7·2	19·48	35·19	—	—
2475	,,	,, 26	noon . .	61 28	6 28	6·5	19·44	35·12	—	—
2476	,,	,, 26	8 P.M. .	61 0	5 17	7·0	19·44	35·12	26·30	—
2477	,,	,, 27	4 A.M. .	60 28	4 20	7·2	19·50	35·23	—	—
2478	,,	,, 27	noon . .	59 31	3 5	7·5	19·45	35·14	—	—
2479	Teutonic. .	,, 11	midnight	51 5	14 20	7·2	19·76	35·70	—	—
2480	,,	,, 12	noon . .	50 39	20 42	9·4	19·76	35·70	—	—
2481	,,	,, 12	midnight	49 28	26 40	10·0	19·69	35·58	—	—
2482	,,	,, 13	noon . .	48 33	32 4	8·9	19·89	35·93	—	—
2483	,,	,, 13	midnight	47 5	37 4	8·9	19·94	36·02	27·02	—
2484	,,	,, 14	noon . .	45 31	41 29	13·3	19·81	35·79	—	—
2485	,,	,, 14	midnight	44 16	45 19	4·4	19·70	35·59	—	—
2486	,,	,, 15	noon . .	42 30	49 31	9·4	19·29	34·86	—	—
2487	,,	,, 15	midnight	41 57	53 50	5·6	19·40	35·05	—	—
2488	,,	,, 16	noon . .	41 40	56 27	9·4	19·70	35·59	—	—
2489	,,	,, 16	midnight	41 21	60 25	12·2	20·14	36·37	27·26	—
2490	,,	,, 17	noon . .	40 56	64 21	7·8	19·87	35·89	—	—
2491	,,	,, 17	midnight	40 41	68 43	3·3	18·23	32·95	24·50	—
2492	,,	,, 24	,,	40 12	70 32	5·6	18·03	32·59	24·30	—
2493	,,	,, 25	noon . .	40 34	65 13	13·3	19·57	35·36	—	—
2494	,,	,, 25	midnight	40 54	60 17	13·3	19·80	35·77	—	—
2495	,,	,, 26	noon . .	41 12	55 23	13·3	19·80	35·77	—	—
2496	,,	,, 26	midnight	41 51	50 33	12·2	19·69	35·58	—	—
2497	,,	,, 27	noon . .	43 21	46 0	16·7	20·06	36·23	27·06	—
2498	,,	,, 27	midnight	45 3	41 18	13·3	19·63	35·47	—	—
2499	,,	,, 28	noon . .	46 51	36 15	12·2	19·73	35·64	—	—
2500	,,	,, 28	midnight	48 8	30 49	11·1	19·73	35·64	—	—
3501	,,	,, 29	noon . .	49 21	25 42	11·1	19·70	35·59	—	—
2502	,,	,, 29	midnight	50 7	20 7	11·1	19·66	35·52	—	—
2503	,,	,, 30	noon . .	50 54	14 44	11·7	19·68	35·56	—	—
2504	Corean . .	,, 10	,,	51 41	7 54	7·8	19·67	35·54	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
2505	Corean . .	Mar. 11	noon . .	50° 58′	14° 6′	9·4	19·82	35·80	—	—
2506	,,	,, 12	,,	50 14	20 11	10·6	19·78	35·73	—	—
2507	,,	,, 13	,,	49 30	25 58	10·0	19·89	35·93	—	—
2508	,,	,, 14	,,	49 3	28 20	10·6	19·70	35·59	—	—
2509	,,	,, 15	,,	48 14	31 52	12·2	19·74	35·66	—	—
2510	,,	,, 16	,,	47 27	35 17	10·0	19·76	35·70	—	—
2511	,,	,, 17	,,	47 37	35 45	8·9	19·55	35·32	—	—
2512	,,	,, 18	,,	45 53	39 37	11·7	19·93	36·00	—	—
2513	,,	,, 19	,,	43 36	44 37	15·0	20·36	36·77	27·45	—
2514	,,	,, 20	,,	42 36	49 0	5·6	19·16	34·62	—	—
2515	,,	,, 21	,,	42 49	55 36	3·9	18·73	33·85	—	—
2516	,,	,, 22	,,	43 47	62 6	0·6	17·72	32·04	—	—
2517	,,	,, 24	,,	43 54	63 55	0·0	17·50	31·65	23·34	—
2518	,,	,, 25	,,	40 36	67 22	2·8	18·21	32·91	—	—
2519	,,	,, 26	,,	39 49	69 36	6·1	18·81	34·00	—	—
2520	,,	,, 27	,,	off Delaware R.		5·0	16·51	29·87	21·97	—
2521	,,	Apr. 2	,,	39° 6′	70° 18′	11·1	19·61	35·43	—	—
2522	,,	,, 3	,,	39 58	64 38	14·4	20·21	36·49	—	—
2523	,,	,, 4	,,	40 43	59 35	17·2	20·18	36·44	27·10	—
2524	,,	,, 5	,,	41 19	53 41	13·9	19·89	35·93	—	—
2525	,,	,, 6	,,	42 13	47 49	10·0	19·03	34·39	—	—
2526	,,	,, 7	,,	44 58	42 17	16·1	20·15	36·39	27·06	—
2527	,,	,, 8	,,	47 44	37 12	12·8	19·89	35·93	—	—
2528	,,	,, 9	,,	50 12	31 32	8·9	19·63	35·47	—	—
2529	,,	,, 10	,,	52 25	25 40	9·4	19·62	35·45	—	—
2530	,,	,, 11	,,	54 4	18 47	9·4	19·67	35·54	—	—
2531	,,	,, 12	,,	55 14	11 17	10·0	19·75	35·68	—	—
2532	Loughrigg Holme	Jan. 20	,,	51 20	5 54	9·1	19·77	35·72	—	—
2533	,,	,, 20	midnight	49 34	6 36	8·8	19·49	35·21	—	—
2534	,,	,, 21	noon . .	47 50	7 21	10·4	19·68	35·56	—	—
2535	,,	,, 21	midnight	46 11	8 3	10·8	19·65	35·50	—	—
2536	,,	,, 22	noon . .	44 25	8 28	12·1	19·81	35·79	—	—
2537	,,	,, 22	midnight	42 40	9 29	11·7	19·45	35·14	26·73	—
2538	,,	,, 23	noon . .	40 50	9 30	13·3	19·84	35·84	—	—
2539	,,	,, 23	8 P.M. .	39 39	9 33	12·9	19·90	35·95	27·21	—
2540	,,	Mar. 7	noon . .	41 19	9 49	13·8	19·87	35·89	27·27	—
2541	,,	,, 7	midnight	42 50	9 39	12·8	19·82	35·80	—	—
2542	,,	,, 8	noon . .	44 17	9 11	11·9	19·75	35·68	—	—
2543	,,	,, 8	midnight	45 45	8 33	10·8	19·72	35·63	—	—
2544	,,	,, 9	noon . .	47 17	8 1	11·3	19·77	35·72	—	—
2545	,,	,, 9	midnight	48 45	7 23	10·0	19·70	35·59	—	—
2546	,,	,, 10	noon . .	50 8	6 50	9·1	19·64	35·48	—	—
2547	,,	,, 10	midnight	51 33	5 55	7·8	19·57	35·36	—	—
2548	,,	,, 22	,,	48 40	5 48	10·3	19·76	35·70	—	—
2549	,,	,, 23	noon . .	47 0	5 0	11·4	19·69	35·58	—	—
2550	,,	,, 23	midnight	45 12	4 0	12·3	19·75	35·68	—	—
2551	,,	,, 24	noon . .	43 28	2 59	13·2	19·45	35·14	26·52	—
2552	,,	Apr. 6	midnight	44 15	3 33	12·1	19·77	35·72	—	—
2553	,,	,, 7	noon . .	45 24	4 15	11·7	19·75	35·68	—	—
2554	,,	,, 7	midnight	46 21	4 46	11·3	19·77	35·72	—	—
2555	,,	,, 8	noon . .	46 54	5 8	11·3	19·77	35·72	—	—
2556	,,	,, 8	midnight	48 22	5 59	11·1	19·75	35·68	—	—
2557	Anchoria .	Mar. 20	noon . .	54 50	12 41	11·1	17·59	31·81	—	—
2558	,,	,, 21	,,	53 46	19 44	10·6	19·63	35·47	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
2559	Anchoria .	Mar. 22	noon . .	52° 30′	26° 26′	9·7	17·54	31·71	—	—
2560	,,	,, 23	,,	50 31	32 47	10·0	19·59	35·40	—	—
2561	,,	,, 24	,,	48 13	39 11	11·1	19·60	35·41	—	—
2562	,,	,, 25	,,	45 19	45 2	13·3	19·68	35·56	—	—
2563	,,	,, 26	,,	42 25	50 20	-1·1	18·08	32·68	—	—
2564	,,	,, 27	,,	42 53	57 14	4·4	18·82	34·01	—	—
2565	,,	,, 28	,,	41 53	63 36	1·7	18·14	32·79	24·25	—
2566	,,	,, 29	,,	40 35	70 8	4·4	18·12	32·75	—	—
2567	,,	Apr. 4	,,	40 27	68 31	5·0	18·16	32·82	—	—
2568	,,	,, 5	,,	40 41	62 7	20·0	19·96	36·05	—	—
2569	,,	,, 6	,,	41 18	55 41	14·4	19·93	36·00	26·78	—
2570	,,	,, 7	,,	41 50	48 52	17·2	20·19	36·46	27·07	—
2571	,,	,, 8	,,	44 45	43 8	14·4	19·90	35·95	—	—
2572	,,	,, 9	,,	47 34	37 26	13·3	19·80	35·77	—	—
2573	,,	,, 10	,,	49 56	30 45	10·3	19·63	35·47	—	—
2574	,,	,, 11	,,	52 4	24 5	8·9	19·62	35·45	—	—
2575	,,	,, 12	,,	54 44	16 43	10·6	19·75	35·68	—	—
2576	,,	,, 13	,,	55 6	9 29	8·9	19·69	35·58	—	—
2577	Teutonic. .	,, 8	midnight	51 1	14 54	11·1	19·81	35·79	—	—
2578	,,	,, 9	noon . .	50 36	20 46	10·0	19·83	35·82	—	—
2579	,,	,, 9	midnight	49 33	26 39	12·2	19·81	35·79	—	—
2580	,,	,, 10	noon . .	48 36	31 55	7·8	19·75	35·68	—	—
2581	,,	,, 10	midnight	47 30	35 40	6·7	19·78	35·73	—	—
2582	,,	,, 11	noon . .	46 7	40 11	14·4	19·95	36·04	—	—
2583	,,	,, 11	midnight	44 33	43 59	13·9	19·67	35·54	—	—
2584	,,	,, 12	noon . .	42 59	48 24	14·4	19·72	35·63	—	—
2585	,,	,, 12	midnight	42 23	52 33	1·7	18·47	33·39	—	—
2586	,,	,, 13	noon . .	41 49	58 15	5·0	19·47	35·17	—	—
2587	,,	,, 13	midnight	41 16	63 56	10·6	20·11	36·31	—	—
2588	,,	,, 14	noon . .	40 40	69 19	10·0	18·27	33·02	24·56	—
2589	,,	,, 21	midnight	40 10	69 39	11·1	18·06	32·64	—	—
2590	,,	,, 22	noon . .	40 10	64 50	10·0	19·62	35·45	—	—
2591	,,	,, 22	midnight	40 10	59 56	14·4	20·04	36·19	—	—
2592	,,	,, 23	noon . .	40 10	54 53	14·4	20·13	36·35	—	—
2593	,,	,, 23	midnight	40 25	48 52	15·6	20·26	36·58	—	—
2594	,,	,, 24	noon . .	41 58	45 33	13·9	20·20	36·47	27·32	—
2595	,,	,, 24	midnight	44 9	40 44	11·1	20·05	36·21	—	—
2596	,,	,, 25	noon . .	45 55	36 2	9·4	19·83	35·82	—	—
2597	,,	,, 25	midnight	47 24	31 7	10·6	19·83	35·82	—	—
2598	,,	,, 26	noon . .	48 46	25 56	10·0	19·79	35·75	—	—
2599	,,	,, 26	midnight	50 0	20 51	12·2	19·90	35·95	—	—
2600	,,	,, 27	noon . .	50 51	15 29	11·7	19·84	35·84	—	—
2601	California .	Feb. 21	,,	36 31	7 52	13·9	20·11	36·31	—	—
2602	,,	,, 21	midnight	37 16	10 16	13·9	20·05	36·21	—	—
2603	,,	,, 22	noon . .	38 1	12 41	13·3	20·02	36·16	—	—
2604	,,	,, 22	midnight	38 44	15 14	13·3	19·98	36·09	—	—
2605	,,	,, 23	noon . .	39 28	17 47	13·3	19·93	36·00	—	—
2606	,,	,, 23	midnight	40 4	20 10	13·3	19·97	36·07	—	—
2607	,,	,, 24	noon . .	40 41	22 47	13·3	19·90	35·95	—	—
2608	,,	,, 24	midnight	41 13	25 14	13·3	19·22	35·98	—	—
2609	,,	,, 25	noon . .	41 44	27 41	13·3	19·87	35·89	—	—
2610	,,	,, 25	midnight	41 58	29 28	13·3	19·93	36·00	—	—
2611	,,	,, 26	noon . .	42 12	31 15	13·3	19·94	36·02	—	—
2612	,,	,, 26	midnight	42 28	33 18	13·9	19·99	36·11	—	—
2613	,,	,, 27	noon . .	42 44	35 22	14·4	19·85	35·86	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3
		1897.		N.	W.					
2614	California .	Feb. 27	midnight	43° 2′	37° 38′	14·4	19·88	35·91	—	—
2615	,,	,, 28	noon . .	43 20	39 54	14·4	19·86	35·88	—	—
2616	,,	,, 28	midnight	43 16	42 3	13·3	19·84	35·84	—	—
2617	,,	Mar. 1	noon . .	43 11	44 12	13·3	20·12	36·33	27·15	—
2618	,,	,, 1	midnight	43 12	44 52	13·3	19·97	36·07	—	—
2619	,,	,, 2	noon . .	43 14	46 20	13·3	19·85	35·86	—	—
2620	,,	,, 2	midnight	43 22	48 24	6·7	18·88	34·12	—	—
2621	,,	,, 3	noon . .	43 30	50 29	0·0	17·96	32·48	—	—
2622	,,	,, 3	midnight	43 12	53 3	1·11	18·29	33·06	—	—
2623	,,	,, 4	noon . .	42 54	55 40	5·6	18·44	33·34	—	—
2624	,,	,, 4	midnight	42 38	56 40	2·8	18·38	33·22	24·72	—
2625	,,	,, 5	noon . .	42 17	57 41	7·8	19·41	35·07	—	—
2626	,,	,, 5	midnight	42 10	59 30	11·1	19·46	35·15	—	—
2627	,,	,, 6	noon . .	42 1	61 42	11·1	18·57	33·56	—	—
2628	,,	,, 6	midnight	41 33	63 45	3·3	17·88	32·33	23·97	—
2629	,,	,, 7	noon . .	41 11	66 0	4·4	19·62	35·45	—	—
2630	,,	,, 7	midnight	40 50	68 30	4·4	18·09	32·70	—	—
2631	,,	,, 17	,,	40 27	71 4	4·4	18·13	32·77	—	—
2632	,,	,, 18	noon . .	40 27	68 41	4·4	18·09	32·70	24·40	—
2633	,,	,, 18	midnight	40 45	66 20	5·6	18·71	33·82	25·16	—
2634	,,	,, 19	noon . .	41 5	63 58	13·3	19·56	35·34	—	—
2635	,,	,, 19	midnight	41 17	61 43	13·3	20·02	36·16	—	—
2636	,,	,, 20	noon . .	41 30	59 28	14·4	20·15	36·39	27·19	—
2637	,,	,, 20	midnight	41 39	57 45	9·4	19·42	35·08	—	—
2638	,,	,, 21	noon . .	41 48	56 1	11·1	19·65	35·50	—	—
2639	,,	,, 21	midnight	42 0	53 49	8·9	19·49	35·21	—	—
2640	,,	,, 22	noon . .	42 13	51 37	4·4	19·68	35·56	—	—
2641	,,	,, 22	midnight	42 21	49 19	5·6	18·85	34·07	25·39	—
2642	,,	,, 23	noon . .	42 29	47 1	13·3	19·82	35·80	—	—
2643	,,	,, 23	midnight	42 29	45 1	15·6	20·09	36·29	—	—
2644	,,	,, 24	noon . .	42 30	42 31	15·6	19·97	36·07	—	—
2645	,,	,, 24	midnight	42 27	40 0	13·3	19·99	36·11	—	—
2646	,,	,, 25	noon . .	42 24	37 29	14·4	20·07	36·25	—	—
2647	,,	,, 25	midnight	42 24	35 4	14·4	20·04	36·19	—	—
2648	,,	,, 26	noon . .	42 29	32 39	14·4	20·03	36·18	—	—
2649	,,	,, 26	midnight	42 19	30 22	13·9	19·94	36·02	—	—
2650	,,	,, 27	noon . .	42 8	27 54	14·4	19·90	35·95	—	—
2651	,,	,, 27	midnight	41 34	25 23	14·4	19·99	36·11	—	—
2652	,,	,, 28	noon . .	41 1	22 51	14·4	19·95	36·04	—	—
2653	,,	,, 28	midnight	40 10	20 31	14·4	19·50	35·23	—	—
2654	,,	,, 29	noon . .	39 38	18 12	15·0	20·10	36·30	—	—
2655	,,	,, 29	midnight	38 54	15 56	14·4	20·05	36·21	—	—
2656	,,	,, 30	noon . .	38 10	13 41	14·4	20·11	36·31	—	—
2657	,,	,, 30	midnight	37 32	11 33	14·4	20·20	36·48	27·10	—
2658	,,	,, 31	noon . .	36 56	9 25	14·4	20·01	36·14	—	—
2659	,,	,, 31	midnight	36 18	7 0	14·4	20·16	36·40	—	—
2660	Laura . . .	Apr. 26	noon . .	59 56	3 18	8·0	19·56	35·34	—	—
2661	,,	,, 26	8 P.M. .	60 50	5 8	7·2	19·52	35·27	—	—
2662	,,	,, 27	4 A.M. .	61 55	6 32	7·0	19·43	35·10	—	—
2663	,,	,, 27	noon . .	62 1	7 40	7·3	19·45	35·14	—	—
2664	,,	,, 27	8 P.M. .	62 20	10 22	8·0	19·50	35·23	—	—
2665	,,	,, 28	4 A.M. .	62 22	13 14	8·2	19·49	35·21	—	—
2666	,,	,, 28	noon . .	62 28	15 46	8·5	19·50	35·23	—	—
2667	,,	,, 28	8 P.M. .	62 55	17 54	7·3	19·51	35·25	—	—
2668		,, 29	4 A.M. .	63 10	19 14	7·3	19·51	35·25	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
2669	Laura . .	Apr. 29	noon . .	63° 30′	21° 41′	7·3	18·96	34·26	—	—
2670	,,	May 5	8 P.M. .	64 55	24 16	4·5	19·30	34·87	—	—
2671	,,	,, 7	4 A.M. .	65 24	23 0	4·0	19·24	34·76	—	—
2672	,,	,, 7	noon . .	65 26	24 30	4·5	19·30	34·87	—	—
2673	,,	,, 7	8 P.M. .	65 57	23 55	3·5	19·04	34·40	—	—
2674	,,	,, 8	noon . .	65 59	23 59	3·0	18·91	34·17	25·54	—
2675	,,	,, 9	,,	66 15	23 27	2·8	19·15	34·60	—	—
2676	,,	,, 10	,,	65 39	24 36	2·9	19·15	34·60	—	—
2677	,,	,, 10	8 P.M. .	65 0	24 29	4·9	19·42	35·08	—	—
2678	,,	,, 11	4 A.M. .	64 19	22 24	5·3	19·09	34·49	25·81	—
2679	,,	,, 13	,,	64 7	23 0	5·2	19·27	34·82	—	—
2680	,,	,, 13	noon . .	63 30	21 19	6·5	19·44	35·12	—	—
2681	,,	,, 13	8 P.M. .	63 0	18 25	7·5	19·51	35·25	—	—
2682	,,	,, 14	4 A.M. .	62 49	15 34	8·0	19·50	35·23	26·31	—
2683	,,	,, 14	noon . .	62 40	12 27	8·2	19·51	35·25	—	—
2684	,,	,, 14	8 P.M. .	62 30	9 43	8·2	19·49	35·21	—	—
2685	,,	,, 15	4 A.M. .	62 25	7 15	8·2	19·52	35·27	—	—
2686	,,	,, 16	8 P.M. .	60 57	5 17	8·5	19·58	35·38	26·47	—
2687	,,	,, 17	4 A.M. .	59 57	3 18	9·3	—	—	—	—
2688	Anchoria .	Apr. 18	noon . .	55 12	10 9	9·4	19·71	35·61	—	—
2689	,,	,, 19	,,	54 2	17 33	10·0	19·70	35·59	—	—
2690	,,	,, 20	,,	52 33	24 33	10·0	19·61	35·43	—	—
2691	,,	,, 21	,,	50 22	31 18	10·0	19·65	35·50	—	—
2692	,,	,, 22	,,	47 47	37 22	9·7	19·64	35·48	—	—
2693	,,	,, 23	,,	44 36	42 40	14·4	20·18	36·44	—	—
2694	,,	,, 24	,,	42 33	48 13	8·9	19·15	34·60	—	—
2695	,,	,, 25	,,	42 0	54 55	11·7	19·26	34·80	—	—
2696	,,	,, 26	,,	41 39	61 33	7·2	17·85	32·26	—	—
2697	,,	,, 27	,,	40 49	67 14	6·1	17·90	32·37	24·04	—
2698	,,	,, 28	,,	40 27	73 30	6·7	17·45	31·56	23·42	—
2699	,,	May 9	,,	40 28	70 23	7·8	18·06	32·64	—	—
2700	,,	,, 10	,,	40 30	64 30	18·9	19·88	35·91	—	—
2701	,,	,, 11	,,	40 43	58 42	17·2	19·68	35·56	—	—
2702	,,	,, 12	,,	40 33	52 45	20·6	20·08	36·27	26·94	—
2703	,,	,, 13	,,	41 28	46 42	18·3	20·07	36·25	27·05	—
2704	,,	,, 14	,,	43 32	41 27	16·1	19·90	35·95	—	—
2705	,,	,, 15	,,	46 19	36 50	15·0	19·99	36·11	—	—
2706	,,	,, 16	,,	48 48	31 29	11·1	19·65	35·50	—	—
2707	,,	,, 17	,,	50 52	25 55	12·2	19·69	35·58	—	—
2708	,,	,, 18	,,	52 39	20 3	11·7	19·63	35·47	—	—
2709	,,	,, 19	,,	54 12	13 50	11·1	19·67	35·54	—	—
2710	Teutonic . .	,, 6	midnight	50 11	14 18	11·7	19·83	35·82	—	—
2711	,,	,, 7	noon . .	50 11	20 40	12·8	19·63	35·47	—	—
2712	,,	,, 7	midnight	48 59	26 37	13·3	19·73	35·64	—	—
2713	,,	,, 8	noon . .	47 35	32 25	16·1	19·81	35·79	—	—
2714	,,	,, 8	midnight	45 49	37 39	16·7	19·98	36·09	—	—
2715	,,	,, 9	noon . .	44 11	42 41	16·7	20·12	36·33	—	—
2716	,,	,, 9	midnight	42 9	46 59	11·1	20·12	36·33	—	—
2717	,,	,, 10	noon . .	41 0	51 48	10·6	19·92	35·98	—	—
2718	,,	,, 10	midnight	40 47	57 4	15·0	20·21	36·49	—	—
2719	,,	,, 11	noon . .	40 49	62 1	18·3	19·64	35·48	—	—
2720	,,	,, 11	midnight	40 41	67 8	8·9	17·77	32·13	23·91	—
2721	,,	,, 12	noon . .	40 32	72 5	10·6	17·82	32·22	—	—
2722	,,	,, 19	midnight	40 8	69 41	10·0	17·88	32·33	23·97	—
2723	,,	,, 20	noon . .	40 12	64 40	16·1	19·81	35·79	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
2724	Teutonic .	May 20	midnight	40° 12′	59° 44′	19·4	19·20	34·69	—	—
2725	,,	,, 21	noon . .	40 12	54 54	20·0	19·58	35·38	—	—
2726	,,	,, 21	midnight	40 26	50 1	20·0	19·52	35·27	—	—
2727	,,	,, 22	noon . .	42 13	45 27	20·6	20·12	36·33	—	—
2728	,,	,, 22	midnight	44 8	40 49	17·8	19·95	36·04	—	—
2729	,,	,, 23	noon . .	45 55	35 55	18·3	19·92	35·98	—	—
2730	,,	,, 23	midnight	48 25	30 46	12·8	19·79	35·75	—	—
2731	,,	,, 24	noon . .	48 53	25 18	13·9	19·80	35·77	—	—
2732	,,	,, 24	midnight	50 8	19 31	10·6	19·72	35·63	—	—
2733	,,	,, 25	noon . .	50 59	13 36	12·2	19·76	35·70	—	—
2734	Loughrigg Holme	,, 31	noon . .	47 43	46 12	5·6	18·57	33·56	25·03	—
2735	,,	,, 31	midnight	48 3	44 15	7·1	19·02	34·37	25·65	—
2736	,,	June 1	noon . .	48 24	42 5	13·2	19·62	35·45	—	—
2737	,,	,, 1	midnight	48 49	39 22	13·4	19·56	35·34	—	—
2738	,,	,, 2	noon . .	49 14	36 37	13·3	19·75	35·68	26·63	—
2739	,,	,, 2	midnight	49 36	33 58	12·8	19·75	35·68	26·63	—
2740	,,	,, 3	noon . .	49 52	31 18	12·6	19·68	35·56	—	—
2741	,,	,, 3	midnight	50 7	28 56	11·7	19·63	35·47	—	—
2742	,,	,, 4	noon . .	50 16	26 36	11·7	19·60	35·41	—	—
2743	,,	,, 4	midnight	50 22	25 18	11·7	19·63	35·47	—	—
2744	,,	,, 5	noon . .	50 27	22 54	12·5	19·67	35·54	—	—
2745	,,	,, 5	midnight	50 30	20 28	12·8	19·69	35·58	26·60	—
2746	,,	,, 6	noon . .	50 33	18 3	12·5	19·63	35·47	—	—
2747	,,	,, 6	midnight	50 22	16 2	13·1	19·61	35·43	—	—
2748	,,	,, 7	noon . .	50 9	13 45	13·1	19·64	35·48	—	—
2749	,,	,, 7	midnight	50 4	10 34	13·9	19·66	35·52	—	—
2750	,,	,, 8	noon . .	49 59	7 24	15·1	19·62	35·45	—	—
2751	,,	Apr. 25	midnight	55 24	7 50	8·1	19·35	34·96	—	—
2752	,,	,, 26	noon . .	55 29	11 2	10·2	19·65	35·50	—	—
2753	,,	,, 26	midnight	55 19	14 13	10·3	19·62	35·45	—	—
2754	,,	,, 27	noon . .	55 9	17 22	10·4	19·63	35·47	26·46	—
2755	,,	,, 27	midnight	54 54	20 30	9·9	19·64	35·48	—	—
2756	,,	,, 28	noon . .	54 40	23 11	10·1	19·57	35·36	—	—
2757	,,	,, 28	midnight	54 17	25 55	9·8	19·54	35·30	—	—
2758	,,	,, 29	noon . .	53 56	28 17	9·3	19·49	35·21	—	—
2759	,,	,, 29	midnight	53 23	30 47	8·3	19·46	35·15	—	—
2760	,,	,, 30	noon . .	52 48	33 17	7·9	19·36	34·98	—	—
2761	,,	,, 30	midnight	52 2	36 1	8·2	19·36	34·98	—	—
2762	,,	May 1	noon . .	51 17	38 46	9·3	19·50	35·23	—	—
2763	,,	,, 1	midnight	50 26	41 34	10·1	19·59	35·40	—	—
2764	,,	,, 2	noon . .	49 38	44 10	6·8	19·11	34·53	—	—
2765	,,	,, 2	midnight	48 56	46 16	4·7	19·16	34·62	—	—
2766	,,	,, 3	noon . .	48 21	47 32	0·3	17·81	32·21	23·95	—
2767	,,	,, 3	midnight	48 0	47 55	0·0	17·98	32·51	—	—
2768	,,	,, 4	noon . .	47 34	49 28	1·1	17·98	32·51	—	—
2769	,,	,, 4	midnight	46 57	51 17	0·0	18·17	32·84	—	—
2770	,,	,, 5	noon . .	46 20	53 15	1·8	18·01	32·56	—	—
2771	,,	,, 6	2 A.M. .	46 17	56 14	2·2	17·87	32·31	—	—
2772	,,	,, 6	noon . .	46 53	58 4	2·2	17·30	31·29	—	—
2773	,,	,, 27	,,	46 43	57 27	4·9	17·06	30·86	—	—
2774	,,	,, 27	midnight	46 17	55 24	4·3	17·83	32·24	—	—
2775	,,	,, 28	noon . .	46 17	53 25	4·6	17·98	32·51	—	—
2776	,,	,, 28	midnight	46 22	52 24	4·5	17·99	32·53	—	—
2777	,,	,, 29	noon . .	46 40	50 56	4·3	18·09	32·70	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
2778	Loughrigg Holme	May 29	midnight	47° 5′	49° 2′	4·2	18·03	32·59	—	—
2779	,,	,, 30	noon . .	47 21	47 52	3·7	17·93	32·42	—	—
2780	,,	,, 30	midnight	47 30	47 10	3·2	17·98	32·51	—	—
2781	Anchoria .	,, 29	noon . .	54 24	14 7	11·1	19·63	35·47	—	—
2782	,,	,, 30	,,	52 48	20 44	10·6	19·59	35·40	—	—
2783	,,	,, 31	,,	51 22	27 15	12·2	19·68	35·56	—	—
2784	,,	June 1	,,	49 7	34 35	12·8	19·64	35·48	—	—
2785	,,	,, 2	,,	46 9	41 11	12·8	18·69	33·78	—	—
2786	,,	,, 3	,,	43 28	46 8	17·2	19·68	35·56	—	—
2787	,,	,, 4	,,	41 0	50 58	15·0	18·81	34·00	—	—
2788	,,	,, 5	,,	41 37	56 58	17·8	19·70	35·59	26·59	—
2789	,,	,, 6	,,	41 30	63 50	18·3	19·50	35·23	—	—
2790	,,	,, 7	,,	40 31	70 39	11·7	17·86	32·30	24·00	—
2791	,,	,, 13	,,	40 39	61 32	11·1	18·03	32·59	24·21	—
2792	,,	,, 14	,,	42 7	62 52	10·6	17·68	31·97	23·78	—
2793	,,	,, 15	,,	44 13	57 22	9·4	18·12	32·75	—	—
2794	,,	,, 16	,,	46 55	51 37	4·4	18·07	32·66	—	—
2795	,,	,, 17	,,	49 28	45 29	8·3	18·77	33·92	—	—
2796	,,	,, 18	,,	51 34	38 24	12·2	19·38	35·01	—	—
2797	,,	,, 19	,,	53 3	30 43	12·2	19·35	34·96	—	—
2798	,,	,, 20	,,	54 13	22 44	12·2	19·57	35·36	—	—
2799	,,	,, 21	,,	54 51	14 5	12·2	19·59	35·40	—	—
2800	Teutonic. .	,, 3	midnight	50 53	14 49	13·3	19·77	35·72	—	—
2801	,,	,, 4	noon . .	50 25	21 27	11·1	19·68	35·56	—	—
2802	,,	,, 4	midnight	49 0	27 10	11·1	19·79	35·75	—	—
2803	,,	,, 5	noon . .	47 35	32 27	13·3	20·00	35·12	—	—
2804	,,	,, 5	midnight	45 58	37 24	12·8	19·99	36·11	—	—
2805	,,	,, 6	noon . .	44 14	42 33	12·2	20·23	36·53	27·37	—
2806	,,	,, 6	midnight	42 7	47 1	10·0	19·39	35·03	—	—
2807	,,	,, 7	noon . .	40 59	52 7	11·7	19·77	35·72	—	—
2808	,,	,, 7	midnight	40 48	57 31	16·1	19·16	34·62	—	—
2809	,,	,, 8	noon . .	40 31	63 6	16·7	20·15	36·39	27·27	—
2810	,,	,, 8	midnight	40 34	68 34	13·9	18·45	33·36	24·84	—
2811	,,	,, 14	,,	40 10	69 41	15·0	19·36	34·98	—	—
2812	,,	,, 15	noon . .	40 30	64 46	13·3	20·02	36·16	—	—
2813	,,	,, 15	midnight	40 54	59 27	14·4	19·50	35·23	—	—
2814	,,	,, 16	noon . .	41 22	53 53	17·8	19·31	34·89	—	—
2815	,,	,, 16	midnight	42 9	48 54	14·4	18·44	33·34	—	—
2816	,,	,, 17	noon . .	44 12	44 12	17·8	19·62	35·45	—	—
2817	,,	,, 17	midnight	45 51	39 11	17·8	19·97	36·07	—	—
2818	,,	,, 18	noon . .	47 31	33 56	15·6	19·71	35·61	—	—
2819	,,	,, 18	midnight	48 46	28 26	15·6	19·63	35·47	—	—
2820	,,	,, 19	noon . .	49 47	22 45	15·6	19·68	35·56	—	—
2821	,,	,, 19	midnight	50 36	16 52	14·4	19·63	35·47	—	—
2822	,,	,, 20	noon . .	51 15	10 53	13·9	19·63	35·47	—	—
					E.					
2823	Monarch . .	Jan. 6	,,	50 59	1 45	6·7	18·98	34·30	—	—
2824	,,	,, 7	,,	50 59	1 45	6·1	18·94	34·23	—	—
2825	,,	,, 8	,,	51 29	1 31	6·7	19·17	34·63	—	—
2826	,,	,, 9	,,	52 55	4 12	2·2	19·31	34·89	—	—
2827	,,	,, 10	,,	53 36	6 14	1·1	17·47	31·60	23·54	—
2828	,,	,, 12	,,	53 48	6 43	0·0	17·50	31·65	—	—
2829	,,	,, 13	,,	53 46	6 42	1 7	17·54	31·71	—	—
2830	,,	,, 14	,,	53 50	6 36	1·7	17·53	31·70	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	E.					
2831	Monarch . .	Jan. 15	noon . .	53° 48′	6° 41′	1·7	17·57	31·77	—	—
2832	,,	,, 16	,,	52 44	3 28	5·6	19·35	34·96	—	—
2833	,,	Apr. 12	,,	52 24	4 21	7·8	16·78	30·37	22·59	—
2834	,,	,, 13	,,	52 24	4 23	7·8	17·07	30·87	22·97	—
2835	,,	,, 15	,,	52 11	3 11	7·8	19·31	34·89	—	—
2836	,,	May 3	,,	Calais	Roads	10·0	18·89	34·14	—	—
					W.					
2837	,,	,, 15	,,	50° 19′	4° 33′	10·8	19·49	35·21	—	—
2838	,,	,, 16	,,	52 16	5 26	9·4	19·32	34·91	—	—
2839	,,	,, 17	,,	55 55	6 6	8·3	18·90	34·16	—	—
2840	,,	,, 18	,,	Castle Bay,	Barra Id.	9·4	19·22	34·72	—	—
2841	,,	,, 19	,,	,,	,,	9·4	19·19	34·67	—	—
2842	,,	,, 20	,,	Oban H	arbour	9·4	18·58	33·58	—	—
2843	,,	,, 21	,,	54° 3′	4°47′50″	11·1	19·03	34·39	—	—
2844	,,	,, 23	,,	Holyhead	Harb.	10·0	19·12	34·54	—	—
2845	,,	,, 25	,,	51° 25′	9° 28′	12·2	19·42	35·08	—	—
2846	,,	,, 26	,,	Crook	haven	11·7	19·49	35·21	26·28	—
2847	,,	June 3	,,	51° 23′ 30″	9° 38′	11·7	19·43	35·10	—	—
2848	,,	,, 4	,,	Fastnet L.H. dist., cables	Rock S.55°E. 2·3	11·1	19·45	35·14	—	—
2849	,,	,, 5	,,	51° 27′	9° 6′	14·4	19·42	35·08	—	—
2850	,,	,, 6	,,	51 53	6 22	12·2	19·41	35·07	—	—
2851	,,	,, 12	,,	51 26	5 13	13·3	19·18	34·65	—	—
2852	,,	,, 13	,,	50 25	1 55	13·3	19·49	35·21	26·27	—
					E.					
2853	California .	Apr. 2	5 P.M. .	36 8	5 10	15·0	20·45	36·93	—	—
2854	,,	,, 2	8 P.M. .	36 14	4 38	15·0	20·21	36·49	—	—
2855	,,	,, 2	midnight	36 21	3 56	15·0	20·45	36·93	—	—
2856	,,	,, 3	4 A.M. .	36 29	3 0	14·4	20·79	37·53	28·11	—
					W.					
2857	,,	,, 3	noon . .	37 3	1 26	14·4	20·66	37·30	—	—
2858	,,	,, 3	8 P.M. .	37 52	0 18	14·4	20·55	37·10	—	—
					E.					
2859	,,	,, 4	4 A.M. .	38 58	0 40	13·9	20·84	37·62	—	—
2860	,,	,, 4	noon . .	40 6	1 36	13·3	20·91	37·75	—	—
2861	,,	,, 4	8 P.M. .	41 10	2 36	13·3	20·86	37·65	—	—
2862	,,	,, 5	4 A.M. .	42 6	3 22	13·3	20·81	37·57	—	—
2863	,,	,, 5	noon . .	42 47	4 30	13·3	20·78	37·51	—	—
2864	,,	,, 13	,,	43 54	8 28	13·3	21·01	37·93	—	—
2865	,,	May 2	,,	39 25	10 30	15·0	21·06	38·02	—	—
2866	,,	,, 2	midnight	38 42	8 10	15·6	20·78	37·51	—	—
2867	,,	,, 3	noon . .	38 11	5 35	15·6	20·44	36·91	—	—
2868	,,	,, 3	midnight	37 44	3 11	15·0	20·70	37·37	27·87	—
2869	,,	,, 4	noon . .	37 18	0 48	15·0	20·76	37·47	—	—
2870	,,	,, 4	midnight	36 52	2 22	15·0	20·78	37·51	—	—
					W.					
2871	,,	,, 5	noon . .	36 26	4 4	15·6	20·33	36·70	27·46	—
2872	,,	,, 7	,,	36 47	9 3	15·0	19·88	35·91	—	—
2873	,,	,, 7	midnight	37 25	11 24	15·0	19·97	36·07	—	—
2874	,,	,, 8	noon . .	38 3	13 45	15·0	20·03	36·18	—	—
2875	,,	,, 8	midnight	38 42	16 21	15·0	19·96	36·05	—	—
2876	,,	,, 9	noon . .	39 22	19 0	15·0	19·86	35·88	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
2877	California .	May 9	midnight	39° 53′	21° 37′	15·0	19·88	35·91	—	—
2878	,,	,, 10	noon . .	40 24	24 18	15·0	19·97	36·07	—	—
2879	,,	,, 10	midnight	40 33	27 7	15·0	19·90	35·95	—	—
2880	,,	,, 11	noon . .	40 41	29 56	14·4	19·86	35·88	—	—
2881	,,	,, 11	midnight	40 47	32 23	14·4	19·96	36·05	—	—
2882	,,	,, 12	noon . .	40 53	35 14	14·4	20·18	36·44	—	—
2883	,,	,, 12	midnight	40 59	37 49	15·0	20·03	36·18	—	—
2884	,,	,, 13	noon . .	41 6	40 25	15·6	20·10	36·30	—	—
2885	,,	,, 13	midnight	41 5	42 43	16·7	20·08	36·27	—	—
2886	,,	,, 14	noon . .	41 3	44 51	17·2	20·05	36·21	—	—
2887	,,	,, 14	midnight	41 3	47 20	16·7	19·95	36·04	—	—
2888	,,	,, 15	noon . .	41 1	49 50	16·7	19·64	35·48	—	—
2889	,,	,, 15	midnight	41 4	52 10	14·4	19·32	34·91	—	—
2890	,,	,, 16	noon . .	41 7	54 31	14·4	18·70	33·80	—	—
2891	,,	,, 16	midnight	41 7	56 46	15·6	19·88	35·91	—	—
2892	,,	,, 17	noon . .	41 7	59 2	16·7	18·70	33·80	—	—
2893	,,	,, 17	midnight	41 9	61 27	16·7	19·49	35·21	—	—
2894	,,	,, 18	noon . .	41 11	63 53	16·7	19·83	35·82	—	—
2895	,,	,, 18	midnight	40 56	66 23	8·3	17·97	32·49	—	—
2896	,,	,, 19	noon . .	40 42	68 58	7·8	18·05	32·63	—	—
2897	,,	,, 19	midnight	40 30	71 28	7·8	17·70	32·01	—	—
2898	,,	,, 29	,,	40 23	72 0	8·9	17·15	31·02	—	—
2899	,,	,, 30	noon . .	40 21	70 1	11·1	17·87	32·31	24·08	—
2900	,,	,, 30	midnight	40 24	67 48	11·1	18·30	33·08	—	—
2901	,,	,, 31	noon . .	40 28	65 35	15·6	17·68	31·97	23·86	—
2902	,,	,, 31	midnight	40 35	63 26	16·7	18·74	33·87	—	—
2903	,,	June 1	noon . .	41 26	61 16	13·9	18·53	33·49	—	—
2904	,,	,, 1	midnight	41 27	58 52	18·9	19·51	35·25	—	—
2905	,,	,, 2	noon . .	41 28	56 28	16·7	19·08	34·47	—	—
2906	,,	,, 2	midnight	41 23	54 1	17·8	19·44	35·12	—	—
2907	,,	,, 3	noon . .	41 17	51 34	16·7	19·67	35·54	—	—
2908	,,	,, 3	midnight	41 44	49 22	15·6	19·15	34·60	—	—
2909	,,	,, 4	noon . .	42 19	47 10	10·0	19·13	34·56	—	—
2910	,,	,, 4	midnight	43 25	45 14	14·4	19·10	34·51	—	—
2911	,,	,, 5	noon . .	44 32	43 17	15·6	20·06	36·23	27·02	—
2912	,,	,, 5	midnight	45 54	41 28	15·6	20·07	36·25	—	—
2913	,,	,, 6	noon . .	46 16	39 39	14·4	19·80	35·77	—	—
2914	,,	,, 6	midnight	47 19	37 40	14·4	19·75	35·68	—	—
2915	,,	,, 7	noon . .	48 22	35 40	14·4	19·78	35·73	—	—
2916	,,	,, 7	midnight	49 18	32 28	13·3	19·59	35·40	—	—
2917	,,	,, 8	noon . .	50 14	31 15	11·1	19·57	35·36	—	—
2918	,,	,, 8	midnight	50 59	28 44	12·2	19·50	35·23	—	—
2919	,,	,, 9	noon . .	51 45	26 12	12·2	19·62	35·45	—	—
2920	,,	,, 9	midnight	52 29	23 53	12·2	19·60	35·41	—	—
2921	,,	,, 10	noon . .	53 13	21 34	12·2	19·59	35·40	—	—
2922	,,	,, 10	midnight	53 49	18 45	11·1	19·63	35·47	—	—
2923	,,	,, 11	noon . .	54 24	15 56	11·1	19·63	35·47	—	—
2924	,,	,, 11	midnight	54 57	18 57	11·1	19·59	35·40	—	—
2925	,,	,, 12	noon . .	55 30	9 54	11·1	19·57	35·36	—	—
2926	Siberian . .	,, 21	,,	51 22	14 47	13·9	19·68	35·56	26·58	—
2927	,,	,, 22	,,	51 32	21 20	13·9	19·64	35·48	—	—
2928	,,	,, 23	,,	51 43	28 40	13·9	19·57	35·36	—	—
2929	,,	,, 24	,,	51 15	35 48	12·2	19·49	35·21	—	—
2930	,,	,, 25	,,	52 18	42 16	12·8	19·14	34·58	—	—
2931	,,	26	,,	48 39	48 32	5·6	18·32	33·11	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
2932	Siberian . .	June 27	noon . .	47° 54′	51° 46′	6·7	18·31	33·10	—	—
2933	,,	,, 29	,,	45 48	55 55	8·3	17·89	32·35	—	—
2934	,,	,, 30	,,	Egg I., N.Scotia 12 miles		12·2	17·79	32·17	—	—
2935	,,	July 2	,,	41° 45′	66° 15′	12·2	17·97	32·49	—	—
2936	,,	,, 3	,,	39 33	71 42	21·7	18·20	32·90	—	—
2937	,,	,, 10	,,	39 28	71 49	25·0	19·06	34·44	—	—
2938	,,	,, 11	,,	41 3	66 48	22·2	19·11	34·53	—	—
2939	,,	,, 12	,,	42 30	61 30	15·6	18·19	32·88	—	—
2940	,,	,, 13	,,	44 41	56 29	14·4	17·29	31·27	—	—
2941	,,	,, 15	,,	49 23	47 58	10·0	18·58	33·53	—	—
2942	,,	,, 16	,,	51 25	41 50	13·9	19·31	34·89	—	—
2943	,,	,, 17	,,	52 55	36 2	12·2	19·29	34·86	—	—
2944	,,	,, 18	,,	54 3	29 43	13·3	19·47	35·17	—	—
2945	,,	,, 19	,,	55 2	22 42	13·9	19·52	35·27	—	—
2946	,,	,, 20	,,	55 35	14 55	15·6	19·57	35·36	26·32	—
2947	,,	,, 21	,,	N. coast	Ireland	15·0	19·29	34·86	—	—
2926A	Laura . .	June 6	,,	59° 32′	2° 24′	9·5	19·35	34·96	—	—
2927A	,,	,, 6	8 P.M. .	60 17	3 42	9·3	19·58	35·38	—	—
2928A	,,	,, 7	4 A.M. .	61 10	5 32	8·3	19·61	35·43	26·42	—
2929A	,,	,, 9	,,	62 29	8 2	8·5	19·56	35·34	26·40	—
2930A	,,	,, 9	noon . .	62 42	11 11	9·5	19·51	35·25	—	—
2931A	,,	,, 9	8 P.M. .	62 55	14 4	9·2	19·49	35·21	—	—
2932A	,,	,, 10	4 A.M. .	63 9	17 6	8·8	19·50	35·23	—	—
2933A	,,	,, 10	noon . .	63 23	19 46	9·5	19·49	35·21	—	—
2934A	,,	,, 10	8 P.M. .	63 34	21 49	9·0	19·47	35·17	—	—
2935A	,,	,, 20	4 A.M. .	64 8	22 54	7·5	19·03	34·39	25·73	—
2936A	,,	,, 20	noon . .	63 33	20 45	9·0	19·42	35·08	—	—
2937A	,,	,, 20	8 P.M. .	63 14	18 45	9·0	19·35	34·96	—	—
2938A	,,	,, 21	4 A.M. .	62 57	16 30	8·8	19·50	35·23	26·37	—
2939A	,,	,, 21	noon . .	62 42	14 27	9·5	19·50	35·23	—	—
2940A	,,	,, 21	8 P.M. .	62 37	11 52	9·0	19·49	35·21	—	—
2941A	,,	,, 22	4 A.M. .	62 34	9 25	9·0	19·48	35·19	—	—
2942A	,,	,, 22	noon . .	62 26	7 10	8·5	19·45	35·14	—	—
2943A	,,	,, 24	4 A.M. .	61 40	6 40	8·0	19·46	35·15	—	—
2944A	,,	,, 24	noon . .	61 5	5 28	10·0	19·47	35·17	—	—
2945A	,,	,, 24	8 P.M. .	60 8	3 25	11·0	19·55	35·32	—	—
2948	Teutonic .	July 1	midnight	51 3	14 40	13·9	19·58	35·38	—	—
2949	,,	,, 2	noon . .	50 25	20 47	15·0	19·61	35·43	—	—
2950	,,	,, 2	midnight	49 30	26 39	15·6	19·70	35·59	—	—
2951	,,	,, 3	noon . .	48 22	32 24	15·6	19·56	35·34	—	—
2952	,,	,, 3	midnight	46 31	38 0	16·7	19·53	35·28	—	—
2953	,,	,, 4	noon . .	45 13	42 51	17·8	19·92	35·98	—	—
2954	,,	,, 4	midnight	43 17	47 56	12·2	18·16	32·82	—	—
2955	,,	,, 5	noon . .	42 10	53 23	17·8	19·36	34·98	—	—
2956	,,	,, 5	midnight	41 40	58 53	18·3	19·54	35·30	—	—
2957	,,	,, 6	noon . .	41 10	64 18	15·6	17·56	31·75	—	—
2958	,,	,, 6	midnight	40 37	69 14	16·1	17·63	31·88	—	—
2959	,,	,, 14	,,	40 12	69 35	21·1	18·38	33·22	—	—
2960	,,	,, 15	noon . .	40 31	65 20	26·1	19·85	35·86	—	—
2961	,,	,, 15	midnight	40 53	60 30	18·9	18·08	32·68	—	—
2962	,,	,, 16	noon . .	41 16	55 37	23·3	19·03	34·39	—	—
2963	,,	,, 16	midnight	41 37	50 41	20·0	18·92	34·19	—	—
2964	,,	,, 17	noon . .	43 8	46 17	20·0	19·41	35·07	—	—
2965	,,	,, 17	midnight	44 55	41 40	17·8	19·93	36·00	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
2966	Teutonic . .	July 18	noon . .	46° 35′	37° 7′	15·6	19·69	35·58	—	—
2967	,,	,, 18	midnight	47 57	32 0	15·6	20·09	36·29	—	—
2968	,,	,, 19	noon . .	49 12	26 57	16·1	19·69	35·58	—	—
2969	,,	,, 19	midnight	50 5	21 24	16·1	19·62	35·45	—	—
2970	,,	,, 20	noon . .	50 57	15 39	16·1	19·59	35·40	—	—
2971	Anchoria .	June 27	,,	55 16	11 59	12·8	19·55	35·32	—	—
2972	,,	,, 28	,,	54 51	20 17	12·2	19·57	35·36	—	—
2973	,,	,, 29	,,	53 45	28 22	12·8	19·58	35·38	—	—
2974	,,	,, 30	,,	52 39	35 42	12·2	19·39	35·03	—	—
2975	,,	July 1	,,	50 39	42 12	15·0	19·78	35·73	26·79	—
2976	,,	,, 2	,,	48 20	48 3	6·1	18·23	32·95	—	—
2977	,,	,, 3	,,	45 43	53 45	7·8	18·03	32·59	—	—
2978	,,	,, 4	,,	43 23	59 39	12·8	18·09	32·70	—	—
2979	,,	,, 5	,,	41 19	65 45	15·0	18·17	32·84	—	—
2980	,,	,, 6	,,	40 30	71 36	21·1	17·55	31·73	23·58	—
2981	,,	,, 11	,,	40 34	69 20	17·2	17·68	31·97	—	—
2982	,,	,, 12	,,	41 50	63 11	16·1	17·62	31·86	23·64	—
2983	,,	,, 13	,,	44 9	57 27	13·9	17·84	32·26	—	—
2984	,,	,, 14	,,	46 30	52 5	·8·9	17·89	32·35	—	—
2985	,,	,, 15	,,	49 9	46 13	11·1	18·85	34·07	—	—
2986	,,	,, 16	,,	51 20	39 40	12·8	19·29	34·86	—	—
2987	,,	,, 17	,,	52 43	33 2	12·2	19·39	35·03	—	—
2988	,,	,, 18	,,	53 55	26 5	13·9	19·57	35·36	—	—
2989	,,	,, 19	,,	54 49	18 27	13·9	19·60	35·41	—	—
2990	,,	,, 20	,,	55 21	11 6	15·0	19·63	35·47	26·44	—
2991	Laura . .	,, 15	4 A.M. .	59 38	2 45	10·7	19·26	34·80	—	—
2992	,,	,, 15	noon . .	60 34	4 34	12·5	19·53	35·28	26·25	—
2993	,,	,, 15	8 P.M. .	61 16	6 15	12·5	19·47	35·17	—	—
2994	,,	,, 18	4 A.M. .	62 26	8 58	11·2	19·50	35·23	—	—
2995	,,	,, 18	noon . .	62 37	11 22	10·7	19·50	35·23	—	—
2996	,,	,, 18	8 P.M. .	62 51	14 11	11·0	19·49	35·21	—	—
2997	,,	,, 19	4 A.M. .	63 20	16 45	11·5	19·45	35·14	—	—
2998	,,	,, 19	noon . .	63 24	18 40	12·2	19·48	35·19	—	—
2999	,,	,, 19	8 P.M. .	63 24	20 3	11·5	19·29	34·86	—	—
3000	,,	,, 20	4 A.M. .	63 26	21 40	11·7	18·69	33·78	—	—
3001	,,	,, 20	noon . .	63 57	23 1	12·0	18·57	33·56	—	—
3002	,,	,, 22	8 P.M. .	64 25	22 19	11·5	18·27	33·02	24·59	—
3003	,,	,, 23	4 A.M. .	65 8	23 45	10·5	19·11	34·53	—	—
3004	,,	,, 23	8 P.M. .	65 29	24 36	10·5	18·51	33·46	—	—
3005	,,	,, 25	,,	66 14	23 40	9·5	18·80	33·98	—	—
3006	,,	,, 26	noon . .	66 7	23 42	9·0	19·07	34·46	—	—
3007	,,	,, 26	8 P.M. .	65 20	24 30	11·5	19·15	34·60	—	—
3008	,,	,, 27	4 A.M. .	64 26	22 58	11·5	18·99	34·32	—	—
3009	,,	,, 30	,,	64 14	22 33	11·5	18·54	33·51	24·97	—
3010	,,	,, 30	noon . .	63 32	21 6	11·0	19·14	34·58	—	—
3011	,,	,, 30	8 P.M. .	63 17	18 58	11·0	19·45	35·14	—	—
3012	,,	,, 31	4 A.M. .	62 48	16 30	11·7	19·49	35·21	—	—
3013	,,	,, 31	noon . .	62 36	13 39	12·5	19·41	35·07	—	—
3014	,,	,, 31	8 P.M. .	62 32	10 45	12·0	19·46	35·15	—	—
3015	,,	Aug. 1	4 A.M. .	62 24	8 1	12·0	19·53	35·28	—	—
3016	,,	,, 1	noon . .	62 24	6 47	11·5	19·45	35·14	—	—
3017	,,	,, 1	8 P.M. .	62 8	6 30	11·5	19·45	35·14	—	—
3018	,,	,, 2	,,	61 15	5 21	11·8	19·40	35·05	—	—
3019	,,	,, 3	4 A.M. .	60 12	3 34	12·7	19·15	34·60	—	—
3020	,,	,, 3	noon . .	59 35	2 26	13·0	19·36	34·98	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO$_3$.
		1897.		N.	W.					
3021	Para . . .	Jan. 28	noon . .	47° 48′	7° 11′	11·1	20·01	36·14	—	—
3022	,,	,, 29	,,	44 29	13 40	11·1	20·03	36·18	—	—
3023	,,	,, 30	,,	41 22	19 53	13·9	20·45	36·93	—	—
3024	,,	Mar. 14	,,	42 41	24 42	13·1	20·15	36·39	—	—
3025	,,	,, 15	,,	45 32	17 49	12·2	20·23	36·53	—	—
3026	,,	,, 16	,,	47 37	10 24	11·1	20·41	31·86	27·37	—
3027	,,	Apr. 8	,,	48 2	7 0	10·6	20·03	36·18	—	—
3028	,,	,, 9	,,	45 6	12 55	12·2	20·09	36·29	—	—
3029	,,	,, 10	,,	41 31	18 48	12·8	20·02	36·16	—	—
3030	,,	May 23	,,	41 51	27 50	15·0	20·12	36·33	—	—
3031	,,	,, 24	,,	43 53	21 23	13·9	19·78	35·73	26·77	—
3032	,,	,, 25	,,	46 22	14 58	13·3	19·90	35·95	—	—
3033	,,	,, 26	,,	49 6	7 38	12·8	19·85	35·86	—	—
3034	,,	June 17	,,	47 51	6 36	15·0	19·85	35·86	—	—
3035	,,	,, 18	,,	44 38	12 47	17·8	19·84	35·84	26·72	—
3036	,,	,, 19	,,	43 12	15 34	18·9	19·91	35·97	—	—
3037	,,	,, 20	,,	40 50	19 43	19·4	19·86	35·88	—	—
3038	,,	Aug. 2	,,	40 24	25 30	21·1	19·89	35·93	—	—
3039	,,	,, 3	,,	43 4	20 0	19·4	19·90	35·95	—	—
3040	,,	,, 4	,,	45 38	13 53	18·3	19·88	35·91	—	—
3041	,,	,, 5	,,	48 29	7 44	18·3	19·78	35·73	26·61	—
3042	Loughrigg Holme	July 3	midnight	58 40	5 20	10·6	19·36	34·98	—	—
3043	,,	,, 4	noon . .	58 32	8 22	11·9	19·56	35·34	—	—
3044	,,	,, 4	midnight	58 22	11 50	11·1	19·53	35·28	—	—
3045	,,	,, 5	noon . .	58 14	14 5	11·0	19·50	35·23	—	—
3046	,,	,, 5	midnight	57 41	16 44	10·8	19·54	35·30	—	—
3047	,,	,, 6	noon . .	57 9	19 20	11·7	19·50	35·23	—	—
3048	,,	,, 6	8 P.M. .	56 53	21 27	11·6	19·57	35·36	—	—
3049	,,	,, 6	midnight	56 44	22 35	11·1	19·53	35·28	—	—
3050	,,	,, 7	4 A.M. .	56 36	23 38	11·4	19·51	35·25	—	—
3051	,,	,, 7	noon . .	56 22	25 27	11·4	19·46	35·15	—	—
3052	,,	,, 7	8 P.M. .	56 4	26 52	10·8	19·44	35·12	—	—
3053	,,	,, 7	midnight	55 54	27 40	10·3	19·44	35·12	26·14	—
3054	,,	,, 8	8 A.M. .	55 36	29 16	11·8	19·47	35·17	—	—
3055	,,	,, 8	noon . .	55 27	29 59	10·6	19·33	34·93	—	—
3056	,,	,, 8	8 P.M. .	55 7	31 36	10·5	19·39	35·03	—	—
3057	,,	,, 8	midnight	54 56	32 27	10·7	19·42	35·08	—	—
3058	,,	,, 9	8 A.M. .	54 35	34 14	10·6	19·30	34·87	—	—
3059	,,	,, 9	noon . .	54 23	35 12	10·4	19·22	34·72	—	—
3060	,,	,, 9	8 P.M. .	54 8	36 38	10·7	19·27	34·82	—	—
3061	,,	,, 9	midnight	53 55	37 9	10·0	19·30	34·87	25·90	—
3062	,,	,, 10	8 A.M. .	53 24	38 22	9·7	19·20	34·69	—	—
3063	,,	,, 10	noon . .	53 3	39 8	11·2	19·31	34·89	—	—
3064	,,	,, 10	midnight	52 0	41 38	11·2	19·22	34·72	—	—
3065	,,	,, 11	noon . .	50 49	44 26	14·2	19·44	35·12	26·13	—
3066	,,	,, 11	midnight	49 32	46 59	11·1	19·08	34·47	—	—
3067	,,	,, 12	noon . .	48 17	49 27	5·8	17·99	32·53	—	—
3068	,,	,, 12	midnight	47 20	50 42	7·3	17·99	32·53	—	—
3069	,,	,, 13	noon . .	46 43	52 0	8·7	17·98	32·51	24·03	—
3070	,,	,, 13	midnight	46 27	55 13	10·0	17·84	32·26	—	—
3071	,,	,, 14	noon . .	46 21	57 52	12·8	17·79	32·17	—	—
3072	,,	Aug. 6	6 A.M. .	51 48	55 38	6·9	16·25	29·42	—	—
3073	,,	,, 6	noon . .	52 4	54 26	9·3	16·45	29·77	—	—
3074	,,	,, 6	8 P.M. .	52 37	52 52	8·0	17·74	32·07	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO$_3$.
		1897.		N.	W.					
3075	Loughrigg Holme	Aug. 7	4 A.M. .	53° 9′	51° 20′	8·0	18·58	33·58	24·95	—
3076	,,	,, 7	noon . .	53 38	49 56	11·0	18·92	34·19	—	—
3077	,,	,, 7	8 P.M. .	54 9	48 12	10·6	19·06	34·44	—	—
3078	,,	,, 8	4 A.M. .	54 40	46 28	9·6	19·16	34·62	—	—
3079	,,	,, 8	noon . .	55 10	44 43	11·3	19·16	34·62	25·76	—
3080	,,	,, 8	8 P.M. .	55 34	42 46	10·0	19·23	34·74	—	—
3081	,,	,, 9	4 A.M. .	55 59	40 47	9·5	19·26	34·80	—	—
3082	,,	,, 9	noon . .	56 24	38 52	9·3	19·30	35·87	—	—
3083	,,	,, 9	8 P.M. .	56 38	36 43	9·8	19·40	35·05	26·04	—
3084	,,	,, 10	4 A.M. .	56 52	34 30	10·0	19·39	35·03	—	—
3085	,,	,, 10	noon . .	57 7	32 25	11·1	19·44	35·12	—	—
3086	,,	,, 10	8 P.M. .	57 20	30 11	11·8	19·47	35·17	—	—
3087	,,	,, 11	4 A.M. .	57 33	27 58	12·1	19·47	35·17	26·14	—
3088	,,	,, 11	noon . .	57 47	25 52	12·3	19·45	35·14	—	—
3089	,,	,, 11	8 P.M. .	57 57	23 42	12·3	19·43	35·10	—	—
3090	,,	,, 12	4 A.M. .	58 6	21 48	12·9	19·49	35·21	—	—
3091	,,	,, 12	noon . .	58 12	20 33	12·9	19·48	35·19	26·16	—
3092	,,	,, 12	8 P.M. .	58 18	18 23	13·2	19·56	35·34	—	—
3093	,,	,, 13	4 A.M. .	58 24	16 25	13·4	19·54	35·30	—	—
3094	,,	,, 13	noon . .	58 31	14 20	13·9	19·52	35·27	—	—
3095	,,	,, 13	8 P.M. .	58 34	12 6	14·3	19·59	35·40	—	—
3096	,,	,, 14	4 A.M. .	58 36	9 54	14·4	19·57	35·36	—	—
3097	,,	,, 14	noon . .	58 39	7 38	14·3	19·50	35·23	—	—
3098	,,	,, 14	8 P.M. .	58 39	5 8	14·3	19·34	34·94	—	—
		1896.								
3099	Minia . . .	Oct. 24	noon . .	50 0	7 52	11·6	18·93	34·21	—	—
3100	,,	,, 26	,,	51 20	13 23	11·4	19·57	35·36	—	—
3101	,,	,, 27	,,	51 29	19 32	12·2	—	—	—	—
3102	,,	,, 28	,,	51 16	26 0	12·8	19·16	34·62	—	—
3103	,,	,, 29	,,	50 52	32 28	13·0	—	—	—	—
3104	,,	,, 30	,,	50 37	38 23	14·4	19·35	34·96	—	—
3105	,,	,, 31	,,	49 35	44 26	12·3	17·90	32·37	—	—
3106	,,	Nov. 1	,,	48 25	49 45	6·8	17·23	31·17	23·16	—
3107	,,	,, 9	8 A.M. .	46 32	59 30	8·3	16·26	29·44	21·75	—
3108	,,	,, 13	,,	44 54	61 21	9·2	16·30	29·50	—	—
3109	,,	Dec. 22	6 A.M. .	49 0	50 27	−0·6	18·15	32·80	—	—
		1897.								
3110	,,	Jan. 22	noon . .	42 42	68 54	4·7	—	—	—	—
3111	,,	May 16	8 A.M. .	48 52	50 35	0·6	—	—	—	—
3112	,,	,, 24	4 P.M. .	48 19	48 16	2·8	18·03	32·59	—	—
3113	,,	,, 25	noon . .	48 21	44 22	6·2	17·96	32·48	—	—
3114	,,	,, 26	8 A.M. .	48 19	40 24	14·0	17·97	32·49	24·19	—
3115	,,	June 1	,,	48 22	39 30	13·3	18·94	34·23	—	—
3116	,,	July 19	1 P.M. .	47 44	43 53	11·1	19·62	35·45	—	—
3117	,,	,, 20	noon . .	48 18	39 49	16·9	19·63	35·47	—	—
3118	,,	,, 21	,,	49 20	36 24	14·2	18·60	33·62	24·97	—
3119	,,	,, 20	,,	48 18	39 49	16·9	19·72	35·63	26·60	—
3120	,,	,, 21	,,	49 20	36 24	14·2	19·41	35·07	—	—
3121	,,	,, 22	1 P.M. .	50 20	31 30	15·0	19·43	35·10	—	—
3122	,,	,, 23	noon . .	51 21	26 12	15·8	19·61	35·43	—	—
3123	,,	,, 25	,,	51 53	14 57	16·4	—	—	—	—
3124	Traveller. .	Apr. 8	,,	59 8	5 1	7·2	19·48	35·19	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3125	Traveller .	Apr. 9	noon . .	59° 45′	7° 1′	8·3	19·61	35·43	—	—
3126	,,	,, 10	,,	58 56	7 50	8·9	19·61	35·43	—	—
3127	,,	,, 11	,,	59 28	11 9	8·8	19·59	35·40	—	—
3128	,,	,, 12	,,	60 8	13 9	8·3	19·57	35·36	26·21	—
3129	,,	,, 13	,,	60 28	16 13	8·3	19·55	35·32	—	—
3130	,,	,, 14	,,	60 18	16 13	8·4	19·64	35·48	—	—
3131	,,	,, 15	,,	60 9	14 20	8·3	19·55	35·32	—	—
3132	,,	,, 16	,,	59 32	13 11	8·4	19·59	35·40	—	—
3133	,,	,, 17	,,	58 39	12 16	8·3	19·60	35·41	—	—
3134	,,	,, 18	,,	57 32	12 10	8·9	19·59	35·40	—	—
3135	,,	,, 19	,,	58 3	12 40	8·8	19·57	35·36	—	—
3136	,,	,, 20	,,	58 2	14 0	8·9	19·54	35·30	26·25	—
3137	,,	,, 21	,,	57 50	18 40	9·4	19·54	35·30	—	—
3138	,,	,, 22	,,	58 17	22 56	8·9	19·54	35·30	—	—
3139	,,	,, 23	,,	58 34	25 32	8·3	19·52	35·27	—	—
3140	,,	,, 24	,,	58 44	26 47	8·3	19·53	35·28	—	—
3141	,,	,, 25	,,	58 37	29 33	7·2	19·45	35·14	—	—
3142	,,	,, 26	,,	57 47	30 31	7·4	19·46	35·15	26·18	—
3143	,,	,, 27	,,	56 55	30 50	8·4	19·46	35·15	—	—
3144	,,	,, 28	,,	57 35	30 34	7·9	19·44	35·12	—	—
3145	,,	,, 29	,,	58 23	28 36	8·1	19·54	35·30	—	—
3146	,,	,, 30	,,	58 59	27 55	7·8	19·47	35·17	—	—
3147	,,	May 1	,,	59 38	26 55	7·2	19·53	35·28	—	—
3148	,,	,, 2	,,	60 3	25 59	6·1	19·53	35·28	26·31	—
3149	,,	,, 3	,,	59 44	26 52	7·5	19·49	35·21	—	—
3150	,,	,, 4	,,	59 16	27 26	7·3	19·54	35·30	—	—
3151	,,	,, 5	,,	58 28	27 46	7·3	19·52	35·27	—	—
3152	,,	,, 6	,,	58 25	27 22	7·6	19·44	35·12	—	—
3153	,,	,, 7	,,	59 1	27 37	7·4	19·46	35·15	26·19	—
3154	,,	,, 8	,,	59 1	27 37	7·6	19·44	35·12	—	—
3155	,,	,, 9	,,	58 3	27 57	7·8	19·41	35·07	—	—
3156	,,	,, 10	,,	58 41	30 36	7·3	19·40	35·05	—	—
3157	,,	,, 11	,,	58 50	33 41	6·3	19·36	34·98	—	—
3158	,,	,, 12	,,	59 6	36 15	5·6	19·45	35·14	—	—
3159	,,	,, 13	,,	59 59	35 44	4·7	19·35	34·96	—	—
3160	,,	,, 14	,,	59 6	36 0	6·2	19·46	35·15	26·10	—
3161	,,	,, 15	,,	58 20	37 37	5·3	19·35	34·96	—	—
3162	,,	,, 16	,,	58 8	38 57	4·7	19·36	34·98	—	—
3163	,,	,, 17	,,	58 2	41 25	3·1	18·72	33·83	25·28	—
3164	,,	,, 18	,,	58 23	42 42	3·3	18·94	34·23	—	—
3165	,,	,, 19	,,	58 28	45 38	3·2	19·05	34·42	—	—
3166	,,	,, 20	,,	59 28	47 58	2·8	18·99	34·32	—	—
3167	,,	,, 21	,,	60 58	48 28	0·3	18·50	33·44	—	—
3168	,,	,, 22	,,	61 6	48 40	0·6	17·20	31·11	—	—
3169	,,	July 12	,,	58 19	49 58	5·8	19·10	34·51	—	—
3170	,,	,, 13	,,	58 11	47 9	5·7	19·04	34·40	—	—
3171	,,	,, 15	,,	57 57	41 2	6·7	19·26	34·80	—	—
3172	,,	,, 16	,,	58 10	39 13	7·1	19·26	34·80	—	—
3173	,,	,, 17	,,	58 19	38 5	7·5	19·28	34·84	—	—
3174	,,	,, 18	,,	58 14	34 49	8·9	19·29	34·86	—	—
3175	,,	,, 19	,,	58 23	32 22	10·0	19·31	34·89	—	—
3176	,,	,, 20	,,	58 52	31 16	10·0	19·32	34·91	—	—
3177	,,	,, 21	,,	59 4	30 32	10·6	19·29	34·86	—	—
3178	,,	,, 22	,,	59 8	30 26	11·1	19·31	34·89	—	—
3179	,,	,, 23	,,	58 46	28 44	11·4	19·45	35·14	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3180	Traveller .	July 24	noon . .	58° 29′	27° 44′	12·2	19·45	35·14	—	—
3181	,,	,, 25	,,	58 26	26 55	12·8	19·45	35·14	—	—
3182	,,	,, 26	,,	58 34	24 13	12·5	19·47	35·17	26·11	—
3183	,,	,, 27	,,	58 21	21 37	12·8	19·47	35·17	—	—
3184	Thorwaldsen	Apr. 5	8 A.M. .	59 48	2 18	6·7	19·55	35·32	—	—
3185	,,	,, 5	noon . .	59 50	2 53	6·5	19·53	35·28	—	—
3186	,,	,, 5	4 P.M. .	59 52	3 36	6·8	19·56	35·34	—	—
3187	,,	,, 6	8 A.M. .	60 7	8 17	7·7	10·57	35·36	26·28	—
3188	,,	,, 6	noon . .	60 1	9 26	7·6	19·57	35·36	—	—
3189	,,	,, 6	4 P.M. .	59 55	10 46	7·8	19·63	35·47	—	—
3190	,,	,, 7	noon . .	59 57	11 59	8·2	19·59	35·40	—	—
3191	,,	,, 8	,,	59 51	14 8	7·8	19·56	35·34	26·21	—
3192	,,	,, 9	,,	59 30	13 33	8·0	19·57	35·36	—	—
3193	,,	,, 10	,,	59 11	13 52	8·5	19·59	35·40	—	—
3194	,,	,, 11	,,	59 11	16 26	8·6	19·58	35·38	—	—
3195	,,	,, 12	,,	59 22	15 44	8·3	19·57	35·36	26·20	—
3196	,,	,, 13	,,	59 33	17 45	8·5	19·57	35·36	—	—
3197	,,	,, 14	,,	58 51	17 25	8·2	19·56	35·34	—	—
3198	,,	,, 21	8 A.M. .	58 25	20 33	9·0	19·54	35·30	—	—
3199	,,	,, 21	noon . .	58 16	20 50	9·2	19·56	35·34	26·23	—
3200	,,	,, 21	4 P.M. .	58 13	21 8	9·2	19·57	35·36	—	—
3201	,,	,, 22	8 A.M. .	57 49	24 1	8·7	19·40	35·05	—	—
3202	,,	,, 22	noon . .	57 48	24 46	8·8	19·54	35·30	—	—
3203	,,	,, 22	4 P.M. .	57 50	25 8	8·8	19·56	35·34	26·20	—
3204	,,	,, 23	8 A.M. .	57 25	25 38	7·0	19·50	35·23	—	—
3205	,,	,, 23	noon . .	57 17	25 39	6·8	19·44	35·12	—	—
3206	,,	,, 23	4 P.M. .	57 6	25 36	7·5	19·45	35·14	—	—
3207	,,	,, 24	8 A.M. .	57 13	26 37	7·5	19·51	35·25	—	—
3208	,,	,, 24	noon . .	57 10	27 32	8·5	19·52	35·27	—	—
3209	,,	,, 24	4 P.M. .	57 11	27 42	8·2	19·46	35·15	—	—
3210	,,	,, 25	8 A.M. .	57 6	28 28	6·7	19·41	35·07	—	—
3211	,,	,, 25	noon . .	56 57	28 43	6·6	19·43	35·10	—	—
3212	,,	,, 25	4 P.M. .	56 47	28 59	6·6	19·46	35·15	—	—
3213	,,	,, 26	8 A.M. .	55 55	29 52	7·0	19·42	35·08	—	—
3214	,,	,, 26	noon . .	56 9	30 30	6·8	19·41	35·07	—	—
3215	,,	,, 26	4 P.M. .	56 26	30 35	7·0	19·43	35·10	—	—
3216	,,	,, 27	8 A.M. .	56 24	30 40	7·4	19·45	35·14	—	—
3217	,,	,, 27	noon . .	56 29	31 16	7·2	19·45	35·14	26·06	—
3218	,,	,, 27	4 P.M. .	56 40	30 47	7·5	19·43	35·10	—	—
3219	,,	,, 28	8 A.M. .	56 35	32 10	6·8	19·44	35·12	—	—
3220	,,	,, 28	noon . .	56 27	32 14	7·0	19·43	35·10	—	—
3221	,,	,, 28	4 P.M. .	56 11	32 25	7·0	19·37	35·00	25·92	—
3222	,,	,, 29	8 A.M. .	56 16	33 23	7·0	19·40	35·05	—	—
3223	,,	,, 29	noon . .	56 27	33 21	7·2	19·45	35·14	—	—
3224	,,	,, 29	4 P.M. .	56 39	33 19	7·2	19·44	35·12	—	—
3225	,,	,, 30	8 A.M. .	56 25	33 6	7·5	19·39	35·03	—	—
3226	,,	,, 30	noon . .	56 18	33 7	7·5	19·43	35·10	—	—
3227	,,	,, 30	4 P.M. .	56 5	33 5	7·5	19·39	35·03	—	—
3228	,,	May 7	noon . .	56 56	34 46	5·7	19·40	35·05	—	—
3229	,,	,, 8	,,	56 17	36 22	5·8	19·35	34·96	—	—
3230	,,	,, 9	8 A.M. .	56 40	38 34	5·0	19·24	34·76	—	—
3231	,,	,, 9	noon . .	56 47	39 42	5·5	19·32	34·91	—	—
3232	,,	,, 9	4 P.M. .	56 53	40 37	4·5	19·30	34·87	—	—
3233	,,	,, 10	8 A.M. .	57 5	43 36	3·8	19·30	34·87	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3234	Thorwaldsen.	May 10	noon . .	57° 28′	44° 30′	3·8	19·29	34·86	—	—
3235	,,	,, 10	4 P.M. .	57 33	45 29	4·0	19·22	34·72	—	—
3236	,,	,, 11	8 A.M. .	57 46	47 17	2·7	19·15	34·60	—	—
3237	,,	,, 11	noon . .	57 51	47 23	3·0	19·16	34·62	—	—
3238	,,	,, 11	4 P.M. .	58 3	48 4	3·5	19·29	34·86	—	—
3239	,,	July 31	noon . .	60 18	7 16	12·0	19·44	35·12	—	—
3240	,,	Aug. 1	,,	60 9	5 47	13·8	19·44	35·12	—	—
3241	,,	,, 2	8 A.M. .	59 44	3 59	13·7	19·61	35·43	26·30	—
3242	,,	,, 2	4 P.M. .	59 36	2 55	12·8	19·54	35·30	26·20	—
3243	,,	July 16	noon . .	56 4	33 20	10·7	19·37	35·00	—	—
3244	,,	,, 10	4 P.M. .	57 55	49 17	5·5	19·18	34·65	—	—
3245	,,	,, 11	8 A.M. .	57 19	47 1	5·5	19·19	34·67	—	—
3246	,,	,, 11	noon . .	57 8	46 31	6·0	19·21	34·71	—	—
3247	,,	,, 11	4 P.M. .	57 8	46 14	6·0	19·25	34·78	25·81	—
3248	,,	,, 12	8 A.M. .	57 8	43 38	5·5	19·28	34·84	—	—
3249	,,	,, 12	noon . .	56 52	42 57	6·5	19·25	34·78	—	—
3250	,,	,, 13	8 A.M. .	56 59	39 50	6·4	19·33	34·93	—	—
3251	,,	,, 13	noon . .	57 1	39 15	6·5	19·28	34·84	25·88	—
3252	,,	,, 13	4 P.M. .	57 2	38 35	7·3	19·27	34·82	—	—
3253	,,	,, 14	8 A.M. .	57 5	35 57	8·8	19·30	34·87	—	—
3254	,,	,, 14	noon . .	56 51	35 20	9·0	19·35	34·96	—	—
3255	,,	,, 14	4 P.M. .	56 51	35 3	9·7	19·30	34·87	25·71	—
3256	,,	,, 15	8 A.M. .	56 19	34 9	9·5	19·39	35·03	—	—
3257	,,	,, 15	noon . .	56 12	33 51	10·7	19·45	35·14	—	—
3258	,,	,, 15	4 P.M. .	56 5	33 29	10·7	19·47	35·17	—	—
3259	,,	,, 21	noon . .	57 10	32 53	11·5	19·41	35·07	—	—
3260	,,	,, 25	,,	57 9	28 27	14·0	19·56	35·34	—	—
3261	,,	,, 26	8 A.M. .	57 48	25 57	12·4	19·53	35·28	—	—
3262	,,	,, 26	noon . .	57 49	24 43	12·8	19·56	35·34	—	—
3263	,,	,, 26	4 P.M. .	57 59	24 2	13·0	19·36	34·98	—	—
3264	,,	,, 27	8 A.M. .	58 30	21 52	12·2	19·50	35·23	—	—
3265	,,	,, 27	noon . .	58 36	20 58	13·6	—	—	—	—
3266	,,	,, 27	4 P.M. .	58 48	20 2	13·0	19·53	35·28	—	—
3267	,,	,, 28	8 A.M. .	59 7	17 34	13·2	—	—	—	—
3268	,,	,, 28	noon . .	59 11	16 52	13·1	—	—	—	—
3269	,,	,, 28	4 P.M. .	59 15	16 6	13·0	—	—	—	—
3270	,,	,, 29	8 A.M. .	59 29	13 11	12·5	—	—	—	—
3271	,,	,, 29	noon . .	59 30	13 4	13·0	—	—	—	—
3272	,,	,, 29	4 P.M. .	59 34	12 31	13·0	—	—	—	—
3273	,,	,, 30	noon . .	59 55	9 51	12·7	—	—	—	—
3271A	Teutonic . .	Aug. 5	midnight	51 28	15 18	16·1	19·63	35·47	—	—
3272A	,,	,, 6	noon . .	51 30	21 0	15·6	19·61	35·43	—	—
3273A	,,	,, 6	midnight	50 46	27 25	15·0	19·66	35·52	—	—
3274	,,	,, 7	noon . .	50 0	34 4	15·0	19·47	35·17	—	—
3275	,,	,, 7	midnight	48 34	40 7	15·6	19·53	35·28	—	—
3276	,,	,, 8	noon . .	47 10	45 58	11·7	18·28	33·04	—	—
3277	,,	,, 8	midnight	45 32	51 25	13·3	18·01	32·56	—	—
3278	,,	,, 9	noon . .	43 53	56 55	17·8	18·11	32·73	—	—
3279	,,	,, 9	midnight	42 33	62 22	18·9	17·57	31·77	23·52	—
3280	,,	,, 10	noon . .	41 11	67 57	16·7	17·90	32·37	24·03	—
3281	,,	,, 10	midnight	off Fire	Island	21·7	17·02	30·78	—	—
3282	,,	,, 18	,,	40° 10′	70° 0′	22·2	18·66	33·73	—	—
3283	,,	,, 19	noon . .	41·16	65 12	22·2	19·00	34·33	—	--

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3284	Teutonic . .	Aug. 19	midnight	42° 14′	60° 26′	19·4	17·69	31·99	—	—
3285	,,	,, 20	noon . .	43 27	55 41	20·6	18·50	33·44	—	—
3286	,,	,, 20	midnight	44 48	50 55	16·7	17·98	32·51	—	—
3287	,,	,, 21	noon . .	46 11	45 40	13·9	17·95	32·46	—	—
3288	,,	,, 21	midnight	47 41	40 14	15·0	18·45	33·36	—	—
3289	,,	,, 22	noon . .	49 5	35 1	17·2	19·68	35·56	—	—
3290	,,	,, 22	midnight	50 8	30 0	15·6	19·69	35·58	—	—
3291	,,	,, 23	noon . .	50 47	23 33	15·0	19·67	35·54	—	—
3292	,,	,, 23	midnight	51 10	17 27	15·0	19·65	35·50	—	—
3293	,,	,, 24	noon . .	51 20	11 29	14·4	19·63	35·47	—	—
3294	Anchoria .	July 31	,,	55 10	14 4	15·0	19·63	35·47	—	—
3295	,,	Aug. 1	,,	54 36	22 11	14·4	19·59	35·40	—	—
3296	,,	,, 2	,,	53 39	29 47	13·3	19·49	35·21	—	—
3297	,,	,, 3	,,	52 9	37 0	14·4	19·53	35·28	—	—
3298	,,	,, 4	,,	50 10	43 29	15·6	18·73	33·85	—	—
3299	,,	,, 5	,,	47 45	49 57	11·7	17·52	31·68	—	—
3300	,,	,, 6	,,	45 4	55 26	16·1	18·11	32·73	—	—
3301	,,	,, 7	,,	42 36	61 37	17·2	17·67	31·95	23·60	—
3302	,,	,, 8	,,	40 59	68 34	15·0	18·00	32·54	—	—
3303	,,	,, 15	,,	40 36	69 23	15·0	18·01	32·56	—	—
3304	,,	,, 16	,,	42 2	63 40	18·9	17·56	31·75	23·42	—
3305	,,	,, 17	,,	44 6	57 37	17·2	17·38	31·43	23·36	—
3306	,,	,, 18	,,	46 30	51 45	13·9	17·93	32·42	—	—
3307	,,	,, 19	,,	49 6	45 49	12·2	18·61	33·64	—	—
3308	,,	,, 20	,,	50 53	39 14	14·4	19·46	35·15	—	—
3309	,,	,, 21	,,	52 30	32 0	12·2	19·42	35·08	26·08	—
3310	,,	,, 22	,,	54 2	24 33	12·8	19·55	35·32	—	—
3311	,,	,, 23	,,	54 50	16 29	13·9	19·61	35·43	26·30	—
3312	,,	,, 24	,,	55 16	8 26	13·9	19·47	35·17	—	—
3313	Siberian . .	,, 1	,,	off Kinsale Hd., Ireland		17·8	19·44	35·12	—	—
3314	,,	,, 2	,,	51° 36′	15° 56′	18·3	19·72	35·63	—	—
3315	,,	,, 3	,,	51 44	23 23	15·6	19·26	34·80	—	—
3316	,,	,, 4	,,	51 46	29 51	14·4	19·82	35·80	26·58	—
3317	,,	,, 5	,,	51 18	37 0	15·0	19·68	35·56	—	—
3318	,,	,, 6	,,	50 1	43 35	16·7	18·79	33·96	—	—
3319	,,	,, 7	,,	48 10	49 48	11·1	17·46	31·57	23·54	—
3320	,,	,, 9	,,	off C. Spear, Newfoundland		16·1	17·40	31·47	—	—
3321	,,	,, 10	,,	45° 30′	58° 15′	16·7	16·78	30·37	22·42	—
3322	,,	,, 13	,,	41 17	66 53	15·6	18·09	32·70	—	—
3323	,,	,, 14	,,	39 8	72 18	23·9	18·93	34·21	—	—
3324	,,	,, 20	,,	38 52	74 3	23·9	17·20	31·11	—	—
3325	,,	,, 21	,,	40 7	69 5	22·2	17·21	31·13	—	—
3326	,,	,, 22	,,	41 38	64 0	19·4	18·97	34·28	—	—
3327	,,	,, 23	,,	43 36	59 7	20·0	17·46	31·57	23·39	—
3328	,,	,, 24	,,	45 58	54 0	15·0	17·52	31·68	—	—
3329	,,	,, 25	,,	48 12	51 5	11·7	17·43	31·52	—	—
3330	,,	,, 26	,,	50 12	45 32	14·4	19·25	35·78	—	—
3331	,,	,, 27	,,	52 3	39 15	12·2	19·45	35·14	—	—
3332	,,	,, 28	,,	53 25	32 23	11·7	19·56	35·34	26·28	—
3333	,,	,, 29	,,	54 25	25 20	16·1	19·64	35·48	—	—
3334	,,	,, 30	,,	55 11	17 55	15·6	19·60	35·41	—	—
3335	,,	,, 31	,,	55 19	10 10	13·9	—	—	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3336	Siberian . .	Sept. 1	2 A.M. .	off Ails	a Craig	13·9	18·58	33·58	—	—
3337	Laura . .	Aug. 22	8 P.M. .	59° 28′	3° 3′	12·3	19·57	35·36	—	—
3338	,,	,, 23	4 A.M. .	60 6	3 43	12·0	19·60	35·41	—	—
3339	,,	,, 23	noon . .	60 55	5 29	12·0	19·49	35·21	—	—
3340	,,	,, 24	4 A.M .	61 56	6 45	10·3	19·46	35·14	26·12	—
3341	,,	,, 25	8 P.M. .	62 24	7 26	11·3	19·45	35·14	—	—
3342	,,	,, 26	4 A.M. .	62 31	10 24	11·5	19·41	35·07	—	—
3343	,,	,, 26	noon . .	62 41	13 32	12·0	19·47	35·17	—	—
3344	,,	,, 26	8 P.M. .	62 48	16 28	12·2	19·46	35·14	26·06	—
3345	,,	,, 27	4 A.M. .	62 50	19 18	11·5	19·54	35·30	—	—
3346	,,	,, 27	noon . .	63 37	21 14	11·5	19·20	34·69	—	—
3347	,,	,, 27	8 P.M. .	64 14	22 34	11·0	18·98	34·30	25·53	—
3348	,,	Sept. 2	4 A.M. .	64 8	23 0	10·2	19·16	34·62	—	—
3349	,,	,, 2	noon . .	63 30	20 52	11·0	18·78	33·94	25·21	—
3350	,,	,, 2	8 P.M. .	63 20	19 33	10·5	18·81	34·00	—	—
3351	,,	,, 3	4 A.M. .	63 0	16 51	10·3	19·47	35·17	—	—
3352	,,	,, 3	noon . .	62 42	14 9	10·5	19·46	35·15	26·12	—
3353	,,	,, 3	8 P.M. .	62 36	11 24	10·0	19·51	35·25	—	—
3354	,,	,, 4	4 A.M. .	62 25	8 46	9·5	19·53	35·28	—	—
3355	,,	,, 4	noon . .	62 20	7 0	9·5	19·53	35·28	—	—
3356	,, .	,, 5	4 A.M. .	61 56	6 35	9·5	19·52	35·27	—	—
3357	,,	,, 5	noon . .	61 10	5 12	10·0	19·50	35·23	—	—
3358	,,	,, 6	4 A.M. .	60 12	3 33	11·0	19·47	35·17	—	—
3359	,,	,, 6	noon . .	59 41	2 40	11·5	19·60	35·41	—	—
3360	Teutonic . .	,, 2	midnight	51 24	15 0	14·4	19·78	35·73	—	—
3361	,,	,, 3	noon . .	51 19	21 19	13·3	19·81	35·79	26·60	—
3362	,,	,, 3	midnight	50 37	27 46	12·2	19·72	35·63	—	—
3363	,,	,, 4	noon . .	50 0	33 55	16·7	19·60	35·41	—	—
3364	,,	,, 4	midnight	48 45	39 50	18·9	19·14	34·58	—	—
3365	,,	,, 5	noon . .	47 10	45 51	13·3	18·35	33·17	—	—
3366	,,	,, 5	midnight	45 3	51 0	14·4	18·00	32·54	—	—
3367	,,	,, 6	noon . .	43 47	57 11	17·8	16·72	30·24	22·22	—
3368	,,	,, 6	midnight	42 23	62 40	20·0	18·03	32·59	—	—
3369	,,	,, 7	noon . .	40 58	67 56	16·7	18·04	32·61	—	—
3370	,,	,, 7	midnight	off Fire	Island	20·0	17·11	30·95	22·92	—
3371	,,	,, 15	,,	40° 9′	69° 55′	18·3	18·73	33·85	—	—
3372	,,	,, 16	noon . .	41 4	65 4	17·8	17·67	31·95	23·65	—
3373	,,	,, 16	midnight	42 15	61 10	16·7	18·03	32·59	—	—
3374	,,	,, 17	noon . .	43 30	55 23	17·8	17·90	32·37	—	—
3375	,,	,, 17	midnight	44 48	50 33	15·0	17·98	32·51	—	—
3376	,,	,, 18	noon . .	46 2	45 40	15·0	18·24	32·97	—	—
3377	,,	,, 18	midnight	47 35	40 33	18·9	19·44	35·12	—	—
3378	,,	,, 19	noon . .	49 5	35 22	17·2	19·60	35·41	—	—
3379	,,	,, 19	midnight	49 56	29 50	16·7	19·56	35·34	—	—
3380	,,	,, 20	noon . .	50 35	23 34	15·6	19·67	35·54	26·34	—
3381	,,	,, 20	midnight	51 2	17 22	15·0	19·64	35·48	—	—
3382	,,	,, 21	noon . .	51 24	11 12	14·4	19·65	35·50	—	—
					E.					
3383	Balaena . .	Apr. 19	4 P.M. .	64 40	2 0	7·8	19·45	35·14	—	—
3384	,,	,, 20	noon . .	66 22	2 40	6·1	19·41	35·07	—	—
3385	,,	,, 21	,,	68 40	2 50	5·0	19·40	35·05	26·04	—
3386	,,	,, 22	,,	71 40	2 50	3·1	19·36	34·98	—	—
3387	,,	,, 23	,,	74 5	3 0	2·2	19·25	34·78	—	—
3388	,,	,, 24	,,	76 40	2 30	2·2	19·35	34·96	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	E.					
3389	Balaena . .	Apr. 25	noon . .	77° 40′	1° 0′	0·6	19·31	34·89	—	—
3390	,,	,, 26	,,	78 30	0 0	-1·1	19·13	34·56	—	—
3391	,,	,, 27	,,	79 10	1 0	0·0	19·00	34·33	—	—
3392	,,	,, 28	,,	79 0	1 30	0·3	19·06	34·44	25·58	—
3393										
					W.					
3394	,,	May 1	1 P.M. .	78 30	0 30	0·0	18·78	33·94	—	—
3395	,,	,, 19	,,	77 55	3 30	0·0	19·13	34·96	—	—
3396	,,	,, 20	,,	77 6	4 30	-0·6	18·97	34·28	—	—
3397	,,	,, 21	,,	76 25	6 20	-0·1	18·99	34·32	—	—
3398	,,	,, 22	,,	76 20	5 0	-0·2	18·94	34·23	25·40	—
3399	,,	,, 23	,,	77 30	3 30	0·3	19·19	34·67	—	—
					E.					
3400	,,	,, 24	,,	78 30	0 20	0·6	19·38	35·01	—	—
3401	,,	,, 25	,,	79 20	1 30	0·1	19·21	34·71	—	—
3402	,,	,, 26	,,	79 45	2 0	0·4	19·02	34·37	—	—
3403	,,	,, 27	,,	78 40	0 0	0·6	19·08	34·47	25·58	—
3404	,,	,, 30	,,	78 35	0 20	0·0	19·08	34·47	—	—
					W.					
3405	,,	June 1	,,	78 11	2 0	0·3	19·09	34·49	—	—
3406	,,	,, 2	,,	78 30	1 0	0·0	18·89	34·14	—	—
					E.					
3407	,,	,, 3	,,	79 20	3 10	0·0	18·88	34·12	—	—
3408	,,	,, 4	,,	79 10	2 30	-0·6	18·82	34·01	—	—
3409	,,	,, 5	,,	78 50	1 20	-0·6	18·92	34·19	—	—
3410	,,	,, 6	,,	78 40	0 50	0·3	19·10	34·51	25·50	—
3411	,,	,, 7	,,	78 38	0 0	0·0	18·96	34·26	—	—
					W.					
3412	,,	,, 8	,,	78 32	0 50	0·6	18·96	34·26	—	—
3413	,,	,, 10	,,	77 15	1 20	0·6	19·11	34·53	—	—
3414	,,	,, 11	,,	76 0	6 20	1·1	19·03	34·39	—	—
3415	,,	,, 12	,,	74 45	10 0	0·6	18·98	34·30	—	—
3416	,,	,, 13	,,	74 50	12 0	0·0	18·73	33·85	25·12	—
3417	,,	,, 15	,,	74 45	13 0	-0·6	18·62	33·66	—	—
3418	,,	,, 17	,,	74 24	13 0	-0·1	18·53	33·49	—	—
3419	,,	,, 18	,,	74 4	14 40	0·0	18·55	33·53	—	—
3420	,,	,, 22	,,	75 10	12 0	0·3	18·67	33·75	—	—
3421	,,	,, 24	,,	73 36	15 30	0·8	18·38	33·22	—	—
3422	,,	,, 25	,,	73 5	13 0	2·8	18·67	33·75	—	—
3423	,,	,, 26	,,	73 10	6 0	1·1	18·75	33·89	—	—
3424	,,	,, 27	,,	73 10	0 0	2·5	19·05	34·42	—	—
					E.					
3425	,,	,, 28	,,	73 30	8 0	5·6	19·35	34·96	—	—
3426	,,	,, 29	,,	74 0	15 0	6·1	19·40	35·05	26·04	—
3427	,,	,, 30	4 P.M. .	75 0	20 0	0·6	18·58	33·58	—	—
3428	,,	July 2	1 P.M. .	75 50	28 0	2·2	18·60	33·62	24·98	—
3429	,,	,, 3	,,	76 2	31 0	0·6	17·57	31·77	—	—
3430	,,	,, 4	,,	76 0	37 30	0·0	17·73	32·06	—	—
3431	,,	,, 5	,,	76 40	44 0	0·6	17·84	32·26	—	—
3432	,,	,, 6	,,	78 20	47 0	0·6	18·60	33·62	—	—
3433	,,	,, 7	,,	79 35	50 0	0·8	18·70	33·80	—	—
3434	,,	,, 11	,,	80 0	48 0	1·4	18·48	33·41	24·67	—
3435	,,	Aug. 13	,,	77 50	34 0	0·0	16·64	30·11	22·20	—
3436	,,	,, 14	,,	76 30	31 0	1·7	17·65	31·92	—	—
3437	,,	,, 15	,,	76 0	31 30	2·8	17·99	32·53	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	E.					
3438	Balaena . .	Aug. 16	1 P.M. .	75° 30′	32° 30′	4·4	19·21	34·71	—	—
3439	,,	,, 17	,,	73 30	36 0	6·4	19·45	35·14	—	—
3440	,,	,, 18	,,	71 40	29 0	8·9	19·06	34·44	—	—
3441	,,	,, 20	,,	70 20	23 0	10·0	18·48	33·41	—	—
3442	,,	,, 21	,,	70 10	17 10	10·0	19·12	34·54	—	—
3443	,,	,, 22	,,	68 40	12 0	12·2	18·81	34·00	—	—
3444	,,	,, 23	,,	67 0	9 10	13·3	18·66	33·73	—	—
3445	,,	,, 24	,,	65 20	7 30	12·8	18·60	33·62	—	—
3446	,,	,, 25	,,	64 6	6 30	13·3	18·65	33·71	—	—
3447	,,	,, 26	,,	62 48	4 0	13·6	18·82	34·01	—	—
					W.					
3448	Anchoria. .	Sept. 4	noon . .	55 14	14 11	13·3	19·78	35·73	—	—
3449	,,	,, 5	,,	54 42	21 41	13·3	19·65	35·50	—	—
3450	,,	,, 6	,,	53 40	29 33	12·8	19·62	35·45	—	—
3451	,,	,, 7	,,	51 57	37 23	12·8	19·59	35·40	—	—
3452	,,	,, 8	,,	49 57	43 49	15·0	19·50	35·23	—	—
3453	,,	,, 9	,,	47 34	49 52	11·7	17·52	31·68	—	—
3454	,,	,, 10	,,	45 2	55 58	13·9	17·97	32·49	—	—
3455	,,	,, 11	,,	42 34	62 17	17·8	17·81	32·21	23·79	—
3456	,,	,, 12	,,	40 51	69 0	17·2	18·30	33·08	—	—
3457	,,	,, 19	,,	40 28	68 45	16·1	17·95	32·46	—	—
3458	,,	,, 20	,,	41 54	63 0	20·0	18·64	33·69	—	—
3459	,,	,, 21	,,	43 59	57 24	15·6	17·94	32·44	—	—
3460	,,	,, 22	,,	46 35	51 32	11·7	17·40	31·47	—	—
3461	,,	,, 23	,,	49 7	46 3	11·1	18·81	34·00	—	—
3462	,,	,, 24	,,	51 16	39 26	11·7	19·18	34·65	—	—
3463	,,	,, 25	,,	52 57	32 16	11·7	19·45	35·14	26·10	—
3464	,,	,, 26	,,	54 9	24 33	12·8	19·61	35·43	—	—
3465	,,	,, 27	,,	54 51	16 14	12·8	19·70	35·59	—	—
3466	,,	,, 28	,,	55 4	8 41	13·9	19·45	35·14	—	—
3467	Traveller. .	July 28	,,	57 46	17 31	13·6	19·55	35·32	—	—
3468	,,	,, 29	,,	57 38	13 28	13·6	19·52	35·27	—	—
3469	,,	,, 30	,,	58 7	11 14	13·4	19·58	35·38	—	—
3470	,,	,, 31	,,	58 28	10 38	13·9	19·56	35·34	26·23	—
3471	,,	Aug. 1	,,	58 36	7 56	13·9	19·57	35·36	—	—
3472	,,	,, 2	,,	58 36	5 30	13·8	19·43	35·10	—	—
3473	,,	,, 30	,,	58 41	7 2	12·8	19·55	35·32	—	—
3474	,,	,, 31	,,	59 25	12 16	12·2	19·50	35·23	26·23	—
3475	,,	Sept. 1	,,	59 7	18 30	12·2	19·44	35·12	—	—
3476	,,	,, 2	,,	58 40	22 46	12·2	19·51	35·25	—	—
3477	,,	,, 3	,,	58 31	25 59	11·9	19·50	35·23	—	—
3478	,,	,, 4	,,	59 33	28 54	10·6	19·42	35·08	—	—
3479	,,	,, 5	,,	59 50	30 18	10·0	19·39	35·03	—	—
3480	,,	,, 6	,,	59 25	32 44	10·0	19·33	34·93	—	—
3481	,,	,, 7	,,	59 6	35 0	9·7	19·31	34·89	—	—
3482	,,	,, 8	,,	58 32	39 27	9·3	19·28	34·84	—	—
3483	,,	,, 9	,,	58 22	44 40	8·1	19·17	34·63	25·75	—
3484	,,	,, 10	,,	58 55	46 38	8·1	19·21	34·71	—	—
3485	,,	,, 11	,,	59 30	47 35	7·3	19·15	34·60	—	—
3486	,,	,, 12	,,	61 0	48 40	1·4	17·03	30·80	22·93	—
3487	,,	,, 24	,,	59 40	49 13	6·7	19·15	34·60	—	—
3488	,,	,, 25	,,	58 27	45 55	7·1	19·35	34·96	—	—
3489	,,	,, 26	,,	58 8	43 40	7·1	19·27	34·82	—	—
3490	,,	,, 27	,,	58 1	38 38	8·1	19·31	34·89	—	—
3491	,,	,, 28	,,	57 48	34 36	9·2	19·36	34·98	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3492	Traveller .	Sept. 29	noon . .	57° 47′	33° 12′	10·1	19·51	31·25	—	—
3493	,,	,, 30	,,	58 28	31 46	9·7	19·33	34·93	—	—
3494	,,	Oct. 1	,,	58 40	30 23	10·0	19·36	34·98	—	—
3495	,,	,, 2	,,	58 40	25 45	11·1	19·50	35·23	—	—
3496	,,	,, 3	,,	58 47	21 41	11·6	19·51	35·25	—	—
3497	,,	,, 4	,,	59 7	18 19	11·4	19·49	35·21	—	—
3498	,,	,, 5	,,	59 37	16 31	11·1	19·45	35·14	—	—
3499	,,	,, 6	,,	58 57	12 14	11·7	19·53	35·28	—	—
3500	,,	,, 7	,,	58 18	8 4	11·9	19·39	35·03	25·95	—
3501	,,	,, 8	,,	Pentland Firth		11·8	19·21	34·71	—	—
					E.					
3502	,,	,, 9	,,	58° 2′	0° 52′	11·1	19·57	35·36	—	—
3503	,,	,, 10	,,	57 54	4 33	11·3	18·89	34·14	—	—
					W.					
3504	Siberian . .	Sept. 12	,,	51 40	8 22	14·4	19·59	34·50	—	—
3505	,,	,, 13	,,	51 44	15 33	15·0	19·67	35·54	—	—
3506	,,	,, 14	,,	51 48	22 22	15·0	19·56	35·34	—	—
3507	,,	,, 15	,,	51 37	28 26	13·9	19·59	35·40	—	—
3508	,,	,, 16	,,	51 5	35 20	13·9	19·44	35·12	—	—
3509	,,	,, 17	,,	50 7	42 3	15·6	19·40	35·05	—	—
3510	,,	,, 18	,,	48 47	48 32	10·6	18·71	33·82	—	—
3511	,,	,, 21	,,	off Ferryland Point, Newfoundland		12·8	17·05	30·83	22·77	—
3512	,,	,, 22	,,	45° 42′	55° 39′	12·2	17·82	32·22	—	—
3513	,,	,, 23	,,	off Green Id. Nova Scotia		15·0	16·34	29·58	—	—
3514	,,	,, 25	,,	42° 21′	65° 30′	15·0	17·45	31·56	23·25	—
3515	,,	,, 26	,,	39 51	70 20	19·4	19·23	34·74	—	—
3516	,,	Oct. 3	,,	39 6	72 35	18·9	18·98	34·30	—	—
3517	,,	,, 4	,,	40 20	68 50	15·6	18·33	33·13	—	—
3518	,,	,, 5	,,	41 42	64 12	15·6	17·85	32·28	—	—
3519	,,	,, 6	,,	43 28	59 9	13·3	17·03	30·80	—	—
3520	,,	,, 7	,,	45 50	54 56	12·2	17·87	32·31	—	—
3521	,,	,, 9	,,	49 17	48 20	6·7	18·76	33·90	—	—
3522	,,	,, 10	,,	51 17	42 19	15·6	19·57	35·36	—	—
3523	,,	,, 11	,,	52 30	36 24	10·0	19·32	34·91	26·00	—
3524	,,	,, 12	,,	53 49	29 43	11·1	19·54	35·30	—	—
3525	,,	,, 13	,,	55 0	22 50	13·3	19·68	35·56	—	—
3526	,,	,, 14	,,	55 20	19 0	7·8	19·67	35·54	—	—
3527	,,	,, 15	,,	55 33	16 25	11·1	19·64	35·48	—	—
3528	,,	,, 16	,,	56 0	13 30	11·1	19·64	35·48	—	—
3529	,,	,, 17	,,	off Mull of Kintyre		13·9	18·79	33·96	—	—
3530	Teutonic .	Sept. 30	midnight	51° 23′	15° 30′	13·9	19·67	35·54	—	—
3531	,,	Oct. 1	noon . .	51 22	21 14	15·6	19·91	35·97	—	—
3532	,,	,, 1	midnight	50 13	27 5	15·0	19·73	35·64	—	—
3533	,,	,, 2	noon . .	50 5	33 27	14·4	19·50	35·23	—	—
3534	,,	,, 2	midnight	48 50	38 20	16·1	19·75	35·68	—	—
3535	,,	,, 3	noon . .	47 34	44 19	10·6	18·69	33·78	25·10	—
3536	,,	,, 3	midnight	46 0	49 10	12·2	18·03	32·59	—	—
3537	,,	,, 4	noon . .	44 26	54 56	14·4	18·21	32·91	—	—
3538	,,	,, 4	midnight	42 50	60 26	12·8	17·27	31·23	—	—
3539	,,	,, 5	noon . .	41 36	65 56	16·7	18·21	32·91	—	—
3540	,,	,, 5	midnight	off Fire Id.		16·1	18·34	33·15	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3541	Teutonic	Oct. 13	midnight	40° 10′	70° 0′	20·0	19·68	35·56	—	—
3542	,,	,, 14	noon	40 58	65 17	16·7	18·22	32·93	—	—
3543	,,	,, 14	midnight	42 10	60 10	15·6	17·89	32·35	—	—
3544	,,	,, 15	noon	43 23	55 40	15·6	18·40	33·26	—	—
3545	,,	,, 15	midnight	44 47	50 3	12·2	18·01	32·56	—	—
3546	,,	,, 16	noon	46 12	45 51	11·1	18·49	33·43	—	—
3547	,,	,, 16	midnight	47 32	40 50	17·8	19·66	35·52	—	—
3548	,,	,, 17	noon	48 51	35 39	16·7	19·76	35·70	—	—
3549	,,	,, 17	midnight	49 48	29 55	15·0	19·78	35·73	—	—
3550	,,	,, 18	noon	50 45	24 37	15·6	19·76	35·70	—	—
3551	,,	,, 18	midnight	51 3	18 30	15·0	19·74	35·66	—	—
3552	,,	,, 19	noon	51 20	12 23	15·0	19·72	35·63	—	—
3553	Laura	Sept. 27	4 A.M.	59 36	2 28	11·0	19·39	35·03	—	—
3554	,,	,, 27	noon	60 20	3 57	10·0	19·60	35·41	—	—
3555	,,	,, 27	8 P.M.	61 21	5 52	9·5	19·47	35·17	—	—
3556	,,	,, 28	4 A.M.	61 57	6 45	9·5	19·49	35·21	26·17	—
3557	,,	,, 30	,,	62 24	6 50	9·0	19·46	35·15	—	—
3558	,,	,, 30	noon	62 29	8 25	9·0	19·44	35·12	—	—
3559	,,	,, 30	8 P.M.	62 35	10 53	9·0	19·50	35·23	—	—
3560	,,	Oct. 1	4 A.M.	62 39	13 32	9·0	19·46	35·15	26·16	—
3561	,,	,, 1	noon	63 2	16 56	9·3	19·34	34·94	—	—
3562	,,	,, 1	8 P.M.	63 16	18 46	8·5	18·53	33·49	—	—
3563	,,	,, 2	4 A.M.	63 18	21 16	8·5	19·26	34·80	—	—
3564	,,	,, 2	noon	64 12	22 33	8·5	18·31	33·10	24·61	—
3565	,,	,, 6	8 P.M.	64 33	24 10	9·0	19·43	35·10	—	—
3566	,,	,, 7	4 A.M.	65 45	24 55	8·0	18·38	35·01	—	—
3567	,,	,, 10	noon	66 2	23 53	7·3	18·96	34·26	25·33	—
3568	,,	,, 15	4 A.M.	65 8	23 26	5·2	19·30	34·87	—	—
3569	,,	,, 15	noon	65 18	23 12	6·3	19·19	34·67	—	—
3570	,,	,, 18	,,	64 58	24 10	6·2	19·26	34·80	—	—
3571	,,	,, 18	8 P.M.	64 15	22 26	7·7	19·33	34·93	—	—
3572	,,	,, 20	,,	64 11	22 26	7·5	19·20	34·69	—	—
3573	,,	,, 21	4 A.M.	63 40	21 47	7·0	19·43	35·10	—	—
3574	,,	,, 21	8 P.M.	63 6	19 36	8·4	19·50	35·23	—	—
3575	,,	,, 22	4 A.M.	62 52	16 54	9·0	19·44	35·12	—	—
3576	,,	,, 22	noon	62 42	14 8	9·0	19·46	35·15	26·11	—
3577	,,	,, 22	8 P.M.	62 33	11 14	9·0	19·49	35·21	—	—
3578	,,	,, 23	4 A.M.	62 23	8 25	7·5	19·43	35·10	—	—
3579	,,	,, 23	noon	62 38	7 5	8·0	19·54	35·30	—	—
3580	,,	,, 24	,,	61 56	6 41	8·0	19·57	35·36	—	—
3581	,,	,, 24	8 P.M.	61 3	5 15	8·0	19·45	35·14	—	—
3582	,,	,, 25	4 A.M.	60 15	3 47	8·7	19·57	35·36	—	—
3583	Anchoria	,, 9	noon	55 23	14 7	12·2	19·64	35·48	—	—
3584	,,	,, 10	,,	54 41	21 51	12·8	19·56	35·34	—	—
3585	,,	,, 11	,,	53 36	30 21	12·2	19·47	35·17	—	—
3586	,,	,, 12	,,	51 56	38 0	11·7	19·24	34·76	25·73	—
3587	,,	,, 13	,,	49 48	44 17	11·1	18·77	33·92	—	—
3588	,,	,, 14	,,	47 47	50 13	6·7	17·54	31·71	—	—
3589	,,	,, 15	,,	44 47	55 58	10·6	17·89	32·35	—	—
3590	,,	,, 16	,,	42 32	62 21	13·3	17·42	31·51	23·24	—
3591	,,	,, 17	,,	40 48	68 11	12·8	18·02	32·58	—	—
3592	,,	,, 24	,,	40 40	68 54	12·8	17·93	32·42	—	—
3593	,,	,, 25	,,	41 55	63 6	12·8	18·26	33·00	—	—
3594	,,	,, 26	,,	44 11	57 47	6·7	18·00	32·54	24·08	—
3595	,,	,, 27	,,	46 24	52 10	5·0	17·75	32·09	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3596	Anchoria .	Oct. 28	noon . .	48° 55′	42° 26′	7·2	18·87	34·10	—	—
3597	,,	,, 29	,,	51 4	39 51	11·1	19·16	34·62	—	—
3598	,,	,, 30	,,	52 44	32 14	10·6	19·34	34·94	25·90	—
3599	,,	,, 31	,,	54 6	24 14	12·2	19·58	35·38	—	—
3600	,,	Nov. 1	,,	54 57	16 0	12·8	19·61	35·43	—	—
3601	,,	,, 2	,,	55 22	7 39	12·2	19·46	35·15	—	—
3602	Teutonic . .	Oct. 28	midnight	51 25	14 0	15·0	19·67	35·54	—	—
3603	,,	,, 29	noon . .	51 25	20 45	14·4	19·64	35·48	—	—
3604	,,	,, 29	midnight	50 50	26 48	14·4	19·67	35·54	—	—
3605	,,	,, 30	noon . .	50 15	32 51	13·9	19·51	35·25	—	—
3606	,,	,, 30	midnight	48 21	38 21	15·0	19·57	35·36	—	—
3607	,,	,, 31	noon . .	47 33	43 49	9·4	18·72	33·83	—	—
3608	,,	,, 31	midnight	45 57	49 52	8·3	17·89	32·35	—	—
3609	,,	Nov. 1	noon . .	44 24	55 54	9·4	18·33	33·13	—	—
3610	,,	,, 1	midnight	43 2	60 50	13·9	18·28	33·04	—	—
3611	,,	,, 2	noon . .	41 40	65 45	12·2	17·44	31·54	23·35	—
3612	,,	,, 2	midnight	50 miles W. Nantucket		13·9	17·90	32·37	—	—
3613	,,	,, 10	,,	40° 10′	70° 0′	13·9	18·40	33·26	—	—
3614	,,	,, 11	noon . .	41 1	64 56	13·3	18·11	32·73	—	—
3615	,,	,, 11	midnight	42 14	59 50	12·8	18·03	32·59	—	—
3616	,,	,, 12	noon . .	43 26	55 26	12·8	18·33	33·13	—	—
3617	,,	,, 12	midnight	44 56	50 40	8·3	17·84	32·26	23·93	—
3618	,,	,, 13	noon . .	46 15	45 50	8·3	18·40	33·26	—	—
3619	,,	,, 13	midnight	47 35	40 53	14·4	18·96	34·26	—	—
3620	,,	,, 14	noon . .	48 48	35 57	15·6	19·57	35·36	—	—
3621	,,	,, 14	midnight	49 55	30 37	14·4	19·47	35·17	—	—
3622	,,	,, 15	noon . .	50 35	25 15	14·4	19·50	35·23	—	—
3623	,,	,, 15	midnight	50 58	19 50	13·9	19·61	35·43	—	—
3624	,,	,, 16	noon . .	51 14	13 39	14·4	19·53	35·28	—	—
3625	Eclipse . .	Apr. 20	,,	58 57	8 0	9·7	19·40	35·05	26·08	—
3626	,,	,, 21	,,	59 2	12 5	9·4	19·57	35·36	—	—
3627	,,	,, 22	,,	59 16	16 13	9·1	19·62	35·45	—	—
3628	,,	,, 23	,,	59 51	21 25	9·2	19·57	35·36	—	—
3629	,,	,, 24	,,	59 25	23 21	8·7	19·53	35·28	26·21	—
3630	,,	,, 25	,,	59 10	26 38	7·7	19·49	35·21	—	—
3631	,,	,, 26	,,	59 17	29 26	7·2	19·50	35·23	—	—
3632	,,	,, 27	,,	58 34	30 12	4·4	19·49	35·21	—	—
3633	,,	,, 28	,,	58 12	31 20	7·5	19·46	35·15	—	—
3634	,,	,, 29	,,	56 36	31 32	7·2	19·51	35·25	26·14	—
3635	,,	,, 30	,,	56 24	32 0	7·5	19·52	35·27	—	—
3636	,,	May 1	,,	57 30	31 20	6·8	19·64	35·48	—	—
3637	,,	,, 2	,,	57 50	30 32	7·2	19·54	35·30	—	—
3638	,,	,, 3	,,	58 15	30 27	7·2	19·57	35·36	—	—
3639	,,	,, 4	,,	58 47	30 0	7·5	19·56	35·34	—	—
3640	,,	,, 5	,,	58 49	30 45	6·7	19·49	35·21	—	—
3641	,,	,, 6	,,	59 7	32 43	6·8	19·47	35·17	—	—
3642	,,	,, 7	,,	60 51	35 1	5·3	19·40	35·05	—	—
3643	,,	,, 8	,,	—	—	6·0	19·48	35·19	26·11	—
3644	,,	,, 9	,,	60 31	40 39	4·3	19·19	34·67	—	—
3645	,,	,, 10	,,	58 26	42 22	3·1	19·05	34·42	—	—
3646	,,	,, 11	,,	57 49	43 31	3·9	19·32	34·91	—	—
3647	,,	,, 12	,,	58 30	46 36	2·9	19·15	34·60	25·83	—
3648	,,	,, 13	,,	—	—	1·9	19·00	34·33	—	—
3649	,,	,, 14	,,	58 24	—	3·2	19·38	35·01	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3650	Eclipse . .	May 15	noon . .	58° 36′	53° 44′	3·2	19·39	35·03	—	—
3651	,,	,, 16	,,	—	—	-1·0	18·66	33·73	25·01	—
3652	,,	,, 17	,,	60 53	57 55	-1·1	18·72	33·83	—	—
3653	,,	,, 18	,,	61 18	58 51	-1·0	18·74	33·87	—	—
3654	,,	,, 19	,,	62 0	58 15	-1·1	18·63	33·67	—	—
3655	,,	,, 20	,,	62 46	57 5	-1·1	18·52	33·48	—	—
3656	,,	,, 21	,,	63 10	56 30	-1·1	18·45	33·36	—	—
3657	,,	,, 22	,,	64 19	54 52	0·0	18·61	33·64	—	—
3658	,,	,, 23	,,	65 45	53 22	1·7	18·73	33·85	—	—
3659	,,	,, 24	,,	—	—	1·7	18·59	33·60	24·87	—
3660	,,	,, 25	,,	—	—	1·7	18·66	33·73	—	—
3661	,,	,, 26	,,	65 53	54 30	0·3	18·81	34·00	—	—
3662	,,	,, 27	,,	66 6	56 34	0·2	18·65	33·71	—	—
3663	,,	,, 28	,,	—	—	-0·6	18·59	33·60	—	—
3664	,,	,, 29	,,	67 51	55 10	-1·0	18·37	33·20	—	—
3665	,,	,, 30	,,	68 38	54 10	0·0	18·31	33·10	—	—
3666	,,	June 1	,,	—	—	0·8	18·02	32·58	—	—
3667	,,	,, 2	,,	70 38	54 50	-0·6	18·37	33·20	—	—
3668	,,	,, 3	,,	70 57	54 15	0·0	18·24	32·97	—	—
3669	,,	,, 4	,,	71 6	55 3	0·1	17·86	32·30	—	—
3670	,,	,, 5	,,	72 7	56 3	-0·9	18·49	33·43	—	—
3671	,,	,, 6	,,	—	—	-0·7	18·32	33·11	—	—
3672	,,	,, 7	,,	—	—	0·0	18·45	33·36	—	—
3673	,,	,, 8	,,	73 40	57 20	-1·0	18·34	33·15	24·50	—
3674	,,	,, 9	,,	74 24	58 0	—	18·38	33·22	—	—
3675	,,	,, 10	,,	75 33	65 0	-0·6	18·47	33·39	—	—
3676	,,	,, 11	,,	76 15	70 0	—	18·53	33·49	—	—
3677	,,	,, 12	,,	—	—	5·0	18·52	33·48	—	—
3678	,,	,, 13	,,	75 11	73 20	-1·1	18·56	33·55	—	—
3679	,,	,, 14	,,	75 0	78 0	1·0	18·02	32·58	24·11	—
3680	,,	,, 15	,,	74 30	75 0	1·6	17·95	32·46	—	—
3681	,,	,, 16	,,	74 10	74 30	—	—	—	—	—
3682	,,	,, 17	,,	74 2	74 19	0·9	17·77	32·13	—	—
3683	,,	,, 18	,,	—	—	1·7	17·86	32·30	—	—
3684	,,	,, 19	,,	—	—	1·7	17·82	32·22	—	—
3685	,,	,, 20	,,	—	—	—	—	—	—	—
3686	,,	,, 21	,,	—	—	0·1	17·56	31·75	—	—
3687	,,	,, 22	,,	73 53	—	0·2	17·42	31·51	—	—
3688	,,	,, 23	,,	—	—	4·4	16·98	30·72	—	—
3689	,,	,, 24	,,	73 53	73 47	1·7	17·27	31·23	—	—
3690	,,	,, 25	,,	—	—	4·9	17·95	32·46	—	—
3691	,,	,, 26	,,	—	—	3·6	17·26	31·21	—	—
3692	,,	,, 27	,,	—	—	0·3	17·97	32·49	—	—
3693	,,	,, 28	,,	—	—	0·1	17·88	32·33	—	—
3694	,,	,, 29	,,	74 10	81 30	-1·1	17·90	32·37	—	—
3695	,,	,, 30	,,	—	—	-1·1	15·44	27·97	—	—
3696	,,	July 1	,,	74 10	81 30	-0·6	13·83	25·07	—	—
3697	,,	,, 2	,,	—	—	0·4	4·91	7·11	—	—
3698	,,	,, 3	,,	—	—	0·6	5·05	9·21	6·04	—
3699	,,	,, 4	,,	—	—	0·6	6·88	12·54	8·63	—
3700	,,	,, 5	,,	—	—	0·0	9·88	17·95	—	—
3701	,,	,, 6	,,	—	—	-0·3	16·56	29·96	—	—
3702	,,	,, 7	,,	73 45	77 50	0·6	17·72	32·04	—	—
3703	,,	,, 8	,,	72 45	76 4	0·8	18·29	33·06	—	—
3704	,,	,, 9	,,	73 22	73 3	1·1	17·56	31·75	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO$_3$.
		1897.		N.	W.					
3705	Eclipse . .	July 10	noon . .	73° 30′	72° 50′	0·0	16·74	30·28	—	—
3706	,,	,, 11	,,	72 50	75 0	1·1	17·37	31·41	—	—
3707	,,	,, 12	,,	72 45	—	1·1	16·40	29·68	—	—
3708	,,	,, 13	,,	—	—	1·1	16·04	29·04	—	—
3709	,,	,, 14	,,	72 50	76 12	1·7	17·20	31·11	—	—
3710	,,	,, 15	,,	72 30	76 12	1·7	16·87	30·52	—	—
3711	,,	,, 16	,,	72 0	73 30	1·4	17·29	31·27	23·12	—
3712	,,	,, 17	,,	—	—	0·8	16·15	29·23	—	—
3713	,,	,, 18	,,	71 48	73 34	0·4	7·29	13·28	—	—
3714	,,	,, 19	,,	70 50	69 50	0·6	9·61	17·46	—	—
3715	,,	,, 20	,,	—	—	1·1	3·59	6·55	—	—
3716	,,	,, 21	,,	—	—	1·7	13·86	25·13	—	—
3717	,,	,, 22	,,	—	—	1·4	4·38	7·99	—	—
3718	,,	,, 23	,,	—	—	1·7	1·84	3·36	—	—
3719	,,	,, 24	,,	—	—	1·1	1·97	3·60	1·75	—
3720	,,	,, 25	,,	—	—	4·8	2·39	4·37	—	—
3721	,,	,, 26	,,	70 47	68 18	2·7	12·73	23·09	—	—
3722	,,	,, 27	,,	—	—	1·1	9·29	16·90	—	—
3723	,,	,, 28	,,	—	—	1·3	2·94	5·37	3·11	—
3724	,,	,, 29	,,	—	—	0·9	4·17	7·61	—	—
3725	,,	,, 30	,,	—	—	0·8	4·61	8·41	—	—
3726	,,	Aug. 1	,,	69 47	67 18	0·9	15·61	28·27	20·74	—
3727	,,	,, 2	,,	—	—	1·2	13·89	25·18	—	—
3728	,,	,, 3	,,	—	—	0·8	10·85	19·70	—	—
3729	,,	,, 4	,,	—	—	0·6	15·46	28·02	—	—
3730	,,	,, 5	,,	—	—	0·6	16·63	30·09	22·13	—
3731	,,	,, 6	,,	70 20	68 4	0·9	7·19	13·10	—	—
3732	,,	,, 7	,,	70 27	68 50	0·6	14·83	26·87	—	—
3733	,,	,, 8	,,	—	—	0·7	8·93	16·24	11·51	—
3734	,,	,, 9	,,	—	—	4·3	7·21	13·13	—	—
3735	,,	,, 10	,,	—	—	1·7	12·61	22·87	—	—
3736	,,	,, 11	,,	—	—	3·9	8·88	16·15	—	—
3737	,,	,, 12	,,	70 36	68 28	3·3	13·29	24·11	—	—
3738	,,	,, 13	,,	—	—	5·0	4·64	8·46	5·61	—
3739	,,	,, 14	,,	—	—	—	—	—	—	—
3740	,,	,, 15	,,	—	—	5·8	4·77	8·70	—	—
3741	,,	,, 16	,,	70 40	71 10	5·8	11·09	20·14	—	—
3742	,,	,, 17	,,	—	—	5·7	0·46	0·84	99·72	—
3743	,,	,, 18	,,	—	—	5·8	—	—	—	—
3744	,,	,, 19	,,	—	—	5·8	0·06	0·11	—	—
3745	,,	,, 20	,,	—	—	5·7	1·37	2·50	—	—
3746	,,	,, 21	,,	—	—	5·7	—	—	—	—
3747	,,	,, 22	,,	—	—	5·8	0·16	0·29	—	—
3748	,,	,, 23	,,	70 50	—	4·7	10·28	18·68	—	—
3749	,,	,, 24	,,	70 57	70 41	4·3	11·38	20·66	—	—
3750	,,	,, 25	,,	71 13	70 39	3·9	15·79	28·60	—	—
3751	,,	,, 26	,,	71 45	73 40	4·3	12·65	22·94	—	—
3752	,,	,, 27	,,	—	—	4·2	13·11	23·78	17·25	—
3753	,,	,, 28	,,	—	—	4·1	13·16	23·87	—	—
3754	,,	,, 29	,,	—	—	5·1	12·53	22·73	—	—
3755	,,	,, 30	,,	—	—	3·9	15·77	28·56	—	—
3756	,,	Sept. 1	,,	71 55	74 35	3·3	14·37	26·05	—	—
3757	,,	,, 2	,,	—	—	0·1	14·43	26·16	—	—
3758	,,	,, 3	,,	—	—	—	—	—	—	—
3759	,,	,, 4	,,	—	—	2·9	14·83	26·87	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3760	Eclipse . .	Sept. 5	noon . .	—	—	2·7	14·83	26·87	—	—
3761	,,	,, 6	,,	—	—	2·1	15·32	27·76	—	—
3762	,,	,, 7	,,	—	—	2·3	15·07	27·31	—	—
3763	,,	,, 8	,,	—	—	1·6	16·76	30·32	—	—
3764	,,	,, 9	,,	—	—	2·7	15·85	28·70	—	—
3765	,,	,, 10	,,	—	—	2·2	15·74	28·51	—	—
3766	,,	,, 11	,,	—	—	—	—	—	—	—
3767	,,	,, 12	,,	—	—	—	—	—	—	—
3768	,,	,, 13	,,	—	—	—	16·01	28·99	—	—
3769	,,	,, 14	,,	—	—	0·1	15·85	28·70	—	—
3770	,,	,, 15	,,	—	—	—	—	—	—	—
3771	,,	,, 16	,,	—	—	0·0	16·04	29·04	—	—
3772	,,	,, 17	,,	71° 50′	73° 56′	0·4	16·13	29·20	—	—
3773	,,	,, 18	,,	—	—	0·6	16·13	29·20	—	—
3774	,,	,, 19	,,	—	—	0·3	16·69	30·19	—	—
3775	,,	,, 20	,,	—	—	—	16·40	29·68	—	—
3776	,,	,, 20	,,	—	—	—	16·25	29·42	—	—
3777	,,	,, 20	,,	—	—	—	—	—	—	—
3778	,,	,, 23	,,	72 2	—	0·0	16·22	29·36	—	—
3779	,,	,, 24	,,	—	—	—	—	—	—	—
3780	,,	,, 25	,,	—	—	-0·3	15·21	26·94	—	—
3781	,,	,, 26	,,	71 44	73 40	0·1	16·27	29·46	—	—
3782	,,	,, 27	,,	—	—	0·6	16·42	29·72	—	—
3783	,,	,, 28	,,	70 40	69 58	0·0	16·00	28·97	—	—
3784	,,	,, 29	,,	—	—	—	—	—	—	—
3785	,,	,, 30	,,	70 20	68 8	0·2	16·75	30·30	—	—
3786	,,	Oct. 1	,,	70 17	68 9	0·1	16·69	30·19	22·27	—
3787	,,	,, 2	,,	—	—	—	—	—	—	—
3788	,,	,, 3	,,	70 4	67 15	0·3	16·77	30·34	—	—
3789	,,	,, 4	,,	69 32	67 42	0·1	16·93	30·62	—	—
3790										
3791										
3792	,,	,, 7	,,	69 26	66 10	-0·7	17·15	31·02	22·93	—
3793										
3794	,,	,, 9	,,	69 50	67 10	-0·7	16·80	30·40	—	—
3795	,,	,, 10	,,	70 17	68 9	-0·1	16·87	30·52	—	—
3796	,,	,, 11	,,	—	—	-0·6	16·75	30·30	—	—
3797	,,	,, 12	,,	70 4	67 15	-0·3	16·89	30·55	—	—
3798	,,	,, 13	,,	69 30	66 12	-0·6	17·72	32·04	—	—
3799	,,	,, 14	,,	—	—	-0·6	17·00	30·75	—	—
3800	,,	,, 15	,,	—	—	-0·3	17·06	30·85	—	—
3804	,,	,, 19	,,	69 10	66 40	-1·1	17·14	31·00	—	—
3808	,,	,, 23	,,	68 0	65 45	-1·1	17·41	31·49	—	—
3811	,,	,, 26	,,	66 25	62 0	-1·3	17·40	31·47	—	—
3813	,,	,, 28	,,	64 26	57 55	0·6	18·53	33·49	—	—
3814	,,	,, 29	,,	—	—	1·1	19·10	34·51	—	—
3815	,,	,, 30	,,	61 21	54 36	1·4	18·23	32·95	24·38	—
3816	,,	Nov. 1	,,	58 10	44 18	5·3	19·35	34·96	—	—
3817	,,	,, 2	,,	58 29	39 0	6·1	19·30	34·87	—	—
3818	,,	,, 3	,,	58 44	34 52	7·3	19·35	34·96	—	—
3819	,,	,, 4	,,	58 32	31 18	6·7	19·64	35·48	—	—
3820	,,	,, 5	,,	—	—	9·4	19·49	35·21	—	—
3821	,,	,, 6	,,	—	—	8·9	19·42	35·08	—	—
3822	,,	,, 7	,,	—	—	8·9	19·42	35·08	—	—
3823	,,	,, 8	,,	60 27	28 11	9·4	19·44	35·12	26·09	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO$_3$.
		1897.		N.	W.					
3824	Eclipse . .	Nov. 9	noon . .	60° 29′	24° 44′	8·9	19·50	35·23	—	—
3825	,,	,, 10	,,	60 8	20 39	10·4	19·55	35·32	—	—
3826	,,	,, 11	,,	59 36	15 17	10·6	19·55	35·32	—	—
3827	,,	,, 12	,,	59 6	13 0	10·1	19·52	35·27	26·18	—
3828	,,	,, 13	,,	58 10	12 11	10·8	19·63	35·47	—	—
3829	,,	,, 14	,,	57 27	10 32	10·2	19·62	35·45	—	—
3831	,,	,, 16	,,	57 48	6 15	11·0	19·23	34·74	25·75	—
3832	California .	Oct. 13	,,	36 33	8 31	18·9	20·37	36·78	—	—
3833	,,	,, 13	midnight	37 15	11 4	18·9	20·07	36·25	27·18	—
3834	,,	,, 14	noon . .	37 58	13 36	18·9	20·01	36·14	—	—
3835	,,	,, 14	midnight	38 28	15 38	18·9	20·08	36·27	—	—
3836	,,	,, 15	noon . .	38 54	17 25	18·9	20·05	36·21	—	—
3837	,,	,, 15	midnight	39 24	18 51	18·3	20·07	36·25	—	—
3838	,,	,, 16	noon . .	39 55	20 18	18·3	20·04	36·19	—	—
3839	,,	,, 16	midnight	40 14	21 58	18·3	20·10	36·30	—	—
3840	,,	,, 17	noon . .	40 33	23 43	18·3	20·06	36·23	—	—
3841	,,	,, 17	midnight	41 1	26 7	17·8	19·91	35·97	26·96	—
3842	,,	,, 18	noon . .	41 29	28 32	17·8	19·97	36·07	—	—
3843	,,	,, 18	midnight	41 57	31 2	17·8	19·95	36·04	—	—
3844	,,	,, 19	noon . .	42 24	33 32	17·8	19·81	35·79	—	—
3845	,,	,, 19	midnight	42 36	35 39	17·8	19·87	35·89	—	—
3846	,,	,, 20	noon . .	42 52	37 48	17·8	19·81	35·79	—	—
3847	,,	,, 20	midnight	42 57	39 38	17·2	19·87	35·89	—	—
3848	,,	,, 21	noon . .	43 3	41 28	17·2	19·71	35·61	—	—
3849	,,	,, 21	midnight	43 10	43 56	17·2	19·67	35·54	—	—
3850	,,	,, 22	noon . .	43 18	46 25	19·4	19·65	35·50	—	—
3851	,,	,, 22	midnight	43 22	49 5	11·1	18·57	33·56	—	—
3852	,,	,, 23	noon . .	43 25	51 44	8·9	17·94	32·44	23·99	—
3853	,,	,, 23	midnight	43 1	54 33	14·4	19·03	34·39	25·52	—
3854	,,	,, 24	noon . .	42 37	57 22	13·3	18·67	33·75	—	—
3855	,,	,, 24	midnight	42 15	60 9	12·2	18·68	33·76	—	—
3856	,,	,, 25	noon . .	41 53	62 56	13·3	18·42	33·30	—	—
3857	,,	,, 25	midnight	41 22	65 48	11·1	17·53	31·70	—	—
3858	,,	,, 26	noon . .	40 51	68 44	11·1	17·94	32·44	—	—
3859	,,	,, 26	midnight	40 30	71 24	11·1	18·54	33·51	—	—
3860	,,	Nov. 5	,,	40 15	72 0	11·1	18·28	33·04	—	—
3861	,,	,, 6	noon . .	40 21	69 36	11·1	17·95	32·46	—	—
3862	,,	,, 6	midnight	40 36	67 2	11·1	17·37	31·41	—	—
3863	,,	,, 7	noon . .	40 46	64 28	16·7	19·77	35·72	—	—
3864	,,	,, 7	midnight	41 9	61 53	18·9	19·60	35·41	—	—
3865	,,	,, 8	noon . .	41 32	59 22	19·4	19·50	35·23	—	—
3866	,,	,, 8	midnight	42 3	56 59	19·4	19·15	34·60	—	—
3867	,,	,, 9	noon . .	42 34	54 15	17·8	18·89	34·14	—	—
3868	,,	,, 9	midnight	42 56	51 40	4·4	18·40	33·26	—	—
3869	,,	,, 10	noon . .	43 18	49 5	5·6	18·35	33·17	—	—
3870	,,	,, 10	midnight	43 25	46 12	15·6	19·61	35·43	—	—
3871	,,	,, 11	noon . .	43 34	43 19	16·7	19·53	35·28	—	—
3872	,,	,, 11	midnight	43 17	40 23	16·7	19·87	35·89	—	—
3873	,,	,, 12	noon . .	43 1	37 28	15·6	19·91	35·97	—	—
3874	,,	,, 12	midnight	42 36	34 54	15·0	19·91	35·97	—	—
3875	,,	,, 13	noon . .	42 11	32 8	14·4	20·07	36·25	26·92	—
3876	,,	,, 13	midnight	41 46	29 27	14·4	19·93	36·00	—	—
3877	,,	,, 14	noon . .	41 21	26 46	14·4	19·94	36·02	—	—
3878	,,	,, 14	midnight	40 51	23 18	14·4	20·11	36·31	—	—
3879	,,	,, 15	noon . .	40 20	21 50	14·4	19·90	35·95	26·78	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	${}_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3880	California .	Nov. 15	midnight	39° 47′	19° 27′	15·6	20·21	36·49	—	—
3881	,,	,, 16	noon . .	39 14	17 4	15·6	20·16	36·40	—	—
3882	,,	,, 16	midnight	38 46	14 42	15·6	20·00	36·12	—	—
3883	,,	,, 17	noon . .	37 59	12 22	15·6	20·13	36·35	27·13	—
3884	,,	,, 17	midnight	37 14	10 5	15·6	20·15	36·39	—	—
3885	,,	,, 18	noon . .	36 29	7 48	15·6	20·19	36·46	—	—
3886	,,	,, 18	midnight	36 0	5 20	15·6	20·27	36·60	—	—
3887	Siberian . .	Oct. 24	noon . .	Galley Head, Ireland, bearing N.W.		14·4	19·66	35·52	—	—
3888	,,	,, 25	,,	51° 43′	15° 47′	13·3	19·63	35·47	—	—
3889	,,	,, 26	,,	51 48	22 38	13·3	19·61	35·43	—	—
3890	,,	,, 27	,,	51 37	29 25	13·9	19·58	35·38	—	—
3891	,,	,, 28	,,	51 8	35 35	12·8	19·42	35·08	—	—
3892	,,	,, 29	,,	50 34	41 8	14·4	19·56	35·34	—	—
3893	,,	,, 30	,,	49 12	45 55	7·8	18·82	34·01	—	—
3894	,,	,, 31	,,	47 49	51 29	4·4	17·82	32·22	—	—
3895	,,	Nov. 2	,,	46 1	55 12	6·1	17·72	32·04	—	—
3896	,,	,, 3	,,	44 57	61 0	8·9	16·91	30·59	—	—
3897	,,	,, 5	,,	43 36	64 15	9·4	16·85	30·48	—	—
3898	,,	,, 6	,,	40 24	68 48	13·3	17·99	32·53	—	—
3899	,,	,, 7	,,	38 59	73 0	16·1	19·29	34·86	—	—
3900	,,	,, 13	,,	39 44	70 16	16·7	19·62	35·45	—	—
3901	,,	,, 14	,,	41 8	66 57	12·8	18·01	32·56	—	—
3902	,,	,, 15	,,	44 1	64 0	8·3	17·20	31·11	—	—
3903	,,	,, 16	,,	off Chebrieto Hd., Halifax, Nova Scotia		7·2	17·25	31·19	—	—
3904	,,	,, 17	,,	42° 25′	58° 14′	6·1	17·42	31·51	—	—
3905	,,	,, 18	,,	off C. Race, Newfoundland		3·9	17·49	31·63	—	—
3906	,,	,, 19	,,	48° 33′	49° 55′	1·1	18·08	32·68	—	—
3907	,,	,, 20	,,	50 22	44 49	11·7	19·24	34·76	—	—
3908	,,	,, 21	,,	52 4	39 16	9·4	19·30	34·87	—	—
3909	,,	,, 22	,,	53 2	34 53	10·0	19·29	34·86	—	—
3910	,,	,, 23	,,	53 28	31 27	10·6	19·35	34·96	—	—
3911	,,	,, 24	,,	54 21	25 20	11·1	19·49	35·21	—	—
3912	,,	,, 25	,,	55 4	18 14	11·7	19·61	35·43	26·37	—
3913	,,	,, 26	,,	55 17	11 0	11·1	19·64	35·48	—	—
3914	,,	,, 27	,,	Mull of Galloway		11·7	18·91	34·17	—	—
3915	,,	,, 28	,,	in Crosby Channel Liverpool		10·6	18·34	33·15	—	—
3916	,,	,, 29	,, .	off Portpatrick, Scotland		10·0	19·14	34·58	—	—
3917	Anchoria .	,, 13	,,	55° 19′	14° 40	11·7	19·52	35·27	26·21	—
3918	,,	,, 14	,,	54 34	22 51	12·2	19·58	35·38	—	—
3919	,,	,, 15	,,	53 24	30 49	10·6	19·41	35·07	—	—
3920	,,	,, 16	,,	51 51	38 23	9·4	19·28	34·84	—	—
3921	,,	,, 17	,,	50 0	44 11	11·7	19·63	35·47	26·47	—
3922	,,	,, 18	,,	48 22	48 22	5·0	18·92	34·19	—	—
3923	,,	,, 19	,,	45 41	54 1	3·3	17·74	32·07	—	—
3924	,,	,, 20	,,	43 14	60 9	5·0	18·12	32·75	—	—
3925	,,	,, 21	,,	41 13	66 31	8·3	17·53	31·70	23·42	—
3926	,,	,, 22	,,	40 25	73 7	11·7	18·19	32·88	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3927	Anchoria .	Nov. 28	noon . .	40° 44′	68° 50′	9·4	18·05	32·63	—	—
3928	,,	,, 29	,,	41 56	63 27	6·1	17·52	31·68	—	—
3929	,,	,, 30	,,	43 42	58 17	6·1	17·94	32·44	—	—
3930	,,	Dec. 1	,,	46 16	53 11	1·7	17·76	32·11	—	—
3931	,,	,, 2	,,	48 46	47 10	3·9	19·10	34·51	—	—
3932	,,	,, 3	,,	50 59	40 25	12·2	19·52	35·27	—	—
3933	,,	,, 4	,,	52 8	35 55	7·2	19·20	34·69	—	—
3934	,,	,, 5	,,	53 12	33 43	7·2	19·45	35·14	26·14	—
3935	,,	,, 6	,,	53 45	30 54	9·4	19·41	35·07	—	—
3936	,,	,, 7	,,	54 31	23 35	10·0	19·65	35·50	—	—
3937	,,	,, 8	,,	54 48	15 32	10·6	19·68	35·56	—	—
3938	,,	,, 9	,,	55 17	8 5	10·0	19·47	35·17	—	—
3939	Laura . .	Nov. 16	4 A.M. .	60 10	3 32	8·0	19·44	35·12	—	—
3940	,,	,, 16	noon . .	61 8	5 31	7·5	19·45	35·14	—	—
3941	,,	,, 18	4 A.M. .	61 55	6 33	7·0	19·42	35·08	—	—
3942	,,	,, 23	8 P.M. .	62 27	8 51	6·5	19·48	35·19	26·10	—
3943	,,	,, 24	4 A.M. .	62 36	11 12	7·5	19·46	35·15	—	—
3944	,,	,, 24	noon . .	62 45	14 31	7·0	19·49	35·21	—	—
3945	,,	,, 24	8 P.M. .	62 47	17 22	8·0	19·49	35·21	—	—
3946	,,	,, 25	4 A.M. .	62 50	19 48	8·0	19·39	35·03	26·12	—
3947	,,	,, 25	noon . .	63 23	20 0	7·0	19·30	34·87	—	—
3948	,,	,, 25	8 P.M. .	63 32	20 57	7·0	19·63	35·47	—	—
3949	,,	,, 26	4 A.M. .	63 30	22 10	7·0	19·67	35·54	—	—
3950	,,	,, 26	noon . .	63 45	22 48	6·0	19·33	34·93	—	—
3951	,,	Dec. 8	8 P.M. .	63 46	22 47	5·5	19·21	34·71	—	—
3952	,,	,, 9	4 A.M. .	63 29	20 23	5·5	19·37	35·00	—	—
3953	,,	,, 9	noon . .	63 20	18 50	5·5	19·16	34·62	—	—
3954	,,	,, 9	8 P.M. .	62 53	16 16	7·5	19·51	35·25	—	—
3955	,,	,, 10	4 A.M. .	62 47	13 35	7·2	19·49	35·21	—	—
3956	,,	,, 10	noon . .	62 31	11 2	7·2	19·50	35·23	—	—
3957	,,	,, 10	8 P.M. .	62 26	8 16	7·3	19·48	35·19	—	—
3958	,,	,, 13	4 A.M. .	61 16	6 7	7·5	19·49	35·21	—	—
3959	,,	,, 13	noon . .	60 31	4 33	7·8	19·49	35·21	—	—
3960	,,	,, 13	8 P.M. .	60 5	3 37	8·5	19·56	35·34	—	—
3961	,,	,, 14	4 A.M. .	59 38	2 38	9·0	19·56	35·34	—	—
3962	Teutonic . .	,, 2	midnight	51 23	14 30	12·8	19·71	35·61	—	—
3963	,,	,, 3	noon . .	51 23	20 46	13·9	19·64	35·48	—	—
3964	,,	,, 3	midnight	50 48	26 56	13·9	19·62	35·45	—	—
3965	,,	,, 4	noon . .	50 14	33 7	10·0	19·34	34·94	—	—
3966	,,	,, 4	midnight	49 8	37 33	9·4	19·44	35·12	—	—
3967	,,	,, 5	noon . .	48 2	42 0	7·2	19·63	35·47	—	—
3968	,,	,, 5	midnight	46 48	46 38	6·1	18·82	34·01	—	—
3969	,,	,, 6	noon . .	45 34	51 17	6·1	17·86	32·30	—	—
3970	,,	,, 6	midnight	44 9	56 37	5·0	17·98	32·51	—	—
3971	,,	,, 7	noon . .	42 45	61 58	8·3	18·26	33·00	—	—
3972	,,	,, 7	midnight	41 37	67 32	7·2	17·96	32·48	—	—
3973	,,	,, 8	noon . .	40 29	73 7	10·0	18·11	32·73	—	—
3974	,,	,, 15	midnight	40 12	69 57	9·4	17·73	32·06	—	—
3975	,,	,, 16	noon . .	41 6	65 14	12·2	18·71	33·82	—	—
3976	,,	,, 16	midnight	42 15	60 50	10·6	18·47	33·39	—	—
3977	,,	,, 17	noon . .	43 25	55 38	10·0	18·37	33·20	—	—
3978	,,	,, 17	midnight	44 50	50 39	5·0	17·90	32·37	—	—
3979	,,	,, 18	noon . .	46 16	45 41	5·0	18·69	33·78	—	—
3980	,,	,, 18	midnight	47 43	40 43	7·2	18·69	33·78	—	—
3981	,,	,, 19	noon . .	49 11	35 46	12·8	19·66	35·52	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
3982	Teutonic	Dec. 19	midnight	49° 57′	30° 13′	13·3	19·82	35·80	—	—
3983	,,	,, 20	noon	50 43	24 40	13·9	19·70	35·59	—	—
3984	,,	,, 20	midnight	51 1	18 42	15·6	19·73	35·64	—	—
3985	,,	,, 21	noon	51 20	12 44	12·8	19·69	35·58	—	—
3986	Loughrigg Holme	Aug. 28	,,	58 42	3 38	14·7	19·27	34·82	—	—
3987	,,	,, 28	8 P.M.	58 41	6 10	13·7	19·36	34·98	—	—
3988	,,	,, 29	4 A.M.	58 38	8 39	13·9	19·56	35·34	—	—
3989	,,	,, 29	noon	58 34	10 38	13·7	19·54	35·30	26·23	—
3990	,,	,, 29	9 P.M.	58 27	13 14	12·9	19·49	35·21	—	—
3991	,,	,, 30	4 A.M.	58 20	15 15	12·9	19·54	35·30	—	—
3992	,,	,, 30	noon	58 12	17 47	12·9	10·45	35·14	—	—
3993	,,	,, 30	8 P.M.	57 57	20 11	13·0	19·49	35·21	26·13	—
3994	,,	,, 31	4 A.M.	57 43	22 33	12·2	19·51	35·25	—	—
3995	,,	,, 31	noon	57 29	24 55	12·9	19·47	35·17	—	—
3996	,,	,, 31	8 P.M.	57 11	27 15	12·1	19·49	35·21	—	—
3997	,,	Sept. 1	4 A.M.	56 53	29 29	11·5	19·46	35·15	—	—
3998	,,	,, 1	noon	56 35	31 50	11·1	19·46	35·15	—	—
3999	,,	,, 1	8 P.M.	56 16	34 3	10·7	19·42	35·08	—	—
4000	,,	,, 2	4 A.M.	55 57	36 19	10·0	19·46	35·15	—	—
4001	,,	,, 2	noon	55 37	38 34	10·6	19·25	34·78	—	—
4002	,,	,, 2	8 P.M.	55 11	40 27	10·4	19·22	34·72	—	—
4003	,,	,, 3	4 A.M.	54 44	42 23	10·3	19·19	34·67	25·72	—
4004	,,	,, 3	noon	54 16	44 27	11·4	19·53	35·28	—	—
4005	,,	,, 3	8 P.M.	53 47	46 30	11·2	19·09	34·49	—	—
4006	,,	,, 4	4 A.M.	53 18	48 32	11·2	19·52	35·27	—	—
4007	,,	,, 4	noon	52 47	50 35	8·0	18·57	35·56	—	—
4008	,,	,, 4	8 P.M.	52 14	52 29	8·2	17·74	32·07	—	—
4009	,,	,, 5	4 A.M.	52 0	54 30	8·1	16·69	30·19	—	—
4010	,,	,, 5	8 A.M.	51 44	55 26	9·2	16·60	30·03	—	—
4011	,,	,, 5	noon	51 30	56 22	13·0	16·77	30·34	—	—
4012	,,	,, 5	4 P.M.	51 11	57 15	13·2	16·54	29·93	—	—
4013	,,	,, 5	8 P.M.	50 48	58 3	12·7	17·10	30·93	—	—
4014	,,	Nov. 28	8 A.M.	46 6	56 55	3·5	17·77	32·13	—	—
4015	,,	,, 28	noon	46 6	56 0	3·2	17·85	32·28	—	—
4016	,,	,, 28	8 P.M.	46 15	54 30	2·3	17·63	31·88	23·58	—
4017	,,	,, 29	4 A.M.	46 26	52 51	1·6	17·64	31·90	—	—
4018	,,	,, 29	noon	46 42	51 14	2·4	17·92	32·40	—	—
4019	,,	,, 29	8 P.M.	47 2	49 38	0·6	18·14	32·79	—	—
4020	,,	,, 30	4 A.M.	47 22	48 0	0·8	18·24	32·97	—	—
4021	,,	,, 30	noon	47 44	46 26	2·9	18·81	34·00	—	—
4022	,,	,, 30	8 P.M.	48 2	44 52	5·1	18·81	34·00	—	—
4023	,,	Dec. 1	4 A.M.	48 20	43 24	7·2	18·75	33·89	—	—
4024	,,	,, 1	noon	48 29	42 23	7·8	18·74	33·87	—	—
4025	,,	,, 1	8 P.M.	48 47	41 8	10·8	19·15	34·60	—	—
4026	,,	,, 2	4 A.M.	49 5	39 27	11·6	19·63	35·47	—	—
4027	,,	,, 2	noon	49 22	37 42	9·9	19·23	34·74	—	—
4028	,,	,, 2	8 P.M.	49 36	35 49	12·9	19·60	35·41	—	—
4029	,,	,, 3	4 A.M.	49 50	33 58	11·8	19·48	35·19	—	—
4030	,,	,, 3	noon	50 4	32 8	11·3	19·38	35·01	—	—
4031	,,	,, 3	8 P.M.	50 13	30 20	10·2	19·59	35·40	26·29	—
4032	,,	,, 4	4 A.M.	50 22	28 29	11·8	19·52	35·27	—	—
4033	,,	,, 4	noon	50 30	26 37	12·2	19·59	35·40	—	—
4034	,,	,, 4	8 P.M.	50 31	24 42	12·6	19·60	35·41	—	—
4035	,,	,, 5	4 A.M.	50 32	22 56	12·9	19·74	35·66	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
4036	Loughrigg Holme	Dec. 5	noon . .	50° 32′	21° 6′	12·2	19·78	35·73	—	—
4037	,,	,, 5	8 P.M. .	50 29	19 17	11·7	19·62	35·45	—	—
4038	,,	,, 6	4 A.M. .	50 26	17 25	11·2	19·71	35·61	—	—
4039	,,	,, 6	noon . .	50 23	15 45	12·5	19·62	35·45	—	—
4040	,,	,, 6	8 P.M. .	50 16	13 49	12·3	19·63	35·47	—	—
4041	,,	,, 7	4 A.M. .	50 8	11 54	12·7	19·67	35·54	—	—
4042	,,	,, 7	noon . .	50 0	9 58	11·6	19·56	35·34	—	—
4043	,,	,, 7	8 P.M. .	49 42	8 6	12·3	19·63	35·47	—	—
4044	,,	,, 8	4 A.M. .	49 23	6 14	11·8	19·66	35·52	—	—
4045	,,	,, 8	noon . .	49 48	4 35	12·2	19·66	35·52	—	—
4046	Siberian . .	,, 5	,,	off Tuskar Rk. L. Ho.		11·1	19·40	35·05	—	—
4047	,,	,, 6	,,	51° 27′	12° 0′	11·1	19·65	35·50	—	—
4048	,,	,, 7	,,	51 37	16 1	12·2	19·67	35·54	—	—
4049	,,	,, 8	,,	51 40	17 45	11·7	19·67	35·54	—	—
4050	,,	,, 9	,,	51 40	20 40	11·1	19·72	35·63	—	—
4051	,,	,, 10	,,	51 44	24 0	11·7	19·67	35·54	—	—
4052	,,	,, 11	,,	51 44	29 25	10·0	19·65	35·50	—	—
4053	,,	,, 12	,,	51 26	35 2	9·4	19·51	35·25	—	—
4054	,,	,, 13	,,	51 17	36 55	7·8	19·33	34·93	—	—
4055	,,	,, 14	,,	50 46	41 49	7·8	19·16	34·62	—	—
4056	,,	,, 15	,,	49 27	47 18	5·0	18·98	34·30	—	—
4057	,,	,, 16	,,	4 miles N. of St. John's, N.F.		0·6	17·72	32·04	—	—
4058	,,	,, 17	,,	off Bay of Bulls, N. F.		1·1	17·80	32·19	—	—
4059	,,	,, 18	,,	45° 47′	56° 45′	1·7	17·80	32·19	—	—
4060	,,	,, 19	,,	44 49	60 10	-1·7	16·80	30·40	—	—
4061	,,	,, 20	,,	44 25	62 50	1·7	17·14	31·00	—	—
4062	,,	,, 21	,,	43 58	63 53	3·3	17·18	31·07	—	—
4063	,,	,, 22	,,	40 33	68 5	7·8	17·98	32·51	—	—
4064	,,	,, 23	,,	38 50	73 15	11·1	19·02	34·37	—	—
4065	,,	,, 31	,,	39 17	72 50	11·1	19·04	34·40	—	—
		1898.								
4066	,,	Jan. 1	,,	40 36	68 3	7·2	18·00	32·54	—	—
4067	,,	,, 2	,,	42 3	63 6	6·1	18·20	32·90	—	—
4068	,,	,, 3	,,	43 53	58 10	1·7	17·66	31·93	—	—
4069	,,	,, 4	,,	46 18	53 35	-1·1	17·70	32·01	—	—
4070	,,	,, 5	,,	47 52	51 39	-1·7	18·08	32·68	—	—
4071	,,	,, 6	,,	49 33	46 13	3·9	18·99	34·32	—	—
4072	,,	,, 7	,,	50 40	43 43	10·0	19·59	35·40	—	—
4073	,,	,, 8	,,	51 51	38 5	7·2	19·28	34·84	—	—
4074	,,	,, 9	,,	53 7	33 8	7·2	19·41	35·07	—	—
4075	,,	,, 10	,,	54 9	26 53	8·9	19·48	35·19	—	—
		1897.								
4076	Anchoria .	Dec. 20	,,	55 9	15 43	10·6	19·67	35·54	—	—
4077	,,	,, 21	,,	54 31	24 11	10·6	19·54	35·30	—	—
4078	,,	,, 22	,,	53 34	31 26	8·3	19·42	35·08	—	—
4079	,,	,, 23	,,	52 13	36 42	6·7	19·24	34·76	—	—
4080	,,	,, 24	,,	50 26	42 16	6·7	19·02	34·37	—	—
4081	,,	,, 25	,,	48 28	47 44	-0·6	18·57	33·56	—	—
4082	,,	,, 26	,,	47 5	50 33	0·6	18·23	32·95	—	—

Table I. (*continued*).

Lab. No.	Ship.	Date.	Hour.	Lat.	Long.	Temp.	χ.	p. from χ.	$_4S_{15}$ Sprengel.	SO_3.
		1897.		N.	W.					
4083	Anchoria .	Dec. 27	noon . .	45° 10′	55° 20′	1·7	17·97	32·49	—	—
4084	,,	,, 28	,,	42 35	61 15	6·1	18·32	33·11	—	—
4085	,,	,, 29	,,	40 53	67 22	7·2	18·01	32·56	—	—
4086	,,	,, 30	,,	40 30	73 37	7·2	17·98	32·51	—	—
		1898.								
4087	,,	Jan. 4	,,	40 27	73 6	5·6	18·01	32·56	—	—
4088	,,	,, 5	,,	40 49	67 17	7·2	17·99	32·53	—	—
4089	,,	,, 6	,,	42 14	61 10	2·8	17·58	31·79	—	—
4090	,,	,, 7	,,	44 9	55 24	2·8	18·38	33·22	—	—
4091	,,	,, 8	,,	46 18	49 58	1·1	18·11	32·73	—	—
4092	,,	,, 9	,,	48 32	44 54	3·3	19·10	34·51	—	—
4093	,,	,, 10	,,	50 53	38 12	8·9	19·44	35·12	—	—
4094	,,	,, 11	,,	52 34	30 47	7·8	19·29	34·86	—	—
4095	,,	,, 12	,,	54 2	23 3	10·6	19·66	35·52	—	—
4096	,,	,, 13	,,	54 53	14 46	10·6	19·63	35·47	—	—
		1897.								
4097	Loughrigg Holme	Dec. 25	,,	49 45	5 59	11·7	19·66	35·52	—	—
4098	,,	,, 25	8 P.M. .	48 49	6 33	11·3	19·67	35·54	—	—
4099	,,	,, 26	4 A.M. .	47 51	7 7	12·7	19·77	35·72	—	—
4100	,,	,, 26	noon . .	46 53	7 42	13·6	19·80	35·77	—	—
4101	,,	,, 26	8 P.M. .	45 52	8 18	13·8	19·80	35·77	—	—
4102	,,	,, 27	4 A.M. .	45 2	8 47	14·2	19·88	35·91	—	—
4103	,,	,, 27	noon . .	44 28	9 5	13·6	19·75	35·68	—	—
4104	,,	,, 27	8 P.M. .	43 48	9 27	13·3	19·79	35·75	—	—
4105	,,	,, 28	4 A.M. .	42 54	9 44	14·1	20·03	36·18	—	—
4106	,,	,, 28	noon . .	41 48	9 47	15·0	20·02	36·16	—	—
4107	,,	,, 28	8 P.M. .	40 39	9 47	16·6	20·04	36·19	—	—
4108	,,	,, 28	midnight	40 4	9 47	14·6	20·08	36·27	—	—
		1898.								
4109	,,	Jan. 23	noon . .	40 38	9 33	14·9	19·96	36·05	—	—
4110	,,	,, 23	8 P.M. .	41 57	9 26	14·4	19·95	36·04	—	—
4111	,,	,, 24	4 A.M. .	43 13	9 27	13·4	19·93	36·00	—	—
4112	,,	,, 24	noon . .	44 24	8 56	13·3	19·92	35·98	—	—
4113	,,	,, 24	8 P.M. .	45 35	8 24	12·7	19·81	35·79	—	—
4114	,,	,, 25	4 A.M. .	46 52	7 51	12·2	19·80	35·77	—	—
4115	,,	,, 25	noon . .	48 1	7 16	11·9	19·67	35·54	—	—
4116	,,	,, 25	11 P.M. .	49 45	6 47	11·2	19·62	35·45	—	—

TABLE II.—Chlorines and Densities used as Standards. The Densities in the Second Column are calculated from the Chlorines by Equation (2), page 76.

Cl.	${}_4S_{15}$ calc.	${}_4S_{15}$ found.	Difference.
19·517	26·30	26·29	+0·01
19·415	26·16	26·14	+0·02
13·335	26·05	26·12	−0·07
19·171	25·82	25·82	0·00
18·320	24·64	24·68	−0·04
17·040	22·87	22·90	−0·03
17·005	22·82	22·85	−0·03
16·277	21·80	21·75	+0·05
15·421	20·62	20·68	−0·06
14·220	18·95	18·97	−0·02
12·928	17·15	17·19	−0·04
12·628	16·74	16·75	−0·01
15·571	16·65	16·65	0·00
11·263	14·84	14·84	0·00
9·475	12·35	12·37	−0·02
7·067	9·01	9·03	−0·02

TABLE III.—Differences of the Monthly Means of Atmospheric Pressure and Temperature during 1896 and 1897, from the Averages for the Fifteen Years 1870–84.

Differences of Monthly Means of Pressure during 1896 from the Averages (millims.).

	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Sumburgh Head	+ 7·7	+5·9	−7·5	+0·1	+7·3	−0·5	+3·1	+2·3	−4·3	− 2·7	+7·1	+0·4
Stornoway	+ 9·6	+6·4	−7·1	+3·0	+9·1	−0·6	+3·2	+3·6	−5·0	− 1·4	+7·6	−2·2
Aberdeen	+ 9·3	+7·8	−6·3	+2·9	+8·5	−0·7	+3·2	+3·3	−5·3	− 2·8	+7·9	−1·6
Leith	+ 9·2	+8·1	−5·9	+1·4	+8·5	−1·0	+2·9	+3·3	−6·0	− 3·2	+7·9	−2·9
Spurn Head	+ 9·3	+9·1	−4·7	+5·2	+7·2	−1·4	+2·4	+2·3	−6·2	− 4·4	+7·0	−3·7
Yarmouth	+ 8·5	+8·9	−4·1	+5·3	+6·0	−1·4	+1·9	+1·8	−5·4	− 4·4	+5·9	−3·8
Holyhead	+10·3	+8·9	−3·8	+7·4	+8·7	−1·0	+2·6	+4·0	−6·2	− 3·1	+8·0	−4·4
Valencia	+10·5	+7·5	−2·6	+9·8	+9·3	−1·2	+2·7	+6·0	−6·0	− 0·8	+7·4	−5·2
Scilly	+ 9·0	+7·5	−1·7	+9·5	+7·0	−1·4	+1·9	+3·7	−5·4	− 2·8	+5·5	−5·5
Nantes	+ 6·2	+7·4	−0·4	+8·5	+2·4	−1·3	+0·5	+1·3	−2·1	− 3·4	+1·5	−2·7
Paris	+ 6·2	+7·9	−1·7	+7·4	+3·9	−0·9	+0·8	+1·0	−3·5	− 4·1	+2·6	−4·1
Moscow	+ 0·3	−1·0	+2·6	+2·7	−0·5	−0·1	−0·7	+1·1	+2·6	+ 4·2	−1·6	−3·6
St. Petersburg	− 1·1	+2·5	+2·2	+2·9	+1·8	+1·2	+1·8	+1·8	+1·3	+ 1·8	+2·6	+6·2
Archangel	− 3·2	+2·0	+8·5	+4·0	+2·1	+1·6	+2·3	+3·9	+2·3	+ 0·6	−0·3	−3·3
Haparanda	+ 3·4	−0·3	−2·8	−0·2	+1·8	+0·3	+2·2	+1·7	+0·3	− 2·0	+1·2	+4·5
Stockholm	+ 1·8	+5·5	−2·9	+0·4	+3·5	−0·4	+2·5	+0·5	−0·5	− 0·2	+5·8	+4·3
Copenhagen	+ 4·8	+8·0	−4·6	+1·6	+3·1	−0·1	+1·9	+0·5	−1·9	− 1·7	+8·9	+2·0
Skagen	+ 3·8	+7·2	−5·2	+0·4	+3·5	+0·1	+2·5	+1·2	−1·9	− 2·6	+7·0	+3·1
Bergen	+ 5·9	+7·9	−5·7	+0·3	+6·0	+0·4	+3·1	+0·7	−2·8	− 3·0	+7·3	+2·4
Christiansund	+ 5·2	+6·3	−5·4	−0·3	+6·2	+2·2	+4·4	+2·5	−2·2	− 0·6	+6·3	+4·1
Vardö	− 2·7	+0·4	+8·5	+1·6	+1·9	+3·7	+4·9	+4·3	+3·1	− 0·4	+0·9	+5·7
Thorshavn	+ 8·2	+3·5	−8·5	−1·3	+6·5	+2·0	+3·5	+2·2	−1·6	+ 1·9	+5·0	−1·5
Berufjord	+ 6·9	−2·2	−7·2	−3·5	+2·0	+3·2	−0·2	+3·4	−1·5	+ 6·9	+1·9	−4·4
Stykkisholm	+ 7·9	−5·4	−5·6	−3·9	+0·8	+2·2	−1·8	+4·3	+0·5	+10·9	−1·4	−8·0
Grimsey	+ 7·7	−3·8	−4·9	−0·7	+1·3	+3·0	−1·2	+4·2	−0·3	+10·2	−0·8	−6·7
Ivigtut	+ 7·2	−1·0	−1·0	+0·4	−0·4	+3·0	−2·9	+3·6	−1·4	+12·3	−2·2	−7·3
Jacobshavn	+ 8·0	+2·3	+0·5	+3·8	+0·3	+3·6	−1·9	+2·8	−0·2	+11·4	−0·9	−6·3
Upernivik	+ 7·4	+2·3	+0·9	+2·8	+0·2	+3·1	−1·9	+2·1	+0·2	+10·2	−0·7	−6·1
St. John's, N.F.	+ 2·0	−1·0	+8·4	+2·3	+5·6	+2·8	+2·3	+4·6	+4·8	+ 5·8	+6·6	+3·0
Halifax	+ 0·8	−3·3	−0·2	+5·3	+2·3	−0·2	+1·5	+1·5	+1·0	+ 0·5	+5·6	+3·0
Boston	+ 1·8	−5·0	−1·3	+4·8	+0·9	+0·4	+1·4	+0·5	−0·7	− 1·0	+3·1	+2·1
New York	+ 2·6	−2·9	−1·0	+1·7	+1·8	−0·1	+0·4	+0·8	−1·7	− 1·7	+1·8	+1·0

Differences of Monthly Means of Pressure during 1897 from the Averages (millims.).

	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Sumburgh Head	+ 5·4	−0·4	−7·3	− 1·9	−1·0	+1·9	+1·5	−2·2	−1·3	+6·7	+ 7·6	−1·1
Stornoway	+ 5·6	+0·5	−9·2	− 2·5	−0·3	+2·1	+2·0	−4·2	0·0	+6·1	+ 7·4	−3·2
Aberdeen	+ 3·7	+1·4	−9·0	− 1·4	−0·1	+1·9	+2·3	−3·0	−0·3	+7·4	+ 7·6	−1·8
Leith	+ 2·4	+1·5	−9·5	− 1·5	0·0	+1·2	+2·4	−3·7	−0·3	+6·9	+ 7·2	−2·5
Spurn Head	− 0·8	+2·3	−8·4	− 1·2	−0·3	+0·3	+2·2	−2·9	−0·5	+7·7	+ 7·3	−0·6
Yarmouth	− 2·4	+2·2	−7·2	− 1·3	−0·7	+0·4	+1·7	−2·5	0·0	+7·3	+ 7·3	+0·2
Holyhead	+ 0·3	+2·5	−8·7	− 1·4	+0·5	+0·7	+2·6	−3·9	+0·9	+6·5	+ 7·5	−2·3
Valencia	+ 0·2	+2·0	−8·7	− 2·2	+0·4	+0·7	+2·1	−5·2	+1·9	+3·9	+ 6·2	−4·2
Scilly	− 2·9	+3·0	−6·0	− 1·3	+0·2	+0·5	+1·8	−3·7	+1·4	+4·8	+ 5·8	−2·5
Nantes	− 6·5	+5·0	−3·1	0·0	−1·2	+0·7	+0·1	−1·5	+1·8	+4·9	+ 4·6	+0·2
Paris	− 6·6	+4·0	−3·3	− 0·1	−0·3	+0·9	+0·6	−1·3	+0·5	+5·3	+ 6·3	+0·8
Moscow	—	—	—	—	—	—	—	—	—	—	—	—
St. Petersburg	—	—	—	—	—	—	—	—	—	—	—	—
Archangel	—	—	—	—	—	—	—	—	—	—	—	—
Haparanda	+ 9·6	−6·5	+5·4	+ 4·0	+2·4	+0·4	+0·1	+2·2	−4·2	+4·9	− 1·8	+3·8
Stockholm	+ 4·9	−3·8	−1·4	+ 0·5	+2·0	+2·4	+0·7	+1·5	−2·6	+7·2	+ 4·6	+4·5
Copenhagen	− 0·3	+0·9	−5·1	− 0·2	−0·1	+3·0	0·0	+0·5	−0·8	+8·4	+10·3	+3·5
Skagen	—	—	—	—	—	—	—	—	—	—	—	—
Bergen	+ 4·6	−0·4	−5·8	− 0·5	−0·4	+2·8	+1·3	−1·2	−1·6	+7·6	+ 7·4	+1·2
Christiansund	+ 7·2	−2·9	−2·6	+ 0·9	+1·1	+3·9	+2·8	+0·2	−1·9	+7·7	+ 6·3	+3·0
Vardö	+12·2	−1·2	+8·7	+ 7·4	+3·9	+1·2	+0·8	+3·8	−2·4	+5·2	− 2·4	+6·4
Thorshavn	+ 8·7	−1·2	−5·5	− 3·6	−0·8	+3·9	+2·1	−3·5	−0·4	+6·6	+ 6·4	−3·0
Berufjord	+10·0	−2·7	−1·5	− 8·1	−2·3	+6·1	+1·1	−2·2	+0·2	+5·0	+ 4·2	−3·4
Stykkisholm	+11·4	−2·3	−1·5	−10·0	−0·1	+7·6	+0·4	−1·4	+1·5	+3·7	+ 1·7	−3·6
Grimsey	+10·3	−2·5	−0·3	− 9·3	−0·5	+7·3	+0·4	−1·0	+0·9	+5·2	+ 2·1	−3·5
Ivigtut	+ 8·0	+4·6	−0·6	− 2·8	+1·9	+7·6	−0·2	+3·5	−0·6	−1·1	+ 1·0	−2·6
Jacobshavn	—	—	—	—	—	—	—	—	—	—	—	—
Upernivik	—	—	—	—	—	—	—	—	—	—	—	—
St. John's, N.F.	+ 1·5	+2·5	+5·6	+ 3·3	+5·1	0·0	+1·8	—	+0·2	−0·5	+ 1·5	+0·2
Halifax	0·0	+2·0	+3·3	+ 5·1	+2·5	−0·2	+3·0	0·0	0·0	+3·0	+ 1·3	+0·2
Boston	—	—	—	—	—	—	—	—	—	—	—	—
New York	− 0·1	+1·1	+0·8	− 0·1	−1·0	−0·9	−0·5	−0·3	+1·1	+1·7	− 0·7	−0·7

Differences of Monthly Means of Temperature during 1896 from the Averages. ° C.

	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Sumburgh Head	+0·3	+2·1	+1·0	+1·4	+1·5	+0·1	+0·2	−0·7	+0·1	−1·2	+0·7	+0·6
Stornoway	+1·2	+2·8	+0·6	+1·2	+2·4	+1·0	−0·2	−0·5	+0·2	−2·1	+1·2	+0·1
Aberdeen	+1·2	+2·1	+1·1	+2·0	+1·9	+0·6	−0·3	−1·1	−0·6	−1·8	−0·1	+0·8
Leith	+1·6	+1·7	+0·8	+2·6	+2·4	+0·3	+0·2	−0·4	−0·2	−2·1	+0·1	+0·3
Spurn Head	+1·4	+0·4	+1·4	+1·5	+0·4	+1·9	+0·6	−0·8	−0·1	−1·5	−0·6	+0·4
Yarmouth	+0·9	+0·2	+1·6	+0·9	0·0	+1·7	+0·5	−1·1	−0·2	−1·6	−1·4	+0·3
Holyhead	+0·7	+0·8	+0·9	+1·1	+1·2	+1·4	−0·2	−1·1	−0·6	−2·4	−1·6	−0·5
Valencia	+0·7	+1·4	+1·0	+0·7	+0·9	+0·7	+0·1	−0·8	−0·1	−3·0	−1·8	−0·8
Scilly	+0·9	+1·1	+1·8	+1·3	+1·8	+1·9	+1·2	−0·2	+0·1	−2·1	−1·6	+0·1
Nantes	−0·3	−2·8	+0·9	−0·9	0·0	+0·8	+1·4	−0·7	−0·4	−2·1	−2·8	+0·5
Paris	+0·7	−1·0	+2·7	−0·3	+0·1	+0·9	+1·0	−1·8	+0·1	−1·0	−3·4	+1·5
Breslau	−1·2	+0·9	+3·4	−1·3	−0·4	+1·9	+0·7	−0·9	+0·1	+2·4	−2·4	+0·6
Moscow	−2·7	−0·5	+0·5	+1·6	+0·4	+1·9	+0·6	+1·9	+1·6	+4·6	−2·8	−2·2
St. Petersburg	+1·8	−0·2	+1·3	+0·6	+1·3	+2·6	+1·9	−0·1	−0·3	+3·4	−1·7	−1·1
Archangel	+0·3	−1·5	+1·5	+1·4	+2·6	+0·3	−0·1	+0·2	+0·6	+2·8	−1·2	+2·8
Haparanda	+2·9	+5·0	+2·2	+1·1	+1·9	+0·1	+3·3	+0·1	+0·6	+0·1	+0·7	+1·6
Stockholm	+0·4	+2·8	+2·6	+0·7	+1·1	+4·1	+2·2	−0·4	+0·6	+1·7	−0·7	+0·8
Swinemünde	+0·6	+0·7	+2·7	−1·1	−1·3	+1·6	−0·8	−1·7	−0·5	+1·6	−2·1	−0·7
Copenhagen	+0·3	+2·3	+1·8	+0·3	+1·6	+3·8	+1·9	−0·5	−0·1	+0·6	−1·0	+0·2
Skagen	+1·1	+3·1	+1·0	+0·2	+1·3	+2·4	+0·6	−0·6	0·0	−0·1	−0·6	+0·4
Cologne	+0·1	−1·1	+2·0	−1·5	−0·4	+1·9	−0·4	−2·2	−0·5	−0·4	−2·7	−0·7
Hamburg	+0·1	+0·2	+1·8	−0·6	−0·3	+2·3	+0·1	−1·9	−0·2	+0·7	−2·3	−0·4
Bergen	+0·5	+3·0	+1·5	+1·0	+0·4	+1·1	−0·3	−1·3	+0·5	+0·5	+1·5	+1·1
Christiansund	+1·1	+2·7	+2·3	+1·9	+0·4	−0·6	−0·2	−1·1	+1·4	−0·9	+1·5	+0·8
Vardö	−1·8	+1·6	+1·6	+2·6	+2·3	+1·0	+1·1	+0·3	+1·1	−0·6	−1·1	+1·2
Thorshavn	+0·3	+2·6	+0·1	+0·9	+2·2	−0·4	+0·4	+0·1	−0·1	−2·1	+1·6	+1·7
Berufjord	+0·3	+3·5	+0·6	+1·6	+3·2	−0·7	+0·4	+0·7	+0·2	−3·0	+0·2	+3·1
Stykkisholm	+0·1	+2·4	+0·6	+1·1	+1·5	−0·9	+0·1	−0·4	−0·8	−2·1	−0·1	+2·3
Grimsey	—	—	—	—	—	−4·5	−3·1	−2·8	−2·2	−2·9	−0·9	+2·5
Ivigtut	0·0	−1·4	−1·6	−2·7	−2·9	−0·1	−0·6	0·0	−1·1	+1·2	+1·3	+6·7
Jacobshavn	−5·9	−9·7	−5·4	−9·8	−4·5	−0·3	+0·9	+1·4	+0·9	+2·3	−2·7	−2·9
Upernivik	−5·4	−6·2	−6·0	−8·1	−3·8	+0·4	+0·4	+2·6	+1·2	+3·1	−2·6	+0·7
St. John's, N.F.	+1·6	+0·2	+1·8	−1·4	−2·2	−1·1	+0·5	−1·3	+0·8	+1·6	−1·0	−1·9
Halifax	−0·4	+0·3	+1·6	+0·8	+0·7	+0·5	+0·4	−0·8	+0·3	+0·8	+1·7	−0·3
Boston	−2·4	−1·1	−2·2	+1·4	+2·1	−0·9	−0·3	−0·5	−0·9	−1·8	+2·7	−1·3
New York	−0·3	−0·2	−2·8	+0·8	+2·6	+0·3	+0·7	+1·3	−0·1	+2·2	+1·9	−1·3

Differences of Monthly Means of Temperature during 1897 from the Averages. °C.

	Jan.	Feb.	March.	April.	May.	June.	July.	Aug.	Sept.	Oct.	Nov.	Dec.
Sumburgh Head	−1·1	+0·1	+0·3	−0·5	−0·7	−0·4	+0·4	+1·7	−0·6	+1·1	+0·8	+1·6
Stornoway	−1·4	+0·4	+0·2	−0·8	−1·4	+0·2	+0·9	+1·6	−0·8	+1·1	+1·3	+0·3
Aberdeen	−1·3	+0·2	+0·7	−0·8	−1·0	−0·6	+0·3	+1·1	−1·0	+1·3	+2·1	+1·1
Leith	−1·4	+1·2	+0·6	−0·4	−0·9	−1·0	+0·3	+1·2	−0·3	+0·7	+2·3	+0·7
Spurn Head	−0·9	+1·1	+1·6	+0·1	−0·7	+0·4	−0·2	+1·2	−0·9	+1·2	+1·5	+1·2
Yarmouth	−1·4	+0·9	+1·6	−0·2	−0·7	+0·7	−0·2	+0·8	−1·2	+0·6	+1·1	+1·1
Holyhead	−1·7	+0·8	+0·3	−1·2	−0·9	−0·2	+0·2	+0·4	−1·3	+1·0	+1·3	+0·7
Valencia	−2·2	+1·7	−0·1	−0·7	−0·5	+0·2	+0·8	−0·1	−1·1	+1·7	+1·3	+0·9
Scilly	−1·2	+1·4	+0·9	−0·1	−0·2	+0·8	+1·2	+0·8	−0·8	+1·3	+1·7	+1·1
Nantes	−0·1	+2·2	+1·1	0·0	−1·4	+1·4	+1·2	−0·2	−1·0	−0·6	+0·7	+0·8
Paris	+0·4	+3·2	+2·8	−0·4	−0·9	+1·8	+0·6	+0·3	−0·9	+0·2	−0·4	+1·2
Breslau	−1·8	+1·3	+3·4	+1·1	+0·8	+1·9	−0·6	+1·4	−0·2	−0·5	−1·2	+1·4
Moscow	—	—	—	—	—	—	—	—	—	—	—	—
St. Petersburg	—	—	—	—	—	—	—	—	—	—	—	—
Archangel	—	—	—	—	—	—	—	—	—	—	—	—
Haparanda	+1·0	−0·6	−4·9	+3·1	+5·6	+0·5	0·0	0·0	+1·6	+2·1	+3·8	+3·2
Stockholm	−1·9	−1·5	+1·0	+1·4	+3·3	+1·6	+0·5	+2·2	−0·2	+0·3	+0·2	+3·5
Swinemünde	−2·5	−1·3	+1·9	+0·1	−1·9	0·0	−1·5	+1·0	−0·7	−0·5	−0·1	+1·9
Copenhagen	−2·2	0·0	−0·9	+0·5	+1·6	+2·3	−0·1	+2·1	−0·4	−0·8	+0·7	+2·6
Skagen	—	—	—	—	—	—	—	—	—	—	—	—
Cologne	−2·1	+0·8	+1·3	−0·6	−0·2	+1·9	−0·7	+0·7	−1·1	−0·7	−0·7	+0·6
Hamburg	−3·5	−0·6	+1·2	+0·2	+0·1	+1·8	−1·2	+1·3	−0·8	−0·9	−0·1	+2·3
Bergen	−2·6	+0·8	+1·6	+0·7	+0·5	−0·1	+0·3	+0·8	−1·2	+1·4	+0·7	+3·1
Christiansund	−1·4	+1·2	+1·2	+1·8	+1·4	−1·5	−0·5	+1·1	0·0	+1·4	+1·5	+2·3
Vardö	+1·9	−1·0	−1·4	+4·2	+1·2	+0·4	+0·9	0·0	+1·3	+1·8	+0·7	+0·5
Thorshavn	−1·4	−0·2	+0·2	+0·2	−0·7	−0·5	+0·3	+0·6	−0·5	+0·8	+0·9	+1·8
Berufjord	+0·9	+0·6	+0·5	+1·4	+0·1	+0·7	+1·2	+0·9	−0·1	+0·9	+0·2	+2·3
Stykkisholm	+1·3	−0·4	+1·6	+1·2	−0·6	−0·7	+0·1	−0·4	0·0	+1·4	+0·7	+2·2
Grimsey	+1·7	−0·4	+2·4	+1·3	−1·8	−3·2	−1·9	−3·2	−2·5	+0·5	−1·0	+2·8
Ivigtut	+1·1	+0·3	+1·1	−6·4	−2·1	+1·4	−0·8	−0·8	+0·7	+0·5	—	—
Jacobshavn	—	—	—	—	—	—	—	—	—	—	—	—
Upernivik	—	—	—	—	—	—	—	—	—	—	—	—
St. John's, N.F.	+0·6	−2·1	−1·2	+0·9	+2·3	−3·1	−1·6	—	−0·9	−1·6	−1·2	−5·5
Halifax	+0·5	+0·1	+0·6	+1·6	+0·7	−1·1	+0·6	−0·5	+0·2	+0·7	+1·0	−3·2
Boston	—	—	—	—	—	—	—	—	—	—	—	—
New York	+0·3	+0·5	+1·7	−0·2	−0·5	−1·0	+0·2	−0·3	−0·2	+3·7	−0·2	+0·9

APPENDICES.

Appendix I.—List of Observers.

Ship.	Observer's Name.	No. of Boxes supplied.
SS. Ethiopia	Captain J. Wilson, R.N.R.	18
R.M.S. Teutonic	,, J. G. Cameron, R.N.R.	25
R.M.S. Para	,, W. H. Milner	4
Sch. Traveller	,, A. Simpson	7
H.M.T.S. Monarch	,, T. Alford	3
R.M.S. Moor	,, E. J. Griffin, R.N.R.	1
SS. Loughrigg Holme	,, J. W. Millican	14
SS. Otra	,, Christopherson	1
SS. Longhirst	,, O. E. Anderson	3
SS. Laura	,, F. P. Christiansen	16
SS. Wydal	,, J. H. Gibson	1
SS. Active	,, T. Robertson	5
SS. Balaena	,, ,, ,,	5
SS. Eclipse	,, W. F. Miln	15
SS. Aldgate	,, G. H. Cheshire	1
SS. Castor	Mr. Barentz (mate)	(*a*)
SS. Frolic	Captain T. Bryant	3 (*b*)
SS. Hercules	,, J. Wilson	1 (*b*)
SS. Capricornus	,, W. Carrington	3 (*b*)
SS. Jamesia	,, A. Neale	1 (*b*)
SS. China	,, A. Christiansen	1 (*b*)
SS. America	,, C. Venables	1 (*b*)
S. Thorwaldsen	,, Jensen	5
H.M.S. Thyra	,, Garde	1 (*c*)
SS. Corean	,, W. S. Main	7
SS. Siberian	,, ,, ,,	5
SS. California	,, G. Mitchell	13
SS. Granuaile	Mr. R. M. Barrington	1
S.Y. Otaria	Sir G. Baden-Powell	2
SS. Minia	Captain Trott	1
		Total 164

(*a*) Lost with all hands, October, 1896.

(*b*) Through the kindness of Mr. Ç. M. Mundahl, Grimsby, owner.

(*c*) Royal Danish Navy.

APPENDIX II.—Set of Forms supplied.

A.

INSTRUCTIONS.

(*a.*) Fill one of the bottles about half full with the sample in the bucket, shake it, and throw away the contents. Then fill the bottle almost but not quite full with the sample, and cork *as tightly as possible.*

Note the number on the bottle, and opposite the same number fill in the particulars required in Sheet B.

(*b.*) On reaching England send the box to Mr. DICKSON, Chemical Laboratory, University Museum, Oxford, unless it contains enough empty bottles for another outward and homeward voyage. (Send by passenger train, and do *not* pay carriage.)

The filled up sheets, with request form C, should be *posted* to Mr. DICKSON as soon as possible after arrival.

N.B.—It is requested that samples be collected at least once daily, at the same hour each day (noon if possible); but only in the Atlantic north of 40° N. lat. The collection should be made from a point as far forward as possible, well clear of all discharge pipes. Samples which have been contaminated by water from the ship are of no value.

B.

Sample of Surface Water collected on board SS. ____________________

Captain ____________________ . during voyage from ____________________

to ____________________.

No. of Sample.	Date.	Hour.	Lat.	Long.	Temperature.	Remarks.
1						
2						
3						
4						
5						
6						
7						
8						
9						
10						
11						
12						
13						
14						
15						
16						
17						
18						
19						
20						
21						
22						
23						
24						
25						
26						
27						
28						
29						
30						

____________________ Observer.

C.

SS...

....

Dear Sir,

Please send......................................*boxes of bottles for samples of sea water to me here not later than*..

Yours faithfully,

....

Commander.

APPENDIX III.—Numbers of the Samples in Table I., which contain less than 17 grammes of chlorine per litre, and of which the densities have been determined.

266, 267, 282, 1109, 1135, 1136, 1283, 1293, 1775, 1776, 1777, 1778, 1787, 1790, 1791, 1792, 1793, 1794, 1795, 1796, 1797, 1798, 1799, 1800, 1801, 1803, 1805, 1809, 1817, 1830, 1832, 1833, 1834, 1840, 1841, 2520, 2833, 3107, 3321, 3367, 3435, 3698, 3699, 3719, 3723, 3726, 3730, 3733, 3738, 3742, 3752, 3786.

DISTRIBUTION OF TEMPERATURE IN THE SURFACE WATERS OF THE NORTH ATLANTIC DURING THE YEAR 1896.

The Edinburgh Geographical Institute

J.G.Bartholomew.

DISTRIBUTION OF SALINITY IN THE SURFACE WATERS OF THE NORTH ATLANTIC DURING THE YEAR 1896.

Edinburgh Geographical Institute
J.G.Bartholomew

Phil. Trans., A. vol. 196. Plate 3.

H. N. Dickson.

DISTRIBUTION OF TEMPERATURE IN THE SURFACE WATERS OF THE NORTH ATLANTIC DURING THE YEAR 1897.

The Edinburgh Geographical Institute

J. G. Bartholomew.

H. N. Dickson.

DISTRIBUTION OF SALINITY IN THE SURFACE WATERS OF THE NORTH ATLANTIC DURING THE YEAR 1897.

Phil. Trans., A. vol. 196, Plate 4.

The Edinburgh Geographical Institute

J. G. Bartholomew

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Phil. Trans., A, vol. 196, 1901, pp. 205–223.

IV. *On a New Manometer, and on the Law of the Pressure of Gases between* 1·5 *and* 0·01 *Millimetres of Mercury.*

By LORD RAYLEIGH, *F.R.S.*

Received January 15,—Read February 21, 1901.

Introduction.

THE behaviour of air and other gases at low densities is a subject which presents peculiar difficulties to the experimenter, and highly discrepant results have been arrived at as to the relations between density and pressure. While MENDELEEF and SILJERSTRÖM have announced considerable deviations from BOYLE'S law, AMAGAT* finds that law verified in the case of air to the full degree of accuracy that the observations admit of. In principle AMAGAT'S method is very simple. The reservoir consists mainly of two nearly equal bulbs, situated one above the other and connected by a comparatively narrow passage. By the rise of mercury from a mark below the lower bulb to another on the connecting passage, the volume is altered in a known ratio which is nearly that of 2 : 1. The corresponding pressures are read with a specially constructed differential manometer. Of this the lower part which penetrates the mercury of the cistern is single. Near the top it divides into a U, widening at the level of the surface of the mercury into tubes of 2 centims. diameter. Higher up again these tubes re-unite and by means of a three-way tap can be connected either with an air-pump or with the upper bulb. Suitable taps are provided by which the two branches can be isolated from one another. During the observations one branch is vacuous and the other communicates with the enclosed gas, so that the difference of levels represents the pressure. This difference is measured by a cathetometer.

It is evident that when the pressure is very low the principal difficulty relates to the measurement of this quantity, and that the errors to be feared in respect to volume and temperature are of little importance. AMAGAT, fully alive to this aspect of the matter, took extraordinary pains with the manometer and with the cathetometer by which it was read. An insidious error may enter from the refraction of the walls of the tubes through which the mercury surfaces are seen. But after all his precautions AMAGAT found that he could not count upon anything less than

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* 'Ann. de Chimie,' vol. 28, p. 480, 1883.

$\frac{1}{100}$ millim., even in the means of several readings. It may be well to give his exact words (p. 494) :—" Dans les expériences dont je donnerai plus loin les résultats numériques, les déterminations sont faites en général en alternant cinq fois les lectures sur chaque menisque ; les lectures étaient faites au demi-centième, et les divergences dans les séries régulières oscillent ordinairement entre un centième et un centième et demi ; en prenant la moyenne, il ne faut pas compter sur plus d'un centième ; et cela, bien entendu, sans tenir compte des causes d'erreur indépendantes de la lecture cathétométrique. . . . . . . Les résultats numériques consignés aux Tableaux que je vais donner maintenant sont eux-mêmes la moyenne de plusieurs expériences ; car, outre que les lectures ont été faites en général cinq fois en alternant, on est toujours, après avoir réduit le volume à moitié, revenu au volume primitif, puis au volume moitié : chaque expérience a donc été faite aux moins deux fois, et souvent trois et quatre."

The following are the final results for air :—

Pression initiale en millims.	$\frac{pv}{p'v'}$.	Pression initiale en millims.	$\frac{pv}{p'v'}$.
millims.		millims.	
12·297	0·9986	1·898	1·0050
12·260	1·0020	1·852	0·9986
10·727	0·9992	1·751	[1]·0030
7·462	1·0013	1·457	1·0150
7·013	1·0015	1·414	1·0143
6·210	1·0021	1·377	1·0042
6·160	1·0025	1·316	1·0137
4·946	1·0010	1·182	1·0030
4·275	1·0048	1·140	1·0075
3·841	1·0027	1·100	0·9999
3·770	1·0019	0·978	1·0160
3·665	0·9999	0·958	1·0100
3·165	1·0015	0·860	1·0045
2·531	1·0013	0·295	0·9680
2·180	1·0015		

Since, as it would appear, the "initial" pressure is the smaller of a pair, the lowest pressure concerned is about ·3 millim. of mercury, and the error at this stage is about 3 per cent. It is not quite clear which is which of pv and $p'v'$. For while it is expressly stated that p is smaller than p', the nature of v'/v is given at 2·076. I think that this is really the value of v/v'. But any lingering doubt that may be felt upon this point is of no consequence here, inasmuch as AMAGAT's comment upon the tabular numbers is "On ne saurait donc se prononcer, ni sur les sens ni même sur l'existence de ces écarts."

After such elaborate treatment by the greatest authority in these matters, the question would probably have long remained where AMAGAT left it, had not C. BOHR found reason to suspect the behaviour of *oxygen* at low pressures. This led to a prolonged and apparently very careful investigation, of which the conclusion was that at a pressure of ·7 millim. of mercury the law connecting pressure and volume is subject to a *discontinuity*.

“ 1. Bei einer Temperatur zwischen 11° und 14° C. weicht der Sauerstoff innerhalb der beobachteten Druckgrenzen von der Boyle-Mariotte'schen Gezetze ab. Die Abhängigkeit zwischen Volumen und Druck für einen Werth des letztgenannten, grösser als 0·70 mm., kann man annähernd durch die Formel

$$(p + 0{\cdot}109)\, v = k$$

ausdrücken, während die Formel für Werthe der Drucke, welche kleiner als 0·70 mm. sind:

$$(p + 0{\cdot}070)\, v = k$$

ist.

2. Sinkt der Druck unterhalb 0·70 mm., so erleidet der Sauerstoff eine Zustandsveränderung; er kann wieder durch ein Erhöhen des Druckes bis über 0·70 mm die ursprüngliche Zustandsform übergeführt werden.”*

Fig. (1) is a reproduction of one of Bohr's curves, in which the ordinate represents pv and the abscissa represents p on such a scale that 1 millim. of mercury

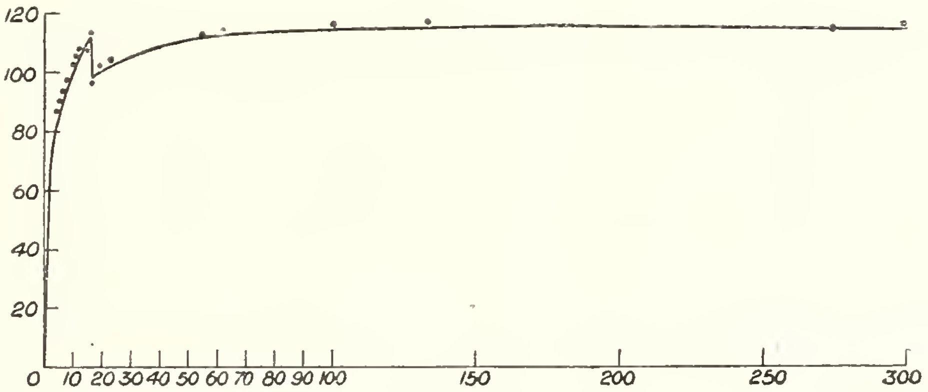

Fig. 1.

corresponds to the number 20. It will be seen that at the place of discontinuity a change of pv to no less than $\frac{1}{10}$ of its amount occurs with no perceptible concomitant change in the value of p. In the neighbourhood of the discontinuity the pressure is uncertain. Thus (p. 475) “ Wenn man bei einer gewissen Sauerstoffmenge im Rohre α das Quecksilber erst in der Art einstellt, dass der Druck einen etwas geringeren Werth als 0·70 millim. hat, und dann durch Verringern des Volumens den Druck über 0·70 millim. steigert (z.B. bis 0·8 millim.), so zeigt sich, dass dieser Druck nicht constant bleibt, sondern im Verlaufe von 3—5 Stunden bis zu einem Werthe sinkt, der ungefähr 10 Proc. kleiner ist, als der ursprüngliche.”

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* ‘Wied. Ann.,’ vol. 27, p. 479, 1886.

So far as I am aware, no attempt to repeat BOHR's difficult and remarkable experiments has been recorded, but some confirmation of anomalous behaviour of oxygen in this region of pressure is afforded by the observations of RAMSAY and BALY.[*] SUTHERLAND[†] interprets the results as a "Spontaneous Change of Oxygen into Ozone and a Remarkable Type of Dissociation," and connects therewith some observations of CROOKES relating to radiometer effects in oxygen gas. On the other hand, chemical tests applied by Professor THRELFALL and Miss MARTIN[‡] failed to indicate the presence of ozone in suitably expanded oxygen.

Improved Apparatus for Measuring very small Pressures.

In spite of the interest attaching to the anomaly encountered by BOHR, I should hardly have ventured to attack the question experimentally myself, had I not seen my way to what promised to be an improved method of dealing with very small pressures. In operations connected with the weighing of gases, extending over a series of years, I have had much experience of a specially constructed manometric gauge in which an iron rod provided above and below with suitable points is actually applied to the two mercury surfaces arranged so as to be situated in the same vertical line.[§] Although *two* variable quantities had to be adjusted—the pressure of the gas *and* the supply of mercury—no serious difficulty was encountered; and the delicacy obtained in the observation of the approximation of a point and its image in the mercury surface with the assistance of an eye-lens of 25 millims. focus was very satisfactory. In order to get actual measures of the delicacy, a hollow glass apparatus in the form of a fork was mounted upon a levelling table. The stalk below was terminated with a short length of rubber tubing compressible by a screw. This allowed the supply of mercury to be adjusted. The mercury surfaces in the U were about 20 millims. in diameter, and were exposed to the air. They were to be adjusted to coincidence with needle points, rigidly connected to the glass-work, by suitable use of the compressor and of the screw of the levelling table. Readings of the latter in successive and independent settings showed that a degree of accuracy was attainable much superior to the limit fixed by AMAGAT for the best work with the cathetometer. It is unnecessary to record the numbers obtained at this stage of the work, inasmuch as the final results to be given below prove that the errors of setting are considerably less than $\frac{1}{1000}$ millim.

It will now be possible to form a preliminary idea of the proposed manometer. The readings of the levelling screw, obtained as above, may be regarded as corre-

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[*] 'Phil. Mag.,' vol. 38, p. 301, 1894.

[†] 'Phil. Mag.,' vol. 43, p. 201, 1897.

[‡] 'Proc. Roy. Soc. of New South Wales,' 1897.

[§] "On the Densities of the Principal Gases," 'Proc. Roy. Soc.,' vol 53, p. 134, 1893.

sponding to the zero of pressure, or rather of pressure difference. If the pressures operative upon the mercury surfaces be slightly different, the setting is disturbed; and the change of reading at the screw required to re-establish the adjustment represents the difference of pressures. In order to interpret the result absolutely it is only necessary to know further the pitch of the levelling screw, the leverage with which it acts, and the distance between the points to which the mercury surfaces are set. If the space over one mercury surface be vacuous, the change of reading at the levelling screw represents the absolute pressure in the space over the other mercury surface.

The difficulty, which will at once present itself to the mind of the reader, in the use of a manometer on this plan, is the necessity for a flexible connection between the instrument and the rest of the apparatus, such as the air-pump and the vessel in which the pressure is required to be known. With the aid of short lengths of rubber tubing this requirement could be easily met, but the class of work for which such a manometer is wanted would usually preclude the use of rubber. In my apparatus the requisite flexibility is obtained by the insertion of considerable lengths (3 metres) of glass tubing between the manometer and the parts which cannot turn with it. Although the adjustment was made by the screw of a levelling table as described, the actual readings were taken by the mirror method, the supports of the mirror being connected as directly as possible with the points whose angular motion is to be registered. In this way we become independent of the rigidity of the glass-work, and are permitted to use wood freely in the levelling table and in its supports. It frequently happened that an adjustment left correct was found to be out after an interval. The screw had not been moved, but the mirror-reading was altered. On resetting by use of the screw, the original mirror-reading was recovered within the limits of error.

The essential parts of the manometer, as finally employed, are shown (fig. 2) in elevation and plan, and the general scheme of the mounting is indicated in fig. 3. At A is the stalk of the glass fork, of such length that the mercury in the hose below is always at a pressure above atmosphere; B, B are bulbs of about 25 millims. diameter, at

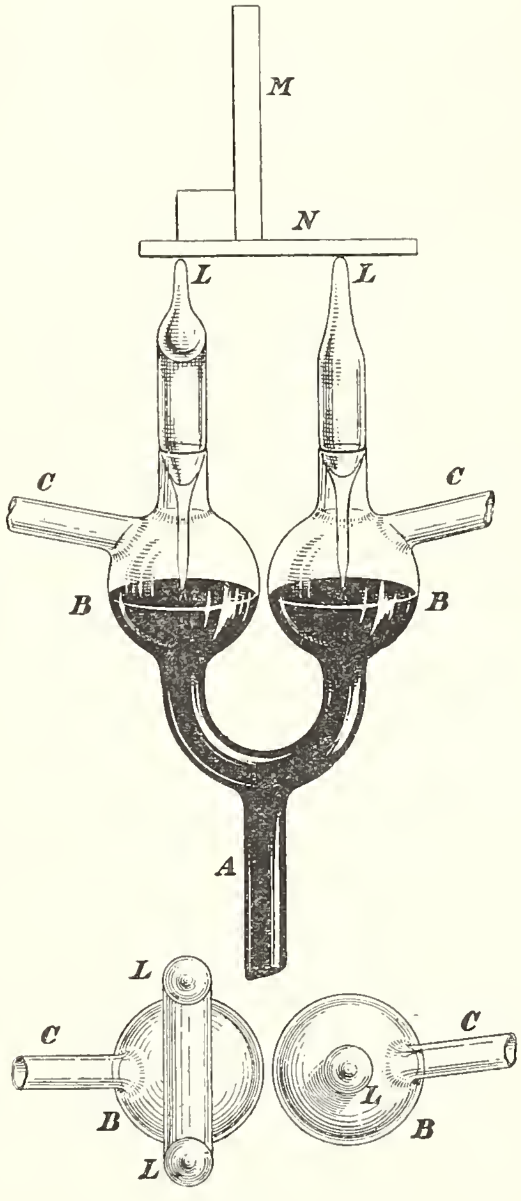

Fig. 2.

the centres of which are situated the *points*. These are of glass,[*] which need not be opaque; and they must be carefully finished upon a stone. A considerable degree of sharpness is desirable, but *similarity* is more important than the extreme of sharpness. In the actual apparatus complete similarity was not attained, and in the first trials the difference was rather embarrassing. However, after a little practice the eye becomes educated to set the mercury to each point in a constant manner, and this is all that is really required. The same consideration shows that minute outstanding capillary differences should not lead to error. It may be remarked that the mercury is always on the rise at the time of adjustment, and in fact it was found best to make it a rule not to allow the points to be drowned at any time when it could be avoided. After such a drowning it was usually (perhaps always) found that the mercury surface was disturbed by the *proximity* of the points without actual contact, an effect attributed to electrification.

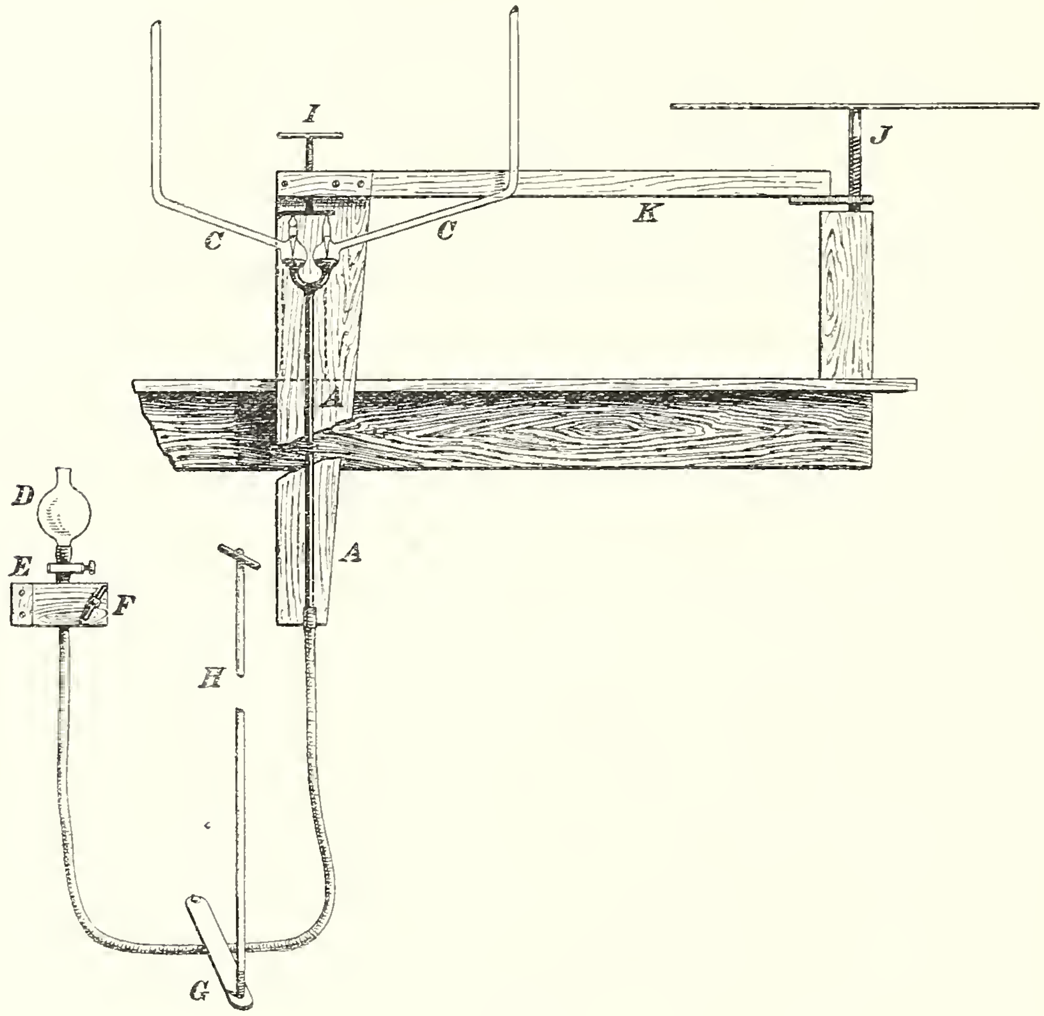

Fig. 3.

The presentation of the point to the mercury, or rather of the point to its image as seen by reflection in the mercury, was examined with the aid of two similar eye-lenses (not shown) of 22 millims. focus. The illumination, from a small gas flame

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[*] At first iron needle points were tried.

suitably reflected by mirrors, was from behind, and it and the lenses were so arranged that both points could be seen without a motion of the head. Precautions were required to prevent the radiation from the gas flame and from the observer from producing disturbance. especially by unequal heating of the two limbs of the U. The U itself was well bandaged up, and between it and the observer were interposed sheets of copper and of insulating material so as to ensure that at all events there should be no want of symmetry in any heating that might take place.

The adjustment itself is a *double* one, requiring both the use of the levelling screw J and an accurate feed of mercury. The hose terminates as usual in a small mercury reservoir D. This facilitates the preliminary arrangements, but in use the reservoir is cut off by a screw clamp E just below it. The rough adjustment of the supply of mercury is effected by a large wooden compressor F. The fine adjustment required for the actual setting is a more delicate matter. The first attempts were by fine screw compressors acting upon the pendent part of the hose, but the tremors thence arising were found very disturbing. A remedy was eventually applied by operating upon the part of the hose which lies flat upon the floor or rather on the bottom of a mercury tray. The compressor is shown at G, fig. 3; the screw being provided with a long handle H to bring it within convenient reach. The advantage accruing from this small device would scarcely be credited.

The glass-work is attached by cement to a board, which hangs downwards in face of the observer and is itself fixed rigidly to the levelling stand K. This is supported at two points I, which define the axis of rotation, and by a finely adjustable screw J, within reach of the observer. The whole stands in a very steady position upon the floor of an underground cellar in my country house.

The arrangements for the connection of the mirror must now be described. The glass stems, whose lower extremities form the "points," are prolonged upwards by substantial tubing, and terminate above in three slightly rounded ends, L, L, suitable for the support of the mirror platform N. The two supports necessary on the left are obtained by a symmetrical branching of the tube on that side. The platform is of worked glass, so that a slight displacement of the contacts has no effect on the slope of the mirror. The latter is of worked glass silvered in front. Suitable stops are provided to guide the mirror platform into the right position and to prevent accidents, but these exercise no constraint.

The axis I I about which the apparatus rotates is horizontal and parallel to the face of the mirror, so that the sine of the angle θ of rotation from the zero position represents the difference of levels of the mercury surfaces. The axis I I lies approximately in the mirror surface and at about the middle of the height of the operative part. The rotation of the mirror is observed in the usual way by means of a telescope and vertical millimetre scale. The aperture of the object-glass is 30 millims., and the distance from the mirror 3150 millims. The readings can be taken to about ·1 millim.

In many kinds of observation the zero can only be verified at intervals, as it requires the pressures over the mercury to be equalised. On the whole the zero was tolerably constant to within two or three-tenths of a millimetre of the scale. A delicate level was attached to the telescope to give warning of any displacement of the stand (all of metal) or of the ground.

The differences of pressure to be evaluated are not quite in simple proportion to the scale reading from zero. The latter varies as tan 2θ, while the former varies as sin θ. The correcting factor is therefore

$$\frac{\sin\theta}{\frac{1}{2}\tan 2\theta} = 1 - \tfrac{3}{2}\theta^2 \quad \text{approximately.}$$

If the zero reading (in millimetres) be a, and the current reading x, D the distance between telescope and mirror,

$$\theta = \frac{x - a}{2\mathrm{D}} \quad \text{approximately;}$$

so that the correcting factor is

$$1 - \tfrac{3}{2}\frac{(x - a)^2}{4\mathrm{D}^2}.$$

The actual correction to be applied to $(x - a)$ is thus

$$-\tfrac{3}{2}\frac{(x - a)^3}{4\mathrm{D}^2}.$$

In practice $(x - a)$ rarely exceeded 350, for which the correction would be $-1\cdot6$. When $(x - a)$ falls below 120, the correction is insensible.

The next question is the reduction to absolute measure. What (corrected) scale-reading corresponds to 1 millim. actual difference of mercury levels? The distance between the points is 27·3 millims., so that 1 millim. mercury corresponds to 231 millims. of the telescope scale. The highest pressure that could be dealt with is about 1½ millims. of mercury.

The above reckoning proceeds upon the supposition that the distance between the points can be regarded as invariable. Certain small discrepancies manifested at the higher slopes of the apparatus induced me to examine the question more particularly, for it seemed not impossible that owing to the bending of the glass-work some displacement might occur. But a rather troublesome measurement of the actual distance in various positions by means of microscopes negatived this idea. I would however recommend that this point be kept specially in view in the design of any subsequent apparatus of this kind.

Experiments to determine the Relation of Pressure and Volume at given Temperature.

In order to test BOYLE'S law one of the lateral branches C is connected to the air-pump and the other to the chamber in which the gas is contained. The pump is of the Toepler form, and is provided with a bulb containing phosphoric anhydride. No tap or contracted passage intervenes between the pump-head and B. A lateral channel communicates with a three-way tap, by which this side of the apparatus can be connected with the gas-generating vessel. The third way leads to a blow-off under mercury more than a barometer-height below.

The two sides of the apparatus are connected by a cross-tube which can be closed or opened by means of a tap. The plug of this tap is provided with a wide bore. When it is intended to read the zero, the tap is open. If desired, the mercury may be raised in the Toepler so as to prevent the penetration of gas into the pump-head. When pressures are to be observed, the tap of the cross-tube is closed, and a good vacuum is made on the pump side. No particular difficulty was experienced with the vacuum. In the use of the Toepler the mercury was allowed to flow out below, and was transferred at intervals to the movable reservoir. The latter was protected from atmospheric moisture by a chloride of calcium tube. When, after standing five or ten minutes, the mercury was put over, and, on impact, gave a hard metallic sound with inclusion of no more than a small speck of gas, the vacuum was nearly sufficient, and no further change could be detected at the manometer. The capacity of the pump-head was two or three times that of the remaining space to be exhausted.

In the earlier experiments the gas-containing tube, placed vertically, was graduated to 50 cub. centims. at intervals of 10 cub. centims. Prolonged below by more than a barometer-height of smaller tubing, it terminated in a hose and mercury reservoir, the latter protected by chloride of calcium. In order to get rid of most of the adherent moisture and carbonic anhydride, the tubes on both sides of the apparatus were heated pretty strongly in a vacuous condition. The first trial was with oxygen in the hope of at once obtaining a confirmation of BOHR'S anomaly; but not succeeding in this, I fell back upon nitrogen and hydrogen. With a vacuum on the pump side, readings of pressure were taken with the mercury in the chamber at 0 and at 50 cub. centims., and the ratio of pressures (about 2 : 1) was deduced. When this had been repeated, some of the gas was allowed to escape by opening the cross-tap, the zero was again observed, and the vacuum re-established on the pump side. Another ratio of pressures could now be obtained, corresponding to the same (unknown) volumes as before, but to a different total pressure.

In utilising the ratios of pressure thus obtained, it was of course necessary to consider how far the temperature could be assumed to be unchanged within each pair of pressures brought into comparison. The general temperature of the cellar was extremely uniform, and no difference could be read upon a thermometer worth taking into account. Passing over this question for the present, we may consider how far the results conformed to BOYLE's law. The agreement of the ratios, except, perhaps, at the highest pressures of about $1\frac{1}{2}$ millims. of mercury, was sufficiently good, and of itself goes a long way to confirm BOYLE's law. In strictness, all that the constancy of the ratio can prove is that the relation between pressure (p) and density (ρ) is of the form

$$p = \kappa\rho^n \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (1),$$

where n is some numerical quantity. To limit n to the value unity, the constancy of the ratios might be followed up into the region of pressure for which BOYLE's law is known to hold, but this can scarcely be said to have been done here. Otherwise, we need to know what the ratio of densities in the two positions of the mercury really is, and not merely that it remains constant.

In the case of the original volume chamber the first was the method employed. The smaller volume, defined by the upper mark in the volume tube and by the "point" in the manometer, was fitted with dry air at a known atmospheric pressure. The included air was then isolated and expanded until it occupied the larger (approximately double) volume, and the new pressure determined by observation of the difference of levels in the tube and in a mercury reservoir similarly fashioned. The operation was rather a difficult one, and the result was only barely accurate enough. The ratio of volumes thus determined by use of BOYLE's law, as applied to air at atmospheric and half atmospheric pressures, agreed sufficiently well with the ratio of pressures found by the manometer for rare hydrogen and nitrogen; and thus BOYLE's law may be considered to be extended to these rare gases. The rarefaction was carried down to a total pressure of only ·02 millim. At this stage discrepancies of the order of 5 per cent. are to be expected.

Having obtained fairly satisfactory results with hydrogen and nitrogen, I returned to oxygen, fully expecting to verify the anomalous behaviour described by BOHR. In this I have totally failed. The gas was prepared by heating permanganate of potash, dried by phosphoric anhydride, and may be regarded as fairly pure. The region of pressure round ·7 millim. was carefully examined, use being made of the intermediate divisions of the 50 cub. centims. range of volume. No unsteadiness of the kind indicated by BOHR, or appreciable departure from BOYLE's law, was detected. And when the pressures were diminished down to a few hundredths of a millimetre, there was no falling off in the product of pressure and volume. The observations were repeated a second time with a fresh supply of oxygen.

The experience gained up to this date (August, 1900) showed that the manometer worked well, and that there was no difficulty about the vacuum, but I was not altogether satisfied with the way in which the volumes had been determined. There was some want of elegance, to say the least, in using BOYLE's law for this purpose, and barely adequate accuracy in the application itself. The latter objection might have been overcome by the use of a suitable cathetometer, but such was not to hand. The most direct method by actually gauging with mercury the spaces concerned being scarcely feasible, I devised another method which has the advantage of easy execution and is practically independent of the assumption of Boyle's law. The opportunity was taken to increase the range over which the volume could be varied.

The new chamber, composed mainly of tubing of 18 millims. diameter, is graduated at intervals of 10 cub. centims. over a total range of 200 cub. centims. It is prolonged above and below by narrow tubing in order to connect it with the sloping manometer bulb and with the hose and mercury reservoir as before. The zero mark is situated on the upper tube a few centimetres above its junction with the wider one. It is scarcely necessary to say that no rubber was employed except for the hoses, and that these were always occupied by mercury under a pressure above atmosphere. The mercury reservoirs themselves were protected against damp by chloride of calcium.

If we call the ungauged volume (from the zero mark to the bulb of the sloping manometer with "point" set) V, and the gauged volume v, the total volume occupied by the gas is $V + v$; and the problem is how to determine V. If we may assume the correctness of Boyle's law for rare gases and may rely upon the sloping manometer, the process is simple enough. We have only to find the pressures exerted by the included gas at volumes V and $V + v$, whence by Boyle's law the ratio of these volumes is known and thus V determined in terms of v. In order to avoid the use of Boyle's law, further observations are necessary.

The requisite data can be obtained by changing the quantity of gas. Suppose that with the original quantity of gas certain pressures, P, P′, correspond to total volumes, $V + v_1$, $V + v_2$, and that with a reduced amount of gas the *same* pressures are recorded with volumes $V + v_3$, $V + v_4$. Since the pressure is a function of the density, whether Boyle's law be applicable or not, it must follow that

$$\frac{V + v_1}{V + v_2} = \frac{V + v_3}{V + v_4} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (2),$$

whence V is determined in terms of the known volumes v_1, v_2, v_3, v_4. It may be remarked that this argument does not assume even the correctness of the scale of pressures.

In carrying out the method practically it was necessary to work to the fixed marks of the volume chamber, and thus the same pressures could not be recovered *exactly*. But the use of Boyle's law in order to make what is equivalent to small corrections is unobjectionable.

With this explanation it may suffice to give the details of an actual determination executed with nitrogen. With the original quantity of gas, volumes $V + 70$, $V + 170$ gave pressures proportional to 345·4, 184·9. Sufficient gas was now removed to allow the remainder to give nearly the same higher pressure as before with $v = 0$. Thus, corresponding to volumes V + 0, V + 40 the pressures were 344·9, 183·3 We have now only to calculate V from the equation

$$\frac{V + 40}{V} = \frac{344 \cdot 9}{183 \cdot 3} \, \frac{184 \cdot 9}{345 \cdot 4} \, \frac{V + 170}{V + 70},$$

or $$V^2 + 110\, V + 2800 = 1 \cdot 0072\, (V^2 + 170\, V):$$

whence $$V = 45 \cdot 5 \text{ cub. centims.}$$

The adopted value, derived from observations upon nitrogen and hydrogen, is

$$V = 45 \cdot 6 \text{ cub. centims.}$$

In charging the apparatus, the first step is to make a good vacuum throughout, the cross-tap being open. The gas supply being started, the first portions are allowed to blow off from under mercury, and then, by use of the three-way tap, a sufficiency is introduced into the apparatus to an absolute pressure of, perhaps, 10 centims. of mercury. The gas-leading tube would then be sealed off. Ultimately the remainder of the supply tube and the blow-off tube were exhausted to diminish the risk of leakage.

The "nitrogen" was prepared from air by passage over red-hot copper and desiccation with phosphoric anhydride. Accordingly it contained argon to the amount of about 1 per cent.

In taking a set of observations the procedure would be as follows. Assurance having been obtained that the vacuum was good, the next step would be to set the mercury in the volume chamber so that $v = 190$ cub. centims., then after a few minutes to adjust the sloping manometer and to read the telescope scale. It was of course necessary to ensure that sufficient time was allowed for uniformity of pressure to establish itself, and observations were frequently renewed after a quarter of an hour or longer. In the case of oxygen, to be considered later, several hours were sometimes allowed. If operations were leisurely conducted, with first a rough setting of the volume and then a rough setting of the manometer followed by accurate settings in the same order, little or no change could afterwards be detected. Indeed I was rather surprised to find how rapidly equilibrium seemed to be established. The next smaller volume, *e.g.*, $v = 150$, would then be observed, and so on until $v = 40$. In observations to be used for the examination of BOYLE's law v was not further reduced, as too much stress might thereby be thrown upon the accuracy of V. The same observations were then repeated in reverse order and the mean taken. The numbers recorded are thus the mean of two settings only of the manometer.

The next step was to allow about half the gas to escape. The mercury at the pump was allowed to rise so as to cut off the pump-head and V + v was so adjusted as to be equal to the volume remaining upon the other side, about 130 cub. centims. The cross-tap was then opened, and after a sufficient interval of time the zero, corresponding to no pressure, was read. In the course of the observations upon nitrogen, extending over ten days, the zero varied from 43·5 to 43·8. Whenever possible the zero used for a set was the mean of values found before and after.

The annexed tables give the results for nitrogen in detail. In Table I., dealing with the highest quantity of gas, the first column gives the volume ($V = 45·6$ cub. centims.); the second represents the pressure, being the mean of the two actually read numbers (expressing millimetres of telescope scale) less the zero reading 43·7 and corrected to infinitely small arcs as already explained. The third column is the logarithm of the product of the first two, and should be constant if BOYLE's law holds. The fourth column gives the approximate value of the pressure in millimetres of mercury; the fifth the deviation of pv from the mean taken as unity. In the sixth column is shown the amount by which the observed value of p exceeds that requisite in order to make pv constant, expressed in millimetres of mercury.

TABLE I.—Nitrogen.

November 9–11, Zero = 43·7.

Volume in cub. centims.	Pressure in scale divisions.	Log. product.	Pressure in millims. Hg.	Deviation of pv.	Error of p in millims.
V + 70	345·4	·6013	1·49	+ ·0002	+ ·0003
V + 80	318·3	·6018	1·38	+ ·0014	+ ·0019
V + 90	294·1	·6007	1·27	− ·0012	− ·0015
V + 110	256·8	·6016	1·11	+ ·0009	+ ·0010
V + 130	227·4	·6013	·98	+ ·0002	+ ·0002
V + 150	203·7	·6004	·88	− ·0018	− ·0016
V + 170	184·9	·6005	·80	− ·0014	− ·0011
V + 190	169·8	·6021	·73	+ ·0021	+ ·0015
		·6012			

TABLE II.—Nitrogen.

November 11–12, Zero = 43·7.

Volume in cub. centims.	Pressure in scale divisions.	Log. product.	Pressure in millims. Hg.	Deviation of pv.	Error of p in millims.
V + 0	344·9	·1966	1·49	+ ·0007	+ ·0010
V + 10	282·3	·1958	1·22	− ·0012	− ·0015
V + 20	239·5	·1962	1·04	− ·0002	− ·0002
V + 40	183·3	·1956	·79	− ·0016	− ·0013
V + 60	148·8	·1963	·64	·0000	·0000
V + 80	125·2	·1966	·54	+ ·0007	+ ·0004
V + 110	101·1	·1968	·44	+ ·0012	+ ·0005
V + 150	80·2	·1955	·35	− ·0018	− ·0006
V + 190	66·9	·1976	·29	+ ·0030	+ ·0009
		·1963			

Table III.—Nitrogen.

November 13, Zero = 43·6.

Volume in cub. centims.	Pressure in scale divisions.	Log. product.	Pressure in millims. Hg.	Deviation of *pv*.	Error of *p* in millims.
V + 40	91·1	·892	·394	·000	·0000
V + 60	73·9	·892	·320	·000	·0000
V + 80	62·3	·893	·269	+ ·002	+ ·0005
V + 110	50·2	·893	·217	+ ·002	+ ·0004
V + 150	39·6	·889	·171	– ·007	– ·0012
V + 190	33·1	·892	·143	·000	·0000
		·892			

Table IV.—Nitrogen.

November 14, Zero = 43·5.

Volume in cub. centims.	Pressure in scale divisions.	Log. product.	Pressure in millims. Hg.	Deviation of *pv*.	Error of *p* in millims.
V + 40	46·0	·595	·199	+ ·005	+ ·0010
V + 60	37·1	·593	·160	·000	·0000
V + 80	31·1	·592	·135	– ·002	– ·0003
V + 110	25·1	·592	·109	– ·002	– ·0002
V + 150	20·1	·595	·087	+ ·005	+ ·0004
V + 190	16·5	·590	·071	– ·007	– ·0005
		·593			

Table V.—Nitrogen.

November 16, Zero = 43·5.

Volume in cub. centims.	Pressure in scale divisions.	Log. product.	Pressure in millims. Hg.	Deviation of *pv*.	Error of *p* in millims.
V + 40	22·8	·290	·099	·000	·0000
V + 60	18·6	·293	·081	+ ·007	+ ·0006
V + 80	15·6	·292	·067	+ ·005	+ ·0003
V + 110	12·7	·296	·055	+ ·014	+ ·0008
V + 150	9·9	·287	·043	– ·007	– ·0003
V + 190	8·15	·283	·035	– ·016	– ·0006
		·290			

TABLE VI.—Nitrogen.

November 17–18, Zero = 43·7.

Volume in cub. centims.	Pressure in scale divisions.	Log. product.	Pressure in millims. Hg.	Deviation of *pv*.	Error of *p* in millims.
V + 40	11·40	·989	·049	+ ·005	+ ·0002
V + 60	9·10	·983	·039	– ·009	– ·0004
V + 80	7·65	·983	·033	– ·009	– ·0003
V + 110	6·25	·988	·027	+ ·002	+ ·0001
V + 150	5·10	·999	·022	+ ·028	+ ·0006
V + 190	4·05	·980	·017	– ·016	– ·0003
		·987			

TABLE VII.—Nitrogen.

November 18–19, Zero = 43·8.

Volume in cub. centims.	Pressure in scale divisions.	Log. product.	Pressure in millims. Hg.	Deviation of *pv*.	Error of *p* in millims.
V + 40	5·90	·703	·026	+ ·014	+ ·0004
V + 60	4·60	·686	·020	– ·026	– ·0005
V + 80	4·15	·717	·018	+ ·047	+ ·0008
V + 110	3·10	·683	·013	– ·033	– ·0004
V + 150	2·55	·698	·011	+ ·002	·0000
		·697			

In the second set the quantity of gas had been adjusted to give a suitable pressure with $v = 0$. It is from it and from Table I. that the data were obtained for the calculation of V already given.

These tables give a fairly complete account of the behaviour of nitrogen from a pressure of about 1·5 millims. down to ·01 millim. of mercury. In each set the range of pressure is nearly in the ratio of 3 : 1, and overlaps the range of the preceding and following sets. An examination of the fifth column shows no indication of departure from BOYLE'S law. The sixth column allows a judgment to be formed of the degree of accuracy to which the law is verified. It gives the amount by which p exceeds the value necessary in order that pv should be absolutely constant, expressed in millimetres of mercury. The errors thus exhibited include not only those arising in the setting of the manometer and the reading of the telescope, but also those entailed in the measurements of volume, and in consequence of fluctuations of temperature. The latter source of error is of course more important at the higher pressures. It will be seen that the accuracy attained is very remarkable. Even at the higher pressures the mean error is only about ·001 millim., while at the lower pressures of Tables III.—VII. the mean error is less than ·0004 millim. And it must be remembered that the numbers to which these errors relate are the means of *two* observations only.

As a means of dealing with very small pressures, the sloping manometer has proved itself in a high degree satisfactory, the performance being some twenty-five times better than AMAGAT's standard. It could hardly have been expected that the mean error would prove to be less than one wave-length of yellow light.* Considered as a pressure, the mean error corresponds to the change of barometric pressure accompanying an elevation of 4 millims.

On hydrogen more than one series of observations have been carried out. The specimen that will be given is not in some respects the most satisfactory, but it is chosen as having been pursued to the greatest rarefactions. The gas was dried carefully with phosphoric anhydride and was introduced into the apparatus as already described. It is thought sufficient to record only numbers corresponding to the three last columns of Tables I.—VII., the first column giving the pressure in millims. of mercury, the second the deviation of pv from the mean value of the set taken as unity, the third the error in p from what would be required to make pv absolutely constant.

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* I had at one time contemplated an apparatus from which a further ten-fold increase in accuracy might be expected. Two beams of light, reflected nearly perpendicularly from the mercury surfaces, would be brought to interference by an arrangement similar to that used in investigating the refractivity of gases ('Proc. Roy. Soc.,' vol. 59, p. 200, 1896; vol. 64, p. 97, 1898). Preliminary trials proved that the method is feasible; but the delicacy is excessive in view of the fact that according to HERTZ the pressure of mercury vapour at common temperatures itself amounts to ·001 millim.

TABLE VIII.—Hydrogen. October–November, 1900.

Pressure in millims. Hg.	Deviation of *pv*.	Error of *p* in millims.	Pressure in millims Hg.	Deviation of *pv*.	Error of *p* in millims.
1·43	+ ·0025	+ ·0036	1·44	+ ·0018	+ ·0026
1·31	+ ·0030	+ ·0039	1·18	− ·0005	− ·0006
1·20	+ ·0002	+ ·0002	1·00	+ ·0009	+ ·0009
1·11	− ·0012	− ·0013	·87	+ ·0007	+ ·0006
·97	− ·0005	− ·0005	·77	+ ·0005	+ ·0004
·86	− ·0002	− ·0002	·62	·0000	·0000
·77	− ·0016	− ·0012	·57	− ·0028	− ·0016
·70	− ·0025	− ·0017	·52	− ·0009	− ·0005
·64	+ ·0007	+ ·0004	·48	− ·0018	− ·0009
—	—	—	·42	+ ·0018	+ ·0008
·769	+ ·0021	+ ·0016	·386	·0000	·0000
·624	+ ·0028	+ ·0017	·315	+ ·0044	+ ·0014
·524	·0000	·0000	·264	+ ·0023	+ ·0006
·423	+ ·0002	+ ·0001	·213	+ ·0014	+ ·0003
·335	− ·0037	− ·0012	·168	− ·0072	− ·0012
·279	− ·0018	− ·0005	·140	− ·0014	− ·0002
·196	+ ·0079	+ ·0015	·098	− ·009	− ·0009
·158	+ ·0046	+ ·0007	·080	·000	·0000
·133	+ ·0053	+ ·0007	·068	+ ·005	+ ·0003
·106	− ·0053	− ·0006	·055	+ ·007	+ ·0004
·085	− ·0037	− ·0003	·044	+ ·007	+ ·0003
·070	− ·0083	− ·0006	·036	− ·005	− ·0002
·051	+ ·007	+ ·0004	·027	− ·047	− ·0013
·041	+ ·002	+ ·0001	·023	+ ·016	+ ·0004
·034	− ·009	− ·0003	·018	− ·054	− ·0010
·027	− ·023	− ·0006	·016	+ ·021	+ ·0003
·023	+ ·036	+ ·0009	·013	+ ·040	+ ·0005
·018	− ·014	− ·0003	·010	+ ·019	+ ·0002

In several of the sets of observations recorded in Table VIII., there would seem to be a tendency for the positive errors to concentrate towards the beginning, *i.e.*, for *pv* to diminish slightly with *p*. It was at this stage that a suspicion arose that the distance between the glass points of the manometer might not be quite constant, but, as has been related, the suspicion was not verified. It is just possible that at the higher pressures and smaller volumes the temperature changes were not insensible. It is probable that they would operate in the direction mentioned, inasmuch as at the smaller volumes a larger proportion of the gas would be in the connecting tubes at a higher level in the room, and therefore warmer. Considerable precaution was taken, and I was not able to satisfy myself that disturbance due to temperature really existed. In another series of observations on hydrogen the tendency was scarcely apparent, and it remains doubtful whether there is any real indication of departure from BOYLE's law. It may be noted that interest was concentrated rather upon the lower pressures, and that perhaps less pains were taken over the readings of the higher pressures, where in any case the error would be a smaller proportion of the whole. Also some of the observations were not repeated. Another point that may be noted is that the means are chosen with respect to the values of *pv*, and that a different choice would in many cases materially reduce the mean error in the last column.

Having thoroughly tested the apparatus and the method of experimenting with hydrogen and nitrogen, I returned with curiosity to the case of oxygen. Special pains were taken to ensure that the gas should be pure and above all dry. To this end glass tubes were prepared containing permanganate of potash and phosphoric anhydride, and these were connected by sealing to one of the branches of the 3-way tap. A high vacuum having been made throughout, heat was gradually applied, and some of the oxygen allowed to blow off. The phosphoric tube (of considerable capacity) was then allowed to stand full of gas for some little time, after which the necessary gas to a pressure of about 10 centims. was allowed to enter the apparatus by means of the 3-way tap. With regard to the maintenance of the purity of the gas under rarefaction, it may be remarked that the method of experimenting was favourable, inasmuch as the last stages were not reached until the apparatus had been exposed to the gas under trial for a week or two. Any contamination that might be communicated from the glass during the first few days would for the most part be removed before the final stages were reached.

Before the regular series was commenced, special observations extending over several days were made in the region of pressure (from 1 millim. to ·5 millim.) where BOHR found anomalies. No unsteadiness could be detected. Whatever reading was obtained within a few minutes of a change of pressure was confirmed after an interval of an hour or more. For example, on November 29, at $12^{h}\ 25^{m}$ the pressure which had stood for some time at ·80 millim. was lowered to ·65 millim. At $8^{h}\ 0^{m}$ the pressure was unaltered. In no case was the behaviour in any way different to that which had been observed with the other gases. It is true that when the observations were reduced one preliminary set showed an excess of pressure at the smaller volumes similar to that recorded in the case of hydrogen, but the tendency is scarcely visible in the regular series now to be given, which extended from November 27 to December 9.

An examination of the numbers in the Table IX. shows that BOYLE'S law was observed, practically up to the limits of the accuracy of the measurements, and in particular that there was no such falling off in the value of *pv* at low pressures as was encountered by BOHR. What can be the cause of the difference of our experiences I am at a loss to conjecture. I can only suppose that it must be connected somehow with the quality of the gas, complicated perhaps by interaction with the glass or with the mercury.

TABLE IX.—Oxygen.

Pressure in millims. Hg.	Deviation of *pv*.	Error of *p* in millims.	Pressure in millims. Hg.	Deviation of *pv*.	Error of *p* in millims.
1·53	+ ·0016	+ ·0024	·580	– ·0035	– ·0020
1·17	– ·0012	– ·0014	·472	+ ·0005	+ ·0002
·95	+ ·0005	+ ·0005	·396	– ·0007	– ·0003
·80	+ ·0007	+ ·0006	·321	+ ·0016	+ ·0005
·65	+ ·0012	+ ·0008	·255	+ ·0012	+ ·0003
·57	– ·0009	– ·0005	·212	+ ·0016	+ ·0003
·51	– ·0014	– ·0007	—	—	—
·47	– ·0014	– ·0007	—	—	—
·43	+ ·0009	+ ·0004	—	—	—
·288	+ ·002	+ ·0007	·142	+ ·005	+ ·0007
·233	·000	·0000	·115	+ ·009	+ ·0011
·196	·000	·0000	·094	– ·019	– ·0018
·159	+ ·005	+ ·0008	·077	·000	·0000
·125	– ·002	– ·0003	·062	+ ·012	+ ·0007
·103	– ·009	– ·0010	·051	– ·012	– ·0006
·068	– ·002	– ·0002	·034	·000	·0000
·056	+ ·005	+ ·0003	·029	+ ·059	+ ·0017
·048	+ ·019	+ ·0009	·022	– ·042	– ·0009
·038	+ ·009	+ ·0004	·019	+ ·023	+ ·0004
·029	– ·019	– ·0005	·014	– ·035	– ·0005
·025	– ·009	– ·0002	—	—	—

The final result of the observations on the three gases may be said to be the full confirmation of BOYLE's law between pressures of 1·5 millims. and ·01 millim. of mercury. If there is any doubt, it relates to the case of hydrogen, which appears to press somewhat in excess at the highest pressures. But when we consider the smallness of the amount and the various complications to which it may be due, as well as *à priori* probabilities, we may well hesitate to accept the departure from BOYLE's law as having a real existence.

So far as the present results can settle the question, they justify to the full the ordinary use of McLEOD's gauge within the limits of pressure mentioned and for nitrogen and hydrogen gases. The same might be said for oxygen; but until the discrepancy with the conclusions of BOHR can be explained, the necessity for some reserves must be admitted.

In any case the new manometer has done its work successfully, and is proved to be capable of measuring small pressures to about $\frac{1}{2000}$ of a millimetre of mercury. It was constructed under my direction by Mr. GORDON.

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V. *Data for the Problem of Evolution in Man.—VI. A First Study of the Correlation of the Human Skull.*

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NOTE.

The substance of this paper was presented by Miss LEE as a thesis for the London D.Sc. in March, 1899. After its presentation Miss LEE asked me to criticise and revise it with a view to publication. Illness in the spring of 1899 and later pressure of other work prevented my completing this revision until now. When Miss LEE started her work practically nothing had been published on the correlation of the parts of the skull; since then an interesting paper has appeared by Dr. FRANZ BOAS. To this reference is made in the footnotes at points where there is agreement or disagreement with his conclusions. The subject is of such great scientific interest, and anthropologically of such importance, that I urged Miss LEE to somewhat enlarge her original thesis by a series of additional investigations now incorporated in this paper. I have further rearranged a good deal of her material and reworded some of her conclusions, but the reduction of the material and the inferences drawn from it are substantially her work. My task has been that of an editor, who wished to mould the author's researches into a component part of a wider series dealing generally with the quantitative data for the problem of evolution in man. Such is the limit of my revision. I have passed of course nothing which did not seem to me valid, and have suggested to the author some lacunæ which could be filled up by a consideration of additional data.—KARL PEARSON.

CONTENTS.

(1.) THE reconstruction of an organism from a knowledge of some only of its parts is a problem which has occupied the attention of biologists for many years past. CUVIER was the first to introduce in his 'Discours sur les Révolutions de la Surface du Globe,' 1812,[*] the idea of correlation. He considered that a knowledge of the size of a shoulder blade, leg, or arm might make it possible to reconstruct the whole individual to which the bone had belonged. The conception was taken up by OWEN, but has fallen into discredit owing to the many errors made in attempts from a wide but only *qualitative* knowledge of the skeleton, to reconstruct forms the appreciation of which depends really on *quantitative* measurement and an elaborate quantitative theory. Such a theory having now been developed, and anatomists having provided large series of measurements, it has become possible to reconsider the problem on a sounder basis, and to determine more closely the limits under which our modern methods may be safely applied.

The three fundamental problems of the subject are: (i.) The reconstruction of an individual, of whom one or more organs only are known, when a series of organs for individuals of the same local race have been measured and correlated.

As illustration, one may take the reconstruction of the probable stature of an individual for medico-legal purposes when a limb only has been found.

(ii.) The reconstruction of the mean type of a local race from a knowledge of a series of one or more organs in that race, when a wide series of these and other organs have been measured in other races.

As illustration, we may consider the reconstruction of the stature of prehistoric and defunct races from the measurement of their long bones, when the correlations between stature and long bones for some modern race have been determined from measurements made in the dissecting room.[†]

An important question in all researches of this kind is the legitimacy of applying results obtained for one local race to a second. We know that the variability and

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[*] Page 98 of the edition of 1830, the earliest in our Library.

[†] See PEARSON, "On the Stature of Prehistoric Races," 'Phil. Trans.,' A, vol. 192, pp. 169–244. An attempt is now being made by Professors WINDLE and PEARSON to collect data from English dissecting rooms, and an elaborate series of measurements with the like end in view are now being made in Strasburg on German material.

correlation are not constant for all the local races of a species; some of the limits of this legitimacy will be considered in this paper. A very full discussion of the matter for the regression equations of the long bones in the case of twenty local races in man by Mr. LESLIE BRAMLEY-MOORE is nearly completed.

(iii.) The reconstruction of an organ in the living individual not measurable during life, from a determination of the size of accessible organs, and a knowledge of the correlation between these organs and the inaccessible organ obtained from measurements made on individuals of the same race after death.

As an illustration, we may take the determination of the skull capacity from measurements made on the head of living individuals.

In all the three problems cited above, we can only obtain *probable* results, *i.e.*, we obtain the average value—generally not very far from the modal value of the second organ in a group of individuals with their first organ equal to that of the particular individual measured. The closeness of the result obtained is determined fairly accurately by the probable error of the array or group of individuals above referred to. If, instead of reconstructing an individual, we reconstruct a local race from a fairly large number of organs, this probable error will be at once largely reduced; but in doing this we assume the legitimacy of applying results obtained from one local race to a second local race.

(2.) The whole theory of reconstruction is summed up in the determination of the regression equations. It has been shown* that the most probable value of an organ, B, reconstructed from n organs A_1, A_2 . . . . A_n, is given by the expression

$$B - m_0 = -\left\{\frac{R_{01}}{R_{00}}\frac{\sigma_0}{\sigma_1}(A_1 - m_1) + \frac{R_{20}}{R_{00}}\frac{\sigma_0}{\sigma_2}(A_2 - m_2) + \ldots + \frac{R_{0n}}{R_{00}}\frac{\sigma_0}{\sigma_n}(A_n - m_n)\right\} \quad . \text{ (i.)},$$

with a probable error $= \cdot 67449\sigma_0\sqrt{R/R_{00}}$ . . . . (ii.),

where r_{0q} = correlation coefficient of B and A_q,

$r_{q'q} = r_{qq'}$ = ,, ,, ,, A_q ,, $A_{q'}$,

σ_0 = standard deviation of B,

σ_q = ,, ,, ,, A_q,

m_0 = mean of B,

m_q = ,, A_q,

$-\frac{R_{0q}}{R_{00}}\frac{\sigma_0}{\sigma_q}$ = partial regression coefficient of B from A_q,

and R is the following determinant, R_{pq} the minor corresponding to r_{pq}:

$$\begin{vmatrix} 1, & r_{01}, & r_{02}, & \ldots & r_{0n} \\ r_{10}, & 1, & r_{12}, & \ldots & r_{1n} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ r_{n0}, & r_{n1}, & r_{n2} & \ldots & 1 \end{vmatrix}$$

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* 'Phil. Trans.,' A, vol. 192, p. 172.

All the regression equations in the present paper have been worked from the above results, the most lengthy being those which depended on the evaluation of the numerical magnitude of the above determinant for the correlation of four organs.

(3.) The special object of this investigation is to apply the theory given in the last section to the reconstruction of skull capacity—to determine which of the measurements, length, breadth, height, or cephalic index of skull, or which combination of these measurements, will give the best result. In carrying out this special investigation, all the three fundamental problems considered in section (1) naturally arise, and will be referred to below. Further, certain problems regarding variation and correlation in man and woman also occur, and will be considered in their places.

The problem of the determination of the capacity of the skull has been one which has long occupied the attention of craniologists and anthropologists, and a great variety of methods have been considered and have found acceptance from one or another authority. The ideal method has not, however, been yet discovered, and in the well-known 'Frankfurter Verständigung' of the German craniologists, the matter was reserved for "further consideration," and has remained for a number of years in that unsettled state. From a fairly elaborate system of skull capacity measurements made at University College by Miss C. D. Fawcett, B.Sc., it would appear that the same experimenter may, with very slight practice, reach surprisingly close results for the capacity by very diverse methods; but that two different experimenters may give a mean skull capacity for a series which differs by 15 to 40 cub. centims. This of course only denotes about 1 to 3 per cent. of personal equation; but it appears large when read in gross. I cannot think that any conclusions as to relative racial differences ought to be based solely on divergencies in skull capacity of less than 40 cub. centims. when the two or more series under consideration have been measured by different observers. The knowledge of this divergence arising from the personal equation of different observers has led certain craniologists to suggest formulæ for calculating the capacity of the skull without measuring its contents, but from measurements of its girth, its height, length, or breadth. These formulæ seem to be unsatisfactory because they have not been based on a knowledge of the mathematical theory of correlation. It will be shown in the sequel that a formula can be found which gives the average capacity of a series of skulls from their mean height, length, and breadth with a fair degree of accuracy. In view of this it is a matter for consideration whether its use might not effectively replace the laborious and unsatisfactory methods of determining capacity by seed, shot, or sand. These could always be fallen back upon should any suspicion arise that the formula in question was being applied to a too widely divergent local race.

(4.) In selecting material for this investigation, I had to bear in mind results already reached by my co-workers at University College, but only in part at present published. In particular, that there was comparatively small correlation between the parts of the skull usually measured, and, further, that such correlation as actually exists varies enormously, even in sign, as we pass from one local race to another. This want of correlation, or want of steady correlation, in the parts of the skull, as compared with the correlation exhibited by the long bones, or by parts of the hand, is extremely interesting from the standpoint of evolution. It would appear to be much easier to modify a single character of the skull by selection without altering other characters than can be the case with other parts of the skeleton.

The measurements considered in this paper are: L the greatest length, B the greatest breadth of the skull, H the height measured from the auricular line, I the cephalic index = B/L, and C the capacity. In choosing the material several points had to be borne in mind:

(i.) A sufficiently large series must be used.

As a matter of fact, 50 to 100 skulls are considered by craniologists to be a fair series, but such numbers are small from the mathematician's standpoint.

(ii.) Material must be drawn from as widely different races as possible, if we are to measure the legitimacy of applying results obtained from one local race to another.

(iii.) The capacities must have been determined by competent observers using approximately like methods of measurement.

The data which seemed to me to approximately fulfil these conditions are the following:—

(*a*.) A series of Bavarian (*Alt-Baierische*) skulls measured by Professor J. RANKE, and given in his 'Beiträge zur physischen Anthropologie der Bayern.' In this case there were 100 ♂ and 99 ♀ with L, B, H, I, and C available.

(*b*.) A series of Aino skulls measured by KOGANEI, a craniologist trained in German schools, and given in the 'Mittheilungen aus der Medicinischen Facultät der Kaiserlich-Japanischen Universität,' Tokio, Bd. ii., 1894. In this case L, B, H, I, and C are given for 76 ♂ and 52 ♀ skulls, and there are 11 ♂ and 11 ♀ skulls for which L, B, H, and I only are given.

(*c*.) A series of Naqada skulls discovered in Egypt by Professor FLINDERS PETRIE, and measured by Miss C. D. FAWCETT, B.Sc., on the basis of the 'Frankfurter Verständigung.' I have to thank her for allowing me to use her results before publication. In this case L, B, H, and C were available for 69 ♂ and 98 ♀ skulls, and L, B, H only for 76 ♂ and 100 ♀ skulls.

As supplementary and test series, I have used primarily—

(*d*.) 201 ♂ and 96 ♀ skulls of ancient Egyptians. This series consists of mummies from Thebes in the Mook collection at Leipzig.

(*e*.) 76 ♂ and 23 ♀ skulls of modern Egyptians in a *Privat-Sammlung* at Leipzig.

The details of both these series are taken from the great craniological catalogue of the German Anthropological Society.*

(5.) Starting with the series (*a*) and (*b*), I have obtained for their means and

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* The parts are published separately as off-prints from the 'Archiv für Anthropologie.'

variabilities the results in Table I. below. In this case the Aino may be looked upon as a primitive uncivilised and the Germans as a highly-developed civilised race. An examination of this table shows that the Germans while gaining in breadth have lost in length, the mean auricular height for both sexes in both races remaining fairly stationary. Thus the brachycephalic tendency is in this case accompanied by a loss of length, and is not merely a gain in breadth.

TABLE I.

		Mean.	Standard deviation.	Coefficient of variation.
Aino, male . .	Length	185·82 millims.	5·936	3·195
	Breadth	141·23	3·897	2·759
	Height	119·32	4·377	3·668
	Capacity. . . .	1461·64 cub. centims.	100·605	6·883
	Cephalic index .	76·50	2·392	3·127
Aino, female .	Length	177·17 millims.	5·453	3·077
	Breadth	136·79	3·662	2·677
	Height	114·97	3·651	3·175
	Capacity . . .	1307·69 cub. centims.	89·751	6·864
	Cephalic index .	77·40	2·440	3·152
German, male .	Length	180·58 millims.	6·088	3·371
	Breadth	150·47	5·849	3·887
	Height	120·75	5·397	4·469
	Capacity . . .	1503·72 cub. centims.	116·890	7·773
	Cephalic index .	83·30	3·500	4·201
German, female	Length	173·59 millims.	6·199	3·571
	Breadth	144·11	4·891	3·394
	Height	114·17	4·463	3·909
	Capacity . . .	1337·15 cub. centims.	108·730	8·131
	Cephalic index .	83·10	2·973	3·578

It will further be seen that the Aino are less variable than the Germans in *all* the characters under discussion,* and in both sexes. The increase in skull capacity of the Germans on the Aino is less for the female than for the male, whilst in the variation of this character the change is greater for the female than the male. Further, the variability of the two sexes is more nearly equal in the Ainos than in the Germans.

These results are in good accord with those obtained by KARL PEARSON in his paper on "Variation in Man and Woman," and by him and myself in our paper "On the Relative Variation and Correlation in Civilised and Uncivilised Races," the conclusions there reached being—

(*a*.) Civilised races are more variable than uncivilised races.

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* It must be noted that the Germans are not a town population, but skulls from the churchyard mortuary chapels (*Gebein-Häuser*) of a limited rural district.

(*b*.) There is greater equality of variation for the two sexes in uncivilised than in civilised races.

(*c*.) Man tends with advance in civilisation to gain in size on woman.

(*d*.) Woman tends with advance in civilisation to gain in variability on man.

(6.) Turning to the correlations we obtain for Aino and Germans the results given in Table II. The correlation-coefficients are clearly very different for the two races. Putting aside the somewhat erratic correlation of capacity and cephalic index, we note that for the Aino the female correlations are all less than the male, but for the German the female are all greater than the male. Further, with the same omission in five out of the six cases, the Aino male is more highly correlated than the German male, and in four out of the six cases the German female is more highly correlated than the Aino female. This is again in general agreement with the results suggested in the second paper cited above, namely :—*

TABLE II.—Coefficients of Correlation.

Organs.	Male Aino.		Female Aino.	
Capacity and length $= r_{01}$.	·8928 ± ·0157	No. = 76	·6627 ± ·0525	No. = 52
Capacity and breadth $= r_{02}$.	·5606 ± ·0531		·5021 ± ·0700	
Capacity and height $= r_{03}$.	·5444 ± ·0544		·5210 ± ·0681	
Capacity and cephalic index .	− ·3069 ± ·0701		− ·2466 ± ·0878	
Length and breadth $= r_{12}$.	·4316 ± ·0588	No. = 87	·3765 ± ·0729	No. = 63
Length and height $= r_{13}$.	·5008 ± ·0542		·3489 ± ·0746	
Breadth and height $= r_{23}$.	·3454 ± ·0637		·1778 ± ·0823	
	Male German.		Female German.	
Capacity and length $= r_{01}$.	·5152 ± ·0495	No. = 100	·6873 ± ·0366	No. = 99
Capacity and breadth $= r_{02}$.	·6720 ± ·0370		·7068 ± ·0339	
Capacity and height $= r_{03}$.	·2431 ± ·0635		·4512 ± ·0540	
Capacity and cephalic index .	·2022 ± ·0647		− ·0307 ± ·0677	
Length and breadth $= r_{12}$.	·2861 ± ·0619		·4876 + ·0517	
Length and height $= r_{13}$.	− ·0975 ± ·0668		·3136 ± ·0611	
Breadth and height $= r_{23}$.	·0715 ± ·0671		·2764 ± ·0626	

(*a*.) That correlation is more nearly equal for the two sexes in uncivilised than in civilised races, and

(*b*.) That woman tends with advance to gain in correlation on man.

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[*] This confirmation of the results of the above paper is of interest, since they have been called in question by E. T. BREWSTER ('Proc. Boston Soc., Nat. Hist.' vol. 29; pp. 45–61). His series, however, are extremely small and his treatment of them not entirely satisfactory.

In the Aino race the length is more highly correlated with the capacity than the other dimensions are. In the German race, on the other hand, it is the breadth. Thus we shall find for the Ainos that formulæ involving the length, and for the Germans that formulæ involving the breadth, give the least probable error in the reconstruction of the capacity. It would be of interest to investigate whether this result is a distinguishing mark of dolichocephalic and brachycephalic races.

The correlation of capacity and cephalic index is, as I have said, somewhat erratic. For the Aino male and female it is quite sensible but negative. In other words, in a dolichocephalic race, it would appear as if dolichocephaly tended towards greater skull capacity. On the other hand, among the brachycephalic Germans, there is for the males a sensible correlation of a positive kind between capacity and brachycephaly. For the German women, however, we find this correlation less than half the probable error, and thus practically non-existent.

In order to throw, if possible, more light on this point the results in Table III. were worked out for two races, one of which is rather more dolichocephalic than the Aino. In this case very little stress can be laid on the ♀ modern Egyptians; they are far too few in number. The ♀ Theban mummies give a sensibly zero correlation, but in the three other cases the correlation is clearly negative. Thus there appears to be little doubt that in dolichocephalic races those who possess the race character most markedly have the greater skull capacity.

TABLE III.

	Mean.	Standard deviation.	Correlation.	Number.
Male Thebans (Mummies):				
Capacity	1393·6	120·80	− ·1480 ± ·0482	187
Cephalic index	74·8	3·17		
Female Thebans (Mummies):				
Capacity	1248·2	102·02	+ ·0080 ± ·0736	84
Cephalic index	76·3	3·70		
Male Modern Egyptians:				
Capacity	1356·5	116·55	− ·1410 ± ·0883	56
Cephalic index	77·3	5·42		
Female Modern Egyptians:				
Capacity	1195·8	85·74	− ·4960 ± ·1060	23
Cephalic index	76·7	5·10		

In Table IV. will be found similar data for three fairly brachycephalic races :—

TABLE IV.

Race.	Mean.	Standard deviation.	Correlation.	Number.
Male French:				
Capacity	1473·05	107·33	·1437 ± ·0883	56
Cephalic index	79·8	4·078		
Male Malays:				
Capacity	1429·76	100·243	·0331 ± ·0773	76
Cephalic index	81·9	5·127		
Male Etruscans:				
Capacity	1455·9	135·87	·2157 ± ·0729	78
Cephalic index	78·5	3·322		
Female Etruscans:				
Capacity	1323·6	110·77	·1443 ± ·1071	38
Cephalic index	78·3	3·300		

We see that the correlation is in all cases positive, but it is small, and in three of the cases given is hardly sensible considering the size of the probable errors. On the whole, I think we must conclude that while there is only a small relationship between cephalic index and capacity, yet that in brachycephalic races greater roundness points to greater capacity, and in dolichocephalic races less roundness points to greater capacity. In either case the emphasis of the racial character denotes an increase of capacity.

Accordingly, while we have been able to draw some interesting general conclusions as to the relationship of brachycephaly and capacity, it will be clear that the correlation here is far too uncertain to base any reliable reconstruction formula upon it. The regression formula for capacity in this case will be found to have, on the whole, the largest probable error, and to give the worst results when applied to test cases selected at random.*

(7.) I turn to the general regression formulæ for the determination of capacity. These are given for the Aino and Germans of both sexes in Tables V. to VIII. It will be seen from these tables that the reconstruction formulæ based on the cephalic index has in each case the largest probable error. Further, a very slight examination of the tables confirms the remark already made that for the Aino the length is a more important factor than the breadth, and that for the Germans the breadth is more important than the length as far as capacity is concerned. In the former race,

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* The general result as to cephalic index agrees with that obtained by Dr. FRANZ BOAS, 'American Anthropologist,' N.S., July, 1899, "The Cephalic Index," p. 448.

formulæ involving the length give, on the whole, a lower probable error in the value calculated for the capacity; in the latter race we must replace in this statement length by breadth.[*]

TABLE V.

	Formulæ for Aino males.	Probable error of mean.
(1)	$C = 15{\cdot}130\ L - 1349{\cdot}95$	$\frac{30{\cdot}57}{\sqrt{n}}$
(2)	$C = 14{\cdot}472\ B - 582{\cdot}24$	$\frac{56{\cdot}19}{\sqrt{n}}$
(3)	$C = 12{\cdot}511\ H - 31{\cdot}21$	$\frac{56{\cdot}92}{\sqrt{n}}$
(4)	$C = -12{\cdot}907\ I + 2449{\cdot}00$	$\frac{64{\cdot}58}{\sqrt{n}}$
(5)	$C = 13{\cdot}555\ L + 5{\cdot}562\ B - 1842{\cdot}61$	$\frac{27{\cdot}58}{\sqrt{n}}$
(6)	$C = 14{\cdot}029\ L + 2{\cdot}984\ H - 1501{\cdot}23$	$\frac{29{\cdot}61}{\sqrt{n}}$
(7)	$C = 10{\cdot}921\ B + 9{\cdot}153\ H - 1172{\cdot}95$	$\frac{50{\cdot}14}{\sqrt{n}}$
(8)	$C = 12{\cdot}857\ L + 5{\cdot}171\ B + 2{\cdot}190\ H - 1919{\cdot}24$	$\frac{27{\cdot}02}{\sqrt{n}}$
(9)	$C = {\cdot}000328\ (L \times B \times H) + 430{\cdot}30$	$\frac{42{\cdot}89}{\sqrt{n}}$

Capacity is measured in cubic centims., and length, breadth, and height in millims.
n = number from which C is determined.
I = cephalic index = 100 B/L

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[*] Dr. Franz Boas (*loc. cit.*, p. 461) states: "The relation between capacity and head-diameters is found to be of fundamental importance, and among these the relation between transversal diameter and capacity is most significant." This, I think, is only true for fairly brachycephalic races. He is dealing with 87 Sioux Indian skulls with cephalic index = 79.

Table VI.

	Formulæ for Aino females.	Probable error of mean.
(1)	$C = 10{\cdot}908\,L - 624{\cdot}86$	$\frac{45{\cdot}33}{\sqrt{n}}$
(2)	$C = 12{\cdot}334\,B - 375{\cdot}63$	$\frac{52{\cdot}35}{\sqrt{n}}$
(3)	$C = 12{\cdot}809\,H - 164{\cdot}95$	$\frac{51{\cdot}67}{\sqrt{n}}$
(4)	$C = -9{\cdot}071\,I + 2028{\cdot}00$	$\frac{58{\cdot}67}{\sqrt{n}}$
(5)	$C = 9{\cdot}084\,L + 7{\cdot}210\,B - 1288{\cdot}10$	$\frac{42{\cdot}22}{\sqrt{n}}$
(6)	$C = 9{\cdot}013\,L + 8{\cdot}112\,H - 1221{\cdot}74$	$\frac{41{\cdot}29}{\sqrt{n}}$
(7)	$C = 10{\cdot}363\,B + 10{\cdot}961\,H - 1370{\cdot}10$	$\frac{45{\cdot}12}{\sqrt{n}}$
(8)	$C = 7{\cdot}379\,L + 6{\cdot}795\,B + 7{\cdot}752\,H - 1820{\cdot}40$	$\frac{46{\cdot}42}{\sqrt{n}}$
(9)	$C = {\cdot}000400\,(L \times B \times H) + 187{\cdot}80$	$\frac{37{\cdot}90}{\sqrt{n}}$

Table VII.

	Formulæ for German males.	Probable error of mean.
(1)	$C = 9{\cdot}892\,L - 282{\cdot}55$	$\frac{67{\cdot}58}{\sqrt{n}}$
(2)	$C = 13{\cdot}432\,B - 517{\cdot}34$	$\frac{58{\cdot}39}{\sqrt{n}}$
(3)	$C = 5{\cdot}264\,H + 868{\cdot}05$	$\frac{76{\cdot}48}{\sqrt{n}}$
(4)	$C = 6{\cdot}754\,I + 941{\cdot}11$	$\frac{77{\cdot}22}{\sqrt{n}}$
(5)	$C = 6{\cdot}752\,L + 11{\cdot}421\,B - 1434{\cdot}06$	$\frac{51{\cdot}99}{\sqrt{n}}$
(6)	$C = 10{\cdot}446\,L + 6{\cdot}414\,H - 1157{\cdot}17$	$\frac{63{\cdot}46}{\sqrt{n}}$
(7)	$C = 13{\cdot}152\,B + 4{\cdot}245\,H - 987{\cdot}76$	$\frac{55{\cdot}47}{\sqrt{n}}$
(8)	$C = 7{\cdot}348\,L + 10{\cdot}898\,B + 5{\cdot}228\,H - 2094{\cdot}31$	$\frac{75{\cdot}97}{\sqrt{n}}$
(9)	$C = {\cdot}000332\,(L \times B \times H) + 415{\cdot}34$	$\frac{55{\cdot}41}{\sqrt{n}}$

TABLE VIII.

	Formulæ for German females.	Probable error of mean.
(1)	$C = 12{\cdot}055\,L - 755{\cdot}53$	$\frac{53{\cdot}27}{\sqrt{n}}$
(2)	$C = 15{\cdot}716\,B - 927{\cdot}66$	$\frac{51{\cdot}88}{\sqrt{n}}$
(3)	$C = 10{\cdot}993\,H + 82{\cdot}13$	$\frac{65{\cdot}45}{\sqrt{n}}$
(4)	$C = -1{\cdot}125\,I + 1430{\cdot}60$	$\frac{73{\cdot}31}{\sqrt{n}}$
(5)	$C = 7{\cdot}884\,L + 10{\cdot}842\,B - 1593{\cdot}96$	$\frac{43{\cdot}16}{\sqrt{n}}$
(6)	$C = 10{\cdot}618\,L + 6{\cdot}366\,H - 1232{\cdot}85$	$\frac{58{\cdot}70}{\sqrt{n}}$
(7)	$C = 14{\cdot}014\,B + 6{\cdot}749\,H - 1452{\cdot}89$	$\frac{48{\cdot}06}{\sqrt{n}}$
(8)	$C = 7{\cdot}065\,L + 10{\cdot}126\,B + 4{\cdot}848\,H - 1902{\cdot}02$	$\frac{40{\cdot}76}{\sqrt{n}}$
(9)	$C = {\cdot}000383\,(L \times B \times H) + 242{\cdot}19$	$\frac{42{\cdot}58}{\sqrt{n}}$

It will be noticed that a formula, No. (9), not hitherto referred to, has been introduced into these tables. As capacity is of three dimensions, an attempt has been several times made by anatomists to determine a relation between capacity and the product, L × B × H. This attempt seems to me to have failed because it has been based on a relation of the kind

$$\text{capacity} = \text{constant} \times (L \times B \times H),\text{*}$$

whereas the mathematical theory shows that we should rather expect a relation of the type

$$\text{capacity} = \text{constant} + \text{constant} \times (L \times B \times H),$$

Of course, if L, B, and H differ only by small quantities, x_1, x_2, x_3, from their means, the last relation may be written

$$\text{capacity} = \gamma_0 + \gamma_1 (m_1 + x_1)(m_2 + x_2)(m_3 + x_3),$$

where γ_0 and γ_1 represent constants, or

$$\begin{aligned} C &= \gamma_0 + \gamma_2 x_1 + \gamma_3 x_2 + \gamma_4 x_3 + \text{products of small quantities} \\ &= \gamma_5 + \gamma_2\,L + \gamma_3\,B + \gamma_4 H, \end{aligned}$$

where γ_5, γ_2, γ_3, γ_4 are constants, if we neglect terms of the order $x_1/m_1 \times x_2/m_2$ as compared with x_1/m_1 and x_2/m_2, &c.

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* Relations of the form: capacity = const. × $(L + B + H)^3$ have also been suggested.

This would simply throw us back on the regression formula (8) of our tables. Now the order of x/m is of the order of σ_i/m, or $\frac{1}{100}$ the coefficient of variation, say, ·03 to ·04. But deviations equal to 3σ may and do occur; hence, in individual cases an error of 9 to 12 per cent. might arise, if we were to assume that formula (9) can be replaced in all cases by (8). Accordingly, at Mr. YULE's suggestion, I formed the product of L × B × H, and correlated this product with the capacity. In the following Table IX. I give the data for Aino, German, and Naqada races :—

TABLE IX.

	Mean L × B × H.	Standard deviation.	Coefficient of variation.	Coefficient of correlation L × B × H and capacity.
Male, Aino	3144286·72	237683·63	7·559	·7949 ± ·0389
Female, Aino . . .	2797031·90	174791·20	6·249	·7797 ± ·0367
Male, German . . .	3282337·66	246992·49	7·525	·7006 ± ·0343
Female, German . .	2860212·85	231245·01	8·084	·8142 ± ·0229
Male, Naqada . . .	2881136·61	199446·14	6·922	·6736 ± ·0443
Female, Naqada . .	2619630·70	179387·60	6·864	·7934 ± ·0253

I supplement this table by the remaining data required for the three races from Egypt :—

TABLE X.

Race.	No.	Capacity.		L × B × H.	
		Mean.	Standard deviation.	Mean.	Standard deviation.
Ancient Egyptians, .	201	1389·6	120·80*	2859374·1	†
Ancient Egyptians, ♀ .	96	1253·7	102·02*	2589814·6	†
Modern Egyptians, ♂ .	76	1354·5	116·55*	2801989·8	†
Modern Egyptians, ♀ .	23	1195·8	85·74*	2424920·4	†
Naqada, ♂	69	1386·6	104·36	2881136·6	199446·1
Naqada, ♀	98	1279·3	94·03	2619630·7	179387·6

For the Naqada race we deduce the regression formulæ from the above results :—

Males . . . . $C = {\cdot}000352 \times (L \times B \times H) + 372{\cdot}39.$

Females . . . $C = {\cdot}000416 \times (L \times B \times H) + 189{\cdot}81.$

The probable errors for reconstruction by aid of these are :—

Males . . $\dfrac{52{\cdot}0228}{\sqrt{n}}$. Females . . $\dfrac{38{\cdot}6026}{\sqrt{n}}$.

* Values cited from the smaller series in Table III.

† Not calculated.

We have now all the data necessary for reconstructing the skull capacity, and it remains for us to consider how we can apply these to our three fundamental problems.

(8.) *First Fundamental Problem. The Reconstruction of the Individual from the known Formulæ for his own Race.*

In order to illustrate the degree of exactness with which we can reconstruct the individual from their own racial data, a perfectly random selection of twenty skulls was taken out of those of each sex for the Ainos and Germans, and their capacities reconstructed from each of the nine regression formulæ given in Tables V. to VIII. respectively. The results are tabulated in the following Tables XI. to XIV., and will enable the reader to appreciate the degree of exactness with which it would be possible to reconstruct the capacity of an individual skull from any one or more measurements made upon it.

These results show us at once that the last five formulæ are, when available, by far the best to use. (3) and (4), namely, reconstruction from the auricular height and the cephalic index, give occasionally very poor results. The latter formula, while of much interest from the racial standpoint,* need never be used for reconstruction, for the knowledge of the cephalic index means a knowledge of L and B, and accordingly we can always use (5) if not (8).

An examination of the actual mean error made when we use all nine formulæ and take the mean of their results shows that, as a rule, we shall obtain less error by selecting one good formula like (8) or (9) and using that only than if we attempt to use them all. In round numbers we see that the mean error made in reconstruction by these formulæ is about 5 per cent., but if we use (8) or (9) the mean error will lie between 3 and 4 per cent. The maximum error reached by a good formula like (8) or (9) is upwards of 10 per cent., but its occurrence is infrequent. On the whole, I consider this reconstruction of the individual from data for his own race fairly satisfactory. It is practically nearly as good as we get in the reconstruction of stature from the long bones.† I would also remind the reader that the theory of correlation shows that we cannot hope to get better results. We have solved the problem as closely as it can be solved, so long as the skull is a variable organ. From a knowledge of the degrees of variation and correlation of an extended number of parts of the skull (unpublished data), I feel fairly confident that no external measurements can be taken upon it which will give substantially better results than those already considered.‡ When we bear in mind that two different observers, using even

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* If we wish to identify criminals, we select characters to be measured and indexed which exhibit the least correlation. In the same way to differentiate and specify races, it is best to select a group of characters having the least correlation; one such is certainly the cephalic index.

† See PEARSON: "On the Reconstruction of the Stature of Prehistoric Races" ('Phil. Trans.,' A, vol. 192, pp. 188–189).

‡ An appendix is devoted to a consideration of the horizontal and vertical girths.

TABLE XI.—Aino Male.

Table of Differences of Actual and of Reconstructed Skull Capacity.

No.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	Actual mean error.	Mean error calculated from probable error.
(1)	−6	+79	+47	−26	+14	+85	−67	+40	+119	+45	+49	−7	+120	+115	−87	+65	−85	−46	+23	+45	58·50	36·16
(2)	−85	+117	+167	−27	−21	−170	+33	−12	+115	−22	+58	+78	−36	−11	+14	+26	−123	−96	+50	−39	65·00	66·47
(3)	−65	+110	+218	−17	+130	+138	−5	−62	+158	−47	+55	+103	−82	+30	+3	+68	−115	+8	+33	−35	74·10	67·34
(4)	+18	+60	+242	−20	+12	+192	−5	+11	+171	+11	+43	+47	+44	+64	−2	−26	−138	−40	+112	−66	66·20	76·40
(5)	−31	+95	+31	−26	+6	+79	−57	+34	+106	+37	+53	+2	+95	+93	−81	+76	−82	−60	+9	+52	55·25	32·63
(6)	−17	+87	+51	−26	+45	+74	−69	+22	+121	+33	+51	+5	+93	+110	−85	+81	−82	−30	+11	+49	58·60	35·03
(7)	−100	+131	+153	−19	+81	+123	+4	−50	+123	−42	+51	+94	−86	+3	−3	+83	−105	−36	+7	−9	65·65	59·32
(8)	−37	+100	+31	−26	+28	+79	−60	+22	+108	+29	+54	+9	+76	+91	−81	+67	−80	−67	+1	+54	55·15	31·96
(9)	−78	+117	+105	−27	+82	+81	−34	−34	+116	−22	+55	+58	−35	+41	−43	+100	−92	−29	−6	+25	59·00	50·74
Mean deviation	−49	+100	+116	−24	+42	+76	−29	−3	+126	+2	+53	+43	+21	+60	−41	+65	−100	−46	+60	+8	53·20	
Actual capacity	1485	1385	1190	1475	1465	1245	1425	1470	1300	1480	1415	1380	1465	1440	1430	1490	1610	1525	1350	1555		

TABLE XII.—Aino Female.

Table of Differences of Actual and of Reconstructed Skull Capacity.

No.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	Actual mean error.	Mean error calculated from probable error.
(1)	−133	−23	+8	+57	−75	−19	+70	+63	−20	+18	+38	+77	+41	+143	−85	−35	+93	−55	−4	+97	57·70	53·63
(2)	−99	−63	+6	+91	−56	−48	+119	−4	−149	−104	+107	−3	+18	+39	−116	−69	−11	−18	−47	+26	59·65	61·93
(3)	−78	−32	+26	−42	−100	+9	+36	−47	−154	−96	+113	+41	+47	+7	−46	−49	+21	+64	+79	−6	54·65	61·12
(4)	−139	+1	+5	+51	−52	+16	+133	+18	−133	−64	+131	+49	+132	+45	−42	−58	+59	+18	+47	+27	61·00	69·41
(5)	−112	−46	+13	+78	−71	−39	+69	+57	−25	+3	+34	+59	+2	−29	−110	−35	+66	−64	0	+77	49·45	49·95
(6)	−94	−25	+29	−1	−97	−2	+17	+32	−25	+10	+39	+89	+19	−48	−66	−19	+87	−10	+23	+91	41·15	48·85
(7)	−52	−61	+33	+5	−93	−24	+32	−31	−126	−88	+87	+28	−16	−4	−89	−42	0	+28	+67	−8	45·70	53·38
(8)	−75	−45	+33	+22	−93	−22	+18	+29	−30	−3	+36	+72	−18	−46	−93	−20	+62	−20	+25	+72	41·70	54·91
(9)	−76	−56	+33	+4	−102	−27	+6	+11	−37	−15	+35	+64	−30	−50	−97	−22	+46	−17	+27	+59	40·70	44·84
Mean deviation	−95	−39	+21	+29	−82	−17	+56	+14	−78	−38	+69	+53	+22	+6	−83	−39	+47	−8	+24	+48	43·4	
Actual capacity	1450	1340	1320	1260	1370	1325	1170	1330	1500	1430	1170	1305	1210	1250	1380	1395	1300	1295	1255	1340		

TABLE XIII.—German Male.

Table of Differences of Actual and Reconstructed Skull Capacity.

No.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	Actual mean error.	Mean error deduced from probable error.
(1)	−187	− 52	− 53	− 48	−82	− 94	+ 86	−67	+18	− 27	+63	+68	−11	+65	+122	+ 83	+172	+ 79	+ 5	+159	77·05	79·94
(2)	−181	− 36	− 62	+ 7	−62	−101	− 49	+ 3	−97	− 82	+36	−30	+88	−16	+ 32	+ 39	−146	+164	+ 78	+ 63	68·60	69·07
(3)	−179	−160	− 93	− 62	−90	− 20	− 18	−30	+26	+ 15	+14	+29	+60	+72	+ 54	+105	+119	+177	+196	+187	85·30	90·48
(4)	−197	−181	−142	−109	−91	− 70	− 96	−10	−28	+ 5	+21	+16	+85	+48	+ 53	+ 87	+ 41	+171	+208	+219	93·90	91·35
(5)	−181	− 68	− 16	+ 38	−58	−114	+ 47	−27	−72	− 98	+58	− 2	+38	− 7	+ 69	+ 37	− 75	+117	− 25	+ 32	58·95	61·50
(6)	−170	− 52	+ 5	+ 28	−73	− 36	+140	−67	+46	− 34	+47	+51	− 7	+66	+ 94	+ 90	+164	+115	+ 16	+ 85	69·30	75·07
(7)	−163	−139	− 28	+ 52	−58	− 51	− 18	+ 3	−76	− 83	+24	−40	+92	+13	+ 13	+ 45	−148	+190	+ 94	+ 21	67·56	65·62
(8)	−160	− 67	+ 29	+ 96	−52	− 66	+ 93	−29	−45	−100	+45	−13	+49	− 3	+ 49	+ 45	+ 71	+145	+ 15	− 22	59·70	89·87
(9)	−149	+ 89	+ 37	+106	−59	− 21	+105	−33	− 6	− 68	+27	− 6	+32	+22	+ 36	+ 68	− 11	+169	+ 49	+ 15	55·40	65·55
Mean deviation	−174	− 74	− 36	+ 12	−69	− 64	+ 32	−29	−26	− 52	+37	+ 8	+47	+29	+ 58	+ 66	+ 21	+147	+ 71	+ 84	56·8	
Actual capacity	1705	1660	1640	1625	1600	1572	1560	1535	1500	1485	1475	1460	1450	1433	1425	1405	1375	1360	1325	1260		

TABLE XIV.—German Female.

Table of Differences of Actual and Reconstructed Skull Capacity.

No.	1.	2.	3.	4.	5.	6.	7.	8.	9.	10.	11.	12.	13.	14.	15.	16.	17.	18.	19.	20.	Actual mean error.	Mean error deduced from probable error.
(1)	− 70	− 76	− 90	− 77	− 1	−34	−40	−20	+ 1	+ 4	+20	+102	−10	+60	+84	−31	+129	+ 68	+ 85	− 95	54·85	63·02
(2)	− 92	−107	− 61	+ 43	− 48	+23	−27	−42	−35	− 15	+13	− 91	+71	+71	+48	−15	− 16	+ 39	+103	+141	55·05	61·37
(3)	−207	− 89	− 43	−186	− 58	−33	+ 1	+17	+ 2	+ 22	−20	+ 79	+44	+65	+80	+72	+ 76	+111	+162	+136	75·15	77·43
(4)	−172	−152	−108	−100	−117	−55	−45	−24	−16	+ 7	+16	+ 45	+52	+71	+85	+96	+117	+136	+151	+224	89·45	86·72
(5)	− 45	− 71	− 64	+ 19	− 7	+19	−83	−34	−11	+ 56	+17	− 8	+23	− 4	+62	−64	+ 57	+ 25	+ 74	− 47	39·50	51·06
(6)	− 97	− 48	− 55	−122	+ 2	−16	−86	+ 4	+19	+103	− 1	+119	−10	+60	+85	−29	+101	+ 62	+ 41	−114	58·70	59·23
(7)	−115	− 73	− 27	− 47	− 30	+28	− 3	−15	−21	− 1	− 8	− 51	+62	−20	+51	−17	+ 2	+ 35	+ 53	+ 90	37·45	56·85
(8)	− 67	− 50	− 39	− 28	− 5	+23	−60	−15	− 3	+ 59	+ 2	+ 12	+21	0	+63	−60	+ 40	+ 23	+ 42	− 64	36·05	48·22
(9)	−100	− 41	− 20	− 98	− 5	+17	−42	+ 5	+ 5	+ 63	−15	+ 42	+16	+15	+66	−40	+ 34	+ 35	+ 19	− 48	36·30	50·38
Mean deviation	−107	− 79	− 56	− 66	− 30	− 3	−13	−14	− 6	+ 33	+ 3	+ 28	+30	+35	+69	−10	+ 60	+ 59	+ 81	+ 25	41·8	
Actual capacity	1520	1490	1444	1433	1415	1390	1378	1362	1355	1335	1322	1300	1280	1270	1255	1240	1225	1202	1185	1100		

the same process—if they have not worked together, watching and comparing each other's methods—may easily differ by 20 to 40 cubic centims. in their determination of skull capacity for the same skull, we appreciate that the errors made by our reconstruction formulæ are not much greater than the personal equation of two observers.

We may then conclude that our formulæ will allow us to make from the usual measurements of L, B, and H a fair estimate of the capacity of a skull, which is too fragile or too imperfect to have its capacity determined directly.

(9.) The next problem under this section is: *The determination of the individual capacity from data drawn from a second race.* This really involves the second fundamental problem, but for purposes of practical convenience I consider it here, justifying my application later. I found very poor results arose when I calculated individual Germans from Aino formulæ except in the case of formula (9). This on the other hand gave almost as good results, as if the individual Germans had been determined directly from their own racial formula (9). To illustrate this, I give in Tables XV. and XVI. the reconstruction for the selection made at random of forty German skulls, and further, a reconstruction for forty Naqada skulls also taken at random. In both cases I calculated the capacity from the Aino formula (9). German formula (9) applied to the Naqada gave very nearly identical results.

TABLE XV.—German Capacity calculated from Aino Formula (9).

Male.			Female.		
Actual capacity.	Calculated.	Difference.	Actual capacity.	Calculated.	Difference.
1705	1558	−147	1520	1417	−103
1660	1573	− 87	1490	1458	− 32
1640	1678	+ 38	1444	1422	− 22
1625	1733	+108	1433	1329	−104
1600	1545	− 55	1415	1407	− 8
1572	1554	− 18	1390	1405	+ 15
1560	1667	+107	1378	1330	− 48
1535	1505	− 30	1362	1362	0
1500	1497	− 3	1355	1356	+ 1
1485	1421	− 64	1335	1395	+ 60
1475	1506	+ 31	1322	1300	− 22
1460	1457	− 3	1300	1337	+ 37
1450	1496	+ 46	1280	1289	+ 9
1433	1459	+ 26	1270	1276	+ 6
1425	1465	+ 40	1255	1314	+ 59
1405	1476	+ 71	1240	1188	− 52
1375	1369	− 6	1225	1250	+ 25
1360	1532	+172	1202	1226	+ 24
1325	1388	+ 63	1185	1192	+ 7
1260	1280	+ 20	1100	1034	− 66
	Mean error = 56·7			Mean error = 35·0	

TABLE XVI.—Naqada Capacity calculated from Aino Formula (9).

Male.			Female.		
Actual capacity.	Calculated.	Difference.	Actual capacity.	Calculated.	Difference.
1448	1418	− 30	1266	1271	+ 5
1354	1375	+ 21	1174	1171	− 3
1354	1379	+ 25	1148	1146	− 2
1260	1351	+ 91	1195	1213	+ 18
1481	1502	+ 21	1160	1228	+ 68
1232	1285	+ 53	1120	1223	+103
1335	1329	− 6	1248	1209	− 39
1388	1430	+ 42	1451	1383	− 68
1326	1288	− 38	1160	1268	+108
1338	1348	+ 10	1290	1276	− 14
1305	1413	+108	1106	1124	+ 18
1224	1366	+142	1214	1159	− 55
1368	1380	+ 12	1120	1249	+129
1328	1321	− 7	1190	1280	+ 90
1475	1435	+ 40	1304	1276	− 28
1281	1305	+ 24	1173	1215	+ 42
1440	1426	− 14	1152	1137	− 15
1174	1252	+ 78	1135	1173	+ 38
1292	1321	+ 29	1299	1285	− 14
1253	1374	+121	1158	1152	− 6
	Mean error = 45·6			Mean error = 43·15	

Now Tables XIII. and XIV. show that the mean errors made for the 20 ♂ and 20 ♀ German skulls, reconstructed by the German formulæ (9) were respectively 55·4 and 36·3 cub. centims. The same skulls reconstructed from the Aino formulæ (9) give mean errors of 56·7 and 35·0 cub. centims.; while the Naqada skulls have mean errors of 45·6 and 43·1 cub. centims. respectively. We may thus conclude that within the limits of error occurring in reconstructing capacity, formula (9) as found for any race may be safely used to calculate the capacity of an individual of a different race. This is a very important result, and its basis will be further considered in the next section of this paper. We conclude that an average error of, say, 3 to 4 per cent. is all we shall make in applying (9) to determine the skull capacity of any individual not necessarily of the same race.

(10.) *Second Fundamental Problem. On the determination of the mean skull capacity of any local race of man from the regression formulæ of a second race.*

Professor KARL PEARSON has shown in a memoir, not yet published, that a general theorem holds for the influence of selection on the regression formulæ. A statement of this theorem is given by him in the 'Phil. Trans.,' A, vol. 192, p. 177. It may be summed up as follows: If selection has differentiated local races, then the regression formulæ will in general change from local race to local race, but that certain *indirectly* selected organs, when they have their values expressed in terms of *all* the directly selected organs, and any number of indirectly selected organs will have regression formulæ the same for the differentiated races. Further, if size be the character chiefly selected, then the changes in the constants of the regression formulæ will only be of the second or third order.

Without entering into a discussion of this and allied theorems by which Professor Pearson hopes to quantitatively attack the problem of the evolutionary relationship of local races, I would note that for our immediate purposes we seek a formula which will apply to all local races, and that the best formula will be one that is sensibly identical in its results for extremely different types of life.

Now a very short inspection of Tables V. to VIII. shows that for neither sex are the constants for any one of the first eight regression formulæ approximately alike. It seems therefore absolutely impossible to apply successfully any one of these to any other local race. On the other hand, considering the comparative paucity of the skulls dealt with, there is a remarkable agreement between the constants of formula (9) for both races. This agreement for different races again receives striking confirmation when we examine the results for the Naqada race given on p. 237. I reproduce the whole series here :—

Table XVII.—Reconstruction Formula (9).

Males.

German formula . . .	$C = {\cdot}000332 \times L \times B \times H + 415{\cdot}34$
Aino formula	$C = {\cdot}000328 \times L \times B \times H + 430{\cdot}30$
Naqada formula . . .	$C = {\cdot}000352 \times L \times B \times H + 372{\cdot}39$
Mean formula	$C = {\cdot}000337 \times L \times B \times H + 406{\cdot}01$

Females.

German formula . . .	$C = {\cdot}000383 \times L \times B \times H + 242{\cdot}19$
Aino formula	$C = {\cdot}000400 \times L \times B \times H + 187{\cdot}80$
Naqada formula . . .	$C = {\cdot}000416 \times L \times B \times H + 189{\cdot}81$
Mean formula	$C = {\cdot}000400 \times L \times B \times H + 206{\cdot}60$

We could hardly have selected three more diverse races than German, Aino, and Naqada, and yet we have reached for sparse material a surprising identity of results! If we want the mean skull capacity of any race for which L, B, and H are known, we have only to select the closest race out of Table XVII., or, failing knowledge of racial relationships, the mean formula, and we shall obtain a result well within the error of the personal equation of two observers, or the differences arising from using different methods of directly determining capacity. These points we shall now demonstrate numerically.

In Table XVIII. a comparative illustration is given of the accuracy of formula (9), and the failure of formulæ (5) to (8) when they are applied from one local race to a second; formula (9) alone comes out and comes out triumphantly from the test. The errors made are from 2 to 5 cubic centims. on a total of 1300 to 1500, or the largest error is less than ·45 per cent.

Table XVIII.

Formula.	Mean capacity of Germans calculated from Ainos.	Actual value.	Mean capacity of Ainos calculated from Germans.	Actual value.
	Male		Male	
(5)	1442·07		1433·59	
(6)	1392·44		1549·29	
(7)	1575·56		1376·13	
(8)	1445·00		1432·61	
(9)	1506·91	1503·72	1459·14	1461·64
	Female		Female	
(5)	1327·82		1285·92	
(6)	1268·97		1380·24	
(7)	1374·73		1240·01	
(8)	1324·77		1292·20	
(9)	1331·89	1337·15	1313·45	1307·69

Table XIX.—Reconstruction of Local Races from Formulæ (9).

Race.	Sex.	German formula.		Aino formula.		Naqada formula.		Mean formula.		Actual value.
		Value.	Error.	Value.	Error.	Value.	Error.	Value.	Error.	
German	♂	—	—	1507	+ 3	1528	+24	1512	+ 8	1504
Aino	♂	1459	− 3	—	—	1479	+17	1466	+ 4	1462
Naqada	♂	1372	−15	1375	−12	—	—	1377	−10	1387
Theban Mummies . .	♂	1368	−22	1365	−25	1379	−11	1370	−20	1390
Modern Egyptians . .	♂	1346	− 9	1349	− 6	1359	+ 4	1350	− 5	1355
Ancient Etruscans . .	♂	1427	−29	1430	−26	1445	−11	1433	−23	1456
German	♀	—	—	1332	− 5	1380	+43	1351	+14	1337
Aino	♀	1313	+ 5	—	—	1353	+45	1325	+17	1308
Naqada	♀	1246	−33	1236	−43	—	—	1255	−25	1279
Theban Mummies . .	♀	1235	−19	1224	−30	1267	+13	1243	−11	1254
Modern Egyptians . .	♀	1171	−25	1158	−38	1199	+ 3	1177	−19	1196
Ancient Etruscans . .	♀	1294	−30	1287	−37	1332	+ 8	1305	−19	1324

In Table XIX. a more elaborate investigation is made of formula (9) only, using the four forms given in Table XVII. and tabulating the errors made. We see that the maximum error of the mean formula is under 2 per cent., and the average error under 1 per cent. These errors appear to me less than the personal equation of two observers, measuring the same series of skulls. In fact, I am inclined to think that the errors of the mean formulæ may be as much due to the different observers who have determined the "actual" values as to defects in the formulæ themselves. The close association of the Aino and German results is specially noteworthy.

The results for the correlation and regression, not only of the skull, but of the long bones of the Ainos, show a relation much closer to modern Europeans (French and German) than the latter bear to the Naqadas. The primitive Aino race appears to be in some manner much more closely related to the evolutionary source of the Aryan races than either are to the Naqada.

On the other hand, it will be seen that the Naqada formula while giving bad results for German and Aino gives much better results than they do for both the ancient and modern Egyptians. Its maximum error as applied to the Egyptian races is only slightly over 1 per cent., while its average error as applied to all three Egyptian races is under ·4 per cent.

The mean formula over-estimates the Aino and German, and under-estimates the Egyptian races.*

The general rule for deducing the best result, would clearly be to work with the formula for the most closely associated race. But if no association can be predicted, then we shall hardly have an error as large as 2 per cent. if we use the mean formula. As this error is less than that frequently obtained by different observers for the same series, I conclude that a fairly satisfactory formula has been reached for the reconstruction of skull capacity from external measurements.

(11.) At this point it seems desirable to say a few words about the errors made by different observers in estimating skull capacity. I believe the differences of the same observer using different methods on the same skull can be reduced to a very few cubic centimetres, but the personal equation of two observers using different or even nominally the same methods on the same skulls will be very considerable. When the observers have been trained in different schools and use different methods the divergences may be very great. The value of the capacity depends so largely on the amount of "packing" both in the skull and in the measuring glass. Thus I found with two very careful investigators measuring about fifty skulls, their averages differed by about 30 cubic centims., and this difference was approximately constant for each skull. Three measurers using the same process with great care got results for individual skulls occasionally differing by even as much as 40 cubic centims.! On the

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* It should be noticed that the German formulæ give better results than the Aino for the Naqadas, although in cephalic index the Aino is much closer to the Naqada than the German is.

other hand one measurer using the same method soon obtained practically identical results in making re-measurements, and even one measuring in three different ways.

If the reader will merely look at the Table XX. which follows, giving the capacity and chief dimensions of the skull for a number of races, he will easily convince himself that the differences in capacity must be largely due to the differences of personal equation and of method and not to the thicknesses of bone in the crania. Take the French (P) skulls; they are not decisively the largest in the series and yet they are credited with capacities which easily head the list. For relative purposes it is almost impossible to credit different series with a correctness within 30 cubic centims. of the true value. Hence such deviations as we find in the second column of Table XIX. seem well within the observational accuracy attainable, and I think it quite possible that if we had some further large series of L, B, H, and C, determined by careful observers, we should have a formula giving more trustworthy results for the mean capacity than could be obtained by the direct measurement of an individual observer. The averaging of a number of series would tend to eliminate the large personal equations which I feel sure exist in measurements of this kind.

TABLE XX.—General Table of Skull-dimensions for divers Races.

Race.	Number.	Sex.	L.	B.	H.	H′.	C.
Aino	76	♂	185·82	141·23	119·32	139·50	1462
Malay†	76	♂	174·33	142·36	116·88	140·68	1430
Negro*	54	♂	185·04	135·20	[115·17?]§	134·77	1430
Bavarian	100	♂	180·58	150·47	120·75	133·78	1503
Badenser†	78	♂	181·50	148·60	113·40	132·50	1525
French (M)‡	56	♂	179·96	143·41	112·86	128·95	1473
French (P)*	77	♂	182·69	145·22	[117·71?]§	132·01	1560
Egyptians, ancient†	201	♂	181·83	137·14	114·28	135·94	1390
Egyptians, modern†	76	♂	179·11	136·51	115·42	137·50	1355
Naqada	69	♂	185·13	134·87	115·59	135·21	1387
Etruscans†	78	♂	182·88	143·53	115·90	139·20	1456
Aino	52	♀	177·17	136·79	114·97	135·10	1308
Negro*	23	♀	174·52	130·52	[106·51?]§	126·91	1256
Bavarian	100	♀	173·59	144·11	114·17	128·01	1337
Badenser†	45	♀	172·20	141·30	107·70	124·90	1339
French (P)*	41	♀	174·34	135·49	[112·10?]§	125·10	1338
Egyptians, ancient†	96	♀	175·92	134·16	110·25	130·64	1254
Egyptians, modern†	23	♀	175·04	131·00	107·65	130·81	1196
Naqada	98	♀	177·48	131·61	113·11	129·55	1279
Etruscans†	38	♀	177·47	138·81	111·34	133·71	1324

* Extracted for Professor PEARSON from BROCA'S manuscript registers at Paris, by the courtesy of M. MANOUVRIER.

† From the German Anthropological Catalogue.

‡ Skulls of French prisoners who died in Munich during the Franco-German war. (German Anthropological Catalogue.)

§ Rough estimate, as data were wanting.

While I believe strongly in picking out the formula for the most closely associated race, I give the value of the constants for the male and female formulæ as obtained not by correlation, but by the method of least squares from the results in Table XX.

For the males excluding the negroes, I find for the ten races:[*]

$$♂ \quad C = \cdot 000365\, L \times B \times H + 359 \cdot 34 \quad . \quad . \quad . \quad . \quad . \quad (10).$$

For the females excluding the negroes, I find for the eight races:

$$♀ \quad C = \cdot 000375\, L \times B \times H + 296 \cdot 40 \quad . \quad . \quad . \quad . \quad . \quad (11)$$

Table XXI. gives the capacities of the races as found from (10) and (11).

Such equations should, I think, only be used when we have no knowledge of the evolutionary history of the race, which would lead us to adopt one of the special equations of Table XVII.

(12.) In attempting to use the formulæ given in this section, the reader must bear in mind that they can only be applied when the maximum length, maximum breadth, and the *auricular* height are known. The latter measurement unfortunately is occasionally omitted in series of skull measurements. If the total height of the skull H′ be given, then the following formulæ can be used, which have been calculated by the method of least squares from all the results in Table XX.

$$♂ \quad C = \cdot 000266\, L \times B \times H' + 524 \cdot 6 \quad . \quad . \quad . \quad . \quad . \quad (12),$$

$$♀ \quad C = \cdot 000156\, L \times B \times H' + 812 \cdot 0 \quad . \quad . \quad . \quad . \quad . \quad (13).$$

The following table includes results from (12) and (13) as well as from (10) and (11):—

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[*] Excluding for various reasons the unsatisfactory French (P), the French (M), and the Badenser, I find

$$C = \cdot 000370\, L \times B \times H + 321 \cdot 16 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (10)\ \textit{bis}.$$

TABLE XXI.—Calculated and Observed Capacity for 10 Races.

Race.	Sex.	From L, B, and H.	From L, B, and H′.	Observed.
Aino	♂	+40 (+17)	+ 36	1462
Malay	♂	−12 (−35)	+ 23	1430
Negro	♂	[−19?]	− 9	1430
Bavarian	♂	+54 (+32)	− 11	1503
Badenser	♂	−50	− 50	1525
French (M)	♂	−51	− 63	1473
French (P)	♂	−61	−104	1560
Ancient Egyptian	♂	+10 (−15)	+ 36	1390
Modern Egyptian	♂	+34 (+10)	+ 64	1355
Naqada	♂	+23 (+2)	+ 36	1387
Etruscan	♂	+14 (−9)	+ 40	1456
Mean deviation	—	34·8 (17·1)	42·9	—
Aino	♀	+34	+15	1308
Negro	♀	[−50?]	+ 7	1256
Bavarian	♀	+31	−25	1337
Badenser	♀	−60	−53	1339
French (P)	♀	−49	−65	1338
Ancient Egyptian	♀	+18	+39	1254
Modern Egyptian	♀	+26	+82	1196
Naqada	♀	− 1	+ 5	1279
Etruscan	♀	+ 1	− 6	1324
Mean deviation	—	24·6	33·0	—

The table illustrates several important points, thus:

(i.) We obtain less average error by estimating with H than H′, or the capacity of the skull is better determined from the auricular height, than from the total height of the skull.

(ii.) If we exclude the series for which the values of the capacity seem to be doubtful, *i.e.*, the Badenser and French, we obtain (bracketed numbers from (10) *bis*, footnote, p. 247) a mean error of about 1·2 per cent. and a maximum error of 2·5 per cent. For the series as a whole we have a mean error of about 2·4 per cent. with a maximum error of 4 per cent.

The latter occurs in the case of the Parisian French; but I have not the least hesitation in asserting that the capacities of the French skulls as determined in France, are quite incomparable with the capacities as determined by German investigators. I believe the French capacity is 60 to 80 cubic centims. beyond its true value, and I hold that my formula determines that value far more closely than the mean of the numbers (1560) given by BROCA's MS. registers. I do not think it can differ by more than a few cubic centimetres from 1499, and this difference is probably in defect. It will be seen that the Munich French skulls are somewhat smaller than the Parisian French skulls, but this does not account for the whole difference of 87 cubic centims. found by German and French determinations. It is largely a question of method. Again Mr. HERBERT THOMPSON found for the capacities of 39 ♂ and 55 ♀ Naqada skulls, 1339 and 1243 cubic centims. respectively, but Miss C. D. FAWCETT using a different method on 69 ♂ and 98 ♀ skulls obtained 1387 and 1279 cub. centims. respectively. Something here is due to the difference of the samples, but as in the previous case the personal equation is the chief source of difference. Now if differences of sample, of observer and of method will lead to determinations of racial capacity differing by 3 to 6 per cent., is not a great deal to be said for a formula which when applied to a series of results of a uniform character (like those of the best German determinations given above) leads to an error of only 2·5 per cent. as a maximum? I should personally feel as content with the results in Table XXI. of my mean regression formulæ and of the least square formulæ of p. 247, as with the average found for a race after days of laborious determination of capacity by aid of shot, seed, or sand. If the reader be not content with this degree of approximation, then I think no formula will satisfy him; *for nature being inherently variable, the capacity is no definite function of any dimensions of the skull, it is only moderately correlated with these dimensions, and the probable error of the determination cannot be reduced beyond quite sensible limits.*

The alternative to a formula is, of course, to make direct determination more uniform and exact. Now I believe two observers may be trained to get fairly accordant results, but will these results be the *real* capacity of the skull? May not the reality lie more nearly in the mean of the determinations of a number of careful observers measuring independently? Their errors may fall on either side of the truth, whereas a systematised procedure may give their errors a common bias. Hence a formula based upon a fairly wide set of results by different, but careful, observers may after all be more trustworthy than direct determination by a conventional method. It might, of course, be possible to reduce the conventional method to physical exactness; but I do not think this exactness is reached by the construction of control skulls (*Normalschädel*, *Crâne étalon*), which cannot cover all types; it might possibly be done by opening each skull (allowing for the thickness of the saw cut), and then filling either half. But such a process is laborious, it destroys the skull for some other purposes, and when the true capacity has been found we should have only the *average of a sample.* With the size of cranial samples at present available, the mean errors of the means amount to about 12 cubic centims., or are of the order of the errors of a good formula. Hence physical exactness (which would also improve the constants of the formula) is not all that is wanted.

(13.) Accepting the product formula as a working result, a further question may still arise as to whether it is needful to form the mean product of L × B × H or whether we may content ourselves with the product of the mean values of L, B, and H for the race.

The following table indicates the order of error made by using the product of means for the mean product:—

TABLE XXII.

Race.	Mean product.	Product of means.
Etruscan ♂	3,046,886	3,042,232
Etruscan ♀	2,746,817	2,742,818
German ♂	3,282,338	3,280,662
German ♀	2,860,213	2,856,635
Naqada ♂	2,881,137	2,886,107
Naqada ♀	2,619,631	2,642,039
Aino ♂	3,144,287	3,129,831
Aino ♀	2,797,032	2,786,983
Thebans ♂	2,859,374	2,849,705
Thebans ♀	2,589,815	2,602,057
Modern Egyptians ♂ . .	2,801,990	2,822,055
Modern Egyptians ♀ . .	2,424,920	2,468,440

It will be found that whether we use the mean product or the product of the means will make only a few cubic centimetres difference in the estimate, something under the 1 per cent., within which we cannot suppose our results to be correct. Hence for practical purposes we may content ourselves with using the product of the means, the determination of which is far less laborious. Our least square formulæ have all been based on the product of the means.

(14.) *Third Fundamental Problem. To reconstruct from external measurements an organ not measurable on the living organism,* i.e., *the skull capacity from measurements on the living head.*

It has been shown by KARL PEARSON ('Phil. Trans.,' A, vol. 192, p. 183) that if x and y be two characters and m, n, m', n' four constants, then the correlation coefficient of $mx + n$ and $m'y + n'$ is the same as that of x on y. The regression coefficient will be the same if $m = m'$. Now in the case of length, breadth, height, l, b, and h measured on the living head we have differences from their values as measured on the skull depending on the thickness of the living tissues covering the skull. These tissues of course vary from individual to individual, but as the thickness of the tissues themselves are of the second order of small quantities as compared with the length, breadth, and height of the skull, we may safely assume that their variations will be of the like order compared to those of l, b, and h. We shall thus obtain a very fair approximation to the regression coefficients connecting the skull capacity with head-length, breadth, and auricular height, by using those already found for the like quantities measured on the skull. Thus we should have a formula (9) of the form

$$C - C_0 = \alpha(l - l_0) + \beta(b - b_0) + \gamma(h - h_0) \quad . \quad . \quad . \quad . \quad . \quad (A)$$

where l_0, b_0, h_0 are the mean length, breadth, and auricular height of the living head, and C_0, α, β, γ constants to be determined from the measurement of skulls.

Further, formula (9) takes the form

$$C = \epsilon(l - \delta_1)(b - \delta_2)(c - \delta_3) + \eta \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (B)$$

where ϵ and η are to be determined from skull measurements, and δ_1, δ_2, δ_3 give the mean differences between head and skull measurements. What values are to be given to these quantities?

As we have seen, the constants ϵ and η of (B) do not vary very much from local race to local race, while, on the other hand, α, β, γ of (A) differ very considerably from race to race. We shall hardly expect, therefore, to obtain as good results from (A) as from (B). Let us accordingly take (B) first, and consider δ_1, δ_2, δ_3.

H. WELCKER[*] gives the following measurements for an average of thirteen males in middle life:—

Thickness of flesh at back of head = 6·8 millims.
,, ,, middle of forehead = 4·3 millims.
,, ,, top of crown = 5·9 millims.

The values at the side of the head and on the auricular orifices are not given. But the results seem to show an average of 11 to 12 millims. to be subtracted from the head measurements when we wish to get those of the skull.

MERKEL[†] gives 6 millims. as an average thickness of the tissues covering the skull. Thus WELCKER and MERKEL are in good agreement.

We can consider this matter from another standpoint. I can find no head measurements from Bavarians or Badenser to compare with my skull measurements in Table XX., but the following table gives some results from English sources:—

TABLE XXIII.—Mean Head Measurements.

Organ.	Male.			Female.	
	B.A.	Anatomists.	U.C. Staff.	B.A.	B.C. Students.
l_0	198·1	198·4	196·38	185·6	189·71
b_0	155·0	157·2	153·48	147·3	146·78
h_0	130·9	133·1	134·78	128·4	132·73
$\frac{1}{3}(l_0+b_0+h_0)$	161·3	162·9	161·55	153·8	156·41

The British Association measurements are averages obtained by myself from the values given for several years in the "Reports of the Anthropometric Committee" which are published in the 'Transactions.'[‡] They cover quite a long series. The "anatomists" are the head measurements of thirty-five of the anatomists attending

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[*] 'Schillers Schädel und Todtenmaske,' Braunschweig, 1883.

[†] 'Handbuch der topographischen Anatomie,' Bd. I., p. 12.

[‡] Reports of Committee, 1889 . . . 1893.

the meeting of the Anatomical Society in Dublin, June 10, 1898. They were measured in the Anthropometrical Laboratory of Trinity College, and the data were published in the 'Journal of Anatomy' in 1898. The University College staff consist of twenty-five members only of the staff of University College, London, measured by Professor KARL PEARSON. The Bedford College students were measured by Miss C. D. FAWCETT, B.Sc., and myself, and were thirty in number. In all these cases there were undoubtedly a good many heads not of English origin, but this was especially the case at the Anatomical Congress, where a number of foreign *savants* were present. I should consider the British Association returns the most homogeneous and reliable for working with, but it is noteworthy to what an extent the Bedford College women exceed in size of head the women attending the British Association meetings.

Now it would be impossible to compare the l_0, b_0, h_0 of the British Association measurements directly with the L_0, B_0, H_0 of the Bavarians, for the latter belong to a far more brachycephalic race. But if we compare $\frac{1}{3}(l_0 + b_0 + h_0)$ with $\frac{1}{3}(L_0 + B_0 + H_0)$ we find a difference of 10·7 for ♂ and 9·8 for ♀. If we compare the corresponding results for the Aino, a race with somewhat the same degree of dolichocephaly, we find differences of 12·5 and 10·8 respectively. Although we cannot lay much stress on this reasoning which supposes $\frac{1}{3}(L_0 + B_0 + H_0)$ approximately constant for local races, still it confirms WELCKER and MERKEL'S results so far as it goes. I think, without differentiating between the sexes, we shall obtain reasonable results by considering that we must subtract about 11 millims. from the head measurements in order to obtain the corresponding skull measurements. This being so, we have the following fundamental equations deduced from the mean equation of p. 243, to find the capacity from measurements on the living head :—

$$\left.\begin{array}{ll} ♂ & C = \cdot 000337\,(l - 11)\,(b - 11)\,(h - 11) + 406\cdot 01 \\ ♀ & C = \cdot 000400\,(l - 11)\,(b - 11)\,(h - 11) + 206\cdot 60 \end{array}\right\} \quad . \quad . \quad . \quad (14)$$

If we use the British Association mean values in (14), we find that for the mean skull capacity of the British—no doubt English in the bulk—the values

♂ 1495 cubic centims. ♀ 1323·5 cubic centims.

There appears at present to be no satisfactory determination of the skull capacity of English men and women, and these results are, I believe, as reliable as any estimates yet formed.[*] The ratio of ♂ to ♀ skull capacity would thus be 1·13, corresponding well with the ratio of brain weights, 1·12, as determined by REID and PEACOCK, but considerably higher than the ratio for brain weights given by CLENDINNING and SIMS.

A rough sort of control formula for comparison with (14) may be obtained by substituting the British Association values for C_0, l_0, b_0, and h_0 in the equation

$$C - C_0 = \epsilon\,(l \times b \times h - l_0 \times b_0 \times h_0).$$

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[*] See PEARSON, "Variation in Man and Woman," 'The Chances of Death,' vol. I., p. 328.

In this way we find :

$$\left.\begin{array}{ll}\text{♂} & C = \cdot 000{,}337\, l \times b \times h + 140\cdot 13 \\ \text{♀} & C = \cdot 000{,}400\, l \times b \times h - 80\cdot 62\end{array}\right\} \quad . \; . \; . \; . \; . \; . \quad (15).$$

This formula merely assumes that the factor multiplying the product of length, breadth, height remains the same, whether these quantities are measured on the head or the skull.

We now turn to the discovery of linear formulæ corresponding to (8) of pp. 234, 236. Here we are met by the very obvious difficulty that unlike formula (9) the constants of formula (8) change much from local race to local race. If we take the formula for the Germans as being nearest akin to the English, we are met by the obvious fact that the constants change widely when we pass from a brachycephalic to a dolichocephalic race; the English, indeed, have a cephalic index nearer to the Ainos than to the Germans. Accordingly, in default of more ample data for striking a mean formula, I have inserted in (A) of p. 250, the mean values of the German and Aino constants. We thus have :—

$$\begin{array}{ll}\text{♂} & C - C_0 = 10\cdot 1025\,(l - l_0) + 8\cdot 0345\,(b - b_0) + 3\cdot 709\,(h - h_0), \\ \text{♀} & C - C_0 = 7\cdot 222\,\;(l - l_0) + 8\cdot 4605\,(b - b_0) + 6\cdot 300\,(h - h_0).\end{array}$$

Inserting the British Association mean values for l_0, b_0, and h_0, as well as the mean capacities found from (14), we have :—

$$\left.\begin{array}{ll}\text{♂} & C = 10\cdot 1025\, l + 8\cdot 0345\, b + 3\cdot 709\, h - 2237\cdot 52 \\ \text{♀} & C = 7\cdot 222\, l \;\; + 8\cdot 4605\, b + 6\cdot 300\, h - 2071\cdot 22\end{array}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \quad (16).$$

Another linear formula may be obtained in an entirely different manner by taking the tangent plane at the mean to the surface in (14). Thus the skull measurement surface is :—

$$C = \epsilon LBH + \eta,$$

and the tangent plane is

$$C - C_0 = \epsilon L_0 B_0 H_0 \left\{ \frac{L - L_0}{L_0} + \frac{B - B_0}{B_0} + \frac{H - H_0}{H_0} \right\}.$$

Now introduce the British Association values, remembering that $L_0 = l_0 - 11$, $B_0 = b_0 - 11$, $H_0 = H_0 - 11$, and we find :—

$$\left.\begin{array}{ll}\text{♂} & C = 5\cdot 8185\, l + 7\cdot 5600\, b + 9\cdot 0796\, h - 2017\cdot 96 \\ \text{♀} & C = 6\cdot 4006\, l + 8\cdot 1992\, b + 9\cdot 5192\, h - 2294\cdot 46\end{array}\right\} \quad . \; . \; . \; . \; . \quad (17).$$

Equation (17) will be found to give results excellently in accord with (14); it is the linear formula most comparable with (14), yet the coefficients differ very widely from those of (16), the height which is least influential in (16) being now the most influential factor. It would have been satisfactory to find (17) more closely in agreement with (16), but the universality of (14) on which (17) is based, is quite wanting in (16).

Lastly, we may place here the linear formula found by taking the value of the German coefficients of formula (8), (pp. 234 and 235), and using British Association mean values, we have :—

$$\left.\begin{array}{ll} \male & C = 7{\cdot}348\, l + 10{\cdot}898\, b + 5{\cdot}228\, h - 2334{\cdot}17 \\ \female & C = 7{\cdot}065\, l + 10{\cdot}126\, b + 4{\cdot}848\, h - 2101{\cdot}81 \end{array}\right\} \quad . \quad . \quad . \quad . \quad . \qquad (18).$$

The following table illustrates the degree of closeness of these various formulæ as applied to 17 selected heads of very different sizes. We observe that while the formulæ give considerable differences in the absolute capacities, especially in the case of the macrocephalic heads, the relative order of the heads as determined by all the formulæ is the same with but two exceptions. In the first place (14), (15), (17) and (18) give a relative order entirely the same, except for the slight displacement of Professor Howes under (18). For the females (16) is also entirely in accord with (14), (15), (17) and (18). The second displacement is that of Professor Weldon's head under (16), which alters its place by two. I can only account for this by the fact that Professor Weldon has a high cephalic index (82·7), and therefore the German formula was likely to give a better result than one based on the average of the German and of a less brachycephalic race like the Aino.

Table XXIV.—Skull Capacities from Living Head by Various Formulæ.

Name.	Formula.				
	14.	17.	15.	16.	18.
J. Lynn Thomas	1813	1789	1861	1785	1773
W. F. R. Weldon	1616	1616	1632	1533	1579
W. Ramsay	1581	1579	1594	1569	1572
A. Keith	1530	1530	1536	1557	1548
A. Platt	1501	1502	1501	1479	1481
G. B. Howes	1483	1485	1481	1458	1496
K. Pearson	1452	1454	1444	1398	1410
E. Barclay Smith	1408	1407	1396	1365	1396
J. Kollmann	1373	1370	1353	1332	1369
♀ Student 1	1647	1620	1697	1593	1587
♀ Student 4	1514	1507	1543	1471	1458
♀ Student 8	1488	1481	1512	1453	1440
♀ Student 12	1450	1447	1471	1442	1430
♀ Student 16	1368	1368	1376	1384	1388
♀ Student 20	1321	1321	1320	1318	1307
♀ Student 24	1302	1305	1299	1303	1284
♀ Student 28	1230	1227	1214	1225	1216

Thus of the two exceptions to complete accordance we see that only Professor WELDON's head in the case of formula (16) presents any serious disturbance of the relative order.

On the whole, my methods will, I think, determine within reasonable limits the relative order of skull capacity from measurements on the living head. It is noteworthy that except for the macrocephalic heads of Mr. LYNN THOMAS and Bedford College student No. 1, formulæ (17) and (14) give sensibly identical results, or there is one linear formula which gives results sensibly identical with those of the product formula. This shows us that the surface represented by (14) is sensibly plane for the range of skull measurements actually occurring. On consideration accordingly we may conclude that (14) (or its linear form (17)) gives the best results; (15) gives a good control formula; while of formulæ directly obtained from the regression equation for length, breadth, and height, the German appears best for the males, the mean of the German and Aino best for the females. For the remainder of my investigations on the capacity of the living head I shall accordingly use only the formulæ (14) and (16) or (18) for comparison.

I propose first to investigate whether there is any obvious relationship between skull capacity and current appreciation of intellectual ability.

My first series is contained in Table XXV. We have here the estimated skull capacities of thirty-five living anatomists. The list contains the names of many of great scientific reputation, and of others of less distinction. It will be seen that about the middle of the list, if we divide at D. HEPBURN, the eighteenth man, certain transfers would occur from the upper to the lower half, and *vice versâ*, if we judged by formula (18) and not (14). But these transfers are of men having roundly about the same skull capacity, and I think that generally we may feel quite satisfied with the accordance of the two series.* Now the average capacity of the first eighteen anatomists is 1601 cub. centims., and of the last seventeen anatomists is 1468 cub. centims. There is thus a most substantial difference.† Yet he would be a bold man who would assert that there is a substantial average intellectual superiority in the first half. In fact, a number of most capable men fall into the last nine, and J. KOLLMANN, one of the ablest living anthropologists, has absolutely the smallest skull capacity!

My second list contains the estimated skull capacity of twenty-five members of the teaching staff of University College, London. I give here the actual head measurements, as possibly of service in the future; those of the anatomists are published in the 'Journal of Anatomy' (see above). Here the first thirteen have a mean skull capacity of 1579 cubic centims. and the last twelve of 1436 cubic centims.—again a

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* We must always remember that (14) is *à priori* to be considered a much better formula than (18), for the change of its constants from race to race is far less.

† The mean of the whole series as given by (14) is 1537, and by (18) is 1534, a remarkable accordance in the average results of the two methods.

Table XXV.—Estimated Skull Capacity of 35 Anatomists.

Name.	Formula (14).	Formula (18).
J. Lynn Thomas	1813	1773
A. H. Young	1656	1640
B. A. Windle	1649	1605
D. G. Cunningham	1635	1600
Hector Leboucq	1631	1654
C. de Bruyne	1616	1636
T. Symington	1604	1627
A. M. Paterson	1595	1616
E. H. Taylor	1593	1624
Wilhelm His	1587	1556
C. R. Browne	1585	1578
G. Elliott Smith	1573	1570
C. D. Marshall	1570	1561
F. Frohse	1569	1625
A. F. Dixon	1541	1513
R. J. Berry	1539	1538
A. Robinson	1538	1532
D. Hepburn	1531	1537
Arthur Keith	1530	1548
Anonymous	1520	1524
Robert Howden	1511	1498
G. Disse	1507	1519
T. H. Bryce	1507	1491
Hans Gadow	1506	1483
Stanley Boyd	1499	1466
James Cantlie	1486	1496
G. B. Howes	1483	1496
Sir Wm. Turner	1469	1473
A. Macalister	1456	1458
W. Spalteholtz	1455	1524
G. D. Thane	1443	1413
James Musgrove	1425	1445
E. Barclay Smith	1408	1396
Peter Thompson	1385	1318
J. Kollmann	1372	1369

very sensible difference.* The only differentiation I feel able to make between the two divisions here is that six out of the second twelve are mathematicians, and no mathematician has here a head above the average. In the first group we find not the exact but the descriptive sciences and the arts. No generalisation can be drawn, however, from such narrow data. We have only the suggestion of a field for further enquiry.†

The agreement in Table XXVI. between the results of formulæ (14) and (18) is not so good as in the case of Table XXV., but the approximate general order is

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* The mean of the whole table is 1511, which may be compared with the 1537 of the anatomists. Both are sensibly larger than the British Association mean.

† The data for 1000 Cambridge men classified according to head measurements, branch of study and academic distinction, are at present being investigated.

maintained, and only one interchange between the first and second groups would take place.

TABLE XXVI.—Head Measurements and Estimated Skull Capacity of certain Members of the Teaching Staff of University College.

Name.	Head Measurements.			Estimated Capacity.	
	L.	B.	H.	(14)	(18)
H. TONKS	201	154	145	1633	1579
F. W. GOODBODY	203	160	137	1621	1617
T. G. FOSTER	201	159	139	1619	1602
W. F. R. WELDON	193·5	160	143	1616	1579
M. TRAVERS	199	158	140	1607	1582
F. G. DONNAN	197	155	143	1597	1550
W. RAMSAY	202	157	136	1581	1572
A. W. PORTER	199	154	140	1575	1535
J. SULLY	202	156	135	1563	1556
H. R. KENWOOD	194	162	135	1561	1563
R. RUSSELL	202	155	134	1546	1540
W. A. OSBORNE	197	150	138	1513	1470
A. PLATT	197	153	134	1501	1481
E. H. STARLING	204	149	131	1483	1473
L. N. G. FILON	201	151	130	1473	1468
W. P. KER	190	154	134	1467	1441
E. C. C. BALY	201	144	135	1462	1418
K. PEARSON	191	150	135	1452	1410
M. J. M. HILL	193	152	132	1452	1430
G. E. PETAVEL	192	155	130	1451	1445
G. THANE*	195	150	130·5	1436	1415
H. T. HARRIS	188	154	131	1430	1410
G. H. FOWLER	187	153	128	1391	1376
SWALE VINCENT	193	153	123	1381	1394
G. U. YULE	187	144	131	1352	1294

My third and last series, that of Table XXVII., contains the estimated skull capacities of thirty women students of Bedford College. I arranged these students on a considerable personal experience of their work into three classes of ten each, representing clever, medium, and dull students. I then divided the skull capacity list into three sections—large, medium, and small capacity. I was totally unable to find any correspondence between these two divisions into three classes.

I have used in this case formulæ (14) and (16). They give results generally in very good agreement, the general order not being substantially modified when we pass from one series to the other. The mean found from (14) is 1390 cubic centims., and

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* The values for L, B, H differ somewhat from those determined at the Dublin Anatomists' Congress, but they are, I believe, correct.

from (16) is 1376 cubic centims. These are in fairly good agreement. The average capacity is thus very sensibly larger than that of the British Association women (p. 251).

Table XXVII.—Head Measurements and Estimated Skull Capacity of 30 Bedford College Women Students.

Students.	L.	B.	H.	Formula (14).	Formula (16).
No. 1	200	157	141·5	1647	1593
„ 2	198	154	138	1565	1531
„ 3	196·5	149	140	1527	1491
„ 4	190	151·5	141	1514	1471
„ 5	187	151	143	1508	1458
„ 6	189	151	141·5	1507	1463
„ 7	195	144	142	1489	1450
„ 8	191	150	139	1488	1453
„ 9	200	145	135	1463	1450
„ 10	195	149	134	1456	1442
„ 11	194·5	144	139	1456	1427
„ 12	199	146	133·5	1450	1442
„ 13	190	150	135·5	1446	1424
„ 14	190	149	131	1393	1387
„ 15	192	155	124	1385	1408
„ 16	194	149	126	1368	1384
„ 17	187	148	130	1354	1350
„ 18	188	147	130	1352	1349
„ 19	180	152	129	1331	1327
„ 20	189	142·5	130	1321	1318
„ 21	186	147	128	1320	1322
„ 22	184	148	127	1306	1310
„ 23	187	145	127·5	1306	1309
„ 24	192	138	130	1302	1303
„ 25	187	137	133	1289	1276
„ 26	187	142	127	1276	1281
„ 27	187	138·5	127	1248	1251
„ 28	180	141	127·5	1230	1225
„ 29	186	135	127	1213	1214
„ 30	170	148	125	1200	1196

From my Tables XXV. to XXVII. I conclude that there is certainly no *marked* correlation between skull capacity and intellectual ability.

There is another standpoint, however, from which these things may be considered. I know of no measurements upon which a direct determination of the correlation of brain *weight* and skull capacity could be made. Of course, the two are not proportional; still, there can hardly be a doubt that they are highly correlated. Now, if two quantities are correlated with a third, it does not invariably follow that they will be correlated with each other.* Yet I take it that it is rather quantity than density of brain stuff which is at the basis of the current belief that size of brain is closely related to intellectual ability, and that any illustration of the absence of

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* A child is correlated with both parents, but, unless there be sexual selection, the parents are not correlated with each other.

sensible correlation between skull capacity and intellectual ability will tend to weaken current conceptions as to a relationship between brain weight and intellectual ability. The whole problem of the relation of size of head to intellectual distinction as judged by popular standards is under investigation from wider data ; meanwhile, I think we may conclude—

(i.) That there is no marked correlation between skull capacity and intellectual power in the case of either sex alone.

(ii.) That brain weight must have a very considerable correlation with skull capacity, and, therefore, our data present nothing to encourage the belief that there is a relation between brain weight and brain power.

(iii.) That arguments based on the relative brain weight of the two sexes as showing relative brain power require a more solid quantitative basis than they at present exhibit.*

(iv.) That such arguments as those of A. R. WALLACE against the evolution of man's intellectual powers by aid of natural selection turn wholly on the size of the brain. But it would not appear from the above results that skull capacity at any rate is a character closely correlated with intellectual ability in the individual, and, therefore it is quite conceivably not correlated with racial ability.

So soon as data are forthcoming connecting the skull capacity with brain weight, or still better, brain weight with head measurements, we shall be in a position to reconstruct brain weight from head measurements. I do not see that the error of the determination is likely to be much larger than that found in the case of skull capacity, but if it reached 8 to 9 instead, say, of 3 to 4 per cent., it would still be sufficiently approximate to give quite reasonable results for large numbers of individuals classified into big groups according to their ability. It is, I hold, only by such methods that we can hope to reach any quantitative certainty of a relation between brain power and brain size. Personally I am inclined to hold with Professor PEARSON that the complexity of the convolutions of the brain, and the variety and efficiency of its commissures, rather than its actual size, are the characters we might expect to differentiate race from race and sex from sex, and to have developed with man's civilisation.†

I am not unaware that a correlation has often been asserted between brain weight and ability on the ground of the actual measurement of the brain weights of a number of men of genius. But what is the average of such brains compared with ? The average brain weight of the bodies which reach the dissecting rooms of our hospitals, a large proportion of which belong to the emaciated and worn out. Probably on the same basis a correlation between genius and body-weight could

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* Before questioning whether man or woman (relatively to stature, body weight, or other character) has the greater brain weight, it seems desirable to settle whether brain weight in either sex alone, absolutely, or relatively to some other character, has anything to do with intellectual ability.

† 'Grammar of Science,' 2nd ed., p. 539.

easily be demonstrated. Or again because English women have a mean brain weight of 1235 grs. and French women of 1142 grs., are we to argue that English women are intellectually abler than French women? The fact is that to solve a problem of this kind we ought to keep within one fairly equally nourished class, and within one local race and actually correlate brain size and ability. I do not see how this can be done for brain weight, but it seems to me quite possible for the capacity of the brain chamber estimated from external measurements.

(15.) *Conclusions.* I have now completed the discussion of the three problems I proposed to investigate. It will be seen that the accuracy of predictions depends sensibly on two factors: (i) the existence of suitable data upon which the regression formulæ can be based and (ii) the number of measurements used to form an estimate. Thus in the third fundamental problem we do not get as good absolute results as we might hope to do, because we have not really at present available the best data possible. Again in the first fundamental problem we cannot expect to reconstruct the capacity of the individual skull without a fairly large average error. For it is of the very essence of the principle of variation, on which evolution itself depends, that in any population we should have an array of skulls with the same length, breadth, and height, and yet having within certain limits a variety of capacities. All we can hope to say is, that with such a length, breadth and height such a capacity is most frequent. When we come to averaging a series, then we shall determine with far greater accuracy the mean of an array. Here the nature of the problem is, however, modified. The question is now how far can we apply results deduced from one local race to a second. We want in fact a "panracial" regression formula to replace our intraracial regression formula. As it is impossible to find such a regression formula for the primitive stock from which man may be supposed to be derived, we are compelled to take the regression formulæ which are least changed as we pass from race to race. The mean formula thence derived appears to give excellent results, when applied to determine the capacity of very diverse races. While I do not profess to have solved the problems proposed to the degree of accuracy which might be obtained with wider data and measurements made *ad hoc* in the anatomical school, I yet consider that I have given practical solutions to the following problems:—

(i.) The reconstruction of the capacity of the individual skull, when this cannot be measured directly. This is done with a mean error of 3 to 4 per cent.

(ii.) The determination of the mean skull capacity of a race without the use of sand, seed, or shot, to a degree of accuracy comparable with that of the direct method owing to the personal equation of the measurers even when using the same method of direct determination.

(iii.) The determination of the skull capacity of living individuals with a degree of accuracy sufficient to determine whether skull capacity is or is not closely correlated with intellectual power.

APPENDIX.

On the Correlation of Skull Capacity with Circumferential Measurements on the Skull.

It may have occurred to some readers that other measurements of the skull beside length, breadth, and auricular height would give effective means of reconstructing the capacity. The two that most readily suggest themselves are the horizontal circumference, U say, and the vertical circumference, from the top rim of one auricular passage over the top of the skull to the other, V say. The following are the values for C, U, and V for the Naqada skulls as measured and calculated by Miss C. D. FAWCETT, B.Sc., who has most kindly placed them at my disposal.

Naqada Skulls, ♂.

Organ.	Mean.	S. D.	Correlation.
U . . .	509·170	12·178	r_{UC} = ·6803
C . . .	1379·23	109·213	r_{UV} = ·5116
V . . .	304·423	9·850	r_{VC} = ·6736

Naqada Skulls, ♀.

U . . .	492·759	11·958	r_{UC} = ·6588
C . . .	1283·238	86·902	r_{UV} = ·4519
V . . .	296·615	8·430	r_{VC} = ·5821

The units are millims. for U and V and cubic centims. for C. From these the following equations for the reconstruction of C in terms of U and V result :—

$$\left.\begin{array}{l}\text{For males:}\\ \qquad C = 3{\cdot}5035\,U + 2{\cdot}7789\,V - 1250{\cdot}604\\ \text{For females:}\\ \qquad C = 3{\cdot}2244\,U + 3{\cdot}2859\,V - 1280{\cdot}286\end{array}\right\} \quad . \ . \ . \ . \ . \ . \quad (19).$$

I have worked out somewhat fuller data for the collection of skulls of *Theban Mummies* at Leipzig, the measurements of which are given in the German Anthropological Catalogue.

Ancient Egyptians, ♂, 202 Skulls.

Organ.	Mean.	S. D.	Correlation.
U . . .	511·722	14·010	$r = ·8133 \pm ·0161$
C . . .	1391·54	121·616	$r_{UV} = ·6651 \pm ·0265$
V . . .	306·703	8·204	$r_{VC} = ·7876 \pm 0176$

Ancient Egyptians, ♀, 96 Skulls.

U . . .	495·104	14·116	$r_{UC} = ·8262 \pm ·0218$
C . . .	1251·98	102·063	$r_{UV} = ·6246 \pm ·0420$
V . . .	296·073	8·414	$r_{VC} = ·6731 \pm ·0377$

From these data I have deduced the following equations for reconstruction :—

For males :

$$C = 7·060\ U - 2220·98 \qquad \text{p.e.} = \frac{47·72}{\sqrt{n}} \quad . \quad . \quad . \quad . \quad (20)^a.$$

$$C = 11·676\ V - 2189·61 \qquad \text{p.e.} = \frac{50·54}{\sqrt{n}} \quad . \quad . \quad . \quad . \quad (21)^a.$$

$$C = 4·505\ U + 6·559\ V - 2925·31 \qquad \text{p.e.} = \frac{39·28}{\sqrt{n}} \quad . \quad . \quad . \quad . \quad (22)^a.$$

For females :

$$C = 5·974\ U - 1705·73 \qquad \text{p.e.} = \frac{38·78}{\sqrt{n}} \quad . \quad . \quad . \quad . \quad (20)^b.$$

$$C = 8·165\ V - 1165·66 \qquad \text{p.e.} = \frac{50·91}{\sqrt{n}} \quad . \quad . \quad . \quad . \quad (21)^b.$$

$$C = 4·811\ U + 3·124\ V - 2054·94 \qquad \text{p.e.} = \frac{36·23}{\sqrt{n}} \quad . \quad . \quad . \quad . \quad (22)^b.$$

Now, although the Naqada and Theban skulls have in some cases very close mean values—and it is impossible not to consider the races very closely related—yet the reconstruction equations for C from U and V differ very widely. It is true that the Theban skull capacity calculated from the Naqada formula or the Naqada capacity calculated from the Theban formula do not give very bad results :

	Actual.	From Theban formula (22).
Naqada ♂ . .	1379	1365
Naqada ♀ . .	1281	1242
	Actual.	From Naqada.
Theban ♂ . .	1391	1394
Theban ♀ . .	1251	1285

But this agreement does not arise from any real accordance in the formulæ, but from the fact of the close equality of the Naqada and Theban mean values for U, V and C.

To test the applicability of these circumferential formulæ when extended from one race to a second, I take the following data :—

Organ.	Race.		
	♂ Aino.	♀ Aino.	♂ French.*
U	522·5	501·7	527·6
V	328·5	317·1	317·9
C	1462	1308	1475

These lead to the following results for capacity :—

Race.	Actual.	From Naqada formula (19).	From Theban formula (22).
Aino ♂ . . .	1462	1493	1583
Aino ♀ . . .	1308	1379	1350
French ♂ . .	1475	1482	1537

We see :

(i.) That the Naqada and Theban formulæ, although deduced from kindred races and from very considerable numbers, lead to widely divergent results.

(ii.) That the Naqada, which is for Aino ♂ and French ♂ better than the Theban formula, gives results worse than the formulæ based upon L × B × H previously discussed.

We conclude, therefore, that it appears unlikely that a reconstruction formula, based on the circumferential measurements of the skull, can be found which will give good results, if extended from one local race to another.

If we apply these formulæ to reconstruct the capacity within the race, (20) and (21) give differences much of the order of the earlier reconstruction formulæ (1) to (8), while (22) gives results as good as (9).

The following table gives the errors made in estimating the capacity of forty Theban skulls, twenty of either sex, chosen at random. It will be seen that the errors can be fairly large when we use circumferential measurements.

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* From the skulls of 50 French prisoners who died at Munich during the Franco-German war. Data given in the German Anthropological Catalogue.

Actual Values compared with Values Predicted from Circumferential Measurements in the case of 40 Theban Skulls.

No.	Male skulls.				Female skulls.			
	Actual value.	Error by $(20)^a$.	Error by $(21)^a$.	Error by $(22)^a$.	Actual value.	Error by $(20)^b$.	Error by $(21)^b$.	Error by $(22)^b$.
1	1480	− 2	− 50	− 10	1280	+ 1	+ 37	+ 20
2	1383	+ 32	− 23	+ 5	1337	− 74	− 69	− 70
3	1563	− 42	− 203	− 107	1356	− 15	− 48	− 11
4	1380	+ 70	− 7	+ 39	1253	+ 22	− 10	+ 14
5	1543	− 79	− 8	− 25	1413	− 90	− 23	− 51
6	1390	− 32	+ 28	− 4	1420	− 109	− 103	− 96
7	1310	+ 21	+ 74	+ 38	1227	+ 24	+ 65	+ 39
8	1355	− 61	− 18	+ 7	1120	+ 23	+ 66	+ 19
9	1353	− 44	− 60	− 98	1220	+ 121	+ 80	+ 122
10	1407	+ 29	− 35	− 2	1270	− 54	+ 14	− 35
11	1250	+ 143	− 77	+ 20	1333	− 142	− 8	− 102
12	1550	− 86	− 143	− 104	1330	− 133	− 62	− 116
13	1430	− 12	0	− 6	1260	+ 93	− 17	+ 70
14	1435	+ 22	− 17	+ 14	1390	+ 22	− 147	− 12
15	1493	− 50	− 40	− 34	1165	− 3	+ 94	+ 17
16	1250	+ 31	+ 75	+ 33	1195	− 3	+ 24	− 5
17	1290	+ 69	+ 23	+ 36	1093	− 31	+ 60	+ 24
18	1443	− 56	− 48	− 53	1347	− 6	− 22	+ 4
19	1170	+ 19	− 67	− 70	1245	+ 24	+ 31	+ 30
20	1525	+ 10	− 130	− 40	1250	− 76	+ 83	− 29
Mean error	—	45·5	56·3	37·25	—	53·3	53·15	44·3

Finally for our third problem—that of reconstructing the capacity of the living head—there appears no obvious method of allowing for the difference between the circumferential measurements with and without the living tissues. Of course such measurements as those now being made at Strasburg in the Anatomical Institute may surmount this difficulty and enable us to predict capacity from measurements on the living head.

It would thus seem that, as far as the present investigations go, circumferential measurements do not present great advantages over those discussed in the body of this paper, although the correlations between capacity and these measurements appear, as far as yet has been investigated, to have high values.*

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* This is directly opposed to the view of Dr. FRANZ BOAS ('American Anthropologist,' N.S., vol. I., p. 461). He holds that: "It would seem that circumferences are the most available means of judging cranial size." He does not appear, however, to have correlated the circumferential measurements with capacity, and seen how widely the resulting equations differ from race to race.

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VI. *The Theory of the Double Gamma Function.*

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§ 1. INTRODUCTION.

THE present paper continues my researches in the theory of gamma functions. Previously to a certain extent I obtained known results by new methods: none of the succeeding investigations however have, I believe, been undertaken or suggested by other mathematicians.

In the first paper* published in the connection I attempted to give a homogeneous theory of the ordinary gamma function, considered from the point of view of WEIERSTRASS' function theory. I introduced a parameter ω, and showed that the theory was subordinate to that of a function satisfying the difference equation

$$f(z+\omega) - f(z) = z^s,$$

s being any complex quantity.

That theory led naturally to the consideration† of the G function, satisfying the difference equation

$$G(z+1) = \Gamma(z)G(z),$$

and substantially a function all of whose properties could be obtained by differentiating the simple gamma function with respect to the parameter.

I next considered‡ an extended function $G(z/\tau)$ satisfying the two functional relations

$$f(z+1) = \Gamma\left(\frac{z}{\tau}\right)f(z); \qquad f(z+\tau) = \Gamma(z)\,(2\pi)^{\frac{\tau-1}{2}}\,\tau^{-z+\frac{1}{2}}f(z),$$

and reducible to the G function when $\tau = 1$. Several points in that paper suggested the formation of a symmetrical double gamma function, in which τ should be replaced by the quotient of two parameters ω_1 and ω_2. In the present investigation such a function is defined, and its theory developed in, I hope, complete detail. The function is the natural extension to two parameters of the simple gamma function $\Gamma_1(z \mid \omega)$.

It is necessary for a complete exposition of the theory to consider the properties of what I propose to call double Bernoullian numbers and functions: functions which are the natural extension to two parameters of the simple Bernoullian functions, considered in Part II. of the earliest paper of the series. Such a theory is developed in Part I. of the present paper.

In Part II. I consider the elementary theory of the double gamma function. It is shown that certain symmetrical modular constants arise as finite terms of asymptotic expansions in a manner exactly analogous to the origin of EULER'S constant γ.

Such considerations lead naturally to Part III., in which are deduced from a contour integral, which is a double generalisation of RIEMANN'S ζ function, certain noteworthy asymptotic approximations, of which the most important is an extension

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* BARNES, "The Theory of the Gamma Function," 'Messenger of Mathematics,' vol. 29, pp. 64–128.

† BARNES, "The Theory of the G Function," 'Quarterly Journal of Mathematics,' vol. 31, pp. 264–314.

‡ BARNES, "Genesis of the Double Gamma Function," 'Proceedings of the London Mathematical Society,' vol. 31, pp. 358–381.

of STIRLING's theorem. By the aid of this theory it is possible to express the logarithm of the double gamma function and the double gamma modular constants as contour integrals similar to those given in Part III. of the 'Theory of the Gamma Function.'

In Part IV. I consider the multiplication and transformation theories of double gamma functions as well as certain curious integral formulæ, which correspond to RAABE's theorem for the simple gamma function, and are elementary cases of a general theorem connecting successive similar transcendents of higher orders.

In Part V. the asymptotic expansion of the double gamma function is obtained, and it is shown that the function cannot arise as the solution of a differential equation whose coefficients are more simple transcendents.

There exist similar functions of any number of parameters, and these transcendents I propose to call multiple gamma functions. I reserve the formal expression of their properties for publication elsewhere. I have worked out the theory for double gamma functions independently inasmuch as, the complex variable being two dimensional, there are many points in which a higher analogy breaks down: and also, since many proofs in the higher theory are, in their simplest form, inductive and, to be rigorous, require a knowledge of the theorem for the two simplest cases. Not only so, but in the case of the double gamma functions it is possible to give easily an algebraical theory (such as that worked out in Part II.), which is more simple than if one derived all the formulæ from the fundamental consideration of certain contour integrals.

I append a statement of the notation adopted in this paper, mentioning the place in the present series of investigations where such notation is used for the first time.

Derivation.	Name.	Symbol.	First occurrence.
Algebraic solution of $f(a + \omega) - f(a) = a^n$	Simple Bernoullian function	$S_n(a \mid \omega)$	"Gamma Function," § 11.
$\frac{S'_n(0 \mid \omega)}{n}$	Simple Bernoullian number	${}_1B_n(\omega)$	"Gamma Function," § 15.
Algebraic solution of $f(a + \omega_1) - f(a)$ $= S_n(a \mid \omega_2) + \frac{S'_{n+1}(0 \mid \omega_2)}{n+1}$	Double Bernoullian function	${}_2S_n(a \mid \omega_1, \omega_2)$	For the case of equal parameters: "G Function," § 15. In general: "Double Gamma Function," § 2.
$\frac{{}_2S'_n(0 \mid \omega_1, \omega_2)}{n}$	Double Bernoullian number	${}_2B_{n+1}(\omega_1, \omega_2)$	"Double Gamma Function," § 7.
$\omega^{\frac{z}{\omega} - 1} \Gamma\left(\frac{z}{\omega}\right)$	Simple gamma function	$\Gamma_1(z \mid \omega)$	"Gamma Function," § 2.

Derivation.	Name.	Symbol.	First occurrence.
$\frac{d^r}{dz^r}\log\Gamma_1(z\,\vert\,\omega)$	Logarithm derivative of simple gamma function	$\psi_1^{(r)}(z\,\vert\,\omega)$	"Gamma Function," § 2.
Solution of $f(z+1) = \Gamma(z)f(z)$	G function	$G(z)$	"G Function," § 3.
Solution of $f(z+1) = \Gamma\left(\frac{z}{\tau}\right)f(z)$	Unsymmetrical double gamma function	$G(z\,\vert\,\tau)$	"Genesis," &c., § 1.
Vide §§ 18–24	Double gamma function	$\Gamma_2(z\,\vert\,\omega_1, \omega_2)$	"Double Gamma Function," § 19.
$\frac{d^r}{dz^r}\log\Gamma_2(z\,\vert\,\omega_1, \omega_2)$	Logarithm derivatives of double gamma function	$\psi_2^{(r)}, (z\,\vert\,\omega_1, \omega_2)$	"Double Gamma Function," § 19.
$\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-az}}{1-e^{-\omega z}}(-z)^{s-1}dz$	Simple Riemann ζ (zeta) function	$\zeta(s,a,\omega)$	"Gamma Function," § 23.
$\frac{\iota\Gamma(1-s)}{2\pi}e^{2\mathrm{M}s\pi i}\int_{\mathrm{L}}\frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}$	Double Riemann ζ function	$\zeta_2(s, a\,\vert\,\omega_1, \omega_2)$	For equal parameters: "G Function," § 23. In general: "Double Gamma Function," § 39.
$-\frac{\gamma}{\omega}+\frac{\log\omega}{\omega}$	Simple gamma modular form	$\gamma_{11}(\omega)$	"Gamma Function," § 2.
Finite terms of certain asymptotic limits	Unsymmetrical double gamma modular forms	$C(\tau)$ $D(\tau)$	"Genesis," &c., §§ 3 and 4.
Do. do.	Symmetrical double gamma modular forms	$\gamma_{21}(\omega_1, \omega_2)$ $\gamma_{22}(\omega_1, \omega_2)$	"Double Gamma Function," §§ 21 and 23.
Do. do.	Glaisher-Kinkelin constant	A	"G Function," § 3.
$\sqrt{\frac{2\pi}{\omega}}$	Simple Stirling modular form	$\rho_1(\omega)$	"Gamma Function," § 31.
Limit of a certain definite integral	Double Stirling modular form	$\rho_2(\omega_1, \omega_2)$	"Double Gamma Function," § 43.
Constants which take the values $0_1 \pm 1$, according to the distribution of ω_1 and ω_2	. . . .	$\left.\begin{matrix} m \\ m' \end{matrix}\right\}$ M	"Double Gamma Function," § 21. "Double Gamma Function," § 39.

The symbolic notation by which $F_2[f(z+\omega)]$ is written for

$$f(z+\omega_1+\omega_2)-f(z+\omega_1)-f(z+\omega_2)$$

is introduced in § 49.

PART I.

The Theory of Double Bernoullian Functions and Numbers.

§ 2. In the "Theory of the Gamma Function," Part II., we have defined the simple Bernoullian function $S_n(a\,|\,\omega)$ as that solution of the difference equation

$$f(a+\omega) - f(a) = a^n,$$

where n is a positive integer, which is such that it is an algebraical polynomial and $S_n(o\,|\,\omega) = 0$. And it was proved that such a solution does exist.

In exactly the same manner it may be proved that the difference equation

$$f(a+\omega_1) - f(a) = S_n(a\,|\,\omega_2) + \frac{S'_{n+1}(o\,|\,\omega_2)}{n+1}$$

has an algebraic solution, which is a rational integral polynomial of degree $n+2$.

The difference between any two solutions will be a simply periodic function of period ω, and will therefore be a constant if the solutions are both algebraic polynomials.

There thus exists a unique algebraical polynomial of degree $n+2$, which is a solution of the difference equation.

$$f(a+\omega_1) - f(a) = S_n(a\,|\,\omega_2) + \frac{S'_{n+1}(o\,|\,\omega_2)}{n+1},$$

with the condition $f(o) = 0$.

This solution we call the double Bernoullian function of a with parameters ω_1 and ω_2, and we denote it by ${}_2S_n(a\,|\,\omega_1,\omega_2)$. By symmetry with this notation the simple Bernoullian function would be denoted by ${}_1S_n(a\,|\,\omega)$.

We shall often omit the parameters ω_1 and ω_2, when there is no doubt as to their existence, and write the function simply ${}_2S_n(a)$.

§ 3. We now proceed to show that the double Bernoullian function of a of order n is also the unique algebraical polynomial which is the solution of the difference equation

$$f(a+\omega_2) - f(a) = S_n(a\,|\,\omega_1) + \frac{S'_{n+1}(o\,|\,\omega_2)}{n+1},$$

with the condition $f(o) = 0$.

For since

$${}_2S_n(a+\omega_1) - {}_2S_n(a) = S_n(a\,|\,\omega_2) + \frac{S'_{n+1}(o\,|\,\omega_2)}{n+1} \quad . \quad . \quad . \quad . \quad . \quad . \quad (1.)$$

we have at once

$$\begin{aligned} &{}_2S_n(a+\omega_1+\omega_2) - {}_2S_n(a+\omega_2) - {}_2S_n(a+\omega_1) + {}_2S_n(a) \\ &= S_n(a+\omega_2\,|\,\omega_2) - S_n(a\,|\,\omega_2) = a^n = S_n(a+\omega_1\,|\,\omega_1) - S_n(a\,|\,\omega_1). \end{aligned}$$

And therefore if we put

$$f_n(a) = {}_2S_n(a + \omega_2) - {}_2S_n(a) - S_n(a \mid \omega_1)$$

we shall have

$$f_n(a + \omega_1) - f_n(a) = 0.$$

Now ${}_2S_n(a)$ and $S_n(a/\omega_1)$ are algebraical polynomials of a, and therefore $f_n(a)$ is also such a function. And, therefore, since it is simply periodic of period ω_1, it must be a constant. Thus we have

$$_2S_n(a + \omega_2) - {}_2S_n(a) = S_n(a \mid \omega_1) + \text{constant} \quad . \quad . \quad . \quad . \quad . \quad (2.)$$

Again, integrating the relation (1) with respect to a between o and ω_2, we have

$$\int_0^{\omega_1 + \omega_2} {}_2S_n(a)\,da - \int_0^{\omega_1} {}_2S_n(a)\,da - \int_0^{\omega_2} {}_2S_n(a)\,da = 0,$$

since

$$\int_0^{\omega_2} S_n(a \mid \omega_2)\,da = -\omega_2 \frac{S'_{n+1}(o \mid \omega_2)}{n+1}.$$

Integrating the relation (2) in the same manner between o and ω_1, we obtain for the value of the constant

$$-\frac{1}{\omega_1}\int_0^{\omega_1} S_n(a \mid \omega_1)\,da = \frac{S'_{n+1}(o \mid \omega_1)}{n+1}.$$

And thus ${}_2S_n(a)$ is the unique algebraic solution of the equation

$$f(a + \omega_2) - f(a) = S_n(a \mid \omega_1) + \frac{S'_{n+1}(o \mid \omega_1)}{n+1},$$

with the condition $f(o) = 0$.

From the symmetrical nature of the equations which give ${}_2S_n(a \mid \omega_1, \omega_2)$, we see that this function itself must be symmetrical in ω_1 and ω_2.

§ 4. If now we assume

$$_2S_n(a \mid \omega_1, \omega_2) = \alpha_{n+2}\, a^{n+2} + \alpha_{n+1}\, a^{n+1} + \ldots + \alpha_1 a,$$

the calculation of the highest coefficients may be readily effected.

For we have

$$_2S_n(a + \omega_1) - {}_2S_n(a) = S_n(a \mid \omega_2) + \frac{S'_{n+1}(o \mid \omega_2)}{n+1}$$

$$= \frac{a^{n+1}}{(n+1)\,\omega_2} - \frac{a^n}{2} + \binom{n}{1}\frac{B_1}{2}\, a^{n-1}\omega_2 - \binom{n}{3}\frac{B_2}{3}\, a^{n-3}\omega_2^{\,3} + \ldots$$

Hence if we substitute the assumed expansion for ${}_2S_n(a)$ and equate corresponding powers of a, we find

$$(n + 2)\,\omega_1\,\alpha_{n+2} = \frac{1}{(n+1)\,\omega_2},$$

$$\frac{(n+2)(n+1)}{1 \,.\, 2}\,\omega_1^{\,2}\,\alpha_{n+2} + (n+1)\,\omega_1\,\alpha_{n+1} = -\tfrac{1}{2},$$

$$\frac{(n+2)(n+1)n}{3\,!}\,\omega_1^{\,3}\,\alpha_{n+2} + \frac{(n+1)n}{2\,!}\,\omega_1^{\,2}\,\alpha_{n+1} + n\omega_1\,\alpha_n = \binom{n}{1}\frac{B_1}{2}\,\omega_2,$$

and so on.

On solving these equations successively we readily obtain

$$\alpha_{n+2} = \frac{1}{(n+1)(n+2)\,\omega_1\omega_2},$$

$$\alpha_{n+1} = -\frac{\omega_1+\omega_2}{2(n+1)\,\omega_1\omega_2},$$

and, since $B_1 = \frac{1}{6}$,

$$\alpha_n = \frac{1}{12}\,\frac{\omega_1^{\,2}+\omega_2^{\,2}+3\omega_1\omega_2}{\omega_1\omega_2}.$$

Thus

$${}_2S_n(a\,|\,\omega_1, \omega_2) = \frac{a^{n+2}}{(n+1)(n+2)\,\omega_1\omega_2} - \frac{a^{n+1}(\omega_1+\omega_2)}{2(n+1)\omega_1\omega_2} + a^n\,\frac{\omega_1^{\,2}+\omega_2^{\,2}+3\omega_1\omega_2}{12\omega_1\omega_2} + \ldots$$

Further terms can be calculated if necessary. It will be seen, however, that they form what we propose to call double Bernoullian numbers, whose properties may be investigated without the necessity of their formal evaluation.

Corollary. We note that

$${}_2S_1(a\,|\,\omega_1, \omega_2) = \frac{a^3}{6\omega_1\omega_2} - \frac{a^2(\omega_1+\omega_2)}{4\omega_1\omega_2} + a\,\frac{\omega_1^{\,2}+\omega_2^{\,2}+3\omega_1\omega_2}{12\omega_1\omega_2}.$$

And hence

$${}_2S'_1(a\,|\,\omega_1, \omega_2) = \frac{a^2}{2\omega_1\omega_2} - \frac{a(\omega_1+\omega_2)}{2\omega_1\omega_2} + \frac{\omega_1^{\,2}+\omega_2^{\,2}+3\omega_1\omega_2}{12\omega_1\omega_2},$$

$${}_2S_1^{(2)}(a\,|\,\omega_1, \omega_2) = \frac{a}{\omega_1\omega_2} - \frac{\omega_1+\omega_2}{2\omega_1\omega_2},$$

$${}_2S_1^{(3)}(a\,|\,\omega_1, \omega_2) = \frac{1}{\omega_1\omega_2}.$$

It will be found that these expressions are of constant occurrence in the course of the present investigation.

Note also that

$${}_2S_0(a\,|\,\omega_1, \omega_2) = \frac{a^2}{2\omega_1\omega_2} - \frac{a(\omega_1+\omega_2)}{2\omega_1\omega_2}.$$

§ 5. We will now prove that, if $n-k \geqslant 0$, $k > 0$,

$${}_2S_n^{(k)}(a\,|\,\omega_1, \omega_2) = \frac{n\,!}{(n-k)\,!}\,{}_2S_{n-k}(a\,|\,\omega_1, \omega_2) + {}_2S_n^{(k)}(o\,|\,\omega_1, \omega_2).$$

We have, when $n-k \geqslant 0$ and $k < 0$,

$$\begin{aligned}{}_2S_n^{(k)}(a+\omega_1) - {}_2S_n^{(k)}(a) &= S_n^{(k)}(a\,|\,\omega_2) \\ &= S_n^{(k)}(o\,|\,\omega_2) + \frac{n\,!}{(n-k)\,!}S_{n-k}(a\,|\,\omega_2)\end{aligned}$$

("Theory of the Gamma Function," § 14).

and therefore

$$\begin{aligned}&{}_2S_n^{(k)}(a+\omega_1) - {}_2S_n^{(k)}(a)\\ &= S_n^{(k)}(o\,|\,\omega_2) + \frac{n\,!}{(n-k)\,!}\left[{}_2S_{n-k}(a+\omega_1) - {}_2S_{n-k}(a) - \frac{S_{n-k+1}(o\,|\,\omega_2)}{n-k+1}\right].\end{aligned}$$

Thus if we write

$$f(a) = {}_2S_n^{(k)}(a) - \frac{n\,!}{n-k\,!}\,{}_2S_{n-k}(a)$$

we shall have

$$\begin{aligned}f(a+\omega_1) - f(a) &= S_n^{(k)}(o\,|\,\omega_2) - \frac{n\,!}{(n-k+1)\,!}\,S'_{n-k+1}(o\,|\,\omega_2)\\ &= 0 \text{ (“Gamma Function,” § 15).}\end{aligned}$$

Similarly

$$f(a+\omega_2) - f(a) = 0.$$

and therefore since $f(a)$ is an algebraical polynomial in a, it is a constant.

On making $a = 0$ and remembering that ${}_2S_n(o) = 0$, we obtain

$${}_2S_n^{(k)}(a) - \frac{n\,!}{(n-k)\,!}\,{}_2S_{n-k}(a) = {}_2S_n^{(k)}(o),$$

which is the required result.

§ 6. We are now in a position to prove that

$$\int_0^{\omega_1} {}_2S_n(a)\,da = -\frac{\omega_1}{n+1}\,{}_2S'_{n+1}(o\,|\,\omega_1,\omega_2) + \frac{S'_{n+2}(o\,|\,\omega_2)}{(n+1)\,.\,(n+2)}$$

$$\int_0^{\omega_2} {}_2S_n(a)\,da = -\frac{\omega_2}{n+1}\,{}_2S'_{n+1}(o\,|\,\omega_1,\omega_2) + \frac{S'_{n+2}(o\,|\,\omega_1)}{(n+1)\,.\,(n+2)}$$

and at the same time the important relation

$${}_2S_{n+k}^{(k)}(o\,|\,\omega_1,\omega_2) = \frac{(n+k)\,!}{(n+1)}\,{}_2S'_{n+1}(o\,|\,\omega_1,\omega_2).$$

Since ${}_2S_n(o) = 0$ we see from the fundamental difference equation that

$${}_2S_n(\omega_1) = \frac{S'_{n+1}(o\,|\,\omega_2)}{n+1}.$$

Hence, if we put $a = \omega$, in § 5, we see that when $n-k \geqq 0$, and $k > 0$,

$$\begin{aligned}{}_2S_n^{(k)}(\omega_1) - {}_2S_n^{(k)}(o) &= \frac{n\,!}{(n-k)\,!}\,{}_2S_{n-k}(\omega_1)\\ &= \frac{n\,!}{(n+1-k)\,!}\,S'_{n-k+1}(o\,|\,\omega_2).\end{aligned}$$

Take now the relation

$${}_2S_n^{(k)}(a) = \frac{n\,!}{(n-k)\,!}\,{}_2S_{n-k}(a) + {}_2S_n^{(k)}(o),$$

and integrate with respect to a between 0 and ω_1, we obtain

$${}_2S_n^{(k-1)}(\omega_1) - {}_2S_n^{(k-1)}(o) = \frac{n\,!}{(n-k)\,!}\int_0^{\omega_1} {}_2S_{n-k}(a)\,da + \omega_1\,{}_2S_n^{(k)}(o),$$

so that

$$\frac{n\,!}{(n-k)\,!}\int_0^{\omega_1} {}_2S_{n-k}(a)\,da = -\,\omega_1\,{}_2S_n^{(k)}(o) + \frac{n\,!}{(n+2-k)\,!}\,S'_{n+2-k}(o\,|\,\omega_2).$$

Write now n for $(n-k)$, as is evidently allowable, since both n and k are positive integers, and we have

$$\int_0^{\omega_1} {}_2S_n(a)\,da = -\,\omega_1\frac{n\,!}{(n+k)\,!}\,{}_2S^{(k)}_{n+k}(o) + \frac{S'_{n+2}(o\,|\,\omega_2)}{(n+1)\,.\,(n+2)}.$$

We thus see that

$$\frac{n\,!}{(n+k)\,!}\,{}_2S^{(k)}_{n+k}(o\,|\,\omega_1,\,\omega_2)$$

is independent of k, since this is the only time in the relation just obtained which depends on k.

Putting then $k = 1$, we have when $k > 0$ and $n \overset{=}{>} k$.

$${}_2S^{(k)}_{n+k}(o\,|\,\omega_1,\,\omega_2) = \frac{(n+k)\,!}{(n+1)\,!}\,{}_2S'_{n+1}(o\,|\,\omega_1,\,\omega_2),$$

which is one of the relations required.

And also

$$\int_0^{\omega_1} {}_2S_n(a)\,da = -\,\frac{\omega}{n+1}\,{}_2S'_{n+1}(o) + \frac{S'_{n+2}(o\,|\,\omega_2)}{(n+1)\,.\,(n+2)},$$

another of the given relations. The second integral formula of course may be written down by symmetry.

We notice that in the notation formerly introduced ("Gamma Function," § 15) we have

$$\frac{S'_{n+2}(o\,|\,\omega_2)}{(n+1)\,.\,(n+2)} = \frac{{}_1B_{n+2}(\omega_2)}{n+2},$$

and therefore that each of these expressions

$$= 0 \qquad \text{when } n \text{ is odd,}$$

$$= \frac{(-)^{\frac{n}{2}}B_{\frac{n}{2}+1}\,\omega_2^{\,n+1}}{(n+1)\,.\,(n+2)} \qquad \text{when } n \text{ is even.}$$

Thus we see that when n is odd

$$\int_0^{\omega_1} {}_2S_n(a)\,da = -\,\frac{\omega_1}{n+1}\,{}_2S'_{n+1}(o).$$

§ 7. We now introduce double Bernoullian numbers analogous to the simple Bernoullian numbers introduced in the theory of the gamma function.

In that theory the simple nth Bernoullian number was defined by the relation

$${}_1B_n(\omega) = \frac{S'_n(o \mid \omega)}{n},$$

and now the nth double Bernoullian number is given by

$${}_2B_n(\omega_1, \omega_2) = \frac{{}_2S'_n(o \mid \omega_1, \omega_2)}{n}.$$

We note that by the theorem of § 6 we may put

$${}_2B_{n+1}(\omega_1, \omega_2) = \frac{n\,!}{(n+k)\,!}\,{}_2S^{(k)}_{n+k}(o \mid \omega_1, \omega_2),$$

and therefore

$${}_2S_n^{(k)}(o) = \frac{n\,!}{(n-k)}\,{}_2B_{n-k+1}(\omega_1, \omega_2).$$

§ 8. At this point we may conveniently note the reduction which takes place in the double Bernoullian functions and numbers when the parameters are equal to one another.

If we put $\omega_1 = \omega_2 = \omega$ we have as the single difference equation of the nth double Bernoullian function the relation

$$f(a+\omega) - f(a) = S_n(a \mid \omega) + {}_1B_{n+1}(\omega),$$

and the function is now defined as the algebraical solution of this equation with the condition ${}_2S_n(o \mid \omega, \omega) = 0$.

Put now

$$f(a) = -\,S_{n+1}(a \mid \omega) + (a-\omega)\,S_n(a \mid \omega) + a\,\frac{S'_{n+1}(o \mid \omega)}{n+1},$$

and we have

$$f(a+\omega) - f(a) = \omega S_n(a \mid \omega) + \frac{\omega S'_{n+1}(o \mid \omega)}{n+1}.$$

Hence, the other definition conditions being satisfied, we see that

$${}_2S_n(a \mid \omega, \omega) = \frac{a-\omega}{\omega}\,S_n(a \mid \omega) - \frac{1}{\omega}\,S_{n+1}(a \mid \omega) + \frac{a}{\omega}\,\frac{S'_{n+1}(o \mid \omega)}{n+1},$$

that is, the double Bernoullian function when the parameters are equal reduces to simple Bernoullian functions and numbers.

It will be seen later that it is for this reason that it was possible to obtain all the expansions in the theory of the G function in terms of simple Bernoullian functions.

Note that the above relation may be written

$${}_2S_n(a \mid \omega, \omega) = \frac{a-\omega}{\omega}\,S_n(a \mid \omega) - \frac{1}{\omega}\,S_{n+1}(a \mid \omega) + \frac{a}{\omega}\,{}_1B_{n+1}(\omega).$$

On differentiation we have

$$ {}_2S'_n(a\,|\,\omega,\omega) = S_n(a\,|\,\omega) + \frac{a-\omega}{\omega}S'_n(a\,|\,\omega) - \frac{1}{\omega}S'_{n+1}(a\,|\,\omega) + \frac{1}{\omega}\,{}_1B_{n+1}(\omega), $$

so that
$$ {}_2S'_n(o\,|\,\omega,\omega) = -S'_n(o\,|\,\omega) - \frac{n}{\omega}\,{}_1B_{n+1}(\omega), $$

and therefore
$$ {}_2B_n(\omega,\omega) = -\,{}_1B_n(\omega) - \frac{1}{\omega}\,{}_1B_{n+1}(\omega). $$

§ 9. We now see at once that

$$ {}_2S_n(a\,|\,\omega_1,\omega_2) = \frac{a^{n+2}}{(n+1)(n+2)\,\omega_1\,\omega_2} - \frac{\omega_1+\omega_2}{2\,(n+1)\,\omega_1\,\omega_2}\,a^{n+1} + {}_2B_1(\omega_1,\omega_2)\,a^n $$
$$ + \binom{n}{1}\,{}_2B_2(\omega_1,\omega_2)\,a^{n-1} + \binom{n}{2}\,{}_2B_3(\omega_1,\omega_2)\,a^{n-2} + \ldots $$

and so complete the expansion of § 4.

For by Maclaurin's theorem we have

$$ {}_2S_n(a\,|\,\omega_1,\omega_2) = a\,{}_2S'_n(o) + \frac{{}_2S_n^{(2)}(o)}{2\,!}\,a^2 + \ldots \qquad + \frac{{}_2S_n^{(n+2)}(o)}{(n+2)\,!}\,a^{n+2}, $$

since the higher differentials vanish.

From the few terms found in § 4 we see that

$$ {}_2S_n^{(n+2)}(o) = \frac{n\,!}{\omega_1\omega_2}, $$
$$ {}_2S_n^{(n+1)}(o) = -\,\frac{n\,!\,(\omega_1+\omega_2)}{2\omega_1\omega_2}. $$

Now when $n \geqq k$ and $k > 0$, we have

$$ {}_2S_n^{(k)}(o) = \frac{n\,!}{(n-k)\,!}\;{}_2B_{n-k+1}(\omega_1,\omega_2), $$

and thus we have the expansion in question.

§ 10. It may now be shown that

$$ {}_2S_n(a) = (-)^n\,{}_2S_n(\omega_1+\omega_2-a) + \frac{(-)^{n-1}}{n+1}\left[S'_{n+1}(o\,|\,\omega_1) + S'_{n+1}(o\,|\,\omega_2)\right], $$

or, as we may write it,

$$ {}_2S_n(a) = (-)^n\,{}_2S_n(\omega_1+\omega_2-a) + (-)^{n-1}\left[{}_1B_{n+1}(\omega_1) + {}_1B_{n+1}(\omega_2)\right]. $$

Remembering the value of, ${}_1B_{n+1}(\omega)$ we thus prove that

${}_2S_n(a) = {}_2S_n(\omega_1+\omega_2-a)$, when n is even,

${}_2S_n(a) = -\,{}_2S_n(\omega_1+\omega_2-a) + \dfrac{(-)^{\frac{n-1}{2}}}{n+1}\,B_{\frac{n+1}{2}}(\omega_1^{\,n}+\omega_2^{\,n})$, when n is odd.

Substitute $\omega_2 - a$ for a in the fundamental difference equation and we find

$$ {}_2S_n(\omega_1 + \omega_2 - a) - {}_2S_n(\omega_2 - a) = S_n(\omega_2 - a \mid \omega_2) + {}_1B_{n+1}(\omega_2), $$

and therefore, since $S_n(\omega_2 - a \mid \omega_2) = (-)^{n+1} S_n(a \mid \omega_2)$, we have

$$ {}_2S_n(\omega_1 + \omega_2 - a) - {}_2S_n(\omega_2 - a) = (-)^{n-1} S_n(a \mid \omega_2) + {}_1B_{n+1}(\omega_1). $$

If therefore we put

$$ f(a) = (-)^n \left[{}_2S_n(\omega_1 + \omega_2 - a) + \frac{a}{\omega_1}\, {}_1B_{n+1}(\omega_2) \right] + \frac{a}{\omega_1}\, {}_1B_{n+1}(\omega_2), $$

we see that $f(a)$ is an algebraic solution of the difference equation

$$ f(a + \omega_1) - f(a) = S_n(a \mid \omega_2) + {}_1B_{n+1}(\omega_2), $$

and therefore can only differ by a constant from ${}_2S_n(a)$.

Determine this constant by making $a = 0$ and we have the relation

$$ {}_2S_n(a) = (-)^n \left[{}_2S_n(\omega_1 + \omega_2 - a) - {}_2S_n(\omega_1 + \omega_2) \right] + \frac{(-)^n a}{\omega_1}\, {}_1B_{n+1}(\omega_2) $$
$$ + \frac{a}{\omega_1}\, {}_1B_{n+1}(\omega_2). $$

When n is odd the last two terms cancel each other, and when n is even ${}_1B_{n+1}(\omega_2)$ vanishes.

Hence $$ {}_2S_n(a \mid \omega_1, \omega_2) = (-)^n \left[{}_2S_n(\omega_1 + \omega_2 - a) - {}_2S_n(\omega_1 + \omega_2) \right]. $$

From the fundamental difference equations we see at once that

$$ {}_2S_n(\omega_1 + \omega_2) = \frac{S'_{n+1}(o \mid \omega_1)}{n+1} + \frac{S_{n+1}(o \mid \omega_2)}{n+1} $$
$$ = {}_1B_{n+1}(\omega_1) + {}_1B_{n+1}(\omega_2), $$

and therefore we have the relation stated.

§ 11. We may now show that, when n is even,

$$ {}_2B_n(\omega_1, \omega_2) = (-)^{\frac{n}{2}} \frac{B_{n/2}}{2n} (\omega_1^{\,n-1} + \omega_2^{\,n-1}), $$

a simple expression for the even double Bernoullian numbers which corresponds in some degree to the fact that the even simple Bernoullian numbers vanish.

On differentiating with regard to a the result of the previous paragraph we find

$$ {}_2S'_n(a \mid \omega_1, \omega_2) + (-)^n\, {}_2S'_n(\omega_1 + \omega_2 - a) = 0. $$

From the fundamental difference equations we have

$$ {}_2S_n(a + \omega_1 + \omega_2) - {}_2S_n(a + \omega_1) - {}_2S_n(a + \omega_2) + {}_2S_n(a) = a^n, $$

and hence
$$
\begin{aligned}
{}_2S'_n(\omega_1 + \omega_2) &= {}_2S'_n(\omega_1) + {}_2S'_n(\omega_2) - {}_2S'_n(o) \\
&= {}_2S'_n(o) + S'_n(o \mid \omega_2) + S'_n(o \mid \omega_1).
\end{aligned}
$$

Thus, since ("Gamma Function," § 15)
$$
\begin{aligned}
S'_n(o \mid \omega) &= 0 \quad \text{when } n \text{ is odd} \\
&= (-)^{\frac{n}{2}-1} B_{\frac{n}{2}} \, \omega_1^{n-1} \quad \text{when } n \text{ is even},
\end{aligned}
$$

we see that
$$
{}_2S'_n(\omega_1 + \omega_2) = {}_2S'_n(o), \text{ when } n \text{ is odd};
$$

and
$$
{}_2S'_n(\omega_1 + \omega_2) = {}_2S'_n(o) + (-)^{\frac{n}{2}-1} B_{\frac{n}{2}} (\omega_1^{n-1} + \omega_2^{n-1}), \text{ when } n \text{ is even.}
$$

But our former relation gives us, when n is even,
$$
{}_2S'_n(\omega_1 + \omega_2) = - {}_2S'_n(o).
$$

Hence, when n is even,
$$
{}_2S'_n(o) = (-)^{\frac{n}{2}} \cdot \frac{B_{\frac{n}{2}}}{2} \cdot (\omega_1^{n-1} + \omega_2^{n-1}),
$$

which is equivalent to the relation required.

§ 12. We now proceed to show that
$$
\int_0^a {}_2S_n(a)\, da = \frac{{}_2S_{n+1}(a)}{n+1} - a \, {}_2B_{n+1}(\omega_1, \omega_2).
$$

We have
$$
{}_2S_n(a + \omega_1) - {}_2S_n(a) = S_n(a \mid \omega_2) + \frac{S'_{n+1}(o \mid \omega_2)}{n+1}.
$$

Hence, integrating with respect to a
$$
\int_0^{a+\omega_1} {}_2S_n(a)\, da - \int_0^a {}_2S_n(a)\, da = \int_0^{\omega_1} {}_2S_n(a)\, da + \int_0^a S_n(a \mid \omega_2)\, da + a \frac{S'_{n+1}(o \mid \omega_2)}{n+1}
$$

But ("Gamma Function," § 19)
$$
\int_0^a S_n(a \mid \omega_2)\, da + a \frac{S'_{n+1}(o \mid \omega_2)}{n+1} = \frac{S_{n+1}(a \mid \omega_2)}{n+1}.
$$

so that, if $f(a) = \int_0^a {}_2S_n(a)\, da$, this function is an algebraic solution of the difference equation
$$
f(a + \omega_1) - f(a) = \int_0^{\omega_1} {}_2S_n(a)\, da + \frac{S_{n+1}(a \mid \omega_2)}{n+1}.
$$

But this difference equation is evidently satisfied by
$$
\begin{aligned}
&\frac{a}{\omega_1} \left[\int_0^{\omega_1} {}_2S_n(a)\, da - \frac{S'_{n+2}(o \mid \omega_2)}{(n+1)(n+2)} \right] + \frac{1}{n+1} \, {}_2S_{n+1}(a) \\
&= \frac{a}{\omega_1} \left[- \frac{\omega_1}{n+1} \, {}_2S'_{n+1}(o) \right] + \frac{{}_2S_{n+1}(a)}{n+1}. \qquad \text{by § 6,} \\
&= \frac{{}_2S_{n+1}(a)}{n+1} - a \, {}_2B_{n+1}(\omega_1, \omega_2).
\end{aligned}
$$

Hence as these two solutions both vanish when $a = 0$, we have

$$\int_0^a {}_2S_n(a)\,da = \frac{{}_2S_{n+1}(a)}{n+1} - a\,{}_2B_{n+1}(\omega_1, \omega_2).$$

As a *corollary* we have on differentiation

$${}_2S'_{n+1}(a) = (n+1)\,{}_2S_n(a) + {}_2S'_{n+1}(o).$$

§ 13. The multiplication theory of double Bernoullian functions may be conveniently expressed by the formula

$${}_2S_n(ma \mid \omega_1, \omega_2) = m^n\,{}_2S_n\left(a \;\middle|\; \frac{\omega_1}{m}, \frac{\omega_2}{m}\right).$$

From the fundamental difference relation we have

$${}_2S_n\left\{m\left(a+\frac{\omega_1}{m}\right)\middle|\,\omega_1, \omega_2\right\} = {}_2S_n\,\{ma \mid \omega_1, \omega_2\} + m^n\left[S_n\left(a\,\middle|\,\frac{\omega_2}{m}\right) + \frac{S'_{n+1}\left(\frac{\omega_2}{m}\right)}{n+1}\right].$$

Hence $\dfrac{1}{m^n}\,{}_2S_n(ma \mid \omega_1, \omega_2)$ satisfies the difference equation

$$f\left(a+\frac{\omega_1}{m}\right) - f(a) = S_n\left(a\,\middle|\,\frac{\omega_2}{m}\right) + \frac{S'_{n+1}\left(\frac{\omega_2}{m}\right)}{n+1},$$

and is the only algebraical solution such that $f(o) = 0$.

Hence $$\frac{1}{m^n}\,{}_2S_n(ma \mid \omega_1, \omega_2) = {}_2S_n\left(a\,\middle|\,\frac{\omega_1}{m}, \frac{\omega_2}{m}\right),$$

which is the relation required.

As a *corollary* we see that the nth double Bernoullian number is homogeneous and of degree $(n-1)$ in the ω's.

For in § 9 we have seen that in the expansion of

$${}_2S_n(a \mid \omega_1, \omega_2),$$

the part of the coefficient of a^{n-s+1} which involves the ω's is ${}_2B_s(\omega_1, \omega_2)$.

§ 14. The transformation of the parameters of the double Bernoullian function is given by the relation

$$\begin{aligned}{}_2S_n\left(a\,\middle|\,\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \sum_{k=0}^{p-1}\sum_{l=0}^{q-1}{}_2S_n\left(a+\frac{k\,\omega_1}{p}+\frac{l\omega_2}{q}\,\middle|\,\omega_1, \omega_2\right)& \\ + pq\,{}_2B_{n+1}(\omega_1, \omega_2) - {}_2B_{n+1}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)&,\end{aligned}$$

as we proceed to prove.

Let $$f(a) = \sum_{k=0}^{p-1}\sum_{l=0}^{q-1}{}_2S_n\left(a+\frac{k\omega_1}{p}+\frac{l\omega_2}{q}\,\middle|\,\omega_1, \omega_2\right),$$

then we obtain at once

$$f\left(a+\frac{\omega_1}{p}\right)-f(a)=\sum_{l=0}^{q-1}\left[{}_2\mathrm{S}_n\left(a+\frac{l\omega_2}{q}+\omega_1\right)-{}_2\mathrm{S}_n\left(a+\frac{l\omega_2}{q}\right)\right]$$

$$=\sum_{l=0}^{q-1}\left[\mathrm{S}\left(a+\frac{l\omega_2}{q}\,\middle|\,\omega_2\right)+{}_1\mathrm{B}_{n+1}(\omega_2)\right]$$

$$=\mathrm{S}_n\left(a\,\middle|\,\frac{\omega_2}{q}\right)+{}_1\mathrm{B}_{n+1}\left(\frac{\omega_2}{q}\right)\text{ (" Gamma Function," § 18).}$$

Thus $f(a)$ is an algebraic solution of the difference equation satisfied by ${}_2\mathrm{S}_n\left(a\,\middle|\,\frac{\omega_1}{p},\frac{\omega_2}{q}\right)$. The two solutions can then only differ by a constant, and thus

$${}_2\mathrm{S}_n\left(a\,\middle|\,\frac{\omega_1}{p},\frac{\omega_2}{q}\right)=\sum_{k=0}^{p-1}\sum_{l=0}^{q-1}{}_2\mathrm{S}_n\left(a+\frac{k\omega_1}{p}+\frac{l\omega_2}{q}\,\middle|\,\omega_1,\omega_2\right)+\mathrm{R}_n,$$

where R_n is independent of a.

To determine this constant, let us integrate between 0 and $\frac{\omega_1}{p}$. We find

$$\int_0^{\frac{\omega_1}{p}}{}_2\mathrm{S}_n\left(a\,\middle|\,\frac{\omega_1}{p},\frac{\omega_2}{q}\right)da=\sum_{l=0}^{q-1}\int_0^{\omega_1}{}_2\mathrm{S}_n\left(\zeta+\frac{l\omega_2}{q}\,\middle|\,\omega_1,\omega_2\right)d\zeta+\frac{\omega_1}{p}\mathrm{R}_n,$$

and therefore by § 6

$$-\frac{\omega_1}{p}\,{}_2\mathrm{B}_{n+1}\left(\frac{\omega_1}{p},\frac{\omega_2}{q}\right)+\frac{{}_1\mathrm{B}_{n+2}\left(\frac{\omega_2}{q}\right)}{n+1},$$

$$=-q\omega_1\,{}_2\mathrm{B}_{n+1}(\omega_1,\omega_2)+\sum_{l=0}^{q-1}\frac{{}_2\mathrm{S}_{n+1}\left(\omega_1+\frac{\omega}{p}\right)-{}_2\mathrm{S}_{n+1}\left(\frac{l\omega_2}{q}\right)}{n+1}+\frac{\omega_1}{p}\mathrm{R}_n,\text{ by § 12.}$$

$$=-q\omega_1\,{}_2\mathrm{B}_{n+1}(\omega_1,\omega_2)+\left(\frac{1}{q^{n+1}}-q\right)\frac{{}_1\mathrm{B}_{n+2}(\omega_2)}{n+1}+q\,\frac{{}_1\mathrm{B}_{n+2}(\omega_2)}{n+1}+\frac{\omega_1}{p}\mathrm{R}_n$$

(" Gamma Function," § 18).

And therefore

$$-\frac{\omega_1}{p}\,{}_2\mathrm{B}_{n+1}\left(\frac{\omega_1}{p},\frac{\omega_2}{q}\right)=-q\omega_1\,{}_2\mathrm{B}_{n+1}(\omega_1,\omega_2)+\frac{\omega_1}{p}\mathrm{R}_n,$$

so that

$$\mathrm{R}_n=pq\,{}_2\mathrm{B}_{n+1}(\omega_1,\omega_2)-{}_2\mathrm{B}_{n+1}\left(\frac{\omega_1}{p},\frac{\omega_2}{q}\right).$$

On substituting this value we have the theorem enunciated.

As a *corollary* we have on making $a=0$.

$$\sum_{k=0}^{p-1}\sum_{l=0}^{q-1}{}_2\mathrm{S}_n\left(\frac{k\omega_1}{p}+\frac{l\omega_2}{q}\,\middle|\,\omega_1,\omega_2\right)={}_2\mathrm{B}_{n+1}\left(\frac{\omega_1}{p},\frac{\omega_2}{q}\right)pq\,{}_2\mathrm{B}_{n+1}(\omega_1,\omega_2),$$

§ 15. We now proceed to prove the expansion of fundamental importance in the theory of double Bernoullian functions :—

$$\frac{ze^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}=\frac{{}_2\mathrm{S}_1^{(3)}(a)}{z}-{}_2\mathrm{S}_1^{(2)}(a)+\frac{{}_2\mathrm{S}_1'(a)}{1\,!}z+,\,\ldots+\frac{(-)^{n-1}{}_2\mathrm{S}_n'(a)}{n\,!}z^n+\ldots$$

In this expansion the double Bernoullian functions have ω_1 and ω_2 for parameters, and the expansion is valid provided $|z|$ is less than the smaller of the two quantities $\left|\frac{2\pi\iota}{\omega_1}\right|$. $\left|\frac{2\pi\iota}{\omega_2}\right|$.

For within a circle whose radius is less than this quantity, the function

$$\frac{z^2 e^{-az}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})}$$

has no poles, and hence it is expansible in a Taylor's series of powers of z.

Thus we may assume, for all finite values of a,

$$\frac{ze^{-az}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} = \frac{A_1(a)}{z} - A_0(a) + \frac{f_1(a)}{1\,!} z + \ldots + (-)^{n-1} \frac{f_n(a)}{n\,!} z^n + \ldots$$

where it is obvious that the functions of a which enter as coefficients are all algebraical polynomials.

Change now a into $a + \omega_1$, and subtract the expansion so obtained from the one just written.

We find

$$\frac{ze^{-az}}{1 - e^{-\omega z}} = \frac{A_1(a) - A_1(a + \omega_1)}{z} - A_0(a) + A_0(a + \omega_1) + \frac{f_1(a) - f_1(a + \omega_1)}{1\,!} z + \ldots$$

$$\ldots + (-)^{n-1} \frac{f_n(a) - f_n(a + \omega_1)}{n\,!} z^n + \ldots$$

But in the "Theory of the Gamma Function," § 20, we obtained the expansion which may be written

$$\frac{- ze^{-az}}{1 - e^{-\omega_2 z}} = - S_1^{(2)}(a \mid \omega_2) + \frac{S_1'(a \mid \omega_2)}{1\,!} z + \ldots + (-)^{n-1} \frac{S'_n(a \mid \omega_2)}{n\,!} z^n + \ldots$$

Equating coefficients in these two expansions, we obtain

$$A_1(a + \omega_1) - A_1(a) = 0$$

$$A_0(a + \omega_1) - A_0(a) = S_1^{(2)}(a \mid \omega_2)$$

$$f_1(a + \omega_1) - f_1(a) = S'_1(a \mid \omega_2)$$

$$\cdots\cdots\cdots\cdots\cdots\cdots$$

$$f_n(a + \omega_1) - f_n(a) = S'_n(a \mid \omega_2)$$

and it is obvious that a similar set of equations hold in which ω_1 and ω_2 are interchanged.

Hence $A_1(a)$ is a constant whose value, from the first term of the expansion, is

$$\frac{1}{\omega_1 \omega_2} = {}_2S_1^{(3)}(a \mid \omega_1, \omega_2) \quad \text{by § 4.}$$

Again $A_0(a) = {}_2S_1^{(2)}(a \mid \omega_1, \omega_2)$, for these two expressions can only differ by a constant which by § 4 and the actual expansion is at once seen to vanish.

Finally for all positive integral values of n, $f_n(a)$ can only differ from ${}_2S'_n(a \mid \omega_1, \omega_2)$ by a constant,—*i.e.*, let us say,

$$f_n(a) = {}_2S'_n(a \mid \omega_1, \omega_2) + \mu_n.$$

Differentiate the expansion thus obtained with respect to a, and we find

$$\frac{-z^2 e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = {}_2S_1^{(3)}(a) - \frac{{}_2S_1^{(2)}(a)}{1!}z + \ldots + (-)^{n-1}\frac{{}_2S_n^{(2)}(a)}{n!}z^n + \ldots$$

and hence we have finally (§ 6)

$$\frac{z e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{{}_2S_1^{(3)}(a)}{z} - {}_2S_1^{(2)}(a) + \frac{{}_2S'_1(a)}{1!}z + \ldots + (-)^{n-1}\frac{{}_2S'_n(a)}{n!}z^n + \ldots$$

which is the expansion required.

This expansion may be used to define the double Bernoullian numbers, and all their properties may be deduced from it. A procedure analogous to the one here suggested will be the one employed in the general theory of multiple Bernoullian functions.

§ 16. Several expansions of constant occurrence may be deduced from the one just obtained.

In the first place, note that we may write the expansion in the form

$$\frac{z e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{1}{\omega_1\omega_2 z} + \frac{\omega_1+\omega_2}{2\omega_1\omega_2} - \frac{a}{\omega_1\omega_2} + \frac{{}_2S'_1(a)}{1!}z + \ldots$$
$$+ (-)^{n-1}\frac{{}_2S'_n(a)}{n!}z^n + \ldots$$

Put now $a = 0$, and we have by definition of the double Bernoullian numbers,

$$\frac{z}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{1}{\omega_1\omega_2 z} + \frac{\omega_1+\omega_2}{2\omega_1\omega_2} + {}_2B_1(\omega_1, \omega_2)z + \ldots$$
$$+ (-)^{n-1}\frac{{}_2B_n(\omega_1, \omega_2)}{(n-1)!}z^n + \ldots$$

We thus have (§ 11)

$$\frac{z}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{1}{\omega_1\omega_2 z} + \frac{\omega_1+\omega_2}{2\omega_1\omega_2} + \frac{{}_2B_1(\omega_1, \omega_2)}{0!}z + \frac{{}_2B_3(\omega_1, \omega_2)}{2!}z^3 + \ldots$$
$$+ \frac{{}_2B_{2n+1}(\omega_1, \omega_2)}{(2n)!}z^{2n+1} + \ldots$$
$$+ \sum_{m=1}^{\infty} (-)^{m-1}\frac{B_m(\omega_1^{2m-1} + \omega_2^{2m-1})z^{2m}}{2\,.\,(2m)!}.$$

And the last series may be written

$$\tfrac{1}{2}\left[\frac{z}{1-e^{-\omega_1 z}} + \frac{z}{1-e^{-\omega_2 z}} - \frac{1}{\omega_1} - \frac{1}{\omega_2} - z\right].$$

Hence we find

$$\frac{z}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{1}{2}\frac{z}{(1-e^{-\omega_1 z})} - \frac{z}{2(1-e^{-\omega_2 z})} + \frac{z}{2}$$
$$= \frac{1}{\omega_1\omega_2 z} + \sum_{n=0}^{\infty} \frac{{}_2B_{2n+1}(\omega_1, \omega_2)}{(2n)!} z^{2n+1}$$

as the expansion from which the odd double Bernoullian numbers may be derived.

Finally if we integrate the fundamental expansion of § 15 with respect to a between 0 and a, we obtain

$$\frac{1-e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{{}_2S_1^{(2)}(a) - {}_2S_1^{(2)}(o)}{z} - [{}_2S'_1(a) - {}_2S'_1(o)] + \frac{{}_2S_1(a)}{1!}z + \dots$$
$$\dots + \frac{(-)^{n-1}{}_2S_n(a)}{n!}z^n + \dots$$

or, as we may write it,

$$\frac{1-e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{a}{\omega_1\omega_2 z} - {}_2S_0(a) + \frac{{}_2S_1(a)}{1!}z + \dots + \frac{(-)^{n-1}{}_2S_n(a)}{n!}z^n + \dots$$

as the expansion from which the double Bernoullian functions themselves are at once obtained.

All these expansions are valid within the circle whose radius is the smaller of the two quantities $\left|\frac{2\pi\iota}{\omega_1}\right|$ and $\left|\frac{2\pi\iota}{\omega_2}\right|$.

§ 17. Hitherto we have considered the double Bernoullian function as defined by one of two difference equations, each of which involves the simple Bernoullian function.

We proceed now to prove that, to a linear function of a, ${}_2S_n(a)$ is the only rational integral algebraic function of a satisfying the difference equation

$$f(a+\omega_1+\omega_2) - f(a+\omega_1) - f(a+\omega_2) + f(a) = a^n.$$

In the first place it is at once evident that ${}_2S_n(a)$ does satisfy this equation.

Again the difference of any two solutions is a solution of

$$f(a+\omega_1+\omega_2) - f(a+\omega_1) - f(a+\omega_2) + f(a) = 0.$$

Putting $\qquad f(a+\omega_1) - f(a) = \phi(a),$

we have $\qquad \phi(a+\omega_2) - \phi(a) = 0.$

Hence, if $f(a)$ is the difference of two algebraic solutions of the original equation, $\phi(a)$ will be an algebraic simply periodic function, and therefore a constant.

And thus we shall have

$$f(a+\omega_1) - f(a) = \text{constant},$$

so that if $f(a)$ is to be an algebraic polynomial, it must be of the form

$$\lambda a + \mu,$$

where λ and μ are constant with respect to a.

Thus the difference of any two rational integral algebraic solutions of

$$f(a + \omega_1 + \omega_2) - f(a + \omega_1) - f(a + \omega_2) + f(a) = a^n$$

is of the form $\lambda a + \mu$. Whence the theorem in question.

[Dr. E. W. Hobson has kindly pointed out to me that the analysis of the preceding paragraphs would be much simplified by starting from the direct definition of the double Bernoullian function in § 17.

We should thus define the double Bernoullian function by the expansion

$$\frac{1-e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 x})} = \frac{a}{\omega_1 \omega_2 z} - {}_2S_0(a) + \ldots + (-)^{n-1} z^n \frac{{}_2S_n(a \mid \omega_1, \omega_2)}{n!} + \ldots$$

On differentiating with respect to a, we get the expression of § 15.

From the relation

$$\frac{1-e^{-(a+\omega_1)z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{1-e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{e^{-az}}{1-e^{-\omega_2 z}},$$

we obtain the fundamental difference relations for the double Bernoullian function.

The result of § 10 follows from the identity

$$\frac{1-e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{1-e^{(\omega_1+\omega_2-a)z}}{(1-e^{\omega_1 z})(1-e^{\omega_2 z})} = \frac{1}{1-e^{-\omega_1 z}} - \frac{1}{(1-e^{-\omega_2 z})} - 1,$$

and that of § 14 from

$$\sum_{k=0}^{p-1} \sum_{l=0}^{q-1} \left\{1 - e^{-\left(a + \frac{k\omega_1}{p} + \frac{l\omega_2}{q}\right)z}\right\} = pq - \frac{e^{-az}(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}{(1-e^{-\frac{\omega_1 z}{p}})(1-e^{-\frac{\omega_2 z}{q}})}.$$

Inasmuch as theoretically the properties of an algebraical polynomial should not be derived from consideration of the coefficients of an infinite series, the original investigation has been retained. I had already proposed to myself to work out the theory of multiple Benoullian functions by a method closely allied to that suggested by Dr. Hobson.—*Note added July* 3, 1900.]

Part II.

The Double Gamma Function $\Gamma_2(a \mid \omega_1, \omega_2)$ *and its Elementary Properties.*

§ 18. In the elementary consideration of the simple gamma function it was found to be necessary to rely on two algebraical limit theorems:—

(1) Euler's theorem $\underset{n=\infty}{Lt}\left[1 + \frac{1}{2} + \ldots + \frac{1}{n} - \log n\right] = \gamma.$

(2) Stirling's theorem $\underset{n=\infty}{Lt}\left[\frac{n!}{n^{n+\frac{1}{2}}e^{-n}}\right] = \sqrt{2\pi}.$

In an analogous treatment of the double gamma function we may expect that similar limit theorems will be required. This, in fact, is the case; but for our present purpose it is sufficient to take particular cases of the asymptotic expansion for $\log \Gamma(z)$.

To make use of this approximation we need only remember that ("Theory of the Gamma Function," § 39) if z and ω be any finite complex quantities, and n a positive integer,

$$\log \prod_{m=0}^{n} (z + m\omega) = \log \Gamma_1 [z + (n+1)\omega \mid \omega] - \log \Gamma_1 (z \mid \omega).$$

We suppose that such values of the logarithms are chosen that additive terms involving $2\pi\iota$ do not enter. In other words, we shall say that the logarithms have their absolute values, the formula just written being merely a convenient way of writing the identity

$$\prod_{\omega=0}^{n} (z + m\omega) = \frac{\Gamma_1[z + (n+1)\omega]}{\Gamma_1(z)}.$$

On differentiating this identity with respect to z we have

$$\sum_{\omega=0}^{n} \frac{1}{(z + m\omega)} = \psi_1^{(1)}[z + (n+1)\omega \mid \omega] - \psi_1^{(1)}(z \mid \omega)$$

with the notation of § 2 of the "Theory of the Gamma Function."

§ 19. The double gamma function of z with parameters ω_1 and ω_2 we write

$$\Gamma_2 (z \mid \omega_1, \omega_2).$$

When there is no doubt as to their presence the parameters are omitted. From this function we form the subsidiary system

$$\psi_2^{(1)} (z \mid \omega_1, \omega_2) = \frac{d}{dz} \log \Gamma_2 (z \mid \omega_1, \omega_2).$$

$$\psi_2^{(2)} (z \mid \omega_1, \omega_2) = \frac{d^2}{dz^2} \log \Gamma_2 (z \mid \omega_1, \omega_2).$$

and so on.

As a definition we assume

$$\psi_2^{(3)}(z \mid \omega_1, \omega_2) = -2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(z + m_1\omega_1 + m_2\omega_2)^3}.$$

This double series is, by EISENSTEIN'S Theorem,* absolutely convergent, provided the ratio of ω_1 to ω_2 is not real and negative. This limitation on the parameters holds throughout the whole theory of the double gamma functions. It corresponds to the limitation in WEIERSTRASS' theory of elliptic functions that Γ must not be a real quantity.

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* *v.* FORSYTH, "Theory of Functions," § 56.

We shall show that by successive integrations we may determine $\Gamma_2(z \mid \omega_1, \omega_2)$ as a function symmetrical in ω_1 and ω_2 such that

$$\frac{\Gamma_2^{-1}(z + \omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_2)}{\rho_1(\omega_2)} e^{-2m\pi\iota S'_1(z \mid \omega_2)},$$
$$\frac{\Gamma_2^{-1}(z + \omega_2)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_1)}{\rho_1(\omega_1)} e^{-2m'\pi\iota S'_1(z \mid \omega_1)},$$

where $\rho_1(\omega) = \surd(2\pi/\omega)$ ("Theory of the Gamma Function," § 31), and m and m' are integers (unity or zero), to be determined in accordance with the detailed theory which we proceed to give.

And the function so determined will be unique, provided

$$\underset{z=0}{\mathrm{L}t}[z\Gamma_2(z \mid \omega_1, \omega_2)] = 1.$$

§ 20. We readily see that the function

$$\psi_2^{(3)}(z \mid \omega_1, \omega_2) = -2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(z + \Omega)^3},$$

where $\Omega = m_1\omega_1 + m_2\omega_2$ satisfies the two difference equations

$$\psi_2^{(3)}(z + \omega_1) = \psi_2^{(3)}(z) - \psi_1^{(3)}(z \mid \omega_2)$$
$$\psi_2^{(3)}(z + \omega_2) = \psi_2^{(3)}(z) - \psi_1^{(3)}(z \mid \omega_1),$$

where here, as always, we suppress the parameters of the functions $\psi_2^{(r)}(z)$ $(r = 1, 2, \ldots)$ and $\Gamma_2(z)$ when these parameters are supposed to exist in perfectly general form.

For we have at once from the definition-series

$$\psi_2^{(3)}(z + \omega_1) - \psi_2^{(3)}(z) = -2\left[-\sum_{m_2=0}^{\infty} \frac{1}{(z + m_2\omega_2)^3}\right]$$
$$= -\psi_1^{(3)}(z \mid \omega_2). \qquad \text{("Gamma Function," § 2)}$$

Next, we may show that the function

$$\psi_{21}^{(2)}(z \mid \omega_1\ \omega_2) = -\gamma_{21}(\omega_1, \omega_2) + \frac{1}{z^2} + \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty}{}' \left[\frac{1}{(z + \Omega)^2} - \frac{1}{\Omega^2}\right]$$

satisfies the two difference equations

$$\psi_2^{(2)}(z + \omega_1) = \psi_2^{(2)}(z) - \psi_1^{(2)}(z \mid \omega_2)$$
$$\psi_2^{(3)}(z + \omega_2) = \psi_2^{(2)}(z) - \psi_1^{(2)}(z \mid \omega_1),$$

whatever be the value of the constant $\gamma_{21}(\omega_1, \omega_2)$.

For the series for $\phi_2^{(2)}(z)$ is absolutely convergent so long as

$$\frac{1}{(z + \Omega)^2} - \frac{1}{\Omega^2}$$

is regarded as one term, and we may subtract two absolutely convergent series by a term-by-term process. Hence we have immediately

$$\psi_2^{(2)}(z+\omega_1) - \psi_2^{(2)}(z) = -\sum_{m_2=0}^{\infty} \frac{1}{(z+m_2\omega_2)^2},$$

and therefore ("Theory of the Gamma Function," § 2)

$$\psi_2^{(2)}(z+\omega_1) - \psi_2^{(2)}(z) = -\psi_1^{(2)}(z\,|\,\omega_2).$$

Similarly,

$$\psi_2^{(2)}(z+\omega_2) - \psi_2^{(2)}(z) = -\psi_1^{(2)}(z\,|\,\omega_1).$$

§ 21. It may now be shown that the function

$$-\psi_2^{(1)}(z\,|\,\omega_1, \omega_2) = z\gamma_{21}(\omega_1, \omega_2) + \gamma_{22}(\omega_1, \omega_2) + \tfrac{1}{2} + \sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}{}' \left[\frac{1}{z+\Omega} - \frac{1}{\Omega} + \frac{z}{\Omega^2}\right]$$

satisfies the two difference relations

$$\psi_2^{(1)}(z+\omega_1) - \psi_2^{(1)}(z) = -\psi_1^{(1)}(z\,|\,\omega_2) + \frac{2m\pi\iota}{\omega_2},$$

$$\psi_2^{(1)}(z+\omega_2) - \psi_2^{(1)}(z) = -\psi_1^{(1)}(z\,|\,\omega_1) + \frac{2m'\pi\iota}{\omega_1},$$

for certain values of the numbers m and m', provided

$$\gamma_{21}(\omega_1, \omega_2) = -\underset{n=\infty}{\mathrm{L}t}\left[\sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \frac{1}{\Omega^2} - \frac{1}{\omega_1\,\omega_2}\log n + \frac{1}{\omega_1\,\omega_2}\left\{\log(\omega_1+\omega_2) - \log\omega_1 - \log\omega_2\right\}\right],$$

the principal values of the logarithms being taken.

We may write $\psi_2{}^{(1)}(z)$ in the form

$$-\underset{n=\infty}{\mathrm{L}t}\left[z\gamma_{21}(\omega_1, \omega_2) + \gamma_{22}(\omega_1, \omega_2) + \frac{1}{z} + \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \left\{\frac{1}{z+\Omega} - \frac{1}{\Omega} + \frac{z}{\Omega^2}\right\}\right],$$

and now we obtain at once

$$\psi_2^{(1)}(z+\omega_1) - \psi_2^{(1)}(z)$$
$$= -\omega_1\gamma_{21}(\omega_1, \omega_2) + \underset{n=\infty}{\mathrm{L}t}\left[\sum_{m_2=0}^{n}\frac{1}{z+m_2\,\omega_2} - \sum_{m_2=0}^{n}\frac{1}{z+(n+1)\,\omega_1+m_2\,\omega_2} - \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \frac{\omega_1}{\Omega_2}\right].$$

Hence we may take

$$\psi_2^{(1)}(z+\omega_1) - \psi_2^{(1)}(z) = -\psi_1^{(1)}(z\,|\,\omega_2) + 2\frac{m\pi\iota}{\omega_2},$$

provided

$$\omega_1\gamma_{21}(\omega_1, \omega_2) = \psi_1^{(1)}(z\,|\,\omega_2) - 2\frac{m\pi\iota}{\omega_2}$$
$$+\underset{n=\infty}{\mathrm{L}t}\left[\sum_{m_2=}^{n}\frac{1}{z+m_2\omega_2} - \sum_{m_2=0}^{n}\frac{1}{z+m_2\omega_2+(n+1)\,\omega_1} - \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \frac{\omega_1}{\Omega^2}\right].$$

or, utilising § 19 Corollary, provided

$$\omega_1\gamma_{21}(\omega_1, \omega_2) + 2\frac{m\pi i}{\omega_2} = \underset{n=\infty}{\mathrm{Lt}}\Big[\psi_1^{(1)}[z + (n+1)\,\omega_2\,|\,\omega_2] + \psi_1^{(1)}[z + (n+1)\,\omega_1\,|\,\omega_2]$$
$$- \psi_1^{(1)}[z + (n+1)\,.\,(\omega_1 + \omega_2)\,|\,\omega_2] - \sum_{m_1=0}^{n}\,\sum_{m_2=0}^{n}{}' \frac{\omega_1}{\Omega^2}\Big].$$

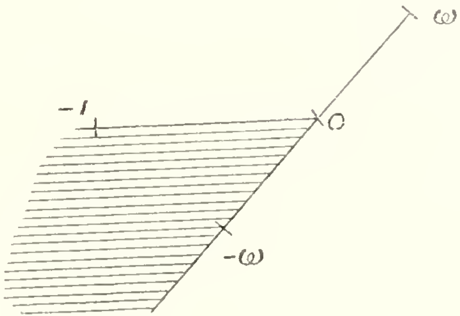
But ("Gamma Function," Part IV.) we know that when $|z|$ is very large, and $\frac{z}{\omega}$ not real and negative,

$$\log\Gamma_1(z + a\,|\,\omega) = \left(\frac{z+a}{\omega} - \tfrac{1}{2}\right)\left\{\log\frac{z}{\omega} + \log\omega\right\} - \frac{z}{\omega} + \tfrac{1}{2}\log\frac{2\pi}{\omega}$$
$$+ \text{ terms which vanish when } |z| \text{ becomes infinite.}$$

In every case the principal value of the logarithm is to be taken, *i.e.*, that value whose amplitude lies between $-\pi$ and π.

Now
$$\log\frac{z}{\omega} + \log\omega = \log z$$

in all cases except when z lies in the region formed by lines from the origin to the points $-\omega$ and -1 (shaded in the figure).



When z does lie within this region, we readily see that

$$\log\frac{z}{\omega} + \log\omega = \log z + 2\pi\iota$$

if $\mathrm{I}(\omega)$, the imaginary part of ω, is positive, and

$$\log\frac{z}{\omega} + \log\omega = \log z - 2\pi\iota$$

if $\mathrm{I}(\omega)$, the imaginary part of ω, is negative.

Thus

$$\log\Gamma_1(z + a\,|\,\omega) = \left(\frac{z+a}{\omega} - \tfrac{1}{2}\right)\{\log z + 2k\pi\iota\} - \frac{z}{\omega} + \tfrac{1}{2}\log\frac{2\pi}{\omega}$$
$$+ \text{ terms which vanish when } |z| \text{ is infinite,}$$

where $k = 0$, unless z lie within the region between the axes to -1 and $-\omega$, and where $k = \pm 1$, the upper or lower sign being taken according as $I(\omega)$ is positive or negative, when z does lie within this region.

On differentiating, we have the derived expansion*

$$\psi_1^{(1)}(z + a \mid \omega) = \frac{1}{\omega}(\log z + 2k\pi\iota)$$

$$+ \text{ terms which vanish when } |z| \text{ becomes infinite.}$$

the principal value of the logarithm being again taken, and k being determined as before.

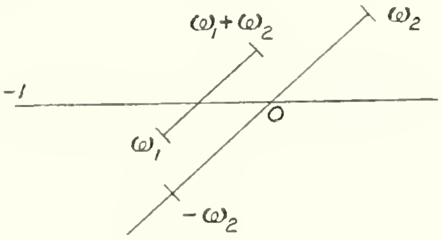
Substituting in the expression for $\gamma_{21}(\omega_1, \omega_2)$, we have

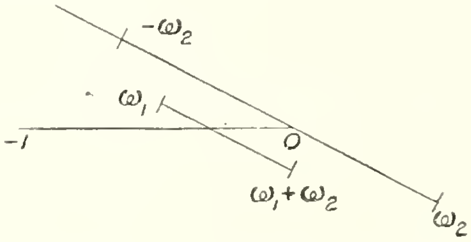
$$\omega_1\gamma_{21}(\omega_1, \omega_2) + 2\frac{m\pi i}{\omega_2} = \underset{n=\infty}{\mathrm{Lt}}\left[\frac{1}{\omega_2}\log n\omega_2 + \frac{1}{\omega_2}\log n\omega_1\right.$$
$$\left. - \frac{1}{\omega_2}\log n\,.\,(\omega_1 + \omega_2) - \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \frac{\omega_1}{\Omega^2} + \frac{2k\pi\iota}{\omega_2}\right].$$

where $k = 0$, unless $\left.\begin{matrix}\omega_1 \text{ does} \\ (\omega_1 + \omega_2) \text{ does not}\end{matrix}\right\}$ or $\left.\begin{matrix}(\omega_1 + \omega_2) \text{ does} \\ \omega_1 \text{ does not}\end{matrix}\right\}$

lie in the region bounded by lines from the origin to -1 and $-\omega_2$. [It is understood, of course, that the principal values of the logarithms are to be taken.]

When, as in the figures $\left.\begin{matrix}\omega_1 \text{ does} \\ (\omega_1 + \omega_2) \text{ does not}\end{matrix}\right\}$ lie within the region of exception, $k = \pm 1$, the upper or lower sign being taken according as $I(\omega_2)$ is positive or negative.





From the diagrams, we see at once that it is impossible that $\left.\begin{matrix}(\omega_1 + \omega_2) \text{ should} \\ \omega_1 \text{ should not}\end{matrix}\right\}$ lie within the region bounded by the lines from the origin to -1 and ω_2.

Take now $m = k$, that is to say, let m be such that we have $m = 0$, unless $\left.\begin{matrix}\omega_1 \text{ does} \\ (\omega_1 + \omega_2) \text{ does not}\end{matrix}\right\}$ lie in the region of exception, and $m = \pm 1$ according as $I(\omega_2)$ is positive or negative, when this exceptional circumstance takes place.

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* According to M. Poincaré, we may not in general differentiate an asymptotic expression. The one in question, however, may be readily established by the methods employed for $\log \Gamma_1(z + a \mid \omega)$.

Then

$$\gamma_{21}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\left[\frac{1}{\omega_1\omega_2}\{\log n\omega_2 + \log n\omega_1 - \log n(\omega_1 + \omega_2)\} - \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}\frac{1}{\Omega^2}\right].$$

But this expression is symmetrical in ω_1 and ω_2; and we must therefore have the analogous relation

$$\psi_2^{(1)}(z + \omega_2) - \psi_2^{(1)}(z) = -\psi_1^{(1)}(z\,|\,\omega_2) + \frac{2m'\pi\iota}{\omega_1},$$

where $m' = 0$, unless $\left.\begin{matrix}\omega_2 \text{ does} \\ (\omega_1 + \omega_2) \text{ does not}\end{matrix}\right\}$ lie within the region bounded by the axes from the origin to $-\omega_1$ and -1, in which case $m' = \pm 1$, the upper or lower sign being taken as I (ω_1) is positive or negative.

Provided therefore that

$$\gamma_{21}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\left[\frac{1}{\omega_1\omega_2}\log n - \sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}'\frac{1}{\Omega^2} + \frac{1}{\omega_1\omega_2}\{\log\omega_1 + \log\omega_2 - \log(\omega_1 + \omega_2\}\right],$$

we have, with the assigned values of m and m', the two difference relations

$$\psi_2^{(1)}(z + \omega_1) - \psi_2^{(1)}(z) = -\psi_1^{(1)}(z\,|\,\omega_2) + \frac{2m\pi\iota}{\omega_2},$$

$$\psi_2^{(1)}(z + \omega_2) - \psi_2^{(1)}(z) = -\psi_1^{(1)}(z\,|\,\omega_1) + \frac{2m'\pi\iota}{\omega_1}.$$

The function γ_{21}, (ω_1, ω_2) we propose to call the first double gamma modular form. It will subsequently be expressed in terms of the function D(τ) introduced into the theory of the functions G $(z\,|\,\tau)$ ("Genesis of the Double Gamma Function," § 4).

It will be seen later that the algebra of the double gamma function would have been slightly simplified had a modified value been taken for this function $\gamma_{21}(\omega_1, \omega_2)$, and the analogue shortly to be considered, $\gamma_{22}(\omega_1, \omega_2)$. I did not observe this fact until the theory had been completely developed, and the matter is scarcely of sufficient importance to demand the labour which such a change would entail.

Corollary.—Notice that it has been proved incidentally that

$$\sum_{m_1=0}^{n}\sum_{m=\infty}^{n}{}'\frac{1}{(m_1\omega_1 + m_2\omega_2)^2}$$

is infinite, when n is infinite, like $\dfrac{1}{\omega_1\,\omega_2}\log n$.

§ 22. As the numbers in m and m' enter constantly into the analysis, it is necessary to consider their properties.

Suppose that the functions $\log z$, $\log_{\omega_1} z$, $\log_{\omega_2} z$ are natural logarithms (with e as base), which are real when z is real and positive, and which are rendered uniform by cross-cuts along the axes of -1, $-\omega_1$ and $-\omega_2$ respectively.

Then it is readily seen that

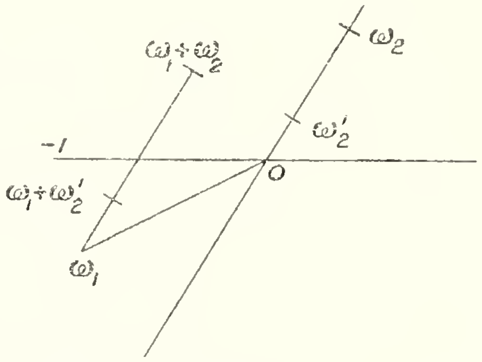
$$\log_{\omega_1}(\omega_1 + \omega_2) - \log_{\omega_1}\omega_1 = \log(\omega_1 + \omega_2) - \log\omega_1 - 2m\pi\iota$$
$$\log_{\omega_1}(\omega_1 + \omega_2) - \log_{\omega_1}\omega_2 = \log(\omega_1 + \omega_2) - \log\omega_2 - 2m'\pi\iota$$
$$\log_{\omega_2}(\omega_1 + \omega_2) - \log_{\omega_2}\omega_2 = \log(\omega_1 + \omega_2) - \log\omega_2 - 2m'\pi\iota$$
$$\log_{\omega_2}(\omega_1 + \omega_2) - \log_{\omega_2}\omega_1 = \log(\omega_1 + \omega_2) - \log\omega_1 - 2m\pi\iota.$$

By inspection of a diagram we see that m and m' both vanish if the difference of the amplitudes of ω_1 and ω_2 is less than π, these amplitudes being measured between 0 and $\pm\ \pi$ positively or negatively from the positive half of the real axis. In particular when the real parts of ω_1 and ω_2 are both positive, m and m' both vanish. Not only so, but in all cases either m or m' must vanish.

Again, if the difference of the amplitudes of ω_1 and ω_2 is greater than π, m and m cannot both vanish. In fact, in this case we have the important relation

$$m - m' = \pm 1,$$

the upper or lower sign being taken according as $I\left(\frac{\omega_2}{\omega_1}\right)$ is negative or positive. This result is intuitive geometrically; in the figure, for instance, two cases are indicated in which $I\left(\frac{\omega_2}{\omega_1}\right)$ is negative.



For corresponding to the unaccented value of ω_2,

$$\left.\begin{array}{l} m = 1 \\ m' = 0 \end{array}\right\},$$

and corresponding to the accented value of ω_2,

$$\left.\begin{array}{l} m = 0 \\ m' = -1 \end{array}\right\}.$$

Thus in both cases $m - m' = 1$.

No such simple expression can be given for $m + m'$, a number which is of constant occurrence in the higher theory.

However, from the values for m and m' previously given, we see that when the axes of ω_1 and ω_2 include the axis of -1 within an angle less than two right angles, the values of $(m + m')$ are given by the table

$(m + m')$	$\mathrm{I}(\omega_1)$	$\mathrm{I}(\omega_1 + \omega_1)$
1	+ *ve*	+ *ve*
− 1	+ *ve*	− *ve*
− 1	− *ve*	+ *ve*
1	− *ve*	− *ve*

and therefore $m + m' = \pm 1$, the upper or lower signs being taken according as $\mathrm{I}(\omega_1 + \omega_2)$ and $\mathrm{I}(\omega_1)$ have the same or opposite signs.

§ 23. It may now be shown that, if C have any arbitrary value, the function

$$\Gamma_2^{-1}(z \mid \omega_1, \omega_2) = \mathrm{C}\, e^{\frac{z^2}{2}\gamma_{21}(\omega_1, \omega_2) + z\gamma_{22}(\omega_1, \omega_2)} \,.\, z \times \prod_{m_1=0}^{\infty} \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right) e^{-\frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2}}\right],$$

where $\Omega = m_1\omega_1 + m_2\omega_2$, will satisfy the two difference relations

$$\frac{\Gamma_2^{-1}(z + \omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_2)}{\sqrt{\dfrac{2\pi}{\omega_2}}}\, e^{-2m\pi\iota\left(\frac{z}{\omega_2} - \frac{1}{2}\right)};$$

$$\frac{\Gamma_2^{-1}(z + \omega_2)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_1)}{\sqrt{\dfrac{2\pi}{\omega_1}}}\, e^{-2m'\pi\iota\left(\frac{z}{\omega_1} - \frac{1}{2}\right)};$$

where m and m' are the numbers previously specified, and

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\left[\sum_0^n \sum_0^n{}' \frac{1}{\Omega} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log n - \frac{n+1}{\omega_2}\log\left(1 + \frac{\omega_2}{\omega_1}\right) - \frac{n+1}{\omega_1}\log\left(1 + \frac{\omega_1}{\omega_2}\right) + \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\left\{\log(\omega_1 + \omega_2) - \log\omega_1 - \log\omega_2\right\}\right],$$

the principal values of the logarithms being taken.

Observe that with the notation previously introduced ("Theory of the Gamma Function," §§ 16 and 31) we may write these difference relations in the form

$$\frac{\Gamma_2^{-1}(z + \omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_2)}{\rho_1(\omega_2)}\, e^{-2m\pi\iota\, S'_1(z \mid \omega_2)}$$

$$\frac{\Gamma_2^{-1}(z + \omega_2)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z \mid \omega_1)}{\rho_1(\omega_1)}\, e^{-2m'\pi\iota\, S'_1(z \mid \omega_1)}.$$

The proof, to which we now proceed, is exactly analogous to the one just given.

We have

$$\frac{\Gamma_2^{-1}(z + \omega_2)}{\Gamma_2^{-1}(z)} = e^{\frac{\gamma_{21}}{2}\omega_1^2 + \gamma_{21}z\omega_1 + \gamma_{22}\omega_1} \,.\, \frac{z + \omega_1}{z} \,.\, \underset{n=\infty}{\mathrm{L}t}\left[\prod_{m_1=0}^{n} \prod_{m_2=0}^{n}{}' \left\{\frac{z + (m_1 + 1)\omega_1 + m_2\omega_2}{z + m_1\omega_1 + m_2\omega_2} \times e^{-\frac{\omega_1}{\Omega}} + \frac{2z\omega_1 + \omega_1^2}{2\Omega^2}\right\}\right].$$

where γ_{21} and γ_{22} are understood, as always, to mean $\gamma_{21}(\omega_1, \omega_2)$ and $\gamma_{22}(\omega_1, \omega_2)$.

Substitute now the value of $\gamma_{21}(\omega_1, \omega_2)$ obtained in § 21, and we find

$$\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)} = \exp.\left[\frac{2z\omega_1+\omega_1^2}{2\omega_1\omega_2}\{\log\,\omega_2+\log\,\omega_1-\log\,(\omega_1+\omega_2)\}+\omega_1\gamma_{22}\right].\frac{z+\omega_1}{z}.$$

$$\times \underset{n=\infty}{\text{L}t}\left[\exp.\left\{\frac{2z\omega_1+\omega_1^2}{2\omega_1\omega_2}\log\,n\right\}\prod_{m_1=0}^{n}\prod_{m_2=0}^{n}{}'\left\{\frac{z+(m_1+1)+m_2\omega_2}{z+m_1\omega_1+m_2\omega_2}e^{-\frac{\omega_1}{\Omega}}\right\}\right],$$

$$= \exp.\left[\frac{2z\omega_1+\omega_2}{2\omega_1\omega_2}\{\log\,\omega_1+\log\,\omega_2-\log\,(\omega_1+\omega_2)\}+\omega_1\gamma_{22}\right].\Gamma_1(z\,|\,\omega_2)$$

$$\times \underset{n=\infty}{\text{L}t}\left[\exp.\left\{\frac{2z\omega_1+\omega_1^2}{2\omega_1\omega_2}\log n-\sum_0^n\sum_0^n{}'\frac{\omega_1}{\Omega}\right\}.\frac{\Gamma_1[z+(n+1).(\omega_1+\omega_2)\,|\,\omega_2]}{\Gamma_1[z+(n+1)\,\omega_1\,|\,\omega_2]\,\Gamma_1[z+(n+1)\omega_2\,|\,\omega_2]}\right]$$

by the employment of the identity of § 18. Their principal values must throughout be assigned to the logarithms.

But, as has been seen in § 21, from the formula obtained in the "Theory of the Gamma Function," § 41, we have when $|z|$ is large and z not in the vicinity of the axis of $-\omega$,

$$\log\Gamma_1(z+a\,|\,\omega) = \left(\frac{z+a}{\omega}-\tfrac{1}{2}\right)\{\log z+2k\pi\iota\}-\frac{z}{\omega}+\log\rho_1(\omega)$$

$$+ \text{ terms which vanish when } |z| \text{ is infinite,}$$

where $k = 0$, unless z lies within the region between the axes of -1 and $-\omega$, in which case $k = \pm 1$, the upper or lower sign being taken as $I(\omega)$ is positive or negative. The principal value of $\log z$ is to be taken, and the prescription to be given to $\log\Gamma_1(z+a\,|\,\omega)$ is left indeterminate: it is obvious that we only get additive terms involving $2\pi\iota$, which vanish in the sequel.

Inasmuch as when n is large, none of the points

$$z+(n+1)(\omega_1+\omega_2),\quad z+(n+1)\,\omega_1,\ \text{and}\ z+(n+1)\,\omega_2$$

lie in the vicinity of the negative direction of the axis of ω_2, we may substitute the values given by the asymptotic expansion in the expression for $\dfrac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)}$.

We shall find

$$\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)\,\Gamma_1(z\,|\,\omega_2)} = \exp.\underset{n=\infty}{\text{L}t}\left[\frac{2z\omega_1+\omega_1^2}{z\omega_1\omega_2}\{\log\,n\omega_2+\log\,\omega_1-\log\,(\omega_1+\omega_2)\}+\omega_1\gamma_{22}\right.$$

$$-\sum_0^n\sum_0^n{}'\frac{\omega_1}{\Omega}+\left(\frac{z+(n+1)(\omega_1+\omega_2)}{\omega_2}-\tfrac{1}{2}\right)\log n(\omega_1+\omega_2)-\frac{n(\omega_1+\omega_2)}{\omega_2}$$

$$-\log\rho_1(\omega_2)-\left(\frac{z+(n+1)\,\omega_2}{\omega_2}-\tfrac{1}{2}\right)\log n\omega_2$$

$$-\left(\frac{z+(n+1)\,\omega_1}{\omega_2}-\tfrac{1}{2}\right)\log\,n\omega_1+\frac{n(\omega_1+\omega_2)}{\omega_2}$$

$$\left.+\,2\left(\frac{z+(n+1)(\omega_1+\omega_2)}{\omega_2}-\tfrac{1}{2}\right)k_1\pi\iota-2\left(\frac{z+(n+1)\,\omega_1}{\omega_2}\right)-\tfrac{1}{2}k_2\pi\iota\right]$$

In this expression their principal values are throughout to be assigned to the logarithms, and the numbers k are to be such that

$k_1 = 0$, unless $(\omega_1 + \omega_2)$ has within the region bounded by axes to -1 and $-\omega_2$, in which case

$k_1 = \pm 1$, the upper or lower sign being taken as $I(\omega_2)$ is positive or negative, while

$k_2 = 0$, unless ω_1 lies within the region bounded by axes to -1 and $-\omega_2$, in which case

$k_2 = \pm 1$, the upper or lower sign being taken as $I(\omega_2)$ is positive or negative.

On reduction we now see that

$$\begin{aligned}\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)\Gamma_1(z\,|\,\omega_2)} = \exp.\ \underset{n=\infty}{\mathrm{L}t}\Big[&\omega_1\gamma_{22}(\omega_1, \omega_2) + \sum_0^n \sum_0^{n}{}' \frac{\omega_1}{\Omega} + (n+\tfrac{1}{2})\frac{\omega_1+\omega_2}{\omega_1}\log n(\omega_1+\omega_2) \\ &- \frac{(2n+1)\,\omega_2-\omega_1}{2\omega_2}\log n\omega_2 - \frac{(2n+1)\,\omega_1-\omega_2}{2\omega_2}\log n\omega_1 - \log \rho_1(\omega_2) \\ &+ \left(\frac{z+(n+1)(\omega_1+\omega_2)}{\omega_2} - \tfrac{1}{2}\right)2k_1\pi i - \left(\frac{z+(n+1)\,\omega_1}{\omega_2} - \tfrac{1}{2}\right)2k_2\pi\iota\Big].\end{aligned}$$

We must consider the three possible cases in which k_1 and k_2 do not both vanish.

(1) Firstly, when $\left\{\begin{matrix}\omega_1 \text{ does} \\ (\omega_1+\omega_2) \text{ does not}\end{matrix}\right\}$ lie within the region bounded by the axes to -1 and $-\omega_2$.

In this case $\left\{\begin{matrix} k_1 = 0 \\ k_2 = \pm 1\end{matrix}\right\}$ the upper or lower sign being taken as $I(\omega_2)$ is positive or negative.

And we have

$$\begin{aligned}\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)\Gamma_1(z\,|\,\omega_2)}\sqrt{\frac{2\pi}{\omega_2}} = \underset{n=\infty}{\mathrm{L}t}\ \exp.\Big[\omega_1\Big\{&\gamma_{22} - \sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} + (n+\tfrac{1}{2})\frac{\omega_1+\omega_2}{\omega_1\omega_2}\log n(\omega_1+\omega_2) \\ &- \frac{(2n+1)\,\omega_2-\omega_1}{2\omega_1\omega_2}\log n\omega_2 - \frac{(2n+1)\,\omega_1-\omega_2}{2\omega_2}\log n\omega_1 \\ &\pm 2\frac{(n+1)\,\pi\iota}{\omega_2}\Big\} \mp 2\left(\frac{z}{\omega_2} - \tfrac{1}{2}\right)\pi\iota\Big].\end{aligned}$$

the upper or lower sign being taken as $I(\omega_2)$ is positive or negative.

But in this case $m = \pm 1$, the signs being chosen in the same way.

If then we take

$$\begin{aligned}\gamma_{22}(\omega_1, \omega_2) = \sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - (n+\tfrac{1}{2})\frac{\omega_1+\omega_2}{\omega_1\omega_2}\log n(\omega_1+\omega_2) + \frac{(2n+1)\,\omega_2-\omega_1}{2\omega_1\omega_2}\log n\omega_2 \\ + \frac{(2n+1)\,\omega_1-\omega_2}{2\omega_2}\log n\omega_1 \pm 2\frac{(n+1)\,\pi\iota}{\omega_2}\end{aligned}$$

we shall have

$$\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z\,|\,\omega_2)}{\sqrt{(2\pi/\omega_2)}}e^{-2m\left(\frac{z}{\omega_2}-\frac{1}{2}\right)\pi\iota} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (1).$$

(2) Secondly, when $\left\{\begin{matrix}\omega_1 \text{ does} \\ (\omega_1+\omega_2) \text{ does}\end{matrix}\right\}$ lie within the region bounded by axes to -1 and $-\omega_2$.

In this case $\begin{matrix} k_1 = \pm 1, \\ k_2 = \pm 1, \end{matrix} \quad m = 0.$

We shall have then the same relation

$$\frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z)\omega_2}{\sqrt{(2\pi/\omega_2)}}\,e^{-2m'\pi\iota\left(\frac{z}{\omega_2}-\frac{1}{2}\right)},$$

provided we take

$$\gamma_{22}(\omega_1, \omega_2) = \sum_0^n \sum_0^n{}' \frac{1}{\Omega} - (n+\tfrac{1}{2})\frac{\omega_1+\omega_2}{\omega_1\omega_2}\log n(\omega_1+\omega_2) + \frac{(2n+1)\,\omega_2-\omega_1}{2\omega_1\omega_2}\log n\omega_2$$
$$+ \frac{(2n+1)\,\omega_1-\omega_2}{2\omega_2}\log n\omega_1 \mp 2\frac{(n+1)\pi\iota}{\omega_1},$$

the upper or lower sign being taken according as $I(\omega_2)$ is positive or negative.

(3) The third case, when $\left\{\begin{matrix}\omega_1 \text{ does not} \\ (\omega_1+\omega_2) \text{ does}\end{matrix}\right\}$ lie within the region bounded by axes -1 and $-\omega_2$, is easily seen to be impossible.

In all other cases we shall have the relation (1). provided

$$\gamma_{22}(\omega_1, \omega_2) = \sum_0^n \sum_0^{n_0}{}' \frac{1}{\Omega} - (n+\tfrac{1}{2})\frac{\omega_1+\omega_2}{\omega_1\omega_2}\log n(\omega_1+\omega_2) + \frac{(2n+1)\,\omega_2-\omega_1}{2\omega_1\omega_2}\log n\omega_2$$
$$+ \frac{(2n+1)\,\omega_1-\omega_2}{2\omega_2}\log n\omega_1 \quad . \quad . \quad . \quad . \qquad (2).$$

Suppose now that we had investigated similarly the quotient

$$\frac{\Gamma_2^{-1}(z+\omega_2)}{\Gamma_2^{-1}(z)},$$

we should have obtained the difference equation

$$\frac{\Gamma_2^{-1}(z+\omega_2)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z\,|\,\omega_1)}{\sqrt{(2\pi/\omega_1)}}e^{-2m'\pi\iota\left(\frac{z}{\omega_2}-\frac{1}{2}\right)},$$

where $\gamma_{21}(\omega_1, \omega_2)$ has the value D, let us say, given by equation (2), except in two cases.

(1)′ When $\left\{\begin{matrix}\omega_2 \text{ does} \\ (\omega_1+\omega_2) \text{does not}\end{matrix}\right\}$ lie within the region bounded by the axes to -1 and $-\omega_1$, in which case

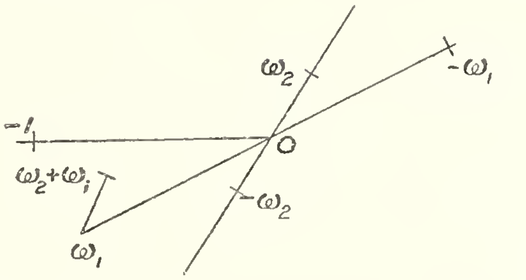
$$\gamma_{22}(\omega_1, \omega_2) = D \pm 2\frac{(n+1)\pi\iota}{\omega_1},$$

the upper or lower sign being taken according as $I(\omega_1)$ is positive or negative.

(2)′ When $\left\{\begin{matrix}\omega_2 \text{ does} \\ (\omega_1 + \omega_2) \text{ does}\end{matrix}\right\}$ lie within the region bounded by the axes to -1 and $-\omega_1$, in which case

$$\gamma_{22}(\omega_1, \omega_2) = \mathrm{D} \mp 2\frac{(n+1)\pi\iota}{\omega_2}.$$

But the cases (1) and (2)′ are precisely the same, as an inspection of the figure shows at once, except that $\mathrm{I}(\omega_2)$ is positive when $\mathrm{I}(\omega_1)$ is negative, and *vice versâ*.



And, similarly, the cases (2) and (1)′ are the same, with a similar change.

Hence the values which must be assigned to $\gamma_{22}(\omega_1, \omega_2)$, in order that the equations

$$\left\{\begin{aligned} \frac{\Gamma_2^{-1}(z+\omega_1)}{\Gamma_2^{-1}(z)} &= \frac{\Gamma_1(z|\omega_2)}{\sqrt{2\pi/\omega_2}}\, e^{-2m\pi\iota\left(\frac{z}{\omega_2}-\frac{1}{2}\right)} \\ \frac{\Gamma_2^{-1}(z+\omega_2)}{\Gamma_2^{-1}(z)} &= \frac{\Gamma_1(z|\omega_1)}{\sqrt{2\pi/\omega_1}}\, e^{-2m'\pi\iota\left(\frac{z}{\omega_1}-\frac{1}{2}\right)} \end{aligned}\right.$$

may co-exist, are precisely the same.

We shall have then these two equations, provided

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\, \mathrm{D},$$

where D stands for

$$\sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - \left(n + \tfrac{1}{2}\right)\frac{\omega_1 + \omega_2}{\omega_1\omega_2}\log\left[n(\omega_1 + \omega_2)\right] + \frac{(2n+1)\omega_2 - \omega_1}{2\omega_1\omega_2}\log n\omega_2 + \frac{(2n+1)\omega_1 - \omega_2}{2\omega_2\omega_1}\log n\omega_1$$

(the principal values of the logarithms being taken), except in two cases,

(1) When $\left\{\begin{matrix}\omega_1 \text{ does} \\ (\omega_1 + \omega_2) \text{ does not}\end{matrix}\right\}$ lie within the region bounded by the axes to -1 and $-\omega_2$, in which case

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\left[\mathrm{D} + 2\frac{m(n+1)\pi\iota}{\omega_2}\right].$$

(2) When $\left\{\begin{matrix}\omega_2 \text{ does} \\ (\omega_1 + \omega_2) \text{ does not}\end{matrix}\right\}$ lie within the region bounded by the axes to -1 and $-\omega_1$, in which case

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\left[\mathrm{D} + 2\frac{m'(n+1)\pi\iota}{\omega_1}\right].$$

Since m or m' always vanishes, we see, on combining these results, that in all cases

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t} \left[\sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - (n + \tfrac{1}{2}) \frac{\omega_1 + \omega_2}{\omega_1 \omega_2} \log n(\omega_1 + \omega_2) \right.$$

$$\left. + \frac{(2n+1)\omega_2 - \omega_1}{2\omega_1\omega_2} \log n\omega_2 + \frac{(2n+1)\omega_1 - \omega_2}{2\omega_1\omega_2} \log n\omega_1 + 2\left(\frac{m}{\omega_2} + \frac{m'}{\omega_1}\right)(n+1)\pi\iota \right]$$

$$= \underset{n=\infty}{\mathrm{L}t} \left[\sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - \frac{\omega_1 + \omega_2}{2\omega_1, \omega_2} \log n \right.$$

$$- (n+1) \frac{1}{\omega_2} \{\log(\omega_1 + \omega_2) - \log \omega_1 - 2m\pi\iota\}$$

$$- (n+1) \frac{1}{\omega_1} \{\log(\omega_1 + \omega_2) - \log \omega_2 - 2m'\pi\iota\}$$

$$\left. + \frac{\omega_1 + \omega_2}{2\omega_1\omega_2} \{\log(\omega_1 + \omega_2) - \log \omega_1 - \log \omega_2\} \right].$$

But

$$\log(\omega_1 + \omega_2) - \log \omega_1 - 2m\pi\iota = \log\left(1 + \frac{\omega_2}{\omega_1}\right)$$

$$\log(\omega_1 + \omega_2) - \log \omega_2 - 2m'\pi\iota = \log\left(1 + \frac{\omega_1}{\omega_1}\right),$$

the principal values of the logarithms being always taken. Hence

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{Lt}} \left[\sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2} \log n \right.$$

$$- \frac{n+1}{\omega_2} \log\left(1 + \frac{\omega_2}{\omega_1}\right) - \frac{n+1}{\omega_1} \log\left(1 + \frac{\omega_1}{\omega_2}\right)$$

$$\left. + \frac{\omega_1 + \omega_2}{2\omega_1\omega_2} \{\log(\omega_1 + \omega_2) - \log \omega_1 - \log \omega_2\} \right].$$

As a *corollary*, we see that, when n is very large,

$$\sum_{m_1=0}^{n} \sum_{m_2=0}^{n}{}' \frac{1}{m_1\omega_1 + m_2\omega_2} \text{ is infinite like } \frac{\omega_1 + \omega_2}{2\omega_1\omega_2} \log n - \frac{n}{\omega_2} \log\left(1 + \frac{\omega_2}{\omega_1}\right)$$

$$- \frac{n}{\omega_1} \log\left(1 + \frac{\omega_1}{\omega_2}\right).$$

§ 24. We now determine the constant C in the expression

$$\Gamma_2^{-1}(z \mid \omega_1, \omega_2) = \mathrm{C}e^{\frac{z^2}{2}\gamma_{21} + z\gamma_{22}} . z . \prod_{m_1=0}^{\infty} \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right) e^{-\frac{z}{\Omega} + \frac{z^2}{2\Omega^2}}\right],$$

by the condition assigned in § 19, that

$$\underset{z=0}{\mathrm{L}t}\, z\Gamma_2(z \mid \omega_1, \omega_2) = 1.$$

This at once gives us C = 1.

It is evident that, with the conditions of § 19, one and only one function can be constructed, and this is the double gamma function $\Gamma_2(z \mid \omega_1, \omega_2)$, which is such that

$$\Gamma_2^{-1}(z\,|\,\omega_1, \omega_2) = e^{\frac{z^2}{2}\gamma_{21} + z\gamma_{22}} \,.\, z \,.\, \prod_{m_1=0}^{\infty} \ \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right) e^{-\frac{z}{\Omega} + \frac{z^2}{2\Omega^2}}\right],$$

where $\gamma_{21}(\omega_1, \omega_2)$ and $\gamma_{22}(\omega_1, \omega_2)$ are two constants, which we call the first and second double gamma modular functions of the parameters ω_1 and ω_2.

These constants are given by the relations

$$\gamma_{21}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{Lt}} \left[\frac{1}{\omega_1\omega_2}\log n - \sum_{m_1=0}^{n} \ \sum_{m_2=0}^{n}{}' \frac{1}{\Omega^2} \right.$$
$$\left. + \frac{1}{\omega_1\omega_2}\{\log\omega_1 + \log\omega_2 - \log(\omega_1 + \omega_2)\}\right]$$

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{Lt}} \left[\sum_{m_1=0}^{n} \ \sum_{m_2=0}^{n}{}' \frac{1}{\Omega} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log n - \frac{n+1}{\omega_2}\log\left(1 + \frac{\omega_2}{\omega_1}\right)\right.$$
$$\left. - \frac{n+1}{\omega_2}\log\left(1 + \frac{\omega_1}{\omega_2}\right) + \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\{\log(\omega_1 + \omega_2) - \log\omega_1 - \log\omega_2\}\right],$$

where the logarithms are such that their principal values must always be taken.

And the theory is the natural extension of that of the simple gamma function $\Gamma_1(z\,|\,\omega_1)$, which is such that

$$\Gamma_1^{-1}(z\,|\,\omega_1) = e^{-\gamma_{11}z} \,.\, z \,.\, \prod_{m_1=1}^{\infty} \left[\left(1 + \frac{z}{m_1\omega_1}\right) e^{-\frac{z}{m_1\omega_1}}\right],$$

where the constant γ_{11} is given by the relation

$$\gamma_{11}(\omega_1) = \underset{n=\infty}{\mathrm{Lt}} \left[\sum_{m_1=1}^{n} \frac{1}{m_1\omega_1} - \frac{1}{\omega_1}\log n\omega_1\right],$$

and the principal value of the logarithm must again be taken.

§ 25. We may now see at once that

$$\Gamma_2(\omega_1\,|\,\omega_1, \omega_2) = \sqrt{(2\pi/\omega_2)} \,.\, e^{-m\pi i},$$
$$\Gamma_2(\omega_2\,|\,\omega_1, \omega_2) = \sqrt{(2\pi/\omega_1)} \,.\, e^{-m'\pi i}.$$

For we have seen that

$$\frac{\Gamma_2^{-1}(z + \omega_1)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z\,|\,\omega_2)}{\sqrt{(2\pi/\omega_2)}}\, e^{-2m\pi i\left(\frac{z}{\omega_2} - \frac{1}{2}\right)},$$

where m is either zero or unity according to the determination of § 22.

But $$\underset{z=0}{\mathrm{Lt}}\,[z\Gamma_2(z)] = 1,$$

and $$\underset{z=0}{\mathrm{Lt}}\,[z\Gamma_1(z\,|\,\omega_2)] = 1,$$

as we see immediately from the product expression for

$$\Gamma_1(z\,|\,\omega_2).$$

Hence, making $z = 0$, we have

$$\Gamma_2^{-1}(\omega_1) = \frac{1}{\sqrt{(2\pi/\omega_2)}} e^{m\pi\iota},$$

which is one of the relations required.

We thus see that $\Gamma_2(\omega_1)$ is independent of ω_1, and not only so, but its value is substantially a quantity that appeared several times in the theory of the simple gamma function.

Thus we saw [§ 4 cor. "Gamma Function"] that

$$\prod_{q=1}^{n-1} \Gamma_1\left(\frac{q\omega}{n}\middle|\,\omega\right) = n^{-\frac{1}{2}}\{\sqrt{(2\pi/\omega)}\}^{n-1};$$

that (§ 8)

$$\int_0^\omega \log\Gamma_1(z\,|\,\omega)dz = \omega\log\sqrt{(2\pi/\omega)};$$

and the most general form of Stirling's theorem was seen to be

$$\begin{aligned}\log\prod_{m_1=0}^{pn}(a + m_1\omega) = p\{n\log n - n\} &+ n\{\mathrm{S}_1^{(2)}(a+\omega)p\omega\log p\omega\}\\ &+ \{1 + \mathrm{S}_1'(a)\}\log n - \log\Gamma_1(a) + \log\sqrt{(2\pi/\omega)}\\ &+ \mathrm{S}_1'(a+\omega)\log p\omega + \sum_{m=1}^{\infty}\frac{(-)^{m-1}}{mn^m}\,\frac{\mathrm{S}m(a+\omega) + {}_1\mathrm{B}_{m+1}}{(p\omega)^m}.\end{aligned}$$

We now see an additional reason why it was proposed to write

$$\rho_1(\omega) = \sqrt{(2\pi/\omega)},$$

and to call $\gamma_{11}(\omega)$ and $\rho_1(\omega)$ the two simple gamma modular forms, the latter being sometimes called the simple Stirling modular form. We shall see that there exist three double gamma modular forms

$$\gamma_{21}(\omega_1, \omega_2),\ \gamma_{22}(\omega_1, \omega_2) \text{ and } \rho_2(\omega_1, \omega_2)$$

of exactly analogous nature.

§ 26. We proceed now to connect the function $\Gamma_2(z\,|\,\omega_1, \omega_2)$ with Alexeiewsky's function $\mathrm{G}(z\,|\,\tau)$, some of whose properties were investigated in "The Genesis of the Double Gamma Functions."

In the first place, we take $\tau = \omega_2/\omega_1$, and then we have

$$\mathrm{G}\left(\frac{z}{\omega_1}\middle|\,\tau\right) = e^{a\frac{z}{\omega_2} + b\frac{z}{2\omega_2^2}} \cdot \frac{z}{\omega_2} \cdot \prod_{m_1=0}^{\infty}\ \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right)e^{-\frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2}}\right],$$

where $\Omega = m_1\omega_1 + m_2\omega_2$,

and wherein

$$a = \frac{\tau}{2}\log 2\pi\tau + \tfrac{1}{2}\log\tau - \gamma\tau - \mathrm{C}(\tau),$$

$$b = -\tau\log\tau - \frac{\pi^2\tau^2}{6} - \tau^2\mathrm{D}(\tau).$$

We also have

$$\Gamma_2{}^{-1}(z) = e^{\gamma_{21}\frac{z^2}{z} + z\gamma_{22}} \,.\, z \,.\, \prod_{m_1=0}^{\infty} \prod_{m_2=0}^{\infty}{}' \left[\left(1 + \frac{z}{\Omega}\right) e^{-\frac{z}{\Omega} + \frac{1}{2}\frac{z^2}{\Omega^2}}\right],$$

so that on comparison of the two products we find

$$\Gamma_2{}^{-1}(z) = \mathrm{K} e^{\alpha z^2 + \beta z} \mathrm{G}\left(\frac{z}{\omega_1}\middle| \tau\right),$$

where α, β, and K are suitable functions of ω_1 and ω_2.

Now $\mathrm{G}(z \mid \tau)$ satisfies the difference equation

$$\mathrm{G}(z \mid \tau) = \Gamma^{-1}\left(\frac{z}{\tau}\right) \mathrm{G}(z + 1 \mid \tau),$$

and hence $\mathrm{G}\left(\frac{z}{\omega}\middle| \tau\right)$ satisfies the relation

$$f(z + \omega_1) = \omega_2{}^{1-\frac{z}{\omega_2}} \Gamma_1(z \mid \omega_2) f(z).$$

Hence a solution of the difference equation

$$f(z + \omega_1) = \sqrt{(\omega_2/2\pi)} \,.\, \Gamma_1(z \mid \omega_2) f(z)\, e^{-2m\pi\iota \mathrm{S}_1'(z \mid \omega_2)}$$

is

$$\mathrm{T}(z) = \omega_2{}^{\frac{z^2}{2\omega_1\omega_2} - \frac{z}{2\omega_2} - \frac{z}{2\omega_1}} (2\pi)^{-\frac{z}{2\omega_1}} \mathrm{G}\left(\frac{z}{\omega_1}\middle| \tau\right) e^{-2m\pi\iota\, [{}_2\mathrm{S}_1'(z \mid \omega_1, \omega_2) - {}_2\mathrm{S}_1'(o)]} \text{ (by § 3).}$$

And it is evident that the coefficient of $2m\pi\iota$ in the last exponential may be written ${}_2\mathrm{S}_0(z \mid \omega_1, \omega_2)$.

The general solution of the difference equation is

$$\Gamma_2{}^{-1}(z) \times p(z \mid \omega_1),$$

where $p(z \mid \omega_1)$ is a function of z simply periodic of period ω_1.

Hence
$$\frac{\Gamma_2{}^{-1}(z)}{\mathrm{T}(z)} = p(z \mid \omega_1).$$

But $\frac{\Gamma_2{}^{-1}(z)}{\mathrm{T}(z)}$ has been seen to be an expression of the form $\mathrm{K}e^{\alpha z^2 + \beta z^2}$, where K, α, and β are independent of z. We thus have

$$e^{\alpha(z + \omega_1)^2 + \beta(z + \omega_1)} = e^{\alpha z_2{}^2 + \beta z},$$

so that $\alpha = 0$ and $\beta = \frac{2r\pi\iota}{\omega_1}$, where r is some integer.

Hence
$$\Gamma_2{}^{-1}(z) = \mathrm{K} e^{\frac{2r\pi\iota z}{\omega_1}} (\omega_2 e^{-m\pi\iota})^{{}_2\mathrm{S}_0(z)} (2\pi)^{\frac{-z}{2\omega_1}} \mathrm{G}\left(\frac{z}{\omega_1}\middle| \tau\right) \quad . \; . \; . \; . \; . \; . \; . \; (1),$$

and since
$$\underset{z=0}{\mathrm{L}t}\, [z\Gamma_2(z)] = 1, \quad \underset{z=0}{\mathrm{L}t} \left[\frac{z}{\mathrm{G}\left(\frac{z}{\omega_1}\middle| \tau\right)}\right] = \omega_2,$$

we have at once
$$\mathrm{K} = \omega_2.$$

Now $G\left(\frac{z}{\omega_1}\middle|\tau\right)$ satisfies the equation*

$$\frac{f(z+\omega_2)}{\Gamma\left(\frac{z}{\omega_1}\right)f(z)} = \tau^{-\frac{z}{\omega_1}+\frac{1}{2}}(2\pi)^{\frac{\tau-1}{2}},$$

where $\tau^{-\frac{z}{\omega_1}+\frac{1}{2}} = e^{\left(-\frac{z}{\omega_1}+\frac{1}{2}\right)\log\tau}$; and the principal value of $\log\tau$ is to be taken. (The same remark, of course, applies to every many-valued expression of this nature which occurs in the course of the investigation.)

Employing the relation (1) in conjunction with this equation and the equation

$$\frac{\Gamma_2^{-1}(z+\omega_2)}{\Gamma_2^{-1}(z)} = \frac{\Gamma_1(z\,|\,\omega_1)}{\sqrt{(2\pi/\omega_2)}}\,e^{-2m'\pi\iota\,S_1(z\,|\,\omega_1)},$$

we obtain

$$\Gamma_1(z\,|\,\omega_1)\,\omega_1^{\frac{1}{2}}(2\pi)^{-\frac{1}{2}}\;e^{-2m'\pi\iota\,S_1'(z\,|\,\omega_1)}$$

$$= e^{2r\pi\iota\tau}\,\omega_2^{\frac{z}{\omega_2}-\frac{1}{2}}(2\pi)^{-\frac{\omega_2}{2\omega_1}}\quad e^{-2m\pi\iota\left(\frac{z}{\omega_1}-\frac{1}{2}\right)}\Gamma(z\,|\,\omega_1)\omega_1^{1-\frac{z}{\omega_1}}\,\tau^{-\frac{z}{\omega_1}+\frac{1}{2}}(2\pi)^{\frac{\tau-1}{\tau}},$$

which reduces to

$$e^{2(m-m')\pi\iota\,S_1'(z\,|\,\omega_1)} = e^{2r\pi\iota\tau+S'_1(z\,|\,\omega_1)\,[\log\omega_2-\log\omega_1-\log\tau]},$$

But we have seen (§ 22) that

$$\log\omega_2-\log\omega_1-\log\tau = 2(m-m')\pi\iota$$

for $m-m'=0$, unless the difference of the amplitudes of ω_1 and ω_2 is greater than π, in which case $m-m'=\pm 1$, according as $-I\left(\frac{\omega_2}{\omega_1}\right)$ is positive or negative.

We thus find $r=0$, and incidentally we obtain a valuable verification of our results.

And now finally

$$\Gamma_2^{-1}(z) = \omega_2(2\pi)^{-\frac{z}{2\omega_1}}\left(\omega_2 e^{-2m\pi\iota}\right)^{{}_2S_0(z)}\,G\left(\frac{z}{\omega_1}\middle|\tau\right),$$

the relation between the two forms of double gamma functions.

§ 27. From the relation just found we may at once express the gamma modular constants $C(\tau)$ and $D(\tau)$ of the former theory in terms of $\gamma_{22}(\omega_1,\omega_2)$ and $\gamma_{21}(\omega_1,\omega_2)$ respectively.

For we have

$$G\left(\frac{z}{\omega_1}\middle|\tau\right) = e^{a\frac{z}{\omega_2}+b\frac{z^2}{2\omega_2^2}}\;.\;\frac{z}{\omega_2}\;.\;\prod_{m_1=0}^{\infty}\prod_{m_2=0}^{\infty}{}'\left[\left(1+\frac{z}{\Omega}\right)e^{-\frac{z}{\Omega}+\frac{z^2}{2\Omega^2}}\right],$$

where

$$a = \frac{\tau}{2}\log 2\pi\tau+\tfrac{1}{2}\log\tau+\gamma\tau-C(\tau),$$

$$b = -\tau\log\tau-\frac{2\pi\tau^2}{6}-\tau^2 D(\tau),$$

and also

$$\Gamma_2^{-1}(z) = e^{2\,\gamma_{21}+z\gamma_{22}}\;.\;z\;.\;\prod_{m_1=0}^{\infty}\prod_{m_2=0}^{\infty}{}'\left[\left(1+\frac{z}{\Omega}\right)e^{-\frac{z}{\Omega}+\frac{z^2}{2\Omega^2}}\right]$$

* "Genesis of the Double Gamma Functions," § 10.

Substituting in the relation

$$\Gamma_2^{-1}(z) = (2\pi)^{-\frac{z}{2\omega_1}\omega_2} \,.\, (\omega_2 e^{-2m\pi\iota})^{{}_2S_0(z)} \,.\, G\left(\frac{z}{\omega_1}\middle|\tau\right),$$

we find

$$\frac{z^2}{2}\gamma_{21}(\omega_1, \omega_2) + z\gamma_{22}(\omega_1, \omega_2) = -\frac{z}{2\omega_1}\log 2\pi + \left(\frac{z^2}{2\omega_1\omega_2} - \frac{z}{2\omega_1} - \frac{z}{2\omega_2} + 1\right)\log \omega_2$$
$$- 2m\pi\iota\left(\frac{z^2}{2\omega_1\omega_2} - \frac{z}{2\omega_1} - \frac{z}{2\omega_2}\right) + a\frac{z}{\omega_2} + b\frac{z^2}{2\omega_2^2}.$$

And hence equating coefficients of z and z^2 we find

$$\frac{\gamma_{21}(\omega_1, \omega_2)}{2} = \frac{1}{2\omega_1\omega_2}\{\log \omega_2 - 2m\pi\iota\} + \frac{b}{2\omega_2^2},$$

$$\gamma_{22}(\omega_1, \omega_2) = -\frac{1}{2\omega_1}\{\log 2\pi\omega_2 - 2m\pi\iota\} - \frac{1}{2\omega_2}\{\log \omega_2 + 2m\pi\iota\} + \frac{a}{\omega_2}.$$

Thus*

$$D(\tau) = -\omega_1^2\gamma_{21}(\omega_1, \omega_2) + \frac{\omega_1}{\omega_2}\{\log \omega_2 - \log \tau\} - \frac{\pi_2}{6} - \frac{\omega_1}{\omega_2}2m\pi\iota,$$

$$C(\tau) = -\omega_1\gamma_{22}(\omega_1, \omega_2) - \left(\tfrac{1}{2} + \frac{\omega_1}{2\omega_2}\right)\{\log \omega_2 - \log \tau - 2m\pi\iota\} + \gamma,$$

which are the relations required.

Since $\qquad \log \omega_2 - \log \tau - 2m\pi\iota = \log \omega_1 - 2m'\pi\iota,$

we may evidently write these relations in the form

$$D(\tau) = -\omega^2{}_1\gamma_{21}(\omega_1, \omega_2) + \frac{\omega_1}{\omega_2}\{\log \omega_1 - 2m'\pi\iota\} - \frac{\pi_2}{6},$$

$$C(\tau) = -\omega_1\gamma_{22}(\omega_1, \omega_2) - \left(\tfrac{1}{2} + \frac{\omega_1}{2\omega_2}\right)\{\log \omega_1 - 2m'\pi\iota\} + \gamma.$$

§ 28. We will now show that, when the parameters ω_1 and ω_2 are equal, we have

$$\gamma_{21}(\omega, \omega) = \frac{1}{\omega^2}\left[\log \omega - 1 - \gamma - \frac{\pi^2}{6}\right],$$

and $\qquad \gamma_{22}(\omega, \omega) = \frac{1}{\omega}[\gamma - \tfrac{1}{2} - \log \omega].$

By the definition formula of $\gamma_{21}(\omega_1, \omega_2)$, we at once have

$$\omega^2\gamma_{21}(\omega, \omega) = -\underset{n=\infty}{Lt}\left[\sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}' \frac{1}{(m_1 + m_2)^2} - \log\frac{n\omega}{2}\right].$$

Group together all terms for which $m_1 + m_2 = \epsilon$, and we have

$$\sum^{n}\sum^{n}{}' \frac{1}{(m_1 + m_2)^2} = \sum_{\epsilon=1}^{n}\frac{\epsilon + 1}{\epsilon^2} + \frac{n}{(n+1)^2} + \frac{n-1}{(n+2)^2} + \ldots + \frac{1}{(n+n)^2},$$

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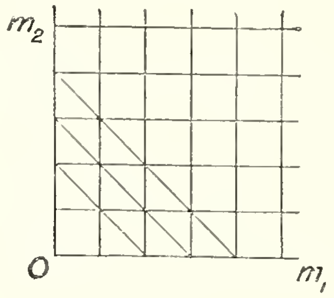
* "Genesis of the Double Gamma Functions," § 6.

for we may suppose that the terms are represented by the corners of the small squares into which the positive quadrant is divided—in the new grouping we take together all terms lying on a line equally inclined to the two axes.



Thus
$$\sum_0^n \sum_0^n{}' \frac{1}{(m_1+m_2)} = \sum_{\epsilon=1}^n \left(\frac{1}{\epsilon}+\frac{1}{\epsilon^2}\right) + \frac{1}{(n+1)^2} + \ldots + \frac{1}{(n+n)^2}$$
$$+ \frac{n-1}{(n+1)^2} + \frac{n-2}{(n+2)^2} + \ldots + \frac{n-n}{(n+n)^2}.$$

So that when n is very large
$$\sum_0^n \sum_0^n{}' \frac{1}{(m_1+m_2)^2} = \log n + \gamma + \frac{\pi^2}{6} + \int_0^1 \frac{1-x}{(1+x)^2} dx$$
$+$ terms which vanish when n becomes infinite
$$= \log n + \gamma + \frac{\pi_2}{6} + 1 - \log 2 + \text{similar terms.}$$

Hence
$$\gamma_{21}(\omega, \omega) = \frac{1}{\omega^2}\left[\log \omega - \frac{\pi_2}{6} - 1 - \gamma\right],$$
which is the first relation,

In the second place we have, when ω_1 and ω_2 are equal,
$$\omega\gamma_{22}(\omega, \omega) = \underset{n=\infty}{\mathrm{Lt}} \left[\sum_0^n \sum_0^n{}' \frac{1}{m_1+m_2} - 2n \log 2 - \log 2n\omega\right],$$
and by the same method of reasoning as before
$$\sum_0^n \sum_0^n{}' \frac{1}{m_1+m_2} = \sum_{\epsilon=1}^n \frac{\epsilon+1}{\epsilon} + \frac{n}{n+1} + \frac{n-1}{n+2} + \ldots + \frac{1}{n+n},$$
$$= \sum_{\epsilon=1}^n \frac{1}{\epsilon} + n + \frac{1}{n+1} + \ldots + \frac{1}{n+n}$$
$$+ n\left\{2\left(\frac{1}{n+1} + \frac{1}{n+2} + \ldots + \frac{1}{n+n}\right) - \frac{n}{n}\right\}$$
$$= n + \log n + \gamma + \log 2 + 2n \log 2 - \tfrac{1}{2} - n,$$
neglecting terms which vanish when n is infinite, and thus
$$\gamma_{22}(\omega, \omega) = \frac{1}{\omega}\{\gamma - \tfrac{1}{2} - \log \omega\}.$$

It is an interesting piece of work to show that these results accord with those previously obtained for the G function of parameter unity.

§ 29. We proceed now to write down the expansion of $\log \Gamma_2(z)$ and the first few of the derived functions $\psi_2^{(r)}(z)$ in the vicinity of $z = 0$.

We have, by definition,

$$\psi_2^{(3)}(z \mid \omega_1, \omega_2) = -2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty}{}' \frac{1}{(z+\Omega)^3}, \text{ where } \Omega = m_1\omega_1 + m_2\omega_2,$$

Since the series on the right-hand side is absolutely convergent, we may expand in the form

$$\psi_2^{(3)}(z) = -\frac{2}{z^3} - 2\left\{\sum_0^{\infty}\sum{}' \frac{1}{\Omega^3} - \sum_0^{\infty}\sum{}' \frac{3z}{\Omega^4} + \sum_0^{\infty}\sum{}' \frac{3 \cdot 4}{1 \cdot 2}\frac{z^2}{\Omega^5} \quad \dots \right\}.$$

Hence, integrating,

$$\psi_2^{(2)}(z) = \frac{1}{z^2} - \gamma_{21}(\omega_1, \omega_2) - 2\sum_0^{\infty}\sum{}' \frac{z}{\Omega^3} + 3\sum_0^{\infty}\sum{}' \frac{z^2}{\Omega^4} - 4\sum_0^{\infty}\sum{}' \frac{z^3}{\Omega^5} + \dots$$

the constant being determined by making $z = o$.

Thus integrating again, and determining the constant in the same manner,

$$\psi_2^{(1)}(z) = -\frac{1}{z} - \gamma_{21}(\omega_1, \omega_2)z - \gamma_{22}(\omega_1, \omega_2) - \sum_0^{\infty}\sum{}' \frac{z^2}{\Omega^3} + \sum_0^{\infty}\sum{}' \frac{z^3}{\Omega^4} - \dots$$

and finally,

$$\log \Gamma_2(z) = -\log z - z\gamma_{22} - \frac{z^2\gamma_{21}}{2} - \sum^{\infty}\sum{}' \frac{z^3}{3\Omega^3} + \sum_0^{\infty}\sum{}' \frac{z^4}{4\Omega^4} - \sum_0^{\infty}\sum{}' \frac{z^5}{5\Omega^5} \quad \dots$$

the expansion holding good within a circle of radius just less than the least value of

$$|m_1\omega_1 + m_2\omega_2| \qquad \left.\begin{matrix} m_1 = 0, 1, 2, \dots \infty \\ m_2 = 0, 1, 2, \dots \infty \end{matrix}\right\} \qquad \left.\begin{matrix} 0 \\ 0 \end{matrix}\right\}\text{excluded.}$$

We note that by EISENSTEIN'S Theorem each coefficient in the series is an absolutely convergent series.

§ 30. We proceed now to the expressions for the double gamma functions as simply infinite products of simple gamma functions.

Consider the product

$$P(z) = \Gamma_1(z \mid \omega_1) \prod_{m_2=1}^{\infty} \left[\frac{\Gamma_1(z + m_2\omega_2 \mid \omega_1)}{\Gamma_1(m_2\omega_2 \mid \omega_1)} e^{-\psi_1^{(1)}(m_2\omega_2 \mid \omega_1) - \frac{z^2}{2}\psi_1^{(2)}(m_2\omega_2 \mid \omega_1)}\right].$$

The typical term may be written

$$e^{\frac{z^3}{3!}\psi_1^{(3)}(m_2\omega_2 \mid \omega_1) + \frac{z^4}{4!}\psi_1^{(4)}(m_2\omega_2 \mid \omega_1) + \dots}$$

and the series $\sum_{m_2=1}^{\infty} \psi_1^{(r)}(m_2\omega_2 \mid \omega_1)$ are absolutely convergent when $r \geqq 3$. The product is therefore in general absolutely convergent.

Again, it has no finite zeros, for its zeros would be those of $\Gamma_1(z + m_2\omega_2 \mid \omega_1)$ for $m_2 = 0, 1, \ldots \infty$. And its poles are given by

$$z + m_1\omega_1 + m_2\,\nu_2 = 0 \quad \begin{cases} m_1 = 0, 1, \ldots \infty. \\ m_2 = 0, 1, \ldots \infty. \end{cases}$$

Thus $\dfrac{\Gamma_2(z)}{\mathrm{P}(z)}$ has no zeros or poles in the finite part of the plane.

Change now z into $z + \omega_1$, and we have

$$\frac{\mathrm{P}(z + \omega_1)}{\mathrm{P}(z)} = z \prod_{m_2=1}^{\infty} \left[\left(1 + \frac{z}{m_2\omega_2}\right) e^{-\frac{z}{m_2\omega_2}} \right]$$
$$\times \prod_{m_2=1}^{\infty} \left[m_2\omega_2 \,.\, e^{-\omega_1\psi_1^{(1)}(m_2\omega_2 \mid \omega_1) - \frac{2\omega_1 z + \omega_1^2}{2}\psi_1^{(2)}(m_2\omega_2 \mid \omega_1) + \frac{z}{m_2\omega_2}} \right].$$

Now the product last written must be convergent, for all other terms of the identity are finite for finite values of $|z|$, and this product is evidently of the form e^{pz+q}, where p and q are functions of ω_1 and ω_2 only.

Hence we must have

$$\mathrm{P}(z + \omega_1) = \mathrm{P}(z)\,\Gamma_1^{-1}(z \mid \omega_2)\, e^{\alpha z + \beta}.$$

Now we have proved that

$$\Gamma_2(z + \omega_1) = \Gamma_2(z)\,\Gamma_1^{-1}(z \mid \omega_2)\,.\left(\frac{2\pi}{\omega_2}\right)^{\frac{1}{2}} e^{2m\pi\iota\left(\frac{z}{\omega_2} - \frac{1}{2}\right)}.$$

Thus if we put

$$f(z) = \frac{\Gamma_2(z)}{\mathrm{P}(z)},$$

we shall have

$$\frac{f(z + \omega_1)}{f(z)} = e^{\alpha_1 z + \beta_1},$$

and similarly we shall have

$$\frac{f(z + \omega_2)}{f(z)} = e^{\alpha_2 z + \beta_2}.$$

Hence $f(z)$ is a doubly periodic function of the third kind, with no finite zeros or poles.

Thus we must have

$$f(z) = \mathrm{C}e^{\mathrm{A}z^2 \times \mathrm{B}z},$$

for in HERMITE's expression of such a function the σ functions are each associated with a finite non-congruent zero.*

To determine C we put $z = 0$, and obtain

$$\mathrm{C} = \underset{z=0}{\mathrm{L}t}\left[\frac{\Gamma_2(z)}{\Gamma_1(z \mid \omega_1)}\right] = 1.$$

Differentiating logarithmically, the identity

$$\Gamma_2(z_1 \mid \omega, \omega_2) = \mathrm{P}(z)\, e^{\mathrm{A}z^2 + \mathrm{B}z},$$

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* FORSYTH, 'Theory of Functions,' § 142.

we find on making $z = 0$,

$$-\gamma_{22}(\omega_1, \omega_2) = \mathrm{B} + \frac{1}{\omega_1}(\log \omega_1 - \gamma),$$

and therefore
$$\mathrm{B} = -\gamma_{22}(\omega_1, \omega_2) + \frac{\gamma}{\omega_1} - \frac{\log \omega_1}{\omega_1}.$$

Differentiating again, we have

$$\psi_2^{(2)}(z \mid \omega_1, \omega_2) = 2\mathrm{A} + \psi_1^{(2)}(z \mid \omega_1) + \sum_{m_2=1}^{\infty} \left[\psi_1^{(2)}(z + m_2\omega_2 \mid \omega_1) - \psi_1^{(2)}(m_2\omega_2 \mid \omega_1)\right].$$

Again, making $z = 0$, we find

$$-\gamma_{21}(\omega_1, \omega_2) = 2\mathrm{A},$$

or
$$\mathrm{A} = -\tfrac{1}{2}\gamma_{21}(\omega_1, \omega_2).$$

Finally, then, we have

$$\Gamma_2(z \mid \omega_1, \omega_2) = e^{-\frac{\gamma_{21}z^2}{2} - z\left\{\gamma_{22} + \frac{1}{\omega_1}\log\omega_1 - \frac{\gamma}{\omega_1}\right\}} \times \Gamma(z \mid \omega_1)$$
$$\times \prod_{m_2=1}^{\infty} \left[\frac{\Gamma_1(z + m_2\omega_2 \mid \omega_1)}{\Gamma_1(m_2\omega_2 \mid \omega_1)} e^{-z\psi_1^{(1)}(m_2\omega_2 \mid \omega_1) - \frac{z^2}{2}\psi_1^{(2)}(m_2\omega_2 \mid \omega_1)}\right].$$

This formula is equivalent to the one obtained in the "Genesis of the Double Gamma Functions," § 2.

It is an interesting verification to actually transform the one formula into the other, making use of the relations established between $\gamma_{21}(\omega_1, \omega_2)$, $\gamma_{22}(\omega_1, \omega_2)$, $\mathrm{C}(\tau)$ and $\mathrm{D}(\tau)$.

On account of the symmetry of the present functions, the formula corresponding to that given in § 8 of the "Genesis of the Double Gamma Functions" may be written

$$\Gamma_2(z \mid \omega_1, \omega_2) = e^{-\gamma_{21}\frac{z}{2} - z\left\{\gamma_{22} + \frac{1}{\omega_2}\log\omega_2 - \frac{\gamma}{\omega_2}\right\}} \,.\, \Gamma_1(z \mid \omega_2)$$
$$\times \prod_{m_1=1}^{\infty} \left[\frac{\Gamma_1(z + m_1\omega_1 \mid \omega_2)}{\Gamma_1(m_1\omega_1 \mid \omega_2)} e^{-z\psi_1^{(1)}(m_1\omega_1 \mid \omega_2) - \frac{z^2}{2}\psi_1^{(2)}(m_1\omega_1 \mid \omega_2)}\right].$$

The product formulæ just obtained correspond to the expression of the σ function as an infinite product of circular functions.

Such a circumstance, of course, at once prompts us to try and find a formula corresponding to the expression of the σ function as a *sum* of exponential functions. But it is readily seen that such is an impossibility. We cannot express the double gamma function as a sum of an infinite series of simple gamma functions of varying arguments.

It is this fact, combined with the absence of any quasi-addition theorem for the double gamma functions, which precludes the possibility of any collection of formulæ rivalling in number and elegance those of the doubly periodic functions.

§ 31. We proceed now to express WEIERSTRASS' elliptic functions in terms of double gamma functions.

In WEIERSTRASS' notation of elliptic functions we have

$$\wp'(z) = -2\sum_{m_1}\sum_{m_2}{}_{-\infty}^{\infty}\frac{1}{(z+\Omega)^3},$$

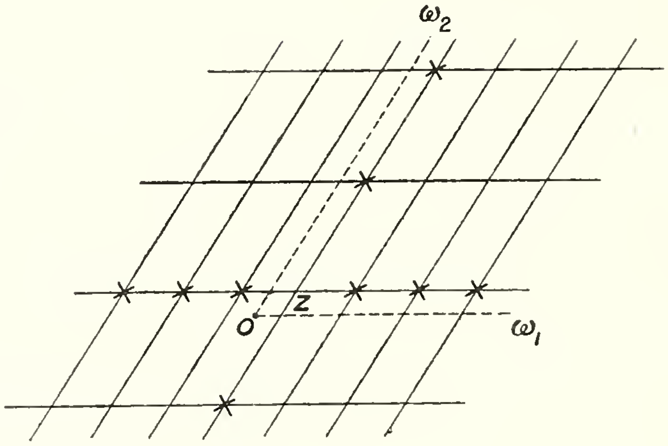
where $\Omega = m_1\omega_1 + m_2\omega_2$, and ω_1 and ω_2 are the periods of $\wp(z)$.

Now by definition

$$\psi_2^{(3)}(z\,|\,\omega_1, \omega_2) = -2\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{1}{(z+\Omega)^3}.$$

Representing the various terms by the corners of the parallelograms of the figure, we readily see that

$$\psi_2^{(3)}(z\,|\,\omega_1, \omega_2) + \psi_2^{(3)}(z\,|-\omega_1, \omega_2) + \psi_2^{(3)}(z\,|\,\omega_1, -\omega_2) + \psi_2^{(3)}(z\,|-\omega_1, -\omega_2)$$
$$= -2\left[\sum\sum_{-\infty}^{\infty}\frac{1}{(z+\Omega)^3} + \sum_{-\infty}^{\infty}\frac{1}{(z+m_1\omega_1)^3} + \sum_{-\infty}^{\infty}\frac{1}{(z+m_2\omega_2)^3} + \frac{1}{z^3}\right].$$



Hence, using the natural summation $\Sigma\psi_2^{(3)}(z\,|\pm\omega_1, \pm\omega_2)$ to express the left-hand side of this relation, we have

$$\Sigma\psi_2^{(3)}(z\,|\pm\omega_1, \pm\omega_2) = \wp'(z) + \Sigma\psi_1^{(3)}(z\,|\pm\omega_1) + \Sigma\psi_1^{(3)}(z\,|\pm\omega_2) + \frac{2}{z^3},$$

and therefore, on integration,

$$\wp(z) = \Sigma\psi_2^{(2)}(z\,|\pm\omega_1, \pm\omega_2) - \Sigma\psi_1^{(2)}(z\,|\pm\omega_1) - \Sigma\psi_1^{(2)}(z\,|\pm\omega_2) + \frac{1}{z^2} + \nu,$$

where ν is constant with respect to z.

Evidently

$$\nu = \Sigma\gamma_{21}(\pm\omega_1, \pm\omega_2) + 2\sum_{1}^{\infty}\frac{1}{(n\omega_1)^2} + 2\sum_{1}^{\infty}\frac{1}{(n\omega_2)^2}.$$

Now

$$\gamma_{21}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{Lt}}\left[\frac{1}{\omega_1\omega_2}\left[\log n\omega_2 + \log n\omega_1 - \log n(\omega_1+\omega_2)\right] - \sum_{m_1=0}^{n}\sum_{m_2=0}^{n'}\frac{1}{(m_1\omega_1+m_2\omega_2)^2}\right],$$

where the principal values of the logarithms are to be taken.

And hence

$$\nu = \underset{n=\infty}{\mathrm{L}t}\left[-\sum_{m_1=-n}^{n}\sum_{m_2=-n}^{n}\frac{1}{\Omega^2}+\frac{1}{\omega_1\omega_2}\left\{\begin{array}{l}\log(\omega_1+\omega_2)+\log[-(\omega_1+\omega_2)]\\-\log(\omega_1-\omega_2)-\log[-(\omega_1-\omega_2)]\end{array}\right\}\right].$$

Now, as may be readily seen by examining the different possible cases in a diagram,

$$\log(\omega_1+\omega_2)+\log[-(\omega_1+\omega_2)]-\log(\omega_1-\omega_2)-\log[-(\omega_1-\omega_2)]$$
$$= 2\log\frac{\omega_1+\omega_2}{\omega_1 \backsim \omega_2},$$

where that value of $\omega_1 \backsim \omega_2$ is to be taken which is on the same side of the real axis as $\omega_1+\omega_2$.

With this proviso,

$$\nu = -\sum_{-\infty}\sum{}'\frac{1}{\Omega^2}+\frac{2}{\omega_1\omega_2}\log\frac{\omega_1+\omega_2}{\omega_1 \backsim \omega_2},$$

the infinities in the double summation being equal in absolute magnitude.

§ 32. We may now express WEIERSTRASS' ζ function in terms of derivatives of simple and double gamma functions. For on integrating the relation obtained in the previous paragraph we find,

remembering that $\qquad \dfrac{d}{dz}\zeta(z) = -\wp(z),$*

$$-\zeta(z) = \Sigma\psi_1^{(1)}(z\,|\pm\omega_1,\pm\omega_2)-\Sigma\psi_1^{(1)}(z\,|\pm\omega_1)-\Sigma\psi_1^{(1)}(z\,|\pm\omega_2)-\frac{1}{z}+\nu z+\mu,$$

where μ is constant with respect to z.

Making then $z = 0$, we find

$$o = -\Sigma\gamma_{22}(\pm\omega_1,\pm\omega_2)-\frac{1}{\omega_1}\{\log(\omega_1)-\gamma\}+\frac{1}{\omega_1}\log(-\omega_1)-\gamma\}$$
$$-\frac{1}{\omega_2}\{\log\omega_2-\gamma\}+\frac{1}{\omega_2}\{\log(-\omega_2)-\gamma\}+\mu.$$

Now by § 23,

$$\Sigma\gamma_{22}(\pm\omega_1,\pm\omega_2) = \frac{\omega_1+\omega_2}{2\omega_1\omega_2}\left\{\begin{array}{l}\log(\omega_1+\omega_2)-\log\omega_1-\log\omega_2\\-\log[-(\omega_1+\omega_2)]+\log(-\omega_1)+\log(-\omega_2)\end{array}\right\}$$
$$+\frac{\omega_1-\omega_2}{2\omega_1\omega_2}\left\{\begin{array}{l}\log(\omega_2-\omega_1)-\log(-\omega_1)-\log\omega_2\\-\log(\omega_1-\omega_2)+\log\omega_1+\log(-\omega_2)\end{array}\right\}$$
$$=\left(\frac{1}{2\omega_1}+\frac{1}{2\omega_2}\right)\{\log(\omega_1+\omega_2)-\log[-(\omega_1+\omega_2)]\}$$
$$+\frac{1}{\omega_1}\{\log(-\omega_1)-\log\omega_1\}+\frac{1}{\omega_2}\{\log(-\omega_2)-\log\omega_2\}$$
$$-\left\{\frac{1}{2\omega_1}-\frac{1}{2\omega_2}\right\}\{\log(\omega_2-\omega_1)-\log(\omega_1-\omega_2)\}.$$

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* JORDAN, 'Cours d'Analyse,' 2nd edition, p. 347. Note that JORDAN uses $2\omega_1$ and $2\omega_2$ instead of ω_1 and ω_2 of this paper.

Hence $$\mu = \left(\frac{1}{2\omega_1} + \frac{1}{2\omega_2}\right)\{\log(\omega_1 + \omega_2) - \log[-(\omega_1 + \omega_2)]\}$$
$$-\left\{\frac{1}{2\omega_1} - \frac{1}{2\omega_2}\right\}\{\log(\omega_2 - \omega_1) - \log(\omega_1 - \omega_2)\}.$$

Now $$\log(\omega_1 + \omega_2) - \log[-(\omega_1 + \omega_2)] = \pm \pi\iota,$$

according as $I(\omega_1 + \omega_2)$ is positive or negative, and
$$\log(\omega_2 - \omega_1) - \log(\omega_1 - \omega_2) = \mp \pi\iota,$$
according as $I(\omega_1 - \omega_2)$ is positive or negative. Therefore the values of μ are given in the following table :—

	$I(\omega_1 + \omega_2)$ positive.	$I(\omega_1 + \omega_2)$ negative.
$I(\omega_1 - \omega_2)$ positive	$\mu = \frac{\pi\iota}{\omega_1}$	$\mu = -\frac{\pi\iota}{\omega_2}$
$I(\omega_1 - \omega_2)$ negative	$\mu = \frac{\pi\iota}{\omega_2}$	$\mu = -\frac{\pi\iota}{\omega_1}$

In other words,

if $|I(\omega_1)| > |I(\omega_2)|$, $\mu = \pm \pi\iota/\omega_1$, the upper or lower sign being taken according as $I(\omega_1)$ is positive or negative, and

if $|I(\omega_2)| > |I(\omega_1)|$, $\mu = \pm \pi\iota/\omega_2$, the upper or lower sign being taken according as $I(\omega_2)$ is positive or negative.

§ 33. The expression for $\sigma(z)$ in terms of simple and double gamma functions is now immediate.

For on integrating the result of the previous paragraph
$$-\log\sigma(z) = \rho + \mu z + \frac{\nu z^2}{2} - \log z + \Sigma\log\Gamma_2(z\,|\pm\omega_1, \pm\omega_2) - \Sigma\log\Gamma_1(z\,|\pm\omega_1)$$
$$- \Sigma\log\Gamma_1(z\,|\pm\omega_2),$$
ρ being the constant of integration.

Make now $z = 0$, and we at once see that $\rho = 0$.

Hence $$\sigma(z) = e^{-\mu z - \nu\frac{z^2}{2}} \,.\, z \,.\, \frac{\Pi\Gamma_2^{-1}(z\,|\pm\omega_1, \pm\omega_2)}{\Pi\Gamma_1^{-1}(z\,|\pm\omega_1)\Pi\Gamma_1^{-1}(z\,|\pm\omega_2)},$$
and in this expression
$$\nu = -\sum_{-\infty}^{\infty}\sum{}' \frac{1}{\Omega^2} + \frac{z}{\omega_1\omega_2}\log\frac{\omega_1 + \omega_2}{\omega_1 \backsim \omega_2},$$

the infinities in the double summation being equal in absolute magnitude, and that value of $\omega_1 \backsim \omega_2$ being taken which is on the same side of the real axis as $\omega_1 + \omega_2$; while

$\mu = \pm \frac{\pi\iota}{\omega_1}$ if $|\,I(\omega_1)| > |\,I(\omega_2)|$, the upper or lower sign being taken according as $I(\omega_1)$ is positive or negative,

and $\mu = \pm \frac{\pi\iota}{\omega_2}$ if $|\,I(\omega_2)\,| > |I(\omega_1)|$, the upper or lower sign being taken according as $I(\omega_2)$ is positive or negative.

§ 34. By means of the preceding paragraphs we may now at once prove WEIERSTRASS' relation*

$$\omega_2\eta_1 - \omega_1\eta_2 = \pm 2\pi\iota,$$

the upper or lower sign being taken according as $I\left(\frac{\omega_2}{\omega_1}\right)$ is positive or negative, where η_1 and η_2 are determined by the relations

$$\zeta(z + \omega_1) = \zeta(z) + \eta_1$$
$$\zeta(z + \omega_2) = \zeta(z) + \eta_2.$$

Take the expression for $\zeta(z)$ given in § 32, and we find by use of the formulæ of § 22 that

$$\zeta(z) - \zeta(z + \omega_1) = \frac{2\pi\iota}{\omega_2}(m - m_1 - m_2 + m_3) + \nu\omega_1,$$

where

$m_1 = 0$, unless $\left.\begin{array}{l}\omega_1 \text{ does} \\ \omega_1 - \omega_2 \text{ does not}\end{array}\right\}$ lie within the region bounded by axes from the origin to -1 and ω_2, in which case $m_1 = \pm 1$, according as $-I(\omega_2)$ is positive or negative, and m_2 and m_3 are obtained by changing the signs of (i) both ω_1 and ω_2 and (ii) ω_1 respectively in this formula.

Thus $$\frac{\eta_1}{\omega_1} = \sum_{-\infty}\sum{}' \frac{1}{\Omega^2} - \frac{2}{\omega_1\omega_2}\log\frac{\omega_1 + \omega_2}{\omega_1 \sim \omega_2} - \frac{2\pi\iota}{\omega_1\omega_2}[m - m_1 - m_2 + m_3],$$

the infinities in the double summation being equal in absolute magnitude, and that value of $\omega_1 \sim \omega_2$ being taken which is on the same side of the real axis as $(\omega_1 + \omega_2)$.

And, similarly,

$$\zeta(z + \omega_2) - \zeta(z) = \eta_2,$$

where $$\frac{\eta_2}{\omega_2} = \sum_{-\infty}^{\infty}\sum{}' \frac{1}{\Omega^2} - \frac{2}{\omega_1\omega_2}\log\frac{\omega_1 + \omega_2}{\omega_1 \sim \omega_2} + \frac{2\pi\iota}{\omega_1\omega_2}[- m' + m_1' + m_2' - m_3'],$$

where m' has its usual meaning, and m_r' is obtained from m' just as is m_r from m.

Hence $$\frac{\eta_1}{\omega_1} - \frac{\eta_2}{\omega_2} = \frac{2\pi\iota}{\omega_1\omega_2}[-(m - m') + (m_1 - m_1') + (m_2 - m_2') - (m_3 - m_3')].$$

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* JORDAN, 'Cours d'Analyse,' p. 351; FORSYTH, 'Theory of Functions,' § 129. Again notice that each of the quantities η and ω is double that usually taken.

But we have seen in § 22 that, if the difference of amplitudes of ω_1 and ω_2 be greater than π,

$$m - m' = \pm 1,$$

the upper or lower sign being taken according as $\mathrm{I}\left(\frac{\omega_1}{\omega_2}\right)$ is positive or negative.

Similarly it may be proved that if the difference of amplitude of ω_1 and $-\omega_2$ is $> \pi$,

$$m_1 - m_1' = \pm 1, \text{ according as } \mathrm{I}\left(\frac{\omega_2}{\omega_1}\right) \text{ is } +ve \text{ or } -ve.$$

And . . . . $-\omega_1$ and ω_2 is $> \pi$.

$$m_2 - m_2' = \pm 1 \quad . \quad . \quad . \quad . \quad \mathrm{I}\left(\frac{\omega_2}{\omega_1}\right) \text{ is } +ve \text{ or } -ve;$$

while, finally, ... $-\omega_1$ and $-\omega_2$ is $> \pi$.

$$m_3 - m_3' = \mp 1 \quad . \quad . \quad . \quad . \quad \mathrm{I}\left(\frac{\omega_3}{\omega_1}\right) \text{ is } +ve \text{ or } -ve.$$

In all other cases the differences between corresponding m's vanish.

But the difference of amplitude of one, and of only one, pair of the set

$$\omega_1, \omega_2;\ \omega_1, -\omega_2;\ -\omega_1, \omega_2;\ -\omega_1, -\omega_2$$

can be greater than π.

Hence $$-(m - m') + (m_1 - m_1') + (m_2 - m_2') - (m_3 - m_3')$$

must always equal ± 1.

And it is easy to see, by taking the particular cases which can arise, that the upper or lower sign must be taken according as $\mathrm{I}(\omega_2/\omega_1)$ is positive or negative.

We have then finally

$$\frac{\eta_1}{\omega_1} - \frac{\eta_2}{\omega_2} = \pm \frac{2\pi\iota}{\omega_1\omega_2},$$

and therefore $$\eta_1\omega_2 - \eta_2\omega_1 = \pm 2\pi\iota,$$

the upper or lower sign being taken according as $\mathrm{I}\left(\frac{\omega_2}{\omega_1}\right)$ is positive or negative.

§ 35. For brevity, we merely indicate the relation of the formulæ which we have found to the known relations:

$$\sigma(z + \omega_1) = -e^{\eta_1\left(z + \frac{\omega_1}{2}\right)}\sigma(z)$$

$$\sigma(z + \omega_2) = -e^{\eta_2\left(z + \frac{\omega_2}{2}\right)}\sigma(z).$$

By § 33

$$\frac{\sigma(z + \omega_1)}{\sigma(z)} = e^{-\mu\omega_1 - \nu\frac{2z\omega_1 + \omega_1^2}{2}} \cdot \frac{z + \omega_1}{z} \cdot \frac{\Pi\left\{\frac{\Gamma_2^{-1}(z + \omega_1 \mid \pm\omega_1, \pm\omega_2)}{\Gamma_2^{-}(z \mid \pm\omega_1, \pm\omega_2)}\right\}}{\Pi\frac{\Gamma_1^{-1}(z + \omega_1 \mid \pm\omega_1)}{\Gamma_1^{-1}(z \mid \pm\omega_1)}\ \Pi\frac{\Gamma_1^{-1}(z + \omega_1 \mid \pm\omega_2)}{\Gamma_1^{-1}(z \mid \pm\omega_2)}};$$

and by the previous paragraph, the expression on the right-hand side of this relation will evidently reduce to the form $e^{\eta_1 z + \delta_1}$, where δ_1 is some constant whose value may be readily seen to be given by

$$\frac{\delta_1 - \frac{1}{2}\eta_1\omega_1}{\omega_1} = -\mu + \pi\iota\,(m - m_3)\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right) + \pi\iota\,(m_2 - m_1)\left(\frac{1}{\omega_2} - \frac{1}{\omega_1}\right).$$

This value simplifies on detailed consideration.

When $|\mathrm{I}(\omega_1)| > |\mathrm{I}(\omega_2)|$,

$$\delta_1 - \tfrac{1}{2}\eta_1\omega_1 = \mp \pi\iota,$$

according as $\mathrm{I}(\omega_1)$ is positive or negative.

When $|\mathrm{I}(\omega_2)| > |\mathrm{I}(\omega_1)|$, there are four subsidiary possibilities ;—

(α) When ω_1 lies within the axes to -1 and $-\omega_2$,
(β) ,, ω_1 ,, ,, ,, -1 and ω_2,
(γ) ,, $-\omega_1$,, ,, ,, -1 and $-\omega_2$,
and (δ) ,, $-\omega_1$,, ,, ,, -1 and ω_2.

In cases (α) and (δ) $\delta_1 - \frac{1}{2}\eta_1\,\omega_1 = \pm\,\pi\iota/\omega_1$, the upper or lower sign being taken according as $\mathrm{I}(\omega_2)$ is positive or negative ; and in cases (β) and (γ) the upper and lower signs are interchanged.

We thus see that in all cases

$$e^{\delta_1} = -\,e^{-\frac{1}{2}\eta_1\omega_2},$$

so that we have the required equation

$$\sigma\,(z + \omega_1) = -\,e^{\eta_1(z + \frac{1}{2}\omega_1)}\,\sigma\,(z).$$

Similarly we find $$\sigma\,(z + \omega_2) = -\,e^{\eta_2(z + \frac{1}{2}\omega_2)}\,\sigma\,(z).$$

The verification of these results affords substantial proof of the general correctness of the signs which are involved in the work.

§ 36. It is interesting finally to notice that just as the gamma functions do not exist when $\tau = \omega_2/\omega_1$ is real and negative, so the elliptic functions demand that τ shall not be real.

The condition that τ must not be real and negative arose explicitly at several stages, and might have been predicted *à priori.*

For, when ω_2/ω_1 is real and negative, it is obvious that

$$\Omega = m_1\omega_1 + m_2\omega_2 \left\{\begin{matrix} m_1 = 0,\ 1,\ \ldots,\ \infty & 0 \\ m_2 = 0,\ 1,\ \ldots,\ \infty & 0 \end{matrix}\right\}\text{excluded.}$$

will have a zero value at least once.

And thus the function

$$\Gamma_2^{-1}(z) = e^{\frac{z^2}{2}\gamma_{21} + z\gamma_{22}}\,.\,z\,.\prod_{m_1=0}^{\infty}\ \prod_{m_2=0}^{\infty}{}'\left[\left(1 + \frac{z}{\Omega}\right)e^{-\frac{z}{\Omega} + \frac{z^2}{2\Omega^2}}\right]$$

will be infinite independently of z ; that is to say, it ceases to exist.

For all other values of ω_1 and ω_2 the function $\Gamma_2^{-1}(z \mid \omega_1, \omega_2)$ exists. But the product

$$\Pi\Gamma_2(z \mid \pm\omega_1, \pm\omega_2),$$

and consequently $\sigma(z)$, will not exist when either ω_2/ω_1 or $-\omega_2/\omega_1$ is real and negative; that is, when τ is real. The criterion for the existence of multiple gamma functions (n-ple where n is greater than 2) is more intricate, and, as we know, n-ply periodic functions ($n > 2$) do not exist.*

PART III.

Contour Integrals connected with the Double Gamma Function. The Double Riemann Zeta Function.

§ 37. In the theory of the simple gamma function it was shown that the intervention of a definite integral, coupled with the theory of asymptotic approximations, it was possible to obtain contour and line integrals to express EULER's constant γ, and the logarithm of the simple gamma function and its derivatives. We now proceed to show that it is possible to extend the method thus previously employed so as to obtain expressions as contour and line integrals for the gamma modular constants γ_{21} and γ_{22}, and the logarithm of the double function and its derivatives. It will be noticed that when the real parts of ω_1 and ω_2 are positive, the numbers m and m' which intervened in Part II. vanish, and there is consequently a noteworthy simplification of the formulæ obtained. This simplification extends also to the definite integral expressions, and consequently we shall first investigate the theory in this simple case, proceeding subsequently to contour integrals of greater complexity. Finally we make use of an extension of MELLIN's method of defining the simple ζ function by a series instead of a contour integral, and we show that there is complete agreement between the formulæ obtained in the different ways.

§ 38. When the real parts of ω_1 and ω_2 are positive, and when in addition the real part of a is positive, we define the double Riemann ζ function

$$\zeta_2(s, a \mid \omega_1, \omega_2)$$

for all values, real or complex of s, by the integral

$$\frac{i\Gamma(1-s)}{2\pi}\int\frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})},$$

wherein $(-z)^{s-1} = e^{(s-1)\log(-z)}$, $\log(-z)$ being real when z is negative and being rendered uniform by a cut along the positive direction of the real axis. The integral is to be taken along a contour enclosing the origin (but no other pole of the

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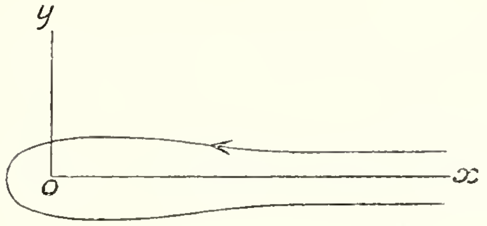
* The existence-criteria for functions which are substantially multiple gamma functions have been discussed by CRANI, 'Batt. Gior.,' vol. 29.

subject of integration), and the positive half of the real axis; and extending from $+\infty$ to $+\infty$ as in the figure.



Under the limitations specified the integral is, in general, finite. Moreover, by a theorem previously obtained,* we have under such limitations

$$\frac{\iota\Gamma(1-s)}{2\pi}\int(-z)^{s-1}e^{-(a+m_1\omega_1+m_2\omega_2)z}dz = \frac{1}{(a+m_1\omega_1+\omega_2\omega_2)^s},$$

the latter expression having its principal value.

And therefore

$$\sum_{m_1=0}^{pn}\sum_{m_2=0}^{qn}\frac{1}{(a+m_1\omega_1+\omega_2\omega_2)^s}$$

$$= \frac{\iota\Gamma(1-s)}{2\pi}\int\frac{1-e^{-(pn+1)\omega_1 z}}{1-e^{-\omega_1 z}}\cdot\frac{1-e^{-(qn+1)\omega_2 z}}{1-e^{-\omega_2 z}}e^{-az}(-z)^{s-1}dz$$

$$= \zeta_2(s,a\,|\,\omega_1,\omega_2) - \frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-[p(n+1)\omega_1+a]z}+e^{-[q(n+1)\omega_2+a]z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}(-z)^{s-1}dz$$

$$+ \frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-[a+p(n+1)\omega_1+q(n+1)\omega_2]z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}(-z)^{s-1}dz,$$

all the integrals being taken along the fundamental contour.

When n is a large positive integer, we proceed to throw the two integrals last written into the form of asymptotic series.

For this purpose consider the expansion obtained in § 16, which may be written

$$\frac{ze^{-(a+\omega_1)z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{1}{\omega_1\omega_2 z} - {}_2S'_0(a+\omega_1) + \frac{{}_2S'_1(a+\omega_1)}{1!}z + \ldots$$

$$+ \frac{(-)^{n-1}{}_2S'_n(a+\omega_1)}{n!}z^n + \ldots$$

We showed that this expansion is valid provided $|z|$ was less than the smaller of the two quantities $\left|\frac{2\pi\iota}{\omega_1}\right|, \left|\frac{2\pi\iota}{\omega_2}\right|$,

Outside this circle the series diverges. But within the region bounded by lines going to infinity from the poles of

$$\frac{ze^{-(a+\omega_1)z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{1}{\omega_1\omega_2 z},$$

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* "Theory of the Gamma Function," §§ 22, 33 and 34.

the series will, in the language of M. BOREL,[*] be summable, that is to say, within this region (which is conveniently bounded by straight lines from the points $\pm\frac{2\pi\iota}{\omega_1}$, $\pm\frac{2\pi\iota}{\omega_2}$ which pass to infinity through the remaining poles) it is possible from the values of terms of the series at any point to obtain, by the employment of intermediate functions, a magnitude, independent of these particular intermediate functions, which is the value of the function at the point.

If then on any term $a_n z^n$ of such a series we perform the operation which is expressed by

$$\frac{\iota\Gamma(1-s)}{2\pi}\int a_n z^n(-z)^{s-1}dz,$$

we shall expect to obtain a quantity which is the n-th term of a possibly, and even probably, divergent sequence, which in turn is, by suitable operations, summable to the value which results from the performance of the fundamental operation on the function of which the original series is the expression.

Such considerations being understood to underlie the operations, we have

$$\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-[p(n+1)\omega_1+a]z}+e^{-[q(m+1)\omega_2+a]z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}(-z)^{s-1}dz,$$

$$\begin{aligned}
&=\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{(-z)^{s-3}}{\omega_1\omega_2}\left[e^{-pn\omega_1 z}+e^{-qn\omega_2 z}\right]dz\\
&+\frac{\iota\Gamma(1-s)}{2\pi}\int(-z)^{s-2}e^{-pn\omega_1 z}\left\{\frac{\omega_1+\omega_2}{2\omega_1\omega_2}+\frac{a-\omega_2}{\omega_1\omega_2}\right\}dz\\
&+\frac{\iota\Gamma(1-s)}{2\pi}\int(-z)^{s-2}e^{-qn\omega_2 z}\left\{\frac{\omega_1+\omega_2}{2\omega_1\omega_2}+\frac{a-\omega_2}{\omega_1\omega_2}\right\}dz\\
&+\frac{\iota\Gamma(1+s)}{2\pi}\sum_{m=1}^{\infty}\int\frac{(-z)^{m+s-2}}{m\,!}\left\{{}_2S'_m(a+\omega_1)e^{-pn\omega_1 z}+{}_2S'_m(a+\omega_2)e^{-qn\omega_2 z}\right\}dz
\end{aligned}$$

$$\begin{aligned}
=\frac{1}{(s-1)(s-2)\omega_1\omega_2}&\left\{\frac{1}{(pn\omega_1)^{s-2}}+\frac{1}{(qn\omega_2)^{s-2}}\right\}-\frac{1}{s-1}\,\frac{\omega_1+\omega_2+2a}{2\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1)^{s-1}}+\frac{1}{(qn\omega_2)^{s-1}}\right\}\\
&+\frac{1}{s-1}\left\{\frac{1}{\omega_1}\cdot\frac{1}{(pm\omega_1)^{s-1}}+\frac{1}{\omega_2}\frac{1}{(qn\omega_2)^{s-1}}\right\}\\
&+\sum_{m=1}^{\infty}(-)^{m-1}\frac{s\,.\,(s+1)\,.\,.\,.\,(s+m-2)}{1\,.\,2\,.\,.\,.\,m}\left\{\frac{{}_2S'_m(a+\omega_1)}{(pn\omega_1)^{m+s-1}}+\frac{{}_2S'_m(a+\omega_2)}{(qn\omega_2)^{m+s-1}}\right\}.
\end{aligned}$$

Now it may be readily deduced from the results obtained in Part I., that

$${}_2S_m'(a)=m[{}_2S_{m-1}(a+\omega_1)+{}_2B_m]$$

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[*] BOREL, 'Liouville,' 5 Sér., vol. 2, pp. 103 *et seq.* 'Annales de École Normale Supérieure,' 3 Sér., vol. 6, pp. 1 *et seq.*

Hence we have

$$-\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-[(pn+1)\omega_1+a]z}+e^{-[(qn+1)\omega_2+a]z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}(-z)^{s-1}dz$$

$$= -\frac{1}{(s-1)(s-2)\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1)^{s-2}}+\frac{1}{(qn\omega_2)^{s-2}}\right\}+\frac{1}{s-1}\cdot\frac{2a+\omega_1+\omega_2}{\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1)^{s-1}}\right.$$

$$\left.+\frac{1}{(qn\omega_2)^{s-1}}\right\}-\frac{1}{s-1}\left\{\frac{1}{\omega_1(pn\omega_1)^{s-1}}+\frac{1}{\omega_2(qn\omega_2)^{s-1}}\right\}-\left\{\frac{{}_2S_1'(a+\omega_1)}{(pn\omega_1)^s}-\frac{{}_2S_1'(a+\omega_2)}{(qn\omega_2)^s}\right\}$$

$$+\sum_{m=1}(-)^{m-1}\binom{s+m-1}{m}\left\{\frac{{}_2S_m(a+\omega_1)+{}_2B_{m+1}}{(pn\omega_1)^{m+s}}+\frac{{}_2S_m(a+\omega_2)+{}_2B_{m+1}}{(qn\omega_2)^{m+s}}\right\}$$

In an exactly similar manner we find, since

$${}_2S_m'(a+\omega_1+\omega_2)=m[{}_2S_{m-1}(a+\omega_1+\omega_2)+{}_2B_m],$$

that

$$\frac{\iota\Gamma(1-s)}{2\pi}\int\frac{e^{-[a+(pn+1)\omega_1+(qn+1)\omega_2]z}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}(-z)^{s-1}dz$$

$$=\frac{1}{(s-1)(s-2)\omega_1\omega_2}\cdot\frac{1}{(pn\omega_1+qn\omega_2)^{s-2}}-\frac{2a+\omega_1+\omega_2}{(s-1)2\omega_1\omega_2}\cdot\frac{1}{(pn\omega_1+qn\omega_2)^{s-1}}$$

$$+\frac{{}_2S_1'(a+\omega_2+\omega_2)}{(pn\omega_2+qn\omega_2)^s}+\sum_{m=1}^{\infty}(-)^m\binom{s+m-1}{m}\frac{{}_2S_m(a+\omega_1+\omega_2)+{}_2B_{m+1}}{(pn\omega_1+qn\omega_2)^s}$$

If now we group together all the results which have been obtained, we find the asymptotic quality, true for all values of s, and for values of a, ω_1, and ω_2, whose real parts are positive,

$$\sum_{m_1=0}^{pn}\sum_{m_2=0}^{qn}\frac{1}{(a+m_1\omega_1+m_2\omega_2)^s}=\zeta_2(s,a\,|\,\omega_1,\omega_2)$$

$$+\frac{1}{(s-1)(s-2)\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1+qn\omega_2)^{s-2}}-\frac{1}{(pn\omega_1)^{s-2}}\quad\frac{1}{(qn\omega_2)^{s-2}}\right\}$$

$$-\frac{2a+\omega_1+\omega_2}{2(s+1)\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1+qn\omega_2)^{s-1}}-\frac{1}{(pn\omega_1)^{s-1}}-\frac{1}{(qn\omega_2)^{s-1}}\right\}$$

$$-\frac{1}{s-1}\left\{\frac{1}{\omega_1(pn\omega_1)^{s-1}}+\frac{1}{\omega_2(qn\omega_2)^{s-1}}\right\}$$

$$+\frac{{}_2S_1'(a+\omega_1+\omega_2)}{(pn\omega_1+qn\omega_2)^s}-\frac{{}_2S_1'(a+\omega_1)}{(pn\omega_1)^s}-\frac{{}_2S_1'(a+\omega_2)}{(qn\omega_2)^s}$$

$$+\sum_{m=1}^{\infty}\frac{(-)^m}{n^{m+s}}\binom{m+s-1}{m}\left\{\frac{{}_2S_m(a+\omega_1+\omega_2)+{}_2B_{m+1}}{(p\omega_1+q\omega_2)^{m+s}}-\frac{{}_2S_m(a+\omega_1)+{}_2B_{m+1}}{(p\omega_1)^{s+m}}\right.$$

$$\left.-\frac{{}_2S_m(a+\omega_2)+{}_2B_{m+1}}{(q\omega_2)^{m+s}}\right\} \quad . \quad . \ (A)$$

It may readily be seen, just as for the case of a single parameter, that the series proceeding by powers of $\frac{1}{n}$ is a series of powers of a real variable, whose line of convergency is of zero length.

This series is summable by an almost evident modification of the application of Borel's ideas, which was employed in the "Theory of the Gamma Function," § 38.

It is thus the asymptotic equivalent of the sum

$$\sum_{m_1=0}^{pn} \sum_{m_2=0}^{qn} \frac{1}{(a + m_1\omega_1 + m_2\omega_2)^s};$$

and it satisfies Poincaré's* criterion for asymptotic equality that the difference between this sum and the first m terms of the series has its absolute value less than a quantity of order $\frac{1}{n^{m+s-2}}$.

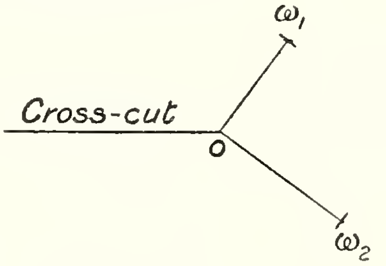
§ 39. The function $\zeta_2(s, a \mid \omega_1, \omega_2)$ has been defined, and the asymptotic equality (A) has been deduced only for the case in which the real parts of a, ω_1, and ω_2 are all positive.

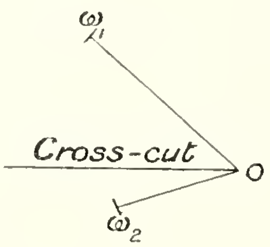
It is natural to try and use the equality to define the function for all values of a, ω_1, and ω_2.

In the first place, it is evident that when $\Re(\omega_1)$ and $\Re(\omega_2)$ are both positive, the equality (A) holds for all values of a, for the various terms of the sum and the equivalent asymptotic series are continuous for all but an enumerable number of values of a, ω_1, and ω_2. Hence in this case $\zeta_2(s, a \mid \omega_1, \omega_2)$ may be defined as the term independent of n in the equality.

So also when s is a real positive or negative integer, the equality will hold for all values of a, ω_1, and ω_2.

But when s is not an integer, the various terms involving s in their index are multiform functions, and to ensure uniformity we have to assign definite cross-cuts to the logarithms which arise in the equivalent exponentials. When, as under the limitations for which the equality (A) has been established, these cross-cuts are formed by a line outside the smaller angle between the axes of ω_1 and ω_2, the expansion is perfectly valid; but when the common cross-cut lies within this angle, terms arise similar to those which occurred in Part II. of this paper, which are multiples of $2\pi\iota$, and involve n.





And, therefore, if we attempted in this case (see the second figure) to define $\zeta_2(s, a \mid \omega_1, \omega_2)$ as the absolute term in an asymptotic equality such as (A) § 38, where for complex values of s the principal value of each term is taken, we should ultimately

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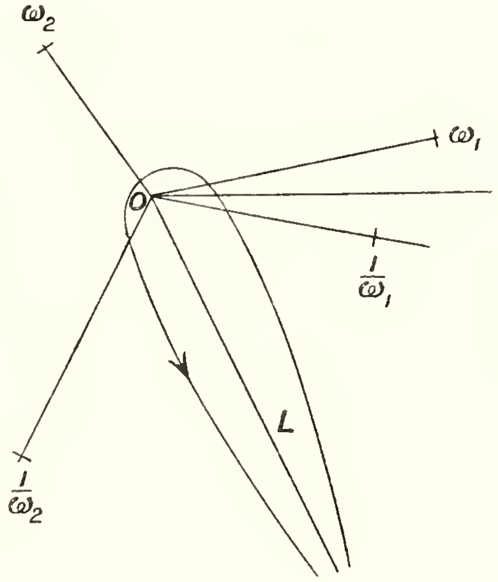
* Poincaré, 'Acta Mathematica,' vol. 8, pp. 295–344; 'Mécanique Céleste,' vol. 2, pp. 12–14.

find that $\zeta_2(s, a \mid \omega_1, \omega_2)$ as so defined would involve n. In other words, we should have made an assumption which could not be justified.

If we wish to obtain an expansion valid for all values of ω_1 and ω_2, we must consider as our starting point the integral

$$\frac{\iota\Gamma(-s)}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})};$$

where $(-z)^{s-1} = e^{(s-1)\log(-z)}$, the logarithm being rendered uniform by a cut along an axis L, coinciding with the bisector of the smaller angle between the axes of $\frac{1}{\omega_1}$ and $\frac{1}{\omega_2}$, and where the integral is taken along a contour having this axis L for axis (as in the figure), and enclosing the origin, but no other possible pole of the subject of integration. That value of $\log(-z)$ is to be taken which is such that the imaginary part of the initial value of $\log(-\mathrm{L})$ lies between 0 and $-2\pi\iota$.



This integral of course is only valid when a lies between the smaller area bounded by the axes of ω_1 and ω_2, or, as we may say, when a is positive with respect to ω_1 and ω_2. We notice that the line L is uniquely defined, since the ratio ω_2/ω_1 cannot be real and negative. The definition of the integral is not complete when ω_1 and ω_2 include and are equally inclined to the axis of -1; in this case we may take L to be a line nearly coinciding with this axis.

We now define the double Riemann ζ function, when the variable a is positive with respect to the ω's, and s, ω_1, and ω_2 have any complex values, by means of the equality

$$\zeta_2(s, a \mid \omega_1, \omega_2) = \frac{\iota\Gamma(1-s)}{2\pi}\, e^{2\mathrm{M}s\pi i}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})},$$

where $M = 0$, unless the axes of $\frac{1}{L}$ and $(\omega_1 + \omega_2)$ include the axis of -1, in which case

$M = \mp 1$, the upper or lower sign being taken as $I(\omega_1 + \omega_2)$ is positive or negative.

§ 40. Let us take now the integral which has just been defined, and apply to it the procedure of § 38.

We shall evidently have to consider integrals of the type

$$\frac{\iota\Gamma(1-s)}{2\pi}\int_L e^{-nz}(-z)^{s-1}dz,$$

where the axis of n lies within the smaller angle between the axes of ω_1 and ω_2. We can at once see that this integral

$$= \frac{\iota\Gamma(1-s)}{2\pi}\, e^{2\mu s\pi\iota}\int_{\frac{1}{n}} e^{-nz}(-z)^{s-1}dz$$

where $\mu = 0$, unless the axes of L and $\frac{1}{n}$ embrace the axis of -1, in which case

$\mu = \pm 1$, the upper or lower sign being taken as $I(n)$ is positive or negative.

Let $n = re^{\iota\theta}$, $\omega_1 = ae^{\iota\alpha}$, $\omega_2 = be^{\iota\beta}$, where θ, α and β are measured between 0 and 2π by rotation in the positive direction from the positive half of the real axis. The axis of L proceeds from the origin to the point $e^{-\frac{1}{2}(\alpha+\beta)\iota}$ and therefore where z is at a distance ρ along the axis of L,

$$nz = r\rho\, e^{\iota\left(\theta - \frac{\alpha+\beta}{2}\right)}.$$

This quantity has its real part positive when

$$-\frac{\pi}{2} < \theta - \frac{\alpha+\beta}{2} < \frac{\pi}{2},$$

a relation which is satisfied when θ lies between α and β, and the difference between α and β is less than π. It is also satisfied when the axis of L proceeds from the origin to the point $e^{-\iota\left(\frac{\alpha+\beta}{2}+\epsilon\right)}$; where ϵ is a quantity less than half the excess of π over $\alpha \backsim \beta$. We see then that the axis of $\frac{1}{n}$ lies within a range of a right angle on either side of the axis of L. Hence by the propositions previously proved ["Theory of the Gamma Function," §§ 33, 34],

$$\frac{\iota\Gamma(1-s)}{2\pi}\int_L e^{-n}(-z)^{s-1}dz = \frac{\iota\Gamma(1-s)}{2\pi}\int_{\frac{1}{n}} e^{-nz}(-z)^{s-1}dz,$$

unless the axes of L and $\frac{1}{n}$ embrace the negative half of the real axis. In this latter case, since the imaginary part of the initial value of $\log(-\mathrm{L})$ lies between 0 and $-2\pi\iota$, as also does that of $\log\left(-\frac{1}{n}\right)$, we are giving a different prescription to the many-valued function which occurs in the subject of integration.

We therefore have

$$\frac{\iota\Gamma(1-s)}{2\pi}\int_{\mathrm{L}} e^{-nz}(-z)^{s-1}dz = \frac{\iota\Gamma(1-s)}{2\pi}\, e^{2\mu s\pi\iota}\int_{\frac{1}{n}} e^{-nz}(-z)^{s-1}dz$$

where μ has the values which have been assigned to it.

Now
$$\frac{\iota\Gamma(1-s)}{2\pi}\int_{\frac{1}{n}} e^{-nz}(-z)^{s-1}dz = e^{s\log\left(n^{-1}\right)},$$

the logarithm having its principal value ("Theory of the Gamma Function," p. 107).

Hence
$$\frac{\iota\Gamma(1-s)}{2\pi}\int_{\mathrm{L}} e^{-nz}(-z)^{s-1}dz = e^{-s[\log n + 2\mu'\pi\iota]},$$

where $\mu' = 0$, unless n and $\frac{1}{\mathrm{L}}$ embrace the axis of -1, in which case $\mu' = \pm 1$, the upper or lower sign being taken as $\mathrm{I}\left(\frac{1}{\mathrm{L}}\right)$ is positive or negative.

Finally then

$$\frac{\iota\Gamma(1-s)}{2\pi}\int_{\mathrm{L}} e^{-nz}(-z)^{s-1}dz = \frac{1}{n^s},$$

where the latter function $= e^{-s\log n}$, where $\log n$ has a cross-cut along the axis of $-\frac{1}{\mathrm{L}}$, and is real when n is real and positive. In other words, $\log n$ has its principal value with respect to the axis of $-\frac{1}{\mathrm{L}}$.

§ 41. If now we apply to the integral

$$\frac{\iota\Gamma(1-s)}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}$$

the procedure of § 38, we shall, for all values of s, a, ω_1, and ω_2, such that a is positive with respect to the ω's, obtain the asymptotic equality

$$\sum_{\omega_1=0}^{pn}\sum_{m_2=0}^{qn}\frac{1}{(a+m_1\omega_1+m_2\omega_2)^s} = \frac{\iota\Gamma(1-s)}{2\pi}\int_{\mathrm{L}}\frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}$$
$$+\frac{1}{n^{s-2}(s-1)(s-2)\,\omega_1\omega_2}\left\{\frac{1}{(p\omega_1+q\omega_2)^{s-2}}-\frac{1}{(p\omega_1)^{s-2}}-\frac{1}{(q\omega_2)^{s-2}}\right\}$$
$$-\frac{2a+\omega_1+\omega_2}{2(s-1)\,\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1+qn\omega_2)^{s-1}}-\frac{1}{(pn\omega_1)^{s-1}}-\frac{1}{(qn\omega_2)^{s-1}}\right\}$$

[OVER]

$$- \frac{1}{s-1}\left\{\frac{1}{\omega_1 (pn\omega_1)^{s-1}} + \frac{1}{\omega_2 (qn\omega_2)^{s-1}}\right\} + \frac{{}_2S'_1(a+\omega_1+\omega_2)}{(pn\omega_1+qn\omega_2)^s} - \frac{{}_2S'_1(a+\omega_1)}{(pn\omega_1)^s} - \frac{{}_2S'_1(a+\omega_2)}{(qn\omega_2)^s}$$

$$+ \sum_{m=1}^{\infty} \frac{(-)^m}{n^{m+s}} \binom{m+s-1}{m} \left\{ \frac{{}_2S_m(a+\omega_1+\omega_2) + {}_2B_{m+1}}{(p\omega_1+q\omega_2)^{m+s}} - \frac{{}_2S_m(a+\omega_1) + {}_2B_{m+1}}{(p\omega_1)^{m+s}} \right.$$

$$\left. - \frac{{}_2S_m(a+\omega_2) + {}_2B_{m+1}}{(q\omega_2)^{m+s}} \right\},$$

wherein all the many-valued functions with s as index have their principal values with respect to the axis of $-\frac{1}{L}$. It proves convenient to consider these functions as having their principal value with respect to the axis of $-(\omega_1 + \omega_2)$. In order that this may be the case, we must multiply the integral by

$$e^{2Ms\pi i}$$

where, as in § 39, $\mu = 0$, unless $(\omega_1 + \omega_2)$ and $\frac{1}{L}$ includes the axis of -1, in which case $M = \pm 1$, as $I(\omega_1 + \omega_2)$ is negative or positive.

Remembering the definition of $\zeta_2(s, a \mid \omega_1, \omega_2)$ given at the end of § 39, we see that we obtain for our fundamental asymptotic equality an expression which in form is identical with (A) § 38, but in which the many-valued functions with s as index have their principal values with respect to the axis of $-(\omega_1 + \omega_2)$. It is evident that the equality will hold for all values of a, and will thus serve to define $\zeta_2(s, a \mid \omega_1, \omega_2)$ for all values of s, a, ω_1, and ω_2.

§ 42. We proceed now to take such particular cases of the general asymptotic equality which has just been obtained as lead to expressions for the logarithm of the double gamma function and its derivatives.

Suppose that s is a positive integer greater than 2; then, making n infinite in the general asymptotic equality, we see that

$$\zeta_2(s, a \mid \omega_1, \omega_2) = \frac{(-)^s}{(s-1)!} \psi_2^{(s)}(a \mid \omega_1, \omega_2),$$

where

$$\psi_2^{(s)}(a \mid \omega_1, \omega_2) = \frac{d^s}{da^s} \log \Gamma_2(a \mid \omega_1, \omega_2).$$

This relation is true for all values of a, ω_1 and ω_2; it is the first of a series connecting the double zeta and double gamma functions.

Let us next put $s + \epsilon$ for s, where ϵ is a small real quantity and s is, as before, a positive integer greater than 2. Then, provided a is positive with respect to ω_1 and ω_2,

$$\zeta_2(s+\epsilon, a \mid \omega_1, \omega)_2 = \frac{\iota\Gamma(1-s-\epsilon)}{2\pi} e^{2M\pi\iota(s+\epsilon)} \int_L \frac{e^{-az}(-z)^{s-1+\epsilon}\,dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}$$

the integral being taken along the L-contour, and M being the integer defined at the end of § 39.

Hence if $\log(-z)$ is real when z is negative, and is rendered uniform by a cut along the positive direction of the axis L,

$$\begin{aligned}&\zeta_2(s+\epsilon, a \mid \omega_1, \omega_2)\\ &= -\frac{\iota}{2\pi}\cdot\frac{(-)^{s-1}}{(s-1)\,!}\cdot\frac{1}{\epsilon}\int_{\mathrm{L}} dz\,\frac{e^{-az}(-z)^{s-1}e^{2\mathrm{M}s\pi\iota}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\left\{1+\gamma\epsilon+\ .\ .\ .\ \right\}\\ &\quad\times\left\{1-\epsilon\left(\frac{1}{1}+\ .\ .\ .\ +\frac{1}{s-1}\right)-\ .\ .\ .\ \right\}\left\{1+\epsilon\log(-z)+\ .\ .\ .\ \right\}\\ &\quad\times\left\{1+2\mathrm{M}\pi\iota\epsilon+\ .\ .\ .\right\},\end{aligned}$$

for
$$\begin{aligned}\Gamma(\epsilon-s+1)&=\frac{\Gamma(\epsilon)}{(\epsilon-s+1)(\epsilon-s+2)\ .\ .\ .\ (\epsilon-1)}\\ &=\frac{(-)^{s-1}}{(s-1)\,!\,\epsilon}\left\{1-\gamma\epsilon+\ .\ .\ .\ \right\}\left\{1+\epsilon\left(\frac{1}{1}+\ .\ .\ .\ +\frac{1}{s-1}\right)+\ .\ .\ .\ \right\}.\end{aligned}$$

Now when s is an integer greater than 2,

$$\int_{\mathrm{L}}\frac{e^{-az}(-z)^{s-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}=0,$$

for the integral may be reduced to two line integrals which destroy one another, and an integral round a small circle enclosing the origin whose value is zero.

We have then, on making $\epsilon = 0$,

$$\zeta(s, a \mid \omega_1, \omega_2)=\frac{\iota}{2\pi}\frac{(-)^{s-1}}{(s-1)\,!}\int_{\mathrm{L}}\frac{e^{-az}(-z)^{s-1}\left\{-\log(-z)+2\mathrm{M}\pi\iota+\frac{1}{1}+\ .\ .\ .\ +\frac{1}{s-1}-\gamma\right\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}dz.$$

But, when s is an integer greater than 2,

$$\int_{\mathrm{L}}\frac{e^{-az}(-z)^{s-1}\left\{2\mathrm{M}\pi\iota+\frac{1}{1}+\ .\ .\ .\ +\frac{1}{s-1}-\gamma\right\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}dz$$

vanishes for the reason just assigned.

We see, then, that when a is positive with respect to ω_1 and ω_2, and s is a positive integer greater than 2,

$$\zeta_2(s, a \mid \omega_1, \omega_2)=(-)^{s}\frac{\iota}{2\pi}\frac{1}{(s-1)\,!}\int_{\mathrm{L}}\frac{e^{-az}(-z)^{s-1}\log(-z)}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}dz,$$

and therefore under the same conditions,

$$\psi_2^{(s)}(a \mid \omega_2, \omega_2)=\frac{\iota}{2\pi}\int_{\mathrm{L}}\frac{e^{-az}(-z)^{s-1}\log(-z)}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}dz.$$

§ 43. Put now in this result $s = 3$; then with the assigned limitations

$$\psi_2^{(3)}(a \mid \omega_1, \omega_2)=\frac{\iota}{2\pi}\int_{\mathrm{L}}\frac{e^{-az}(-z)^{2}\log(-z)}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}dz,$$

and it is obvious that we may replace $\log(-z)$ in the subject of integration by $\log(-z) + \gamma$ without altering the value of the integral. Integrate successively with respect to a, and we obtain

$$\log \Gamma_2(a \mid \omega_1, \omega_2) = \frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\, dz + (1, a)^2,$$

where the coefficients of the additive quadratic form are constants with respect to a.

Remember that

$$\underset{a=0}{\mathrm{L}t}\,[a\Gamma_2(a \mid \omega_1, \omega_2)] = 1,$$

then we evidently have

$$\log \Gamma_2(a \mid \omega_1, \omega_2) = \frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\, dz + \lambda_1 a + \lambda_2 a^2$$
$$- \underset{a=0}{\mathrm{L}t}\left[\frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{\omega_1 z})(1-e^{-\omega_2 z})} + \log a\right].$$

We now define the third double gamma modular form $\rho_2(\omega_1, \omega_2)$ by the relation

$$\log \rho_2(\omega_1, \omega_2) = -2\mathrm{M}\pi\iota\, {}_2\mathrm{S}'_1(o \mid \omega_1, \omega_2)$$
$$- \underset{a=0}{\mathrm{L}t}\left[\frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\, dz + \log a\right],$$

and we proceed to show that the constants λ_1 and λ_2 are such that

$$\log \frac{\Gamma_2(a \mid \omega_1, \omega_2)}{\rho_2(\omega_1, \omega_2)} = {}_2\mathrm{S}_0(a)\,(\mathrm{M} + m + m')2\pi\iota + {}_2\mathrm{S}'_1(o)2\mathrm{M}\pi\iota$$
$$+ \frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\, dz,$$

where the numbers m and m' have the values assigned in Part II.

If this relation is true we shall have

$$\log \frac{\Gamma_2^{-1}(a+\omega_1)}{\Gamma_2^{-1}(a)} = -\,\mathrm{S}'_1(a \mid \omega_2)\,(\mathrm{M} + m + m')2\pi\iota + \frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{1-e^{-\omega_2 z}}\, dz.$$

§ 44. Let us now consider this integral.

It is to be taken along a contour embracing the axis L, which we take to be the bisector of the smaller angle between the axes of $\frac{1}{\omega_1}$ and $\frac{1}{\omega_2}$, unless such bisector should be the axis of -1, in which case we take it to be nearly coincident with this line. And in the subject of integration that value of $\log(-z)$ is to be taken which is real when z is real and negative, and is limited by a cross-cut along the axis of L.

Let us consider the relation of this integral to the integral

$$\frac{\iota}{2\pi}\int_{\frac{1}{\omega_2}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_2 z})}\, dz,$$

which is defined in the same way with reference to the axis of $1/\omega_2$.

We proceed to show that

$$\frac{\iota}{2\pi}\int_{\mathrm{L}} = \frac{\iota}{2\pi}\int_{\frac{1}{\omega_2}} - 2\pi i\,\mu'\,\mathrm{S}'_1(a\,|\,\omega_2),$$

where $\mu' = 0$, unless the axes L and $1/\omega_2$ do not enclose the axis of -1, in which case

$$\mu' = \pm 1, \text{ according as } \mathrm{I}(\omega_2) \text{ is positive or negative.}$$

For take the integral along the contour embracing the axis L, and suppose the contour to expand so that it embraces also the axis of $1/\omega_2$.

Then since a is positive with respect to ω_1 and ω_2, and since the angle between the axes of L and $1/\omega_2$ is less than $\frac{1}{2}\pi$, the value of the integral will be unaltered, for its value along the part of the great circle at infinity between the axes of L and $1/\omega_2$ is zero.

Suppose now that the contour is taken to lie on the infinite-sheeted Neumann sphere, whose sheets intersect in the cross-cut from 0 to ∞, on which the subject of integration of the integral is uniform. We may, without altering the value of the integral, deform the cross-cut so as to take up a position along the axis of $1/\omega_2$, instead of along the axis L, provided that in doing so we do not give a new specification to the logarithm. The latter phenomenon will occur when $1/\omega_2$ and L embrace the axis of -1, in which case we take the first contour in a sheet in which z can assume real values, while the second is taken in one in which $\log(-z)$ for real negative values of z is equal to $\pm 2\pi\iota$.

After deformation of the cross-cut we may compress the contour so that it embraces the axis of $1/\omega_2$. It is easy to see by this repetition of the argument previously employed in the "Theory of the Gamma Function," that we have

$$\begin{aligned}
&\frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{1-e^{-\omega_2 z}}\,dz \\
&= \frac{\iota}{2\pi}\int_{\frac{1}{\omega_2}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+2\mu'\pi\iota+\gamma\}}{1-e^{-\omega_2 z}}\,dz \\
&= \frac{\iota}{2\pi}\int_{\frac{1}{\omega_2}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{1-e^{-\omega_2 z}}\,dz - 2\mu'\pi\iota\,\mathrm{S}'_1(a\,|\,\omega_2),
\end{aligned}$$

where μ' has the value previously given.

§ 45. The assumption made in § 43 for the values of the constants λ_1 and λ_2 will therefore lead to the relation

$$\log\frac{\Gamma_2^{-1}(a+\omega_1)}{\Gamma_2^{-1}(a)} = \log\frac{\Gamma_1(a\,|\,\omega_2)}{\rho_1(\omega_2)}\,\mathrm{S}'_1(a\,|\,\omega_2)\,\{\mathrm{M}+m+m'+\mu'\}\,2\pi\iota,$$

for ("Theory of the Gamma Function," § 37)

$$\log\frac{\Gamma_1(a\,|\,\omega)}{\rho_1(\omega)} = \frac{\iota}{2\pi}\int_{\frac{1}{\omega}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{1-e^{-\omega z}}\,dz.$$

Now, by considering the various cases which can arise, it may be readily seen that

$$\mathrm{M} + m' + \mu' = 0.$$

These constants all vanish unless ω_1 and ω_2 embrace the axis of -1. When this take place, suppose that ω_1 lies above, and ω_2 below, the real axis.

Then

$$\left.\begin{array}{l}\mu' = 0\\ m' = 1\\ \mathrm{M} = 1\end{array}\right\}\text{when } (\omega_1 + \omega_2) \text{ lies above and } 1/\mathrm{L} \text{ below the real axis.}$$

$$\left.\begin{array}{l}\mu' = 0\\ m' = 0\\ \mathrm{M} = 0\end{array}\right\}\text{when } (\omega_1 + \omega_2) \text{ and } 1/\mathrm{L} \text{ both lie below the real axis.}$$

$$\left.\begin{array}{l}\mu' = -1\\ m' = 1\\ \mathrm{M} = 0\end{array}\right\}\text{when } (\omega_1 + \omega_2) \text{ and } 1/\mathrm{L} \text{ both lie above the real axis.}$$

$$\left.\begin{array}{l}\mu' = -1\\ m' = 0\\ \mathrm{M} = 1\end{array}\right\}\text{when } (\omega_1 + \omega_2) \text{ lies below and } 1/\mathrm{L} \text{ above the real axis.}$$

We get similar sets of values when the imaginary parts of ω_1 and ω_2 have opposite signs to those just assumed.

In all cases

$$\mathrm{M} + m' + \mu' = 0,$$

and, therefore, with the values assumed for λ_1 and λ_2,

$$\frac{\Gamma_2^{-1}(a + \omega_1)}{\Gamma_2^{-1}(a)} = \frac{\Gamma_1(a \mid \omega_2)}{\rho_1(\omega_2)} e^{-2m\pi\iota \mathrm{S}_1'(a \mid \omega_2)}.$$

Similarly we should find

$$\frac{\Gamma_2^{-1}(a + \omega_2)}{\Gamma_2^{-1}(a)} = \frac{\Gamma_1(a \mid \omega_1)}{\rho_1(\omega_1)} e^{-\mathrm{S}m'\pi\iota \mathrm{S}_1'(a \mid \omega_1)}.$$

But these are identically the fundamental formulæ for the double gamma function found in § 23.

The values assumed for λ_1 and λ_2 are therefore correct.

We have, then, the two important formulæ

$$\log \frac{\Gamma_2(a \mid \omega_1, \omega_2)}{\rho_2(\omega_1, \omega_2)} = {}_2\mathrm{S}_0(a)(\mathrm{M} + m + m')\, 2\pi\iota + {}_2\mathrm{S}'_1(o)\, 2\mathrm{M}\pi\iota$$
$$+ \frac{\iota}{2\pi} \int_{\mathrm{L}} \frac{e^{-az}(-z)^{-1}\{\log(-z) + \gamma\}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})}\, dz,$$

and $$\log \rho_2(\omega_1, \omega_2) = -2\mathrm{M}\pi\iota\, {}_2\mathrm{S}'_1(o \mid \omega_1, \omega_2)$$
$$- \underset{=0}{\mathrm{L}t}\left[\frac{\iota}{2\pi} \int_{\mathrm{L}} \frac{e^{-az}(-z)^{-1}\{\log(-z) + \gamma\}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})}\, dz + \log a\right],$$

which express as contour integrals the double gamma function, and the third double gamma modular form. The first relation only holds when a is positive with respect to the ω's. The second is valid for all values of ω_1 and ω_2, subject of course to the dominant condition that ω_2/ω_1 is not real and negative.

It is worth noticing that the first formulæ may also be written

$$\log \frac{\Gamma_2(a \mid \omega_1, \omega_2)}{\rho_2(\omega_1, \omega_2)} = {}_2S_0(a)(m + m')\, 2\pi\iota + {}_2S'_1(a)\{2M\pi\iota + \gamma\}$$
$$+ \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\log(-z)}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\, dz.$$

§ 46. Subject to the condition that the real parts of a and L are positive, we may now express our contour-integrals as line-integrals.

Consider the integral

$$\frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\log(-z)}{1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\, dz.$$

By hypothesis the logarithm has a cross-cut along the axis of L, the initial value of its imaginary part lying between 0 and $-2\pi\iota$. Hence if the contour of the integral be reduced to a straight line from ∞ to ϵ, where ϵ is a point on the axis of L very near the origin, a circle of small radius $|\epsilon|$ round the origin, and a straight line from ϵ back again to $+\infty$, we shall have

$$\frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\log(-z)}{(1-e^{-\omega_1 z}(1-e^{-\omega_2 z})}\, dz = -\int_\epsilon^\infty (L) \frac{e^{-az}(-z)^{-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}$$
$$+ \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-a\epsilon e^{\iota\theta}}\{\log\epsilon + \iota(\theta-\pi)\}}{(1-e^{-\omega_1 \epsilon e^{\iota\theta}})(1-e^{-\omega_2 \epsilon e^{\iota\theta}})}\, d\theta.$$

The logarithm in this second integral, which results from the small circular contour surrounding the origin, has its principal value. The integral itself is evidently equal to (§ 15)

$$\frac{1}{2\pi}\int_0^{2\pi} d\theta[\log\epsilon + \iota(\theta-\pi)]\left[\frac{e^{-2\iota\theta}}{\omega_1\omega_2\epsilon^2} - \frac{{}_2S_1^{(2)}(a)}{\epsilon}e^{-\iota\theta} + \frac{{}_2S'_1(a)}{1\,!}\right.$$
$$\left. + \text{ terms involving positive powers of } \epsilon\right]$$
$$= {}_2S'_1(a)\log\epsilon - \frac{1}{2\omega_1\omega_2\epsilon^2} + \frac{{}_2S_1^{(2)}(a)}{\epsilon} + \text{ terms which vanish with } |\epsilon|.$$

Thus
$$\frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\log(-z)}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\, dz$$
$$= -\int_\epsilon^\infty (L)\frac{e^{-az}(-z)^{-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} + {}_2S'_1(a)\log\epsilon - \frac{1}{2\omega_1\omega_2\epsilon^2} + \frac{{}_2S_1^{(2)}(a)}{\epsilon}$$
$$+ \text{ terms which vanish with } |\epsilon|.$$

Now, as has been seen in the "Theory of the Gamma Function," § 28 cor., when $\mathfrak{R}(\mathrm{L})$ is positive,

$$\int_{\epsilon}^{\infty}(\mathrm{L})\, e^{-z}\frac{dz}{z} = -\log\epsilon - \gamma + \text{terms which vanish with } |\epsilon|.$$

And evidently

$$\int_{\epsilon}^{\infty}(\mathrm{L})\frac{dz}{z^2} = \frac{1}{\epsilon}, \qquad \int_{\epsilon}^{\infty}(\mathrm{L})\frac{dz}{z^3} = \frac{1}{2\epsilon^2}.$$

The integral under consideration is thus equal to

$$\int_{\epsilon}^{\infty}(\mathrm{L})\frac{dz}{z}\left\{\frac{e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{1}{\omega_1\omega_2 z^2} + \frac{{}_2\mathrm{S}_1^{(2)}(a)}{z} - e^{-z}{}_2\mathrm{S}'_1(a)\right\} - \gamma\,{}_2\mathrm{S}'_1(a)$$
$$+ \text{terms which vanish with } |\epsilon|.$$

If now we make ϵ coincide with the origin, the integral last written remains finite and we have

$$\frac{\iota}{2\pi}\int_{\mathrm{L}}\frac{e^{-az}(-z)^{-1}\log(-z)\,dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}$$
$$= \int_{0}^{\infty}(\mathrm{L})\frac{dz}{z}\left\{\frac{e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{1}{\omega_1\omega_2 z^2} + \frac{{}_2\mathrm{S}_1^{(2)}(a)}{z} - e^{-z}{}_2\mathrm{S}'_1(a)\right\} - \gamma\,{}_2\mathrm{S}'_1(a).$$

This equality may equally be written

$$\log\frac{\Gamma_2(a\,|\,\omega_1,\omega_2)}{\rho_2(\omega_1,\omega_2)} - {}_2\mathrm{S}_0(a)(\mathrm{M}+m+m')\,2\pi\iota + {}_2\mathrm{S}'_1(o)\,2\mathrm{M}\pi\iota$$
$$= \frac{\iota}{2\pi}\int_{\mathrm{L}}^{\infty}\frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz$$
$$= \int_{0}^{\infty}(\mathrm{L})\frac{dz}{z}\left\{\frac{e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \frac{{}_2\mathrm{S}_1^{(3)}(a)}{z^2} + \frac{{}_2\mathrm{S}_1^{(2)}(a)}{z} - e^{-z}{}_2\mathrm{S}'_1(a)\right\}$$

under the assigned conditions that a is positive with respect to the ω's, and that the real part of L is positive. We thus express the logarithm of the double gamma function as a line-integral.

In order to obtain a line integral for $\log\rho_2(\omega_1,\omega_2)$, we notice that we have

$$\log a = \int_{0}^{\infty}(e^{-z} - e^{-az})\frac{dz}{z},$$

and therefore

$$\frac{\iota}{2\pi}\int_{\mathrm{L}}\frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}\,dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} + \log a$$
$$= \int_{0}^{\infty}(\mathrm{L})\frac{dz}{z}\left\{\frac{e^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - e^{-az} - \frac{{}_2\mathrm{S}_1^{(3)}(a)}{z^2} + \frac{{}_2\mathrm{S}_1^{(2)}(a)}{z} + e^{-z}\{1 - {}_2\mathrm{S}'_1(a)\}\right\}$$

Therefore on making $a = 0$, we have by § 45,

$$\log \rho(\omega_1, \omega_2) = -2\mathrm{M}\pi\iota\, {}_2\mathrm{B}_1(\omega_1, \omega_2)$$

$$- \int_0^\infty (\mathrm{L})\, \frac{dz}{z} \left\{ \frac{1}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} - 1 - \frac{{}_2\mathrm{S}_1^{(3)}(o)}{z^2} + \frac{{}_2\mathrm{S}_1^{(2)}(o)}{z} + e^{-z}[1 - {}_2\mathrm{S}'_1(a)] \right\}.$$

On differentiating the formulæ which express the logarithm of the double gamma function as line and contour integrals, we obtain

$$\psi'_2(a \mid \omega_1, \omega_2) = \frac{\iota}{2\pi} \int_{\mathrm{L}} \frac{e^{-az} \log(-z)\, dz}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} + {}_2\mathrm{S}'_0(a) \Big[\gamma + (\mathrm{M} + m + m') \,.\, \pi\iota \Big]$$

$$= \int_0^\infty (\mathrm{L}) \frac{dz}{z} \left\{ \frac{-ze^{-az}}{(1 - e^{-\omega_2 z})(1 - e^{-\omega_2 z})} + \frac{{}_2\mathrm{S}_0^{(2)}(a)}{z} - e^{-z}\, {}_2\mathrm{S}'_0(a) \right\} + {}_2\mathrm{S}'_0(a)[(\mathrm{M} + m + m')2\pi\iota].$$

Similarly, again differentiating,

$$\psi_2^{(2)}(a \mid \omega_1, \omega_2) = \frac{\iota}{2\pi} \int_{\mathrm{L}} \frac{e^{-az}(-z)\{\log(-z)\} dz}{-e^{-\omega_1 z})(1 - e^{-\omega_2 z})} + {}_2\mathrm{S}'^{(2)}_0(a) \Big[\gamma + 2\pi\iota(\mathrm{M} + m + m' \Big]$$

$$= - \int_0^\infty (\mathrm{L}) z dz \left\{ \frac{-e^{-az}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} + \frac{e^{-z}}{z^2}\, {}_2\mathrm{S}_0^{(2)}(a) \right\} + {}_2\mathrm{S}_0^{(2)}(a) \Big[(\mathrm{M} + m + m') \,.\, 2\pi\iota \Big].$$

And, if s be greater than 2,

$$\psi^{(s)}(a \mid \omega_1, \omega_2) = \frac{\iota}{2\pi} \int_{\mathrm{L}} \frac{e^{-az}(-z)^{s-1} \log(-z)\, dz}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} = - \int_0^\infty (\mathrm{L}) \frac{e^{-az}(-z)^{s-1}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})}\, dz.$$

Notice that, when we have the more narrow restrictions, the real parts of a, ω_1, and ω_2 are all positive, the constants m, m', and M all vanish, and there is a substantial simplification in the formulæ.

§ 47. We may now deduce expressions as line and contour integrals for the first and second double gamma modular forms

$$\gamma_{21}(\omega_1, \omega_2) \text{ and } \gamma_{22}(\omega_1, \omega_2).$$

We have seen (§ 22) that

$$-\psi_2^{(1)}(a \mid \omega_1, \omega_2) = \gamma_{22}(\omega_1, \omega_2) + a\gamma_{21}(\omega_1, \omega_2) + \frac{1}{a}$$

$$+ \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty}{}' \left[\frac{1}{a + m_1\omega_1 + m_2\omega_2} - \frac{1}{m_1\omega_2 + m_2\omega_2} + \frac{a}{(m_1\omega_1 + m_2\omega_2)^2} \right],$$

and, therefore, on making $a = 0$,

$$\gamma_{22}(\omega_1, \omega_2) = - \left[\psi_2^{(1)}(a \mid \omega_1, \omega_2) + \frac{1}{a} \right]_{a=0},$$

so that, by the last paragraph,

$$\gamma_{22}(\omega_1, \omega_2) = \int_0^\infty (\mathrm{L})\, dz \left\{ \frac{1}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} - 1 - \frac{{}_2\mathrm{S}_1^{(3)}(o)}{z^2} + \frac{e^{-z}}{z}\, {}_2\mathrm{S}_1^{(2)}(o) \right\}$$

$$- {}_2\mathrm{S}'_0(o)\, 2\pi\iota(\mathrm{M} + m + m').$$

The additive term will of course vanish when the real parts of ω_1 and ω_2 are positive.

Similarly, we have

$$\gamma_{21}(\omega_1, \omega_2) = -\left[\psi_2^{(2)}(a \mid \omega_1, \omega_2) - \frac{1}{a^2}\right]_{a=0}$$

$$= \underset{a=0}{\mathrm{L}t}\left[-\frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz + \frac{1}{a^2}\right] - {}_2\mathrm{S}_0^{(2)}(a)2\pi\iota(\mathrm{M}+m+m'),$$

so that the first double gamma modular form is expressed as a line integral by the formula

$$\gamma_{21}(\omega_1, \omega_2) = \int_0^\infty (-z)\,dz\left\{\frac{1}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - 1 - \frac{e^{-z}}{z^2}\,{}_2\mathrm{S}_1^{(3)}(o)\right\}$$
$$- {}_2\mathrm{S}_1^{(3)}(o)2\pi\iota(\mathrm{M}+m+m').$$

It will be noted that for the modular forms

$$\rho_2(\omega_1, \omega_2), \qquad \gamma_{21}(\omega_1, \omega_2), \qquad \gamma_{22}(\omega_1, \omega_2),$$

we have, by making a vanish, obtained line integrals which are in general finite, although in our fundamental formulæ the restriction was made that the real part of a should be positive.

This restriction was necessary to ensure that the contour integral should be finite at infinity. It is clear from the mode of generation of the line integrals, that the process which has been carried out is perfectly valid, since by the introduction of the terms $\log a$, $-\frac{1}{a^2}$, $\frac{1}{a}$ allowance has been made for the manner in which the contour integral tends to an infinite value as a tends to zero.

§ 48. At the beginning of § 43 we entered on the investigation which has just been given by integrating with respect to a under the sign of contour integration, and in this way we deduced the contour integral for $\log \Gamma_2(a)$ from that for $\psi_2^{(3)}(a)$.

We now proceed to show how the contour integral for $\log \Gamma_2(a)$ may be obtained without the employment of this process.

For this purpose we take the fundamental asymptotic equality of § 38, valid for all values of s, a, ω_1 and ω_2. the many-valued functions with s as index having their principal values with respect to the axis of $-(\omega_1 + \omega_2)$.

$$\sum_{m_1=0}^{pn}\sum_{m_2=0}^{qn}\frac{1}{(a+m_1\omega_1+m_2\omega_2)^s} = \zeta_2(s, a \mid \omega_1, \omega_2) + \frac{1}{(s-1)(s-2)\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1+qn\omega_2)^{s-2}}\right.$$
$$\left. - \frac{1}{(pn\omega_1)^{s-2}} - \frac{1}{(qn\omega_2)^{s-2}}\right\}$$
$$- \frac{2a+\omega_1+\omega_2}{2(s-1)\omega_1\omega_2}\left\{\frac{1}{(pn\omega_1+qn\omega_2)^{s-1}} - \frac{1}{(pn\omega_1)^{s-1}} - \frac{1}{(qn\omega_2)^{s-1}}\right\} - \frac{1}{s-1}\left\{\frac{1}{\omega_1(pn\omega_1)^{s-1}}\right.$$
$$\left. + \frac{1}{\omega_2(qn\omega_2)^{s-1}}\right\}$$

[OVER]

$$+ \frac{{}_2\mathrm{S}_1'(a + \omega_1 + \omega_2)}{(pn\omega_1 + qn\omega_2)^s} - \frac{{}_2\mathrm{S}_1'(a+\omega_1)}{(pn\omega_1)^s} - \frac{{}_2\mathrm{S}_1'(a + \omega_2)}{(qn\omega_2)^s}$$

$$+ \sum_{m=1}^{\infty} \frac{(-)^m}{n^{m+s}} \binom{m + s - 1}{m} \left\{ \frac{{}_2\mathrm{S}_m(a + \omega_1 + \omega_2) + {}_2\mathrm{B}_{m+1}}{(p\omega_1 + q\omega_2)^{m+s}} - \frac{{}_2\mathrm{S}_m(a + \omega_1) + {}_2\mathrm{B}_{m+1}}{(p\omega_1)^{m+s}} \right.$$
$$\left. - \frac{{}_2\mathrm{S}_m(a + \omega_1) + {}_2\mathrm{B}_{m+1}}{(q\omega_2)^{m+s}} \right\}$$

where, if a is positive with respect to the ω's, $\zeta(s, a \mid \omega_1, \omega_2)$ may be expressed by the integral

$$e^{2\mathrm{M}s\pi i} \cdot \frac{\iota}{2\pi} \Gamma(1 - s) \cdot \int_{\mathrm{L}} \frac{e^{-az}(-z)^{s-1} dz}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} \quad (\S\ 39.)$$

Make now $s = \epsilon$, where ϵ is very small; expand the various terms of the asymptotic equality in powers of ϵ, neglecting those higher than the first, and we obtain, if the real part of a is positive,

$$(pn + 1)(qn + 1) - \epsilon \log \prod_{m_1=0}^{pn} \prod_{m_2=0}^{qn} (a + m_1\omega_1 + m_2\omega_2)$$

$$= \frac{\iota}{2\pi} \int_{\mathrm{L}} \frac{e^{-az}(-z)^{-1}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} \{1 + \epsilon \log(-z)\} \{1 + \gamma\epsilon\} \{1 + 2\mathrm{M}\epsilon\pi\iota\}\, dz$$

$$+ \frac{1}{2\omega_1\omega_2} \{1 + \epsilon(\tfrac{1}{1} + \tfrac{1}{2})\} \{n^2(p\omega_1 + q\omega_2)^2 [1 - \epsilon \log(p\omega_1 n + q\omega_2 n)]$$
$$- (np\omega_1)^2 [1 - \epsilon \log pn\omega_1] - (nq\omega_2)^2 [1 - \epsilon \log qn\omega_2]\}$$

$$+ {}_2\mathrm{S}_1^{(2)}(a + \omega_1 + \omega_2)(1 + \epsilon) \Big[n(p\omega_1 + q\omega_2)[1 - \epsilon \log(pn\omega_1 + qn\omega_2)]$$
$$- np\omega_1 [1 - \epsilon \log pn\omega_1] - nq\omega_2 [1 - \epsilon \log qn\omega_2] \Big]$$

$$+ (1 + \epsilon)[pn(1 - \epsilon \log pn\omega_1) + qn(1 - \epsilon \log qn\omega_2)]$$

$$+ {}_2\mathrm{S}_1'(a + \omega_1 + \omega_2)[1 - \epsilon \log(pn\omega_1 + qn\omega_2)] - {}_2\mathrm{S}_1'(a + \omega_1)[1 - \epsilon \log pn\omega_1]$$
$$- {}_2\mathrm{S}_1'(a + \omega_2)[1 - \epsilon \log qn\omega_2]$$

$$+ \sum_{m=1}^{\infty} \frac{(-)^m}{n^m} \frac{\epsilon}{m} \left[\frac{{}_2\mathrm{S}_m(a + \omega_1 + \omega_2) + {}_2\mathrm{B}_{m+1}}{(p\omega_1 + q\omega_2)^m} - \frac{{}_2\mathrm{S}_m(a + \omega_1) + {}_2\mathrm{B}_{m+1}}{(p\omega_1)^m} - \frac{{}_2\mathrm{S}_m(a + \omega_2) + {}_2\mathrm{B}_{m+2}}{(q\omega_1)^m} \right].$$

This equality will hold for all values of s, a, ω_1, ω_2 if the integral be replaced by $\zeta_2(\epsilon, a \mid \omega_1, \omega_2)$, the logarithms having their principal value with respect to $-(\omega_1 + \omega_2)$.

Equate now the absolute terms in this asymptotic equality, and we find, if a is positive with respect to the ω's,

$$1 - \frac{\iota}{2\pi} \int_{\mathrm{L}} \frac{e^{-az}(-z)^{-1}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})}\, dz + {}_2\mathrm{S}_1'(a + \omega_1 + \omega_2) - {}_2\mathrm{S}_1'(a + \omega_1) - {}_2\mathrm{S}_1'(a + \omega_2).$$

But we have seen (§ 17) that

$${}_2\mathrm{S}_1'(a + \omega_1 + \omega_2) - {}_2\mathrm{S}_1'(a + \omega_1) - {}_2\mathrm{S}_1'(a + \omega_2) = -{}_2\mathrm{S}_1(a) + 1.$$

Hence, provided a is positive with respect to the ω's,

$$ {}_2S_1'(a) = \frac{\iota}{2\pi} \int_L \frac{e^{-az}(-z)^{-1}}{(1-e^{-m_1 z})(1-e^{-\omega_2 z})}\, dz, $$

and for all values of a, ω_1 and ω_2

$$ \zeta_2(o, a \mid \omega_1, \omega_2) = {}_2S_1'(a \mid \omega_1, \omega_2). $$

§ 49. Secondly, equate terms involving the first power of ϵ.

We obtain the asymptotic equality

$$ \begin{aligned} &- \log \prod_{m_1=0}^{pn} \prod_{m_2=0}^{qm} (a + m_1\omega_1 + m_2\omega_2) \\ &= \left[\frac{\partial}{\partial s} \zeta_2(s, a \mid \omega_1, \omega_2)\right]_{s=0} + n^2 \left(\tfrac{1}{1} + \tfrac{1}{2}\right) pq - n^2 \log n \, {}_2S_1^{(3)}(o) \left[p\omega_1 + q\omega_2\right)^2 \\ &\qquad\qquad - (p\omega_1)^2 - (q\omega_2)^2] \\ &\quad - n^2 {}_2S_1^{(3)}(o) \tfrac{1}{2} [(p\omega_1 + q\omega_2)^2 \log (p\omega_1 + q\omega_2) - (p\omega_1)^2 \log (p\omega_1) - (q\omega_2)^2 \log q\omega_2] \\ &\quad + n(p+q) - n \log n \, [{}_2S_1^{(2)}(a + \omega_1 + \omega_2)(p\omega_1 + q\omega_2) - {}_2S_1^{(2)}(a + \omega_1) p\omega_1 \\ &\qquad\qquad - {}_2S_1^{(2)}(a + \omega_2) p\omega_2] \\ &\quad - n [{}_2S_1^{(2)}(a + \omega_1 + \omega_2)(p\omega_1 + q\omega_2) \log (p\omega_1 + q\omega_2) - {}_2S_1^{(2)}(a + \omega_1) p\omega_1 \log p\omega_1 \\ &\qquad\qquad - {}_2S_1^{(2)}(a + \omega_2) q\omega_2 \log q\omega_2] \\ &\quad - [{}_2S_1'(a + \omega_1 + \omega_2) \log (p\omega_1 + q\omega_2) - {}_2S_1'(a + \omega_1) \log p\omega_1 \\ &\qquad\qquad - {}_2S_1'(a + \omega_2) \log q\omega_2] \\ &\quad - \log n \, [{}_2S_1'(a + \omega_1 + \omega_2) - {}_2S_1'(a + \omega_1) - {}_2S_1'(a + \omega_2)] \\ &\quad + \sum_{m=1}^{\infty} \frac{(-)^m}{(m+1) m n^m} \left[\frac{{}_2S'_{m+1}(a + \omega_1 + \omega_2)}{(p\omega_1 + q\omega_2)^m} - \frac{{}_2S'_{m+1}(a + \omega_1)}{(p\omega_1)^m} - \frac{{}_2S'_{m+1}(a + \omega_2)}{(p\omega_2)^m}\right], \end{aligned} $$

valid for all values of s, a, ω_1 and ω_2, provided the logarithms have their principal values with respect to the axis of $-(\omega_1 + \omega_2)$.

In order that the labour of writing down cumbrous formulæ like the one just obtained may be diminished as much as possible, we propose to introduce a symbolic notation suggested by CAYLEY's notation of matrices.*

If $f(z)$ be any function of z, we shall represent symbolically

$$ f(z + \omega_1 + \omega_2) - f(z + \omega_1) - f(z + \omega_2) $$

by $F_2[f(z + \omega)]$, the suffix 2 denoting that we are dealing with two parameters.

Thus the difference equation for double Bernoullian numbers (§ 17) is written

$$ F_2[{}_2S_n(z + \omega)] = -{}_2S_n(z) + z^n. $$

Similarly $\quad F_2[{}_2S_1^{(2)}(a + \omega) p\omega \log p\omega]$ denotes the function

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* CAYLEY, 'Collected Works,' vol. 2. The corresponding theory for multiple gamma functions will be developed by employing a symbolic notation *ab initio*.

$$ {}_2S_1^{(2)}(a + \omega_1 + \omega_2)(p\omega_1 + q\omega_2)\log(p\omega_1 + q\omega_2) - {}_2S_1^{(2)}(a + \omega_1)p\omega_1 \log p\omega_1 $$
$$ - {}_2S_1^{(2)}(a + \omega_2)q\omega_2 \log q\omega_2. $$

[The analogy with the matrix notation would be more complete if p and q were replaced by p_1 and p_2. The convention adopted here is, however, quite natural.]

And now our asymptotic equality may be written

$$ \log \prod_{m_1=0}^{pn} \prod_{m_2=0}^{qn} (a + m_1\omega_1 + m_2\omega_2) $$
$$ = -\left[\frac{\partial}{\partial s}\zeta_2(s, a \mid \omega_1, \omega_2)\right]_{s=0} + pq\left[n^2\log n - n^2\left(\tfrac{1}{1} + \tfrac{1}{2}\right)\right] + (p+q)\left[n\log n - n\right] $$
$$ + \frac{n^2}{2}F_2\left[{}_2S_1^{(3)}(\omega)(p\omega)^2\log(p\omega)\right] + nF_2\left[{}_2S_1^{(2)}(a+\omega)(p\omega)\log(p\omega)\right] $$
$$ + F_2\left[{}_2S_1'(a+\omega)\log(p\omega)\right] + \log n\left[1 - {}_2S_1'(a)\right] $$
$$ + \sum_{m=1}^{\infty}\frac{(-)^{m-1}}{mn^m}F_2\left\{\frac{{}_2S_m(a+\omega) + {}_2B_{m+1}}{(p\omega)^m}\right\}, $$

for, as may be readily proved,

$$ {}_2S_1^{(2)}(a + \omega_1 + \omega_2)(p\omega_1 + q\omega_2) - {}_2S_1^{(2)}(a + \omega_1)p\omega_1 - {}_2S_1^{(2)}(a + \omega_2)p\omega_2 = p + q. $$

When the variable a is positive with respect to the ω's, we note that the part of the absolute term in this asymptotic equality which is equal to

$$ -\left[\frac{\partial}{\partial s}\zeta_2(s, a \mid \omega_1, \omega_2)\right]_{s=0} $$

may be written

$$ \frac{-\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\gamma + \log(-z)\}\,dz}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} - 2M\pi\iota\,{}_2S_1'(a), $$

which is the expression which has been proved equal to ${}_2S_0(a)\ 2(m + m')\pi\iota - \log\dfrac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)}$ by the process of differentiation under the sign of contour integration.

§ 50. But if we take the expression for $\log \Gamma_2(a)$ which has been obtained in § 24,

$$ -\log\Gamma_2(a) = \frac{a^2}{2}\gamma_{21}(\omega_1, \omega_2) + a\gamma_{22}(\omega_1, \omega_2) + \log a $$
$$ + \sum_{m_1=0}^{\infty}\ {\sum_{m_2=0}^{\infty}}'\left[\log(a + \Omega) - \log\Omega - \frac{a}{\Omega} + \frac{a^2}{2\Omega^2}\right], $$

and write it

$$ -\log\Gamma_2(a) = \frac{a^2}{2}\gamma_{21}(\omega_1, \omega_2) + a\gamma_{22}(\omega_1, \omega_2) + \log a $$
$$ + \underset{n=\infty}{Lt}\sum_{m_1=0}^{pn}\ {\sum_{m_2=0}^{qn}}'\left[\log(a + \Omega) - \log\Omega - \frac{a}{\Omega} + \frac{a^2}{2\Omega^2}\right], $$

we may obtain this expression independently.

For putting $a = 0$ in the asymptotic expansion of

$$\log \prod_{m_1=0}^{pn} \prod_{m_2=0}^{qn} (a + \Omega)$$

we find

$$\log \prod_{m_1=0}^{pn} \mathrm{H}_{m_2=0}^{qn} (m_1\omega_1 + m_2\omega_2)$$

$$= - \underset{\substack{a=0\\ =0}}{\mathrm{Lt}} \left[\frac{\partial}{\partial s} \zeta_2 (s, a \,|\, \omega_1, \omega_2) + \log a\right] + pq \left[n^2 \log n - n^2 \left(\tfrac{1}{1} + \tfrac{1}{2}\right)\right.$$

$$+ (p + q)[n \log n - n] + n^2 \mathrm{F}_2 [{}_2\mathrm{S}_1{}^{(3)} (\omega)\, p\omega \log p\omega] + n\mathrm{F}_2 [{}_2\mathrm{S}_1{}^{(2)} (\omega)\, p\omega \log p\omega]$$

$$+ \mathrm{F}_2 [{}_2\mathrm{S}_1{}' (\omega) \log p\omega] + \log n \,[1 - {}_2\mathrm{S}_1{}' (0)]$$

$$+ \sum_{m=1}^{\infty} \frac{(-)^{m-1}}{mn^m} \mathrm{F}_2 \left\{\frac{{}_2\mathrm{S}_m (\omega) + {}_2\mathrm{B}_{m+1}}{(p\omega)^m}\right\} \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad (1).$$

This is the extension of STIRLING's Theorem to two parameters. If for all values of ω_1 and ω_2, we put

$$\log \rho_2 (\omega_1, \omega_2) = - \underset{\substack{a=0\\ s=0}}{\mathrm{Lt}} \left[\frac{\partial}{\partial s} \zeta (s, a \,|\, \omega_1, \omega_2) + \log a\right],$$

we may call $\rho_2 (\omega_1, \omega_2)$ the double Stirling function of the parameters ω_1 and ω_2. It is the same as the third double gamma modular form previously defined. For we have by § 43,

$$\log \rho_2 (\omega_1, \omega_2) = \underset{a=0}{\mathrm{Lt}} \left[\frac{\iota}{2\pi} \int \frac{e^{-az} (-z)^{-1} \{\log (-z) + \gamma\} dz}{(1 - e^{-\omega_1 z}) (1 - e^{-\omega_2 z})} - \log a\right] - 2\mathrm{M}\pi\iota {}_2\mathrm{S}_1{}' (0 \,|\, \omega_1, \omega_2) .$$

$$= \underset{\substack{a=0\\ s=0}}{\mathrm{Lt}} \left[-\frac{\partial}{\partial s} \zeta (s, a \,|\, \omega_1, \omega_2) - \log a\right] \quad \text{by § 42.}$$

We now see the exact analogy between the function $\rho_2 (\omega_1, \omega_2)$ and the simple Stirling function $\rho_1 (\omega) = \sqrt{(2\pi/\omega)}$ as defined in § 31 of the "Theory of the Gamma Function."

For a brief inspection shows us that the result of § 30 of that theory may be written

$$\log \prod_{m_1=1}^{pn} (m_1 \omega) = p (n \log n - n) + n\{\mathrm{S}_1{}^{(2)} (\omega)\, p\omega \log p\omega\}$$

$$+ [1 + \mathrm{S}'_1 (o)] \log n + \log \rho_1 (\omega) + \mathrm{S}_1{}' (\omega) \log p\omega$$

$$+ \sum_{m=1}^{\infty} \frac{(-)^{m-1}}{mn^m} \frac{\mathrm{S}_m (\omega) + {}_1\mathrm{B}_{m+1}}{(p\omega)^m}$$

which is the complete form of STIRLING's Theorem for a single parameter.

The analogy between this asymptotic expansion and

$$\log \prod_{m_1=0}^{pn} \prod_{m_2=0}^{qn} (m_1\omega_1 + m_2\omega_2) = pq\left[n^2 \log n - n^2\left(\tfrac{1}{1} + \tfrac{1}{2}\right)\right] + n^2\mathrm{F}_2\left[{}_2\mathrm{S}_1^{(3)}(\omega)\, p\omega \log p\omega\right]$$
$$+ (p+q)\left[n \log n - n\right] + n\mathrm{F}_2\left[{}_2\mathrm{S}_1^{(2)}(\omega)\, p\omega \log p\omega\right]$$
$$+ \left[1 - {}_2\mathrm{S}'_1(o)\right] \log n + \log \rho_2(\omega_1, \omega_2) + \mathrm{F}_2\left[{}_2\mathrm{S}'_1(\omega) \log p\omega\right]$$
$$+ \sum_{m=1}^{\infty} \frac{(-)^{m-1}}{mn^m}\, \mathrm{F}_2\left\{\frac{{}_2\mathrm{S}_m(\omega) + {}_2\mathrm{B}_{m+1}}{(p\omega)^m}\right\}$$

is so evident as to determine the nomenclature.

Note in the second place that the fundamental asymptotic expansion (A) of § 38 may, with the symbolic notation subsequently introduced, be written

$$\sum_{m_1=0}^{pn} \sum_{m_2=0}^{qn} \frac{1}{(a + m_1\omega_1 + m_2\omega_2)^s} = \zeta_2(s, a \mid \omega_1, \omega_2) + \frac{1}{(s-1)(s-2)\,.\,n^{s-2}}\, \mathrm{F}_2\left\{\frac{{}_2\mathrm{S}_1^{(3)}(\omega)}{(p\omega)^{s-2})}\right\}$$
$$- \frac{1}{(s-1)\,.\,n^s}\, \mathrm{F}_2\left\{\frac{{}_2\mathrm{S}_1^{(2)}(a+\omega)}{(p\omega)^{s-1}}\right\} + \frac{1}{n^s}\, \mathrm{F}_2\left\{\frac{{}_2\mathrm{S}'_1(a+\omega)}{(p\omega)^s}\right\}$$
$$+ \sum_{m=1}^{\infty} \frac{(-)^m}{n^{m+s}} \binom{m+s-1}{m}\, \mathrm{F}_2\left\{\frac{{}_2\mathrm{S}_m(a+\omega) + {}_2\mathrm{B}_{m+1}}{(p\omega)^{m+s}}\right\}$$

§ 51. If in this equality, true for all values of s, a, ω_1 and ω_2, we make $s = 1$, we obtain, since

$$\mathrm{F}_2\left\{{}_2\mathrm{S}_1^{3)}(\omega)\,(p\omega)\right\} = 0$$
$$\mathrm{F}_2\left\{{}_2\mathrm{S}_1^{(2)}(a+\omega)\right\} = -\,{}_2\mathrm{S}_1^{(2)}(a)\ .$$

$$\sum_{m_1=0}^{pn} \sum_{m_2=0}^{qn} \frac{1}{a + m_1\omega_1 + m_2\omega_2} = \underset{s=1}{\mathrm{L}t}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{{}_2\mathrm{S}_1^{(2)}(a)}{s-1}\right] + \frac{1}{n}\, \mathrm{F}_2\left\{\frac{{}_2\mathrm{S}'_1(a+\omega)}{p\omega}\right\}$$
$$+ n\mathrm{F}_2\left\{{}_2\mathrm{S}_1^{(3)}(\omega)\, p\omega \log p\omega\right\} - {}_2\mathrm{S}_1^{(2)}(a) \log n + \mathrm{F}_2\left\{{}_2\mathrm{S}_1^{(2)}(a+\omega) \log p\omega\right\}$$
$$+ \sum_{m=1}^{\infty} \frac{(-)^m}{n^{m+1}}\, \mathrm{F}_2\left\{\frac{{}_2\mathrm{S}_m(a+\omega) + {}_2\mathrm{B}_{m+1}}{(p\omega)^{m+1}}\right\} ;$$

which is equivalent to the result of differentiating the asymptotic expansion of § 49.

If now we make $a = 0$, we obtain

$$\sum_{m_1=0}^{pn} {\sum_{m_2=0}^{qn}}' \frac{1}{m_1\omega_1 + m_2\omega_2} = \underset{\substack{a=0\\ s=1}}{\mathrm{L}t}\left[\zeta(s, a \mid \omega_1, \omega_2) + \frac{{}_2\mathrm{S}_1^{(2)}(a)}{s-1} - \frac{1}{a}\right]$$
$$+ n\mathrm{F}_2\left\{{}_2\mathrm{S}_1^{(3)}(\omega)\, p\omega \log p\omega\right\} - {}_2\mathrm{S}_1^{(2)}(o) \log n + \mathrm{F}_2\left\{{}_2\mathrm{S}_1^{(2)}(\omega) \log p\omega\right\}$$
$$+ \sum_{m=0}^{\infty} \frac{(-)^m}{n^{m+1}}\, \mathrm{F}_2\left\{\frac{{}_2\mathrm{S}_m(\omega) + {}_2\mathrm{B}_{m+1}}{(p\omega)^{m+1}}\right\} .$$

We now see that

$$\underset{n=\infty}{\mathrm{L}t}\left[\sum_{m_1=0}^{pn} {\sum_{m_2=0}^{qn}}' \frac{1}{m_1\omega_1 + m_2\omega_2} + {}_2\mathrm{S}_1^{(2)}(o) \log n - \mathrm{F}\left\{{}_2\mathrm{S}_1^{(2)}(\omega) \log p\omega\right\}\right.$$
$$\left. - n\mathrm{F}_2\left\{{}_2\mathrm{S}_1^{(3)}(\omega)\, p\omega \log p\omega\right\}\right]$$
$$= \underset{\substack{a=0\\ =0}}{\mathrm{L}t}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{{}_2\mathrm{S}_1^{(2)}(a)}{s-1} - \frac{1}{a}\right],$$

which is a quantity independent of p and q.

But when $p = q = 1$, the expression on the left-hand side may be written

$$\underset{n=\infty}{\mathrm{L}t}\left[\sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log n + \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\left\{\log(\omega_1 + \omega_2) - \log\omega_1 - \log\omega_2\right\}\right.$$
$$\left. - (n+1)\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\log(\omega_1 + \omega_2) + \frac{n+1}{\omega_1}\log\omega_2 + \frac{n+1}{\omega_2}\log\omega_1\right].$$

Since the principal values of the logarithms with respect to the axis of $-(\omega_1 + \omega_2)$ are in all cases to be taken, we see from § 23 that this expression is equal to

$$\gamma_{22}(\omega_1, \omega_2) + {}_2\mathrm{S}_1^{(2)}(o)\, 2(m + m')\pi\iota.$$

For denoting by a capital letter the logarithm which has its principal value with respect to the axis of -1, we have

$$\log(\omega_1 + \omega_2) = \mathrm{Log}(\omega_1 + \omega_2),$$
$$\log\omega_1 = \mathrm{Log}\,\omega_1 + 2m\pi\iota,$$
$$\log\omega_2 = \mathrm{Log}\,\omega_2 + 2m'\pi\iota.$$

And therefore

$$\underset{\substack{a=0\\s=1}}{\mathrm{L}t}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{{}_2\mathrm{S}_1^{(2)}(a)}{s-1} - \frac{1}{a}\right]$$
$$= \underset{n=\infty}{\mathrm{L}t}\left[\sum_0^n \sum_0^{n}{}' \frac{1}{\Omega} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log n + \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\left\{\log(\omega_1 + \omega_2) - \log\omega_1 - \log\omega_2\right\}\right.$$
$$- (n+1)\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\log(\omega_1 + \omega_2) + \frac{n+1}{\omega_1}\log\omega_2 + \frac{n+1}{\omega_2}\log\omega_1$$
$$\left. + \frac{n+1}{\omega_1}2m\pi\iota + \frac{n+1}{\omega_2}2m'\pi\iota\right] - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}2(m + m')\pi\pi i$$
$$= \gamma_{21}(\omega_1, \omega_2) + {}_2\mathrm{S}_1^{(2)}(o)\, 2(m + m')\pi\iota.$$

We notice that this formula agrees with our previous results. For by § 47

$$\gamma_{21}(\omega_1, \omega_2) + {}_2\mathrm{S}'_1(o)[(m + m')2\pi\iota]$$
$$= - \underset{a=0}{\mathrm{L}t}\left[\frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}\{\log(-z) + \gamma\}dz}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} + \frac{1}{a}\right] - 2\mathrm{M}\pi\iota\,{}_2\mathrm{S}_1(o),$$

and by § 42 this last expression

$$= \underset{\substack{a=0\\s=1}}{\mathrm{L}t}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{{}_2\mathrm{S}_1^{(2)}(a)}{s-1} - \frac{1}{a}\right].$$

It is worth noticing that incidentally $\gamma_{22}(\omega_1, \omega_2)$ has been obtained as a limit in the more general form

$$\gamma_{22}(\omega_1, \omega_2) = \underset{n=\infty}{\mathrm{L}t}\left[\sum_0^{pn}\sum_0^{qn}\frac{1}{\Omega} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log n\right.$$
$$+ \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\{\log(p\omega_1 + q\omega_2) - \log p\omega_1 - \log q\omega_2\}$$

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$$-\left(\frac{pn+1}{\omega_2}+\frac{qn+1}{\omega_1}\right)\log(p\omega_1+q\omega_2)+\frac{pn+1}{\omega_2}(\log q\omega_1+2m\pi\iota)$$
$$+\frac{qn+1}{\omega_1}(\log q\omega_2+2m'\pi\iota)\Big],$$

where $\log(p\omega_1+q\omega_2)$ has its principal value with respect to the axis of $-(\omega_1+\omega_2)$.

§ 52. Let us next put $s=2$ in the final equality of § 50.

We shall obtain, in the limit when n is infinite,

$$\sum_{m_1=0}^{pn}\sum_{\omega_2=0}^{qn}\frac{1}{(a+m_1\omega_1+m_2\omega_2)^2}=\underset{s=2}{\mathrm{L}t}\left[\zeta_2(s,a\,|\,\omega_1,\omega_2)-\frac{{}_2\mathrm{S}_1^{(3)}(a)}{s-2}\right]+(1+\log n)\,{}_2\mathrm{S}_1^{(3)}(o)$$
$$-\mathrm{F}_2\{{}_2\mathrm{S}_1^{(3)}(\omega)\log p\omega\},$$

the logarithms having their principal values with respect to the axis of $-(\omega_1+\omega_2)$.

Thus
$$\underset{\substack{s=2\\a=0}}{\mathrm{L}t}\left[\zeta_2(s,a\,|\,\omega_1,\omega_2)-\frac{{}_2\mathrm{S}_1^{(3)}(0)}{s-2}-\frac{1}{a^2}\right]+\frac{1}{\omega_1\omega_2}$$
$$=\sum_0^{pn}\sum_0^{qn}{}'\frac{1}{(m_1\omega_1+m_2\omega_2)^2}-\frac{1}{\omega_1\omega_2}[\log n-\log(p\omega_1+q\omega_2)+\log p\omega_1+\log q\omega_2].$$

Now the left-hand side of this relation is independent of p and q, and therefore we see, by § 22, on putting $p=q=1$, that

$$\underset{\substack{s=2\\a=0}}{\mathrm{L}t}\left[\zeta_2(s,a\,|\,\omega_1,\omega_2)-\frac{{}_2\mathrm{S}_1^{(3)}(o)}{s-2}-\frac{1}{a^2}\right]+\frac{1}{\omega_1\omega_2}=-\gamma_{21}(\omega_1,\omega_2)-2(m+m')\pi\iota\,{}_2\mathrm{S}_1^{(3)}(o).$$

This formula again agrees with results which can be deduced from the integral formulæ. Incidentally $\gamma_{21}(\omega_1,\omega_2)$ has been expressed as the more general limit

$$\gamma_{21}(\omega_1,\omega_2)=\frac{1}{\omega_1\omega_2}\log n-\sum_0^{pn}\sum_0^{qn}\frac{1}{(m_1\omega_1+m_2\omega_2)^2}$$
$$-\frac{1}{\omega_1\omega_2}[\log(p\omega_1+q\omega_2)-\log p\omega_1-\log q\omega_2],$$

where $\log(p\omega_1+q\omega_2)$ has its principal value with respect to the axis of $-(\omega_1+\omega_2)$.

§ 53. We can now finish the investigation indicated at the beginning of § 50, and obtain the expression for $\log\dfrac{\Gamma_2(a)}{\rho_2(\omega_1,\omega_2)}$ without integrating under the sign of contour integration. We have

$$-\log\Gamma_2(a\,|\,\omega_1,\omega_2)=\frac{a^2}{2}\gamma_{21}+a\gamma_{22}+\log a$$
$$+\underset{n=\infty}{\mathrm{L}t}\sum_{\substack{0\\m_1}}^{pn}\sum_{\substack{0\\m_2}}^{qn}{}'\left[\log(a+\Omega)-\log\Omega-\frac{a}{\Omega}+\frac{a^2}{2\Omega^2}\right],$$

where $\Omega=m_1\omega_1+m_2\omega_2$.

Utilising the results of §§ 49–52, we find, when a is positive with respect to the ω's,

$$\begin{aligned}-\log \Gamma_2(a) = \underset{n=\infty}{\mathrm{L}t} \Bigg\{ & \frac{-\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz - 2\mathrm{M}\pi\iota\,{}_2\mathrm{S}_1'(a) \\ & + n\mathrm{F}_2\{[{}_2\mathrm{S}_1^{(2)}(a+\omega) - {}_2\mathrm{S}_1^{(2)}(\omega)]\,p\omega\log p\omega\} + \mathrm{F}_2\{{}_2\mathrm{S}_0(a+\omega)\log p\omega\} \\ & + \log n\{{}_2\mathrm{S}_1'(o) - {}_2\mathrm{S}_1'(a)\} - \log\rho_2(\omega_1, \omega_2) \\ & + a\big[-{}_2\mathrm{S}_1^{(2)}(o)\,2(m+m')\pi\iota + {}_2\mathrm{S}_1^{(2)}(o)\log n - \mathrm{F}_2\{{}_2\mathrm{S}_1^{(2)}(\omega)\log p\omega\} \\ & \qquad\qquad - n\mathrm{F}_2\{{}_2\mathrm{S}^{(3)}(\omega)\,p\omega\log p\omega\}\big] \\ & + \frac{a^2}{2}\big[-{}_2\mathrm{S}_1^{(3)}(o)\,2(m+m')\pi\iota + {}_2\mathrm{S}_1^{(3)}(o)\log n - \mathrm{F}_2\{{}_2\mathrm{S}_1^{(3)}(\omega)\log p\omega\}\big]\Bigg\},\end{aligned}$$

the logarithms having their principal values with respect to $-(\omega_1+\omega_2)$.

Now
$$ {}_2\mathrm{S}_1'(a) = {}_2\mathrm{S}_1'(o) + a\,{}_2\mathrm{S}_1^{(2)}(o) + \frac{a^2}{2}\,{}_2\mathrm{S}_1^{(3)}(o),$$

for the higher terms in TAYLOR's expansion all vanish. Hence

$$\begin{aligned}\log\frac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)} = \frac{\iota}{2\pi}\int_{\mathrm{L}} & \frac{e^{-az}(-z)^{-1}\{\log(-z)+\gamma\}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz \\ & + {}_2\mathrm{S}_0(a)\,2(\mathrm{M}+m+m')\pi\iota + {}_2\mathrm{S}_1'(o)\,2\mathrm{M}\pi\iota,\end{aligned}$$

which is the expression which has been previously obtained in § 45.

§ 54. We next proceed to consider the values of the double Riemann ζ function, $\zeta_2(s, a \mid \omega_1, \omega_2)$, when s is a negative integer.

By the definition of § 39,

$$\zeta_2(s, a \mid \omega_1, \omega_2) = \frac{\iota\Gamma(1-s)}{2\pi}\,e^{2\mathrm{M}s\pi\iota}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{s-1}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz,$$

a being positive with respect to the ω's.

Now in § 15 it has been proved that

$$\begin{aligned}\frac{ze^{-az}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = \frac{{}_2\mathrm{S}_1^{(3)}(a)}{z} - {}_2\mathrm{S}_1^{(2)}(a) + \frac{{}_2\mathrm{S}_1'(a)}{1\,!}z + \dots \\ \dots + (-)^{n-1}\frac{{}_2\mathrm{S}_n'(a)}{n\,!}z^n + \dots\end{aligned}$$

Therefore

$$\begin{aligned}\zeta_2(s, a \mid \omega_1, \omega_2) = -\frac{\iota\Gamma(1-s)}{2\pi}\,e^{2\mathrm{M}s\pi\iota}\int(-z)^{s-2}\,dz\,\Bigg\{ & \frac{1}{\omega_1\omega_2 z} - {}_2\mathrm{S}_1^{(2)}(a) \\ & + \frac{{}_2\mathrm{S}_1'(a)}{1\,!}z \dots + (-)^{n-1}\frac{{}_2\mathrm{S}_n'(a)}{n\,!}z^n + \dots\Bigg\},\end{aligned}$$

the latter integral being taken along a small circle surrounding the origin. [Since s is an integer, the two line integrals in the standard reduction of § 46 destroy one another.]

Hence if $s = -m$, when m is a positive integer greater than or equal to zero,

$$\zeta_2(s, a \mid \omega_1, \omega_2) = \Gamma(1+m) \frac{{}_2S'_{m+1}(a)}{(m+1)!}.$$

Thus, when s is a negative integer,

$$\zeta_2(s, a \mid \omega_1, \omega_2) = \frac{{}_2S_{1-s}(a)}{1-s}.$$

But by § 12 corollary,
$$\frac{{}_2S'_{n+1}(a)}{n+1} = {}_2B_{n+1}(\omega_1, \omega_2) + {}_2S_n(a),$$

and therefore finally, when s is a negative integer,

$$\zeta_2(s, a \mid \omega_1, \omega_2) = {}_2S_{-s}(a) + {}_2B_{1-s}(\omega_1, \omega_2).$$

When a is not positive with respect to the ω's this formula continues to hold, as is immediately evident by the theory of the function $\zeta_2(s, a \mid \omega_1, \omega_2)$, to which we shall shortly proceed.

§ 55. We proceed next to find the values of $\zeta_2(s, a \mid \omega_1, \omega_2)$ for positive integral (including zero) values of s.

We have seen, in § 48, that when a is positive with respect to the ω's,

$$\frac{\iota}{2\pi}\int(\mathrm{L})\frac{e^{-az}(-z)^{-1}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = {}_2S'_1(a),$$

so that
$$\zeta_2(o, a \mid \omega_1, \omega_2) = {}_2S'_1(a).$$

Differentiate with regard to a, and we find

$$\frac{\iota}{2\pi}\int(\mathrm{L})\frac{e^{-az}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = {}_2S'^{(2)}_1(a).$$

Now by § 39, when ϵ is a small quantity,

$$\zeta_2(1-\epsilon, a \mid \omega_1, \omega_2) = \frac{\iota\Gamma(\epsilon)}{2\pi}\, e^{-2\mathrm{M}\epsilon\pi\iota}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{-\epsilon}}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz$$

$$= \frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}\{1-\epsilon\log(-z)+\ldots\}\{1-\gamma\epsilon+\ldots\}\{1-2\mathrm{M}\epsilon\pi\iota\ldots\}}{\epsilon(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz,$$

so that, neglecting powers of ϵ above the first,

$$\epsilon\zeta_2(1-\epsilon, a \mid \omega_1, \omega_2) = {}_2S_1^{(2)}(a)\{1-\gamma\epsilon-2\mathrm{M}\epsilon\pi\iota\}$$
$$-\frac{\iota}{2\pi}\,\epsilon\int_{\mathrm{L}}\frac{\log(-z)e^{-az}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}.$$

But we immediately deduce from § 53 that

$$-\frac{d}{da}\log\Gamma_2(a) = \frac{\iota}{2\pi}\int_{\mathrm{L}}\frac{e^{-az}\log(-z)\,dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} - \{2\pi\iota(\mathrm{M}+m+m')+\gamma\}\,{}_2S_1^{(2)}(a).$$

Thus
$$\underset{s=1}{\mathrm{L}t}\left[\zeta_2(s, a\,|\,\omega_1, \omega_2) + \frac{{}_2\mathrm{S}_1^{(2)}(a)}{s-1}\right] = -\psi'_2(a) + 2(m+m')\pi\iota\,{}_2\mathrm{S}_1^{(2)}(a).$$

Again differentiating, we have

$$\frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)\,dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})} = {}_2\mathrm{S}_1^{(3)}(a).$$

Also, if ϵ be small, we have

$$\zeta_2(2-\epsilon, a\,|\,\omega_1, \omega_2) = \frac{\iota\Gamma(\epsilon-1)}{2\pi}\,e^{-2\mathrm{M}\epsilon\pi\iota}\int_{\mathrm{L}} \frac{e^{-az}(-z)^{1-\epsilon}dz}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})},$$

and
$$\Gamma(\epsilon-1) = \frac{\Gamma(\epsilon)}{\epsilon-1} = -\frac{(1-\gamma\epsilon\ \ldots)(1+\epsilon+\ \ldots)}{\epsilon} = -\frac{1}{\epsilon}\{1+(1-\gamma)\epsilon\ \ldots\}.$$

Thus
$$\zeta_2(2-\epsilon, a\,|\,\omega_1, \omega_2) = -\frac{1}{\epsilon\omega_1\omega_2} + \frac{\gamma-1+2\mathrm{M}\pi\iota}{\omega_1\omega_2} + \frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)\log(-z)}{(1-e^{-\omega_1 z})(1-e^{-\omega_2 z})}\,dz,$$

so that, since

$$\psi_2^{(2)}(a\,|\,\omega_1, \omega_2) = \frac{\iota}{2\pi}\int_{\mathrm{L}} \frac{e^{-az}(-z)\log(-z)\,dz}{(1-e^{\omega_1 z})(1-e^{-\omega_2 z})} + [2(\mathrm{M}+m+m')\pi\iota+\gamma]\,{}_2\mathrm{S}_1^{(3)}(a),$$

we have

$$\underset{s=2}{\mathrm{L}t}\left[\zeta_2(s, a\,|\,\omega_1, \omega_2) - \frac{1}{(s-2)\,\omega_1\omega_2}\right] = \psi_2^{(2)}(a) - \frac{1}{\omega_1\omega_2} - 2(m+m')\pi\iota\,{}_2\mathrm{S}_1^{(3)}(a).$$

Tabulating our results, we see that

$$\begin{aligned}
\zeta_2(s, a\,|\,\omega_1, \omega_2) &= \frac{(-)^s}{(s-1)!}\,\psi_2^{(s)}(a), && \text{when } s>2\,;\\
&= \frac{{}_2\mathrm{S}_1^{(3)}(a)}{s-2} + \psi_2^{(2)}(a) - \frac{1}{\omega_1\omega_2} - 2(m+m')\pi\iota\,{}_2\mathrm{S}_1^{(3)}(a), && s=2\,;\\
&= -\frac{{}_2\mathrm{S}_1^{(2)}(a)}{s-1} - \psi_2^{(1)}(a) + 2(m+m')\pi\iota\,{}_2\mathrm{S}_1^{(2)}(a), && s=1\,;\\
&= {}_2\mathrm{S}'_1(a) && ,\ s=0\,;\\
&= {}_2\mathrm{S}_{-s}(a) + {}_2\mathrm{B}_{1-s}(\omega_1, \omega_2) && ,\ s<0.
\end{aligned}$$

These formulæ hold for all values of the variable a, though we have only established them for the case when a is positive with respect to the ω's. They evidently agree with the results established for the case $\omega_1 = \omega_2$ in "The Theory of the G Function," § 34.

§ 56. In a note appended to the "Theory of the Gamma Function," it was stated that a theory of the simple Riemann ζ function had been developed by MELLIN.[*] It

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[*] MELLIN, 'Acta Societatis Fennicae,' vol. 24, No. 10, 1899.

was, however, published after my paper had been sent to press, and I was therefore ignorant of his elegant results. Expressed in the notation which I have adopted, his method is as follows.

It is evident from the expression of the function $\zeta(s, a, \omega)$ as an asymptotic limit, that its importance lies in the fact that it is a solution of the difference equation

$$f(a + \omega) - f(a) = -\frac{1}{a^s},$$

when s has any value, real or complex. [From this result we see at once that we should expect that, when s is a positive integer, the simple ζ function should be substantially a derivative of the gamma function, and when s is a negative integer, a Bernoullian function.] Now when $\mathfrak{R}(s) > 1$, the simplest solution of our difference equation is evidently

$$\zeta(s, a, \omega) = \sum_{n=0}^{\infty} \frac{1}{(a + n\omega)^s}.$$

This solution becomes nugatory when $\mathfrak{R}(s) \not> 1$, but MELLIN has succeeded in finding a modified solution by the following ingenious modification of MITTAG-LEFFLER'S process.

We construct the function

$$S_{-s,k}(a \mid \omega)$$

when $1 > \mathfrak{R}(s) < -k$, by writing $-s$ in place of m in the m^{th} simple Bernoullian function

$$S_m(a \mid \omega) = \frac{a^{m+1}}{(m+1)\omega} - \frac{a^m}{2} + \ldots + \binom{m}{r} a^{m-r} {}_1B_{r+1}(\omega) \ldots,$$

and taking the sum of the first $k + 1$ terms, k being of course a positive integer.

Thus
$$S_{s,-k}(a \mid \omega) = \frac{a^{1-s}}{(1-s)\omega} - \frac{a^{-s}}{2} + \sum_{r=1}^{k=1} \binom{-s}{r} a^{-s-r} {}_1B_{1+r}(\omega).$$

And now $\zeta(s, a, \omega)$ is defined by the relation

$$\begin{aligned}\zeta(s, a, \omega)& \\ = - S_{-s,k}(a \mid \omega) &+ \sum_{n=0}^{\infty} \left[\frac{1}{(a + n\omega)^s} - S_{-s,k}\{a + (n+1)\omega \mid \omega\} + S_{-s,k}(a + n\omega \mid \omega)\right].\end{aligned}$$

We readily see that this function formally satisfies our fundamental difference equation, and we may at once prove that the series does define a function existing over the whole plane.

For when $$s = 0, -1, \ldots, -k,$$

it is evident that $$S_{-s,k}(a \mid \omega) = S_{-s}(a \mid \omega) + \text{constant},$$

and therefore $$S_{-s,k}(a \mid \omega) - S_{-s,k}(a) - a^{-s} = 0.$$

When a is large and s has any value, the left-hand side of this relation may be expanded in the form

$$\frac{P_0(s)}{a^s} + \frac{P_1(s)}{a^{s+1}} + \ldots + \frac{P_r(s)}{a^{r+s}} + \ldots,$$

where the P's are integral polynomials of s of degree indicated by their suffices. As they vanish for $(k+1)$ values of s, we must have

$$S_{-s,k}(a+\omega) - S_{-s,k}(a) - a^{-s} = \frac{P_{k+1}(s)}{a^{s+k+1}} + \frac{P_{k+2}(s)}{a^{s+k+2}} + \ldots$$

Thus $\zeta(s, a, \omega)$ is convergent with $\sum\limits_{n=1}^{\infty} \frac{1}{(a+n\omega)^{s+k+1}}$, and is therefore convergent, provided

$$\mathfrak{R}(s+k+1) > 1, \qquad \text{or } \mathfrak{R}(s) > -k,$$

which is the condition with which we started.

There is one point which does not arise in the work of MELLIN, who takes the case $\omega = 1$. It is that throughout we must work with many-valued functions with s as index, which have their principal values with respect to the axis of $-\omega$. For in expanding $S_{-s,k}(a+\omega) - S_{-s,k}(a) - a^{-s}$, where a is large, we have tacitly assumed that $\log(a+\omega) = \log a + \log\left(1 + \frac{\omega}{a}\right)$, which, unless ω is real, is not the case when a is large and nearly real and negative, so that a and $a+\omega$ lie on opposite sides of the axis of -1, if this axis is the axis of the logarithms.

§ 57. It is now possible to construct the double Riemann ζ function by extending the previous analysis. The function so constructed might be made fundamental in the theory of double gamma functions and double Bernoullian numbers, these functions arising for particular values of the variable s. We will indicate the development of the theory from this point of view, for brevity establishing only the principal results, or those which, as in §§ 54, 55, have been established only over part of the a plane.

The double Riemann ζ function $\zeta_2(s, a \mid \omega_1, \omega_2)$ is the simplest solution of the difference equation

$$f(a+\omega_1+\omega_2) - f(a+\omega_1) - f(a+\omega_2) + f(a) = \frac{1}{a^s},$$

a, s, ω_1 and ω_2 having any complex values such that ω_2/ω_1 is not real and negative. The determination of a^{-s} will appear in the course of the investigation.

In the first place, it is evident that when $\mathfrak{R}(s) > 2$, a solution is given by the series

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(a + m_1\omega_1 + m_2\omega_2)^s},$$

which will then, by EISENSTEIN'S theorem, be convergent.

When $2 > \mathfrak{R}(s) > -k$, k being some positive integer, we form a modified solution by the introduction of the function ${}_2S_{-s,k}(a \mid \omega_1, \omega_2)$, formed as follows. We take the mth double Bernoullian function, ${}_2S_m(a \mid \omega_1, \omega_2)$, write $-s$ in place of m, and then take the sum of the first $(k + 2)$ terms.

We thus have

$$ {}_2S_{-s,k}(a \mid \omega_1, \omega_2) = \frac{1}{(1-s)(2-s)\,\omega_1\omega_2\,a^{s-2}} - \frac{\omega_1 + \omega_2}{2(1-s)\,\omega_1\omega_2\,a^{s-1}} + \frac{{}_2B_1}{a^s} + \sum_{r=0}^{k-1} \binom{-s}{r} \frac{{}_2B_{r+1}}{a^{s+r}}, $$

and then $\zeta_2(s, a \mid \omega_1, \omega_2)$ is given by

$$ \begin{aligned} {}_2S_{-s,k}(a) - \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \big[& {}_2S_{-s,k}\{a + (m_1+1)\omega_1 + (m_2+1)\omega_2\} \\ & - {}_2S_{-s,k}\{a + (m_1 + 1)\,\omega_1 + m_2\omega_2\} \\ & - {}_2S_{-s,k}\{a + m_1\omega_1 + (m_2 + 1)\,\omega_2\} + {}_2S_{-s,k}\{a + m_1\omega_1 + m_2\omega_2\} \\ & - (a + m_1\omega_1 + m_2\omega_2)^{-s} \big], \end{aligned} $$

an expression which for shortness we shall sometimes write

$$ {}_2S_{-s,k}(a) - \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \chi(a + m_1\omega_1 + m_2\omega_2 \mid s, k). $$

It is at once evident that the function so defined formally satisfies the fundamental difference equation, and we may readily prove that the series is in general convergent.

For when $s = 0, -1, -2, \ldots, -k, -(k+1)$, obviously

$$ {}_2S_{-s,k}(a \mid \omega_1, \omega_2) = {}_2S_{-s}(a \mid \omega_1, \omega_2) + \lambda a + \mu $$

where λ and μ are constants.

And therefore

$$ \chi(z \mid s, k) \text{ vanishes when } s = 0, -1, -2, \ \ldots, -k, -(k+1). $$

When z is large this expression admits of expansion in the form

$$ \frac{P_0(s)}{z^s} + \frac{P_1(s)}{z^{s+1}} + \ldots $$

where $P_0(s)$, $P_1(s)$, ... are integral polynomials in s of degree indicated by their suffixes, provided that the logarithms which intervene in defining the many-valued functions with s as index are such that, when ϵ is small compared with z,

$$ \log z + \log\left(1 + \frac{\epsilon}{z}\right) = \log(z + \epsilon). $$

If the axes of ω_1 and ω_2 include the axis of -1, this will not be the case for terms of the double series, for which the numbers m_1 and m_2 in the term $z = a + m_1\omega_1 + m_2\omega_2$ are large, unless the logarithms have their principal value with respect to some line between the axes of $-\omega_1$ and $-\omega_2$. We take this line to be $-(\omega_1 + \omega_2)$.

And now, since $\chi(z \mid s, k)$ vanishes for $k + 2$ values of s, we see that its expansion when z is large must be

$$\frac{P_{k+2}(s)}{z^{s+k+2}} + \frac{P_{k+3}(s)}{z^{s+k+3}} + \ldots .$$

The series for $\zeta_2(s, a \mid \omega_1, \omega_2)$ is therefore convergent with

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(a + m_1\omega_1 + m_2\omega_2)^{s+k+2}} .$$

It is then convergent when $s + k + z > 2$ or $\mathfrak{R}(s) > -k$, $|s|$ being finite.

We have then obtained a solution of the difference equation

$$f(a + \omega_1 + \omega_2) - f(a + \omega_1) - f(a + \omega_2) + f(a) = a^{-s},$$

where a^{-s} has its principal value with respect to the axis of $-(\omega_1 + \omega_2)$, which is valid for all values of s, a, ω_1 and ω_2.

§ 58. The identity of the function $\zeta_2(s, a \mid \omega_1, \omega_2)$ just defined with that previously employed is easily seen.

From the mode of formation of ${}_2S_{-s,k}(a \mid \omega_1, \omega_2)$ it is evident that when a is positive with respect to the ω's, we have

${}_2S_{-s,k}(a \mid \omega_1, \omega_2) =$ the sum of the first $(k + 2)$ terms in the expansion in powers of $\frac{1}{a}$ of

$$\frac{\iota\Gamma(1 - s)}{2\pi} e^{2M\pi\iota s} \int_L \frac{e^{-az}(-z)^{s-1}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} dz ,$$

and therefore

$${}_2S_{-s,\infty}(a \mid \omega_1, \omega_2) = \frac{\iota\Gamma(1 - s)}{2\pi} e^{2M\pi\iota s} \int_L \frac{e^{-az}(-z)^{s-1}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} dz .$$

Therefore when n is large we have the asymptotic expansion

$$\begin{aligned} {}_2S_{-s,\infty}\{a + (pn + 1)\omega_1 \mid \omega_1, \omega_2\} &= \frac{1}{(s - 1)(s - 2)\omega_1\omega_2(pn\omega_1)^{s-2}} - \frac{1}{s - 1} \cdot \frac{2a + \omega_1 + \omega_2}{2\omega_1\omega_2(pn\omega_1)^{s-1}} \\ &+ \frac{1}{s - 1 \,.\, \omega_1} \cdot \frac{1}{(pn\omega_1)^{s-1}} + \sum_{m=1}^{\infty} (-)^{m-1} \frac{s(s + 1)\ldots(s + m - 2)}{m\,!} \frac{{}_2S'_m(a + \omega_1)}{(pn\omega_1)^{m+s-1}} , \end{aligned}$$

a formula which may be proved to be true for all values of a by a term-by-term expansion of the series for ${}_2S_{-s,\infty}\{a + (pn + 1)\omega_1 \mid \omega_1, \omega_2\}$.

Now from the expression for $\zeta_2(s, a \mid \omega_1, \omega_2)$ given in § 57, we see on taking the $(pn + 1)$ first values of m_1 and the $(qn + 1)$ first values of m_2, that

$$\begin{aligned} \zeta_2(s, a \mid \omega_1\omega_2) = \underset{n=\infty}{Lt} \Bigg[&\sum_{m_1=0}^{pn} \sum_{m_2=0}^{qn} \frac{1}{(a + m_1\omega_1 + m_2\omega_2)^s} - {}_2S_{-s,k}\{a + (pn + 1)\omega_1 + (qn + 1)\omega_2\} \\ &+ {}_2S_{-s,k}\{a + (pn + 1)\omega_1\} + {}_2S_{-s,k}\{a + (qn + 1)\omega_2\} \Bigg] . \end{aligned}$$

Putting $k = \infty$, and employing the asymptotic expansion just obtained for ${}_2S_{-s,\infty}\{a + (pn + 1\}\,\omega_1]$ we obtain $\zeta_2(s, a \mid \omega_1, \omega_2)$ from the same asymptotic equality as that by which it has previously been defined in §§ 39 and 41.

§ 59. As an example of the way in which we should proceed in a theory based on the double gamma function as defined in § 57, we will prove the relation

$$\underset{s=1}{\mathrm{Lt}}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{1}{s-1}\,{}_2S_0'(a)\right] = -\,\psi_2'(a \mid \omega_1, \omega_2) + 2(m + m')\pi\iota\,{}_2S_0\,(a)$$

established in § 55 for the case when a is positive with respect to the ω's.

In the first place, when $s = 1 + \epsilon$ and ϵ is small, we see that

$$\begin{aligned}{}_2S_{-s,k}(a \mid \omega_1, \omega_3) &= {}_2S_{-1-\epsilon,0}(a \mid \omega_1, \omega_2)\\ &= \frac{1}{-\,\epsilon(1-\epsilon)\omega_1\omega_2 a^{\epsilon-1}} + \frac{\omega_1 + \omega_2}{2\epsilon\omega_1\omega_2}\,a^{-}\\ &= \left(-\frac{1}{\epsilon} + \log a\right){}_2S_0'(a) - \frac{a}{\omega_1\omega_2} + \text{higher powers of } \epsilon.\end{aligned}$$

Therefore taking $\zeta_2(s, a \mid \omega_1, \omega_2)$ as the limit, when n is infinite, of the sum obtained by taking the first $(pn + 1,\ qn + 1)$ terms of the double series

$$\begin{aligned}&\zeta_2(1 + \epsilon, a) + \frac{1}{\epsilon}\,{}_2S_0'(a)\\ &= \underset{n=\infty}{\mathrm{Lt}}\Bigg[\sum_{m_1=0}^{pn}\sum_{m_2=0}^{qn}\frac{1}{a + m_1\omega_1 + m_2\omega_2} - \frac{a}{\omega_1\omega_2}\\ &\qquad - {}_2S_0'[a + (pn+1)\omega_1 + (qn+1)\omega_2]\log[a + (pn+1)\omega_1 + (qn+1)\omega_2]\\ &\qquad + {}_2S_0'[a + (pn+1)\omega_1]\log[a + (pn+1)\omega_1]\\ &\qquad + {}_2S_0'[a + (qn+1)\omega_2]\log[a + (qn+1)\omega_2]\Bigg],\end{aligned}$$

the logarithms having their principal values with respect to the axis of $-\,(\omega_1 + \omega_2)$.

Putting $a = 0$, $p = q = 1$, we find

$$\begin{aligned}&\underset{\substack{a=0\\ s=1}}{\mathrm{Lt}}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{1}{s-1}\,{}_2S_0'(a) - \frac{1}{a}\right]\\ &= \sum_0^n\sum{}'\frac{1}{m_1\omega_1 + m_2\omega_2} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log n - (n+1)\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\log(\omega_1 + \omega_2)\\ &\qquad + \frac{n+1}{\omega_2}\log\omega_1 + \frac{n+1}{\omega_1}\log\omega_2 + \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\{\log(\omega_1 + \omega_2) - \log\omega_1 - \log\omega_2\}\\ &= \gamma_{22}(\omega_1, \omega_2) - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\,2\pi\iota(m + m').\end{aligned}$$

And now, in the limit where n is infinite, we find

$$\underset{s=1}{\mathrm{L}t}\left[\zeta_2(s, a \mid \omega_1, \omega_2) + \frac{1}{s-1}\,{}_2S_0'(a)\right]$$

$$= \sum_0^n\sum \frac{1}{a + m_1\omega_1 + m_2\omega_2} - \left\{\frac{a + (n+1)\omega_1 + (n+1)\omega_2}{\omega_1\omega_2} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\right\}\log n(\omega_1 + \omega_2)$$

$$+ \left\{\frac{a + (n+1)\omega_2}{\omega_1\omega_2} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\right\}\log n\omega_2 + \left\{\frac{a + (n+1)\omega_1}{\omega_1\omega_2} - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\right\}\log \omega_1$$

$$= \frac{1}{a} + \sum_0^n\sum{}'\left[\frac{1}{a + \Omega} - \frac{1}{\Omega} + \frac{a}{\Omega^2}\right] + a\gamma_{21} + \gamma_{22} + 2(m + m')\pi\iota\,.\,\frac{2a - \omega_1 - \omega_2}{2\omega_1\omega_2}$$

where $\Omega = m_1\omega_1 + m_2\omega_2$,

$$= -\psi_2^{(1)}(a \mid \omega_1, \omega_2) + {}_2S_0'(a)\,2(m + m')\pi\iota,$$

the complete form of the result established in § 55 for the case where a is positive with respect to the ω's.

We may establish the other results of that paragraph in a similar manner.

§ 60. Finally we will briefly consider the reduction of the second form of the double ζ function to the double gamma function in the case when $s = 0$.

If we put $s = \epsilon$, where $|\epsilon|$ is very small, we obtain

$${}_2S_{-s,k}(a \mid \omega_1, \omega_2) = {}_2S_{-\epsilon,1}(a \mid \omega_1, \omega_2)$$

$$= \frac{a^2}{2\omega_1\omega_2}\{1 + \epsilon + \ldots\}\left\{1 + \frac{\epsilon}{2} + \ldots\right\}\{1 - \epsilon\log a + \ldots\}$$

$$- \frac{a(\omega_1 + \omega_2)}{2\omega_1\omega_2}\{1 + \epsilon + \ldots\}\{1 - \epsilon\log a + \ldots\}$$

$$+ {}_2B_1(\omega_1, \omega_2)\{1 - \epsilon\log a + \ldots\}$$

$$= {}_2S_1'(a)[1 - \epsilon\log a] + \epsilon\left[\frac{3a^2}{4\omega_1\omega_2} - a\frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\right] + \text{higher powers of } \epsilon.$$

And therefore, by the second definition of the double ζ function

$$\zeta_2(\epsilon, a \mid \omega_1, \omega_2) = {}_2S_1'(a)[1 - \epsilon\log a] + \epsilon\left[\frac{3a^2}{4\omega_1\omega_2} - a\frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\right]$$

$$- \sum_{m_1=0}\sum_{m_2=0}^{\infty}\Big[{}_2S_1'(a + \Omega + \omega_1 + \omega_2)\{1 - \epsilon\log(a + \Omega + \omega_1 + \omega_2)\ldots\}$$

$$- {}_2S_1'(a + \Omega + \omega_1)\{1 - \epsilon\log(a + \Omega + \omega_1)\ldots\}$$

$$- {}_2S_1'(a + \Omega + \omega_2)\{1 - \epsilon\log(a + \Omega + \omega_2)\ldots\}$$

$$+ {}_2S_1'(a + \Omega)\{1 - \epsilon\log(a + \Omega)\ldots\} - 1 + \epsilon\log(a + \Omega) + \frac{3\epsilon}{2}\Big],$$

where $\Omega = m_1\omega_1 + m_2\omega_2$.

Thus $$\zeta_2(o, a \mid \omega_1, \omega_2) = {}_2S_1'(a)$$

and

$$\underset{\epsilon = o}{\mathrm{L}t}\left[\frac{\zeta_2(\epsilon, a \mid \omega_1, \omega_2) - {}_2S_1'(a)}{\epsilon}\right] = -\sum_{m_1=0}^{pn}\sum_{m_2=0}^{qn}\log(a + m_1\omega_1 + m_2\omega_2) +$$

[OVER]

$$\begin{aligned}&+ {}_2\mathrm{S}_1'[a + (pn+1)\omega_1 + (qn+1)\omega_2]\log[a + (pn+1)\omega_1 + (qn+1)\omega_2]\\&- {}_2\mathrm{S}_1'[a + (pn+1)\omega_1]\log[a + (pn+1)\omega_1]\\&- {}_2\mathrm{S}_1'[a + (qn+1)\omega_2]\log[a + (qn+1)\omega_2]\\&+ \frac{3a^2}{4\omega_1\omega_2} - \frac{a}{2}\cdot\frac{\omega_1+\omega_2}{\omega_1\omega_2} - \frac{3}{2}(pn+1)(qn+1)\end{aligned}$$

in the limit when n is made infinite.

Substituting the value of ${}_2\mathrm{S}_1'(a)$ in the various terms, expanding the logarithms, and re-arranging the result in powers of n, we find, with the symbolic notation of § 49,

$$\underset{s=0}{\mathrm{L}t}\left[\frac{\zeta_2(s, a \mid \omega_1, \omega_2) - {}_2\mathrm{S}_1{}'(a)}{s}\right]$$

$$\begin{aligned}&= -\log \prod_{\substack{0\\m_1}}^{pn}\prod_{\substack{0\\m_2}}^{qn}(a+\Omega) + pq[n^2\log n - n^2(\tfrac{1}{1}+\tfrac{1}{2})] + (p+q)[n\log n - n]\\&\quad+ \frac{n^2}{2}\mathrm{F}_2[{}_2\mathrm{S}_1{}^{(3)}(\omega)(p\omega)^2\log p\omega] + n\mathrm{F}_2[{}_2\mathrm{S}_1{}^{(2)}(a+\omega)p\omega\log p\omega]\\&\quad+ \mathrm{F}_2[{}_2\mathrm{S}_1{}'(a+\omega)\log p\omega] + [1 - {}_2\mathrm{S}_1(a)]\log n\end{aligned}$$

in the limit when n is made infinite.

Since the left-hand side of this equality is by the definition of $\zeta_2(s, a \mid \omega_1, \omega_2)$ finite unless a is at one of the points

$$-(m_1\omega_1 + m_2\omega_2) \qquad \left.\begin{aligned}m_1 &= 0, 1, 2, \ldots, \infty\\ m_2 &= 0, 1, 2, \ldots, \infty\end{aligned}\right\},$$

we see that we have thus been led in a purely algebraical manner to the determination of the dominant terms of the fundamental expansion of § 49.

If we make $a = 0$, and remember the definition of § 50, viz. :—

$$-\log \rho_2(\omega_1, \omega_2) = \underset{\substack{s=0\\a=0}}{\mathrm{L}t}\left[\frac{\zeta_2(s, a \mid \omega_1, \omega_2) - {}_2\mathrm{S}_1{}'(a)}{s} + \log a\right],$$

we arrive at the dominant terms of the extension of STIRLING's theorem to two parameters.

If we utilise this result in conjunction with the one just obtained, we find

$$\underset{s=0}{\mathrm{L}t}\left[\frac{\zeta_2(s, a) - {}_2\mathrm{S}_1{}'(a)}{s}\right] - \log\rho_2(\omega_1, \omega_2)$$

$$\begin{aligned}&= -\log a + \log\prod_0^{pn}\prod_0^{qn}{}'\,\Omega - \log\prod_0^{pn}\prod_0^{qn}{}'(a+\Omega) + \frac{na}{\omega_1\omega_2}\mathrm{F}_2[p\omega\log p\omega]\\&\quad+ \mathrm{F}_2[\{{}_2\mathrm{S}_0(a+\omega) - {}_2\mathrm{S}_0(\omega)\}\log p\omega] - {}_2\mathrm{S}_0(a)\log n,\end{aligned}$$

and by the definition of the double gamma function the expression last written reduces to

$$\log \Gamma_2 (a) - \left[\frac{a^2}{\omega_1\omega_2} - a\,\frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\right] 2(m + m')\,\pi\iota$$

Thus
$$\exp.\left\{\underset{s=0}{\mathrm{L}t}\,\frac{\zeta_2(s,a) - {}_2\mathrm{S}_1'(a)}{s}\right\} = \frac{\Gamma_2(a \mid \omega_1, \omega_2)}{\rho_2(\omega_1, \omega_2)}\,e^{-{}_2\mathrm{S}_0(a)\,2\,(m + m')\,\pi\iota}.$$

We may utilise this formula to establish the fundamental difference equations for the double gamma function.

By the definition of the double ζ function of § 57

$$\zeta_2(a + \omega_1) - \zeta_2(a) = \underset{n=\infty}{\mathrm{L}t}\left[{}_2\mathrm{S}_{-s,k}[a + \omega_1 + (n+1]\,\omega_2) - {}_2\mathrm{S}_{-s,k}[a + (n+1)\,\omega_2] - \sum_{m_2=0}^{n} \frac{1}{(a + m_2\omega_2)^s}\right]$$

Therefore, in the limit when n is infinite,

$$\underset{s=0}{\mathrm{L}t}\left[\frac{\zeta_2(a + \omega_1) - {}_2\mathrm{S}_1'(a + \omega_1)}{s} - \frac{\zeta_2(a) - {}_2\mathrm{S}_1'(a)}{s}\right]$$
$$= \sum_{m_2=0}^{n} \log(a + m_2\omega_2) - {}_2\mathrm{S}_1'[a + \omega_1 + (n+1)\,\omega_2]\log[a + \omega_1 + (n+1)\,\omega_2]$$
$$+ {}_2\mathrm{S}_1'[a + (n+1)\,\omega_2]\log[a + (n+1)\,\omega_2] + \frac{3\omega_1}{4\omega_2} + \frac{3}{2\omega_2}[a + (n+1)\,\omega_2] - \frac{\omega_1 + \omega_2}{2\omega_2}.$$

On reduction we obtain

$$\log\left\{\frac{\Gamma_2(a + \omega_1)}{\Gamma_2(a)}\,e^{-2\,(m + m')\,\pi\iota\,\mathrm{S}_1'(a \mid \omega_2)}\right\}$$
$$= \underset{n=\infty}{\mathrm{L}t}\left[\sum_{m_2=0}^{n} \log(a + m_2\omega_2) - \left(\frac{a + (n+1)\,\omega_2}{\omega_2} - \tfrac{1}{2}\right)\log n\omega_2 + n\right].$$

This latter expression is, by the expansion obtained in the "Theory of the Gamma Function," § 30, equal to

$$-\log\frac{\Gamma_1(a \mid \omega_2)}{\rho_1(\omega_2)} - 2m'\pi\iota\,\mathrm{S}_1'(a \mid \omega_2).$$

[The term $2m'\pi\iota\,(n + 1)$ which arises is absorbed by the identities which change $\log(a + m_2\omega_2)$ into $\log m_2\omega_2 + \log\left(1 + \frac{a}{m_2\omega_2}\right)$. The prescription of the absolute logarithms has been throughout left indeterminate.]

We thus have

$$\frac{\Gamma_2^{-1}(a + \omega_1)}{\Gamma_2^{-1}(a)} = \frac{\Gamma_1(a \mid \omega_2)}{\rho_1(\omega_2)}\,e^{-2m\pi\iota\,\mathrm{S}_1'(a \mid \omega_2)}.$$

one of the fundamental formulæ of § 23.

Sufficient indication has perhaps now been given of the alternative development of the theory of the double ζ function.

Note.—The asymptotic expansions of this part were obtained in my original cast of this theory, to which reference has already been made in the note which follows the "Theory of the Gamma Function," by the assumption that they would involve merely powers of n and $\log n$ coupled with inductive processes. Such a method, though long, is, could the fundamental assumption be justified, probably the most elementary way of obtaining these results.

[*Additional Note added July* 5, 1900.—Dr. HOBSON has pointed out that, if we admit the validity of the application of the calculus of operators to a parameter in the subject of integration of a contour integral, the theory from § 57 onwards may be developed in a very elegant manner.

We take the formula

$$f(a + \omega_1 + \omega_2) - f(a + \omega_1) - f a + \omega_2) + f(a) = \frac{1}{a^s},$$

and the known theorem

$$\frac{1}{a^s} = \frac{\iota\Gamma(1 - s)}{2\pi} \int e^{-az} (-z)^{s-1} dz,$$

and deduce

$$f(a) = \frac{\iota\Gamma(1 - s)}{2\pi} \frac{1}{(e^{\omega_1 \frac{d}{da}} - 1)(e^{\omega_2 \frac{d}{da}} - 1)} \int e^{-az} (-z)^{s-1} dz$$

$$= \frac{\iota\Gamma(1 - s)}{2\pi} \int \frac{e^{-az} (-z)^{s-1}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} dz.]$$

PART IV.

The Multiplication, Transformation, and Integral Formulæ for the Double Gamma Function.

§ 61. After the developments of Part III., we now return to the pure theory of the double gamma function. As regards the multiplication and transformation theories, two distinct courses are open to us. We may either proceed entirely algebraically, making use of the limit theorems which have been established, and so deduce the required results without the intervention of contour integrals at any stage, *or* we may directly utilise these latter to obtain the formulæ in question. The former course is, on abstract grounds, preferable :* we ought to deduce algebraical results by algebraical processes. But it is open to the fatal objection of leading to very lengthy algebra. We will employ the two methods, side by side, to deduce the multiplication formulæ, and it will be observed that the second method is both more elegant and more speedy. For the sake of brevity, the results of the transformation

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* In the first sketch of this theory, before the discovery of the contour integral expressions, all the results were obtained in this way.

theory are obtained solely by this course. Inasmuch as the function $\Gamma_2(mz \mid \omega_1, \omega_2)$ can be expressed in terms of the function $\Gamma_2\left(z \,\middle|\, \frac{\omega_1}{m}, \frac{\omega_2}{m}\right)$, the multiplication theory can be deduced from the theory of transformation. As the work of obtaining the new expression is in every case almost equal to that of obtaining the results *ab initio*, we adopt the latter course.

Multiplication Theory.

§ 62. We have from the definition (§ 19)

$$\psi_2^{(3)}(mz) = -2\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{1}{(mz+m_1\omega_1+m_2\omega_2)^3},$$

and therefore

$$m^3\psi_2^{(3)}(mz) = -2\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{1}{\left(z+\dfrac{m_1\omega_1}{m}+\dfrac{m_2\omega_2}{m}\right)^3}$$

$$= -2\sum_{r=0}^{m-1}\sum_{s=0}^{m-1}\sum_{m_1=0}^{\infty}\sum_{m_2=0}^{\infty}\frac{1}{\left(z+\dfrac{r\omega_1}{m}+\dfrac{s\omega_2}{m}+m_1\omega_1+m_2\omega_2\right)^3}$$

$$= \sum_{r=0}^{m-1}\sum_{s=0}^{m-1}\psi_2^{(3)}\left(z+\frac{r\omega_1+s\omega_2}{m}\right),$$

the parameters being understood to be ω_1 and ω_2 when not explicitly written.

Integrate with respect to z and we obtain

$$m^2\psi_2^{(2)}(mz) = \sum_{p=0}^{m-1}\sum_{q=0}^{m=1}\psi_2^{(2)}\left(z+\frac{p\omega_1+q\omega_2}{m}\right)+r \quad . \quad . \quad . \quad . \quad . \quad (\text{i.}),$$

where r is constant with respect to z.

Now

$$\psi_2^{(2)}(z) = -\gamma_{21}(\omega_1, \omega_2)+\frac{1}{z^2}+\sum_0^{\infty}\sum{}'\left\{\frac{1}{(z+\Omega)^2}-\frac{1}{\Omega^2}\right\}.$$

Substitute from this relation in the identity (1), take the same number of terms involving z on both sides of the equality, and remember that $\frac{1}{(z+\Omega)^2}-\frac{1}{\Omega^2}$ is always to be regarded as a single entity. We find that, in the limit when n is infinite, we have

$$-r+m^2\sum_{\substack{0\\m_1}}^{n}\sum_{\substack{0\\m_2}}^{n}{}'\frac{1}{\Omega^2} = m^2\sum_{\substack{0\\m_1}}^{mn+m-1}\sum_{\substack{0\\m_2}}^{mn+m-1}{}'\frac{1}{\Omega^2}$$

where $\Omega = m_1\omega_1+m_2\omega_2$.

Now we have seen that (§ 22), in the limit when n is infinite,

$$\sum_{m_1=0}^{n}\sum_{m_2=0}^{n}{}'\frac{1}{\Omega^2} = -\gamma_{21}(\omega_1, \omega_2) + \frac{1}{\omega_1\omega_2}\log n + \frac{1}{\omega_1\omega_2}[\log\omega_1 + \log\omega_2 - \log(\omega_1 + \omega_2)]$$

the principal values of the logarithms being taken.

Therefore $$r = -\frac{m^2}{\omega_1\omega_2}\log m,$$
so that we have

$$m^2\psi_2^{(2)}(mz) = \sum_{p=0}^{m-1}\sum_{q=0}^{m-1}\psi_2^{(2)}\left(z + \frac{p\omega_1 + q\omega_2}{m}\right) - \frac{m^2}{\omega_1\omega_2}\log m.$$

Integrate again with respect to z and we find

$$m\psi_2^{(1)}(mz) = \sum_{p=0}^{m-1}\sum_{q=0}^{m-1}\psi_2^{(1)}\left(z + \frac{p\omega_1 + q\omega_2}{m}\right) - \frac{m^2 z}{\omega_1\omega_2}\log m + s \quad . \quad . \quad . \quad \text{(ii.)},$$

where s is constant with respect to z.

Now we have, in the limit when n is infinite,

$$\psi_2^{(1)}(z) = -z\gamma_{21}(\omega_1, \omega_2) - \gamma_{22}(\omega_1, \omega_2) - \frac{1}{z} - \sum_{\substack{0\\m_1}}^{n}\sum_{\substack{0\\m_2}}^{n}{}'\left[\frac{1}{z+\Omega} - \frac{1}{\Omega} + \frac{z}{\Omega^2}\right].$$

Hence we find, on equating the irresoluble terms involving z in the same way on both sides of the identity (ii.), that, in the limit when n is infinite,

$$m\left\{\sum_{\substack{0\\m_1}}^{mn+m-1}\sum_{\substack{\\m_2}}{}'\frac{1}{\Omega} - \gamma_{22}(\omega_1, \omega_2)\right\} = s + m^2\left\{\sum_{\substack{0\\m_1}}^{n}\sum_{\substack{\\m_2}}{}' - \gamma_{22}(\omega_1, \omega_2)\right\}$$
$$- \left[\gamma_{21} + \sum_{\substack{0\\m_1}}^{n}\sum_{\substack{\\m_2}}{}'\frac{1}{\Omega^2}\right]\sum_{\substack{0\\p}}^{m-1}\sum_{\substack{\\q}}{}'\left(\frac{p\omega_1 + q\omega_2}{m}\right),$$

and, therefore, when n is infinite,

$$s = m\left\{\sum_{0}^{mn+m-1}\sum{}'\frac{1}{\Omega} - \gamma_{22}(\omega_1, \omega_2)\right\} - m^2\left\{\sum_{0}^{n}\sum{}'\frac{1}{\Omega} - \gamma_{22}(\omega_1, \omega_2)\right\}$$
$$+ m(m-1)\frac{\omega_1 + \omega_2}{2}\left\{\sum_{0}^{n}\sum{}'\frac{1}{\Omega^2} + \gamma_{21}(\omega_1, \omega_2)\right\}.$$

Now in § 23 we have seen that

$$\sum_{\substack{0\\m_1}}^{n}\sum_{\substack{\\m_2}}{}'\frac{1}{\Omega} - \gamma_{22} = \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log n + (n+1)\left[\left(\frac{1}{\omega_1} + \frac{1}{\omega_2}\right)\log(\omega_1 + \omega_2) - \frac{1}{\omega_2}\log\omega_1\right.$$
$$\left. - \frac{1}{\omega_1}\log\omega_2 - \frac{2m\pi\iota}{\omega_2} - \frac{2m'\pi\iota}{\omega_1}\right] - \frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\{\log(\omega_1 + \omega_2) - \log\omega_1 - \log\omega_2\},$$

and, therefore, after some reduction, we see that

$$s = m\frac{\omega_1 + \omega_2}{2\omega_1\omega_2}\log m.$$

We thus have

$$m\,\psi_2^{(1)}(mz) = \sum_{p=0}^{m-1}\sum_{q=0}^{m-1}\psi_2^{(1)}\left(z + \frac{p\omega_1 + q\omega_2}{m}\right) - \frac{m^2 z}{\omega_1\omega_2}\log m + m\,\frac{(\omega_1+\omega_2)}{2\omega_1\omega_2}\log m.$$

Integrate again with respect to z, determine the constant by making $z = 0$, and remember that $\underset{z=0}{\mathrm{Lt}}\,\dfrac{\Gamma_2(mz)}{\Gamma_2(z)} = \dfrac{1}{m}$: we obtain the formula

$$m\,\Gamma_2(mz) = \frac{\displaystyle\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}\Gamma_2\left(z + \frac{r\omega_1 + s\omega_2}{m}\right)}{\displaystyle\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}{}'\,\Gamma_2\left(\frac{r\omega_1 + s\omega_2}{m}\right)}\,e^{-{}_2S_0(mz)\log m},$$

the principal value of the logarithm being taken.

§ 63. By means of the extension of STIRLING's theorem to two parameters (§ 50), it is possible to express the form $\displaystyle\prod_{r=0}^{m-1}\prod_{s=o}^{m-1}{}'\,\Gamma_2\left(\frac{r\omega_1+s\omega_2}{m}\right)$ which has thus arisen in terms of the double Stirling function $\rho_2(\omega_1, \omega_2)$.

For, in the limit when n is infinite,

$$\Gamma_2^{-1}(z) = e^{\frac{z^2}{2}\gamma_{21} + z\gamma_{22}}\,.\,z\,.\prod_{m_1=0}^{n}\prod_{m_2=0}^{n}{}'\left[\left(1+\frac{z}{\Omega}\right)e^{-\frac{z}{\Omega}+\frac{z^2}{2\Omega^2}}\right],$$

and therefore, under the same condition, we obtain from the result of the previous paragraph

$$mz\,e^{\gamma_{21}\frac{m^2z^2}{2} + mz\gamma_{22}}\prod_{m_1=0}^{mn+m-1}\prod_{m_2}{}'\left[\left(1+\frac{mz}{\Omega}\right)e^{-\frac{mz}{\Omega}+\frac{1}{2}\frac{m^2z^2}{\Omega^2}}\right]$$

$$= \frac{m e^{{}_2S_0(mz)\log m}}{\displaystyle\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}{}'\,\Gamma_2^{-1}\left(\frac{r\omega_1+s\omega_2}{m}\right)}\,.\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}\left[\left(z+\frac{r\omega_1+s\omega_2}{m}\right)e^{\gamma_{21}\frac{\left(z+\frac{r\omega_1+s\omega_2}{m}\right)^2}{2} + \gamma_{22}\left(z+\frac{r\omega_1+s\omega_2}{m}\right)}\right.$$

$$\left.\times\prod_{m_1=0}^{n}\prod_{m_2=0}^{n}\left\{\left(1+\frac{z+\frac{r\omega_1+s\omega_2}{m}}{\Omega}\right)e^{-\frac{z+\frac{r\omega_1+s\omega_2}{m}}{\Omega}+\frac{1}{2}\frac{\left(z+\frac{r\omega_1+s\omega_2}{m}\right)^2}{\Omega^2}}\right\}\right]$$

Make now $z = 0$, and we find

$$\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}{}'\,\Gamma_2^{-1}\left(\frac{r\omega_1+s\omega_2}{m}\right) = \prod_{r=0}^{m-1}\prod_{s=0}^{m-1}\left\{\left(\frac{r\omega_1+s\omega_2}{m}\right)^1\prod_{m_1=0}^{n}\prod_{m_2=0}^{n}{}'\left(1+\frac{\frac{r\omega_1+s\omega_2}{m}}{\Omega}\right)\right.$$

$$\left.\times\,\mathrm{Exp.}\left[-\frac{r\omega_1+s\omega_2}{m}\left(\sum_{m_1=0}^{n}\sum_{m_2}{}'\frac{1}{\Omega}-\gamma_{22}\right)+\frac{1}{2}\left(\frac{r\omega_1+s\omega_2}{m}\right)^2\left(\sum_{m_1=0}^{n}\sum_{m_2}{}'\frac{1}{\Omega^2}+\gamma_{21}\right)\right]\right\}$$

where $\left(\dfrac{r\omega_1+s\omega_2}{m}\right)^1$ denotes that in the product the term for which $\left.\begin{matrix} r = 0\\ s = 0\end{matrix}\right\}$ simultaneously is to be excluded.

We have then, in the limit when n is infinite,

$$\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}{}' \Gamma_2^{-1}\left(\frac{r\omega_1 + s\omega_2}{m}\right) = \frac{\mathop{\Pi\;\Pi'}\limits_{\substack{0\\ m_1\;\; m_2}}^{mn+m-1} \dfrac{1}{\Omega}}{\left[\mathop{\Pi\;\Pi'}\limits_{\substack{0\\ m_1\;\; m_2}}^{n} \dfrac{1}{\Omega}\right]^{m^2}} \cdot m^{-m^2n^2-2m^2n-m^2+1}.$$

$$\times\ \mathrm{Exp.}\left[-m(m-1)\frac{\omega_1+\omega_2}{2}\left(\mathop{\Sigma\;\Sigma'}\limits_{\substack{0\\ m_1\;\; m_2}}^{n}\frac{1}{\Omega}-\gamma_{22}\right)+\left\{\frac{(m-1)(2m-1)}{12}(\omega_1^2+\omega_2^2)\right.\right.$$

$$\left.\left.+\frac{(m-1)^2}{4}\omega_1\omega_2\right\}\left\{\mathop{\Sigma\;\Sigma'}\limits_{\substack{0\\ m_1\;\; m_2}}^{n}\frac{1}{\Omega^2}+\gamma_{21}\right\}\right].$$

Utilise now the extension of STIRLING'S theorem and the limit formulæ for $\mathop{\Sigma\,\Sigma'}\limits_{0}^{n}\dfrac{1}{\Omega}$ and $\mathop{\Sigma\,\Sigma'}\limits_{0}^{n}\dfrac{1}{\Omega^2}$. We find that

$$\log\prod_{r=0}^{m-1}\prod_{s=0}^{m-1}{}' \Gamma_2^{-1}\left(\frac{r\omega_1+s\omega_2}{m}\right) = \{-m^2n^2-2m^2n-m^2+1\}\log m$$

$$+(mn+m-1)^2\log(mn+m-1)-\tfrac{3}{2}(mn+m-1)^2$$

$$+(mn+m-1)^2 F_2[{}_2S_1^{(3)}(\omega)\,\omega\log\omega]+2(mn+m-1)\log(mn+m-1)$$

$$-2(mn+m-1)+(mn+m-1)\,F_2[{}_2S_1^{(2)}(\omega)\,\omega\log\omega]$$

$$+[1-{}_2S_1'(o)]\log(mn+m-1)+(1-m^2)\log\rho_2(\omega_1,\omega_2)+F_2[{}_2S_1'(\omega)\log\omega]$$

$$-m^2\left[n^2\log n-\frac{3n^2}{2}\right]-m^2n^2F_2[{}_2S_1^{(3)}(\omega)\,\omega\log\omega]-2m^2[n\log n-n]$$

$$-m^2n\,\mathrm{Fe}[{}_2S_1^{(3)}(\omega)\,\omega\log\omega]-m^2[1-{}_2S_1'(o)]\log n-m^2F_2[{}_2S_1'(\omega)\log\omega]$$

$$-m(m-1)\frac{\omega_1+\omega_2}{2}\left\{{}_2S_1^{(2)}(o)\,2(m+m')\pi\iota+nF_2[{}_2S_1^{(3)}(\omega)\,\omega\log\omega]-{}_2S_1^{(2)}(o)\log n\right.$$

$$\left.+F_2[{}_2S_1^{(2)}(\omega)\log\omega]\right\}+\left[\frac{2m^2-3m+1}{12}(\omega_1^2+\omega_2^2)+\frac{m^2-2m+1}{4}\omega_1\omega_2\right]$$

$$\times\left\{-2(m+m')\pi\iota\,{}_2S_1^{(3)}(o)+\frac{1}{\omega_1\omega_2}\log n-F_2[{}_2S_1^{(3)}(\omega)\log\omega]\right\},$$

where the logarithms have their principal values with respect to the axis of $-(\omega_1+\omega_2)$.

The labour of reducing such an expression is evidently very great. It is diminished by observing that the result must be independent of n, so that we may neglect all terms which involve this letter; but even then it is only after several steps that we prove that the right-hand side is equal to

$$[1-{}_2S_1'(o)]\log m+(1-m^2)\log\rho_2(\omega_1,\omega_2)$$

$$-2(m+m')\pi\iota\left[-(m^2-m)\frac{(\omega_1+\omega_2)^2}{4\omega_1\omega_2}+\frac{2m^2-3m+1}{12}\cdot\frac{\omega_1^2+\omega_2^2}{\omega_1\omega_2}+\frac{m^2-2m+1}{4}\right]$$

$$=[1-{}_2S_1'(o)]\log m+(1-m^2)\log\rho_2(\omega_1,\omega_2)+(m^2-1)\,2(m+m')\pi\iota\,{}_2S_1(o)$$

We thus see that

$$\prod_{r=0}^{m-1} \prod_{s=0}^{m-1}{}' \Gamma_2\left(\frac{r\omega_1 + s\omega_2}{m}\right) = [\rho_2(\omega_1, \omega_2)]^{m^2-1} \,.\, m^{{}_2S_1'(o)-1}, \; e^{(1-m^2)2(m+m')\pi\iota_2S_1'(o)}.$$

§ 64. In the case when $m = 2$ the preceding result has an especial importance. It is convenient to write

$$\Gamma_2\left(z + \frac{\omega_1}{2} \middle| \omega_1, \omega_2\right) = \gamma_1(z \mid \omega_1, \omega_2),$$

$$\Gamma_2\left(z + \frac{\omega_2}{2} \middle| \omega_1, \omega_2\right) = \gamma_2(z \mid \omega_1, \omega_2),$$

$$\Gamma_2\left(z + \frac{\omega_1 + \omega_2}{2} \middle| \omega_1, \omega_2\right) = \gamma_3(z \mid \omega_1, \omega_2),$$

and in accordance with this notation we put

$$\Gamma_2(z \mid \omega_1, \omega_2) = \gamma_0(z \mid \omega_1, \omega_2).$$

These functions evidently correspond to the functions

$$\sigma(z), \; \sigma_1(z), \; \sigma_2(z), \; \sigma_3(z),$$

in WEIERSTRASS' theory of elliptic functions.

Omitting the zero argument, we take

$$\Gamma_2\left(\frac{\omega_1}{2} \middle| \omega_1, \omega_2\right) = \gamma_1(\omega_1, \omega_2) \text{ and two similar equations,}$$

so that

$$\prod_{r=0}^{2-1} \prod_{s=0}^{2-1}{}' \Gamma_2\left(\frac{r\omega + s\omega_2}{2}\right) = \gamma_1\gamma_2\gamma_3,$$

the parameters ω_1 and ω_2 being omitted.

And now, from the result of the preceding paragraph,

$$\gamma_1\gamma_2\gamma_3 = \rho_2{}^3(\omega_1, \omega_2)\, 2^{{}_2S_1'(o)-1} \; e^{-3\,.\,2(m+m')\pi\iota_2S_1'(o)},$$

so that

$$\rho_2(\omega_1, \omega_2) = \sqrt[3]{(\gamma_1\gamma_2\gamma_3)}\, 2^{\frac{1}{3}[1-{}_2S_1'(o)]} \; e^{2(m+m')\pi\iota_2S_1'(o)}.$$

We thus express the double Stirling function of ω_1 and ω_2 in terms of the product of double gamma functions whose argument is a half quasi-period.

We have previously seen in the theory of the simple gamma function that

$$\rho_1(\omega) = 2^{\frac{1}{2}}\Gamma_1\left(\frac{\omega}{2}\right),$$

and the formula just obtained is the natural extension of this result.

§ 65. From the results of §§ 62 and 63 we see that we may express the multiplication formula for $\Gamma_2(z)$ in the form

$$\Gamma_2(mz) = \frac{\prod\limits_{r=0}^{m-1}\prod\limits_{s=0}^{m-1}\Gamma_2\left(z + \frac{r\omega_1 + s\omega_2}{m}\right)}{\rho_2^{m^2-1}(\omega_1, \omega_2)} e^{-{}_2S_1'(mz)\log m + (m^2-1)2(m+m')\pi\iota_2 S_1'(o)}.$$

We now proceed to obtain this result at once from the expression of $\log \dfrac{\Gamma_2(z)}{\rho_2(\omega_1, \omega_2)}$ as a contour integral.

We have seen (§ 45) that, when a is positive with respect to the ω's,

$$\log \frac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)} = {}_2S_0(a)(M + m + m')2\pi\iota + {}_2S_1'(o)\,2M\pi\iota$$
$$+ \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z) + \gamma\}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})}dz,$$

and, therefore, m being a positive integer,

$$\log \prod_{r=0}^{m-1}\prod_{s=0}^{m-1}\left\{\frac{\Gamma_2\left(a + \dfrac{r\omega_1 + s\omega_2}{m}\right)}{\rho_2(\omega_1, \omega_2)}\right\} = (M + m + m')\,2\pi\iota \sum_{r=0}^{m-1}\sum_{s=0}^{m-1} {}_2S_0\left(a + \frac{r\omega_1 + s\omega_2}{m}\right)$$
$$+ m^2\,{}_2S_1'(o)2M\pi\iota + \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z) + \gamma\}\sum\limits_{r=0}^{m-1}\sum\limits_{s=0}^{m-1} e^{-z\cdot\frac{r\omega_1 + s\omega_2}{m}}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})}.$$

But
$$1 - e^{-z\frac{\omega_1}{m}} + \ldots + e^{-z\frac{(m-1)\omega_1}{m}} = \frac{1 - e^{-\omega_1 z}}{1 - e^{-\frac{\omega_1 z}{m}}}$$

and (§ 14)

$$\sum_{r=0}^{m-1}\sum_{s=0}^{m-1} {}_2S_0\left(a + \frac{r\omega_1 + s\omega_2}{m}\right) = {}_2S_0\left(a \,\middle|\, \frac{\omega_1}{m}, \frac{\omega_2}{m}\right) - m^2\,{}_2B_1(\omega_1, \omega_2) + {}_2B_1\left(\frac{\omega_1}{m}, \frac{\omega_2}{m}\right)$$
$$= {}_2S_0(ma \,|\, \omega_1, \omega_2) + (1 - m^2)\,{}_2B_1(\omega_1, \omega_2).$$

Therefore when a is positive with respect to the ω's,

$$\log \prod_{r=0}^{m-1}\prod_{s=0}^{m-1}\left\{\frac{\Gamma_2\left(a + \dfrac{r\omega_1 + s\omega_2}{m}\right)}{\rho_2(\omega_1, \omega_2)}\right\} = (M + m + m')2\pi\iota\,{}_2S_0(ma \,|\, \omega_1, \omega_2) + 2M\pi\iota\,{}_2S_1'(o)$$
$$+ (m + m')2\pi\iota(1 - m^2)\,{}_2B_1(\omega_1, \omega_2) + \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z) + \gamma\}dz}{(1 - e^{-\frac{\omega_1 z}{m}})(1 - e^{-\frac{\omega_2 z}{m}})}.$$

Since m is a positive integer, the axis L defined with reference to the parameters ω_1, ω_2 is the same as that defined with reference to the parameters $m\omega_1$, $m\omega_2$ for the lines representing these two sets of parameters are coincident. If then we change z into mz, the integral last written becomes one which (§ 45) is equal to

$$\log \frac{\Gamma_2(ma)}{\rho_2(\omega_1, \omega_2)} + {}_2S_1'(ma)\{\log m - 2M\pi\iota\} - {}_2S_0(ma)(m + m')2\pi\iota,$$

the arithmetic value of log m being taken.

And therefore

$$\log \prod_{r=0}^{m=1} \prod_{s=0}^{m=1} \left\{ \frac{\Gamma_2\left(a + \frac{r\omega_1 + s\omega_2}{m}\right)}{\rho_2(\omega_1, \omega_2)} \right\} = (m + m')2\pi\iota(1 - m^2)\,{}_2S_1'(o)$$
$$+ {}_2S_1'(ma)\log m + \log \frac{\Gamma_2(ma)}{\rho_2(\omega_1, \omega_2)}\,.$$

which is the result required. This result has of course only been proved by means of the contour integral, under the assumption that a is positive with respect to the ω's. To establish it in general we should appeal to the principle of continuity.

§ 66. Before concluding the multiplication theory, we deduce expressions for the values of

$$\sum_{p=0}^{m=1} \sum_{q=0}^{m=1}{}' \psi_2^{(r)}\left(\frac{p\omega_1 + q\omega_2}{m}\right), \quad \text{where } r = 1, 2, 3, \ldots$$

We recall from § 29, that within a circle of sufficiently small radius surrounding the origin, we have

$$\log \Gamma_2(z) = -\log z - z\gamma_{22} - \frac{z^2\gamma_{21}}{2} - \sum_0^\infty \sum{}' \frac{z^3}{3\Omega^3} + \sum_0^\infty \sum{}' \frac{z^4}{4\Omega^4} - \ldots$$

Again, from the multiplication theorem of the preceding paragraph, we have, by a similar expansion,

$$\log \Gamma_2(mz) = -(m + m')2\pi\iota(1 - m^2)\,{}_2S_1'(o) + (1 - m^2)\log \rho_2(\omega_1, \omega_2)$$
$$- \log m\left[{}_2S_1'(o) + mz\,{}_2S_1^{(2)}(o) + \frac{m^2z^2}{2}\,{}_2S_1^{(3)}(o)\right] + \log \Gamma_2(z)$$
$$+ \log \prod_{p=0}^{m-1} \prod_{q=0}^{m-1}{}' \Gamma_2\left(\frac{p\omega_1 + q\omega_2}{m}\right) + \sum_{p=0}^{m-1} \sum_{q=0}^{m-1}{}' \left[z\psi_2^{(1)}\left(\frac{p\omega_1 + q\omega_2}{m}\right)\right.$$
$$\left. + \frac{z^2}{2\,!}\psi_2^{(2)}\left(\frac{p\omega_1 + q\omega_2}{m}\right) + \ldots\right].$$

Combine these two theorems, and equate coefficients of various powers of z in the resulting identity. We find

$$\gamma_{22}(\omega_1, \omega_2) = \frac{m}{m-1}\,{}_2S_0'(o)\log m - \frac{1}{m-1} \sum_{p=0}^{m-1} \sum_{q=0}^{m=1}{}' \psi_2^{(1)}\left(\frac{p\omega_1 + q\omega_2}{m}\right),$$
$$\gamma_{21}(\omega_1, \omega_2) = \frac{m^2}{m^2-1}\,{}_2S_0^{(2)}(o)\log m - \frac{1}{m^2-1} \sum_{p=0}^{m-1} \sum_{q=0}^{m-1}{}' \psi_2^{(2)}\left(\frac{p\omega_1 + q\omega_2}{m}\right)$$

and, when r is greater than 2,

$$\sum_{p=0}^{m-1} \sum_{q=0}^{m-1}{}' \psi_2^{(r)}\left(\frac{p\omega_1 + q\omega_2}{m}\right) = (-)^r(r-1)\,!\,(m^r - 1) \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty}{}' \frac{1}{(m_1\omega_1 + m_2\omega_2)^r}\,.$$

Transformation Theory.

§ 67. We shall now consider the theory of the transformation of the parameters of the double gamma function. It must not be supposed that we intend to consider the general linear transformation.

There exists no such theory for the present functions—at any rate, no theory having the simplicity and elegance which is characteristic of the elliptic functions, and the reason is obvious—the change of ω_1 into $\omega_1 + \omega_2$ makes no difference of form in such a series as

$$\sum_{m_1=-\infty}^{\infty} \sum_{m_2=-\infty}^{\infty} \frac{1}{(z + m_1\omega_1 + m_2\omega_2)^3},$$

but it makes a change of comparatively great complexity in such a series as

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(z + m_1\omega_1 + m_2\omega_2)^3}.$$

The former series is the basis of those occurring in the theory of elliptic functions, the latter of those occurring in double gamma functions. We shall then limit our consideration to transformations which result from the change of ω_1 and ω_2 into ω_1/p and ω_2/q respectively, p and q being positive integers.

By definition we have

$$\psi_2^{(3)}(z \mid \omega_1, \omega_2) = -2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{(z + \Omega)^3}$$

where $\Omega = m_1\omega_1 + m_2\omega_2$.

Hence
$$\psi_2^{(3)}\left(z \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = -2 \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{1}{\left(z + \dfrac{m_1\omega_1}{p} + \dfrac{m_2\omega_2}{q}\right)^3}$$

$$= -2 \sum_{r=0}^{p-1} \sum_{s=0}^{q-1} \left[\sum_{m_1}^{\infty} \sum_{m_2}{}_{0} \frac{1}{\left(z + \dfrac{r\omega_1}{p} + \dfrac{s\omega_2}{q} + m_1\omega_1 + m_2\omega_2\right)^3} \right]$$

$$= \sum_{s=0}^{p-1} \sum_{s=0}^{q-1} \psi_2^{(3)}\left(z + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right),$$

it being understood that the parameters when not expressed are always ω_1 and ω_2.

On integrating successively three times with respect to z, we shall find

$$\log \Gamma_2\left(z \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \chi_2(z) + \log \prod_{r=0}^{p-1} \prod_{s=0}^{q-1} \Gamma_2\left(z + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)$$

where $\chi_2(z)$ is an algebraical polynomial in z of order 2.

As has been stated, it is possible to obtain $\chi_2(z)$ by purely algebraical processes, use being made of the limit theorems previously established. We may, however, obtain its value as follows :—

In the relation which has been obtained, change z into $z + \omega_1/p$ and subtract the first result from the one so formed. We find by § 23,

$$\log \frac{\Gamma_1\left(z \,\middle|\, \frac{\omega_2}{q}\right)}{\rho_1\left(\frac{\omega_3}{q}\right)} - 2m_{p,q}\,\pi\iota\, S_1'\left(z \,\middle|\, \frac{\omega_2}{q}\right) = \chi_2(z) - \chi_2\left(z + \frac{\omega_1}{p}\right)$$

$$+ \log \prod_{s=0}^{q-1} \left\{ \frac{\Gamma_1\left(z + \frac{s\omega_2}{q} \,\middle|\, \omega_2\right)}{\rho_1(\omega_2)} \right\} - 2m\,\pi\iota \sum_{s=0}^{q-1} S_1'\left(z + \frac{s\omega_2}{q} \,\middle|\, \omega_2\right).$$

where m has the value asigned in § 22, and where $m_{p,q} = 0$, unless $\left\{\begin{array}{l}\frac{\omega_1}{p} \text{ does} \\ \frac{\omega_1}{p} + \frac{\omega_2}{q} \text{ does not}\end{array}\right.$ lie between -1 and $-\frac{\omega_2}{q}$. in which case $m_{p\cdot q} = \pm 1$ as $I\left(\frac{\omega_2}{q}\right)$ is positive or negative Now ("Theory of the Gamma Function," § 7)

$$\log \prod_{s=0}^{q-1} \Gamma_1\left(z + \frac{s\omega_2}{q}\right) = \log \Gamma_1\left(z \,\middle|\, \frac{\omega_2}{q}\right) + q \log \rho_1(\omega_2) - \log \rho_1\left(\frac{\omega_2}{q}\right),$$

and by § 18 of the same paper,

$$S_1'\left(z \,\middle|\, \frac{\omega_2}{q}\right) = \sum_{s=0}^{q-1} S_1\left(z + \frac{s\omega_2}{q} \,\middle|\, \omega_2\right).$$

We therefore have

$$\chi_2\left(z + \frac{\omega_1}{p}\right) - \chi_2(z) = 2\pi\iota\,(m_{p,q} - m)\, S_1'\left(z \,\middle|\, \frac{\omega_2}{q}\right).$$

Similarly,

$$\chi_2\left(z + \frac{\omega_2}{q}\right) - \chi_2(z) = 2\pi\iota\,(m_{p,q} - m')\, S_1'\left(z \,\middle|\, \frac{\omega_1}{p}\right),$$

where m' has the value assigned in § 21, and $m'_{p,q}$ differs from it in that $p\omega_2 + q\omega_1$ must in the definition be substituted for $\omega_1 + \omega_2$.

Now we have seen that (§ 22) $m - m' = \pm 1$, the upper or lower sign being taken as $I\left(\frac{\omega_2}{\omega_1}\right)$ is negative or positive; and, since p and q are positive integers, the same is true of $m_{p,q} - m'_{p,q}$.

Thus

$$m_{p,q} - m = m'_{p,q} - m' = \mu_{p,q} \text{ (say)}.$$

and now $\chi_2(z)$ satisfies the two difference relations

$$\chi_2\left(z + \frac{\omega_1}{p}\right) - \chi_2(z) = 2\pi\iota\,\mu_{p,q}\, S_1'\left(z \,\middle|\, \frac{\omega_2}{q}\right)$$

$$\chi_2\left(z + \frac{\omega_2}{q}\right) - \chi_2(z) = 2\pi\iota\,\mu_{p,q}\, S_1'\left(z \,\middle|\, \frac{\omega_1}{p}\right);$$

and, therefore, since $\chi_2(z)$ is a quadratic polynomial in z, we must have

$$\chi_2(z) = 2\pi\iota\, \mu_{p,q}\ {}_2S_1'\left(z \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) + \text{constant}.$$

If we determine the constant by making $z = 0$, we have finally

$$\Gamma_2\left(z \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \frac{\prod\limits_{r=0}^{p-1} \prod\limits_{s=0}^{q-1} \Gamma_2\left(z + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)}{\prod\limits_{r=0}^{p-1} \prod\limits_{s=0}^{q-1}{}' \Gamma_2\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)} e^{2\pi\iota\mu_{p,q}\ {}_2S_0\left(\ \middle|\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)}.$$

From the values of $m_{p,q}$ and m, it is readily seen that $\mu_{p,q} = 0$, unless the axes to $(\omega_1 + \omega_2)$ and $(q\omega_1 + p\omega_2)$ include the axis of -1, in which case

$$\mu_{p,q} = -1 \text{ if } I(\omega_1 + \omega_2) \text{ is } + ve \text{ and } I(q\omega_1 + p\omega_2) - ve$$
$$\mu_{p,q} = +1 \text{ if } I(\omega_1 + \omega_2) \text{ is } - ve \text{ and } I(q\omega_1 + p\omega_2) + ve.$$

§ 68. The constant which enters into the transformation formula of the preceding paragraph can be expressed in terms of $\rho_2(\omega_1, \omega_2)$ and $\rho_2\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)$. For this purpose we consider the contour integral which represents the double gamma function.

Since p is a positive integer,

$$\frac{1 - e^{-\omega_1 a}}{1 - e^{-\frac{\omega_1 a}{p}}} = \sum_{r=0}^{p-1} e^{-a\frac{r\omega_1}{p}}.$$

And therefore if the integral, its contour, and the logarithm which occurs in the subject of integration, be defined as in Part III., we have

$$\frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z) + \gamma\}\, dz}{(1 - e^{-\frac{\omega_1 z}{p}})(1 - e^{-\frac{\omega_2 z}{q}})}$$
$$= \frac{\iota}{2\pi}\int_L \frac{e^{-az}(-z)^{-1}\{\log(-z) + \gamma\}\sum\limits_{r=0}^{p-1}\sum\limits_{s=0}^{q-1} e^{-z\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)}}{(1 - e^{-\omega_1 z})(1 - e^{-\omega_2 z})} dz$$

for the bisector of the angle between the axes of ω_1/p and ω_2/q is the same line as the bisector of the angle between the axes of $1/\omega_1$ and $1/\omega_2$. We therefore have by § 45, when a is positive with respect to the ω's,

$$\log\frac{\Gamma_2\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right)}{\rho_2\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)} - {}_2S_0\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right)(m_{p,q} + m'_{p,q})\, 2\pi\iota - {}_2S_1'\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) 2\mu_{p,q}\pi\iota$$
$$= \sum_{r=0}^{p-1}\sum_{s=0}^{q-1} \log\left\{\frac{\Gamma_2\left(a + \frac{r\omega_1}{p}, \frac{s\omega_2}{q}\right)}{\rho_2(\omega_1, \omega_2)}\right\} - (m + m')\, 2\pi\iota \sum_{r=0}^{p-1}\sum_{s=0}^{q-1} {}_2S_0\left(a + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)$$
$$- \sum_{r=0}^{p-1}\sum_{s=0}^{q-1} {}_2S_1'\left(a + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right) 2M\pi\iota.$$

Now by § 39 $M = 0$, unless the axes of $\frac{1}{L}$ and $(\omega_1 + \omega_2)$ include the axis of -1, in which case

$$M = \mp 1 \text{ as } I(\omega_1 + \omega_2) \text{ is positive or negative.}$$

Therefore $M_{p,q} = 0$, unless the axes of $\frac{1}{L}$ and $(q\omega_1 + p\omega_2)$ include the axis of -1, in which case

$$M_{p,q} = \mp 1, \text{ as } I(q\omega_1 + p\omega_2) \text{ is positive or negative.}$$

Again, by § 14,

$${}_2S_0\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \sum_{r=0}^{p-1} \sum_{s=0}^{q-1} {}_2S_0\left(a + \frac{r\omega_1}{p} + \frac{\omega_2}{q}\right) + pq\, {}_2B_1(\omega_1, \omega_2) - {}_2B_1\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right).$$

We therefore have

$$\log \frac{\Gamma_2\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right)}{\rho_2\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)} = \sum_{r=0}^{p-1} \sum_{r=0}^{q-1} \log \frac{\Gamma_2\left(a + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)}{\rho_2(\omega_1, \omega_2)}$$

$$+ \left[pq\,{}_2B_1(\omega_1, \omega_2) - {}_2B_1\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)\right](m + m')\, 2\pi\iota + {}_2S'_1\left(o \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) 2\pi\iota\,(M_{p,q} - M)$$

$$+ {}_2S_0\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) 2\pi\iota\,[m_{p,q} + m'_{p,q} + M_{p,q} - m - m' - M].$$

But from the values which have just been given, it is clear that

$$M_{p,q} - M = -\mu_{p,q}.$$

We thus have, when a is positive with respect to the ω's,

$$\log \Gamma_2\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \sum_{r=0}^{p-1} \sum_{s=0}^{q-1} \log \Gamma_2\left(a + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right) + 2\pi\iota\mu_{p,q}\,{}_2S_0\left(a \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right)$$

$$+ \log \rho_2\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) - pq \log \rho_2(\omega_1, \omega_2) + \left[pq\,{}_2B_1(\omega_1, \omega_2) - {}_2B_1\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)\right](m + m')\, 2\pi\iota$$

$$- {}_2B_1\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) 2\pi\iota\mu_{p,q}.$$

This result agrees with that of § 67, and on comparison of the absolute term we see that

$$\log \prod_{r=0}^{p-1} \prod_{s=0}^{q-1}{}' \Gamma_2\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right) = pq\left\{\log \rho_2(\omega_1, \omega_2) - (m + m')\, 2\pi\iota\, {}_2B_1(\omega_1, \omega_2)\right\}$$

$$- \left\{\log \rho_2\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) - (m + m' + \mu_{p,q})\, 2\pi\iota\, {}_2B_1\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)\right\}.$$

This may be regarded as the transformation formula for the double Stirling function $\rho_2(\omega_1, \omega_2)$.

Notice that when $p = q = m$, $\mu_{p,q} = 0$, ${}_2\mathrm{B}_1\left(\frac{\omega_1}{m}, \frac{\omega_2}{m}\right) = {}_2\mathrm{B}_1(\omega_1, \omega_2)$, and therefore the preceding formula becomes

$$\log \prod_{r=0}^{p-1} \prod_{s=0}^{q-1}{}' \Gamma_2\left(\frac{r\omega_1 + s\omega_2}{m}\right) = m^2 \log \rho_2(\omega_1, \omega_2) - \log \rho_2\left(\frac{\omega_1}{m}, \frac{\omega_2}{m}\right)$$
$$+ (1 - m^2)\, 2\pi\iota\,(m + m')\, {}_2\mathrm{B}_1(\omega_1, \omega_2).$$

Comparing this with the result obtained in § 63, we find

$$\log \rho_2\left(\frac{\omega_1}{m}, \frac{\omega_2}{m}\right) = \log \rho_2(\omega_1, \omega_2) + [1 - {}_2\mathrm{S}_1'(o)] \log m.$$

This result may also be obtained by the transformation of the line integral which expresses $\log \rho_2(\omega_1, \omega_2)$.

§ 69. We have still to consider the transformation of the first and second double gamma modular forms

$$\gamma_{21}(\omega_1, \omega_2) \quad \text{and} \quad \gamma_{22}(\omega_1, \omega_2).$$

With this object we write symbolically

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty}{}' \frac{1}{\left(m_1 \frac{\omega_1}{p} + m_2 \frac{\omega_2}{q}\right)^m} = \frac{1}{v^m},$$

and we write

$$\mathrm{S}_m = \sum_{r=0}^{p-1} \sum_{s=0}^{q-1}{}' \psi_2^{(m)}\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right), \text{ where } m \geqq 1\,;$$

so that S_m is a form analogous to the modular forms introduced into the theory of elliptic functions by ABEL.

By § 29 we know that within a circle of sufficiently small radius there exists the expansion

$$\log \Gamma_2(z) = -\log z - z\gamma_{22} - \frac{z^2}{2}\gamma_{21} - \sum_0^{\infty}\sum{}' \frac{z^3}{3\Omega^3} + \sum_0^{\infty}\sum{}' \frac{z^4}{4\Omega^4} - \ldots$$

Take now the formula of § 67

$$\log \Gamma_2\left(z \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \sum_{r=0}^{p-1} \sum_{s=0}^{q-1} \log \Gamma_2\left(z + \frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right) - \sum_{r=0}^{p-1} \sum_{s=0}^{q-1}{}' \log \Gamma_2\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right)$$
$$+ 2\pi\iota\mu_{p,q}\, {}_2\mathrm{S}_0\left(z \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right),$$

and expand in powers of z.

We evidently obtain

$$-\log z - \gamma_{22}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) z - \gamma_{21}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right)\frac{z^2}{2} - \frac{z^3}{3v^3} + \frac{z^4}{4v^4} - \ldots$$

$$= -\log z - \gamma_{22}(\omega_1, \omega_2) z - \gamma_{21}(\omega_1 \ \omega_2)\frac{z^2}{2} - \frac{z^3}{3\omega^3} + \frac{z^4}{4\omega^4} - \ldots$$

$$+ z\mathrm{S}_1 + \frac{z^2}{2\,!}\mathrm{S}_2 + \ldots + 2\pi\iota\mu_{p,q}\left[{}_2\mathrm{S}_0\left(o \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) + z\,{}_2\mathrm{S}_0{}'\left(o \,\middle|\, \frac{\omega_1}{p}; \frac{\omega_2}{q}\right)\right.$$

$$\left. + \frac{z^2}{2}\,{}_2\mathrm{S}_0^{(2)}\left(o \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right) + \ldots\right],$$

where we put symbolically

$$\sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty}{}' \frac{1}{(m_1\omega_1 + m_2\omega_2)^m} = \frac{1}{\omega^m}.$$

And now, equating coefficients of the various powers of z,

$$\gamma_{22}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \gamma_{22}(\omega_1, \omega_2) - \sum_{r=0}^{p-1} \sum_{s=0}^{q-1}{}' \psi_2{}'\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right) - 2\pi\iota\mu_{p,q}\,{}_2\mathrm{S}_0{}'\left(o \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right)$$

$$\gamma_{21}\left(\frac{\omega_1}{p}, \frac{\omega_2}{q}\right) = \gamma_{21}(\omega_1, \omega_2) - \sum_{r=0}^{p-1} \sum_{s=0}^{q-1} \psi_2^{(2)}\left(\frac{r\omega_1}{p} + \frac{s\omega_2}{q}\right) - 2\pi\iota\mu_{p,q}\,{}_2\mathrm{S}_0^{(2)}\left(o \,\middle|\, \frac{\omega_1}{p}, \frac{\omega_2}{q}\right),$$

the transformation equations for the first and second gamma modular forms.

Note that we also have, where $m > 2$,

$$\sum_{r=0}^{p-1} \sum_{s=0}^{q-1}{}' \psi_2^{(m)}\left(\frac{r\omega_1}{p}, \frac{s\omega_2}{q}\right) = (-)^m (m-1)\,!\left\{\frac{1}{v^m} - \frac{1}{\omega^m}\right\}.$$

If $\tau = \omega_2/\omega_1$ we may put $\omega^2\, \gamma_{21}(\omega_1, \omega_2) = g_{21}(\tau)$; and now, putting $p = 1$, $q = n$, we have

$$g_{21}\left(\frac{\tau}{n}\right) - g_{21}(\tau) = -\,\omega_1{}^2 \sum_{s=1}^{n-1} \psi_2^{(2)}\left(\frac{s\omega_2}{n}\right) - 2\pi\iota\mu_{1,n}\,\frac{n}{\tau}.$$

And putting $p = n$, $q = 1$,

$$n^2 g_{21}(n\tau) - g_{21}(\tau) = -\,\omega_1{}^2 \sum_{r=1}^{n-1} \psi_2^{(2)}\left(\frac{r\omega_1}{n}\right) - 2\pi\iota\mu_{n,1}\,\frac{n}{\tau}.$$

We get analogous formulæ by writing $\omega_1\, \gamma_{22}(\omega_1, \omega_2) = g_{22}(\tau)$.

The analogy between these results and those obtained in the analysis of elliptic functions is obvious. We cannot obtain, however, results which will connect such expressions as

$$g_{21}(\tau + 1) \text{ and } g_{21}(\tau).$$

Integral Formulæ.

§ 70. In the general theory of multiple gamma functions the fundamental integral formula expresses the fact that the integral of the n-ple gamma function can be expressed in terms of $(n+1)$-ple gamma functions of specialised parameters. As we have not yet defined the treble gamma function, we cannot prove this theorem for the case when $n = 2$. In the case when $n = 1$, this proposition reduces to ALEXEIEWSKY's theorem ("Theory of the G Function," § 13). We proceed first to translate this theorem into the notation of the present paper, and then to give an alternative proof capable of extension to the n-ple gamma functions.

The G function is substantially the double gamma function with equal parameters, the two being connected by the relation

$$\Gamma_2^{-1}(z \mid \omega, \omega) = \mathrm{G}\left(\frac{z}{\omega}\right)(2\pi)^{-\frac{z}{2\omega}}\,\omega^{z\,.\,\frac{z-2\omega}{2\omega^2}+1}.$$

["Theory of the G Function," § 29.]

By differentiating ALEXEIEWSKY's theorem we obtain

$$o = \tfrac{1}{2}\log 2\pi - z - a + \tfrac{1}{2} + (z + a - 1)\frac{d}{dz}\log\Gamma(z+a) - \frac{d}{dz}\log \mathrm{G}(z+a),$$

and therefore, writing z for $z + a$, and substituting from the relation just quoted, we find

$$o = -\frac{z}{\omega} + \tfrac{1}{2} + \left(\frac{z}{\omega} - 1\right)\omega\frac{d}{dz}\log\Gamma\left(\frac{z}{\omega}\right) + \omega\frac{d}{dz}\log\Gamma_2(z \mid \omega) + \frac{z-\omega}{\omega}\log\omega.$$

But
$$\log\Gamma\left(\frac{z}{\omega}\right) = \log\Gamma_1(z \mid \omega) - \left(\frac{z}{\omega} - 1\right)\log\omega.$$

We thus have
$$z\,\psi_1'(z \mid \omega) + \omega\,\psi_2'(z + \omega \mid \omega) = \mathrm{S}_1'(z \mid \omega). \qquad \text{(i.)}$$

On integration we have

$$\int_0^a \log\Gamma_1(a \mid \omega)\,da = a\log\Gamma_1(a \mid \omega) - \mathrm{S}_1(a \mid \omega) + \omega\log\frac{\Gamma_2(a + \omega \mid \omega)}{\Gamma_2(\omega \mid \omega)}.$$

We may put $z + a$ in place of z, so that

$$(z+a)\frac{d}{dz}\log\Gamma_1(z + a \mid \omega) + \omega\frac{d}{dz}\log\Gamma_2(z + a + \omega \mid \omega) = \mathrm{S}_1'(z + a \mid \omega).$$

And now, on integration with respect to z between the limits 0 and z, we obtain the extended formula

$$\int_0^z \log\Gamma_1(z + a \mid \omega)\,dz = (z+a)\log\Gamma_1(z + a \mid \omega) + \omega\log\frac{\Gamma_2(z + a + \omega \mid \omega)}{\Gamma_2(a + \omega \mid \omega)}$$
$$- \mathrm{S}_1(z + a \mid \omega) + \mathrm{S}_1(a \mid \omega) - a\log\Gamma_1(a \mid \omega).$$

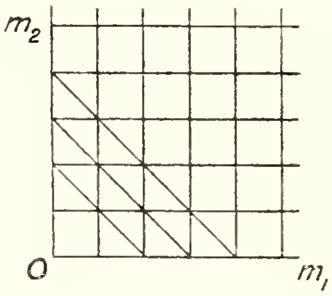
§ 71. There are two alternative methods of obtaining this formula, or rather of obtaining the fundamental relation (1).

Firstly, we may directly transform the series which expresses $\psi_2'(z\,|\,\omega, \omega)$.

We have, from the definition,

$$-\psi_2'(z\,|\,\omega, \omega) = z\gamma_{21}(\omega, \omega) + \gamma_{22}(\omega, \omega) + \frac{1}{z}$$
$$+ \sum_{0}^{\infty}\sum_{m_1\ m_2}{}' \left[\frac{1}{z + (m_1 + m_2)\,\omega} - \frac{1}{(m_1 + m_2)\,\omega} + \frac{z}{(m_1 + m_2)^2\,\omega^2}\right].$$

Put now $m_1 + m_2 = \epsilon$, a change which is equivalent to grouping together terms corresponding to points on the cross lines of the figure.



There we have

$$-\psi_2^{1}(z\,|\,\omega) = \gamma_{22}(\omega) + z\gamma_{21}(\omega) + \frac{1}{z} + \sum_{\epsilon=1}^{\infty}\left[\frac{\epsilon+1}{z+\epsilon\omega} - \frac{\epsilon+1}{\epsilon\omega} + \frac{(\epsilon+1)\,z}{\epsilon^2\omega^2}\right]$$
$$= \gamma_{22}(\omega) + z\gamma_{21}(\omega) + \frac{1}{z} + \sum_{\epsilon=1}^{\infty}\left[\frac{1}{z+\epsilon\omega} - \frac{1}{\epsilon\omega}\right]\left(1 - \frac{z}{\omega}\right) + \frac{z}{\omega^2}\cdot\frac{\pi^2}{6}.$$

Now we have seen in § 28 that

$$\gamma_{21}(\omega) = \frac{1}{\omega^2}\left[\log\omega - 1 - \gamma - \frac{\pi^2}{6}\right].$$

$$\gamma_{22}(\omega) = \frac{1}{\omega}\left[\gamma - \tfrac{1}{2} - \log\omega\right].$$

We therefore have

$$-\psi_2'(z\,|\,\omega) = \left(\frac{z}{\omega} - 1\right)\psi_1'(z - \omega\,|\,\omega) - \frac{1}{\omega}\,S_1'(z - \omega\,|\,\omega),$$

which is equivalent to the former relation.

Secondly, we may make use of the contour-integral expression in the following manner.

We have ("Theory of the Gamma Function," § 30), when a is positive with respect to ω,

$$\psi_1^{(2)}(a\,|\,\omega) = \frac{\iota}{2\pi}\int_{\frac{1}{\omega}} \frac{e^{-az}(-z)\{\log(-z) + \gamma\}}{1 - e^{-\omega z}}\,dz.$$

Therefore, under the same limitation,

$$\frac{\partial}{\partial\omega}\psi_1^{(2)}(a\,|\,\omega) = \frac{\iota}{2\pi}\int_{\frac{1}{\omega}} \frac{e^{-(a+\omega)z}(-z)^2\{\log(-z)+\gamma\}}{(1-e^{-\omega z})^2}\,dz = \psi_2^{(3)}(a+\omega\,|\,\omega).$$

Now, since

$$\psi_1^{(2)}(a\,|\,\omega) = \sum_{m=0}^{\infty}\frac{1}{(a+m\omega)^2},$$

it is evident that $\psi_1^{(2)}(a\,|\,\omega)$ is homogeneous of degree -2 in a and ω.

Therefore, by **Euler's** theorem,

$$\left(a\frac{\partial}{\partial a} + \omega\frac{\partial}{\partial\omega}\right)\psi_1^{(2)}(a\,|\,\omega) = -2\psi_1^{(2)}(a\,|\,\omega),$$

so that

$$a\psi_1^{(3)}(a\,|\,\omega) + \omega\psi_2^{(3)}(a+\omega\,|\,\omega) = 2\psi_1^{(2)}(a\,|\,\omega).$$

On integrating twice with respect to a, we obtain

$$a\psi_1'(a\,|\,\omega) + \omega\psi_2'(a+\omega\,|\,\omega) = \chi_1(a\,|\,\omega),$$

where $\chi_1(a\,|\,\omega)$ is a linear function of a.

Changing a into $a+\omega$, and subtracting, we see that $\chi_1(a\,|\,\omega)$ satisfies the difference equation characteristic of $S_1'(a\,|\,\omega)$, so that it can only differ from this function by a constant, which will vanish, as we see by making $a = 0$.

We therefore obtain again the relation required.

§ 72. We proceed now to the analogues of **Raabe's** formula. This formula may be written ("Theory of the Gamma Function," § 8)

$$\int_0^{\omega}\log\Gamma_1(z+a\,|\,\omega)\,dz = a\log a - a + \frac{\omega}{2}\log\frac{2\pi}{\omega}.$$

We will evaluate

$$\int_0^{\omega_1}\log\Gamma_2(z+a\,|\,\omega_1,\omega_2)\,dz \quad\text{and}\quad \int_0^{\omega_2}\log\Gamma_2(z+a\,|\,\omega_1\,\omega_2)\,dz.$$

The method which will be employed is the same as that by which **Raabe's** formula itself was originally obtained; it was, in fact, first invented for the proof of the present theorem.

Let

$$f(a) = \int_0^{\omega_1}\log\Gamma_2(z+a\,|\,\omega_1,\omega_2)\,dz.$$

Then

$$\begin{aligned}\frac{df(a)}{da} &= \int_0^{\omega_1}\frac{\Gamma_2'(z+a)}{\Gamma_2(z+a)}\,dz\\ &= \log\frac{\Gamma_2(a+\omega_1)}{\Gamma_2(a)}\\ &= -\log\Gamma_1(a\,|\,\omega) + \log\rho_1(\omega_2) + 2m\pi\iota\,S_1'(a\,|\,\omega_2),\end{aligned}$$

and, therefore, on integration,

$$\int_0^{\omega_1} \log \Gamma_2 (z + a \,|\, \omega_1, \omega_2) dz = - \int_0^a \log \Gamma_1 (a \,|\, \omega_2)\, da + a \log \rho_1 (\omega_2)$$
$$+ 2m\pi\iota\, S_1 (a \,|\, \omega_2) + \int_0^{\omega_1} \log \Gamma_2 (z \,|\, \omega_1, \omega_2)\, dz.$$

§ 73. We proceed to evaluate the constant term

$$\int_0^{\omega_1} \log \Gamma_2 (z \,|\, \omega_1, \omega_2)\, dz,$$

by an application of the multiplication theory for the case $m = 2$. We have

$$\int_0^{\omega_1} \log \Gamma_2 (2z)\, dz = \tfrac{1}{2} \int_0^{2\omega_1} \log \Gamma_2 (z)\, dz,$$

and therefore, by § 65,

$$\int_0^{\omega_1} dz \left\{ \log \Gamma_2 (z) + \log \Gamma_2 \left(z + \frac{\omega_1}{2} \right) + \log \Gamma_2 \left(z + \frac{\omega_2}{2}\right) + \log \Gamma_2 \left(z + \frac{\omega_1 + \omega_2}{2} \right) \right.$$
$$\left. - 3 \log \rho_2 (\omega_1, \omega_2) - {}_2S_1{}' (2z) \log 2 + 3 (m + m') 2\pi\iota\, {}_2S_1{}' (o) \right\}$$
$$= \tfrac{1}{2} \int_0^{\omega_1} dz \{ 2 \log \Gamma_2(z) - \log \Gamma_1(z \,|\, \omega_2) + \log \rho_1 (\omega_2) + 2m\pi\iota\, S_1{}'(z \,|\, \omega_2) \} .$$

Put $a = \frac{1}{2}\omega_1$, $\frac{1}{2}\omega_2$, and $\frac{1}{2} (\omega_1 + \omega_2)$ successively in the formula which we have just obtained for $\int_0^{\omega_1} \log \Gamma_2 (z + a)\, dz$, and substitute in the formula just written. We obtain

$$3\int_0^{\omega_1} \log \Gamma_2(z) dz$$
$$= \left[\int_0^{\frac{\omega_1}{2}} + \int_0^{\frac{\omega_2}{2}} + \int_0^{\frac{\omega_1 + \omega_2}{2}} - \tfrac{1}{2} \int_0^{\omega_1} \right] \{ \log \Gamma_1(z \,|\, \omega_2) dz \} - \left(\frac{\omega_1}{2} + \omega_2\right) \log \rho_1(\omega_2)$$
$$+ 2m\pi\iota \left[\tfrac{1}{2}S_1 (\omega_1 \,|\, \omega_2) - S_1\left(\frac{\omega_1}{2} \,\middle|\, \omega_2\right) - S_1 \left(\frac{\omega_2}{2} \,\middle|\, \omega_2\right) - S_1 \left(\frac{\omega_1 + \omega_2}{2} \,\middle|\, \omega_2\right) \right]$$
$$+ 3\omega_1 \log \rho_2(\omega_1, \omega_2) + \tfrac{1}{2}{}_2S_1 (2\omega_1) \log 2 - 3\omega_1 (m + m') 2\pi\iota\, {}_2S_1{}'(o).$$

By means of the formula

$$\int_0^a \log \Gamma_1(a \,|\, \omega) da = a \log \Gamma_1 (a \,|\, \omega) - S_1 (a \,|\, \omega) + \omega \log \frac{\Gamma_1(a + \omega \,|\, \omega)}{\rho_1(\omega)} ,$$

we see that

$$\left[\int_0^{\frac{\omega_1}{2}} + \int_0^{\frac{\omega_2}{2}} + \int_0^{\frac{\omega_1+\omega_2}{2}} - \tfrac{1}{2}\int_0^{\omega_1}\right] \log \Gamma_1(z \mid \omega_2)\,dz$$

$$= \frac{\omega_1}{2} \log \Gamma_1\left(\frac{\omega_1}{2} \,\middle|\, \omega_2\right) + \frac{\omega_2}{2} \log \Gamma_1\left(\frac{\omega_2}{2} \,\middle|\, \omega_2\right) + \frac{\omega_1+\omega_2}{2} \log \Gamma_1\left(\frac{\omega_1+\omega_2}{2} \,\middle|\, \omega_2\right)$$

$$- \frac{\omega_1}{2} \log \Gamma_1(\omega_1 \mid \omega_2) + \tfrac{1}{2} S_1(\omega_1 \mid \omega_2) - S_1\left(\frac{\omega_1}{2} \,\middle|\, \omega_2\right) - S_1\left(\frac{\omega_2}{2} \,\middle|\, \omega_2\right) - S_1\left(\frac{\omega_1+\omega_2}{2} \,\middle|\, \omega_2\right)$$

$$+ \omega_2 \log \left\{ \frac{\Gamma_2\left(\omega_2 + \frac{\omega_1}{2} \mid \omega_2\right) \Gamma_2\left(\omega_2 + \frac{\omega_2}{2} \mid \omega_2\right) \Gamma_2\left(\omega_2 + \frac{\omega_1+\omega_2}{2} \mid \omega_2\right)}{\Gamma_1^{\frac{1}{2}}(\omega_1 + \omega_2 \mid \omega_2)\, \rho_1^{\frac{5}{2}}(\omega_2)} \right\},$$

and this expression in turn, by utilising the multiplication formulæ when $m = 2$ for the simple and double gamma functions and the simple Bernoullian function, is equal to

$$\left(-2\omega_2 + \frac{\omega_1}{2}\right) \log \rho_1(\omega_2) + \left[\frac{\omega_1}{4} - \frac{\omega_2}{4} - \frac{\omega_1^2}{2\omega_2} + \frac{\omega_2}{2}\, {}_2S_1'(\omega_1 + \omega_2 \mid \omega_2, \omega_2)\right] \log 2$$

$$+ \tfrac{3}{4} S_2'(o \mid \omega_2) - S_1\left(\frac{\omega_2}{2} \,\middle|\, \omega_2\right) + \frac{\omega_2}{2} \log \Gamma_1\left(\frac{\omega_1+\omega_2}{2} \,\middle|\, \omega_2\right)$$

$$+ \frac{\omega_2}{2} \log \left\{ \frac{\Gamma_2^2\left(\frac{\omega_2}{2} \,\middle|\, \omega_2\right) \rho_1^3(\omega_2)}{\Gamma_1\left(\frac{\omega_1+\omega_2}{2} \,\middle|\, \omega_2\right) \Gamma_1^2\left(\frac{\omega_2}{2} \,\middle|\, \omega_2\right)} \right\} + \frac{3\omega_2}{2} \log \rho_2(\omega_2, \omega_2).$$

Now, on making $\omega_1 = \omega_2$, we obtain from the multiplication formula

$$\log\left[\Gamma_2^2\left(\frac{\omega_2}{2} \,\middle|\, \omega_2\right) \rho_1(\omega_2)\right] = 3 \log \rho_2(\omega_2, \omega_2) + [{}_2S_1'(o \mid \omega_2) - 1] \log 2.$$

Therefore the expression which we have just found reduces to

$$\left(-2\omega_2 + \frac{\omega_2}{2}\right) \log \rho_1(\omega_2) + \left\{\frac{\omega_1}{4} - \frac{\omega_2}{4} - \frac{\omega_1^2}{2\omega_2} + \frac{\omega_2}{2}\, {}_2S_1'(\omega_1 + \omega_2 \mid \omega_2, \omega_2)\right.$$

$$\left. + \frac{\omega_2}{2}\, {}_2S_1'(o \mid \omega_2, \omega_2)\right\} \log 2$$

$$+ \tfrac{3}{4} S_2'(o \mid \omega_2) - S_1\left(\frac{\omega_2}{2} \,\middle|\, \omega_2\right) + 3\omega_2 \log \rho_2(\omega_2, \omega_2).$$

And therefore

$$3\int_0^{\omega_1} \log \Gamma_2(z \mid \omega_1, \omega_2)\,dz$$

$$= 3\omega_1 \log \rho_2(\omega_1, \omega_2) + 3\omega_2 \log \rho_2(\omega_2, \omega_2) - 3\omega_2 \log \rho_1(\omega_2)$$

$$- 3\omega_1(m + m')2\pi\iota\, {}_2S_1(o) + (1 + 2m\pi\iota)\left[- S_1\left(\frac{\omega_2}{2} \,\middle|\, \omega_2\right) + \tfrac{3}{4} S_2'(o \mid \omega_2)\right],$$

since the coefficient of log 2 is equal to

$$-\frac{\omega_1^2}{2\omega_2}+\frac{\omega_1}{4}-\frac{\omega_2}{2}+\frac{\omega_2}{2}\,{}_2S_1'(\omega_1+\omega_2\,|\,\omega_2,\omega_2)+\frac{\omega_2}{2}\,{}_2S_1'(o\,|\,\omega_2,\omega_2)+\tfrac{1}{2}\,{}_2S_1(2\omega_1\,|\,\omega_1,\omega_2)=0.$$

We have, then, finally,

$$\int_0^{\omega_1}\log\Gamma_2(z)dz=\omega_1[\log\rho_2(\omega_1,\omega_2)-(m+m')2\pi\iota\,{}_2S_1'(o)]+\omega_2\log\frac{\rho_2(\omega_2,\omega_2)}{\rho_1(\omega_2)}$$
$$+\frac{\omega_2}{12}(1+2m\pi\iota).$$

We thus see that substantially the double Stirling function of ω_1 and ω_2 is expressed by $\frac{1}{\omega_1}\int_0^{\omega_1}\log\Gamma_2(z)\,dz$, or, by symmetry, by $\frac{1}{\omega_2}\int_0^{\omega_2}\log\Gamma_2(z)\,dz$. We have, in fact, the relation

$$\rho_2(\omega_1,\omega_2)-(m+m')2\pi\iota\,{}_2S_1'(o)=\frac{1}{\omega_1}\int_0^{\omega_1}\log\Gamma_2(z)dz-\frac{\omega_2}{\omega_1}\log A$$
$$+\frac{\omega_2}{12\omega_1}[\log\omega_2-2m\pi\iota],$$

where A is the Glaisher-Kinkelin constant ("Theory of the G Function," § 3); and therefore

$$\rho_2(\omega_1,\omega_2)-(m+m')\,2\pi\iota\,{}_2S_1'(o)$$
$$=\frac{\omega_1}{\omega_1^2-\omega_2^2}\int_0^{\omega_1}\log\Gamma_2(z)\,dz-\frac{\omega_2}{\omega_1^2-\omega_2^2}\int_0^{\omega_2}\log\Gamma_2(z)\,dz+\frac{\omega_1\omega_2}{12(\omega_1^2-\omega_2^2)}\log\frac{\omega_2}{\omega_1};$$

for, by § 22, $\log\omega_2-\log\omega_1-2(m-m')\pi\iota=\log\frac{\omega_2}{\omega_1}$.

§ 74. We may readily prove these results by the relation which exists between A and $\rho_2(\omega)$. [The latter is a convenient way of writing $\rho_2(\omega,\omega)$.]

For, when each of the parameters is equal to ω, we have ("Theory of the G Function," § 29)

$$\Gamma_2^{-1}(z\,|\,\omega)=G\left(\frac{z}{\omega}\right)(2\pi)^{-\frac{z}{2\omega}}\,\omega^{\frac{(z-\omega)^2}{2\omega^2}+\frac{1}{2}},$$

and therefore

$$G(\tfrac{1}{2})=\Gamma_2^{-1}\left(\frac{\omega}{2}\,|\,\omega\right)(2\pi)^{\frac{1}{4}}\,\omega^{-\frac{5}{8}}.$$

Now from the multiplication theorem for the double gamma functions, when $m=2$, we have

$$\Gamma_2^2\left(\frac{\omega}{2}\,|\,\omega\right)\left(\frac{2\pi}{\omega}\right)^{\frac{1}{2}}=\rho^3(\omega)\,2^{-\frac{7}{12}}.$$

Hence

$$G(\tfrac{1}{2})=\frac{(2\pi)^{\frac{1}{2}}}{\omega^{\frac{7}{8}}}\,\rho_2^{-\frac{3}{2}}(\omega)\,2^{\frac{7}{24}}.$$

Now it has been seen that ("Theory of the G Function," § 17)

$$\mathrm{A} = \frac{2^{\frac{1}{36}} e^{\frac{1}{12}}}{\pi^{\frac{1}{6}} \mathrm{G}^{\frac{2}{3}}(\frac{1}{2})},$$

and therefore

$$\mathrm{A} = \frac{\rho_2(\omega)}{\rho_1(\omega)} (e\omega)^{\frac{1}{12}},$$

which is the relation which was used at the end of the preceding paragraph.

§ 75. It is interesting as a verification of the algebra to notice that Alexeiewsky's analogue of Raabe's formula ("Theory of the G Function," § 16) yields the result of § 73 in the case where the parameters are equal to one another.

This theorem is expressed by the formula

$$\int_0^1 \log \mathrm{G}(z+1)\, dz = \tfrac{4}{3} \log \mathrm{G}(\tfrac{1}{2}) + \tfrac{7}{12} \log \pi + \tfrac{7}{36} \log 2 - \tfrac{1}{12},$$

and therefore, utilising the relation between the G and double gamma functions,

$$\int_0^\omega \log \Gamma_2(a+\omega \mid \omega)\, da = -\frac{4\omega}{3} \log \mathrm{G}(\tfrac{1}{2}) + \frac{\omega}{6} \log 2\pi + \frac{7\omega}{18} \log 2 + \frac{\omega}{12} - \frac{2\omega}{3} \log \omega .$$

If now we express $\log \mathrm{G}(\frac{1}{2})$ in terms of $\log \rho_2(\omega)$ by the formula of § 74, we find

$$\int_0^\omega \log \Gamma_2(a+\omega \mid \omega)\, da = \omega \log \rho_2(\omega) + \omega \log \frac{\rho_2(\omega)}{\rho_1(\omega)} + \frac{\omega}{12},$$

a formula which is equivalent to the result of § 73.

§ 76. By combining the results of §§ 72 and 73, we may now write down the value of

$$\int_0^{\omega_1} \log \Gamma_2(z + a \mid \omega_1, \omega_2)\, dz .$$

For we have seen in § 72 that this integral is equal to

$$-\int_0^a \log \Gamma_1(a \mid \omega_2) + a \log \rho_1(\omega_2) + 2m\pi\iota\, \mathrm{S}_1(a \mid \omega_2) + \int_0^{\omega_1} \log \Gamma_2(z \mid \omega_1, \omega_2)\, dz,$$

which expression in turn is equal to

$$\begin{aligned} -a \log \Gamma_1(a \mid \omega_2) &+ \mathrm{S}_1(a \mid \omega_2)[1 + 2m\pi\iota] - \omega_2 \log \Gamma_2(a + \omega_2 \mid \omega_2) + (a + \omega_2) \log \rho_1(\omega_2) \\ &+ \omega_1 [\log \rho_2(\omega_1, \omega_2) - (m + m')\, 2\pi\iota\, {}_2\mathrm{S}_1'(o)] + \omega_2 \log \frac{\rho_2(\omega_2, \omega_2)}{\rho_1(\omega_2)} + \frac{\omega_2}{12}(1 + 2m\pi\iota). \end{aligned}$$

Thus

$$\begin{aligned} \int_0^{\omega_1} &\log \Gamma_2(z + a \mid \omega_1, \omega_2)\, dz \\ &= -a \log \frac{\Gamma_1(a \mid \omega_2)}{\rho_1(\omega_2)} + \omega_1 [\log \rho_2(\omega_1, \omega_2) - (m + m')\, 2\pi\iota\, {}_2\mathrm{S}_1'(o)] \\ &+ (1 + 2m\pi\iota) \left\{ \mathrm{S}_1(a \mid \omega_2) + \frac{\omega_2}{12} \right\} - \omega_2 \log \frac{\Gamma_2(a + \omega_2 \mid \omega_2)}{\rho_2(\omega_2, \omega_2)} \end{aligned}$$

Since, by § 64,

$$\log \rho_2(\omega_1, \omega_2) - (m + m')\, 2\pi\iota\, {}_2S_1'(o) = \tfrac{1}{3} \log \gamma_1\gamma_2\gamma_3 + \tfrac{1}{3}[1 - {}_2S_1'(o)] \log 2,$$

we see that this formula may equally be written

$$\int_0^{\omega_1} \log \Gamma_2(z + a \mid \omega_1, \omega_2)\, dz$$

$$= -\, a \log \frac{\Gamma_1(a \mid \omega_2)}{\rho_1(\omega_2)} + \frac{\omega_1}{3} \log \gamma_1\gamma_2\gamma_3 + \frac{\omega_1}{3}[1 - {}_2S_1'(o)] \log 2$$

$$+ (1 + 2m\pi\iota) \left\{ S_1(a \mid \omega_2) + \frac{\omega_2}{12} \right\} - \omega_2 \log \frac{\Gamma_2(a + \omega_2 \mid \omega_2)}{\rho_2(\omega_2)}.$$

This and the corresponding formula, obtained by the interchange of ω_1 and ω_2, m and m', are the analogues for double gamma functions of RAABE's formula for simple gamma functions.

§ 77. In the particular case when a is positive with respect to the ω's, it is possible to obtain more simply the value of

$$\int_0^{\omega_1} \log \Gamma_2(a \mid \omega_1, \omega_2)\, da$$

by means of the contour integrals investigated in Part III.

We give this method of proof as it leads incidentally to an expression as a contour integral for $\log \dfrac{\rho_2(\omega_2)}{\rho_1(\omega_2)}$.

We find, on integrating the expression for $\log \dfrac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)}$, given in § 45,

$$\int_0^{\omega_1} \log \Gamma_2(a)\, da - \omega_1 \log \rho_2(\omega_1, \omega_2)$$

$$= (M + m + m')\, 2\pi\iota \int_0^{\omega_1} {}_2S_0(a)\, da + \omega_1\, {}_2S_1'(o)\, 2M\pi\iota - \frac{\iota}{2\pi} \int_L \frac{(-z)^{-2}\{\log(-z) + \gamma\}}{(1 - e^{-\omega_2 z})}\, dz,$$

and the right-hand side, by an application of the formulæ of §§ 6 and 44, becomes

$$(M + m + m')\, 2\pi\iota \left[-\, \omega_1\, {}_2S_1'(o) + \frac{S_2'(o \mid \omega_2)}{2} \right] + \omega_1\, {}_2S_1'(o)\, 2M\pi\iota$$

$$- \frac{\iota}{2\pi} \int_{\frac{1}{\omega_2}} \frac{(-z)^{-2}\{\log(-z) - (M + m')\, 2\pi\iota + \gamma\}}{(1 - e^{-\omega_2 z})}\, dz.$$

Now, reducing the contour to a small circle round the origin, we see that

$$\frac{\iota}{2\pi}(M + m')\, 2\pi\iota \int_{\frac{1}{\omega_2}} \frac{(-z)^{-2} dz}{1 - e^{-\omega_2 z}} = -\, \frac{S_2'(o \mid \omega_2)}{2} (M + m')\, 2\pi\iota,$$

and therefore

$$\int_0^{\omega_1} \log \Gamma_2(a)\,da - \omega_1 \log \rho_2(\omega_1, \omega_2) + \omega_1 (m + m')\, 2\pi\iota\, {}_2S_1'(o) - m 2\pi\iota \frac{S_2'(o\,|\,\omega_2)}{2}$$
$$= - \frac{\iota}{2\pi} \int_{\frac{1}{\omega_2}} \frac{(-z)^{-2}\{\log(-z) + \gamma\}}{1 - e^{-\omega_2 z}}\,dz\,.$$

Since $\frac{S_2'(o\,|\,\omega_2)}{2} = \frac{\omega_2}{12}$, we see that, to establish the formula of § 73, it is necessary to show that

$$-\frac{\iota}{2\pi} \int_{\frac{1}{\omega}} \frac{(-z)^{-2}\{\log(-z) + \gamma\}}{1 - e^{-\omega z}}\,dz = \omega \log \frac{\rho_2(\omega)}{\rho_1(\omega)} + \frac{\omega}{12}.$$

This may be readily done as follows :—

We have

$$\log \frac{\Gamma_2(a\,|\,\omega)}{\rho_2(\omega)} = \frac{\iota}{2\pi} \int_{\frac{1}{\omega}} \frac{e^{-az}(-z)^{-1}\{\log(-z) + \gamma\}}{(1 - e^{-\omega z})^2}\,dz\,.$$

Integrate with respect to a between ω and 2ω, and we find

$$\int_{\omega}^{2\omega} \log \Gamma_2(a\,|\,\omega)\,da - \omega \log \rho_2(\omega) = -\frac{\iota}{2\pi} \int_{\frac{1}{\omega}} \frac{e^{-\omega z}(-z)^{-2}\{\log(-z) + \gamma\}}{1 - e^{-\omega z}}\,dz$$
$$= -\frac{\iota}{2\pi} \int_{\frac{1}{\omega}} \frac{(-z)^{-2}\{\log(-z) + \gamma\}}{1 - e^{-\omega z}}\,dz\,,$$

for it may be readily seen that $\frac{\iota}{2\pi} \int_{\frac{1}{\omega}} \frac{\log(-z) + \gamma}{z^2}\,dz = 0.$

[This vanishing contour integral occurs when Raabe's formula is proved by means of the expression of the simple gamma function as a contour integral.]

But as we have deduced in § 75, from Alexeiewsky's theorem,

$$\int_{\omega}^{2\omega} \log \Gamma_2(a\,|\,\omega)\,da - \omega \log \rho_2(\omega) = \omega \log \frac{\rho_2(\omega)}{\rho_1(\omega)} + \frac{\omega}{12}.$$

The contour integral has, therefore, the required value.

We here conclude our investigations of the algebra of the double gamma functions. It is evident that the formulæ admit of still further development ; they lead, for instance, to many curious relations between the integrals obtained in Part III. Such considerations are, however, foreign to our immediate purpose. The development of the integral formulæ and the theories of multiplication and transformation in the case of the double Riemann ζ function is interesting in that we thus combine many of the formulæ which have been obtained separately for Bernoullian and gamma functions, and the algebra by which such developments are obtained by the extension of Mellin's definition of the simple ζ function is in many ways attractive. Owing, however, to the length of this paper, we do not propose to consider it in this place.

PART V.

The Asymptotic Expansion and Transcendentally-transcendental Nature of the Double Gamma Function.

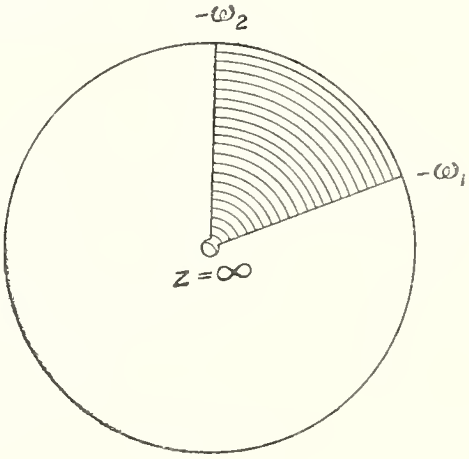
§ 78. There remains now the consideration of two more general characteristics of the double gamma function :—

1. It admits over part of the region near infinity an asymptotic expansion in powers of the variable.
2. It cannot be obtained as the solution of a differential equation whose coefficients involve exclusively more simple functions.

It will afterwards be seen that these characteristics are common to all gamma functions.

Let us consider first the behaviour of $\Gamma_2(z)$ near infinity. We know that its poles are given by

$$z + m_1\omega_1 + m_2\omega_2 = 0 \qquad \left.\begin{array}{l} m_1 = 0, 1, \ldots \infty \\ m_2 = 0, 1, \ldots \infty \end{array}\right\}$$



Therefore near $z = \infty$ the poles of $\Gamma_2(z)$ are massed together between and on the negative axes of ω_1 and ω_2, so as to form a lacunary space on the equivalent portion of the Neumann sphere. Between these axes, therefore, an asymptotic expansion cannot represent the function. We have to consider whether such an expansion can exist outside this lacunary area, within, that is to say, the non-shaded portion of the figure.

We shall in the first place proceed entirely algebraically. It will be proved that within this non-shaded area a quasi-Laurent asymptotic expansion of the form

$$(1, z)^2 \log z + (1, z)^2 + \sum_{r=1}^{\infty} \frac{\lambda_r}{z^r}$$

exists, and then it will be shown that, the possibility of such an expansion being established, its actual form is readily obtained by a process of difference-integration. Subsequently we shall verify the results by an alternative proof by means of contour integration, this proof being the natural extension to double gamma functions of the one employed for functions of a single parameter ("Theory of the Gamma Function," Part IV.).

It should be noticed that the expansion under consideration differs materially from the expansions obtained in Part III. In those expansions the limits of the number of terms of the series and products, the quantities pn and qn, formed the infinite basis terms; but in the present case that basis is the variable itself.

§ 79. We first write down the asymptotic expansion for

$$\log \Gamma_2(z + a \mid \omega, \omega).$$

We have obtained ("Theory of the G Function," § 15) the asymptotic expansion

$$\log G(z + a) = \frac{1}{12} - \log A + \frac{z + a - 1}{a} \log 2\pi + \left(\frac{(z + a - 1)^2}{2} - \frac{1}{12}\right) \log z$$
$$- \frac{3z^2}{4} - (a - 1)z + \sum_{n=1}^{\infty} \frac{(-)^{n-1}}{nz^n} S_n(a \mid 1, 1) + \sum_{n=1}^{\infty} \frac{(-)^n B_{n+1}}{2n(2n + 2)z^{2n}}$$
$$+ \sum_{n=1}^{\infty} \frac{(-)^n B_n}{2n(2n - 1)z^{2n-1}},$$

this expansion being valid for all values of a and z such that $|z|$ is large, and the principal value of $\log z$ being taken. There is, in the language previously employed, a barrier-line along the axis of -1.

Now in § 70 it has been seen that

$$\log G\left(\frac{z}{\omega}\right) = -\log \Gamma_2(z \mid \omega) + \frac{z}{2\omega} \log 2\pi - \left(\frac{(z - \omega)^2}{2\omega^2} + \frac{1}{2}\right) \log \omega,$$

and

$$\log A - \frac{1}{12} = \log \frac{\rho_2(\omega)}{\rho_1(\omega)} + \frac{1}{2} \log \omega.$$

Therefore

$$- \log \frac{\Gamma_2(z + a \mid \omega)}{\rho_2(\omega)} = \left(\frac{(z + a - \omega)^2}{2\omega^2} - \frac{1}{12}\right) \log z - \frac{3z^2}{4\omega^2} - z\left(\frac{a}{\omega^2} - \frac{1}{\omega}\right)$$
$$+ \sum_{n=1}^{\infty} \frac{(-)^{-m}}{mz^m} \left\{ {}_2S_m(a \mid \omega) + \frac{{}_2S'_{m+1}(o)}{m + 1} \right\},$$

where now there is a barrier-line along the axis of $-\omega$.

But by § 5

$$\frac{{}_2S'_{m+1}(a \mid \omega)}{m + 1} = {}_2S_m(a \mid \omega) + \frac{{}_2S'_{m+1}(o)}{m + 1},$$

so that we have finally

$$\log \frac{\Gamma_2(z+a\,|\,\omega)}{\rho_2(\omega)} = -\,{}_2S_1'(z+a\,|\,\omega)\log z + \frac{z}{1}\,{}_2S_1^{(2)}(a\,|\,\omega) + \frac{3z^2}{4\omega^2} + \sum_{m=1}^{\infty} \frac{(-)^m {}_2S'_{m+1}(a\,|\,\omega)}{m\,(m+1)z^m},$$

with a barrier-line along the axis of $-\,\omega$.

When $a = \omega$, we see that we have the asymptotic expansion

$$\log \frac{\Gamma_2(z+\omega\,|\,\omega)}{\rho_2(\omega)} = \left(\frac{1}{12} - \frac{z^2}{2\omega^2}\right)\log z + \frac{3z^2}{4\omega^2} + \sum_{m=1}^{\infty} \frac{(-)^{m-1},\ B_{m+2}(\omega)}{m\omega z^m}.$$

§ 80. We will now prove that, provided z be positive with respect to the ω's, there exists an asymptotic expansion for $\log \Gamma_2(z)$ of the form

$$(1, z)^2 \log z + (1, z)^2 + \sum_{\lambda=1}^{\infty} \frac{\lambda r}{z^r},$$

where $(1, z)^2$ denotes a quadratic function of z.

In the first place it is evident that we have

$$z = a + n_1\omega_1 + n_2\omega_2,$$

where a is some finite quantity, and n_1 and n_2 are singly or together large positive integers.

Now, from the fundamental difference equations of the double gamma function, it is at once seen that

$$\begin{aligned}
&\log \Gamma_2(a) - \log \Gamma_2(a + n_1\omega_1 + n_2\omega_2) \\
&= \log \prod_{m_1=0}^{n_1} \prod_{m_2=0}^{n_2} (a + m_1\omega_1 + m_2\omega_2) + \log \prod_{m_1=0}^{n_1} \frac{\Gamma_1(a + m_1\omega_1\,|\,\omega_2)}{\rho_1(\omega_2)} \\
&\quad + \log \prod_{m_2=0}^{n_2} \frac{\Gamma_1(a + m_2\omega_2\,|\,\omega_1)}{\rho_1(\omega_1)} - 2m\pi\iota \sum_{m_1=0}^{n_1} S_1'(a + m_1\omega_1\,|\,\omega_2) \\
&\quad - 2m'\pi\iota \sum_{m_2=0}^{n_2} S_1'(a + m_2\omega_2\,|\,\omega_2).
\end{aligned}$$

A term has been neglected which is an integral multiple of $2\pi\iota$, and which is therefore absorbed by a suitable specification of the logarithms involved. The above formula may be rewritten

$$\begin{aligned}
\log \Gamma_2(a + n_1\omega_1 + n_2\omega_2) &= \left[\log \Gamma_2(a) - \log \prod_{m_1=0}^{n_1} \frac{\Gamma_1(a + m_1\omega_1\,|\,\omega_2)}{\rho_1(\omega_2)}\right] \\
&\quad + \left[\log \Gamma_2(a) - \log \prod_{m_2=0}^{n_2} \frac{\Gamma_1(a + m_2\omega_2\,|\,\omega_1)}{\rho_1(\omega_1)}\right] \\
&\quad - \left[\log \Gamma_2(a) + \log \prod_{m_1=0}^{n_1} \prod_{m_2=0}^{n_2} (a + m_1\omega_1 + m_2\omega_2)\right] + a\,(1, n)^2 + (1, n)^2.
\end{aligned}$$

In the first place, if we put n_1 in place of pn and n_2 in place of qn in the procedure of the paragraphs leading up to § 49, we evidently obtain for

$$\log \Gamma_2(a) + \log \prod_{m_1=0}^{n_1} \prod_{m_2=0}^{n_2} (a + m_1\omega_1 + m_2\omega_2),$$

an expansion in powers of $\frac{1}{n_1}$ and $\frac{1}{n_2}$, every term of which involves a algebraically, and of which the non-ultimately-vanishing terms are typified by $(1, n)^2 \log n + (1, n)^2$.

In the second place, consider

$$\log \Gamma_2(a) - \log \prod_{m_1=0}^{n_1} \frac{\Gamma_1(a + m_1\omega_1 \mid \omega_2)}{\rho_1(\omega_2)}.$$

We have seen, in § 30, that

$$\Gamma_2(a) = e^{-\gamma_{21}\frac{a^2}{2} - a\left\{\gamma_{22} + \frac{1}{\omega_2}\log\omega_2 - \frac{\gamma}{\omega_2}\right\}} \Gamma_1(a \mid \omega_2)$$
$$\times \prod_{m_1=1}^{\infty} \left[\frac{\Gamma_1(a + m_1\omega_1 \mid \omega_2)}{\Gamma_1(m_1\omega_1 \mid \omega_2)} e^{-a\psi_1^{(1)}(m_1\omega_1 \mid \omega_2) - \frac{a^2}{2}\psi_1^{(2)}(m_1\omega_1 \mid \omega_2)}\right],$$

the product being absolutely convergent.

The typical term may be written

$$\text{Exp.}\left[\frac{a^3}{3!}\psi_1^{(3)}(m_1\omega_1 \mid \omega_2) + \frac{a^4}{4!}\psi_1^{(4)}(m_1\omega_1 \mid \omega_2) + \ldots\right]$$

Therefore ("Genesis of the Double Gamma Function," §§ 4 and 5)

$$\log \Gamma_2(a) - \log \prod_{m_1=0}^{n_1} \Gamma_1(a + m_1\omega_1 \mid \omega_2)$$

admits an asymptotic expansion of the form

$$(1, n_1^2) \log n_1 + (1, n_1)^2 + \sum_{r=1}^{\infty} \frac{\lambda_r}{n_1^r},$$

each term of which involves a algebraically.

Combining these results we see that

$$\log \Gamma_2(a + n_1\omega_1 + n_2\omega_2)$$

admits an asymptotic expansion in powers of $\frac{1}{n_1}$ and $\frac{1}{n_2}$, each term of which involves a algebraically, and of which the terms which do not ultimately vanish are typified by $(1, n)^2 \log n + (1, n)^2$.

But $\log \Gamma_2(a + n_1\omega_1 + n_2\omega_2)$ is a function of $a + n_1\omega_1 + n_2\omega_2$. It must then be capable of an asymptotic expansion in powers of $\frac{1}{n_1\omega_1 + n_2\omega_2}$, each term of which involves a algebraically, and of which the terms which do not ultimately vanish are typified by

$$(1,\, n_1\omega_1 + n_2\omega_2)^2 \log (n_1\omega_1 + n_2\omega_2) + (1,\, n_1\omega_1 + n_2\omega_2)^2 .$$

And now by mere re-arrangement we may include a with the term $n_1\omega_1 + n_2\omega_2$ each time that the latter occurs, and we obtain for $\log \Gamma_2(z)$, where $z = a + n_1\omega_1 + n_2\omega_2$, an asymptotic expansion of the form

$$(1,\, z)^2 \log z + (1,\, z)^2 + \sum_{\lambda=1}^{\infty} \frac{\lambda_r}{z^r} .$$

§ 81. We can readily extend the previous proof to the case where z lies between the axes of $-\omega_1$ and ω_2, so that it is given by

$$z = a - n_1\omega_1 + n_2\omega_2 .$$

By writing the fundamental difference equation in the form

$$\Gamma_2(z - \omega_1) = \Gamma_2(z) \frac{\Gamma_1(z - \omega_1 \,|\, \omega_2)}{\rho_1(\omega_2)} e^{-2m\pi\iota S_1'(z - \omega_1 \,|\, \omega_2)}$$

we readily see that

$$\begin{aligned}
&\log \Gamma_2 (z - n_1\omega_1 + n_2\omega_2) \\
&\quad = \left[\log \Gamma_2 (z) - \sum_{m_2=0}^{n_2-1} \log \frac{\Gamma_1(z + m_2\omega_2 \,|\, \omega_1)}{\rho_1(\omega_1)}\right] \\
&\qquad + \left[\log \Gamma_2(z - \omega_1 \,|\, -\omega_1, \omega_2) + \sum_{m_1=1}^{n_1} \sum_{m_2=0}^{n_2-1} \log (z - m_1\omega_1 + m_2\omega_2)\right] \\
&\qquad + \left[-\log \Gamma_2 (z - \omega_1 \,|\, -\omega_1, \omega_2) + \sum_{m_1=1}^{n_1} \log \frac{\Gamma_1(z - m_1\omega_1 \,|\, \omega_2)}{\rho_1(\omega_2)}\right] \\
&\qquad - \sum_{m_1=1}^{n_1} 2m\pi\iota S_1'(z - m_1\omega_1 \,|\, \omega_2) + \sum_{m_2=0}^{n_2-1} 2m'\pi\iota\, S_1'(z + m_2\omega_2 \,|\, \omega_1) .
\end{aligned}$$

But by the theorems just quoted in the previous paragraph the three expressions in the square brackets severally admit of asymptotic expansions in powers of $\frac{1}{n_1}$ and $\frac{1}{n_2}$, whose terms which do not ultimately vanish are typified by

$$(1,\, n)^2 \log n + (1,\, n)^2$$

and whose coefficients all involve a algebraically.

Thus, by a repetition of the previous argument, $\log \Gamma_2(a - n_1\omega_1 + n_2\omega_2)$ admits when $z = a - n_1\omega_1 + n_2\omega_2$, an asymptotic expansion of the form

$$(1,\, z)^2 \log z + (1,\, z)^2 + \sum_{\lambda=1}^{\infty} \frac{\lambda_r}{z^r} .$$

In an exactly similar manner we may show that $\log \Gamma_2(z)$ will admit of an asymptotic expansion of the same form when $|z|$ is large and lying between the axes of $-\omega_2$ and ω_1.

§ 82. But when we come to the case of the negative quasi-quadrant given by

$$z = a - n_1\omega_1 - n_2\omega_2,$$

it is interesting to notice that the above proof breaks down.

As before, by the use of the fundamental difference equation, we obtain the relation

$$\begin{aligned}
&\log \Gamma_2(a - n_1\omega_1 - n_2\omega_2) - \log \Gamma_2(a) - \log \Gamma_2(a - \omega_1 \,|\, -\omega_1, \omega_2) \\
&\qquad - \log \Gamma_2(a - \omega_2 \,|\, \omega_1, -\omega_2) - \log \Gamma_2(a - \omega_1 - \omega_2 \,|\, -\omega_1, -\omega_2) \\
&= \left[- \log \Gamma_2(a - \omega_1 \,|\, -\omega_1, \omega_2) + \sum_{m_1=1}^{n_1} \log \frac{\Gamma_1(a - m_1\omega_1 \,|\, \omega_2)}{\rho_1(\omega_2)} \right] \\
&\quad + \left[- \log \Gamma_2(a - \omega_2 \,|\, \omega_2, -\omega_1) + \sum_{m_2=1}^{n_2} \log \frac{\Gamma_1(a - m_2\omega_2 \,|\, \omega_1)}{\rho_1(\omega_2)} \right] \\
&\quad - \left[\log \Gamma_2(a - \omega_1 - \omega_2 \,|\, -\omega_1, -\omega_2) + \sum_{m_2=1}^{n_2} \sum_{m_1=1}^{n_2} \log (a - m_1\omega_1 - m_2\omega_2) \right] \\
&\quad - \sum_{m_1=1}^{n_1} 2m\pi\iota\, S_1'(a - m_1\omega_1 \,|\, \omega_2) - \sum_{m_2=1}^{n_2} 2m'\pi\iota\, S_1'(a - m_2\omega_2 \,|\, \omega_1) \Big].
\end{aligned}$$

The several expressions bracketed on the right-hand side of this identity admit of asymptotic expansions in powers of $\frac{1}{n_1}$ and $\frac{1}{n_2}$, of which the terms involve a algebraically; and therefore the whole of the right-hand side admits of an expansion of this form. But there remain the non-algebraic terms

$$\begin{aligned}
\log \Gamma_2(a) + \log \Gamma_2(a - \omega_1 \,|\, -\omega_1, \omega_2) + \log \Gamma_2(a - \omega_2 \,|\, \omega_1, -\omega_2)& \\
+ \log \Gamma_2(a - \omega_1 - \omega_2 \,|\, -\omega_1, -\omega_2)&,
\end{aligned}$$

and when we seek to group $-n_1\omega_1 - n_2\omega_2$ with a, we are forced back on the original function $\Gamma_2(a - n_1\omega_1 - n_2\omega_2)$. Thus as regards the possibility of an expansion, when z is negative with regard to the ω's, our results are, as we should expect from § 78, entirely negative. The region between the axes of $-\omega_1$ and $-\omega_2$ is a *barrier-region* for the asymptotic expansion of the double gamma function. When $\omega_1 = \omega_2$ this region closes up into the barrier-line which occurs for the G and simple gamma functions.

§ 83. We can now find the asymptotic expansion of

$$\log \frac{\Gamma_2(z + a \,|\, \omega_1, \omega_2)}{\Gamma_2(z \,|\, \omega_1, \omega_2)},$$

for large values of $|z|$ which are such that z does not lie in the barrier-region, a being any complex quantity of finite modulus.

For such values of z and a we have the expansion

$$\log \frac{\Gamma_2(z+a)}{\Gamma_2(z)} = [f_1(a)z + f_2(a)]\log z + \phi_1(a)z + \phi_2(a) + \sum_{r=1}^{\infty} \frac{(-)^r \chi_r(a)}{r(r+1)z^r},$$

where $f(a)$ and $\phi(a)$ are algebraical polynomials of degree indicated by their suffixes, and $\chi_r(a)$ is, so long as r is finite, likewise an algebraical function.

Now by the fundamental difference equation

$$\log \frac{\Gamma_2(z+a+\omega_1)}{\Gamma_2(z)} = \log \frac{\Gamma_2(z+a)}{\Gamma_2(z)} - \log \frac{\Gamma_1(z+a\,|\,\omega_2)}{\rho_1(\omega_2)} + 2m\pi\iota\, S_1'(z+a\,|\,\omega_2).$$

Again ("Theory of the Gamma Function," § 41), we have the asymptotic expansion

$$\log \frac{\Gamma_1(z+a\,|\,\omega_2)}{\rho_1(\omega_2)} = S_1'(z+a\,|\,\omega_2)\log_{\omega_2} z - z\, S_1^{(2)}(z+a\,|\,\omega_2) + \sum_{m=1}^{\infty} \frac{(-)^{m+1} S'_{m+1}(a\,|\,\omega_2)}{m(m+1)z^m},$$

where $\log_{\omega_2} z$ is that natural logarithm of z which has its principal value with respect to the axis of $-\omega_2$. It is thus equal to

$$\log z - 2m'\pi\iota,$$

the latter logarithm having its principal value with respect to the axis of $-(\omega_1+\omega_2)$.

We have then, if $\log z$ have its principal value with respect to the axis of $-(\omega_1+\omega_2)$,

$$\begin{aligned}
&[f_1(a+\omega_1)z + f_2(a+\omega_1) - f_1(a)z - f_2(a)]\log z \\
&\qquad + z\{\phi_1(a+\omega_1) - \phi_1(a)\} + \phi_2(a+\omega_1) - \phi_2(a) + \sum_{r=1}^{\infty} (-)^r \frac{\chi_r(a+\omega_1) - \chi_r(a)}{r(r+1)z^r} \\
&= - S_1'(z+a\,|\,\omega_2)[\log z - 2(m+m')\pi\iota] + z\, S_1^{(2)}(z+a\,|\,\omega_2) + \sum_{r=1}^{\infty} \frac{(-)^r S'_{r+1}(a\,|\,\omega_2)}{r(r+1)z^r}.
\end{aligned}$$

If we equate corresponding powers of z on both sides of this result, we find

$$\chi_r(a+\omega_1) - \chi_r(a) = S'_{r+1}(a\,|\,\omega_2),$$

and similar relations among the f's and ϕ's.

We shall get, in like manner, another set of relations in which ω_1 and ω_2 are interchanged. Remembering that the f's, ψ's, and χ's are all algebraical polynomials which vanish with a, we thus prove that

$$\begin{aligned}
f_1(a)z + f_2(a) &= {}_2S_0(z) - {}_2S_0(z+a) \\
\phi_1(a)z + \phi_2(a) &= z\{{}_2S_0'(z+a) - {}_2S_0'(z)\} \\
\chi_r(a) &= {}_2S'_{m+1}(a) - {}_2S'_{m+1}(o).
\end{aligned}$$

By this process, which may appropriately be called a process of finite integration, we obtain the asymptotic expansion

$$\log\frac{\Gamma_2(z+a)}{\Gamma_2(z)} = [{}_2S_0(z) - {}_2S_0(z+a)]\,[\log z - 2(m+m')\pi\iota]$$
$$+ z[{}_2S_0'(z+a) - {}_2S_0'(z)] + \sum_{m=1}^{\infty}\frac{(-)^m\,{}_2S_m(a)}{mz^m},$$

log z having its principal value with respect to the axis of $-(\omega_1 + \omega_2)$.

§ 84. We may now obtain the asymptotic expansion for $\log\Gamma_2(z+a)$, under the limitations assigned at the commencement of the preceding paragraph.

For this purpose integrate the relation just obtained with respect to a between the limits o and ω_1.

Then, by the formulæ of §§ 12 and 76, we find

$$-z\log\frac{\Gamma_1(z\,|\,\omega_2)}{\rho_1(\omega_2)} + \omega_1[\log\rho_2(\omega_1, \omega_2) - (m+m')\,2\pi\iota\,{}_2S_1'(o)]$$
$$+ (1+2m\pi\iota)\left\{S_1(z\,|\,\omega_2) + \frac{\omega_2}{12}\right\} - \omega_2\log\frac{\Gamma_2(z+\omega_2\,|\,\omega_2)}{\rho_2(\omega_2)} - \omega_1\log\Gamma_2(z)$$
$$= [{}_2S_1(z) - {}_2S_1(z+\omega_1) + \omega_1\,{}_2S_1'(z)]\,[\log z - 2(m+m')\pi\iota]$$
$$+ z[{}_2S_0(z+\omega_1) - {}_2S_0(z) - \omega_1\,{}_2S_0'(z)] + \sum_{m=1}^{\infty}\frac{(-)^m[{}_2S_{m+1}(\omega_1) - \omega_1\,{}_2S'_{m+1}(o)]}{m(m+1)z^m}.$$

Substitute now the asymptotic expansions for $\log\dfrac{\Gamma_1(z\,|\,\omega_2)}{\rho_1(\omega_2)}$ and $\log\dfrac{\Gamma_2(z+\omega_2\,|\,\omega_2)}{\rho_2(\omega_2)}$, of which the former has been quoted in the preceding paragraph, and the latter obtained in § 79.

Then we find

$$-\omega_1\log\left[\frac{\Gamma_2(z\,|\,\omega_1, \omega_2)}{\rho_2(\omega_1\,\omega_2)}\,e^{(m+m')2\pi\iota\,{}_2S_1'(o\,|\,\omega_1,\,\omega_2)}\right]$$
$$= -(1+2m\pi\iota)\left[S_1(z\,|\,\omega_2) + \frac{\omega_2}{12}\right] - \left[S_1(z\,|\,\omega_2) + \frac{S_2'(o\,|\,\omega_2)}{2} - \omega_1\,{}_2S_1'(z)\right]$$
$$\{\log_{\omega_1+\omega_2} z - 2(m+m')\pi\iota\}$$
$$+ z\{S_0(z\,|\,\omega_2) + S_1'(0\,|\,\omega_2) - \omega_1\,{}_2S_0'(z)\} + z\left(\frac{z}{\omega_2} - \tfrac{1}{2}\right)\log_{\omega_2} z - \frac{z^2}{\omega_2} + \frac{S_2'(o\,|\,\omega_2)}{2}$$
$$+ \sum_{n=1}^{\infty}\frac{(-)^n}{(n+1)\,z^n}\,{}_1B_{n+2}(\omega_2) - \omega_2\log_{\omega_2} z\left\{\frac{z^2}{2\omega_2^{\,2}} - \frac{1}{12}\right\} + \frac{3z^2}{4\omega_2}.$$
$$+ \sum_{n=1}^{\infty}\frac{(-)^{n-1}\,{}_1B_{n+2}(\omega_2)}{nz^n} + \sum_{n=1}^{\infty}(-)^n\,\frac{{}_1B_{n+2}(\omega_2) - \omega_1\,{}_2S'_{\omega+1}(o)}{n(n+1)\,z^n}.$$

Remember that $\log_{\omega_2} z = \log_{\omega_1+\omega_2} z - 2m'\pi\iota$; then we obtain by an easy reduction, the asymptotic expansion

$$\log\frac{\Gamma_2(z)\,e^{2\pi\iota\,(m+m')\,{}_2S_1'(o)}}{\rho_2(\omega_1, \omega_2)}$$
$$= -\,{}_2S_1'(z)\,\{\log_{\omega_1+\omega_2} z - 2(m+m')\,\pi\iota\} + z\,{}_2S_1^{(2)}(o) + \frac{z^2}{2\,!}\,{}_2S_1^{(3)}(o)\,(\tfrac{1}{1} + \tfrac{1}{2})$$
$$+ \sum_{m=1}^{\infty}\frac{(-)^m\,{}_2S'_{m+1}(o)}{m(m+1)\,z^m},$$

which is the complete asymptotic expansion for $\log \Gamma_2(z)$ when $|z|$ is large, and z does not lie within the barrier region negative with respect to the axes of $-\omega_1$ and $-\omega_2$.

If we combine this result with that obtained in § 83 we find the more general formula

$$\log \frac{\Gamma_2(z+a)\, e^{2\pi\iota(m+m')\,{}_2S_1'(o)}}{\rho_2(\omega_1, \omega_2)}$$
$$= -{}_2S_1'(z+a)\,\{\log_{\omega_1+\omega_2} z - 2(m+m')\pi\iota\} + z\,{}_2S_1^{(2)}(a) + \frac{z^2}{2\,!}\,{}_2S_1^{(3)}(a)\left(\tfrac{1}{1}+\tfrac{1}{2}\right)$$
$$+ \sum_{m=1}^{\infty} \frac{(-)^m\,{}_2S'_{m+1}(a)}{m(m+1)\,z^m},$$

valid under the assigned limitations.

The expansion is written in the precise form adopted, in order that the analogy with the corresponding formula in the theory of multiple gamma functions may be more clearly displayed.

§ 85. We might now conclude this investigation. Since, however, this would appear to be the first time in analysis in which an asymptotic expansion with a barrier region has been obtained, it seems better to give an alternative proof which shall not need the difficult argument of §§ 80–82. This proof is the direct extension of that previously given for the case of the simple gamma function.* We therefore proceed as briefly as possible.

In the investigation of § 57 it was shown that when $|s|$ is finite and $\mathfrak{R}(s) > -k$, where k is a positive integer, the series for $\zeta_2(s, a\,|\,\omega_1, \omega_2)$ is absolutely convergent.

Suppose now that $s = \sigma + i\tau$, where $\sigma > -k$, and suppose further that z does not lie within the region bounded by axes to $-\omega_1$ and $-\omega_2$, and that a is positive with respect to the ω's.

Then, since

$${}_2S_{-s,k}(a) = \frac{1}{(1-s)(2-s)\,\omega_1\,\omega_2\,a^{s-2}} - \frac{\omega_1+\omega_2}{2(1-s)\,\omega_1\,\omega_2\,a^{s-1}} + \frac{{}_2B_1}{a^s} + \sum_{r=0}^{k-1} \binom{-s}{r} \frac{{}_2B_{r+1}}{a^{s+r}},$$

it is evident that, if p be any positive integer, the absolute value of each term of the expression

$$\frac{s^p\, z^s}{\sin \pi s}\, {}_2S_{-s,k}\,(a\,|\,\omega_1, \omega_2)$$

tends to zero as $|\tau|$ tends to infinity. For, by the restrictions on z and a,

$$\frac{z^s}{a^s} = (r\,e^{\iota\psi})^{\sigma+\iota\tau}, \qquad \text{where } o < \psi < \pm\pi,$$

and therefore

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* "Theory of the Gamma Function," Part IV. I regret to say that the Lemma of § 40 is faulty; the theorem is evidently only true when a/ω is real. A slight modification will, however, establish the truth of the main proposition under the conditions enunciated.

$$\left| \frac{z^s}{a^{s+l}} \frac{s^p}{\sin \pi s} \right| = e^{-\psi\tau - \pi |\tau|} \, . \, \tau^p \, \mathrm{M},$$

where M is finite, however large $|\tau|$ may be.

Hence the absolute value of

$$\frac{s^p z^s}{\sin \pi s} \, {}_2\mathrm{S}_{-s, k}(a \mid \omega_1, \omega_2)$$

tends to zero as $|\tau|$ tends to infinity; and this theorem is true if a is replaced by $a + m_1\omega_1 + m_1\omega_2$, where m_1 and m_2 are positive integers.

Hence the absolute value of

$$\frac{s^p z^s}{\sin \pi s} \chi (a + m_1\omega_1 + m_2\omega_2 \mid s, k)$$

(where χ is the function introduced for brevity in § 57) tends to zero as $|\tau|$ tends to infinity.

But

$$\zeta_2(s, a \mid \omega_1, \omega_2) = {}_2\mathrm{S}_{-s,k}(a) - \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \chi (a + m_1\omega_1 + m_2\omega_2 \mid s, k),$$

and therefore

$$\frac{s^p z^s}{\sin \pi s} \zeta_2 (s, a \mid \omega_1, \omega_2) = \frac{s^p z^s {}_2\mathrm{S}_{-s, k}(a)}{\sin \pi s} - \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{s^p z^s}{\sin \pi s} \chi (a + m_1\omega_1 + m_2\omega_2 \mid s, k).$$

Now the double series on the right-hand side is absolutely convergent for all finite values of $|\tau|$, and the absolute value of each term tends to zero as $|\tau|$ tends to infinity.

Therefore
$$\left| \frac{s^p z^s \zeta_2 (s, a \mid \omega_1, \omega_2)}{\sin \pi s} \right|$$

remains finite as $|s|$ tends to infinity, $\mathfrak{R}(s)$ being finite and not greater than 2.

When $\mathfrak{R}(s)$ is greater than 2, we have

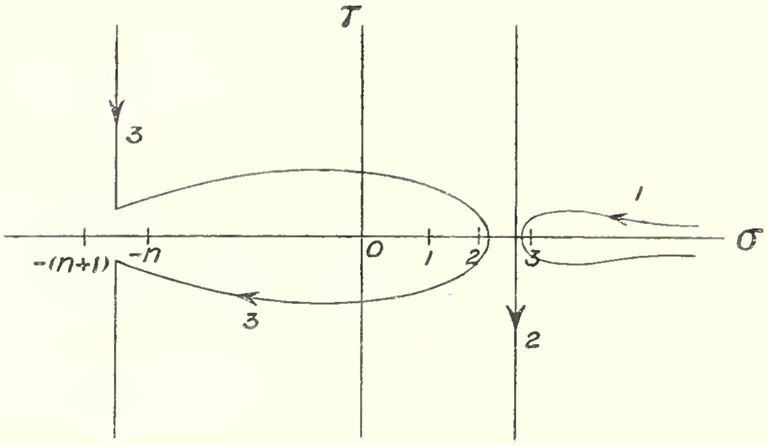
$$\left| \frac{s^p z^s \zeta_2 (s, a \mid \omega_1, \omega_2)}{\sin \pi s} \right| = \left| \sum_{m_1=0}^{\infty} \sum_{m_2=0}^{\infty} \frac{z^s s^p}{(a + m_1\omega_1 + m_2\omega_2)^s \sin \pi s} \right|,$$

and therefore the expansion on the left-hand side is finite however large $|s|$ may be, provided $|z| < 1$.

§ 86. Consider now the integral

$$\frac{1}{2\pi\iota} \int z^s \, . \, \frac{\pi \zeta_2 (s, a \mid \omega_1, \omega_2)}{s \sin \pi s} \, ds.$$

The subject of integration is a uniform function of s, wherein z^s is to have its principal value with respect to the axis of $-(\omega_1 + \omega_2)$, z is to lie within the region bounded by axes to $-\omega_1$ and ω_2, and a is to be positive with respect to the ω's.



In the first place let the contour be taken to lie along the real axis, passing from $+\infty$ to $+\infty$, and cutting the axis between the points $\sigma = 2$ and $\sigma = 3$, as the contour 1 of the figure. This is equivalent to taking the integral round a contour enclosing the points 3 and $+\infty$.

When $|z| < 1$ the integral is, by the theorem of the preceding paragraph, finite; and by CAUCHY's theorem it will be equal in value to the sum of the residues inside the contour.

Now by § 53, when $s = 2 + k$, where k is an integer,

$$\zeta_2(s, a) = \frac{(-)^k}{(k+1)!}\psi_2^{(2+k)}(a).$$

Hence the value of the integral along the contour 1 is

$$\sum_{k=0}^{\infty} \frac{z^{2+k}}{(k+2)!}\frac{d^{2+k}}{da^{2+k}}\log\Gamma_2(a),$$

and by TAYLOR's theorem this expression is, under the assigned limitations, equal to

$$\log\frac{\Gamma_2(z+a)}{\Gamma_2(a)} - z\psi_2^{(1)}(a).$$

Let us now make the contour expand until it becomes a straight line perpendicular to the axis of σ, cutting the axis between the points 2 and 3, and a half circle at infinity. The value of the integral will be unaltered, since the contour in expanding passes over no poles of the subject of integration. And by the theorem of the previous paragraph the part of the integral which is taken along the semicircle at infinity vanishes. Hence the integral along the perpendicular line (the contour numbered 2 in the figure) is equal to

$$\log\frac{\Gamma_2(z+a)}{\Gamma_2(a)} - z\psi_2^{(1)}(a), \text{ when } |z| < 1.$$

But the integral and this expression both remain continuously finite when $|z|$ becomes greater than unity. They are therefore equal to one another for all values of $|z|$.

Let now the perpendicular contour be distorted into a contour which encloses the points 2, 1, 0, . . . $-n$, and which after the point $\sigma = -n$ again goes off to infinity perpendicularly to the real axis.

So far as the value of the integral is concerned this contour will differ from the second contour only by two strips at infinity of length less than $(n + 3)$ parallel to the real axis: and by the previous paragraph the integral along these strips will vanish.

By CAUCHY's theorem the value of the integral along this third contour will be equal to minus the sum of its residues at the points 2, 1, 0, . . . $-n$, together with the integral along the perpendicular line cutting the axis of σ between the points $-n$ and $-(n + 1)$.

Now, when $s = 1$ or 2, the residue of the integral is equal to the coefficient of $1/\epsilon$ in

$$\frac{(-)^s}{\epsilon}\left(1 - \frac{\epsilon}{s} + \ldots\right)\left\{\frac{(-)^{s+1}}{\epsilon\,(s-1)!}\,{}_2S_1^{(s+1)}(a) + \frac{(-)^{s+1}}{(s-1)!}\left(\frac{1}{1} + \ldots + \frac{1}{s-1}\right){}_2S_1^{(s+1)}(a)\right.$$
$$\left. + \frac{(-)^s}{(s+1)!}\,\psi_2^{(s)}(a) + (-)^{s-1}\,2(m + m')\,\pi\iota\,{}_2S_1^{(s+1)}(a)\right\}\frac{z^s}{s}\,\{1 + \epsilon\log z + \ldots\}$$

by § 53,

and is therefore equal to

$$\frac{-z^s}{s!}\left[\frac{1}{1} + \ldots + \frac{1}{s} - \log z\right]{}_rS_1^{(s+1)}(a) + \frac{z^s}{s!}[\psi_2^{(s)}(a) - 2(m + m')\,\pi\iota\,{}_2S_1^{(s+1)}(a)],$$

where the logarithm has its principal value with respect to the axes of $-(\omega_1 + \omega_2)$.

When $\epsilon = 0$, the residue is the coefficient of $1/\epsilon$ in

$$\frac{1 + \epsilon\log z + \ldots}{\epsilon^2}\left\{{}_2S'_1(a) + \epsilon\log\frac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)}\,e^{-{}_2S_0(a)\,2(m + m')\pi\iota}\right\} \qquad \text{by § 60.}$$

and is therefore equal to

$${}_2S'_1(a)\log z + \log\frac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)}\,e^{-{}_2S_0(a)\,2(m + m')\pi\iota}.$$

When $s = -m$, the residue is

$$\frac{(-)^{m+1}\,{}_2S_{m+1}(a)}{m\,(m + 1)\,z^m}.$$

We therefore have

$$\log\frac{\Gamma_2(z + a)}{\Gamma_2(a)} - \frac{z}{1!}\psi_2^{(1)}(a)$$
$$= -\log\left\{\frac{\Gamma_2(a)}{\rho_2(\omega_1, \omega_2)}\,e^{\,{}_2S'_1(o)\,2(m + m')\pi\iota}\right\} - {}_2S'_1(a)\,[\log z - 2(m + m')\pi\iota]$$
$$- \sum_{s=1}^{2}\frac{z^s}{s!}\left[\psi_2^{(s)}(a) + {}_2S_1^{(s+1)}(a)\left\{\log z - 2(m + m')\,\pi\iota - \frac{1}{1} - \ldots - \frac{1}{s}\right\}\right]$$
$$+ \sum_{m=1}^{n}\frac{(-)^m\,{}_2S'_{m+1}(a)}{m\,(m + 1)\,z^m} + J_n(z, a \mid \omega_1, \omega_2)$$

where the quantity $\log z$ has its principal value with respect to the axis of $-(\omega_1 + \omega_2)$ and $J_n(z, a \mid \omega_1, \omega_2)$ is equal to the fundamental integral taken along a perpendicular contour cutting the axis of σ between $-n$ and $-(n+1)$. It is evident that the integral when $|z|$ is large is of an order of magnitude less than $\dfrac{1}{|z|^n}$.

We therefore have the asymptotic expansion, when $|z|$ is large,

$$\log \frac{\Gamma_2(z+a)\, e^{2\pi\iota(m+m')_2 S_1'(o)}}{\rho_2(\omega_1, \omega_2)} = -\frac{z^2}{2\,!}\, {}_2S_1^{(3)}(a)\left[\log z - 2(m+m')\pi\iota - \tfrac{1}{1} - \tfrac{1}{2}\right]$$
$$- \frac{z}{2\,!}\, {}_2S_1^{(2)}(a)\left[\log z - 2(m+m')\pi\iota - \tfrac{1}{1}\right]$$
$$- {}_2S_1'(a)\left[\log z - 2(m+m')\pi\iota\right] + \sum_{m=1}^{\infty} \frac{(-)^m\, {}_2S'_{m+1}(a)}{m(m+1)\, z^m}$$

and the residue after n terms of the final series have been taken is of the same order of magnitude as the final term taken.

This expansion is evidently the same as that previously obtained. The limitation that a must be positive with respect to the ω's may evidently be removed by employing the fundamental difference relations for the double gamma function and the asymptotic expansion for $\log \Gamma_1(z+a)$. We are finally left with the essential limitation that z shall not lie within the barrier region bounded by the axis to $-\omega_1$ and $-\omega_2$.

The Transcendentally-transcendental Nature of $\Gamma_2(z)$.

§ 87. We finally prove the theorem that the double gamma function cannot arise as the solution of a differential equation whose coefficients are not generated from the function itself. Modifying slightly the nomenclature introduced by MOORE,* we may say that $\Gamma_2(z)$ is a transcendentally-transcendental function. The proof is a slight modification of that given for the G function (§ 30), which in turn was similar to the investigation of Part V. of the "Theory of the Gamma Function."

In the first place it may be proved exactly as before that if the theorem is true for $\dfrac{d^2}{dz^2}\log\Gamma_2(z)$, it is true for $\Gamma_2(z)$. We shall therefore confine ourselves to the consideration of the function

$$\phi(z) = -\frac{d^2}{dz^2}\log\Gamma_2(z).$$

By the fundamental difference equations of § 20, we have

$$\phi(z+\omega_1) - \phi(z) = \psi(z \mid \omega_2)$$
$$\phi(z+\omega_2) - \phi(z) = \psi(z \mid \omega_1)$$

where, for convenience, we put $\psi(z) = \dfrac{d^2}{dz^2}\log\Gamma_1(z)$.

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* MOORE, 'Math. Ann.,' vol. 48, pp. 49 *et seq.* MOORE uses the term only to describe functions which cannot be generated by a differential equation with algebraic coefficients.

Suppose that

$$y = \phi(x) \text{ satisfies the differential equation}$$
$$f(x, y, y', \ldots y^{(n)}) = 0,$$

so transformed that it is rational and integral in y and its derivatives.

Let the terms of class s be symbolically

$$\mathrm{R}_0(x)\,\mathrm{Q}_s^0(y),\ \mathrm{R}_1(x)\,\mathrm{Q}'_s(y),\ \ldots\ ,\ \mathrm{R}_k(x)\,\mathrm{Q}_s^k(y)\,;$$

in terms of class $(s-1)$ being

$$\mathrm{S}_0(x)\,\mathrm{Q}^0_{s-1}(y),\ \ldots\ldots\ldots\ ,\ \mathrm{S}_l(x)\,\mathrm{Q}^l_{s-1}(y),$$

and the functions $\mathrm{R}(x)$, $\mathrm{S}(x)$ being holomorphic.

If $\phi(x)$ satisfies the differential equation, $\phi(x) + \psi(x\,|\,\omega_2)$ will satisfy the equation in which $(x+\omega_1)$ is written for x and $\phi(x) + \psi(x\,|\,\omega_1)$ the equation in which $(x+\omega_2)$ is written for x.

Make the first substitution, divide the equations by $\mathrm{R}_0(x)$ and $\mathrm{R}_0(x+\omega_1)$ respectively, and subtract one from the other. We find

$$\frac{\mathrm{R}_1(x+\omega_1)}{\mathrm{R}_0(x+\omega_1)}\mathrm{Q}'_s[\phi(x)+\psi(x\,|\,\omega_2)] + \ldots + \frac{\mathrm{R}_k(x+\omega_1)}{\mathrm{R}_0(x+\omega_1)}\mathrm{Q}_s^k[\phi(x)+\psi(x\,|\,\omega_2)]$$
$$-\left\{\frac{\mathrm{R}_1(x)}{\mathrm{R}_0(x)}\mathrm{Q}_s'[\phi(x)] + \ldots + \frac{\mathrm{R}_k(x)}{\mathrm{R}_0(x)}\mathrm{Q}_s^k[\phi(x)]\right\} + \mathrm{Q}_s^0[\phi(x)+\psi(x\,|\,\omega_2)] - \mathrm{Q}_s^0[\phi(x)]$$
$$+ \text{ terms of lower class} = 0\,.$$

But
$$\mathrm{Q}_s^0[\phi(x)+\psi(x\,|\,\omega_2)] - \mathrm{Q}_s^0[\phi(x)]$$

consists solely of terms of lower class than s.

Hence either the equation which has been obtained vanishes identically, or we can reduce the equation for y to one in which there are fewer terms of class s.

The equation cannot vanish identically unless the coefficients of the various terms of class s all vanish, which necessitates that the ratios

$$\frac{\mathrm{R}_1(x)}{\mathrm{R}_0(x)},\ \ldots\ ,\ \frac{\mathrm{R}_k(x)}{\mathrm{R}_0(x)}$$

are doubly periodic functions of x of periods ω_1 and ω_2.

The equation for y can thus be always reduced to one of the form

$$\mathrm{R}(x)\,[p_0(x)\,\mathrm{Q}_s^0(y) + \ldots + p_k(x)\mathrm{Q}_s^k(y)]$$
$$+ \mathrm{S}_0(x)\mathrm{Q}_{s-1}^0(y) + \ldots + \mathrm{S}_l(x)\mathrm{Q}_{s-1}^l(y)$$
$$+ \text{ terms of lower class} = 0.$$

where all the coefficients are holomorphic functions, and, in addition, the functions $p(x)$ are doubly periodic of periods ω_1 and ω_2.

Divide the equation by R(x) and subtract it from the equation which results from changing x into $x + \omega_1$. We obtain

$$\begin{aligned} p_0(x)[\mathrm{Q}_s^0\{\phi(x) + \psi(x\,|\,\omega_2)\} - \mathrm{Q}_s^0\{\phi(x)\}] + \ldots + p_k(x)[\mathrm{Q}_s^k\{\phi(x) + \psi(x\,|\,\omega_2)\} \\ - \mathrm{Q}_s^k\{\phi(x)\}] \\ + \frac{\mathrm{S}_0(x+\omega_1)}{\mathrm{R}(x+\omega_1}\mathrm{Q}^0_{s-1}\{\phi(x) + \psi(x\,|\,\omega_2)\} - \frac{\mathrm{S}_0(x)}{\mathrm{R}(x)}\mathrm{Q}^0_{s-1}\{\phi(x)\} \\ + \ldots\ldots\ldots\ldots \\ + \frac{\mathrm{S}_l(x+\omega_1)}{\mathrm{R}\,(x+\omega_1)}\mathrm{Q}^l_{s-1}\{\phi(x) + \psi(x\,|\,\omega_2)\} - \frac{\mathrm{S}_l(x)}{\mathrm{R}(x)}\mathrm{Q}^l_{s-1}\{\phi(x)\} \\ + \text{terms of lower class} = 0. \end{aligned}$$

This equation will not vanish identically unless the functions

$\frac{\mathrm{S}_l(x)}{\mathrm{R}(x)}, \ldots \frac{\mathrm{S}_l(x)}{\mathrm{R}(x)}$ all satisfy relations of the form

$$f(x+\omega_1) - f(x) = \sum_{k=0}^{n} p_k(x)\,\mathrm{Q}_1^k[\psi(x\,|\,\omega_2)]$$

and therefore by symmetry relations of the form

$$f\,(x+\omega_2) - f\,(x) = \sum_{k=0}^{n} p_k\,(x)\;\mathrm{Q}_1^k\left[\;\psi(x\,|\,\omega_1)\right]$$

The quantity $(n+1)$ which gives the number of terms on the right-hand side of these two relations will not vanish unless the original equation can be reduced to the form

$$\begin{aligned} \mathrm{R}\,(x)\,[\,p_{1,0}\,(x)\;\mathrm{Q}_s^0\,(y) + \ldots + p_{1,k}\,(x)\;\mathrm{Q}_s^k\,(y) \\ + p_{2,0}\,(x)\;\mathrm{Q}^0_{s-1}\,(y) + \ldots + p_{2,\,l}\,(x)\;\mathrm{Q}^l_{s-1}\,(y)] \\ + \mathrm{T}_0\,(x)\;\mathrm{Q}^0_{s-2}\,(y) + \ldots + \mathrm{T}_m\,(x)\;\mathrm{Q}^{(m)}_{s-2}\,(y) \\ + \text{terms of lower class} = 0 \quad \ldots\ldots\ldots \quad (1), \end{aligned}$$

where the coefficients are holomorphic functions of x and the p's are doubly periodic functions of periods ω_1 and ω_2.

Either then the original equation can be reduced to this form, or at least one of the ratios $\frac{\mathrm{S}_0(x)}{\mathrm{R}(x)}, \ldots, \frac{\mathrm{S}_l(x)}{\mathrm{R}(x)}$ is composed of an additive number of solutions of difference equations of the form

$$\begin{cases} f\,(x+\omega_1) - f\,(x) = p\,(x)\;\psi\,(x\,|\,\omega_2) \\ f\,(x+\omega_2) - f\,(x) = p\,(x)\;\psi\,(x\,|\,\omega_1) \end{cases}$$

But the most general solution of such a pair of equations is

$$q\,(x)\;\phi\,(x) + r\,(x),$$

where $q(x)$ and $r(x)$ are doubly periodic functions of x of periods ω_1 and ω_2. And therefore one of the ratios $\frac{S_0(x)}{R(x)}, \ldots, \frac{S_l(x)}{R(x)}$ must be a function generated from the function $\Gamma_2(x \mid \omega_1, \omega_2)$.

The original equation therefore either contains the double gamma function implicitly among its coefficients, or it is reducible to the form (1).

Continue our former procedure, and we see that either at least one of the ratios $\frac{T_0(x)}{R(x)}, \ldots \frac{T_m(x)}{R(x)}$ is composed of an additive number of equations of the type

$$f(x+\omega_1) - f(x) = p(x)\, Q_2 \{\psi(x \mid \omega_2)\}$$
$$f(x+\omega_2) - f(x) = p(x)\, Q_2 \{\psi(x \mid \omega_1)\}$$

and is therefore generated from the double gamma function, or the original equation is reducible to one in which the ratios of terms of the three highest classes are doubly periodic functions of x of periods ω_1 and ω_2.

The successive repetitions of the argument are now evident. Ultimately we reduce the equation to one in which either all the coefficients are doubly periodic functions (which is absurd), or to one in which the last term is generated from the double gamma function.

Thus the proposition is established. The double gamma function cannot satisfy a differential equation in which the coefficients are finite combinations of, *e.g.*,

(1) Rational or irrational algebraic functions of x,
(2) Simply or doubly periodic functions,
(3) Simple gamma functions,
(4) G functions,
(5) Theta functions,

or, in fact, of any functions which are not substantially reducible to or compounded of the double gamma function itself.

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Penrose, F. C. Phil. Trans., A, vol. 196, 1901, pp. 389–395.

VII. *Some Additional Notes on the Orientation of Greek Temples; being the Result of a Journey to Greece and Sicily in April and May,* 1900.

By F. C. Penrose, M.A., F.R.S.

Received January 17,—Read February 14, 1901.

A journey to Greece and Sicily in the spring of last year enabled me to obtain a few more examples of the orientation of Greek temples, and to make a correction in the case of one of those previously published.

Delphi.

At Delphi the complete clearance of the site of the Temple of Apollo allowed of the measures of the existing foundations being taken by direct observation, and I found the orientation angle to be 227° 8′; that is, east amplitude + 42° 52′. I also examined more particularly the openings between the mountains which would be available for the sunrise. Of these there are two, one with amplitude — 7° 42′ E. has the altitude above the temple floor of 3° 8′, reduced as respects the sun by refraction to 2° 49′. The other at amplitude — 23° 16′ E. with altitude 2° 6′, reduced to 1° 40′.

The temple seems to have been rebuilt at least twice. Of the middle building (if the number was three) there is historical record. The terrace on which the temple area stands is supported by a wall of polygonal masonry, covered with inscriptions, from which the orientation of the present foundations differs by about 4°: I have presumed that this marks the line of probably the earliest foundation. The amplitude of —23° 16′ seems to agree best with this, and that of — 7° 42′ with the existing foundations. I have accordingly given the elements of the two separately.

Delphi. Latitude 38° 27′ 33″.

Name of temple.	Orientation angle.	Ref. letter.	Details	Stellar elements.	Solar elements.	Name of star.
Ancient temple of Apollo.	231° 17′	A	Amplitude of star or sun	– 42° 12′ W.	– 23° 16′ E.	ε Canis Majoris; setting towards south-western axis.
		B	Corresponding altitude	3° 30′	1° 40′	
		C	Declination	– 29° 9′	– 16° 55′	
		D	Hour angles	$3^h\ 51^m$	$5^h\ 58^m$	
		E	Depression of sun when star heliacal	—	10°	
		F	R. A.	$5^h\ 5^m$	$14^h\ 54^m$	
		G	Approximate date—930 B.C., November 8.			

29.4.1901

Delphi. Latitude 38° 27′ 33″—*continued.*

Name of temple.	Orientation angle.	Ref. letter.	Details.	Stellar elements.	Solar elements.	Name of star.
Later temple of Apollo.	227° 8′	A	Amplitude of star or sun	− 44° 1′ W.	− 7° 42′ E.	β Lupi; setting.
		B	Corresponding altitude	3° 30′	2° 49′	
		C	Declination.	− 30° 18′	− 4° 15′	
		D	Hour angles	$3^h\ 45^m$	$7^h\ 8^m$	
		E	Depression of sun when star heliacal	—	16°	
		F	R. A.	$12^h\ 28^m$	$23^h\ 21^m$	
		G	Approximate date—585 B.C., March 9.			

Delos.

The Isle of Delos, besides a group of five (or more) temples of the classical period, is remarkable for containing a very ancient religious structure, certainly one of the most ancient shrines now remaining in Greece or her colonies. A short stay in the sland enabled me to examine this example, but I was obliged to confine myself to this one.

This Cynthian Grotto has no structural flank walls, but two masses of natural rock are connected by a solid roof formed by large wrought stones placed rafterwise, and well jointed at their meeting line at the apex. This meeting line is the only leading architectural line in the whole structure. The access was from the east, where are remains of a wall of polygonal masonry.

The west end of the grotto was closed : near it there seems to have been a statue. The northern jamb of the doorway in the eastern wall remains. The return or southern face of this door jamb is not parallel with the line given by the roof stones. As it is probable that this obliquity may have had some astronomical reference to the star which seems to have been connected with the line derived from the roof, I have added elements on the hypothesis of such an intention.

This doorway is to all appearance of later work than the roof, and this seems to point out that a structural alteration was made so as to follow the star.

Delos. Latitude 37° 22′

Name of temple.	Orientation ange .	Ref. letter.	Details.	Stellar elements.	Solar elements.	Name of star.
Grotto on Mt. Cynthus sacred to Apollo.	276° 2′	A	Amplitude of star or sun	+ 0° 13′ E.	− 6° 2′	α Libræ; rising.
		B	Corresponding altitude	3° 30′	0 0	
		C	Declination.	+ 2° 18′	+ 4° 47′	
		D	Hour angles	$5^h\ 49^m$	$6^h\ 46^m$	
		E	Depression of sun when star heliacal	—	12°	
		F	R. A.	$11^h\ 47^m$	$12^h\ 44^m$	
		G	Approximate date—1550 B.C., October 5.			

The same Temple. Elements calculated for a Line parallel to the Southern Face of the Door Jamb.

Orientation angle.	Ref. letter.	Details.	Stellar elements.	Solar elements.	Name of star.
282° 38′	A	Amplitude of star or sun	− 1° 44′ E.	− 12° 38′ E.	α Libræ; rising.
	B	Corresponding altitude	3° 30′	0 0	
	C	Declination.	− 3° 10′	− 10° 27′	
	D	Hour angles	$5^{h}\ 34^{m}$	$6^{h}\ 39^{m}$	
	E	Depression of sun when star heliacal	—	14°	
	F	R. A.	$12^{h}\ 35^{m}$	$13^{h}\ 40^{m}$	
	G	Approximate date—580 B.C., October 19.			

Syracuse.

In the temple at Syracuse, vulgarly attributed to Diana, but really, as determined by an inscription, sacred to Apollo, I found that both the architectural remains and the character of the inscription referred to, required a much earlier date than I had previously derived from the orientation as measured along the axis.* I have, therefore, changed the elements to those which are given by the northern limit of the eastern intercolumniation—an alternative not always possible, but not unfrequently found to be the only satisfactory one, having respect to the proper depression of the sun.

Syracuse. Latitude 37° 3′ 30″.

Name of temple.	Orientation angle.	Ref. letter.	Stellar elements.	Solar elements.	Name of star.
Temple of Apollo (incorrectly called Temple of Diana).	271° 45′; the northern limit of east opening 269° 53′.	A	+ 2° 30′ E.	+ 0° 7′	Spica; rising.
		B	3° 0′	0 0	
		C	+ 4° 0′	+ 0° 6′	
		D	$5^{h}\ 56^{m}$	$6^{h}\ 50^{m}$	
		E	—	10°	
		F	$11^{h}\ 6^{m}$	$11^{h}\ 59^{m}$	
		G	700 B.C., September 21.		

The date so derived is 30 years subsequent to the Hellenic foundation of the city. This improved determination led to the examination of the elements of the Syracusan temple on the site of the present *duomo*, of which the architectural detail points out a date decidedly later than the Temple of Apollo; whereas the orientation date† already published is nearly 100 years earlier than the arrival of the Greeks; the

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* 'Phil. Trans.,' A, vol. 190 (1897), p. 59.

† *Ibid.*, p. 58.

result was the discovery of an error I had made in working out the example, which showed that the orientation angle should have been 271° 2′ instead of 269° 12′. With this alteration, and taking as usual the line of the axis, and with the same star, the approximate date becomes 550 B.C. instead of 815. If in this case the northern limit of the intercolumniation had been used instead of the axis, it would have made the date still earlier.

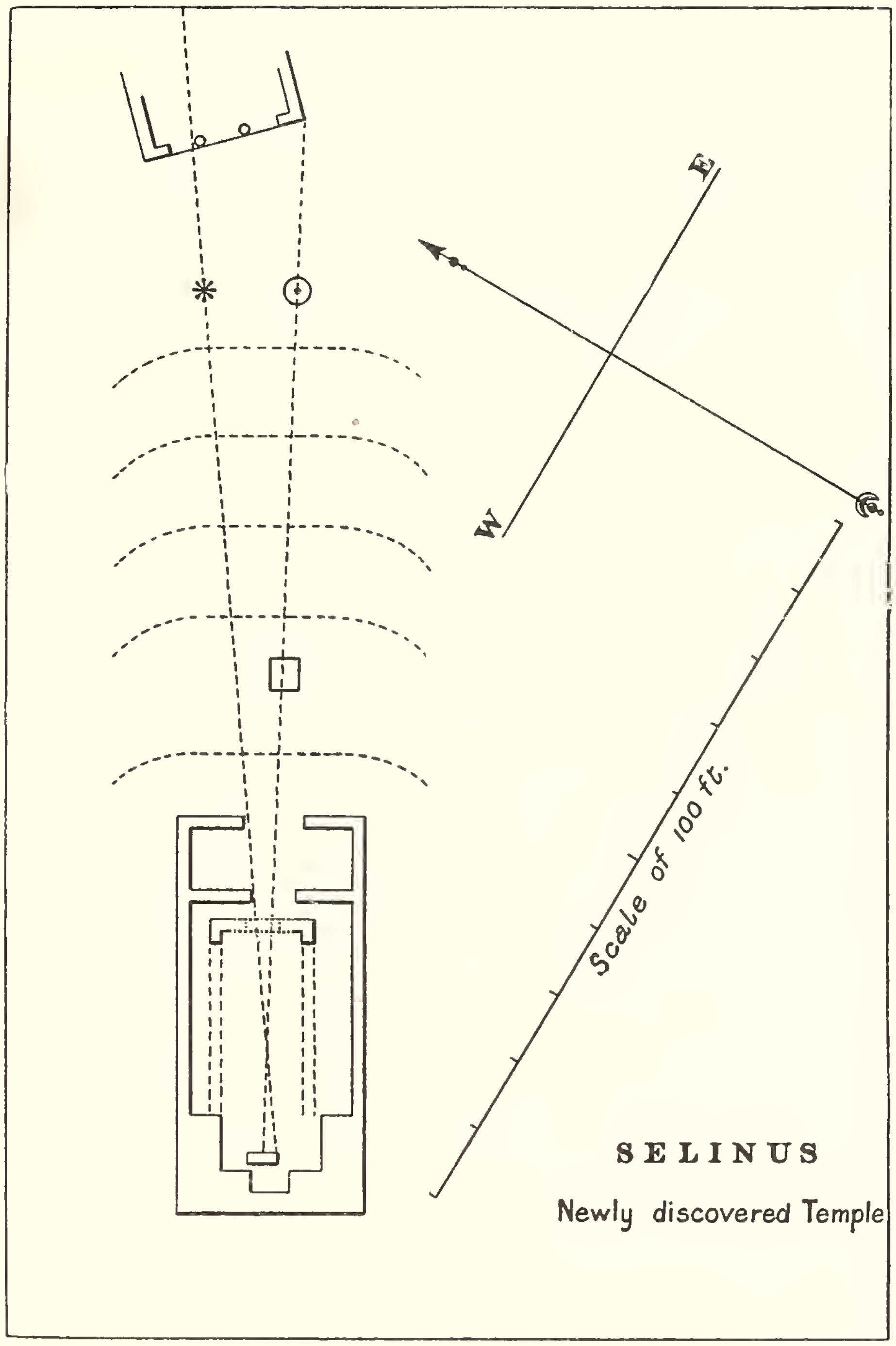
Selinus (a newly discovered Temple named in KALLOWAY *and* PUCHSTEIN'S *Work 'Megaron of Demeter'). See Plan.*

This small temple, which lies apart from the two great groups of temples, and near the right bank of the little stream (the Selinus), must have been of great sanctity, judging from the great number of votive offerings which have been found near it. Its plan is remarkable, and would be difficult to explain, except on the astronomical theory of its orientation.

It is clear from the angle of its orientation that, if intended for a solar temple, it could only have been so at the summer solstice. The eastern axis has the amplitude of + 30° 11′. That of the sunrise in that latitude and at the presumed date, hereafter given, and on a level site, would have been + 30° 35′, but the eastern horizon in this case is not level, and when first shining into the temple the sun would have had the amplitude of 28° 16′. To account for the presumed error made in setting out the work, let it be assumed that the normal amplitude at the solstice of $30\frac{1}{2}$° had been ascertained elsewhere, and that at some other time of year the lines of the temple had been laid down on the actual site to that angle (for the meridian could be ascertained very nearly at any time), and that some progress had been made with the work, when the solstice came round and the error was noticed. Instead of taking the work down and beginning again, other means were taken to meet the difficulty; what these were is very clearly pointed out by the plan.

It would have been observed that the sun's first ray entering centrally the eastern door would have fallen considerably to the north of the niche which seems to have been provided for the statue, which niche is centrally placed between the flank walls. To meet this difficulty, a narrower *naos* was constructed, of which the foundations remain, hugging the north wall of the temple; the centre of which, marked also by a foundation-stone for the statue, would have received the ray in the desired manner. What further marks the incident is that the southernmost angle of the Propylæa, which gives access to the temple area, is exactly kept clear of the line required for the sunrise. The altar, which stood nearly in front of the eastern door, could not have interfered with it, being itself low, and standing on rather lower ground than the temple.

The warning star would, indeed, have to be seen over the roof of the Propylæa, but there is sufficient difference between the level of the ground at the two sites to allow this to have been done. The star at the place calculated for it would have the advantage of about a degree of elevation more than the sun (more could have been given if it had been necessary), and the detail of the Propylæa is sufficiently well preserved to enable its height to be computed with sufficient accuracy.



Selinus. Latitude 37° 35′.

Ref. letter.	Stellar elements.	Solar elements.	Name of star.
A	+35° 29′ E.	+28° 16′	
B	3° 30′	2° 38′	β Geminorum; rising.
C	+29° 46′	+23° 46′	
D	7h 23m	8h 15m	
E	—	10° 30′	
F	5h 8m	6h 0m	
G	520 B.C., June 21.		

Taormina.

Lately, some foundations of a small Greek temple have been discovered adjoining the celebrated Theatre of Taormina. Although the architecture of the theatre now visible is evidently Roman work, it is clear from some remains of foundations and an inscription that originally it was truly Greek. But these do not supply distinct information as to the antiquity of the first construction; but the orientation of the temple seems to do so. The important city of Naxos, the earliest of the Hellenic colonies in Sicily (founded 735 B.C.), lay immediately below the site, but this city was utterly destroyed by the Syracusans about 400 B.C. Some remnants, however, of the population were collected about fifty years afterwards by a leader named Andromachus and established in a new city on a ridge adjoining this theatre, which is about 800 feet above the site of the city of Naxos, and he named it Tauromenium.

The very great scale of the theatre might, indeed, of itself suggest that it had been the work of the flourishing population of the ancient city in the palmy days of Greek civilisation, and not that of a town inhabited by returned exiles, and nearer to the days of Grecian decadence. And it would not have been out of parallel with Greek habits (as seen, for instance, at Segesta) for a theatre to be placed at a great height above the inhabited parts of a town, and in this case on a site so remarkable for its beauty.

But, in addition to such arguments, the temple seems to give convincing evidence that the Taormina Theatre is not that of Tauromenium but of Naxos. Its situation is exactly where a small temple dedicated to Bacchus is found in connection with several of the Greek theatres. The orientation angle lies between 281° and 282°. Owing to a slip in my record, I cannot be positive as to the minutes, but 281° 25′ which I have adopted must be very near the mark, and I make out the elements to that figure. Towards whichever of the limits it should be placed, it points out α Libræ as the star, and the date derivable as at least 300 years earlier than the foundation of Tauromenium.

Taormina. Latitude 37° 48′.

Name of temple.	Orientation angle.	Ref. letter.	Stellar elements.	Solar elements.	Name of star.
Supposed Temple of Bacchus.	281° 25′ (approximate)	A	– 5° 41′ E.	– 11° 25′	α Libræ; rising.
		B	3° 30′	0 0	
		C	– 2° 20′	– 9° 0′	
		D	5^h 35^m	6^h 33^m	
		E	—	12°	
		F	12^h 28^m	13^h 26^m	
		G	715 B.C., October 16.		

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VIII. *The Distribution of Molecular Energy.*

By J. H. Jeans, *B.A., Scholar of Trinity College, and Isaac Newton Student in the University of Cambridge.*

Communicated by Professor J. J. Thomson, *F.R.S.*

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Introduction.

§ 1. This paper is primarily an attempt to deal with certain points connected with the application to the Kinetic Theory of Gases of Boltzmann's Theorem on the partition of energy in a dynamical system.

It is found by experiment that the ratio of the two specific heats of certain monatomic gases (*e.g.*, mercury, argon) is $1\frac{2}{3}$. If we admit that the energy of these gases is distributed in the manner indicated by Boltzmann's Theorem, then this theorem leaves no escape from the conclusion that the molecules of these gases must be rigid and geometrically perfect spheres. A similar difficulty arises in connection with other gases: the number of degrees of freedom which a consideration of the ratio in question leads us to expect a molecule of a gas to possess, is always less than the number which the spectrum of the glowing gas shows to actually exist. Further, Boltzmann's Theorem excludes the possibility of the ratio of the two specific heats having any values except one of a certain series of values, whereas experiment shows that the ratio is not always equal to one of this series, although it is generally very near to such a value. Finally Boltzmann's Theorem leaves no room for a variation of this ratio with the temperature, although such a variation is known to exist.

In the present paper I have tried to suggest a way by which it is possible to escape from this dilemma. As there is not sufficient known about the constitution of a molecule to enable it to be completely specified as a dynamical system, the paper is limited to the consideration of two imaginary types of molecules.

The conclusions arrived at are the same in each case. In the first place the distribution of energy which is given by Boltzmann's Theorem is the only distribution which is permanent *under the conditions postulated by this theorem.* And in the second place, this law of distribution may break down entirely as soon as we admit an interaction, no matter how small, between the molecules and the surrounding ether. That such an interaction must exist is shown by the fact that a gas is capable of radiating energy. In fact, BOLTZMANN'S Theorem rests on the assumption that the molecules of a gas form a conservative dynamical system, and it will appear that the introduction of a small dissipation function may entirely invalidate the conclusions of the theorem.[*] Thus we may regard the BOLTZMANN distribution as unstable, in the sense that a slight deviation from perfect conservation of energy may result in a complete redistribution of the total energy, and it will appear that this new distribution of energy will lead to values for the ratios of the two specific heats which are not open to the objections mentioned above.

§ 2. A second difficulty, of a mathematical rather than physical nature, may be mentioned here, as it will occur incidentally in the course of the analysis. It is well illustrated by the consideration of an imaginary type of molecule which has been suggested by BOLTZMANN.[†] A loaded sphere, that is to say, a sphere of which the centre of gravity is at a small distance r from the geometrical centre, will, for the present purpose, possess five degrees of freedom, and this is true, however, small r may be. The ratio of the specific heats of an ideal gas whose molecules are of this type ought accordingly to be $1\frac{2}{5}$. If, however, r actually vanishes, the molecules must be regarded as completely symmetrical, and possessing only three degrees of freedom, so that the ratio ought now to be $1\frac{2}{3}$. There is thus seen to be a discontinuity when r has a zero value, and this requires investigation.

It must be borne in mind that a degree of freedom, for purposes of BOLTZMANN'S Theorem, is not idendical with the usual dynamical degree of freedom. In the strict dynamical sense every sphere possesses six degrees of freedom, the principal momenta being the momenta of the centre of gravity in three rectangular directions and the three rotatory momenta about the principal axes of inertia. But if the sphere is perfectly smooth, rigid, and symmetrical, it is impossible to change the three latter momenta by the application of any forces which are at our disposal in the kinetic theory of gases, and for this reason the corresponding degrees of freedom must be left out of account, when applying BOLTZMANN'S Theorem. Similar remarks apply, in the case of the loaded sphere, to the degree of freedom which arises from rotation about the axis of symmetry, so that the loaded sphere must be supposed to possess five degrees of freedom, and not six.

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[*] The matter may be looked at from a slightly different point of view as follows: If an interaction between matter and ether exists, no matter how small this interaction may be, the complete dynamical system will consist of the molecules of the gas, together with the ether, and must therefore be regarded as a system possessing an infinite number of degrees of freedom. Applying BOLTZMANN'S Theorem to this system we are merely led to the conclusion that no steady state is possible until all the energy of the gas has been dissipated by radiation into the ether. This application of the theorem may or may not be legitimate, but it is, I think, certain that no other application is legitimate.

[†] 'Vorlesungen über Gastheorie,' Part II., p. 129.

It is obvious that the degrees of freedom which have been rejected do not in any way influence the motion of the sphere, whereas if they are counted as separate degrees of freedom, the series of states through which the sphere is made to pass by varying all the co-ordinates, is no longer such as to satisfy MAXWELL'S condition of "Continuity of Path." (See § 11 of the present paper.)

PART I.

THE DISTRIBUTION OF ENERGY IN A GAS OF WHICH THE MOLECULES ARE LOADED SPHERES.

The Transfer of Energy caused by Collisions.

§ 3. We may begin with the consideration of a gas of which the molecules are loaded spheres of the kind mentioned in the last section. These spheres are to be perfectly elastic, each of radius a, and the centre of gravity of each is to be at a small distance r from the geometrical centre.

We require to find equations giving the transfer of energy between the various degrees of freedom in such a gas. If we know the law of distribution of the various co-ordinates of the molecules, we shall be able, upon making the usual assumptions of the kinetic theory, to calculate the number of collisions which are such that the values of the variables, which are required to completely specify a collision, lie within certain specified small ranges of value. At each of these collisions the transfer of energy is the same, a function of the variables which specify the collision; so that by multiplying this quantity by the number of collisions of the kind under consideration which occur during the interval dt, and integrating over all possible values of the variables which specify the collision, we shall obtain an expression for the transfer of energy during the time dt.

Let us denote the mean energy of translation of all the molecules at any specified instant by K, the mean energy of rotation by H. If we regard the number of molecules in the gas as infinite, we may regard the quantities H and K as varying continuously with the time, and the expressions which have just been found for the transfer of energy will enable us to calculate $d\text{H}/dt$ and $d\text{K}/dt$ corresponding to any state of the gas. The values of $d\text{H}/dt$, $d\text{K}/dt$ will in general depend on r, a, ρ (the density of the gas) and the coefficients which occur in the various laws of distribution of co-ordinates.

If we consider the case in which $r = 0$, we find that no transfer of energy is possible, so that $d\text{H}/dt$, $d\text{K}/dt$ must vanish with r. We further notice that these differential coefficients must remain unaltered, if $-r$ be written for r, so that, assuming for the moment that they can be expanded in ascending powers of r, we see that the lowest power of r which can occur is r^2. We shall suppose r to be so small that terms in r^4 may be neglected in comparison with terms containing r^2.

Since the total energy remains unaltered at every collision, we must have

$$dK/dt = -dH/dt;$$

it is therefore only necessary to calculate one of these differential coefficients.

§ 4. The state of a molecule at any instant will be determined by the following 12 variables,

- (i.) The co-ordinates in space of its centre of gravity x, y, z, and their time-rate of change u, v, w.
- (ii.) Any three independent variables, e, f, g, specifying the orientation in space of the molecule.
- (iii.) ϖ_1, ϖ_2, ϖ_3, the rotations about three principal axes, the last of these being the rotation about the axis of symmetry of the molecule.

Let accented letters refer to a second molecule; then a collision between these two molecules, if possible, is completely specified by the whole 24 variables, but these are not all independent, and the collision will, as regards transfer of energy, be sufficiently specified by the independent variables

$$u - u',\ v - v',\ w - w',$$
$$\varpi_1,\ \varpi_2,\ \varpi_3,\ \varpi'_1,\ \varpi'_2,\ \varpi'_3,$$

and six other variables to determine the direction in space of the axes of the molecules, and the line of centres.

Let the variables after collision be distinguished from those before collision by placing a bar over them, then we can from the ordinary equation of impact calculate the value of

$$\bar{c}^2 + \bar{c}'^2 - (c^2 + c'^2)$$

where $c^2 = u^2 + v^2 + w^2$, in terms of the variables before collision.

This expression must be a quadratic function of the velocities, and ϖ_3, ϖ'_3 cannot enter. If we write

$$(u - u')^2 + (v - v')^2 + (w - w')^2 = V^2,$$
$$\varpi_1^2 + \varpi_2^2 = \varpi^2,$$

it is easily seen that the expression must be of the form

$$c^2 + \bar{c}'^2 - (c^2 + c'^2) = \alpha_1 V^2 + \beta_1 (\varpi^2 + \varpi'^2) \quad . \quad . \quad . \quad . \quad . \quad \text{(i.)},$$

where α_1, β_1 are functions of the six variables determining orientations in space, and are algebraical functions of r, in which the lowest power is r^2.

§ 5. In calculating the number of collisions of this kind which are to be expected in the interval of time dt, a consideration enters, which does not enter in the simpler case in which the spheres are symmetrically loaded.

From the co-ordinates of the two molecules just before collision, we can trace back, as far as the previous collisions, the paths by which the molecules arrived at this position. If these paths are such that the spaces occupied by the two molecules, at any two corresponding points of these paths, are found to overlap, then it is clear that a collision of the kind we are investigating can only occur, either when the same two molecules have previously collided, or when one of them has collided with a third molecule within a certain small interval previous to the collision in question. In either case it would be wrong to calculate the probability of such a collision upon the assumption that the molecules of the gas are, in BOLTZMANN'S sense, *ungeordnet*.

When, however, terms of degree higher than r^2 are neglected, it will be legitimate to ignore this consideration altogether. For the number of collisions to which it applies will vanish with r, so that if equation (i.) be summed over all collisions, the terms on the right-hand side which are influenced by this consideration will be of a higher order in r than r^2, and may accordingly be inaccurately calculated, without invalidating the result as far as terms in r^2.

§ 6. When we agree to ignore this consideration, we may at once average equation (i.) over all values of the six variables of orientation. The probability of these variables having specified values at a collision is not independent of the velocities of the collision, but will be the same for all collisions such as we are now considering, in which these velocities have specified values. In this way we find that the mean increase in $c^2 + c'^2$ at a collision at which the velocities are u, v, w, ϖ, u', v', w', ϖ', is of the form

$$r^2\{\alpha_2 V^2 + \beta_2(\varpi^2 + \varpi'^2)\} + \text{terms of a higher order in } r^* \quad . \quad . \quad . \text{(ii.)},$$

in which α_2, β_2 are constants.

Now if we suppose that the gas has reached its present state through a series of natural processes, the law of distribution of velocities will depend only on c^2 and ϖ^2. In the case in which $r = 0$, this law is known to be

$$e^{-hmc^2} f(\varpi)\, du\, dv\, dw\, d\varpi \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \text{(iii.)}.$$

Hence in the case in which r is small, it may be taken to be

$$\mathrm{F}(c, \varpi)\, du\, dv\, dw\, d\varpi. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \text{(iv.)},$$

where F is a function of which the coefficients involve r, but is such that (iv.) reduces to (iii.) when $r = 0$.

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[*] Direct calculation shows that the values of α_2, β_2 are $\alpha_2 = -\frac{2}{3\kappa^2}$, $\beta_2 = \frac{4}{3}$, where κ is the radius of gyration of a molecule about a line perpendicular to the axis of symmetry.

Calculated upon the usual assumption, the number of collisions which occur in a volume Ω of the gas and within a time dt, between pairs of molecules of which the velocities lie within a range $du\,dv\,dw\,d\varpi\,du'\,dv'\,dw'\,d\varpi'$ surrounding the values $u, v, w, \varpi, u', v', w', \varpi'$, is

$$\frac{\pi a^2 \mathrm{V}\,dt}{\Omega}\,\mathrm{F}(c, \varpi)\,\mathrm{F}(c', \varpi')\,du\,dv\,dw\,d\varpi\,du'\,dv'\,dw'\,d\varpi'.$$

Hence referring to expression (ii.) we see that the total increase in the translational velocity of the gas, in time dt, is

$$\tfrac{1}{2}\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int \frac{\pi a^2 \mathrm{V}\,dt}{\Omega}\,\mathrm{F}(c, \varpi)\,\mathrm{F}(c', \varpi')\,[r^2\alpha_2\mathrm{V}^2 + r^2\beta_2(\varpi^2 + \varpi'^2) + \ldots\ldots]$$
$$du\,dv\,dw\,d\varpi\,du'\,dv'\,dw'\,d\varpi'.$$

If we reject all terms of a degree higher than r^2 in r, this expression becomes

$$\frac{\pi a^2 dt}{2\Omega}\,r^2\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int\!\!\int \mathrm{V}e^{-hm(c^2+c'^2)}f(\varpi)f(\varpi')\,[\alpha_2\mathrm{V}^2 + \beta_2(\varpi^2 + \varpi'^2)]\,du\,dv\,dw\,d\varpi\,du'\,dv'\,dw'\,d\varpi' \quad \ldots\ldots \text{(v.)}.$$

Now the functional form represented by f is unknown, but the part of the above integral which contains α_2 depends only upon $\int f(\varpi)\,d\varpi$ and this can be seen to be proportional to ρ and to involve h. Let us denote $\int f(\varpi)\,d\varpi$ by $\rho\mathrm{I}$, so that I is a function of h only; then the part of (v.) which contains α_2, contains $\rho^2\mathrm{I}^2$ multiplied by $\frac{\pi a^2 dt}{2\Omega}\,r^2$ and a function of h.

The part of (v.) which contains β_2 depends on I and also on $\int f(\varpi)\,\varpi^2\,d\varpi$. If we write $\int f(\varpi)\,\varpi^2\,d\varpi = \rho\mathrm{I}\Psi$ (so that Ψ is the mean value of ϖ^2 taken over all the molecules of the gas), then this part of (v.) will be $\rho^2\mathrm{I}^2\Psi$ multiplied by $\frac{\pi a^2 dt}{2\Omega}\,r^2$ and by a function of h.

Hence determining the functions of h from a consideration of dimensions, we find

$$\rho\frac{d\mathrm{K}}{dt} = \alpha_2\rho^2 h^{-3/2} + \beta_3\rho^2\psi h^{-1/2} \quad \ldots\ldots\ldots \text{(vi.)},$$

in which α_3, β_3 are constants; or in terms of H and K,

$$\frac{d\mathrm{K}}{dt} = \alpha\rho\mathrm{K}^{3/2} + \beta\rho\mathrm{K}^{1/2}\mathrm{H}^{*} \quad \ldots\ldots\ldots \text{(vii.)},$$

in which the constants α, β do not in any way depend upon the law of distribution of velocities.

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[*] Using the values for α, β given in the footnote on p. 401, we can find for α, β the values

$$\alpha = -\frac{32}{9}\,\frac{a^2r^2}{m\mathrm{K}^2}\sqrt{\frac{2\pi}{3m}}, \qquad \beta = \frac{16}{3}\,\frac{a^2r^2}{m\mathrm{K}^2}\sqrt{\frac{2\pi}{3m}}.$$

In this way we can prove the relation $\mathrm{H} = \frac{2}{3}\mathrm{K}$, instead of assuming it.

In BOLTZMANN's steady state, $dK/dt = 0$, and $H = \frac{2}{3}K$, since the energy is equally divided between the five degrees of freedom. This leads to the relation

$$\alpha = -\tfrac{2}{3}\beta,$$

and equation (vii.) may be written

$$\frac{dK}{dt} = \beta\rho\sqrt{K}\,(H - \tfrac{2}{3}K) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(viii.)}.$$

The rate of variation of H on account of collisions is therefore given by

$$\frac{dH}{dt} = -\beta\rho\sqrt{K}\,(H - \tfrac{2}{3}K) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(ix.)}.$$

Introduction of Dissipation.

§ 7. If the changes in the values of H and K arise solely from collisions, the equations just found will enable us to trace these changes, starting from any initial values. Let us, however, suppose that there is a second cause of change in H and K; suppose that the angular velocity ϖ is retarded by a frictional reaction between the molecules and the surrounding medium, the amount of this retardation being $\epsilon\varpi$. It is not suggested that a reaction of this type exists in nature, but this reaction may be taken as being the simplest reaction possible, just as a loaded sphere has been taken as the simplest dynamical system which will serve our purpose. Due to a frictional reaction of this kind, the mean rotational energy H will suffer a decrease of amount $2\epsilon H$ per unit time.

The equations expressing the rates of change of H and K will therefore be

$$\frac{dK}{dt} = \beta\rho\sqrt{K}\,(H - \tfrac{2}{3}K) \quad . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(x.)},$$

$$\frac{dH}{dt} = -2\epsilon H - \beta\rho\sqrt{K}\,(H - \tfrac{2}{3}K) \; . \; . \; . \; . \; . \; . \quad \text{(xi.)}.$$

§ 8. A steady state is now impossible, on account of the dissipation of energy which takes place. If, however, we suppose the translational energy to be in some way increased at such a rate as exactly to counterbalance the loss of energy, the gas will rapidly assume a steady state given by $dH/dt = 0$, or, by equation (xi.),

$$2\epsilon H + \beta\rho\sqrt{K}\,(H - \tfrac{2}{3}K) = 0 \quad . \; . \; . \; . \; . \; . \; . \; . \quad \text{(xii.)}.$$

The value of H corresponding to a given value of K is therefore

$$H = \tfrac{2}{3}K\Big/\left(1 + \frac{2\epsilon}{\beta\rho\sqrt{K}}\right) \; . \; . \; . \; . \; . \; . \; . \; . \; . \; . \quad \text{(xiii.)}.$$

Hence if ϵ, β, K have values such that ϵ is very great in comparison with $\beta\sqrt{K}$, then H will be very small in comparison with K. The rate at which the external agency introduces energy is $2\epsilon H$, and this may be made very small by supposing ϵ to be very small, although great in comparison with $\beta\sqrt{K}$. If this agency is removed the state of the gas will only change very slowly, so that the state specified by equation (xiii.) may be appropriately described as the "approximately steady" state. When H vanishes in comparison with K, the equation giving this state takes the simpler form

$$\epsilon \mathrm{H} = \tfrac{1}{3}\beta\rho \mathrm{K}^{3/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xiv.})$$

The conditions which have been found to be necessary in order that this state may exist, are that ϵ, β should be small, and that K should be so small that $\frac{\epsilon}{\beta\sqrt{\mathrm{K}}}$ is very great. Thus the steady state will be possible for all temperatures below a certain temperature, namely, the temperature at which $\beta\sqrt{\mathrm{K}}$ begins to be comparable with ϵ. Below this temperature H vanishes in comparison with K, and the rate of dissipation of energy is a small quantity of the second order.

It follows that if experiments are conducted at temperatures so low as to be below this critical temperature, no value of γ can possibly be observed except

$$\gamma = 1\tfrac{2}{3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xv.}).$$

At higher temperatures, there is no definite ratio between H and K which tends to establish itself. In fact, if experiments are conducted with a view to determining γ, the value observed will depend on the past history of the gas and the duration of the experiment, so that γ may have any value between $1\frac{2}{3}$ and $1\frac{2}{5}$.

Thus it appears that if, under the conditions we are now considering, a consistent value is obtained for γ from experiments on the gas in question, this value can be no other than $1\frac{2}{3}$, and the temperature at which the experiments are conducted must be what has been referred to as a low temperature. It must be particularly noticed, that this temperature is only low relatively to the other temperatures considered: no knowledge as to its absolute value is possible so long as ϵ and r/a remain unknown quantities. If, however, for the moment, we assume that the present molecules are a fair representation of the molecules of an actual gas, and that the dissipation of energy caused by our assumed frictional reactions supplies a true analogy to radiation of energy in nature, then we can form some estimate as to what a "low" temperature must mean. It is a temperature at which H, and therefore the radiation, is inappreciable; that is to say, it is a temperature at which the gas is non-incandescent.

The Distribution of Energy in the approximately Steady State.

§ 9. To sum up, it appears that if we are willing to admit that our present dynamical system supplies a sufficiently good analogue to a real gas, then the introduction of a dissipation function will supply an explanation of the difficulties mentioned in the introduction, at any rate for the case of a non-luminous gas. Part II. of this paper consists of an effort to show that our present system is a fair analogy, if not of an actual molecule, yet at any rate, of a dynamical system which contains all the features we believe to be essential to a molecule.

Referring back to equations (x.), (xi.) and (xiv.), it is now clear that, for a non-luminous gas, the equations determining H and K will be

$$dK/dt = -\tfrac{2}{3}\beta\rho K^{3/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xvi.)},$$

and

$$\epsilon H = \tfrac{1}{3}\beta\rho K^{3/2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xvii.)}.$$

Equation (xvii.) is the relation between H and K which must now replace the equation of MAXWELL and BOLTZMANN, viz. :

$$H = \tfrac{2}{3}K.$$

It therefore appears that, in the present case, the total radiation will be proportional to $K^{3/2}$, and in the more general case discussed in Part II., the radiation will be seen to increase still more rapidly with the temperature. Thus it is easy to see how it is possible for the total radiation to increase very rapidly near the temperature of incandescence, whereas if we supposed the energy divided in any invariable ratio between the different degrees of freedom, it is difficult to see how the radiation could be anything but directly proportional to the temperature.

Extension of the foregoing Theory.

§ 10. It is possible, under certain conditions, to apply the above methods to a more general type of molecule.

Let the energy of the molecule consist partly of translational energy, and partly of various kinds of internal energy, potential as well as kinetic. The only case considered will be that in which the internal energy is small : the potential energy will therefore arise from small oscillations about a position of equilibrium, and these oscillations will be of definite period, and such as may be supposed to result in the emission of light possessing a line-spectrum. Thus the total energy corresponding to any such principal mode of vibration, will, when averaged over a large number of molecules, be half potential and half kinetic.

It is necessary for the success of the present method that the probability of a collision between two molecules should depend solely on their relative velocity, and not on their internal co-ordinates. Now a rotation is to be regarded as internal energy, and a rapid rotation will be equivalent to an increase of volume, and will therefore increase the probability of a collision unless the molecules are spheres of invariable radius, and of which the centres move in straight lines. Thus the molecules must either be spheres of which the centre of gravity and the geometrical centre coincide, or else as in the former case, they must differ by so little from this, that the divergence has no effect on the final result (*cf.* § 5).

We shall further suppose for the present that the internal energies are only slightly altered by collision, as was the case in the former problem.

The law of distribution of internal energy will now be independent of the velocity of translation, so that the mean value of the internal energy of any specified mode, will be the same whether the average is taken over all collisions or over all molecules.

The same notation as before will be used in connection with the motion of translation. The energies of the various internal modes will be denoted by $e_1, e_2 \ldots .$ and their mean values taken over all molecules by $E_1, E_2 \ldots .$ If t_1, r_1 are the potential and kinetic energies of which the sum is e_1 then the mean value of t_1 will be $\frac{1}{2}E_1$.

We begin by calculating the increase at any single collision in $(c^2 + c'^2)$, $(e_1 + e'_1)$, &c. Each of these quantities will be a quadratic function of the velocities concerned, and is symmetrical as regards the two molecules. We next assume the law of distribution of translational velocities to be

$$\phi(u, v, w) = Ae^{-hmc^2},$$

and average the values we have found over all collisions, the procedure being exactly identical with that already followed in the former problem.

For given velocities we arrive at an equation similar to equation (i.), p. 400. It is to be particularly noticed that the translational velocities can only enter through the term V^2. We now continue in the manner of § 6. The factor V again occurs multiplying every integrand, and giving rise to the term $\sqrt{K}$ in the final result.

By this means we arrive at equations similar to equations (viii.) and (ix.), p. 403, giving the rate of change of $K, E_1, E_2, \ldots$ arising from collisions. By what has been already said, these must be of the form—

$$dE_1/dt = \rho\sqrt{K}\{a_{11}E_1 + a_{12}E_2 + \ldots + b_1K\}$$
$$dE_2/dt = \rho\sqrt{K}\{a_{21}E_1 + a_{22}E_2 + \ldots + b_2K\},$$

and similar equations, together with

$$dK/dt = \rho\sqrt{K}\{c_1E_1 + c_2E_2 + \ldots + cK\} \quad . \quad . \quad . \quad . \quad \text{(xviii.)}.$$

It is here assumed that a specified value of any internal velocity is just as probable as the equal negative velocity, otherwise the mean value of products of different velocities could not be supposed to vanish. The equations determining the steady state are

$$a_{11}E_1 + a_{12}E_2 + \ldots + b_1K = 0,$$
$$a_{21}E_1 + a_{22}E_2 + \ldots + b_2K = 0,$$
$$\cdots\cdots$$
$$c_1E_1 + c_2E_2 + \ldots . + cK = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad \text{(xix.)}.$$

These equations are not independent; when they are added together the resulting equation vanishes identically in virtue of the fact that the mean total energy is, under all circumstances, unchanged by collisions.

Hence the equations can all be satisfied when the variables $E_1, E_2 \ldots K$ are in a definite ratio. The distribution of energy indicated by this ratio will therefore be permanent, and since the equations which determine it are linear, it will be unique. This is the distribution discovered by BOLTZMANN, in which the energy is equally divided between the various degrees of freedom.

Continuity of Path.

§ 11. Any of the coefficients in the above system of equations may vanish; so that it will be possible for the equations to fall into two groups, in such a way that no variable occurs in both groups. The motion will in this case be steady provided all the variables of the first group are in a given ratio, and all the variables of the second group are in a given ratio, but there need be no fixed ratio between the two groups.

Thus the total energy of the first group will be divided according to BOLTZMANN'S Law, and the same applies to the second group, but the distribution between the two groups will not follow this law.

This is the analytical expression of MAXWELL'S condition as to "Continuity of Path."*

The Two Kinds of Internal Co-ordinates.

§ 12. Let us suppose, as before, that certain velocities are subject to a retardation proportional to the velocities. The mean energies arising from these degrees of freedom will be denoted by $F_1, F_2, \ldots$, the letters $E_1, E_2, \ldots$ being reserved for those energies which are not dissipated by friction.

The system of equations (xviii., p. 406) must now be replaced by

$$dE_1/dt = \rho\sqrt{K}\,\{\Sigma a_{1s}E_s + \Sigma p_{1s}F_s + b_1K\},$$

$$dF_1/dt = \rho\sqrt{K}\,\{\Sigma q_{1s}E_s + \Sigma r_{1s}F_s + b'_1K\} - \epsilon_1F_1,$$

$$dK/dt = \rho\sqrt{K}\,\{\Sigma c_sE_s + \Sigma c_s'F_s + eK\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xx.}).$$

If we suppose that at a collision only a small amount of energy can be exchanged between the F modes and the remaining modes, then all the coefficients p, q, b', and c' will be small.

It is immediately obvious that equations (xx.) may be treated exactly as equations

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* MAXWELL, 'Camb. Phil. Soc. Trans.,' vol. 12, p. 548; or 'Collected Works,' vol. 2, p. 714.

(x.) and (xi.) were treated. At low temperatures the ratios F/E all tend to zero, and the equations may be replaced by

$$dE_1/dt = \rho\sqrt{K}\,\{\Sigma a_{1s}E_s + b_1K\},$$

$$\rho\sqrt{K}\,\{\Sigma q_{1s}E_s + \Sigma r_{1s}F_s + b'_1K\} - \epsilon_1F_1 = 0,$$

and

$$dK/dt = \rho\sqrt{K}\,\{\Sigma c_sE_s + eK\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxi.}).$$

A steady state would be possible, if we could simultaneously satisfy all equations of the type,

$$\Sigma a_{1s}E_s + b_1K = 0,$$

together with

$$\Sigma c_sE_s + eK = 0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxii.}),$$

by values of K, E_1, E_2, &c., which were different from zero.

Let us use S to denote summation with respect to degrees of freedom. Then since we know the solution of equations (xix., p. 406) we can, by substitution, arrive at a system of relations between the coefficients.

Making allowance for the alteration in notation (*cf.* equations xx., p. 407) these relations can be written in the form

$$Sa_{1s} + Sp_{1s} + 3b_1 = 0,$$

and

$$Sc_s + Sc'_s + 3e = 0.$$

If we attribute an amount of energy λ to every degree of freedom, we have

$$\Sigma a_{1s}E_s + b_1K = \lambda\,(Sa_{1s} + 3b_1) = -\lambda Sp_{1s}$$

and

$$\Sigma c_sE_s + eK = \lambda\,(Sc_s + 3e) = -\,\lambda Sc'_s.$$

Thus the equations (xxii.) are very approximately satisfied, in virtue of the smallness of the p and c' coefficients. The solution we have found will therefore give a state which is approximately steady.

§ 13. Thus at a sufficiently low temperature the energy of the gas which we have been considering will distribute itself in such a manner that an equal amount of energy will correspond to each degree of freedom which is not retarded by friction. The amount of energy corresponding to a degree of freedom which is retarded by friction will be vanishingly small. The amount of such energy is given by the equations,

$$F_1 = \rho\,\frac{\sqrt{K}}{\epsilon_1}\,\{\Sigma q_{1s}E_s + b'_1K\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxiii.}).$$

If the degrees of freedom included in the E's are n in number, and those counted in the F's are m in number, it is obvious that the ratio of the specific heats must be taken to be

$$\gamma = 1 + \frac{2}{n+3} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xxiv.}),$$

whereas BOLTZMANN'S theorem would lead to

$$\gamma = 1 + \frac{2}{m + n + 3}.$$

The equation (xxiv.) will only hold in the limit when the temperature $\theta = 0$. At other temperatures γ will have a slightly different value, since the F energies cannot be entirely neglected. Our results as before only hold up to the temperature at which the gas begins to emit an appreciable amount of radiant energy, and this temperature may be supposed to be somewhat above the point of incandescence.

Up to this point, F_1, F_2, . . . will always be in the same proportion to one another, so that the brightness of the various lines in the spectrum will be in a constant ratio, each being proportional to $\theta^{3/2}$.

§ 14. We have been working on the assumption that there is a complete absence of frictional forces acting on K, and on E_1, E_2, ... These assumptions, however, are not necessary. In the steady state we have from equation (xxiii.),

$$F_1 = \frac{\beta_1}{\epsilon_1} \rho K^{3/2},$$

where β_1 is a quantity which depends only on the construction of the molecule.* The temperatures which have been considered have been those for which $\frac{\beta_1}{\epsilon_1}\sqrt{K}, \frac{\beta_2}{\epsilon_2}\sqrt{K}$, &c., are all very small. But if for any single degree of freedom, say that for which the energy co-ordinate is F_1, either ϵ_1 is exceptionally small or β_1 exceptionally great, the range of temperature will be greatly restricted on this account. At temperatures at which $\frac{\beta_1}{\epsilon_1}\sqrt{K}$ is large while the remaining similar quantities are small, it is clear that F_1 must be treated as an E co-ordinate.

At zero temperature all the energy co-ordinates to which friction corresponds must be regarded as F co-ordinates. As the temperature increases we must suppose these co-ordinates one by one to change from being F co-ordinates, and after occupying a position intermediate between that of an F and that of an E co-ordinate to finally become E co-ordinates. If there is a co-ordinate for which ϵ is extremely small, or β very great, that is to say, a co-ordinate corresponding to a degree of freedom which is only very slightly retarded by friction, or to one from which energy passes freely, then such a co-ordinate will become an E co-ordinate at such a low temperature that it may be regarded as always being an E co-ordinate.

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[*] It may be noticed that the value of β_1 supplies a measure of the facility with which energy is exchanged between the F_1 mode and the other modes. If $\beta_1 = 0$, it is impossible for such an exchange to take place, and the F_1 mode does not satisfy the condition of continuity of path. Thus if friction dissipates the energy of the F_1 mode, the value of F_1 will finally be zero. If, however, we have $\beta_1 = 0$, together with $\epsilon_1 = 0$, the value of F_1 is indeterminate. The rotation of the loaded sphere about the axis of symmetry supplied a good illustration of a mode of energy for which $\beta = 0$.

Variation of γ with Temperature.

§ 15. Thus our conclusion is that it is not permissible to count the degrees of freedom; they must be weighted as well. We can write

$$\gamma = 1 + \frac{2}{3 + \Sigma a_s},$$

where a_s is the "weight" of the sth degree of freedom. So long as $\frac{\beta_s}{\epsilon_s}\sqrt{\mathrm{K}}$ is small, we may put $a_s = 0$. When this quantity is very great, either owing to the greatness of K or the absence of ϵ_s, we put $a_s = 1$. For intermediate values of $\frac{\beta_s}{\epsilon_s}\sqrt{\mathrm{K}}$, a_s will be a proper fraction, the value of which depends not only on the temperature but also on the series of changes through which the gas has passed.

As a consequence of this, it is clear that γ may be expected to vary with the temperature, and that it is no longer restricted to having one of the values given by the formula $1 + 2/n$. It would be going too far to expect any agreement with experiment at present, since we are considering a purely arbitrary type of molecule such as certainly does not exist in nature.

Case of a Diatomic Molecule.

§ 16. As an illustration of the foregoing theory, we may examine the case of a molecule which is composed of two atoms held together by an attractive force. The atoms can execute internal vibrations giving rise to the emission of light; for these vibrations we can suppose $\frac{\beta}{\epsilon}\sqrt{\mathrm{K}}$ to be very small, so that $a = 0$. The molecule can rotate about its axis of symmetry, but we can suppose β to be zero, so that again $a = 0$. For the rotation about the two remaining axes β will be large, so that we may take $a = 1$. We can suppose that the oscillations of the atoms as a whole about their position of equilibrium are dissipated by radiation so that $a = 0$.*

This leads us to the value $\gamma = 1\frac{2}{5}$, a value which it is impossible to arrive at by means of BOLTZMANN's theorem when we are considering a molecule which is made up of two separable parts, but which is nevertheless known to be the true value for many diatomic gases, such as hydrogen, nitrogen, and oxygen. If the present theory in

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* Radiation of this kind would give rise to definite lines in the spectrum of a frequency which might, and probably would, be very different from the frequencies of the light vibrations given out by the internal vibrations of the atoms. In this connection it is of interest to remember that experiments with Hertzian vibrators have demonstrated the existence, in certain substances, of free periods of which the frequency is only about 1/1,000,000th of the frequency of the sodium lines. (P. DRUDE, 'Wied. Ann.,' vol. 58, p. 1; vol. 59, p. 17; vol. 64, p. 131.)

any way corresponds to the facts, the molecules of these gases must possess a symmetry similar to that possessed by figures of revolution.

PART II.

THE DISTRIBUTION OF ENERGY IN A GAS OF WHICH THE MOLECULES ARE OF A MORE GENERAL TYPE.

Statement of Problem to be Discussed.

§ 17. Having discovered, by means of the simple dynamical illustration discussed in Part I., what sort of results are to be expected, it now becomes possible to examine the case in which the molecules form a more complex dynamical system, and as this may be done by an entirely different method from that previously followed, it is now possible to remove the restrictions which it was previously found necessary to impose on the nature of the molecules.

The molecules are, as before, supposed to be all exactly similar, but intermolecular forces are no longer excluded, and the radiation is supposed to be of a more general type.

Let us suppose that each molecule is a dynamical system, possessing in itself $k + n$ degrees of freedom in addition to the freedom of the molecule to move in space. There will therefore be $2(k + n) + 3$ co-ordinates required to specify the condition of a molecule apart from its position in space, and $4(k + n) + 8$ quantities are required to specify a collision.

The co-ordinates of position of any molecule will be

x, y, z, the co-ordinates of its centre of gravity referred to axes fixed in space,

$p_1, p_2, \ldots p_n$, the co-ordinates which do not occur in the expression for the potential energy; as, for example, the co-ordinates which determine the orientation of a rigid body.

$r_1, r_2, \ldots r_\kappa$, the co-ordinates which do occur in the expression for the potential energy.

The co-ordinates of velocity will be

u_1, u_2, u_3 the time-differentials of x, y, z.

$q_1, q_2, \ldots q_n$,, ,, ,, $p_1, p_2, \ldots p_n$.

$s_1, s_2, \ldots s_\kappa$,, ,, ,, $r_1, r_2, \ldots r_\kappa$.

We shall write c^2 for $u^2 + v^2 + w^2$, and it will frequently be necessary, for the sake of brevity, to denote all co-ordinates of the same type by a single representative letter without a suffix.

Thus
$$f(p) = f(p_1, p_2, \ldots p_n)$$
$$du = du_1\, du_2\, du_3, \text{ \&c.}$$

We shall assume that these co-ordinates are principal co-ordinates of the system, so that both the kinetic and potential energies will be the sums of squares. We may, in fact, write

$$2V = c_1 r_1^2 + c_2 r_2^2 + \ldots + c_\kappa r_\kappa^2$$

$$2T = mc^2 + 2Q + 2S$$

where

$$2Q = b_1 q_1^2 + b_2 q_2^2 + \ldots + b_n q_n^2$$

$$2S = d_1 s_1^2 + d_2 s_2^2 + \ldots + d_n s_n^2.$$

We shall suppose the oscillations of the r, s co-ordinates to be so small as to be isochronous, and in this case the c and d coefficients will be constants. Since the r co-ordinates are to be very small, the "configuration" of a molecule may be supposed to be determined by its p co-ordinates.

With a view to simplifying subsequent analysis, we shall assume that the b's also are constants. It will be seen that the character of our results is not materially modified by this simplification, and the assumption is, of course, legitimate if we suppose the molecule, except as regards small oscillations, to behave like a rigid body, the atoms never moving far from certain equilibrium positions. We shall suppose that the vibrations of the molecule result in a radiation of energy, and we accordingly assume a dissipation function G. This will be supposed to be a quadratic function of the s co-ordinates with constant coefficients; it will not in general be reducible to the sum of squares. The existence of G implies an interaction between matter and ether. The assumption that G contains no terms in u or q is in strictness only legitimate if we suppose the u and q velocities to be uninfluenced by the ether, but it is easy to see, as in § 14, that even if these velocities are acted upon by the ether, the neglect of these actions is of no importance so long as they are sufficiently small.

We have spoken of T and V as kinetic and potential energy, but there is no reason why these energies should not be regarded as electro-magnetic and electrostatic energy, or indeed as energy of any other kind, provided only that it is always possible to deduce the equations of motion from the energy function by LAGRANGE'S method. But it is probably best to regard the system just specified as simply a dynamical system in the strictest sense, this system being capable of *illustrating* all the properties which experiment shows to be possessed by a molecule.

§ 18. Corresponding to a collision in the case of two spheres, we shall suppose that it is possible for an action to take place between two molecules, and this action will be spoken of as an "encounter." For the present, it is not necessary to specify the exact nature of an encounter, but it will be supposed

(i.) that the duration of an encounter is infinitesimal, so that an encounter causes no direct change in the co-ordinates of position of a molecule, (ii.) that an encounter is similar to an impact in the former problem, in that it may or may not entirely change the translational velocities of the two molecules concerned, but that the internal velocities are only changed by a small amount.

The sudden increase in any quantity ξ consequent on an encounter will be denoted by $\Delta\xi$.

In virtue of the above assumptions

$$\Delta p = 0, \quad \Delta r = 0,$$

Δq and Δs are small, and Δu will in general be comparable with u.

A "collision" will be a special case of an encounter, and may be described as follows. Suppose that every molecule is surrounded by a small sphere, of which the centre coincides with the centre of gravity of the molecule, and which moves as though it were rigidly attached to the molecule. The radius of the sphere is not yet fixed, but it must be such that the sphere entirely encloses the matter of which the molecule is composed. Then a collision will be defined as an encounter which is such that the spheres of the two molecules which are engaged, intersect; the "duration" of a collision will be taken to be sufficiently long to include the whole interval from the instant at which the spheres first intersect to the instant at which they separate. The assumptions as to the nature of the gas, which are usually expressed by saying that the gas is *molekular-ungeordnet*, and that the number of collisions in which three or more molecules are engaged, is infinitely small in comparison with the number of binary collisions, will be replaced by the following assumptions:

(i.) The duration of a collision is so short, that the positional co-ordinates may be treated as constant throughout the collision, while the velocity co-ordinates are abruptly changed.

(ii.) The chance of any molecular sphere intersecting two other spheres at once, vanishes in comparison with the chance of its intersecting one other sphere.

(iii.) The chance that a molecule A is found with all its co-ordinates within certain small ranges of values, which are such that the sphere of the molecule does not intersect any other sphere, depends solely upon the co-ordinates of the molecule A, and upon the potential upon A of the field of intermolecular force; it does not depend upon the arrangement of the other molecules.

The Characteristic Equation.

§ 19. Starting from the state of the gas at the time $t = 0$, we can arrive at the state after an interval dt, by imagining the following succession of events. Suppose in the first place that each molecule is allowed to move under no force except its own internal forces for a time dt, and let all the collisions which would occur in this time be supposed to occur. After this imagine the molecules divided into pairs in every possible way and suppose an encounter to occur between the two molecules of every such pair. The duration of the encounter is to be dt; during it the co-ordinates of each molecule are to change only on account of the forces of the encounter; that is, on account of the intermolecular forces existing between the two molecules under consideration. These encounters are to take place consecutively, not simultaneously.

It is easily seen that each molecule has now been acted upon by exactly the same forces by which in the actual course of events it would have been acted upon in the interval dt. Hence, since there is no limit to the smallness of dt, the final state of the gas is independent of the order in which this series of events takes place, and is identical with the state in which the gas would have been found if the forces had acted simultaneously.

Two points deserve attention in connection with this argument. Firstly, it might be objected that the changes in the co-ordinates of molecules which experience an actual collision are not additive, inasmuch as one of these changes is not infinitesimal. It is, however, clear that there is no necessity to take the *infinitesimal* changes into account at all in the case of these molecules, for the number of these molecules vanishes in comparison with the total number when dt is made to vanish. Secondly, it is true that the number of molecules within any specified limits will not always consist of the same individual molecules. But it is a fundamental assumption of the kinetic theory that any N molecules which have nothing in common except that certain co-ordinates have specified values, will behave exactly like N other similarly conditioned molecules.

We can therefore reduce the continuous changes of the co-ordinates of molecules which arise from the action of intermolecular forces, to a series of encounters of the kind described in § 18.

§ 20. To completely specify an encounter, we require the values of all the co-ordinates enumerated in § 17, of both molecules. It will, however, be convenient to write

$$x'' = x' - x,$$

with a similar notation for y, z, u, v, w, and to specify a collision by the values of x'', . . . instead of the values of x' . . .

As regards the law of distribution of the various co-ordinates, we notice that whatever the values of the q, r, s, and u co-ordinates may be, the probability that the p co-ordinates lie within the limits dp may, in the absence of external forces, be taken to be $f(p)\,dp$ where $\int f(p)\,dp = 1$. Since the internal energy is only slightly changed by encounters, we shall again suppose that the distribution of internal energy is independent of the translational energy, and hence we may suppose that the number of molecules per unit volume which lie within a range $dp\,dq\,dr\,ds\,du$ is

$$\Phi(u)\,\mathrm{F}(q, r, s)\,f(p)\,dp\,dq\,dr\,ds\,du,$$

or, as it will be frequently written,

$$\Phi \mathrm{F} f\,dp\,d\mathrm{P}\,du,$$

$d\mathrm{P}$ being written for $dq\,dr\,ds$.

We shall suppose that the gas has adjusted itself so that the distribution of the energy of translation is the permanent distribution. Thus at a point at which the potential energy of a molecule is ψ we shall have

$$\Phi(u) = e^{-h(mc^2+2\psi)}.$$

As we are going to admit the existence of intermolecular forces, the potential of a molecule at a point will depend on the co-ordinates of the molecules as well as the position of the point. Thus ψ will in general be a function of x, y, z, p and r.

Let molecules of which the co-ordinates lie within limits $dp\,d\mathrm{P}\,du$ be called molecules of class α; if the limits are $dp'\,d\mathrm{P}'\,du'$, let the molecules be described as molecules of class β. Each of these classes will consist of a number of molecules which is indefinitely small in comparison with the total number of molecules present.

At any moment, let us imagine all the molecules placed in position, except those belonging to one or other of these two classes. Let them produce a field of force such that if a molecule of class α is placed with its centre at the point x, y, z, then the potential of this molecule will be Ω.

Then the probability that a molecule of class α will be found with its centre within an element $dx\,dy\,dz$ at $x\,y\,z$ is

$$e^{-h(mc^2+2\Omega)}\mathrm{F} f\,dp\,d\mathrm{P}\,du\,dx\,dy\,dz.$$

Hence the total number of such molecules to be found in the whole unit volume may be obtained by integrating this expression over the whole volume, and may be written as

$$\mathrm{N}_\alpha = e^{-h(mc^2+2\Psi)}\mathrm{F} f\,dp\,d\mathrm{P}\,du \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{i.}),$$

where

$$e^{-2h\Psi} = \iiint e^{-2h\Omega}\,dx\,dy\,dz \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{ii.}).$$

The integral is taken over the whole unit volume, since the integrand is supposed to vanish if the point x, y, z is such that the centre of a molecule of class α cannot be found there.

The quantity Ψ will be called the mean intermolecular potential for a molecule of class α. It is clearly a function of all the coefficients which occur in the law of distribution as well as of the co-ordinates of molecules of class α. If we remember that the r co-ordinates are supposed to be very small, it is clear that it will be sufficient to imagine that Ψ is a function of

(i.) h, on account of the way in which this coefficient enters in equation (ii.),
(ii.) the coefficients occurring in $f(p)$ (these have, however, already been assumed to be invariable),
(iii.) the p co-ordinates of molecules of class α.

Thus, for our purpose, ψ is a function of h and p only.

Fixing our attention on any one of these molecules of class α, the probability that the centre of a molecule of class β may be found within the limits $dx'' \, dy'' \, dz''$ measured relatively to the first molecule will be

$$e^{-h(mc'^2+2\Omega')}\mathrm{F}'f' \, dp' \, d\mathrm{P}' \, du' \, dx'' \, dy'' \, dz'',$$

where Ω' is the potential of a molecule of class β at this point. Now Ω' can be made up of two parts, Ω_1 the part due to the presence of the single molecule of the first class, and Ω_2 the part due to all the other molecules combined.

It is clear that Ω_1 will only depend on the two molecules of the encounter, and is therefore a function of $p \; p' \; r \; r' \; x'' \; y''$ and z''.

The total number of encounters of the type we are now considering, namely those within limits

$$dp \, d\mathrm{P} \, du \, dp' \, d\mathrm{P}' \, du' \, dx'' \, dy'' \, dz'' \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(iii.)},$$

will be

$$\Sigma e^{-h(mc'^2+2\Omega_1+2\Omega_2)}\mathrm{F}'f' \, dp' \, d\mathrm{P}' \, du' \, dx'' \, dy'' \, dz'',$$

where the summation extends to all the molecules of class α.

This number may be written as

$$e^{-h(mc'^2+2\Omega_1)}\mathrm{F}'f' \, dp' \, d\mathrm{P}' \, du' \;\; \Sigma e^{-2h\Omega_2} \, dx'' \, dy'' \, dz''.$$

Now it is obvious that the mean value of $e^{-2h\Omega_2}$ taken over all the elements of volume included in the summation, will be $e^{2h\Psi'}$, where Ψ' is the mean intermolecular potential of a molecule of class β, and is therefore a function of h and p' only.

Thus, since the summation extends to N_a elements of volume,

$$\Sigma e^{-2h\Omega_2} \, dx'' \, dy'' \, dz'' = \mathrm{N}_a \, e^{-2h\Psi'} \, dx'' \, dy'' \, dz''.$$

This gives us for the total number of encounters of the type we are considering,

$$e^{-h(m(c^2+c'^2)+2\Psi+2\Psi'+2\Omega_1)}\mathrm{FF}'ff' \, dp \, d\mathrm{P} \, du \, dp' \, d\mathrm{P}' \, du' \, dx'' \, dy'' \, dz'';$$

or, if we write

$$\eta \equiv h(m(c^2 + c'^2) + 2\Psi + 2\Psi' + 2\Omega_1) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(iv.)},$$

the number is

$$e^{-\eta} \, \mathrm{FF}'ff' \, dp \, d\mathrm{P} \, du \, dp' \, d\mathrm{P}' \, du' \, dx'' \, dy'' \, dz'' \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(v.)}.$$

Let us call encounters of this type, encounters of class A, and denote their number (expression (v.)) by N_A.

There will be a second class of encounters which will be called class B, such that the co-ordinates *after* the encounter lie within the limits (iii.), (p. 416). The co-ordinates before the encounter will accordingly lie within certain other limits,

$$d\bar{p}\, d\bar{P}\, d\bar{u}\, d\bar{p}'\, d\bar{P}'\, d\bar{u}'\, d\bar{x}''\, d\bar{y}''\, d\bar{z}'' \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad \text{(vi.)},$$

surrounding certain values $\bar{p}$, $\bar{P}$ . . . &c. By LIOUVILLE's Theorem, the complete differential (vi.) is equal to the complete differential (iii.), hence the number of encounters of class B will, by comparison with (v.), be seen to be

$$e^{-\bar{\eta}}\,\overline{FF'}ff'\,dp\,dP\,du\,dp'\,dP'\,du'\,dx''\,dy''\,dz'',$$

the positional co-ordinates and therefore also f, f', remaining unaltered by the encounter.

Let N_0 be the total number of molecules lying within a range dP. Then a certain number of encounters of class A will result in a unit loss to N_0, a certain number of encounters of class B in a unit gain. Thus, if for one at least of the co-ordinates which are changed by the encounter, say ξ, we have $\Delta\xi > d\xi$, where Δ denotes an increase due to an encounter, then it is certain that the co-ordinate ξ will be placed outside the limit $d\xi$, and N_0 will accordingly be diminished by unity.

It is, however, conceivable that for every co-ordinate we may have $\Delta\xi < d\xi$, and in this case there is a probability λ that no single co-ordinate passes outside its limits, and therefore that the molecule after encounter must still be counted in N_0. It is easy to see that the probability that the ξ co-ordinate remains with the limits $d\xi$ is $\left(1 - \frac{\Delta\xi}{d\xi}\right)$, and therefore that

$$\lambda = \Pi\left(1 - \frac{\Delta\xi}{d\xi}\right),$$

where Π denotes continued multiplication extending to all the q, s co-ordinates. The loss experienced by N_0 on account of encounters of class A will therefore be $(1 - \lambda)\,N_A$ where $\lambda = 0$ for certain values of $x''\ y''\ z''$, and is a proper fraction over the remainder, and the boundary of these regions depends on the co-ordinates of encounter.

It follows that as the result of encounters of classes A and B combined, there is a net gain to N_0 of

$$[(1 - \bar{\lambda})e^{-\bar{\eta}}\bar{F}\bar{F}' - (1 - \lambda)e^{-\eta}FF']ff'\,dp\,dP\,du\,dp'\,dP'\,du'\,dx''\,dy''\,dz''.$$

The expression in square brackets may be written as $\Delta\,\{(1 - \lambda)e^{-\eta}FF'\}$. Hence the total gain to N_0 arising from all classes of encounters will be

$$dP\int\Delta\{(1 - \lambda)e^{-\eta}FF'\}ff'\,dp\,du\,dp'\,dP'\,du'\,dx''\,dy''\,dz''. \quad . \quad . \quad . \qquad \text{(vii.)}.$$

Here, as throughout the paper, a single tall integral sign denotes integration over all values of the variables, of which the differentials occur after the sign of integration.

For those values of x'', y'', z'', for which λ is different from zero, we have seen that $\Delta\xi$ is, for every co-ordinate ξ of the same order of small quantities as $d\xi$, and hence it follows that

$$\int \Delta\{\lambda e^{-\eta}\,\mathrm{FF}'\}ff'\,dp\,du\,dp'\,d\mathrm{P}'\,du'\,dx''\,dy''\,dz''$$

is of the same order of small quantities.

In the limit, when the differentials such as $d\xi$ are supposed to vanish, this integral will vanish also, so that we may put $\lambda = 0$ in expression (vii.).

The total gain to N_0 from all encounters is therefore $\mathrm{I}d\mathrm{P}$, where

$$\mathrm{I} = \int \Delta\{e^{-\eta}\,\mathrm{FF}'\}ff'\,dp\,du\,dp'\,d\mathrm{P}'\,du'\,dx''\,dy''\,dz'' \quad . \quad . \quad . \quad . \quad \text{(viii.)}.$$

§ 21. Now we have seen (§ 19) that the change in N_0 in time dt will arise from three causes—

(α) the change arising from collisions,
(β) the change arising from encounters other than collisions,
(γ) the change arising from the change in the co-ordinates of the various molecules during the time dt, which would occur if there was no interaction between different molecules.

Now any molecule, A, will collide with a second molecule, B, in the interval of time dt, provided that at the beginning of this time the centre of A lies within a certain region of space; provided, that is, that $x''\,y''\,z''$ lie within certain limits. It is easily seen that these limits are arrived at by writing $dx'' = u_1''\,dt$, and allowing for $y''\,z''$ a range of values corresponding to points inside a certain circle of diameter equal that of a molecular sphere. Hence the increase in N_0 arising from cause (α) will be $\mathrm{J}d\mathrm{P}\,dt$ where

$$\mathrm{J} = \int \Delta\{e^{-\eta}\,\mathrm{FF}'\}ff'\,dp\,du\,dp'\,d\mathrm{P}'\,du'\,u_1''\,dy''\,dz'' \quad . \quad . \quad . \quad . \quad \text{(ix.)},$$

and Δ denotes the increase due to a collision.

The increase arising from encounters other than collisions has already been found to be $\mathrm{I}d\mathrm{P}$ (see viii.), where Δ must denote the increase due to an encounter of which the duration is dt. Since dt is to be very small, we may replace Δ by $dt\,\mathrm{D}/\mathrm{D}t$, where $\mathrm{D}/\mathrm{D}t$ denotes the rate of increase under the influence of intermolecular forces only. The increase to N_0 arising from cause (β) will therefore be $\mathrm{K}\,d\mathrm{P}\,dt$, where

$$\mathrm{K} = \int \frac{\mathrm{D}}{\mathrm{D}t}\{e^{-\eta}\mathrm{FF}'\}ff'\,dp\,du\,dp'\,d\mathrm{P}'\,du'\,dx''\,dy''\,dz'' \quad . \quad . \quad . \quad . \quad \text{(x.)}.$$

§ 22. In addition to the gain or loss which has just been calculated, N_0 will experience a further gain or loss on account of the gradual changes in the co-ordinates of the various molecules.

Let ξ be a typical co-ordinate, and ξ_0 and $\xi_0 + \delta\xi_0$ the limits of the values of ξ for molecules counted in N_0. After a time dt, ξ will have increased to $\xi + \frac{d\xi}{dt}dt$, so that some molecules will enter within the above limits, and others will pass out, in course of the time dt.

The molecules which enter within the limits will be those for which, at the beginning of the interval, ξ had a value which was between $\xi_0 - \left(\frac{d\xi}{dt}\right)_{\xi_0} dt$ and ξ_0. The number of such molecules is accordingly

$$\frac{1}{\delta\xi_0}\left(N_0\frac{d\xi}{dt}\right)_{\xi_0} dt.$$

Similarly the number of molecules which escape from between these limits is

$$\frac{1}{\delta\xi_0}\left(N_0\frac{d\xi}{dt}\right)_{\xi_0+\delta\xi_0} dt,$$

so that the resultant gain to N_0 on account of changes in the ξ co-ordinates, the other co-ordinates being supposed to remain constant, will be

$$-\frac{\partial}{\partial\xi}\left(N_0\frac{d\xi}{dt}\right)dt.$$

There is no limit to the smallness of dt, so that if all the co-ordinates vary simultaneously, the gain to N_0 will be

$$-dt\,\Sigma\frac{\partial}{\partial\xi}\left(N_0\frac{d\xi}{dt}\right)$$

or

$$-dt\,\Sigma\frac{\partial N_0}{\partial\xi}\frac{d\xi}{dt} - N_0\,dt\,\Sigma\frac{\partial}{\partial\xi}\left(\frac{d\xi}{dt}\right)$$

in which the summation extends to all the co-ordinates q, r, and s.

The value of $d\xi/dt$ must be found from the equations of motion of the molecule when under the action of no external forces. If η be any co-ordinate of position, and E be written for the total energy $T + V$, then these equations will be of the form

$$\frac{d}{dt}\left(\frac{\partial E}{\partial\dot{\eta}}\right) = \frac{\partial E}{\partial\eta} - \frac{\partial G}{\partial\dot{\eta}}.$$

This leads to the following scheme of values for the various time-differentials

$$\frac{dp_1}{dt} = q_1, \qquad \frac{dq_1}{dt} = 0, \qquad \frac{dr_1}{dt} = s_1,$$

$$\frac{ds_1}{dt} = -\frac{c_1}{d_1}r_1 - \frac{1}{d_1}\frac{dG}{ds_1}, \qquad \frac{du_1}{dt} = 0.$$

Now it appears from the scheme of values just found that terms of the form $\frac{\partial}{\partial\xi}\left(\frac{d\xi}{dt}\right)$ can only arise in connection with the s co-ordinates, so that $\Sigma\frac{\partial}{\partial\xi}\left(\frac{d\xi}{dt}\right)$ may be replaced by $-\epsilon$ where

$$\epsilon = \frac{1}{d_1}\frac{\partial^2 G}{\partial s_1^{\,2}} + \frac{1}{d_2}\frac{\partial^2 G}{\partial s_2^{\,2}} + \ldots$$

Thus the gain to N_0 arising from the cause (γ) will be

$$-\,dt\,\Sigma\frac{\partial N_0}{\partial\xi}\frac{d\xi}{dt} + \epsilon N_0\,dt \quad \ldots\ldots\ldots \quad \text{(xi.)}.$$

We have now counted up all sources of change in N_0; we therefore have, as the equation to be satisfied by N_0,

$$\frac{\partial N_0}{\partial t}dt = J\,dP\,dt + K\,dP\,dt + \epsilon N_0\,dt - dt\,\Sigma\frac{\partial N_0}{\partial\xi}\frac{d\xi}{dt}. \quad \ldots \quad \text{(xii.)}.$$

§ 23. We have found the equation ((i.), p. 415),

$$N_\alpha = e^{-h(mc^2+\Psi 2)}\,Ff\,dp\,dP\,du$$

and

$$N_0 = \Sigma N_\alpha$$

where the summation is such as to cover all ranges of p and u.

We may therefore write

$$N_0 = HF\,dP$$

where H is given by

$$H = \int e^{-h(mc^2+2\Psi)}f\,dp\,du \quad \ldots\ldots\ldots\ldots \quad \text{(xiii.)},$$

and is therefore a function of h only.

Substituting this value of N_0 in equation (xii.), we have the equation

$$\frac{\partial}{\partial t}(HF) = J + K + \epsilon HF - H\Sigma'\frac{\partial F}{\partial\xi}\frac{d\xi}{dt} \quad \ldots\ldots \quad \text{(xiv.)}$$

Let us write $F = e^{-\chi}$ and substitute this value for F in the integrals J and K. We have

$$\Delta\{e^{-\eta}FF'\} = \Delta e^{-(\eta+\chi+\chi')}$$

$$= e^{-(\eta+\chi+\chi')}\{e^{-\Delta\,\eta+\chi+\chi')} - 1\}$$

Referring to equation (iv.) we find that

$$\eta + \chi + \chi' = h\{m(c^2 + c'^2) + 2\Psi + 2\Psi' + 2\Omega\} + \chi + \chi'.$$

From the equation of energy (see p. 412).

$$\Delta \{ m(c^2 + c'^2) + 2(Q + S + Q' + S') \} = 0.$$

Hence
$$\Delta(\eta + \chi + \chi') = -\Delta\zeta,$$
where
$$\zeta = 2h(Q + S + Q' + S') - \chi - \chi'.$$

In virtue of the assumptions which have been made, ζ is a small quantity, so that we may put

$$e^{\Delta\zeta} - 1 = \Delta\zeta,$$

and therefore
$$\Delta\{e^{-\eta}FF'\} = Fe^{-(\eta+\chi)}\Delta\zeta.$$

The same transformation holds if Δ is replaced by D/Dt. Hence we have (see equations (ix.) and (x.), p. 418).

$$J = FL, \quad K = FM,$$

where
$$L = \int e^{-(\eta+\chi')} \Delta\zeta ff' \, dp \, du \, dp' \, dP' \, du' \, u_1'' \, dy'' \, dz'' \quad . \quad . \quad . \quad . \quad \text{(xv.)},$$

$$M = \int e^{-(\eta+\chi')} \frac{D\zeta}{Dt} ff' \, dp \, du \, dp' \, dP' \, du' \, dx'' \, dy'' \, dz'' \quad . \quad . \quad . \quad . \quad \text{(xvi.)}.$$

Making these substitutions, equation (xiv.) becomes

$$\frac{\partial H}{\partial t} F - HF \frac{\partial \chi}{\partial t} = F(L + M) + \epsilon HF + HF \Sigma \frac{\partial \chi}{\partial \xi} \frac{d\xi}{dt},$$

or dividing throughout by HF,

$$\frac{\partial \chi}{\partial t} = \frac{1}{H} \frac{dH}{dt} - \epsilon - \frac{1}{H}(L + M) - \Sigma \frac{\partial \chi}{\partial \xi} \frac{d\xi}{dt} \quad . \quad . \quad . \quad . \quad \text{(xvii.)}.$$

This is the characteristic equation satisfied by χ.

Form of General Solution.

§ 24. We must first examine in what way the integrals L and M involve q, r, and s. In L these co-ordinates are only involved through the factor $\Delta\xi$ which occurs in the integrand.

Now $\Delta r = 0$, and Δq, Δs can, from the equations of impact, be expressed as linear functions of all the velocities concerned. The coefficients will be functions of p, p', r, r', but we may as usual put r, $r' = 0$.

It follows that $\Delta\xi$ is a function of q, r, and s of degree equal to that of ξ, and will involve p, p', u, u' as well as these variables. If, then, ξ is of degree n in q, r, and s, we may regard $\Delta\xi$ as a function of q, r, and s of degree n, of which the coefficients are functions of the variables with respect to which integration is performed in evaluating L. Hence after integration we shall have L as a function of q, r, and s of degree n.

In the same way, the variables q, r, and s only enter the integral M, through the term $D\xi/Dt$, and we may write

$$\frac{D\zeta}{Dt} = \Sigma \frac{\partial \zeta}{\partial \xi} \frac{D\xi}{Dt}.$$

Now $D\xi/Dt$ denotes the rate of increase in ξ owing to the action of forces of which the potential energy is Ω_1.

Write $E = T + T' + \Omega_1$, then

$$\frac{D}{Dt}\left(\frac{dE}{d\dot{\eta}}\right) = \frac{dE}{d\eta},$$

in which η is any one of the p or r co-ordinates, and $\dot{\eta}$ is the corresponding velocity co-ordinate.

Hence $$\frac{Dq}{Dt} = \frac{1}{b_1} \frac{d\Omega_1}{dp_1}; \qquad \frac{Dr_1}{Dt} = 0; \qquad \frac{Ds_1}{Dt} = \frac{1}{d_1} \frac{d\Omega_1}{dr_1}.$$

The right-hand members of these equations will be functions of the p, p', r, r' co-ordinates, but it is clearly legitimate to put r, r' all equal to zero, and regard the expressions as functions of p and p' only.

It therefore appears that $D\chi/Dt$ will be a function of q, r, and s, of which the degree is $(n - 1)$, and upon integration, that M is a function of q, r, and s of the same degree.

The terms $\frac{\partial \chi}{\partial t}$ and $\Sigma \frac{\partial \chi}{\partial \xi} \frac{d\xi}{dt}$ which occur in equation (xvii.) will be functions of q, r, and s of degree n.

§ 25. It is therefore clear that the correct form to assume for χ is a rational integral algebraic function of the co-ordinates q, r, and s.

If we assume χ to be the most general function of degree n in these co-ordinates, the coefficients being functions of the time, and if we substitute this assumed value for χ in equation (xvii.), we shall get, on each side of the equation, a function of q, r, and s of degree n.

If therefore we equate the coefficients of every term on the two sides of the equation, we shall have found a solution of equation (xvii.), p. 421, inasmuch as this equation is now satisfied identically for every value of q, r, and s.

The process of equating these coefficients leads to a series of differential equations, in which the time-rate of increase of every coefficient is given explicitly in terms of the other coefficients and of h. If, therefore, we suppose the coefficients to vary with the time in the manner given by these equations, the value of χ so obtained will be a solution of equation (xvii.) for all time. Since the equations involve h, and h varies with the time, one further equation is required before we can express the co-ordinates at any time in terms of the initial values of the co-ordinates and the time. This additional equation is supplied by the fact that N, the total number of molecules, remains constant.

We have (see § 23, equation (xiii.)),

$$N = \int HF\,dP = \int e^{-h(mc^2+2\Psi)} f F\,dp\,dP\,du \quad . \quad . \quad . \quad . \quad . \quad \text{(xviii.)},$$

and the equation $dN/dt = 0$ is the equation required. With the help of the other equations, it can be written so as to give dh/dt explicitly as a function of h and the other co-ordinates.

Thus if we have the initial form of χ given, we have obtained sufficient equations to enable us theoretically to determine χ at any subsequent time. It is not proposed to attempt the solution of the system of equations in the most general case; the discussion is confined to the modified forms which these equations assume in the two states of which the physical interest is greatest, namely the steady state, and the state in which the gas is non-luminous.

Solution in Steady State.

§ 26. The mathematical condition that a steady state may be possible, is that it may be possible for the time rates of variations of the coefficients to vanish simultaneously. From the equations found by equating to zero the time rates of all the coefficients except h, it is possible to find these coefficients in terms of h so that the condition for a steady state to be possible is that the function of h obtained by substituting these values in the expression for dh/dt shall vanish identically for all values of h. It is, however, known that the condition that a steady state shall be possible is that G shall be identically zero, and we may therefore begin by putting $G = 0$ and neglecting the equation $dh/dt = 0$.

Thus all the equations necessary are contained in the characteristic equation satisfied by χ, and this is now (*cf.* equation xvii., p. 421)

$$\frac{1}{H}(L + M) + \sum_{1}^{\kappa} \frac{\partial\chi}{d\xi}\frac{\partial\xi}{dt} = 0,$$

or substituting for $d\xi/dt$ from the scheme of p. 419,

$$\frac{1}{H}(L + M) + \sum_{1}^{\kappa}\left(\frac{\partial\chi}{\partial r_1}s_1 - \frac{c_1}{d_1}\frac{\partial\chi}{\partial s_1}r_1\right) = 0 \quad . \quad . \quad . \quad . \quad . \quad \text{(xix.)}.$$

Let us assume as a possible value for χ the most general expression of degree n in q, r and s, the coefficients now being independent of the time.

Consider the system of equations which is obtained when we equate to zero the coefficients of terms of degree n in equation (xix.). The terms of degree n in the integral L arise entirely from the terms of degree n in χ (see § 24). These terms will each be multiplied by linear functions of the coefficients of terms of degree n in χ, and the whole expression will be multiplied by λ where

$$\lambda = \int e^{-(\eta+\chi')} ff' \, dp \, du \, dp' \, d\mathrm{P}' \, du' \, u'' \, dx'' \, dy''.$$

Now $\lambda \mathrm{F} \, d\mathrm{P}$ is the number of collisions per unit time which are experienced by all the molecules of which the q, r, s, co-ordinates lie within limits $d\mathrm{P}$. Since the probability of a collision does not depend on the q, r, s co-ordinates, it follows that the total number of collisions per unit time is $\frac{1}{2}\lambda \int \mathrm{F} \, d\mathrm{P}$. It follows that λ is a function of h only, and does not depend on the coefficients which occur in χ.

In the second integral M, terms of degree n do not occur at all.

In the remaining terms of equation (xix.) terms occur of degree n in q, r, s, the coefficients being of the same form as those occurring in L except that the factor λ does not occur.

We have thus found as many equations as there are coefficients; in these equations every term is a coefficient multiplied by a constant. The only solution of this system of equations is that every coefficient vanishes.

This result depends on the assumption that n is greater than two. Hence in the steady state χ will contain no terms of a degree higher than the second in q, r and s. At the same time χ can contain no terms which are linear in q, r or s. The introduction of these terms would give a law of distribution such that an infinite number of molecules would have an infinite value for $\pm q$, $\pm r$, and $\pm s$.

We may, therefore, suppose that, except for an additive constant, χ is a quadratic function of q, r and s, in which only square terms occur.

The equations between the coefficients of this quadratic expression must be linear, since they are the coefficients of the terms of highest degree in χ, and hence must lead to unique value for these coefficients.

Thus for the type of molecule which we have been considering, there is only one steady state possible on the assumptions we have made.

It is easily verified that

$$\chi = \mathrm{Q} + \mathrm{S} + \mathrm{V} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (\text{xx.})$$

is a solution of equation (xix.). For, with this value for χ, $\mathrm{D}\xi/\mathrm{D}t$ and $\Delta\xi$ both vanish, so that L and M both vanish, and we have also $\frac{\partial\chi}{\partial r_1} s_1 = \frac{c_1}{d_1} \frac{\partial\chi}{\partial s_1} r_1$. Hence every term of equation (xix.) vanishes separately, and the value of χ given by equation (xx.) supplies a solution which, as we have seen, must be unique. This is the solution of MAXWELL and BOLTZMANN.*

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* The possibility of χ being an infinite series has been disregarded in the above sections. We may, however, consider that a series which was divergent for any finite values of the variables would lead to an impossible law of distribution, whilst a series which is convergent for all finite values, may be treated as the limit of a finite series in which the number of terms is made infinite.

Solution for Low Temperatures.

§ 27. In the approximately steady state which was found to be possible at low temperatures, for the previous system of molecules, it was found that the various mean energies varied very slowly with the time. But if such a state were possible for the present system, it would not follow that χ would only vary slowly with the time, so that an approximate solution of $d\chi/dt = 0$, even if it could be found, would be useless.

Let us, however, examine under what circumstances we could have $d\chi/dt$ equal to zero, without approximation. From the remarks at the beginning of the last section, it is clear that the coefficients in χ can be so chosen as to make $d\chi/dt$ equal to zero, but that it is only in the event of G being absent, that these values will also make $dh/dt = 0$.

Let us suppose that by some external agency h is caused to increase uniformly throughout the whole gas, at a rate exactly equal to the rate at which it decreases in consequence of the value of dh/dt, found in the manner described at the beginning of § 26, being different from zero. Then a completely steady state will have become possible, and this is because the imaginary agency introduces exactly sufficient energy to compensate that lost by radiation. In a state such as that which, in the first part, was described as approximately steady, the radiation was very small. If a similar state can be shown to be possible in the present case, the radiation will be very small, and therefore the energy introduced from outside will be very small. Hence it will be legitimate to describe the state which would be arrived at by checking the external flow of energy as approximately steady.

The equation which leads to such a state is $d\chi/dt = 0$, or

$$\epsilon + \frac{1}{\mathrm{H}}(\mathrm{L} + \mathrm{M}) + \sum_{1}^{\kappa}\left(\frac{\partial\chi}{\partial r} s_1 - \frac{c_1}{d_1}\frac{\partial\chi}{\partial s_1} r_1\right) - \sum_{1}^{\kappa} \frac{1}{d_1}\frac{\partial\chi}{\partial s_1}\frac{\partial \mathrm{G}_1}{\partial s_1} = 0 \quad . \quad . \quad . \quad \text{(xxi.)}.$$

The last term in equation (xxi.) is of the same degree as χ, so that just as in the last section it may be shown that χ cannot contain terms of a degree higher than the second. And for the same reason as before, χ cannot contain terms linear in any of the coefficients, so that we may assume

$$\chi = \Sigma\beta_1 q_1^2 + \Sigma\gamma_1 r_1^2 + \Sigma\delta_1 s_1^2 + \kappa_1.$$

Now at low temperatures all the terms in equation (xxi.) are small, except terms of the form

$$\frac{\partial\chi}{\partial r_1} s_1 - \frac{c_1}{d_1}\frac{\partial\chi}{\partial s_1} r_1 .$$

These terms must therefore vanish approximately for all values of the variable, and this requires the relation

$$\gamma_1/\delta_1 = c_1/d_1.$$

This merely shows that, neglecting small quantities of the order of the coefficients in G, the energy of any of the r, s modes of vibration must be equally divided between kinetic and potential energy.

It is now clear, that at high temperatures the last term in equation (xxi.) is small in comparison with the others, so that the law of distribution will be very little altered by the presence of a dissipation function, whereas at sufficiently low temperatures, the term arising from the dissipation function becomes as important as any other term in the equation, and, therefore, the presence of a dissipation function, however small, will be sufficient to entirely alter the law of distribution.

And without investigating the solution of the system of equations which determine the coefficients in χ, it is clear that since they are all linear, every coefficient must be a single-valued function of h only. Hence, as before, there is only one approximately steady state for a given temperature, but it is no longer true that the various lines of the spectrum increase in brightness in the same ratio when the temperature is increased. Since the u, q modes of energy suffer very little loss of energy, and since energy passes freely between these modes, it follows that the energies of these modes will very approximately be distributed according to BOLTZMANN's Law. Hence all that was said about the ratio of the two specific heats with reference to the former type of molecule considered, will apply also to molecules of the type which we have just been discussing.

PART III.

PHYSICAL CONSEQUENCES OF THE FOREGOING THEORY.

§ 28. We have thus been led to the same results in both parts of this paper. It seems natural to suppose that results which are qualitatively the same will be found to be valid for any gas, and, assuming this to be the case, to examine some of the consequences of these results.

Radiation at a given Temperature.

§ 29. In the first dynamical illustration, the radiation from each degree of freedom at temperature T was found to be proportional to $T^{3/2}$. In the second illustration, the expression for the radiation would have been too complicated for the calculation of it to have been profitable.

A modification of the system discussed in Part II., leads to an interesting expression for the radiation. In this modified system the molecules are to be spheres with modes of internal vibration to and from which energy only passes with difficulty. Each sphere is surrounded by a field of force, such that when two molecules are in contact in any position, their potential has a constant value Ω. When two molecules are not very near, let us suppose that the transfer of energy between different r, s modes may be neglected, so that the main transfer of energy takes place through collisions.*

At any collision, the r, s energy of the colliding molecules will be small, so that the amount by which the r, s energy is increased will depend only on the u and q energies of the colliding molecules; that is to say, on the average, on h.

Thus if n collisions take place per unit time, the total transfer from the u, q modes of energy to any specified r, s mode may be taken to be $\mathrm{C}n/h$, where C is a constant for any particular r, s mode.

The sum of such amounts must be equal to the amount radiated per unit time, and may therefore be supposed to be roughly proportional to the energies of the r, s modes. It follows that for a single molecule the r, s energy is proportional to $n/\mathrm{N}h$.

Now we may take for the number of collisions

$$n = \frac{\mathrm{AN}^2 e^{-h\Omega}}{\sqrt{h}},$$

where A is a constant.

Hence
$$\frac{n}{\mathrm{N}h} = \frac{\mathrm{AN}e^{-h\Omega}}{h^{3/2}},$$

and if ρ be the density, the radiation will be proportional to $\rho h^{-3/2}\, e^{-h\Omega}$, or, in terms of the temperature, to $\rho\mathrm{T}^{3/2}\, e^{-\frac{c\Omega}{\mathrm{T}}}$.

This example is of such a special nature that not much importance can be attached to the actual result obtained. It is, however, of interest, as showing that it is at any rate possible for the radiation to increase very rapidly with the temperature. A comparison with the result of § 9, shows that the introduction of a field of force has introduced a factor $e^{-\frac{c\Omega}{\mathrm{T}}}$ into the expression for the radiation, and a factor of this form figures in every formula for radiation.†

The presence of the factor ρ multiplying the expression for the radiation, is an essential feature of the present theory.‡ If the exponential factor changes very rapidly with the temperature, so that the point of incandescence is sharply defined, then this point will clearly be almost independent of ρ, and variations in radiation on

---

* This assumption, although not stated explicitly in the investigation of Part II., is implied in the assumptions made there.

† WIEN and PLANCK give for the radiation in the part of the spectrum between λ and $\lambda + \delta\lambda$, the formula $c_1\lambda^{-5}\, e^{-\frac{c_2}{\lambda\mathrm{T}}}\, d\lambda$. LORD RAYLEIGH, on theoretical grounds, suggests as an emendation $c_1\mathrm{T}\lambda^{-4}\, e^{-\frac{c_2}{\lambda\mathrm{T}}}\, d\lambda$. ('Phil. Mag.,' June, 1900.)

‡ [Added March 19, 1901.—I was not aware, when writing this, that the presence of the factor ρ had been detected experimentally. See LIVEING and DEWAR, 'Roy. Soc. Proc.,' 49, p. 217, or KAYSER, 'Handbuch der Spectroscopie' (1900), I., p. 143.]

account of density will be unnoticeable in comparison with variations on account of temperature.

It is, however, clear that in considering radiations from gases of great density, such as for example occur in the sun, the factor ρ would be of considerable importance.

Generalised Theory of Temperature.

§ 30. The state of a gas may be regarded as depending upon a principal temperature T, and also on a number of subsidiary temperatures $\tau_1, \tau_2, \ldots$, each of these temperatures corresponding to one (or possibly more) of the degrees of freedom of the molecule. The principal temperature is to correspond to the three degrees of freedom implied by the possibility of translation through the ether, and to any other degrees of freedom which are such that their mean energy is at all temperatures equal to a third of the mean energy of translation.

The principal temperature is to be proportional to the mean energy of translation of a molecule and each subsidiary temperature proportional to the mean energy of each of the modes to which it corresponds. Thus two modes can only have the same subsidiary temperature when their mean energies are, under all circumstances, equal, as, for example, when they are the kinetic and potential energies of the same vibration. When the energy is equally distributed between all the degrees of freedom all these temperatures are to become equal.

We have found that at temperatures below the temperature of incandescence there is an approximately steady state in which

$$\tau_1 = \rho f_1(\mathrm{T}), \qquad \tau_2 = \rho f_2(\mathrm{T}), \qquad \&c.,$$

where $f_1(\mathrm{T})$, $f_2(\mathrm{T})$, &c., are functions of T, which at these temperatures are very small in comparison with T.

At higher temperatures we have not investigated the forms of $\tau_1, \tau_2, \ldots$, but at infinite temperatures,

$$\tau_1 = \tau_2 = \quad . \quad . \quad . \quad . \quad = \mathrm{T}.$$

§ 31. The steady state specified above was arrived at on the assumption that external agencies could only influence the energy of translation, and that the other energies were only influenced indirectly through changes in the energy of translation.

Thus the above equations will not hold in the presence of agencies which exert a direct influence on the subsidiary temperatures. Such influences may be looked for in the forces of chemical action, disturbances in the ether, and possibly in the cathode rays, if we suppose these rays to be streams of charged ions which are so small as to penetrate inside a molecule rather than act on the molecule as a whole.

When such agencies are present, the above equations must give place to others. The subsidiary temperatures which are most directly concerned may attain to abnormally high values, and this may result in the phenomena of phosphorescence, chemi-luminescence, &c. To take a definite instance, suppose that τ_1 corresponds to a vibration in the molecule of frequency p. If a ray of light passes through the substance, those components of this wave of which the frequency is nearly equal to p will supply energy to the mode τ_1 of the molecules, and this energy will be distributed from the τ_1 mode to the other modes, and so through the substance. Thus the result is a heating of the substance, and an absorption band in the spectrum of the light transmitted through it. The illustration might be varied by supposing that energy could not easily distribute itself from τ_1 to all the other temperatures, but that it passed freely to a second temperature τ_2. In this case the temperature τ_2 might conceivably attain to such a high value as to emit its own spectrum, and so set up fluorescence or calorescence.

The spectrum of the gas in any condition whatever will be arrived at by the superposition of the various spectra of the subsidiary temperatures, and the state of the gas as regards the emission of radiation will be completely specified by the values of the various subsidiary temperatures.

Thermodynamics.

§ 32. At temperatures at which the gas is dark, we may take

$$\tau_1 = \tau_2 = \ldots = 0.$$

Thus at these temperatures we are only concerned with the principal temperature, and the total energy of the gas is proportional to this temperature. If n degrees of freedom correspond to this temperature, the ratio of the specific heats will be

$$1 + 2/n,$$

both specific heats being constant as regards the temperature. The view which we have put forward does not clash with the ordinary thermodynamics as regards dark gases.

When the subsidiary temperatures begin to have appreciable values the case is different. The total internal energy is now given by

$$W = C\{nT + \Sigma\kappa_1\tau_1\},$$

where C is a constant, and κ_1 is the number of modes of energy of which the subsidiary temperature is τ_1. The specific heat at constant volume is given by

$$C_1 = dW/dT = C\{n + \Sigma\rho\kappa_1 f_1'(T)\},$$

and therefore depends on both the temperature and density.

If a quantity dQ of work be absorbed by a gas,

$$
\begin{aligned}
dQ &= NdW - pdv \\
&= CN\left\{ndT + \Sigma\kappa_1 d\,(\rho f_1\,(T))\right\} - T\rho R d\left(\frac{1}{\rho}\right).
\end{aligned}
$$

If the gas be made to pass through any succession of states so as to return to the same temperature and density,

$$
\int\frac{dQ}{T} = CN\Sigma\kappa_1\int\frac{d(\rho f_1\,(T))}{TdT}\,dT.
$$

Thus the second law of thermodynamics will only hold in special cases for a gas which is emitting radiation of any kind. This part of the subject of luminosity has, however, been developed by WIEDEMANN,* so that it seems unnecessary to pursue it any further here.

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* E. WIEDEMANN, "Mechanics of Luminosity," 'Phil. Mag.,' vol. 28, 1889, p. 152, or 'Wied. Annalen vol. 37, p. 181.

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IX. *Über Reihen auf der Convergenzgrenze.*

Von Emanuel Lasker, *Dr. Phil.*

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1. Es seien $x_1 \ldots x_r$ r complexe Variabele, welche in einem $2r$-dimensionalen fictiven Raume als Coordinaten seiner reellen Punkte dienen mögen. Das System der Werte $x_1 \ldots x_r$ sei kurz mit "Punkt" $x_1 \ldots x_r$ bezeichnet. Sind $e_1 \ldots e_r$ r positive endliche im übrigen beliebig kleine Zahlen, und nimmt η_i alle complexen Werte an, die der Ungleichung

$$|\eta_i| \leqq e_i$$

genügen, so heisse das Aggregat aller Punkte

$$x_1 + \eta_1, \; x_2 + \eta_2, \ldots \; x_r + \eta_r$$

die *Nachbarschaft* des Punktes $x_1 \ldots x_r$.

2. Es sei ferner $u_1, u_2 \ldots u_n \ldots$ eine Folge unendlich vieler Functionen von $x_1 \ldots x_r$ und

$$u_1 + u_2 + \ldots + u_n + \ldots$$

absolut und gleichmässig convergent für die Nachbarschaft eines Punktes $P \equiv x_1 \ldots x_r$. P wird dann *innerer Convergenzpunkt* der Reihe $\Sigma_1^\infty u_n$ genannt werden. Die Gesammtheit aller inneren Convergenzpunkte der Reihe bilden ihren *inneren Convergenzbereich*, die Begrenzung dieses Bereiches ihre *Convergenzgrenze*. Ist die Reihe absolut, aber nicht gleichmässig in der Nachbarschaft von P convergent; oder convergiert sie in P absolut, jedoch nicht mehr in der Nachbarschaft von P, so heisst P "äusserer Convergenzpunkt" der Reihe. Das Aggregat der äusseren Convergenzpunkte der Reihe formt geometrische Gebilde, die "äussere Convergenzgebilde" der Reihe genannt werden mögen.

Wir werden im folgenden nur von den inneren Convergenzbereichen und der Convergenzgrenze reden, viele der folgenden Untersuchungen bleiben aber anwendbar auch auf äussere Convergenzgebilde und deren Grenzen.

3. P durchlaufe eine continuierliche oder discontinuierliche Folge von Lagen

$$P_1, \; P_2 \ldots P_n \ldots$$

im inneren Convergenzbereiche der Reihe, derart dass die Folge der P nur *einen* Grenzpunkt L zulasse, welcher auf der Convergenzgrenze gelegen ist. Die Summe $f = u_1 + u_2 + \ldots + u_n + \ldots$ stellt eine Function von P dar, und die den Lagen $P_1 \ldots P_n \ldots$ entsprechenden Werte seien mit $f_1, f_2 \ldots f_n \ldots$ bezeichnet. Die folgende Untersuchung beschäftigt sich dann mit dem Verhältnis von $\lim_{n=\infty} f_n$ zu dem Werte, den $u_1 + \ldots + u_n + \ldots$ in $P = L$ annimmt, und mit der asymptotischen Darstellung von f in L bei zu Grundelegung des Grenzüberganges

$$P \underset{n=\infty}{\equiv} P_1, P_2, P_3 \ldots P_n \ldots$$

überhaupt.

4. Es sei *in* L die Reihe convergent und ihr Wert mit f_L bezeichnet. Alsdann ist

$$\lim_{n=\infty} f_n = f_L$$

immer dann, wenn die Reihe

$$u_1 + u_2 + \ldots + u_n + \ldots$$

bei zu Grundelegung des Grenzüberganges

$$P \equiv P_1, P_2, P_3 \ldots P_n \ldots$$

gleichmässig convergiert, d. h. wenn sich bei vorgegebener Zahl δ Indices N, M angeben lassen, so dass

$$|u_{N+1} + u_{N+2} + \ldots| < \delta,$$

sobald als P irgend eine der Lagen $P_M, P_{M+1}, P_{M+2} \ldots$ einnimmt. Dieser Satz ist wohlbekannt. Ein Specialfall desselben sagt aus: Ist die Reihe

$$u_1 + u_2 + \ldots + u_n + \ldots$$

in L absolut convergent, und convergiert auch die Reihe

$$U_1 + U_2 + \ldots + U_n + \ldots,$$

wo U_h den Maximalwert des Moduls von u_h bezeichnet, wenn P die Lagen $P_1, P_2 \ldots P_n \ldots$ durchläuft, so ist

$$\lim_{n=\infty} f_n = f_L.$$

Der Satz kann dazu dienen, bis zu einem gewissen Grade Aufschluss zu geben über das asymptotische Verhalten von $u_1 + \ldots + u_n + \ldots$, wenn P die Lagen P_i durchläuft. Es sei z. B. angesetzt

$$F(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n + \ldots$$

Der Convergenzradius der Reihe sei = 1. Die Reihe multiplicieren wir mit $(1-x)^\lambda$, wo λ eine positive Grösse bezeichnet, und erhalten

$$F(x) \,.\, (1-x)^\lambda = \Sigma_0^\infty c_n x^n \,.\, (1-x)^\lambda.$$

Wir lassen nun x die positiven Werte zwischen 0 und 1 durchlaufen und wenden obiges Criterion an. Der Maximalwert von $x^n(1-x)^\lambda$ unter den obigen Umständen ist $= \frac{n^n \,.\, \lambda^\lambda}{(n+\lambda)^{n+\lambda}}$. f_L ist offenbar $= 0$, da jedes Glied der Reihe in $x = 1$ verschwindet. Mithin ist

$$\lim_{x=1} F(x) \,.\, (1-x)^\lambda$$

immer dann $= 0$, wenn $\Sigma \frac{c_n}{n^\lambda}$ absolut convergiert; und dasselbe gilt offenbar auch für

$$\lim_{x=j} F(x) \,.\, (x-j)^\lambda,$$

wo j irgend ein Punkt des Convergenzkreises.

Ist umgekehrt bekannt, dass für einen bestimmten Wert von λ und irgend einen Punkt des Convergenzkreises j $\lim_{x=i} F(x) \,.\, (x-j)^\lambda$ *nicht* gleich 0 ist, so folgt daraus die Divergenz von

$$\Sigma_1^\infty \frac{c_n}{n^\lambda}.$$

Genau analoge Untersuchungen lassen sich vielfach anstellen, z. B. bei Annahme von Potenzreihen von mehr als 1 Veränderlichen, bei zu Grundelegung von Reihen der Form $\Sigma_1^\infty \frac{1}{x+a_n}$, bei Betrachtung von Integralen etc. Ergänzt, und in gewissem Sinne erweitert, wird der Satz durch die nun folgende Überlegung.

5. Es sei in $P \equiv L$ die Reihe

$$u_1 + u_2 + u_3 + \ldots + u_n + \ldots$$

divergent. Gesucht ist ein Criterion, welches anzeigt, dass daraufhin $\lim_{n=\infty} f_n = \infty$.

Wir behaupten, dies sei immer der Fall, wenn sich irgend eine endliche Zahl c angeben lässt, derart dass die Ungleichung

$$|u_1 + u_2 + \ldots + u_n + \ldots| > c \,.\, (|u_1| + |u_2| + \ldots + |u_n| + \ldots)$$

gültig bleibt für *alle* Lagen von P, deren Index grösser ist als eine angebbare endliche Zahl N. Dies Criterion sei das Criterion K genannt.

Um seine Existenz zu beweisen schliessen wir wie folgt. Es ist für alle Lagen von P

$$\begin{aligned} F &= |u_1| + |u_2| + |u_3| + \ldots + |u_n| + \ldots \\ &> |u_1| + |u_2| + |u_3| + \ldots + |u_h|, \end{aligned}$$

wo h irgend eine endliche Zahl. Nun ist

$$|u_1| + |u_2| + \ldots + |u_n| + \ldots$$

in $P = L$ divergent; mithin lässt sich ein Index h angeben, derart dass in $P = L$

$$|u_1| + |u_2| + \ldots + |u_h| > \eta,$$

wie gross η auch sei. Andererseits sind die u_i in den Lagen $P_1, P_2, P_3 \ldots P_n \ldots$ stetig und endlich (oder diese Untersuchung würde gar keinen Sinn haben), und dasselbe gilt somit für die $|u_i|$. Ist daher δ eine beliebig kleine endliche vorgegebene Zahl, so ist es möglich, einen Index M zu finden, derart dass für alle Lagen von P deren Index $\geqq$ M, der Wert von

$$|u_1| + |u_2| + \ldots + |u_h|$$

in $P = L$ sich von dem Wert desselben Ausdruckes für obige Lagen von P um weniger als δ unterscheide. Somit ist der Wert von F in $P = P_n$, wo $n > M$ ist, grösser als $\eta - \delta$. Dies besagt dass F, wenn P die Lagen $P_1, P_2 \ldots P_n \ldots$ durchläuft, über alle Grenzen wächst. Es ist somit

$$\lim_{n=\infty} F = \infty,$$

und somit infolge der Ungleichung

$$|u_1 + u_2 + \ldots + u_n + \ldots| > c. (|u_1| + |u_2| + \ldots + |u_n| + \ldots)$$

auch

$$\lim_{n=\infty} f_n = \infty \quad q.\ e.\ d.$$

6. Es seien nun

$$f = u_1 + u_2 + \ldots + u_n + \ldots$$
$$\psi = v_1 + v_2 + \ldots + v_n + \ldots$$

zwei Reihen, von denen bekannt sei, dass die zweite dem Criterion K genüge, und dass, wenn P die Lagen $P_1 \ldots P_n \ldots$ durchläuft,

$$\lim_{P=L,\, n=\infty} \frac{u_n}{v_n} \textit{ gleichmässig} = \text{einer endlichen Grösse } \rho.$$

Alsdann genügt auch die erste Reihe dem Criterion K; es ist also

$$\lim_{P=L} f = \infty, \qquad \lim_{P=L} \psi = \infty,$$

und ferner findet sich

$$\lim_{P=L} \frac{f}{\psi} = \lim_{P=L,\, n=\infty} \frac{u_n}{v_n} = \rho.$$

Dieser Satz heisse das Theorem T.

Der Grenzübergang, in welchem P die Lagen $P_1 \ldots P_n \ldots$ durchläuft, sei kurz der Grenzübergang G.

$u_n - \rho . v_n$ sei $= w_n$. Dann ist

$$\lim_{P=L,\, n=\infty} \frac{w_n}{v_n} = 0.$$

Mithin ist auch
$$\lim_{P=L,\, n=\infty} \frac{|w_n|}{|v_n|} = 0.$$

Es sei τ irgend eine vorgegebene beliebig kleine endliche positive Zahl. Es lässt sich dann immer ein Index N und eine Lage Q von P angeben, so dass wenn $n > N$ und P beim Grenzübergang G die Lage Q passiert hat,

$$\left|\frac{w_n}{v_n}\right| < \frac{\tau}{2}, \text{ da ja nach Voraussetzung}$$

$$\lim_{P=L,\, n=\infty} \left|\frac{w_n}{v_n}\right| \textit{ gleichmässig} = 0.$$

Der Ausdruck

$$\Sigma_1^\infty |w_n| - \tau . |v_n|$$

zerfällt in drei Teile A, B, C.

Im ersten Teile A stehen alle Glieder, deren Index $< N$. Derselbe ist immer kleiner als $\Sigma_1^N |w_n|$, also kleiner als eine angebbare endliche Grösse.

Im zweiten Teile B stehen alle Glieder

$$\Sigma |w_n| - \frac{\tau}{2} |v_n|,$$

deren Index $> N$. Derselbe ist für jede Lage von P jenseits Q negativ.

Der dritte Teil C ist $\Sigma - \frac{\tau}{2} |v_n|$ und nähert sich, da nach Voraussetzung die Reihe Σv_n dem Criterion K genügt, dem Werte $-\infty$.

Somit lässt sich eine Lage Q′ von P angeben, so dass beim Grenzübergang G bei einer Lage von P jenseits Q′

$$\Sigma_1^\infty |w_n| - \tau . |v_n|$$

immerfort negativ bleibt. Daraus folgt dann, da τ eine beliebige Grösse war,

$$\lim_{P=L} \frac{\Sigma_1^\infty |w_n|}{\Sigma_1^\infty |v_n|} = 0;$$

also *a fortiori*
$$\lim_{P=L} \frac{\Sigma_1^\infty w_n}{\Sigma_1^\infty |v_n|} = 0.$$

Andererseits, da $\Sigma_1^\infty v_n$ dem Criterion K genügt, lässt sich eine endliche Zahl c finden, so dass beim Grenzübergang G gleichmässig

$$\Sigma v_n > c \,.\, \Sigma |v_n| .$$

Somit ist
$$\lim_{P=L} \frac{\Sigma_1^\infty w_n}{\Sigma_1^\infty v_n} = 0 ,$$

und daher
$$\lim_{P=L} \frac{\Sigma_1^\infty u_n}{\Sigma_1^\infty v_n} = \rho . \qquad \text{Q. e. d.}$$

Das Theorem T bleibt noch gültig, wenn die u_n in L selbst unendlich gross werden, nur muss dann das Criterion K durch ein anderes ergänzt werden, welches wir das Criterion K′ nennen mögen. Dasselbe besagt, dass, wenn η eine vorgegebene beliebig grosse Zahl sei, wenn ferner ein Index M gegeben sei. sich immer eine Lage R von P finden lasse, so dass beim Grenzübergang G jenseits R immerfort

$$\eta \,.\, (|u_1| + |u_2| + |u_3| + \; . \; . \; . \; + |u_M|) < |u_{M+1} + u_{M+2} + u_{M+3} + \; . \; . \; . \; . \; | .$$

Der Beweis ist dann ganz genau analog dem obigen, d.h., basiert darauf, dass bei beliebig vorgegebenem τ $\Sigma_1^\infty \; (|w_n| - \tau . |v_n|)$ schliesslich immerfort negativ wird.

Es ist klar, dass sich die obigen Bemerkungen ohne weiteres auf bestimmte Integrale zwischen positiven Grenzen und bei positiver Bahn erweitern lassen.

Auch für unendliche Producte existiert ein Theorem T. Sei

$$\Pi = (1 + u_1) \; . \; . \; . \; (1 + u_2) \; . \; . \; . \; (1 + u_n) \; . \; . \; .$$

ein absolut convergentes Product innerhalb eines inneren Convergenzbereiches Sei $\mathfrak{G}$ die Convergenzgrenze, und sei der Grenzübergang G wie oben konstruiert. Die u_i erfüllen das Criterion K, und ausserdem sei noch

$$\lim_{n=\infty \; P=L} u_n \text{ gleichmässig} = 0.$$

Alsdann ist $\log \Pi = \Sigma \log (1 + u_n)$;

ferner
$$\lim_{n=\infty \; P=L} \frac{\log (1 + u_n)}{u_n} \text{ gleichmässig} = 1,$$

nach einer elementaren Eigenschaft der natürlichen Logaritmen, somit nach Theorem T

$$\lim_{P=L} \frac{\log \Pi}{\Sigma u_n} = 1 .$$

7. Wir wollen nun auf einige nahe liegende Anwendungen des Theorems T näher eingehen.

Eine elementare Formel ist

$$\left(\frac{1}{1-x}\right)^{\lambda} = \Sigma_0^{\infty} \frac{\Gamma(n+\lambda)}{\Gamma(n+1)\,\Gamma(\lambda)} x^n ;$$

ebenso

$$\log \frac{1}{1-x} = \Sigma_1^{\infty} \frac{x^n}{n} .$$

x nähere sich durch positive Werte wachsend der 1.

Es sei angesetzt $f(x) = \Sigma_0^{\infty} c_n x^n$ und der Convergenzradius der Potenzreihe sei $= 1$. Wir erhalten dann ohne weiteres

Satz I. Ist $\lim\limits_{n=\infty} c_n \,.\, n^{1-\lambda} = \rho$, so findet sich

$$\lim_{x=1} f(x) \,.\, (1-x)^{\lambda} = \rho \,.\, \Gamma(\lambda)$$

und für $\lambda = 0$

$$\lim_{x=1} \frac{f(x)}{\log\left(\frac{1}{1-x}\right)} = \rho .$$

Satz I gilt auch noch für imaginäre λ, deren reeller Teil positiv ist. Er gilt auch noch, wenn x sich so der 1 nähert, dass sich durch 1 eine gerade Linie legen lässt, die mit der reellen Axe einen spitzen Winkel bildet, innerhalb welchem und dem Convergenzkreis x variiert.

Man kann nämlich nachweisen, dass dann das Criterion K befriedigt ist. Es sei $\lambda = \alpha + i\beta$ und α positiv. Alsdann ist identisch

$$\Sigma_0^{\infty} \left| \frac{\Gamma(n+\lambda)}{\Gamma(n+1)\,.\,\Gamma\lambda} \cdot x^n \right| = \left| \frac{1}{\Gamma\lambda} \right| \Sigma_0^{\infty} \frac{|\Gamma(n+\lambda)|}{\Gamma(n+1)} \cdot |x|^n .$$

Nun ist nach einer Eigenschaft der Γ Function

$$\lim_{n=\infty} n^{-\alpha+1} \,.\, \frac{|\Gamma(n+\lambda)|}{\Gamma(n+1)} = 1 ;$$

somit nach dem evidenten Teil des Satzes I, wenn $|x|$ sich wachsend der Einheit nähert,

$$\lim_{|x|=1} (1-|x|)^{\alpha} \cdot \Sigma_0^{\infty} \frac{|\Gamma(n+\lambda)|}{\Gamma(n+1)} \cdot |x|^n = \Gamma(\alpha) .$$

Ist δ eine beliebig vorgegebene kleine Grösse, so lässt sich also nach obigem immer ein Wert ξ so nahe an 1 finden, dass für alle Lagen von x, für die $|x| > \xi$,

$$\Sigma_0^{\infty} \frac{|\Gamma(n+\lambda)|}{\Gamma(n+1)} \cdot |x|^n < \frac{\Gamma(\alpha)+\delta}{(1-|x|)^{\alpha}} .$$

Ist w der erwähnte spitze Winkel, so ist aus elementar geometrischen Gründen für alle Lagen von x

$$1 - |x| < \cos w \,.\, |1-x| ;$$

somit

$$\Sigma_0^{\infty} \frac{|\Gamma(n+\lambda)|}{\Gamma(n+1)} \cdot |x|^n < \frac{\Gamma(\alpha)+\delta}{(\cos w)^{\alpha}} \cdot \left| \left(\frac{1}{1-x}\right)^{\alpha} \right| .$$

Für alle Lagen x, welche überhaupt in Betracht kommen, übersteigt das Argument von $1 - x$ niemals einen gewissen endlichen Wert c^k; somit ist abgesehen von einem endlichen Factor $e^{k \cdot \beta}$ der Wert von

$$\left|\left(\frac{1}{1-x}\right)^{\lambda}\right|$$

identisch mit $\left|\left(\frac{1}{1-x}\right)^{\alpha}\right|$. Das Criterion K ist demnach in Kraft für die Reihe

$$\Sigma_0^\infty \frac{\Gamma(n+\lambda)}{\Gamma(n+1) \,.\, \Gamma(\lambda)} \cdot x^n$$

unter den oben erwähnten Bedingungen, somit gilt auch das Theorem T unter denselben Bedingungen; *q.e.d.*

Satz I wurde von APPELL gefunden—'Comptes Rendus,' vol. 87 (Sur certaines séries ordonnées par rapport aux puissances croissantes d'une variable)—jedoch mit der Einschränkung, dass λ positiv und kleiner als 1 sei und dass x auf der positiven Axe wachsend sich 1 nähere. Der Satz findet sich in der Appellschen Form auch bei PICARD, 'Traité d'Analyse,' Tome 1, p. 208, 209, 210 aus dem Jahre 1891.

Satz II. Es sei $f(x)$ wiederum $= \Sigma_0^\infty c_n \,.\, x^n$.

Es ist dann identisch

$$\frac{f(x)}{1-x} = \Sigma_0^\infty (c_0 + c_1 + c_2 + \ldots + c_n) x^n .$$

Indem wir auf $\frac{f(x)}{1-x}$ Satz I anwenden, erhalten wir Satz II. Derselbe lautet also: Sind die Coefficienten einer Potenzreihe $\Sigma_0^\infty c_n \,.\, x^n$ derart beschaffen, dass

$$\lim n^{-\lambda} (c_0 + c_1 + \ldots + c_n) = \rho ,$$

so ist

$$\lim_{x=1} f(x) \,.\, (1-x)^\lambda = \rho \,.\, \Gamma(\lambda + 1) .$$

Dabei nähert sich x dem Punkte 1 auf die nämliche Weise wie im Satze I, und λ ist eine complexe Zahl, deren reeller Teil grösser als -1.

Für $\lambda = 0$ ergiebt sich der Abelsche Satz, für $\lambda = 1$ der Satz von FROBENIUS (Über die Leibnitzsche Reihe, 'Crelle,' Bd. 89, Jahr 1880). Auch die Sätze von HÖLDER, 'Math. Annalen,' Jahr 1882 (Über Grenzwerte von Reihen an der Convergenzgrenze) ergeben sich leicht als Specialfälle des obigen durch mehrfache Ausführung der Division durch $1 - x$. Satz II findet sich für den Fall, dass x auf der positiven Axe wachsend sich 1 annähert, im Aufsatz von FRANEL, 'Math. Annalen,' Bd. 52, Jahr 1899.

Wenn im Satz II $\lambda = -1$, so findet sich

$$\lim_{x=1} \frac{f(x)}{(1-x) \log \frac{1}{1-x}} = \rho.$$

Satz III. Es sei wiederum wie im Satz I

$$f(x) = \Sigma_0^\infty c_n x^n$$

und

$$\lim c_n \,.\, n^{1-\lambda} = \rho$$

$$\lambda = \alpha + \beta i,$$

wobei α eine negative Grösse $= -\gamma$.

Die kleinste ganze Zahl grösser als γ sei h.

Alsdann convergieren nach bekannten Criterien die Reihen $\Sigma_0^\infty c_n$, $\Sigma_0^\infty n \,.\, c_n$, $\Sigma_0^\infty n(n-1)c_n \;.\;.\;.\; \Sigma_0^\infty n(n-1) \;.\;.\;.\; (n-h+1)c_n$, welche wir beziehungsweise mit $f(1), f'(1), f''(1) \;.\;.\;.\; f^{(h-1)}_{(1)}$ bezeichnen wollen.

Satz III sagt dann aus, es sei

$$f(x) = f(1) + f'(1) \,.\, (x-1) + \frac{f''(1)}{2\,!} \,.\, (x-1)^2 + \;.\;.\;.\; + \frac{f^{(h-1)}_{(1)}}{(h-1)\,!} \,.\, (x-1)^{h-1}$$

$$+ \rho \,.\, (1-x)^{-\lambda} \,.\, \psi(x),$$

wo

$$\lim_{x=1} \psi(x) = \Gamma(\lambda).$$

Der Beweis beruht auf folgendem Hülfssatze:

Ist $a_1, a_2, \;.\;.\;.\; a_n \;.\;.\;.$ eine Folge derart, dass

$$\lim_{n=\infty} a_n \,.\, n^\mu = \rho,$$

wo μ eine complexe Zahl, deren reeller Teil positiv und grösser als 1 ist, so dass also

$$a_1 + a_2 + \;.\;.\;.\; + a_n + \;.\;.\;.\; \text{convergiert},$$

so ist

$$\lim_{m=\infty} (a_m + a_{m+1} + a_{m+2} + \;.\;.\;.\;) \, n^{\mu-1} = \frac{\rho}{\mu-1}.$$

In der That, es sei

$$a_n = \frac{\rho}{\mu-1}(n^{-\mu+1} - (n+1)^{-\mu+1}) + b_n,$$

also $\lim\limits_{n=\infty} b_n \,.\, n^{\mu'} = 0$, wo μ' der reelle Teil von μ.

Es ist für jedes vorgegebene δ möglich einen Index N zu finden, so dass für $n > \mathrm{N}$

$$|b_n| < \delta(n^{-\mu'+1} - (n-1)^{-\mu'+1}).$$

Wählen wir $m > \mathrm{N}$, so ist also

$$b_m + b_{m+1} + b_{m+2} + \;.\;.\;.\; < |b_m| + |b_{m+1}| + \;.\;.\;.\; < \delta \,.\, m^{-\mu'+1}$$

Daher muss sein

$$\lim_{m=\infty} (b_m + b_{m+1} + b_{m+2} + \;.\;.\;.\;) \,.\, m^{\mu-1} = 0\,;$$

$a_m + a_{m+1} + a_{m+2} + \;.\;.\;.$ ist aber identisch

$$= \frac{\rho}{\mu-1} \,.\, m^{-\mu+1} + b_m + b_{m+1} + b_{m+2} + \;.\;.\;.$$

Der Hülfssatz ist somit erwiesen.

Nun wenden wir uns zum Beweis des Satzes III.

Es war
$$f(x) = \Sigma_0^\infty c_n x^n \text{ und}$$
$$c_0 + c_1 + c_2 + \ldots\ldots + c_n + \ldots = f(1).$$

Wir bilden
$$\frac{f(x) - f(1)}{x - 1} = \Sigma_1^\infty (c_n + c_{n+1} + c_{n+2} + \ldots) x^{n-1};$$

$c_n + c_{n+1} + c_{n+2} + \ldots$ setzen wir $= c'_n$. Da nun $\lim\limits_{n=\infty} c_n \,.\, n^{1-\lambda} = \rho$ war, so ist nach dem Hülfssatz
$$\lim_{n=\infty} c'_n \,.\, n^{-\lambda} = \frac{\rho}{-\lambda}.$$

Somit können wir diese Reihe genau so behandeln, wie die ursprüngliche Reihe; und es ist unschwer zu sehen, dass sich dies Verfahren fortsetzen lässt, so lange als die Summe der Coefficienten der neu gebildeten Reihen convergiert, d. h. h mal. Schliesslich erhalten wir auf der rechten Seite eine Reihe, deren Coefficienten ϕ_n der Bedingung genügen
$$\lim_{n=\infty} \phi_n \,.\, n^{-h+1-\lambda} = \frac{\rho}{-\lambda \,.\, (-\lambda - 1)(-\lambda - 2) \ldots (-\lambda + 1 - h)}.$$

Für die dazugehörige Reihe $\mathrm{F}(x) = \Sigma_0^\infty \phi_n x^n$ gilt aber nach Satz I
$$\lim_{x=1} \mathrm{F}(x) \,.\, (1 - x)^{h+\lambda} = \frac{\rho}{-\lambda \,.\, (-\lambda - 1) \ldots (-\lambda + 1 - h)} \,.\, \Gamma(h + \lambda).$$

Nach dem bekannten Functionaltheorem der Γ-Function; und wenn man berücksichtigt, dass
$$\mathrm{F}(x) = \frac{\dfrac{\dfrac{f(x) - f(1)}{x - 1} - f'(1)}{x - 1} - \dfrac{f''(1)}{2!}}{x - 1} - \text{etc. } (h \text{ mal}),$$
ergiebt sich dann rein aritmetisch die oben gegebene Form des Satzes III.

Für den Fall, dass γ eine ganze Zahl ist und der imaginäre Teil von λ von 0 verschieden ist, hören unsere Beweise auf gültig zu sein, und wir lassen es dahingestellt, was dann eintreten mag. Ist aber λ selbst eine ganze Zahl, so tritt offenbar die logaritmische Modification des Satzes I in Kraft und alles andere bleibt unverändert.

8. Einige kurze Andeutungen, wie man die Sätze I, II, and III ausbeuten mag, mögen hier nicht am unrechten Platze sein.

Es sei etwa angesetzt

$$f(x) = \Sigma_0^\infty x^n \cdot \frac{\Gamma(n+\alpha) \,.\, \Gamma(n+\beta) \,.\, \Gamma(n+\gamma)}{\Gamma(n+\delta) \,.\, \Gamma(n+\epsilon) \,.\, \Gamma(n+\zeta)}.$$

Alsdann genügt $f(x)$ offenbar einer homogenen linearen Differentialgleichung mit rationalen Coefficienten. Der Punkt $x = 1$ ist ein singulärer Punkt derselben. Fragt man nach der Fortsetzung des Integrals $f(x)$ beim Punkte $x = 1$, so lässt die Fuchssche Theorie noch die Frage nach dem Werte einiger Coefficienten offen, die gerade durch Sätze I und III in befriedigender Weise gelöst wird.

Oder sei angesetzt

$$f(x) = \Sigma_0^\infty x^{n^2}$$

und fragen wir nach dem asymptotischen Verhalten der Function, wenn x sich geradlinig vom Nullpunkt einem Punkte j des Convergenzkreises nähert. Ist $j = 1$, so sagt uns Satz II ohne weiteres, dass

$$\lim_{x=1} f(x) \,.\, \sqrt{1-x} = \Gamma(3/2).$$

Ist $j = -1$, so setzen wir $x = -x'$ und lassen x' sich der positiven Einheit nähern. Es ist dann

$$F(x') = f(-x') = \Sigma_0^\infty x'^{4n^2} - \Sigma_0^\infty x'^{(2n+1)^2},$$

$$\frac{F(x')}{1-x'} = \Sigma_0^\infty x'^{4n^2}(1 + x' + \ldots + x'^{4n}), \text{ also}$$

$$\Sigma_0^\infty (4n+1) x'^{4n^2} > \frac{F(x')}{1-x'} > \Sigma_0^\infty (4n+1) x'^{4n^2+4n}.$$

Nach Satz II ist aber

$$\lim_{x'=1} (1-x') \,.\, \Sigma(4n+1) \,.\, x'^{4n^2} = \tfrac{1}{2}$$

und ebenso

$$\lim_{x'=1} (1-x') \,.\, \Sigma(4n+1) \,.\, x'^{4n^2+4n} = \tfrac{1}{2}.$$

Daher ist

$$\lim_{x=-1} f(x) = \tfrac{1}{2}.$$

Es sei $j = e^{2\pi i \cdot \tau}$ und $\tau = a/b$ eine rationale Zahl, insbesondere b eine ungerade Primzahl. Alsdann führt ein Verfahren zum Ziel, welches wir für den besonderen Fall $b = 5$, $a = 1$ durchführen werden. Wir setzen wiederum

$$x = j \,.\, x'$$

und lassen x' sich positiv wachsend der Einheit nähern. Es ist identisch

$$f(x) = \Sigma_0^\infty x'^{(5n)^2} + j \,.\, \Sigma_0^\infty x'^{(5n+1)^2} + x'^{(5n+4)^2}$$
$$+ j^4 \,.\, \Sigma_0^\infty x'^{(5n+2)^2} + x'^{(5n+3)^2}.$$

Nun ist nach Satz II

$$\lim_{x'=1} \Sigma_0^\infty x'^{(5n)^2} . \sqrt{1-x'} = \tfrac{1}{5} . \Gamma 3/2 \,,$$

$$\lim_{x'=1} \left(\Sigma_0^\infty x'^{(5n+1)^2} + x'^{(5n+4)^2}\right) . \sqrt{1-x'} = \tfrac{2}{5} . \Gamma 3/2 \,,$$

$$\lim_{x'=1} \left(\Sigma_0^\infty x'^{(5n+2)^2} + x'^{(5n+3)^2}\right) . \sqrt{1-x'} = \tfrac{2}{5} . \Gamma 3/2 \,.$$

Daraus ergiebt sich

$$\lim_{x'=1} f(x) . \sqrt{1-x'} = \tfrac{1}{5} . \Gamma 3/2 (1 + 2j + 2j^4) \,,$$

$$\lim_{x=j} f(x) . \sqrt{j-x} = \sqrt{j} . \tfrac{1}{5} . \Gamma 3/2 . (1 + 2j + 2j^4) \,.$$

Diese Schlussweise ist ganz allgemein. Es geht aus diesem asymptotischen Gesetz ohne weiteres hervor, dass die singulären Punkte der Function unendlich dicht auf dem Convergenzkreise liegen, dass derselbe also eine natürliche Grenze der Reihe bildet.

9. Es sei $\phi(1), \phi(2) \ldots \phi(n) \ldots$ eine Folge, derart dass

$$\phi(1) + \phi(2) + \ldots + \phi(n) + \ldots\ldots$$

in einem Grenzpunkte der Convergenz L beim Grenzübergang G das Criterion K erfüllt. Ferner sei $\phi(u)$ eine Function der reellen positiven Veränderlichen u, derart dass $\lim\limits_{\mathrm{P}=\mathrm{L}\; u=\infty} \dfrac{\phi(u+h)}{\phi(u)} = 1$ beim Grenzübergang G, und zwar in gleichmässiger Weise. Diese Grenzbeziehung soll Geltung haben für *jeden* endlichen positiven Wert von $h \leqq 1$. Unter diesen Voraussetzungen ist

$$\lim_{\mathrm{P}=\mathrm{L}} \frac{\phi(1) + \phi(2) + \ldots + \phi(n) + \ldots\ldots}{\int_1^\infty \phi(u)\, du} = 1 \,.$$

Dies ist eine unmittelbare Folge des Theorems T. Es ist nämlich identisch

$$\int_1^\infty \phi(u)\, du = \int_1^2 \phi(u)\, du + \int_2^3 \phi(u)\, du \ldots + \int_n^{n+1} \phi(u)\, du + \ldots$$

Ferner ist $\int_n^{n+1} \dfrac{\phi(u) - \phi(n)}{\phi(n)}\, du$, infolge der Grenzbeziehung $\lim\limits_{n=\infty\; \mathrm{P}=\mathrm{L}} \dfrac{\phi(u+h)}{\phi(u)} = 1$, für genügend grosse Werte von u und Lagen von P genügend nahe an L, kleiner als δ, wie klein δ auch sei. Somit ist

$$\lim_{n=\infty\; \mathrm{P}=\mathrm{L}} \frac{\int_n^{n+1} \phi(u)\, du}{\phi(n)} = 1$$

in gleichmässiger Weise, also folgt nach Theorem T die Behauptung.

Ein ganz analoger Satz gilt offenbar auch für mehrfache Summen und mehrfache Integrale.

Die Voraussetzung, dass $\phi(1) + \phi(2) + \ldots + \phi(n) + \ldots$ bei $\lim \mathrm{P} = \mathrm{L}$ das Criterion K erfülle, kann auch durch die andere ersetzt werden, dass

$$\Sigma_1^\infty \int_n^{n+1} \phi(u)\, du$$

das Criterion K erfülle, d. h., dass sich eine Constante c finden lasse, so dass gleichmässig

$$\left| \int_1^\infty \phi(u)\, du \right| > c \,.\, \int_1^\infty |\phi(u)|\, du \,.$$

10. Der Satz 9 ist für die wirkliche Berechnung von Grenzwerten sehr dienlich, da das Rechnen mit bestimmten Integralen viel einfacher ist als das Rechnen mit Reihen.

Es sei z. B. x eine positive reelle Veränderliche und angesetzt $f(x) = \Sigma_1^\infty \frac{1}{x + n^\lambda}$ $\lambda > 1$. Der Grenzübergang G bestehe darin, dass x unendlich gross wird. Es ist das asymptotische Verhalten von $f(x)$ zu untersuchen.

$\lim\limits_{x=\infty} f(x)$ ist offenbar $= 0$, jedoch $\lim\limits_{x=\infty} x \,.\, f(x) = \infty$, da es dem Criterion K genügt. Das Integral

$$\int_0^\infty \frac{x}{x + u^\lambda} \cdot du$$

ist nun leicht zu finden, wenn man

$$u^\lambda = x \,.\, v^\lambda \text{ setzt,}$$

wobei weder Integrationsbahn noch Grenzen geändert werden. Dasselbe ist augenscheinlich

$$= \int_0^\infty \frac{x^{1+1/\lambda}\, dv}{x + x \,.\, v^\lambda} = x^{\frac{1}{\lambda}} \int_0^\infty \frac{dv}{1 + v^\lambda} \,.$$

Somit ist nach dem Satz des Art. 9

$$\lim_{x=\infty} x^{1-1/\lambda} \,.\, f(x) = \int_0^\infty \frac{dv}{1 + v^\lambda} \,.$$

11. Eine Folge von Zahlen $a_1, a_2, a_3 \ldots\ldots a_n \ldots\ldots$, für welche

$$\lim_{n=\infty} \frac{a_n}{n^{\alpha_0} \log n^{\alpha_1} \log\log n^{\alpha_2} \ldots \log\log\log\,(h \text{ mal})\, n^{\alpha_h}} = 1$$

bei irgendwelchen Werten der α, gehöre zum *Bonnetschen Typus* $[\alpha_0, \alpha_1, \alpha_2 \ldots\ldots \alpha_h]$. Eine Folge des Typus $[-1, -1, -1 \ldots\ldots (k+1 \text{ mal})]$ sei kurz $\mathrm{L}_{k+1}^{(n)}$.

Im Zusammenhang hiermit stellen wir nun gewisse Sätze auf.

Satz IV. Ist $c_n = \rho\,[\alpha_0, \alpha_1 \; . \; . \; . \; . \; \alpha_h]$ und ist

$$\alpha_0 > -1\,,$$

ferner $f(x) = \Sigma_0^\infty\, c_n \, . \, x^n$, so findet sich

$$\lim_{x=1} f(x) \, . \, (1-x)^{\alpha_0+1} \, . \left(\log \frac{1}{1-x}\right)^{-\alpha_1} . \left(\log\log \frac{1}{1-x}\right)^{-\alpha_2} \; . \; . \; . \; = \rho \, . \, \Gamma(1+\alpha_0)\,.$$

Ist jedoch $\alpha_0 = -1$ und α_{k+1} die erste der Zahlen α, welche von -1 verschieden ist, so ist

$$\lim_{x=1} f(x) \, . \left(\log\log \; . \; . \; . \; (k+1 \text{ mal}) \frac{1}{1-x}\right)^{\alpha_{k+1}+1} . \left(\log \; . \; . \; . \; (k+2 \text{ mal}) \frac{1}{1-x}\right)^{\alpha_{k+2}} \; . \; . \; .$$

$$= \frac{\rho}{\alpha_{k+1}+1}\,.$$

Dabei nähert sich x durch positive reelle Werte wachsend der Einheit.

Der Beweis beruht auf der Aussage des Art. 9. $f(x)$ lässt sich daraufhin seinem asymptotischen Verhalten nach studieren durch Betrachtung des Integrals

$$\int_c^\infty u^{\alpha_0} (\log u)^{\alpha_1} (\log\log u)^{\alpha_2} \; . \; . \; . \; (\log\log\,(h \text{ mal})\, u)^{\alpha_h} \, . \, x^u \, du\,,$$

wobei die untere Grenze des Integrals irgend eine passend gewählte endliche Constante, die Bahn die positive Axe der u ist.

$x^u = e^{u \, . \log x}$, $\log x = -\frac{1}{p}$. Wir ersetzen u durch pv. Das Integral wird dann

$$\int_{\frac{c}{p}}^\infty e^{-v} \, . \, (p \, . \, v)^{\alpha_0} \, . \, (\log pv)^{\alpha_1} (\log\log pv)^{\alpha_2} \; . \; . \; . \; p \, . \, dv\,.$$

Wir betrachten sein asymptotisches Verhalten, wenn p durch positive reelle Werte über alle Grenzen wächst.

Das Integral ist identisch

$$= p^{\alpha_0+1} (\log p)^{\alpha_1} (\log\log p)^{\alpha_2} \; . \; . \; . \; \mathrm{I},$$

wo

$$\mathrm{I} = \int_{\frac{c}{p}}^\infty e^{-v} \, . \, v^{\alpha_0} \, . \left(\frac{\log pv}{\log p}\right)^{\alpha_1} . \left(\frac{\log\log pv}{\log\log p}\right)^{\alpha_2} \; . \; . \; . \; dv\,.$$

Es ist identisch $\frac{\log p \, . \, v}{\log p} = 1 + \frac{\log v}{\log p}$. Ferner ist $\frac{\log\log pv}{\log\log p} < 1 + \frac{\log\log v}{\log\log p}$, wenn p und v genügend gross gewählt sind, nämlich so, dass $\log p > 2$ und $\log v > 2$, da dann

$$\log p + \log v < \log p \, . \, \log v, \text{ also}$$

$$\log\log\,(p \, . \, v) < \log\log p + \log\log v\,.$$

Überhaupt findet man leicht, dass bei genügend grossen Werten von p und v

$$\frac{\log\log\ .\ .\ .\ (h\text{ mal})\,p\,.\,v}{\log\log\ .\ .\ .\ (h\text{ mal})\,p} < 1 + \frac{\log\log\ .\ .\ .\ (h\text{ mal})\,v}{\log\log\ .\ .\ .\ (h\text{ mal})\,p}.$$

Somit ist für beliebige Werte von $\alpha_1, \alpha_2\ .\ .\ .\ .$

$$\left|\left(\frac{\log\log\ .\ .\ .\ (h\text{ mal})\,p\,.\,v}{\log\log\ .\ .\ .\ (h\text{ mal})\,p}\right)^{\alpha_h}\right|$$

immer zwischen 1 und

$$\left|\left(1 + \frac{\log\log\ .\ .\ .\ (h\text{ mal})\,v}{\log\log\ .\ .\ .\ (h\text{ mal})\,p}\right)^{\alpha_h}\right|$$

gelegen, wenn nur p und v genügend gross gewählt sind.

Das Integral

$$\mathrm{I} = \int_{\frac{c}{p}}^{\infty} \left|e^{-v}\right| . \left|v^{\alpha_0}\right| . \left|\left(1 + \frac{\log v}{\log p}\right)^{\alpha_1}\right| . \left|\left(1 + \frac{\log\log v}{\log\log p}\right)^{\alpha_2}\right|\ .\ .\ .\ dv$$

ist aber *in* $p = \infty$ convergent. Die bereits erwähnte specielle Form des Criterion der Gleichmässigkeit der Convergenz trifft nun nach obigen Ungleichungen hier zu.

Es ist daher

$$\lim_{p=\infty} \mathrm{I} = \int_0^{\infty} e^{-v}\,.\,v^{\alpha_0}\,dv = \Gamma(1 + \alpha_0)$$

für irgendwelche Werte der α, vorausgesetzt nur, dass der reelle Teil von α_0 grösser ist als -1.

Ist andererseits $\alpha_0 = -1$, $\alpha_1 = -1\ .\ .\ .\ .\ \alpha_k = -1$, und α_{k+1} von -1 verschieden, so verfahren wir wie folgt.

$$\frac{f(x)}{1-x} \text{ ist } = \Sigma_0^{\infty}(c_0 + c_1 + \ .\ .\ .\ + c_n)\,x^n\,.$$

Durch ein Verfahren, welches dem im Hülfssatz von Satz III angewandten parallel läuft, beweisen wir leicht, dass die Folge $c'_n = c_0 + c_1 + \ .\ .\ .\ + c_n$ zu dem Bonnetschen Typus $[0,\ 0,\ \underset{(k+1)\text{ mal}}{0\ .\ .\ .}\ ,\ \alpha_{k+1} + 1,\ \alpha_{k+2}\ .\ .\ .\ \alpha_h]$ gehört, genauer das $\frac{1}{\alpha_{k+1}+1}$-fache einer Folge dieses Typus ist. Dieser Fall ist somit auf den ersten Fall reducierbar.

Damit ist im wesentlichen der Beweis vollendet. $-\frac{1}{p}$ war $= \log x$. Da x sich wachsend der Einheit nähert, ist $\lim\limits_{x=1} \frac{\log x}{1-x} = -1$. Also ist p durch $\frac{1}{1-x}$ zu ersetzen und man braucht nur noch Theorem T anzuwenden, um die Behauptung zu erschliessen.

Das Analogon der Sätze II und III ist offenbar auch im hier erörterten weiteren Sinne möglich.

Satz V. Es sei a_n eine Folge positiver Grössen, welche immerfort wachsen, derart dass $\lim\limits_{n=\infty} a_n = \infty$.

Es sei ferner a_n ein Bonnetscher Typus

$$a_n = [\underset{m+1 \text{ mal}}{0, 0, 0 \ldots} \alpha, \beta, \gamma \ldots \delta]$$

$$\alpha > 1$$

und

$$F(x) = \Sigma_1^\infty \frac{L_{m+1}(n)}{x + a_n}.$$

Wir wollen das asymptotische Verhalten von $F(x)$ untersuchen, wenn x durch positive reelle Werte über alle Grenzen wächst.

$x F(x) = \Sigma_1^\infty \frac{x . L_{m+1}(n)}{x + a^n}$ genügt beim Grenzübergang $x = \infty$ offenbar dem Criterion K. Somit können wir die Summe durch das Integral ersetzen

$$\int_c^x \frac{x du}{u \log u \log \log u \ldots \log \log (m \text{ mal})\, u . x + \phi(u)},$$

wo c eine geeignet gewählte Constante und $\phi(u) = (\log \log \ldots (m+1 \text{ mal})\, u)^\alpha (\log \log \ldots (m+2 \text{ mal})\, u)^\beta \ldots$ ist. Nun ist $du\, L_{m+1}(u)$ offenbar das Differential von $\log \log \ldots (m+1 \text{ mal})\, (u)$. Das letztere ersetzen wir durch v; alsdann wird das Integral

$$\int_{c'}^\infty \frac{x dv}{x + v^\alpha . (\log v)^\beta . (\log \log v)^\gamma \ldots},$$

wo c' wiederum eine geeignet gewählte Constante.

Den Ausdruck $v^\alpha (\log v)^\beta (\log \log v)^\gamma \ldots$ setzen wir $= w$, also $dw = w \left(\alpha \frac{dv}{v} + \beta \frac{dv}{v . \log v} + \gamma . \frac{dv}{v . \log v \log \log v} + \ldots \right)$. Das Integral

$$\int^\infty \frac{x v dw}{w(x + w)} \text{ ist} = \alpha \int^x \frac{dv}{x + w} + \beta \int^\infty \frac{dv}{\log v (x + w} +$$

Mithin ist nach Theorem T

$$\lim_{x=\infty} \frac{\int^\infty \frac{x . v . dw}{w(x + w)}}{\int^\infty \frac{dv}{x + w}} = \alpha.$$

Es bleibt uns noch das Integral

$$\int^\infty \frac{x . v . dw}{w(x + w)}$$

zu betrachten. Nach Theorem T können wir für v hier seinen asymptotischen Wert setzen. Da aber

$$v^{\alpha}(\log v)^{\beta}(\log\log v)^{\gamma}\ldots = w,$$

so ist für $w = \infty$ asymptotisch

$$v = \zeta \,.\, w^{\alpha'}(\log w)^{\beta'}(\log\log w)^{\gamma'}\ldots, \quad \text{wo}$$

$$\alpha\alpha' = 1, \quad \alpha\beta' + \beta = 0, \quad \alpha\gamma' + \gamma = 0, \ldots \alpha\delta' + \delta = 0$$

und

$$\zeta^{\alpha} \,.\, \alpha'^{\beta} = 1, \quad \text{d. h.}\ \zeta^{\alpha} = \alpha^{\beta}.$$

Somit kommen wir auf das Studium eines Integrals

$$\zeta \,.\, \int^{\infty} \frac{x \,.\, w^{\frac{1}{\alpha}-1} \,.\, (\log w)^{\beta'} \,.\, (\log\log w)^{\gamma'}}{x + w} \ldots dw \,.$$

In demselben ersetzen wir w durch $x \,.\, t^{\alpha}$.
Es ist dann das Integral

$$\mathrm{I} = \alpha \,.\, \zeta \,.\, x^{\frac{1}{\alpha}} \,.\, \int^{\infty} \frac{(\log x \,.\, t^{\alpha})^{\beta} \,.\, (\log\log x \,.\, t^{\alpha})^{\gamma'} \ldots dt}{1 + t^{\alpha}} \,.$$

Dieses behandeln wir genau so wie das entsprechende Integral im Satz IV. Man findet somit als ersten Term seiner asymptotischen Entwickelung

$$\mathrm{I} \equiv \alpha \,.\, \zeta \,.\, x^{\frac{1}{\alpha}} \,.\, (\log x)^{\beta'} \,.\, (\log\log x)^{\gamma'} \ldots \alpha^{\beta'} \,.\, \int_0^{\infty} \frac{dt}{1 + t^{\alpha}} \,.$$

ζ war $= \alpha^{\frac{\beta}{\alpha}} = \alpha^{-\beta'}$. Somit kommt

$$\mathrm{I} \equiv \alpha \,.\, x^{\frac{1}{\alpha}} \,.\, (\log x)^{\beta'} \,.\, (\log\log x)^{\gamma'} \ldots \int_0^{\infty} \frac{dt}{1 + t^{\alpha}} \,.$$

Es war aber $\lim\limits_{x=\infty} \dfrac{\mathrm{I}}{x \,.\, \mathrm{F}(x)} = \alpha$. Also ist schliesslich

$$\lim_{x=\infty} x^{1-\frac{1}{\alpha}} \,.\, (\log x)^{\frac{\beta}{\alpha}} \,.\, (\log\log x)^{\frac{\gamma}{\alpha}} \ldots \mathrm{F}(x) = \int_0^{\infty} \frac{dt}{1 + t^{\alpha}} \,.$$

Um das Verfahren zu beschreiben, welches im Falle $\alpha = 1$ zum Ziele führt, genügt es, ein concretes Beispiel zu betrachten. Es sei angesetzt

$$\mathrm{F}(x) = \Sigma_2^{\infty} \frac{1}{(\log n)^{\beta} \,.\, (x + n)}$$

und $\beta > 1$. Dies führt uns zur Untersuchung des Integrals

$$\int_c^{\infty} \frac{du}{(\log u)^{\beta} \,.\, (x + u)} \,.$$

Dasselbe ist identisch

$$= -\frac{1}{\beta - 1}\int_c^\infty \frac{u \,.\, d\,(\log u)^{1-\beta}}{x + u}.$$

Es ist nun nach der Formel der partiellen Integration

$$-\int_c^\infty \frac{u}{x+u} \,.\, d\,(\log u)^{1-\beta} + (\log u)^{1-\beta} \,.\, d\,\frac{u}{x+u} = \frac{c}{x+c} \,.\, (\log c)^{1-\beta}.$$

Diese Formel gilt für alle Werte von x. Gehen wir jetzt zur Grenze $\lim x = \infty$ über, so kann man, da

$$\lim_{x=\infty} x \,.\, \mathrm{F}(x) = \infty,$$

die rechte Seite der obigen Identität ausser Spiel lassen. Der asymptotische Ausdruck von $\mathrm{F}(x)$ wird somit

$$\frac{1}{\beta - 1}\int_c^\infty x\,(\log u)^{1-\beta}\,\frac{du}{(x+u)^2}.$$

Ersetzen wir hier u durch $x \,.\, v$ und gehen wir dann zur Grenze $x = \infty$ über, so wird nach dem früheren Verfahren gezeigt, dass der Ausdruck asymptotisch

$$\equiv \frac{1}{\beta - 1}\,(\log x)^{1-\beta}\int_0^\infty \frac{dv}{(1+v)^2} \qquad \equiv \frac{1}{\beta - 1} \,.\, (\log x)^{1-\beta}.$$

Somit ist

$$\lim_{x=\infty} \mathrm{F}(x) \,.\, (\log x)^{\beta-1} = \frac{1}{\beta - 1}.$$

Danach lautet der Satz V wie folgt:

Es sei $\mathrm{L}_{m+1}\ (n)$ irgend ein Bonnetscher Typus $[\underbrace{-1, -1, -1 \ldots -1}_{m+1 \text{ mal}}]$ und a_n ein Bonnetscher Typus $[\underbrace{0, 0, 0 \ldots 0}_{m+1 \text{ mal}}, \alpha, \beta, \gamma \ldots \delta]$, derart dass $\Sigma_1^\infty \frac{\mathrm{L}_{m+1}(n)}{a_n}$ absolut convergiert. Es sei angesetzt

$$\mathrm{F}(x) = \Sigma_1^\infty \frac{\mathrm{L}_{m+1}(n)}{x + a_n}.$$

Ist dann $\alpha > 1$, so findet sich

$$\lim_{x=\infty} x^{1-\frac{1}{\alpha}} \,.\, (\log x)^{\frac{\beta}{\alpha}}\,(\log\log x)^{\frac{\gamma}{\alpha}} \ldots \mathrm{F}(x) = \int_0^\infty \frac{dt}{1+t^\alpha}.$$

Ist jedoch $\alpha = 1$, $\beta = 1 \ldots$ und ϵ die erste dieser Grössen, die von 1 verschieden ist, so findet sich

$$\lim_{x=\infty} (\log\log \ldots x)^{\epsilon-1} \,.\, (\log\log \ldots x)^{\eta} \,.\, (\log\log \ldots x)^{\zeta} \ldots (\log\log \ldots x)^{\delta} \,.\, \mathrm{F}(x)$$
$$= \frac{1}{\epsilon - 1},$$

wo nun η, ζ . . . δ die auf ϵ folgenden Werte der obigen Zahlenreihe bedeuten. Dabei ist der Logaritme so oft iteriert als der Platz des dazugehörigen Buchstaben (ϵ, η, ζ . . . δ) in der Zahlenreihe α, β, γ . . . δ anzeigt, wobei von α mit 0 zu zählen angefangen wird.

12. Aus dem Satze V kann man leicht Folgerungen ziehen, welche für die Theorie der ganzen (transcendenten) Functionen nicht ohne Interesse sind. Einzelne, doch nicht alle, der folgenden Resultate sind anticipiert von verschiedenen Autoren, hauptsächlich LAGUERRE, POINCARÉ, HADAMARD, BOREL, VON SCHAPER. Man findet eine sehr eingehende Besprechung der Literatur des Gegenstandes in BOREL's 'Leçons sur la Théorie des Fonctions entières,' 1900.

Es sei r_1, r_2 . . . r_n . . . eine Folge immerfort wachsender positiver Grössen, derart dass $\lim_{n=\infty} r_n = \infty$. Hat die Reihe

$$r_1^x + r_2^x + r_3^x + \ldots$$

einen inneren Convergenzbereich, so werden wir sagen, dass die Folge r_n zur Bonnetschen Classe 1 gehört. Hat diese Reihe keinen Convergenzbereich, jedoch die andere

$$\Sigma_1^\infty n^{-1} \,.\, r_n^x,$$

so gehört die Folge zur Bonnetschen Classe 2. Hat auch diese Reihe keinen Convergenzbereich, jedoch

$$\Sigma_3^\infty n^{-1} \,.\, (\log n)^{-1} r_n^x,$$

so gehört die Folge zur Bonnetschen Classe 3, u. s. f.

Die Reihen convergieren nur für complexe Werte von x, deren reeller Teil negativ ist. Die reelle Convergenzgrenze der Reihe $r_1^x + r_2^x + \ldots$ hat VON SCHAPER *Convergenzexponent*, BOREL *Ordnung*, genannt. Wir wollen beide Namen benutzen.

Es sei a_1, a_2, a_3 . . . a_n . . . eine Folge complexer Zahlen, deren Moduln $|a_n| = r_n$ immerfort wachsen, so dass $\lim_{n=\infty} r_n = \infty$. Alsdann kann man beliebig viele ganze Functionen bilden, die nur in den a_i, und zwar dort einfach, verschwinden. Dazu kann man die Metode von WEIERSTRASS benutzen, oder auch wie folgt verfahren. Man bestimmt irgend eine Folge ganzer Functionen von x

$$g_1(x),\ g_2(x),\ g_3(x)\ \ldots\ g_n(x)\ \ldots$$

derart, dass

$$\Sigma_1^\infty \frac{g_n(x)}{g_n(a_n) \,.\, a_n}$$

für jeden Wert von x absolut und gleichmässig convergiert, und setzt dann

$$\frac{G'(x)}{G(x)} = \Sigma_1^\infty \frac{g_n(x)}{g_n(a_n) \,.\, (x - a_n)}.$$

Diese Differentialgleichung definiert offenbar eine ganze Function G (x), welche die gestellte Forderung erfüllt. Das Integral der Gleichung erhält man in der Form

$$G(x) = G(o) \,.\, \Pi_1^\infty \left(1 - \frac{x}{a_n}\right) e^{\int_0^x \frac{g_n(x) - g_n(a_n)}{x - a_n} dx},$$

denn dieses Product ist absolut und gleichmässig convergent. Das letztere zeigt sich sehr leicht wie folgt. Da $\Sigma_1^\infty \frac{g_n(x)}{g_n(a_n) \, . \, a_n}$ absolut und gleichmässig für jeden endlichen Wert von x convergiert, so auch

$$\Sigma_1^\infty \frac{g_n(x)}{g_n(a_n) \, . \, (x - a_n)}$$

mit Ausnahme der Pole. Ist x_0 ein gegebener Wert und B irgend ein endlicher gegebener Bereich, der 0 und x_0 einschliesst, ist ferner eine kleine Zahl δ vorgegeben, so können wir einen Index m' bestimmen, so dass

$$\Sigma_m^{m+p} \left| \frac{g_n(x)}{g_n(a_n)(x - a_n)} \right| < \delta,$$

wie gross auch p sei, wenn nur x in B gelegen ist und $m \geqq m'$ ist.

Somit ist

$$\Sigma_m^{m+p} \left| \int_0^{x_0} \frac{g_n(x)}{g_n(a_n) \, . \, (x - a_n)} \, dx \right| < \delta \, . \, |x_0| \, .$$

Diese Ungleichung schliesst aber die absolute und gleichmässige Convergenz des obigen Productes in sich ein.

Wenn nun die $g_1(x) \; . \; . \; . \; g_n(x) \; . \; . \; .$ in zweckmässiger Weise gewählt sind, so erhalten wir ganze Functionen $\mathrm{G}(x)$, welche mannigfache interessante und wichtige Eigenschaften besitzen.

13. Ein Satz von SCHOU (‘Comptes Rendus,’ t. CXXV, p. 763) besagt folgendes: Ist der Modul einer ganzen Function $\mathrm{G}(x)$, wenn $|x| = r$ ist, kleiner als $e^{\mathrm{V}(r)}$, so ist die Anzahl N der Wurzeln von $\mathrm{G}(x) = 0$, deren Modul kleiner ist als r

$$\mathrm{N} < \frac{\mathrm{V}(s \, . \, r)}{\log (s - 1)},$$

wo s irgend eine Zahl grösser als 2 bedeutet.*

Ist $\mathrm{V}(r) = r$ für irgend einen endlichen Wert von α, so heisst $\mathrm{G}(x)$ nach LAGUERRE eine ganze Function eines endlichen genre und nach v. SCHAPER eine Hadamardsche Function. Ist $\mathrm{V}(r) = e^{r^a}$, so wollen wir $\mathrm{G}(x)$ eine Borelsche Function 2^{ter} Classe nennen. Ist $\mathrm{V}(r) = e^{e^{r^a}}$, so heisse $\mathrm{G}(x)$ eine Borelsche Function dritter Classe, u. s. f. Die Moduln der Wurzeln einer Borelschen Function h^{ter} Classe bilden eine Folge, die zur h^{ten} Bonnetschen Classe gehört, wie sich aus der Ungleichung von SCHOU ergiebt. Man braucht nur $|a_{\mathrm{N}}| < r$ mit der obigen Ungleichung zu verbinden, um dies sofort einzusehen.

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* Man könnte unschwer den Schouschen Satz wie folgt präcisieren: Es sei $\mathfrak{P}(x)$ eine Potenzreihe, deren Convergenzradius $= \rho$, r irgend eine positive Zahl $< \rho$, N die Anzahl der Wurzeln von $\mathfrak{P}(x) = 0$, deren Modul $\leqq r$, $\mathrm{M}\,\mathfrak{P}_r$ der Maximalwert von $\mathfrak{P}(x)$ für $|x| = r$, s eine Zahl grösser als 1 und kleiner als $\frac{\rho}{r}$; alsdann gilt $\mathrm{N} . \log s < \log \mathrm{M}\,\mathfrak{P}_{s \, . \, r}$, während der Schousche Satz nur besagt $\mathrm{N} . \log (s) < \log \mathrm{M}\,\mathfrak{P}_{(s+1)} \, r$, also r nicht gestattet, über die Hälfte von ρ hinauszugehen.

Jetzt nehmen wir insbesondere an, dass die Folge der Wurzeln von $G(x)$ einen Bonnetschen Typus bildet, und bringen dadurch unsere Untersuchung in Berührung mit der Aussage des Satzes V.

14. Gehört a_n zur ersten Bonnetschen Classe, und ist k die kleinste ganze Zahl, für welche $\Sigma |a_n|^{-k}$ noch convergiert, so definiert

$$\frac{G'(x)}{G(x)} = \frac{x^{k-1}}{a_n^{k-1}(x - a_n)}$$

eine Function $G(x)$, deren "genre" nach LAGUERRE $= k - 1$ ist. Ist z. B. $a_n = n^\alpha$, α keine ganze Zahl und

$$(k-1)\alpha < 1 < k\alpha ,$$

so definiert die Gleichung

$$\frac{\psi_\alpha'(x)}{\psi_\alpha(x)} = \frac{x^{k-1}}{n^{\alpha k(-1)}(x - n^\alpha)}$$

eine ganze Function ψ_α vom genre $k - 1$. Von dieser Function wissen wir, nach Satz V, dass für negative Werte von x, wenn $|x|$ über jede Grenze wächst,

$$\lim_{x=-\infty} \left| \frac{\psi_\alpha'(x)}{x^{k-1} \cdot \psi_\alpha(x)} \right| \cdot |x|^{k-\frac{1}{\alpha}} = \int_0^\infty \frac{du}{u^{\alpha(k-1)}(1 + u^\alpha)} \text{ ist.}$$

Das letztere Integral wollen wir C_α nennen. Dieselbe Grenzbeziehung gilt, wenn ψ_α durch eine Function f ersetzt wird, deren Wurzeln a_n eine Folge bilden, die zum Typus n^α gehört. Dies Resultat ergiebt sich aus der nun folgenden Überlegung, welche den Beweis des Satzes V nachträglich ergänzt.

Es sei
$$p = \Sigma_1^\infty \frac{c_n}{x + a_n} \qquad q = \Sigma_1^\infty \frac{c_n'}{x + a_n'} .$$

Die c_n, c_n', a_n, a_n' seien positive Grössen, derart dass $\Sigma_1^\infty \frac{c_n}{a_n}$ und $\Sigma \frac{c_n'}{a_n'}$ convergieren, aber $\Sigma_1^\infty c_n$, $\Sigma_1^\infty c_n'$ divergieren. Ferner sei $\lim\limits_{n=\infty} a_n = \infty$ $\lim\limits_{n=\infty} a_n' = \infty$ und

$$\lim_{n=\infty} \frac{a_n}{a_n'} = 1 \qquad \lim_{n=\infty} \frac{c_n}{c_n'} = 1 .$$

Alsdann ist, wenn x durch positive Werte über jede Grenze wächst,

$$\lim_{x=\infty} \frac{p}{q} = 1 .$$

Es divergieren nämlich die Reihen

$$x \cdot p = \Sigma_1^\infty \frac{c_n \cdot x}{x + a_n}, \qquad x \cdot q = \Sigma_1^\infty \frac{c_n' x}{x + a_n'}$$

in $x = \infty$ und genügen unter den gemachten Voraussetzungen dem Criterion K. Bilden wir nun

$$x \cdot (p - q) = \Sigma_1^\infty x \cdot \frac{x(c_n - c_n') + a_n' c_n - a_n c_n'}{(x + a_n)(x + a_n')}$$

und vergleichen wir den n^{ten} Term dieser Reihe mit dem n^{ten} Term der Reihe

$$x \cdot q = \Sigma_1^\infty \frac{x \cdot c_n'}{x + a_n'},$$

so erhalten wir als Quotient den Ausdruck

$$\frac{x(c_n - c_n') + a_n' c_n - a_n c_n'}{c_n'(x + a_n)},$$

welcher unter den zugestandenen Voraussetzungen offenbar, für $\lim x = \infty$ und $\lim n = \infty$, gleichmässig gegen 0 convergiert. Somit haben wir nach Theorem T

$$\lim_{x=\infty} \frac{x(p-q)}{x \cdot q} = 0,$$

$$\text{d.h. } \lim_{x=\infty} \frac{p}{q} = 1. \qquad \text{Q. e. d.}$$

15. Nun ist es für uns von Wert, für die Function ψ_a viel genauere asymptotische Entwickelungen als die obige zu gewinnen. Zu diesem Zwecke machen wir von einem Calcül Gebrauch, den wir in folgendem den Calcül C nennen werden, und den wir zunächst ganz im allgemeinen characterisieren werden.

Es sei $\phi(0) + \phi(1) + \phi(2) + \phi(3) + \ldots + \phi(n) + \ldots$ eine Summe derart, wie wir sie im Anfang betrachtet haben. Wir zerspalten jedes Glied der Summe in seinen reellen und imaginären Bestandteil und betrachten die Summe der reellen und imaginären Bestandteile gesondert. Wir können somit die $\phi(n)$ als reelle Grössen ansetzen. Im inneren Convergenzbereiche der Summe ziehen wir eine stetige Curve $\mathfrak{C}$, welche in einem Grenzpunkt L der Convergenz mündet, oder betrachten eine Folge discontinuierlicher Lagen des Punktes P, welche einen Grenzpunkt L der Convergenz zum Grenzpunkt hat. Wir nehmen an, dass wir eine reelle Function $\phi(u)$ der reellen positiven Variabelen u und des Punktes P kennen, deren Wert für $u = n$ mit $\phi(n)$ identisch ist.

Für irgend eine gegebene Lage P_0 des Punktes P auf $\mathfrak{C}$ oder der Folge P_i sei nun $\phi(u)$ monoton abnehmend oder zunehmend in den Intervallen

$$u = 0 \ldots \alpha_1, \quad u = \alpha_1 \ldots \alpha_2, \quad u = \alpha_2 \ldots \alpha_3, \quad \ldots \quad u = \alpha_k \ldots \alpha_{k+1}, \quad \ldots .$$

Der Sinn des Wachsens von $\phi(u)$ ändere sich daher in der Folge wachsender positiver Zahlen $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_k \ldots$ und sonst nirgends. Rückt P_0 beim Grenzübergang nach L, so mögen die α sämmtlich ins ∞ fallen, oder auch nicht. Es ist dies für das folgende irrelevant.

Es seien $m_1, m_2 \ldots m_k \ldots$ ganze positive Zahlen, derart dass $m_1 \leqq \alpha_1 < m_1 + 1$, $m_2 \leqq \alpha_2 < m_2 + 1$, u. s. f. Um den Ideen eine feste Richtung zu geben

wachse $\phi(u)$ zwischen $u = 0$ und $u = m_1$,
nehme ab zwischen $u = m_1 + 1$ und $u = m_2$,
wachse wiederum zwischen $u = m_2 + 1$ und $u = m_3$,
u. s. f.

Alsdann ist
$$\phi(0) > \int_0^1 \phi(u)\,du > \phi(1),$$
$$\phi(1) > \int_1^2 \phi(u)\,du > \phi(2),$$
$$\cdots\cdots\cdots$$
$$\phi(m_1 - 1) > \int_{m_1-1}^{m_1} \phi(u)\,du > \phi(m_1),$$
mithin
$$\int_0^{m_1} \phi(u)\,du + \phi(m_1) > \phi(0) + \phi(1) + \ldots + \phi(m_1) > \int_0^{m_1} \phi(u)\,du + \phi(0).$$

Genau so ist
$$\int_{m_1+1}^{m_2} \phi(u)\,du + \phi(m_1+1) < \phi(m_1+1) + \ldots + \phi(m_2) < \int_{m_1+1}^{m_2} \phi(u)\,du + \phi(m_2);$$
dann wieder
$$\int_{m_2+1}^{m_3} \phi(u)\,du + \phi(m_2+1) > \phi(m_2+1) + \ldots + \phi(m_3) > \int_{m_2+1}^{m_3} \phi(u)\,du + \phi(m_3),$$
u. s. f.

Mithin
$$\int_0^\infty \phi(u)\,du - \Sigma_h \int_{m_h}^{m_{h+1}} \phi(u)\,du + \phi(m_1) + \phi(m_2) + \phi(m_2+1) + \phi(m_4) + \phi(m_4+1) + \phi(m_6) + \ldots.$$
$$> \phi(0) + \phi(1) + \phi(2) + \ldots + \phi(n) + \ldots$$
$$> \int_0^\infty \phi(u)\,du - \Sigma \int_{m_h}^{m_{h+1}} \phi(u)\,du + \phi(0) + \phi(m_1+1) + \phi(m_3) + \phi(m_3+1) + \ldots$$

Wir gehen nun zur Grenze lim P = L über. Beim Grenzübergang G verschieben sich (allgemein gesprochen) die Lagen der α, und wir wollen annehmen, dass eine endliche bestimmte Anzahl derselben unter einer bestimmbaren endlichen Grenze a bleibt, während die übrigen α sämmtlich ins ∞ rücken. Dann betrachten wir nicht die Summe $\phi(0) + \phi(1) + \ldots$, sondern nur den Teil $\phi(a) + \phi(a+1) + \phi(a+2) + \ldots$ derselben. Die oben entwickelte Ungleichung giebt uns dann asymptotisch diese Summe in der Gestalt eines Integrals mit Zusatzgliedern der Form
$$\int_{m_h}^{m_{h+1}} \phi(u)\,du \quad \text{und} \quad \phi(m_h),\ \phi(m_{h+1}),$$
welche das eigenartige haben, dass alle m_h ins Unendliche rücken.

Ohne Zweifel kann man für den Ausdruck
$$\int_0^\infty \phi(u)\,du - \phi(0) - \phi(1) - \ldots - \phi(n) - \ldots$$

noch viele andere asymptotische Entwickelungen finden, z. B. dadurch, dass man den Ausdruck in Form eines Integrals schreibt, und für dies Integral den Mittelwertsatz oder andere Sätze ähnlicher Art anwendet. Für die Zwecke, welche wir hier im Auge haben, ist der Calcül C ausreichend und vielleicht sogar unersetzlich. Dennoch ist es ausserordentlich fraglich, ob dem Calcül C eine allgemeinere Bedeutung zukommt.

16. Es sei $\alpha > 1$, also $k = 1$, und

$$\frac{\psi_\alpha'}{\psi_\alpha} = \Sigma_1^\infty \frac{1}{x - n^\alpha}.$$

Wir betrachten zunächst negative Werte von x. Der Calcül C giebt dann sogleich,

wenn $$\phi(u) = \frac{1}{y + n^\alpha}, \quad \text{wo} \quad y = -x,$$

$$\int_0^\infty \phi(u)\,du + \frac{1}{y} > -\frac{\psi_\alpha'(-y)}{\psi_\alpha(-y)} > \int_0^\infty \phi(u)\,du\,.$$

Nun ist identisch $$\int_0^\infty \frac{du}{y + u^\alpha} = y^{\frac{1}{\alpha} - 1}\,.\,C_\alpha,$$

somit durch Integration zwischen 2 positiven Grenzen y_1, y_2

$$(y_2^{\frac{1}{\alpha}} - y_1^{\frac{1}{\alpha}})\,\alpha\,.\,C_\alpha + \log\frac{y_2}{y_1} > \log\frac{\psi_\alpha(-y_2)}{\psi_\alpha(-y_1)} > (y_2^{\frac{1}{\alpha}} - y_1^{\frac{1}{\alpha}})\,\alpha\,.\,C_\alpha,$$

wo $$y_2 > y_1,$$

oder $$e\,(y_2^{\frac{1}{\alpha}} - y_1^{\frac{1}{\alpha}})\,\alpha\,.\,C_\alpha\,.\,\frac{y_2}{y_1} > \frac{\psi_\alpha(-y_2)}{\psi_\alpha(-y_1)} > e\,(y_2^{\frac{1}{\alpha}} - y_1^{\frac{1}{\alpha}})\,\alpha\,.\,C_\alpha\,.$$

Setzen wir insbesondere $y_1 = 1$

$$e y_2^{\frac{1}{\alpha}}\,.\,\alpha\,C_\alpha\,.\,y_2 > \frac{e^{\alpha\,.\,C_\alpha}}{\psi_\alpha(-1)}\,.\,\psi_\alpha(-y_2) > e y_2^{\frac{1}{\alpha}}\,.\,\alpha\,.\,C_\alpha$$

für jeden Wert von $y_2 > 1$.

Für einen complexen Wert von x sei

$$x = v + i\,.\,w\,.,$$

also $$\frac{1}{x - n^\alpha} = \frac{v - n^\alpha}{(v - n^\alpha)^2 + w^2} - i\,.\,\frac{w}{(v - n^\alpha)^2 + w^2}.$$

Wir setzen $$\mathrm{P} = \Sigma_1^\infty \frac{v - n^\alpha}{(v - n^\alpha)^2 + w^2}, \qquad \mathrm{Q} = \Sigma_1^\infty \frac{w}{(v - n^\alpha)^2 + w^2}.$$

Es sei n_1 eine positive ganze Zahl, derart dass

$$n_1^\alpha < v < (n_1 + 1)^\alpha$$

also $n_1 = 0$, wenn v negativ, oder positiv aber kleiner als 1. Für Q haben wir dann offenbar 2 Intervalle anzusetzen, nämlich

$$u = 1 \ldots n_1 \quad \text{und} \quad u = n_1 + 1 \ldots\ldots \infty .$$

Im ersten Intervall nimmt $\phi(u)$ zu, im zweiten ab. Wir haben daher durch Calcül C

$$\int_0^\infty \frac{w \,.\, du}{(v-u^a)^2+w^2} - \int_{n_1}^{n_1+1} \frac{w\,du}{(v-u^a)^2+w^2} + \frac{w}{(v-n_1{}^a)^2+w^2} > Q > \int_0^\infty \frac{w\,du}{(v-u^a)^2+w^2} - \int_{n_1}^{n_1+1} \frac{w\,du}{(v-u^a)^2+w^2} .$$

Es ist aber identisch

$$\int_0^\infty \frac{v-u^a-i\,.\,w}{(v-u^a)^2+w^2} \,.\, du = (v+i\,.\,w)^{\frac{1}{a}-1} \,.\, C_a ;$$

somit
$$\int_0^\infty \frac{(v-u^a)\,du}{(v-u^a)^2+w^2} = \Re\,(v+i\,.\,w)^{\frac{1}{a}-1} \,.\, C_a ,$$

und
$$\int_0^\infty \frac{w\,du}{(v-u^a)^2+w^2} = \Re\, i\,(v+i\,.\,w)^{\frac{1}{a}-1} \,.\, C_a ,$$

wo, wie gewöhnlich, $\Re\,(\xi)$ den reellen Teil von ξ bedeutet.

Setzen wir $v + i \,.\, w = r \,.\, e^{\phi \,.\, i}$, so kommt

$$\int_0^\infty \frac{w\,du}{(v-u^a)^2+w^2} = -\, r^{\frac{1}{a}-1} \,.\, \sin\left(\frac{1}{\alpha} - 1\right)\phi \,.\, C_a .$$

Andererseits $\frac{v-u^a}{(v-u^a)^2+w^2}$ nimmt zu zwischen $u = 0 \ldots v - w$ (w positiv gedacht),
nimmt ab zwischen $v - w \ldots v + w$,
und wieder zu zwischen $v + w \ldots \infty$.

Ist $v < w$, so fällt das erste Intervall aus der Betrachtung.

Ist $w = 0$, so fällt das 2te Intervall fort. $\frac{1}{v-u^a}$ nimmt sowohl im ersten als 2ten Intervall zu.

Es sei $w = 0$ und v positiv, ferner

$$n_1{}^a < v < (n_1 + 1)^a.$$

Durch Calcül C wird

$$\int_1^\infty \frac{du}{v-u^a} - \int_{n_1}^{n_1+1} \frac{du}{v-u^a} + \frac{1}{v-n_1{}^a} > P > \int_0^\infty \frac{du}{v-u^a} - \int_{n_1}^{n_1+1} \frac{du}{v-u^a} + \frac{1}{v} + \frac{1}{v-(n_1+1)^a} .$$

Da die Bahn durch einen Pol von $\frac{1}{v-u^a}$ geht, so schreiben wir statt

$$\int_0^\infty \frac{du}{v-u^a} - \int_{n_1}^{n_1+1} \frac{du}{v-u^a} \quad \text{besser} \quad \int_0^{n_1} \frac{du}{v-u^a} - \int_{n_1+1}^\infty \frac{du}{u^a-v}$$

Substituieren wir $s^a = v \qquad u' = s \,.\, u$, so wird

$$\int_0^{n_1} \frac{du}{v - u^a} = s^{1-a} \,.\, \int_0^{\frac{n_1}{s}} \frac{du}{1 - u^a} \qquad \text{und}$$

$$\int_{n_1+1}^{\infty} \frac{du}{u^a - v} = s^{1-a} \,.\, \int_{\frac{n_1+1}{s}}^{\infty} \frac{du}{u^a - 1}.$$

Die Integrale $\int_0^{\frac{n_1}{s}} \frac{du}{1 - u^a}$ und $\int_{\frac{n_1+1}{s}}^{\infty} \frac{du}{u^a - 1}$ haben offenbar einen bestimmten Wert für jeden positiven Wert von $v > 1$, der keine ganze Zahl ist. Bezeichnen wir die Integrale kurz mit b_v und c_v, so ist

$$\int_0^{n_1} \frac{du}{v - u^a} - \int_{n_1+1}^{\infty} \frac{du}{u^a - v} = v^{\frac{1}{a} - 1}(b_v - c_v).$$

Wir wollen einen Moment abschweifen. Haben wir ein Integral $\int_0^{1-\eta} \frac{f(u)\,du}{1 - u}$, wo $f(u)$ im Intervall $0 \;.\;.\;.\; 1$ endlich und stetig und bei $u = 1$ in eine convergente Potenzreihe entwickelbar ist, so ist offenbar

$$\int_0^{1-\eta} \frac{f(u)}{1 - u} du = f(1) \,.\, \log \frac{1}{\eta} + \int_0^{1-\eta} \frac{f(u) - f(1)}{1 - u} du$$

und $$f(1) \,.\, \log \frac{1}{\eta} + \int_0^1 \frac{f(u) - f(1)}{1 - u} du - \eta \,.\, m > \int_0^{1-\eta} \frac{f(u)}{1 - u} du > f(1) \,.\, \log \frac{1}{\eta}$$
$$+ \int_0^1 \frac{f(u) - f(1)}{1 - u} du - \eta \,.\, M,$$

wo m den Minimalwert, M den Maximalwert der Function

$$\frac{f(u) - f(1)}{1 - u}$$

im Intervall $1 - \eta \;.\;.\;.\; 1$ bezeichnet.

Es sei nun $s = n_1 + \epsilon$, $\frac{\epsilon}{n_1 + \epsilon} = \eta_1$, $\frac{1 - \epsilon}{n_1 + 1} = \eta_2$.

$$b_v = \int_0^{\frac{n_1}{s}} \frac{du}{1 - u^a} = \int_0^{1 - \eta_1} \frac{du}{1 - u^a} = \frac{1}{\alpha} \,.\, \log \frac{1}{\eta_1} + K + \eta_1 \,.\, H,$$

wo K eine Constante ist, und für *jeden* Wert von ϵ endlich und stetig ist, und wo $\lim\limits_{n_1 = \infty} H =$ einer bestimmten Constanten ist.

Ähnlich ist $c_v = \int_{\frac{n_1+1}{s}}^{\infty} \frac{du}{u^a - 1}$, wenn u durch $\frac{1}{u}$ ersetzt wird,

$$c_v = \int_0^{1 - \eta_2} \frac{du}{u^{2-a}(1 - u^a)} = \frac{1}{\alpha} \,.\, \log \frac{1}{\eta_2} + K' + \eta_2 \,.\, H',$$

wo K′ und H′ eine ähnliche Bedeutung wie K und H oben haben.

Schliesslich wird $$b_v - c_v = \frac{1}{\alpha} \,.\, \log \frac{\eta_2}{\eta_1} + \mathrm{K}'' + \eta_1 \,.\, \mathrm{H} - \eta_2 \,.\, \mathrm{H}'$$
$$= \frac{1}{\alpha} \,.\, \log \frac{\epsilon}{1-\epsilon} + \mathrm{L} + \frac{\mathrm{M}}{n_1},$$

wo L eine Constante und M eine für alle in Betracht kommenden Werte von ϵ endliche und stetige Function ist, die für $n = \infty$ einen endlichen Grenzwert hat.

Somit erhalten wir

$$v^{\frac{1}{a}-1}\left(\frac{1}{\alpha} \log \frac{\epsilon}{1-\epsilon} + \mathrm{L} + \frac{\mathrm{M}}{n_1}\right) + \frac{1}{v - n_1{}^a} > \frac{\psi_a'(v)}{\psi_a(v)} > v^{\frac{1}{a}-1}\left(\frac{1}{\alpha} \log \frac{\epsilon}{1-\epsilon} + \mathrm{L} + \frac{\mathrm{M}}{n_1}\right)$$
$$+ \frac{1}{v} + \frac{1}{v - (n_1+1)^a},$$

wenn $$v = (n_1 + \epsilon)^a, \qquad 0 < \epsilon < 1.$$

Wenn wir ϵ constant halten, v dagegen unendlich gross wird, so giebt uns diese Ungleichung eine sehr genaue asymptotische Beziehung für $\frac{\psi_a'(v)}{\psi_a(v)} \,.\, v^{1-\frac{1}{a}}$, derzufolge

$$\lim_{v=\infty} \frac{\psi_a'(v)}{\psi_a(v)} \,.\, v^{1-\frac{1}{a}} = \frac{1}{\alpha} \log \frac{\epsilon}{1-\epsilon} + \mathrm{L}.$$

Die Wurzeln, für grosse Werte von v, von $\psi_a' = 0$ erhält man also durch $\frac{1}{\alpha}. \log \frac{\epsilon}{1-\epsilon} + \mathrm{L} = 0$, natürlich asymptotisch. Ist z. B. $\alpha = 2$, so ist $\psi_a(x) = \frac{\sin(\pi\sqrt{x})}{\pi\sqrt{x}}$ und $\mathrm{L} = 0$, woraus $\epsilon = \frac{1}{2}$.

Ist w von Null verschieden, so sind $v - w$ und $v + w$ die Werte, in denen der Sinn des Wachsens von

$$\frac{v - u^a}{(v - u^a)^2 + w^2}$$

sich ändert. Ist sowohl $v - w$ wie $v + w$ negativ, so folgt daher

$$\int_0^\infty \frac{v - u^a}{(v - u^a)^2 + w^2}\, du > \mathrm{P} > \int_0^\infty \frac{v - u^a}{(v - u^a)^2 + w^2}\, du - \frac{v}{v^2 + w^2}.$$

Ist $v - w$ negativ, dagegen $v + w$ positiv und n_1 die ganze Zahl, für welche $n_1{}^a < v + w < (n_1 + 1)^a$, so ist

$$\int_0^\infty \frac{v - u^a}{(v - u^a)^2 + w^2}\, du - \int_{n_1}^{n_1+1} \frac{v - u^a}{(v - u^a)^2 + w^2}\, du + \frac{v}{v^2 + w^2} > \mathrm{P} > \int_0^\infty \frac{v - u^a}{(v - u^a)^2 + w^2}\, du$$
$$- \int_{n_1}^{n_1+1} \frac{v - u^a}{(v - u^a)^2 + w^2}\, du + \frac{v - n_1{}^a}{(v - n_1{}^a)^2 + w^2} + \frac{v - (n_1 + 1)^a}{[v - (n_1 + 1)^a]^2 + w^2}.$$

Sind schliesslich $v - w$ wie $v + w$ positiv, und

$$n_1{}^a < v - w < (n_1 + 1)^a \qquad n_2{}^a < v + w < (n_2 + 1)^a$$

so wird

$$\int_0^\infty \frac{v - u^\alpha}{(v - u^\alpha)^2 + w^2}\,du - \int_{n_1}^{n_1+1} \frac{v - u^\alpha}{(v - u^\alpha)^2 + w^2}\,du - \int_{n_2}^{n_2+1} \frac{v - u^\alpha}{(v - u^\alpha)^2 + w^2}\,du + \frac{v - n_1{}^\alpha}{(v - n_1{}^\alpha)^2 + w^2}$$
$$+ \frac{v - (n_1 + 1)^\alpha}{[v - (n_1 + 1)^\alpha]^2 + w^2} > \mathrm{P} >$$
$$\int_0^\infty \frac{v - u^\alpha}{(v - u^\alpha)^2 + w^2}\,du - \int_{n_1}^{n_1+1} \frac{v - u^\alpha}{(v - u^\alpha)^2 + w^2}\,du - \int_{n_2}^{n_2+1} \frac{v - u^\alpha}{(v - u^\alpha)^2 + w^2}\,du + \frac{v}{v^2 + w^2}$$
$$+ \frac{(v - n_2{}^\alpha)}{(v - n_2{}^\alpha)^2 + w^2} + \frac{v - (n_2 + 1)^\alpha}{[v - (n_2 + 1)^\alpha]^2 + w^2}.$$

In jedem Falle ist $\mathrm{P} + i\,.\,\mathrm{Q}$ darstellbar durch $\int_0^\infty \frac{du}{v + iw - u^\alpha}$ und eine Anzahl Zusatzglieder. Ist $w \neq 0$, so ist das Integral identisch $= (v + i.w)^{\frac{1}{\alpha}-1}\,.\,\mathrm{C}_\alpha$. Somit wird schliesslich eine Beziehung der Form erhalten

$$(\theta) \quad (v + i\,.\,w)^{\frac{1}{\alpha}-1}\,.\,\mathrm{C}_\alpha + \mathrm{A} > \frac{\psi_\alpha'(v + i\,.\,w)}{\psi_\alpha(v + i\,.\,w)} > (v + i\,.\,w)^{\frac{1}{\alpha}-1}\,.\,\mathrm{C}_\alpha + \mathrm{B},$$

wo A und B gewisse Zusatzglieder darstellen, $w \neq 0$, und die Zeichen $>$ sich sowohl auf reellen wie imaginären Bestandteil der betreffenden Grössen beziehen.

Sind x_2 und x_1 die Endpunkte einer Geraden $\mathfrak{G}$, innerhalb deren, wenn man sich von x_1 nach x_2 bewegt, in $dx = dv + i\,.\,dw$, dv wie dw positiv sind, so kann man obige Ungleichung mit $dv + i\,.\,dw$ multiplicieren und zwischen x_1 und x_2 integrieren. Dadurch wird

$$\alpha\,.\,(x_2{}^{\frac{1}{\alpha}} - x_1{}^{\frac{1}{\alpha}})\,\mathrm{C}_\alpha + \int_{(\mathfrak{G})} \mathrm{A}\,(dv + idw) > \log\frac{\psi_\alpha(x_2)}{\psi_\alpha(x_1)} > \alpha\,.\,(x_2{}^{\frac{1}{\alpha}} - x_1{}^{\frac{1}{\alpha}})\,\mathrm{C}_\alpha$$
$$+ \int_{(\mathfrak{G})} \mathrm{B}\,(dv + idw).$$

Ist dagegen auf der Geraden dv positiv, aber dw negativ, so muss man in der Ungleichung (θ) i durch $-i$ ersetzen und erhält dann eine Ungleichung derselben Art wie im ersten Fall.

Die Grössen $\int_{(\mathfrak{G})} \mathrm{A}\,(dv + i\,dw)$ und $\int_{(\mathfrak{G})} \mathrm{B}\,(dv + idw)$ sind nun sämmtlich der Art, dass, wenn x_2 beliebig wächst, die Integrale kleiner bleiben als angebbare Multipla des Logaritmen von $w^2 + v^2$. Denn da wir uns auf einer Geraden ins ∞ bewegen sollen, so ist w asymptotisch ein Multiplum von v, und alle Grössen, die A und B ausmachen, sind Multipla von $1/v$. Wir kommen daher leicht zu dem Resultat: Bewegt sich x auf einer Geraden nach ∞, so ist der asymptotische Ausdruck für $\psi_\alpha(x)$

$$\psi_\alpha(x) = x^\lambda e^{x^{\frac{1}{\alpha}}\,.\,\alpha\,.\,\mathrm{C}_\alpha},$$

wo λ eine Function von x ist, die immerfort zwischen angebbaren endlichen Grenzen bleibt. Voraussetzung ist nur, dass die Gerade nicht mit der positiven Axe coincidiert (oder ihr parallel laufend in demselben Sinne ins ∞ geht).

Auch Grenzübergänge anderer Art lassen sich durch obige Ungleichungen durchführen. Für die Zwecke dieser Arbeit ist die genaue Ausführung der angeregten Untersuchungen irrelevant, und wir wollen daher nur noch einige Worte über die Borelschen Functionen, deren Wurzeln einen Bonnetschen Typus bilden, zufügen.

Die Function $f(x)$ habe die Wurzeln $a_n = (\log n)^\mu$.

Ist μ eine rationale Zahl, $= \frac{a}{b}$, so setzen wir, wenn p die kleinste ganze in μ enthaltene Zahl und ρ die a^{te} Einheitswurzel bezeichnet,

$$g_n(x) = \frac{\left(e^{x^{\frac{b}{a}}} + e^{x^{\frac{b}{a}}} \cdot \rho + e^{x^{\frac{b}{a}}} \cdot \rho^2 + \ldots + e^{x^{\frac{b}{a}}} \cdot \rho^{a-1}\right)}{x^{p-1}} = \mathrm{H}(x),$$

die eventuell auftretenden negativen Potenzen von x vernachlässigend. Die ganze Function $\mathrm{H}(x)$ ist dann offenbar der Art, dass $\Sigma \frac{1}{\mathrm{H}(a_n)}$ divergiert, aber $\Sigma \frac{1}{\mathrm{H}(a_n) \cdot a_n}$ convergiert.

$$\frac{f'(x)}{f(x)} = \Sigma_1^\infty \frac{\mathrm{H}(x)}{\mathrm{H}(a_n) \cdot x - a_n}$$

definiert eine Borelsche Function $f(x)$, denn es lässt sich leicht zeigen (in der Art wie es am Ende des 2^{ten} Capitels ausgeführt wird), dass der Maximalwert von $f(x)$, wenn $|x| = r$, $< e^{e^{r^{\frac{b}{a}}} \cdot c}$, wo c eine angebbare Constante. $\frac{f'(x)}{\mathrm{H}(x) \cdot f(x)}$ ist eine der Reihen, wie sie im Satze V untersucht worden sind, wenn man x negativ wählt. Für diesen Fall kann man auch ganz leicht den Calcül C anwenden.

Ist μ keine rationale Zahl, aber > 1, so setzen wir

$$\mathrm{H}(x) = \psi_\mu(-x).$$

Nach den asymptotischen Beziehungen, welche für $\psi_\mu(-x)$ entwickelt waren, divergiert $\Sigma_1^\infty \frac{1}{\psi_\mu(-a_n)}$, aber $\Sigma_1^\infty \frac{1}{\psi_\lambda(-a_n) \cdot a_n}$ convergiert. Im übrigen behalten die obigen Bemerkungen ihre Gültigkeit.

Auch wenn $\mu < 1$, kann man ähnlich verfahren, indem man $\mathrm{H}(x) = \psi_\mu(\rho \cdot x)$ setzt, wo $|\rho| = 1$, im übrigen eine passend gewählte Grösse, und $\psi'_\mu(x)$ die n^μ zu Nullpunkten hat, so dass $\frac{\psi_\mu'}{\psi_\mu} = \frac{x^{k-1}}{x^{(k-1)\mu}(x - n^\mu)}$, $(k-1)\,\mu < 1 < k \cdot \mu$.

Man kann ganz genau so zu Borelschen Functionen höherer Classe aufsteigen. Die so gebildeten ganzen Functionen sind interessant, weil sie mancherlei Beziehungen zur Theorie der algebraischen Differentialgleichungen mit rationalen Coefficienten haben, wie auch zu ganzen Functionen, die durch bestimmte Integrale entstehen (nach Art der Inversen der Γ-Function oder der Riemannschen Primzahl-Function). Sie sind ausserdem für die allgemeine Theorie der ganzen Functionen sehr dienlich als Vergleichsfunctionen. Sehr häufig kann man nämlich mit Hülfe einer Ungleichung, welche wir jetzt noch entwickeln werden, Theoreme allgemeiner Art zurückführen auf den besonderen Fall der Functionen, die oben betrachtet wurden.

17. Ist $\alpha_1, \alpha_2 \ldots \alpha_n \ldots$ eine Folge positiver Grössen, deren Summe convergiert, und $a_1, a_2 \ldots a_n \ldots$ eine Folge positiver Grössen, die monoton wachsend ∞ zur Grenze haben, so ist immer

$$\lim_{n=\infty} \frac{\alpha_1 a_1 + \alpha_2 a_2 + \ldots + \alpha_n a_n}{a_n} = 0.$$

Dies folgt sogleich aus der Ungleichung

$$\begin{aligned} \frac{\alpha_1 a_1 + \ldots + \alpha_n a_n}{a_n} &= \frac{\alpha_1 a_1 + \ldots + \alpha_m a_m}{a_n} + \frac{\alpha_{m+1} a_{m+1} + \ldots + \alpha_n a_n}{a_n} \\ &< \frac{\alpha_1 a_1 + \ldots + \alpha_m a_m}{a_n} + (\alpha_{m+1} + \ldots + \alpha_n) \\ &< \frac{a_m}{a_n} \Sigma_1^\infty \alpha_i + (\alpha_{m+1} + \ldots + \alpha_n), \end{aligned}$$

wenn

$$m < n,$$

in Verbindung mit $\lim\limits_{n=\infty} a_n = \infty$ und der Convergenz der Reihe.

Divergente Reihen besitzen diese Eigentümlichkeit nicht. Ist $\beta_1 + \beta_2 + \ldots + \beta_n + \ldots$ divergent, so kann man bei vorgegebener Zahl n eine Zahl p finden, so dass

$$\beta_n + \beta_{n+1} + \ldots + \beta_{n+p} > 1.$$

Wählen wir daher, für irgend eine Folge $n_1, n_2, n_3 \ldots$ immerfort und über alle Grenzen wachsender Indices die a_n so, dass

$$\begin{gathered} a_{n_1} = a_{n_1+1} = \ldots a_{n_1+p_1}, \\ a_{n_2} = a_{n_2+1} = \ldots a_{n_2+p_2} \\ \text{etc.}, \end{gathered}$$

so ist für diese Folge a_n (welche für wachsende n immerfort und über jede Grenze wachsen soll) $\lim\limits_{n=\infty} \dfrac{\beta_1 a_1 + \ldots + \beta_n a_n}{a_n}$ offenbar nicht $= 0$.

18. Sind die α_n ihrer Grösse nach geordnet, so kann man $a_n = \dfrac{1}{\alpha_n}$ wählen, und erhält dann

$$\lim_{n=\infty} n \,.\, \alpha_n = 0,$$

den Pringsheimschen Satz. Ist auch $\alpha_n \,.\, n$ eine geordnete Folge, so kann man $a_n = \dfrac{1}{\alpha_n \,.\, n}$ wählen, und es kommt $\lim\limits_{n=\infty} n \,.\, \log n \,.\, \alpha_n = 0$. Ist auch $n \log n \,.\, \alpha_n$

eine geordnete Folge, so kann man $a_n = \frac{1}{\alpha_n \,.\, n \,.\, \log n}$ setzen, was $\lim\limits_{n=\infty} n \,.\, \log n \,.\, \log \log n \,.\, a_n = 0$ ergiebt, u. s. f.

Es sei nun $r_1, r_2 \ldots r_n \ldots$ eine Folge immerfort wachsender positiver Grössen, deren Grenzwert $= \infty$. Die Folge gehöre zur h^{ten} Bonnetschen Classe, es sei also (θ) $\Sigma_1^\infty n^{-1} \,.\, (\log n)^{-1} \,.\, (\log \log n)^{-1} \ldots (\log \log \ldots h-1 \text{ mal } (n))^{-1} \,.\, r_n^{-\lambda}$ für endliche positive Werte von λ convergent. Ist dann σ der Grenzpunkt der Convergenz (der Convergenzexponent h^{ter} Classe, oder die Ordnung h^{ter} Classe) und die Reihe in $\lambda = \sigma$ noch convergent, so haben wir, wenn wir $\alpha_n = n^{-1} \,.\, (\log n)^{-1} \ldots r_n^{-\sigma}$ setzen, da die Folge der α_n, wie der $n \,.\, \alpha_n$, wie auch der $n \,.\, \log n \,.\, \alpha_n \ldots$ und schliesslich der $n \,.\, \log n \ldots \log \log (h-1) \text{ mal } n \,.\, \alpha_n$ eine monoton abnehmende ist,

$$\lim_{n=\infty} \log \log \ldots h \text{ mal } (n) \,.\, r_n^{-\sigma} = 0.$$

Dies lässt sich auch schreiben

$$r_n > \eta \,.\, (\log \log \ldots h \text{ mal } (n))^{\frac{1}{\sigma}},$$

wo n einen nur von η abhängigen Index übersteigt. Divergiert die Reihe (θ) in $\lambda = \sigma$, so ersetzen wir σ durch $\sigma + \epsilon$, wo ϵ beliebig klein; die Schlussfolgerung bleibt dann erhalten.

Offenbar ist σ der kleinste Wert, für welchen eine solche Ungleichung bestehen kann.

19. Die Betrachtungen dieses ersten Capitels waren in ihrer Mehrheit derart, dass sie mehr oder weniger interessante Ergebnisse liefern, ohne aber zu einer allgemeinen Theorie zu führen. Wie man leicht die hier angeführten Beispiele, in denen die discutierten Metoden fruchtbar sind, noch bedeutend erweitern kann, so ist es auf der anderen Seite leicht, Beispiele von Functionen zu bilden, zu welchen die hier angeführten Sätze nur in einer sehr losen Beziehung stehen. Zu einer Vertiefung der Metoden gelangt man erst durch Einführung eines neuen Begriffes. Dieser Begriff wird seine Definition und Erläuterung in den nachfolgenden Seiten finden.

CAPITEL 2.

1. Es sei $u_1 + u_2 + \ldots + u_n + \ldots$ eine convergente Summe, welche eine Function einer complexen Variabelen x definiere. Innerhalb eines Bereiches B habe u_n die singulären Punkte $a_{n,m}$. Die $a_{n,m}$ mögen in ihrer Gesammtheit eine Folge bilden, welche den einen Grenzpunkt a zulässt. In a sei jede der u_i regulär und habe dort den Wert 0. Verlangt ist, die Bedingungen aufzustellen, denen eine in B verlaufende und in a mündende Curve $\mathfrak{C}$ genügen muss, damit der Wert der Summe $\Sigma_1^\infty u_i$ bei Annäherung auf $\mathfrak{C}$ an $x = a$ die 0 zur Grenze habe.

Dies Problem führt zur Betrachtung gewisser Bereiche, welche wie folgt construiert sind. In einen einfach zusammenhängenden Bereich B werden unendlich viele Löcher $L_1, L_2, L_3 \ldots L_n \ldots$ gebohrt. Jedes Loch L_i stelle einen einfach zusammenhängenden Bereich dar. Keines der L_i schliesse den Punkt a ein, mit wachsendem n komme L_n aber dem Punkte a beliebig nahe. Der so hergestellte Bereich L werde asymptotischer Bereich um den Punkt a genannt. Die Gesammtheit der Bereiche L_i heisse der asymptotische Nichtbereich um a.

Im folgenden werden wir das oben angegebene Problem nicht in seiner Allgemeinheit angreifen, wir werden jedoch besondere Fälle von genügender Wichtigkeit und Ausdehnung in Angriff nehmen. Es wird sich zeigen, dass die Curve $\mathfrak{C}$ der Bedingung genügen muss, innerhalb eines asymptotischen Bereiches zu verlaufen, dessen asymptotischer Nichtbereich die $a_{n,m}$ einschliesst. Die Ausdehnung der Löcher L_i wird dabei nicht gleichgültig sein, sondern wird erst durch bestimmte Verfahrungsarten festzustellen sein.

Ohne von vornherein die Begriffe zu sehr complicieren zu wollen, werden wir doch noch zwei Bedingungen erwähnen, welchen die asymptotischen Bereiche genügen sollen. Schlägt man um a mit einem sehr kleinen Radius ρ einen Kreis, so muss die Summe des Flächeninhalts aller in diesem Kreise liegenden Löcher L_i im Verhältnis zum Flächeninhalt des Kreises mit ρ beliebig klein werden. Und ferner muss eine Contour C um a innerhalb des asymptotischen Bereiches möglich sein, deren Maximalabstand von a kleiner als $h\rho$, deren Minimalabstand von a grösser als $k\rho$ ist, wo h, k zwei beliebig vorgegebene Grössen kleiner als 1.

Es geht hieraus hervor, dass eine endliche Anzahl asymptotischer Bereiche um a einen asymptotischen Bereich um a gemein haben.

2. Im folgenden nehmen wir an, dass der Punkt a im Unendlichen liege, construieren für eine besondere Gattung von Reihen $\Sigma_1^\infty u_n$ den oben erwähnten asymptotischen Bereich, und zeigen seine Bedeutung für die Characterisierung der durch $\Sigma_1^\infty u_n$ definierten Function.

Es sei $u_n = \frac{1}{x - a_n}$. $\Sigma \frac{1}{|a_n|}$ sei convergent und $\frac{|a_n|}{n}$ sei $\geqq \frac{|a_{n'}|}{n'}$, wenn $n > n'$. Der asymptotische Bereich, in welchem $\Sigma_1^\infty \frac{1}{x - a_n}$ für $\lim x = \infty$ sich der 0 nähert, wird dann wie folgt construiert. Wir schlagen um a_n als Centrum einen Kreis L_n mit dem Radius r_n. $\lim\limits_{n=\infty} r_n$ sei $= \infty$, jedoch $\lim\limits_{n=\infty} \frac{r_n}{a_{n+1} - a_n} = 0$. Die Gesammtheit der Kreise L_n bildet den asymptotischen Nichtbereich.

Dies ergiebt sich leicht aus den Voraussetzungen. Es sei

$$f(x) = \Sigma_1^\infty \frac{1}{x - a_n},$$

alsdann ist

$$|f(x)| \leqq \Sigma_1^\infty \frac{1}{|x - a_n|}.$$

Es sei $|x|$ irgend eine Zahl grösser als $|a_m|$ und kleiner als $|a_{m+1}|$. Dann ist

$$\Sigma_1^{\infty}\frac{1}{|x-a_n|} = \Sigma_1^{m-1}\frac{1}{|x-a_n|} + \frac{1}{|x-a_m|} + \frac{1}{|x-a_{m+1}|} + \Sigma_{m+2}^{\infty}\frac{1}{|x-a_n|}.$$

Ferner ist

$$\Sigma_1^{m-1}\frac{1}{|x-a_n|} < \Sigma_1^{m-1}\frac{1}{|x|-|a_n|} < \Sigma_1^{m-1}\frac{1}{|a_m|-|a_n|}.$$

Setzen wir $a_n = n\,.\,b_n$, so ist $|b_n|$ nach Voraussetzung eine Folge niemals abnehmender Grössen. Somit ist

$$\Sigma_1^{m-1}\frac{1}{|a_m|-|a_n|} < \Sigma_1^{m-1}\frac{1}{m\,.\,|b_m|-n\,.\,|b_m|} < \frac{1}{|b_m|}\,.\,\Sigma_1^{m-1}\frac{1}{m-n} < \frac{\log m}{|b_m|}.$$

Nach der in 18 erörterten Erweiterung des Pringsheimschen Satzes ist dieser Ausdruck mit wachsendem m kleiner als jede beliebige Grösse. Mithin *a fortiori*

$$\lim_{m=\infty}\Sigma_1^{m-1}\frac{1}{x-a_n} = 0.$$

Andererseits ist

$$\Sigma_{m+2}^{\infty}\frac{1}{|x-a_n|} < \Sigma_{m+2}^{\infty}\frac{1}{|a_n|-|x|} < \Sigma_{m+2}^{\infty}\frac{1}{|a_n|-|a_{m+1}|}$$

$$= \Sigma_{m+2}^{\infty}\frac{1}{|a_n|} + \Sigma_{m+2}^{\infty}\frac{|a_{m+1}|}{(|a_n|-|a_{m+1}|)|a_n|} < \Sigma_{m+2}^{\infty}\frac{1}{|a_n|} + \Sigma_{m+2}^{\infty}\frac{(m+1)\,|b_{m+1}|}{(n\,|b_{m+1}|-(m+1)\,|b_{m+1}|)\,n\,.\,|b_{m+1}|},$$

da ja $|b_n| \geqq |b_{m+1}|$ war,

$$= \Sigma_{m+2}^{\infty}\frac{1}{|a_n|} + \frac{1}{|b_{m+1}|}\,\Sigma_{m+2}^{\infty}\frac{m+1}{n\,(n-m-1)}.$$

Es ist aber $\dfrac{m+1}{n\,(n-m-1)}$ identisch $= \dfrac{1}{n-m-1} - \dfrac{1}{n}$,

somit $\Sigma_{m+2}^{\infty}\dfrac{m+1}{n\,(n-m-1)}$ asymptotisch $= \log m$.

Mithin schliesslich

$$\Sigma_{m+2}^{\infty}\frac{1}{|x-a_n|} < \Sigma_{m+2}^{\infty}\frac{1}{|a_n|} + \frac{\log m}{|b_{m+1}|}.$$

Aus der Convergenz von $\dfrac{1}{|a_n|}$ und der bereits angewandten Erweiterung des Pringsheimschen Satzes folgt demnach

$$\lim_{m=\infty}\Sigma_{m+2}^{\infty}\frac{1}{|x-a_n|} = 0\,.$$

Liegt x im asymptotischen Bereich, so ist

$$|x - a_m| \geqq r_m,$$
$$|x - a_{m+1}| \geqq r_{m+1}.$$

Auch war
$$\lim_{m=\infty} r_m = \infty .$$

Daraus geht dann zur Evidenz hervor, dass, wenn x im asymptotischen Bereiche sich dem Punkte $x = \infty$ annähert (d. h $|x|$ von irgend einer angebbaren Lage von x auf der Annäherungscurve niemals abnimmt),

$$\lim_{x=\infty} f(x) = 0 . \qquad \text{Q. e. d.}$$

3. Ist
$$\phi(x) = \Pi_1^\infty \left(1 - \frac{x}{a_n}\right), \text{ so ist}$$

$$\frac{\phi'(x)}{\phi(x)} = f(x).$$

Einen wie oben konstruierten asymptotischen Bereich wollen wir L nennen. Von einer ganzen Function F(x) (des genre 0), deren logaritmischer Differentialquotient in einem asymptotischen Bereich $\mathfrak{A}$ sich der 0 nähert, wollen wir sagen, sie *gehöre* zu dem asymptotischen Bereiche $\mathfrak{A}$. Alsdann können wir zeigen, dass zu L alle ganzen Functionen gehören, die sich in der Form schreiben lassen

$$c_0 . \phi(x) + c_1 . \phi'(x) + c_2 . \phi''(x) + \ldots + c_n . \phi^{(n)}(x) + \ldots ,$$

wenn die c der Bedingung Ω genügen, dass die Reihe $\Sigma\, c_n . u^n$ einen von 0 verschiedenen Convergenzradius hat und $c_0 \neq 0$ ist.

Zunächst wollen wir nachweisen, dass, wenn die c der Bedingung Ω genügen, die Reihe $\Sigma c_n . \phi^{(n)}(x)$ eine ganze Function $\psi(x)$ von x darstellt. Nachdem wollen wir zur Evidenz bringen, dass $\lim\limits_{x=\infty} \dfrac{\psi'(x)}{\psi(x)} = 0$, in L, und dass $\psi(x)$ eine Function des genre 0 ist, deren Nullpunkte im Nichtbereich von L liegen.

Nach einem Satze von Poincaré ist die n^{te} Wurzel aus dem $n!$fachen des n^{ten} Coeffizienten der Taylorschen Entwickelung einer Function des genre 0 in der Grenze für $n = \infty$ beliebig klein. Mit anderen Worten, es ist

$$|\phi^{(n)}(x)| < \delta^n ,$$

wo δ eine beliebig vorgegebene Zahl ist, wenn nur n genügend gross gewählt wird. Der Wert von x ist dabei beliebig. Somit convergiert $\Sigma_1^\infty c_n . \phi^{(n)}(x)$ für jeden endlichen Wert von x, und stellt demnach eine ganze Function von x dar.

Es sei $\gamma_1, \gamma_2 \ldots \gamma_n \ldots$ eine Folge positiver Grössen, derart dass $\Sigma \dfrac{1}{\gamma_n}$ convergiert. Es sei ferner

$$\vartheta(x) = \Pi_1^\infty \left(1 + \frac{x}{\gamma_n}\right) = \Sigma_1^\infty k_n . x^n .$$

Alsdann ist $k_1, \frac{2\,.\,k_2}{k_1}, \frac{3\,.\,k_3}{k_2}, \;.\;.\;.\; \frac{n\,.\,k_n}{k_{n-1}}, \;.\;.\;.$ eine Folge positiver immerfort abnehmender Grössen, deren Grenzwert die 0 ist. Denn es ist identisch

$$\frac{\vartheta'(x)}{\vartheta(x)} = \Sigma \frac{1}{x+\gamma_n},$$

$$\frac{d}{dx}\frac{\vartheta'(x)}{\vartheta(x)} = \Sigma - \frac{1}{(x+\gamma_n)^2} = \frac{\vartheta''(x)}{\vartheta(x)} - \left(\frac{\vartheta'(x)}{\vartheta(x)}\right)^2.$$

Mithin für jeden positiven Wert von x

$$\frac{\vartheta''(x)}{\vartheta(x)} < \left(\frac{\vartheta'(x)}{\vartheta(x)}\right)^2 \qquad \text{oder} \qquad \frac{\vartheta'(x)}{\vartheta(x)} > \frac{\vartheta''(x)}{\vartheta'(x)}.$$

Nach einem Satze von HERMITE ist

$$\vartheta'(x) = \vartheta'(0)\,.\,\Pi\left(1+\frac{x}{\gamma_n'}\right),$$

wo die γ_n' den γ_n zwischengelagert sind. Es ist somit auch

$$\frac{\vartheta''(x)}{\vartheta'(x)} > \frac{\vartheta'''(x)}{\vartheta''(x)} > \;.\;.\;.\; > \frac{\vartheta^{(n+1)}(x)}{\vartheta^{(n)}(x)}.$$

Nach dem Satze von POINCARÉ zeigt sich, dass der Grenzwert der Folge 0 ist. Setzen wir noch $x = 0$, so haben wir damit die Behauptung verificiert.

Es sei nun angesetzt $\gamma_n = |x - a_n|,$

$$\vartheta(t) = \Pi_1^\infty\left(1+\frac{t}{|x-a_n|}\right) = \Sigma_1^\infty k_n\,.\,t^n.$$

Es ist identisch

$$\Pi_1^\infty\left(1+\frac{t}{x-a_n}\right) = \frac{\phi(x+t)}{\phi(x)} = \Sigma_1^\infty \frac{\phi^{(n)}x}{n!\,\phi(x)}\,.\,t^n.$$

Danach ist evident

$$\left|\frac{\phi^{(n)}x}{n!\,\phi\,x}\right| < k_n \qquad \text{und} \qquad \left|\frac{\phi^{(n)}(x)}{\phi(x)}\right| \leqq k_1^n$$

für jeden Wert von n und x. Mithin ist

$$\left|\frac{\psi(x)}{\phi(x)}\right| \leqq \Sigma\,|c_n|\,.\,k_1^n.$$

Nähert sich innerhalb L x dem Punkte $x = \infty$, so war aber

$$\lim_{x=\infty} k_1 = 0.$$

Daraus kommt dann leicht, dass innerhalb L

$$\lim_{x=\infty}\left|\frac{\psi(x)}{\phi(x)}\right| = c_0.$$

Auch ist
$$\lim_{x=\infty}\left|\frac{\psi'(x)}{\phi(x)}\right| = 0.$$

Mithin
$$\lim_{x=\infty}\frac{\psi'(x)}{\psi(x)} = 0, \quad \text{da ja } c_0 \neq 0.$$

Ein Satz, der, wie es scheint, von Hermite und Hurwitz unabhängig von einander aufgefunden wurde, besagt folgendes: Ist innerhalb eines einfach zusammenhängenden Bereiches B eine Function u eindeutig und ohne wesentliche Singularitäten, und ist auf dem Rande von B überall $|u| < 1$, so nimmt u innerhalb B genau so oft den Wert 1 an als die Function dort den Wert ∞ annimmt. Der Beweis ruht darauf, dass $\int^{(R)}\frac{du}{1-u} = 0$, da bei Beschreibung des Randes R, wegen $|u| < 1$, $1-u$ eine Contour beschreibt, die den Punkt $u = 0$ ausschliesst.

Wir beschreiben nun um jeden der Kreise L_n eine Contour C_n, die innerhalb des Bereiches L liegt. Wählen wir n genügend gross, so ist wegen
$$\lim_{n=\infty}\frac{\psi(x) - c_0\,.\,\phi(x)}{\phi(x)} = 0$$
auf C_n
$$\frac{\psi(x) - c_0\,.\,\phi(x)}{\phi(x)} < \delta.$$
Somit hat die Gleichung
$$\frac{\psi(x) - c_0\,.\,\phi(x)}{\phi(x)} = -\,c_0$$
innerhalb C_n genau so viel Wurzeln, als die linke Seite dort den Wert ∞ annimmt, d. h. genau eine. Die Wurzeln von $\psi(x)$ liegen daher im Nichtbereich von L, und zwar je eine ihrer Wurzeln in L_n, sobald n einen angebbaren Wert c übersteigt, der nur von den Werten $c_0, c_1, c_2 \ldots c_n, \ldots$ abhängig ist. Es ist danach $\psi(x)$ offenbar eine Function des genre 0, denn die Möglichkeit der Existenz eines störenden Factor der Form $e^{q(x)}$ ist nach den entwickelten Ungleichungen überhaupt ausgeschlossen.

4. Wir werden jetzt daran gehen, den allgemeineren Satz zu erweisen, den wir den Satz S nennen wollen.

Satz S: Ist $f(x) = \Sigma_1^\infty \frac{c_n}{x-a_n}$, wo die c_n und a_n der Bedingung Ω unterworfen sind, dass

1) $\Sigma_1^\infty \frac{c_n}{a_n}$ absolut convergiert,

2) $|a_n| \geqq |a_{n'}|$ wenn $n \geqq n'$,

3) $\lim_{n=\infty} a_n = \infty$,

so giebt es einen asymptotischen Bereich, in welchem
$$\lim_{x=\infty} f(x) = 0.$$

Wollte man den Beweis des Satzes S in der Art und Weise führen, wie vorhin den Beweis des in 2 gegebenen Satzes, so würde man zu keinem Ergebnis gelangen. Der zu construierende asymptotische Bereich hängt ganz und gar von der Lagerung der a_n und c_n ab, und über diese kann man ohne besondere Annahmen ausserordentlich wenig sagen. Was sich allgemein sagen lässt, ist wenig mehr als in der Grenzbeziehung $\lim\limits_{n=\infty} \dfrac{|c|_1 + |c_2| + \ldots + |c_n|}{a_n} = 0$ enthalten ist, die aus den Betrachtungen des Art. 18 folgt; denn es lässt sich leicht zeigen, dass man Folgen c_n, a_n construieren kann, welche den Bedingungen Ω genügen und für die

$$\lim_{n=\infty} \frac{|c_1| + |c_2| + \ldots + |c_n|}{a_n} \,.\, r_n \quad \text{nicht mehr} = 0,$$

wo r_n irgend eine vorgegebene über alle Grenzen wachsende Folge darstellt.

Es ist
$$|f(x)| < \Sigma_1^\infty \frac{|c_n|}{|x - a_n|} < \Sigma_1^\infty \frac{|c_n|}{\big|\,|x| - |a_n|\,\big|}.$$

Wir wollen zunächst zeigen, dass es eine Folge von Werten $|x|$ giebt, so dass $\lim |x| = \infty$ und $\lim\, \Sigma_1^\infty \dfrac{|c_n|}{\big|\,|x| - |a_n|\,\big|} = 0$. In dem Ausdruck $\Sigma_1^\infty \dfrac{|c_n|}{\big|\,|x| - |a_n|\,\big|}$ fassen wir alle Glieder zusammen, für welche der Modul des Pols $|a_n|$ denselben Wert hat. Dadurch erhalten wir einen Ausdruck genau derselben Art. Da wir zunächst nur von positiven Grössen sprechen werden, schreiben wir bis auf weiteres statt $|a_n|$, $|c_n|$ und $|x|$ kurz a_n, c_n, x.

Es sei x grösser als a_m, jedoch kleiner als a_{m+1}. $a_{m+1} - a_m$, nach obigem eine positive von 0 verschiedene Grösse, sei mit b_m bezeichnet und

$$x = a_m + \eta \,.\, b_m = a_{m+1} - \eta' \,.\, b_m$$

gesetzt, wo η, η' irgend 2 positive von 0 verschiedene Grössen, die der Beziehung genügen

$$\eta + \eta' = 1.$$

Es ist nun
$$\Sigma_1^\infty \frac{c_n}{|x - a_n|} = \Sigma_1^m \frac{c_n}{x - a_n} + \Sigma_{m+1}^\infty \frac{c_n}{a_n - x},$$

$$\Sigma_1^m \frac{c_n}{x - a_n} = \frac{c_m}{\eta \,.\, b_m} + \frac{c_{m-1}}{a_m - a_{m-1} + \eta \,.\, b_m} + \frac{c_{m-2}}{a_m - a_{m-2} + \eta \,.\, b_m} + \ldots,$$

$$\eta \,.\, \Sigma_1^m \frac{c_n}{x - a_n} = \frac{c_m}{a_{m+1} - a_m} + \frac{c_{m-1}}{\frac{1}{\eta}(a_m - a_{m-1}) + b_m} + \frac{c_{m-2}}{\frac{1}{\eta}(a_m - a_{m-2}) + b_m}$$

$$< \frac{c_m}{a_{m+1} - a_m} + \frac{c_{m-1}}{a_{m+1} - a_{m-1}} + \frac{c_{m-2}}{a_{m+1} - a_{m-2}} + \ldots = p_m.$$

Ähnlich ist

$$\Sigma_{m+1}^{\infty} \frac{c_n}{a_n - x} = \frac{c_{m+1}}{\eta' \,.\, b_m} + \frac{c_{m+2}}{a_{m+2} - a_{m+1} + \eta' \,.\, b_m} + \frac{c_{m+3}}{a_{m+3} - a_{m+1} + \eta' \,.\, b_m} + \dots$$

und $$\eta' \,.\, \Sigma_{m+1}^{\infty} \frac{c_n}{a_n - x} < \frac{c_{m+1}}{a_{m+1} - a_m} + \frac{c_{m+2}}{a_{m+2} - a_m} + \frac{c_{m+3}}{a_{m+3} - a_m} + \dots = q_m.$$

Wenn es Indices m giebt, für welche p_m und q_m kleiner wird als eine beliebig vorgegebene Grösse, so wird es auch eine Folge über alle Grenzen wachsender Grössen $|x|$ geben, für welche $f(x)$ kleiner als eine beliebig vorgegebene Grösse wird.

$$p_m \text{ war } = \sum_0^{m-1}{}_h \frac{c_{m-h}}{a_{m+1} - a_{m-h}} \quad \text{und} \quad q_m = \sum_1^{\infty}{}_h \frac{c_{m+h}}{a_{m+h} - a_m}.$$

Wenn $a_{m+h'} > 2 \,.\, a_m$ wird, so ist

$$\sum_{h'}^{\infty}{}_h \frac{c_{m+h'}}{a_{m+h'} - a_m} < 2 \,.\, \sum_{h'}^{\infty}{}_h \frac{c_{m+h'}}{a_{m+h'}},$$

und diese Summe wird für genügend grosse m kleiner als eine beliebig vorgegebene Grösse, da nach Voraussetzung die Summe $\sum_1^{\infty}{}_h \frac{c_h}{a_h}$ convergiert. Somit setzen wir

$$q_{m'} = \sum_1^{h'}{}_h \frac{c_{m+h}}{a_{m+h} - a_m},$$

wo h' definiert ist als der kleinste Index, für den $a_{m+h'} \geqq 2 \,.\, a_m$, und stellen uns nun die Aufgabe, zu erweisen, dass die 0 einer der Grenzpunkte der Folge $p_m + q_m'$ ist. Der Beweis ist möglich nach der hier befolgten Metode.

Zunächst machen wir eine Voraussetzung V über die Folge der a_n, welche aber die Allgemeinheit in keiner Weise einschränkt. Wir nehmen an, dass $a_{n+1} - a_n$ kleiner sei als eine Constante ϵ, und dass $\lim\limits_{n=\infty} a_{n+1} - a_n = 0$. Trifft für die thatsächlich gegebene Folge diese Voraussetzung nicht zu, so steht es uns frei, der Function $f(x)$ soviel Glieder der Form $\frac{c}{x - a}$ zuzusetzen, dass für die so condensierte Folge der a die Voraussetzung zutrifft. Die c der Zusatzglieder wählen wir sämmtlich $= 0$, oder, wenn wir wollen, von 0 verschieden und so klein, dass die Summe $\Sigma \frac{c}{a}$ immer noch convergiert. Ist für die erweiterte Function der zu beweisende Satz richtig, so ist er es auch gewiss für $f(x)$.

Wir definieren nun vier Functionen einer reellen positiven Variabelen ξ durch die Gleichungen

$$\begin{aligned} \text{A}(\xi) &= \Sigma_1^{\infty} (a_{n+1} - a_n)\, \xi^{a_{n+1}}, \\ \text{B}(\xi) &= \Sigma_1^{\infty} c_n \,.\, \xi^{a_n}, \\ \text{C}(\xi) &= \Sigma_1^{\infty} p_n \,.\, (a_{n+1} - a_n) \,.\, \xi^{a_{n+1}}, \\ \text{D}(\xi) &= \Sigma_1^{\infty} q_n' (a_{n+1} - a_n) \,.\, \xi^{a_{n+1}}. \end{aligned}$$

Diese vier Functionen, wie sich sehr bald zeigen wird, sind durch obige Reihen erklärt für die Werte $\xi < 1$.

Es handelt sich nun um die asymptotische Darstellung der vier Functionen bei $\xi = 1$. Der Kürze wegen werden wir statt des häufig wiederkehrenden

$$\lim_{\xi = 1} \frac{u}{v} = 1$$

einfacher schreiben $u \equiv v$. Ferner werden wir einen öfter wiederkehrenden Prozess kurz $\mathfrak{P}$ nennen. Der Prozess ist auf $\mathrm{B}(\xi)$ angewandt der folgende. Es ist

$$\mathrm{B}(\xi) = \Sigma_1^\infty c_n \,.\, \xi^{a_n}, \quad \text{also, wenn } \xi < 1,$$

$$\Sigma_1^\infty c_n \,.\, \xi^{\mathrm{E}(a_n)} > \mathrm{B}(\xi) > \Sigma_1^\infty c_n \,.\, \xi^{\mathrm{E}(a_n)+1}\,.$$

Nun fassen wir alle Glieder zusammen, die mit demselben Exponenten ξ^m multipliciert sind, und nennen den resultierenden Coefficienten von ξ^m c_m'.

$$\Sigma_1^\infty c_m' \,.\, \xi^m = \mathrm{L}(\xi).$$

Es ist also
$$\mathrm{L}(\xi) > \mathrm{B}(\xi) > \xi \,.\, \mathrm{L}(\xi)$$
und
$$\mathrm{B}(\xi) \equiv \mathrm{L}(\xi)\,.$$

Offenbar ist $\Sigma_1^\infty \frac{c_m'}{m}$ convergent, daher auch

$$\int_0^1 \mathrm{L}(\xi) \,.\, d\xi\,.$$

Auf $\mathrm{A}(\xi)$ angewandt liefert $\mathfrak{P}$ nach Theorem T und V

$$\mathrm{A}(\xi) \equiv \frac{1}{1-\xi}.$$

Ehe wir $\mathfrak{P}$ auf $\mathrm{C}(\xi)$ und $\mathrm{D}(\xi)$ anwenden können, müssen wir dieselben transformieren. Wir ordnen $\mathrm{C}(\xi)$ und $\mathrm{D}(\xi)$ um, indem wir alle Glieder, die mit demselben Coefficienten c_n multipliciert sind, zusammenfassen. Dadurch wird

$$\mathrm{C}(\xi) = \Sigma_1^{\infty}{}_n\, c_n \, \Sigma_1^{\infty}{}_h \frac{a_{n+h} - a_{n+h-1}}{a_{n+h} - a_n} \,.\, \xi^{a_{n+h}}\,.$$

Setzen wir
$$\Sigma_1^{\infty}{}_h \frac{a_{n+h} - a_{n+h-1}}{a_{n+h} - a_n} \,.\, \xi^{a_{n+h}} = \xi^{a_n} \,.\, \mathrm{E}_n(\xi), \quad \text{so ist}$$

$$\xi \,.\, \frac{\partial}{\partial\xi}(\mathrm{E}_n) = \Sigma_1^{\infty}{}_h (a_{n+h} - a_{n+h-1}) \,.\, \xi^{a_{n+h} - a_n}\,,$$

was nach $\mathfrak{P}$ und Theorem T und V sich verhält wie $\frac{1}{1-\xi}$. Mithin

$$\mathrm{E}_n \equiv \log\left(\frac{1}{1-\xi}\right), \quad \text{und nach Theorem T}$$

$$\mathrm{C}(\xi) \equiv \mathrm{B}(\xi) \,.\, \log\left(\frac{1}{1-\xi}\right).$$

$\int_0^1 \frac{C(\xi)}{\log\left(\frac{1}{1-\xi}\right)} d\xi$ ist also convergent. Wir werden nachweisen, dass dasselbe richtig ist für $\int_0^1 \frac{D(\xi)}{\log\left(\frac{1}{1-\xi}\right)} d\xi$. Die Notwendigkeit dieses Nachweises war auch der Grund, warum wir statt der q_n die q_n' einführen mussten; denn $\int_0^1 \frac{d\xi \,.\, \Sigma_1^\infty q_n (a_{n+1} - a_n) \xi^{a_{n+1}}}{\log\left(\frac{1}{1-\xi}\right)}$ wäre im allgemeinen nicht mehr convergent.

Es ist, wenn wir in $D(\xi)$, wie es vorhin bei $C(\xi)$ gethan wurde, die Glieder zusammenfassen, welche zu demselben Coefficienten c_n gehören,

$$D(\xi) = \Sigma_1^\infty{}_n \, c_n \, \Sigma_0^k{}_h \, \frac{a_{n-h+1} - a_{n-h}}{a_n - a_{n-h}} \, \xi^{a_{n-h+1}}.$$

Dabei ist k die grösste ganze Zahl, für welche die Ungleichung gilt

$$a_{n-k} > \tfrac{1}{2} \,.\, a_n,$$

also für jedes gegebene n bestimmt. Nun ist evident, nach V,

$$\Sigma_0^k{}_h \, \frac{a_{(n-h+1)} - a_{n-h}}{a_n - a_{n-h}} \,.\, \xi^{a_{n-h+1}} < \Sigma_0^k{}_h \, \frac{a_{n-h+1} - a_{n-h}}{a_n - a_{n-h}} \,.\, \xi^{a_{n-h}}, \qquad \text{jedoch}$$

$$> x^\epsilon \, \Sigma_0^k{}_h \, \frac{a_{n-h+1} - a_{n-h}}{a_n - a_{n-h}} \,.\, \xi^{a_{n-h}}.$$

Andererseits ist

$$-\xi \,.\, \frac{\partial}{\partial \xi}\left[\xi^{-a_n} \,.\, \Sigma_0^k{}_h \, \frac{a_{n-h+1} - a_{n-h}}{a_n - a_{n-h}} \,.\, \xi^{a_{n-h}}\right] = \Sigma_0^k{}_h \, (a_{n-h+1} - a_{n-h}) \, \xi^{a_{n-h}},$$

somit nach $\mathfrak{P}$ und V

$$\left[\xi^{-a_n} \,.\, \Sigma_0^k{}_h \, \frac{a_{n-h+1} - a_{n-h}}{a_n - a_{n-h}} \,.\, \xi^{a_{n-h+1}}\right] \equiv \frac{\xi^{-1}}{2} + \frac{\xi^{-2}}{3} + \frac{\xi^{-3}}{4} + \ldots + \frac{\xi^{-\frac{\nu}{2}}}{\frac{\nu}{2}},$$

$$\text{wo } \nu = E(a_n),$$

und nach $\mathfrak{P}$ und Theorem T

$$D(\xi) \equiv \Sigma_1^\infty{}_n \, c_n \left(\frac{\xi^{\nu-1}}{2} + \frac{\xi^{\nu-2}}{3} + \ldots + \frac{\xi^{\frac{\nu}{2}}}{\frac{\nu}{2}}\right) \equiv \Sigma_1^\infty{}_m \, c_m' \left(\frac{\xi^{m-1}}{2} + \frac{\xi^{m-2}}{3} + \ldots + \frac{\xi^{\frac{m}{2}}}{\frac{m}{2}}\right).$$

Es ist aber offenbar nach $\mathfrak{P}$

$$\Sigma_1^\infty{}_m \, c_m' \left(\tfrac{1}{2} + \ldots + \frac{1}{\frac{m}{2}}\right) \,.\, \xi^{m-1} < D(\xi) < \Sigma_1^\infty{}_m \, c_m' \left(\tfrac{1}{2} + \ldots + \frac{1}{\frac{m}{2}}\right) \,.\, \xi^{\frac{m}{2}}$$

oder, das Zeichen $<$ asymptotisch aufgefasst, nach Theorem T

$$\Sigma_1^\infty{}_m\, c'_m \,.\, \log m \,.\, \xi^m \leqq \mathrm{D}(\xi) \leqq \Sigma_1^\infty{}_m\, c'_m \,.\, \log m \,.\, \xi^{\frac{m}{2}}.$$

Auch ist

$$\Sigma_1^\infty{}_m\, c'_m \,.\, \xi^m \left(\xi + \frac{\xi^2}{2} + \ldots + \frac{\xi^m}{m}\right) < (\Sigma_1^\infty{}_m\, c'_m \,.\, \xi^m) \log\left(\frac{1}{1-\xi}\right), \qquad \text{also}$$

$$\Sigma_1^\infty{}_m\, c'_m \,.\, \log m \,.\, \xi^{2m} \leqq \Sigma_1^\infty{}_m\, c'_m \,.\, \xi^m \,.\, \log\left(\frac{1}{1-\xi}\right).$$

Daraus geht dann zur Evidenz hervor, dass die Operation $\int_0^1 \frac{d\xi}{\log \frac{1}{1-\xi}} \,.\, \vartheta(\xi)$, auf

$$\vartheta(\xi) = \mathrm{D}(\xi)$$

angewandt, convergent sein muss.

Wäre nun $p_n + q_n'$ für jeden Wert von n grösser als eine angebbare endliche, von Null verschiedene positive Zahl δ, so wäre

$$\Sigma_1^\infty (p_n + q_{n'})(a_{n+1} - a_n)\, \xi^{a_{n+1}} > \delta \mathrm{A}(\xi),$$

also

$$\frac{\mathrm{C}(\xi) + \mathrm{D}(\xi)}{\log\left(\frac{1}{1-\xi}\right)} > \frac{\delta \,.\, \mathrm{A}(\xi)}{\log\left(\frac{1}{1-\xi}\right)}.$$

Nun war aber

$$\int_0^1 \frac{\mathrm{C}(\xi) + \mathrm{D}(\xi)}{\log\left(\frac{1}{1-\xi}\right)}\, d\xi \quad \text{convergent, während}$$

wegen

$$\mathrm{A}(\xi) \equiv \frac{1}{1-\xi}$$

$$\int_0^1 \frac{\mathrm{A}(\xi)}{\log\left(\frac{1}{1-\xi}\right)}\, d\xi \quad \text{divergiert.}$$

Somit kann eine solche Zahl δ nicht existieren. Wie klein auch eine vorgegebene Zahl δ sei, die Folge $p_n + q_n'$ enthält immer Glieder, welche kleiner sind als δ.

Dasselbe ist auch wahr für $p_n + q_n$, also auch für p_n wie q_n individuell. Nun hatten wir aber

$$|f(x)| < \frac{p_m}{\eta} + \frac{q_m}{\eta'}.$$

Somit erhalten wir: Ist $f(x) = \Sigma_1^\infty \frac{c_n}{x - a_n}$ gegeben, so bilden wir alle Ausdrücke p_n und q_n mit Hülfe der Folgen $|c_n|$ und $|a_n|$. Wir bilden dann $p_n + q_n$ und bestimmen diejenige Reihe der Indices

$$1, n_1, n_2, n_3 \ldots n_j \ldots\ldots$$

für welche $p_{n_j} + q_{n_j}$ kleiner ist als jedes der vorhergehenden Glieder der Folge $p_n + q_n$. Diese Folge von Indices existiert nach obigem und erstreckt sich ins Unendliche. Die $|a_{n_j}|\,|a_{n_j+1}|$ nennen wir die *ausgezeichneten Intervalle.*

Die Werte von p_n nennen wir δ_j und diejenigen
von q_{n_j} . . . . . ϵ_j .

Es ist dann $\lim\limits_{j=\infty} \delta_j = 0$ und $\lim\limits_{j=\infty} \epsilon_j = 0$. Alsdann bestimmen wir irgend eine Folge positiver Zahlen

$$\eta_j,\ \eta_j'$$

derart, dass beide von 0 verschieden sind, dass

ferner
$$\eta_j + \eta_j' = 1,$$

und dass
$$\lim_{j=\infty} \frac{\delta_j}{\eta_j} + \frac{\epsilon_j}{\eta_j'} = 0\,.$$

Schlagen wir dann mit $r_j = |a_{n_j}| + \eta_j(|a_{n_j+1} - a_{n_j}|)$ Kreise C_j um den Nullpunkt, so ist für jeden Punkt x_j auf dem Kreise C_j

$$|f(x_j)| < \frac{\delta_j}{\eta_j} + \frac{\epsilon_j}{\eta_j'}$$

und
$$\lim_{j=\infty} |f(x_j)| = 0\,.$$

Alle Pole a, deren Modul grösser als $|a_{n_j}|$ und kleiner als, oder $=$ $|a_{n_{j+1}}|$ ist, fassen wir in eine Gruppe G_j zusammen. Dieselbe umgeben wir mit einer Contour L_j, die derart construiert ist, dass die Summe $\Sigma\left|\frac{c}{x-a}\right|$, erstreckt über die Punkte a der Gruppe G_j, für *alle* Punkte x ausserhalb L_j kleiner ist als die grössere der beiden Zahlen

$$\frac{\delta_j}{\eta_j} + \frac{\epsilon_j}{\eta_j'} \quad \text{und} \quad \frac{\delta_{j+1}}{\eta_{j+1}} + \frac{\epsilon_{j+1}}{\eta'_{j+1}}\,.$$

Die L_j bilden dann den asymptotischen Nichtbereich des zu construierenden asymptotischen Bereiches L. Es ist offenbar in L $\lim\limits_{x=\infty} f(x) = 0$. Dass L die beiden in 1 erwähnten Bedingungen erfüllt, erhellt aus der Thatsache, dass eine beliebige endliche Anzahl solcher Bereiche L wiederum einen solchen Bereich L gemein haben; und dass Kreise in L möglich sind, deren Radius beliebig gross wird.

5. Der Satz S lässt sich mit den bereits verwandten Hülfsmitteln erweitern auf Functionen

$$f(x) = \Sigma \frac{c_n x + c_n'}{(x-a_n)^2}\,,$$

derart dass $\Sigma \frac{c_n}{a_n}$ und $\Sigma \frac{c_n'}{a_n^2}$ absolut convergieren.

Es tritt nur an Stelle der einfachen Integration nach ξ eine zweifache Integration nach ξ. Überhaupt liesse er sich aussprechen für Summen $\Sigma_1^\infty u_n$, derart dass in $x = \infty$ $u_n = 0$, und dass u_n eine rationale Function von x ist, deren Nenner niemals einen angebbaren Grad überschreitet. Ohne Zweifel kann man ihn auch erweitern auf Summen $\Sigma_1^\infty u_n$, deren Terme u_n analytische Functionen von $\frac{1}{x}$ sind, die bei $\frac{1}{x} = 0$ regulär sich verhalten. Nur erfordert diese Erweiterung eine ungemein feine Differentiation in den Begriffsbildungen, und der Beweis dieses Satzes dürfte an Schwierigkeit den des Satzes S bedeutend überschreiten.

Der Satz S lässt sich sehr leicht nach einer anderen Richtung hin erweitern, welche für die Theorie der ganzen Functionen sehr wichtig ist. Nichts hindert uns im voranstehenden Beweis anzunehmen, dass die c_n irgendwie von x abhängen, wenn nur eine Constante c sich finden lässt, so dass für jede in Betracht kommende Lage von x $\Sigma_1^\infty \frac{|c_n|}{|a_n|} < c$. Daher kann man ohne weiteres den Satz S′ aussprechen, der lautet:

Satz S′: Ist $f(x) = \Sigma_1^\infty \frac{g_n(x)}{g_n(a_n)(x - a_n)}$, so giebt es einen asymptotischen Bereich L, in welchem

$$\lim_{x=\infty} \frac{f(x)}{\mathrm{M}(x)} = 0,$$

wo
$$\mathrm{M}(x) = \left|\frac{g_1(x)}{a_1 \cdot g_1(a_1)}\right| + \left|\frac{g_2(x)}{g_2(a_2) \cdot a_2}\right| + \left|\frac{g_3(x)}{g_3(a_3) \cdot a_3}\right| + \ldots,$$

denn dann ist die obenerwähnte Zahl $c = 1$.

Es sei nun $\frac{\mathrm{G}'(x)}{\mathrm{G}(x)} = \Sigma_1^\infty \frac{g_n(x)}{g_n(a_n) \cdot (x - a_n)}$, wo $\mathrm{G}(x)$ eine ganze Function.

$$\left|\frac{g_1(x)}{g_1(a_1)}\right| + \left|\frac{g_2(x)}{g_2(a_2)}\right| + \ldots + \left|\frac{g_n(x)}{g_n(a_n)}\right| + \ldots$$

sei wiederum mit $\mathrm{M}(x)$ bezeichnet. Integrieren wir zwischen irgend 2 Punkten x_2, x_1 eines im asymptotischen Bereiche liegenden Kreises mit Radius r um 0, so wird

$$\log \frac{\mathrm{G}(x_2)}{\mathrm{G}(x_1)} = \int_{x_1}^{x_2} \Sigma_1^\infty \frac{g_n(x)}{g_n(a_n) \cdot (x - a_n)} \cdot dx.$$

Ist $\delta \cdot \mathrm{M}(r)$ das Maximum von $\left|\frac{\mathrm{G}'(x)}{\mathrm{G}(x)}\right|$ auf jenem Kreise, so ist also

$$\left|\log \frac{\mathrm{G}(x_2)}{\mathrm{G}(x_1)}\right| < \mathrm{C} \cdot r \cdot \delta \cdot \mathrm{M}(r),$$

wo $\mathrm{C} \cdot r$ die Länge des Bogens zwischen x_1 und x_2 misst, C also eine Grösse $< 2\pi$ ist. Aus dieser Beziehung fliesst sogleich

1) Das Maximum von $|G(x)|$ auf jenem Kreise ist $< e^{2\pi \,.\, \delta \,.\, r \,.\, M(r)}$;

2) Das Minimum von $|G(x)|$ auf jenem Kreise ist $> e^{-2\pi \,.\, \delta \,.\, r \,.\, M(r)}$,

$$\text{da ja } \left|\log \frac{G(x_2)}{G(x_1)}\right| > \log \left|\frac{G(x_2)}{G(x_1)}\right| \text{ ist.}$$

Ferner folgt noch: Das Wachstum des imaginären Teiles von $\log G(x)$, wenn x von x_1 nach x_2 sich continuierlich auf jenem Kreise bewegt, ist an absolutem Werte

$$< C \,.\, r \,.\, \delta \,.\, M(r)\,.$$

6. Ist insbesondere $G(x)$ eine Function des genre l, so ist $g_n(x) = x^l$, und es ergiebt sich daher aus 5 Satz: Ist $G(x)$ eine Function des genre l, so kann man eine Folge von Kreisen C_1, C_2, C_3 . . . C_n . . . um den Nullpunkt mit den Radien R_1, R_2 . . . R_n . . . bestimmen, derart dass für Punkte x auf C_n

$$|G(x)| > e^{-\delta \,.\, R_n{}^{l+1}},$$

wenn nur n genügend gross gewählt wird, wie klein die vorgegebene positive Zahl δ auch sei.

In HADAMARD's berühmten preisgekrönten Abhandlung spielt ein Satz der obigen Art eine sehr wichtige Rolle. Ist ρ nach BOREL die Ordnung von $G(x)$, so ist zufolge des Satzes von HADAMARD eine Folge von Kreisen obiger Art möglich, auf der die Ungleichung besteht

$$|G(x)| > e^{-R^{\rho+\epsilon}},$$

wo ϵ eine beliebig kleine positive von 0 verschiedene Grösse bedeutet. Ist ρ keine ganze Zahl, so ist $\rho < l + 1$ und somit geht der Satz von HADAMARD dann weiter als der obige. Ist umgekehrt $\rho = l + 1$, so ist der hier ausgesprochene Satz der weitergehende.

Der Hadamardsche Satz lässt sich übrigens unschwer aus Satz S ableiten. Es sei $\Sigma_1^\infty |a_n|^{-\lambda}$ convergent, $l \leqq \lambda \leqq l + 1$. $l + 1 - \lambda$ setzen wir $= \sigma$. Dann ist in einem asymptotischen Bereiche L,

$$\text{wenn wieder} \qquad \frac{G'(x)}{x^l \,.\, G(x)} = f(x) = \Sigma_1^\infty \frac{1}{a_n{}^l(x - a_n)} \quad \text{gesetzt ist,}$$

$$\lim_{x=\infty} x^\sigma \,.\, f(x) = 0\,.$$

$$\text{Es ist nämlich} \qquad |x^\sigma \,.\, f(x)| < \Sigma_1^\infty \frac{|x^\sigma|}{|a_n|^l \,.\, \big|\,|x| - |a_n|\,\big|}\,.$$

$$\text{Ist} \qquad |a_m| < |x| < |a_{m+1}|, \quad \text{so ist}$$

$$\Sigma_1^\infty \frac{|x^\sigma|}{|a_n^l|\,\big|\,|x| - |a_n|\,\big|} = \Sigma_1^m \frac{|x^\sigma|}{|a_n|^l(|x| - |a_n|)} + \Sigma_{m+1}^\infty \frac{|x^\sigma|}{a_n^l(|a_n| - |x|)}\,.$$

$$\text{Das Glied} \qquad \Sigma_{m+1}^\infty \frac{|x^\sigma|}{a_n^l(|a_n| - |x|)} \quad \text{ist} \quad < \Sigma_{m+1}^\infty \frac{|a_n|^\sigma}{|a_n|^l(|a_n| - |x|)},$$

was nach Satz S für geeignete Lagen von x in den ausgezeichneten Intervallen von $\Sigma_1^\infty \frac{a_n^{\sigma-l}}{x-a_n}$ kleiner wird als jede beliebig vorgegebene Grösse. Andererseits ist

$$\Sigma_{m'}^{m} \frac{|x^\sigma|}{a_n^l(|x|-|a_n|)} - \Sigma_{m'}^{m} \frac{|a_n|^\sigma}{a_n^l(|x|-|a_n|)},$$

da

$$\frac{|x|-|a_n|}{|x|^\sigma-|a_n|^\sigma} < \frac{1}{\sigma} \cdot |a_n|^{1-\sigma}, \quad \text{wenn} \quad |x| > |a_n|$$

kleiner als

$$\Sigma_{m'}^{m} \frac{\sigma \, . \, |a_n|^{\sigma-1}}{|a_n|^l}.$$

Hier ist m' irgend eine Zahl kleiner als m. Die letztere Summe ist ein Teil einer convergenten Summe, ist daher für genügend grosse Werte von m', die im Vergleich zu m aber noch sehr klein sein können, beliebig klein. Die Summe

$$\Sigma_1^{m'} \frac{|x|^\sigma}{|a_n|(|x|-|a_n|)}$$

aber ist augenscheinlich, wenn m im Vergleich zu m' sehr gross genommen wird, wegen

$$|x| > (a_m)$$

beliebig klein zu machen.

Somit ist nachgewiesen, dass ein asymptotischer Bereich L existiert, innerhalb dessen

$$\lim_{x=\infty} x^\sigma . f(x) = 0,$$

und da $f(x) = \frac{G'(x)}{x^l \, . \, G(x)}$ war, so erhalten wir durch die bereits angewandte Integration den Satz von Hadamard. Das obige Verfahren erweitert den Satz von Hadamard noch in soweit, dass es auch noch die Folgerung $G(x) > e^{-\delta \, . \, R^\rho}$ zulässt, wenn

$$\Sigma |a_n|^{-\rho} \text{ noch convergiert,}$$

wo, wie früher, ρ den Convergenzexponent der Folge $|a_n|$ bedeutet.

Der obige Beweis des Hadamardschen Satzes lässt sich übrigens unschwer nach verschiedenen Seiten hin zu einer Erweiterung des Satzes S benutzen. Doch ist es für den Augenblick unnötig, darauf näher einzugehen.

7. Man könnte die oben entwickelten Ungleichungen benutzen, um, dem schönen Ideengange der Hadamardschen und Borelschen Untersuchungen folgend, eine Theorie der ganzen Functionen aufzustellen. Es würde sich dabei empfehlen, den Begriff einer *Gattung* von ganzen Functionen einzuführen, wobei ein und derselben Gattung alle diejenigen Functionen angehören würden, deren $g_1(x) \ . \ . \ . \ g_n(x) \ . \ . \ .$ gegeben ist. Adoptiert man den Borelschen Standpunkt, dem derselbe bereits in seiner, in der 'Acta Mathematica,' 1896, erschienenen Abhandlung Ausdruck verliehen hat, so würde man

$$g_n(x) = x^{\rho_n}$$

setzen, wo ρ_n eine passend gewählte Folge ganzer Zahlen ist, die mit wachsendem n niemals abnehmen. Setzt man insbesondere

$$\rho_n = \mathrm{E}\left(\lambda \cdot \frac{\log n}{\log \log n}\right),$$

so würde die entstehende Gattung einen Teil der Borelschen Functionen 2ter Classe umfassen, nämlich jenen Teil, deren *Ordnung 2ter Classe* kleiner als die Constante λ sein würde. Für die Borelschen Functionen dritter Classe, deren Ordnung $< \lambda$, wäre

$$\rho_n = \mathrm{E}\left(\lambda . \frac{\log n}{\log \log \log n}\right) \text{ u.s.f.}$$

Man könnte sich mancherlei Aufgaben stellen, insbesondere, zu erweisen, wenn G_1 und G_2 einer Gattung G angehören, dass auch $G_1 + G_2$ wie $\frac{\partial}{\partial x} G_1$ ihr angehören, oder dass, wenn $f(x) = \Sigma \frac{c_n}{x - a_n}$ und $G(x)$ in den a_i verschwindet, $G(x) . f(x)$ derselben Gattung angehört wie $G(x)$. Wir können allerdings nicht, ohne die Grenzen dieser Abhandlung ungebührlich auszudehnen, auf die angeregten Probleme näher eingehen. Wir werden nur noch einen Satz entwickeln, der in gewissem Sinne die Umkehrung bildet zu Satz S und die Betrachtungen dieses Capitels zu einem vorläufigen Abschluss führt. Derselbe lautet:

Satz U: Ist $f(x)$ eine eindeutige analytische Function, welche innerhalb eines Bereiches B erklärt ist, und existiert in B ein asymptotischer Bereich L um einen singulären Punkt a von $f(x)$, so dass in L

$$\lim_{x=a} f(x) = 0,$$

so ist $f(x)$ entwickelbar in eine convergente Summe $\Sigma_1^\infty u_n(x)$, vermehrt um eine in $x = a$ convergente Potenzreihe $\mathfrak{P}(x - a)$, und zwar derart

1) dass die $u_n(x)$ durch die Singularitäten von $f(x)$ in B definiert sind;
2) dass alle $u_n(x)$ in $x = a$ regulär sind und dort den Wert 0 haben;
3) dass auch $\mathfrak{P}(x - a)$ in $x = a$ den Wert 0 hat.

Um den Satz zu erweisen, schlagen wir innerhalb L um a mit den Radien $\rho_1, \rho_2, \rho_3 \ldots . \rho_n \ldots .$ Kreise $C_1, C_2 \ldots . C_n \ldots .$ Die ρ_i bilden eine stetig abnehmende Reihe von Grössen, so dass $\lim_{n=\infty} \rho_n = 0$. (Statt der Kreise könnte man übrigens auch andere Contouren wählen, deren Maximalabstand von $a < h . \rho_n$, deren Minimalabstand von $a > k . \rho_n$, wo h, k vorgegebene Constanten. Die Möglichkeit solcher Contouren in L war in der Definition von L Bedingung.) Es sei z irgend ein nicht singulärer Punkt von $f(x)$ in C_1.

Es sei dann
$$\frac{1}{2\pi i}\int_{(C_n)} \frac{f(x)}{(x-a)(x-z)}\, dx = I_n(z)$$

und
$$I_n(z) - I_{n+1}(z) = v_n(z).$$

$v_n(z)$ ist also das über die beiden Kreise C_n, C_{n+1} erstreckte Integral von $\frac{f(x)}{(x-a)(x-z)}dx$, dividiert durch $2\pi i$. Daher ist $v_n(z)$ eine eindeutige Function von z, welche nur von der Art der Singularitäten von $f(x)$, die zwischen C_n und C_{n+1} gelegen sind, bestimmt ist. Offenbar ist wegen

$$\lim_{x=a} f(x) = 0$$

auch

$$\lim_{n=\infty} I_n = 0,$$

mithin $\Sigma_1^\infty v_n(z)$ convergent für jede Lage von z in C_1.

z rücke zwischen C_m und C_{m+1}. Alsdann ist $I_m(z) - I_{m+1}(z)$ nicht mehr $= v_m(z)$, sondern, da $x = z$ einen Pol von $\frac{f(x)}{(x-a)(x-z)}$ bildet,

offenbar $$= -\frac{f(z)}{z-a} + v_m(z).$$ Aus $\lim\limits_{n=\infty} I_n = 0$

kommt $$(I_1 - I_2) + (I_2 - I_3) + (I_3 - I_4) + \ldots\ldots = I_1,$$

mithin $$\Sigma_1^\infty v_n(z) = \frac{f(z)}{z-a} + I_1.$$

I_1 ist nun augenscheinlich eine eindeutige Function von z, welche für *jede* Lage von z innerhalb C_1 endlich ist. Mithin ist I_1 in eine Potenzreihe nach aufsteigenden Potenzen von $z - a$ entwickelbar, welche in C_1 convergiert. Setzen wir nun

$$\mathfrak{P}(z-a) = -(z-a)I_1(z)$$

und

$$u_n(z) = (z-a)v_n(z),$$

so ist der Satz U verificiert.

Aus Satz U folgt z. B.:—Liegen in dem asymptotischen Bereich L keine Singularitäten von $f(x)$, so ist $f(x)$ in L und dem Nichtbereich von L regulär, und es ist überhaupt $\lim\limits_{x=a} f(x) = 0$. Oder auch: Hat $f(x)$ in L keine wesentlichen Singularitäten, sondern nur Pole $a_1, a_2 \ldots, a_n \ldots$ und ist in a_n

$$f(x) - \left(\frac{c_{1,n}}{x-a_n} + \frac{c_{2,n}}{(x-a_n)^2} + \ldots + \frac{c_{h,n}}{(x-a_n)^h}\right)(x-a)$$

regulär, so ist identisch

$$f(x) = \Sigma_1^\infty (z-a)\left[\frac{c_{1,n}}{z-a_n} + \frac{c_{2,n}}{(z-a_n)^2} + \ldots + \frac{c_{h,n}}{(z-a_n)^h}\right] - \mathfrak{P}(z-a).$$

Dabei ist die Summe $\Sigma_1^\infty \ldots\ldots$ für jeden Wert von z convergent, doch nicht absolut, sondern bedingt. Man muss, um die Convergenz der Summe zu sichern, immer diejenigen Terme zusammenfassen, welche den singulären Punkten a_i entsprechen, die gemeinsam zwischen 2 aufeinanderfolgenden Contouren C liegen. Selbst bei dieser Zusammenfassung ist die absolute Convergenz noch nicht erwiesen.

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X. *An Investigation of the Spectra of Flames resulting from Operations in the Open-hearth and "Basic" Bessemer Processes.*

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Contents.

In the 'Philosophical Transactions' for 1894 were published Parts I. and II. of "Flame Spectra at High Temperatures, the Spectrum of Metallic Manganese, &c.," and Part III. "The Spectroscopic Phenomena and Thermochemistry of the Bessemer Process." The results obtained by one of us up to 1895 had reference to the phenomena observed in flames seen to issue from the vessels in which are carried on the operations of the "acid" Bessemer process. The following account deals with an investigation of the Thomas-Gilchrist or "basic" process.

Works in various parts of England were visited in order to ascertain where suitable observations could be made without incurring serious personal risks or damage to instruments, and to obtain permission to carry on these investigations.

5.6.1901.

The Cleveland district of Yorkshire was chosen as the principal centre, owing to the interest taken in the work by Mr. ARTHUR COOPER, Managing Director of the North-Eastern Steel Company, and, in consequence of the courtesy and attention shown us, the North-Eastern Steel Works at Middlesbrough were selected.

It was found necessary at the outset to have three observers at work simultaneously. Twenty-eight plates of photographs and copious notes were taken successfully, with photographed spectra to the number of about 140.

According to previous experience it was found advisable to obtain photographs of the flame, sparks and fume at different stages of the "blow," as well as photographs of spectra. These were executed with a small Anschütz camera and Goertz lens, giving excellent pictures with very rapid exposures. Some of the photographs of spectra could not be surpassed with the dispersion of the instrument employed, but the work was not accomplished without some difficulty, which was occasioned by the large quantity of lime dust blown into the air.

The results were quite different from those previously obtained; for instance, many lines and bands quite new to the Bessemer flame have been recognised in addition to the spectra of the common alkali metals, iron, and manganese.

Thus rubidium, cæsium, calcium, copper, silver, and gallium have been identified. Very careful analyses of the crude metal, the ore, and lime, and also of the rail steel manufactured, were made for the purpose of separating some of these substances and identifying them subsequently by a spectrographic observation of the separated bases.

The technical importance of being able to ascertain the amount of phosphorus in the metal during the process of "blowing" made us very desirous, if possible, of determining this. Notwithstanding that the work has been continuously in progress since 1895, and that every line and band in the different spectra photographed has been accounted for, our observations so far have failed to yield information or even any indication of when the blast should be stopped. The cause is not far to seek if we bear in mind that phosphorus existed in the pig-iron probably as a ferrous phosphide, which, according to previous experiments, is not volatilised, and that at a high temperature in contact with a powerful base like lime and in presence of air it becomes a non-volatile calcium phosphate or a ferrous phosphate. Nevertheless, some insight into the chemistry of the process of the "blow" has been obtained. The greatest interest is attached to the knowledge it has given us of flame spectra under variations of temperature and of the wide distribution of many of the rarer elements in minute proportions in ores and common minerals ('Roy. Soc. Proc.,' vol. 60, pp. 35 and 393; 'Chem. Soc. Trans.,' 1897, pp. 533 and 547).

As we have had the honour of laying before the Royal Society and also the Chemical Society accounts of some of our results, there is no necessity to make more than an incidental allusion to them.

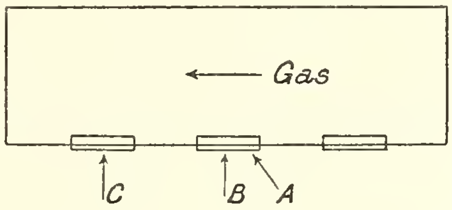
We have also sought for the spectra of metals in the open hearth steel furnace, and propose to very briefly record here our want of success in this direction.

The Siemens-Martin Furnace.

The investigation was made, by the kindness of Mr. F. W. WEBB, in the steel works of the London and North-Western Railway Company, Crewe, in the first week of January, 1895.

The steel melting furnace No. 3 had been charged with 15 tons of old rails at 11 P.M. on January the 2nd. The observations were made from about 10 A.M. on the 3rd January. Eye observations were first made with a direct-vision spectroscope through one of the charging doors of the furnace ; the charge was perfectly fluid, and was covered with slag, through which bubbles of gas escaped. The charge was ready for tapping about 3 P.M. on the same day. The only line seen in the spectrum of the light emitted by the gas above the charge and the furnace wall beyond was the yellow line of sodium seen as one bright line ; the continuous spectrum was very strong. Photographs of the spectrum were then taken to obtain a permanent record and to discover how far the spectrum extended towards the ultra-violet.

Plate 1.—A Cadett plate stained with erythrosin after MCCLEAN'S method.



Four exposures were made with the instrument pointing as indicated in the above plan of the furnace, and about 3 yards away from the door. The spectrum extended to the wave-lengths here recorded.

		λ.
1.	Pointed as in A, ½ min. . . .	3800
2.	,, ,, ,, 2 mins. . .	3540
3.	,, ,, B, 1 min. . . .	3720
4.	,, ,, ,, 3 mins. . .	3470

Plate 2.—THOMAS'S "Sandall" plate stained with erythrosin. Same direction as B, the slit being about 4 feet away from the door in the first, and about 3 yards in the second, third, and fourth photographs.

		h.	m.	s.		h.	m.	s.	λ.
1.	Exposed from	11	27	10	to	11	29	10	3590
2.	,, ,,	11	31	0	,,	11	35	0	3565
3.	,, ,,	11	36	50	,,	11	38	50	3700
4.	,, ,,	11	39	10	,,	11	39	25	—

5. Solar spectrum for reference.

For 3 and 4 the spectrograph was pointed as in C.

Plate 1 was very sensitive to the greenish-yellow and yellow rays, but it shows no trace of the sodium lines. The spectra are very strong in the blue and violet, but fade gradually towards the extreme violet, and thence to the wave-lengths recorded above.

It must be remembered that the flame playing across the hearth of the furnace, although at a very high temperature, is oxidising, and, as will be seen later (p. 498), it should, at the most, give only feeble spectra of such elements as lithium, potassium, calcium, and, we should infer, iron and probably manganese. The carbon monoxide as it escapes from the bath of metal and slag would, however, tend to intensify the spectra of these metals, and care was taken to turn the collimator in a direction parallel to the surface of the bath and as near to its surface as the opening for the door would allow.

The temperature of the gas must be higher than the temperature of the walls of the furnace which are heated by it, hence the fact of the sodium line being bright is in accordance with KIRCHHOFF's law.

Observations on the Flames from a Cupola Furnace.

While at Crewe Works, eye observations were made on the spectra of the gases passing the opening of a cupola through which the charge was introduced. Pig-iron was being melted in the cupola for the "acid" Bessemer process, while the blast was turned on the flame was bluish, and lines of sodium, lithium, and potassium were detected; the latter were very weak. When the Root's blower stopped, the flame became smaller and whiter, and the lines of the above elements became stronger; the ends of the two strongest bands of manganese were also seen. There was, doubtless, a large proportion of reducing gas in the flame in the latter case.

Description of the "Blow" and "Over-blow" in the "Basic" Bessemer Process at Middlesbrough.

The vessel is first charged with about 2 tons of lime in lumps, and then with 12 tons of fluid "mixer metal." The blast is turned on, and the vessel rotated into a nearly vertical position.

The iron used for the basic Bessemer process in the works is a mixture of fluid metals coming direct from the blast furnaces with molten pig-iron from a cupola, the two fluids being completely mixed and in such proportions that the composition of the mixture is practically constant; it is thus rendered suitable for uniform treatment. This, which is technically known as "mixer metal," is poured out of a Hoerder mixer into a ladle, which delivers it into the converter; this "mixer metal" usually contains—

Manganese	about	1·0	per cent.
Silicon	,,	0·6	,,
Phosphorus . . .	,,	2·5	,,
Sulphur	,,	0·08	,,
Carbon	,,	3·4	,,

The blow may be divided into three stages. The first stage ends when the flame drops, indicating that the carbon has burnt out. The second stage ends when the vessel is turned down for a sample of the metal to be taken and the slag poured off. More lime is then added, and the blow is continued for a few seconds to complete the removal of the phosphorus; this forms the third stage.

First Stage.—The average duration of this was 12 minutes and 20 seconds. The blow begins with the expulsion of a large quantity of lime dust, and as soon as this has cleared away a flame is visible. It has a yellow or yellowish-red colour, the colour being probably caused by hydrocarbons from the tarry matter in the lining of the vessel. The flame grows larger and brighter, and assumes the usual appearance observed in the "acid" process. Sparks are thrown off, which in turn throw off other sparks; these are succeeded by larger showers of sparks and a louder roaring of the flame. When, after a time, the flame gets thin, some fume is seen; then the flame drops and the sparks do not throw off scintillations.

Second Stage.—This, technically known as the "over-blow," lasts about 5½ minutes. In the earlier part the flame is very short; it gradually extends, and the fume arises very copiously. It was noticed that the cloud of fume was yellower than that seen at Crewe and Dowlais in the "acid" process. On one occasion the fume was observed to have a rich purple colour when viewed by transmitted light, which colour faded gradually and became more like brown umber; the flame gradually increased in density until it became very brilliant indeed, but its colour was always of a golden-yellow, until sometimes it resembled phosphorus burning in oxygen. Slag is thrown out in large quantities, and with a shower of sparks, just prior to the converter being turned down.

Owing to the exigencies in the arrangement of the Bessemer plant and the nature of the operations carried on, it was found that the only point of observation available was a gallery at one end of a row of four converters, and when the most distant of these vessels was in use nearly all the details of the "blow" as photographed elsewhere in the "acid" process were absent.

For instance, the manganese bands were not defined or even clearly visible. Then again, the spectra were complicated by a superposed feeble solar spectrum. Only those vessels nearest the instrument yielded satisfactory photographs. It was proved eventually that the lime dust and other dust and smoke in the air reflected the sun's rays and caused an overlying solar spectrum to appear. To get rid of this it became necessary to make all the observations during the evening after dark.

Having described the course of the "blow" in the Thomas-Gilchrist process, it will be noticed that we have to observe the "blow" in different stages, and also the "over-blow"; it is desirable that some general account of the photographs which were taken at different times be recorded, and that the details of some of the spectra be given when of particular interest.

In all some twenty-six plates were developed, some few of which were failures owing to circumstances already described, but Nos. 4, 14, 15, 17, 23, and 26 were particularly valuable. A full account of all the details of the operations witnessed with a precise description of the spectra on each plate being considered unnecessary, the particulars of No. 14 only are here given, followed by a summary of all the spectra on the other plates. In making the observations it was found necessary at first for one person to attend to the spectrograph, a second to watch the "blow" and observe the flame with a direct-vision hand spectroscope, and obtain a record of what was taking place with the Anschütz camera, while the third recorded the time and entered the notes and remarks made to him. Subsequently, two observers were engaged—one in managing the spectrograph, the other in entering all details in the note-book. When a succession of thirteen spectra are photographed in 15 minutes, there is but little opportunity for entering notes of every detail. The omission of these details greatly diminishes the value of the observations; on the other hand, photographs of the operations greatly enhance their value.

Plate No. 14. A series of six spectra which are in very sharply defined focus in the green. The following table gives the intensities of the lines and bands in the different spectra from I. to VI., with the period of exposure in each case, the periods being consecutive.

Note.—There were 140 spectra on the twenty-six plates. The plates were numbered consecutively and the spectra on each plate were numbered in the order of their exposure. The references in the text to numbered plates relate to the *photographs* and not to illustrations, which have been omitted.

Wave-lengths.	Element.	Intensities numbered 1 to 10.						Description.
		I.	II.	III.	IV.	V.	VI.	
5893	Na	7	8	9	9	10	10	Centre of lines.
5832	FeK	—	—	—	1	2	2	Line and m.r.* edge of band.
5802	K	—	—	1	2	3	2	Sharp line.
5794	Fe	—	—	—	1	3	3	Sharp line.
5764	Mn	—	—	—	—	2	—	Sharp line.
5756	Mn	—	—	—	—	2	—	Sharp line.
5720	Mn	—	—	—	—	2	2	Broad line.
5690	Na	—	—	—	—	2	2	Line.
5609	Mn	—	—	—	—	—	—	Centre of dark space.
5582	Mn	1	2	4	8	8	7	m.r. edge of band.
5547 5539	Ca CaFe	—	—	1	2	3	3	Weak lines which merge into a band.
5439	Mn	—	—	2	3	4	3	Line on band.
5430	Mn	—	—	—	2	3	2	m.r. edge of band.
5408	Mn	—	—	—	—	1	—	Doubtful line.
5401	Mn	—	—	2	3	4	3	Line near edge of band.
5375	Mn	—	—	—	—	1	—	Doubtful line.
5364	Mn	—	1	2	3	4	3	m.r. edge of band.
5347	K	—	—	2	3	3	2	Line.
5331	K	—	—	—	—	1	—	Doubtful line.
5272	Mn	—	—	—	1	3	3	Line.
5242	Mn	—	—	—	1	3	2	Band.
5193	Mn	—	—	—	1	3	—	Line or edge of band.
5179	Mn	—	—	—	1	3	2	Edge of band.
5108	KFe	—	—	—	1	3	—	Line.
4644	K	—	—	1	3	3	1	Line.
4557	Cs	—	—	—	1	2	—	Line.
4485	Fe	—	—	—	—	2	3	Line.
4463	Fe	—	—	—	1	2	3	Line.
4429	Fe	—	—	—	1	3	3	Line.
4407	Fe	—	—	—	—	1	1	Line.
4385	Fe	—	—	—	—	1	2	Line.
4377	Fe	—	—	—	1	3	3	Line.
4227	Ca	—	—	1	3	4	3	Line.
4216	Rb	—	2	3	3	3	3	Line.
4202	Rb	1	3	5	5	5	4	Line.
4172	Ga	—	2	3	3	3	3	Line.
4048 4044	K K	6	7	8	9	9	7	Line.
4032	Mn	1	3	5	8	8	7	Middle of group.
Exposure periods . .		m. s. 2 0	m. s. 2 0	m. s. 2 0	m. s. 2 2	m. s. 2 3	m. s. 0 54	

Intervals.—Between commencement of blowing and first exposure 3 minutes 35 seconds.

Between commencement of blowing and dropping of the flame 14 minutes 40 seconds.

Between first rising of fume from the over-blow and the end of the operation 4 minutes 25 seconds.

Duration of blow 22 minutes 35 seconds.

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* m.r. = more refrangible.

The Cause of the Continuous Spectrum, which is a Marked Feature of the Flame from the "Basic" Process.

It has already been stated in the description of the "blow" as observed at the North-Eastern Steel Company's Works at Middlesbrough, that a flame is seen issuing from the mouth of the vessel from the first moment after the dispersion of the lime dust, a feature which is quite characteristic of the "basic" process. This flame is yellow and highly illuminating, but not to such an extent as to be visible on the slit plate of the instrument. It is evident, therefore, that some gas is evolved, the origin of which can only be the tarry matter used in the lining of the vessel. The smaller quantity of silicon in the iron is very quickly oxidised in this process. In the spectrum of the flame at this period it is observed that the continuous rays are generally stronger in the series of photographs taken at Middlesbrough than in those from Crewe and Dowlais, where the "acid" process was in use. This might be accounted for in part by the presence of larger quantities of the alkalies, which emit strong continuous flame spectra accompanied by lines, and partly by the presence of lime dust from the lime used in the "basic" process, which becomes incandescent in the flame. At this early period of the "blow" in the "acid" process only the lines of the alkali metals and of manganese appear, and they are very weak. The continuous spectrum cannot, however, be wholly attributed to the alkalies without further inquiry, for both iron and manganese or their compounds yield marked continuous spectra in the oxyhydrogen flame, which, however, is hotter than that of the flame of the converter at this stage. Moreover, in the "basic" as in the "acid" process, especially in the latter periods, this is caused by the flame of carbon monoxide. The visible spectra at Middlesbrough shortly after blowing has been started show the red and green bands of calcium oxide, and a minute later the lines of sodium. For instance, in the notes of our first series of photographs it is recorded that a flame was visible from the commencement, but it could not be seen upon the slit; there were showers of sparks and flashes of red at intervals during the first three minutes. The bands of manganese, two red lines, and the yellow sodium lines were visible. The manganese bands were mainly those in the violet. A subsequent examination of the photographs showed a band in the green due to calcium oxide, the lines of potassium 4047·36 and 4044·29, as also the manganese lines; hence one of the red lines belonged undoubtedly to calcium oxide, for this would certainly be present along with the green calcium oxide band; the other red line is more likely to be that of lithium than potassium, owing to the former being so much the brighter, and lithium we know by eye observations is always conspicuous in these spectra a few minutes before the potassium line appears. Lastly, we have the sodium rays in the yellow.

With reference to Plate 19, we have it recorded that the red and green bands of calcium oxide were seen from the very first appearance of flame and for about two minutes of the "blow"; but the lithium and potassium lines appear rather more than a minute after the beginning. On Plate No. 17, which is a very complete series of thirteen photographs taken during a "blow" lasting 15 minutes, and commencing directly the converter was turned up, it was recorded that the red lines of lithium and potassium, with the yellow line of sodium, were plainly visible with the direct vision spectroscope as soon as the shower of lime dust cleared away after the "blow" commenced, that is to say, from the first appearance of the flame.

The bands of manganese were seen very early, but they were not sharp during the first 4 minutes; this refers to the bands in the green.

There can be no doubt that the immediate production of a flame is caused not by the oxidation of material in the bath, but by carbonaceous matter in the lining of the vessel; that its luminosity is due partly to the volatilisation of alkalies, but certainly, anterior to this effect, by the ignition of lime dust carried up by the blast into the flame.

Characteristics of the Spectra of the "Basic" Bessemer Flame.

A flame spectrum is observed shortly after the commencement of the "blow," consisting of a band of continuous rays of the red and green bands of calcium oxide. One minute later the red lines of lithium and potassium are seen with the lines of sodium. There are about fifty-three lines of iron as in the flame from the "acid" process, but the lines are of lesser intensity, and some of those near to, but more refrangible than, 4306·7 (the solar line G) are absent.

A number of lines which do not occur in the spectra of the flames from the "acid" process have been observed in most, if not all, of the spectra of the "basic" process. They have been traced partly to elements present in the lime, but the lines are chiefly caused by elements introduced with the iron, and are hitherto unrecognised constituents of the iron ores from which the crude metal is smelted for use in the Bessemer process. Rubidium and cæsium undoubtedly accompany the lime, while as certainly gallium, copper, and silver, and some rubidium, enter with the iron. Lead lines were not observed in the flame, neither were nickel nor cobalt lines.

Another feature of these spectra is the less prominent appearance of the manganese bands, and lines of manganese, with the greater intensity of the lines of potassium.

The description of six spectra on Plate No. 14 (see p. 485) shows the increase in temperature of the flame with the progress of the "blow" until that period when the flame drops.

The Identification of Lines and Bands observed in the Spectra of the Flames from the "Basic" Process.

In the region of the rays more refrangible than the strong manganese line λ 4030, no uncertainty is felt as to any line or band. The continuous spectrum extends throughout the whole of this portion to the edge of the plate, about λ 3210. It fades gradually, except at two places, from λ 3705 to 3600 and from λ 3581 to 3500, where two broad bands occur. These bands were observed in the Crewe Bessemer spectra, and in the spectra of manganese and its compounds heated in the oxyhydrogen flame (Flame Spectra), 'Phil. Trans.,' A, vol. 185, p. 1029. Between λ 4030 and 3448 the only lines detected are the iron lines, which are present without any considerable variation of intensity in the oxyhydrogen spectra of iron and its compounds. The lines just beyond λ 3448 are the pair of potassium lines, λ 3447·49 and 3446·49, the latter one slightly stronger than the former, and both of them stronger than the iron line 3440·7.

The iron line λ 3443·96 is present in some of the spectra. Between this point and the edge of the plate, about λ 3210, no iron lines are found either in the Middlesbrough or the oxyhydrogen spectra. Lines of other elements are visible on the photographs of several of the Middlesbrough spectra, the strongest being the sodium line λ 3303, which is really a pair of lines, but they are not distinctly separated unless the quantity of substance is very small. The silver line λ 3383·0 is present in several spectra, and on Plate 26 the second silver line λ 3280·8 is seen. On Plate 26 the second spectrum indicated by 26^2, in which the lines are very sharp near the edges of the plate, the two copper lines λ 3274·1 and 3247·7 are present, and beyond these again is the lithium line λ 3233 and the potassium line λ 3217·5. There is no indication of the spectrum of water vapour; the strongest lines in this spectrum would not, however, fall on the photographic plate used by us.

The identification of the lines and bands in the visible portion of the spectrum is much more difficult.

(1.) *Lines.*—Near the more refrangible end of the visible spectrum the manganese triplet occurs, each line being very strong; the neighbouring potassium lines λ 4044 and 4047 are also very strong. Next in order are four strong lines with wave-lengths 4171·5, 4201·9, 4216·0, and 4226·4.

These lines are not present in the Bessemer spectra from the "acid" process which we had hitherto examined, and their identification was not readily accomplished. The first fact noticed was that the first and last lines varied in strength independently of all other lines in the spectra, and that of the two intermediate lines, 4201·9 was always stronger than 4216·0. These two were, therefore, identified with rubidium, as they were the strongest lines in the oxyhydrogen flame spectrum of that element. The line 4226·4 was identified with calcium, this being decidedly the strongest line in the oxyhydrogen flame spectrum of calcium oxide. It corresponds with a strong Fraunhofer line (best seen in Plate No. 4).

The oxyhydrogen spectrum of rubidium is such that no further lines of this element were expected, and none were found. In the case of the calcium oxide, the band in the green (the wave-length of its centre is about 5530) might be expected, and a band does occur, but this will be discussed later.

The line λ 4171·5 was identified only after a prolonged study. Of the many elements and compounds examined in the oxyhydrogen flame, none gave a strong line of this wave-length. Reference to published tables of arc and spark spectra indicated that it might be the strongest of the gallium lines.

Lecocq de Boisbaudran gives the wave-length of the strongest of the lines of this element as 4170·4 on Ångström's scale, or 4171·0 on Rowland's. The second line he gives as 4031·9 (Ångström), 4032·6 (Rowland); this latter would lie between the two closely adjacent lines in the manganese triplet, and it would, therefore, be quite impossible to distinguish it when the manganese lines are so strong. No other strong lines are recorded, and this indication of the origin of the line could be confirmed only by analyses of the materials composing the lining of the converters, and of the "basic" earth and metals contained in the charge. By spectrographic analysis the same line was found in the oxyhydrogen spectrum of the crude iron ("mixer metal"). By chemical analysis the metals forming sesquioxides were freed from manganese, and the sesquioxides in this condition showed by spectrographic analysis the *two strong lines*, that with wave-length 4032·7 being weaker than 4171·5. The presence of gallium was proved by separating the pure oxide from a weighed quantity of metal, and determining the amount present. The purity of the oxide separated was established by spectrographic analysis ('Proc. Royal Society,' vol. 60, pp. 35 and 393).

Of the lines next in order, ten are identical with lines in the spectra of flames from the "acid" process and with lines in the oxyhydrogen flame spectrum of iron and its compounds. It must be remarked, however, that we now meet with notable differences in the *relative intensities* of these lines in the various Bessemer and oxyhydrogen flame spectra.

The differences, we believe, result wholly from differences in temperature, and our reasons for this we propose now to state.

The iron lines in this region of the spectrum of the oxyhydrogen flame appear on the following table, with varying intensities :—

VARIATIONS in the Intensity of Iron Lines in Spectra from various Sources.

Wave-lengths.		Intensity.									
							Bessemer flame spectra.				
							Crewe.		Middlesbrough.		
							Temperature.				
Oxyhydrogen flame.	Identified in solar spectrum.	Solar.	Spark.	Arc.	Oxy-hydrogen flame.	CO + O flame.	Low.	High.	I.	II.	III.
4063·75	4063·759	20	10	10	3	1	½	3	…	…	½
4071·92	4071·908	15	10	10	2	1	½	2½	…	…	½
4132·24	4132·235	10	8	10	1	…	½	1			
4144·06	4144·038	15	7	10	1	…	½	1½			
4202·20	4202·198	8	9	10	1	…	…	1			
*4216·39	4216·351	3	2	6	1	½	2	4			
4250·93	4250·945	8	8	10	1	…	…	½			
4271·93	4271·934	15	10	10	4	2	1	4	½	…	½
4308·06	4308·081	6	10	10	4	2	1	4	1	…	½
4325·94	4325·939	6	10	10	4	1	1	4	1	…	½
*4376·03	4376·107	6	4	8	2	2	3	5	4	2	3
4383·71	4383·720	15	10	10	8	4	3	6	4	1	3
4404·94	4404·927	10	10	10	3½	2	2	4	2	1	1
4415·29	4415·293	8	8	10	1	…	…	1	…	…	1
*4427·46	4427·482	5	3	8	2	2	3	5	5	2	3½
*4461·78	4461·818	4	3	6	1	1	2½	4	2	1	2
*4482·34	4482·338	5	4	8	1	…	1½	3	1	1	1
*4489·86	4489·911	4	1	4	½	…	…	1			

When the intensity is not stated in these columns the lines are either absent or exceedingly feeble, so that no value could be attached to them.

I. Plate 17, spectrum (8). II. Plate 14, spectra (5) and (6), and Plate 23 (2.) III. Plate 26, spectrum (2).

A line at 4045 was hidden by overlying potassium lines. Two lines less refrangible than the above appear in seven of the Crewe spectra on Plate 8, and their intensity is 1.

The lines with wave-lengths 4215·0, 4375·19, 4426·7, 4461·5, and 4481·7 may all be classed as strong lines in the Bessemer flame spectra. In the spectrum of the flame during the earlier period of the blow they are practically the strongest lines of the above series. As the blow proceeds—that is to say, as the temperature increases—other lines, notably those with wave-lengths 4270·6, 4306·7, 4324·3, 4383·3 (this especially), and 4405·0, become stronger, while those just indicated are not strengthened in the same proportion.

The series of photographs taken at the Crewe works of the London and North-

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* By interpolation. The other lines are coincident with strong lines in the spark spectrum.

Western Railway during the month of January, 1895, from which the details of two spectra are taken for comparison of the changes of intensity in the lines of iron, as given in the preceding table, show the gradual progress of these changes in a very marked degree. The solar spectrum lines are represented by Rowland's figures for the comparative intensities, and lines of the other spectra are compared directly with these as photographed by us with the instrument which was used for the Bessemer spectra.

There can be but little doubt, if any, that the arc and solar spectra are more nearly alike in the intensity of their lines than any other two; next to the arc, we have the oxyhydrogen flame spectra and Bessemer spectra. The change in intensity is much more striking in the spark spectrum of iron. In this the lines of the first group are either absent or present only as very weak lines, while those of the second group are very strong lines.

We have clearly identified all the lines under discussion by making more accurate observations of the lines in the flame spectrum of iron. These latter measurements were made in the Chemical Laboratory of the University of Cambridge with a Rowland's plane grating and spectrometer belonging to Professor Liveing.

Oxide of iron heated in the carbon monoxide and oxygen flame gives a spectrum intermediate between the Bessemer and the oxyhydrogen flame spectra.

The next line beyond this group of iron lines is one of wave-length 4555·1. It is best seen on the third spectrum on the Plate 15 taken at Middlesbrough, but occurs also in other spectra. It is seen best in the earlier spectra of a blow, and is invisible in those which come later. It is, therefore, caused by a small quantity of a volatile element, unless, perhaps, the stronger continuous spectrum of the later periods of a blow masks the weaker lines. In this event, the weak lines of iron should also disappear, but they do not; so the first inference is probably correct. The strongest cæsium line has, according to Kayser and Runge, the wave-length 4555·44 in the arc spectrum, and the line in question is doubtless this line, which we know to be by far the strongest in the oxyhydrogen flame spectrum of cæsium. Rubidium and other alkali metals are present, and the presence of cæsium may therefore be expected.

On a New Line in the Spectrum of Potassium.

In many of the spectra a sharp line occurs near wave-length 4642. This line also becomes weaker, and on some plates disappears, during the later periods of the blow It was not until after much careful study that it was finally traced to potassium. It is not recorded on any hitherto published arc, spark, or flame spectra of this element.

It occurs in the oxyhydrogen spectrum when potassium compounds are used, but it does not appear with the same intensity as in the Bessemer flame. Comparative experiments were made in which, first, a very small quantity of the potassium compound was introduced into the oxyhydrogen flame; secondly, the potassium salt was volatilised in the oxygen and carbon monoxide flame. The line was of greater relative intensity in the latter spectrum, but it still did not equal that observed in the Bessemer flame.

There are two possible explanations of the reason for the disappearance of this line: first, its emissive power and its chemical action may be greater at lower temperatures, and, therefore, its photographic intensity greater; and, secondly, it may be that when the flame is less dense, so that the vapour pressure of the metal is decreased, the molecules are endowed with greater freedom of motion.

The following photographs were taken with the object of ascertaining whether the density of the vapour in the flame really affected the intensity of the line 4642:—

Plate 103, *Spectrum* (2.)—Paper moistened with a solution containing 0·1 gramme of potassium chloride. Made into a coil, and burnt in the oxyhydrogen flame.

Spectrum (3).—The same as (2), but the paper was burnt in thin strips.

Spectrum (4).—Similar to (3), but 0·2 gramme of potassium chloride was present.

Plate 104, *Spectrum* (3).—Potassium chloride as in 103, Spectrum 3, but about one-half the quantity taken.

Spectrum (4).—Potassium chloride, about one-quarter of the quantity used for photographing Spectrum (3) on Plate 103. The vapour of potassium in the flame is known to be proportionally increased with the larger quantity of substance volatilised, and the intensity of the chemical action or emissive power of the continuous rays in these spectra is observed to be in the inverse proportion to the vapour pressure of the substance. With reduction of vapour pressure there is a diminution of chemical action exerted by continuous rays over a wide range of differences in wave-length, and this is accompanied by a greater intensity of chemical action or emissive power of the molecules for the ray λ 4642.

The question of the temperature was next examined, as up to this time the line had not been observed in a Bessemer flame.

Plate 373, *Spectrum* (1).—Spark spectra of metallic lines of known wave-lengths for measurements. Exposure, 30 seconds.

Spectra (2), (3), *and* (4).—Potassium chloride heated on a support of cyanite.

(2.) In the flame of hydrogen. Exposure, $1\frac{1}{4}$ hours.

(3.) In air and hydrogen, using the same burner as that for the oxyhydrogen flame. Exposure, 15 minutes.

(4.) In the oxyhydrogen flame. Exposure, $1\frac{1}{2}$ minutes.

(5.) In the oxyhydrogen flame. Exposure, 5 minutes.

Two red lines were seen by using a small direct-vision spectroscope. These are doubtless the doublets $\left\{\begin{matrix}7699{\cdot}3\\7665{\cdot}6\end{matrix}\right\}$ and $\left\{\begin{matrix}6938{\cdot}8\\6911{\cdot}2\end{matrix}\right\}$, which have been observed in the arc spectrum of potassium.

The line 4642 was visible in all the above spectra. It was stronger in (2) than in (3), but strongest of all in (4).

The density of (5) was much greater than in any of the other photographs, but the intensity of the line was not proportionately increased, for there appeared to be very little difference between its intensity in Spectra (4) and (5).

Provided the temperature is sufficiently high to cause the emission of a ray with wave-length 4642, its brilliancy or intensity of chemical action is increased by diminishing the quantity of vapour or vapour-pressure in the flame, and thereby permitting the molecules so great a freedom of internal motion that it becomes possible for them to vibrate in the particular manner which causes the emission of this ray.

Taking a survey of all the plates upon which the line 4642 has been observed, and of those in which it is absent, and also having regard to the conditions under which the spectra in each case were produced, it is not so much reduction of temperature as reduction of quantity of vapour in the flame which increases its intensity.

The lines next in order are two of potassium with wave-lengths 5112 and 5098, which occur also in the oxyhydrogen flame, the arc, and spark spectra of potassium.

In the case of a line with wave-length 5112 it is probably intensified in some of the Bessemer flame spectra by a closely adjacent iron line, wave-length 5109, observed in the oxyhydrogen spectrum of iron.

About wave-length 5269 there is an iron line and at the more refrangible edge a manganese band. The iron line wave-length 5268·9 is the strongest in this region in the oxyhydrogen spectrum of iron.

At wave-length 5328 there is a line due in part each to iron and potassium; at λ 5343 a potassium line is seen, and at λ 5361 a line of potassium and the edge of a band of manganese.

In the earlier part of the blow a diffuse band occurs about 5540, and towards the end of the first part of the blow it appears to be replaced by two sharp lines. The change is best seen in the spectra on Plate No. 14. The band is due to calcium, but the lines are not calcium lines, for in the sixth spectrum on this plate the violet calcium line λ 4227 is much weaker than in the fifth spectrum, while the two lines are stronger.

The sixth spectrum is remarkable for an increase in the strength of the lines of iron throughout the whole spectrum, while the lines and bands of potassium and calcium are weaker. It is probable, therefore, that they are really iron lines. They are present in some Bessemer flame spectra photographed at Crewe in January, 1895, and occur in these only when the lines of iron are strongest. In the oxyhydrogen spectra of iron and its compounds the continuous spectrum is very strong in this region, but on very careful examination of a number of spectra we find a few in which traces of these two lines are present on the strong continuous spectrum. The relative intensity of the lines as compared with the other lines is, however, not equal to what obtains in the Bessemer flame spectra. Failing to find corresponding lines in the spectrum of any other element, we have no hesitation, on the above evidence, in attributing them to iron. A line λ 5792 was identified in a similar way.

The line λ 5803 is a potassium line, which is seen as a strong line in the oxyhydrogen spectrum of potassium. The line λ 5833 is also present in the oxyhydrogen flame spectrum of potassium, and is, in some of the Bessemer flame spectra, intensified by an iron line, 5834·5, observed in oxyhydrogen flame spectra.

In the region of the spectrum just considered there are many bands, and these are almost wholly due to manganese or compounds of manganese. They coincide with bands in the oxyhydrogen flame spectra of manganese and its compounds; they are, in both series of spectra, all degraded towards the red, and the more refrangible edges are sharp. The measurement of the more refrangible edges only are given in the tables, and these measurements vary on different photographs as the intensities of the bands vary. Sometimes when there is a strong continuous spectrum present, the sharp edge of a band has the appearance of a broad or nebulous line. The three more refrangible bands terminate about wave-lengths 4561, 4402, and 4245, and near these positions there are lines in the Bessemer spectra which, according to their intensities, modify the appearance of the edges of the bands. The line near 4556 is the cæsium line 4555, and in the other two cases, iron lines of wave-lengths 4404 and 4251.

Lines and Bands Less Refrangible than the D Lines.

We have been able to identify the lines and bands less refrangible than the D lines (which were photographed when using Lumière B plates) only by direct comparison with other spectra. We have no lines of reference on the Middlesbrough plates from which to draw a curve. A series of spectra was photographed at Crewe in January, 1895, on a plate stained with cyamine, and this includes both solar and Bessemer flame spectra. A curve of wave-lengths was drawn and the wave-lengths of the unknown lines and bands in the Bessemer flame spectrum were determined from it. By direct comparison of the Middlesbrough spectra with this plate, and by measurements of the oxyhydrogen flame spectra of the metals or compounds of lithium, manganese, and iron, we have identified the lines and bands in the Middlesbrough spectra. They were recorded in the analytical table of the second spectrum of Plate No. 26, and call for no special remark beyond the fact that the bands and lines are due chiefly to manganese, but are modified by bands and lines of iron, and possibly by the orange lithium line 6108.

The red lithium line 6708, and the red potassium lines 6938 and 6911, do not appear on the photographs, but were seen by the eye observations.

The Spectrum of the "Over-blow."

During the earlier part of the "over-blow" the flame is very short and not very luminous; the light emitted acts but feebly on a photographic plate. There is more fume expelled during this stage than during the first or carbon period of the "blow," and the quantity increases as the "over-blow" proceeds; the flame during this time increases in size and brilliancy, and towards the end emits a brilliant golden-yellow coloured light. There is a notable increase in the quantity of fume expelled after the "over-blow" has proceeded for 2 minutes.

The spectrum of the flame during these 2 minutes is very feeble, but during the latter 3½ minutes it is much stronger. It extends from the red end of the spectrum to about wave-length 4000, in the violet; it is strongest in the green about wave-length 5500, but fades rapidly towards the violet.

The following lines have been observed in the photographs:—

Wave-length.	
5893.	A line present in all the spectra, due to sodium.
5432.	Sharp, observed only on one plate, No. 16.
5394.	Sharp, observed on two plates, Nos. 16 and 17.
4047.	Lines present in all spectra due to potassium.
4044.	
4034.	Lines present in all spectra, due to manganese.
4033.	
4030.	

These lines are not as sharp on the photographs as the corresponding lines in the spectra photographed during the first period of the "blow."

The two lines, wave-lengths 5432 and 5394, coincide with two lines near the edges of the bands in the flame spectra of the manganese and its compounds. It was not apparent at first why these lines should be present in the spectra of some flames and not in the spectra of others, and they were not observed in the eye observations made during our first visit to Middlesbrough. In August, 1898, on a second visit to the same works, eye observations were made with the object of solving the problem. Three vessels, Nos. 2, 3, and 4, were in use on the day in question, and it was found that these two lines were strongest in the spectrum of the flame issuing during the "blow" from vessel No. 4, and also that the edge of the strongest green band was also seen.

The lines were sharp, and, when using a very narrow slit, stood out clearly on the strong continuous spectrum. The two lines were stronger in the spectrum of the flame issuing from No. 3 vessel than in that from No. 2 vessel. They were observed in two or three "blows" in each vessel and with similar results each time.

It was found from the inquiries made of the officials that whilst all three vessels were receiving similar charges of metal and lime, the lining of vessel No. 4 was almost new, and a new bottom had been fitted to it shortly before the time of observation. About forty charges had been blown in vessel No. 3, and about seventy in No. 2. The intensity of the line diminishes therefore with the age of the lining.

The tar in the lining is not wholly decomposed when the lining is worn out, but the rate of the decomposition and the quantity of the volatile products must be less than in a new lining, and it appears to us that it is the volatile products of the decomposing tar which cause these lines to appear in the spectra of the flames. In the "acid" process the flame drops as soon as the carbon is oxidised; the bands and lines in the spectrum immediately become diffuse, and the spectrum presents more the appearance of a continuous band of rays. According to Sir I. LOWTHIAN BELL, carbon monoxide is much more stable in presence of iron at high temperatures than carbon dioxide. SNELUS ('Chem. News,' vol. 24, p. 159) has shown that the proportion of the former oxide exceeds that of the latter in the gases issuing from the converter in the "acid" process during the latter two-thirds of the time, and towards the end there is no carbon dioxide present; so that all the carbon in the volatile products from the tar will exist in the flame during the "over-blow" as carbon monoxide. The hydrogen in these products and in the water vapour in the blast will pass out of the converter in the elementary state. These reducing gases acting on the compounds of manganese in the slag reduce some of the metal, which then passes off as vapour. The combustion of the hot gases, carbon monoxide, and hydrogen outside the vessel will give a hotter flame, and the increased temperature, together with the reducing gases in the flame, will account for the increased intensity of the manganese spectrum.

There are two points regarding the flame of the "over-blow" which require explanation. First, the great brilliancy of the flame during the latter part of the blow; secondly, the absence of the spectra of iron and calcium, and the feeble character of the spectra of potassium, sodium, and manganese.

Immediately the carbon is burnt out of the iron the flame drops to about one-sixth of its height; its luminosity also diminishes, and the quantity of fume increases. There can be no considerable variation in the temperature of the fluid charge during the few moments in which this great change takes place; the gases will therefore leave the converter at about the same temperature throughout this time.

It is probable that the impurities in the iron are oxidised chiefly by secondary reactions, but a portion of them is oxidised directly by the oxygen of the air. The volume of the iron in the converter is very much greater than the volume of the manganese, and yet almost the whole of the latter is oxidised and passes into the slag in the first few minutes of the "blow." The fact that the flame maintains its size until the sudden drop confirms this hypothesis. If this hypothesis is true the greater portion of the oxygen of the air enters into combination with iron, forming magnetic oxide of iron which is probably reduced to a lower oxide. This oxide of iron will be reduced to the metal by the silicon, manganese, and carbon present in the iron, and by the phosphorus or ferrous phosphide in presence of the lime in the order stated, and if the time of oxidation were extended indefinitely each one would be oxidised almost completely in turn. In the converter the oxidation is so rapid that all the reactions go on together; but the complete oxidation of each impurity is achieved in the order named.

Any phosphorus oxidised in the early part of the "blow" in contact with and entering into combination with the lime is finally removed from the iron, as the resulting compound is not reduced by the reducing agents present. About one-half the phosphorus is oxidised in this way before the flame drops ('J. I. and S. Inst.,' 1896, No. 1, p. 465).

The chief cause of the flame in the "acid" process has been shown to be carbon monoxide which, leaving the vessel at a high temperature, accompanied by vapours of iron, manganese, &c., is burnt by the oxygen of the air. The hottest part of this flame is the outer layer, where the temperature is increased by the combustion of the carbon monoxide and hydrogen. When the carbon is completely removed from the iron in the "basic" process, there is still the appearance of a flame which increases in length as the "over-blow" proceeds; the quantity of fume expelled increases rapidly at this period also. The short flame is bright, and may consist principally of finely divided particles of oxide of iron heated to the temperature of the bath of metal and slag, and by the heat evolved by the combustion of metallic vapour inside the vessel. The particles of matter expelled from the vessel are of two kinds: one consists of larger particles of slag produced by the violent agitation of the fluid charge inside the converter; these quickly fall to the ground. The other kind is the fume proper, composed of the products of combustion of the metallic vapours. The particles forming this fume are undoubtedly very small; this is proved by the fact that they scatter the light falling on them, and cast[*] a brown shadow, and also by the great height to which the cloud ascends. These very small particles will be kept at the temperature of the escaping gases.

The spectrum during the "over-blow" is very feeble in the violet and does not extend beyond 4000. The spectrum of the lime-light extends into the ultra-violet, so also does the light emitted mainly from the furnace walls, through the working doors of a Siemens-Martin steel-melting furnace.

It appears, therefore, that the temperature of the source of light in the flame of the "over-blow" is comparatively low, approaching that of a yellowish-white heat, and that the light of what is apparently a flame emanates from a torrent of very small particles, liquid or solid, heated to a temperature of such luminosity.

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[*] Mr. C. H. RIDSDALE first remarked the colour of this shadow (see 'Journ. Iron and Steel Institute,' No. 2, 1895, p. 35).

The temperature of the flame of the "over-blow" is therefore much lower than during the first period, when highly heated carbon monoxide is undergoing combustion. This will, at least in part, account for the feeble character of the line spectra of the alkalies, &c., at this stage. Another reason for this is discovered in the work of GOUY ('C. R.,' vol. 83, pp. 70-2; 'Phil. Mag.,' 1877, No. 2, p. 156), who found that with salts of lithium, calcium, strontium, and barium the maximum intensity of their flame spectra is reached "before the flame ceases to be reducing (for a copper wire), and is followed by a rapid diminution. With a large excess of air the spectrum disappears." The facts appear unfavourable to the opinion which attributes the bands of their spectra to oxides. "With sodium, on the contrary, the brightness of the flame augments rapidly as it becomes less reducing; the maximum is produced at the instant it ceases to be so, and is followed by a reduction of brightness much slower than with other metals."

Now without the decomposition products of the tar in the lining of the converter the flame cannot be more than feebly reducing; the only reducing gas is the hydrogen resulting from the decomposition of the water vapour in the air. GOUY's results, therefore, also in part explain the absence of the line and bands of calcium, and account for the feeble character of the other lines and for the changes in the spectrum described below.* The quantity of fume expelled is very great in the latter part of the "over-blow"; it is mainly composed of oxide of iron, and yet no bands nor lines of iron are present in the spectrum.

The brilliancy of the flame is so great towards the end of the "blow" that we were led to suppose that it might be due in part to an oxide of phosphorus. It is practically impossible to collect a sample of the fume free from the slag, and hence no direct evidence can be obtained on this point. F. E. THOMPSON states ('J. I. and S. Inst.,' 1896, No. 1, p. 464) that "phosphorus towards the end of 'hot blows' does not pass readily into the slag, and scrap must be added," that is, to cool the bath. This indicates that at the highest temperatures, in presence of a large excess of molten iron, phosphorus pentoxide is not formed as readily as at lower temperatures; the less acid lower oxide of phosphorus would probably pass away with the nitrogen and be converted into phosphorus pentoxide by the external air.

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* In the blow recorded on Plate 17 the red lines of the lithium and potassium with the yellow lines of sodium were seen with the direct-vision spectroscope as soon as the vessel was turned up. The bands of manganese appeared very early, but they were not sharp during the first four minutes; they increased in brilliancy throughout the first part of the blow, that is to say, until the flame dropped. The lithium and potassium lines continued very bright throughout the whole of this period. At the dropping of the flame the bands all disappeared, as practically also did the lithium and potassium lines, the yellow sodium lines alone remaining visible in the weak continuous spectrum. The flame at this time was very short and feeble. It lengthened slowly as the blow proceeded, until it became about one-fourth to one-third the length of the longest flame of the first period, and the lithium and potassium lines became stronger. When the fume began to issue in quantity it had at first a rich purple colour by transmitted light, which colour faded gradually, then became like brown umber, while the flame turned denser and denser until before long it had become very brilliant indeed. The appearance during the "after-blow" (third period) was similar to that at the end of the second period.

LINES and Bands observed in the Spectra of "Basic" Bessemer Flames and the Lines and Bands in Oxyhydrogen Flame Spectra with which they have been identified.

Wave-lengths.	Element.	Oxyhydrogen flame spectra.	
		Wave-length.	Description.
6290	Fe	6290·0	Beginning of a band.
6240	Mn	6240·0	Broad line.
6236	Fe	6236·0	m.r. edge of band.
6211	Mn	6211·0	Broad line.
6182	Mn	6182·0	,,
6160	Mn	6160·0	,,
6103	Fe	6103·0	Line.
	Li	6104·0	,,
6092	Fe	6092·0	,,
5972	Mn	5972·0	Line or edge of band.
	Fe	5981·0	Line.
5945	Mn	5945·0	Line or edge of band.
	Fe	5940·0	Line.
5893	Na	5893·2	Middle of D. and D.
5864	Mn	5854·0	m.r. edge of band.
	Fe	5870·0	,, ,,
5833	K	5834·5	Line.
	Fe	5831·7	,,
5803	K	5799·6	Strong line.
5793	Fe	5789·8	Line.
5764	Mn	5765·6	Line on band.
5756	Mn	5753·8	,, ,,
5720	Mn	5716·5	,, ,,
5690	Na	5686·0	Broad line on band.
5614	Mn	5612·0	Middle of broad line on band.
5581	Mn	5583·7	m.r. edge of very strong band.
5547	Ca	5543·5	(Eder and Valenta line.)
5520	Ca	5517·0	,, ,, ,,
5439	Mn	5434·7	Line on band.
5430	Mn	5432·2	,, ,,
5408 ?	Fe	5403·2	Weak line.
5401	Mn	5394·0	Middle of line on band.
5387	Mn	5390·5	,, ,, ,,
5361	Mn	5359·5	m.r. edge of band.
	K	5360·0	Strong line.
5343	K	5340·2	,, ,,
5328	K	5324·0	Weak line.
	Fe	5326·9	Line.
5269	Mn	5266·5	m.r. edge of band.
	Fe	5268·9	Line.
5232	Mn	5230·0	m.r. edge of band.
5193	Mn	5192·0	,, ,,
5161	Mn	5158·0	,, ,,
5110	K	5111·4	Weak line.
	Fe	5109·0	Very weak line.
5097	K	5098·4	Weak line.
5090	Mn	5090·4	m.r. edge of band.
5052	Mn	5051·0	,, ,,
5017	Mn	5012·5	,, ,,
	Fe	5010·5	Very weak line.
4980	Mn	4978·2	m.r. edge of band,

Lines and Bands observed in the Spectra of "Basic" Bessemer Flames and the Lines and Bands in Oxyhydrogen Flame Spectra with which they have been identified—*continued.*

Wave-lengths.	Element.	Oxyhydrogen flame spectra.	
		Wave-length.	Description.
4935	Mn	4933·2	m.r. edge of band.
4890	Mn	4894·0	,, ,,
4853	Mn	4854·5	,, ,,
4820	Mn	4825·0	Line.
4786	Mn	4784·5	,,
4735	Mn	4739·0	m.r. edge of band.
4643	K	4642·4	Line.
4556	Cs	4555·4	Very strong line.
	Mn	4561·3	m.r. edge of band.
4483	Fe	4482·34	Very weak line.
4463	Fe	4461·78	,, ,,
4428	Fe	4427·46	,, ,,
4406	Fe	4404·94	Weak line.
	Mn	4402·2	m.r. edge of band.
4384	Fe	4383·71	Line.
4377	Fe	4376·03	Weak line.
4327	Fe	4325·94	,, ,,
4307	Fe	4308·06	,, ,,
4272	Fe	4271·93	,, ,,
4250	Fe	4250·93	Very weak line.
	Mn	4245·4	m.r. edge of band.
4227	Ca	4226·93	Very strong line.
4217	Rb	4215·9	Strong line.
	Fe	4216·39	Weak line.
4202	Rb	4202·0	Very strong line.
	Fe	4202·20	Weak line.
4172	Ga	4171·6	Very strong line.
	Fe	4144·06	Weak line.
	Fe	4132·24	,, ,,
In Spectrum.			
26^2	Fe	4071·92	Weak line.
,,	Fe	4063·75	,, ,,
4047·4	K	4047·4	Strong line.
	Fe	4058·8	Line.
4043·8	K	4044·0	Very strong line.
4034·0	Mn	4034·5	,, ,,
4033·1	Mn	4033·2	,, ,,
4030·5	Mn	4031·0	,, ,,
3929·6	Fe	3930·5	Strong line.
3927·4	Fe	3928·0	,, ,,
3923·3	Fe	3922·8	,, ,,
3920·7	Fe	3920·0	,, ,,
3907·5	Fe	3906·4	Weak line.
3899·5	Fe	3899·9	Strong line.
3895·5	Fe	3895·9	,, ,,
3886·4	Fe	3886·2	Very strong line.
3878·7	Fe	3878·9	,, ,,
3860·0	Fe	3860·1	,, ,,
3856·3	Fe	3856·6	,, ,,

Lines and Bands observed in the Spectra of "Basic" Bessemer Flames and the Lines and Bands in Oxyhydrogen Flame Spectra with which they have been identified—*continued.*

Wave-lengths.	Element.	Oxyhydrogen flame spectra.	
		Wave-length.	Description.
3840·0	Fe	3840·2	Weak line.
3833·7	Fe	3834·2	,, ,,
3825·4	Fe	3825·9	Line.
3824·1	Fe	3824·5	Very strong line.
3820·7	Fe	3820·6	,, ,,
3766·7	Fe	3767·2	Weak line.
3763·5	Fe	3763·8	,, ,,
3758·6	Fe	3758·4	Line.
3749·4	Fe	3749·3	Very strong line.
3748·6	Fe	3748·2	,, ,,
3746·0	Fe	3745·8	,, ,,
3737·4	Fe	3737·4	Very strong line.
3735·2	Fe	3735·1	Strong line.
3733·7	Fe	3733·3	Line.
3726·0	Fe	3727·9	Very weak line.
3722·8	Fe	3722·5	Strong line.
3720·1	Fe	3719·9	Very strong line.
3706·0	Fe	3705·7	,, ,,
3680·0	Fe	3679·9	Strong line.
3648·0	Fe	3647·8	Weak line.
3633·0	Fe	3631·2	,, ,,
3619·0	Fe	3618·7	,, ,,
3609·0	Fe	3609·0	,, ,,
3581·5	Fe	3581·5	Strong line.
3570·1	Fe	3570·1	Weak line.
3566·0	Fe	3565·5	,, ,,
3526·0	Fe	3525·9	,, ,,
3523·0	Fe	3521·2	Very weak line.
3498·0	Fe	3497·6	Weak line.
3491·0	Fe	3490·7	Line.
3477·0	Fe	3476·6	Very weak line.
3475·5	Fe	3475·4	Line.
3466·0	Fe	3466·0	,,
3447·0	K	3447·5	Strong line.
3446·0	K	3446·4	,, ,,
3445·0	Fe	3443·8	Very weak line.
3441·0	Fe	3440·7	Very strong line.
3382·0	Ag	3383·0	,, ,,
3303·0	Na	3303·0 }	Strong line } a doublet.
		3302·5 }	,, ,, }
3283·0	Ag	3281·0	Very strong line.
3275·0	Cu	3274·0	,, ,,
3247·0	Cu	3247·7	,, ,,
3228·0	Li	3233·0	Strong line.
3221·0	K	3217·0	,, ,,

In order to confirm the results obtained by observations made on the flame from the converters, we decided to examine the metal with which the vessels are charged.

A tabulated statement of the spectrographic analysis of the metal is published in the 'Transactions of the Chemical Society,' vol. 79, p. 61, 1901. An account of the chemical analysis and the separation of the different constituent substances has already been communicated to the Society ('Proc. Roy. Soc.,' vol. 60, p. 393).

Analysis of the Crude Metal and Raw Materials used in the Manufacture of Steel.

The mixture of raw iron from the blast furnaces with molten pig iron from a cupola, and technically termed "mixer metal," was submitted to spectrographic analysis. The identity of such lines as were due to the presence of foreign metals was proved by carefully executed chemical analyses and by measurements from the photographs of the spectra of the various precipitated substances which were separated from the iron.

Mr. C. H. RIDSDALE very kindly supplied us with samples of the following raw materials:—

Roasted Cleveland ore, manganese ore, tap cinder, mixer metal, and burnt lime. Also a sample of flue dust from a blast furnace plant and a sample of the old lining of a converter.

These were analysed by heating each of them on a cyanite support in the oxyhydrogen flame for about half an hour, and photographing the spectrum of the flame coloured by the vapours of substances volatilised from them. The results are as follows:—

Roasted Cleveland ore contains iron, sodium, potassium, manganese, chromium, copper, gallium, lead, calcium.

Manganese ore contains manganese, iron, sodium, potassium, silver, copper, lead, indium, and calcium.

Tap cinder contains iron, sodium, potassium, manganese, copper, and lead.

Mixer metal contains, besides iron, sodium, potassium, nickel, copper, chromium, gallium, and manganese; also lead, cobalt, and silver in smaller proportions.

Burnt lime contains calcium, strontium, manganese, magnesium, iron, potassium, and sodium.

Flue dust contains iron, sodium, potassium, nickel, copper, silver, chromium, gallium, lead, manganese, calcium, and rubidium.

Old converter lining contains calcium, magnesium, iron, potassium, sodium, and strontium.

Some analyses have been made on larger quantities of materials, and the presence of some other elements detected. Rubidium, strontium, and magnesium occur in the part of the Cleveland ore which is insoluble in hydrochloric acid. Rubidium also occurs in the manganese ore.

The tap cinder contains traces of gallium and chromium. It is well known that vanadium is present both in the Cleveland ore, in the iron smelted from it, and in the basic slag. The basic Bessemer steel appears to be free from vanadium. This element gives no spectrum in the oxyhydrogen flame by which its presence may be recognised, but we have detected it by chemical reactions in solutions of the iron and of the basic slag.

The oxyhydrogen flame spectra of all the elements named above have been studied in the course of this investigation, as well as those of nearly all of the other elements.

The basic steel contains the following elements: Iron, gallium, manganese, lead, copper, nickel, cobalt, silver, chromium, calcium, sodium, and potassium. The chromium, cobalt, and silver are present in minute traces.

The Bessemer flame spectrum does not contain any lines of strontium, chromium, nickel, cobalt, or lead, elements present in the raw materials which are easily recognised by lines in the oxyhydrogen flame spectra. These are doubtless present in too small quantities to give lines strong enough to appear on the strong continuous spectrum; some of the weaker iron lines are also lost in the continuous spectrum in many of the Bessemer flame spectra photographed.

Some attempts were made to reproduce the more complex part of the spectrum of the Bessemer flame in the laboratory. Rolls of filter paper containing a powdered mixture of compounds of iron, manganese, potassium, sodium, and calcium were burnt in the oxyhydrogen flame, and the spectrum photographed in the same instrument as was used at Middlesbrough. After a few experiments, in which different proportions of the constituents were used, a mixture was arrived at which gave a very fair copy of the Bessemer flame spectrum.

Conclusions.

(1.) *Line spectra are not observed in the open-hearth furnace.*

The open-hearth furnace, though at a temperature as high as that of the Bessemer converter, does not show in a photograph the spectra of any of the metals known to be present, but a band of continuous rays extending from the red into the ultra-violet. The reason is that the atmosphere of the furnace is an oxidising one, and, moreover, the oxidation becomes greater when the furnace doors are opened. This was shown by observations on a cupola furnace; the metallic lines were visible only when the blast was turned off, that is to say, when the atmosphere had a reducing action.

(2.) *The phenomena of the "basic" Bessemer blow differ considerably from those of the "acid" process.*

First, from the very commencement of blowing a luminous flame appears. It is visible as soon as the lime dust has been blown out of the vessel. We have proved that it might proceed from possibly three sources: ignited lime dust, ignited alkali metals, which give a continuous spectrum, and hydrocarbons from the tar mixed with the "basic" lining of the vessel. Such ignition could only take place at this stage in consequence of the combustion of highly heated carbon monoxide or a hydrocarbon, and the presence of this latter substance is easily accounted for.

We conclude that the immediate production of a flame is caused not by the oxidation of material in the bath of metal, but by carbonaceous matter in the lining of the converter; that its luminosity is due partly to the volatilisation of the alkalies or alkali metals, but certainly prior to this effect to the incandescence of a cloud of lime dust carried up by the blast into the flame.

Secondly, volatilisation of metals occurs largely at an early period in the blow, and is due to the difference in the composition of the metal blown. In the "acid" process the crude metal contains much silicon, carbon, and manganese. The combination of the silicon and manganese with oxygen is the cause of the high temperature attained without any indication of the flame which subsequently appears, for both the products of combustion of silicon and manganese are liquid slags at that high temperature. In the "basic" process the combustion of carbon takes place at a much earlier period, because there is practically no distinct period when siliceous slags are formed; with the carbon monoxide the metals are volatilised, the atmosphere being a reducing one.

Thirdly, a very large amount of fume is formed at the close of the second period. This arises from the oxidation of metal and of phosphorus in the iron phosphide being productive of a high temperature, but little or no carbon remaining in the bath; the flame is comparatively short, and the metallic vapours carried up are burnt by the blast.

Fourthly, the over-blow is characterised by a very powerful illumination, from what appears to be a brilliant yellow flame which is generated within the vessel. During this over-blow there is a copious production of dense fume, composed of oxidised metallic vapours, chiefly iron. The particles forming the fume are undoubtedly of very minute dimensions, as is proved by the fact that they scatter the light which falls upon them and cast a brown shadow, and the cloud of fume ascends to a great height before it is dispersed. Such small particles while within the vessel are, doubtless, at the same temperature as the escaping gases. The spectrum of the over-blow is continuous and very feeble in the ultra-violet, as it does not extend beyond wave-length 4000. This indicates that the source of light is at a comparatively low temperature, approaching that of a yellowish-white heat. We conclude, therefore, that the source of light, or what apparently is a flame, really emanates from a torrent of very small particles, liquid or solid, at a yellowish-white heat. From the spectrum measurements we have evidence that the source of light during the over-blow is at a much lower temperature than that which prevails during the first period, when the highly heated carbon monoxide is undergoing combustion. This is sufficient to account for the feeble character of the line spectra of the alkali metals, &c., seen at this stage in the band of continuous rays. But a second reason for this may be found in the observation of GOUY, that a large excess of air causes the spectra of lithium, calcium, strontium, and barium salts to disappear, and the maximum intensity of their flame spectra is attained before the flame ceases to exert a reducing action. The spectrum of sodium is not affected in the same way, but, on the contrary, the intensity of its emitted rays increases rapidly as the flame diminishes in reducing power, and is at its maximum at the instant it ceases to be a reducing flame. The subsequent reduction in brightness is much slower than with the other metals. We believe, therefore, that GOUY'S results, in part, offer a satisfactory explanation of the absence of the lines and bands of calcium, and account for the feeble character of the other lines and for the changes in the spectrum already described (see pp. 485 and 498).

Fifthly, the spectra of flames from the "basic" process differ from those of the "acid" process in this respect that the manganese bands are almost entirely absent, but lines of elements not usually associated with Bessemer metal, and which have not appeared in previous photographs, are seen in these. We have proved that the difference in the spectra is due in the first place to a difference in the composition of the crude blast-furnace iron, which is best adapted to the working of the basic process. A second reason is the use of charges of "basic" materials placed in the vessel with the metal. The potassium, sodium, lithium, rubidium, and cæsium have been traced to the lime; the manganese, copper, silver, and gallium to the ore.

Other constituents, such as vanadium and titanium, are not in evidence, because they do not yield flame spectra, and, furthermore, because they pass in an oxidised state along with the phosphorus into the "basic" cinder.

(3.) *Differences in the Intensity of Metallic Lines.*

Iron lines to the number of fifty-three have been measured, as in the spectra of flames from the "acid" process, but they are of lower intensity, and some of those near the position of the solar line G are absent.

There is a less prominent appearance of the manganese lines and bands, the cause of which has already been stated; but there is a greater intensity of the lines of potassium and calcium, and to these the elements—rubidium, cæsium, gallium, copper, and silver, must be added.

The variations in intensity of the iron lines in the "basic" Bessemer flame are due to differences in temperature solely.

There are two lines which in some spectra appear to have become intensified, while the calcium band wave-length about 5540 has become greatly enfeebled. It seemed as if they had arisen out of the calcium band, but they are not calcium lines.

They have been observed in many different spectra of Bessemer flames and of iron and its compounds in the oxyhydrogen flame, and they are strongest where the iron lines are strongest. As no corresponding lines have been found in the spectrum of any other element, we believe we are justified in attributing them to iron.

Another line, 5792, was identified with iron in a similar manner.

Lines of potassium and the edges of manganese bands are shown to have been intensified by the proximity of iron lines in some cases. The following are examples: λ 5864, Fe λ 5870, Mn band 5269, Fe λ 5268·9. The line λ 5361 is common to both potassium and manganese, and 5343 to potassium and iron. A similar case has already been observed by us with two rubidium lines, 4216·3 and 4203·2, and two iron lines.

(4.) *A New Line of Potassium with Variable Intensity.*

The new line in the spectrum of potassium, wave-length approximately 4642, varies in intensity or disappears altogether; but provided the temperature is sufficiently high to cause the element to emit this ray, its brilliancy or intensity of chemical action is increased by diminishing the quantity of vapour in the flame, and so giving greater freedom of motion to the molecules. From a minute examination of all the plates on which the line 4642 has been photographed, and of those in which it is absent, and also having regard to the conditions under which the spectra in each case were produced, we conclude that it is not so much reduction of temperature, as reduction of the quantity of vapour in the flame, which increases its intensity.

This much is quite certain, that increase of intensity does not in this instance indicate a higher temperature.

We desire to record our sincere thanks for permission and facilities given to us to visit the works for the purpose of carrying out these researches, first to Mr. Arthur Cooper, Managing Director of the North-Eastern Steel Company, and to Mr. F. W. Webb, Chief Engineer of the Locomotive Department of the London and North-Western Railway; for assistance cordially rendered by Mr. C. H. Ridsdale, the chemist to the North-Eastern Steel Company, and by our friend Mr. E V. Clark, A.R.S.M., sometime Demonstrator of Chemistry and Assaying in the Royal College of Science, Dublin.

We have also gratefully to acknowledge that the expenses up to the end of 1895 were defrayed out of a sum voted upon the recommendation of the Government Grant Committee of the Royal Society.

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XI. *On the Conductivity of Gases under the Becquerel Rays.*

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§ 1. *Introduction.*

It has been known almost from the first discovery of the Becquerel Rays that these mysterious emanations caused gases through which they passed to conduct electricity. But though careful measurements have been made of the relative conductivity* of gases under the Röntgen rays (J. J. Thomson, 'Proc. Camb. Phil. Soc.,' vol. 10, p. 9) and under cathode rays (MacLennan, 'Proc. Roy. Soc.,' vol. 66, p. 375), little seems to have been done in this direction for the Becquerel rays. This paper deals with measurements of the kind in question, and with the conclusions which may be drawn from them. It will be first desirable to state clearly what conditions must be complied with in order that the quantities measured may have an intelligible meaning.

In the first place, the E.M.F. applied to the conducting gas must suffice to "saturate" the current. In other words, it must be so great that a further increase will not increase the current appreciably (see J. J. Thomson, 'Phil. Mag.,' November, 1896). In the second place, it is essential to make certain that the layer of gas employed is so far rarefied that the absorption of the radiation by it is inappreciable. If this condition is not complied with, the layers of gas more remote from source of radiation are less powerfully affected by it than the nearer ones. The effective strength of the radiation will thus depend on the absorbing power of the gas at the particular pressure, and the observed ratio of conductivities of two gases at the same pressure will not represent the ratio of their conductivities under radiation of a given strength.

It is universally recognised that the conductivity of gases under Röntgen and Becquerel rays is due to the production in them of positive and negatively charged

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* It is hardly necessary to state that the word "conductivity" applied to an ionised gas is used in quite a different sense from the same word when applied to a metal. The expression "conductivity" of a gaseous conductor is used to denote the current which it will carry under a saturating electromotive force.

ions by the rays. When the current is saturated, the ions travel across from one electrode to the other so quickly that there is no time for recombination to take place to any appreciable extent. Thus the current measures the amount of ionisation produced, supposing the strength of radiation to remain constant. In order to determine whether or not the radiation was sensibly absorbed, the conductivity was observed at different pressures. When the conductivity was proportional to the pressure, it was concluded that no sensible absorption took place in the layer of gas at any pressure within the range of the experiment. When, on the other hand, the conductivity did not increase so fast as the law of proportionality to the pressure would require, it was clear that the absorption was appreciable. In such a case, the gas was rarefied far enough to ensure that the law of proportionality should be obeyed.

§ 2. *Radio-active Substances.*

The radiation from various radio-active bodies has been investigated. These included—

(1.) A preparation obtained from DE HAEN, of Hamburg (see 'Wied. Ann.,' vol. 68, p. 902). This substance, as a few simple chemical tests showed, consists principally of barium carbonate, and, no doubt, its very strong radiating power is due to the presence of the new metal, radium, discovered by Madame CURIE in pitchblende, which is separated, together with barium, in the analysis of that mineral.

This radium preparation gave out radiations of at least two distinct kinds: one easily absorbable by solids or gases, and, as CURIE has shown ('Comptes Rendus,' vol. 130, p. 73), not deflected by magnetic force. The other, more penetrating, and deflectable by the magnet. The relative conductivities in gases due to each of these kinds has been investigated.

Since, when investigating the conductivity due to the non-deflectable kind, the other variety were also present, it is important to inquire whether the proportion is large enough to vitiate the results.

In CURIE's paper the following numbers are given, showing what proportion of the conductivity is due to the non-deflectable rays at various distances from the source:—

Distance centim. . .	7·1	6·9	6·5	6·0	5·1	3·4
Percentage	0	0	11	33	56	74

The absorption of the air, of course, accounts for the small proportion at long distances. A rough extra-polation from these numbers shows that close up to the substance only about 10 per cent. of the ionisation is due to deflectable rays. The conductivity ratio to be measured did not differ in any case more than about 30 per cent. for the two kinds of rays. Thus the error due to the presence of deflectable rays is only of the order of 3 per cent., an amount not well outside the errors of experiment.

(2.) A preparation containing the other radio-active constituent of pitchblende, polonium. This body I obtained as follows :—

Pitchblende was dissolved in dilute nitric acid. The filtered liquid was treated with sulphuretted hydrogen. The precipitate was found to contain antimony, arsenic, copper, and bismuth, the polonium being associated with the last. To remove antimony and arsenic, the precipitate was digested with ammonium sulphide. The liquid was filtered off, and the remaining precipitate, containing copper and bismuth, with polonium, was dissolved in nitric acid, and excess of ammonia added. The resulting precipitate, consisting of bismuth and polonium hydroxide, was tested for radio-activity, and found to be fairly active. Some of it was fused with an excess of potassium cyanide with a view to reducing it to the metallic state. The contents of the crucible, digested with water, and filtered, gave a black insoluble powder, not a coherent button of metal as had been anticipated. This black powder presumably consisted of the reduced metal. It was many times more active than the original oxidised product, and was accordingly used in the experiments.*

(3.) Another specimen of polonium was used which I owe to the kindness of Sir WILLIAM CROOKES. I have no exact account of the method of preparation, but Sir W. CROOKES tells me that it was, in outline, the same as that described by CURIE ('Comptes Rendus,' vol. 127, p. 175). This substance was considerably more powerful than the polonium which I prepared.

(4) Ordinary uranium compounds, as met with in commerce, have a feeble radio-activity. Indeed, it was in these that BECQUEREL first detected the effect. Sir W. CROOKES has shown ('Proc. Roy. Soc.,' vol. 66, p. 409) that the activity is due, not to the uranium itself, but to a powerfully radiating body accompanying it in small quantities, and probably distinct from both radium and polonium. He has shown (*loc. cit.*) that if ordinary commercial crystallised uranium nitrate be dissolved in ether, the resulting liquid divides into two layers—the one consisting mainly of an aqueous, the other of an ethereal, solution of uranium nitrate. The water, of course, is supplied by the water of crystallisation of the salt. The solid obtained by evaporating the aqueous solution is distinctly more powerfully radio-active than that from the ethereal one. In my experiments a product was used which had been several times concentrated in this way.

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* The improvement in the polonium due to treatment with potassium cyanide at a red heat was so marked, that it does not seem possible to explain it by the greater concentration of the material when in the metallic condition. The observation is of some interest, and I hope to investigate the matter further; but it bears only indirectly on the subject of the present paper.

Uranium, as well as radium, gives off two types of radiation, one much more penetrating than the other. (See RUTHERFORD, 'Phil. Mag.,' January, 1899.) BECQUEREL has shown ('Comptes Rendus,' vol. 130, p. 1583) that some at least of the uranium rays are deflected by the magnet, but he does not seem to have made any experiments to decide whether both types are so, or only one. In fact, the radiation is so feeble as to make such experiments very difficult. But it seems probable that, as in the case of radium, the penetrating type of rays are deflected, while the others are not so.

My experiments were exclusively concerned with the rays from the bare compound, which consist for the most part of the easily absorbed type. Only a small fraction of the total conductivity is due to the other kind, so small, in fact, as to make any determination of the conductivities difficult, unless some more powerful preparation than mine were available. The experiments were accordingly confined to the easily absorbed type of radiation.

§ 3. *Method of Experimenting.*

The gas of which the conductivity was to be determined occupied the space between two parallel plate electrodes. One of these was maintained at a high potential, the other connected to the one pair of quadrants of an electrometer, the other pair being to earth. The rate of movement of the needle then gave a measure of the current through the gas.

When the rays which had penetrated through a considerable thickness of metal were to be investigated, the arrangement was as follows:—

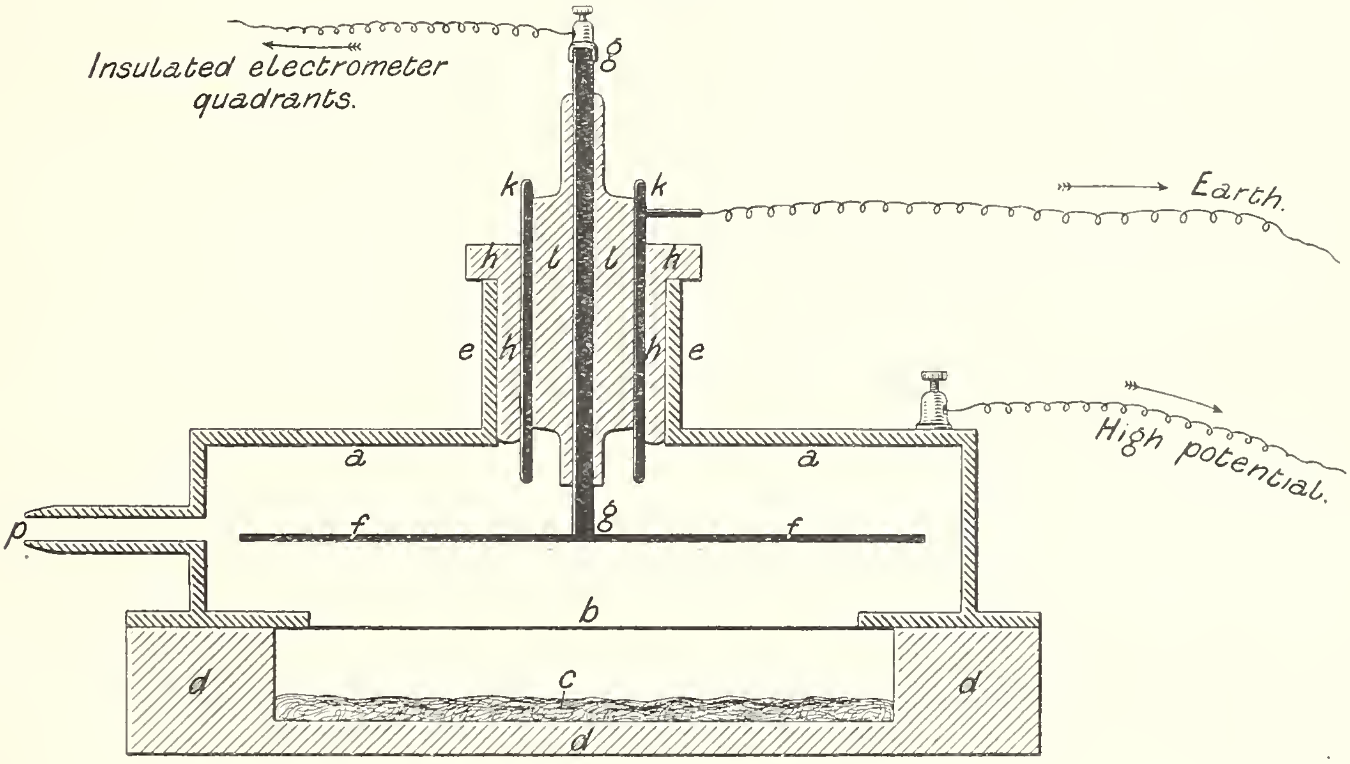
The gas of which the conductivity was to be determined was contained in the apparatus represented in section by fig. 1. It consists of an air-tight cylindrical brass box *a*, about 1 inch long, 3 inches diameter, provided with a bottom of thin copper sheet *b*, soldered on, through which the rays penetrated into it. The bottom was ·007 centim. thick. Inside this cylinder, and parallel to its ends, was the insulated metal plate *f*, carried on the rod *g*. This plate was used as the low-potential electrode.

The arrangements for insulating the plate *f* from the outside cylindrical brass box, which served as the high potential electrode, were somewhat special.

At *e e* is seen the brass neck of the main vessel. Inside it is fixed an ebonite collar *h*, and inside this again a second brass tube *k*. This inside brass tube carries the ebonite stopper *l*, up the middle of which the brass wire *g* passes. This latter carries the electrode *f*, and is connected outside to the electrometer. The intermediate brass tube *k* is put to earth. Finally, the outside vessel is put to the + terminal of a battery of 100 storage cells, the other terminal of which is to earth.

The ebonite collar *h* has thus to sustain an E.M.F. of 200 volts; but the ebonite stopper *l* has only to sustain 3 or 4 volts, since the potential to which *f* was allowed to rise, in order that a suitable reading of the rate of charge might be obtained, never exceeded this value. Now a slight leakage through *h* is of no importance, its only effect being to take a small current from the battery. On the other hand, a failure of the insulation of the stopper *l* would vitiate the measurements. But the danger of such a failure is but small, because of the smallness of the E.M.F. to which this insulation is exposed. If a simple ebonite stopper were used to carry the electrode, these more complicated arrangements being dispensed with, it would be essential that its insulation should be perfect even when exposed to the whole 200 volts.

Fig. 1.



The brass vessel was screwed down to a block of lead, *d*, by suitable thumbscrews. A circular cavity, *c*, was turned in this block, and served to contain the radio-active material.

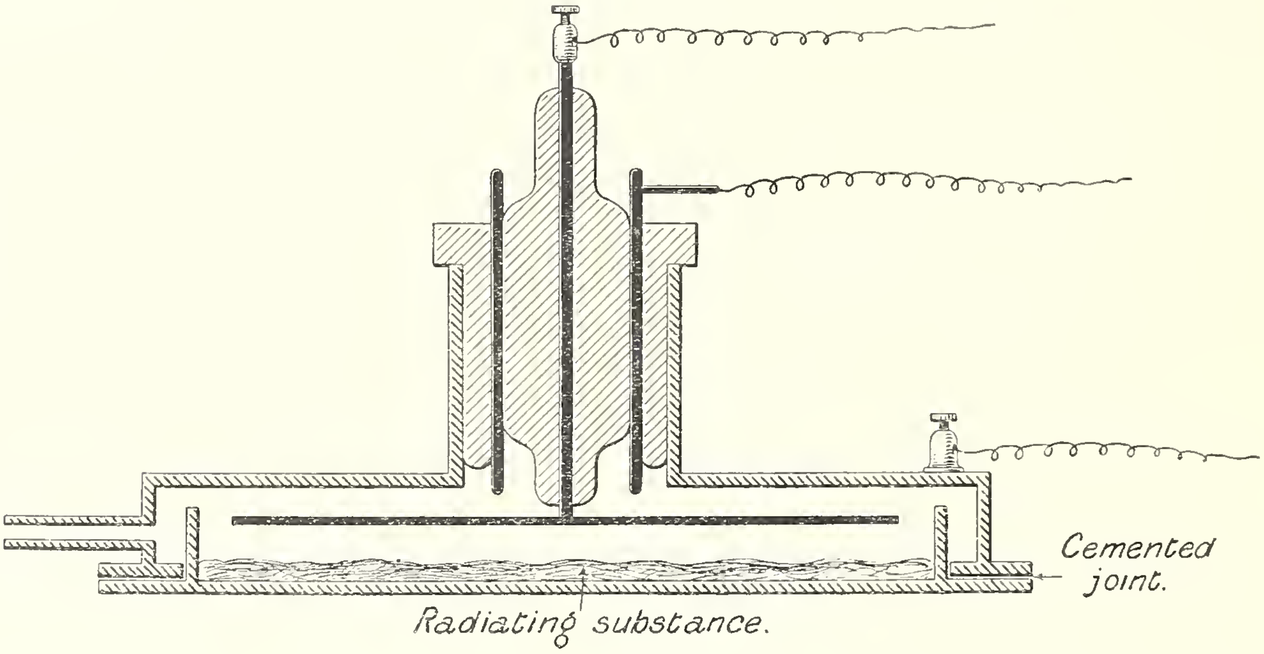
At *p* a side tube was soldered in, through which the apparatus could be exhausted and gas admitted.

For investigating the easily absorbable radiation, a slightly different arrangement was employed. Instead of the thin copper bottom soldered on, a thick circular brass plate was used. On this was laid a lead tray containing the radio-active body; the brass cover, in this case provided with a flange round the bottom edge, was then cemented on with the soft cement known as Prout's Elastic Glue.

Fig. 2 will make the arrangement clear. The vessel inside was connected to a water-pump, which would exhaust to a pressure of 15 millims., and to a manometer.

The procedure in investigating each kind of radiation was as follows :—First, the rate of leak was taken in air at different pressures; the limit of pressure within which the rate of leak was sensibly proportional to the pressure was thus ascertained. And in the subsequent measurements care was taken to be well inside this limit.

Fig. 2.



In comparing the different gases, air was in all cases taken as the standard. The rate of leak in air was measured at a pressure such as to give this rate a convenient value. The air was then removed by the water-pump, and the gas under investigation admitted. The apparatus was several times exhausted and refilled to ensure purity. Finally, the pressure was adjusted to give about the same rate of leak as that previously measured in air, and the exact rate carefully determined. If p p' were the pressures of the gas under investigation and of the air respectively, i i' the observed rates of leak, then the relative saturation conductivity of the gas was given by the fraction $\frac{i\,p'}{i\,p}$, air at the same pressure and under radiation of the same strength being taken as unity.

The electromotive force used was 200 volts, amply sufficient to produce saturation under the conditions of the experiments.

In investigating the vapours of volatile liquids, such as methyl iodide, it was necessary to take care that the vapour should not be so nearly saturated as to deviate sensibly from BOYLE's law. The rates of leak through vapours were accordingly taken at a pressure of not more than half that which would have been in equilibrium over the liquid at the same temperature. These vapours gave large conductivities, and the smallness of the pressure at which it was necessary to work did not cause any inconvenience.

I shall now mention the methods of preparation of the gases employed. It would not have been worth while, in view of the limited accuracy of the electrometer measurements, to have taken more than ordinary care to secure purity.

The gases were in all cases dried by passing them slowly over phosphorus pentoxide.

The *hydrogen* was prepared from sulphuric acid and zinc, and purified by occlusion in palladium foil, from which it was subsequently expelled by heat.

The *oxygen* was obtained by heating potassium permanganate.

The *hydrochloric acid* was made by the action of pure sulphuric acid on rock salt.

The *cyanogen* was obtained by heating mercury cyanide.

The *carbon dioxide* by the action of pure hydrochloric acid on white marble.

The *sulphur dioxide* from the liquefied gas commercially supplied, in a "syphon."

The *chloroform*, *methyl iodide*, and *carbon tetrachloride* vapours were from the pure liquids commercially procured.

§ 4. *Experiments on the Penetrating Radiation from the Active Barium Compound.*

The vessel used to contain the gas was that with the thin copper bottom, ·007 centim. thick, through which the radiation had to pass. As any easily absorbable radiation had already been filtered out by the copper bottom of the vessel, it was not to be expected that the rate of leak would be otherwise than simply proportional to the pressure; but it was thought best to test this experimentally, partly as a guarantee of the accuracy of the method of measurement.

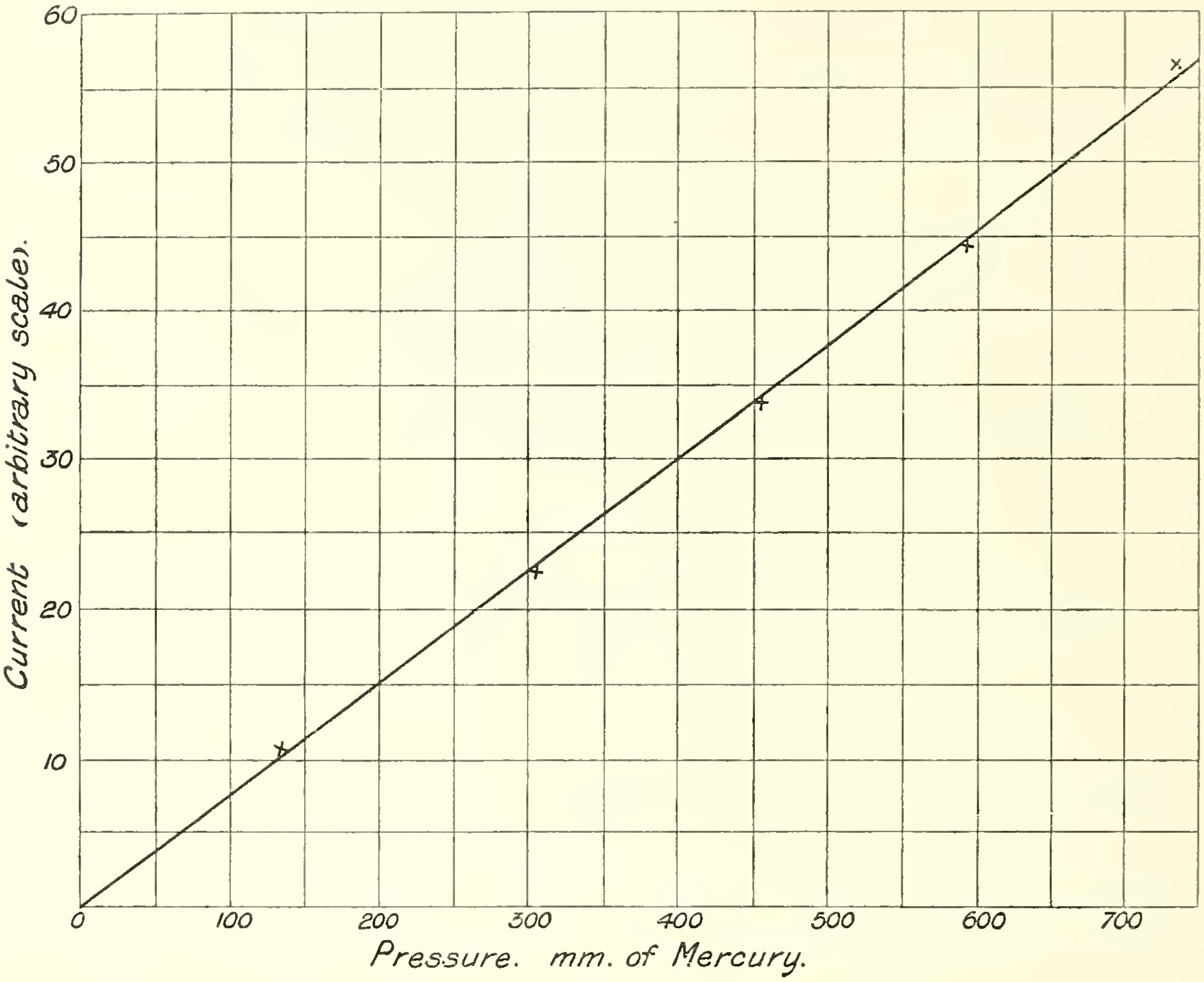
The following measurements were obtained :—

Pressure of air, millims. of mercury.	Time taken for electrometer needle to pass over 50 scale divisions.	Current (arbitrary scale).
	seconds.	
735	17·7	565
591	22·6	443
454	29·4	340
306	44·6	224
136	95·6	105

The measurements of the rate of leak were in each case the mean of about six observations. The results are plotted on diagram No. 1, and it will be seen that the rate of leak is closely proportional to the pressure up to atmospheric pressure, as was anticipated. It was safe, then, to employ any pressure within that range.

The first experiments were made on hydrogen. These will be given in full as specimens. In the other cases only the results of each determination will be given.

Diagram No. 1.—Radium. Penetrating Radiation.



Hydrogen.—Three entirely separate determinations were made, with different samples of gas.

(1.) Air. Pressure 144 millims.

30 scale divisions of electrometer passed over in—

55·5 57 55 57 57 59 55·5 seconds.

Mean 56·4 seconds.

Hydrogen pressure 727 millims.

30 scale divisions in—

73·5 71·5 73·0 73·5 71·0.

Mean 72·5 seconds.

$$\text{Conductivity of hydrogen} = \frac{56{\cdot}4 \times 144}{727 \times 72{\cdot}5} = {\cdot}154 \text{ (air} = 1).$$

(2.) Air. Pressure 123 millims.

30 scale divisions in—

66·5 64·5 63·5 65 66 seconds.

Mean 65·1 seconds.

Hydrogen pressure 696·5 millims.

30 scale divisions in—

72·5 74·5 73·0 75·0.

Mean 73·75 seconds.

$$\text{Conductivity of hydrogen } \frac{65{\cdot}1 \times 123}{696{\cdot}5 \times 73{\cdot}75} = {\cdot}156 \text{ (air} = 1).$$

(3.) Hydrogen. Pressure 347 millims.

30 scale divisions in—

137 133 134 132·5.

Mean 134·1 seconds.

Air pressure 129·3 millims.

30 scale divisions in—

57·5 57 58·5 58.

Mean 57·75 seconds.

$$\text{Conductivity of hydrogen} = \frac{57{\cdot}8 \times 129{\cdot}3}{134{\cdot}1 \times 347} = {\cdot}161 \text{ (air} = 1).$$

We have then as the results of these three determinations—

Conductivity of hydrogen ·154, ·156, ·161.

The final mean value being ·157.

The results for the other gases will now be given in a tabular form, since no good purpose would be served by writing them out at full length. To make the table complete, the results for hydrogen will be repeated.

Nature of gas.	Relative conductivity (air = 1).			
	Separate determinations.			Mean.
Hydrogen	·161	·154	·156	·157
Oxygen	1·22	1·23	1·28	1·21
Hydrochloric acid	1·48	1·52	1·38	1·46
Carbonic acid	1·61	1·57	1·54	1·57
Cyanogen	1·82	1·81	1·84	1·82
Sulphurous acid	2·35	2·33	2·27	2·32
Chloroform	4·83	4·98	4·87	4·89
Methyl iodide	5·22	5·33	4·98	5·18
Carbon tetrachloride	5·93	5·78	—	5·85

The observations on hydrochloric acid gas are not to be relied on. This gas rapidly attacked the metal of which the apparatus was made. The chlorine no doubt combined with the metal, setting the hydrogen free. The pressure rapidly diminished if the apparatus was left standing full of the gas. No doubt the difficulty might have been avoided by using an apparatus of platinum, but it was thought hardly worth while to go to the trouble and expense which this would have involved. It is to be regarded as a matter of accident that the separate determinations agree with one another as well as they do. In the other cases I believe the mean results are not, for the most part, more than 2 per cent. from the truth, if so much.

The discussion of these results, as well as those that follow, will be deferred till the end of the paper.

It will be worth while to mention that the conductivities of hydrogen and air were compared when an additional sheet of copper, equal in thickness to that which formed the bottom of the vessel (·007 centim.), was used to cut down the radiation. Almost exactly the same ratio as before was found. So far as this property is concerned, the radiation which penetrates one sheet of copper ·007 centim. in thickness is homogeneous.

§ 5. *Easily Absorbable Rays from Radio-active Barium Carbonate.*

To investigate the easily absorbable rays, the second form of apparatus was used, the radiating body being in contact with the gas under investigation, without the interposition of any solid partition. The radiation from the barium compound is enormously reduced by the interposition of so thin a screen as an ordinary piece of tinfoil; these "soft" rays accordingly form much the greater part of the whole. They cannot be conveniently separated from the penetrating rays used in the above experiments, but these latter are present in so small a proportion that they do not much affect the results (see above, p. 508). As is there mentioned, there is a difference in kind as well as in degree between the behaviour of the two types of rays. The penetrating rays are deflected by the magnet; the others are not so.

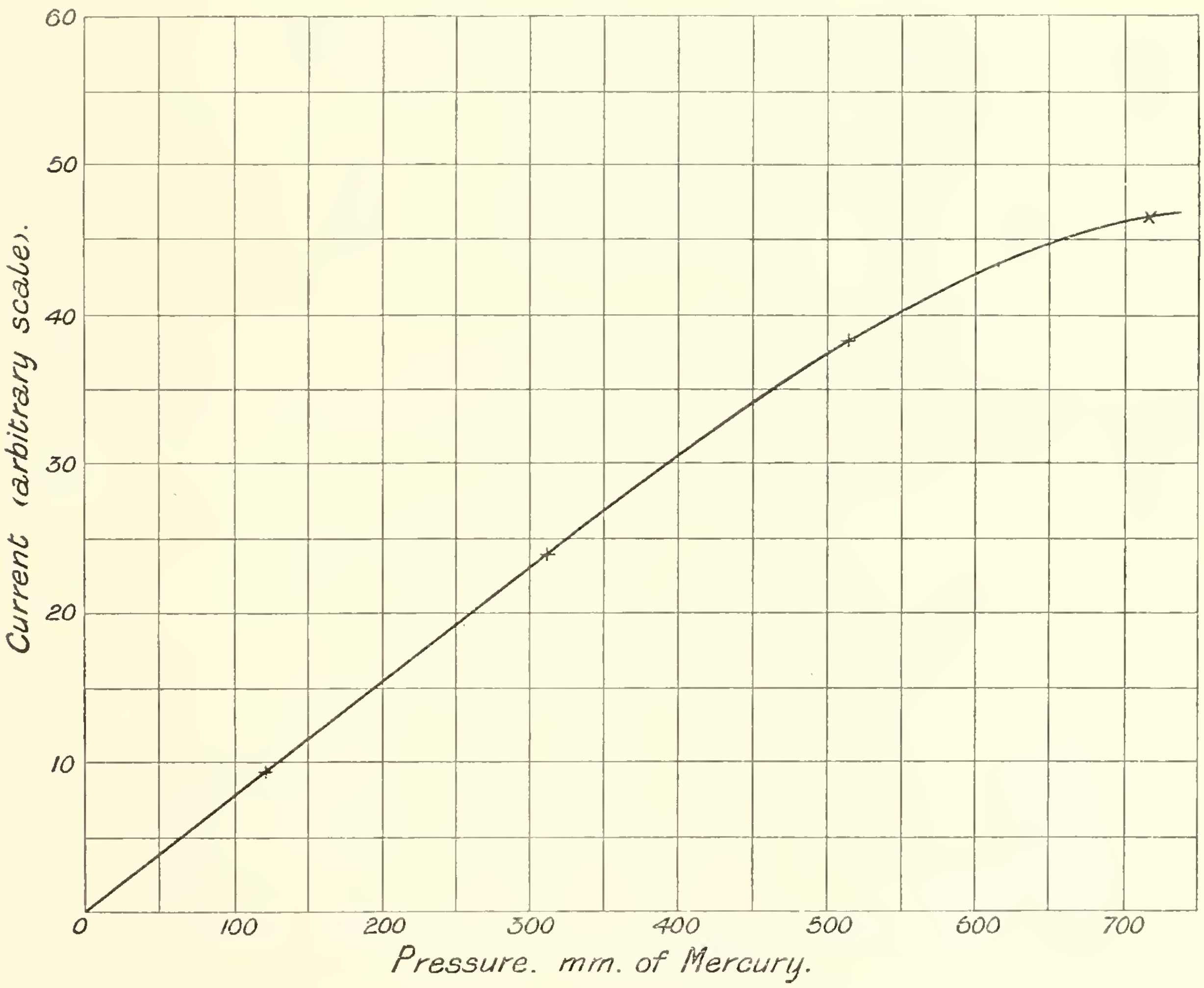
The barium compound was so very active that even when a very small quantity was taken, and the air pressure considerably reduced, the rate of leak was much too large for convenient observation. A parallel plate condenser was accordingly connected to the electrometer so as to increase its capacity. The distance between the plates was adjusted till the rate of leak had a convenient value, and was not afterwards altered. This plan is much better in practice than diminishing the sensitiveness of the electrometer by lowering the potential of the needle.

The rate of leak in air was determined at various pressures.

Pressure, millims.	Time taken to pass over 100 scale divisions (mean of 5 observations).	Current (arbitrary scale).
719	21·4	467
517	26·1	383
313	41·6	240
121	101·0	83

The results are plotted on diagram No. 2. Here the absorption of the radiation by the air is quite perceptible. The current is sensibly proportional to the pressure for pressures less than half that of the atmosphere. In comparing different gases, this limit of pressure was not exceeded.

Diagram No. 2.—Radium. Easily absorbed Radiation.



The measurements are given in the following table :—

Gas.	Conductivity (air = 1).				
	Separate determinations.				Mean.
Hydrogen	·218	·224	—	·212	·218
Sulphur dioxide . .	1·89	2·00	1·92	2·10	1·98
Methyl iodide . . .	3·78	3·72	3·82	36·3	3·74

§ 6. *Radiation from Polonium.*

This radiation is not deflected by the magnet* (CURIE, 'Comptes Rendus,' vol. 130, p. 73), and, so far at least, resembles the "soft" radiation from the active barium. The first set of experiments was made with the sample of polonium prepared by myself. The following numbers give the rate of leak at various pressures; about five observations were made in each case, and the mean taken.

Pressure, millims.	Time taken to pass over 100 divisions.	Current (arbitrary scale).
740	19·2	521
477	27·3	366
315	39·3	255
180	76·9	130
46	302·0	33

The results are plotted on diagram No. 3. There is not much indication of absorption until the pressure is comparable with that of the atmosphere. For the various gases, the following results were obtained :—

Gas.	Relative conductivity (air = 1).				
	Separate determinations.				Mean.
Hydrogen	·215	·226	·242	·215	·226
Oxygen	1·13	1·16	—	1·18	1·16
Carbon dioxide . .	1·53	1·55	—	1·54	1·54
Cyanogen	1·95	1·93	—	—	1·94
Chloroform	4·42	4·45	—	—	4·44
Sulphur dioxide . .	1·96	2·02	2·14	—	2·04
Methyl iodide . . .	3·32	3·88	3·43	3·42	3·51
Carbon tetrachloride.	5·30	5·37	—	—	5·34

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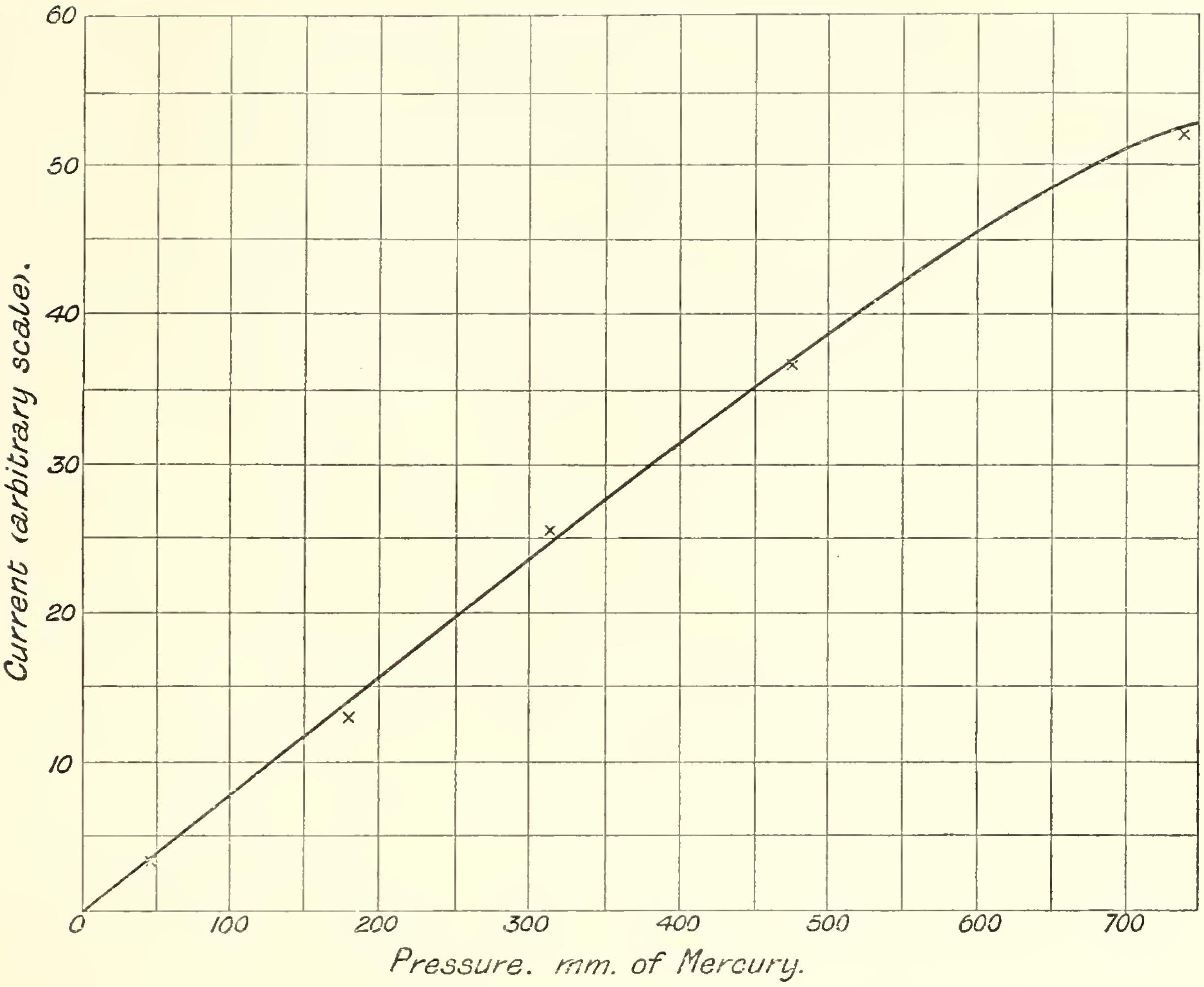
* This result is not in agreement with that arrived at by GIESEL ('Wied. Ann.,' vol. 69, p. 834). But my own experience confirms CURIE's conclusion.

Diagram No. 3.—Polonium, prepared by Author.



These experiments, as I have said, were made with polonium of my own preparation. Some of the experiments were repeated, using the more powerful polonium preparation of Sir W. Crookes.

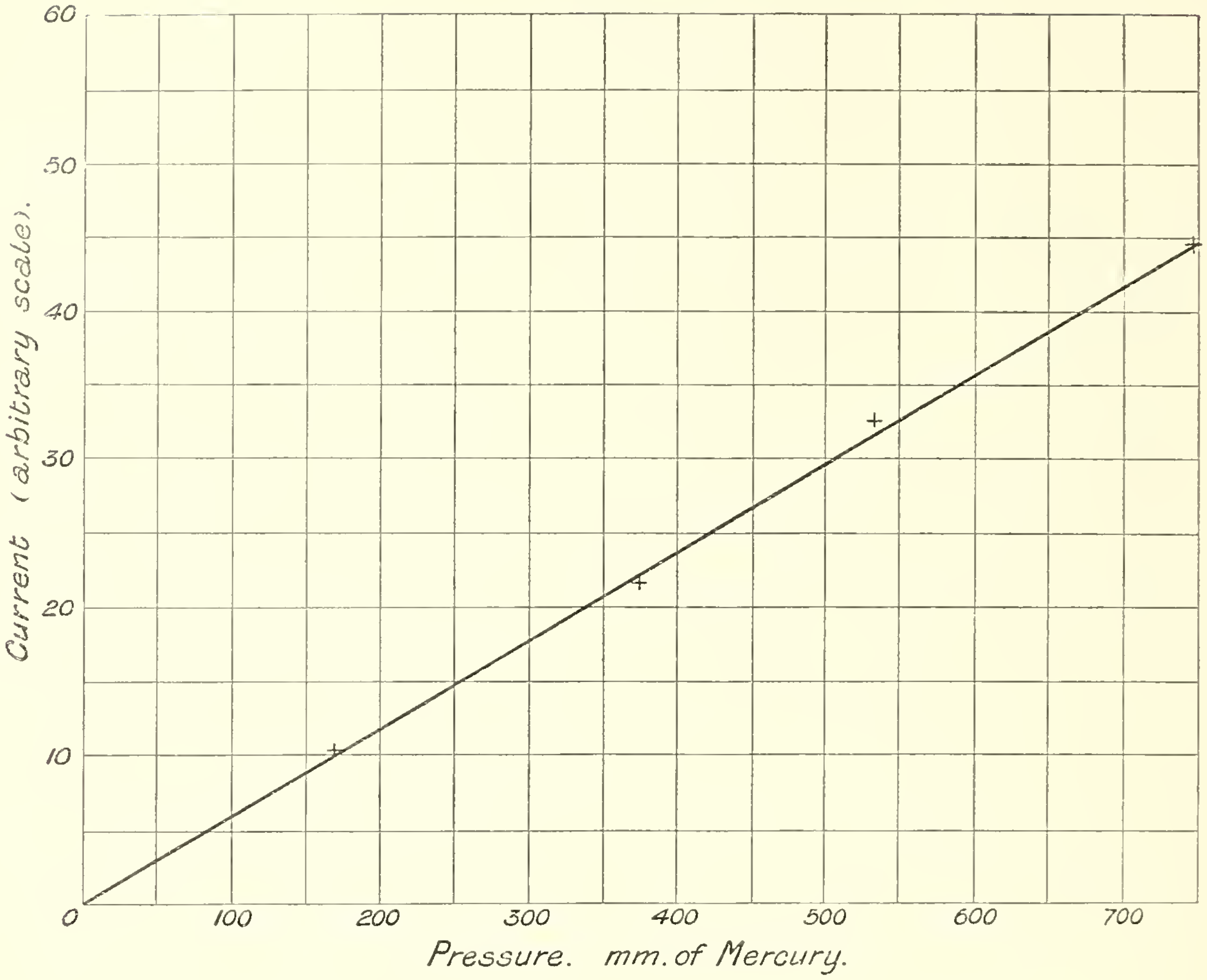
The rates of leak at different pressures were determined, with the following result:—

Pressure, millims.	Time taken to pass over 100 electrometer divisions.	Current (arbitrary scale).
	seconds.	
749	22·4	446
535	31·7	316
375	45·8	218
170	98·4	102

See diagram No. 4.

The rate of leak is here sensibly proportional to the pressure throughout the entire

Diagram No. 4.—Polonium, prepared by Sir W. CROOKES.



range. The radiation from this polonium preparation appears to be more penetrating than that from the other.

The next table gives the relative conductivities found with this radiation.

Gas.	Relative conductivity (air = 1).				
	Separate determinations.				Mean.
Hydrogen	·232	·222	·213	·210 ·216	·219
Sulphur dioxide . .	1·99	2·15	—	2·12	2·03
Methyl iodide . . .	3·44	5·48	—	3·51	3·47

The mean values do not differ much from those found for the other sample of polonium. There is not, I think, any evidence that the relative conductivities are not exactly equal in the two cases.

§ 7. *Radiation from Uranium Salt.*

The uranium salt having been placed in the apparatus, the rates of leak were observed at various pressures, with the following results :—

Pressure, millims.	Time taken for electrometer needle to pass over 100 divisions.	Current (arbitrary scale).
	seconds.	
726	27·7	361
565	32·6	307
372	44·5	225
176	81·4	123
116	116	86

Diagram No. 5.—Uranium.

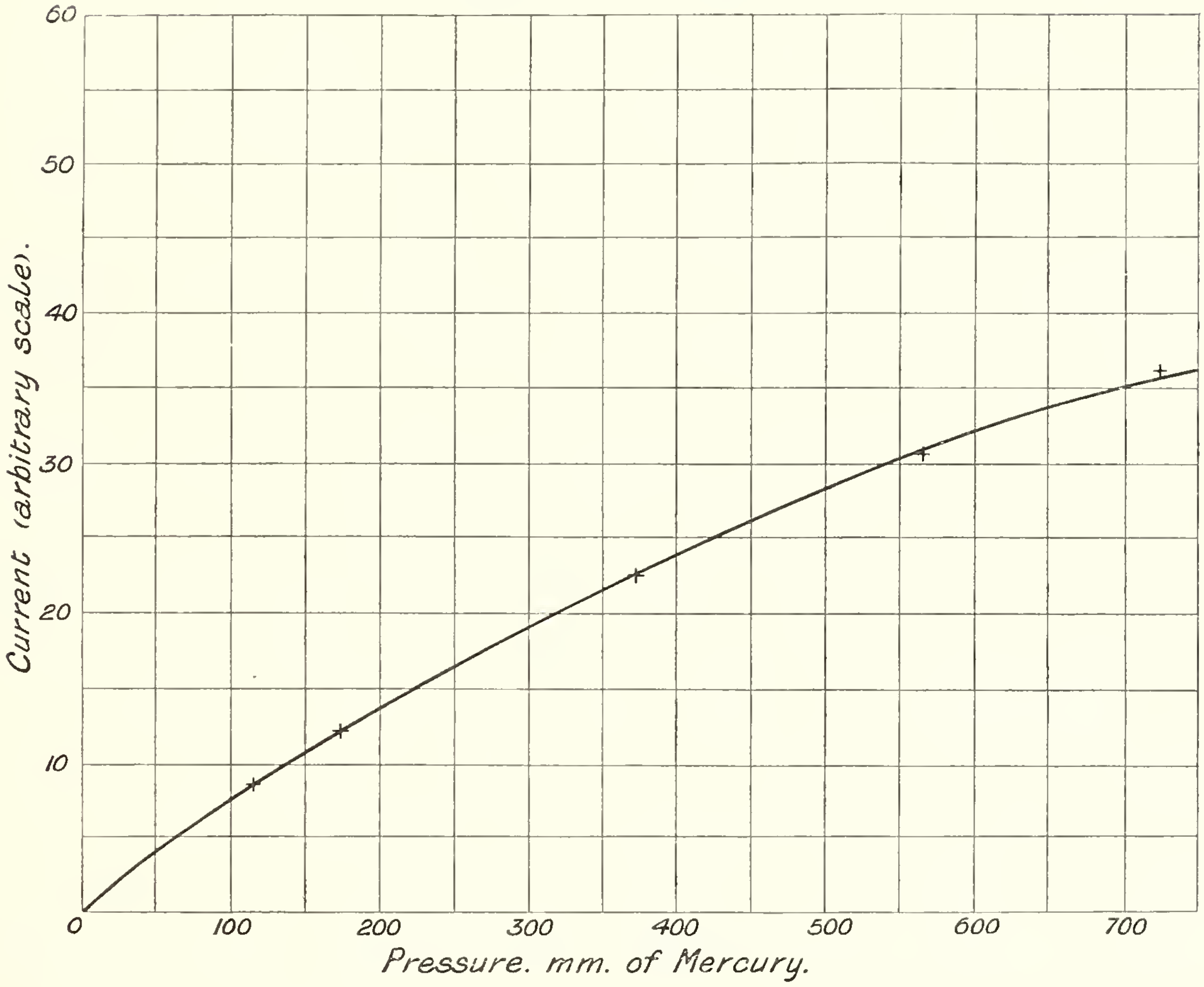


Diagram No. 5, which exhibits these results graphically, shows that for this type of radiation the current begins to deviate appreciably from that value which proportionality to the pressure would require, when the pressure only amounts to about 150 millims. This type of radiation is, therefore, more rapidly absorbed by the air than any of the kinds which have already been discussed. The rate of leak at pressures so low was rather too small for convenient measurement. A somewhat different method of comparing the gases was used, which avoided this difficulty. The rate of leak was taken in air, at some convenient pressure, usually about 250 millms., and afterwards at a slightly greater pressure, perhaps 260 millims. The gas for comparison having been introduced into the vessel. its pressure was adjusted till the rate of leak was intermediate between those given by air at the two slightly different pressures. The pressure at which the air would have given the exact rate of leak observed in the other gas was determined by interpolation, on the sufficiently accurate assumption that the rate of leak could be represented between these narrow limits as a linear function of the pressure.

In this way, then. were determined the pressures at which the two gases gave exactly the same rate of leak.

If now we assume that the absorption of the radiation by a gas is proportional to the saturation conductivity produced in the gas, it follows that if two gases are adjusted to give equal conductivity. the radiation is absorbed to an equal extent in each. Consequently the correction for absorption of the radiation is eliminated, and the relative conductivities are inversely proportional to the observed pressures.

The assumption that the absorption by a gas is proportional to the conductivity produced in it, is justified by RUTHERFORD's experiments ('Phil. Mag.,' Jan., 1899, p. 137). He found that if practically the whole of the radiation was absorbed, all the gases tried gave nearly the same rate of leak under a saturating electromotive force. This implies the truth of the relation in question. The next table gives the values found for uranium radiation, with the different gases.

Gas.	Relative conductivity (air = 1).				
	Separate determinations.				Mean.
Hydrogen	·208	·209	·214	·222	·213
Sulphur dioxide . .	2·14	2·06	2·08	2·03	2·08
Methyl iodide . . .	3·48	3·41	3·69	3·62	3·55

§ 7. *Summary of Results.*

The following table collects the final results of the experiments described in this paper. It includes also the results obtained by J. J. THOMSON ('Proc. Camb. Phil. Soc.,' vol. 10, p. 9) and PERRIN ('Rayons Cathodiques et Rayons de Röntgen') for relative conductivities under Röntgen rays, and by MACLENNAN, for cathode rays. It is convenient to quote these here for the sake of comparison.

Gas.	Relative density.	Relative conductivity.							
		Röntgen rays.		Cathode rays.	Radium rays.		Polonium rays.		Uranium rays.
		J. J. Thomson.	Perrin.		Penetrating type.	Absorbable type.	I.	II.	
Hydrogen . .	·0693	·33	·026	·069	·157	·218	·226	·219	·213
Air	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00	1·00
Oxygen . . .	1·11	—	—	1·106	1·21	—	1·16	—	—
Hydrochloric acid . . .	1·27	8·9	8	—	1·46 ?	—	—	—	—
Carbonic acid . . .	1·53	1·4	1·34	1·53	1·57	—	1·54	—	—
Cyanogen . .	1·86	1·05	—	—	1·86	—	1·94	—	—
Sulphur dioxide . .	2·19	6·4	6	—	2·32	1·92	2·04	2·03	2·08
Chloroform .	4·32	—	—	—	4·89	—	4·44	—	—
Methyl iodide	5·05	—	—	—	5·18	3·74	3·51	3·47	3·55
Carbon tetrachloride	5·31	—	—	—	5·83	—	5·34	—	—

§ 8. *Discussion of Results.*

It remains to be considered what conclusions can be drawn from the measurements.

In the first place, let us consider the penetrating radiation from the radium preparation. It will be seen, that in all cases except hydrogen, the conductivity is nearly *proportional to the density of the gas.* Hydrogen is the only case in which the departure from this law is considerable, its conductivity being about double what the law would require. But, since the conductivity of hydrogen is small, the *absolute* difference between the observed conductivity and that which the law would require is not larger than in the other cases. And it is possible that this is the right way of regarding the matter. This aspect of the question is brought out by a graphic representation of conductivity as a function of density. (See diagram No. 6.*) I think that it is impossible to doubt, in view of the results exhibited on this diagram, that the conductivity depends mainly on the density of the gas.

The much greater relative departure from the law in the case of hydrogen naturally raised the question of whether the experiments were open to criticism. They were carefully repeated, taking additional precautions against contamination by impurities. But the accuracy of the former determinations was completely confirmed. There is, I think, no reason whatever to doubt their substantial correctness.†

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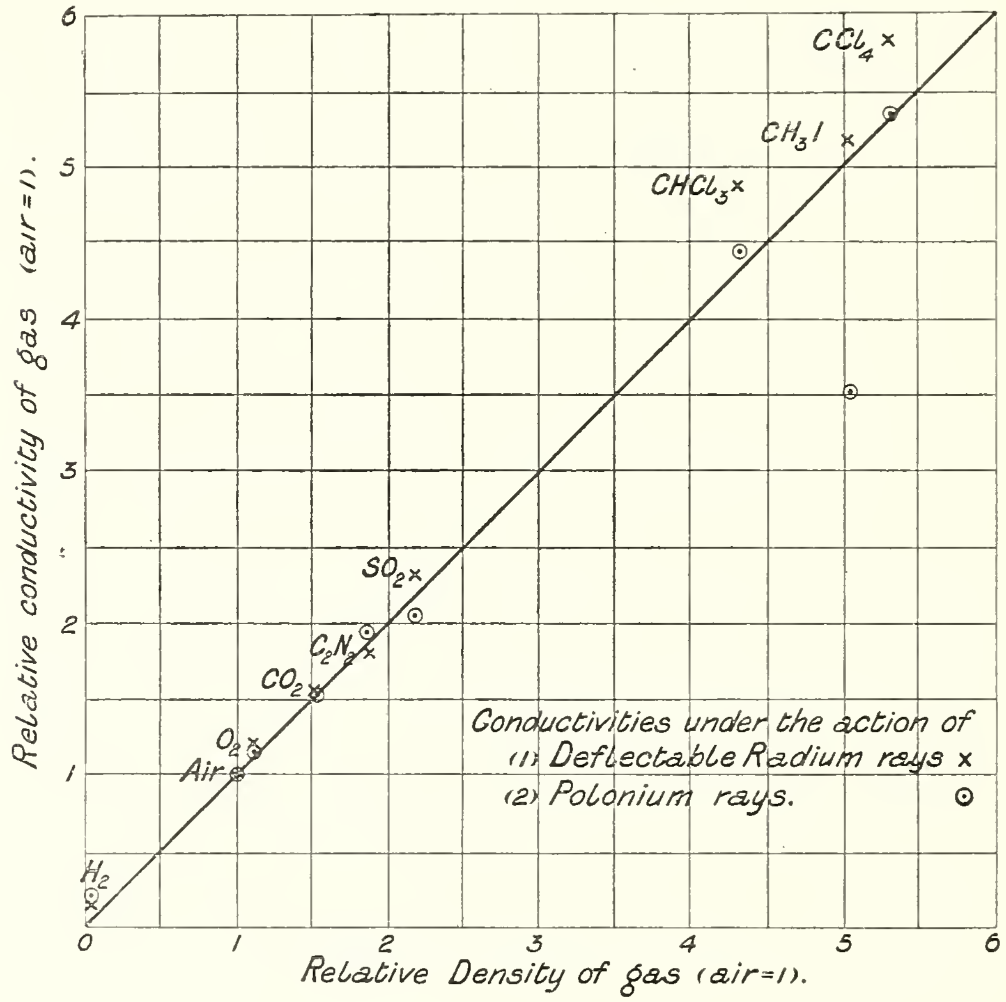
* The determination for hydrochloric acid is not represented in the diagram, because of its uncertainty, mentioned above.

† It is worthy of note, in connection with the large departure of hydrogen from the law of density so

It is certain, therefore, that the relative conductivity of gases under the penetrating type of Becquerel rays are mainly, though not wholly, dependent on their densities.

This result is of considerable interest; the investigations of BECQUEREL, GIESEL, and others have made it clear that rays of this type resemble the cathode rays in many respects. Both are deflected by a magnetic and by an electrostatic field; and carry an electric charge. Here we have one more property in common. For the experiment of MACLENNAN, quoted in the table, established the conclusion that the cathode rays too produce a conductivity in gases proportional to the density. There is not, however, the same departure from the law in the case of hydrogen. Under cathode rays the conductivity of hydrogen relative to the other gases is exactly what the law of density would require.

Diagram No. 6.



So much for the rays deflected by a magnet. We have still to consider the rays which are not deflectable.

well obeyed by the other gases, that there is large and unexplained discrepancy between the values found for the relative conductivity of hydrogen under Röntgen rays by J. J. THOMSON and PERRIN respectively, although their determinations agree fairly well in most other cases.

In the first place, it will be seen that all the compounds examined gave very nearly the same values for the relative conductivities. There is, in fact, no clear evidence of any difference between them in this respect. This result naturally suggests the conclusion that all these substances give absorbable rays of the same physical nature. Secondly, it will be seen that the non-deflectable rays give somewhat different results from the deflectable ones; these differences being well outside the errors of experiment. For the former, the departure from the law of proportionality is decidedly more marked than for the latter.

A reference to the table will show that these rays give results much closer to the deflectable rays than to the Röntgen rays. The very large conductivities characteristic of compounds of sulphur and the halogens under Röntgen rays are not met with in the case of Becquerel rays of any kind.

It is now very generally agreed that the deflectable Becquerel rays consist of a stream of negative ions proceeding from the radio-active body with enormous velocities (Becquerel, 'Comptes Rendus,' vol. 130, p. 109). But this theory gives no account of the nature of the other variety of Becquerel rays. I wish to make some mention of a possible solution of the question, indicating how far it appears to fall into line with the known facts.

Let us, then, imagine that the absorbable rays consist of a stream of positive ions moving from the radio-active body.

Now we know that the positive ions in gases carry the same charge as the negative, and that they have an enormously greater mass (J. J. Thomson, 'Phil. Mag.,' vol. 48, p. 547). Unless, therefore, their velocity is smaller out of all proportion than the negative ions, it is to be expected that they will be much less easily deflected by the magnet. This theory indicates, then, that by applying a very powerful field, the "soft" rays would be deflected. It would be well worth while to experiment in this direction.

Next, it may be noticed that the smaller penetrating power would be well accounted for by the size of the positive ions, which would, of course, make more collisions with the molecules of the surrounding gas than the much smaller negative ions.

Lastly, the experiments described in this paper seem to indicate that the deflectable rays produce conductivity in gases by the same kind of process as the others—a process quite different from that by which the Röntgen rays produce conductivity. This is in accordance with the suggestion as to the nature of the rays, conductivity being supposed to be produced in each case by the collision of the moving ions with the molecules, and the consequent splitting up of the latter into new ions.

In this investigation I have received much kind encouragement from Professor J. J. Thomson, and I wish to express my best thanks to him.

Appendix.

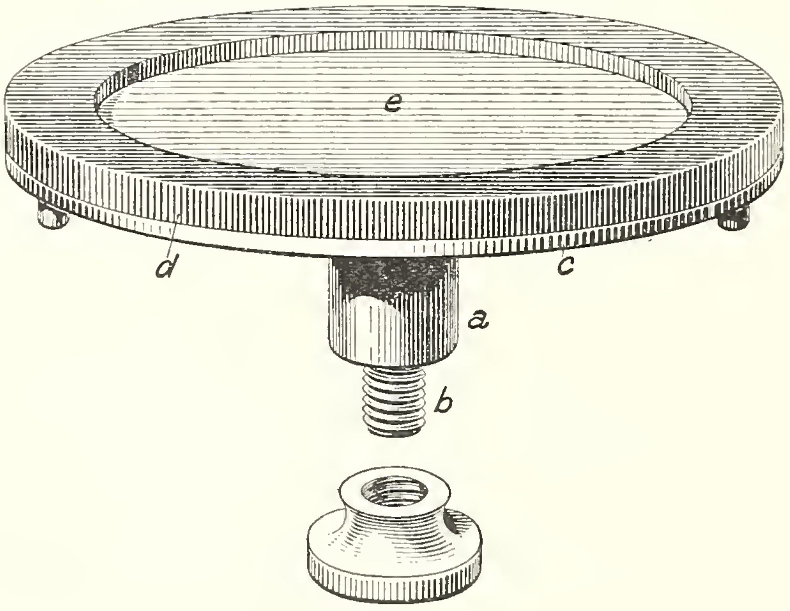
Note on a Practical Point in connection with the Quadrant Electrometer.

Most of those who have had occasion to work with an electrometer of the Elliot pattern must have experienced the difficulty of obtaining a jar of glass that would insulate in a sufficiently satisfactory manner. A jar cannot be considered even tolerably satisfactory unless the charge takes at least a week to leak down to half its original value. Such jars may occasionally be met with, but it is by no means easy to obtain one. It sometimes happens that a jar will satisfactorily retain a small charge for a much longer period than that mentioned, but, if it be charged more highly, the charge rapidly leaks down to a certain small value, after which the loss becomes very much less rapid. But, if the electrometer is required to be highly sensitive, such a jar is of course useless.

The "White Pattern" electrometers are much less subject to this difficulty, because their construction allows of a great length along the glass surface between the charged acid and the outside coating of the jar connected to earth. And it is the surface leakage alone that is practically to be feared.

After trying many jars and failing to obtain a satisfactory one, I adopted an arrangement which has proved very convenient as a substitute. A short brass pillar *a* (fig. 3) carries a circular brass disc *c*. On to *c* a round ebonite plate of the same diameter is fixed by means of three screws. The heads of these screws are on the underside of the brass plate, and they pass into tapped holes in the ebonite, which do not pass right through it. The outside of this ebonite disc is $\frac{1}{4}$ inch thick,

Fig. 3.



but the inside is turned down until it is very thin—less than $\frac{1}{2}$ millim. Into the circular recess thus formed is dropped a flat brass plate, *e*, of such a size as to fit it loosely. The whole arrangement is fixed horizontally between the pillars carrying the quadrants. For fixing it a screw, *b*, is provided, as a prolongation of the pillar, *a*. This screw passes through a hole drilled centrally in the base of the instrument, and a nut fixes the whole in position.

On *e* is placed a platinum crucible containing strong sulphuric acid, into which the prolonged axis of the needle dips as usual. A charge is given to the top brass plate, which is, of course, in conducting communication with the needle. The instrument can then be used as usual.

The insulation of this condenser has been found very satisfactory, the charge not diminishing by more than 20 per cent. of its original value in a week, and this when the charge was sufficient to make the sensitiveness very high.

One or two remarks may be made in conclusion. If the insulation shows a tendency to deteriorate with time, it can be made as good as ever by removing the surface of the thick rim of ebonite in the lathe, with emery paper, subsequently polishing the surface with bath-brick, applied on felt.

A platinum cup was employed for the acid, with the idea that with it there would be no tendency to any creeping of electrification, and consequent uncertainty in the potential of the needle. But very probably glass would be practically as good. A platinum wire dipping in the acid would then have to be connected to the upper brass plate.

Although there is no novelty in principle in this contrivance, yet it has proved so convenient in practice that it has been thought worth while to describe it.

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XII. *The Thermo-chemistry of the Alloys of Copper and Zinc.*

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Communicated by Professor J. H. POYNTING, *F.R.S.*

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Introduction.

IT has long been known that the alloying of certain metals with each other is accompanied by evolution or absorption of heat, but in very few instances have any measurements of the thermal changes been recorded.

There is now no reason to doubt the existence of definite atomic combinations between metals, and it has therefore become a matter of considerable interest and importance to ascertain the heat of formation of these inter-metallic compounds.

LAURIE,[*] CHARPY,[†] and others have adduced evidence pointing to the occurrence of chemical combination between copper and zinc in the making of brass, and an investigation of the thermo-chemistry of their alloys appeared particularly desirable, because a peculiar interest attaches to them in connection with Lord KELVIN'S calculation of molecular dimensions.

During the progress of this work papers, dealing wholly or in part with the same subject, were published by Dr. GALT[‡] and by M. HERSCHKOWITSCH[§]; further reference to these will be made in another place.

Choice of Method.

In making brass by the usual method, solid zinc is pushed beneath the surface of molten copper.

A violent action is frequently observed, and this is often cited as indicating the occurrence of chemical action between the metals.

Now copper melts at about 1080° C., but its temperature is doubtless higher when the zinc is added in the ordinary process of brass making, and since zinc boils at

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[*] 'Trans. Chem. Soc.,' 1888, p. 104.

[†] 'Bull. Soc. d'Encouragement,' Feb. 1896, pp. 33–34.

[‡] 'Rep. Brit. Assoc.,' 1898 and 1899.

[§] 'Zeits. f. Physik. Chemie,' Nov. 1898.

940° C., the violent action, before mentioned, is not necessarily the consequence of chemical union, but may be due to volatilisation of the zinc.

The difficulty of making a quantitative determination of heat evolution under such conditions appeared too great to be overcome.

The only recorded instance of such an attempt is to be found in Lord KELVIN's lecture at the Royal Institution in May, 1897, on "Contact Electricity in Metals," where it is stated that Sir W. C. ROBERTS-AUSTEN observed an evolution of 36 calories per gramme of brass formed by the mixture of 30 parts of zinc with 70 parts of molten copper.

No details of the experiment are given.

A more promising method consists in the determination of the differences between the heats of dissolution, in a suitable solvent, of various alloys of copper and zinc and of the free metals when merely mixed in the same proportions.

This method has been adopted in the present paper.

Preparation of Alloys.

The *copper* employed in making the alloys was the best electrolytic, which had been melted, and then rolled into sheet.

Analysis showed it to be free from arsenic and bismuth, but it contained about ·15 per cent. of lead and a trace of iron.

The *zinc* used had been re-distilled. It was free from arsenic and cadmium, but contained ·06 per cent. of iron and a trace of lead.

It dissolved in hot dilute sulphuric acid only after very prolonged immersion.

Alloys containing more than 30 per cent. of copper were made by adding the requisite weight of zinc to the copper, which had been previously melted in a carbon crucible under a layer of powdered charcoal, and thorough admixture was secured by vigorous stirring with a charred stick.

In making alloys containing less than 30 per cent. of copper it was found advantageous to melt the zinc and add the solid copper to it. The latter soon dissolved if the crucible was kept at a suitable temperature.

It was easier to make an alloy, rich in zinc, of a pre-determined composition in this manner than by adding zinc to molten copper.

The alloys were cast in an iron mould yielding plates about 7 centims. × 2·5 centims. × ·5 centim. and weighing some 70 grammes.

Their outer surfaces were then removed by filing and the central portions of the ingots were used in the experiments.

The composition of each alloy was carefully determined by estimating the copper electrolytically.

In certain experiments, afterwards abandoned, Dutch metal leaf and bronze powders were used.

In order to remove the oil, which the latter always contain, they were washed in carbon bisulphide, ether, and absolute alcohol, and were then dried at 90° C.

It was hoped that a considerable range of alloys would be covered by these materials, as their colours varied greatly from reddish-yellow to pale yellow; but analysis showed that none contained less than 67 or more than 87 per cent. of copper, and it was finally concluded that they were mixtures of various alloys.

Throughout this investigation it was found absolutely necessary to use the metals in a state of very fine division in order that the time occupied by their dissolution might be kept within reasonable limits.

Those alloys which contained between 20 and 40 per cent. of copper were easily shattered by a few heavy blows in a steel mortar, and the rough powder obtained was then ground in an agate mortar.

The remaining alloys were not sufficiently brittle to be thus treated, and the tedious process of reducing them to powder with a very fine file was employed.

These filings were then sifted through the finest linen, and only the small fraction which passed was used.

This powder was then thoroughly stirred with a magnet to remove any fragments of iron which might have come from the file.

In almost every case the latter precaution was found to be unnecessary.

It has been objected that the fragments of a crushed alloy, having been strained, possess more energy than before crushing, and that this may affect the heat of dissolution. ('Report on the Chemical Compounds contained in Alloys,' Brit. Assoc., 1900.)

In various experiments the author has employed a particular alloy, sometimes in the form of rolled sheet and sometimes as filings, or, if sufficiently brittle, then either as a finely crushed powder or as filings, and in no instance has any difference of the nature suggested above become evident.

Choice of Solvent.

Those solvents which most readily suggest themselves are chlorine-water, bromine-water, and nitric acid.

The first and second are not only disagreeable to work with but are also ineffective, unless the metal to be dissolved is in a state of extremely fine division. This necessarily limits their application to the small group of very brittle alloys. A number of experiments were performed with chlorine-water as solvent.

Turning now to nitric acid, Dr. GLADSTONE[*] has quite recently shown that the chemical action of a mere mixture of copper and zinc upon it is different from that of the alloy containing the metals in the same proportion.

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[*] 'Phil. Mag.,' vol. 50, p. 231.

Since this, in itself, must introduce a difference in the heat of dissolution according as the metals are mixed or alloyed, the use of nitric acid is unsuited to the purpose in view.

Dr. GALT'S* experiments, mentioned on the first page, were made with this solvent, and the conclusions drawn from them are, therefore, open to grave doubt.

After very numerous experiments, two solvents were found which gave very satisfactory results and were not open to the objections urged against nitric acid. They are, respectively, mixtures of solutions of ferric chloride and ammonium chloride, and of cupric chloride and ammonium chloride.

Both attack the metals and their alloys rapidly, and no gas is evolved. The reactions involved are simple and give rise to a moderate evolution of heat only.

The latter point is one of considerable importance, because the rather small differences between the heats of dissolution of the metals when merely mixed and when alloyed thereby assume greater relative values, and are consequently less affected by experimental errors.

This matter is further referred to on p. 544.

Apparatus.

The *calorimeter* consisted of a thin glass beaker of rather more than 500 cub. centims. capacity.

This was supported within a highly-polished nickeled vessel by a stout rubber ring of square section, which encircled the beaker just below its rim and rested on a gallery soldered inside the metal vessel.

The latter was screened from external fluctuations of temperature by a surrounding copper water-jacket furnished with a stirrer, the air-space between the nickeled vessel and the water-jacket being covered in with a copper lid.

The beaker itself was covered with a thin ebonite sheet pierced with holes for the stirrer and thermometer.

The *stirrer* consisted of a thin sheet of mica perforated with several holes and carried by a thin glass tube, which could be actuated from a distance by a cord passing over a pulley.

The *thermometer*, by GEISSLER, of Bonn, was divided into ·01° C., and it was easy to read to ·001° C.

The position of the mercury thread was so adjusted that at the beginning of an experiment it always stood as nearly as possible at the same division, and since the rise in temperature during an experiment was usually about 1° C., the same portion of the scale was always used.

This 1° interval was found sensibly correct when compared with a standardised thermometer.

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* 'Rep. Brit. Assoc.,' 1898 and 1899.

The *water equivalent* of the calorimeter, stirrer, and thermometer was determined by the method of mixture, an apparatus similar to that used by THOMSEN in measuring heat of neutralisation being employed.

Warm water was placed in an upper protected vessel, and when its temperature had been found, a plug was removed from the bottom, thus permitting the water to flow directly into cold water of known temperature contained in the calorimeter.

In three successive experiments the following values were obtained for the water equivalent :—

15·92 grammes, 15·6 grammes, 16·3 grammes.

The mean value of 16 grammes was adopted.

In most experiments a rise in temperature of about 1° C. was arranged, as this was a sufficiently large interval to be read accurately, and yet was not large enough to necessitate the introduction of a considerable correction for radiation during the one or two minutes occupied by the process of dissolution.

All the determinations were made with the room, as nearly as possible, at 18° C., and the temperature of the calorimeter and its water-jacket was always carefully adjusted to that of the room before beginning an experiment.

SERIES I.—CHLORINE-WATER AS SOLVENT.

Although the use of this solvent was ultimately abandoned, the actual procedure in performing an experiment was exactly similar to that employed in Series II. and III., and it will therefore be described in this place.

An aqueous solution of chlorine containing 5·3 grammes per litre was prepared ; its heat capacity was found to be sensibly equal to that of the water it contained. It was first sought to ascertain the heat of dissolution of copper.

The *copper* employed was prepared—

(*a.*) By dissolution of pure copper in nitric acid, ignition of the nitrate, and reduction of the resulting oxide by hydrogen.

(*b.*) By precipitation of copper from pure copper sulphate solution by means of redistilled zinc.

The copper powder was heated in dilute sulphuric acid, washed repeatedly with water, and then with absolute alcohol.

After drying, it was heated to redness in a current of hydrogen to reduce the small quantity of copper oxide which had formed during the washing and drying processes.

Copper, thus reduced in hydrogen, retains ·6 volume of that gas, but the heat developed by the union with chlorine of the quantity of hydrogen contained in the small weight of copper used is negligible.

No difference in the thermal behaviour of the specimens of copper obtained by the different processes was observed.

The powdered copper was sifted through the finest linen obtainable, and was used at once.

It was found necessary to moisten the powder before immersing it in the liquid in the calorimeter, because entangled air prevented actual contact between them, thus preventing or seriously delaying dissolution of the metal.

For this purpose the copper was weighed on a paraffined tissue-paper tray, measuring about 3 centims. square, and a few drops of water were then thoroughly incorporated with it by means of a stirrer made of the same paper.

Since not more than ·3 gramme of water was used, its introduction produced no appreciable change in the heat capacity of the calorimeter.

[The foregoing method of preparing the copper for dissolution was applied to the zinc and all the alloys employed in this work, and was absolutely essential to success.]

Blank experiments were made to ascertain the temperature corrections to be applied for radiation, and for cooling consequent on the escape of chlorine from the free surface of the solution.

Notwithstanding every care and precaution, it was only found possible to deal with the limited number of alloys which could be easily powdered by blows in a mortar, and many of these were so difficult to dissolve that the corrections for radiation and cooling by loss of chlorine became too significant and masked any relations existing between the differences of observed and calculated heats of dissolution.

It was, however, discovered that the observed heat of dissolution of an alloy was always less than that of a mere mixture of the same composition as calculated from the ascertained heats of dissolution of copper and zinc, and that these differences were too great to be attributed to error of experiment.

This evidence of the existence of heat of combination in copper-zinc alloys induced the author to seek a more suitable solvent.

[Note on the *Heat of Solution of Chlorine in Water*.

It was easy to measure the heat of dissolution of copper powder in chlorine-water with considerable accuracy, as the process did not occupy more than one or one and a half minutes.

Four experiments, none of which differed more than ·25 per cent. from the mean, gave

$$\mathrm{Cu} \,.\, \mathrm{Cl_2Aq} = 58{,}315$$

$$\text{THOMSEN gives } \mathrm{Cu} \,.\, \mathrm{Cl_2} \,.\, \mathrm{Aq} = 62{,}710$$

$$\left.\begin{array}{r}\text{Therefore heat expended in withdrawing } \mathrm{Cl_2} \\ \text{from aqueous solution by copper} \quad . \quad . \quad .\end{array}\right\} = \quad 4{,}395$$

But the total heat evolved on the dissolution of chlorine gas in water will be greater than this by the thermal equivalent of the work done on it by the atmosphere, viz., 582 calories (at 18° C.).

$$i.e., \ Cl_2 . Aq = 4977 \text{ (at } 18° \text{ C.)}.$$

$$\text{THOMSEN* gives } Cl_2 . Aq = 2600.$$

On communicating with Professor THOMSEN it appeared that the latter statement is incorrect, and he kindly gave the results of two unpublished experiments whose mean, 4870, is not very different from the author's determination.]

SERIES II.—MIXED AMMONIUM CHLORIDE AND FERRIC CHLORIDE SOLUTIONS AS SOLVENT.

If finely divided copper is immersed in a mixture of NH_4Cl and Fe_2Cl_6 solutions it is rapidly attacked with formation of cupric and ferrous chlorides.

Similarly, zinc dissolves with production of zinc chloride and ferrous chloride.

In neither case is there an evolution of gas, and the reaction takes place quantitatively.

This point was tested by dissolving weighed quantities of copper, zinc, mixtures, and alloys of the metals in separate portions of the solution, and then estimating the ferrous chloride produced.

The presence of ammonium chloride appears necessary, in practice, for the rapid dissolution of the metals and their alloys.

It may be that a thin film of cuprous chloride is first formed over the immersed metal, and that the conversion of this into the soluble cupric salt is hastened by the solvent action of the ammonium salt.

A very strong solution of the chlorides was found essential to success, and that finally employed contained in each litre 182 grammes Fe_2Cl_6 and 107 grammes NH_4Cl.

With this solution the disappearance of the metallic powders seldom occupied more than two minutes, and very frequently not more that one and a-half minutes.

Specific Heat of the Solution.

A mercurial thermometer was constructed with a cylindrical bulb of some 15 cub. centims. capacity. Two marks on the stem corresponded to 25° C. and 95° C., and in order to reduce its length, a small reservoir had been blown between the marks.

The quantity of heat evolved by this thermometer during the fall of the mercury from mark to mark was determined by immersing its bulb in 500 grammes of water in

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* 'Thermochemische Untersuchungen,' vol. 2, p. 400.

the calorimeter at the instant the mercury passed the upper mark, and withdrawing it at the instant it passed the lower.

It was found to evolve 457·4 calories.

A similar experiment was now performed with the substitution of 550 grammes of the solution for water, and the specific heat was calculated.

The results of three determinations were :—

(i.) ·752 (ii.) ·755 (iii.) ·755.

The mean of these values is ·754.

The water equivalent of the calorimeter and all its contents was therefore

$$(550 \times \cdot 754) + 16 = 430{\cdot}7 \text{ grammes.}$$

Correction for Radiation.

Since the excess of temperature of the calorimeter above its surroundings at the end of an experiment was usually about 1° C. only, the correction necessary for radiation during the short period occupied by the dissolution of the metal was always very small.

The temperature was read every half minute from the beginning of an experiment. The maximum was usually reached after one and a-half or two minutes had elapsed, the greater part of the rise in temperature occurring during the first half minute.

The readings were then continued for several minutes, as the temperature slowly fell, and the rate of cooling per half minute was found.

It was assumed that during the first half minute of the rise in temperature there would be no sensible radiation loss, as a certain interval of time would elapse before the exterior of the calorimeter attained the temperature of its contents.

The radiation correction was therefore calculated by multiplying the rate of cooling by the number of half minutes, less one, occupied by the rise in temperature.

This correction was usually less than ·5 per cent. of the observed rise in temperature.

Heat of Dissolution of Copper in Ammonium Ferric Chloride Solution.

The copper was prepared exactly as for the experiments with chlorine water, and was introduced into the calorimeter on a paraffined-paper tray after it had been moistened with a very little water.

TABLE I.—Water Equivalent of Calorimeter and Contents = 430·7 grammes.

Weight of copper in grammes.	Rise in temperature ° C.	Calories evolved per gramme of copper.
·672	·279	178·3
1·7907	·751	179·8

Mean value = 179.

Heat of Dissolution of Zinc in Ammonium Ferric Chloride Solution.

Carefully sifted filings were employed in a manner similar to that described for copper.

TABLE II.—Water Equivalent of Calorimeter and Contents = 430·7 grammes.

Weight of zinc in grammes.	Rise in temperature ° C.	Calories evolved per gramme of zinc.
·4545	·9768	926
·512	1·1025	928

Mean value = 927.

In some early experiments* the author found that when nitric acid was used as solvent, a distinct difference existed between the heat of dissolution of a mixture of Cu and Zn, as determined by actual experiment, and the heat of dissolution *calculated* from the observed values for Cu and Zn when dissolved separately.

It is probable that this peculiarity may have been caused by the voltaic action of the metallic powders, in contact with each other and the liquid, influencing the mode of decomposition of the nitric acid. It was therefore deemed advisable to test this point with the present solvent.

Heat of Dissolution of Mixtures of Copper and Zinc in Ammonium Ferric Chloride.

The two metallic powders were weighed side by side, but not touching, on the same paraffined-paper tray, and after being moistened were placed in the calorimeter in the usual manner.

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* 'Proc. Chem. Soc.,' Sept. 1899.

The subjoined Table proves that if any voltaic action occurs in this case it does not appreciably affect the course of the reaction, for calculated and observed values differ by less than 1 per cent.

TABLE III.—Water Equivalent of Calorimeter and Contents = 430 7 grammes.

Weight of mixture of Cu and Zn in grammes.	Percentage of Cu in mixture.	Rise in temperature ° C.	Calories evolved per gramme.	Calories evolved calculated from Tables I. and II.
·5023	10·2	·989	850	851
·5419	18·2	·9925	789	791
·5836	32·05	·9365	690	687
·633	39·95	·926	632	628
·8203	49·4	1·068	561	558
1·00	62·27	1·067	459	461
1·2016	79·6	·9305	334	332

Heat of Dissolution of Alloys of Copper and Zinc in Ammonium Ferric Chloride.

The method of preparation and mode of dealing with the alloys was exactly similar to that previously described.

The quantity of water used to moisten the powdered alloys was so small that its effect on the water equivalent of the whole was negligible.

If we subtract the number of calories evolved by the dissolution of 1 gram of any alloy from the number calculated for a mere mixture of the same composition by the aid of Tables I. and II., the difference represents the heat of formation of 1 gramme of the alloy.

The seventh column in the following Table gives the numbers so obtained for twenty-one different alloys.

TABLE IV.—Water Equivalent of Calorimeter and Contents = 430·7 grammes.

Percentage of copper in alloy.	Weight of alloy in grammes.	Rise in temperature ° C.	Calories per gramme of alloy.	Mean calories per gramme of alloy.	Calories per gramme of mixture calculated from Tables I. and II.	Heat of formation of one gramme of alloy.
6·73	·4286 ·4878	·863 ·9836	868 869	868·5	876·5	8
12·0	·5801 ·5629	1·103 1·073	820 821	820·5	837·5	17
18·17	·6152 ·5894	1·086 1·043	761 763	762	791	29

TABLE IV.—*continued.*

Percentage of copper in alloy.	Weight of alloy in grammes.	Rise in temperature ° C.	Calories per gramme of alloy.	Mean calories per gramme of alloy.	Calories per gramme of mixture calculated from Tables I. and II.	Heat of formation of one gramme of alloy.
24·32	·7314 ·4842	1·201 ·797	707 709	708	745	37
28·18	·4496 ·5418	·702 ·849	673 675	674	716	42
32·05	·5964 ·6745	·879 ·987	635 631	633	687	54
35·53	·6846 ·7266	·969 1·0275	610 609	609·5	661·5	52
36·0	·5983 ·7822	·845 1·109	609 611	610	658	48
38·7	·8118 ·8053	1·1065 1·101	587 589	588	638	50
40·4	·41 ·6105	·5514 ·816	580 576	578	625	47
41·8	·5872 ·7318	·779 ·9608	572 569	570·5	614·5	44
42·7	·6746 ·6612	·882 ·858	563 560	561·5	607·5	46
43·9	·5193 ·4933	·669 ·633	555 553	554	599	45
45·29	·5615 ·7075	·7 ·89	537 542	539	588	49
47·85	·7126 ·8039	·866 ·979	524 525	524·5	569	44·5
50·91	·5037 ·6872	·5826 ·797	498 499	498·5	546·3	48
62·27	·6733 ·5558	·6646 ·543	424·4 419·5	422	461	39
66·8	·8669 ·9568	·7815 ·88	389 396	392·5	427·5	35
72·05	·7952 ·9008	·654 ·7315	354 349	351·5	387·5	36
82·18	1·4359 ·844	·954 ·564	286 288	287	312	25
88·92	1·2681 ·8568	·715 ·489	243 246	244·5	262	17·5

The numbers in the last column have been plotted as ordinates against percentages of copper as abscissæ, and will be discussed later.

Although the results arrived at appeared very satisfactory, it was considered advisable to confirm them by using another solvent, and a mixture of ammonium chloride and cupric chloride solutions was chosen.

SERIES III.—MIXED AMMONIUM CHLORIDE AND CUPRIC CHLORIDE SOLUTIONS AS SOLVENT.

Each litre of solution contained 1 gramme-molecule of cupric chloride and 2 gramme-molecules of ammonium chloride.

Copper is converted into cuprous chloride by this solution at the expense of the cupric salt, which suffers reduction to the cuprous condition.

The ammonium chloride serves, by its solvent action, to keep the surface of the metallic particles free from this cuprous chloride, which would otherwise delay dissolution.

Zinc placed in the solution precipitates finely divided copper from the cupric chloride, and this copper is rapidly re-dissolved, as explained above.

The solution was found to dissolve the alloys rapidly unless the percentage of copper was greater than 60, and, owing to this circumstance, the experiments were not continued beyond the alloy containing 62·2 per cent. of copper.

This difficulty arose from the fact that it was impossible to obtain those alloys which are rich in copper in a sufficiently comminuted state.

Specific Heat of the Solution.

This was found in a manner similar to that previously described.

In three successive experiments the following values were obtained :—

(i.) ·8102 (ii.) ·8102 (iii.) ·812.

Mean value = ·8108.

In the experiments which follow, the calorimeter always contained 520 grammes of the solution, and the water equivalent was

$$(520 \times \cdot 8108) + 16 = 437\cdot 6 \text{ grammes.}$$

Experience showed that it was usually necessary to moisten the metallic powders with rather more water than when ferric chloride was the solvent, or the rate of dissolution was apt to be slow.

In the following Tables the weight of water used for this purpose has been added to the water equivalent given above.

Heat of Dissolution of Copper in Ammonium Cupric Chloride Solution.

The heat evolution in this reaction was so small that exceptionally large quantities of copper had to be employed. This necessitated the addition of some 3 grammes of water to the copper powder to bring it into a suitable condition for rapid dissolution.

Both circumstances tend to render the accuracy of the determination less than usual, but this is of small moment in the final results sought, as by far the greater portion of the heat of dissolution of the alloys is due to the zinc.

TABLE V.

Weight of copper.	Rise in temperature ° C.	Water equivalent of calorimeter and contents.	Calories evolved per gramme of copper.
3·2254	·44	440	60
4·0741	·545	440·6	59

Mean value = 59·5.

Heat of Dissolution of Zinc in Ammonium Cupric Chloride Solution.

TABLE VI.

Weight of zinc.	Rise in temperature ° C.	Water equivalent of calorimeter and contents.	Calories evolved per gramme of zinc.
·5285	·972	438·5	806·4
·5001	·9214	438·5	807·8

Mean value = 807.

Heat of Dissolution of Mixtures of Copper and Zinc in Ammonium Cupric Chloride Solution.

A few experiments were made with mixtures of copper and zinc powders for the reason stated on p. 537.

The agreement between observed and calculated values in Table VII. is sufficiently close to render it unlikely that any notable thermal effect is produced by voltaic action between the metallic particles and the solution.

TABLE VII.

Percentage of copper in mixture.	Weight of mixture.	Rise in temperature ° C.	Water equivalent.	Calories per gramme of mixture.	Calories per gramme calculated from Tables V. and VI.
30·06	·7366	·984	438·4	584·5	583
43·86	·967	1·049	438·8	476	479
50·06	1·0022	·987	438·6	432	433·5

Heat of Dissolution of Alloys of Copper and Zinc in Ammonium Cupric Chloride Solution.

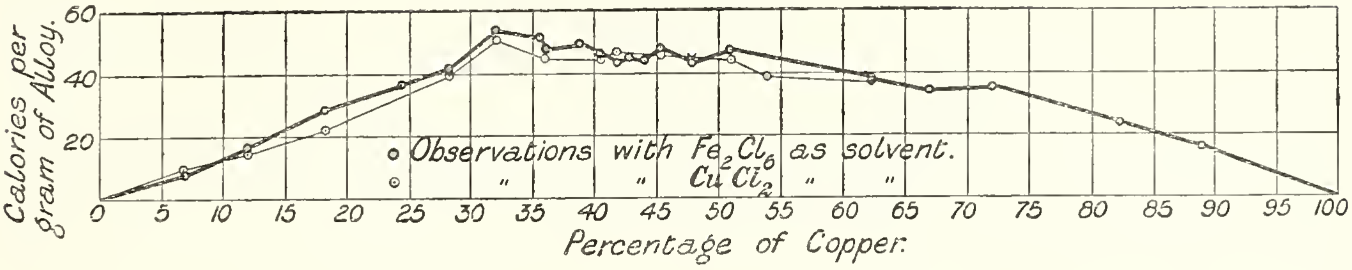
The results of these experiments are given in Table VIII., which is arranged on the plan adopted for the parallel observations with ferric chloride as solvent.

TABLE VIII.

Percentage of copper in alloy.	Weight of alloy.	Rise in temperature ° C.	Water equivalent.	Calories per gramme of alloy.	Mean value of calories per gramme.	Calories per gramme of mixture calculated from Tables V. and VI.	Heat of formation of 1 gramme of alloy.
6·73	·6201 ·591	1·06 1·007	438·3 438·3	749 747	748	757	9
12·0	·5545 ·5656	·887 ·907	438·4 438·4	701 703	702	717	15
18·17	·6377 ·6861	·942 1·018	438·7 438·3	648 650	649	671	22
28·18	·6565 ·7503	·832 ·95	438·5 438·6	556 557	556·5	596·5	40
32·05	·7755 ·7513	·912 ·89	438 438·4	515 518	516·5	567·5	51
36·0	·7843 ·8015	·884 ·901	438·2 438·2	494 492	493	538	45
40·4	·8935 ·8983	·939 ·941	438·3 438·5	461 459	460	505	45
41·8	·9383	·9595	438·5	448	448	495	47
45·29	1·0005 ·9994	·96 ·965	438·6 438·5	421 423	422	468	46
47·85	·7557 ·9838	·703 ·906	438·6 438·7	408 404	406	450	44
50·91	·9973 1·0452	·869 ·908	438·8 438·9	382 381·5	381·7	426·5	45
54·07	1·075 1·0302	·889 ·855	438·5 438·6	362·6 364	363·3	402·6	39
62·27	·9919 1·1792	·694 ·811	438·6 438·4	307 302	304·5	342	37·5

Discussion of Results.

When the results of Table VIII. are plotted in the same manner as those of Table IV., the two series of experiments strikingly confirm each other.



The small divergences from actual coincidence are not surprising, when it is remembered that each result is obtained as the difference between two large and nearly equal numbers.

It is evident at a glance that a clearly-defined maximum heat of formation is to be found in the case of the alloy containing about 32 per cent. of copper. From this point the curve drops until the alloy containing some 40 per cent. of copper is reached, whence it becomes nearly horizontal until the 50 per cent. ordinate is crossed.

Beyond this the curve drops at once, and proceeds with considerable regularity to the axis of percentages.

The significance of the foregoing lies in the coincidence of the maximum heat of formation with the percentage composition which represents an alloy of the atomic constitution $CuZn_2$.

The existence of this compound is now pretty generally accepted owing to the work of LAURIE, CHARPY, and HERSCHKOWITSCH,[*] and the author's results appear to confirm this conclusion.

It is also worthy of note that the upward trend of the curve, as traced from the copper end, ceases temporarily when the alloy containing about 51 per cent. of copper is reached.

This alloy corresponds very nearly with one possessing the atomic formula, CuZn. The author does not wish to lay too much stress upon this circumstance, as considerable irregularity appears to characterise the observations in the region lying between the 40 and 50 per cent. ordinates.

It was, however, the occurrence of this apparent sub-maximum, near the alloy containing 50 per cent. of copper, which led the author to test so many alloys in this region.

Taking the mean of the results afforded by Tables IV. and VIII. we may conclude that the heat of formation of 1 gramme of the compound $CuZn_2$ is about 52·5 calories.

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[*] See page 529.

This assumes that the alloy possessing that composition consists entirely of that compound, and this is perhaps open to question, although its physical properties and microscopic structure seem to point to its perfect homogeneity.

Since the alloy corresponding to CuZn does not show the simplicity of physical structure appertaining to the alloy $CuZn_2$, it is not justifiable to say that the heat of formation of 1 gramme of the *compound* CuZn is 46 calories, although we may make that statement with respect to the *alloy* containing copper and zinc in those proportions.

It does not appear to be known at what stage, in the making of brass, the evolution of heat of formation occurs.

Suppose that on mixing molten copper and zinc in the proportion of two parts of copper to one part of zinc (ordinary yellow brass), the heat of formation, amounting to 36 calories per gramme, is then manifested. Taking the specific heat of the molten metals as ·1 (?) we should obtain a rise in temperature of 360° C. as a result of chemical combination.

Suppose the molten copper is originally at 1200° C., that the zinc added is at 0° C., and that the latent heat of fusion of zinc is 28 (Person); then the temperature of the mixture, neglecting heat of formation, is given by

$$2\,(1200 - t)\cdot 1 = \cdot 1\,t + 28$$

whence

$$t = 707^\circ.$$

Adding the rise in temperature due to heat of formation, we obtain a final temperature $= (707^\circ + 360^\circ) = 1067^\circ$ C. This is well above the boiling point of zinc (940° C.), and may possibly be the explanation of the violent action often observed in making brass, although, as mentioned on p. 530, the direct volatilisation of zinc when placed in a crucible much above 1000° C. may be competent to produce the effect.

The previously mentioned work of Herschkowitsch was carried out by the dissolution of copper-zinc alloys in a water solution of potassium bromide and bromine.

His measurements were confined to two alloys containing 43·95 and 72·67 per cent. of copper, and he gives 9 calories and 12 calories respectively as their heat of formation per gramme.

As these quantities were in each case only about 1 per cent. of the heat of dissolution of the alloy, Herschkowitsch was inclined to the view that the latter possessed no heat of formation.

On reference to Tables IV. and VIII. it is apparent in many cases that the heat of formation of an alloy is about 10 per cent. of its heat of dissolution, and as the determinations are not likely to be more than 1 per cent. in error, it is quite certain that Herschkowitsch's results are inconclusive.

Molecular Dimensions of Copper and Zinc.

In Lord KELVIN's paper on "Contact Electricity of Metals" (Royal Institution, May 21, 1897), the question of the determination of the limits of the molecular dimensions of copper and zinc, by his method described in 'Nature' (p. 551, 1870), is re-opened.

On the assumption that "the work done by the attraction of two metallically connected sheets of copper and zinc when allowed to approach through any distance towards contact is the dynamical equivalent of the portion of their heat of combination due to the approach towards complete chemical combination constituted by the diminution of distance between them," Lord KELVIN showed that the molecular dimensions were less than 10^{-8} and greater than 10^{-9} centim.; for the union of plates of copper and zinc 10^{-8} centim. thick and 10^{-8} centim. apart would result in the production of 7·4 calories per gramme of brass formed.

If the distance apart were reduced to 10^{-9} centim. and the plates were each 10^{-9} centim. thick, then their union would produce 740 calories per gramme of brass formed.

Since the author finds a maximum heat of formation per gramme of brass of 52·5 calories, the limits of the molecular dimensions of copper and zinc above given would be confirmed.

The foregoing assumption that the Volta effect is capable of explanation as the result of the chemical affinity existing between copper and zinc has been subjected to much criticism (LODGE, 'Phil. Mag.,' vol. 49, 1900, p. 372).

There seems, however, to have been a tendency to under-estimate the heat of combination of these metals, for the value obtained by the author, viz., 52·5 calories per gramme of $CuZn_2$, is equivalent to 10,143 calories per gramme-molecule—a quantity quite comparable with the difference between the heats of oxidation of copper and of zinc (about 49,000 calories) on which Principal LODGE bases his calculation of the Volta effect.

Electro-deposition of Brass.

Another matter of interest lies in the bearing of these results on the electro-deposition of brass from a solution such as that of the mixed potassium-cuprous and potassium-zinc cyanides.

Here the metals unite on the cathode, and energy must be liberated.

It would probably be difficult to recognise the production of heat from this source owing to the slow rate of deposition of the alloy and to the complication introduced by the generation of heat due to the resistance of the electrolyte.

It certainly appears that a complete theory of the electro-deposition of brass must recognise the phenomenon.

Summary.

1. The production of an alloy of copper and zinc is always accompanied by evolution of heat.
2. The maximum heat of formation belongs to the alloy whose composition corresponds to the atomic formula $CuZn_2$, and amounts to 52·5 calories per gramme of alloy formed.

   This coincidence confirms the existence of a definite chemical compound, $CuZn_2$, whose heat of formation would therefore be 10,143 calories per gramme-molecule.
3. There is some indication of the possible existence of another definite compound, CuZn, having a heat of formation per gramme of alloy of about 45 calories, but the evidence is not decisive.
4. If Lord KELVIN's calculation of the molecular dimensions of copper and zinc is accepted, the results of this paper agree with his estimate of 10^{-8} centim. and 10^{-9} centim. as limits.
5. The energy liberated in the union of copper with zinc must be taken into account in a complete theory of the electro-deposition of brass.

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