An Investigation of Comparative Deflections of Steel Arch Ribs with Three, Two and No Hinges

AN ABSTRACT OF A THESIS PRESENTED TO THE FACULTY OF THE GRADUATE SCHOOL FOR THE DEGREE OF DOCTOR OF PHILOSOPHY. : : :

By PHOO HWA CHEN

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DEFLECTION OF A THREE-HINGED ARCH

PREFACE

Early in boyhood, before the writer began the study of Civil Engineering, he took a special interest in arches for artistic reasons. While in college, his natural fondness for such structures led him to become interested in the advantages and disadvantages of one type over another in stiffness and economy. The object of this thesis is to investigate the relative stiffness of steel arch ribs with three, two and no hinges. Since hingeless arches are not built with the spandrel-braced form on account of the difficulty of fixing the ends, the rib type is chosen for this investigation.

An investigation of the comparative deflections of steel arch ribs is a very complicated problem. The design of a three-hinged arch is not affected by temperature nor by rib-shortening. The effect of temperature and rib-shortening on arches with two or no hinges varies for different ratios of rise to span. Finally, different designers may assume different ranges of temperature and percentages of over-stress. Realizing these complications, the writer paid special attention to finding the easiest methods of computation for the benefit of future investigators, rather than to compute only the value for deflection.

The general process of finding the easiest methods of computation used in this thesis is to analyze general equations into a number of contributing factors; then to treat each factor separately. The advantages of this process are: (1) each factor may have a very simple solution; (2) it gives the computer a clearer conception of the problem; (3) it offers an opportunity to study the relative importance of different contributing factors and to neglect some of the factors; (4) it may suggest the easiest solution by omitting certain negligible factors.

The special features of this thesis are: (1) the method of stress computation for the three-hinged arch; (2) the method of computing deflections for the three-hinged arch; (3) the method of computing deflections due to axial thrust for two and no hinged arches; (4) the assumption of moment of inertia for the preliminary design of the hingeless arch; (5) the method of stress computation for the hingeless arch. Though no special merit is claimed, these few points are believed to be new.

The writer wishes to express his appreciation and gratitude for valuable suggestions and encouragement received from Prof. H. S. Jacoby, chairman of the committee.

GENERAL CONSIDERATIONS

The arches chosen for this investigation give some advantage to two-hinged and no-hinged arches in stiffness; because the lower the rise, the greater the effect of rib-shortening and temperature upon cross-section area, and hence the smaller the deflection. The reason for choosing arches with a low rise is to bring out the good points of the three-hinged arch even under unfavorable conditions and a desire to utilize arches previously designed. The arch span is 258 feet long, has 20 panels, and a rise of 26 feet. The dead load was assumed to be 59 kips and the live load 18½ kips. The effective depth of the arch rib was assumed to be five feet throughout, and the unit-stress 15,000 pounds per square inch. The discovery of some errors led the writer to revise the designs of the three-hinged and no-hinged arches, and thereby to find some new features of computation, including the moment table for the three-hinged arch, the assumption of moments of inertia for the preliminary design of the hingeless arch, and the effect of varying the end cross-section.

The rise and fall of temperature were assumed to be 75° Fahrenheit. Thirty per cent. overstress was allowed for the hingeless arch, but no allowance was made for the old design of the two-hinged arch.

THE DESIGN

The method of stress computation used in this design is different from what is generally employed. The writer believes that the new process is quicker and easier. In Chapter V of Roofs and Bridges, Part IV, by Professors Merriman and Jacoby, the working equations for three-hinged arches are:

$$V_1 = P(1-K) \qquad V_2 = PK$$

$$H = \frac{PKl}{2h} \qquad M = P(1-K)x - \frac{PKly}{2h} \qquad (a)$$

Equation (a) is for a single concentrated load. As explained in the same text, each leaf of the arch acts both as a simple beam and a strut. By observing this simple treatment, much labor can be saved in computing the bending moment at different sections by equation (b).

$$M = M' - M'' \qquad (b)$$

Where M' is the bending moment as a simple beam, and M'' is the bending moment as a strut. The identity of equations (a) and (b) may be proved in the following way:

$$V_1' = P(1-2K); \qquad V_2' = 2PK = 2V_2; \quad M' = P(1-2K)x$$

$$H = \frac{PKl}{2h}; \qquad M'' = H\left(y - \frac{2hx}{l}\right) = \frac{PKl}{2h}\left(y - \frac{2hx}{l}\right)$$

$$M' - M'' = P(1 - 2K)x - \frac{PKly}{2h} + PKx$$

$$\therefore M' - M'' = P(1 - K)x - \frac{PKly}{2h} = M$$

M' is easily computed for panel loads. If M'' due to a unit load at the crown hinge is known, the complete moment table may be filled in by mere inspection.

For a unit load at the crown, $H = \frac{l}{4h} = 2.48077$. The moment at different sections is shown in table 1.

TABLE 1

Point	$y - \frac{2hx}{l}$	$M''_c = H\left(y - \frac{2hx}{l}\right)$
1	2.34	5.805
2	4.16	10.320
3	5.46	13.545
4	6.24	15.480
5	6.50	16.125
6	6.24	15.480
7	5.46	13.545
8	4.16	10.320
9	2.34	5.805
10	0.00	0.000

For any other position of the load, M'' will be a simple ratio of M''_c. For example, when $k = 0.25$, $M'' = 0.5M_c''$; when $k = 0.2$, $M'' = 0.4\,M''_c$. The bending moment diagram for a simple beam is a triangle. If the moment under the load is known, the rest is also a simple ratio, as shown in table 2, because the panel lengths are all equal.

TABLE 2. M' TABLE

Load at point	Maximum Moment	Coefficient for moment at section 2	4	6	8
2	M'_2	1	3/4	1/2	1/4
4	M'_4	1/2	1	2/3	1/3
6	M'_6	1/3	2/3	1	1/2
8	M'_8	1/4	1/2	3/4	1

TABLE 3. M'' TABLE

Load at point	Moment at section 2	4	6	8
Crown	M''_2	M''_4	M''_6	M''_8
2	$0.2M''_2$	$0.2M''_4$	$0.2M''_6$	$0.2M''_8$
4	$0.4M''_2$	$0.4M''_4$	$0.4M''_6$	$0.4M''_8$
6	$0.6M''_2$	$0.6M''_4$	$0.6M''_6$	$0.6M''_8$
8	$0.8M''_2$	$0.8M''_4$	$0.8M''_6$	$0.8M''_8$

TABLE 4. *M′* TABLE

Load at point	Moment at section 2	4	6	8
1	10.32	7.74	5.16	2.58
2	20.64	15.48	10.32	5.16
3	18.06	20.22	15.48	7.74
4	15.48	30.96	20.64	10.32
5	12.90	25.80	25.80	12.90
6	10.32	20.64	30.96	15.48
7	7.74	15.48	20.22	18.06
8	5.16	10.32	15.48	20.64
9	2.58	5.16	7.74	10.32

TABLE 5. *M″* TABLE

Load at point	Moment at section 2	4	6	8
1	1.32	1.55	1.55	1.32
2	2.06	3.10	3.10	2.06
3	3.10	4.64	4.64	3.10
4	4.13	6.19	6.19	4.13
5	5.16	7.74	7.74	5.16
6	6.19	9.29	9.29	6.19
7	7.22	10.84	10.84	7.22
8	8.26	12.38	12.38	8.26
9	9.29	13.93	13.93	9.29

NOTE—The zigzag line indicates the points of division.

MAXIMUM BENDING MOMENTS

Having Tables 4 and 5 completed, the limiting position for positive or negative moments can be decided at once by comparing *M′* and *M″*. This is indicated by zigzag lines in tables 4 and 5. The maximum moment is obtained by summing up the moments in tables 4 and 5, and multiplying by the panel load. The results are expressed in kip-feet.

TABLE 6

Section	Max. L. L. Moment	D. L. Moment	Total
2	1211.7	1.8	1213.5
4	1747.2	1.8	1749.0
6	1662.8	3.0	1665.7
8	1033.5	3.0	1036.5

FLANGE AREA

The effective depth of arch rib is assumed to be five feet and the unit-stress 15,000 pounds per square inch.

TABLE 7. FLANGE AREA

Section	Max. Moment	Thrust	Area Required
0	0.0	2050	68.0
2	1213.5	1862	78.2
4	1749.0	1832	84.4
6	1665.7	1770	81.2
8	1036.5	1715	71.0
10	0.0	1915	64.0

TABLE 8. COMPOSITION OF FLANGES

Section	Composition	Area	I
	6Ls 6″ x 6″ x $\frac{9}{16}$″	38.58 sq. in.	inch[4]
0–2	3Pls. 14″ x ½″	21.00	
	1 Pl. 16″ x ¾″	12.00	
	1 Pl. 16″ x $\frac{7}{16}$″	7.00	
		78.58	155,496
2–6	6 Ls 6″ x 6″ x $\frac{9}{16}$″	38.58	
	3 Pls. 14″ x ½″	21.00	
	1 Pl. 16″ x ¾″	12.00	
	1 Pl. 16″ x $\frac{7}{16}$″	7.00	
	1 Pl. 16″ x ⅜″	6.00	
		84.58	173,162
6–8	6 Ls 6″ x 6″ x $\frac{9}{16}$″	38.58	
	3 Pls. 14″ x ½″	21.00	
	1 Pl. 16″ x ¾″	12.00	
	1 Pl. 16″ x ⅝″	10.00	
		81.58	163,652
	6 Ls 6″ x 6″ x $\frac{9}{16}$″	38.58	
8–10	3 Pls. 14″ x ½″	21.00	
	1 Pl. 16″ x ¾″	12.00	
		71.58	136,912

DEFLECTION

There is hardly any method in existence which is satisfactory for computing the deflections of three-hinged arch ribs. The new method to be applied here is the only one which is simple and suitable for computing the deflection of various points along the axis of the arch rib.

The general scheme is to separate the total deflection into its component parts. The total deflection is the sum of the deflections in the arch rib itself, plus that due to the crown hinge movement. Again, the deflection of the arch rib itself is the algebraic sum of the deflections due to positive bending moment as a simple beam, and due to negative bending moment as a strut. The movement of the crown hinge has three contributing factors: the deflection due to negative bending moment in both leaves of the rib; the deflection due to positive bending moment in either leaf of the arch rib; and the deflection due to rib shortening. The deflections due to the different contributing factors can be easily obtained either graphically or analytically. The above statement is now arranged in a different way to make it clearer.

1. Deflection due to positive bending moment as a simple beam.
2. Deflection due to negative bending moment as a strut.
3. Influence of crown movement on account of negative bending moment in struts.
4. Influence of crown movement on account of positive bending moment in either leaf as a simple beam.
5. Influence of crown movement on account of rib shortening.

Figure I.

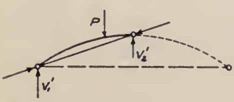


Fig. 2

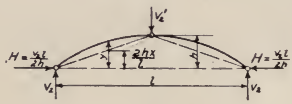


Fig. 3.

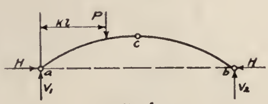


Fig. 4.

Three-Hinged Arch

The truth of the above method may be proved in the following way. The influence of the work of shear is neglected. The deflection is:

$$\triangle = \int_a^b \frac{Mm\,ds}{E\,I} + \int_a^b \frac{T\,t\,ds}{E\,A}$$

The first term contains the first four contributing factors. The second term contains the fifth contributing factor. Since $M = M' - M''$ and $m = m' - m''$, the first term becomes

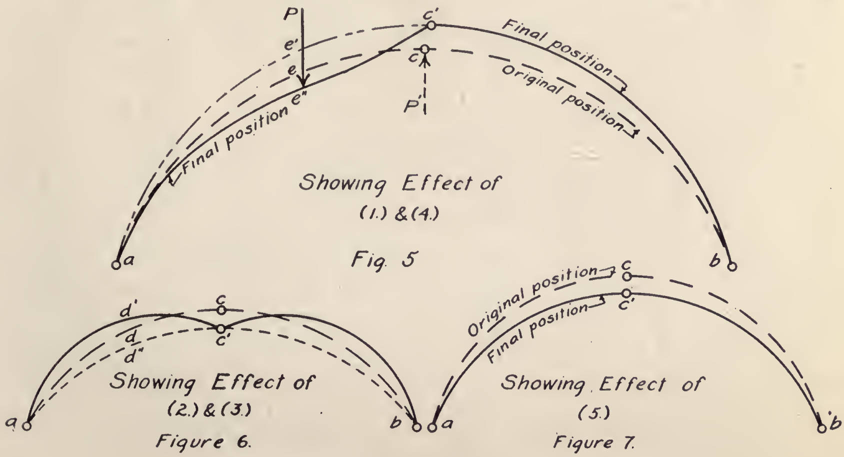
$$\int_a^b \frac{M\,m\,d\,s}{E\,I} = \int_a^b (M' - M'')\,(m' - m'')\,\frac{ds}{E\,I}$$

$$= \int_a^c \frac{M'm'ds}{E\,I} - \int_a^c \frac{M''m'ds}{E\,I} - \int_a^c \frac{M'm''ds}{E\,I} + \int_a^b \frac{M''m''ds}{E\,I}$$

The first three terms have the limits a and c, because M' and m' do not extend to the other half of the arch. By mere inspection, the four terms are the first four contributing factors. The expanded equation is,

$$\triangle = \int_a^c \frac{M'm'ds}{EI} - \int_a^c \frac{M''m'ds}{EI} - \int_a^c \frac{M'm''ds}{EI} + \int_a^b \frac{M''m''ds}{EI} + \int_a^b \frac{T\,t\,ds}{EA} \quad (c)$$

Equation (*c*) may be changed into a more convenient form for use. If f equals the number of panel points from the nearest support to where the deflection is sought; and g equals the number of panel points from the nearest support to where the load is applied; M''_c and m''_c are the



Deflection Diagrams with Scales Exaggerated

bending moments due to a load unity at the crown instead of a partial load transferred to the crown; then equation (*c*) becomes,

$$\triangle = \underset{(1)}{\int_a^c \frac{M'm'ds}{EI}} - g \underset{(2)}{\int_a^c \frac{M_c''m'ds}{EI}} + fg \underset{(3)}{\int_a^b \frac{M_c''m_c''ds}{EI}}$$

$$- f \underset{(4)}{\int_a^c \frac{M''m_c''ds}{EI}} + \underset{(5)}{\int_a^b \frac{T\,t\,ds}{AE}}$$

GRAPHICAL REPRESENTATION OF CONTRIBUTING FACTORS OF DEFLECTION

In Fig. 5 the left leaf of the original arch rib is *a e c*. When a load *P* is applied, the leaf deflects as indicated by *ae''c'* in exaggerated scale.

The ordinates between $ae'c'$ and $ae''c'$ represent the deflections as a simple beam; and this is called the contributing factor (1).

When a load P is applied at point 6, $0.6\ p$ is transferred to the crown hinge which causes negative bending moment. In order to represent the effect of positive bending moment alone, a load $P' = 0.6\ p$ is supposed to act upwards at the crown hinge to counteract the effect of negative bending moment. On account of factor (1), the left leaf lengthens and the crown hinge moves upwards and towards the right. The final position of the arch rib is $ae''c'b$. The ordinates between $ae'c'b$ and $aecb$ represent the effect of contributing factor (4). The ordinates between $aecb$ and $ae''c'b$ represent the combined effect of (1) and (4) as in Fig. 5.

When a partial load is transferred to the crown hinge, it causes negative bending moment and each leaf acts as a strut. On account of the negative bending moment, the original leaf adc in Fig. 6 deflects as $ad'c'$. The ordinates between $ad'c'$ and $ad''c'$ represent the deflection due to factor (2). On account of negative bending moment, each leaf shortens and the crown falls to its new position c'. The ordinates between $adcb$ and $ad''c'b$ represent the deflection due to factor (3) as shown in Fig. 6. The ordinates between $adcb$ and $ad'c'b$ represent the combined effect of (2) and (3).

The effect of axial thrust is simply the shortening of each leaf which causes the crown to fall to its new position. In Fig. 7 the ordinates between acb and $ac'b$ represent the deflection due to factor (5).

As may be readily seen from the figures, factors (3), (4) and (5) are due to crown movement and can be combined in computing the deflections at various points for any given loading, when the movement of the crown is obtained.

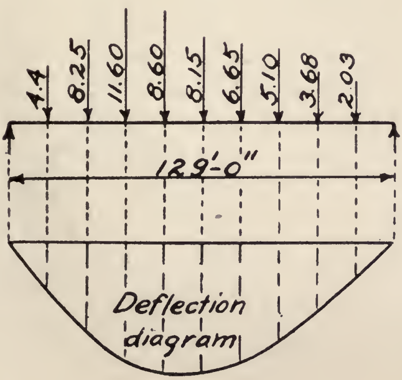
RECIPROCAL DEFLECTIONS

By Maxwell's law of reciprocal deflections, the computation is greatly simplified. For example, factor (4) is obtained directly from factor (3), because $\int M'M''ds/EI$ and $\int M''M'ds/EI$ are identical. Furthermore, Maxwell's law can be used as a good check on the final results. The following table shows the scope of reciprocal deflections. The figures 1, 2, 3, etc., indicate reciprocal deflections. It is seen that symmetrical figures on both sides of these inclined lines are equal.

LOAD AT SECTION

Defl. at.	II	IV	VI	VIII	X
2		1	2	3	4
4	1		8	9	10
6	2	8		13	14
8	3	9	13		16
10	4	10	14	16	
12	5	11	15		16
14	6	12		15	14
16	7		12	11	10
18		7	6	5	4

DEFLECTION AS A SIMPLE BEAM BY THE METHOD OF ELASTIC MOMENTS



As explained on page 223 of *Bridge Engineering* by Dr. J. A. L. Waddell, the deflection of any point of a simple beam with respect to its supports is equal to the moment which would occur if the beam were conceived to be simply supported at the ends and loaded with Mds/EI. Factor (1) can be easily computed by this method as follows. All deflections are to be computed for one kip.

ELASTIC LOAD FOR LOAD OF ONE KIP AT III

$(Mds/EI) \times 10^6$.

Section	I	II	III	IV	V	VI	VII	VIII	X
Elastic load	4.40	8.25	11.60	8.60	8.15	6.65	5.10	3.68	2.03

TABLE 9

Section	ds	$I/1000$	$(ds/I) \times 10^3$	$(ds/EI) \times 10^9$	A.
1	13.72	155.50	.0082	3.39	78.58
2	13.54	164.33	.0824	3.17	81.58
3	13.39	173.16	.0773	2.97	84.58
4	13.26	173.16	.0766	2.95	84.58
5	13.15	173.16	.0759	2.92	84.58
6	13.05	168.41	.0776	2.99	83.08
7	12.98	163.65	.0793	3.05	81.58
8	12.94	150.28	.0861	3.32	76.58
9	12.92	136.91	.0945	3.67	71.58
10	12.90	136.91	.0943	3.66	71.58

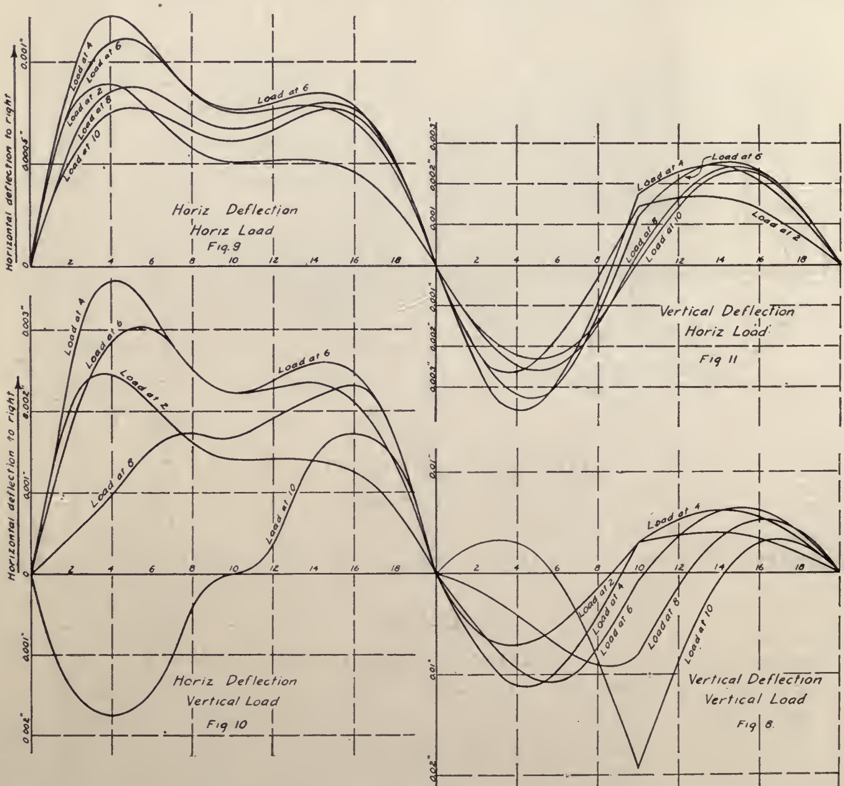
DEFLECTION OF CROWN DUE TO THRUST

Thrust is equal to $H\cos\theta + V\sin\theta$. For a load unity at crown, $H = 2.48$ and $V = 0.5$. The deflection is to be computed as it affects all points along the arch rib.

The result is

$$\int_a^b \frac{t^2\,ds}{2\,A\,E} = 2\int_a^c \frac{t^2\,ds}{2\,A\,E} = 0.002417'' \times 2 = 0.004834''$$

For any other position of loading, an assumption is made that only a portion of the load carried up to the crown hinge causes crown deflection due to rib-shortening. This assumption is justified as the deflection due



Deflections of Three-Hinged Arch

to rib-shortening is only a small portion of the total and it does not make any appreciable error. The greatest difference occurs when the load is at quarter point. The difference in thrust is $(V_2 - V_1)\sin\theta$.

TABLE 10. LOAD AT 5

Section	$(V_2 - V_1)\,sin\theta$	$\frac{ds \times 10^7}{2AE}$	$\frac{T^2 ds \times 10^8}{2AE}$
0–1	.19	408	147
1–2	.17	402	116
2–3	.15	367	83
3–4	.14	364	71
4–5	.12	361	52
			469.0

Total deflection due to thrust = 0.00483400″
Deflection due to $(V_2 - V_1)\,sin\theta$ = 0.00000469″
This comparison indicates that the difference is a negligible quantity.

Factors (1) and (2) are computed by the method of elastic moments and checked by graphical methods. In order to describe the method, the deflections for a load of one kip at point IV are given below. Complete computations are filed in the C.E. Library of Cornell University.

1. Compute factors (2), (3) and (5) for a load unity at the crown. These values are used for any position of a load with a corresponding ratio.
2. Compute factor (1) for load at IV.
3. Compute factor (4) for load at IV.

TABLE 11. DEFLECTION IN 1000TH INCHES

Point	(1)	(2)	(3), (4), and (5)	(1–5)
2	10.61	—2.728	—0.6178	7.26
4	16.91	—4.340	—1.2356	11.33
6	15.94	—4.356	—1.8534	9.73
8	9.45	—2.760	—2.4712	4.22
10		0.000	—3.0890	—3.09
12		—2.760	—2.4712	—5.23
14		—4.356	—1.8534	—6.21
16		—4.340	—1.2356	—5.58
18		—2.728	—0.6178	—3.35

$$(3), (4) \;\&\; (5) = \underset{(4)}{10.849} - 0.4\,(\underset{(3)}{14.566} - \underset{(5)}{4.834}) = 3.089$$

HORIZONTAL DEFLECTION DUE TO VERTICAL LOAD

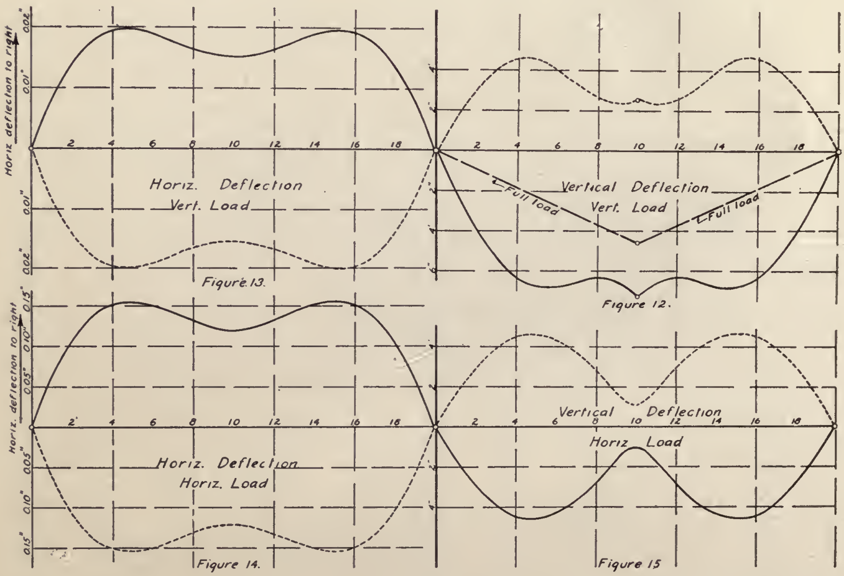
An identical method is used for computing horizontal deflections except that *m* stands for a bending moment due to a horizontal load of unity and *t* for the thrust due to the same loading. The general equation is,

$$\Delta h = \int_a^b \frac{M'm'ds}{EI} - \int_a^b \frac{M''m'ds}{EI} - \int_a^b \frac{M'm''ds}{EI} + \int_a^b \frac{M''m''ds}{EI} + \int_a^b \frac{Ttds}{AE}$$

$$= \int_a^c \frac{M'm'ds}{EI} - \int_a^c \frac{M''m'ds}{EI} - \int_a^c \frac{M'm''ds}{EI} + oo \qquad + oo$$

(1) (2) (3) (4) (5)

(4) and (5) are equal to zero, because the summation for the right leaf of the arch is equal and opposite to the summation for the left leaf. The slight variation of thrust due to $(V_2 - V_1)\ sin\theta$ is a negligible quantity. Deflections as a simple beam and as a strut, are computed by the method of elastic moments.



Deflections of Three-Hinged Arch; Minimum Deflections shown by Dotted Lines.

HORIZONTAL DEFLECTION DUE TO HORIZONTAL LOADING

The same method is used here for computing the horizontal loading. In this case all terms are positive. The deflection due to thrust is found to be very small and that it is beyond the degree of precision required. The general formula is,

$$\Delta h = \int_a^c \frac{M'm'ds}{EI} + \int_a^c \frac{M''m'ds}{EI} + \int_a^c \frac{M'm''ds}{EI} + 2\int_a^c \frac{M''m''ds}{EI}$$

(1) (2) (3) (4)

Factors (2) and (3) have a reciprocal relation. When one is known, the other can be obtained by a simple ratio. Factor (4) needs to be computed only once, as the rest are obtained by applying a simple ratio. The work for computing the deflection is thus greatly simplified.

VERTICAL DEFLECTION DUE TO HORIZONTAL LOADING

Vertical deflections due to horizontal loading are obtained by Maxwell's law of reciprocal deflections, because the vertical deflection due to horizontal loading is equal to the horizontal deflection due to vertical loading.

The general formula for computing deflection is $\int Mmds/EI + \int Ttds/AE$. The deflection due to shear is to be neglected. The first term is the deflection due to bending moment, while the second is due to shortening of the arch rib on account of axial thrust. Here m stands for the bending moment due to a load unity applied at the point where the deflection is sought. Since a two-hinged arch is a combination of a simple beam and strut, the bending moment at different sections is equal to $M'-M''$, M' being the bending moment when acting as a simple beam, while M'' is the bending moment when acting as a strut. Similarly m equals $m'-m''$. Substituting these values in the first term, it becomes $\int(M'-M'')\ (m'-m'')\ ds/EI = \int M'm'ds/EI - \int M''m'ds/EI - \int M'm''ds/EI + \int M''m''ds/EI$. The sum of the last two terms equals zero, because they represent the vertical deflection due to a movement of the end hinges. Since the hinges do not move, their sum must be zero. As the formula shows, the first term of the expanded formula is the deflection when acting as a simple beam; the second term is the deflection when acting as a strut due to H; the third term is the deflection due to a horizontal movement of the hinge on account of M'; the fourth term is the deflection due to a horizontal movement of the hinge on account of M''. Since the last two drop out, the total deflection of the arch rib due to flexure equals its deflection as a simple beam minus its deflection as a strut. These deflections can be obtained easily either analytically or graphically. The complete equation becomes:

$$\text{Deflection} = \underset{(1)}{\int M'm'\frac{ds}{EI}} - \underset{(2)}{\int M''m'\frac{ds}{EI}} + \underset{(3)}{\int Tt\frac{ds}{AE}}$$

TABLE 12

Section	ds	y	I	ds/I	yds/I	y^2ds/I
0	6.90	2.50	7.34	0.94	2.35	5.88
1	13.72	4.94	7.34	1.87	9.23	45.60
2	13.54	9.36	7.96	1.70	15.92	149.00
3	13.39	13.26	8.59	1.56	20.69	274.10
4	13.26	16.64	8.78	1.51	25.12	418.10
5	13.15	19.50	8.97	1.47	28.60	557.50
6	13.05	21.84	8.97	1.46	31.80	694.00
7	12.98	23.66	8.99	1.44	34.15	808.00
8	12.94	24.96	8.80	1.47	36.70	915.00
9	12.92	25.74	8.59	1.51	38.75	999.00
10	12.90	26.00	8.59	1.50	39.05	1015.00

TABLE 13. VERTICAL DEFLECTION AS A SIMPLE BEAM. (IN INCHES)

LOAD AT SECTION

Point	II	IV	VI	VIII	X
2	0.018	0.0300	0.0370	0.0395	0.0395
4	0.0300	0.0550	0.0700	0.0760	0.0752
6	0.0375	0.0700	0.0935	0.1050	0.1050
8	0.0405	0.0760	0.1045	0.1220	0.1250
10	0.0395	0.0750	0.1050	0.1250	0.1327
12	0.0354	0.0765	0.0950	0.1150	0.1250
14	0.0285	0.0550	0.0780	0.0955	0.1050
16	0.0200	0.0390	0.0550	0.0680	0.0752
18	0.0100	0.0200	0.0290	0.0355	0.0395

TABLE 14. VERTICAL DEFLECTION DUE TO H = UNITY. (IN INCHES)

Point	Yds/I	Deflection
1	9.23	
2	15.92	0.0210
3	20.69	
4	25.12	0.0500
5	28.60	
6	31.80	0.0545
7	34.15	
8	36.70	0.0637
9	38.75	
10	39.05	0.0670

VERTICAL DEFLECTION DUE TO THRUST FOR A VERTICAL LOAD

The general formula for the deflection due to thrust is $\int Ttds/AE$ in which T is the thrust due to the applied load P', while t is the thrust due to a load unity P'' applied where the deflection is sought. Since the deflection due to axial thrust is small as compared with that of the bending moment, and the axial thrust due to V is small as compared with that of H, the effect of axial thrust due to V may be neglected without appreciable error. Let H' and H'' be the horizontal reactions due to P' and P'' respectively. Then the formula becomes:

$$\text{Deflection due to thrust} = \int_a^b H'\cos\theta H''\cos\theta\, ds/AE$$

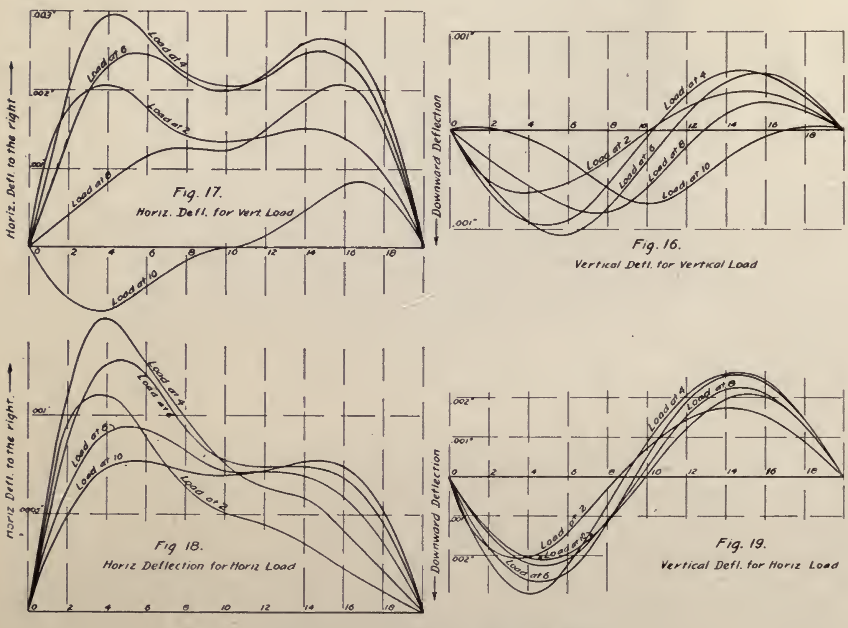
$$= \int_a^b H'H''\cos^2\theta\, ds/AE$$

With a systematic arrangement, the computation becomes very simple. As seen from the following table, there are many reciprocal products.

TABLE 15. $H'H''$

Point	II	IV	VI	VIII	X	
	Load at Section					
	H_2	H_4	H_6	H_8	H_{10}	
2	H^2_2	H_4H_2	H_6H_2	H_8H_2	$H_{10}H_2$	H_2
4	H_2H_4	H^2_4	H_6H_4	H_4H_8	$H_{10}H_4$	H_4
6	H_2H_6	H_4H_6	H^2_6	H_8H_6	$H_{10}H_6$	H_6
8	H_2H_8	H_4H_8	H_6H_8	H^2_8	$H_{10}H_8$	H_8
10	H_2H_{10}	H_4H_{10}	H_6H_{10}	H_8H_{10}	H^2_{10}	H_{10}

There is a very simple way to compute the deflection due to an axial thrust. As it can be readily observed from the table, the deflection



Deflections of a Two-Hinged Arch

due to $H \cos \theta$ is proportional to H in both directions. Now it is only necessary to compute one point for all deflections under any conditions of loading. This is the deflection of the crown due to a load unity at the crown. It is surprising what a great amount of labor is thus saved. All deflections due to axial thrust therefore, can be obtained from a simple ratio of H. As stated elsewhere the deflection due to $V \sin \theta$ may be neglected. If more accurate results are desired, the effect of V should be computed separately. All deflections due to this factor are approximately proportional to x.

The method of combining different factors is illustrated in table 16, where factor (1) stands for deflection as a simple beam, factor (2) for deflection as a strut, and factor (3) due to rib-shortening. The results are plotted for panel loads of one kip. Horizontal deflections are computed by identical methods except that P'' is to act horizontally at the point where the horizontal deflection is sought.

TABLE 16. TOTAL VERTICAL DEFLECTION DUE TO A VERTICAL LOAD AT VI. (IN INCHES)

Point	Factor (1)	Factor —(2)	Factor (3)	Total
2	.0375	.0329	.0006	.0052
4	.0700	.0620	.0012	.0092
6	.0935	.0845	.0016	.0106
8	.1045	.0938	.0020	.0077
10	.1050	.1040	.0020	—.0030
12	.0950	.0988	.0020	—.0018
14	.0780	.0845	.0016	—.0048
16	.0550	.0620	.0012	—.0058
18	.0285	.0329	.0006	—.0038

DESIGN OF A HINGELESS ARCH

Various assumptions of the moment of inertia have been tried for the preliminary design, but none of them seems to be adequate. The preliminary design was computed by assuming I to vary as *sec.* θ. The final results are tabulated as follows:

TABLE 17

Load at Section	V_1	V_2	H	Y_1	Y_2
1	.9927	.0077	.0818	—121.3	10.04
2	.9720	.0280	.3013	— 52.0	9.63
3	.9392	.0608	.6026	—28.9	9.18
4	.8960	.1040	.9523	—17.3	8.67
5	.8437	.1563	1.3057	—10.4	8.09
6	.7840	.2160	1.6405	—5.8	7.43
7	.7182	.2818	1.9232	—2.5	6.67
8	.6480	.3520	2.1427	0.00	5.78
9	.5747	.4253	2.2766	1.9	4.73
10	.5000	.5000	2.3250	3.5	3.47

When an allowance is made for 30 per cent. over-stress in temperature, the following flange area is required.

TABLE 18. AREA IN SQUARE INCHES

Section	A_1	A_2	Error
0	119.5	135.0	—15.5
1	93.1	106.5	13.4
2	75.9	83.5	—7.6
3	72.3	74.7	—2.4
4	65.0	68.6	—3.6
5	67.9	59.2	8.7
6	72.8	67.2	5.6
7	76.9	72.8	4.1
8	78.4	74.7	3.7
9	78.3	75.8	2.5
10	77.3	75.2	2.1

A_1 is computed by I varies as *sec.* θ, while, A_2 is computed by true values of I. The error is far too great for the preliminary design. Evidently the assumption I varies as *sec.* θ must be far from the true value. The value of I in the following design is obtained after several trials. The moment of inertia of the web is neglected.

TABLE 19. COMPOSITION OF FLANGES

Section	Composition	Area in²	I
9–10	6—Ls 6″ x 6″ x9″/16	38.58	
	3Pl. 14 x 9/16	23.60	
	3Pl. 18 x 7/8	47.25	
	2Pl. 18 x3/4	27.00	
		136.43	316,900 in.⁴
8-9	6—Ls 6 x 6 x 9/16	38.58	
	3Pl. 14 x 9/16	23.60	
	3Pl. 18 x 7/8	47.25	
		109.43	234,700 in.⁴
6–8	6—Ls 6 x 6 x 9/16	38.58	
	3Pl. 14 x 9/16	23.60	
	1Pl. 18 x 3/4	13.50	
	1Pl. 18 x 9/16	10.13	
		85.80	167,700 in.⁴
4–6	6—Ls 6 x 6 x 9/16	38.58	
	3Pl. 14 x 9/16	23.60	
	1Pl. 18 x 7/8	7.88	
		70.05	125,850 in.⁴
0–4	6—Ls 6 x 6 x 9/16	38.58	
	3Pl. 14 x 9/16	23.60	
	1Pl. 18 x 7/16	7.88	
	1Pl. 18 x 3/8	6.75	
		76.81	143,400 in.⁴

General methods for designing a hingeless arch are extremely laborious. If mistakes are made, it requires a long time to correct them. As stated by Dr. J. A. L. Waddell in *Bridge Engineering*, page 635, "the labor involved in making the computations is excessive." It seems to the writer that the following interesting method could be used to advantage. A hingeless arch is a two-hinged arch with end bending moments added to keep the tangents fixed. Then the different factors which enter the computations may be separated. The general process is: (1) To find V as a simple beam; (2) to find H as a two hinge arch;

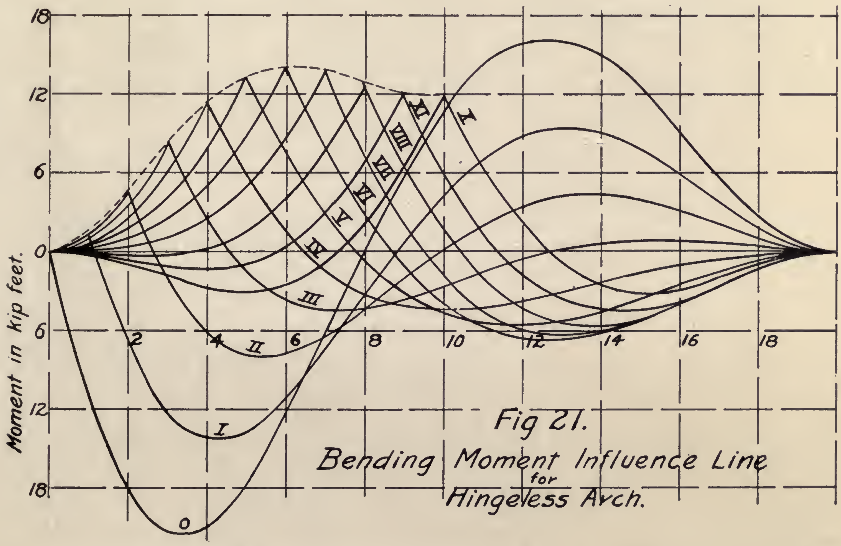


Fig 21.
Bending Moment Influence Line for Hingeless Arch.

(3) to find θ_1 and θ_2 due to the applied load; (4) to find θ due to H unity; (5) to find θ_1 and θ_2 due to M = unity applied at one end; (6) to find H and V due to M = unity applied at one end; and (7) to solve the algebraic equations with the computed data. Factors (1), (2), and (6) are simple, but the rest may need further explanation.

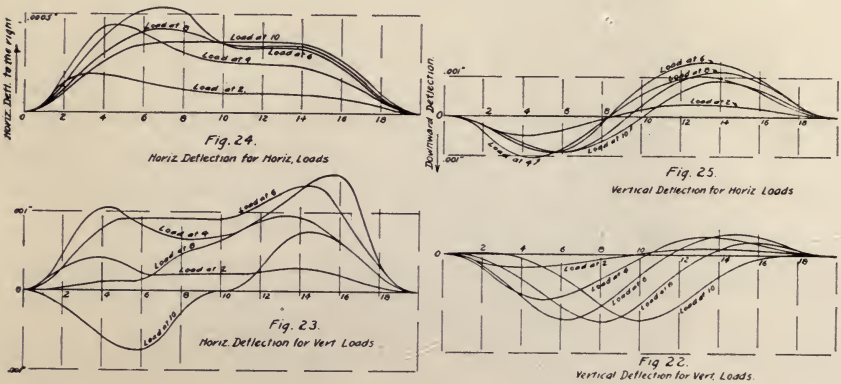
The general equation for θ is, $\theta = \int Mds/EI$. If a simple supported beam is loaded, the point of maximum deflection will be the point of zero shear for elastic loads, Mds/EI. But the change of angle at each end is $\int Mds/EI$ from the end to the point of maximum deflection. Then the angles θ_1 and θ_2 are simply the reactions due to elastic loads. Factors (3), (4) and (5), therefore, can be easily computed. It is to be kept in mind, when M = unity is applied at one end, it causes vertical reactions and the bending moment is diminishing from unity at one end to zero at the other end. Moreover, bending moments will cause horizontal reactions.

The algebraic equations are all simple; for example, if θ_1 and θ_2 are $231/E$ and $42/E$ respectively, and M = unity will cause $\theta_1 = 11.02/E$ and $\theta_2 = 6.2/E$; then

$$11.02\ M_1 + 6.2\ M_2 = 231$$
$$11.02\ M_2 + 6.2\ M_1 = 42$$

M_1 and M_2 being the bending moments at the respective supports.

The advantages of this method are: (1) some factors may be neglected; (2) if mistakes are made, they can be easily detected and corrected. The writer has roughly checked one point with this method by using a slide-rule with fairly good results. Since the writer has directed his attention primarily to deflections, he is not able to spare the time to develop this interesting method. Further investigation is necessary in order to make this method a useful working instrument.



Deflections of a Hingeless Arch

DEFLECTIONS OF A HINGELESS ARCH

The computations of deflections for a hingeless arch are quite simple. There are two main factors: bending moment and axial thrust. Since a hingeless arch is fixed at both ends, the arch is treated as a cantilever for computing deflections due to bending moment. The plan is to start from one end and to use the other as a check. Since their relative position is fixed, the deflection of the other end with respect to the first must be zero, the deflections can be obtained easily either analytically or graphically.

The deflection due to axial thrust is computed in exactly the same way as for a two-hinged arch. The deflection at the crown is first computed, while the rest are obtained by means of a simple ratio of H.

Horizontal deflections are computed by similar methods, except that the load unity is applied horizontally at the point where the deflection is sought.

TABLE 20. VERTICAL DEFLECTION OF A HINGELESS ARCH FOR VERTICAL LOAD AT X

Point	Factor (1)	Factor (2)	Total
2	—.52	.40	—.12
4	—1.08	1.48	.40
6	—.60	2.79	2.19
8	1.28	3.77	5.05
10	2.60	4.13	6.73
12	1.28	3.77	5.05
14	—.60	2.79	2.19
16	—1.08	1.48	.40
18	—.52	.40	—.12

(A) DISCUSSION ON THE DESIGN

The new method of stress computation for a three-hinged arch seems to have advantages over the usual methods because the M' and M'' (tables 4 and 5) tables can be filled in by mere inspection. The point of division can be readily obtained by comparing these two tables. This method can be applied to a spandrel-braced arch as well.

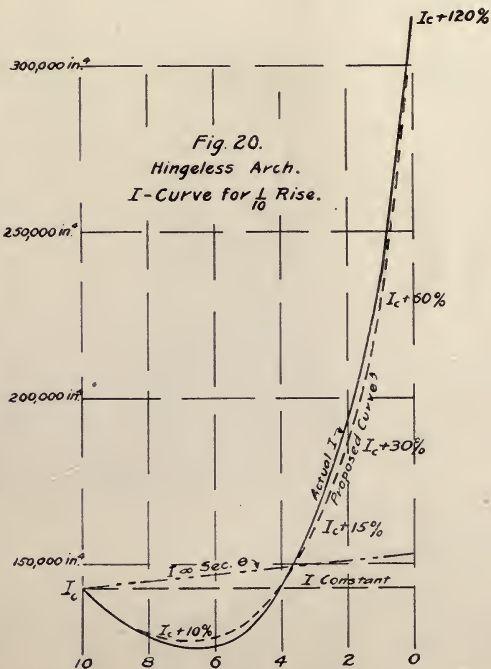
Attention has been called to the assumption of moment of inertia for the preliminary design of a hingeless arch. The writer desired to find an equation for the I-curve with a given rise but on account of the limited time, he was not able to design hingeless arches with different rises. The best that the writer can do at present is to recommend the I-curve for a rise equal to one-tenth of the span. He hopes that someone interested in this problem will discover the necessary equation. The assumption that I varies as $sec\ \theta$ is far from the true value. For a preliminary design of hingeless arches with one-tenth rise, the assumed moment of inertia should be,

Section	10 (Crown)	8	6	4
I	I_c	I_c—10%	I_c—10%	I_c

Section	3	2	1	0
I	I_c+15%	I_c+30%	I_c+60%	I_c+120%

Before choosing the final flange area, $y = \int yds/I \div \int ds/I$ should be investigated to see how much it differs from that for the preliminary design. This is important, because a slight variation will greatly affect the stresses of temperature and rib-shortening.

Change of flang area does not materially affect H nor influence the required area to any great extent, except near the end sections. It is very interesting to note that too much material near the end section may reduce the security of the structure. In designing the hingeless arch, sections 1 and 0 were first assumed to be constant with 136 square inches; and the final required area for section 0 was found to be 142 square inches. When section 1 is reduced to 110 square inches, while the other sections remain the same as before, the final area for section 0 becomes 135 square inches. Too much material near the end section, therefore, is undesirable. The explanation is: a hingeless arch is a combination of a cantilever and a two-hinged arch. The hinges are imaginary and movable. If too much material is added near the end section, it increases the stiffness of the end sections unnecessarily, and in turn it increases the action as a cantilever, thus requiring a larger section near the spring line.

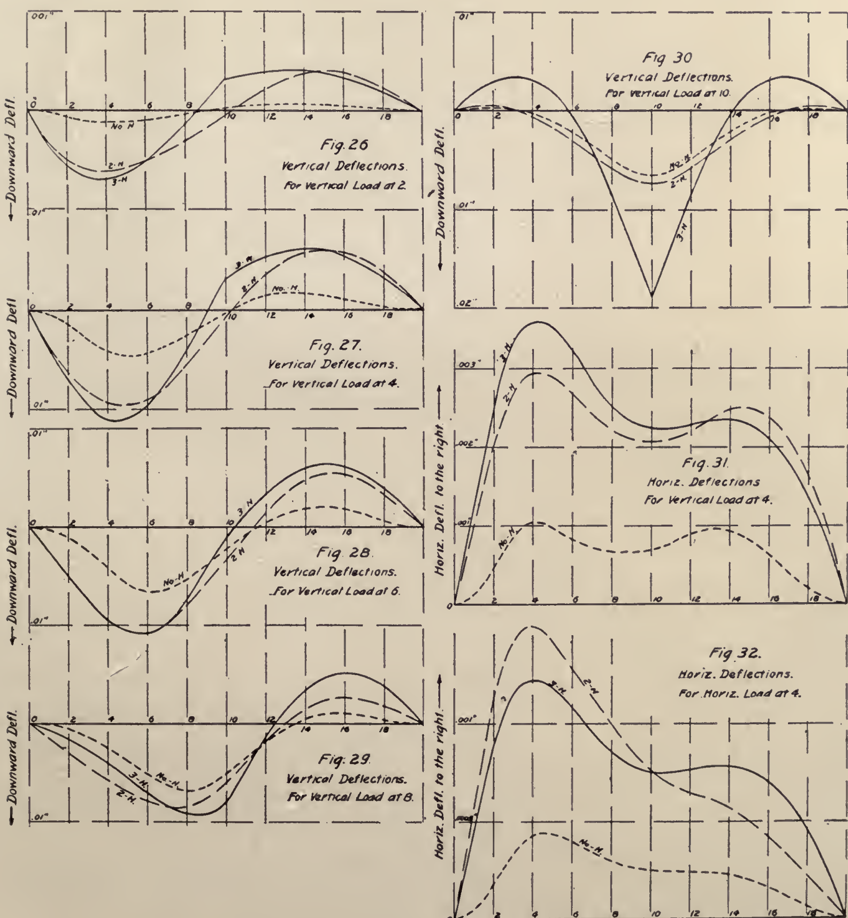


(B) DISCUSSION ON METHODS OF DEFLECTION COMPUTATIONS

The method of computing the deflections for a three-hinged arch is valuable and interesting. The computation has been made as easy as for a simple beam. Moreover, this new method presents a clear conception of the different contributing factors.

The deflection under a full load is chiefly due to axial thrust. It is interesting to note that the deflection curve for a three-hinged arch under full load is composed of two straight lines (see Fig. 33). This clearly shows that the deflection due to axial thrust is proportional to x which in turn is proportional to H, and V. For arches with two and no hinges, the deflections under full load are very nearly proportional to H (see Fig. 33).

The method of computing deflections due to axial thrust for arches with two and no hinges is useful and simple. It is only necessary to compute for one point to obtain all deflections under all kinds of loading.

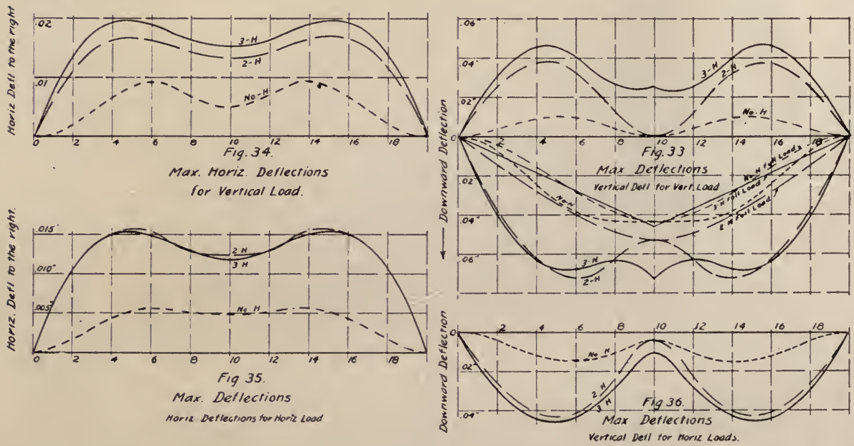


Comparison of Deflections for Hingeless, Two-Hinged and Three-Hinged Arches

(c) DISCUSSION ON DEFLECTION

The relative deflections for load at II, IV, VI, VIII and X are plotted in diagrams 26 to 54 inclusive; and comparisons are made for horizontal and vertical deflections due to horizontal and vertical loads. Finally comparisons are made for loads giving maximum and minimum deflections and for full loads.

So far as the deflection of the crown is concerned, a two-hinged arch is stiffer than a three-hinged arch for full and partial loads. At the quarter points, however, the reverse is true. The hingeless arch is the stiffest of all under all conditions.



Comparison of Deflections for Hingeless, Two-Hinged and Three-Hinged Arches

When the arches are under horizontal loads, the relative deflections of arches with two and three hinges remain the same as before stated. While under full loads, there is practically no difference in deflection for these two types of arches. The deflection for a hingeless arch due to horizontal loads is a negligible quantity; while for arches with two and three hinges, the horizontal deflection due to horizontal loads is about 15 per cent. of Vertical Deflection at the quarter points.

The hingeless arch is the stiffest both under horizontal and vertical loads. It has very small deflections due to horizontal loads, and hence the vibration due to a fast railroad train is greatly reduced. In the writer's opinion, with a favorable location, the hingeless arch would be the first choice if the greatest stiffness is required. Under ordinary conditions, a three-hinged arch is preferable.