TRIGONOMETRY

NEWCOMB'S MATHEMATICAL COURSE

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ELEMENTS OF GEOMETRY.

ELEMENTS OF PLANE AND SPHERICAL TRIGONOMETRY.

LOGARITHMIC AND OTHER MATHEMATICAL TABLES.

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ELEMENTS

OF

PLANE AND SPHERICAL TRIGONOMETRY

BY

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PREFACE.

The distinctive features of the following work belong partly to the course of which it forms a part, and need but a brief statement.

I. The device by which mathematical teaching is to be most promoted is, the author conceives, to be found in the minute subdivision of subjects, and the drill of the student in the separate details before combining them into a whole. The system to which we are thus led is seen in the arrangement of Chapters I., II., and V.

II. By exercises in which the subject is taken up in a concrete form, the formation of mathematical conceptions is greatly facilitated. An application of this principle is seen in the cases where the student is exercised in finding the values of trigonometric functions by construction and measurement.

III. The problems for exercise are quite varied in their character, and are intended to test not only the student's knowledge of the usual methods of computation, but his ability to grasp them and trace them out in the numerous forms they may assume in practical applications.

IV. In the arrangement, strictly logical order has been subordinated to order of teaching. In accordance with this principle, all the simpler applications of the trigonometric functions have been disposed of before their complex relations.

V. The scope of the work is generally limited to the subjects and treatment necessary in the fullest course of mathematics usually taught in our colleges and technological schools. The concluding chapter of each part perhaps exceeds the limit thus set. That of Part I. is an introduction to the employment of imaginary quantities in trigonometric developments, while that of Part II. is an introduction to the higher forms of solid geometry.

VI. To the usual list of subjects treated, has been added a chapter on the theory of polygons. This theory is closely connected with a variety of subjects, including geometry, quaternions, mechanics, graphical statics, surveying, and navigation, and therefore deserves a more prominent place than has hitherto been assigned to it.

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ELEMENTS OF TRIGONOMETRY

PART I.

PLANE TRIGONOMETRY.

CHAPTER I.

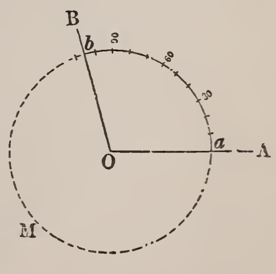
OF GONIOMETRY, OR THE MEASURE OF ANGLES.

1. *Definition.* **Trigonometry** is that branch of geometry in which the relations of lines and angles are treated by algebraic methods.

2. *Def.* An **angle** is the figure formed by two straight lines emanating from the same point, called the **vertex** of the angle.

Def. The lines which form an angle are called its **sides.**

3. *Measures of Angles.* An angle is measured by the length of a circular arc having its centre at the vertex of the angle and its ends on the sides of the angle.



If the angle to be measured is AOB, we conceive that with an arbitrary radius Oa an arc is drawn from a to b.

We regard as the positive direction that in which the arc is described by a motion opposite to that of the hands of a watch, and as the negative direction that in which the hands move.

Hence we may consider the angle as measured either by the arc ab considered as positive, or the conjugate arc aMb considered as negative. The numerical sum of these two arcs is equal to a circumference.

As an example of the use of algebraic signs, we may mention their application to the latitude of places to distinguish them as *north* and *south.* Thus, a city in 42° north latitude is said to have a latitude of $+ 42°$, and one 42° south of the equator is said to be in latitude $- 42°$.

The absolute length of the arc will depend not only upon the magnitude of the angle, but upon the radius with which the arc is drawn. To avoid ambiguity from this cause, the unit of arc is supposed to be some fixed fraction of the circumference, and therefore greater the greater the radius. The arc is then indicated by the number of units and parts of a unit which it contains, and this number is the same for the same angle whatever the radius may be.

To indicate the angle corresponding to any arc we call it the *angle of the arc*, or, for brevity, the *angle-arc.*

4. *The Sexagesimal Division.* The following is the usual division:

The circumference is divided into 360 units, called degrees;

Each degree is divided into 60 minutes;

Each minute is divided into 60 seconds.

Then

$$1 \text{ circumference} = 360° = 21\,600' = 1\,296\,000'';$$

$$1 \text{ quadrant or right angle} = 90°.$$

This is called the **sexagesimal** division of the circle.

5. *The Centesimal Division.* The sexagesimal division of the circle is by no means so convenient as one in which each unit is 10 times or 100 times greater than the next smaller unit. The centesimal division was introduced by the French geometers at the time of the Revolution. In this system

The circumference is divided into 400 *grades;*

The grade is divided into 100 *minutes;*

The minute is divided into 100 *seconds.*

Hence 1 circumference = 400 grades = 40 000 min. = 4 000 000 sec.; which is commonly written

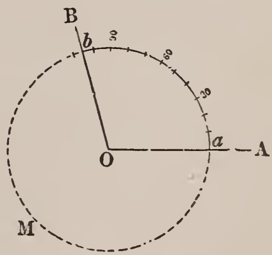
$$400^{gr.} = 40\,000' = 4\,000\,000''.$$

Notwithstanding its greater convenience, this system never came into general use, owing to the difficulty of changing all the mathematical tables to correspond with it.

6. *Decimals of Degrees or Minutes.* Sometimes, instead of seconds, decimals of a minute are used. Both minutes and seconds may be dispensed with and decimals of a degree be used in their place.

7. *General Measure of an Angle.* The best way of thinking of angular measure is to conceive the side OA of the angle to turn round on O until it reaches the position OB. In thus turning, a point A upon it will describe the circular arc which measures the angle AOB. The length of this arc will then be proportional to the amount by which OB turns in passing from OA to OB.

The side may pass from OA to OB not only by describing the arc ab, but by moving through a whole revolution *plus* the arc ab, or through any number of revolutions *plus* the arc ab. When we consider the angle in the most general way, all these motions will equally measure the angle. Hence we may suppose, indifferently,



Angle AOB = angle-arc ab,

or Angle AOB = angle-arc $ab + 360°$,

or Angle AOB = angle-arc $ab + 720°$,

etc. etc.

If we put C for 360°, or the circumference, the general measure of the angle is

$$\text{Angle } AOB = iC + \text{angle-arc } ab, \quad (1)$$

in which i may be any integer whatever, positive or negative.

We may consider this same form to include the negative measure aMb. For, since $ab + aMb = C$, we have

$$\text{Arc } ab = C - \text{arc } aMb.$$

By substituting this value in (1) it becomes

$$\text{Angle } AOB = (i + 1)C - \text{arc } aMb.$$

Since we have $i = \ldots\ldots -2, -1, 0, 1, 2, 3,$ etc., *ad infinitum*, $i+1$ may go through the same system of integral values as i.

In general, if an angle is $n°$ less than a circumference, we may call it, indifferently,

$$\text{Angle of } (360 - n)° \text{ or angle of } -n°.$$

The general measure of the angle expressed in the form (1) has its most convenient application in Astronomy. The heavenly bodies perform unceasing revolutions, and thus describe continually increasing angles; but each revolution brings them back to what we may consider the same position relative to the centre of motion.

8. In order to give entire algebraic precision to the measure of an angle, we must suppose a distinction between the side *from* which we measure and the side *to* which we measure. In all the preceding examples we have supposed the measure to be *from OA to OB.* Had we measured from OB to OA, the arc ab would have been described in the negative direction, or aMb would have been described in the positive direction. Hence we should have had

$$\text{Angle } BOA = -\text{arc } ab \text{ or } +\text{arc } aMb,$$

which is the negative of the corresponding measure from OA to OB. Hence:

By interchanging the sides we change the algebraic sign of the angle.

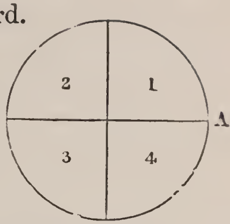
To give uniformity to this mode of measurement, the side OA, from which we measure, is supposed fixed, while the other side varies in direction according to the magnitude of the angle.

When angles are represented in a general way, the side OA may be conceived as extending out horizontally towards the right. Then the other side, OB, will have a definite direction for every angle we choose to assign. For example:

For 90°	the side OB will point	upward.
" 180°	" " " " "	to the left.
" 270°	" " " " "	downward.
" 360°	" " " " "	to the right.
" 450°	" " " " "	upward.
	etc.	etc.

Counting the angles in the negative direction,

For	$-90°$	the side OB will point	downward.
"	$-180°$	" " " " "	to the left.
"	$-270°$	" " " " "	upward.
	etc.		etc.



9. *Division into Quadrants.* The circle which measures angles is, for convenience, supposed to be divided into quadrants, as in the figure.

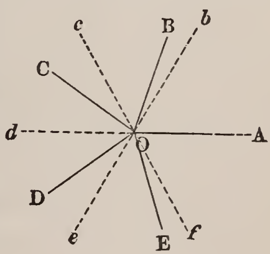
An angle between	0° and	90°	is in the first	quadrant.
" " "	90° "	180°	" " second	"
" " "	180° "	270°	" " third	"
" " "	270° "	360°	" " fourth	"

Counting the angles negatively from OA,

An angle between	0° and	$-90°$	is in the *fourth*	quadrant.
" " "	$-90°$ "	$-180°$	" " *third*	"
" " "	$-180°$ "	$-270°$	" " *second*	"
" " "	$-270°$ "	$-360°$	" " *first*	"

Exercises.

1. From the point O emanate a set of 5 lines making equal angles with each other, and another set of 6 lines making equal angles with each other, the line OA being common to the two sets. Compute the values in degrees of the ten angles AOb, bOB, BOc, etc., to fOA.



2. What is the value of that angle whose negative measure is numerically double its positive measure?

3. If a side starting from the zero point move through $-1905°$, in what quadrant will it be found, and what will be the smallest positive measure of the angle?

4. Two arms start together from the same position OA to turn round O, the one going in the positive direction, so as to revolve

in 60 seconds, the other in the negative direction, so as to revolve in 36 seconds. At what angle and in what time will they meet?

5. If two revolving arms start out together from the position 0° in the same direction, the one going 5° a minute and the other 8° a minute, through what arc will each have moved when they again come together? At what angle will they meet? If they continue turning, after how many revolutions of each will they be together at their starting point?

6. Four lines, a, b, c, d, emanate from the same point o, making angle $boc = 2aob$, $cod = 2boc$, $doa = 2cod$. What are the values of the four angles which they form?

7. If an angle of 140° is multiplied successively by -2, -3, -4, -5, -6, -7, in what quadrants will the respective multiples fall, and what will be the smallest positive measures of the several angles formed?

8. Show that the following pairs of angles are supplementary:

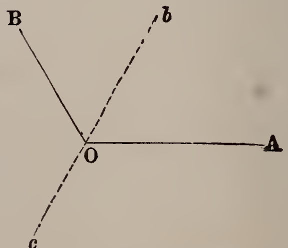
$$90° + x \text{ and } 90° - x;$$
$$270° - x \text{ and } 270° + x;$$
$$60° - x \text{ and } 120° + x.$$

9. Show that the following pairs are complementary:

$$45° - x \text{ and } 45° + x;$$
$$225° - x \text{ and } 225° + x;$$
$$60° - x \text{ and } 30° + x.$$

The Division of Angles.

10. *Bisection.* If the angle $AOB = n°$, and if Ob is its bisector, then $AOb = \frac{1}{2}n°$.



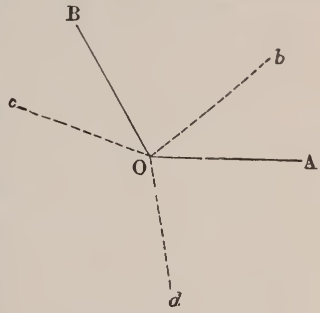
If the side OB revolves about O, and the side Ob also revolves in the same direction half as fast, then Ob will continually bisect the angle AOB.

When OB completes a revolution, returning to the position OB, the bisector Ob will have moved through 180°, and will therefore lie in the opposite direction, Oc. Another revolution of OB will bring the bisector to the position Ob again, yet another to Oc, and so on. Hence:

The general measure of an angle has two bisectors 180° *apart.*

11. *Trisection.* If the side Ob is to continually measure one third the angle AOB as OB revolves, then we must have

$$AOb = \tfrac{1}{3}AOB.$$



If OB, starting from the position in the figure, goes through one revolution, Ob will go through 120° to the position Oc. A second revolution of OB will bring Ob 120° farther, to Od, and a third to its first position, Ob, after which it will repeat its movements. Hence:

One third the general measure of an angle has three special angular values differing by 120°.

12. *Division into n Parts.* If, as OB revolves, Ob continually measures $\frac{1}{n}$ of it, then every revolution of OB will turn Ob through $\frac{1}{n}$ of a revolution. Hence:

The nth part of the general measure of an angle has n special angular values.

13. *Analytic Deduction.* It will be remarked that, in the preceding sections, what we take the nth part of is not the angle AOB, but the general measure of this angle. This will be clear from the following analytic deduction of the same result.

Let the smallest measure of the angle AOB be α. Then the other measures of this angle (§ 7) will be

$$\alpha + C, \ \alpha + 2C, \ \alpha + 3C \ . \ . \ . \ \alpha + iC.$$

Dividing these quantities by n, the quotients will be

$$\frac{\alpha}{n}, \ \frac{\alpha}{n} + \frac{1}{n}C, \ \frac{\alpha}{n} + \frac{2}{n}C, \text{ etc.} \qquad (1)$$

The $(n+1)$th angle of this series will be

$$\frac{\alpha}{n} + \frac{n}{n}C = \frac{\alpha}{n} + C,$$

which will correspond to the same position of OB as $\frac{\alpha}{n}$ does. The continuation of the series will be

$$\frac{\alpha}{n} + C + \frac{1}{n}C, \; \frac{\alpha}{n} + C + \frac{2}{n}C, \text{ etc.,}$$

showing that the positions will be continually repeated in regular order.

EXERCISES.

1. If $AOB = 30°$, or $30° + C$, or $30° + 2C$, etc., at what angles will $\frac{1}{3}AOB$ fall?

2. How many degrees between the minute-lines on a clock-face?

3. At what angles with XII. are the hour and minute hands of a clock together?

4. If the hour-hand is so displaced that when the minute-hand is at XII. the hour-hand is 2^m past XII., at what angles will the hands be together?

5. What values may two thirds of the general measure of an angle of 105° have?

6. If an angle is in the third quadrant, what are the limits between which its bisectors must fall?

7. Between what three sets of limits must α be contained in order that 3α may fall in the fourth quadrant?

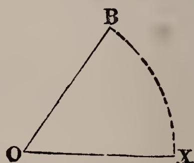
8. Show that while one sixth of the general measure of an angle has six different values, two sixths has only three values, and three sixths only two values. Show that this diminution arises from several values falling together when multiplied by 2 or 3. As an example, take the case when $a = 48°$; $\frac{1}{6}a = 8°, 68°$, etc.

14. *Natural Measure of Angles.* The division of the circumference into 360° is entirely arbitrary, and any other angle than the degree may be taken as the unit.

In purely mathematical investigations, where no division into degrees is required, the length of the radius is taken as the unit of measure.

This unit is called the **radian.**

The radian is therefore the angle subtended by an arc whose length is equal to the radius.



An angle of one radian in which arc XB = radius OX.

To find the relation of this unit of angle to the degree, minute, and second, we note that the ratio of the entire circumference to the diameter is 3.141 592 65, etc. (Geom., Book VI., § 5.)

Hence its ratio to the radius is double this number, or 6.283 185 3, etc. Since the circumference measures 360°, the unit radius will measure $\frac{360°}{6.283\,185\,3}$, or 57°.295 779 5 . . . . Hence

$$\begin{aligned} 1 \text{ radian} &= 57.295\,779\,5 \ldots \text{ degrees.} \\ &= 3437.746\,77 \ldots \text{ minutes.} \\ &= 206\,264.806 \ldots \text{ seconds.} \end{aligned}$$

In mathematics we use the symbol

$$\pi = \frac{\text{circumference}}{\text{diameter}} = \frac{\frac{1}{2}\text{ circumference}}{\text{radius}} = 3.141\,592\,65 \ldots$$

Hence, when we take the radian as the unit,

$\frac{1}{2}\pi$ represents an angle of 90°;
π " " " " 180°;
2π " " " " 360° = circumference;
$2n\pi$ " " " " n circumferences.

Exercises.

Considering the radius of the circle as unity, what is the length of circular arcs subtending the following angles?

1. 28° 17′ 15″.6. Ans. 0.493 72.
2. 14° 8′ 37″.8 " 0.246 85.
3. 22° 25′ 53″.4 " 0.391 51.
4. 90° " 1.570 80.

Note. In these exercises the angle is first to be reduced to a common denomination of measure, either degrees, minutes, or seconds. For instance,

28° 17′ 15″.6 = 101 835″.6 = 1697′.26 = 28°.287 67.

If the radius is 100 metres, how many degrees and minutes will arcs of the following lengths subtend?

5. 100 metres. Ans. 57° 17′ 44″.8.
6. 72 metres. " 41° 15′.1.
7. 310 metres. " 177° 37′.

With what radius will—

8. Arc of 32 metres' length subtend an angle of 32°?

Ans. $57^m.296$.

9. Arc of 32 metres' length subtend an angle of 32′?

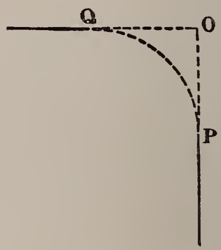
10. " " 32 " " " " " " 32″?

11. " " 1 metre " " " " " " 67″?

Ans. $3078^m.6$.

12. Two railways met at right angles at O. They are to be connected by a quadrant PQ, of which the inner rail shall be 600 metres in length. What is the common distance OP and OQ of the switches from the point O in which the two inner rails would meet?

Ans. $381^m.97$.



13. In the preceding case, if the rails are 5 feet apart, how much longer will the outer rail of the curve be than the inner one?

Ans. 7.854 feet.

14. Show that if three circles, equal or unequal, mutually touch each other externally in the points A, B, and C, the sum of the three included arcs $AB + BC + CA$, expressed in angular measure, is equal to a certain constant. What is this constant?

15. If the two lesser circles, still touching each other, touch the greater one internally, show that the sum of their arcs *minus* the arc of the greater circle, expressed in angular measure, is equal to the same constant as that of the preceding problem.

16. The earth's equatorial diameter being 12,756 kilometres, what is the length of one degree of the equator in kilometres and in miles, assuming 1 metre = 39.37 inches.

17. Explain why a degree of latitude is greater at the poles than at the equator, although the radius of the earth is less.

Remark.—At this stage of his progress, if not sooner, the student should be familiarized with the use of the logarithmic and trigonometric tables, and should employ them in all computations in which they are applicable

CHAPTER II.

THE TRIGONOMETRIC FUNCTIONS.

The Sine, Tangent, and Secant.

15. To investigate the numerical relations between the sides and angles of geometric figures, certain functions of angles are employed in trigonometry. These functions are defined in the following way:

Let OX be that side of the angle from which we measure, the length OX being taken as the radius of a circle. Also, suppose

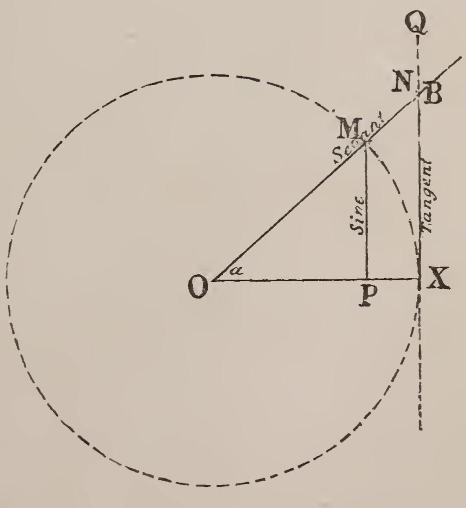
OB, the side to which we measure;

M, the point in which OB intersects the circle;

XQ, the line tangent to the circle at X;

N, the point at which this tangent meets the side OB;

MP, the perpendicular from M upon OX.



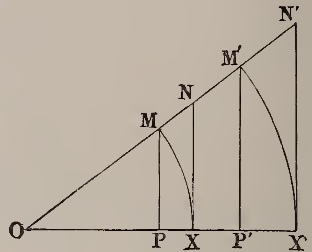
Then taking the radius OX as unity, and expressing other lengths in terms of this unit:

I. The length MP is called the **sine** of the angle XOB.

II. The length NX is called the **tangent** of the angle XOB.

III. The length ON is called the **secant** of the angle XOB.

The absolute lengths of the lines representing the sine, tangent, and secant, considered as lines, will vary with the radius of the circle. This is avoided by taking for the sine, tangent, and secant, not the lines which represent them, but the *ratios* of the lines to the radius of the circle, which ratios will be pure numbers.



We have now to prove that these numbers are the same for the same angle whatever be the radius.

Let $X'ON'$ be the angle.

From the vertex O draw the two arcs XM and $X'M'$ with any two radii OX and OX'.

Erect the respective sines and tangents $PM, XN, P'M', X'N'$.

Then because the triangles OPM, OXN, $OP'M'$, and $OX'N'$ have the angle at O common, and the respective angles at P, X, P', and X' all right angles, and therefore equal, these triangles are equiangular and similar.

Comparing the sides about the equal angles we have the ratio

$$\left.\begin{aligned} PM : OM &= P'M' : OM'; \\ XN : OX &= X'N' : OX'; \\ ON : OX &= ON' : OX'; \end{aligned}\right\} \qquad (a)$$

Because $OM = OX =$ radius of inner circle,

and $OM' = OX' =$ radius of outer circle,

we have, by definition,

$$PM : OM = \text{sine of } POM;$$
$$XN : OX = \text{tangent of } POM;$$
$$ON : OX = \text{secant of } POM.$$

The equations (a) now show that the sine, tangent, and secant of the angle will be represented by the same numbers whether we measure them in the inner or outer circle. Therefore:

To each angle of a definite magnitude corresponds
One definite number, called the sine of the angle;
Another definite number, called the tangent of the angle;
Another definite number, called the secant of the angle.

17. *Notation.* If we call a any angle,

Its sine is written .... $\sin a$;
" tangent " .... $\tan a$;
" secant " .... $\sec a$.

NOTE. The representation of the trigonometric functions by lines is for the sake of clearness. They are not really lines, but ratios of lines which are pure numbers. But in studying these numbers the ideas are fixed by representing them by lines, as is done in some departments of algebra. We have only to remember that the lines are not the functions themselves, but lengths proportional to the functions, and therefore admitting of being used to represent the functions. These lengths are, however, really equal to the products of the radius by the corresponding functions. For example, if

$$\sin XOM = \frac{PM}{\text{Radius}},$$

then

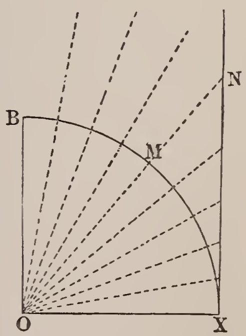
$$\sin XOM \,.\, \text{Radius} = PM.$$

18. *Remark.* The sine of an angle is equal to half the chord of twice the arc of the angle, the radius being supposed unity. Hence:

Any chord in a circle is equal to the radius multiplied by twice the sine of half the angle subtended by the chord.

EXERCISE.

Let the student find by actual measurement with dividers and scale the sine, tangent, and secant of every 10° from 0° to 90° in the following way:



With a radius equal to some unit or some whole number of units on a scale, describe the quadrant XB. Either 4 inches, 5 inches, or a decimetre would be a convenient radius.

Divide the quadrant into 9 arcs of 10° each. Through each point of division, M for instance, draw a radius and continue it until it intersects the tangent at N. Then measure—

1. The distance of each division-point on the arc from the line OX, which distance, divided by the radius, will give the *sine* of the corresponding angle.

2. The distance of each point of intersection, N, from X, which being divided by the radius will give the *tangent* of the angle.

3. The length of each ON, which being divided by the radius will give the *secant* of the corresponding angle.

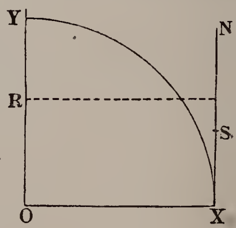
The results should all be expressed in decimals to three places, exhibited in a little table in the following form, and afterward compared with the values found in the trigonometric tables:

Angle.	Sine.	Tangent.	Secant.
0°	. . .	. . .	. . .
10°	. . .	. . .	. . .
20°	. . .	. . .	. . .
30°	. . .	. . .	. . .
etc.	. . .	. . .	. . .

With care the average deviation of the measures from the truth ought not to exceed .005, except in the cases of the tangent and secant of 70° and 80°, which are so great that they cannot be easily found in this way.

19. *To find, by measurement, the angle corresponding to a given sine, tangent, or secant.*

Analysis. If a sine is given, the end of the arc corresponding to the required angle must be at a distance from the line OX equal to the given sine, the radius being unity. Therefore if we take on the perpendicular OY a distance OR equal to the product of the radius by the given sine, and through R draw a parallel to OX, the point in which it intersects the arc will give the required angle.



To find the angle corresponding to a given tangent we take a distance XS equal to the product of the given tangent into the radius. Join OS. The angle XOS will be that required.

For a secant we take the product of the radius by the secant in the dividers, and from O as a centre draw an arc cutting XN in a point N. Joining ON, the angle XON will be that required.

EXERCISES.

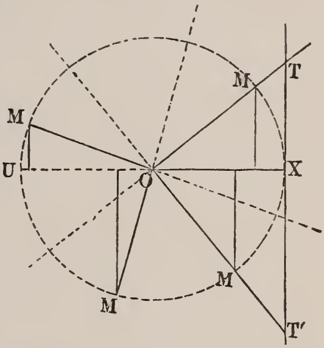
1. Find by measurement the angles of which the sines are $\frac{1}{4}$, 0.3, 0.4, 0.6.

2. Find arc-tan $\frac{1}{2}$, 1, 1.5, 3.

The expression arc-tan is used for brevity to mean the arc corresponding to a given tangent.

3. Find arc-secant 1.5, 2.

20. *Functions of unlimited angles.* Thus far we have considered only the sines, tangents, and secants of angles less than 90°; that is, of angles in the first quadrant (§ 9). As our angle increases to an entire circumference, the functions are determined by the same construction modified to suit the case.



The following are the general definitions:

We first generalize the construction.

On the sides of the angle we take equal lengths OX and OM as the unit of measurement and radius of the circle.

At X we erect a line TXT' perpendicular to OX, extending indefinitely in both directions.

We also suppose the revolving side OM to be produced indefinitely in both directions, and XO to be produced so as to form the diameter XU.

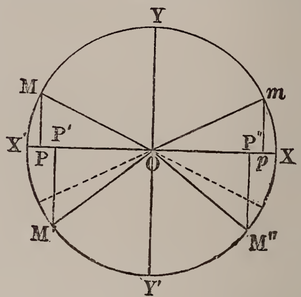
Then, however the side OM may revolve—

I. *The sine of the angle XOM is always represented by the perpendicular from M upon the line OX.* The sine is positive or negative according as M is above or below OX.

II. *The tangent of the angle XOM is always represented by the distance from X to the point in which the side OM produced intersects the line TXT'.* The tangent is positive or negative according as the point of intersection is above or below X.

III. *The secant of XOM is the length of OM produced intercepted between O and the vertical line TXT'.* The secant is positive or negative according as it is in the direction from O toward M or in the opposite direction. The positive direction of OM is supposed to revolve with the side OM, and therefore to be always from O toward M.

21. *Changes in the value of the sine.* If we suppose the side Om of the angle XOm to revolve around O, carrying the sine mp with it, the latter will increase to its maximum value, equal to radius unity, when m reaches Y, and angle XOm = 90°. Hence



$$\sin 90° = +1.$$

As m moves from Y to X', the sine will diminish from 1 to zero. Because angle $XOX' = 180°$,

$$\sin 180° = 0.$$

If m passes over X' into the third quadrant, the perpendicular $M'P'$ will be below the line $X'OX$. This change of direction is expressed by changing the algebraic sign of the perpendicular from + to −. This is in accordance with the following general principle:

Whenever distances measured in one direction are considered positive, those in the opposite direction are negative.

Hence also:

In the third quadrant the sine is negative.

When the point m reaches the position Y' it will have moved through three quadrants or 270°, and the sine will coincide with the radius OY' of length unity. Hence

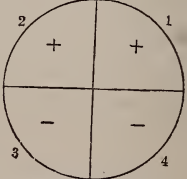
$$\sin 270° = -1. \quad (1)$$

As m moves from Y' to X, the sine will increase from −1 to 0. Hence:

I. *In the fourth quadrant the sine is negative.*

II. $\sin 360° = \sin 0° = 0.$

The changes of algebraic sign as the angle goes through the four quadrants are shown in the annexed diagram.



Algebraic signs of the *sine*.

Angles having equal sines. If angle $X'OM = XOm$, the two angles XOm and XOM will be supplementary. Also in this case the triangles XOm and $X'OM$ will be identically equal; so that

$$PM = pm.$$

Now PM represents by construction the sine of XOM, and pm the sine of XOm. Hence:

The sines of supplementary angles are equal.

In symbolic language this theorem is expressed thus: If α be any angle, then

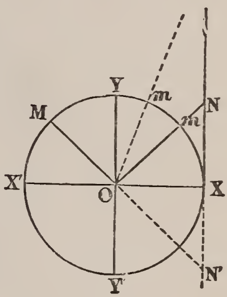
$$\sin(180° - \alpha) = \sin \alpha \qquad (2)$$

and $$\sin(90° + \alpha) = \sin(90° - \alpha).$$

If the points M' and M'' are equally distant from Y', so that angle $M'OY' =$ angle $Y'OM''$, which angle call y, the sines $P'M'$ and $P''M''$ will be equal. Hence, whatever be y,

$$\sin(270° - y) = \sin(270° + y).$$

22. *Changes in the tangent.* As the line Om revolves round O and m approaches Y, the point of intersection N will move upwards without limit. As m reaches Y, Om will become parallel to the tangent line, and the point N will recede to infinity. Hence:



The tangent of 90° *is infinite.*

When m is in the second quadrant, suppose in the position M, the revolving side OM will not intersect the tangent line at all in the positive direction OM. We must therefore suppose the revolving line to be produced in the negative direction ON' so as to intersect the tangent line at N' below X. The distance XN' is then to be regarded as negative. Hence:

In the second quadrant the tangent is negative.

Following the motion, we see that when m reaches X', N' reaches X and the tangent becomes zero. Hence

$$\tan 180° = 0.$$

When m is in the third quadrant, N passes above X and the tangent is positive, so that

In the third quadrant the tangent is positive.

Continuing the reasoning, we see that the tangent of 270° is infinite, and that in the fourth quadrant the tangent is negative.

23 *Changes in the secant.* The secant is defined as the distance ON from O to the point N, in which the revolving side intersects the tangent line XN.

When m falls on X and the angle is zero, the secant is equal to $0X$, or unity. Hence

$$\sec 0° = 1. \tag{3}$$

As m moves from X to Y, the secant increases without limit and becomes infinite when m reaches Y. Hence

$$\sec 90° = \infty. \tag{4}$$

As m moves from Y through X' to Y', the intersection of the revolving line with the tangent line falls in the negative part of OM, or in the direction ON'. Hence:

In the second and third quadrants the secant is negative.

At Y', when the angle is 270°, the secant again becomes infinite.

Between Y' and X, or in the fourth quadrant, it is again positive.

24. If we suppose the revolving line to make an integral number of revolutions from any point, it will return to its original position, and all the trigonometric functions will have the values corresponding to that position. Hence, if C is a circumference and n any integer,

$$\left.\begin{aligned} \sin(nC + \alpha) &= \sin\alpha; \\ \tan(nC + \alpha) &= \tan\alpha; \\ \sec(nC + \alpha) &= \sec\alpha. \end{aligned}\right\} \tag{5}$$

In other words,

The values of the trigonometric functions are not altered by increasing the angle by any integral number of circumferences.

If the angle is increased indefinitely, the values of these functions continually repeat themselves. This fact is expressed by saying that these functions are *periodic*.

Exercises.

Prove the following expressions for the trigonometrical functions of angles of more than 90° by the necessary diagrams. The angle x may be supposed less than 90°, though this restriction is not necessary to the validity of the formulæ.

$$\left.\begin{array}{ll} 1. & \sin(90° + x) = \sin(90° - x); \\ 2. & \sin(180° + x) = -\sin x; \\ 3. & \sin(270° + x) = -\sin(90° - x). \end{array}\right\} \quad (6)$$

$$\left.\begin{array}{ll} 4. & \tan(90° + x) = -(\tan 90° - x); \\ 5. & \tan(180° + x) = \tan x; \\ 6. & \tan(270° + x) = -(\tan 270° - x). \end{array}\right\} \quad (7)$$

$$\left.\begin{array}{ll} 7. & \sec(90° + x) = -\sec(90° - x); \\ 8. & \sec(180° + x) = -\sec x; \\ 9. & \sec(270° + x) = -\sec(270° - x). \end{array}\right\} \quad (8)$$

$$\left.\begin{array}{ll} 10. & \sin(-x) = -\sin x; \\ 11. & \tan(-x) = -\tan x; \\ 12. & \sec(-x) = \sec x. \end{array}\right\} \quad (9)$$

Note. When we have the values of the trigonometric functions from 0 to 90°, we can by these formulæ find the values for all angles.

The Cosine, Cotangent, and Cosecant.

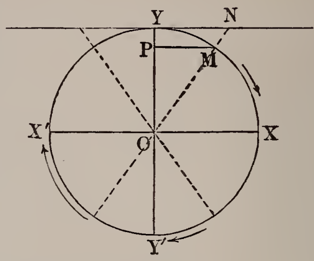
25. In the preceding sections we have supposed the side of the angle from which we count the degrees to go out toward the right, and the positive direction of motion to be opposite to that of the hands of a watch. But this restriction is only to fix the thought. We may suppose the angle to have any situation and to be counted in any direction without changing the values of the sine, tangent, and secant, provided that we reckon the lengths of the lines representing the functions in the right way.

Let us count the angle from OY in the direction toward OX. The tangent line must then touch the circle at Y, and its positive direction must be toward the right.

Then (radius $OY = 1$) the lines

$$\left.\begin{array}{l} PM = \sin POM, \\ YN = \tan POM, \\ ON = \sec POM, \end{array}\right\} \quad (a)$$

will have the same values as in an angle equal to POM counted in the usual way from OX toward Y.



Moreover, the changes of sign will be the same as before through the whole circle, namely:

From X to Y' (now the second quadrant) the sine PM will be positive because it is measured to the right.

From Y' to X' (the third quadrant) it will be negative because it is measured toward the left.

It will also be negative from X' to Y (now the fourth quadrant).

The corresponding propositions can be shown for the tangent and secant.

Now because angle $XOM +$ angle $MOY =$ angle $XOY = 90°$, MOY is the *complement* of XOM. Therefore the equations (a) may be written

$$PM = \text{sin comp. of } XOM;$$
$$YN = \text{tan comp. of } XOM;$$
$$ON = \text{sec comp. of } XOM.$$

Because when we have an angle its complement can always be determined by subtracting it from 90°, we can always find the sine, tangent, and secant of the complement when we know the angle. Therefore the sine, tangent, and secant of the complement of an angle may be regarded as three additional trigonometrical functions of the angle itself. They are named thus:

The sine of the complement is called the **cosine** of the angle.

The tangent of the complement is called the **cotangent** of the angle.

The secant of the complement is called the **cosecant** of the angle.

Thus the new functions are defined in the following way:

$$\left.\begin{aligned} \text{cosine } \alpha &= \sin(90° - \alpha); \\ \text{cotang } \alpha &= \tan(90° - \alpha); \\ \text{cosecant } \alpha &= \sec(90° - \alpha). \end{aligned}\right\} \qquad (10)$$

The words cosine, cotangent, and cosecant are abbreviated to cos, cot, cosec, respectively.

The forms (10) enable us to find the cosine, cotangent, and cosecant of an angle when we know the sine, tangent, and secant of its complement. Thus if the cosine of 60° is required, we have

$$\cos 60° = \sin (90° - 60°) = \sin 30°.$$

Also, by substituting $90° - \alpha$ for α, we find

$$\left.\begin{aligned}\sin \alpha &= \cos (90° - \alpha);\\ \tan \alpha &= \cot (90° - \alpha);\\ \sec \alpha &= \operatorname{cosec} (90° - \alpha).\end{aligned}\right\} \qquad (11)$$

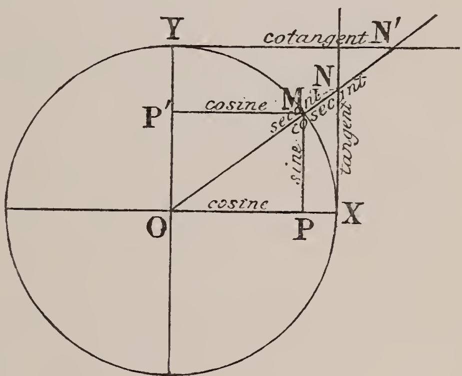
The versed-sine and co-versed-sine. Besides these six functions, two others, the *versed-sine* and *co-versed-sine*, are sometimes used. Their definitions are:

$$\text{Versed-sine} = PX = 1 - \text{cosine};$$
$$\text{Co-versed-sine} = OY - PM = 1 - \text{sine}.$$

26. The six trigonometrical functions may be represented on a single diagram. The functions as written are all those of the angle XOM. For the secant and cosecant we have

$$ON = \sec XOM,$$
$$ON' = \operatorname{cosec} XOM,$$

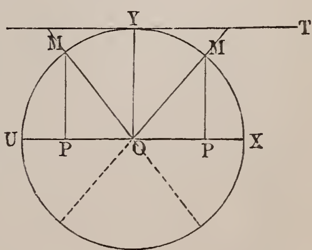
because XOM is the complement of MOY.



Because $PM \parallel OP'$ and $OP \parallel P'M$, therefore $OP = P'M$, so that we may take either OP or $P'M$ as the cosine of XOM.

27. The general definitions of § 20 may be extended to complementary functions, thus:

Having $OY \perp OX$ and $OY = OX = 1$, we draw through Y an indefinite line YT parallel to OX. Then—

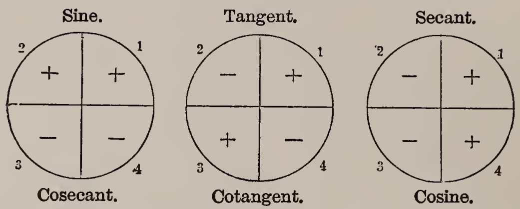


IV. *The cosine of any angle XOM is represented by the distance OP to the foot of the sine, positive or negative according to its direction.*

V. *The cotangent of any angle XOM is represented by the length from Y to the point in which OM produced intersects YT.*

VI. *The cosecant of XOM is represented by the length of the side OM, intercepted between O and the line YT.*

The algebraic signs of the several functions in the four quadrants are shown in the following diagram.



28. The following are the limiting values of the trigonometrical functions:

I. *Sine and cosine.* The sine MP and cosine OP are necessarily not greater in absolute value than $OM = 1$. The limits of these functions are therefore $+1$ and -1.

II. *Secant and cosecant.* Since the tangent line lies without the circle, a secant can never be less than unity in absolute magnitude. But we have found that it may increase to infinity in either the positive or negative direction. Hence the limits of the secant and cosecant are 1 and infinity, and -1 and $-\infty$.

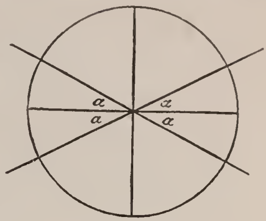
III. *Tangent and cotangent.* The limits of the tangent are easily seen to be $-\infty$ and $+\infty$, or a tangent and cotangent may have any value whatever.

29. When we know the numerical values of the sine, tangent, and secant of all angles from 0° to 90°, we have the values of all six functions of any angle whatever, because as we go around the circle the values of the functions are simply repetitions of the values between 0° and 90°.

Let α be any angle less than 90°. Then any angle in the first quadrant may be represented by α.

In the second quadrant it may be represented by $180° - \alpha$.
" third " " " " $180° + \alpha$.
" fourth " " " " $360° - \alpha$.

The student should now have no difficulty in demonstrating the following relations by completing the construction indicated in the margin and attending to the general definitions §§ 20, 27. Some of these relations have already been given or explained.



Second Quadrant.

$$\left.\begin{aligned}\sin(180^\circ-\alpha)&=\sin\alpha;\\ \cos(180^\circ-\alpha)&=-\cos\alpha=-\sin(90^\circ-\alpha);\\ \tan(180^\circ-\alpha)&=-\tan\alpha;\\ \cot(180^\circ-\alpha)&=-\cot\alpha=-\tan(90^\circ-\alpha);\\ \sec(180^\circ-\alpha)&=-\sec\alpha;\\ \operatorname{cosec}(180^\circ-\alpha)&=\operatorname{cosec}\alpha=\sec(90^\circ-\alpha).\end{aligned}\right\}\qquad(12)$$

Third Quadrant.

$$\left.\begin{aligned}\sin(180^\circ+\alpha)&=-\sin\alpha;\\ \cos(180^\circ+\alpha)&=-\cos\alpha=-\sin(90^\circ-\alpha);\\ \tan(180^\circ+\alpha)&=\tan\alpha;\\ \cot(180^\circ+\alpha)&=\cot\alpha=\tan(90^\circ-\alpha);\\ \sec(180^\circ+\alpha)&=-\sec\alpha;\\ \operatorname{cosec}(180^\circ+\alpha)&=-\operatorname{cosec}\alpha=-\sec(90^\circ-\alpha).\end{aligned}\right\}\qquad(13)$$

Fourth Quadrant.

$$\left.\begin{aligned}\sin(360^\circ-\alpha)&=-\sin\alpha;\\ \cos(360^\circ-\alpha)&=\cos\alpha=\sin(90^\circ-\alpha);\\ \tan(360^\circ-\alpha)&=-\tan\alpha;\\ \cot(360^\circ-\alpha)&=-\cot\alpha=-\tan(90^\circ-\alpha);\\ \sec(360^\circ-\alpha)&=\sec\alpha;\\ \operatorname{cosec}(360^\circ-\alpha)&=-\operatorname{cosec}\alpha=-\sec(90^\circ-\alpha).\end{aligned}\right\}\qquad(14)$$

We may equally express the six functions of all angles in terms of the six functions of angles not greater than 45°. Let γ represent any angle not greater than 45°. We may then represent

Any angle from 0° to 45° by γ;
" " 45° " 90° " $90^\circ-\gamma$;
" " 90° " 135° " $90^\circ+\gamma$;

Any angle from 135° to 180° by $180° - \gamma$;
" " 180° " 225° " $180° + \gamma$;
" " 225° " 270° " $270° - \gamma$;
" " 270° " 315° " $270° + \gamma$;
" " 315° " 360° " $360° - \gamma$.

Then, in addition to the relations (12), (13), and (14), which will remain true when we write γ instead of α, we shall have the following, which the student should prove.

To do this let the student suppose that in the diagram § 26 angle $XOM = \gamma$, and let him construct the six functions for angles of $90° + \gamma$, $270° - \gamma$, etc., and compare the lines representing them with the lines on the diagram of § 26.

The set corresponding to the first quadrant are already given in (10) and (11).

Second Quadrant.

$$\left.\begin{aligned} \sin(90° + \gamma) &= \cos\gamma; \\ \cos(90° + \gamma) &= -\sin\gamma; \\ \tan(90° + \gamma) &= -\cot\gamma; \\ \cot(90° + \gamma) &= -\tan\gamma; \\ \sec(90° + \gamma) &= -\operatorname{cosec}\gamma; \\ \operatorname{cosec}(90° + \gamma) &= \sec\gamma. \end{aligned}\right\} \quad (15)$$

Third Quadrant.

$$\left.\begin{aligned} \sin(270° - \gamma) &= -\cos\gamma; \\ \cos(270° - \gamma) &= -\sin\gamma; \\ \tan(270° - \gamma) &= \cot\gamma; \\ \cot(270° - \gamma) &= \tan\gamma; \\ \sec(270° - \gamma) &= -\operatorname{cosec}\gamma; \\ \operatorname{cosec}(270° - \gamma) &= -\sec\gamma. \end{aligned}\right\} \quad (16)$$

Fourth Quadrant.

$$\left.\begin{aligned} \sin(270° + \gamma) &= -\cos\gamma; \\ \cos(270° + \gamma) &= \sin\gamma; \\ \tan(270° + \gamma) &= -\cot\gamma; \\ \cot(270° + \gamma) &= -\tan\gamma; \\ \sec(270° + \gamma) &= \operatorname{cosec}\gamma; \\ \operatorname{cosec}(270° + \gamma) &= -\sec\gamma. \end{aligned}\right\} \quad (17)$$

Among the preceding forms of this chapter, the following are of especially frequent application:

$$\left.\begin{array}{l}\sin\alpha = \cos(90° - \alpha) = \quad\cos(\alpha - 90°) = -\cos(\alpha + 90°);\\ \cos\alpha = \sin(90° - \alpha) = -\sin(\alpha - 90°) = +\sin(\alpha + 90°).\end{array}\right\} \quad (18)$$

Exercises.

1. Express the six functions of the following angles in terms of the three functions sine, tangent, and secant of angles less than 90°:

sin 105°; 200°; 295°.
cos 105°; 200°; 295°.
tan 105°; 200°; 295°.
cot 105°; 200°; 295°.
sec 105°; 200°; 295°.
cosec 105°; 200°; 295°.

2. The following table shows the values of four of the functions for every 10° of the first 40° to two places of decimals. By means of these values extend the table to 360°, showing the values of all four functions for each angle:

Angle.	Sin.	Tan.	Cot.	Cos.
0°	0·00	0·00	∞	+1·00
10°	+0·17	+0·18	+5·67	+0·98
20°	+0·34	+0·36	+2·75	+0·94
30°	+0·50	+0·58	+1·73	+0·87
40°	+0·64	+0·84	+1·19	+0·77
50°				
60°				
70°				
etc., to 360°				

3. Demonstrate the relations (18) by drawing a diagram showing an arbitrary angle α and an angle 90° greater and less, with the lines representing the sines and cosines.

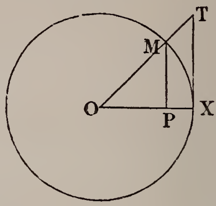
4. What relations subsist between the following pairs of functions?

(*a*) $\sin(45° + a)$ and $\cos(45° - a)$;
(*b*) $\sin(135° + a)$ and $\cos(135° - a)$;
(*c*) $\tan(225° + a)$ and $\cot(225° - a)$;
(*d*) $\sec(315° + a)$ and $\operatorname{cosec}(315° - a)$.

30. *Special values of trigonometric functions.* If angle $XOM = 45°$, we shall also have $OMP = 45°$, and therefore

$$OP^2 + PM^2 = 2\,PM^2 = OM^2,$$

whence $PM = OM\sqrt{\tfrac{1}{2}}.$



Therefore $\sin XOM = \sin 45° = \sqrt{\tfrac{1}{2}}.$ (*a*)

In the same way we find

$$XT = OX,$$

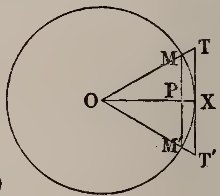
whence $\tan XOM = \tan 45° = 1.$ (*b*)

Again, $OT = \sqrt{OX^2 + XT^2} = \sqrt{2}\,OX;$

whence $\sec XOM = \sec 45° = \sqrt{2}.$ (*c*)

Next, let angle $XOM = 30°$.

Make angle $XOM' = XOM = 30°$.



The triangles $M'OM'$ and TOT' then have each of their angles 60°, and so are equilateral.

Therefore $MP = \tfrac{1}{2}MM' = \tfrac{1}{2}OM = \tfrac{1}{2}OX.$

Hence $\sin 30° = \tfrac{1}{2}.$ (*d*)

In the same way

$$XT = \tfrac{1}{2}OT,$$

$$OT^2 - XT^2 = OX^2 = 1;$$

or $\tfrac{3}{4}OT^2 = 1, \qquad OT = \sqrt{\tfrac{4}{3}} = \dfrac{2}{\sqrt{3}}, \qquad XT = \dfrac{1}{\sqrt{3}}.$

Also, $OP = \sqrt{OP^2 - MP^2} = \sqrt{1 - \tfrac{1}{4}} = \dfrac{\sqrt{3}}{2}.$

Hence $\tan 30° = \dfrac{1}{\sqrt{3}},$ (*e*)

$\sec 30° = \dfrac{2}{\sqrt{3}},$ (*f*)

$\cos 30° = \dfrac{\sqrt{3}}{2}.$ (*g*)

Functions of 18°. It is shown in geometry that if the radius of a circle be divided in extreme and mean ratio, the greater segment will be the chord of 36°; that is, twice the sine of 18°.

Putting 1 for the radius and r for the greater segment, the condition that the division shall be in extreme and mean ratio is

$$1 : r :: r : 1 - r,$$

or, equating the product of the means to that of the extremes,

$$r^2 = 1 - r.$$

The solution of this quadratic equation gives

$$r = \frac{-1 \pm \sqrt{5}}{2}.$$

The positive root is the only one we want. Hence

$$\sin 18° = \frac{\sqrt{5} - 1}{4}.$$

We then find

$$\cos^2 18° = \text{radius}^2 - \sin^2 18° = 1 - \sin^2 18°.$$

Hence

$$\cos 18° = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.$$

31. Angles corresponding to given trigonometric functions. When the value of a trigonometric function is given and the angle is required, there are always two solutions to the problem.

The Sine. It has already been shown that two supplementary angles have the same sine. Hence if α is an angle corresponding to a given sine, $180° - \alpha$ will be another angle equally corresponding to it.

We may also say that if $90° - \beta$ be an angle corresponding to any sine, $90° + \beta$ will also correspond to it, because the sum of these two angles is 180°. The same statement applies to the angles $270° - \beta$ and $270° + \beta$.

Unless there is some restriction upon the angle to be chosen, we cannot decide which angle to take. The most common restriction is that the angle must be between the limits $-90°$ and $+90°$, or must be in either the first or fourth quadrant. There will then be between these limits one angle and only one for a given sine. Since the measurements of latitude on the surface of the earth are restricted between limits $-90°$ and $+90°$, the latitude of a place is completely fixed by its sine.

The Cosine. The construction of the cosine shows that it has equal values for positive and negative angles. Hence if α be an angle corresponding to a given cosine, $-\alpha$ or $360° - \alpha$ will equally correspond to it.

Hence when an angle is determined by its cosine, either of these two angles may be chosen unless some restriction is placed upon the choice. The most common restriction is that the angle shall be positive and less than 180°; that is, in the first or second quadrant. For every cosine there will be one and only one angle between the limits 0° and 180°.

The Tangent. Since two angles which differ by 180° have the same tangent, it follows that if α be an angle corresponding to any tangent, $180° + \alpha$ will equally correspond to it. Hence there is always a choice between these two angles, unless some restriction is placed upon the angle.

The Cotangent. The cotangent being determined, like the tangent, by the intersection of the revolving side with a tangent line, every pair of angles corresponding to the same tangent will also correspond to the same cotangent. Hence α and $180° + \alpha$ always have the same cotangent.

The Secant. We readily see that the angles α and $-\alpha$ or $360° - \alpha$ have the same secant. Hence if α be an angle corresponding to a given secant, $360° - \alpha$ will be another angle corresponding to that same secant.

The Cosecant. From the diagram (§ 27) it is easy to show that any two supplementary angles have the same cosecant.

Exercises.

What other angles have the same sines as the following?

1. 105°; 2. 185°; 3. 290°.

What other angles have the same cosines as the following?

4. 72°; 5. 165°; 6. 320°.

What other angles have the same tangents as the following?

7. 50°; 8. 205°; 9. 355°.

10. A pair of angles having the same sine differ by 24°. What angles are they?

11. A pair of angles having the same cosine differ by 110°. What angles are they?

Note. There are two pairs of angles which answer each of the two last questions.

Find two values of α from each of the following equations:

12. $\sin\alpha = -\cos 23°$. Ans. $\alpha = 247°$ or $293°$.
13. $\cos\alpha = -\sin 23°$.
14. $\sin\alpha = \cos\beta$. Ans. $\alpha = 90° - \beta$ or $90° + \beta$.
15. $\tan\alpha = \cot 72°$.
16. $\cot\alpha = -\tan 175°$.
17. $\tan\alpha = -\cot\beta$.
18. $\sec\alpha = \operatorname{cosec} 32°$.
19. $\operatorname{cosec}\alpha = -\sec 32°$.
20. $\sec\alpha = \operatorname{cosec}\beta$.
21. $\sin\alpha = \sin 23°$.
22. $\sin\alpha = \cos(90° - x)$. Ans. $\alpha = x$ or $180° - x$.
23. $\sin\alpha = \cos(90° + x)$. Ans. $\alpha = -x$ or $180° + x$.
24. $\sin\alpha = \cos(270° + x)$.
25. $\sin\alpha = \cos(270° - x)$.
26. $\cos\alpha = \cos(180° - x)$.
27. $\cos\alpha = \cos(180° + x)$.

32. *Extension to unlimited angles.* Since when any integral number of circumferences is added to an angle its trigonometric functions remain unaltered, we must, to find the most general expression for the angle corresponding to a given function, add an arbitrary number of circumferences to the angles found in the last section. Then the most general expression for angles which have the same sine will be

$$nC + 90° + \alpha \quad \text{and} \quad nC + 90° - \alpha,$$

in which n may take all integral values, positive and negative, including zero, and α must have such a value that $90 - \alpha$ shall correspond to the given sine.

This statement also means that all the angles formed by giving different values to n, while α remains constant, will have the same sine.

Example. Let us suppose the given sine to be that of 65°. Then $\alpha = 25°$, and the pairs of angles

$65°$,	$115°$;	$-\ C + 65°$,	$-\ C + 115°$;
$C + 65°$,	$C + 115°$;	$-2\,C + 65°$,	$-2\,C + 115°$;
$2\,C + 65°$,	$2\,C + 115°$;	etc.	etc.,
etc.	etc.;		

will all have this same sine.

Exercises.

1. Having given

$$\sin 2x = \tfrac{1}{2},$$

find the four corresponding values of x within the first circumference. Note that $\sin 30° = \tfrac{1}{2}$. Ans. 15°, 75°, 195°, 255°.

2. Having given

$$\sin 2\alpha = \sqrt{\tfrac{1}{2}},$$

find the four corresponding values of α.

First find the two values of 2α, and then, by §§ 10–13, the two values of half of each of these angles.

3. Having given

$$\cos 2\beta = \tfrac{1}{2},$$

find the four corresponding values of β.

4. Having given

$$\cos 2\beta = -\tfrac{1}{2},$$

find the four corresponding values of β.

5. If $\tan 2\alpha = -\sqrt{3}$, find four values of α.

6. Show that if the value of $\tan \tfrac{1}{2}\alpha$ be given there will be only one value of α to correspond to it within the first circumference.

7. If $\sin \tfrac{1}{2}x = \sin 15°$,

find what two values may x have, and show that these values will have the same cosine.

8. Form the most general expression for all angles having the same cosine.

9. Form the most general expression for all angles having the same tangent.

Relations between the Six Trigonometric Functions.

33. When we know any one trigonometric function of an angle two definite values of the angle can then be determined, by constructions like those of § 19. Knowing the angle, the values of the other functions can be found. Hence from any one function of an angle all the others can be found. We have now to investigate the algebraic relations by which this may be done, seeking to express each function in terms of the five others.

Let us put

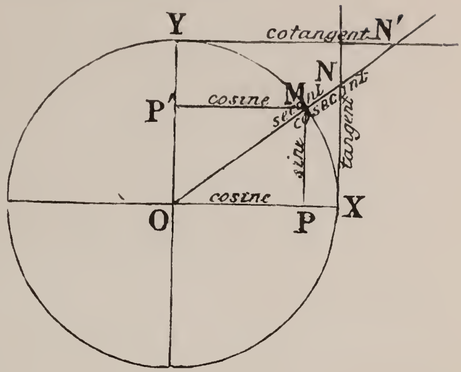
$$\alpha = \text{angle } XOM.$$

Because OPM is a right-angled triangle,

$$OP^2 + PM^2 = OM^2,$$

or

$$\left(\frac{OP}{OM}\right)^2 + \left(\frac{PM}{OM}\right)^2 = 1.$$



Hence

$$\cos^2 \alpha + \sin^2 \alpha = 1,$$

which gives

$$\cos \alpha = \pm \sqrt{1 - \sin^2 \alpha}. \quad (1)$$

The significance of the double sign $\pm$ is this: whenever a sine is given there will be two angles to correspond to it, of which one will be as much less than 90° as the other is greater. The cosines of these angles will be equal with opposite algebraic signs.

The similar triangles OPM and OXN give

$$OP : PM = OX : XN;$$
$$OP : OM = OX : ON.$$

Substituting for the lines their algebraic equivalents, the first proportion gives

$$\cos \alpha : \sin \alpha = 1 : \tan \alpha.$$

Hence $$\tan \alpha = \frac{\sin \alpha}{\cos \alpha} = \frac{\sin \alpha}{\sqrt{1 - \sin^2 \alpha}}, \quad (2)$$

which gives the tangent in terms of the sine and cosine.

The second proportion gives

$$\cos \alpha : 1 = 1 : \sec \alpha,$$

whence $$\sec \alpha = \frac{1}{\cos \alpha} = \frac{1}{\sqrt{1 - \sin^2 \alpha}}. \quad (3)$$

We conclude from this:

The product of the cosine and secant is equal to unity.

In other words, *the secant and cosine are reciprocals of each other.*

By a similar course of reasoning upon the complementary triangle we find that *the cosecant and sine are reciprocals of each other.*

The similar triangles $OP'M$ and OYN' give

$$OP' : P'M = OY : YN';$$

but $$OP' = PM = \sin\alpha,$$

whence $$\sin\alpha : \cos\alpha = 1 : \cot\alpha,$$

and $$\cot\alpha = \frac{\cos\alpha}{\sin\alpha} = \frac{\sqrt{1 - \sin^2\alpha}}{\sin\alpha}. \qquad (4)$$

Comparing with (2) we have the relation: *The product of the tangent and cotangent of any angle is unity.*

In other words, *the tangent and cotangent are reciprocals of each other.*

We thus reach the general conclusion *that the three complementary functions are each the reciprocal of one of the three other functions*, namely:

$$\text{cosine} = \frac{1}{\text{secant}};$$

$$\text{cotangent} = \frac{1}{\text{tangent}};$$

$$\text{cosecant} = \frac{1}{\text{sine}}.$$

EXERCISES.

1. If $\sin\gamma = 0.60$, find $\cos\gamma$, $\tan\gamma$, $\cot\gamma$, $\sec\gamma$, and $\operatorname{cosec}\gamma$

2. Find the values of the same functions when $\cos\gamma = 0.60$.

3. Prove that the mean proportional between $a\cos x$ and $a\sec x$ is a.

4. Prove that the mean proportional between $a\tan\beta$ and $b\cot\beta$ is $\sqrt{ab}$.

34. *Expression of each function in terms of the others.* By means of the relations (1) to (4) any one trigonometric function may be expressed in terms of any other by algebraic substitutions.

The following are the expressions which we thus obtain. All not already given should be deduced by the student as an exercise.

$$\sin\alpha = \sqrt{1 - \cos^2\alpha} = \frac{\tan\alpha}{\sqrt{1 + \tan^2\alpha}} = \frac{1}{\sqrt{1 + \cot^2\alpha}}$$

$$= \frac{\sqrt{\sec^2\alpha - 1}}{\sec\alpha} = \frac{1}{\operatorname{cosec}\alpha}.$$

$$\cos\alpha = \sqrt{1-\sin^2\alpha} = \frac{1}{\sqrt{1+\tan^2\alpha}} = \frac{\cot\alpha}{\sqrt{1+\cot^2\alpha}}$$

$$= \frac{1}{\sec\alpha} = \frac{\sqrt{\operatorname{cosec}^2\alpha - 1}}{\operatorname{cosec}\alpha}.$$

$$\tan\alpha = \frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}} = \frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha} = \frac{1}{\cot\alpha}$$

$$= \sqrt{\sec^2\alpha - 1} = \frac{1}{\sqrt{\operatorname{cosec}^2\alpha - 1}}$$

$$\cot\alpha = \frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha} = \frac{\cos\alpha}{\sqrt{1-\cos^2\alpha}} = \frac{1}{\tan\alpha}$$

$$= \frac{1}{\sqrt{\sec^2\alpha - 1}} = \sqrt{\operatorname{cosec}^2\alpha - 1}.$$

$$\sec\alpha = \frac{1}{\sqrt{1-\sin^2\alpha}} = \frac{1}{\cos\alpha} = \sqrt{1+\tan^2\alpha}$$

$$= \frac{\sqrt{1+\cot^2\alpha}}{\cot\alpha} = \frac{\operatorname{cosec}\alpha}{\sqrt{\operatorname{cosec}^2\alpha - 1}}.$$

$$\operatorname{cosec}\alpha = \frac{1}{\sin\alpha} = \frac{1}{\sqrt{1-\cos^2\alpha}} = \frac{\sqrt{1+\tan^2\alpha}}{\tan\alpha}$$

$$= \sqrt{1+\cot^2\alpha} = \frac{\sec\alpha}{\sqrt{\sec^2\alpha - 1}}.$$

NOTE. In algebraic work of this sort the student will find it convenient, instead of writing $\sin\alpha$, $\cos\alpha$, etc., in full, to use a single symbol for each function; for example, he may put:

$$s = \sin\alpha; \qquad \frac{1}{s} = \operatorname{cosec}\alpha.$$

$$t = \tan\alpha; \qquad \frac{1}{t} = \cot\alpha.$$

$$c = \cos\alpha; \qquad \frac{1}{c} = \sec\alpha.$$

It will be seen that the second members of most of these equations are surds, showing that their values may be either positive or negative. The result may be expressed thus:

Whenever one trigonometric function is given, the four other functions which are not its reciprocal may have either of two equal values with contrary signs.

This arises from the fact that every such function may belong to either of two angles, and affords an example of the correspondence between geometric and algebraic results.

Exercises.

1. From the special values of sin 30° and sin 45° found in § 30, namely,

$$\sin 30° = \tfrac{1}{2},$$
$$\sin 45° = \sqrt{\tfrac{1}{2}},$$

find the values of the other five functions of 30° and of 45°.

Ans. $\cos 30° = \frac{\sqrt{3}}{2}$; $\cos 45° = \sqrt{\tfrac{1}{2}}$.

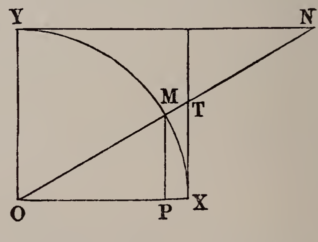
$\tan 30° = \frac{1}{\sqrt{3}}$; $\tan 45° = 1$.

$\cot 30° = \sqrt{3}$; $\cot 45° = 1$.

$\sec 30° = \frac{2}{\sqrt{3}}$; $\sec 45° = \sqrt{2}$.

$\operatorname{cosec} 30° = 2$; $\operatorname{cosec} 45° = \sqrt{2}$.

2. It has been shown that the three products *sin* × *cosec*, *tan* × *cot*, and *cos* × *sec* are each unity. When we replace the functions by the lines which represent them, the products represent the areas of the rectangles contained by the lines, and unity is replaced by the square of the radius. Hence we have the following theorems, which are to be proved by the similar triangles in the accompanying figure, where the construction is that of the trigonometric functions.



I. Rectangle $XT.YN = OX^2$, corresp. to tan × cot = 1.

II. Rectangle $MP.ON = OX^2$, " " sin × cosec = 1.

III. Rectangle $OP.OT = OX^2$, " " cos × sec = 1.

3. From the value of sin 18° in § 30 find the sides of the regular inscribed and circumscribed decagons of a circle of radius *a*.

Prove the following relations:

4. $$1 + \sin x = \frac{\cos^2 x}{1 - \sin x}.$$

5. $$\frac{\sec\theta + \operatorname{cosec}\theta}{\sec\theta - \operatorname{cosec}\theta} = \frac{1 + \cot\theta}{1 - \cot\theta} = \frac{\tan\theta + 1}{\tan\theta - 1}.$$

6. $$\tan\theta + \cot\theta = \frac{\sec^2\theta + \operatorname{cosec}^2\theta}{\sec\theta\operatorname{cosec}\theta}.$$

7. $$\sec\theta + \tan\theta = \frac{1}{\sec\theta - \tan\theta}.$$

8. $$\frac{\tan\theta + \sec\theta - 1}{\tan\theta - \sec\theta + 1} = \tan\theta + \sec\theta.$$

9. $$\frac{\tan\theta - \sin\theta}{\sin^3\theta} = \frac{\sec\theta}{1 + \cos\theta}.$$

10. What angle is that of which the tangent is double the sine?

11. What must be the value of the cosine in order that the tangent may be n times the sine? Ans. $\frac{1}{n}$.

12. Prove $(r\cos x)^2 + (r\sin x\sin u)^2 + (r\sin x\cos u)^2 = r^2$.

13. Prove $(a\sin\gamma)^2 + (a\cos\gamma\sin\delta)^2 + (a\cos\gamma\cos\delta)^2 = a^2$.

14. Prove $(\cos a\cos b - \sin a\sin b)^2$
$$+ (\sin a\cos b + \cos a\sin b)^2 = 1.$$

15. Of what angle is the secant double the sine?

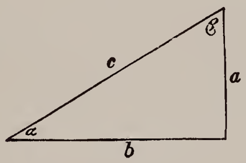
16. Of what angle is the secant four times the cosine?

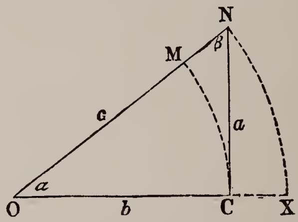
Prove $\sin^2 x - \sin^2 y = \cos^2 y - \cos^2 x$

CHAPTER III.

OF RIGHT TRIANGLES.

35. *Fundamental relations.* Let OCN be a right triangle of which a and b are the sides which contain the right angle, c the hypothenuse, α and β the angles opposite a and b respectively.





If we take ON as a radius and draw the arc NX from the centre O, the side NC will, by definition, represent the sine of XON, and OC its cosine, when the radius is ON. That is,

$$\frac{NC}{ON} = \sin\alpha; \qquad \frac{OC}{ON} = \cos\alpha. \qquad (1)$$

We may show in the same way, by taking N as the centre and NO as the radius,

$$\frac{OC}{ON} = \sin\beta; \qquad \frac{NC}{ON} = \cos\beta. \qquad (2)$$

We might also have deduced these equations from (1), because $\beta = 90° - \alpha$, whence

$$\sin\beta = \sin(90° - \alpha) = \cos\alpha;$$
$$\cos\beta = \cos(90° - \alpha) = \sin\alpha.$$

Again, by taking OC as a radius, we find

$$\frac{NC}{OC} = \tan\alpha = \cot\beta; \qquad \frac{OC}{NC} = \tan\beta = \cot\alpha. \qquad (3)$$

Putting $NC = a$, $OC = b$, $ON = c$, the equations (1), (2), and (3) give the relations

$$\left.\begin{aligned} a &= c \sin\alpha = b\tan\alpha;\\ b &= c\cos\alpha = a\cot\alpha;\\ c &= a\,\mathrm{cosec}\,\alpha = b\sec\alpha. \end{aligned}\right\} \qquad (4)$$

We may express the same relations in terms of β, using the complementary functions, as follows:

$$\left.\begin{aligned} a &= c\cos\beta = b\cot\beta;\\ b &= c\sin\beta = a\tan\beta;\\ c &= a\sec\beta = b\,\mathrm{cosec}\,\beta. \end{aligned}\right\} \qquad (5)$$

These relations may be summed up in the following general theorems:

I. *The hypothenuse of any right triangle is equal to a side into the secant of its adjacent angle or the cosecant of its opposite angle.*

II. *A side is equal to the hypothenuse into the sine of the opposite angle or the cosine of the adjacent angle.*

III. *One side is equal to the other side into the tangent of the angle adjacent to that other side or the cotangent of the angle adjacent to itself.*

EXERCISE.

Show by the above equations how each side will be expressed in terms of the others when $\alpha = 30°$ and when $\alpha = 45°$, using the values of sin α, etc., already found—namely,

$$\sin 45° = \sqrt{\tfrac{1}{2}}; \qquad \tan 45° = 1; \text{ etc.,}$$

and $\quad \sin 30° = \tfrac{1}{2}; \qquad \cos 30° = \dfrac{\sqrt{3}}{2}; \qquad \tan 30° = \dfrac{1}{\sqrt{3}}$; etc.—

and show how the results agree with those of elementary geometry.

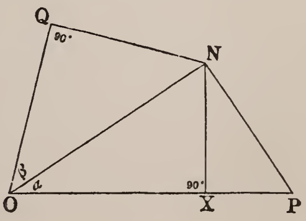
36. *Examples and exercises in expression.* In the accompanying figure OQN, ONP, and OXN are right angles, and we put

$$\alpha = \text{angle } XON;$$
$$\beta = \text{angle } NOQ.$$

It will also be noticed that

$$\text{Angle } XNP = \alpha$$

and Angle $ONX = XPN = 90° - \alpha$.



It is now required to express all the other lines in terms of OX and trigonometric functions of α and β.

Solution. We have

$$ON = OX \sec \alpha;$$
$$XN = OX \tan \alpha;$$
$$OP = ON \sec \alpha = OX \sec^2 \alpha;$$
$$NP = OP \sin \alpha = OX \sin \alpha \sec^2 \alpha = OX \tan \alpha \sec \alpha;$$

or
$$NP = ON \tan \alpha = OX \sec \alpha \tan \alpha \text{ (as before)};$$
$$OQ = ON \cos \beta = OX \sec \alpha \cos \beta;$$
$$NQ = ON \sin \beta = OX \sec \alpha \sin \beta.$$

Exercises.

1. By the same process express OQ, QN, OX, NX, NP, and XP in terms of ON and trigonometric functions.

2. Express the same quantities in terms of NP.

3. Express NX separately in terms of OX and XP, and by multiplying the two values prove the geometric theorem that NX is a mean proportional between OX and XP.

4. In a right triangle the sides which contain the right angle are a and b, ($a > b$), and δ is the difference of the angles at the base (hypothenuse). Express the length of the perpendicular from the vertex upon the base in two ways, and the lengths of the segments into which it divides the base each in one way. The expressions are all to be in terms of a, b and δ.

Ans. (*in part*). One expression for the perpendicular is

$$p = a \sin (45° - \delta).$$

Solution of Right Triangles.

Since in a right triangle one angle—the right angle—is given, only two other independent parts are required to solve the triangle. These two parts may be any two of the sides or one side and one angle. What parts soever are given, the remaining parts may be found by the equations (4) and (5). The following are all the essentially different cases.

37. CASE I. *Given the two sides a and b adjacent to the right angle.**

The first equation (4),

$$a = b \tan \alpha,$$

gives

$$\tan \alpha = \frac{a}{b}.$$

Therefore the quotient of the two sides gives the tangent of the angle opposite the dividend side.

From the tangent the angle α is itself found by the trigonometric tables; then sec α or cos α; then the hypothenuse c from the equation

$$c = b \sec \alpha = \frac{b}{\cos \alpha}.$$

Example. Given $a = 9$ metres, $b = 12$ metres, to find the remaining parts of the triangle.

Solution by numbers and measurement.

$$\tan \alpha = \tfrac{9}{12} = 0.75.$$

On the tangent line XN (§ 22) measure a distance from X equal to 0.75 of the radius OX; join the end of the distance to O, and measure the angle XON which the joining line makes with OX. This angle will be α. The length of the joining line divided by OX will be the secant, which multiplied by $b = 12$ will give the hypothenuse c.

The third angle, $\beta = 90° - \alpha$.

Logarithmic Solution.

log a	0.95 424	cos α	9.90 309
log b	1.07 918	log b	1.07 918
tan α	9.87 506	log c	1.17 609
α,	36° 52′.2	c,	15 metres.
β,	53° 7′.8		

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* It is recommended that in commencing this subject the student first solve a few of the problems by his own process, and without the use of any tables but those he may construct for himself by measurement as described in § 18.

Exercises.

1. Given $a = 4$, $b = 5$; find remaining parts.
2. " $a = 8$, $b = 5$; " "
3. " $a = 43.148$, $b = 84.107$; " "
4. " $a = 2.7938$, $b = 876.59$; " "
5. " $a = 759.28$, $b = 51.85$; " "
6. " $a = 8628$, $b = 27316$; " "

The first two exercises are made purposely simple, that they may be performed by measurement.

38. Case II. *Given the hypothenuse and one side.*

Solution. From the equation

$$c \sin \alpha = a$$

we obtain

$$\sin \alpha = \frac{a}{c},$$

which may be used to find α when a and c are given. Then the remaining side is found by the equation

$$b = c \cos \alpha.$$

Example. Given $a = 13$, $c = 20$, to find the remaining parts.

Solution by numbers and measurement.

$$\sin \alpha = \frac{a}{c} = \frac{13}{20} = 0.65.$$

Take a distance equal to 0.65 of OX in the dividers, and find out what angle it will fit in the diagram (§ 15) to form a sine of an angle. This angle will be α. Measure off $\cos \alpha$ perpendicularly to OB, take its ratio to OX, and multiply it by $c = 5$. This will give the side b.

Logarithmic Solution.

log. 13	1.11 394	$\cos \alpha$	9.88 078
log. 20	1.30 103	log 20	1.30 103
$\sin \alpha$	9.81 291	log b	1.18 181
α,	40° 32′.5	b,	15.199
β,	49° 27′.5		

Exercises.

1. Given $a = 7$, $c = 9$; find remaining parts.
2. " $b = 9$, $c = 16$; " " "
3. " $a = 82.143$, $c = 120.412$; " " "
4. " $b = 2.9235$, $c = 9.827$; " " "

5. A circle of radius r is drawn with its centre at a distance p from a straight line. What length will it cut out from the line? What will be the result if $r < p$?

39. Case III. *Given an angle and one side, as α and a.*

Solution. The equations (a) give

$$b = \frac{a}{\tan \alpha} = a \cot \alpha = a \cot \beta;$$

$$c = \frac{a}{\sin \alpha} = a \operatorname{cosec} \alpha = a \sec \beta.$$

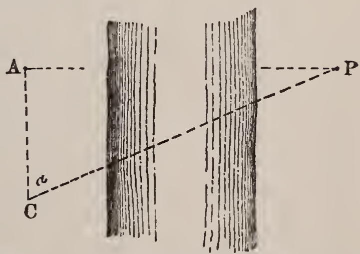
Note. One angle being given, the remaining one may be found from the equation

$$\beta = 90^\circ - \alpha.$$

Exercises.

1. Given $\alpha = 72^\circ\ 39'$, $c = 19.5$; find a and b.
2. " $\alpha = 16^\circ\ 25'.6$, $c = 10.925$; " "
3. " $\beta = 43^\circ\ 28'.4$, $b = 8.1273$; find a and c.
4. " $\beta = 8^\circ\ 29'.2$, $b = 0.9271$; " "

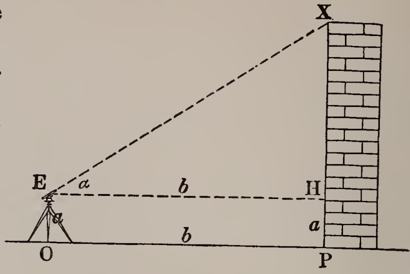
5. An engineer, desiring to find the distance from a point A on one bank of a river to a point P on the other bank, measured off a base line AC b yards in length, in a direction perpendicular to AP. He then measured the angle ACP, and found it to be α. What are the expressions for the distances AP and CP?



What will be the distance if $AC = 80$ yards and $\alpha = 85^\circ\ 22'.5$?

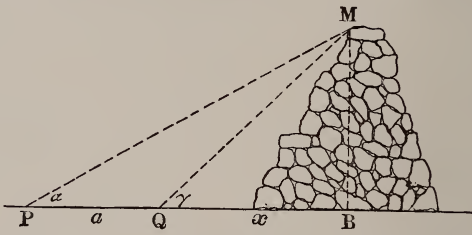
6. An engineer, desiring to determine the height of a vertical

wall, measured off a distance $PO = b$ feet on level ground, and then from a point E, a feet above the ground, measured the angle $HEX = \alpha$ between the line of sight EX and the horizontal line EH. Express the height PX of the wall algebraically in terms of a, b, and α, and compute the height when $b = 400$ feet, $a = 6$ feet, and the angle $\alpha = 22° 17'$.



Method of solution. Find the height HX and add $HP = a$.

7. Desiring to find the height of an inaccessible rock M above a plane, its angle BPM was measured from a point P and found to be α. The observer then advanced a metres towards the rock to Q, and there again measured the angle of elevation and found it to be γ. Express the height BM of the rock.



Method of solution. Let the vertical height $BM = h$ and $QB = x$. Then we have the two right triangles PBM and QBM, which give

$$h = (a + x) \tan \alpha;$$
$$h = x \tan \gamma.$$

From these two equations we obtain the following values of h and x:

$$x = \frac{a \tan \alpha}{\tan \gamma - \tan \alpha};$$

$$h = \frac{a \tan \alpha \tan \gamma}{\tan \gamma - \tan \alpha}.$$

Find the height when $a = 2000$ yards, $\alpha = 30° 28'$, and $\gamma = 40° 53'$.

8. The altitude of a triangle is 7.2648, and the angles at the base are $72° 29.3'$ and $40° 30.5'$ respectively. Compute the base

and sides. Also, find the general expression for the length of the base in terms of altitude and angles at base.

9. From the top of a tower 108 feet high the angles of depression of the top and bottom of another tower standing on the same horizontal plane are found to be 28° 56′ and 53° 41′ respectively. Find the distance between the towers, the height of the second tower, and the distance between the summits of the two towers.

10. In a circle of radius r, express the length of each side, and of the apothegm, of a regular inscribed polygon of n sides. Find first the special values for the triangle, square, pentagon, hexagon, and octagon. *Ans.* For the octagon: side $= 2r \sin 22\frac{1}{2}°$;
apothegm $= r \cos 22\frac{1}{2}°$.

11. If the side of a regular octagon is 10 metres, what are the radii of the inscribed and circumscribed circles?

12. At what altitude is the sun when a tower 20 metres high casts a shadow 75 metres long upon a horizontal plane?

13. It was found that the length of the shadow of a monument upon a horizontal plane diminished 22 metres when the sun's altitude increased from 30° to 45°. What was the height of the monument?

14. If β is one of the acute angles of a right triangle, and c its hypothenuse, express the altitude in terms of c and β.

15. Two lighthouses, each 30 metres above the sea and 500 metres apart, are seen by a ship in line with them to differ 1° in elevation above the horizon. What is the distance from the nearer, supposing the ocean a plane?

16. The great pyramid of Gizeh is 762 feet square at its base, and each side makes an angle of 51° 51′ with the horizon. Find—

(*a*) Its height if continued to its apex.

(*b*) The slope of its edges.

Actually, instead of being continued to its apex, it terminates in a platform 32 feet square. Find—

(*c*) The perpendicular height of this platform above the base.

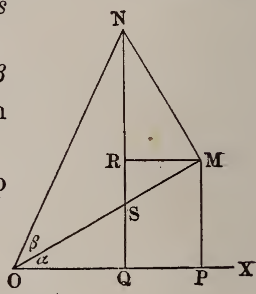
(*d*) The length of each edge from the corner of the platform to the corner of the base.

CHAPTER IV.

RELATIONS BETWEEN FUNCTIONS OF SEVERAL ANGLES.

The Addition and Subtraction Theorems.

40. PROBLEM. *To express the sine and cosine of the sum of two angles in terms of the sines and cosines of the angles themselves.*



Solution. Let $XOM = \alpha$ and $MON = \beta$ be the two angles. $XON = \alpha + \beta$ is then their sum.

Let ON be the unit radius. From N drop

$$NM \perp OM; \qquad NQ \perp OX.$$

From M drop

$$MP \perp OX; \qquad MR \perp NQ.$$

Then
$$\left.\begin{array}{l} \sin(\alpha + \beta) = NQ = NR + MP; \\ \cos(\alpha + \beta) = OQ = OP - MR. \end{array}\right\} \qquad (1)$$

OQS and NMS being right angles, we have

Angle $RNM =$ comp. $RSM =$ comp. $OSQ = SOQ = \alpha$.

By § 35,

$$NM = \sin\beta;$$
$$NR = NM\cos\alpha = \sin\beta\cos\alpha;$$
$$OM = \cos\beta;$$
$$MP = OM\sin\alpha = \cos\beta\sin\alpha;$$

whence, from (1),

$$\sin(\alpha + \beta) = \sin\alpha\cos\beta + \cos\alpha\sin\beta. \qquad (2)$$

We find in the same way

$$OP = OM\cos\alpha = \cos\beta\cos\alpha;$$
$$RM = NM\sin\alpha = \sin\beta\sin\alpha;$$

whence, from (1),

$$\cos(\alpha + \beta) = \cos\alpha\cos\beta - \sin\alpha\sin\beta. \qquad (3)$$

The formulæ (2) and (3) constitute the *addition theorem* of trigonometry.

To find the corresponding subtraction theorem it is only necessary to change the sign of β. We have

$$\sin(-\beta) = -\sin\beta;$$
$$\cos(-\beta) = \cos\beta.$$

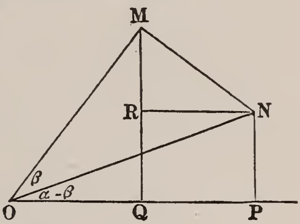
Therefore, changing β to $-\beta$ in (2) and (3), we find

$$\sin(\alpha-\beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta; \qquad (4)$$
$$\cos(\alpha-\beta) = \cos\alpha\cos\beta + \sin\alpha\sin\beta. \qquad (5)$$

It is, however, interesting to show how these equations may be obtained independently by a geometrical construction.

Let POM be the angle α, and NOM the angle β. Then

$$PON = \alpha - \beta.$$



Take ON for the unit radius and drop

$$NP \perp OP; \qquad NM \perp OM;$$
$$MQ \perp OP; \qquad NR \perp MQ.$$

Then $\sin(\alpha-\beta) = PN = QR = MQ - MR;$

$$\cos(\alpha-\beta) = OP = OQ + RN.$$

Because OMN is a right angle,

$$NMR = \text{comp. } OMQ = MOQ = \alpha;$$
$$MN = \sin\beta;$$
$$OM = \cos\beta;$$
$$MR = MN\cos NMR = \sin\beta\cos\alpha;$$
$$MQ = OM\sin MOQ = \cos\beta\sin\alpha;$$
$$OQ = OM\cos MOQ = \cos\beta\cos\alpha;$$
$$RN = MN\sin NMR = \sin\beta\sin\alpha.$$

Making these substitutions, we have the results (4) and (5) for $\sin(\alpha-\beta)$ and $\cos(\alpha-\beta)$.

41. *Sine and cosine of twice an angle.* If we suppose $\beta = \alpha$, we have from (2) and (3) expressions for the sine and cosine of the double of an angle, namely:

$$\sin 2\alpha = \sin\alpha\cos\alpha + \sin\alpha\cos\alpha, \qquad (6)$$

or

$$\sin 2\alpha = 2\sin\alpha\cos\alpha;$$

$$\cos 2\alpha = \cos^2\alpha - \sin^2\alpha = (\cos\alpha + \sin\alpha)(\cos\alpha - \sin\alpha). \qquad (7)$$

Also, by putting for $\cos^2 \alpha$ its value, $1 - \sin^2 \alpha$, and *vice versa*, we have

$$\cos 2\alpha = 1 - 2 \sin^2 \alpha = 2 \cos^2 \alpha - 1. \tag{7'}$$

Exercises.

1. Because $90° = 60° + 30°$, we have, by putting $\alpha = 60°$ and $\beta = 30°$ in the equation (2),

$$\sin 90° = \sin 60° \cos 30° + \cos 60° \sin 30° = 1.$$

It is required to test this equation by substituting the numerical values of the sines and cosines of 30°, 60°, and 90° (§ 30), and to test in the same way the equations obtained by putting $\alpha = 60°$ and $\beta = 30°$ in (3), (4) and (5).

2. Because $\alpha = \alpha - x + x$, we have, from (2),

$$\sin \alpha = \sin (\alpha - x) \cos x + \cos (\alpha - x) \sin x.$$

It is required to write the corresponding equations obtained by making the same substitution in (3), and the equations obtained in the same way from (4) and (5) by the identical equation

$$\alpha = \alpha + x - x.$$

3. Derive the addition theorem for the cosine from that for the sine by substituting $\gamma - 90°$ or $90° - \gamma$ for β in the equation (2), and applying the equations (18) of § 29 (Chap. II.).

4. By means of the addition theorem prove the equations

$$\sin (\alpha + \beta + \gamma) = \sin \alpha \cos \beta \cos \gamma + \cos \alpha \sin \beta \cos \gamma + \cos \alpha \cos \beta \sin \gamma - \sin \alpha \sin \beta \sin \gamma;$$

$$\cos (\alpha + \beta + \gamma) = \cos \alpha \cos \beta \cos \gamma - \sin \alpha \sin \beta \cos \gamma - \sin \alpha \cos \beta \sin \gamma - \cos \alpha \sin \beta \sin \gamma.$$

Note. This is readily done by putting $\alpha + \beta$ for α, and γ for β, in the equations (2) and (3), thus giving

$$\sin (\alpha + \beta + \gamma) = \sin (\alpha + \beta) \cos \gamma + \cos (\alpha + \beta) \sin \gamma;$$

$$\cos (\alpha + \beta + \gamma) = \cos (\alpha + \beta) \cos \gamma - \sin (\alpha + \beta) \sin \gamma;$$

and then developing by the addition theorem.

5. Prove the following values of $\sin 3\alpha$ and $\cos 3\alpha$:

$$\sin 3\alpha = 3 \sin \alpha \cos^2 \alpha - \sin^3 \alpha;$$

$$\cos 3\alpha = \cos^3 \alpha - 3 \sin^2 \alpha \cos \alpha.$$

6. Transform the expression

$$a \cos (\alpha + x) + b \cos (\beta + x) + c \cos (\gamma + x)$$

into

$$\cos x\,(a \cos \alpha + b \cos \beta + c \cos \gamma)$$
$$- \sin x\,(a \sin \alpha + b \sin \beta + c \sin \gamma).$$

7. If we have

$$a \cos \alpha + b \cos \beta + c \cos \gamma = 0$$

and
$$a \sin \alpha + b \sin \beta + c \sin \gamma = 0,$$

prove that we must also have

$$a \sin (\alpha + x) + b \sin (\beta + x) + c \sin (\gamma + x) = 0,$$

whatever be the value of x.

42. *The Addition Theorem for tangents.* Dividing equation (2) by (3), we have

$$\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)} = \tan(\alpha+\beta) = \frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\cos\alpha\cos\beta - \sin\alpha\sin\beta}.$$

Dividing both numerator and denominator of the last member by $\cos\alpha\cos\beta$, the equation becomes

$$\tan(\alpha+\beta) = \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta}. \qquad (8)$$

We obtain in the same way from (4) and (5),

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1 + \tan\alpha\tan\beta}. \qquad (9)$$

Putting $\beta = \alpha$ in (8) we have

$$\tan 2\alpha = \frac{2\tan\alpha}{1 - \tan^2\alpha}. \qquad (10)$$

EXERCISES.

1. Assuming $\beta = 180°$, prove by (8) of this section that

$$\tan(\alpha + 180°) = \tan\alpha.$$

2. Assuming $\alpha = 30°$, substitute in (10) the value of tan 30° given in § 30, and thus obtain the value of tan 60°.

43. *Products of sines and cosines.* Taking the sum and difference of equations (2) and (4) of § 40, and reversing the members of the equation, we find

$$\left.\begin{aligned} 2\sin\alpha\cos\beta &= \sin(\alpha+\beta) + \sin(\alpha-\beta);\\ 2\cos\alpha\sin\beta &= \sin(\alpha+\beta) - \sin(\alpha-\beta).\end{aligned}\right\} \qquad (11)$$

In the same way we find from (3) and (5),

$$\left.\begin{aligned} 2\cos\alpha\cos\beta &= \cos(\alpha+\beta)+\cos(\alpha-\beta);\\ 2\sin\alpha\sin\beta &= -\cos(\alpha+\beta)+\cos(\alpha-\beta).\end{aligned}\right\}\qquad(12)$$

EXERCISES.

1. Prove that if $\alpha-\beta=90^\circ$,

then $\cos(\alpha+\beta)=2\cos\alpha\cos\beta=\sin 2\alpha=-\sin 2\beta.$

2. Prove that if $\alpha+\beta=180^\circ$,

then $\sin(\alpha-\beta)=2\sin\alpha\cos\beta=\sin 2\beta=-\sin 2\alpha.$

44. *Sum of sines and cosines.* If in the four equations (11) and (12) we put

$$\alpha+\beta=x$$

and

$$\alpha-\beta=y,$$

we shall have

$$\alpha=\tfrac{1}{2}(x+y)$$

and

$$\beta=\tfrac{1}{2}(x-y).$$

By substituting these values in (11) and (12) and reversing the members of the equation, we find

$$\left.\begin{aligned} \sin x+\sin y &= 2\sin\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y);\\ \sin x-\sin y &= 2\cos\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y);\\ \cos x+\cos y &= 2\cos\tfrac{1}{2}(x+y)\cos\tfrac{1}{2}(x-y);\\ -\cos x+\cos y &= 2\sin\tfrac{1}{2}(x+y)\sin\tfrac{1}{2}(x-y).\end{aligned}\right\}\qquad(13)$$

Dividing the first of this group of equations by the third, we get

$$\frac{\sin x+\sin y}{\cos x+\cos y}=\tan\tfrac{1}{2}(x+y).\qquad(14)$$

Dividing the second by the fourth, we get

$$\frac{\sin x-\sin y}{\cos y-\cos x}=\cot\tfrac{1}{2}(x+y).\qquad(15)$$

If in the last two equations of (13) we suppose $y=0$, which makes $\cos y=1$, they become

$$\left.\begin{aligned} 1+\cos x &= 2\cos^2\tfrac{1}{2}x;\\ 1-\cos x &= 2\sin^2\tfrac{1}{2}x;\end{aligned}\right\}\qquad(16)$$

a pair of equations which frequently come into use.

If we put $\alpha=\frac{1}{2}x$, these equations become

$$\left.\begin{aligned} 1+\cos 2\alpha &= 2\cos^2\alpha;\\ 1-\cos 2\alpha &= 2\sin^2\alpha;\end{aligned}\right\}\qquad(16')$$

which may also be derived directly from (7′).

Dividing the second of (16) and (16′) by the first, we find

$$\frac{1 - \cos x}{1 + \cos x} = \tan^2 \tfrac{1}{2}x; \tag{17}$$

$$\frac{1 - \cos 2\alpha}{1 + \cos 2\alpha} = \tan^2 \alpha. \tag{17′}$$

45. *The problem of dimidiation.* It is often required to express the sine or cosine of half an angle in terms of the sine or cosine of the entire angle. To effect this let us put in equations (6) and (7) $2\alpha = \gamma$. They then become

$$\left.\begin{aligned} \sin \gamma &= 2 \sin \tfrac{1}{2}\gamma \cos \tfrac{1}{2}\gamma; \\ \cos \gamma &= \cos^2 \tfrac{1}{2}\gamma - \sin^2 \tfrac{1}{2}\gamma, \\ &= 1 - 2 \sin^2 \tfrac{1}{2}\gamma, \\ &= 2 \cos^2 \tfrac{1}{2}\gamma - 1. \end{aligned}\right\} \tag{18}$$

Let our first problem be:

Given, $\cos \gamma$;
Required, $\sin \tfrac{1}{2}\gamma$.

Solving the equation $\cos \gamma = 1 - 2 \sin^2 \tfrac{1}{2}\gamma$, we obtain

$$\sin \tfrac{1}{2}\gamma = \sqrt{\frac{1 - \cos \gamma}{2}}. \tag{19}$$

Let our next problem be:

Given, $\cos \gamma$;
Required, $\cos \tfrac{1}{2}\gamma$.

Solving the equation $\cos \gamma = 2 \cos^2 \tfrac{1}{2}\gamma - 1$, we obtain

$$\cos \tfrac{1}{2}\gamma = \sqrt{\frac{1 + \cos \gamma}{2}}. \tag{20}$$

Let our third problem be:

Given, $\sin \gamma$;
Required, $\sin \tfrac{1}{2}\gamma$.

In the equation

$$\sin \gamma = 2 \sin \tfrac{1}{2}\gamma \cos \tfrac{1}{2}\gamma$$

we put $\cos \tfrac{1}{2}\gamma = \sqrt{1 - \sin^2 \tfrac{1}{2}\gamma}$ and square both members, obtaining

$$\sin^2 \gamma = 4 \sin^2 \tfrac{1}{2}\gamma \,(1 - \sin^2 \tfrac{1}{2}\gamma).$$

Reducing,

$$\sin^4 \tfrac{1}{2}\gamma - \sin^2 \tfrac{1}{2}\gamma + \frac{\sin^2 \gamma}{4} = 0.$$

Considering $\sin \tfrac{1}{2}\gamma$ as an unknown quantity, this equation can be solved as a quadratic. Transposing the last term and adding $\frac{1}{4}$ to each member, it becomes

$$\sin^4 \tfrac{1}{2}\gamma - \sin^2 \tfrac{1}{2}\gamma + \tfrac{1}{4} = \frac{1 - \sin^2 \gamma}{4} = \frac{\cos^2 \gamma}{4};$$

extracting the square root of both members,

$$\sin^2 \tfrac{1}{2}\gamma - \tfrac{1}{2} = \pm \frac{\cos \gamma}{2};$$

whence, by solving,

$$\sin \tfrac{1}{2}\gamma = \sqrt{\frac{1 \pm \cos \gamma}{2}}. \qquad (21)$$

Let our fourth problem be:

Given, $\sin \gamma$;

Required, $\cos \tfrac{1}{2}\gamma$.

In the equation

$$2 \sin \tfrac{1}{2}\gamma \cos \tfrac{1}{2}\gamma = \sin \gamma$$

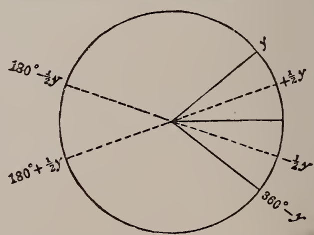
we put $\sin \tfrac{1}{2}\gamma = \sqrt{1 - \cos^2 \tfrac{1}{2}\gamma}$, and then proceed as before.

Solving with respect to $\cos \tfrac{1}{2}\gamma$, we shall find

$$\cos \tfrac{1}{2}\gamma = \frac{\sqrt{1 \mp \cos \gamma}}{2}. \qquad (22)$$

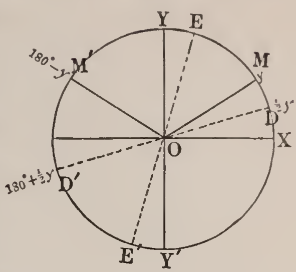
We have now to study the different values which these expressions for $\sin \tfrac{1}{2}\gamma$ and $\cos \tfrac{1}{2}\gamma$ may have in consequence of the double signs of the surds and the double signs under the radical sign.

Equations (19) and (20) show that if the cosine of an angle γ is given, the sine and cosine of half that angle may have either of two opposite values. The geometrical explanation of this is that to $\cos \gamma$ may correspond either of two angles, γ or $360° - \gamma$. The halves of the general measure of γ are $\tfrac{1}{2}\gamma$ and $\tfrac{1}{2}\gamma + 180°$ (§ 10). The halves of the general measure of $360° - \gamma$ are $180° - \tfrac{1}{2}\gamma$ and $-\tfrac{1}{2}\gamma$ (see diagram).



Therefore the half angle may have either of four values. But because $\sin(180° + \tfrac{1}{2}\gamma) = \sin(-\tfrac{1}{2}\gamma)$, and $\sin(180° - \tfrac{1}{2}\gamma) = \sin \tfrac{1}{2}\gamma$, the sines and cosines of these four angles will have only two different values. This result agrees with equations (19) and (20).

But suppose the *sine* of γ to be given. The angle γ may then have either of two supplementary values, γ and $180° - \gamma$. The halves of the general measure of these angles will be $\frac{1}{2}\gamma$, $90° - \frac{1}{2}\gamma$, $180° + \frac{1}{2}\gamma$, and $270° + \frac{1}{2}\gamma$. The sines and cosines of these four angles are all different. Therefore the algebraic expression for the sines ought to be susceptible of four values instead of two, as in the first case. Now this is true of equations (5) and (6), because they show that



$$\sin \tfrac{1}{2}\gamma = +\sqrt{\frac{1 + \cos \gamma}{2}} \text{ or } -\sqrt{\frac{1 + \cos \gamma}{2}}$$

$$\text{or } +\sqrt{\frac{1 - \cos \gamma}{2}} \text{ or } -\sqrt{\frac{1 - \cos \gamma}{2}},$$

and $$\cos \tfrac{1}{2}\gamma = \sqrt{\frac{1 - \cos \gamma}{2}} \text{ or } -\sqrt{\frac{1 - \cos \gamma}{2}}$$

$$\text{or } +\sqrt{\frac{1 + \cos \gamma}{2}} \text{ or } -\sqrt{\frac{1 + \cos \gamma}{2}}.$$

Exercises.

1. We have already found $\sin 30° = \frac{1}{2}$; from this find the sine, cosine, tangent, and cotangent of 15° and of 75°.

2. From the values of the six trigonometric functions for 45° (§ 30) find those for $22\frac{1}{2}°$.

3. From the values for 18° find those for 36°.

46. *Miscellaneous relations.* The following equations are of occasional use in the applications of trigonometry, and can all be derived from the formulæ of the last two chapters. Their derivations are therefore presented as an interesting exercise.

1. $\sin (45° - x) = \cos (45° + x)$.

2. $\sin x = \dfrac{\cos x}{\cot x}$.

3. $\sin x = \sin (60° + x) - \sin (60° - x)$.

4. $\sin x = \frac{2 \tan \frac{1}{2}x}{1 + \tan^2 \frac{1}{2}x}$.

5. $\sin x = \frac{2}{\tan \frac{1}{2}x + \cot \frac{1}{2}x}$.

6. $\sin x = \frac{\sin (30° + x) - \sin (30° - x)}{\sqrt{3}}$.

7. $\sin x = 2 \sin^2 (45° + \frac{1}{2}x) - 1 = 1 - 2 \sin^2 (45° - \frac{1}{2}x)$.

8. $\cos \varepsilon = \sin \varepsilon \cot \varepsilon$.

9. $\cos \varepsilon = \frac{1 - \tan^2 \frac{1}{2}\varepsilon}{1 + \tan^2 \frac{1}{2}\varepsilon}$.

10. $\cos a = \frac{\cot \frac{1}{2}a - \tan \frac{1}{2}a}{\cot \frac{1}{2}a + \tan \frac{1}{2}a}$.

11. $\cos a = \frac{1}{1 + \tan a \tan \frac{1}{2}a}$.

12. $\cos a = 2 \cos (45° + \frac{1}{2}a) \cos (45° - \frac{1}{2}a)$.

13. $\cos a = \frac{2}{\tan (45° + \frac{1}{2}a) + \tan (45° - \frac{1}{2}a)}$.

14. $\tan \theta = \frac{2 \cot \frac{1}{2}\theta}{\cot^2 \frac{1}{2}\theta - 1}$.

15. $\tan \theta = \frac{2}{\cot \frac{1}{2}\theta - \tan \frac{1}{2}\theta}$.

16. $2 \cot 2a = \cot a - \tan a$.

17. $\tan \theta = \frac{\sin 2\theta}{1 + \cos 2\theta}$.

18. $\tan a = \frac{\tan (45° + \frac{1}{2}a) - \tan (45° - \frac{1}{2}a)}{2}$.

19. $\frac{\sin (A + B)}{\sin (A - B)} = \frac{\tan A + \tan B}{\tan A - \tan B} = \frac{\cot B + \cot A}{\cot B - \cot A}$.

20. $\frac{\cos (A + B)}{\cos (A - B)} = \frac{\cot B - \tan A}{\cot B + \tan A} = \frac{\cot A - \tan B}{\cot A + \tan B}$.

21. $\cos a \cos b = \cos^2 \frac{1}{2}(a + b) - \sin^2 \frac{1}{2}(a - b)$.

22. $\sin a \sin b = \cos^2 \frac{1}{2}(a - b) - \cos^2 \frac{1}{2}(a + b)$.

23. $\tan a + \cot a = 2 \operatorname{cosec} 2a$.

24. $\sin^2 \alpha \cos^2 \beta - \cos^2 \alpha \sin^2 \beta = \sin (\alpha + \beta) \sin (\alpha - \beta)$.

25. $\cos^2 \alpha \cos^2 \beta - \sin^2 \alpha \sin^2 \beta = \cos (\alpha + \beta) \cos (\alpha - \beta)$.

CHAPTER V.

TRIGONOMETRIC PROBLEMS.

47. Problem I. *Having given two equations of the form*

$$r \sin \varphi = a,$$
$$r \cos \varphi = b,$$

where a and b are given numbers, it is required to compute r and φ.

Solution. Dividing the first equation by the second, we find

$$\frac{\sin \varphi}{\cos \varphi} = \tan \varphi = \frac{a}{b}.$$

From this value of $\tan \varphi$ we find φ itself, then the sine or cosine of φ, and then r from either of the equations

$$r = \frac{a}{\sin \varphi} = \frac{b}{\cos \varphi}.$$

Example. If $r \sin \varphi = 332.76$, and $r \cos \varphi = 290.08$, it is required to find r and φ.

log $\sin \varphi$, 9.877 25	(5)	log $\cos \varphi$, 9.817 64	(6)
" $r \sin \varphi$, 2.522 13	(1)	" $\tan \varphi$, 0.059 61	(3)
" $r \cos \varphi$, 2.462 52	(2)	φ, 48° 55′.2	(4)
" $r = \log r \sin \varphi - \log \sin \varphi$, 2.644 88	(7)		
" $r = \log r \cos \varphi - \log \cos \varphi$, 2.644 88	(8)		
r, 441.45	(9)		

The numbers in brackets show the order in which the numbers of the computation are written. In writing log $r \sin \varphi$ and $r \cos \varphi$ spaces are left for inserting log $\sin \varphi$ and log $\cos \varphi$ after φ is found, so that either of the latter may be subtracted to obtain r. It is generally best to obtain r from both $r \sin \varphi$ and $r \cos \varphi$, because then if any mistake is made in φ it will be shown by a difference in the results.

On the other hand, a practical computer will not write down either $\sin \varphi$ or $\cos \varphi$, but will subtract in his head and write down $\log r$ only.

EXERCISES.

1. Given $r \sin \varphi = 1.297\,43$, $r \cos \varphi = 6.002\,4$; find r and φ.
2. " " $= 0.082\,19$, " $= 0.128\,8$; " "
3. " " $= 194\,683$, " $= 8460.7$; " "

48. *Distinction of quadrants.* In the preceding examples we have supposed $r \sin \varphi$ and $r \cos \varphi$ to be positive, and have taken φ in the first quadrant. But either or both of these quantities may be negative. Whatever their signs, there are always two values of φ, differing by 180°, corresponding to any given value of $\tan \varphi$ (§ 31). Hence the problem admits of two solutions in all cases. In the one r will be positive, in the other negative.

But in practice only that solution is sought which gives a positive value of r. This being the case, $\sin \varphi$ and $\cos \varphi$ must have the same algebraic signs as the given quantities $r \sin \varphi$ and $r \cos \varphi$ respectively. Now consider each case in order:

I. $r \sin \varphi$ and $r \cos \varphi$ *both positive.* The angle φ must then be taken in the first quadrant, because only in this quadrant are $\sin \varphi$ and $\cos \varphi$ *both* positive.

II. $r \sin \varphi$ *positive* and $r \cos \varphi$ *negative.* Sin φ is positive only in quadrants (1) and (2) (§ 21), and $\cos \varphi$ is negative only in quadrants (2) and (3). Hence the requirement of signs can be fulfilled only in the second quadrant, and

$$90° < \varphi < 180°.$$

III. $r \sin \varphi$ and $r \cos \varphi$ *both negative.* The only quadrant in which sine and cosine are both negative is the third. Hence in this case

$$180° < \varphi < 270°.$$

IV. $r \sin \varphi$ *negative* and $r \cos \varphi$ *positive.* The only quadrant in which $\sin \varphi$ is negative and $\cos \varphi$ positive is the fourth. Hence in this case

$$270° < \varphi < 360°.$$

Exercises.

1. Given $r \sin \varphi = -237.09$, $r \cos \varphi = +192.91$; find r and φ.
2. " " $+2713.8$, " -9269.2; " "
3. " " -1.9634, " -0.09654; " "
4. " " -3.6925, " $+396.72$; " "
5. " " -18.005, " -2.6943; " "
6. Given $$r \sin (\varphi + 47° \, 50') = 7.2693,$$ $$r \cos (\varphi + 47° \, 50') = -12.2916,$$ to find r and φ.

Note. In this last exercise compute the value of $(\varphi + 47° \, 50')$ as if it were one quantity, and subtract the angle $47° \, 50'$ from the result.

7. Given
$$r \sin (\theta + x) = -249.88,$$
$$r \cos (\theta + x) = -92.62,$$
$$u \sin (\theta - x) = 702.02,$$
$$u \cos (\theta - x) = 516.93;$$
find the values of r, u, θ, and x.
8. Given
$$r (\sin \theta + \cos \theta) = 298.07,$$
$$r (\sin \theta - \cos \theta) = 96.04;$$
find r and θ.

49. Problem II. *Having given two equations of the form*
$$x \cos \alpha + y \sin \alpha = p,$$
$$x \sin \alpha - y \cos \alpha = q,$$
it is required to find the values of x and y.

The elimination is conducted by the method of addition and subtraction, as follows:

Multiply the first equation by $\cos \alpha$ and the second by $\sin \alpha$. We thus have
$$x \cos^2 \alpha + y \sin \alpha \cos \alpha = p \cos \alpha;$$
$$x \sin^2 \alpha - y \sin \alpha \cos \alpha = q \sin \alpha.$$
Now adding these equations together and remembering that
$$\sin^2 \alpha + \cos^2 \alpha = 1,$$
we have
$$x = p \cos \alpha + q \sin \alpha,$$
which is the required value of x.

Next multiply the first equation by sin α and the second by $-\cos\alpha$. We have

$$x\sin\alpha\cos\alpha + y\sin^2\alpha = p\sin\alpha;$$
$$-x\sin\alpha\cos\alpha + y\cos^2\alpha = -q\cos\alpha.$$

By addition,

$$y = p\sin\alpha - q\cos\alpha.$$

It will be noticed that in these equations x and y are given in terms of p and q by equations of the same form as the original ones.

EXERCISE.

Find the value of p and q from the equations

$$p\sin\alpha + q\cos\alpha = a;$$
$$p\cos\alpha - q\sin\alpha = b.$$

50. PROBLEM III. *From the equations*

$$r\sin(\beta+\varepsilon) = a,$$
$$r\sin(\beta+\theta) = b,$$

to find the values of r *and* β,—*the other four quantities,* a, b, ε, *and* θ, *being supposed known.*

Solution. Developing the sines of $\beta + a$ and $\beta+\theta$ (§ 40), we have

$$\left.\begin{array}{l} r\sin\beta\cos\varepsilon + r\cos\beta\sin\varepsilon = a; \\ r\sin\beta\cos\theta + r\cos\beta\sin\theta = b. \end{array}\right\} \qquad (a)$$

Regarding $r\sin\beta$ and $r\cos\beta$ as the two unknown quantities, we see that their coefficients are $\cos\varepsilon$, $\sin\varepsilon$, $\cos\theta$ and $\sin\theta$. Multiplying the first equation by $\cos\theta$ and the second by $\cos\varepsilon$, we have

$$r\sin\beta\cos\varepsilon\cos\theta + r\cos\beta\sin\varepsilon\cos\theta = a\cos\theta;$$
$$r\sin\beta\cos\varepsilon\cos\theta + r\cos\beta\cos\varepsilon\sin\theta = b\cos\varepsilon.$$

Subtracting,

$$r\cos\beta(\sin\varepsilon\cos\theta - \cos\varepsilon\sin\theta) = a\cos\theta - b\cos\varepsilon.$$

Noting that the coefficient of $r\cos\beta$ is $\sin(\varepsilon-\theta)$, we find, by division,

$$r\cos\beta = \frac{a\cos\theta - b\cos\varepsilon}{\sin(\varepsilon-\theta)}.$$

To find the value of $r\sin\beta$ we multiply the first equation (a) by $\sin\theta$ and the second by $\sin\varepsilon$, and subtract. We thus find

$$r\sin\beta(\sin\varepsilon\cos\theta - \cos\varepsilon\sin\theta) = b\sin\varepsilon - a\sin\theta.$$

Hence $$r \sin \beta = \frac{b \sin \varepsilon - a \sin \theta}{\sin (\varepsilon - \theta)}.$$

Supposing the numerical values of a, b, ε, and θ to be given, these equations give the values of $r \sin \beta$ and $r \cos \beta$, from which r and β can be computed by Problem I.

Exercises.

Find the values of $r \sin \varphi$ and $r \cos \varphi$ from the following equations by the preceding method:

1. $r \cos (\varphi + \varepsilon) = a$, $r \cos (\varphi - \theta) = b$.

Ans. $$r \sin \varphi = \frac{b \cos \varepsilon - a \cos \theta}{\sin (\varepsilon + \theta)};$$

$$r \cos \varphi = \frac{b \sin \varepsilon + a \sin \theta}{\sin (\varepsilon + \theta)}.$$

2. $r \cos (\varphi + \varepsilon) = a$, $r \sin (\varphi + \theta) = b$.

Ans. $$r \sin \varphi = \frac{b \cos \varepsilon - a \sin \theta}{\cos (\varepsilon - \theta)};$$

$$r \cos \varphi = \frac{b \sin \varepsilon + a \cos \theta}{\cos (\varepsilon - \theta)}.$$

3. $r \sin (\varphi + \varepsilon) = a \cos \gamma$, $r \cos (\varphi - \varepsilon) = a \sin \gamma$.

Ans. $$r \sin \varphi = \frac{a \cos (\gamma + \varepsilon)}{\cos 2\varepsilon};$$

$$r \cos \varphi = \frac{a \sin (\gamma - \varepsilon)}{\cos 2\varepsilon}.$$

4. Find the values of r and φ from the equations

$$r \cos (\varphi + \theta) = 3.790\,8,$$
$$r \cos (\varphi - \theta) = 2.060\,7,$$

when $\theta = 31° \, 27'.4$.

51. Problem IV. *To reduce an expression of the form*

$$a \sin \theta + b \cos \theta$$

to a monomial, a and b being given quantities.

Solution. Determine the values of two auxiliary quantities k and ε from the equations

$$k \cos \varepsilon = a,$$
$$k \sin \varepsilon = b,$$

as in Problem I.

The given expression will then become, by substitution,

$$k \cos \varepsilon \sin \theta + k \sin \varepsilon \cos \theta = k \sin (\theta + \varepsilon).$$

We might equally have supposed $k \sin \varepsilon = a$ and $k \cos \varepsilon = b$, when the given expression would have become $k \cos (\theta - \varepsilon)$.

Example. Reduce the expression

$$1239.3 \sin x - 724.6 \cos x$$

to a monomial.

$k \sin \varepsilon =$	$-\ 724.6$	$\log =$	$-\ 2.860\ 10$
$k \cos \varepsilon =$	1239.3	$\log =$	$3.093\ 18$
		$\log \tan \varepsilon,$	$-\ 9.766\ 92$
		$\varepsilon,$	$-\ 30^\circ\ 18'.8$
		$\log \cos \varepsilon,$	$9.936\ 15$
		$\log k,$	$3.157\ 03$
		$k,$	1435.6

We therefore have

$$1239.3 \sin x - 724.6 \cos x = 1435.6 \sin (x - 30^\circ\ 18'.8).$$

Exercises.

Reduce to monomials the expressions:

1. $27.615 \cos \mu - 23.208 \sin \mu$.
2. $3.600\,3 \sin (\theta - 7^\circ\ 52'.6) + 5.907\,0 \sin (\theta + 53^\circ\ 57'.6)$.
3. Reduce to a monomial the expression

$$92.65 \sin \theta \cos \alpha - 196.23 \cos \theta \sin \alpha,$$

when $\alpha = 162^\circ\ 48'.7$.

4. Reduce to a monomial the expression

$$\cos \alpha \cos \theta + \sin \alpha \sin \theta \cos H,$$

when $\alpha = 62^\circ\ 39'.5$ and $H = 22^\circ\ 36'.8$.

52. Problem V. *To reduce an expression of the form*

$$a \sin x \cos \theta + b \cos x \sin \theta \qquad (1)$$

to a form which shall not contain the product of any two trigonometric functions.

Solution. We have, from § 43,

$$\sin x \cos \theta = \tfrac{1}{2} \sin (x + \theta) + \tfrac{1}{2} \sin (x - \theta);$$
$$\cos x \sin \theta = \tfrac{1}{2} \sin (x + \theta) - \tfrac{1}{2} \sin (x - \theta).$$

Making these substitutions in the expression (1), it becomes

$$\tfrac{1}{2}(a+b)\sin(x+\theta)+\tfrac{1}{2}(a-b)\sin(x-\theta)$$

Exercises.

Clear from products of sines and cosines:

1. $m\cos\alpha\sin\beta - n\sin\alpha\cos\beta$.
2. $a\cos\theta\cos\mu + b\sin\theta\sin\mu$.
3. $a\cos\theta\cos\mu - b\sin\theta\sin\mu$.

53. Problem VI. From the equations

$$\left.\begin{aligned} r\cos\beta\cos\lambda &= a, \\ r\cos\beta\sin\lambda &= b, \\ r\sin\beta &= c, \end{aligned}\right\} \qquad (1)$$

to find the values of r, β, and λ, the values of a, b, and c being given.

Method of solution. Dividing the second equation by the first we obtain

$$\tan\lambda = \frac{b}{a}; \qquad (2)$$

from this equation we find λ, and then $\sin\lambda$ or $\cos\lambda$ from the tables. Then

$$r\cos\beta = \frac{a}{\cos\lambda} = \frac{b}{\sin\lambda} \qquad (3)$$

can be computed. The value of $r\sin\beta$ being given by the third equation (1), the values of r and β are found by Problem I.

Example. Find r, β, and λ from the equations

$$\begin{aligned} r\cos\beta\cos\lambda &= -\ \ 53.953; \\ r\cos\beta\sin\lambda &= +197.207; \\ r\sin\beta &= -\ \ 39.062. \end{aligned}$$

Work

		log cos β,	9.992 21
log $r\cos\beta\cos\lambda$,	− 1.732 01	log $r\cos\beta$,	2.310 60
log $r\cos\beta\sin\lambda$,	2.294 93	log $r\sin\beta$,	− 1.591 75
sin λ,	9.984 33	log tan β,	− 9.281 15
log tan λ,	− 0.562 92	log r,	2.318 39
λ,	105° 18′.0	r,	208.16
log $r\cos\beta$,	2.310 60	β,	− 10° 49′

Exercises.

Find the values of r, β, and λ from:

1. $$r \cos \beta \cos \lambda = 1.271\,83;$$
$$r \cos \beta \sin \lambda = -0.981\,52;$$
$$r \sin \beta = 0.890\,02.$$

2. $$r \sin \beta \sin \lambda = 19.765\,3;$$
$$r \sin \beta \cos \lambda = -7.192\,8;$$
$$r \cos \beta = 12.124\,2.$$

Miscellaneous Exercises.

1. Compute ω and r from the equations
$$1.268\,22 \sin \omega = 0.948\,30 + r \sin (25° \, 27'.2);$$
$$1.268\,22 \cos \omega = 0.281\,16 + r \cos (25° \, 27'.2).$$

First eliminate r.

Ans. $\omega = 60° \, 53'.8;$
$r = 0.371\,7.$

2. Find ω and x from the equations
$$3 \sin \omega + \cos \omega = 2x. \qquad \text{Ans. } \omega = 71° \, 34';$$
$$\sin \omega + 2 \cos \omega = x. \qquad x = \sqrt{\tfrac{5}{2}}.$$

After finding sin ω and cos ω in terms of x employ the equation
$$\sin^2 \omega + \cos^2 \omega = 1.$$

Find x from the following quadratic equations, and express the results without surds:

3. $x^2 + 1 = \dfrac{2x}{\sin \alpha}$. Ans. $x = \cot \frac{1}{2} \alpha$ or $\tan \frac{1}{2} \alpha$.

4. $x^2 + 1 = 2x \sec \alpha$. Ans. $x = \dfrac{1 \pm \sin \alpha}{\cos \alpha}$.

5. $1 - x^2 = 2x \cot \alpha$. Ans. $x = \tan \frac{1}{2}\alpha$ or $-\cot \frac{1}{2}\alpha$.

6. $x^2 - 1 = 2x \tan \alpha$. Ans. $x = \dfrac{\sin \alpha \pm 1}{\cos \alpha}$.

Find θ from the equations:

7. $27.615 \cos \theta - 23.208 \sin \theta = 19.094.$

8. $3.6003 \sin (\theta - 8°) + 5.907 \sin (\theta + 54°) = 2.6253.$

Reduce the first members by Prob. IV.

9. $a \sin \theta + b \cos \theta + c = 0.$

From this last equation find $\sin \theta$, $\cos \theta$, and $\tan \theta$ by separate quadratic equations.

CHAPTER VI.

SOLUTION OF TRIANGLES IN GENERAL.

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54. A plane triangle has six parts, three sides and three angles. Of these parts the three angles are not independent of each other, because when two angles are given the third may be found from the condition that the sum of the three angles is 180°. Hence, if two angles are given, the case is the same as if all three were given.

When any three independent parts are given the remaining three may be found, but in order to be independent one of the three given parts must be a side.

55. The fact that the sum of the three angles of a plane triangle is 180° enables us to express a trigonometric function of any one angle as a similar function of the sum of the other two angles. It has been shown that

$$\begin{aligned} \sin x &= \sin (180° - x); \\ \cos x &= -\cos (180° - x); \\ \tan x &= -\tan (180° - x); \\ \cot x &= -\cot (180° - x); \\ \sec x &= -\sec (180° - x); \\ \operatorname{cosec} x &= -\operatorname{cosec} (180° - x). \end{aligned}$$

If α, β, and γ are the three angles of a triangle, we have

$$\left.\begin{aligned} \alpha &= 180° - (\beta + \gamma); \\ \beta &= 180° - (\gamma + \alpha); \\ \gamma &= 180° - (\alpha + \beta). \end{aligned}\right\} \qquad (a)$$

Therefore

$$\left.\begin{aligned} \sin \alpha &= \sin (\beta + \gamma); \\ \sin \beta &= \sin (\gamma + \alpha); \\ \sin \gamma &= \sin (\alpha + \beta); \\ \cos \alpha &= -\cos (\beta + \gamma); \\ \text{etc.} & \quad \text{etc.} \end{aligned}\right\} \qquad (1)$$

By dividing the equations (a) by 2 we find

$$\left.\begin{aligned}\tfrac{1}{2}\alpha &= 90^\circ - \tfrac{1}{2}(\beta+\gamma);\\ \tfrac{1}{2}\beta &= 90^\circ - \tfrac{1}{2}(\gamma+\alpha);\\ \tfrac{1}{2}\gamma &= 90^\circ - \tfrac{1}{2}(\alpha+\beta).\end{aligned}\right\} \qquad (b)$$

Therefore

$$\left.\begin{aligned}\sin\tfrac{1}{2}\alpha &= \cos\tfrac{1}{2}(\beta+\gamma);\\ \cos\tfrac{1}{2}\alpha &= \sin\tfrac{1}{2}(\beta+\gamma);\\ \tan\tfrac{1}{2}\alpha &= \cot\tfrac{1}{2}(\beta+\gamma);\\ \cot\tfrac{1}{2}\alpha &= \tan\tfrac{1}{2}(\beta+\gamma);\\ \text{etc.} &\qquad \text{etc.}\end{aligned}\right\} \qquad (2)$$

From what has been said the given parts may be

One side and the angles;
Two sides and one angle;
The three sides.

Also, when the sides and one angle are given, this angle may be either that between the given sides or opposite one of them. Hence there are four cases in all to be considered.

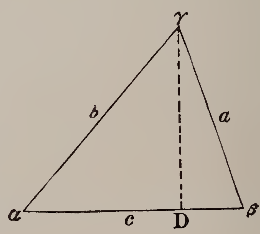
56. Case I. **Given the angles and one side.**

Theorem I. *The sides of a plane triangle are proportional to the sines of their opposite angles.*

Proof. Put

a, b, c, the three sides;

α, β, γ, their opposite angles.



From one angle, as γ, drop a perpendicular γD upon the opposite side, c. Then

$$\gamma D = b\sin\alpha;\ \gamma D = a\sin\beta. \quad (\S 35)$$

Therefore

$$a\sin\beta = b\sin\alpha.$$

Dividing by $\sin\alpha\sin\beta$,

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta}.$$

By dropping a perpendicular from α upon a we should find, in the same way,

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}.$$

Therefore $$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma}, \tag{3}$$

or $$a : b : c = \sin\alpha : \sin\beta : \sin\gamma. \quad \text{Q.E.D.} \tag{4}$$

Def. The common value of the three quotients $\frac{a}{\sin\alpha}$, $\frac{b}{\sin\beta}$, and $\frac{c}{\sin\gamma}$ is called the **modulus** of the triangle.

Theorem II. *The modulus of the triangle is equal to the diameter of the circumscribed circle.*

This theorem may be demonstrated by the student from the property that an inscribed angle is measured by one half the arc on which it stands.

Theorem I. enables us, when the angles and one side of any plane triangle are given, to find the remaining two sides. If the side given is c, we have

$$\left.\begin{aligned} a &= \frac{c\sin\alpha}{\sin\gamma}; \\ b &= \frac{c\sin\beta}{\sin\gamma}. \end{aligned}\right\} \tag{5}$$

or, putting M for the modulus, we have

$$\left.\begin{aligned} a &= M\sin\alpha; \\ b &= M\sin\beta; \\ c &= M\sin\gamma. \end{aligned}\right\} \tag{6}$$

If we put p, p', and p'' for the lengths of the perpendiculars from α, β, and γ respectively upon the opposite sides, we find, from the preceding figure,

$$\left.\begin{aligned} p &= b\sin\gamma = c\sin\beta; \\ p' &= c\sin\alpha = a\sin\gamma; \\ p'' &= b\sin\alpha = a\sin\beta. \end{aligned}\right\} \tag{7}$$

By these equations we may find the lengths of the perpendiculars.

Exercises.

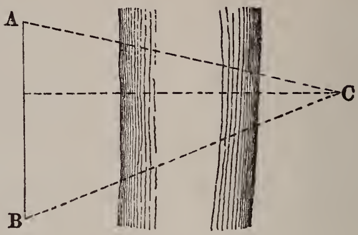
1. Given $\alpha = 78° 23'.2$, $\beta = 52° 16'.3$, $a = 796.25$; find b and c.
2. " $\alpha = 5° 26'.2$, $\beta = 72° 36'.8$, $b = 19.263$; " b and c.
3. " $\alpha = 50° 58'.7$, $\beta = 32° 50'.8$, $c = 169.37$; " a and b.

4. In order to find the distance of a point C across a river from the points A and B, a surveyor measured a base line AB and found it to be 829.72 metres. Placing a theodolite at A, he found

Angle $BAC = 82° 37'.6$.

Carrying his theodolite to B, he found

Angle $ABC = 70° 3'.3$.



Required the distances AC and BC across the river, and the length of the perpendicular from C on AB.

5. In a triangle ABC the angle B exceeds the angle A by 10°, the angle C exceeds the angle B by 20°, and the side AC is 2.72 904 metres. Find the angles and sides of the triangle and the length of the perpendiculars from the angles upon the opposite sides.

6. The base of an isosceles triangle is 132.643 metres, and the angle at the vertex is 32° 53′.7. Find the sides and the altitude.

7. One diagonal of a parallelogram measures 23 metres, and it makes angles of 32° 17′ and 63° 24′ with the sides. Find the lengths of the sides and the angles of the parallelogram.

8. From a point at a distance a from the centre of a circle of radius r tangents are drawn to the circle. Express the lengths of the tangents and the distance between the points of tangency, and compute the result when $r = 7$, $a = 12$.

57. Case II. **Given two sides and the angle opposite one of them.**

Let the given parts be a, b, α. We then compute the parts β, γ, and c by the formulæ (a) and (3) already found.

$$\left.\begin{aligned} \sin\beta &= \frac{b}{a}\sin\alpha; \\ \gamma &= 180° - (\alpha+\beta); \\ c &= \frac{b\sin\gamma}{\sin\beta} = \frac{a\sin\gamma}{\sin\alpha}. \end{aligned}\right\} \quad (8)$$

This case may have two solutions, as is shown in geometry. The two solutions are found in the above equations, because to a given value of $\sin\beta$ corresponds either of two angles β (§ 21), which will be supplements of each other.

But if one solution gives $\alpha + \beta > 180°$ it is not admissible, and only the lesser value is used to give the triangle. This will be the case when $a > b$.

It may also happen that $\sin\beta = \frac{b}{a}\sin\alpha$ comes out greater than unity. There is then no possible triangle which fulfils the conditions.

Example. Given $a = 152.08$, $b = 236.74$, $\alpha = 32° 29'.6$; find the remaining parts.

$\log b$,	2.374 27		
$\log\sin\alpha$,	9.730 14		
$\operatorname{colog} a$,	7.817 93		
$\log\sin\beta$,	9.922 34		
β,	56° 44′.9	or	123° 15′.1
α,	32° 29′.6		32° 29′.6
$\alpha+\beta$,	89° 14′.5		155° 44′.7
	180° 00′.0		180° 00′.0
γ,	90° 45′.5	or	24° 15′.3
$\log\sin\gamma$,	9.999 96	or	9.613 62
$\log\frac{b}{\sin\beta}$,	2.451 93		2.451 93
$\log c$,	2.451 89	or	2.065 55
c,	283.07	or	116.29

Exercises.

1. Given $a = 24$, $b = 33$, $\alpha = 31° 28'$; find the remaining parts.
2. " $a = 34$, $b = 35.79$, $\beta = 17° 59'$; " " "
3. " $a = 29$, $b = 34$, $\alpha = 30° 20'$; " " "
4. " $b = 19$, $c = 18$, $\gamma = 15° 49'$; " " "
5. " $a = 24$, $c = 13$, $\alpha = 115° 0'$; " " "

6. From a point P at a distance a from the centre of a circle of radius r a line is drawn, making the angle β with the line from P to the centre of the circle. At what distances from P will the

line intersect the circle? Compute the distances when $r = 72$, $a = 98$, and $\beta = 28° 56'$.

7. Show that if, in the present case, we take the side which is not given as the base, we can find the altitude of the triangle immediately, and afterwards may find the three required parts of the triangle from the altitude.

CASE III. Given the three sides.

THEOREM III. *In a triangle the square of any side is equal to the sum of the squares of the other two sides minus twice the product of these two sides into the cosine of the angle included by them.*

In symbolic language this theorem is expressed in any of the forms

$$\left.\begin{aligned} a^2 &= b^2 + c^2 - 2bc \cos \alpha, \\ \text{or} \quad b^2 &= a^2 + c^2 - 2ac \cos \beta, \\ \text{or} \quad c^2 &= a^2 + b^2 - 2ab \cos \gamma. \end{aligned}\right\} \qquad (9)$$

Proof. It is shown in geometry that in any triangle the side opposite an acute angle is greater than the sum of the squares of the other two sides by twice the product of one of these sides into the projection of the other side upon it.

If α be the acute angle, we have, by this theorem,

$$a^2 = b^2 + c^2 - 2b \times (\text{projection of } c \text{ on } b).$$

By § 35, II., and the definition of projections.

$$\text{Projection of } c \text{ on } b = c \cos \alpha;$$

substituting this value of the projection,

$$a^2 = b^2 + c^2 - 2bc \cos \alpha. \quad \text{Q.E.D.}$$

The other equations may be proved in exactly the same way.

If the containing angle is obtuse, the square of the opposite side will be greater than the sum of the squares of the containing sides. But this case is included in the trigonometric formula, because then $\cos \alpha$ is negative and $- bc \cos \alpha$ is positive. Hence the formula is applicable to all cases when regard is had to the algebraic sign.

From the first of equations (9) we obtain

$$\cos \alpha = \frac{b^2 + c^2 - a^2}{2bc}, \qquad (10)$$

which with the two companion formulæ enable us to find the angles when the three sides are given.

59. If the angle is small, it cannot be accurately determined by means of its cosine; we therefore transform the expression as follows:

Subtracting each member of the equation from unity, we have

$$1 - \cos\alpha = 1 - \frac{b^2 + c^2 - a^2}{2bc} = \frac{2bc - b^2 - c^2 + a^2}{2bc}.$$

But $1 - \cos\alpha = 2\sin^2\frac{1}{2}\alpha$ (§ 44), and $2bc - b^2 - c^2 = -(b-c)^2$. Therefore

$$2\sin^2\tfrac{1}{2}\alpha = \frac{a^2 - (b-c)^2}{2bc} = \frac{(a+b-c)(a-b+c)}{2bc}.$$

Let us now put s for half the sum of the three sides, so that

$$s = \tfrac{1}{2}(a+b+c).$$

Then

$$a + b - c = 2s - 2c;$$
$$a - b + c = 2s - 2b;$$

and the preceding equation reduces to

$$\sin^2\tfrac{1}{2}\alpha = \frac{(s-b)(s-c)}{bc}. \qquad (11)$$

The expressions for the other two angles are obtained by the same process, the letters a, b, c and α, β, γ being permuted in the orders b, c, a; β, γ, α and c, a, b; γ, α, β. We thus find

$$\left.\begin{aligned}\sin^2\tfrac{1}{2}\beta &= \frac{(s-c)(s-a)}{ca};\\ \sin^2\tfrac{1}{2}\gamma &= \frac{(s-a)(s-b)}{ab}.\end{aligned}\right\} \qquad (12)$$

These equations answer our purpose, but in determining an angle the tangent is the function to be preferred, because an angle can be determined more accurately from its tangent than from its sine or cosine. To obtain expressions for the tangent add unity to both sides of the equation (10). We then have

$$1 + \cos\alpha = 1 + \frac{b^2 + c^2 - a^2}{2bc} = \frac{b^2 + 2bc + c^2 - a^2}{2bc}.$$

Since $1 + \cos\alpha = 2\cos^2\frac{1}{2}\alpha$ (§ 44), this equation reduces to

$$2\cos^2\tfrac{1}{2}\alpha = \frac{(b+c)^2 - a^2}{2bc} = \frac{(b+c+a)(b+c-a)}{2bc}.$$

Whence
$$\cos^2 \tfrac{1}{2}\alpha = \frac{s(s-a)}{bc}. \qquad (13)$$

Dividing (11) by this equation, and writing the corresponding equations for the other angles, we find

$$\left.\begin{aligned} \tan^2 \tfrac{1}{2}\alpha &= \frac{(s-b)(s-c)}{s(s-a)}; \\ \tan^2 \tfrac{1}{2}\beta &= \frac{(s-c)(s-a)}{s(s-b)}; \\ \tan^2 \tfrac{1}{2}\gamma &= \frac{(s-a)(s-b)}{s(s-c)}. \end{aligned}\right\} \qquad (14)$$

The computation will be simplified a little by computing

$$H = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}.$$

We shall then have

$$\left.\begin{aligned} \tan \tfrac{1}{2}\alpha &= \frac{H}{(s-a)}; \\ \tan \tfrac{1}{2}\beta &= \frac{H}{(s-b)}; \\ \tan \tfrac{1}{2}\gamma &= \frac{H}{(s-c)}. \end{aligned}\right\} \qquad (15)$$

By means of these equations we may compute two of the angles, and find the third by subtracting their sum from 180°. But in practice it is better to compute the three angles independently, and check the accuracy of the work by taking their sum.

If this sum comes out materially different from 180°, there is some mistake in the work; if not, it may be presumed correct.

Example. Given $a = 273.960$, $b = 198.632$, $c = 236.914$; find the angles.

$a = 273.960$		
$b = 198.632$	$\log(s-a)$, 1.907 37	
$c = 236.914$	$\log(s-b)$, 2.193 46	$\log\tan\frac{1}{2}\alpha$, 9.903 73
$2s = 709.506$	$\log(s-c)$, 2.071 29	$\log\tan\frac{1}{2}\beta$, 9.617 64
$s = 354.753$	sum of logs, 6.172 12	$\log\tan\frac{1}{2}\gamma$, 9.739 81
$s-a = 80.793$	$\log s$, 2.549 92	
$s-b = 156.121$	$\log H^2$, 3.622 20	
$s-c = 117.839$	$\log H$, 1.811 10	

$$\tfrac{1}{2}\alpha,\ 38^\circ\ 42'.1;\quad \alpha = 77^\circ\ 24'.2$$
$$\tfrac{1}{2}\beta,\ 22^\circ\ 31'.2;\quad \beta = 45^\circ\ \ 2'.4$$
$$\tfrac{1}{2}\gamma,\ 28^\circ\ 46'.8;\quad \gamma = 57^\circ\ 33'.6$$
$$\text{Sum} = 180^\circ\ \ 0'.2 \quad \text{(Check.)}$$

The discrepancy of $0'.2$ is the result of the unavoidable errors from the omission of the decimals of the logarithms beyond the fifth.

Another check on the accuracy of the work is obtained by computing the modulus of the triangle from its three separate expressions (§ 56, 3), and noting whether they agree, thus:

log a,	2.437 69	log b,	2.298 05	log c,	2.374 59
sin α,	9.989 42	sin β,	9.849 79	sin γ,	9.926 32
log modulus,	2.448 27		2.448 26		2.448 27

The three results agree within the unavoidable limits of error.

Exercises.

1. Given $a = 3$, $b = 4$, $c = 5$; find the angles.

2. " $a = 37\,593$, $b = 29\,867$, $c = 40\,005$; " "

3. " $a = 2.796\,1$, $b = 23.928$, $c = 25.046$; " "

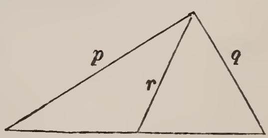
4. The base of a parallelogram is 13, each side is 6, and its lesser diagonal is 12. Find its angles.

5. If the sides of a parallelogram are a and b, and one diagonal is p, express its angles.

6. The parallel sides of a trapezoid are 12 and 17, and the non-parallel sides 6 and 7. Find its four angles.

Suggestion. Divide the trapezoid into a triangle and a parallelogram.

7. In a triangle are given the two sides, p and q, and the medial line r from the vertex to the middle point of the base. Prove Base $= \sqrt{2p^2 + 2q^2 - 4r^2}$.

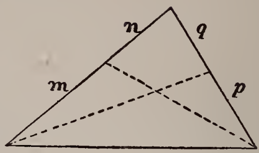


8. Given the three medial lines r, r', r'' of a triangle; find the sides of the triangle from the preceding result.

Ans. $\frac{2}{3}\sqrt{2r^2 + 2r'^2 - r''^2}$; $\frac{2}{3}\sqrt{2r^2 + 2r''^2 - r'^2}$; $\frac{2}{3}\sqrt{2r'^2 + 2r''^2 - r^2}$.

9. Of the bisectors of the angles at the base of a triangle, the one cuts the opposite side in the ratio $m:n$, and the other in the

ratio $p:q$. By means of the equation (10) express the cosine of the angle at the vertex of the triangle.



$$\text{Ans. } \cos\alpha = \frac{p^2n^2 + q^2m^2 - p^2m^2}{2pqmn}.$$

NOTE. In the solution of this question apply the theorem of geometry which defines the ratio in which the bisector of an angle cuts the opposite side.

60. CASE IV. **Given two sides and the included angle.**

This case may be readily solved by Theorem III., because if the given sides are b and c, and the given included angle is α, we have for the third side

$$a = \sqrt{b^2 + c^2 - 2bc\cos\alpha}.$$

Then, having the three sides, the remaining angles may be found as in the last section. But there is a more convenient method founded on the following theorem:

THEOREM IV. *As the sum of any two sides*
is to their difference,
so is the tangent of half the sum of the angles opposite these sides
to the tangent of half their difference.

Proof. From the equation

$$b : c :: \sin\beta : \sin\gamma, \qquad \text{(Th. I.)}$$

we have, by composition and division,

$$b + c : b - c :: \sin\beta + \sin\gamma : \sin\beta - \sin\gamma.$$

But
$$\sin\beta + \sin\gamma = 2\sin\tfrac{1}{2}(\beta+\gamma)\cos\tfrac{1}{2}(\beta-\gamma); \qquad (\S\,44)$$
$$\sin\beta - \sin\gamma = 2\cos\tfrac{1}{2}(\beta+\gamma)\sin\tfrac{1}{2}(\beta-\gamma).$$

Substituting these values, and expressing the proportion as a fraction,

$$\frac{b+c}{b-c} = \frac{\sin\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\beta-\gamma)}{\cos\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}(\beta-\gamma)}$$
$$= \tan\tfrac{1}{2}(\beta+\gamma)\cot\tfrac{1}{2}(\beta-\gamma) \qquad (\S\,33)$$
$$= \frac{\tan\frac{1}{2}(\beta+\gamma)}{\tan\frac{1}{2}(\beta-\gamma)}.$$

Therefore

$$b + c : b - c :: \tan\tfrac{1}{2}(\beta+\gamma) : \tan\tfrac{1}{2}(\beta-\gamma). \quad \text{Q.E.D.} \qquad (16)$$

The solution is now obtained as follows: We have

$$\tfrac{1}{2}(\beta+\gamma) = 90^\circ - \tfrac{1}{2}\alpha;$$

$$\tan\tfrac{1}{2}(\beta+\gamma) = \cot\tfrac{1}{2}\alpha. \qquad (\S\,55)$$

Because the angle α is given, the only unknown term of the proportion is $\tan\frac{1}{2}(\beta-\gamma)$. This is given by the equation

$$\tan\tfrac{1}{2}(\beta-\gamma) = \frac{b-c}{b+c}\tan\tfrac{1}{2}(\beta+\gamma), \qquad (17)$$

which is derived from the proportion (15).

By this equation we obtain $\frac{1}{2}(\beta-\gamma)$, which being added to and subtracted from $\frac{1}{2}(\beta+\gamma)$ gives β and γ. The remaining side of the triangle may then be found by Case I. But when this side as well as the angles are required, a more elegant method may be followed, which will be explained in the next section.

Example. Given $b = 4.567$, $c = 3.456$, $\alpha = 56^\circ\ 7'.8$; find the remaining parts.

b,	4.567		180°
c,	3.456	α,	56° 7′.8
$b-c$,	1.111	$\beta+\gamma$,	123° 52′.2
$b+c$,	8.023	$\frac{1}{2}(\beta+\gamma)$,	61° 56′.1
log $(b-c)$,	0.045 71	$\frac{1}{2}(\beta-\gamma)$,	14° 33′.6
colog $(b+c)$,	9.095 66	β,	76° 29′.7
log tan $\frac{1}{2}(\beta+\gamma)$,	0.273 14	γ,	47° 22′.5
log tan $\frac{1}{2}(\beta-\gamma)$,	9.414 51		

log b,	0.659 63	log c,	0.538 57
log sin β,	9.987 82	log sin γ,	9.866 76
log Mod.,	0.671 81	log Mod.,	0.671 81
$a =$	3.899 8	log sin α,	9.919 23
		log a,	0.591 04

Exercises.

1. Given $a = 12.34$, $b = 43.21$, $\gamma = 34^\circ\ 12'$; find rem. angles.
2. " $b = \sqrt{5}$, $c = \sqrt{3}$, $\alpha = 35^\circ\ 53'$; " "
3. " $a = 35.79$, $c = 1.246\,8$, $\beta = 97^\circ\ 53'$; " "
4. " $a = 189$, $b = 114.75$, $\gamma = 107^\circ\ 48'$; " "

61. If in the present case all three remaining parts are wanted, formulæ may be derived as follows:

From the equations (3) we derive

$$\frac{b}{a} = \frac{\sin\beta}{\sin\alpha}; \qquad \frac{c}{a} = \frac{\sin\gamma}{\sin\alpha}. \qquad (a)$$

Adding these equations we have

$$\frac{b+c}{a} = \frac{\sin\beta + \sin\gamma}{\sin\alpha} = \frac{2\sin\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\beta-\gamma)}{\sin\alpha} \qquad (\S\,44)$$

$$= \frac{2\cos\frac{1}{2}\alpha\cos\frac{1}{2}(\beta-\gamma)}{2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha} \qquad (\S\,55)$$

$$= \frac{\cos\frac{1}{2}(\beta-\gamma)}{\sin\frac{1}{2}\alpha}. \qquad (b)$$

Subtracting the equations (a) we have

$$\frac{b-c}{a} = \frac{\sin\beta - \sin\gamma}{\sin\alpha} = \frac{2\cos\frac{1}{2}(\beta+\gamma)\sin\frac{1}{2}(\beta-\gamma)}{2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}$$

$$= \frac{\sin\frac{1}{2}(\beta-\gamma)}{\cos\frac{1}{2}\alpha}. \qquad (c)$$

From the equations (c) and (b) we obtain

$$\left.\begin{array}{l} a\sin\frac{1}{2}(\beta-\gamma) = (b-c)\cos\frac{1}{2}\alpha, \\ a\cos\frac{1}{2}(\beta-\gamma) = (b+c)\sin\frac{1}{2}\alpha; \end{array}\right\} \qquad (18)$$

which equations are readily solved by Prob. I. Chap. V.

By taking the quotient of these equations we may readily deduce the relation (16).

Exercises.

1. Given $b=2956.2$, $c=9090.8$, $\alpha=98°\ 29'.6$; find β, γ, and a.

2. A surveyor lays off two lines from the same point: the one due north, 279.25 metres, the other east 15° north, 109.262 metres. How far apart are the ends of the lines, and what is the direction of the line joining them?

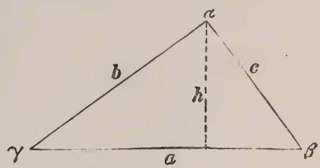
3. The sides of a parallelogram are 26 and 15, and one angle is 126° 52′.2. Find the lengths of the two diagonals and the angles which they make with the sides.

4. Given the two diagonals d and d' of a parallelogram and the angle ε which they form; express the sides and angles of the parallelogram algebraically and compute them for the special case $d = 5$, $d' = 6$, $\varepsilon = 49°\ 18'$.

Areas of Triangles.

62. THEOREM V. *The area of a triangle is equal to half the product of any two sides by the sine of their included angle.*

Proof. It is shown in geometry that the area is half the base into the altitude. Now in the figure,



$$\text{Altitude } h = b \sin \gamma.$$

Therefore, a being the base,

$$\text{Area} = \tfrac{1}{2}ah = \tfrac{1}{2}ab \sin \gamma. \quad \text{Q.E.D.}$$

Cor. 1. *If two triangles have two sides of the one respectively equal to two sides of the other, and the angles which these sides form supplementary, the triangles will be equal in area.*

For the sines of the supplementary angles are equal.

Cor. 2. Since we may take any one side as a base, if we call h, h', h'' the altitudes above the respective sides a, b, and c, we shall have

$$ah = bh' = ch'',$$

and

$$ab \sin \gamma = bc \sin \alpha = ca \sin \beta.$$

For these expressions are each double the area of the triangle.

EXERCISES.

1. Given $a = 75$, $b = 29$, $\beta = 16° 15'.6$; find the remaining parts and the areas of the two triangles which may be formed.

2. Express the area of a parallelogram of which two adjoining sides are of given length, a and b, and make with each other a given angle δ.

3. Express the area of a triangle in terms of a base, c, and the two adjacent angles, α and β.

Ans. $\tfrac{1}{2}c^2 \dfrac{\sin \alpha \sin \beta}{\sin(\alpha + \beta)}$.

4. In a parallelogram is given a diagonal of length d, and the angles θ and φ which the diagonal makes with the two sides adjoining it. Express the area of the parallelogram.

5. Express the area of the parallelogram in terms of the lengths, d and d', of its diagonals, and the angle ε at which they intersect.

6. In a quadrilateral are given the four sides, $a = 25.63$, $b = 24.09$, $c = 9.92$, $d = 29.97$, and the angle, $78° 25'$, which the sides a and b form with each other. Compute the angles and the area of the quadrilateral.

7. A triangle ABC is to be divided into two parts of equal area by a line parallel to AB. What will be the ratios of the segments into which the other two sides are divided? Ans. $\sqrt{\tfrac{1}{2}} : 1 - \sqrt{\tfrac{1}{2}}$.

8. A city lot fronts 60 feet on a street, and the parallel sides run back, the one 100 and the other 135 feet. It is to be divided into two equal parts by a line parallel to its sides. What will be the frontage of each part, and the length of the dividing line?

Remark.—The figure of the lot is a trapezoid, and the problem is that of dividing a trapezoid into two equal portions by a line parallel to the base. Let us put

a, b, the parallel sides;

$m : 1 - m$, the ratio in which the non parallel sides, and therefore the altitude, is divided by the dividing line;

k, the length of the dividing line.

The unknown quantities of the problem are then m and k. If we put h for the altitude of the entire trapezoid, the altitudes of the two parts will be mh and $(1 - m)\,h$ respectively. Therefore the areas will be

$$\frac{a+k}{2}\,mh \quad \text{and} \quad \frac{k+b}{2}\,(1-m)\,h,$$

the equality of which gives the first condition. For the second condition we have the geometrical theorem that the difference between the dividing line and either of the parallel sides is proportional to its distance from such side. This gives the proportion

$$a - k : k - b :: m : 1 - m,$$

whence

$$(k - b)\,m = (a - k)\,(1 - m).$$

The quotient of this equation by the preceding one gives an equation from which m is eliminated, and from which we find the value of k.

$$k = \sqrt{\frac{a^2 + b^2}{2}}.$$

We then find for m the equation

$$m = \frac{a - k}{a - b}.$$

Applying this method to the problem under consideration, we find

$$k = 118.796 \text{ feet};$$

$$m = 0.462\,97;$$

frontages of lots, 27.778 and 32.222 feet.

CHAPTER VII.

THE THEORY OF POLYGONS.

63. A polygon is completely determined when the positions of its vertices, taken in regular order, are given. The polygon may then be formed by joining each pair of consecutive vertices by a straight line.

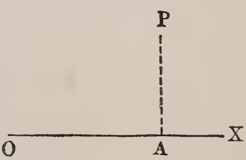
The positions of the vertices may be defined by their co-ordinates, on a system now to be explained.

64. *Co-ordinates of a point.* In geometry the position of a point is fixed by assigning to it certain lines or numbers indicating its situation relative to a fixed line, and a point on that line.

Def. Any numbers or lines which determine the position of a point are called the **co-ordinates** of that point.

Rectangular co-ordinates. Let OX be the fixed line of reference, and O a point of reference on that line from which we measure.

Let P be a point whose position is to be expressed. From P drop a perpendicular PA upon OX. Then:



The line PA is called the **ordinate** of the point P.

The line OA is called the **abscissa** of the point P.

The ordinate and abscissa are called **rectangular co-ordinates** of the point.

The indefinite line OX along which the abscissas are measured is called the **axis of abscissas** or the **axis of X.**

The zero point O from which the co-ordinates are measured is called the **origin.**

When the rectangular co-ordinates are given the position of the point is completely determined.

To find the point when the ordinate and abscissa are given, we measure off from O, on the line OX, a distance equal to the given abscissa.

At the end of this distance we erect a perpendicular equal to the given ordinate.

The end of this perpendicular will be the required position of the point.

The abscissa of a point is represented by the symbol x, the ordinate by the symbol y.

If x is positive, its length is laid off from O toward the right; if negative, toward the left

If the ordinate y is positive, it is measured upward from the axis of X; if negative, downward.

Exercises.

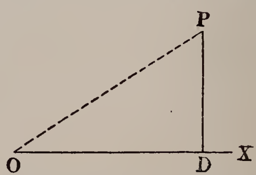
Draw a line OX 4 or 5 inches long as a line of reference, and lay off points having the following co-ordinates from a zero point near the middle of the line:

1.	$x = 1,$	$y = 2$	inches;
2.	$x = 2,$	$y = 1$	"
3.	$x = 2,$	$y = 2$	"
4.	$x = 1,$	$y = 1$	"
5.	$x = 1,$	$y = -2$	"
6.	$x = 2,$	$y = -1$	"
7.	$x = -2,$	$y = -2$	"
8.	$x = -1,$	$y = 2$	"
9.	$x = 0,$	$y = -2$	"

Polar co-ordinates. Draw a line from O to P, and call r its length and φ the angle XOP. Then

$$OD = x = r \cos \varphi,$$
$$DP = y = r \sin \varphi,$$

and Problem I. of Chapter V. reduces to:



Given the co-ordinates x and y of a point, to find the distance and direction of the point from the origin.

Since the quantities r and φ completely determine the position of P, they are also co-ordinates. To distinguish them they are called **polar co-ordinates.**

Exercise.

Eight points have the following several co-ordinates. Find the values of φ and r (the values of r being all equal), and note what relation exists among the values of φ.

1. $x = +4, \quad y = +3;$
2. $x = +3, \quad y = +4;$
3. $x = -3, \quad y = -4;$
4. $x = -4, \quad y = +3;$
5. $x = -4, \quad y = -3;$
6. $x = -3, \quad y = -4;$
7. $x = +3, \quad y = -4;$
8. $x = +4, \quad y = -3.$

65. *Definition of direction of lines and angle between them.* Two finite lines which do not meet are considered to form a certain angle with each other; namely, the angle which would be formed if they were continued until they met, or if a line parallel to the one were drawn through any point of the other.

Since at the point of crossing four angles are formed, we may, in the absence of any convention, regard either of these angles as that between the lines. But as opposite angles are equal, these angles only have two *different* values.

Also, in the absence of a convention, we may regard any angle as either positive or negative. Hence to a given inclination of the lines may be assigned any one of four different values, which values are divisible into two supplementary pairs.

Example. If two lines intersect at an angle of 85°, we may consider their inclination, or the angle which they form, to be

$$\begin{aligned} &\text{either} \quad 85^\circ, \\ &\text{or} \quad 95^\circ, \\ &\text{or} \quad -85^\circ = 275^\circ, \\ &\text{or} \quad -95^\circ = 265^\circ. \end{aligned}$$

This ambiguity is avoided by the following conventions:

1. We assign to each line a positive and a negative direction. The positive direction is that from the beginning to the end of a finite line. The angle they form is then that between their *positive directions.* This is the same as the angle between two lines going out from the same point in the respective positive directions of the lines.

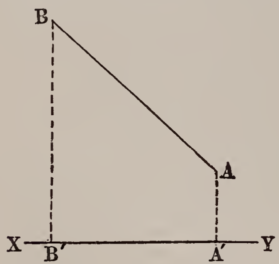
2. We consider one side of the angle as that from which we measure, and we measure the angle to the other side in a positive direction, as explained in §§ 3 and 8.

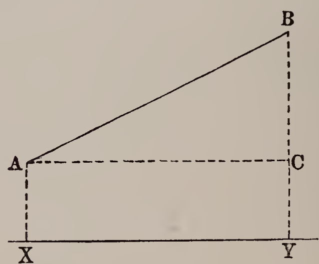
The four values are thus reduced to one.

If two lines are parallel, their angle is 0° or 180° according as they are *similarly directed* or *oppositely directed.*

Projections of Lines.

66. *Def.* The projection of a finite straight line AB upon an indefinite line XY is the distance $A'B'$ between the feet of the perpendiculars from A and B upon the indefinite line.





To find the length of a projection. Through one end of the line, as A, draw AC parallel to XY, meeting BY in C. Then

$$AC = XY;$$
$$AC = AB \cos BAC.$$

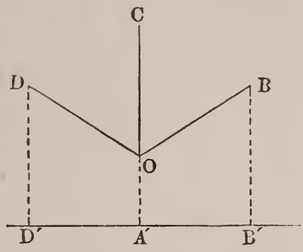
Hence

$$\text{Projection } XY = AB \cos BAC. \qquad (1)$$

That is,

The projection is equal to the length of the line projected into the cosine of the angle which the two lines form with each other.

Algebraic sign of a projection. Let the positive direction of the line OB be from O to B, and let the line turn on O into the successive positions OC and OD.



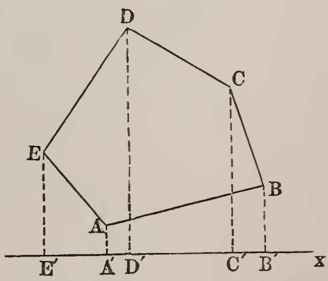
If on the line of projection we regard directions toward the right as positive, the projection $A'B'$ will be positive.

The whole line OC will be projected at the point C; the projection will therefore be zero, a result given by the formula (1), because $\cos 90° = 0$.

The projection $A'D'$ of OD will be negative because it falls in the negative direction. This also corresponds to the formula, because the angle between the two directions OD and $A'B'$ is obtuse.

If we suppose OB to perform a complete revolution around O, we readily see that its projection goes through the same series of changes as the cosine of the angle which it forms with the line of projection.

67. *Projection of sides of a polygon.* Let $ABCDE$ be any polygon the positive directions of whose sides correspond with the circuit we should form in going round the polygon, so as to reach its vertices in alphabetical order. We shall then have, for the projections on the line X,

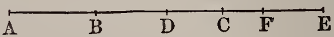


Proj. of $AB = A'B'$, which is $+$;
" $BC = B'C'$, " $-$;
" $CD = C'D'$, " $-$;
" $DE = D'E'$, " $-$;
" $EA = E'A'$, " $+$.

The positive direction being arbitrary, we might equally take the directions AE, ED, DC, etc., as positive. Each of the projections would then have the opposite algebraic sign from that just given.

The student will remark that the projection of the line is positive or negative according as the projection of its end is on the positive or negative side of the projection of its beginning. We wish now to determine the sum of the projections, and for this purpose must understand the algebraic addition of lines.

68. *Algebraic addition of lines.* In geometry the *sum* of two segments AB and BC is defined as the segment AC, formed by putting AB and BC end to end in the same straight line.



In trigonometry and modern geometry we distinguish between the beginning and the end of each segment, and between the positive and negative directions upon the segment; the *positive* direction being *from* the beginning *toward* the end; the *negative*, *from* the end *toward* the beginning.

When this distinction is attended to we must, in designating a segment by letters at its termini, write that letter first which is at the beginning of the segment, so that the letters shall follow each other in the positive order. The segment BA will then be the negative of the segment AB.

We now generalize the definition of the addition of lines as follows:

Def. The algebraic sum of several lines is formed by placing the beginning of each line after the first at the end of the line next preceding. This sum is then the segment from the beginning of the first line to the end of the last one.

Example.—In the preceding figure we have

$$AC + CE = AE,$$

as in geometry, because both segments are positive.

But if we consider the segment CD as beginning at C and ending at D, then, by the above definition, the algebraic sum of the segments AC and CD will be the segment AD, from the beginning of AC to the end of CD. That is, a negative segment will be subtractive in the same way that in algebra the addition of a negative to a positive quantity implies subtraction.

In general, whenever A, B, C, D, E, F represent points upon a straight line, we have

$$AB + BC + CD + DE + EF = AF,$$

however these points may be situated.

If the end of the last line coincides with the beginning of the first, the sum will be zero, by definition. Hence, however the points A, B, and C may be situated, we have

$$AB + BA = 0;$$
$$AB + BC + CA = 0.$$

69. Let us now return to the projected polygon. On the preceding system, the sum of the projections of the several sides upon the line X is

$$A'B' + B'C' + C'D' + D'E' + E'A' = 0.$$

The same thing would be true if we took any other straight line as the line of projection. Hence:

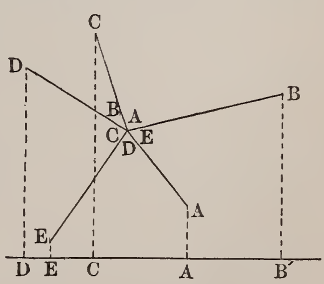
THEOREM I. *The algebraic sum of the projections of the sides of a polygon upon any straight line is zero.*

Since the projection of each side is equal to its length into the cosine of the angle it makes with the line of projection, this theorem may be expressed in the following form:

If the sides of a polygon be a, b, c, etc., and the angles which these sides make with any straight line be α, β, γ, etc., we shall have

$$a \cos \alpha + b \cos \beta + c \cos \gamma + \text{etc.} = 0.$$

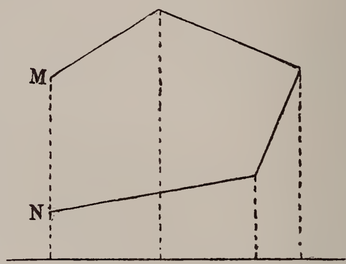
We may imagine the sides of the polygon all taken up and placed with their beginnings at the same point, their length and direction remaining unchanged.



Their several projections will then have the same values as before, and in consequence the algebraic sum of the projections will still be zero.

70. We now have the following theorem, the demonstration of which is left as an exercise for the student:

Theorem II. *If the algebraic sums of the projections of three or more straight lines upon any two non-parallel lines are each zero, these lines when placed end to end without changing their directions will form a polygon, the end of the last line falling upon the beginning of the first.*



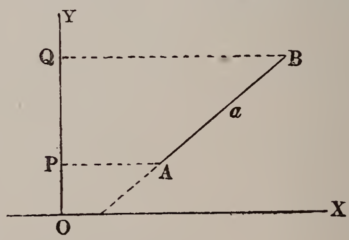
Note. To fix the ideas the student may suppose the lines as first given to all emanate from one point, as in the last figure.

The demonstration is begun by showing that in case the sum of the projections upon a straight line is zero, then, when the lines are placed end to end, the end of the last line and the beginning of the first must lie on the same perpendicular to the line of projection. Thus, in the figure, the sum of the projections of the four unbroken lines is zero, although they do not form a polygon. But, with such a figure, the projections will not be zero on any other non-parallel line.

71. Theorem III. *If a, b, c, etc., be the sides of a polygon, and α, β, γ, etc., the angles which these sides form with any straight line, we shall have*

$$a \sin \alpha + b \sin \beta + c \sin \gamma + \text{etc.} = 0.$$

Proof. Let OX be the base line from which the angles α, β, γ, etc., are counted; AB, any side of the polygon; a, its length; α, the angle which AB makes with X.



Draw OY perpendicular to OX, and let PQ be the projection of AB upon OY. AB will then make with OY an angle $90° - \alpha$, and we shall have

$$\text{Projection } PQ = a \cos (90° - \alpha) = a \sin \alpha.$$

Treating all the other sides in the same way, the algebraic sum of their projections upon OY is found to be

$$a \sin \alpha + b \sin \beta + c \sin \gamma + \text{etc.},$$

which sum is zero by Theorem I.

Theorem IV. *If the sum of the projections of a series of straight lines upon any two non-parallel lines be zero, the sum of their projections upon any third line will be zero.*

NOTE. This theorem follows immediately from Theorems I. and II., but we prove it algebraically in order to show an elegant application of the addition theorem.

Proof. Put

α, β, γ, etc., the angles which the straight lines make with one of the lines of projection;

π, the angle which the first two lines of projection make with each other.

Then $\alpha - \pi$, $\beta - \pi$, $\gamma - \pi$, etc., will be the angles which the lines make with the second line of projection.

By hypothesis we have

$$a \cos \alpha + b \cos \beta + c \cos \gamma + \text{etc.} = 0; \qquad (a)$$

$$a \cos (\alpha - \pi) + b \cos (\beta - \pi) + c \cos (\gamma - \pi) + \text{etc.} = 0.$$

The last equation, by the addition theorem, reduces to

$$\cos \pi (a \cos \alpha + b \cos \beta + c \cos \gamma + \ldots)$$
$$+ \sin \pi (a \sin \alpha + b \sin \beta + c \sin \gamma + \ldots) = 0.$$

The first term of this equation vanishes by (a).

Hence, the whole sum being zero, the second term must also vanish, which requires that we either have

$$\sin \pi = 0,$$

which will give $\pi = 0$ or 180°,—in which case the two lines would be parallel,—or

$$a \sin \alpha + b \sin \beta + c \sin \gamma + \text{etc.} = 0. \qquad (b)$$

Since, by hypothesis, the two lines are not parallel, the equation b must hold true.

Now let θ be the angle which any third line of projection forms with the first line. The angles which the lines a, b, c, etc., form with this third line will then be

$$\alpha - \theta, \ \beta - \theta, \ \gamma - \theta, \ \text{etc.}$$

Therefore the sum of the projections upon this line will be

$$a \cos (\alpha - \theta) + b \cos (\beta - \theta) + c \cos (\gamma - \theta) + \text{etc.},$$

which reduces to

$$\cos \theta (a \cos \alpha + b \cos \beta + c \cos \gamma + \text{etc.})$$
$$+ \sin \theta (a \sin \alpha + b \sin \gamma + c \sin \gamma + \text{etc.}),$$

a sum which vanishes by (a) and (b), whatever be the value of θ, thus proving the theorem.

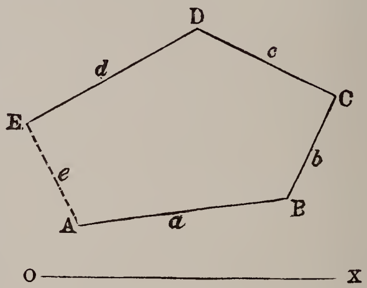
72. *Cor.* From §§ 69 and 71 it follows that if all the sides of a polygon but one are given in length and direction,—a, b, c, etc., being the lengths, and α, β, γ, etc., the angles with a fixed base line,—and if l be the omitted side and ϑ its angle, we shall have

$$\left.\begin{aligned} l \sin \vartheta &= -a \sin \alpha - b \sin \beta - c \sin \gamma - \text{etc.}; \\ l \cos \vartheta &= -a \cos \alpha - b \cos \gamma - c \cos \gamma - \text{etc.}; \end{aligned}\right\} \qquad (2)$$

which will determine l and ϑ, by Prob. I. Chap. V.

Example. A surveyor measures off courses and distances as follows:

I. North 80° 28′ east, 42.68 metres.
II. North 23° 22′ east, 22.79 "
III. North 65° 49′ west, 31.96 "
IV. South 59° 58′ west, 40.13 "



What distance and direction will carry him to his starting-point?

We note that the expression

North $r°$ east

means

$r°$ east of north.

Taking the east and west line OX as the base from which we measure angles, we readily find the angles made by each side with the base, as shown in the following table, which also gives the values of $a \sin \alpha$, $b \sin \beta$, etc., and $a \cos \alpha$, $b \cos \beta$, etc., as computed from the given data:

Side.	Length.	Angle.	$a \sin \alpha$, etc.	$a \cos \alpha$, etc.
a	42.68	9° 32′	+ 7.069	+ 42.090
b	22.79	66° 38′	+ 20.921	+ 9.039
c	31.96	155° 49′	+ 13.093	− 29.155
d	40.13	210° 2′	− 20.085	− 34.742
			+ 20.998	− 12.768

Hence from (2) we have, for the last side,

$$l \sin \vartheta = -20.998;$$
$$l \cos \vartheta = +12.768;$$

from which we find

$$l = 24.575;$$
$$\zeta = 301^\circ\ 18'.1.$$

Expressed in the language of surveyors, the angle ζ indicates the direction, South 31° 18′.2 East.

It will be seen that we have taken as the positive direction of the last line that from the point last reached to the starting-point. This is in accordance with the convention that the positive directions of the several sides of a polygon are so taken that in passing around it the directions shall all be positive or all negative.

But we might equally consider the problem: Having proceeded along a series of connected lines, AB, BC, etc., of which the lengths and directions are given, to E, what is our distance and direction from our starting-point A? It is evident that the distance and direction are the length and direction of the line AE, which is simply the negative of the side EA necessary to complete the polygon. If we call ε the angle of direction of AE, the equations for determining l and ε would be

$$\left.\begin{array}{l} l \sin \varepsilon = a \sin \alpha + b \sin \beta + \text{etc.}; \\ l \cos \varepsilon = a \cos \alpha + b \cos \beta, \text{etc.}; \end{array}\right\} \qquad (3)$$

and, in the preceding example, we should have

$$l \sin \varepsilon = +20.998;$$
$$l \cos \varepsilon = -12.768;$$

which would give

$$l = 24.575;$$
$$\varepsilon = 121^\circ\ 18'.1.$$

EXERCISES.

1. A surveyor ran a course S. 12° 13′ E., 289.26 metres, and thence N. 82° 49′ E., 92.68 metres. What is his direction and distance from his starting-point? What is the direction and distance of the starting-point from him?

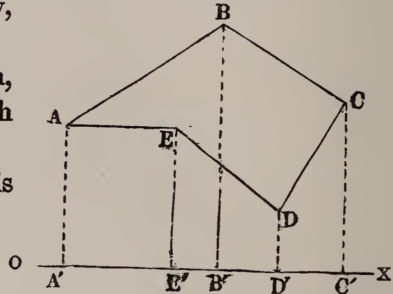
2. Five sides of an irregular hexagon taken in order have the following lengths and directions relative to a fixed line:

a. Length, 297.43 metres; direction, 332° 6′.8;
b. " 606.07 " " 222° 42′.3;
c. " 421.02 " " 157° 59′.8;

d. Length, 343.90 metres; direction, 5° 22′.1;
e. " 40.92 " " 125° 2′.2.

What is the length and direction of the remaining side?

73. *Areas of polygons.* When the sides of a polygon are all given in length and direction, the area may be computed by a process demonstrated in geometry, but which we shall describe here.



Let $ABCDE$ be any polygon, and OX the base line from which we measure angles.

The area of this polygon is equal to

$$\text{Area } A'ABCC'$$
minus Area $A'AEDCC'$.

The first area is equal to the sum of the areas of the two trapezoids

$$A'ABB' \text{ and } B'BCC'.$$

The second is equal to the sum of the areas of the three trapezoids $C'CDD'$, $D'DEE'$, and $E'EAA'$.

It will be noted that there is one trapezoid for each side of the polygon, of which the non-parallel sides are the side of the polygon and its projection upon the base line.

We have for the area of the first trapezoid, noting that the base line OX is perpendicular to the parallel sides,

$$\text{Area } A'ABB' = \tfrac{1}{2}(AA' + BB')\, A'B'.$$

Putting, as before,

a, b, c, etc., for the length of the sides AB, BC, CD, etc.;
α, β, γ, etc., for the angles which they form with OX;

putting, also,

$$p, \text{ the length } AA',$$

we have

$$AA' = p;$$
$$BB' = p + a \sin \alpha;$$
$$A'B' = a \cos \alpha.$$

Substituting these values, we have, for the area of the first trapezoid,

$$\text{Area } A'ABB' = \tfrac{1}{2}(2p + a \sin \alpha)\, a \cos \alpha.$$

Passing on to the other trapezoids, we have, for the lengths of the several perpendiculars,

$$\begin{aligned} BB' &= p + a \sin \alpha; \\ CC' &= p + a \sin \alpha + b \sin \beta; \\ DD' &= p + a \sin \alpha + b \sin \beta + c \sin \gamma; \\ EE' &= p + a \sin \alpha + b \sin \beta + c \sin \gamma + d \sin \delta \\ &= p - e \sin \varepsilon; \qquad (\S\, 71) \\ AA' &= p. \end{aligned}$$

Also, for the altitudes of the trapezoids between their parallel sides,

$$\begin{aligned} A'B' &= a \cos \alpha; \\ B'C' &= b \cos \beta; \\ C'D' &= c \cos \gamma; \\ D'E' &= d \cos \delta; \\ E'A' &= e \cos \varepsilon. \end{aligned}$$

We thus have, for the several areas,

$$\left.\begin{aligned} &\tfrac{1}{2}(2p + a \sin \alpha)\, a \cos \alpha; \\ &\tfrac{1}{2}(2p + 2a \sin \alpha + b \sin \beta)\, b \cos \beta; \\ &\tfrac{1}{2}(2p + 2a \sin \alpha + 2b \sin \beta + c \sin \gamma)\, c \cos \gamma; \\ &\tfrac{1}{2}(2p + 2a \sin \alpha + 2b \sin \beta + 2c \sin \gamma + d \sin \delta)\, d \cos \delta; \\ &\tfrac{1}{2}(2p + 2a \sin \alpha + 2b \sin \beta + 2c \sin \gamma \\ &\qquad\qquad + 2d \sin \delta + e \sin \varepsilon)\, e \cos \varepsilon. \end{aligned}\right\} (a)$$

It has been shown that the required area of the polygon is found by subtracting the last three of these areas from the first two.

But in reaching this conclusion we took no account of algebraic signs, and so virtually considered the areas all positive. Now, as the figure is drawn, $\cos \alpha$ and $\cos \beta$ are positive, and the cosines of γ, δ, and ε are negative. Hence if we put the sign $+$ before each of the areas (a), the subtraction will be indicated by the negative character of those products which have $\cos \gamma$, $\cos \delta$, and $\cos \varepsilon$ as factors, and so the algebraic expression will be correct.

If we add up the quantities (a), beginning with the terms in p, we see that the coefficient of p is

$$a \cos \alpha + b \cos \beta + \ldots + e \cos \varepsilon,$$

which is zero. Hence the quantity p disappears from the expression for the area. Since p is defined as the distance below A at which the base line OX is drawn, this is the same as saying that the area of the polygon is independent of this distance, which evidently must be true. In fact, if we suppose the base line OX to move up or down, remaining parallel to itself, it will add and subtract equal areas to or from the positive and negative trapezoids, and so will leave the algebraic sum of the areas unchanged.

Now let us suppose p zero, and put

$$\left.\begin{array}{l} y_1 = a \sin \alpha; \\ y_2 = 2a \sin \alpha + b \sin \beta = y_1 + a \sin \alpha + b \sin \beta; \\ y_3 = 2a \sin \alpha + 2b \sin \beta + c \sin \gamma = y_2 + b \sin \beta + c \sin \gamma; \\ y_4 = y_3 + c \sin \gamma + d \sin \delta; \\ y_5 = y_4 + d \sin \delta + e \sin \varepsilon. \end{array}\right\} (b)$$

(We remark that y_5 will come out equal to $- e \sin \varepsilon$ if everything is correct.)

We shall then have, for the double of the area of the polygon,

$$2 \text{ Area} = y_1 a \cos \alpha + y_2 b \cos \beta + y_3 c \cos \gamma + y_4 d \cos \delta + y_5 e \cos \varepsilon. \quad (4)$$

As an example, let us compute the area of the polygon investigated in the example of § 72. The following table shows the principal parts of the computation:

$a \sin \alpha$, $b \sin \beta$, etc.	Sums of Pairs.	y_1, y_2, etc.	$a \cos \alpha$, $b \cos \beta$, etc.	Products.
+ 7.069	+ 27.990	+ 7.069	+ 42.090	+ 297.5
+ 20.921	+ 34.014	+ 35.059	+ 9.039	+ 316.9
+ 13.093	− 6.992	+ 69.073	− 29.155	− 2013.9
− 20.085	− 41.083	+ 62.081	− 34.742	− 2156.8
− 20.998		+ 20.998	+ 12.768	+ 268.1
				− 3288.2

The first column gives the values of $a \sin \alpha$, $b \sin \beta$, $c \sin \gamma$, etc., already computed in the preceding example.

The second column gives the values of $a \sin \alpha + b \sin \beta$, $b \sin \beta + c \sin \gamma$, etc., which are added to each value of y to form the value of y next following, as shown in (b).

The third column gives the values of y_1, y_2, y_3, y_4, y_5, computed by the formulæ (b), from the numbers in the first two columns.

The fourth column gives the values of $a \cos \alpha$, $b \cos \beta$, $c \cos \gamma$, etc., already computed.

The fifth column gives the products which enter into the equation (c).

The algebraic sum being twice the area, we have

$$\text{Area of polygon} = 1644.1 \text{ square metres.}$$

74. It will be noticed that the area of the polygon comes out negative, and the question arises, What interpretation is to be put on this result? The answer is that the conventions of positive and negative as employed in this chapter, as applied to lines, express only the directions in which the lines are reckoned. By a change of direction the algebraic signs of all the trapezoids, and therefore of the area of the polygon itself, will be changed. A little consideration will show that this area will come out positive when we go round the polygon in what we have called the negative direction, and *vice versa*.

The algebraic sum of the areas, whether positive or negative, will always be the true area of the trapezoid under one important condition: that none of the sides cross each other. In this case the system of applying the algebraic signs will lead to the areas on the two sides of the point of crossing having opposite signs. Hence the result finally obtained will be the *difference* of the two areas.

Exercises.

1. If the lengths and directions of three of the four sides of a quadrilateral are

$$a = 262.72 \text{ metres}; \quad \alpha = 39^\circ 49';$$
$$b = 109.79 \quad \text{“} \quad \beta = 150^\circ 26';$$
$$c = 300.63 \quad \text{“} \quad \gamma = 242^\circ 52';$$

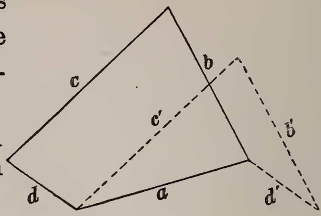
find the remaining side and the area.

2. The sides of a quadrilateral, taken in regular order, have the following lengths and directions, in part:

$$a = 29; \quad \alpha = 12^\circ 26';$$
$$b = 52; \quad \beta = 75^\circ 58';$$
$$c \text{ to be found}; \quad \gamma = 172^\circ 3';$$
$$d = 66; \quad \delta \text{ to be found.}$$

Find c and δ.

3. Prove geometrically that if the sides of a polygon be joined together in any order, their directions remaining unaltered, the end of the last side will still fall upon the beginning of the first.



An example of the construction is shown in the figure, where the sides are changed from the order *abcd* to *adbc*.

4. Express the lengths (x and x') and the directions (θ and θ') of the diagonals of a quadrilateral of which the lengths of the sides, taken in order, are a, b, c, d, and their directions α, β, γ, δ.

It is only necessary to express the values of the quantities $x \sin \theta$, $x \cos \theta$, $x' \sin \theta'$, and $x' \cos \theta'$ in terms of a, b, c, d, α, β, etc. We may suppose x and θ to refer to the diagonal from the beginning of a to the beginning of c, and x' and θ' to run from the end of a to the end of c.

5. Using the same notation as in § 71, prove

$$b \sin(\beta - \alpha) + c \sin(\gamma - \alpha) + d \sin(\delta - \alpha) + \text{etc.} = 0;$$

$$a + b \cos(\beta - \alpha) + c \cos(\gamma - \alpha) + d \cos(\delta - \alpha) + \text{etc.} = 0.$$

6. If α, β, and γ are the angles which the sides a, b, and c of a triangle make with a base line, and A, B, and C are the interior angles of the triangle, it is required—

(1) To show

$$A = 180° + \beta - \gamma; \quad B = 180° + \gamma - \alpha; \quad C = 180° + \alpha - \beta.$$

(2) By combining the equations of Ex. 5, to deduce the law of sines (§ 56, 3) and the fundamental equations (§ 58, 9).

7. From the same point O emanate three lines, OA, OB, OC, of such lengths and directions that the sum of their projections upon any third line vanishes. If we complete the three parallelograms, of each of which two adjacent sides are two of the lines, the areas of these three parallelograms will be equal.

Both a geometric and an algebraic proof may be given; the former from § 70, Th. I., the latter from § 62.

8. From the corners A and B of a pentagonal field $ABCDE$ an engineer measures angles as follows:

Angle $BAC = 79° 23'.6$;	Angle $ABC = 47° 29'.7$;
Angle $BAD = 130° \ 7'.0$;	Angle $ABD = 153° 42'.7$;
Angle $BAE = 152° 40'.2$.	Angle $ABE = 164° \ 0'.8$.

AB measures 192 metres. Find the remaining sides and the area.

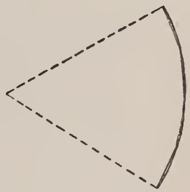
CHAPTER VII.

*TRIGONOMETRIC DEVELOPMENTS.**

75. LEMMA. *When an arc becomes indefinitely small, the ratio of the sine to its arc approaches unity as its limit.*

Remark. In this lemma it is supposed that the arc is expressed in terms of radius as unity.

Proof. It is laid down as an axiom of elementary geometry that when the number of sides of an inscribed regular polygon is indefinitely increased, the perimeter of the polygon approaches the circumference of the circle as its limit. That is, the ratio



$$\frac{\text{perimeter of polygon}}{\text{circumference of circle}} \text{ approaches 1 indefinitely.}$$

Now, if we divide the circumference into n equal arcs, the n chords of these arcs, forming the perimeter of the inscribed polygon, will each be twice the sine of half the arc (§ 18). That is, we shall have

$$\text{Perimeter of polygon} = 2n \times \text{sine of each arc};$$

$$\text{Circumference of circle} = 2n \text{ equal arcs.}$$

Therefore the above ratio is equal to

$$\frac{\text{sine of arc}}{\text{arc itself}},$$

* The study of this chapter requires a knowledge of so much of series and imaginary quantities as is contained in Book XI., Chapters I., II., and V., and Book XII., Chapter I., of the author's "College Algebra." It may be advantageously read in connection with Book XII., Chapter II., of that work. Students taking a partial course may pass to spherical trigonometry without reading this chapter.

which therefore approaches unity as its limit when n is increased indefinitely.

76. Problem. *To develop the sine and cosine of an angle in terms of the ascending powers of the angle.*

Let us suppose

$$\sin x = s_0 + s_1x + s_2x^2 + s_3x^3 + s_4x^4 + s_5x^5 + \text{etc.}; \quad (1)$$

$$\cos x = c_0 + c_1x + c_2x^2 + c_3x^3 + c_4x^4 + c_5x^5 + \text{etc.} \quad (2)$$

in which s_0, s_1, s_2, etc., and c_0, c_1, c_2, etc., are coefficients whose values are to be determined from some known properties of the sine and cosine.

Since sine $(-x) = -\sin x$, the series for $\sin x$ must change its algebraic sign when the algebraic sign of x changes. But only the odd powers of x will then change their sign. Therefore the even powers must not enter into the development, and we must have

$$s_0 = s_2 = s_4 = \text{etc.} = 0.$$

The complete analytic proof of this proposition may be put into the following form. Changing x to $-x$ in the development (1), we have

$$\sin(-x) = s_0 - s_1x + s_2x^2 - s_3x^3 + s_4x^4 - \text{etc.}$$

But we have, by changing the signs of both members of (1),

$$-\sin x = -s_0 - s_1x - s_2x^2 - s_3x^3 - s_4x^4 - \text{etc.}$$

Because these developments must be identically equal, we must have

$$s_0 = -s_0,$$
$$s_2 = -s_2,$$
$$s_4 = -s_4,$$
$$\text{etc.} \quad \text{etc.},$$

which gives $s_0 = s_2 = s_4 = \text{etc.} = 0$.

Therefore the development of the sine becomes

$$\sin x = s_1x + s_3x^3 + s_5x^5 + \text{etc.}$$

Dividing this equation by x, we have

$$\frac{\sin x}{x} = s_1 + s_3x^2 + s_5x^4 + \text{etc.}$$

Now suppose x to approach indefinitely near to zero. The first member will then, according to the lemma, approach unity as its limit, and the second member will approach s_1 as its limit. Therefore we must have $s_1 = 1$, and the development becomes

$$\sin x = x + s_3x^3 + s_5x^5 + \text{etc.} \quad (3)$$

Next take the development (2) for $\cos x$. If we suppose $x = 0$, we have $\cos x = 1$. Therefore, putting $x = 0$, equation (2) will become

$$1 = c_0,$$

which is the required value of c_0.

Again, because $\cos(-x) = \cos x$, the development of $\cos x$ must remain unaltered when we change x into $-x$.

Because this change will reverse the signs of all the odd powers of x, the coefficients of c_1, c_3, etc., of these powers must all vanish, and the development must be

$$\cos x = 1 + c_2x^2 + c_4x^4 + c_6x^6 \ . \ . \ . \qquad (4)$$

77. We must now choose some property of the sine and cosine which will enable us to form equations of condition for the coefficients s_3, s_5 ,etc., and c_2, c_4, c_6, etc. The most simple property for this purpose is that expressed by the addition theorem:

$$\sin(x + y) = \sin x \cos y + \cos x \sin y. \qquad (5)$$

Because the equation (3) is to be true for all values of x, it must remain true when $x + y$ is substituted for x. Making this substitution in (3), we have

$$\sin(x + y) = x + y + s_3(x + y)^3 + s_5(x + y)^5 + \text{etc.}$$

Developing the powers in the second member, and collecting the terms multiplied by the first power of y, we have

$$\left.\begin{aligned} \sin(x + y) = {} & x + s_3x^3 + s_5x^5 + s_7x^7 + \ . \ . \ . \\ & + y(1 + 3s_3x^2 + 5s_5x^4 + 7s_7x^6 + \ . \ . \ .) \\ & + \text{terms} \times y^2, y^3, \text{etc.,} \end{aligned}\right\} \qquad (6)$$

which we need not compute.

From (5) we have, by substituting for $\cos y$ and $\sin y$ their assumed developments, (3) and (4),

$$\left.\begin{aligned} \sin(x + y) &= \sin x(1 + c_2y^2 + c_4y^4 + \ . \ . \ .) \\ &\quad + \cos x(y + s_3y^3 + \text{etc.}) \\ &= \sin x + y \cos x + \text{terms} \times y^2, y^3, \text{etc.} \end{aligned}\right\} \qquad (7)$$

Now the expressions (6) and (7) must be identically equal; therefore the coefficients of each power of y must be identically equal. Equating the coefficients of the first power of y, we have

$$1 + 3s_3x^2 + 5s_5x^4 + 7s_7x^6 + \text{etc.} = \cos x.$$

But we have, by (4),

$$\cos x = 1 + c_2x^2 + c_4x^4 + \text{etc.}$$

This equation must be satisfied for all values of x. Equating the coefficients of like powers of x, we find

$$\left.\begin{aligned} 3s_3 &= c_2; \\ 5s_5 &= c_4; \\ 7s_7 &= c_6; \\ \text{etc.} &\quad \text{etc.} \end{aligned}\right\} \qquad (8)$$

Next consider the addition theorem for the cosine:

$$\cos(x+y) = \cos x \cos y - \sin x \sin y. \qquad (9)$$

We find, by substituting $x+y$ for x in (4),

$$\cos(x+y) = 1 + c_2(x+y)^2 + c_4(x+y)^4 + \text{etc.};$$

from which, developing to the first power of y as before,

$$\left.\begin{aligned} \cos(x+y) = 1 &+ c_2x^2 + c_4x^4 + c_6x^6 + \text{etc.} \\ &+ y(2c_2x + 4c_4x^3 + 6c_6x^5 + \text{etc.}) \\ &+ \text{terms} \times y^2, y^3, \text{etc.} \end{aligned}\right\} \qquad (10)$$

From the second member of (9), by substituting for $\sin y$ and $\cos y$ their developments, namely,

$$\cos y = 1 + c^2y^2 + \text{etc.},$$
$$\sin y = y + s_3y^3 + \text{etc.},$$

we find

$$\cos(x+y) = \cos x + y(-\sin x) + \text{terms} \times y^2, y^3, \text{etc.} \qquad (11)$$

Equating the coefficients of y in (10) and (11), we have

$$2c_2x + 4c_4x^3 + 6c_6x^5 + \text{etc.} = -\sin x = -x - s_3x^3 - s_5x^5 - \text{etc.}$$

Equating the coefficients of like powers of x,

$$\left.\begin{aligned} 2c_2 &= -1; \\ 4c_4 &= -s_3; \\ 6c_6 &= -s_5. \\ \text{etc.} &\quad \text{etc.} \end{aligned}\right\} \qquad (12)$$

The equations (8) and (12), taken alternately, solve our problem.

From $(12)_1$, $c_2 = -\frac{1}{2}$;

" $(8)_1$, $s_3 = \frac{c_2}{3} = -\frac{1}{2.3}$;

" $(12)_2$, $c_4 = -\frac{s_3}{4} = \frac{1}{2.3.4}$;

" $(8)_2$, $s_5 = \frac{c_4}{5} = \frac{1}{2.3.4.5}$;

From $(12)_3$, $c_6 = -\frac{s_5}{6} = -\frac{1}{2.3.4.5.6}$;
etc. etc. etc.

The law of the coefficients is obvious. Substituting them in the developments (3) and (4), we have

$$\left.\begin{aligned} \sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \text{etc.}; \\ \cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \text{etc.} \end{aligned}\right\} \quad (13)$$

78. *Convergence of the series.* These series are convergent for all values of x, a result which may be shown thus:

The ratio of the successive pairs of adjacent terms in the development of the sine are, omitting the minus sign,

$$\frac{x^2}{2.3}, \quad \frac{x^2}{4.5}, \quad \frac{x^2}{6.7}, \quad \frac{x^2}{8.9}, \quad \text{etc.};$$

that is, each term is formed from the preceding one by multiplying by one of these factors. Now, however great may be x, we can continue these fractions so far that their denominators shall become greater than $2x^2$, and so their values less than $\frac{1}{2}$. After this point the sum of all the following terms will be less than that of a geometric progression of which the first term is the term of the development (13), whose quotient by the preceding term is less than $\frac{1}{2}$, and whose ratio is $\frac{1}{2}$. Such a progression has a limit, whence the sum of the series (13) must also have a limit.

The following two applications of these series will be useful as exercises:

1. Square each series, carrying the square as far as the sixth or eighth power of x, and show that the squares fulfil the condition

$$\sin^2 x + \cos^2 x = 1 \textit{ identically.}$$

2. Compute the values of the sine and cosine of 10° and 30° to 5 places of decimals, remembering that we are to take the natural measure of the arc x in radians (§ 14), and compare the result with that in the tables.

We find, from § 14,

$$\text{Arc } 10° = 0.174\,53;$$
$$\text{Arc } 30° = 0.523\,60.$$

79. *Sine and cosine in terms of imaginary exponentials.* It is shown in algebra that if we call e the sum the series,

$$1+1+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\text{etc., } \textit{ad infinitum},$$

or

$$e = 2.718\,281\,828\ \ldots\ ,$$

we shall have

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \text{etc.}$$

Putting i, the imaginary unit, $= \sqrt{-1}$, substituting $xi = x\sqrt{-1}$ for x, and reducing, this equation becomes

$$e^{xi} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \text{etc.} + \left(x - \frac{x^3}{3!} + \frac{x^5}{5!} - \text{etc.}\right)i;$$

or, from the developments (13),

$$e^{xi} = \cos x + i \sin x. \tag{14}$$

Changing xi into $-xi$, we have

$$e^{-xi} = \cos x - i \sin x. \tag{14'}$$

The sum and difference of these equations are

$$\left.\begin{aligned} 2\cos x &= e^{xi} + e^{-xi} = e^{xi} + \frac{1}{e^{xi}};\\ 2i\sin x &= e^{xi} - e^{-xi} = e^{xi} - \frac{1}{e^{xi}},\\ \text{or}\quad 2\sin x &= \frac{1}{i}\left(e^{xi} - e^{-xi}\right)\\ &= -i\left(e^{xi} - e^{-xi}\right). \end{aligned}\right\} \tag{15}$$

For some purposes these equations may be written in the symmetric form

$$\left.\begin{aligned} e^{xi} + e^{-xi} &= \cos x + \cos(-x);\\ e^{xi} - e^{-xi} &= i\sin x - i\sin(-x). \end{aligned}\right\} \tag{15'}$$

These are two of the most celebrated equations in algebraic trigonometry, and are called EULER's equations, after their discoverer, LEONHARD EULER.

80. *Demoivre's theorem.* If in the equation (14) we substitute nx for x it becomes

$$e^{nxi} = \cos nx + i \sin nx.$$

By raising (14) to the nth power we have

$$e^{nxi} = (\cos x + i \sin x)^n.$$

Therefore $(\cos x + i \sin x)^n = \cos nx + i \sin nx,$ (16)
which is known as DEMOIVRE's theorem.

This theorem enables us to develop the sines and cosines of multiples of an angle in powers of the sine or cosine of the simple angle, as follows:

81. PROBLEM. *To develop sin nx and cos nx in powers of sin x and cos x.*

Developing the first member of (16) by the binomial theorem, and substituting for the powers of i their values (Algebra, § 325), namely,

$$\begin{aligned} i^2 &= -1, \\ i^3 &= -i, \\ i^4 &= +1, \\ \text{etc.} &\quad \text{etc.,} \end{aligned}$$

we have*

$$\left.\begin{aligned} (\cos x + i \sin x)^n &= \cos^n x + \binom{n}{1} i \cos^{n-1} \sin x \\ &- \binom{n}{2} \cos^{n-2} x \sin^2 x - \binom{n}{3} i \cos^{n-3} x \sin^3 x \\ &+ \binom{n}{4} \cos^{n-4} x \sin^4 x + \text{etc.} \end{aligned}\right\} \qquad (17)$$

This development being identically equal to the second member of (16), we have, by equating the real terms and putting, for brevity,

$$c = \cos x, \qquad s = \sin x,$$

$$\cos nx = c^n - \binom{n}{2} c^{n-2}s^2 + \binom{n}{4} c^{n-4}s^4 - \binom{n}{6} c^{n-6}s^6 + \text{etc.}$$

This series will go on to infinity unless n is a positive integer,

---

* We here use the very convenient abbreviated notation for the binomial coefficients, namely:

$$\binom{n}{1} = \frac{n}{1} = n; \qquad \binom{n}{3} = \frac{n(n-1)(n-2)}{1 \,.\, 2 \,.\, 3};$$

$$\binom{n}{2} = \frac{n(n-1)}{1 \,.\, 2}; \qquad \binom{n}{s} = \frac{n(n-1)(n-2) \ldots (n-s+1)}{1 \,.\, 2 \,.\, 3 \ldots s}.$$

We then have $$\binom{n}{s} = \binom{n}{n-s}.$$

in which case it will terminate with a coefficient $\left(\frac{n}{j}\right)$, in which $j = n$ when n is even, and $j = n - 1$ when n is odd.

If we suppose n equal to 2, 3, 4, etc., in succession, we have

$$\cos 2x = c^2 - s^2;$$
$$\cos 3x = c^3 - 3cs^2;$$
$$\cos 4x = c^4 - \frac{4 \cdot 3}{1 \cdot 2} c^2s^2 + s^4;$$
$$\cos 5x = c^5 - \frac{5 \cdot 4}{1 \cdot 2} c^3s^2 + \left(\frac{5}{1}\right) cs^4;$$
$$\cos 6x = c^6 - \frac{6 \cdot 5}{1 \cdot 2} c^4s^2 + \frac{6 \cdot 5}{1 \cdot 2} c^2s^4 - s^6.$$

We may make the results more uniform by substituting for the powers of s their values in powers of c, thus:

$$\left.\begin{aligned} s^2 &= 1 - c^2; \\ s^4 &= 1 - 2c^2 + c^4; \\ s^6 &= 1 - 3c^2 + 3c^4 - c^6; \\ &\text{etc.} \qquad\qquad \text{etc.} \end{aligned}\right\} \qquad (18)$$

Making these substitutions, and reducing the numerical coefficients, we find

$$\left.\begin{aligned} \cos 2x &= 2c^2 - 1; \\ \cos 3x &= 4c^3 - 3c; \\ \cos 4x &= 8c^4 - 8c^2 + 1; \\ \cos 5x &= 16c^5 - 20c^3 + 5c; \\ \cos 6x &= 32c^6 - 48c^4 + 18c^2 - 1; \end{aligned}\right\} \qquad (19)$$

Next equating the coefficients of the imaginary terms of (17) to $i \sin nx$ and dividing by i, we find

$$\sin nx = \left(\frac{n}{1}\right) c^{n-1} s - \left(\frac{n}{3}\right) c^{n-3} s^3 + \left(\frac{n}{5}\right) c^{n-5} s^5 - \text{etc.}$$
$$= s\left\{\left(\frac{n}{1}\right) c^{n-1} - \left(\frac{n}{3}\right) c^{n-3} s^2 + \left(\frac{n}{5}\right) c^{n-5} s^4 - \text{etc.}\right\}$$

Supposing $n = 2, 3, 4, 5$, etc., this form gives

$$\left.\begin{aligned} \sin 2x &= 2sc; \\ \sin 3x &= s\{3c^2 - s^2\}; \\ \sin 4x &= s\{4c^3 - 4cs^2\}; \\ \sin 5x &= s\{5c^4 - 10c^2s^2 + s^4\}; \\ \sin 6x &= s\{6c^5 - 20c^3s^2 + 6cs^4\}; \\ &\text{etc.} \qquad\qquad \text{etc.} \end{aligned}\right\} \qquad (20)$$

Substituting for the even powers of s their expressions (18), we find

$$\sin 3x = s\{4c^2 - 1\};$$
$$\sin 4x = s\{8c^3 - 4c\};$$
$$\sin 5x = s\{16c^4 - 12c^2 + 1\};$$
$$\sin 6x = s\{32c^5 - 32c^3 + 6c\}.$$

Instead of substituting for the powers of s their expressions (18) in terms of the powers of c, we might have expressed the powers of c in terms of s, and by substituting them in (20) have developed the multiple sines in powers of $s = \sin x$.

82. *Expression for powers of the cosine.* The reverse problem, to express the powers of the sine or cosine of an angle in terms of simple sines and cosines of multiples of the angle, is of yet more frequent application.

Let us take the first equation (15),

$$2\cos x = e^{xi} + e^{-xi},$$

and raise it to the nth power by the binomial theorem. We shall then have

$$2^n \cos^n x = e^{nxi} + \binom{n}{1} e^{(n-2)xi} + \binom{n}{2} e^{(n-4)xi} + \ldots + e^{-nxi}. \quad (21)$$

To understand this general form let the student take the special cases $n = 4$ and $n = 5$. Then

$$2^4 \cos^4 x = e^{4xi} + 4e^{2xi} + \frac{4.3}{1.2} + 4e^{-2xi} + e^{-4xi};$$

$$2^5 \cos^5 x = e^{5xi} + 5e^{3xi} + \frac{5.4}{1.2} e^{xi} + \frac{5.4}{1.2} e^{-xi} + 5e^{-3xi} + e^{-5xi}.$$

Supposing n to be a positive integer, we see (1) that the coefficients of terms equally distant from the two ends of the series are equal, and (2) that the exponents of e in such terms are equal and of opposite signs. Also, if n is even, the middle term does not contain x; but if n is odd, the terms on each side of the middle will contain x.

Therefore by joining the first and last terms, the term after the first and that before the last, etc., the development (21) may be put into the form

$$2^n \cos^n x = e^{nxi} + e^{-nxi} + \binom{n}{1} \{e^{(n-2)xi} + e^{-(n-2)xi}\}$$
$$+ \binom{n}{2} \{e^{(n-4)xi} + e^{-(n-4)xi}\} + \text{etc.}$$

But by the general equations (15′) we have, putting nx for x,

$$e^{nxi} + e^{-nxi} = 2 \cos nx = \cos nx + \cos (-nx),$$

whatever be the value of n.

Hence, substituting this value of the exponential terms,

$$2^n \cos^n x = \left.\begin{array}{l} \cos nx + \binom{n}{1} \cos (n-2)x + \binom{n}{2} \cos (n-4)x + \text{etc.} \\ + \cos(-nx) + \binom{n}{1} \cos (2-n)x + \binom{n}{2} \cos (4-n)x + \text{etc.,} \end{array}\right\} \quad (22)$$

the terms in the third line forming the end of the series, which is doubled over so that the end comes under the beginning.

Giving to n the successive values 2, 3, 4, etc., we find

$$\left.\begin{array}{l} 2^2 \cos^2 x = \cos 2x + 2 + \cos(-2x); \\ 2^3 \cos^3 x = \cos 3x + (\frac{3}{1}) \cos x + (\frac{3}{2}) \cos(-x) + (\frac{3}{3}) \cos(-3x); \\ 2^4 \cos^4 x = \cos 4x + (\frac{4}{1}) \cos 2x + (\frac{4}{2}) + (\frac{4}{3}) \cos(-2x) \\ \qquad\qquad + (\frac{4}{4}) \cos(-4x); \\ 2^5 \cos^5 x = \cos 5x + (\frac{5}{1}) \cos 3x + (\frac{5}{2}) \cos x \\ \qquad + (\frac{5}{5}) \cos(-5x) + (\frac{5}{4}) \cos(-3x) + (\frac{5}{3}) \cos(-x). \end{array}\right\} \quad (23)$$

We have extended the series in this form that the student may see the law of its formation, which is as follows: The successive coefficients are the binomial coefficients. The coefficient of x in the first term is n, and it diminishes by 2 in each following term, so as to become $-n$ in the last term.

The two well-known formulæ

$$\cos(-nx) = \cos nx,$$
$$\binom{n}{s} = \binom{n}{n-s},$$

will enable us to combine every pair of terms equally distant from the extremes into a single one. For instance, we have

$$\binom{6}{4} = \frac{6.5.4.3}{1.2.3.4} = \frac{6.5}{1.2} = \binom{6}{2};$$
$$\binom{5}{4} = \frac{5.4.3.2}{1.2.3.4} = \binom{5}{1} = 5.$$

Combining them thus and dividing each equation by 2, we find

$$\left.\begin{aligned}
2\ \cos^2 x &= \cos 2x + 1; \\
2^2 \cos^3 x &= \cos 3x + 3 \cos x; \\
2^3 \cos^4 x &= \cos 4x + 4 \cos 2x + \frac{1}{2} \cdot \frac{4 \,.\, 3}{1 \,.\, 2}; \\
2^4 \cos^5 x &= \cos 5x + 5 \cos 3x + \frac{5 \,.\, 4}{1 \,.\, 2} \cos x; \\
2^5 \cos^6 x &= \cos 6x + 6 \cos 4x + \frac{6 \,.\, 5}{1 \,.\, 2} \cos 2x + \frac{1}{2} \cdot \frac{6 \,.\, 5 \,.\, 4}{1 \,.\, 2 \,.\, 3}; \\
&\text{etc.} \qquad \text{etc.} \qquad \text{etc.}
\end{aligned}\right\} \quad (24)$$

A careful study of these forms will show the general law of formation, and enable us to carry the powers forward to any extent.

83. *Powers of the sine.* To find the powers of the sine we take the second of equations (15),

$$2i \sin x = e^{xi} - e^{-xi},$$

and raise it to the nth power by the binomial theorem. We then have

$$\left.\begin{aligned}
2^n i^n \sin^n x = e^{nxi} - \binom{n}{1} e^{(n-2)xi} + \binom{n}{2} e^{(n-4)xi} + \text{etc.} \\
\pm e^{-nxi} \mp \binom{n}{1} e^{-(n-2)xi} \pm \binom{n}{2} e^{-(n-4)xi} + \text{etc.}
\end{aligned}\right\} \quad (25)$$

The bottom line gives the last terms of the series, arranged in the reverse order, so that terms equally distant from the extremes are under each other.

The upper signs are to be used in the bottom line when n is *even;* the lower ones when n is *odd.* This will be seen by forming the developments for $n = 5$ and $n = 6$.

$$\left.\begin{aligned}
2^5 i^5 \sin^5 x &= e^{5xi} - (\tfrac{5}{1})\, e^{3xi} + (\tfrac{5}{2})\, e^{xi} - e^{-5xi} \\
&\qquad + (\tfrac{5}{4})\, e^{-3xi} - (\tfrac{5}{3})\, e^{-xi}; \\
2^6 i^6 \sin^6 x &= e^{6xi} - (\tfrac{6}{1})\, e^{4xi} + (\tfrac{6}{2})\, e^{2xi} + e^{-6xi} \\
&\qquad - (\tfrac{6}{5})\, e^{-4xi} + (\tfrac{6}{4})\, e^{-2xi} - (\tfrac{6}{2}).
\end{aligned}\right\} \quad (25')$$

We first take the case of n even. The nth power of i is then $+1$ or -1 according as $\frac{1}{2}n$ is even or odd. Adding the terms of (25) and substituting for the exponential functions their values in terms of cosines from (15′), we find, when n is even,

$$\left.\begin{aligned}(-1)^{\frac{n}{2}}\, 2^n \sin^n x = \cos nx - \binom{n}{1}\cos(n-2)x \\ + \binom{n}{2}\cos(n-4)x - \text{etc.}\end{aligned}\right\} \quad (26)$$

When n is odd, the sum of each pair of corresponding terms in (25) will give rise to a sine. Making the substitution and dividing by i, we shall have

$$\left.\begin{aligned}2^n i^{n-1} \sin^n x = \sin nx - \binom{n}{1}\sin(n-2)x \\ + \binom{n}{2}\sin(n-4)x - \text{etc.}\end{aligned}\right\} \quad (26')$$

Giving n the successive values 2, 3, 4, etc., and applying the forms (26) and (26′) alternately, and changing all the signs in (26) when $\frac{1}{2}n$ is odd, we find

$$2^2 \sin^2 x = -2\cos 2x + 2.$$

$$2^3 \sin^3 x = -\sin 3x + 3\sin x - 3\sin(-x) + \sin(-3x),$$

or $$2^2 \sin^3 x = -\sin 3x + 3\sin x.$$

$$2^4 \sin^4 x = \cos 4x - (\tfrac{4}{1})\cos 2x + (\tfrac{4}{2}) - (\tfrac{4}{3})\cos(-2x) + (\tfrac{4}{4})\cos(-4x),$$

or $$2^3 \sin^4 x = \cos 4x - 4\cos 2x + 3.$$

$$2^5 \sin^5 x = \sin 5x - (\tfrac{5}{1})\sin 3x + \tfrac{5}{2}\sin x - \sin(-5x) + (\tfrac{5}{4})\sin(-3x) - (\tfrac{5}{3})\sin(-x),$$

or $$2^4 \sin^5 x = \sin 5x - 5\sin 3x + 10\sin x.$$

$$2^6 \sin^6 x = -\cos 6x + (\tfrac{6}{1})\cos 4x - (\tfrac{6}{2})\cos 2x + (\tfrac{6}{3}) - \cos(6x) + (\tfrac{6}{5})\cos(-4x) - (\tfrac{6}{4})\cos(-2x),$$

or $$2^5 \sin^6 x = -\cos 6x + 6\cos 4x - 15\cos 2x + \tfrac{1}{2}\cdot\frac{6.5.4}{1.2.3}.$$

etc. etc.

84. We might have obtained the above results for $\cos x$, $\cos^2 x$, $\cos^3 x$, etc., and $\sin x$, $\sin^2 x$, etc., by consecutive multiplication, and substitution of sines or cosines of sums for products, by § 43. Thus, by § 44, eq. 16′,

$$2\cos^2 x = \cos 2x + 1;$$

$\times\, 2\cos x$,

$$2^2 \cos^3 x = 2\cos x \cos 2x + 2\cos x.$$

Substituting for the first term of the second member its values (12) of § 43, we have

$$2^2 \cos^3 x = \cos 3x + \cos x + 2 \cos x = \cos 3x + 3 \cos x.$$

Multiplying this equation by $2 \cos x$, we should have an expression for $\cos^4 x$, etc. But the use of exponentials enables us not only to obtain the higher powers more expeditiously, but to find the general law of the series, which is not readily done by multiplication.

Exercises.

1. In the expression

$$1 + 2 \cos x + 3 \cos^2 x + 4 \cos^3 x,$$

substitute for the powers of x their values in terms of the multiple of x, and reduce the expression to one containing simple multiples of x.

Solution. From (24),

$$\begin{array}{llllll} 4 \cos^3 x & = \cos 3x & & + 3 \cos x & \\ 3 \cos^2 x & = & + \frac{3}{2} \cos 2x & & + \frac{3}{2} \\ 2 \cos x & = & & + 2 \cos x & \\ 1 & = & & & 1 \\ \hline \text{Sum} & = \cos 3x & + \frac{3}{2} \cos 2x & + 5 \cos x & + \frac{5}{2} \end{array}$$

2. Reduce the expressions

$$\begin{gathered} 4 \cos^3 x - 3 \cos x, \\ 8 \cos^4 x - 8 \cos^2 x + 1, \\ 16 \cos^5 x - 20 \cos^3 x + 5 \cos x, \\ 32 \cos^6 x - 48 \cos^4 x + 18 \cos^2 x - 1, \end{gathered}$$

to terms containing sines and cosines of multiples of x, thus proving eq. (19).

3. Prove that the expression

$$1 - 2\alpha \cos \theta + \alpha^2$$

may be resolved into the two factors $(1 - \alpha e^{\theta i})(1 - \alpha e^{-\theta i})$.

4. Resolve the expression $x^{2n} - 2x^n \cos \theta + 1$ into the product of two factors, as in the last example.

Trigonometric Forms of Imaginary Expressions.

85. It is shown in algebra that an imaginary or complex expression may be reduced to a certain number of real units *plus* a certain number of imaginary units. If we put

i, the imaginary unit, $= \sqrt{-1}$,
a, the number of real units,
b, the number of imaginary units,

the complex expression will be

$$a + bi. \tag{1}$$

We have already shown (§ 47) that, whatever be the numbers a and b, we can find a positive number r and an angle φ, such that

$$r \cos \varphi = a;$$
$$r \sin \varphi = b.$$

If we substitute these values of r and φ in (1) it will become

$$a + bi = r(\cos \varphi + i \sin \varphi).$$

But equation (14) gives

$$\cos \varphi + i \sin \varphi = e^{\phi i}.$$

Therefore $$a + bi = re^{\phi i}. \tag{2}$$

We hence conclude:

Every complex expression can be reduced to the form

$$re^{\phi i},$$

which is called the general form of the complex expression.

The coefficient r is called the **modulus** of the expression.

A yet better term, used by the Germans, is the "absolute value" of the expression.

The angle φ is called the **argument** of the expression.

Example. Reduce the expression

$$-0.9223 + 1.0962i$$

to the general form.

Putting $$r \cos \varphi = -0.9223,$$
$$r \sin \varphi = 1.0962,$$

and applying the process of § 47, we find

$$r = 1.4326;$$
$$\varphi = 130° \; 4'.54.$$

This process being purely algebraic, the angle φ should be expressed in radial units. Reducing to this unit, we find

$$\varphi = 2.2703.$$

Therefore the required general form is

$$-0.9223 + 1.0962i = 1.4326e^{2.2703i}.$$

The student who is acquainted with the geometric representation of imaginary quantities will see that the quantity r corresponds to the modulus and φ to the angle of the complex expression as defined in algebra.

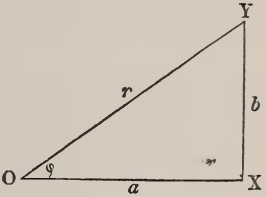
The geometric construction of the expression $a + bi$ is effected by laying off the length a on the axis of X, and at the end of this length erecting a perpendicular equal to b.

If O be the origin, we shall have

$$OX = a;$$
$$XY = b.$$

Then joining OY we shall have

$$OY = r;$$
$$\text{Angle } XOY = \varphi.$$



86. *Multiplication of complex expressions in the general form.* If any two complex expressions are

$$re^{\phi i} \text{ and } qe^{\theta i},$$

we have by multiplying them

$$rqe^{(\phi+\theta)i}.$$

This is another complex expression of the general form of which rq is the modulus and $\varphi + \theta$ the argument. Hence:

The modulus of a product is equal to the product of the moduli of the factors.

The argument of a product is the sum of the arguments of the factors.

If we multiply n equal factors, each represented by $re^{\phi i}$, the result will be

$$(re^{\phi i})^n = r^n e^{n\phi i}.$$

Hence:

The modulus of a power is equal to the corresponding power of the modulus of the root.

The argument of the power is the argument of the root multiplied by the index of the power.

87. *Periodicity of the imaginary exponential.* From the known equations (§ 24)

$$\cos(\varphi + 2\pi) = \cos\varphi,$$
$$\sin(\varphi + 2\pi) = \sin\varphi,$$

and the following equations given by the preceding theory,

$$e^{(\phi + 2\pi)i} = \cos(\varphi + 2\pi) + i\sin(\varphi + 2\pi),$$
$$e^{\phi i} = \cos\varphi + i\sin\varphi,$$

we have

$$re^{(\phi + 2\pi)i} = re^{\phi i};$$

that is:

The value of a complex quantity remains unaltered when we increase its argument by a circumference.

Since the addition of one circumference does not change it, the addition of any number of circumferences will still leave it unchanged. Hence:

If the argument of a complex quantity increases indefinitely, the values of the quantity itself will repeat themselves with every circumference by which the argument increases.

A quantity whose value repeats itself in this way is said to be **periodic.**

88. Let us next inquire for what special values of φ the exponential function $e^{\phi i}$ will be equal to the real or imaginary unit. Considering again the equation

$$e^{\phi i} = \cos\varphi + i\sin\varphi,$$

we notice that $\sin\varphi = 0$ whenever φ is a multiple of 180° or of π. When the multiple of π is even, we have $\cos\varphi = +1$; and when it is odd, $\cos\varphi = -1$. Hence, putting

$$\varphi = \pi,\ 2\pi,\ 3\pi, \text{ etc.},$$

we have

$$\left.\begin{array}{ll} e^{\pi i} = -1; & e^{-\pi i} = -1; \\ e^{2\pi i} = +1; & e^{-2\pi i} = +1; \\ e^{3\pi i} = -1; & e^{-3\pi i} = -1; \\ \text{etc.} & \text{etc.} \end{array}\right\} \qquad (a)$$

In order that $\cos\varphi$ may vanish, the angle φ must be 90°, 270°, 450°, etc.; that is, it must be an odd multiple of $\frac{1}{2}\pi$. Sin φ will then be $+1$ or -1. Putting $\varphi = \frac{1}{2}\pi$, $\varphi = \frac{3}{2}\pi$, $\varphi = \frac{5}{2}\pi$, etc., on both sides of the preceding equation, we have

$$\left.\begin{array}{ll} e^{\frac{1}{2}\pi i} = +i; & e^{-\frac{1}{2}\pi i} = -i; \\ e^{\frac{3}{2}\pi i} = -i; & e^{-\frac{3}{2}\pi i} = +i; \\ e^{\frac{5}{2}\pi i} = +i; & e^{-\frac{5}{2}\pi i} = -i. \end{array}\right\} \qquad (b)$$

By squaring each of these equations we shall reproduce the alternate equations (a).

89. *Roots of unity.* The foregoing theory enables us to find very simple and elegant expressions for the roots of the equation

$$x^n - 1 = 0,$$

or

$$x^n = 1.$$

From the general theory of equations, the equation $x^n - 1 = 0$, being of the nth degree, must have n roots; that is, there are n quantities which, being raised to the nth power, will produce 1.

These quantities are called the nth *roots of unity.*

Because 1^n is always 1, whatever be n, $+1$ is itself one of the nth roots of unity.

Because $(-1)^n = 1$ when n is even, -1 is always an nth root of unity when n is even.

Hence one or two of the n roots of unity, viz. $+1$ and -1, are real; all the others are imaginary.

90. PROBLEM. *To find the nth roots of unity.*

Solution. Let a required root be $re^{\theta i}$, r and θ being quantities to be determined. By the requirements of the problem, the nth power of this quantity must be 1. Its nth power is

$$(re^{\theta i})^n = r^n e^{n\theta i} = r^n (\cos n\theta + i \sin n\theta).$$

In order that this expression may be equal to unity, a real quantity, the coefficient of i must vanish, and we must have

$$\sin n\theta = 0,$$

which gives

$$\cos n\theta = 1.$$

Hence

$$r^n = 1,$$

which is satisfied by supposing

$$r = 1.$$

We must also have

$$n\theta = 0 \text{ or } 2\pi \text{ or } 4\pi \text{ or } 6\pi, \text{ etc.}$$

Dividing by n, we see that θ may have any one of the values

$$\theta = 0,$$

$$\theta = 2\frac{\pi}{n},$$

$$\theta = 4\frac{\pi}{n},$$

$$\theta = 6\frac{\pi}{n},$$

etc.

By substituting, in the assumed expression, $re^{\theta i}$ for the value of the root, we have

$$n\text{th roots of } 1 = 1,\ e^{\frac{2\pi}{n}},\ e^{\frac{4\pi}{n}},\ e^{\frac{6\pi}{n}},\ \text{etc.}$$

Reducing to the trigonometric form, these expressions become

$$1;$$

$$\cos 2\frac{\pi}{n} + i \sin 2\frac{\pi}{n};$$

$$\cos 4\frac{\pi}{n} + i \sin 4\frac{\pi}{n};$$

$$\cos 6\frac{\pi}{n} + i \sin 6\frac{\pi}{n};$$

etc. etc.

The angle increases by $2\frac{\pi}{n}$ with each root, and by writing n consecutive values we shall be carried all round the circle.

The solution which we thus reach may be represented thus:

Divide the circle into n equal arcs.

Let the length of each arc be α, so that $n\alpha = 360° = 2\pi$.

The nth roots of unity will be:

$$\left.\begin{array}{l} \cos 0 \;\; + i \sin 0 = 1; \\ \cos \alpha \;\; + i \sin \alpha; \\ \cos 2\alpha + i \sin 2\alpha; \\ \cos 3\alpha + i \sin 3\alpha; \\ \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ \cos (n-1)\alpha + i \sin (n-1)\alpha. \end{array}\right\} \qquad (3)$$

Example. To find the sixth roots of 1. Here

$$n = 6; \qquad \frac{\pi}{n} = \frac{180°}{n} = 30°; \qquad 2\frac{\pi}{n} = 60° = \alpha.$$

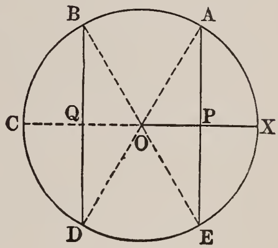
Hence the six roots are

$$\left.\begin{aligned}
&1;\\
&\cos 60^\circ + i \sin 60^\circ = \frac{1}{2} + \frac{\sqrt{3}}{2} i;\\
&\cos 120^\circ + i \sin 120^\circ = -\frac{1}{2} + \frac{\sqrt{3}}{2} i;\\
&\cos 180^\circ + i \sin 180^\circ = -1;\\
&\cos 240^\circ + i \sin 240^\circ = -\frac{1}{2} - \frac{\sqrt{3}}{2} i;\\
&\cos 300^\circ + i \sin 300^\circ = \frac{1}{2} - \frac{\sqrt{3}}{2} i.
\end{aligned}\right\} \quad (4)$$

The result can be readily proved by raising each of these quantities to the sixth power.

The roots may also be constructed as in the anexed figure.

If angle $XOA = AOB =$ etc. $= 60^\circ$, then, since $\frac{PA}{OX} =$ $\sin 60^\circ$ and $\frac{OP}{OX} = \cos 60^\circ$,



$$\frac{OX}{OX} = 1,$$

$$\frac{OP}{OX} + \frac{PA}{OX} i,$$

$$\frac{OQ}{OX} + \frac{BQ}{OX} i,$$

etc. etc.,

are the sixth roots of unity.

Exercises.

1. Find and construct the eighth roots of unity, or the roots of the equation $x^8 - 1 = 0$.

2. Find the roots of the equation $x^{12} - 1 = 0$.

91. *Relations between the roots of unity.* If we represent by x any such quantity as $\cos \alpha + i \sin \alpha$, we have, by what precedes,

$$x^2 = \cos 2\alpha + i \sin 2\alpha;$$
$$x^3 = \cos 3\alpha + i \sin 3\alpha;$$
$$x^n = \cos n\alpha + i \sin n\alpha.$$

Hence the formation of the powers of x may be represented geometrically by laying off equal arcs around a circle.

If x is any nth root of unity, then measuring off its angle α n times will bring us back to the starting-point.

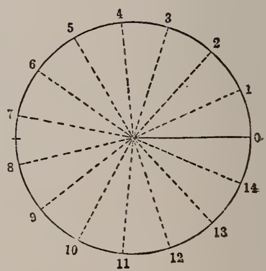
If α is itself the nth part of the circumference, then the remaining roots as given in (3) are the first n powers of x. Hence:

All the roots of unity are powers of the root corresponding to the smallest arc.

From this it follows that if we measure off with a pair of dividers, from 0 to any division-point, the mth, for instance, and repeat the measure n times, the nth measure will end at the zero-point.

This is evident of itself, because n measures of m arcs each will measure off mn arcs; and because n of the arcs make up a circumference, the mn arcs will extend around the circle exactly m times.

But it does not follow that any such series of n measures will include all the roots. Suppose, for example, that in the preceding figure, where $n = 15$, we measure arcs of 6α. The 15 successive points reached with the dividers will then be 0, 6, 12, 3, 9, 0, 6, 12, 3, 9, 0, 6, 12, 3, 9, 0. This series includes only 5 of the points of division, each of these 5 being repeated 3 times, while the remaining 10 have not been included at all.



If we take the measure 4α in our dividers, the points of division included in the series will be

0, 4, 8, 12, 1, 5, 9, 13, 2, 6, 10, 14, 3, 7, 11, 0,

which comprise all n points. Hence in this case all the roots are powers of x^4 or of $\cos 4\alpha + i \sin 4\alpha$.

92. We now have the following proposition, where we put

$$\alpha = \frac{360^\circ}{n}:$$

If m is prime to n, all the nth roots of unity may be represented as powers of $\cos m\alpha + i \sin m\alpha$.

Proof. Starting from any point of the circle, and measuring off equal arcs, each of length $m\alpha$, let p be the smallest number of measures which will bring us back to the starting-point. The total length of arc measured off will then be $pm\alpha$.

Since we are brought back to the starting-point, we must have measured off an entire number of circumferences. Let q be that entire number.

Because each circumference $= n\alpha$, the whole q circumferences measured $= qn\alpha$. Therefore

$$pm\alpha = qn\alpha,$$

and
$$\frac{p}{q} = \frac{n}{m}.$$

Because, by hypothesis, m and n are prime to each other, the fraction $\frac{n}{m}$ is irreducible, and the smallest values of p and q are n and m respectively.

Therefore any n measures will end at n *different* points of the circumference, and will therefore include all n points.

Def. A root of unity whose powers include all other roots is called a *primitive root.*

Cor. If n is a prime number, all the roots are primitive roots.

Exercises.

1. The 15th roots of unity being

$$\cos 24° + i \sin 24°, \qquad \cos 48° + i \sin 48°, \qquad \text{etc.,}$$

it is required to find which of these roots are primitive.

Prove the following propositions:

2. If n is a prime number, all the nth roots of unity are primitive roots.

3. If x be any primitive nth root of unity, and if p be any number prime to n, then x^p will also be a primitive root.

Note. In the preceding theorem this is proved for the case when x is the root corresponding to the smallest angle. The proposition now enunciated extends to the case in which we start from any multiple of this angle prime to n.

PART II.

SPHERICAL TRIGONOMETRY.

CHAPTER I.

FUNDAMENTAL PRINCIPLES.

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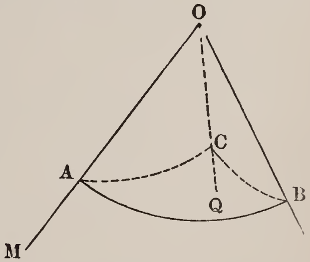
93. *Def.* Spherical trigonometry treats of the relations among the six parts of a trihedral angle.

Def. The six parts of a trihedral angle are its three face-angles and its three edge-angles.

94. *Representation of a trihedral angle by a spherical triangle.* If O be the vertex of a trihedral angle, and OM, OP, and OQ its three edges, we may construct a sphere having its centre at O, and having an arbitrary radius OA. The spherical surface will then cut the edges at the three points A, B, and C equally distant from O.



The three faces OMP, OPQ, OQM will intersect the spherical surface in three arcs of great circles, AB, BC, CA, which arcs form a spherical triangle.

It is shown in geometry that the three angles A, B, and C of the spherical triangle are equal to the respective edge-angles OM, OP, and OQ of the trihedral angle. It is also shown that the arcs AB, BC, and CA, which form the sides of the triangle, measure the respective face-angles MOP, POQ, QOM of the trihedral angle.

Therefore the six parts of the trihedral angle are represented by the corresponding parts of the spherical triangle, and the relations among the parts of the one are the same as the relations among the parts of the other.

The term *spherical trigonometry* is applied because the investigations are generally made by means of the spherical triangle.

A trihedral angle, with its corresponding spherical triangle, may be readily constructed as follows: Cut a circular disk of pasteboard or stiff paper, from four to six inches or more in diameter. From this disk cut out a sector of any magnitude. It will be well to have several disks with sectors ranging from 45° to 200° cut out. Divide the remainder of the disk by two radii into three sectors, such that the greatest shall be less than the sum of the other two. Bend the disk along each of the two dividing radii, cutting the latter part of the way through if necessary, and bring the extreme radii together. We shall then have a figure like *O-ABC* of the preceding diagram, the three plane sides forming the trihedral angle, and the three arcs bounding the edge of the disk forming the spherical triangle.

95. *General remarks upon spherical triangles.* A spherical triangle may be defined as that figure which is formed by joining any three points on the surface of a sphere by arcs of great circles. The three points will then be the vertices of the triangle.

But between any two points we may draw two arcs of a great circle, which together make up a complete great circle through the points. One of these arcs will be less, the other greater, than 180°. To avoid ambiguity, the arc less than 180° is supposed to be taken, unless otherwise expressed. We therefore adopt the rule:

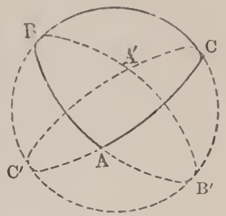
Each side of a spherical triangle is supposed less than 180°, *unless otherwise expressed.*

This rule is a mere convention, which may be set aside whenever we desire to give greater generality to our conclusions. Nothing prevents us from supposing ourselves to pass from one vertex to another by passing several times around the sphere. The corresponding side of the triangle will then consist of several coincident great circles *plus* either of the arcs joining the vertices. If we suppose a to be the shorter arc joining two vertices, the general arc measure of the side through those vertices will be

$$n\,360° + a \qquad \text{or} \qquad (n+1)\,360° - a.$$

96. Every spherical triangle encloses a portion of the spherical surface, forming the area of the triangle. We then have the theorem:

Three great circles divide the surface of the sphere into eight triangular portions.



The eight spherical triangles formed by three great circles.

This is shown as follows: One great circle divides the surface into two equal parts. A second great circle intersects the first in two points, and divides each of those parts into two lunes, so that the whole surface is then divided into four lunes. A third great circle cuts through all four of these lunes, and forms eight spherical triangles.

In the same way, any two planes divide the space around their line of intersection into four parts. A third plane intersecting them divides the space around their point of intersection into eight parts, forming eight trihedral angles.

Remark. The student should guard himself against considering a figure of which either side is a small circle of the sphere as a spherical triangle. For example, the figure formed by two arcs of meridians and a parallel of latitude is not a spherical triangle. Such figures do not represent the parts of a trihedral angle, and so do not correspond to the definition of a spherical triangle. All the important problems connected with them may be reduced to problems of spherical trigonometry, so that there is no need of giving them special consideration.

EXERCISES.

The following exercises are introduced to test the student's fundamental conceptions of spherical geometry, and especially of the relations of great circles of the sphere. Their successful performance will show that he is prepared to take up the subject of spherical trigonometry with advantage. A globe, on which figures may be drawn at pleasure, will be of great service in assisting his conceptions, and should be made use of whenever practicable.

1. A and A' are two opposite points on a sphere. If any third point X be taken on the sphere, to what constant arc will the sum $XA + XA'$ be equal, and what will be the angle AXA'?

NOTE. Opposite points are those at the ends of a diameter.

2. If one side of a spherical triangle be equal to a semicircle, what relations will then subsist between the other two sides? What will be the magnitude of the opposite angle?

3. Let A, B, and C be the three vertices of a spherical triangle; a, b, and c, the sides opposite these vertices respectively; A', B', and C', the points opposite the vertices. It is then required:

(*a*) To express by the letters at their vertices the eight triangles which will be formed when each side of the original triangle ABC is produced into a great circle.

(*b*) To express the sides of each of these eight triangles in terms of a, b, and c, making use of the theorem that any two great circles intersect each other in two opposite points.

(*c*) To express the angles in terms of the angles of the original triangle, which we may represent by the letters A, B, and C marking their vertices.

(*d*) It being found that the eight triangles are divisible into four pairs, such that the sides and angles of each pair are equal, it is required to show the relations of each pair.

4. If one angle of a spherical triangle is A, show that the sum of the other two angles is contained between the limits $180° - A$ and $180° + A$.

NOTE. If the student finds any difficulty in this question he may begin by supposing the triangle to be isosceles, and the two equal sides to increase from 0° to 180°.

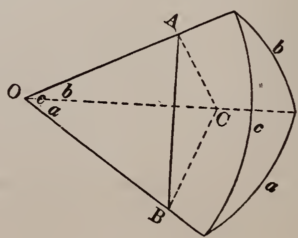
5. Hence show that the spherical excess cannot exceed twice the smallest angle.

6. If the three sides of an equilateral spherical triangle be continually and equally increased, what is the limit of their sum? What is the limit of the angles as the sum approaches its limit?

97. *Fundamental equations.* Let us put

a, b, c, the three face-angles of the trihedral angle—that is, the angles subtended by the three sides of the spherical triangle;

A, B, C, the opposite edge-angles of the trihedral angle, or the angles of the spherical triangle.



Then if O-ABC be any trihedral angle, we shall have

$$a = \text{angle } BOC;$$
$$b = \text{angle } COA;$$
$$c = \text{angle } AOB.$$

Through any point A of OA pass a plane perpendicular to OA, and let B and C be the points in which it meets the other two edges. We shall then have

$$A = \text{angle } BAC,$$

while OAB and OAC will both be right angles.

In the triangle BOC we have

$$BC^2 = OB^2 + OC^2 - 2OB \, . \, OC \cos a. \qquad (\S\ 58)$$

In the triangle BAC we have

$$BC^2 = AB^2 + AC^2 - 2AB \, . \, AC \cos A.$$

Equating these two values of BC^2, and transposing, we find

$$2OB \, . \, OC \cos a = OB^2 - AB^2 + OC^2 - AC^2 + 2AB \, . \, AC \cos A.$$

But, because OAB and OAC are right angles,

$$OB^2 - AB^2 = OA^2;$$
$$OC^2 - AC^2 = OA^2.$$

Substituting these values, and dividing by $2OB \, . \, OC$, we have

$$\cos a = \frac{OA}{OB} \cdot \frac{OA}{OC} + \frac{AB}{OB} \cdot \frac{AC}{OC} \cos A.$$

Now

$$\frac{OA}{OB} = \cos c;$$
$$\frac{OA}{OC} = \cos b;$$
$$\frac{AB}{OB} = \sin c \cdot$$
$$\frac{AC}{OC} = \sin b.$$

Therefore $$\cos a = \cos b \cos c + \sin b \sin c \cos A. \qquad (a)$$

By treating the other edges in order in the same way, we obtain two more equations, which may be written by simply permuting a, b, and c and A, B, and C *circularly*; that is, by substituting for each letter the one next in order, a following c. Thus we have the system of three equations:

$$\left.\begin{aligned} \cos a &= \cos b \cos c + \sin b \sin c \cos A; \\ \cos b &= \cos c \cos a + \sin c \sin a \cos B; \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C. \end{aligned}\right\} \qquad (1)$$

These three equations are the fundamental equations of spherical trigonometry, because by means of them, when three parts are given, the other three may be found. For practical application they are transformed and simplified in numerous ways.

98. *Permutation of parts.* We have deduced the equations (1) from (a) by merely permuting the letters. This process may be applied generally in accordance with the following theorem:

If we have found between the parts of a spherical triangle any equation which is true for all triangles, it will remain true when we permute the sides in any way; provided that we also permute the opposite angles in the same way.

For if we have proved our equation by calling the three sides a, b, and c, and the opposite angles A, B, and C, we could apply the same proof to the other parts of the triangle, substituting

Side a for side b, and *vice versa*,

and

Angle A for angle B, and *vice versa*,

in the demonstration. We should then have a result in which a and b changed places, and A and B changed places.

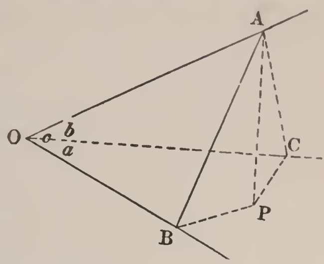
By interchanging a and c, and b and c, with their opposite angles in the same way, we should form all the six equations which could be written by permuting the symbols in the way described.

If, however, we made any supposition respecting any side or angle such that the reasoning applied to it would not apply to the others, then the symbol of this side or angle could not be permuted. For instance, we cannot permute all the parts in a formula true only for right triangles.

It follows from this that any true formula which expresses the value of one part in terms of the two remaining pairs of parts must be symmetrical with respect to the other pairs of parts. For example, equation a remains unchanged when we interchange b and c, else it would be wrong.

99. Theorem of sines. *In a spherical triangle the sines of the sides are proportional to the sines of the opposite angles.*

Proof. Let $O\text{-}ABC$ be the trihedral angle of the spherical triangle, and let A be any point on the edge OA.



Through A pass a plane perpendicular to the edge OB, intersecting the faces AOB and BOC in the lines AB and BP.

Through A pass another plane perpendicular to OC, intersecting the faces AOC and COB in the lines AC and CP.

AP will then be the line of intersection of these planes.

Because the planes ABP and ACP are perpendicular to the lines OB and OC respectively, they are each perpendicular to the plane BOC of these lines (*Geom.*) Therefore their line of intersection, AP, is also perpendicular to this plane, and the triangles APB and APC are right-angled at P. Hence

$$AP = AB \sin ABP = AB \sin B.$$

Also, because AB is perpendicular to OB,

$$AB = OA \sin BOA = OA \sin c.$$

Therefore $AP = OA \sin c \sin B$.

We find in the same way

$$AP = OA \sin b \sin C;$$

whence

$$\sin c \sin B = \sin b \sin C,$$

or

$$\frac{\sin c}{\sin C} = \frac{\sin b}{\sin B}.$$

We may show in the same way, by permuting the parts,

$$\frac{\sin a}{\sin A} = \frac{\sin b}{\sin B} = \frac{\sin c}{\sin C}. \qquad (2)$$

The common value of these three ratios is called the *modulus* of the spherical triangle.

100. The theorem of sines may also be obtained directly from the fundamental equations as follows:

From the first fundamental equation (1) we obtain

$$\cos A = \frac{\cos a - \cos b \cos c}{\sin b \sin c}.$$

Squaring,

$$\cos^2 A = \frac{\cos^2 a - 2\cos a \cos b \cos c + \cos^2 b \cos^2 c}{\sin^2 b \sin^2 c}$$

Hence

$$\sin^2 A = 1 - \cos^2 A =$$

$$\frac{(1-\cos^2 b)(1-\cos^2 c) - \cos^2 a + 2\cos a \cos b \cos c - \cos^2 b \cos^2 c}{\sin^2 b \sin^2 c}$$

$$= \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 b \sin^2 c}.$$

Dividing by $\sin^2 a$,

$$\frac{\sin^2 A}{\sin^2 a} = \frac{1 - \cos^2 a - \cos^2 b - \cos^2 c + 2\cos a \cos b \cos c}{\sin^2 a \sin^2 b \sin^2 c}.$$

The second member of this equation is symmetrical with respect to a, b, and c, and so remains unchanged when these quantities are permuted among themselves. But if we derive the values of $\frac{\sin^2 B}{\sin^2 b}$ and $\frac{\sin^2 C}{\sin^2 c}$ from the last two fundamental equations, the results will be simple permutations of the last equation, and will therefore give the same values of $\frac{\sin^2 B}{\sin^2 b}$ and $\frac{\sin^2 C}{\sin^2 c}$ that we have found for $\frac{\sin^2 A}{\sin^2 a}$.

Hence

$$\frac{\sin^2 A}{\sin^2 a} = \frac{\sin^2 B}{\sin^2 b} = \frac{\sin^2 C}{\sin^2 c}.$$

Extracting the square roots, the general results will have double ($\pm$) algebraic signs; but as the angles are all supposed to be less than 180°, the positive signs are to be taken. Hence

$$\frac{\sin A}{\sin a} = \frac{\sin B}{\sin b} = \frac{\sin C}{\sin c},$$

the reciprocal of the relations (2).

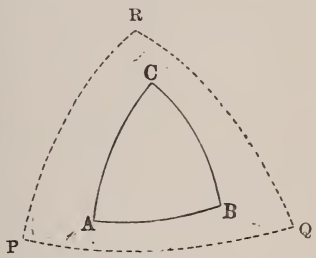
101. *Polar triangles. Def.* When two triangles are so related that the vertices of the one are the poles of the sides of the other, the one is said to be the **polar triangle** of the other.

It is shown in geometry that the relation of a triangle to its polar triangle is reciprocal; that is, if X and Y are two triangles, and Y is the polar triangle of X, then X is the polar triangle of Y. This reciprocity arises from the theorem:

If A, B, and C be the three poles of the sides QR, RP, and PQ of a triangle PQR, then P, Q, and R will be the poles of the sides BC, CA, and AB.

This theorem is readily proved by the geometry of the sphere.

Since every great circle has two poles, one at each end of a diameter, it follows that the three sides of a triangle have six poles in all. We may form a polar triangle to ABC by taking either of the poles of AB, either of the poles of BC, and either of the poles of CA, and joining them by arcs of great circles. Hence there are eight possible polar triangles to every given triangle. To avoid doubt which triangle is to be chosen, we take for each vertex of the polar triangle that pole of each side of the given triangle which is on the side toward the triangle.



For example, if ABC is the given triangle, we take that pole of AB which is on the side toward C, and so with the other sides.

Exercises.

1. What must be the sides and angles of a triangle that it may coincide with its polar triangle?

2. Show that if each side of a triangle is greater than 90° the polar triangle will fall wholly inside of it, and if each side is less than 90° it will be wholly within its polar triangle.

3. If two sides exceed 90° and the third side is less than 90°, what will be the character of the polar triangle, and how will it be situated relatively to the given one?

102. *Use of the polar triangle.* It is shown in geometry that each side of the polar triangle is the supplement of the opposite angle of the other, and *vice versa*. This principle is applied to find new relations between the parts of a triangle in the following way:

1. We imagine ourselves to construct the polar of the given triangle.

2. We write any or all the equations between the parts of the polar triangle.

3. We substitute in these equations the supplementary parts of the given triangle, and thus obtain equations between these parts.

Let us put $$a',\ b',\ c',$$ $$A',\ B',\ C',$$

the sides and opposite angles of the polar triangle. Since the general equations (1) are true for every triangle, they are true of this polar triangle. Hence

$$\cos a' = \cos b' \cos c' + \sin b' \sin c' \cos A'.$$

But the polar triangle is so related to the original triangle that

$$a' = 180° - A, \qquad A' = 180° - a;$$
$$b' = 180° - B, \qquad B' = 180° - b;$$
$$c' = 180° - C, \qquad C' = 180° - c.$$

Therefore

$$\cos a' = -\cos A, \qquad \cos A' = -\cos a;$$
$$\cos b' = -\cos B, \qquad \cos B' = -\cos b;$$
$$\cos c' = -\cos C, \qquad \cos C' = -\cos c;$$

and $$\sin a' = \sin A, \qquad \sin A' = \sin a;$$
etc. etc.

Making these substitutions in the equations (1), we find

$$\left.\begin{aligned} \cos A &= -\cos B \cos C + \sin B \sin C \cos a; \\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b; \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c. \end{aligned}\right\} \qquad (3)$$

This process may be generalized thus:

From every relation between the parts of a spherical triangle we may derive another relation by changing the cosine of each part into the negative of the cosine of the opposite part, and the sine of each part into the sine of the opposite part.

But this relation will not always be different from the original one. If we apply the process to the equations (2), for instance, the same relations will be reproduced, each term being changed to its reciprocal.

It is also to be remarked that the use of the polar triangle is not absolutely necessary to deduce the new relations (3), because they can all be obtained from the fundamental equations (1) by eliminating first b and c, then c and a, then a and b. But the use of the polar triangle gives a much shorter and more elegant mode of deducing them.

103. In the solution of spherical triangles we require formulæ which shall express any three parts in terms of the remaining three; the latter being supposed known, the former unknown.

A set of such equations may be derived from the fundamental equations (1) by substituting in any one the value of the cosine of a side obtained from another. Let us substitute in the third the value of $\cos a$ from the first. We shall have

$$\cos c = \cos^2 b \cos c + \sin b \cos b \sin c \cos A + \sin a \sin b \cos C.$$

Transposing $\cos c$, noting that $1 - \cos^2 b = \sin^2 b$, and dividing by $\sin b$, we have

$$0 = -\sin b \cos c + \cos b \sin c \cos A + \sin a \cos C,$$

which gives

$$\sin a \cos C = \sin b \cos c - \cos b \sin c \cos A. \qquad (4)$$

The theorem of sines (2) also gives

$$\sin a \sin C = \sin c \sin A. \qquad (5)$$

Comparing these two last equations with the first equation (1), we see that they form a set in which the second members contain only the parts b, c, and A, namely, two sides and the included angle; while the first members contain the third side, a, and one of the angles adjacent to it, namely, C.

Hence any two of these three equations may be used to find the side a and the angle C when b, c, and A are given. Since there are three equations where only two are necessary, there must be a relation between them, which we find as follows:

The first members are

$$\cos a;$$
$$\sin a \cos C;$$
$$\sin a \sin C.$$

The sum of the squares of these quantities is

$$\cos^2 a + \sin^2 a(\cos^2 C + \sin^2 C) = \cos^2 a + \sin^2 a = 1.$$

The sum of the squares of the first members being identically equal to unity, the same should be true of the sum of the squares of the second members, which is

$$(\cos b \cos c + \sin b \sin c \cos A)^2$$
$$+ (\sin b \cos c - \cos b \sin c \cos A)^2$$
$$+ \sin^2 c \sin^2 A.$$

Forming these squares, we see that the product of the terms in the squares of the first two quantities cancel each other, leaving as the sum of the squares

$$\cos^2 b \cos^2 c + \sin^2 b \sin^2 c \cos^2 A$$
$$+ \sin^2 b \cos^2 c + \cos^2 b \sin^2 c \cos^2 A$$
$$+ \sin^2 c \sin^2 A,$$

a sum which we readily find to reduce to unity.

104. By permuting the sides b and c and their opposite angles, B and C, we obtain a similar set of equations for a and the adjacent angle B. Repeating the equations already found, we have the following set of five equations from which we may determine a, B, and C, when b, c, and A are given:

$$\left.\begin{aligned} \sin a \sin B &= \sin b \sin A; \\ \sin a \cos B &= \cos b \sin c - \sin b \cos c \cos A; \\ \sin a \sin C &= \sin c \sin A; \\ \sin a \cos C &= \sin b \cos c - \cos b \sin c \cos A; \\ \cos a &= \cos b \cos c + \sin b \sin c \cos A. \end{aligned}\right\} \quad (6)$$

We may write a set similar to this for each of the other sides of the triangle, namely:

$$\left.\begin{aligned} \sin b \sin C &= \sin c \sin B; \\ \sin b \cos C &= \cos c \sin a - \sin c \cos a \cos B; \\ \sin b \sin A &= \sin a \sin B; \\ \sin b \cos A &= \sin c \cos a - \cos c \sin a \cos B; \\ \cos b &= \cos c \cos a + \sin c \sin a \cos B. \end{aligned}\right\} \quad (7)$$

$$\left.\begin{aligned} \sin c \sin A &= \sin a \sin C; \\ \sin c \cos A &= \cos a \sin b - \sin a \cos b \cos C; \\ \sin c \sin B &= \sin b \sin C; \\ \sin c \cos B &= \sin a \cos b - \cos a \sin b \cos C; \\ \cos c &= \cos a \cos b + \sin a \sin b \cos C. \end{aligned}\right\} \quad (8)$$

Then we obtain a similar set for the angles through the intervention of the polar triangle. Applying the set (6) to the polar triangle, it gives

$$\left.\begin{aligned} \sin A \sin b &= \sin B \sin a; \\ \sin A \cos b &= \cos B \sin C + \sin B \cos C \cos a; \\ \sin A \sin c &= \sin C \sin a; \\ \sin A \cos c &= \sin B \cos C + \cos B \sin C \cos a; \\ \cos A &= -\cos B \cos C + \sin B \sin C \cos a. \end{aligned}\right\} \quad (9)$$

Applying these formulæ to each of the angles in succession, we have a set of equations by which, when two angles and the included side are given, the remaining parts may be found. The following are the remaining formulæ obtained by permutation:

$$\left.\begin{aligned} \sin B \sin c &= \sin C \sin b; \\ \sin B \cos c &= \cos C \sin A + \sin C \cos A \cos b; \\ \sin B \sin a &= \sin A \sin b; \\ \sin B \cos a &= \sin C \cos A + \cos C \sin A \cos b; \\ \cos B &= -\cos C \cos A + \sin C \sin A \cos b. \end{aligned}\right\} \quad (10)$$

$$\left.\begin{aligned} \sin C \sin a &= \sin A \sin c; \\ \sin C \cos a &= \cos A \sin B + \sin A \cos B \cos c; \\ \sin C \sin b &= \sin B \sin c; \\ \sin C \cos b &= \sin A \cos B + \cos A \sin B \cos c; \\ \cos C &= -\cos A \cos B + \sin A \sin B \cos c. \end{aligned}\right\} \quad (11)$$

105. If in the three sets of equations (6) to (8) we divide the second equation by the first, and the fourth by the third, and clear of denominators, we shall have the following additional equations:

$$\left.\begin{aligned} \cot B \sin A &= \cot b \sin c - \cos c \cos A; \\ \cot C \sin A &= \cot c \sin b - \cos b \cos A; \\ \cot C \sin B &= \cot c \sin a - \cos a \cos B; \\ \cot A \sin B &= \cot a \sin c - \cos c \cos B; \\ \cot A \sin C &= \cot a \sin b - \cos b \cos C; \\ \cot B \sin C &= \cot b \sin a - \cos a \cos C. \end{aligned}\right\} \quad (12)$$

Treating the equations (9) to (11) in the same way, we shall get a similar set of equations; but on examination they will be found to differ from the equations (12) only in the arrangement of their terms, so that they need not be written.

106. Although the foregoing equations need to be transformed for most of their uses, they may in many cases be applied directly to the solution of spherical triangles. This is especially the case when two sides and the included angle, or two angles and the included side, are given and one or more of the remaining parts required. If, for instance, A, B, and c are given, we may from are the first two of equations (11) obtain the values of C and a by Prob. I. Chap. V. Then from the next two we may obtain c and C, and from the last $\cos C$, and thus a third value of C. If the work is correct, these three values of C will all agree.

Example. Two of the face-angles of a trihedral angle are $a = 132° 46'.7$ and $b = 59° 50'.1$, and the included edge-angle is $C = 56° 28'.4$. Find the remaining parts.

The computation of A, B, and c may be effected by the equations (8), as follows:

(1) $\cos b$,	9.701 13	(6) $\sin b$,	9.936 81
(2) $\sin a$,	9.865 69	(7) $\cos C$,	9.742 19
(3) $\sin C$,	9.920 97	(8) $\cos b$,	9.701 13
(4) $\sin b$,	9.936 81	(9) $\sin a$,	9.865 69
(5) $\cos a$,	− 9.831 98	(11) $\sin b \cos C$,	9.679 00
(12) $\cos a \sin b$,	− 9.768 79	(10) $\cos a$,	− 9.831 98
(13) $-\sin a \cos b \cos C$,	− 9.309 01	(20) $\sin a \cos b$,	9.566 82
(14) Diff.,	0.459 78	(21) $-\cos a \sin b \cos C$,	+ 9.510 98
(15) Add. log.,	0.589 12	(22) Diff.,	0.055 84
(2)+(3) $\sin c \sin A$,	9.786 66	(23) Add. log.,	0.329 84
(13)+(15) $\sin c \cos A$,	− 9.898 13	(26) $\sin B$,	9.857 80
(18) $\cos A$,	9.898 14	(3)+(4) $\sin c \sin B$,	9.857 78
(16) $\tan A$,	− 9.888 53	(21)+(23) $\sin c \cos B$,	9.840 82
(17) A,	142° 16′.4	(24) $\tan B$,	0.016 96
(19) $\sin c$,	9.999 99	(25) B,	46° 7′.1
		(27) $\sin c$,	9.999 98
(31) Subtr. log.,	8.431 7	(30) Diff.,	0.011 58
(28) $\cos a \cos b$,	− 9.533 11	$\cos c$,	7.964 8
(29) $\sin a \sin b \cos C$,	9.544 69	c,	89° 28′.3

NOTE.—We may omit the "log" from the designation of the logarithms of the trigonometric functions whenever no uncertainty will thus arise.

In this computation the numbers in parentheses show the order in which the lines may be written. Lines (15), (23), and (31) are the addition or subtraction logarithms, from the tables for finding the logarithm of the sum or difference of two numbers which are given by their logarithms. The student can equally well find the numbers, add them together, and take the logarithm of their sum.

The agreement of the two values of sin C with each other and with cos C shows the correctness of the calculation.

107. The following transformation, similar to that of Prob. IV. Chap. V., will often render the work convenient. In the second and last of equations (11) let us put

$$\left.\begin{aligned} k \sin K &= \sin A \cos c; \\ k \cos K &= -\cos A. \end{aligned}\right\} \qquad (a)$$

By substitution these equations will then become

$$\left.\begin{aligned} \sin C \cos a &= -k \cos K \sin B + k \sin K \cos B \\ &= k \sin (K - B); \\ \cos C &= k \cos K \cos B + k \sin K \sin B \\ &= k \cos (K - B). \end{aligned}\right\} \qquad (b)$$

To apply these equations we compute k and K from (a), and then sin C cos a and cos C from (b). We complete the work by computing sin C sin a from the first equation of (11).

We may also transform the fourth equation by computing h and H from the equations

$$\begin{aligned} h \sin H &= \sin B \cos c; \\ h \cos H &= -\cos B. \end{aligned}$$

We shall then have

$$\begin{aligned} \sin C \cos b &= h \sin (H - A); \\ \cos C &= h \cos (H - A). \end{aligned}$$

To transform the equations (8) on the same plan, we may compute k, K, h and H from

$$\left.\begin{aligned} k \sin K &= \sin a \cos C; \\ k \cos K &= \cos a; \\ h \sin H &= \sin b \cos C; \\ h \cos H &= \cos b. \end{aligned}\right\} \qquad (c)$$

We then have, in the same way as before,

$$\sin c \cos A = k \sin (b - K); \qquad (d)$$
$$\sin c \cos B = h \sin (a - H);$$
$$\cos C = k \cos (b - K) = h \cos (a - H).$$

We compute the same example as before by these formulæ, as follows:

$\sin a$,	9.865 69
$\cos C$,	9.742 19
$\sin b$,	9.936 81
$k \sin K$,	9.607 88
$k \cos K$,	— 9.831 98
$\cos K$,	— 9.933 82
$\tan K$,	— 9.775 90
K,	149° 10′.0
b,	59° 50′.1
$h \sin H$,	9.679 00
$h \cos H$,	9.701 13
$\cos H$,	9.860 27
$\tan H$,	9.977 87
H,	43° 32′.4
a,	132° 46′.7
$\cos c$,	7.965 1
$\cos c$,	7.964 5
c,	89° 28′.3
B,	46° 7′.1

$\sin a$,	9.865 69
$\sin C$,	9.920 97
$\sin b$,	9.936 81
$b - K$	— 89° 19′.9
$a - H$,	89° 14′.3
$\sin (b - K)$,	— 9.999 97
$\log k$,	9.898 16
$\cos (b - K)$,	8.066 9
$\sin c \sin A$,	9.786 66
$\sin c \cos A$,	— 9.898 13
$\cos H$,	— 9.898 14
$\tan A$,	— 9.888 53
A,	142° 16′.4
$\sin c$,	9.999 99
$\sin (a - H)$,	9.999 96
$\log h$,	9.840 86
	8.123 6
$\sin c \sin B$,	9.857 78
$\sin c \cos B$,	9.840 82
$\tan B$,	0.016 96

Exercises.

1. Transform the equations (6), (7), (9), and (10) in the same way that we have transformed (8) and (11).

2. From the values of A, B, and c, which we have obtained in the last example, find those of a, b, and C with which we started.

3. If m be the arc joining the vertex A to the opposite side, prove

$$\cos b + \cos c = 2 \cos \tfrac{1}{2}a \cos m.$$

CHAPTER II.

RIGHT AND QUADRANTAL TRIANGLES.

Fundamental Definitions and Theorems.

108. *Def.* A right spherical triangle is one which has a right angle.

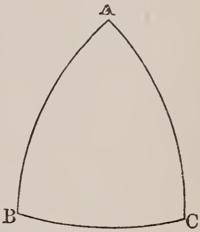
Def. A quadrantal spherical triangle is one which has a side equal to a quadrant.

Def. A trirectangular triangle is one which has three right angles.

Def. A birectangular triangle is one which has two right angles.

Def. A biquadrantal triangle is one which has two sides equal to a quadrant.

THEOREM I. *Every birectangular triangle is also biquadrantal.*



Proof. Let ABC be a spherical triangle in which angle B = angle $C = 90°$. Then:

Because angle B is a right angle, the pole of the great circle BC is on the great circle BA.

Because angle C is a right angle, this pole is on the great circle CA. (*Geom.*)

Therefore the pole of BC is on both BA and CA, and therefore at their point of intersection A.

Because A is the pole of BC, AB and AC are quadrants.

Q.E.D.

THEOREM II. Conversely, *Every biquadrantal triangle is also birectangular.*

Proof. Because every point of the polar circle of the point A is a quadrant distant from A, and because AB and AC are quadrants, this polar circle must pass through both B and C.

But only one great circle can pass through these points.

Therefore BC is the polar circle of A, and A the pole of BC.

Therefore the great circles AB and AC intersect BC at right angles. Q.E.D.

Cor. *Every trirectangular triangle has three quadrants for its sides;* and,

Conversely, *Every triangle having three quadrants for its sides is trirectangular.*

THEOREM III. *In a birectangular triangle the oblique angle is equal to its opposite side.*

Proof. Because the plane of the great circle BC intersects the planes of AB and of AC at right angles, the arc BC measures the dihedral angle between the planes AB and BC.

But the angle A is equal to this same dihedral angle.

Therefore $BC =$ angle A.

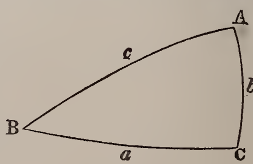
THEOREM IV. *The polar triangle of a right triangle is a quadrantal triangle.*

This follows at once from the fact that the angles of the one triangle are the supplements of the sides of the other.

EXERCISE.

Let the student translate the preceding definitions and theorems into those relating to the face- and edge-angles of a trihedral angle, and, which is the same thing, into those relating to the angles between three lines emanating from a point and the angles between their planes.

109. *Formulæ for right triangles.* Since in a right triangle one of the parts, the right angle, is known in advance, if two other parts be given the remaining three parts may be found.



An equation must therefore exist by which, when any two parts are given, any one of the three remaining parts may be found; hence between every combination of three parts out of the five there must be an equation. The number of combinations of 3 in 5 being $\frac{5 \cdot 4}{1 \cdot 2} = 10$, there must be ten such equations.

To find these equations let C be the right angle, and therefore c the hypothenuse. We seek for those equations in the sets (6) to (12) of the last chapter, in which the angle C enters, and in which the equation contains only three different parts. We then suppose

$$\sin C = 1;$$
$$\cos C = 0;$$
$$\cot C = 0.$$

The set from which the required equations are taken, the number in the set, and the result are shown as follows:

From	$(6)_3$,	$\sin a = \sin c \sin A$;	(1) ⎫
"	$(7)_1$,	$\sin b = \sin c \sin B$;	(2) ⎭
"	$(6)_4$,	$\cos A = \tan b \cot c$;	(3) ⎫
"	$(7)_2$,	$\cos B = \tan a \cot c$;	(4) ⎭
"	$(8)_5$,	$\cos c = \cos a \cos b$;	(5)
"	$(9)_2$,	$\cos B = \cos b \sin A$;	(6) ⎫
"	$(9)_5$,	$\cos A = \cos a \sin B$;	(7) ⎭
"	$(11)_5$,	$\cos c = \cot A \cot B$;	(8)
"	$(12)_5$,	$\cot A = \cot a \sin b$;	(9) ⎫
"	$(12)_6$,	$\cot B = \cot b \sin a$.	(10) ⎭

These ten equations will be found to include all combinations of three out of the five parts a, b, c, A, B. From each of them we may determine any one part in terms of the other two; for example, the first equation gives not only

$$\sin a = \sin c \sin A,$$

but

$$\sin c = \frac{\sin a}{\sin A},$$

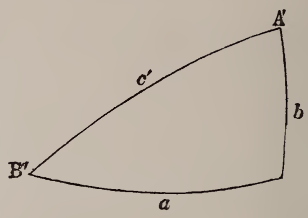
and

$$\sin A = \frac{\sin a}{\sin c}.$$

Properly speaking, only six of these equations are really distinct, as the other four can be derived from them by a mere interchange of letters between corresponding parts. For instance, since the same relation must hold between each oblique angle and its opposite side, the second equation may be derived from the first.

The equations which are thus related are connected by braces in the formulæ above.

110. *Napier's rules.* The six preceding formulæ, which may be found difficult to remember, have been included by Napier in two precepts of remarkable simplicity, and easily remembered.



Let us take for the five parts the sides a and b as before, and, instead of the other three parts, the *complements* of the oblique angles and of the hypothenuse. The fact that the complements are understood is indicated by accenting the letters in the diagram. We suppose

$$B' = 90° - B; \qquad c' = 90° - c; \qquad A' = 90° - a.$$

Omitting the right angle, the five parts a, b, A', c', B' form a continuous series, B' being followed in regular order by a. Now if we select any three of these parts, one of two cases must occur. Either—

(1) The three parts all adjoin each other, as B', a, b; a, b, A', etc., or

(2) Two of the parts adjoin each other and the third is separated from each of them by the remaining intervening parts.

The middle part of the three in the first case, or the separated part in the second, is called the *middle part.*

In the first case the extreme parts of the three are called *adjacent parts.*

In the second case the adjoining parts are called *opposite parts.*

111. Napier's rules are:

I. *The sine of the middle part equals the product of the tangents of the adjacent parts.*

II. *The sine of the middle part equals the product of the cosines of the opposite parts.*

The concurrence of the vowel a in *tangent* and *adjacent,* and of the vowel o in *cosine* and *opposite,* will help in remembering the relations.

Examples. 1. Let the parts be the hypothenuse and the two adjacent angles, or c, A and B.

The middle part is c', and A' and B' are adjacent parts.

By the rule,

$$\sin(90° - C) = \tan(90° - A)\tan(90° - B),$$

or $$\cos c = \cot A \cot B,$$

agreeing with the formula (8).

2. Let the parts be a, A and C. The middle part is then a, and A' and C' are opposite parts. Therefore

$$\sin a = \cos(90° - A)\cos(90° - C)$$
$$= \sin A \sin c,$$

agreeing with the formula (1).

3. Let the three parts be the two sides containing the right angle and one of the oblique angles, say a, b, and A. Then b is the middle part, and the other two adjacent parts. Therefore

$$\sin b = \tan a \tan(90° - A),$$

or $$\sin b \cot a = \cot A,$$

agreeing with the ninth formula.

Exercises.

1. Given $A = 62° 29'.3$, $b = 25° 58'.8$; find a.
2. Given $B = 35° 29'.6$, $a = 75° 5'.3$; find A and b.
3. Given $a = 43° 40'.5$, $c = 98° 29'.1$; find A, B, and b.
4. Given $a = 148° 28'.2$, $A = 101° 3'.9$; find b and c.
5. Given $A = 50° 0'.8$, $B = 79° 57'.3$; find a, b, and c.

112. *Relations between four parts.* Although the preceding formulæ enable us, when two parts are given, to find the remaining three parts, each part has to be found independently by different equations. If all three parts are required, we may determine two of them by a single connected set of operations. For this purpose we select the appropriate equations from the sets (6) to (11) of the preceding chapter, choosing only those in which the angle C enters the second member.

113. Case I. Given *an angle and the adjacent side.*

When $C = 90°$, the last three equations (9) of § 104 are

$$\left.\begin{aligned} \sin A \sin c &= \sin a; \\ \sin A \cos c &= \cos a \cos B; \\ \cos A &= \cos a \sin B. \end{aligned}\right\} \qquad (11)$$

From the first two equations we obtain sin A and c. Since we thus have separate values of sin A and cos A, the agreement of the two values of A serves as a check upon the accuracy of the computation.

If b is also required, the first two equations (9) of § 104 give

$$\left.\begin{aligned}\sin A \sin b &= \sin a \sin B;\\ \sin A \cos b &= \cos B.\end{aligned}\right\} \qquad (12)$$

From which we obtain b and another value of sin A.

By simply reversing equations (11) we obtain a and B when A and c are given.

Example. Given $a = 75°\ 5'.3$, $B = 35°\ 29'.6$, to find the remaining three parts.

$\sin a$,	9.985 12	
$\sin B$,	9.763 88	
$\cos a$,	9.410 49	
$\cos B$,	9.910 73	
$\sin A \sin c = \sin a$,	9.985 12	
$\sin A \cos c = \cos a \cos B$,	9.321 22	
$\tan c$,	0.663 90	$c = 77°\ 46'.0$
$\sin A$,	9.995 10	
$\cos A = \cos a \sin B$,	9.174 37	$A = 81°\ 24'.4$
$\sin A \sin b$,	9.749 00	
$\sin A \cos b$,	9.910 73	
$\tan b$,	9.838 27	$b = 34°\ 34'.2$
$\sin A$,	9.995 10	

Exercises.

1. Given $a = 34°\ 34'.2$, $B = 81°\ 24'.4$; find remaining parts.
2. Given $A = 45°\ 45'.4$, $c = 61°\ 49'.3$; find a and B.
3. Given $a = 120°\ 29'.6$, $B = 22°\ 59'.8$; find c and A.
4. Given $A = 98°\ 0'.4$, $b = 52°\ 7'.8$; find a and B.
5. Given $B = 133°\ 33'.7$, $a = 7°\ 29'.3$; find A and b.

114. Case II. Given the two sides, a and b.

Putting $C = 90°$, the equations (8) of § 104 become

$$\left.\begin{aligned}\sin c \sin A &= \sin a;\\ \sin c \cos A &= \cos a \sin b;\\ \cos c &= \cos a \cos b;\\ \sin c \sin B &= \sin b;\\ \sin c \cos B &= \sin a \cos b.\end{aligned}\right\} \qquad (13)$$

The values of c and A are determined from the first three equations; those of c and B from the last three. The agreement of the two values of sin c with the one value of cosine c affords a check upon the accuracy of the work.

Exercises.

1. Given $a = 39° \; 6'.8$, $b = 82° \, 39'.6$; find A, B, c.
2. Given $a = 103° \, 40'.2$, $b = 62° \, 29'.3$; find A, B, c.
3. Given $a = 172° \; 1'.5$, $b = 158° \; 58'.8$; find A, B, c.

115. Case III. Given the hypothenuse and one angle.

The first three equations of Case I. and the first three of Case II. give

$$\left.\begin{aligned}\cos a \sin B &= \cos A;\\ \cos a \cos B &= \sin A \cos c;\\ \sin a &= \sin A \sin c;\\ \cos a \sin b &= \cos A \sin c;\\ \cos a \cos b &= \cos c.\end{aligned}\right\} \qquad (14)$$

From which a, b, and B may be determined.

Exercises.

In a triangle, right-angled at C, prove the relations:

1. $\sin A \sin 2b = \sin c \sin 2B$.
2. $\sin 2A \sin c = \sin 2a \sin B$.
3. $\sin 2a \sin 2b = 4 \cos A \cos B \sin^2 c$.
4. $\sin^2 \frac{1}{2}c = \sin^2 \frac{1}{2}a \cos^2 \frac{1}{2}b + \sin^2 \frac{1}{2}b \cos^2 \frac{1}{2}a$.
5. $\sin (c - b) = \tan^2 \frac{1}{2}A \sin (b + c)$.
6. $\sin a \cos b = \tan \frac{1}{2}A \sin (b + c)$.
7. In a right triangle of which the oblique angles are

$$A = 69° \, 23'.7, \qquad B = 60° \, 7'.6,$$

find the length of the perpendicular from the right angle upon the base, and the angles which it forms with the sides.

8. In a right triangle is given

$$c = 75^\circ\, 25', \qquad a = 52^\circ\, 16';$$

find the lengths of the segments into which a is divided by the bisector of A.

Exercises in Geometric Application.

9. From a point P above a plane an oblique line PO is drawn, meeting the plane in O and making the angle A with the plane. Let Q be the projection of P upon the plane, so that OQ is the projection of OP. Through O a line OM is drawn, making an angle $QOM = B$ with the projection OQ. It is required to express the angle POM in terms of A and B.

Ans. $\text{Cos } POM = \cos A \cos B$.

10. In the preceding case, if a perpendicular PS be dropped from P upon OM, express the length OS in terms of the angle A and B and the length OP. *Ans.* $OS = OP \cos A \cos B$.

11. Two planes intersect at right angles along a line I. From any point R of I one line is drawn in each plane, making the respective angles A and B with I. Express the angle C between these lines. *Ans.* $\text{Cos } C = \cos A \cos B$.

12. Two planes intersecting at right angles along a line I are intersected by a third plane, making with them the respective angles P and Q. Express the angles which the three lines of intersection make with each other.

Ans. If we put PI for the angle between I and that edge along which the dihedral angle P is formed, etc., we have

$$\cos PI = \frac{\cos Q}{\sin P};$$

$$\cos QI = \frac{\cos P}{\sin Q};$$

$$\cos PQ = \cot P \cot Q.$$

116. *Isosceles triangles.* An isosceles spherical triangle may be divided into two symmetrical right triangles by a perpendicular from its vertex upon its base. If we put

c, each of the equal sides;

C, each of the equal angles at the base;

b, the base, or third side;

B, the angle at the vertex;

P, the middle point of b;

p, the length of the perpendicular BP from B upon b,—
we shall then have two right triangles in each of which the oblique angles are C and $\frac{1}{2}B$, the hypothenuse is c, and the sides containing the right angle are p and $\frac{1}{2}b$. The equations of § 109 will then give

$$\sin \tfrac{1}{2}b = \sin c \sin \tfrac{1}{2}B;$$
$$\sin p = \sin c \sin C;$$
$$\text{etc.} \qquad \text{etc.}$$

Exercises.

1. The equal sides of an isosceles triangle are each 45°, and the angle which they contain is 95°. Find the base and the angles at the base.

2. If the base of an isosceles triangle is 95°, and the angles at the base each 45°, find the remaining parts.

117. *Quadrantal triangles.* Since the polar of a right triangle is a quadrantal triangle, the formulæ for quadrantal triangles may be obtained by applying the formulæ of § 109 to the polar triangle. The side c will then be a quadrant, and the relations among the other parts will be

$$\left.\begin{aligned} \sin A &= \sin C \sin a; \\ \cos a &= -\tan B \cot C; \\ \cos C &= -\cos A \cos B; \\ \cos b &= \cos B \sin a; \\ \cos C &= -\cot a \cot b; \\ \sin B &= \tan A \cot a. \end{aligned}\right\} \qquad (15)$$

If we take, as the five parts of the triangle,

$$A,\ B,\ 90° - a,\ 90° - b,\ C - 90°, \qquad (a)$$

and omit the hypothenuse c, the above formulæ will be expressed by a set of rules identical in expression with those of Napier. For example, let us consider the parts a, b, C. Here C will be a middle part, and a and b adjacent parts. Applying Napier's rules to this case, with the parts (a) we have

$$\sin (C - 90°) = \tan (90° - a) \tan (90° - b);$$

which gives
$$-\cos C = \cot a \cot b,$$
an equation identical with the fifth of the above list.

Exercises.

1. Let the student deduce the six equations (15) by applying Napier's rules to the parts (a).

2. Through the same point there pass two lines intersecting at right angles, and a plane P making the angle α with one of the lines, and the angle β with the other. Express the angle which the plane P forms with the plane of the lines.

Ans. $\text{Sin } A = \sqrt{\sin^2 \alpha + \sin^2 \beta}$.

3. The sides of an obelisk have a slope of 8° from the perpendicular. What is the face-angle at the base of the obelisk, the slope of the edges, and the dihedral angle between two adjacent lateral faces? *Ans.* Face-angle at base, 82° 4′.6; slope, 11° 14′.5; dihedral angle, 91° 6′.6.

In this problem, to reduce to a spherical triangle, consider the centre of the sphere to be at a corner of the obelisk. The slope of the edge will not be represented by either of the six parts of the triangle, but by the complement of the perpendicular from the vertex upon the base.

4. A mason cuts a stone with a rectangular base and four lateral edges, each making an angle of 60° with the base at its corners. What is the inclination of each lateral face to the base, and the dihedral angle between the faces, supposing such inclinations and dihedral angles all equal? *Ans.* 67° 47′.5 and 98° 12′.8.

5. In another stone the base is rectangular; one lateral face makes an angle of 68° 29′ with the base, and the lateral edge bounding this face makes an angle of 52° 15′ with the base. What angles does the adjacent lateral face make with the first face and with the base?

6. When the angular distance of the sun from the south point of the horizon is 75°, and from the west point 60°, what is its altitude above the horizon?

CHAPTER III.

TRANSFORMATION OF THE FORMULÆ OF SPHERICAL TRIGONOMETRY.

118. Although the formulæ already given suffice for the solution of every spherical triangle, there are many transformations which will facilitate the applications of spherical trigonometry, and render the solutions of triangles more accurate and convenient.

Let us first take the fundamental equation (1) of Chapter I.,

$$\cos a = \cos b \cos c + \sin b \sin c \cos A.$$

We may express this in the form

$$\left.\begin{aligned}\cos a &= \cos b \cos c + \sin b \sin c - \sin b \sin c\,(1 - \cos A)\\ &= \cos (b - c) - 2 \sin b \sin c \sin^2 \tfrac{1}{2}A.\end{aligned}\right\}\quad(1)$$

Moreover, because

$$-2 \sin b \sin c = \cos (b + c) - \cos (b - c), \qquad (\S\ 43)$$

we have, by substituting,

$$\left.\begin{aligned}\cos a &= \cos (b - c)\,(1 - \sin^2 \tfrac{1}{2}A) + \cos (b + c) \sin^2 \tfrac{1}{2}A\\ &= \cos (b - c) \cos^2 \tfrac{1}{2}A + \cos (b + c) \sin^2 \tfrac{1}{2}A.\end{aligned}\right\}\quad(2)$$

This last equation may also be derived by the following elegant process. The original fundamental equation may be written

$$\begin{aligned}\cos a = \cos b \cos c\,&(\cos^2 \tfrac{1}{2}A + \sin^2 \tfrac{1}{2}A)\\ + \sin b \sin c\,&(\cos^2 \tfrac{1}{2}A - \sin^2 \tfrac{1}{2}A)\end{aligned}$$

(because $\cos^2 \frac{1}{2}A + \sin^2 \frac{1}{2}A = 1$, and $\cos^2 \frac{1}{2}A - \sin^2 \frac{1}{2}A = \cos A$).

By conjoining the coefficients of $\cos^2 \frac{1}{2}A$ and of $\sin^2 \frac{1}{2}A$, the equation (2) follows by the addition theorem.

By a similar process, from the equation

$$\cos A = -\cos B \cos C + \sin B \sin C \cos a,$$

we obtain

$$\cos A = -\cos (B + C) - 2 \sin B \sin C \sin^2 \tfrac{1}{2}a; \qquad (3)$$

$$\cos A = -\cos (B + C) \cos^2 \tfrac{1}{2}a - \cos (B - C) \sin^2 \tfrac{1}{2}a. \quad (4)$$

By a slight modification of the process employed in forming the equations (1) and (3) we may find

$$\cos a = \cos(b + c) + 2 \sin b \sin c \cos^2 \tfrac{1}{2}A; \qquad (5)$$
$$\cos A = -\cos(B - C) + 2 \sin B \sin C \cos^2 \tfrac{1}{2}a; \qquad (6)$$

which equations the student may prove as an exercise.

119. *Expressions when three sides or three angles are given.* From the last equation (1) we find

$$\sin^2 \tfrac{1}{2}A = \frac{\cos(b - c) - \cos a}{2 \sin b \sin c};$$

by which any angle is expressed in terms of the three sides.

By § 44, 13 we have

$$\cos(b - c) - \cos a = 2 \sin \tfrac{1}{2}(a + c - b) \sin \tfrac{1}{2}(a + b - c). \qquad (a)$$

If we put s for half the sum of the sides, namely,

$$s = \tfrac{1}{2}(a + b + c),$$

we have

$$\left.\begin{aligned} \tfrac{1}{2}(a + c - b) &= s - b; \\ \tfrac{1}{2}(a + b - c) &= s - c. \end{aligned}\right\} \qquad (b)$$

Substituting these values in (a), the expression for $\sin^2 \tfrac{1}{2}A$ becomes

$$\sin^2 \tfrac{1}{2}A = \frac{\sin(s - b)\sin(s - c)}{\sin b \sin c};$$

then, by permutation,

$$\sin^2 \tfrac{1}{2}B = \frac{\sin(s - c)\sin(s - a)}{\sin c \sin a};$$

$$\sin^2 \tfrac{1}{2}C = \frac{\sin(s - a)\sin(s - b)}{\sin a \sin b}. \qquad (7)$$

To find similar expressions for the cosines we take equation (5), which gives

$$\cos^2 \tfrac{1}{2}A = \frac{\cos a - \cos(b + c)}{2 \sin b \sin c}.$$

But

$$\begin{aligned} \cos a - \cos(b + c) &= 2 \sin \tfrac{1}{2}(b + c + a) \sin \tfrac{1}{2}(b + c - a) \\ &= 2 \sin s \sin(s - a) \qquad \text{[from } (b)\text{]}. \end{aligned}$$

Therefore

$$\left.\begin{aligned} \cos^2 \tfrac{1}{2}A &= \frac{\sin s \sin(s - a)}{\sin b \sin c}; \\ \cos^2 \tfrac{1}{2}B &= \frac{\sin s \sin(s - b)}{\sin c \sin a}; \\ \cos^2 \tfrac{1}{2}C &= \frac{\sin s \sin(s - c)}{\sin a \sin b}. \end{aligned}\right\} \qquad (8)$$

Since an angle near 90° cannot be accurately determinea by its sine, nor one near 0° by its cosine, neither of the formulæ (7) or (8) can be advantageously used in all cases. But by taking the quotient of each equation (7) by the corresponding one of (8) we have

$$\left.\begin{aligned} \tan^2 \tfrac{1}{2}A &= \frac{\sin (s-b) \sin (s-c)}{\sin s \sin (s-a)}; \\ \tan^2 \tfrac{1}{2}B &= \frac{\sin (s-c) \sin (s-a)}{\sin s \sin (s-b)}; \\ \tan^2 \tfrac{1}{2}C &= \frac{\sin (s-a) \sin (s-b)}{\sin s \sin (s-c)}. \end{aligned}\right\} \qquad (9)$$

120. Treating the equations (3) and (6) in the same way, and putting

$$S = \tfrac{1}{2}(A + B + C),$$

we find the following expressions for the sides, in terms of the three angles:

$$\left.\begin{aligned} \sin^2 \tfrac{1}{2}a &= -\frac{\cos S \cos (S-A)}{\sin B \sin C}; \\ \sin^2 \tfrac{1}{2}b &= -\frac{\cos S \cos (S-B)}{\sin C \sin A}; \\ \sin^2 \tfrac{1}{2}c &= -\frac{\cos S \cos (S-C)}{\sin A \sin B}. \end{aligned}\right\} \qquad (10)$$

$$\left.\begin{aligned} \cos^2 \tfrac{1}{2}a &= \frac{\cos (S-B) \cos (S-C)}{\sin B \sin C}; \\ \cos^2 \tfrac{1}{2}b &= \frac{\cos (S-C) \cos (S-A)}{\sin C \sin A}; \\ \cos^2 \tfrac{1}{2}c &= \frac{\cos (S-A) \cos (S-B)}{\sin A \sin B}. \end{aligned}\right\} \qquad (11)$$

$$\left.\begin{aligned} \tan^2 \tfrac{1}{2}a &= -\frac{\cos S \cos (S-A)}{\cos (S-B) \cos (S-C)}; \\ \tan^2 \tfrac{1}{2}b &= -\frac{\cos S \cos (S-B)}{\cos (S-C) \cos (S-A)}; \\ \tan^2 \tfrac{1}{2}c &= -\frac{\cos S \cos (S-C)}{\cos (S-A) \cos (S-B)}. \end{aligned}\right\} \qquad (12)$$

For the solution of a triangle in which all three sides or all three angles are given, the equations (9) and (12) are preferable.

For convenience in computation the following slight modification may be made. Put

$$p = \sqrt{\frac{\sin(s-a)\sin(s-b)\sin(s-c)}{\sin s}};$$

then from (9),

$$\tan^2 \tfrac{1}{2}A = \frac{p^2}{\sin^2(s-a)};$$

extracting the square root, and writing the remaining equations,

$$\left.\begin{aligned}\tan \tfrac{1}{2}A &= \frac{p}{\sin(s-a)};\\ \tan \tfrac{1}{2}B &= \frac{p}{\sin(s-b)};\\ \tan \tfrac{1}{2}C &= \frac{p}{\sin(s-c)}.\end{aligned}\right\} \qquad (13)$$

In the same way, if we put

$$P = \sqrt{\frac{-\cos S}{\cos(S-A)\cos(S-B)\cos(S-C)}};$$

we find, from (12),

$$\left.\begin{aligned}\tan \tfrac{1}{2}a &= P\cos(S-A);\\ \tan \tfrac{1}{2}b &= P\cos(S-B);\\ \tan \tfrac{1}{2}c &= P\cos(S-C).\end{aligned}\right\} \qquad (14)$$

Example of Computation.

Given the three sides,

$a = 76° 29'.3$, $b = 93° 18'.6$, $c = 122° 7'.7$;

find the angles.

$a =$ 76° 29′.3	$s =$ 145° 57′.8	cosec 0.252 02
$b =$ 93° 18′.6	$s - a =$ 69° 28′.5	sin 9.971 52
$c =$ 122° 7′.7	$s - b =$ 52° 39′.2	9.900 36
$2s =$ 291° 55′.6	$s - c =$ 23° 50′.1	9.606 49
		$\log p^2$, 9.730 39
9.865 20	9.865 20	$\log p$, 9.865 20
$\sin(s-a)$, 9.971 52	$\sin(s-b)$, 9.900 36	$\sin(s-c)$, 9.606 49
$\tan \frac{1}{2}A$, 9.893 68	$\tan \frac{1}{2}B$, 9.964 84	$\tan \frac{1}{2}C$, 0.258 71
$\frac{1}{2}A$, 38° 3′.35	$\frac{1}{2}B$, 42° 41′.0	$\frac{1}{2}C$, 61° 8′.3
A, 76° 6′. 7	B, 85° 22′.0	C, 122° 16′.6

We may now check the results by computing the va.ues of $\frac{\sin a}{\sin A}, \frac{\sin b}{\sin B}, \frac{\sin c}{\sin C}$, and seeing whether they agree. We find

$$\begin{array}{lll} \sin a = 9.987\,81; & \sin b = 9.999\,27; & \sin c = 9.927\,81 \\ \sin A = 9.987\,11; & \sin B = 9.998\,58; & \sin C = 9.927\,10 \\ \hline .000\,70 & .000\,69 & .000\,71 \end{array}$$

Although the agreement is not perfect, the deviations are no greater than those due to the omission of decimals in the logarithms and angles. The angle B is that for which the check is most doubtful, because it is so near 90° that it may be changed 1′ or 2′ without changing the last figures in log. sin. by more than one or two units in the fifth place. In such a case certainty can be reached only by a duplicate computation.

Exercises.

1. Given $a = 105° 6'.8$, $b = 93° 39'.9$, $c = 50° 20'.3$; find the angles.

2. Given $A = 46° 59'.3$, $B = 122° 32'.6$, $C = 139° 0'.3$; find the sides.

3. Given $A = 78° 40'.7$, $B = 102° 29'.5$, $C = 86° 49'.4$; find the sides.

4. If the sides of a spherical triangle are each 120°, find the following numerical expressions for the sine, cosine, and tangent of each of the angles, and thence, by the aid of the tables, find the angles themselves.

$$\text{sines} = \sqrt{\tfrac{2}{3}};$$
$$\text{cosines} = \sqrt{\tfrac{1}{3}};$$
$$\text{tangents} = \sqrt{2}.$$

121. *Cagnoli's equation.* Cagnoli's equation is

$$\sin a \sin b + \cos a \cos b \cos C = \sin A \sin B - \cos A \cos B \cos c. \quad (15)$$

Proof. Multiplying the third equation (§ 97, 1) by cos C, we have

$$\left.\begin{aligned} \cos c \cos C &= \cos a \cos b \cos C + \sin a \sin b \cos^2 C \\ &= \cos a \cos b \cos C + \sin a \sin b - \sin a \sin b \sin^2 C. \end{aligned}\right\} (a)$$

The equation of sines (§ 99, 2) gives

$$\sin a \sin C = \sin A \sin c;$$
$$\sin b \sin C = \sin B \sin c;$$

whence, by multiplying,

$$\sin a \sin b \sin^2 C = \sin A \sin B \sin^2 c$$
$$= \sin A \sin B - \sin A \sin B \cos^2 c.$$

Substituting this value in (a), and interchanging the terms of the equation,

$$\sin a \sin b + \cos a \cos b \cos C$$
$$= \cos c \cos C + \sin A \sin B - \sin A \sin B \cos^2 c$$
$$= \sin A \sin B + \cos c (\cos C - \sin A \sin B \cos c).$$

From the last equation (11), § 104, we have

$$\cos C - \sin A \sin B \cos c = -\cos A \cos B.$$

Making this substitution, we have the equation (15) as enunciated.

Of course two other similar equations may be obtained by permuting the symbols.

122. *Gauss's equations.* We write the four equations,

(a) Cagnoli's equation,

(b) and (c) the fundamental equations $(1)_3$ and $(3)_3$, and

(d) the identical equation $1 = 1$, as follows:

(a) $\sin a \sin b + \cos a \cos b \cos C = \sin A \sin B - \cos A \cos B \cos c;$

(b) $\cos a \cos b + \sin a \sin b \cos C = \cos c;$

(c) $\cos C = -\cos A \cos B + \sin A \sin B \cos c;$

(d) $1 = 1.$

Taking the sum of the four equations, and substituting

$$\cos (a - b) = \cos a \cos b + \sin a \sin b,$$
$$\cos (A + B) = \cos A \cos B - \sin A \sin B,$$

we have

$$\cos (a - b) + \cos C \cos (a - b) + \cos C + 1 =$$
$$- \cos (A + B) - \cos c \cos (A + B) + \cos c + 1;$$

or

$$(1 + \cos C) [1 + \cos (a - b)] = (1 + \cos c) [1 - \cos (A + B)] \quad (e)$$

If we form the sum $-(a) - (b) + (c) + (d)$, and reduce in the same way, we find

$$(1 + \cos C) [1 - \cos (a - b)] = (1 - \cos c) [1 - \cos (A - B)] \quad (f)$$

The sum $-(a) + (b) - (c) + (d)$ gives

$$(1 - \cos C)\,[1 + \cos(a + b)] = (1 + \cos c)\,[1 + \cos(A + B)] \quad (g)$$

The sum $(a) - (b) - (c) + (d)$ gives

$$(1 - \cos C)\,[1 - \cos(a + b)] = (1 - \cos c)\,[1 + \cos(A - B)] \quad (h)$$

In the equations (e), (f), (g), (h) we substitute the values of $1 \pm \cos$, namely,

$$1 + \cos\, C = 2 \cos^2 \tfrac{1}{2}C;$$
$$1 - \cos\, C = 2 \sin^2 \tfrac{1}{2}C;$$

etc. etc.,

and dividing by 2, we obtain

$$\left.\begin{aligned}
\sin^2 \tfrac{1}{2}C \sin^2 \tfrac{1}{2}(a + b) &= \sin^2 \tfrac{1}{2}c \cos^2 \tfrac{1}{2}(A - B); \; (h) \\
\sin^2 \tfrac{1}{2}C \cos^2 \tfrac{1}{2}(a + b) &= \cos^2 \tfrac{1}{2}c \cos^2 \tfrac{1}{2}(A + B); \; (g) \\
\cos^2 \tfrac{1}{2}C \sin^2 \tfrac{1}{2}(a - b) &= \sin^2 \tfrac{1}{2}c \sin^2 \tfrac{1}{2}(A - B); \; (f) \\
\cos^2 \tfrac{1}{2}C \cos^2 \tfrac{1}{2}(a - b) &= \cos^2 \tfrac{1}{2}c \sin^2 \tfrac{1}{2}(A + B). \; (e)
\end{aligned}\right\} \quad (16)$$

Extracting the square roots, we have

$$\begin{aligned}
\sin \tfrac{1}{2}C \sin \tfrac{1}{2}(a + b) &= \sin \tfrac{1}{2}c \cos \tfrac{1}{2}(A - B); \\
\sin \tfrac{1}{2}C \cos \tfrac{1}{2}(a + b) &= \cos \tfrac{1}{2}c \cos \tfrac{1}{2}(A + B); \\
\cos \tfrac{1}{2}C \sin \tfrac{1}{2}(a - b) &= \sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A - B); \\
\cos \tfrac{1}{2}C \cos \tfrac{1}{2}(a - b) &= \cos \tfrac{1}{2}c \sin \tfrac{1}{2}(A + B).
\end{aligned}$$

In strictness, the second members of this equation may have the negative as well as the positive sign. But it is easy to show that when the sides and angles are all less than 180°, all the members of the equations are positive. Hence the positive sign is the only one necessary to be taken into account.

These equations are applicable when any three consecutive parts of the triangle—two angles and the included side, or two sides and the included angle—are given.

In the first case the three given parts are all in the right-hand members of the equations; in the second case they are all on the left.

These equations are written in the most convenient order for use in the first case; in the second, the following is the order and arrangement:

$$\left.\begin{aligned}
\sin \tfrac{1}{2}c \sin \tfrac{1}{2}(A - B) &= \cos \tfrac{1}{2}C \sin \tfrac{1}{2}(a - b); \\
\sin \tfrac{1}{2}c \cos \tfrac{1}{2}(A - B) &= \sin \tfrac{1}{2}C \sin \tfrac{1}{2}(a + b); \\
\cos \tfrac{1}{2}c \sin \tfrac{1}{2}(A + B) &= \cos \tfrac{1}{2}C \cos \tfrac{1}{2}(a - b); \\
\cos \tfrac{1}{2}c \cos \tfrac{1}{2}(A + B) &= \sin \tfrac{1}{2}C \cos \tfrac{1}{2}(a + b).
\end{aligned}\right\} \quad (17)$$

These are commonly known as *Gauss's equations*, after Gauss, who first introduced them into astronomical computations. They had, however, been previously published anonymously by Delambre.

Example. Given $a = 132° 46'.7$, $b = 59° 50'.1$, $C = 56° 28'.4$, to find the remaining parts.

a,	$132° 46'.7$	$\sin \frac{1}{2}c \sin \frac{1}{2}(A-B)$,	$+\ 9.719\ 08$
b,	$59° 50'.1$	$\sin \frac{1}{2}c \cos \frac{1}{2}(A-B)$,	$+\ 9.672\ 34$
$a+b$,	$192° 36'.8$	$\tan \frac{1}{2}(A-B)$,	$0.046\ 74$
$a-b$,	$72° 56'.6$	$\sin \frac{1}{2}c$,	$9.847\ 48$
$\frac{1}{2}(a+b)$,	$96° 18'.4$	$\cos \frac{1}{2}c \sin \frac{1}{2}(A+B)$,	$9.850\ 32$
$\frac{1}{2}(a-b)$,	$36° 28'.3$	$\cos \frac{1}{2}c \cos \frac{1}{2}(A+B)$,	$-\ 8.715\ 77$
$\frac{1}{2}C$,	$28° 14'.2$	$\tan \frac{1}{2}(A+B)$,	$-\ 1.134\ 55$
		$\cos \frac{1}{2}c$,	$9.851\ 49$

$\sin \frac{1}{2}(a-b)$,	$9.774\ 10$	$\frac{1}{2}(A-B)$,	$48°\ 4'.6$
$\cos \frac{1}{2}C$,	$9.944\ 98$	$\frac{1}{2}(A+B)$,	$94° 11'.7$
$\cos \frac{1}{2}(a-b)$,	$9.905\ 34$	$\frac{1}{2}c$,	$44° 44'.1$
$\sin \frac{1}{2}(a+b)$,	$+\ 9.997\ 37$	A,	$142° 16'.3$
$\sin \frac{1}{2}C$,	$+\ 9.674\ 97$	B,	$46°\ 7'.1$
$\cos \frac{1}{2}(a+b)$,	$-\ 9.040\ 80$	c,	$89° 28'.2$

EXERCISES.

Compute the remaining three parts of each of the following triangles by Gauss's formulæ:

1. $A = 79° 28'.6$, $b = 28° 20'.3$, $c = 112°\ 1'.9$.
2. $A = 32° 58'.5$, $B = 65° 26'.7$, $c = 56° 21'.2$.
3. $a = 112° 12'.6$, $b = 124° 48'.2$, $C = 18° 17'.0$.
4. $a = 52° 22'.2$, $B = 160°\ 0'.8$, $C = 129° 52'.4$.

123. *Napier's analogies.* If, in the preceding problem, only the two remaining sides in the one case, or the two remaining angles in the other, are wanted, the process may be shortened.

Dividing the first of (16) by the second, and the third by the fourth, we have

$$\left.\begin{aligned} \tan \tfrac{1}{2}(a+b) &= \tan \tfrac{1}{2}c \frac{\cos \frac{1}{2}(A-B)}{\cos \frac{1}{2}(A+B)}; \\ \tan \tfrac{1}{2}(a-b) &= \tan \tfrac{1}{2}c \frac{\sin \frac{1}{2}(A-B)}{\sin \frac{1}{2}(A+B)}; \end{aligned}\right\} \qquad (18)$$

from which may be found a and b when A, B, and C are given.

In the same way we find, from (17),

$$\left.\begin{aligned} \tan \tfrac{1}{2}(A - B) &= \cot \tfrac{1}{2}C \frac{\sin \frac{1}{2}(a - b)}{\sin \frac{1}{2}(a + b)}; \\ \tan \tfrac{1}{2}(A + B) &= \cot \tfrac{1}{2}C \frac{\cos \frac{1}{2}(a - b)}{\cos \frac{1}{2}(a + b)}; \end{aligned}\right\} \qquad (19)$$

from which A and B may be found when a, b, and C are given.

The equations (18) and (19) are known as *Napier's analogies.*

CHAPTER IV.

MISCELLANEOUS APPLICATIONS.

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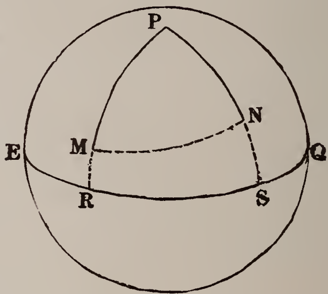
124. *To find the distance between two points on the earth's surface and their direction from each other, when their latitudes and longitudes are given.**



Let M and N be the two points whose latitudes and longitudes are given; P, the pole; and EQ, the equator. Join M to N by the arc of a great circle. Also let

φ = the lat. and λ = the long. of M;
φ' = " " λ' = " " N.

If from the pole P we draw through M and N arcs of great circles PMR and PNS, meeting the equator in R and S, we shall have

$$\varphi = RM = \text{latitude of } M;$$
$$\varphi' = SN = \text{latitude of } N.$$

If we suppose PQ to be the meridian from which we reckon longitudes,

$$\text{Angle } QPM = \lambda;$$
$$\text{Angle } QPN = \lambda'.$$

Then because $PR = PS = 90°$, we have in the triangle MPN

$$PM = 90° - \varphi;$$
$$PN = 90° - \varphi';$$
$$\text{Angle } P = \lambda - \lambda'.$$

---

* It is assumed in the solution of this problem that the earth is a sphere. Although the assumption is not strictly correct, the error to which it can give rise can never amount to more than a few thousandths of the whole distance.

This is a case, therefore, in which we have given two sides and the included angle to find the remaining side. If we put d for the required distance, the general formula gives

$$\left.\begin{aligned}\cos d &= \cos PM \cos PN + \sin PM \sin PN \cos P\\ &= \sin \varphi \sin \varphi' + \cos \varphi \cos \varphi' \cos (\lambda - \lambda').\end{aligned}\right\} \qquad (a)$$

This equation will suffice to determine the distance between the points in arc of a great circle.

To reduce it to statute miles, the degrees must be multiplied by $69\frac{1}{6}$; and to reduce to nautical miles, by 60.

To solve the problem completely we should know not only the distance but the direction, or the angle which the great circle joining the two points makes with the meridian. This angle is different at the two points, being equal to the angle M of the spherical triangle at one point, and to N at the other. Hence the complete solution requires the complete solution of the spherical triangle PMN, for which we may use the Gaussian equations instead of the form (a).

Exercises.

1. Find (1) the distance,* both in degrees and nautical miles, between New York and Liverpool, on an arc of a great circle; (2) the direction in which a ship would sail on leaving the one port for the other, on an arc of a great circle, and the direction in which she would be sailing on approaching her destination. The positions of the cities are:

New York........lat. $+ 40° 42'.7$, long. $74° 0'.0$ west.
Liverpool ....lat. $+ 53° 24'.1$, long. $2° 59'.1$ west.

2. Compute the distance between Liverpool and Rio de Janeiro, the position of the latter being:

Latitude, $- 22° 54'.7$.
Longitude, $43° 9'.0$ west.

Note that the latitude of Rio is algebraically negative, being reckoned *south* from the equator.

3. If a ship sails from New York, starting due east, and continues her course on an arc of a great circle, what will be her latitude when she reaches the meridian of Greenwich, and in what direction will she then be sailing?

---

* By *distance*, as used here, distance on the arc of a great circle is to be understood, unless explicitly stated otherwise.

Geometrical Applications.

125. *Def.* Three straight lines, each perpendicular to the other two and all passing through a common point, are called a **system of rectangular axes.**

Def. The common point of intersection is called the **origin.**

Def. Three planes, each perpendicular to the other two, are called **three rectangular planes.**

Remark. It is shown in geometry that three rectangular planes intersect each other in lines forming a system of rectangular axes.

If a sphere have its centre in the common point of intersection of three rectangular planes, these planes will intersect its surface in three great circles, forming in all eight trirectangular triangles.

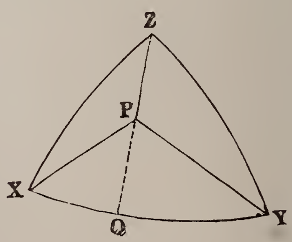
126. THEOREM. *The sum of the squares of the cosines of the three angles which a straight line forms with three rectangular axes is equal to unity.*

This theorem is expressed in trigonometric language thus:

If α, β, *and* γ *be the angles which a straight line forms with three rectangular axes*, then

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \tag{1}$$

Proof. Let the line pass through the common point of intersection of the axes. Let a sphere have its centre at this point, and let X, Y, Z, and P be the points in which the axes and the line intersect the spherical surface.



Join PX, PY, and PZ by arcs of great circles, and produce ZP until it meets XY in Q. Then—

Because the angles α, β, and γ are formed at the centre of the sphere,

Arc PX measures angle α;
" PY " " β;
" PZ " " γ.

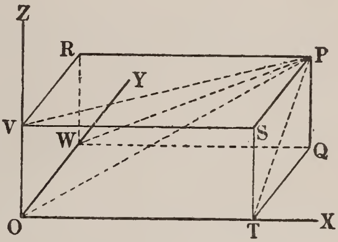
Because XYZ is a trirectangular triangle, Z is the pole of XY, and ZQ is therefore perpendicular to XY. Therefore the spherical triangles XQP and YQP are right-angled at Q, and, by § 109, 5,

$$\cos PX = \cos\alpha = \cos XQ \cos QP;$$
$$\cos PY = \cos\beta = \cos YQ \cos QP.$$

Taking the squares of these equations and adding them to the square of $\cos PZ = \cos\gamma$, we have

$$\begin{aligned}\cos^2\alpha + \cos^2\beta + \cos^2\gamma &= \cos^2 QP(\cos^2 XQ + \cos^2 YQ) + \cos^2 PZ \\ &= \cos^2 QP + \cos^2 PZ \text{ (because } XY = 90^\circ) \\ &= 1 \text{ (because } QZ = 90^\circ). \quad \text{Q.E.D.}\end{aligned}$$

Another proof. Let OX, OY, and OZ be the axes meeting in O, and let OP be the line making with the lines OX, OY, OZ the respective angles



$$POX = \alpha;$$
$$POY = \beta;$$
$$POZ = \gamma.$$

Through P pass three planes parallel to the respective planes formed by the axes, taken two and two, namely,

Plane $PSVR$ || plane XOY;
Plane $PQTS$ || plane YOZ;
Plane $PRWQ$ || plane ZOX.

These planes will then form a rectangular parallelopiped $OTQW$-$VSPR$, of which OP will be a diagonal. By the property of this diagonal (Geom., § 692),

$$OP^2 = OT^2 + OW^2 + OV^2. \qquad (a)$$

Moreover, because these planes are respectively parallel to the three rectangular planes, and because each of the axes is perpendicular to one of these planes, the three planes in question are each perpendicular to one of the axes.

If we join PT, PW, and PV, these lines, being in planes which, as just shown, are perpendicular to the axes OX, OY, and OZ, will be perpendicular to these axes (Geom.), and we shall have

$$\left.\begin{aligned}OT &= OP\cos POX = OP\cos\alpha; \\ OW &= OP\cos POY = OP\cos\beta; \\ OV &= OP\cos POZ = OP\cos\gamma.\end{aligned}\right\} \qquad (b)$$

Taking the sum of the squares of these three equations, and substituting for $OT^2 + OW^2 + OV^2$ its value (a), we have

$$OP^2 = OP^2 (\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma).$$

Dividing by OP^2,

$$\cos^2 \alpha + \cos^2 \beta + \cos^2 \gamma = 1. \quad \text{Q.E.D.} \qquad (2)$$

127. *Corollary.* It is easily shown that the angles α, β, and γ are the complements of the angles which OP forms with the three rectangular planes. For, pass a plane through P and OZ. Because $OZ \perp$ plane XOY, the cutting plane OZP is also perpendicular to XOY, and the line OQ in which it cuts $XOY \perp OZ$. But, by definition of the inclination of a line to a plane (Geom., § 603), the angle POQ will be the inclination of OP to the plane XOY. Therefore, if we call this inclination a, we have

$$a + \alpha = 90°;$$

and in the same way,

$$b + \beta = 90°;$$
$$c + \gamma = 90°;$$

b and c being the inclinations of the line to the two other planes. Therefore $\sin a = \cos \alpha$, $\sin b = \cos \beta$, and $\sin c = \cos \gamma$; whence

$$\sin^2 a + \sin^2 b + \sin^2 c = 1. \qquad (3)$$

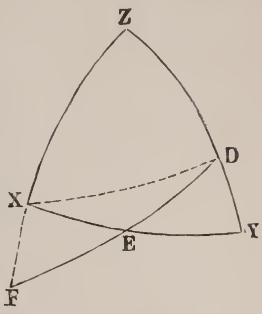
Because parallel lines have equal inclinations to a plane, the angles which any straight line makes with the three planes are equal to those made by a parallel to that line through O. Hence:

The sum of the squares of the sines of the three angles which any straight line makes with three rectangular planes is equal to unity.

128. *Case of a plane cutting three rectangular planes.* We have the following theorem:

THEOREM. *If any plane intersect three rectangular planes, the sum of the squares of the cosines of the three angles which it forms with them is equal to unity.*

This can be demonstrated from the preceding theorem by dropping a perpendicular from the common point of intersection of the three rectangular planes upon the cutting plane. Because the rectangular axes and this line are each perpendicular to one of the four planes, the angles which they form with each other are equal to the angles between the planes. The theorem, having already been proved for the angles formed by the lines, must therefore be true of the equal angles formed by the planes.



But the theorem may also be proved independently as follows:

Pass a sphere around the origin as a centre, and let XY, YZ, ZX be the arcs of great circles in which the rectangular planes intersect its surface.

Pass a plane through the centre parallel to the intersecting plane, and let FD be the great circle in which it intersects the spherical surface.

Put

$$\left.\begin{array}{l}\text{Angle } YDE = \alpha;\\ \text{Angle } XFE = \beta;\\ \text{Angle } XED = \gamma;\end{array}\right\} \text{the three angles which the intersecting plane forms with the three rectangular planes.}$$

Join XD by an arc of a great circle. Then—

Because $XY = XZ = 90°$, X is the pole of YZ, and

$$XD = 90°;$$

$$\text{Angle } ADC = \text{angle } ADB = 90°.$$

Because AFD and AED are triangles of which the side AD is 90°,

$$\cos AFD = \cos\beta = -\cos FAD \cos EDA;$$

$$\cos AED = \cos\gamma = -\cos EAD \cos EDA$$
$$= -\sin FAD \cos EDA$$

(because $FAD = EAD + EAF = EAD + 90°$).

Taking the sum of the squares,

$$\cos^2\beta + \cos^2\gamma = \cos^2 EDA$$
$$= 1 - \sin^2 EDA.$$

Because $EDA = ADB - EDB = 90° - \alpha$,

$$\sin^2 EDA = \cos^2\alpha.$$

Hence

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1. \quad \text{Q.E.D.} \qquad (4)$$

129. *Corollary.* In the same way that we proved the corollary of the last theorem we may show:

If a plane intersects a system of three rectangular axes, forming with them the respective angles a, b, c, then

$$\sin^2 a + \sin^2 b + \sin^2 c = 1. \qquad (5)$$

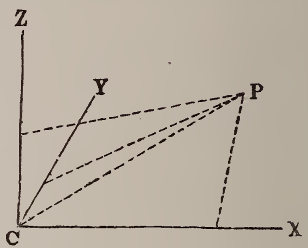
EXERCISES.

1. Having a system of three rectangular axes OX, OY, OZ, a line OP is drawn from O, making

$$\text{Angle } XOP = 60°;$$
$$\text{Angle } YOP = 50°.$$

Find the angle ZOP and the angles which OP makes with the three planes XOY, YOZ, ZOX.

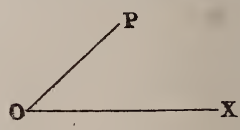
2. Supposing $OP = 32.965$ centimetres, and the angles to have the values in the preceding exercise, find the lengths of the perpendiculars dropped from P upon each of the axes OX, OY, and OZ, and the distances of the feet of these perpendiculars from the origin O.



3. The same thing being supposed, what are the lengths of the respective perpendiculars dropped from P upon the three planes XOY, YOZ, and ZOX?

4. The same thing being supposed, what are the lengths of the projections of OP upon the three planes?

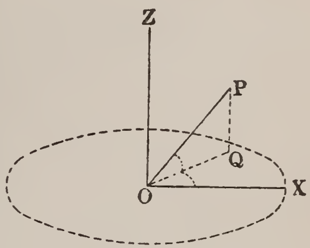
130. *Methods of defining the direction of a line in space.* The direction of a line in a plane is defined by the angle which it makes with some fixed line in the plane. For example, if we have a known fixed line OX, and it is required that another line OP through O shall make an angle $+45°$ with OX, this completely fixes the direction of OP. That is, there is only one line through O which makes an angle of $+45°$ with OX, when we employ the method of measuring angles defined in Plane Trigonometry, Chap. I.



But if OP is not confined to one plane, its position is not fixed by the angle XOP, because any number of lines may be drawn through O, some above the plane of the paper and some below it, all making the same angle with OX. The student will see that these lines are all elements of a cone having O as its vertex and OX as its axis.

Hence at least two conditions are necessary to define the direction of a line in space. These two quantities may be chosen in various ways, of which the following is the most common.

We have (1) a plane of reference, the position of which we suppose fixed, and (2) in this plane we have a fixed line of reference OX. We call the plane of reference the *fundamental plane*.



Let OZ be a line through O, perpendicular to the plane.

Let OP be the line of which the direction is to be defined.

From any point P of this line drop a perpendicular PQ upon the fundamental plane, and join OQ.

The direction of OP is then defined by the following two angles:

(1) The angle POQ which OP forms with its projection OQ; that is, the angle between OP and the plane.

(2) The angle XOQ which the projection of OP makes with OX.

It will be remarked that the planes of these two angles are perpendicular to each other.

To show that these two angles completely fix the direction of OP, we first remark that when the angle XOQ is given the line OQ is fixed.

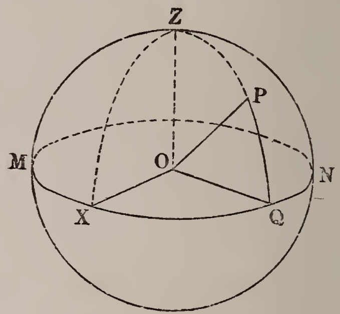
Next, because PQ is perpendicular to the plane, the point P and therefore the line OP must lie in the plane ZOQ, which is fixed because its two lines OZ and OQ are fixed. If the angle QOP in this (vertical) plane is given, there is only one line OP which can form this angle.

Hence *the direction of the line OP is completely determined by the two angles XOQ and QOP.*

The plane XOQ is, when used in this way, the fundamental plane.

131. *Relation of the preceding system to latitude and longitude.* To form another conception of these two angles, pass a sphere around O as a centre, and mark on its surface the points and lines in which the lines and planes belonging to the preceding figure intersect it. Then:

The fundamental plane OXQ intersects the spherical surface in the great circle $MXQN$.



The line OX intersects it in X.

The line OQ intersects it in Q.

The lines OP and OZ intersect it in P and Z.

We therefore have

$$\text{Angle } XOQ = \text{arc } XQ;$$
$$\text{Angle } QOP = \text{arc } QP.$$

If now we imagine this sphere to be the earth, the great circle MN to be its equator, Z to be one of its poles, and P any point on its surface, then—

The arc QP or the angle QOP is the latitude of P.

The arc XQ or angle XOQ is the longitude of P, counted from ZX as a prime meridian.

Thus the angles we have been defining may be described under the familiar forms of longitude and latitude.

132. *Position of a point.* To fix the position of any point relative to a fundamental plane, we must select a point O in that plane and a line OX as a point and line of reference. If P be any point of which we wish to describe the position, we draw the line OP and form the same construction as in article 130. Then the position of P is fixed by the two angles XOQ and QOP, already described, and the distance OP.

Hence *three quantities are required to fix the position of a point in space.*

Def. The quantities which fix or describe the position of a point are called the **co-ordinates** of the point.

The angles XOQ and QOP and the length OP are therefore co-ordinates of the point P, and are distinguished as *polar co-ordinates.*

133. *Polar distance and longitude.* In the preceding figure, because $MXQN$ is a great circle, and Z its pole, we have

$$\text{Angle } XOQ = \text{arc } XQ = \text{angle } XZP;$$
$$\text{Arc } PQ = 90° - \text{arc } ZP.$$

Therefore if we put

p, the arc ZP, which is called the *polar distance* of P;

φ, the arc $PQ =$ angle QOP,

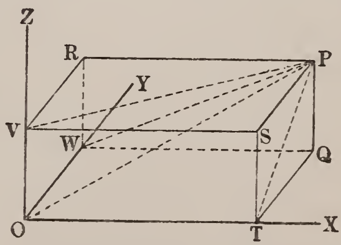
we have

$$\left.\begin{aligned} \sin\varphi &= \cos p; \\ \cos\varphi &= \sin p; \\ \lambda &= \text{angle } XZP. \end{aligned}\right\} \qquad (6)$$

Thus we may define the direction of a line by polar distance and longitude, as well as by latitude and longitude. Applying the same system to positions on the earth's surface, their distance from the north pole of the earth is used instead of their latitude. Thus we should have,

For New York, $p = 49°\ 17'.3$;
For Rio de Janeiro, $p = 112°\ 54'.7$.

134. *Rectangular co-ordinates.* The rectangular co-ordinates of a point are its distances from three rectangular planes. In the figure the lines PQ, PR, and PS are the rectangular co-ordinates of P with respect to the axes OZ, OX, and OY respectively. In other words:



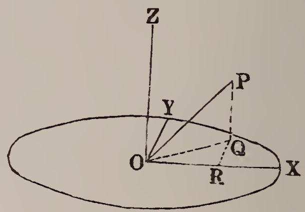
The co-ordinate of a point relative to each axis is the length of the line parallel to that axis from the point to the plane of the two other axes.

We designate these co-ordinates thus: If P is the point whose co-ordinates are x, y and z,

x is the line PR parallel to OX;
y " " PS " " OY;
z " " PQ " " OZ.

135. PROBLEM. *To find the relation between rectangular and polar co-ordinates.*

Let P be the point whose polar co-ordinates are given.



From P drop $PQ \perp$ plane XOY.
From Q drop $QR \perp OX$.
We then have

$$QP = z;$$
$$RQ = y;$$
$$OR = x.$$

The first of these equations follows immediately from the definition of rectangular co-ordinates.

To see the truth of the second we notice that because $PQ \parallel$ plane XOZ, the perpendicular RQ is equal to the perpendicular from P upon the plane XOZ.

Let us now put

φ, the angle QOP, or the elevation of OP, above the plane XOY. We may call this angle the *latitude* of P.

λ, the angle XOQ which OQ, the projection of OP, makes with OX. We may call this angle the *longitude* of P.

r, the length of OP.

We then find

$$\left.\begin{aligned} z &= QP = r \sin \varphi; \\ OQ &= r \cos \varphi; \\ y &= RQ = OQ \sin \lambda = r \cos \varphi \sin \lambda; \\ x &= OR = OQ \cos \lambda = r \cos \varphi \cos \lambda; \end{aligned}\right\} \quad (7)$$

which are the required expressions.

If we take the sum of the squares of these three equations, we find, by simple reductions,

$$x^2 + y^2 + z^2 = r^2. \quad (8)$$

136. The second proof of § 126 affords very simple expressions for the rectangular co-ordinates, in terms of the angles which OP makes with the rectangular axes.

In the figure we have

$$\left.\begin{aligned} OT &= x; \\ TQ &= y; \\ QP &= z. \end{aligned}\right\} \qquad (9)$$

Putting, as before, α, β, and γ for the angles POX, POY, and POZ, which OP makes with the axes OX, OY, and OZ respectively, we have, by (b), § 126,

$$\left.\begin{aligned} x &= r \cos \alpha; \\ y &= r \cos \beta; \\ z &= r \cos \gamma. \end{aligned}\right\} \qquad (10)$$

Comparing these with equations (7), we have the following values of α, β, and γ in terms of φ and λ:

$$\left.\begin{aligned} \cos \alpha &= \cos \varphi \cos \lambda; \\ \cos \beta &= \cos \varphi \sin \lambda; \\ \cos \gamma &= \sin \varphi. \end{aligned}\right\} \qquad (11)$$

The sum of the squares of the second members is identically equal to unity, as it should be.

137. We may find a linear expression for r in terms of the rectangular co-ordinates by multiplying the equations (10) by $\cos \alpha$, $\cos \beta$, and $\cos \gamma$ respectively, and taking the sum of the products, noting equation (2). We thus find

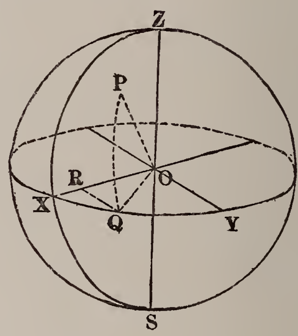
$$x \cos \alpha + y \cos \beta + z \cos \gamma = r.$$

138. The expressions (10) furnish us another definition of the rectangular co-ordinates of a point in space, which will sometimes be useful. Since OTP is a right angle, the point T is the projection of P upon the axis OX. In the same way W and V are the projections of P upon the other axes. Because OT, OW, and OV are the co-ordinates of P, we have the definition:

The rectangular co-ordinates of a point are the distances from the origin to its projections upon the three axes.

139. The preceding formulæ are directly applicable to positions upon the surface of the earth. Let us suppose O to be the centre of the earth; OZ, its axis, Z being the north pole; P, a point on its surface; $OY \perp OX$; the great circle XY the equator, and the plane XOY the plane of the equator.

Also let the great circle ZX be the meridian of Greenwich, from which longitudes are counted, and let PQ be the perpendicular arc of a great circle from P upon the equator.



The three rectangular planes will then be:

Plane XOY, the plane of the earth's equator;

Plane ZOX, the plane passing through the earth's axis and through Greenwich;

Plane YOZ, the plane at right angles to the other two.

The angle $QOP = \varphi$ will then be the latitude of the point P, and $XOQ = \lambda$ will be its longitude.

The rectangular co-ordinates x, y, and z will be the distances of the point P from the respective planes.

Hence if we put

ρ, the radius OP of the earth,

we shall have

$z = \rho \sin \varphi$, distance from plane of equator;

$y = \rho \cos \varphi \sin \lambda$, distance from plane of Greenwich;

$x = \rho \cos \varphi \cos \lambda$, distance from third plane.

Exercises.

1. Assuming the latitude of Washington to be $+ 38° 53'.6$, its longitude west from Greenwich $77° 3'.0$, and its distance from the earth's centre to be 6369 kilometres, it is required to compute:

(*a*) Its distance from the plane of the equator;

(*b*) Its distance from the earth's axis;

(*c*) The circumference of the circle which it describes every day in consequence of the earth's rotation on its axis;

(*d*) Its distance from the plane passing through Greenwich and the axis of the earth.

2. What is the length of the straight line through the earth from Washington to Melbourne? The position of Melbourne is:

Latitude, $-37° 38'.7$;

Longitude, $144° 58'.7$ east of Greenwich;

Distance, 6369 kilometres from earth's centre.

Find the angle between the two radii by § 124, and then the distance by § 61.

3. In the preceding problem, what are the lengths of the segments into which the straight line from Washington to Melbourne is divided by the plane of the equator? Begin by finding the distance of each point from the plane of the equator.

4. How far apart are the feet of the respective perpendiculars from Washington and Melbourne upon the plane of the equator? And how far is each foot from the centre of the earth?

140. *Projection of one line upon another.* In geometry and trigonometry the projection of a finite line a upon an indefinite line X is defined as the distance between the feet of the perpendiculars dropped from the ends of a upon X. Hence it has been shown that if we put

p, the projection,

α, the angle which a makes with X,

we shall have

$$p = a \cos \alpha.$$

In demonstrating this proposition we have supposed a and X to intersect, and therefore to lie in one plane.

When they are not in one plane a general definition of the angle between two such lines is necessary.

Def. The angle between two non-intersecting lines is the same as the angle formed by one of the lines and an intersecting line parallel to the other.

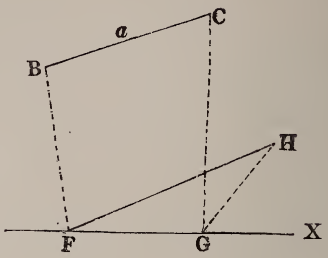
Example. If X and a do not intersect, the angle between X and a is the same as the angle between X and any parallel to a intersecting X.

The theorem which we have proved for a special case is perfectly general, and is as follows:

If α be the angle between any two lines, a the length of one of them, and p the projection of a upon the other, we have

$$p = a \cos \alpha.$$

Proof. Let BC be the projected line; X, the line of projection; BF and CG, the perpendiculars from B and C upon X. (These perpendiculars are not parallel.)



Then FG is by definition the projection of BC upon X.

Through F and G pass planes each perpendicular to the line X.

These planes are parallel (Geom., § 616).

The lines FB and GC being perpendicular to X, lie in these respective planes (Geom., § 586).

Therefore the points B and C lie in these planes.

Through F draw $FH \parallel BC$, and meeting the plane through GC at the point H.

Because the planes and lines are parallel,

$$FH = BC.$$

Join HG. Because H and G are in one plane $\perp$ line X,

$$HG \perp X.$$

Therefore

$$FG = FH \cos HFG;$$

or, because $FH = a$ and angle $HFG = \alpha$,

$$FG = a \cos \alpha. \quad \text{Q.E.D.}$$

141. *Plane triangles in space.* The following problem is of constant occurrence in astronomy:

Given:

(1) The distance and direction of a point B from some point of reference O, the centre of the earth, for example;

(2) The distance and direction of another point C from B;

Required the distance and direction of C from O.

It will be remarked that these distances and directions form the sides of a plane triangle, of which O, B, and C are the vertices.

To attack the problem, we may assume O as the origin of co-ordinates, and find the rectangular co-ordinates of B and C as referred to O.

Let us suppose

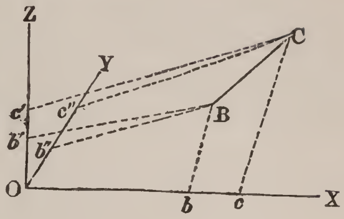
OX, OY, OZ, the three co-ordinate axes;

r, the length OB;*

s, the length BC;

α, β, γ, the angles XOB, YOB, and ZOB, which OB makes with the three co-ordinate axes;

Bb, Bb', Bb'', the perpendiculars from B upon the three axes;



Cc, Cc', Cc'', the perpendiculars from C upon the three axes. Then—

I. Putting x, y, and z for the co-ordinates of B, we have (§ 136)

$$\left.\begin{aligned} x &= Ob = r \cos \alpha; \\ y &= Ob' = r \cos \beta; \\ z &= Ob'' = r \cos \gamma. \end{aligned}\right\} \qquad (12)$$

II. If we designate the angles between BC and the co-ordinate axes by α', β', and γ', we have (§ 140)

$$\left.\begin{aligned} bc &= s \cos \alpha'; \\ b'c' &= s \cos \beta'; \\ b''c'' &= s \cos \gamma'. \end{aligned}\right\} \qquad (13)$$

III. The lines Oc, Oc', Oc'' are by definition the co-ordinates of the point C. Hence if we put $r'' = OC$, and α'', β'', γ'' the angles which OC makes with the three co-ordinate axes, we have

$$\begin{aligned} r'' \cos \alpha'' &= Oc; \\ r'' \cos \beta'' &= Oc'; \\ r'' \cos \gamma'' &= Oc''; \end{aligned}$$

or, comparing with (12) and (13),

$$\left.\begin{aligned} r'' \cos \alpha'' &= r \cos \alpha + s \cos \alpha'; \\ r'' \cos \beta'' &= r \cos \beta + s \cos \beta'; \\ r'' \cos \gamma'' &= r \cos \gamma + s \cos \gamma'. \end{aligned}\right\} \qquad (a)$$

In practice the angles α, β, γ, etc., are not generally convenient, because they are not independent. We therefore substitute for them the polar co-ordinates φ and λ, defined in § 135, putting

* The lines OB and OC are omitted in the figure in order to avoid complexity.

φ and λ, the latitude and longitude expressing the direction of OB.

φ', λ', the same for BC. We may conceive φ' to be the angle which CB produced makes with the plane of XY; or, which is the same thing, the angle between BC and a plane through B parallel to XY. We also put

φ'', λ'', the corresponding quantities for OC.

The general equations (a) combined with (11) then give

$$\left.\begin{aligned} r'' \cos \varphi'' \cos \lambda'' &= r \cos \varphi \cos \lambda + s \cos \varphi' \cos \lambda'; \\ r'' \cos \varphi'' \sin \lambda'' &= r \cos \varphi \sin \lambda + s \cos \varphi' \sin \lambda'; \\ r'' \sin \varphi'' &= r \sin \varphi + s \sin \varphi'. \end{aligned}\right\} \quad (14)$$

When the six quantities which enter into the second members are all given, r'', φ'', and λ'' may be computed by Prob. VI. Chap. IV. But the first two equations may be simplified by the following transformation:

Multiply the first equation by $\sin \lambda'$, and the second by $\cos \lambda'$, and subtract the first product from the second. The remainder reduces to

$$r'' \cos \varphi'' \sin (\lambda'' - \lambda') = r \cos \varphi \sin (\lambda - \lambda').$$

Now multiply the first by $\cos \lambda'$, and the second by $\sin \lambda'$, and add the products. The sum reduces to

$$r'' \cos \varphi'' \cos (\lambda'' - \lambda') = r \cos \varphi \cos (\lambda - \lambda') + s \cos \varphi'.$$

From these two equations and the third of (14) the values of r'', φ'', and $\lambda'' - \lambda'$ may be computed by Prob. VI. Chap. V.

We might equally have effected the transformation by using $\sin \lambda$ and $\cos \lambda$ as multipliers, and proceeding exactly as before. The equations to be solved would then be

$$\left.\begin{aligned} r'' \cos \varphi'' \sin (\lambda'' - \lambda) &= s \cos \varphi' \sin (\lambda' - \lambda); \\ r'' \cos \varphi'' \cos (\lambda'' - \lambda) &= r \cos \varphi + s \cos \varphi' \cos (\lambda' - \lambda); \\ r \sin \varphi'' &= r \sin \varphi + s \sin \varphi'. \end{aligned}\right\} \quad (15)$$