泰和匡文濤編纂

平面三角法講義

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平面三角法講義

序

昔者包犧立厤度。隸首作算數。是數學之發明於吾國。由來久矣。而數學之書。則以周髀九章爲最古。周髀以矩數原始。大開句股之門。九章以測象要終。實妙句股之用。西法之直角三角形。正中法之句股也。其銳鈍二三角。雖不可以句股目之。然可分作二句股以求之。卽今之以八線解三角者。其理亦根源於圜內正方形兩對角線所成之矩形等於兩邊所成矩形之倍。細繹其旨。何莫非句股相等之句股乎。就其圖形玩之。八線組成四種同式句股形。毫無疑義。是以句股概三角可也。善乎梅文鼎先生之論三角形曰。句股雖不能備三角之之形。而能兼三角之理。三角不能出句股之外。而能盡句股之用。一而二。二而一者也。但古法之句股。若易之太極。萬理渾括而無窮。今西法之三角。若易之三百八十四爻。萬象顯呈而無隱。學術者只求其精而捷。不必齗齗於舊法新術也。窮理者宜審其確而當。毋庸期期於中學西藝也。理果精微正確。在古法固不必薄視。術果直捷了當。卽西法亦不必弁髦。此余所以舍句股而言三角。而有上野清平面三角法講義之譯也。是書分十九編。自第一編至第十五編。詳敘初等平面三角之要義。自第十六編至第十九編。略言高等平面三角之一部分。各編說理詳確。取材簡明。按其程度與材料。適足供中等及中等以上學校參攷之用。如中等學校採爲教本。則限於時日。其自第十六編以後。可略而不講。留作生徒自修之資料。以視坊間各教科書。毫無伸縮之餘地者有別。此其可採者一也。數學端題演習。而三角尤甚。千變萬化。不可究詰。公理公式。更僕難終。非有多數

例題以資演習。萬不能收熟則生巧之效。是書舉例三十四。統計一千八百八十九題。以之演習。綽有餘裕。且各題均演有詳草。可爲對照。每題演畢。查與原書相合否。合則審其孰繁孰簡。否則核其孰得孰失。兩兩比較。工拙立見。進步格外神速。此其可採者二也。上野清爲日本有名之數學大家。其所著之平面幾何學講義及立體幾何學講義。經烏程張廷華譯述後。風行海內。是書與前二講義互相關係。如平面幾何學講義第四編。與是書第五編例題二十三及二十四。關係更爲密切。彼此參照。便益甚多。此其可採者三也。今因同人慫恿。付諸刷印。特將三角不外句股。及此講義之優點。誌於簡端。以諗同志。

時在民國五年夏泰和匡文濤自序

原序

予昨年草平面幾何學及立體幾何學講義錄之稿。題數都合二千四十有餘。迨刊行後。正從事增補改正。故未暇及續稿之三角法講義錄。今因書肆需要。舊稿堆積。故特趁早刊行。

但幾何學的例題。比較的爲有限。至三角法例題。實千變萬化。無有限制。今蒐集多數例題。只足充普通研究。頗與前二講義錄略異其趣。然其程度一也。

本書亦如平面幾何學講義錄。分爲五册。第一册至第四册。述初等平面三角法。第五册。述高等平面三角法。至球面三角法及其他高等數學講義錄。則俟諸異日。陸續刊行。

講述者上野清記

平面三角法講義

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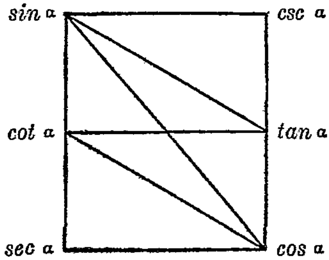
希 臘 文 字

三角法之字母。多用希臘字。故為初學者示其讀法如次。但文字外作( )符號者。用之最多。餘則應用甚少。

(α)	A	Alpha.	阿爾法
(β)	B	Beta.	比達
(γ)	Γ	Gamma.	敢麻
(δ)	Δ	Delta.	笛爾達
(ε)	E	Epsilon.	耶皮西倫
(ζ)	Z	Zeta.	者達
(η)	H	Eta.	耶達
(ϑ)	Θ	Theta.	底達
(ι)	I	Iota.	愛阿達
(κ)	K	Cappa.	卡扒
(λ)	Λ	Lambda.	郎姆塔
(μ)	M	Mu.	紗
(ν)	N	Nu.	紐
ξ	Ξ	Xi.	耶格洒
ο	O	Omiklon.	阿米可倫
(π)	Π	Pi.	排
(ρ)	P	Rho.	簍
(Σ)	Σ	Sigma.	西格馬
τ	T	Tau.	橢
ύ	Υ	Upsilon.	亞皮西倫
(φ)	Φ	Phi.	腓
(χ)	X	Chi.	啓
(ψ)	Ψ	Psi.	撲洗
(ω)	Ω	Omega.	阿蔑嘉

三角函數之關係

1. 橫線上兩函數之積=縱線上三函數之積=1.



例如 $\sin a \csc a = \cot a \tan a = \sec a \cos a$

$= \sin a \cot a \sec a = \csc a \tan a \cos a = 1.$

2. (中間橫線上兩函數之和)×(他之斜線上兩函數之積)=(各端橫線上兩函數之差)÷(他之斜線上兩函數之積)=1.

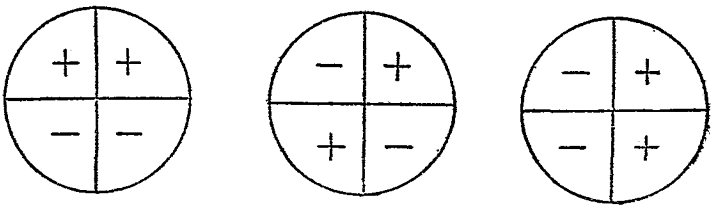
例如 $(\cot a + \tan a)\sin a \cos a$

$= (\csc a - \sin a) \div (\cot a \cos a)$

$= (\sec a - \cos a) \div (\sin a \tan a) = 1.$

三角函數之正負

正弦及餘割　　正切及餘切　　餘弦及正割



代 數 及 三 角 函 數 式

(1)

$$(a+b)(a-b)=a^2-b^2$$

$$\sin(a+b)\sin(a-b)=\sin^2 a-\sin^2 b$$

(2)

$$(a+1)(a^2+1)(a^4+1)\cdots\cdots(a^{n-1}+1)=\frac{a^n-1}{a-1}$$

$$(2\cos a-1)(2\cos 2a-1)(2\cos 2^2a-1)$$

$$\cdots\cdots(2\cos 2^{n-1}a-1)=\frac{2\cos 2^n a+1}{2\cos a+1}$$

(3)

令 $\sin^{-1}a=x$ 則 $a=\sin x$

令 $b^{-1}a=x$ 則 $a=bx$

(4)

設 $a+b+c=0$ 則

$$\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\times\frac{a^2+b^2+c^2}{2}$$

設 $\cos a+\cos b+\cos c=0$ 則

$$\frac{\cos 5a+\cos 5b+\cos 5c}{5}=$$

$$\frac{\cos 3a+\cos 3b+\cos 3c}{3}\times\frac{\cos 2a+\cos 2b+\cos 2c}{2}$$

幾何學及三角法定理

$$c^2=a^2+b^2-2ab\cos C$$

(1)

設 $C=90^\circ$ 則 $c^2=a^2+b^2$

設 $C<90^\circ$ 則 $c^2=a^2+b^2-2ab\cos C<a^2+b^2$

設 $C>90^\circ$ 則 $c^2=a^2+b^2+2ab\cos C>a^2+b^2$

(2)

$$\frac{\sin C}{c}=\frac{\sqrt{(1-\cos^2 C)}}{c}=\frac{\sqrt{\left\{1-\left(\frac{a^2+b^2-c^2}{2ab}\right)^2\right\}}}{c}$$

$$=\frac{\sqrt{(2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4)}}{2abc}=\frac{\sin A}{a}=\frac{\sin B}{b}$$

(3)

$$\frac{\sin A\pm\sin B}{a\pm b}=\frac{\sin C}{c}$$

$\because$ $a+b>c$ $\therefore$ $\sin A+\sin B>\sin C$

$\because$ $a-b<c$ $\therefore$ $\sin A+\sin B<\sin C$

(4)

$$\frac{\sin^2 A\pm\sin^2 B}{a^2\pm b^2}=\frac{\sin^2 C}{c^2}$$

設 $a^2\pm b^2=c^2$ 則 $\sin^2 A\pm\sin^2 B=\sin^2 C$

設 $a^2\pm b^2>c^2$ 則 $\sin^2 A\pm\sin^2 B>\sin^2 C$

設 $a^2\pm b^2<c^2$ 則 $\sin^2 A\pm\sin^2 B<\sin^2 C$

三角函數之虛僞

$$\sin A=\sin(\pi-A)$$

$$\therefore\quad A=\pi-A\quad\therefore\quad A=\tfrac{1}{2}\pi$$

又 $\dfrac{1}{2\sqrt{-1}}(e^{A\sqrt{-1}}-e^{-A\sqrt{-1}})=\dfrac{1}{2\sqrt{-1}}\{e^{(\pi-A)\sqrt{-1}}-e^{-(\pi-A)\sqrt{-1}}\}$

$$\therefore\quad e^{A\sqrt{-1}}-\frac{1}{e^{A\sqrt{-1}}}=e^{(\pi-A)\sqrt{-1}}-\frac{1}{e^{(\pi-A)\sqrt{-1}}}$$

卽 $\{e^{A\sqrt{-1}}-e^{(\pi-A)\sqrt{-1}}\}(e^{\pi\sqrt{-1}}-1)=0$

$\therefore\quad e^{A\sqrt{-1}}-e^{(\pi-A)\sqrt{-1}}=0,\quad e^{A\sqrt{-1}}=e^{(\pi-A)\sqrt{-1}}$

$$\therefore\quad A=\pi-A\quad\therefore\quad A=\tfrac{1}{2}\pi$$

或 $e^{\pi\sqrt{-1}}-1=0,\quad e^{\pi\sqrt{-1}}=1=e^{0}\quad\therefore\quad\pi\sqrt{-1}=0.$

平面三角法講義

第壹編

角之測法

1. 三角法 三角法.西語爲Trigonometry蓋由二希臘語而成.其一表示三角形.又其一表示其測法也.故元來三角法爲推究三角形之邊及角之學科也.此三角形在平面上者.謂之**平面三角法**.在球面上者.謂之**球面三角法**.

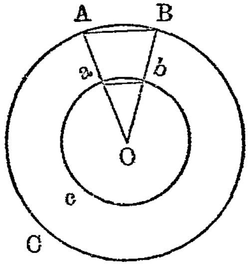
平面三角形.今時有更廣大之意義.卽自三角形有無之關係.以至代數學上之推究.凡含有關於平面角者皆是.

2. 角之測法 測角之法有二.卽**六十分法**與**弧度法**是也.其他百分法.乃用法國度分秒之法.今不用.

3. 六十分法 六十分法.以角之單位爲直角.定直角爲單位.有三便利.第一.凡直角皆相等.第二.直角容易畫得.第三.直角容易識別.

然選定比直角更小之單位.於實際上亦便利.故分直角爲90等分.命其一分爲度.分一度爲60等分.命其一分爲分.分一分爲60等分.命其一分爲秒.而度分秒之記法.則爲(°),(′),(″).故直角$=90°$. $1°=60'$. $1'=60''$.

4. 單位圓 半徑爲1所畫得之圓.謂之單位圓.定單位圓周之長.等於2π.



5. 定理 圓周之變化.如其直徑.

〔證〕 ABC, abc爲同心二圓.O爲其中心.分圓周ABC爲若干等分(n等分).取其一部分之弧AB.連結AO及BO.截圓周abc於a及b.其所截之弧ab.亦爲圓周abc若干等分(n等分)之一部分.由是連結弦AB及ab.則兩三角形ABO, abo.爲相似形.

故令$AO=R$, $aO=r$. 則 弦AB：弦ab :: R：r, 卽

$n.AB : n.ab :: R : r$. 但$n.AB$及$n.ab$.爲兩圓ABC及abc內切正多角形之周邊.而n增大.從而此各周邊近於其圓周.

由是$n.AB$之極限=圓周ABC.(此以C代之) $n.ab$之極限=圓周abc.(此以c代之) 故至極限. $C : c :: R : r :: 2R : 2r$,

由是圓周與其直徑成比例.故云圓周之變化如其直徑.

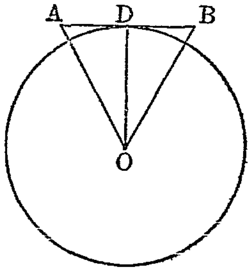
6. 定理 圓周爲C.半徑爲R.則$C=2R\pi$.

〔證〕 於前節之定理.令$r=1$.則$c=2\pi$.(4.節)

故 $c : 2\pi :: R : 1$. $\therefore$ $C=2R\pi$.

7. 定理 圓之面積,等於$R^2\pi$.

〔證〕 令AB爲圓之外切正n多角形之一邊.D爲AB之切點.



則外切正n多角形之面積

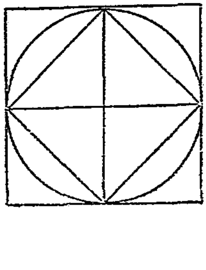
$=n$三角形$ABO=n\frac{1}{2}AB.DO=\frac{1}{2}n.AB.R$.

n增大.從而外切正n多角形之周邊$n.AB$近於圓周.外切正n多角形之面積近於圓之面積.故$n.AB$之極限=圓周$=2R\pi$, 外切正n多角形面積之極限=圓之面積.

由是圓之面積$=\frac{1}{2}(2R\pi)R=R^2\pi$.

8. π之値

求π之値.先令P_n及p_n爲一圓之外切及內切正n多角形之面積.P_{2n}及p_{2n}爲其圓之外切及內切正$2n$多角形之面積.則$p_{2n}=\sqrt{p_n \cdot P_n}$及$P_{2n}=(2p_n P_n)\div(p_n+p_{2n})$(見平面幾何學講義例題1417.)



今作單位圓外切及內切正四角形.則$P_4=2\times2=4$, $p_4=\frac{1}{2}P_4=2$ 故由此公式.得

$p_8=\sqrt{2.4}=2\sqrt{2}=2·8284271$,

$P_8=(2p_4P_4)\div(p_4+P_4)=(2\times2\times4)\div(2+2\sqrt{2})=8(\sqrt{2}-1)$
$=3·3137085$.

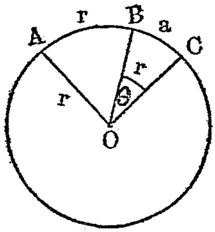
次第施此方法.增多其邊數.從而外切及內切正多角形之面積.互相接近.而近於圓之面積.卽

$p_{16}=3.0614674$	$P_{16}=3.1825979$
$p_{32}=3.1214451$	$P_{32}=3.1517249$
$p_{64}=3.1365485$	$P_{64}=3.1441184$
$p_{128}=3.1403311$	$P_{128}=3.1422236$
$p_{256}=3.1412772$	$P_{256}=3.1417504$
$p_{512}=3.1415138$	$P_{512}=3.1416321$
$p_{1024}=3.1415729$	$P_{1024}=3.1416025$
$p_{2048}=3.1415877$	$P_{2048}=3.1415951$
$p_{4096}=3.1415914$	$P_{4096}=3.1415933$

由是內切及外切正多角形之周邊.至小數5位相合.故單位圓半周之長π.殆等於3.14159.即$\pi=3.14159....$

9. 半徑角

對等於半徑長之弧之中心角.爲半徑角.如圖.令弧AB=r.則



角AOB=半徑角.又令對任意之弧BC之角爲θ.弧BC=a.則中心角與其對弧有比例.

故 半徑角AOB:$\theta::r:a$.

$\therefore$ 半徑角AOB$=\dfrac{\theta r}{a}$.

θ因a之變而變.而$\theta=360°$.

則 a=圓周$=2r\pi$(6.定理)

由是 半徑角AOB$=\dfrac{360°r}{2r\pi}=\dfrac{180°}{\pi}=\dfrac{180°}{3.1416}=57.29578$度.

10. 弧度法

令半徑角爲單位.

於單位圓.半徑角所對之弧之長爲1.180°所對之弧之長爲π.故90°及360°所對之弧之長爲$\frac{1}{2}\pi$及2π.

一般 $\theta=\dfrac{a}{r}$半徑角(前節)

故 $\theta=90°$, 則$a=\frac{1}{2}r\pi$, 故 $90°=\frac{1}{2}\pi\times$半徑角.

$\theta=180°$, 則$a=r\pi$, 故$180°=\pi\times$半徑角.

$\theta=360°$, 則$a=2r\pi$, 故$360°=2\pi\times$半徑角.

是在單位圓90°, 180°, 360°.可用$\frac{1}{2}\pi$, π, 2π代之.

例　題　一

1. 表示半徑角爲度分秒.

2. 示$1°$等於半徑角之項.

3. 示$13°$等於半徑角之小數.

4. 示半徑角之$\frac{\pi}{13}$之度及度之小數.

5. 圓之中心角所對向之弧等於直徑,問直角三分之一,爲此角之何部分.

6. 繫犬以鍊,使犬不出4840平方碼以外之地,問鍊須長若干.

7. 於12時15分之時,示其時計兩針交角之度分秒.

8. 汽車走半徑r哩之圓弧上,每時速率爲a哩,問n秒時走角之何秒.

9. 距塔1哩之遠,其視角爲$1°$,問塔高若干碼.

註. 1 哩=1760 碼.

10. 在1哩之距離對向$1'$之角,其物體之長如何.

11. 直徑8尺之圓之中心角,對向10尺之弧,試求其度分秒.

12. 高6尺之人,視角爲$1'$,其距離如何.

13. 有1哩爲半徑之圓,其中心角所對之弧爲$1''$,求其弧之長爲若干吋.

14. $3°$爲單位之.15,問其單位爲何度,又直角爲其單位之何倍.

例題解自1至14.

(1) 由9節得半徑角$=57^\circ.29578=57^\circ11'44''.8$.

(2) 由10節得$180^\circ=$半徑角$\times\pi$, $\therefore 1^\circ=\frac{\pi}{180}$半徑角$=.01745$半徑角.

(3) $13^\circ=13\times\frac{\pi}{180}$半徑角(前例)$=.22689$半徑角.

(4) $57^\circ.29578\times\frac{\pi}{13}=13^\circ.846153$.

(5) 令中心角爲x. 依題意得$2r\pi:2r::360^\circ:x^\circ$, $\therefore x^\circ=\frac{360^\circ}{\pi}$,

故 $\frac{90^\circ}{3}\div\frac{360^\circ}{\pi}=\frac{\pi}{12}$.

(6) 令鍊長爲x碼. 由下節定理得$x^2\pi=4840$.

$\therefore x=\sqrt{\frac{4840}{\pi}}=\sqrt{\frac{4840}{3.1416}}=39.24$碼.

(7) 於12時15分. 分針之位置. 在12時至15分之處. 又時針之速爲分針之速$\frac{5}{60}$卽$\frac{1}{12}$. 故時針之位置. 在12時至15分$\times\frac{1}{12}$之處. 由是兩針之交角$=15$分-15分$\times\frac{1}{12}=\frac{55}{4}$分. 但時計盤面之一周爲$360^\circ$. 分爲60分. 故1分$=360^\circ\div60=6^\circ$. 故兩針之交角$=\frac{55}{4}\times6^\circ=82^\circ30'$.

(8) 令n秒所走之秒數爲x, 則

$2r\pi:\frac{an}{60\times60}::360\times60\times60'':x$, $\therefore x=\frac{180an}{r\pi}$秒.

(9) 當半徑大而中心角微小時. 則所對向之弦. 殆與弧相等. 故得令塔高爲x. 卽以x爲半徑1哩之圓之中心角所對向之弧考之. 則

$2\times1760\pi:x::360^\circ:1^\circ$, $\therefore x=\frac{88}{9}\pi=30.7179$碼.

(10) 與前例同. 求得答數爲1呎6吋. 但1碼$=3$呎, 1呎$=12$吋.

(11) 令所求之中心角爲x. 則$8\pi:10::360^\circ:x$, $\therefore x=143^\circ14'20''.809$.

(12) 與例題9及10同. 求得答數6875.4碼.

(13) 與前例同. 答.307吋.

(14) $\because .15:1::3^\circ:x^\circ$, $\therefore x^\circ=\frac{3^\circ}{.15}=20^\circ$.

又 $3^\circ:90^\circ::.15:$直角單位之數, $\therefore$ 直角單位之數$=\frac{90\times.15}{3}=4.5$.

15. 200°中心角所對之弧,殆等於半徑$3\frac{1}{2}$倍.試求其π之值.至小數二位止.

16. 設自轉車之直徑爲50吋.則9秒間回轉之數,殆等於1時間所走之哩數.試證之.

17. 圓之半徑20吋.試求其四等分之同心三圓之半徑各如何.

18. 圓之半徑爲1吋.今以等於半徑之弦截取弓形.試求其面積爲若干平方吋.但至小數三位止.

19. 在一定之距離.自地球面上二點下鉛垂線.其交角之度爲m秒.若其點在地球面上h高.其交角之度爲n秒.則地球之半徑等於$\frac{nh}{m-n}$.試證之.

20. 二角之差爲1°.其和之弧度爲1.求此二角之弧度各若干.

21. 地球之直徑爲7920哩.問地球中心角1′所對之弧長若干.

22. 地球之直徑爲7912哩.則其1°之中心角所對向之弧之長,求至小數三位,爲69.045.試證之.

23. 設走地球半徑95000000哩之圓周.須365日.則1秒間殆走19哩.試證之.

24. 地球之直徑(7900哩)向太陽之角爲17″.8.太陽光線達於地球須8分13.3秒.問光線1秒之速如何.

25. 地球與月之距離殆等於地球之半徑60倍.求地球半徑對月中心之角度.

例題解自 15. 至 25.

(15) ∵ $360^\circ:200^\circ::2r\pi:3\frac{1}{2}r$, ∴ $\pi=3.15$. (略近值)

(16) 令 9 秒間所回轉之數為 x, 1 時間之速為 y 哩, 則

$$9:1\times60\times60::50\pi x:1760\times3\times12y,$$

$$\therefore\ \frac{x}{y}=\frac{9\times1760\times3\times12}{1\times60\times60\times50\pi}=\frac{396}{392.7}.$$ 由是知 y 殆等於 x.

(17) 令最小圓之半徑為 x 吋. 順次令其外圓之半徑為 y, z 吋. 則 $4:1::20^2:x^2$, ∴ $x=10$ 吋. 又 $2:1::20^2:y^2$, ∴ $y=14.142$ 吋.

又 $4:3::20^2:z^2$, ∴ $z=17.328$ 吋.

(18) 令 O 為中心. AB 為弦. 則 AO=BO=AB=1 吋.

故中心角 AOB=60°. 由是扇形 AOB 之面積 $=1^2\pi\times\frac{60}{360}=.5236$ 平方吋.

又等邊三角形 AOB 之面積 $=\frac{1\times\sqrt{3}}{2\times2}=.4331$ 平方吋.

故弓形之面積 $=.5236$ 平方吋 $-.4331$ 平方吋 $=.0905$ 平方吋.

(19) 令地球之半徑為 r, 定距離為 x, 而地球面上之垂鉛線, 向地球之中心. 故 $2r\pi:x::360\times60\times60'':m$, 及 $2(r+h)\pi:x::360\times60\times60'':n$,

從此兩比例式消去 r. 即得.

(20) 令兩角之弧度為 x 及 y, 則 $x-y=1\times\frac{2\pi}{360}$, $x+y=1$,

$$\therefore\ x=\frac{1}{2}\left(1+\frac{\pi}{180}\right),\quad y=\frac{1}{2}\left(1-\frac{\pi}{180}\right).$$

(21) 令所求之弧長為 x 哩, 則 $7920\pi:x::360\times60':1'$, ∴ $x=1.15192$ 哩.

(22) ∵ 7912 哩 $:x::360^\circ:1^\circ$, ∴ $x=69.045$ 哩.

(23) 1 秒之運動 $=\frac{2\times95000000\pi}{365\times24\times60\times60}=19$ 哩. (略近值)

(24) 令地球與太陽之距離為 x 哩, 則以 x 為半徑, 太陽之中心為中心, 所畫得之圓. 其對於弧 7900 哩之中心角為 $17''.8$. 故

$2x\pi:7900::360\times60\times60'':17''.8$, 從此即得 x 之值.

而光線 1 秒之速 $=\frac{x}{493.3\text{秒}}=185575$ 哩.

(25) 令地球之半徑 $=r$, 地球與月之距離 $=60r$, 與前例同法, 求得之角令為 x 分; 而 $2\times60r\pi:r::360\times60':x$,

∴ $x=57'.29578$.

26. 月之視直徑爲30′,以直徑6吋之圓板全掩月.問目與圓板之距離須若干.

27. 地球與月兩中心之距離,爲地球半徑59.964倍.而地球半徑爲3963哩.月繞地球一周須27日7時43分11秒.問月每時運動若干哩.

28. 地球之半徑(3963哩)向月之角度爲57′3″.16,求地球與月之距離.

29. 同上.月之直徑向地球之角度爲1868,″求月之直徑若干.

30. 1882年.由測量金星經過太陽面,知地球半徑3963哩.對於太陽成角8″.82.試依此求地球太陽之距離.

例題解自26至30.

(26) 直徑6吋所對之角爲30′,故令所求之距離爲x呎.半徑x呎之圓之中心角爲30′.其角所對之弧爲6吋.

故 $2x\pi:\frac{6}{12}::360\times60':30'$, $\therefore x=57.29578$ 呎.

(27) 27日7時43分11秒$=\frac{2360591}{3600}$時.

$\therefore$ 所求之哩數$=\dfrac{2\times3963\times59.964\times\pi}{\frac{2360591}{3600}}=2276$ 哩.

(28) $\because$ $2x\pi:3963::360\times60':57'\frac{3.16}{60}$, $\therefore x=238793$ 哩.

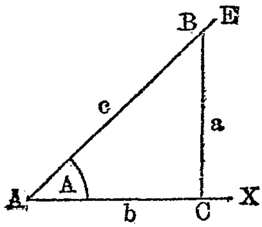
(29) $\because$ $2\times238793\pi:x::360\times60\times60'':1868''$, $\therefore x=2162$ 哩.

(30) $\because$ $2x\pi:3963::360\times60\times60'':8''.82$, $\therefore x=92678844$ 哩.

第貳編

銳角之三角函數

1. 三角函數之定義 任意一銳角EAX.自其一邊AE上任意取一點B.至他一邊AX上.作垂線BC.則得直角三角形BAC.稱此各二邊之比,曰關於其銳角之比.



依幾何學定理.此直角三角形BAC之各邊雖變其大.惟有同一之銳角BAC.則任何亦爲相似形.故知各邊之比.其値不變.今令BA=c, AC=b, CB=a, 及角BAC=A,則關於A角.其各二邊之比有六.如次.

$$\frac{\text{垂線}}{\text{斜線}}=\frac{a}{c}=\textit{sine}\,\text{A}\ (\text{正弦})\ \text{略爲}\ \sin\text{A},$$

$$\frac{\text{底線}}{\text{斜線}}=\frac{b}{c}=\textit{cosine}\,\text{A}\ (\text{餘弦})\ \text{略爲}\ \cos\text{A},$$

$$\frac{\text{垂線}}{\text{底線}}=\frac{a}{b}=\textit{tangent}\,\text{A}\ (\text{正切})\ \text{略爲}\ \tan\text{A},$$

$$\frac{\text{底線}}{\text{垂線}}=\frac{b}{a}=\textit{cotangent}\,\text{A}\ (\text{餘切})\ \text{略爲}\ \cot\text{A},$$

$$\frac{\text{斜線}}{\text{底線}}=\frac{c}{b}=\textit{secant}\,\text{A}\ (\text{正割})\ \text{略爲}\ \sec\text{A},$$

$$\frac{\text{斜線}}{\text{垂線}}=\frac{c}{a}=\textit{cosecant}\,\text{A}\ (\text{餘割})\ \text{略爲}\ \text{cosec}\,\text{A}.$$

又 $1-\cos\text{A}=\textit{versine}\,\text{A}$ (正矢) 略爲 $\text{vers}\,\text{A}$,

$1-\sin\text{A}=\textit{coversine}\,\text{A}$ (餘矢) 略爲 $\text{covers}\,\text{A}$.

以上六比及正矢餘矢.統名三角函數.

2. 三角函數相互之關係 表示加次

〔第一〕 反商之公式

$$\sin A=\frac{1}{\operatorname{cosec} A},\quad \cos A=\frac{1}{\sec A},\quad \tan A=\frac{1}{\cot A}. \qquad (1)$$

(證) 由 1. 節 $\sin A=\frac{a}{c}$, $\operatorname{cosec} A=\frac{c}{a}$,

$$\therefore\ \sin A=\frac{a}{c}=\frac{1}{\frac{c}{a}}=\frac{1}{\operatorname{cosec} A}.$$

又 $\cos A=\frac{b}{c}$, $\sec A=\frac{c}{b}$, $\therefore\ \cos A=\frac{1}{\frac{c}{b}}=\frac{1}{\sec A}$

$$\tan A=\frac{a}{b},\quad \cot A=\frac{b}{a},\quad \therefore\ \tan A=\frac{1}{\frac{b}{a}}=\frac{1}{\cot A}.$$

〔第二〕 比之公式

$$\tan A=\frac{\sin A}{\cos A},\quad \cot A=\frac{\cos A}{\sin A}. \qquad (2)$$

(證) 由 1. 節 $\tan A=\frac{a}{b}$, $\sin A=\frac{a}{c}$, $\cos A=\frac{b}{c}$,

$$\therefore\ \tan A=\frac{a}{b}=\frac{\frac{a}{c}}{\frac{b}{c}}=\frac{\sin A}{\cos A},\quad \cot A=\frac{b}{a}=\frac{\frac{b}{c}}{\frac{a}{c}}=\frac{\cos A}{\sin A}.$$

〔第三〕 各函數之各平方.

$$\sin^2 A+\cos^2 A=1,\ \tan^2 A+1=\sec^2 A,\ \cot^2 A+1=\operatorname{cosec}^2 A. \qquad (3)$$

(證) 由 2. 節 $a^2+b^2=c^2$, 順次以 c^2, b^2, 及 a^2 除之.得

$$\left(\frac{a}{c}\right)^2+\left(\frac{b}{c}\right)^2=1,\qquad \therefore\ \sin^2 A+\cos^2 A=1,$$

$$\left(\frac{a}{b}\right)^2+1=\left(\frac{c}{b}\right)^2,\qquad \therefore\ \tan^2 A+1=\sec^2 A,$$

$$1+\left(\frac{b}{a}\right)^2=\left(\frac{c}{a}\right)^2,\qquad \therefore\ 1+\cot^2 A=\operatorname{cosec}^2 A.$$

(注意) $\sin^2 A$ 本 $(\sin A)^2$ 之略號.餘倣此.

例題二

1. 求次各式之證.

$$\sin\alpha=\sqrt{1-\cos^2\alpha}=\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}=\frac{1}{\sqrt{\cot^2\alpha+1}}=\frac{\sqrt{\sec^2\alpha-1}}{\sec\alpha}=\frac{1}{\operatorname{cosec}\alpha},$$

$$\cos\alpha=\sqrt{1-\sin^2\alpha}=\frac{1}{\sqrt{1+\tan^2\alpha}}=\frac{\cot\alpha}{\sqrt{\cot^2\alpha+1}}=\frac{1}{\sec\alpha}=\frac{\sqrt{\operatorname{cosec}^2\alpha-1}}{\operatorname{cosec}\alpha},$$

$$\tan\alpha=\frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}=\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}=\frac{1}{\cot\alpha}=\sqrt{\sec^2\alpha-1}=\frac{1}{\sqrt{\operatorname{cosec}^2\alpha-1}},$$

$$\cot\alpha=\frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha}=\frac{\cos\alpha}{\sqrt{1-\cos^2\alpha}}=\frac{1}{\tan\alpha}=\frac{1}{\sqrt{\sec^2\alpha-1}}=\sqrt{\operatorname{cosec}^2\alpha-1},$$

$$\sec\alpha=\frac{1}{\sqrt{1-\sin^2\alpha}}=\frac{1}{\cos\alpha}=\sqrt{1+\tan^2\alpha}=\frac{\sqrt{\cot^2\alpha+1}}{\cot\alpha}=\frac{\operatorname{cosec}\alpha}{\sqrt{\operatorname{cosec}^2\alpha-1}},$$

$$\operatorname{cosec}\alpha=\frac{1}{\sin\alpha}=\frac{1}{\sqrt{1-\cos^2\alpha}}=\frac{\sqrt{1+\tan^2\alpha}}{\tan\alpha}=\sqrt{\cot^2\alpha+1}=\frac{\sec\alpha}{\sqrt{\sec^2\alpha-1}}.$$

詳解

(1) 從 2 節第三. $\sin^2\alpha=1-\cos^2\alpha$ $\therefore \sin\alpha=\sqrt{1-\cos^2\alpha}$,

從 2 節第二. $\sin\alpha=\tan\alpha.\cos\alpha=\frac{\tan\alpha}{\sec\alpha}$ (同第一) $=\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}$ (同第三).

$\sin\alpha=\frac{1}{\operatorname{cosec}\alpha}$ (同第一) $=\frac{1}{\sqrt{\cot^2\alpha+1}}$ (同第三),

$\cos\alpha=\sqrt{1-\sin^2\alpha}$ 以下證法.與前同樣.

從 2 節第二. $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}=\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}$,

$\tan\alpha=\frac{1}{\cot\alpha}$ (同第一) $=\frac{1}{\sqrt{\operatorname{cosec}^2\alpha-1}}$ (同第三) $=\sqrt{\sec^2\alpha-1}$ (同第三).

$\cot\alpha=\frac{\sqrt{1-\sin^2\alpha}}{\sin\alpha}$ 以下與前同樣.

$\sec\alpha=\frac{1}{\cos\alpha}=\frac{1}{\sqrt{1-\sin^2\alpha}}=\sqrt{1+\tan^2\alpha}$ (2 節第二第三).

$\operatorname{cosec}\alpha$ 之證法.亦同樣也.

2. 設 $\sin\alpha=\frac{3}{5}$, 則 $\cos\alpha$ 及 $\tan\alpha$ 之值如何.

3. 設 $\tan\alpha=\frac{m}{n}$, 則 $\sin\alpha$ 及 $\cos\alpha$ 之值如何.

4. 設 $\tan\alpha=\frac{2x(x+1)}{2x+1}$, 則 $\sin\alpha$ 及 $\cos\alpha$ 之值如何.

5. 有 $\tan\alpha=1$, 試求他之三角函數之值.

6. 設 $\cot\alpha=\frac{p}{q}$, 則 $\sin\alpha$ 及 $\cos\alpha$ 之值如何.

7. 設 $\sin\alpha=1$, 則 $\cos\alpha+\cot\alpha+\operatorname{cosec}\alpha=1$. 試證之.

8. 有 $\sin\alpha=.012$, 試求他之三角函數之值.

9. 設 $\operatorname{vers}\alpha=\frac{\sqrt{2}-1}{\sqrt{2}}$. 則

$\sin\alpha+\cos\alpha+\tan\alpha+\cot\alpha+\sec\alpha+\operatorname{cosec}\alpha$ 之值如何.

10. 設 $\tan^3\phi=\frac{\alpha}{\beta}$, 則 $\alpha\operatorname{cosec}\phi+\beta\sec\phi=\left(\alpha^{\frac{2}{3}}+\beta^{\frac{2}{3}}\right)^{\frac{3}{2}}$. 試證之.

11. 試以 $\operatorname{vers}\alpha$ 之項. 表示他之角函數.

例題解自2至11.

(2) $\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-\left(\frac{3}{5}\right)^2}=\frac{4}{5}$, $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{\frac{3}{5}}{\frac{4}{5}}=\frac{3}{4}$.

(3) $\sin\alpha=\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}=\frac{\frac{m}{n}}{\sqrt{1+\frac{m^2}{n^2}}}=\frac{m}{\sqrt{m^2+n^2}}$,

$\cos\alpha=\frac{1}{\sqrt{1+\tan^2\alpha}}=\frac{1}{\sqrt{1+\frac{m^2}{n^2}}}=\frac{n}{\sqrt{m^2+n^2}}$.

(4) 同前得 $\sin\alpha=\frac{2x(x+1)}{2x^2+2x+1}$, $\cos\alpha=\frac{2x+1}{2x^2+2x+1}$,

(5) $\sin\alpha=\cos\alpha=\frac{1}{\sqrt{2}}$, $\cot\alpha=1$, $\sec\alpha=\operatorname{cosec}\alpha=\sqrt{2}$,

$\operatorname{vers}\alpha=1-\cos\alpha=1-\frac{1}{\sqrt{2}}=\operatorname{covers}\alpha$.

(6) $\tan\alpha=\frac{1}{\cot\alpha}=\frac{q}{p}$, $\operatorname{cosec}\alpha=\sqrt{1+\cot^2\alpha}=\frac{1}{q}\sqrt{p^2+q^2}$,

$\sin\alpha=\frac{1}{\operatorname{cosec}\alpha}=\frac{q}{\sqrt{p^2+q^2}}$, $\sec\alpha=\sqrt{1+\tan^2\alpha}=\frac{1}{p}\sqrt{p^2+q^2}$,

$\cos\alpha=\frac{1}{\sec\alpha}=\frac{p}{\sqrt{p^2+q^2}}$.

(7) $\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-1}=0$, $\cot\alpha=\frac{\cos\alpha}{\sin\alpha}=0$,

$\operatorname{cosec}\alpha=\frac{1}{\sin\alpha}=1$, $\therefore\ \cos\alpha+\cot\alpha+\operatorname{cosec}\alpha=1$.

(8) $\cos\alpha=\sqrt{1-\sin^2\alpha}=\sqrt{1-.012^2}=.999$, $\tan\alpha=.012$,

$\cot\alpha=83.321$, $\sec\alpha=1.001$, $\operatorname{cosec}\alpha=83.333$.

(9) $\cos\alpha=1-\operatorname{vers}\alpha=1-\frac{\sqrt{2}-1}{\sqrt{2}}=\frac{1}{\sqrt{2}}$, $\sec\alpha=\frac{1}{\cos\alpha}=\sqrt{2}$,

$\sin\alpha=\sqrt{1-\cos^2\alpha}=\frac{1}{\sqrt{2}}$, $\operatorname{cosec}\alpha=\frac{1}{\sin\alpha}=\sqrt{2}$,

$\tan\alpha=\frac{\sin\alpha}{\cos\alpha}=1=\cot\alpha$.

由是原式$=\frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}+1+1+\sqrt{2}+\sqrt{2}=3\sqrt{2}+2$.

(10) $\alpha\operatorname{cosec}\phi+\beta\sec\phi=\alpha\sqrt{1+\cot^2\phi}+\beta\sqrt{1+\tan^2\phi}$

$$=\frac{\alpha(1+\tan^2\phi)^{\frac{1}{2}}}{\tan\phi}+\beta(1+\tan^2\phi)^{\frac{1}{2}}=(1+\tan^2\phi)^{\frac{1}{2}}\left(\frac{\alpha}{\tan\phi}+\beta\right)$$

$$=\left(1+\frac{\alpha^{\frac{2}{3}}}{\beta^{\frac{2}{3}}}\right)^{\frac{1}{2}}\left(\frac{\alpha\beta^{\frac{1}{3}}}{\alpha^{\frac{1}{3}}}+\beta\right)=\frac{1}{\beta^{\frac{1}{3}}}\left(\beta^{\frac{2}{3}}+\alpha^{\frac{2}{3}}\right)^{\frac{1}{2}}\beta^{\frac{1}{3}}\left(\alpha^{\frac{2}{3}}+\beta^{\frac{2}{3}}\right)=\left(\alpha^{\frac{2}{3}}+\beta^{\frac{2}{3}}\right)^{\frac{3}{2}}.$$

(11) $\cos\alpha=1-\operatorname{vers}\alpha$, $\sin\alpha=\sqrt{1-(1-\operatorname{vers}\alpha)^2}=\sqrt{2\operatorname{vers}\alpha-\operatorname{vers}^2\alpha}$,

$\tan\alpha=\frac{\sqrt{2\operatorname{vers}\alpha-\operatorname{vers}^2\alpha}}{1-\operatorname{vers}\alpha}$, $\cot\alpha=\frac{1-\operatorname{vers}\alpha}{\sqrt{2\operatorname{vers}\alpha-\operatorname{vers}^2\alpha}}$,

$\sec\alpha=\frac{1}{1-\operatorname{vers}\alpha}$, $\operatorname{cosec}\alpha=\frac{1}{\sqrt{2\operatorname{vers}\alpha-\operatorname{vers}^2\alpha}}$.

證次之各恆同式.

12. $\tan\alpha\sin\alpha+\cos\alpha\equiv\sec\alpha.$

13. $\cot\alpha\cos\alpha+\sin\alpha\equiv\sec\alpha.$

14. $\cos\alpha\tan\alpha+\sin\alpha\cot\alpha\equiv\sin\alpha+\cos\alpha.$

15. $\tan\alpha+\cot\alpha\equiv\sec\alpha\,\mathrm{cosec}\,\alpha.$

16. $\dfrac{1}{\tan\alpha+\cot\alpha}\equiv\sin\alpha\cos\alpha.$

17. $\dfrac{\sin\alpha+\cos\alpha}{\sec\alpha+\mathrm{cosec}\,\alpha}\equiv\sin\alpha\cos\alpha.$

18. $\sec^2\alpha+\mathrm{cosec}^2\alpha\equiv\sec^2\alpha\,\mathrm{cosec}^2\alpha.$

19. $\tan^2\alpha-\sin^2\alpha\equiv\tan^2\alpha\sin^2\alpha.$

20. $\cot^2\alpha-\cos^2\alpha\equiv\cot^2\alpha\cos^2\alpha.$

21. $(\tan\alpha-\sin\alpha)^2+(1-\cos\alpha)^2\equiv(\sec\alpha-\mathrm{cosec}\,\alpha)^2.$

22. $(\tan\alpha-1)^2+(1-\cot\alpha)^2\equiv(\sec\alpha-\mathrm{cosec}\,\alpha)^2.$

23. $(\sin\alpha+\cos\alpha)^2+(\sin\alpha-\cos\alpha)^2\equiv2.$

24. $\sin^4\alpha-\cos^4\alpha\equiv\sin^2\alpha-\cos^2\alpha.$

25. $\sin^4\alpha+\cos^4\alpha\equiv1-2\sin^2\alpha\cos^2\alpha.$

26. $\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\beta\equiv\sin^2\alpha-\sin^2\beta.$

27. $\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta\equiv\cos^2\alpha-\sin^2\beta.$

28. $\tan^2\alpha+\cot^2\alpha+2\equiv\sec^2\alpha\,\mathrm{cosec}^2\alpha.$

例 題 解 自 12 至 28.

(12) $\tan\alpha\sin\alpha+\cos\alpha=\dfrac{\sin^2\alpha}{\cos\alpha}+\cos\alpha=\dfrac{\sin^2\alpha+\cos^2\alpha}{\cos\alpha}=\sec\alpha.$

(13) 與 12 同樣.

(14) $\cos a \tan a = \sin a$, $\sin a \cot a = \cos a$.

(15) $\tan a + \cot a = \dfrac{\sin a}{\cos a} + \dfrac{\cos a}{\sin a} = \dfrac{\sin^2 a + \cos^2 a}{\cos a \sin a} = \sec a \operatorname{cosec} a.$

(16) 與 15. 同樣.

(17) $\dfrac{\sin a + \cos a}{\sec a + \operatorname{cosec} a} = \dfrac{\sin a \cos a(\sec a + \operatorname{cosec} a)}{\sec a + \operatorname{cosec} a} = \sin a \cos a$

(18) $\sec^2 a + \operatorname{cosec}^2 a = \dfrac{1}{\cos^2 a} + \dfrac{1}{\sin^2 a} = \dfrac{\sin^2 a + \cos^2 a}{\cos^2 a \sin^2 a} = \sec^2 a \operatorname{cosec}^2 a.$

(19) $\tan^2 a - \sin^2 a = \dfrac{\sin^2 a}{\cos^2 a} - \sin^2 a = \dfrac{\sin^2 a(1-\cos^2 a)}{\cos^2 a} = \tan^2 a \sin^2 a.$

(20) 與 19. 同樣.

(21) $(\tan a - \sin a)^2 + (1-\cos a)^2 = \left(\dfrac{\sin a}{\cos a} - \sin a\right)^2 + (1-\cos a)^2$

$= \dfrac{\sin^2 a}{\cos^2 a}(1-\cos a)^2 + (1-\cos a)^2 = \dfrac{(1-\cos a)^2(\sin^2 a + \cos^2 a)}{\cos^2 a}$

$= \left(\dfrac{1-\cos a}{\cos a}\right)^2 = (\sec a - 1)^2.$

(22) $(\tan a - 1)^2 + (1-\cot a)^2 = \left(\dfrac{\sin a}{\cos a} - 1\right)^2 + \left(1 - \dfrac{\cos a}{\sin a}\right)^2$

$= \dfrac{(\sin a - \cos a)^2}{\cos^2 a} + \dfrac{(\sin a - \cos a)^2}{\sin^2 a} = \dfrac{(\sin a - \cos a)^2(\sin^2 a + \cos^2 a)}{\cos^2 a \sin^2 a}$

$= \left(\dfrac{\sin a - \cos a}{\cos a \sin a}\right)^2 = (\sec a - \operatorname{cosec} a)^2.$

(23) $(\sin a + \cos a)^2 + (\sin a - \cos a)^2 = 2(\sin^2 a + \cos^2 a) = 2.$

(24) $\sin^4 a - \cos^4 a = (\sin^2 a + \cos^2 a)(\sin^2 a - \cos^2 a) = \sin^2 a - \cos^2 a.$

(25) $\sin^4 a + \cos^4 a = (\sin^2 a + \cos^2 a)^2 - 2\sin^2 a \cos^2 a.$

$= 1 - 2\sin^2 a \cos^2 a.$

(26) $\sin^2 a \cos^2 \beta - \cos^2 a \sin^2 \beta = \sin^2 a(1-\sin^2\beta) - (1-\sin^2 a)\sin^2\beta$

$= \sin^2 a - \sin^2 \beta$

(27) 與 26. 同樣.

(28) $\tan^2 a + \cot^2 a + 2 = \tan^2 a + \cot^2 a + 2\tan a \cot a = (\tan a + \cot a)^2$

$= \left(\dfrac{\sin a}{\cos a} + \dfrac{\cos a}{\sin a}\right)^2 = \left(\dfrac{\sin^2 a + \cos^2 a}{\cos a \sin a}\right)^2 = (\sec a \operatorname{cosec} a)^2.$

29. $\frac{\tan\alpha}{\tan\alpha-\tan\beta}=\frac{\cot\beta}{\cot\beta-\cot\alpha}$.

30. $\frac{\operatorname{cosec}\alpha+\cot\alpha}{\sec\alpha+\tan\alpha}=\frac{\sec\alpha-\tan\alpha}{\operatorname{cosec}\alpha-\cot\alpha}$.

31. $\sin\alpha(1+\tan\alpha)+\cos\alpha(1+\cot\alpha)=\operatorname{cosec}\alpha+\sec\alpha$.

32. $(\sec\alpha\sec\beta+\tan\alpha\tan\beta)^2-(\tan\alpha\sec\beta+\sec\alpha\tan\beta)^2=1$.

33. $\sin\alpha\cos\alpha\equiv\sqrt{(\sin\alpha-\sin^3\alpha)^2+(\cos\alpha-\cos^3\alpha)^2}$.

34. $2(\sin^6\alpha+\cos^6\alpha)+1=3(\sin^4\alpha+\cos^4\alpha)$.

35. $(\sin\alpha-\operatorname{cosec}\alpha)^2-(\tan\alpha-\cot\alpha)^2+(\cos\alpha-\sec\alpha)^2=1$.

36. $(\tan\alpha+\operatorname{cosec}\beta)^2-(\cot\beta-\sec\alpha)^2$

$=2\tan\alpha\cot\beta(\operatorname{cosec}\alpha+\sec\beta)$.

37. $\frac{1}{\cos\theta+\tan^2\theta\sin\theta}-\frac{1}{\sin\theta+\cot^2\theta\cos\theta}=\frac{\operatorname{cosec}\theta-\sec\theta}{\sec\theta\operatorname{cosec}\theta-1}$.

例題解自 29. 至 37.

(29) $\frac{\tan\alpha}{\tan\alpha-\tan\beta}=\frac{1/\cot\alpha}{1/\cot\alpha-1/\cot\beta}=\frac{\cot\beta}{\cot\beta-\cot\alpha}$.

(30) $\frac{(1+\cos\alpha)/\sin\alpha}{(1+\sin\alpha)/\cos\alpha}=\frac{(1+\cos\alpha)\cos\alpha}{(1+\sin\alpha)\sin\alpha}$

$=\frac{(1+\cos\alpha)\cos^2\alpha\sin\alpha}{(1+\sin\alpha)\sin^2\alpha\cos\alpha}=\frac{(1+\cos\alpha)(1-\sin^2\alpha)\sin\alpha}{(1+\sin\alpha)(1-\cos^2\alpha)\cos\alpha}$

$=\frac{(1-\sin\alpha)\sin\alpha}{(1-\cos\alpha)\cos\alpha}=\frac{(1-\sin\alpha)/\cos\alpha}{(1-\cos\alpha)/\sin\alpha}=\frac{\sec\alpha-\tan\alpha}{\operatorname{cosec}\alpha-\cot\alpha}$.

(31) $\sin\alpha\left(1+\frac{\sin\alpha}{\cos\alpha}\right)+\cos\alpha\left(1+\frac{\cos\alpha}{\sin\alpha}\right)$

$=\frac{\sin\alpha(\cos\alpha+\sin\alpha)}{\cos\alpha}+\frac{\cos\alpha(\sin\alpha+\cos\alpha)}{\sin\alpha}$

$=(\sin\alpha+\cos\alpha)\left(\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}\right)=(\sin\alpha+\cos\alpha)\frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha\cos\alpha}$

(32) $\sec^2\alpha\sec^2\beta+2\sec\alpha\tan\alpha\sec\beta\tan\beta+\tan^2\alpha\tan^2\beta$

$-(\tan^2\alpha\sec^2\beta+2\tan\alpha\sec\alpha\tan\beta\sec\beta+\sec^2\alpha\tan^2\beta)$

$=\sec^2\beta(\sec^2\alpha-\tan^2\alpha)-\tan^2\beta(\sec^2\alpha-\tan^2\alpha)$

$=\sec^2\beta-\tan^2\beta=1.$

(33) $\sin\alpha\cos\alpha=\sqrt{\sin^2\alpha\cos^2\alpha(\sin^2\alpha+\cos^2\alpha)}$

$=\sqrt{\cos^2\alpha\sin^4\alpha+\sin^2\alpha\cos^4\alpha}$

$=\sqrt{\cos^2\alpha(1-\cos^2\alpha)^2+\sin^2\alpha(1-\sin^2\alpha)^2}$

$=\sqrt{(\cos\alpha-\cos^3\alpha)^2+(\sin\alpha-\sin^3\alpha)^2}$

(34) $2(\sin^6\alpha+\cos^6\alpha)+1=2(\sin^6\alpha+\cos^6\alpha)+(\sin^2\alpha+\cos^2\alpha)^3$

$=(\sin^2\alpha+\cos^2\alpha)\{2(\sin^4\alpha-\sin^2\alpha\cos^2\alpha+\cos^4\alpha)+(\sin^2\alpha+\cos^2\alpha)^2\}$

$=3(\sin^4\alpha+\cos^4\alpha).$

(35) $\sin^2\alpha-2+\operatorname{cosec}^2\alpha-(\tan^2\alpha-2+\cot^2\alpha)+(\cos^2\alpha-2+\sec^2\alpha)$

$=\sin^2\alpha+\cos^2\alpha-2+(\operatorname{cosec}^2\alpha-\cot^2\alpha)+(\sec^2\alpha-\tan^2\alpha)$

$=1-2+1+1=1.$

(36) $\tan^2\alpha+2\tan\alpha\operatorname{cosec}\beta+\operatorname{cosec}^2\beta-(\cot^2\beta-2\cot\beta\sec\alpha+\sec^2\alpha)$

$=(\operatorname{cosec}^2\beta-\cot^2\beta)-(\sec^2\alpha-\tan^2\alpha)+\dfrac{2\sin\alpha}{\cos\alpha\sin\beta}+\dfrac{2\cos\beta}{\sin\beta\cos\alpha}$

$=1-1+\dfrac{2\sin\alpha\cos\beta(\sec\beta+\operatorname{cosec}\alpha)}{\cos\alpha\sin\beta}=2\tan\alpha\cot\beta(\operatorname{cosec}\alpha+\sec\beta).$

(37) $\dfrac{1}{\cos\theta+\dfrac{\sin^3\theta}{\cos^2\theta}}-\dfrac{1}{\sin\theta+\dfrac{\cos^3\theta}{\sin^2\theta}}$

$=\dfrac{\cos^2\theta}{\cos^3\theta+\sin^3\theta}-\dfrac{\sin^2\theta}{\sin^3\theta+\cos^3\theta}=\dfrac{\cos^2\theta-\sin^2\theta}{\sin^3\theta+\cos^3\theta}$

$=\dfrac{\cos-\sin\theta}{\cos^2\theta-\cos\theta\sin\theta+\sin^2\theta}=\dfrac{\cos\theta-\sin\theta}{1-\cos\theta\sin\theta}$

$=\dfrac{\dfrac{\cos\theta-\sin\theta}{\cos\theta\sin\theta}}{\dfrac{1-\cos\theta\sin\theta}{\cos\theta\sin\theta}}=\dfrac{\operatorname{cosec}\theta-\sec\theta}{\sec\theta\operatorname{cosec}\theta-1}.$

38. $\tan^2\alpha+\cot^2\alpha+1=(1+\tan\alpha+\tan^2\alpha)(1-\cot\alpha+\cot^2\alpha).$

39. $(1+\cos\alpha-\sin^2\alpha)^2(1-\cos\alpha)^2$

$$+(1+\sin\alpha-\cos^2\alpha)(1-\sin\alpha)^2=\sin^2\alpha\cos^2\alpha.$$

40. 設 $\dfrac{\cos^4\alpha}{\cos^2\beta}+\dfrac{\sin^4\alpha}{\sin^2\beta}=1,$ 則

$\dfrac{\cos^4\beta}{\cos^2\alpha}+\dfrac{\sin^4\beta}{\sin^2\alpha}=1.$ 試證之.

41. 設 $\dfrac{\cos^3\theta}{\cos\alpha}+\dfrac{\sin^3\theta}{\sin\alpha}=1,$ 則

$\left(\dfrac{\cos\alpha}{\cos\theta}-\dfrac{\sin\alpha}{\sin\theta}\right)\left(\dfrac{\cos\alpha}{\cos\theta}+\dfrac{\sin\alpha}{\sin\theta}+1\right)=0.$ 試證之.

42. 設 $\cos\alpha=c,\ \sin\alpha=s,$ 則

$c^{12}+4c^{10}s^2+5c^8s^4-5c^4s^8-4c^2s^{10}-s^{12}=c^2-s^2.$ 試證之.

例題解自 38. 至 42.

(38) $\tan^2\alpha+\cot^2\alpha+1=\tan^2\alpha+\dfrac{1}{\tan^2\alpha}+1=\dfrac{\tan^4\alpha+\tan^2\alpha+1}{\tan^2\alpha}$

$=\dfrac{(\tan^2\alpha+\tan\alpha+1)(\tan^2\alpha-\tan\alpha+1)}{\tan^2\alpha}$

$=(\tan^2+\tan\alpha+1)(1-\cot\alpha+\cot^2\alpha).$

(39) $(1+\cos\alpha-\sin\alpha)^2(1-\cos\alpha)^2+(1+\sin\alpha-\cos^2\alpha)^2(1-\sin\alpha)^2$

$=(\cos\alpha+\cos^2\alpha)^2(1-\cos\alpha)^2+(\sin\alpha+\sin^2\alpha)^2(1-\sin\alpha)^2$

$=\cos^2\alpha(1-\cos^2\alpha)^2+\sin^2\alpha(1-\sin^2\alpha)^2$

$=\cos^2\alpha-\sin^4\alpha+\sin^2\alpha\cos^4\alpha=\sin^2\alpha\cos^2\alpha(\sin^2\alpha+\cos^2)$

$=\sin^2\alpha\cos^2\alpha.$

(40) 去第一分母.得 $\cos^4\alpha\sin^2\beta+\sin^4\alpha\cos^2\beta=\cos^2\beta\sin^2\beta,$

$\cos^2\alpha(1-\sin^2\alpha)\sin^2\beta+\sin^2\alpha(1-\cos^2\alpha)\cos^2\beta=\cos^2\beta\sin^2\beta,$

$\cos^2\alpha\sin^2\beta+\sin^2\alpha\cos^2\beta-\cos^2\alpha\sin^2\alpha(\sin^2\beta+\cos^2\beta)=\cos^2\beta\sin^2\beta,$

即 $\cos^2\alpha\sin^2\beta+\sin^2\alpha\cos^2\beta-\cos^2\beta\sin^2\beta(\sin^2\alpha+\cos^2\alpha)=\cos^2\alpha\sin^2\alpha,$

$$\sin^2\alpha\cos^2\beta(1-\sin^2\beta)+\cos^2\alpha\sin^2\beta(1-\cos^2\beta)=\cos^2\alpha\sin^2\alpha,$$

$$\sin^2\alpha\cos^4\beta+\cos^2\alpha\sin^4\beta=\cos^2\alpha\sin^2\alpha,$$

$$\therefore\ \frac{\cos^4\beta}{\cos^2\alpha}+\frac{\sin^4\beta}{\sin^2\alpha}=1.$$

(41) 於第一之1.順次以 $\cos^2\alpha+\sin^2\alpha$ 及 $\cos^2\theta+\sin^2\theta$ 代之得

$$\frac{\cos^3\theta-\cos^3\alpha}{\cos\alpha}=\frac{\sin^3\alpha-\sin^3\theta}{\sin\alpha},$$

及 $$\frac{\cos^2\theta(\cos\theta-\cos\alpha)}{\cos\alpha}=\frac{\sin^2\theta(\sin\alpha-\sin\theta)}{\sin\alpha},$$

由除法.得 $$\frac{\cos^2\theta+\cos\theta\cos\alpha+\cos^2\alpha}{\cos^2\theta}=\frac{\sin^2\alpha+\sin\alpha\sin\theta+\sin^2\theta}{\sin^2\theta},$$

$$\frac{\cos^2}{\cos^2\theta}-\frac{\sin^2\alpha}{\sin^2\theta}+\frac{\cos\alpha}{\cos\theta}-\frac{\sin\alpha}{\sin\theta}=0,$$

$$\therefore\ \left(\frac{\cos\alpha}{\cos\theta}-\frac{\sin\alpha}{\sin\theta}\right)\left(\frac{\cos\alpha}{\cos\theta}+\frac{\sin\alpha}{\sin\theta}+1\right)=0.$$

(42) $\because\ c^2+s^2=1,\ \therefore\ c^4+s^4=1-2c^2s^2.$

原式 $=c^{12}-s^{12}+4c^2s^2(c^8-s^8)+5c^4s^4(c^4-s^4)$

$=(c^4-s^4)\{c^8+c^4s^4+s^8+4c^2s^2(c^4+s^4)+5c^4s^4\}$

$=(c^2+s^2)(c^2-s^2)\{(c^4+c^2s^2+s^4)(c^4-c^2s^2+s^4)+4c^2s^2(1-2c^2s^2)+5c^4s^4\}$

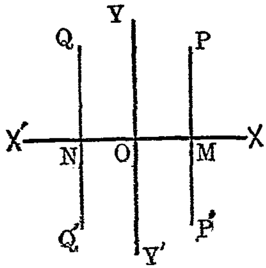
$=(c^2-s^2)\{(1-c^2s^2)(1-3c^2s^2)+4c^2s^2-3c^4s^4\}$

$=(c^2-s^2)(1-4c^2s^2+3c^4s^4+4c^2s^2-3c^4s^4)=c^2-s^2$。

第參編

任意角之三角函數

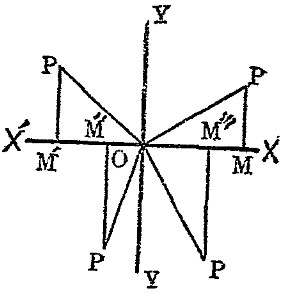
1. 直線之正負 互爲直交之二直線 xox' 及 yoy'. 名橫軸及縱軸. 從其交點 O. 沿 xx' 或與之平行之橫線測之. 其右側橫線爲正. 左側橫線爲負. 又沿 yy' 或與之平行之縱線測之. 其上側之縱線爲正. 下側之縱線爲負.



例如 OM 爲正. ON 爲負. 又 MP 及 NQ 爲正. MP' 及 NQ' 爲負.

2. 象限 如圖. 角 xoy 爲第一象限. 角 yox' 爲第二象限. 角 $x'oy'$ 爲第三象限. 角 $y'ox$ 爲第四象限.

故第一象限內. 橫線及縱線皆爲正.
第二象限內. 橫線爲負, 縱線爲正.
第三象限內. 橫線及縱線皆爲負.
第四象限內. 橫線爲正, 縱線爲負.



3. 三角函數之一般定義 如圖. 令角 MOP$=\alpha$.

〔第一〕 點P在第一象限 xoy 之內. 則 $\alpha<90°$. 故如第二編1. 節.

$$\sin\alpha=\frac{MP}{OP},\ \cos\alpha=\frac{OM}{OP},\ \tan\alpha=\frac{MP}{OM},$$

其他 $\operatorname{cosec}\alpha$, $\sec\alpha$, $\cot\alpha$ 皆爲正.

〔第二〕 點P在第二象限 yox' 之內.

則 $\alpha>90°$, 即 $\alpha<180°$, 故

$$\sin\alpha=\frac{M'P}{OP},\ \cos\alpha=\frac{-OM'}{OP},\ \tan\alpha=\frac{M'P}{-OM'},$$

即 $\sin\alpha$ 爲正. $\cos\alpha$ 爲負. $\tan\alpha$ 爲負. 而其各反商之

$cosec\,a$ 爲正. $sec\,a$ 爲負. $cot\,a$ 爲負.

〔第三〕 P點在第三象限內. 則 $180°<a<270°$.

故 $sin\,a=\frac{-M''P}{OP}$, $cos\,a=\frac{-OM''}{OP}$, $tan\,a=\frac{-M''P}{-OM''}=\frac{M''P}{OM''}$.

∴ $sin\,a$ 爲 $-$, $cos\,a$ 爲 $-$, $tan\,a$ 爲 $+$. 而其反商之

$cosec\,a$ 爲 $-$, $sec\,a$ 爲 $-$, $cot\,a$ 爲 $+$.

〔第四〕 點P在第四象限內. 則 $270°<a<360°$.

故 $sin\,a=\frac{-M'''P}{OP}$, $cos\,a=\frac{OM'''}{OP}$, $tan\,a=\frac{-M'''P}{OM'''}$,

故 $sin\,a$ 爲 $-$, $cos\,a$ 爲 $+$, $tan\,a$ 爲 $-$. 而其反商之

$cosec\,a$ 爲 $-$, $sec\,a$ 爲 $+$, $cot\,a$ 爲 $-$.

4. 各象限內之三角函數值 其界限如次.

先令 $a=0°$. 而 $MP=0$, $OP=OM$,

∴ $sin0°=\frac{MP}{OP}=\frac{0}{OP}=0$, $cos0°=\frac{OM}{OP}=\frac{OP}{OP}=1$,

$tan0°=\frac{MP}{OM}=\frac{0}{OM}=0$, 而其反商爲

$cosec0°=\frac{1}{0}=\infty$, $sec0°=\frac{1}{1}=1$, $cot0°=\frac{1}{0}=\infty$,

又 $vers0°=1-cos0°=1-1=0$, $covers0°=1-sin0°=1-0=1$.

次令 $a=90°$. 則 $OM=0$, $OP=MP$,

∴ $sin90°=\frac{MP}{OP}=\frac{OP}{OP}=1$, $cos90°=\frac{OM}{OP}=\frac{0}{OP}=0$,

$tan90°=\frac{MP}{OM}=\frac{MP}{0}=\infty$. 而其反商爲

$cosec90°=\frac{1}{1}=1$, $sec90°=\frac{1}{0}=\infty$, $cot90°=\frac{1}{\infty}=0$,

又 $vers90°=1-cos90°=1-0=1$, $covers90°=1-sin90°=1-1=0$.

又次令 $a=180°$. 則 $M'P=0$, $OP=OM'$,

故 $sin180°=\frac{M'P}{OP}=\frac{0}{OP}=0$, $cos180°=\frac{-OM'}{OP}=\frac{-OP}{OP}=-1$,

$tan180°=\frac{M'P}{-OM'}=\frac{0}{-OM'}=-0$, 而其反商爲

$cosec180°=\infty$, $sec180°=-1$, $cot180°=-\infty$,

又 $vers\,180°=1-cos\,180°=1-(-1)=2.$

$covers\,180°=1-sin\,180°=1-0=1.$

三次令 $\alpha=270°$. 則 $OM''=0$, $OP=M''P$,

故 $sin\,270°=\frac{-M''P}{OP}=\frac{-OP}{OP}=-1$, $cos\,270°=\frac{-OM''}{OP}=\frac{-0}{OP}=-0$,

$tan\,270°=\frac{-M''P}{-OM''}=\frac{-M''P}{-0}=\infty$, 而其反商爲

$cosec\,270°=-1$, $sec\,270°=-\infty$, $tan\,270°=0$,

又 $vers\,270°=1-cos\,270°=1-(-0)=1$,

$covers\,270°=1-sin\,270°=1-(-1)=2.$

末令 $\alpha=360°$. 則 $M'''P=0$, $OP=OM'''$.

∴ $sin\,360°=\frac{M'''P}{OP}=\frac{0}{OP}=0$, $cos\,360°=\frac{OM'''}{OP}=\frac{OP}{OP}=1$,

$tan\,360°=\frac{M'''P}{OM'''}=\frac{0}{OM'''}=0$. 而其反商爲

$cosec\,360°=\infty$, $sec\,360°=1$, $cot\,360°=\infty$.

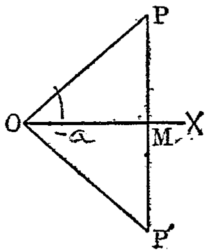
又 $vers\,360°=1-cos\,360°=1-1=0$,

$covers\,360°=1-sin\,360°=1-0=1.$

5. 三角函數之正負及界限

由前兩節所得之三角函數之正負及界限.表示如次.

象限 / 函數	壹	0°	90°	貳	180°	叁	270°	肆	360°
sin	+	0	+1	+	0	−	−1	−	0
cos	+	+1	0	−	−1	−	0	+	+1
tan	+	0	∞	−	0	+	∞	−	0
cosec	+	∞	+1	+	∞	−	−1	−	∞
sec	+	+1	∞	−	−1	−	∞	+	+1
cot	+	∞	0	−	∞	+	0	−	∞
vers	+	0	+1	+	+2	+	+1	+	0
covers	+	+1	0	+	+1	+	+2	+	+1



6. 負角

以橫軸OX爲定線.OP爲動線.最初OP與OX相合.自點O迴轉OP,令左旋成角XOP.此角自0°次第增若干量.此等之角.名正角.反之.自0°右旋之角.名負角.而PM⊥OX.引長PM.令MP′=PM.

角MOP=α.則角MOP′=$-\alpha$.

7. 負角之三角函數

其公式如次.

$$\left.\begin{array}{ll} \sin(-\alpha)=-\sin\alpha, & \operatorname{cosec}(-\alpha)=-\operatorname{cosec}\alpha \\ \cos(-\alpha)=\cos\alpha, & \sec(-\alpha)=\sec\alpha \\ \tan(-\alpha)=-\tan\alpha, & \cot(-\alpha)=-\cot\alpha \end{array}\right\}\quad(1)$$

（證）依前節.令角MOP=α,　角MOP′=α,　∴ OP=OP′.

$$\sin(-\alpha)=\frac{-MP'}{OP'}=\frac{-MP}{OP}=-\sin\alpha,\quad \cos(-\alpha)=\frac{OM}{OP'}=\frac{OM}{OP}=\cos\alpha,$$

$$\tan(-\alpha)=\frac{-MP'}{OM}=\frac{-MP}{OM}=-\tan\alpha,$$ 從此三者之反商.即得 cosec, sec, cot,

8. 任意角之三角函數

第二編之公式(1),(2),(3).其α爲正銳角.假令其α爲任意之角.此等之公式,亦適合也.今證明如次.

例如第二編公式(2).其α在第二象限內.則由3.節第二得

$$\tan\alpha=\frac{M'P}{-OM'}=\frac{M'P/OP}{-OM'/OP}=\frac{\sin\alpha}{\cos\alpha}=\tan\alpha,$$

又α爲負.則 $\tan(-\alpha)=-\tan\alpha=\frac{-\sin\alpha}{\cos\alpha}=\frac{\sin(-\alpha)}{\cos(-\alpha)}$.

又第二編(3).其α在第三象限內.則由3節第三.得

$$\sin^2\alpha+\cos^2\alpha=\left(\frac{-M''P}{OP}\right)^2+\left(\frac{-OM''}{OP}\right)^2=\frac{M''P^2+OM''^2}{OP^2}=\frac{OP^2}{OP^2}=1,$$

又α爲負.則

$$\sin^2(-\alpha)+\cos^2(-\alpha)=(-\sin\alpha)^2+(\cos\alpha)^2=\sin^2\alpha+\cos^2\alpha=1.$$

其他α爲任意之值.亦適合也.

故第二節之公式(1),(2),(3).對於α爲任意之值亦合理.

9. 餘角之公式

如前圖．令角 MOP$=\alpha$, 則

角 MPO$=90^\circ-\alpha$,

$\therefore \quad \sin\alpha=\sin \mathrm{MOP}=\frac{\mathrm{MP}}{\mathrm{OP}}, \quad \cos(90^\circ-\alpha)=\cos \mathrm{MPO}=\frac{\mathrm{MP}}{\mathrm{OP}},$

$\therefore \quad \sin\alpha=\cos(90^\circ-\alpha).$

又 $\cos\alpha=\cos \mathrm{MOP}=\frac{\mathrm{OM}}{\mathrm{OP}}, \quad \sin(90^\circ-\alpha)=\sin \mathrm{MPO}=\frac{\mathrm{OM}}{\mathrm{OP}},$

$\therefore \quad \cos\alpha=\sin(90^\circ-\alpha).$

又 $\tan\alpha=\frac{\sin\alpha}{\cos\alpha}=\frac{\cos(90^\circ-\alpha)}{\sin(90^\circ-\alpha)}=\cot(90^\circ-\alpha)$

由是

$$\left.\begin{array}{ll}\sin\alpha=\cos(90^\circ-\alpha), & \operatorname{cosec}\alpha=\sec(90^\circ-\alpha)\\ \cos\alpha=\sin(90^\circ-\alpha), & \sec\alpha=\operatorname{cosec}(90^\circ-\alpha)\\ \tan\alpha=\cot(90^\circ-\alpha), & \cot\alpha=\tan(90^\circ-\alpha)\end{array}\right\} \quad (2)$$

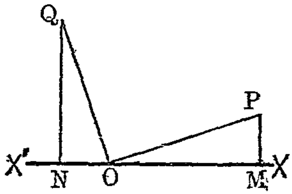
10. 負角之餘角

於公式(2)之α．以$-\alpha$代之．亦合理．

例如令α爲$-\alpha$．則

$\sin(-\alpha)=\cos(90^\circ+\alpha)$．卽從(1)．

$-\sin\alpha=\cos(90^\circ+\alpha)$.



因 PM．QN 爲 XX′ 之垂線．

角 POQ$=90^\circ$．令 OP$=$OQ, PM$=$ON.

則直角三角形 POM, QON 爲全等形．

$\therefore$ OM$=$QN．而令角 MOP$=\alpha$, 則角 MOQ$=90^\circ+\alpha$,

$\therefore \quad \sin\alpha=\frac{\mathrm{PM}}{\mathrm{OP}}=\frac{\mathrm{ON}}{\mathrm{OQ}}, \quad$ 又 $\quad \mathrm{con}(90^\circ+\alpha)=\frac{-\mathrm{ON}}{\mathrm{OQ}},$

由是 $-\sin\alpha=\mathrm{con}(90^\circ+\alpha)$.

由是 ⑶

$$\left.\begin{array}{ll}-\sin\alpha=\cos(90^\circ+\alpha), & -\operatorname{cosec}\alpha=\sec(90^\circ+\alpha)\\ \cos\alpha=\sin(90^\circ+\alpha), & \sec\alpha=\operatorname{cosec}(90^\circ+\alpha)\\ -\tan\alpha=\cot(90^\circ+\alpha), & -\cot\alpha=\tan(90^\circ+\alpha)\end{array}\right\} \quad (3)$$

11 補角之公式

$90^\circ-\alpha$爲銳角．故於(3)之α．以$90^\circ-\alpha$代之．卽得

$-\sin(90°-\alpha)=\cos(90°+90°-\alpha)$, 即 $-\cos\alpha=\cos(180°-\alpha)$.

由是

$$\left.\begin{array}{ll}\sin\alpha=\sin(180°-\alpha), & \operatorname{cosec}\alpha=\operatorname{cosec}(180°-\alpha)\\ -\cos\alpha=\cos(180°-\alpha), & -\sec\alpha=\sec(180°-\alpha)\\ -\tan\alpha=\tan(180°-\alpha), & -\cot\alpha=\cot(180°-\alpha)\end{array}\right\} \quad (4)$$

12. 負角之補角 與10節同法.於公式(4)之 α. 以 $-\alpha$ 代之.

即得

$\sin(-\alpha)=\sin(180°+\alpha)$, 即 $-\sin\alpha=\sin(180°+\alpha)$.

由是

$$\left.\begin{array}{ll}-\sin\alpha=\sin(180°+\alpha), & -\operatorname{cosec}\alpha=\operatorname{cosec}(180°+\alpha)\\ -\cos\alpha=\cos(180°+\alpha), & -\sec\alpha=\sec(180°+\alpha)\\ \tan\alpha=\tan(180°+\alpha), & \cot\alpha=\cot(180°+\alpha)\end{array}\right\} \quad (5)$$

13. 餘角及補角之應用 於(5)之 α. 以 $90°-\alpha$ 代之.

即得次之公式.

$$\left.\begin{array}{ll}-\sin\alpha=\cos(270°-\alpha), & -\operatorname{cosec}\alpha=\sec(270°-\alpha)\\ -\cos\alpha=\sin(270°-\alpha), & -\sec\alpha=\operatorname{cosec}(270°-\alpha)\\ \tan\alpha=\cot(270°-\alpha), & \cot\alpha=\tan(270°-\alpha)\end{array}\right\} \quad (6)$$

又於(6)之 α. 以 $-\alpha$ 代之.則

$$\left.\begin{array}{ll}\sin\alpha=\cos(270°+\alpha), & \operatorname{cosec}\alpha=\sec(270°+\alpha)\\ -\cos\alpha=\sin(270°+\alpha), & -\sec\alpha=\operatorname{cosec}(270°+\alpha)\\ -\tan\alpha=\cot(270°+\alpha), & -\cot\alpha=\tan(270°+\alpha)\end{array}\right\} \quad (7)$$

於(7)之 α. 以 $90°-\alpha$ 代之.則

$$\left.\begin{array}{ll}-\sin\alpha=\sin(360°-\alpha), & -\operatorname{cosec}\alpha=\operatorname{cosec}(360°-\alpha)\\ \cos\alpha=\cos(360°-\alpha), & \sec\alpha=\sec(360°-\alpha)\\ -\tan\alpha=\tan(360°-\alpha), & -\cot\alpha=\cot(360°-\alpha)\end{array}\right\} \quad (8)$$

於(8)之 α. 以 $-\alpha$ 代之.則

$$\left.\begin{array}{ll} \sin\alpha=\sin(360^\circ+\alpha), & \operatorname{cosec}\alpha=\operatorname{cosec}(360^\circ+\alpha) \\ \cos\alpha=\cos(360^\circ+\alpha), & \sec\alpha=\sec(360^\circ+\alpha) \\ \tan\alpha=\tan(360^\circ+\alpha), & \cot\alpha=\cot(365^\circ+\alpha) \end{array}\right\} \quad (9)$$

公式(9)之兩邊.其函數及符號恆相同.因任意之角以360°加之.仍在元之象限之位置.故此再以360°加之.則

$$\sin\alpha=\sin(360^\circ+360^\circ+\alpha)=\sin(720^\circ+\alpha),$$

一般n爲正整數.則$\sin\alpha=\sin(360^\circ n+\alpha)$.

14. 弧度法之公式 前諸節自(2)至(9)之公式之α.以弧度表之如次.

$$\left.\begin{array}{ll} \sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha, & \operatorname{cosec}\left(\frac{\pi}{2}-\alpha\right)=\sec\alpha \\ \cos\left(\frac{\pi}{2}-\alpha\right)=\sin\alpha, & \sec\left(\frac{\pi}{2}-\alpha\right)=\operatorname{cosec}\alpha \\ \tan\left(\frac{\pi}{2}-\alpha\right)=\cot\alpha, & \cot\left(\frac{\pi}{2}-\alpha\right)=\tan\alpha \end{array}\right\} \quad (2')$$

$$\left.\begin{array}{ll} \sin\left(\frac{\pi}{2}+\alpha\right)=\cos\alpha, & \operatorname{cosec}\left(\frac{\pi}{2}+\alpha\right)=\sec\alpha \\ \cos\left(\frac{\pi}{2}+\alpha\right)=-\sin\alpha, & \sec\left(\frac{\pi}{2}+\alpha\right)=-\operatorname{cosec}\alpha \\ \tan\left(\frac{\pi}{2}+\alpha\right)=-\cot\alpha, & \cot\left(\frac{\pi}{2}+\alpha\right)=-\tan\alpha \end{array}\right\} \quad (3')$$

$$\left.\begin{array}{ll} \sin(\pi-\alpha)=\sin\alpha, & \operatorname{cosec}(\pi-\alpha)=\operatorname{cosec}\alpha \\ \cos(\pi-\alpha)=-\cos\alpha, & \sec(\pi-\alpha)=-\sec\alpha \\ \tan(\pi-\alpha)=-\tan\alpha, & \cot(\pi-\alpha)=-\cot\alpha \end{array}\right\} \quad (4')$$

$$\left.\begin{array}{ll} \sin(\pi+\alpha)=-\sin\alpha, & \operatorname{cosec}(\pi+\alpha)=-\operatorname{cosec}\alpha \\ \cos(\pi+\alpha)=-\cos\alpha, & \sec(\pi+\alpha)=-\sec\alpha \\ \tan(\pi+\alpha)=\tan\alpha, & \cot(\pi+\alpha)=\cot\alpha \end{array}\right\} \quad (5')$$

$$\left.\begin{array}{ll} \sin\left(\frac{3\pi}{2}-\alpha\right)=-\cos\alpha, & \operatorname{cosec}\left(\frac{3\pi}{2}-\alpha\right)=-\sec\alpha \\ \cos\left(\frac{3\pi}{2}-\alpha\right)=-\sin\alpha, & \sec\left(\frac{3\pi}{2}-\alpha\right)=-\operatorname{cosec}\alpha \\ \tan\left(\frac{3\pi}{2}-\alpha\right)=\cot\alpha, & \cot\left(\frac{3\pi}{2}-\alpha\right)=\tan\alpha \end{array}\right\} \quad (6')$$

$$\left.\begin{array}{ll} \sin\left(\frac{3\pi}{2}+\alpha\right)=-\cos\alpha, & \operatorname{cosec}\left(\frac{3\pi}{2}+\alpha\right)=-\sec\alpha \\ \cos\left(\frac{3\pi}{2}+\alpha\right)=\ \sin\alpha, & \sec\left(\frac{3\pi}{2}+\alpha\right)=\ \operatorname{cosec}\alpha \\ \tan\left(\frac{3\pi}{2}+\alpha\right)=-\cot\alpha, & \cot\left(\frac{3\pi}{2}+\alpha\right)=-\tan\alpha \end{array}\right\} \quad (7')$$

$$\left.\begin{array}{ll} \sin(2\pi-\alpha)=-\sin\alpha, & \operatorname{cosec}(2\pi-\alpha)=-\operatorname{cosec}\alpha \\ \cos(2\pi-\alpha)=\ \cos\alpha, & \sec(2\pi-\alpha)=\ \sec\alpha \\ \tan(2\pi-\alpha)=-\tan\alpha, & \cot(2\pi-\alpha)=-\cot\alpha \end{array}\right\} \quad (8')$$

$$\left.\begin{array}{ll} \sin(2\pi+\alpha)=\sin\alpha, & \operatorname{cosec}(2\pi+\alpha)=\operatorname{cosec}\alpha \\ \cos(2\pi+\alpha)=\cos\alpha, & \sec(2\pi+\alpha)=\sec\alpha \\ \tan(2\pi+\alpha)=\tan\alpha, & \cot(2\pi+\alpha)=\cot\alpha \end{array}\right\} \quad (9')$$

如前節説明.示(9′)爲一般之式如次.

$$\left.\begin{array}{ll} \sin(2n\pi+\alpha)=\sin\alpha, & \operatorname{cosec}(2n\pi+\alpha)=\operatorname{cosec}\alpha \\ \cos(2n\pi+\alpha)=\cos\alpha, & \sec(2n\pi+\alpha)=\sec\alpha \\ \tan(2n\pi+\alpha)=\tan\alpha, & \cot(2n\pi+\alpha)=\cot\alpha \end{array}\right\} \quad (10)$$

15. 周期 某角 α 之三角函數.等於他之角 $n\pi\pm\alpha$ 之同三角函數則 $n\pi\pm\alpha$ 爲其三角函數之周期.

例如 $\sin(2n\pi+\alpha)=\sin\alpha$.〔公式(10)〕故 $2n\pi+\alpha$ 爲其周期.又從公式(4′). $\sin(\pi-\alpha)=\sin\alpha$.故由公式(10).

$\sin(2n\pi+\pi-\alpha)=\sin\alpha$. 卽 $\sin\{(2n+1)\pi-\alpha\}=\sin\alpha$,

故 n 爲偶數.則 $\sin(n\pi+\alpha)=\sin\alpha$,

n 爲奇數.則 $\sin(n\pi-\alpha)=\sin\alpha$.

由是 $\sin\{n\pi+(-1)^n\alpha\}=\sin\alpha$.

又由(8′). $\cos(2\pi-\alpha)=\cos\alpha$, $\cos(2n\pi-\alpha)=\cos\alpha$,

及由(10). $\cos(2n\pi+\alpha)=\cos\alpha$, $\therefore$ $\cos(2n\pi\pm\alpha)=\cos\alpha$,

又從(5′). $\tan(\pi+\alpha)=\tan\alpha$, $\therefore$ $\tan(2n\pi+\pi+\alpha)=\tan\alpha$,

卽 $\tan\{(2n+1)\pi+\alpha\}=\tan\alpha$, 從(10). $\tan(2n\pi+\alpha)=\tan\alpha$,

故 $\tan(n\pi+\alpha)=\tan\alpha$, 同樣. $\cot(n\pi+\alpha)=\cot\alpha$.

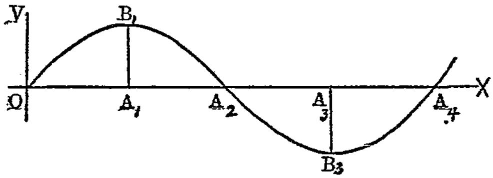
由是正弦之周期角爲 $n\pi+(-1)^n\alpha$, 餘弦之周期角爲 $2n\pi\pm\alpha$, 正切之周期角爲 $n\pi+\alpha$.

$$\left.\begin{array}{ll} \sin\{n\pi+(-1)^n\alpha\} &=\sin\alpha \\ \cos\{2n\pi\pm\alpha\} &=\cos\alpha \\ \tan(n\pi+\alpha) &=\tan\alpha \\ \operatorname{cosec}\{n\pi+(-1)^n\alpha\} &=\operatorname{cosec}\alpha \\ \sec\{2n\pi\pm\alpha\} &=\sec\alpha \\ \cot(n\pi+\alpha) &=\cot\alpha \end{array}\right\} \quad (11)$$

16. 三角函數之圖形

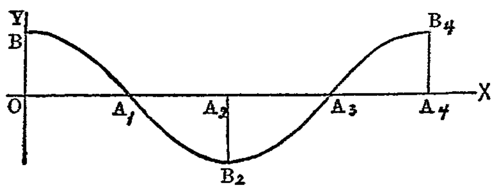
依5.節表示之正負及界限.以圖示之如次.

〔正弦之圖形〕 OX, OY 爲直交二軸.於橫軸OX上.令 $OA_1=A_1A_2=A_2A_3=A_3A_4$. 其 OA_1 爲第一象限. A_1A_2 爲第二象限. A_2A_3 爲第三象限. A_3A_4 爲第四象限.而 $OB_1A_2B_3A_4$ 爲正弦曲線.



於 $\sin\alpha$. 其 $\alpha=0°$, 則 $\sin 0°=0$. 卽在點O. $\alpha=90°$, 則 $\sin 90°=A_1B_1=1$. $\alpha=180°$ 則 $\sin 180°=0$. 卽在點 A_2. $\alpha=270°$. 則 $\sin 270°=-a_3B_3=-1$. $\alpha=360°$, 則 $\sin 360°=0$. 卽在點 A_4.

〔餘弦之圖形〕 如前於OX上取 A_1, A_2, A_3, A_4. 則

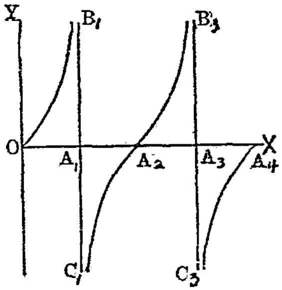


$\alpha=0°$, 則 $\cos 0°=OB=1$. $\alpha=90°$, 則 $\cos 90°=0$. 卽在點 A_1.

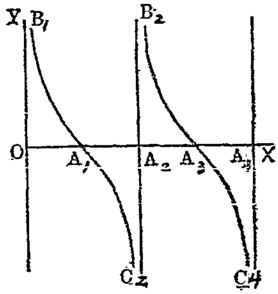
$\alpha=180°$, 則 $\cos 180°=A_2B_2=-1$. $\alpha=270°$, 則 $\cos 270°=0$. 卽在點 A_3.

$\alpha=360°$, 則 $\cos 360°=A_4B_4=1$.

〔正切之圖形〕 正切之曲線.自0°至90°,以正值而增大.即自0至∞.漸漸接近通過A_1之縱線.又自90°至180°,以負值而增大.即自$-\infty$至0而通過A_2點.自180°至270°,以正值而增大.即自0至∞.漸漸接近通過A_3之縱線.又自270°至360°.以負值而增大.即自$-\infty$至0.

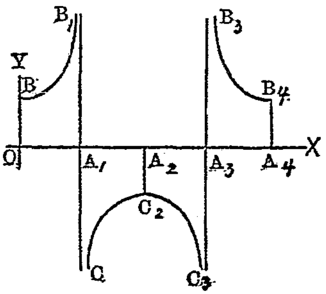


〔餘切之圖形〕 餘切之曲線.自0°至90°,漸漸減小,即自∞至0.自90°至180°,亦漸漸減小,即自0至$-\infty$.自180°至270°又漸漸減小.即自∞至0.自270°至360°,亦漸漸減小,即自0至$-\infty$.

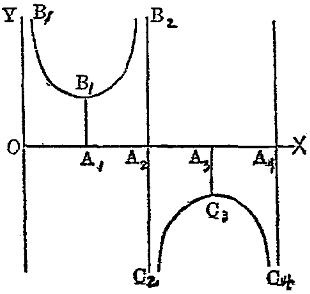


〔正割及餘割之圖形〕 甲圖表正割.乙圖表餘割.

(甲圖)



(乙圖)



例題三

問次列各角在何象限內.

1. $370°$.　2. $420°$.　3. $\frac{7}{3}\pi$.　4. $-40°$.

5. $-100°$.　6. $-365°$.　7. $-750°$.　8 $-\frac{5}{2}\pi$.

次列各角.試變爲比 $45°$ 卽 $\frac{1}{4}\pi$ 小之正角函數.

9. $\sin 740°$.　10. $\cos(-300°)$.　11. $\tan\frac{9}{2}\pi$.

12. $\cot(-\frac{1}{3}\pi)$.　13. $\sec 1120°$.　14. $\operatorname{cosec}(-60°)$.

15. $\operatorname{vers} 100°$　16. $\operatorname{covers}(-100°)$.

次列各式。化爲最簡之式.

17. $a\cos(90°-\alpha)+b\cos(90°+\alpha)$.

18. $\sin\left(\frac{\pi}{2}+\alpha\right)\cos\left(\frac{\pi}{2}+\alpha\right)$.

19. $(a+b)\tan(180°-\alpha)+(a+b)\cot(90°+\alpha)$.

20. $\frac{\sin\alpha\tan(90°+\alpha)}{\tan\alpha\cos(90°-\alpha)}$.

21. $\frac{(a^2-b^2)\cot(\pi-\alpha)}{\cos(\pi+\alpha)}+\frac{(a^2+b^2)\tan\left(\frac{\pi}{2}-\alpha\right)}{\cot(\pi-\alpha)}$

22. $\frac{\sin\left(\frac{\pi}{2}+\alpha\right)\cos\left(\frac{\pi}{2}-\alpha\right)}{\cos(\pi+\alpha)}+\frac{\sin(\pi-\alpha)\cos\left(\frac{\pi}{2}+\alpha\right)}{\sin(\pi+\alpha)}$

例題解自1至22.

(1) $360°+10°$ 故在第一象限.　(2) 第一象限.　(3) 第一象限.
(4) 第四象限.　(5) 第三象限.　(6) 第四象限.　(7) 第四象限.
(8) 在第三第四兩象限之間.　(9) $\sin(2\times 360°+20°)=\sin 20°$.

(10) $\cos(-300°)=\cos 300°=\cos(360°-60°)=\cos 60°$.

(11) ∞.

(12) $-\cot\frac{1}{3}\pi=-\cot\left(\frac{1}{2}\pi-\frac{1}{6}\pi\right)=-\tan\frac{1}{6}\pi$.

(13) $\sec(3\times 360°+40°)=\sec 40°$.

(14) $-\sec 30°$.

(15) $\operatorname{vers} 100°=1-\cos 100°=1-\cos(90°+10°)=1+\sin 10°$.

(16) $\operatorname{covers}(-100°)=1-\sin(-100°)=1+\sin(90°+10°)=1+\cos 10°$.

(17) $a\sin\alpha-b\sin\alpha=(a-b)\sin\alpha$.

(18) $-\cos\alpha\sin\alpha$.

(19) $(a+b)(-\tan\alpha)+(a+b)(-\tan\alpha)=-2(a+b)\tan\alpha$.

(20) $\frac{-\sin\alpha\cot\alpha}{\tan\alpha-\sin\alpha}=-\cot^2\alpha$.

(21) $\frac{-(a^2-b^2)\cot\alpha}{-\cos\alpha}+\frac{(a^2+b^2)\cot\alpha}{-\cot\alpha}=\frac{a^2-b^2}{\sin\alpha}-(a^2+b^2)$.

(22) $\frac{\cos\alpha\sin\alpha}{-\cos\alpha}+\frac{-\sin\alpha\sin\alpha}{-\sin\alpha}=-\sin\alpha+\sin\alpha=0$.

第肆編

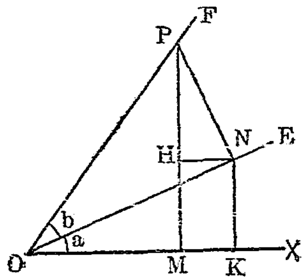
兩角之三角函數

1. 和及差角之正餘絃 其公式如次.

$$\left.\begin{aligned}\sin(\alpha+\beta)&=\sin\alpha\cos\beta+\cos\alpha\sin\beta\\ \cos(\alpha+\beta)&=\cos\alpha\cos\beta-\sin\alpha\sin\beta\\ \sin(\alpha-\beta)&=\sin\alpha\cos\beta-\cos\alpha\sin\beta\\ \cos(\alpha-\beta)&=\cos\alpha\cos\beta+\sin\alpha\sin\beta\end{aligned}\right\}\quad(1)$$

(證) 先以 $\alpha+\beta$ 爲銳角,而令角 $XOE=\alpha$, 角 $EOF=\beta$, 則

角 $XOF=\alpha+\beta$.



從 OF 上任意取一點 P. 引 OX 及 OE 之垂線 PM 及 PN.

又從 N 引 OX 及 PM 之垂線 NK 及 NH, 則

$$\sin(\alpha+\beta)=\sin XOF=\frac{PM}{OP}$$

$$=\frac{KN+HP}{OP}=\frac{KN}{ON}\cdot\frac{ON}{OP}+\frac{HP}{NP}\cdot\frac{NP}{OP}$$

$$=\sin\alpha\cos\beta+\cos HPN\sin\beta=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$$

$$\cos(\alpha+\beta)=\cos XOF=\frac{OM}{OP}=\frac{OK-NH}{OP}=\frac{OK}{ON}\cdot\frac{ON}{OP}-\frac{NH}{NP}\cdot\frac{NP}{OP}$$

$$=\cos\alpha\cos\beta-\sin HPN\sin\beta=\cos\alpha\cos\beta-\sin\alpha\sin\beta.$$

又於 $\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$ 之 α. 以 $90^\circ-\alpha$ 代之. 則

$$\cos(90^\circ-\alpha+\beta)=\cos(90^\circ-\alpha)\cos\beta-\sin(90^\circ-\alpha)\sin\beta,$$

即 $\cos\{90^\circ-(\alpha-\beta)\}=\sin\alpha\cos\beta-\cos\alpha\sin\beta$,

即 $\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta$.

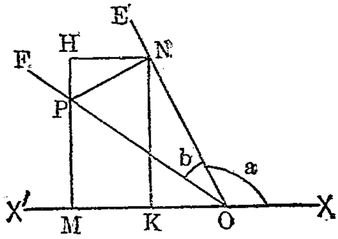
於 $\sin(\alpha+\beta)$ 之 α. 以 $90^\circ-\alpha$ 代之. 則可得第四公式.

次令α, β爲任意之正角. (1)亦合理.

例加α+β爲鈍角, β爲銳角.

如前令角XOE=α,

角EOF=β. 又從OF上任取一點P. 引OX及OE上之垂線PM及PN. 從N引OX及MP之垂線NK及NH, 則



$$\sin(\alpha+\beta)=\sin XOF=\frac{PM}{OP}$$

$$=\frac{KN-HP}{OP}=\frac{KN}{ON}\cdot\frac{ON}{OP}-\frac{HP}{NP}\cdot\frac{NP}{OP}=\sin(180^\circ-\alpha)\cos\beta-\cos NPH\sin\beta$$

$$=\sin\alpha\cos\beta-\cos NOK\sin\beta=\sin\alpha\cos\beta-\cos(180^\circ-\alpha)\sin\beta$$

$$=\sin\alpha\cos\beta+\cos\alpha\sin\beta.$$

終令α及β爲負角. 則

$$\sin\{(-\alpha)+(-\beta)\}=\sin\{-(\alpha+\beta)\}=-\sin(\alpha+\beta)$$

$$=-(\sin\alpha\cos\beta+\cos\alpha\sin\beta)=-\sin\alpha\cos\beta-\cos\alpha\sin\beta$$

$$=\sin(-\alpha)\cos(-\beta)+\cos(-\alpha)\sin(-\beta).$$

由是知(1)對於α及β爲任意之角, 亦合理.

2. 和及差角之正餘切 其公式如次.

$$\left.\begin{aligned}\tan(\alpha+\beta)&\equiv\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\\ \cot(\alpha+\beta)&\equiv\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}\\ \tan(\alpha-\beta)&\equiv\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}\\ \cot(\alpha-\beta)&\equiv\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}\end{aligned}\right\}\quad(2)$$

(證) $\tan(\alpha+\beta)=\dfrac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\dfrac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$

$$=\frac{(\sin\alpha\cos\beta+\cos\alpha\sin\beta)/\cos\alpha\cos\beta}{(\cos\alpha\cos\beta-\sin\alpha\sin\beta)/\cos\alpha\cos\beta}=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}.$$

又 $\cot(\alpha+\beta)=\frac{1}{\tan(\alpha+\beta)}=\frac{(1-\tan\alpha\tan\beta)/\tan\alpha\tan\beta}{(\tan\alpha+\tan\beta)/\tan\alpha\tan\beta}$

$=\frac{\cot\alpha\cot\beta-1}{\cot\beta+\cot\alpha}$. 以下同様

9. 和差及積之正餘弦 其公式如次.

$$\left.\begin{aligned}\sin\theta+\sin\phi&\equiv2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\\\cos\theta+\cos\phi&\equiv2\cos\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}\\\sin\theta-\sin\phi&\equiv2\cos\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\\\cos\phi-\cos\theta&\equiv2\sin\frac{\theta+\phi}{2}\sin\frac{\theta-\phi}{2}\end{aligned}\right\}\quad(3)$$

(證) 令(1)之第一及第三相加,得

$\sin(\alpha+\beta)+\sin(\alpha-\beta)=2\sin\alpha\cos\beta,$

而令 $\alpha+\beta=\theta$, 及 $\alpha-\beta=\phi$, 則 $\alpha=\frac{1}{2}(\theta+\phi)$, $\beta=\frac{1}{2}(\theta-\phi)$,

$\therefore\ \sin\theta+\sin\phi=2\sin\frac{1}{2}(\theta+\phi)\cos\frac{1}{2}(\theta-\phi).$

以下同様.

4. 二倍角之三角函數 其公式如次.

$$\left.\begin{aligned}\sin2\alpha&\equiv2\sin\alpha\cos\alpha\\\cos2\alpha&\equiv\cos^2\alpha-\sin^2\alpha\equiv2\cos^2\alpha-1\equiv1-2\sin^2\alpha\\\tan2\alpha&\equiv\frac{2\tan\alpha}{1-\tan^2\alpha}\end{aligned}\right\}\quad(4)$$

(證) 1.節公式第一.令 $\alpha=\beta$, 則

$\sin2\alpha=\sin\alpha\cos\alpha+\cos\alpha\sin\alpha=2\sin\alpha\cos\alpha.$

又1.節公式第二.令 $\alpha=\beta$, 則

$\cos2\alpha=\cos\alpha\cos\alpha-\sin\alpha\sin\alpha=\cos^2\alpha-\sin^2\alpha.$

又從2.節公式第一.令 $\alpha=\beta$, 則可得 $\tan2\alpha$.

5. 三倍角之三角函數 其公式如次.

$$\left.\begin{aligned} \sin 3\alpha &\equiv 3\sin\alpha - 4\sin^3\alpha \\ \cos 3\alpha &\equiv 4\cos^3\alpha - 3\cos\alpha \\ \tan 3\alpha &\equiv \frac{3\tan\alpha - \tan^3\alpha}{1-3\tan^2\alpha} \end{aligned}\right\} \quad (5)$$

(證) 於 1. 節公式第一. 令 $\beta=2\alpha$, 則

$$\sin 3\alpha = \sin\alpha\cos 2\alpha + \cos\alpha\sin 2\alpha$$

$$= \sin\alpha(1-2\sin^2\alpha) + \cos\alpha(2\sin\alpha\cos\alpha) \quad \text{〔見前節 (4)〕}$$

$$= \sin\alpha(1-2\sin^2\alpha) + 2\sin\alpha(1-\sin^2\alpha) = 3\sin\alpha - 4\sin^3\alpha.$$

於 1. 節公式第二. 令 $\beta=2\alpha$, 則可得第二式.

又於 2. 節公式第一. 令 $\beta=2\alpha$, 則

$$\tan 3\alpha = \frac{\tan\alpha+\tan 2\alpha}{1-\tan\alpha\tan 2\alpha} = \frac{\tan\alpha + 2\tan\alpha/(1-\tan^2\alpha)}{1-\tan\alpha\{2\tan\alpha/(1-\tan^2\alpha)\}} \quad \text{(見前節)}$$

$$= \frac{\tan\alpha(1-\tan^2\alpha)+2\tan\alpha}{1-\tan^2\alpha-2\tan^2\alpha} = \frac{3\tan\alpha-\tan^3\alpha}{1-3\tan^2\alpha}.$$

例 題 四

試證次之各恆同式.

1. $\sin(\alpha+\beta)\sin(\alpha-\beta) \equiv \sin^2\alpha - \sin^2\beta.$

2. $\cos(\alpha+\beta)\cos(\alpha-\beta) \equiv \cos^2\alpha - \sin^2\beta.$

例題解自 1. 至 2.

(1) $\sin(\alpha+\beta)\sin(\alpha-\beta) = (\sin\alpha\cos\beta + \cos\alpha\sin\beta)(\sin\alpha\cos\beta - \cos\alpha\sin\beta)$

$= \sin^2\alpha\cos^2\beta - \cos^2\alpha\sin^2\beta = \sin^2\alpha(1-\sin^2\beta) - (1-\sin^2\alpha)\sin^2\beta$

$= \sin^2\alpha - \sin^2\beta.$

(2) $\cos(\alpha+\beta)\cos(\alpha-\beta) = (\cos\alpha\cos\beta - \sin\alpha\sin\beta)(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

$= \cos^2\alpha\cos^2\beta - \sin^2\alpha\sin^2\beta = \cos^2\alpha(1-\sin^2\beta) - (1-\cos^2\alpha)\sin^2\beta$

$= \cos^2\alpha - \sin^2\beta$

3. $\sin\theta\sin\phi\equiv\sin^2\dfrac{\theta+\phi}{2}-\sin^2\dfrac{\theta-\phi}{2}$.

4. $\cos\theta\cos\phi\equiv\cos^2\dfrac{\theta+\phi}{2}-\sin^2\dfrac{\theta-\phi}{2}$.

5. $\tan\alpha+\tan\beta\equiv\dfrac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.

6. $\tan\alpha-\tan\beta\equiv\dfrac{\sin(\alpha-\beta)}{\cos\alpha\cos\beta}$.

7. $\tan^2\alpha-\tan^2\beta\equiv\dfrac{\sin^2\alpha-\sin^2\beta}{\cos^2\alpha\cos^2\beta}$.

8. $(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2\equiv 4\sin^2\dfrac{\alpha-\beta}{2}$.

9. $(\cos\alpha+\cos\beta)^2-(\sin\alpha+\sin\beta)^2\equiv 4\sin^2\dfrac{\alpha+\beta}{2}$.

10. $\tan\alpha+\cot\alpha\equiv 2\operatorname{cosec}2\alpha$.

11. $\cot\alpha-\tan\alpha\equiv 2\cot 2\alpha$.

12. $\dfrac{\sin\alpha+\sin\beta}{\sin\alpha-\sin\beta}\equiv\dfrac{\tan\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}$.

13. $\dfrac{\cos\alpha+\cos\beta}{\cos\beta-\cos\alpha}\equiv\dfrac{\cot\frac{1}{2}(\alpha+\beta)}{\tan\frac{1}{2}(\alpha-\beta)}$.

14. $\dfrac{\sin\alpha+\sin\beta}{\cos\alpha+\cos\beta}\equiv\tan\frac{1}{2}(\alpha+\beta)$.

15. $\dfrac{\sin\alpha+\sin\beta}{\cos\beta-\cos\alpha}\equiv\cot\frac{1}{2}(\alpha-\beta)$.

16. $\dfrac{\sin\alpha-\sin\beta}{\cos\alpha-\cos\beta}\equiv\tan\frac{1}{2}(\alpha-\beta)$.

17. $\dfrac{\sin\alpha-\sin\beta}{\cos\beta-\cos\alpha}\equiv\cot\frac{1}{2}(\alpha+\beta)$.

18. $\sin 2\alpha\equiv\dfrac{2\tan\alpha}{1+\tan^2\alpha}$.　　19. $\cos 2\alpha\equiv\dfrac{1-\tan^2\alpha}{1+\tan^2\alpha}$.

20. $\sin^2(\alpha+\beta)-\sin^2(\alpha-\beta)\equiv\sin 2\alpha\sin 2\beta$.

21. $\cos^2(\alpha+\beta)-\sin^2(\alpha-\beta)\equiv\cos 2\alpha\cos 2\beta$.

例題解自 3. 至 21.

(3) $\sin\theta\sin\phi=\sin\left(\frac{\theta+\phi}{2}+\frac{\theta-\phi}{2}\right)\sin\left(\frac{\theta+\phi}{2}-\frac{\theta-\phi}{2}\right)$,

令 $\frac{\theta+\phi}{2}=\alpha$, $\frac{\theta-\phi}{2}=\beta$, 則 $=\sin(\alpha+\beta)\sin(\alpha-\beta)$

$=\sin^2\alpha-\sin^2\beta$ (見例題 1.)$=\sin^2\frac{\theta+\phi}{2}-\sin^2\frac{\theta-\phi}{2}$.

(4) 與前例同樣.

(5) $\tan\alpha+\tan\beta=\frac{\sin\alpha}{\cos\alpha}+\frac{\sin\beta}{\cos\beta}=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta}=\frac{\sin(\alpha+\beta)}{\cos\alpha\cos\beta}$.

(6) 與前例同樣. (7) 可從前兩例之恆同式相乘而得.

(8) $(\cos\alpha-\cos\beta)^2+(\sin\alpha-\sin\beta)^2=\cos^2\alpha+\sin^2\alpha+\cos^2\beta+\sin^2\beta$

$-2(\cos\alpha\cos\beta+\sin\alpha\sin\beta)=1+1-2\cos(\alpha-\beta)=2\{1-\cos(\alpha-\beta)\}$

$=2\{1-[1-2\sin^2\frac{1}{2}(\alpha-\beta)]\}=4\sin^2\frac{1}{2}(\alpha-\beta)$.

(9) 與前例同樣.

(10) $\tan\alpha+\cot\alpha=\frac{\sin\alpha}{\cos\alpha}+\frac{\cos\alpha}{\sin\alpha}=\frac{\sin^2\alpha+\cos^2\alpha}{\sin\alpha\cos\alpha}=\frac{2}{2\sin\alpha\cos\alpha}$

$=\frac{2}{\sin 2\alpha}=2\operatorname{cosec}2\alpha$.

(11) $\cot\alpha-\tan\alpha=\frac{\cos^2\alpha-\sin^2\alpha}{\sin\alpha\cos\alpha}=\frac{\cos 2\alpha}{\frac{1}{2}\sin 2\alpha}=2\cot 2\alpha$

(12) 至 (17) 取公式 (3) 之四組施除法. 卽得

(18) $\sin 2\alpha=2\sin\alpha\cos\alpha=\frac{2\sin\alpha\cos^2\alpha}{\cos\alpha}=\frac{2\tan\alpha}{\sec^2\alpha}=\frac{2\tan\alpha}{1+\tan^2\alpha}$.

(19) $\cos 2\alpha=\cos^2\alpha-\sin^2\alpha=\cos^2\alpha\left(1-\frac{\sin^2\alpha}{\cos^2\alpha}\right)$

$=\frac{1}{\sec^2\alpha}(1-\tan^2\alpha)=\frac{1-\tan^2\alpha}{1+\tan^2\alpha}$.

(20) $\sin^2(\alpha+\beta)-\sin^2(\alpha-\beta)=(\sin\alpha\cos\beta+\cos\alpha\sin\beta)^2-(\sin\alpha\cos\beta-\cos\alpha\sin\beta)^2$

$=4\sin\alpha\cos\alpha\sin\beta\cos\beta=\sin 2\alpha\sin 2\beta$.

(21) $\cos^2(\alpha+\beta)-\sin^2(\alpha-\beta)=\frac{1+\cos 2(\alpha+\beta)}{2}-\frac{1-\cos 2(\alpha-\beta)}{2}$

$=\frac{\cos 2(\alpha+\beta)+\cos 2(\alpha-\beta)}{2}=\frac{1}{2}(\cos 2\alpha\cos 2\beta-\sin 2\alpha\sin 2\beta$

$+\cos 2\alpha\cos 2\beta+\sin 2\alpha\sin 2\beta)=\cos 2\alpha\cos 2\beta$.

22. $8\cos^4\alpha \equiv \cos 4\alpha + 4\cos 2\alpha + 3.$

23. $16\cos^5\alpha \equiv \cos 5\alpha + 5\cos 3\alpha + 10\cos\alpha.$

24. $32\cos^6\alpha \equiv \cos 6\alpha + 6\cos 4\alpha + 15\cos 2\alpha + 10.$

25. $64\cos^7\alpha \equiv \cos 7\alpha + 7\cos 5\alpha + 21\cos 3\alpha + 35\cos\alpha.$

26. $128\cos^8\alpha \equiv \cos 8\alpha + 8\cos 6\alpha + 28\cos 4\alpha + 56\cos 2\alpha + 35.$

27. $8\sin^4\alpha \equiv \cos 4\alpha - 4\cos 2\alpha + 3.$

28. $16\sin^5\alpha \equiv \sin 5\alpha - 5\sin 3\alpha + 10\sin\alpha.$

29. $-32\sin^6\alpha \equiv \cos 6\alpha - 6\cos 4\alpha + 15\cos 2\alpha - 10.$

30. $-64\sin^7\alpha \equiv \sin 7\alpha - 7\sin 5\alpha + 21\sin 3\alpha - 35\sin\alpha.$

31. $128\sin^8\alpha \equiv \cos 8\alpha - 8\cos 6\alpha + 28\cos 4\alpha - 56\cos 2\alpha + 35.$

32. $1 + \cos 3\alpha \cos 5\alpha \equiv \cos^2 4\alpha + \cos^2\alpha.$

33. $\cos 5\alpha \equiv 16\cos^5\alpha - 20\cos^3\alpha + 5\cos\alpha.$

34. $\sin 5\alpha \equiv 16\sin^5\alpha - 20\sin^3\alpha + 5\sin\alpha.$

35. $\sin 5\alpha + \cos 5\alpha \equiv (\sin\alpha + \cos\alpha)(2\cos 4\alpha + 2\sin 2\alpha - 1).$

36. $\sin 6\alpha \equiv 2\sin\alpha(16\cos^5\alpha - 16\cos^3\alpha + 3\cos\alpha).$

37. $\sin 7\alpha \equiv 7\sin\alpha - 56\sin^3\alpha + 112\sin^5\alpha - 64\sin^7\alpha.$

38. $\cos 7\alpha \equiv 64\cos^7\alpha - 112\cos^5\alpha + 56\cos^3\alpha - 7\cos\alpha.$

39. $\cos 9\alpha + \cos 7\alpha - 4(\cos 5\alpha + \cos 3\alpha) + 6\cos\alpha$

$$\equiv 256\sin^4\alpha\cos^5\alpha.$$

40. $2\sin^2\alpha\sin^2\beta + 2\cos^2\alpha\cos^2\beta \equiv 1 + \cos 2\alpha\cos 2\beta.$

41. $\sin 3\alpha\,\mathrm{cosec}\,\alpha - \cos 3\alpha\sec\alpha \equiv 2.$

42. $3\sin\alpha - \sin 3\alpha \equiv 2\sin\alpha(1 - \cos 2\alpha).$

例題解自 22. 至 42.

(22) $4\cos^3\alpha=\cos3\alpha+3\cos\alpha$〔公式(5)〕 雙方以 $2\cos\alpha$ 乘之，則

$8\cos^4\alpha=2\cos3\alpha\cos\alpha+6\cos^2\alpha=\cos4\alpha+\cos2\alpha+6\cos^2\alpha$〔公式(3)〕

$=\cos4\alpha+\cos2\alpha+3(\cos2\alpha+1)=\cos4\alpha+4\cos2\alpha+3.$

(23) 試如前例以 $2\cos\alpha$ 乘之，則 $16\cos^5\alpha=2\cos4\alpha\cos\alpha+8\cos2\alpha\cos\alpha+6\cos\alpha$

$=\cos5\alpha+\cos3\alpha+4(\cos3\alpha+\cos\alpha)+6\cos\alpha$〔公式(3)〕

$=\cos5\alpha+5\cos3\alpha+10\cos\alpha.$

(24) 至(26) 順次以 $2\cos\alpha$ 乘之，即得.

(27) $4\sin^3\alpha=-\sin3\alpha+3\sin\alpha$〔公式(5)〕 雙方以 $2\sin\alpha$ 乘之，則

$8\sin^4\alpha=-2\sin\alpha\sin3\alpha+6\sin^2\alpha=\cos4\alpha-\cos2\alpha+6\sin^2\alpha$〔公式(3)〕

$=\cos4\alpha-\cos2\alpha+3(1-\cos2\alpha)=\cos4\alpha-4\cos2\alpha+3.$

(28) 試如前例以 $2\sin\alpha$ 乘之，則 $16\sin^5\alpha=2\cos4\alpha\sin\alpha-8\cos2\alpha\sin\alpha+6\sin\alpha$

$=\sin5\alpha-\sin3\alpha-4(\sin3\alpha-\sin\alpha)+6\sin\alpha$〔公式(3)〕

(29) 至(31) 準前例以 $-2\sin\alpha$ 乘之，即得.

(32) $1+\cos(4\alpha-\alpha)\cos(4\alpha+\alpha)=1+\cos^24\alpha-\sin^2\alpha$ (例題2) $=\cos^24\alpha+\cos^2\alpha$

(33) $\cos5\alpha=\cos(4\alpha+\alpha)=\cos4\alpha\cos\alpha-\sin4\alpha\sin\alpha$

$=(2\cos^22\alpha-1)\cos\alpha-2\sin2\alpha\cos2\alpha\sin\alpha$

$=\{2(2\cos^2\alpha-1)^2-1\}\cos\alpha-4\sin^2\alpha\cos\alpha(2\cos^2\alpha-1)$

$=(8\cos^4\alpha-8\cos^2\alpha+1)\cos\alpha-4(1-\cos^2\alpha)\cos\alpha(2\cos^2\alpha-1).$ (34) 與前同樣.

(35) $\cos(4\alpha+\alpha)+\sin(4\alpha+\alpha)=\cos4\alpha(\sin\alpha+\cos\alpha)-\sin4\alpha(\sin\alpha-\cos\alpha)$

$=\cos4\alpha(\sin\alpha+\cos\alpha)-2\sin2\alpha(\cos^2\alpha-\sin^2\alpha)(\sin\alpha-\cos\alpha)$

$=(\sin\alpha+\cos\alpha)\{\cos4\alpha+2\sin2\alpha(\sin\alpha-\cos\alpha)^2\}$

$=(\sin\alpha+\cos\alpha)\{\cos4\alpha+2\sin2\alpha(1-\sin2\alpha)\}.$

(36) $2\sin3\alpha\cos3\alpha=2(3\sin\alpha-4\sin^3\alpha)(4\cos^3\alpha-3\cos\alpha)$

$=2\sin\alpha\{3-4(1-\cos^2\alpha)\}(4\cos^3\alpha-3\cos\alpha)=2\sin\alpha(4\cos^2\alpha-1)(4\cos^3\alpha-3\cos\alpha).$

(37) $\sin(4\alpha+3\alpha)=\sin4\alpha\cos3\alpha+\cos4\alpha\sin3\alpha.$ (38) 亦同樣.

(39) $\cos9\alpha+\cos7\alpha-4(\cos5\alpha+\cos3\alpha)+6\cos\alpha$

$=2\cos8\alpha\cos\alpha-4\times2\cos4\alpha\cos\alpha+6\cos\alpha=2\cos\alpha(\cos8\alpha-4\cos4\alpha+3)$

$=2\cos\alpha(2\cos^24\alpha-1-4\cos4\alpha+3)=4\cos\alpha(\cos4\alpha-1)^2$ 以下容易.

(40) $2\times\frac{1}{2}(1-\cos2\alpha)\frac{1}{2}(1-\cos2\beta)+2\times\frac{1}{2}(1+\cos2\alpha)\frac{1}{2}(1+\cos2\beta).$

(41) $(3\sin\alpha-4\sin^3\alpha)/\sin\alpha-(4\cos^3\alpha-3\cos\alpha)/\cos\alpha.$

(42) $3\sin\alpha-(3\sin\alpha-4\sin^3\alpha)=2\sin\alpha\{1-(1-2\sin^2\alpha)\}.$

43. $\operatorname{cosec}2\alpha+\cot4\alpha\equiv\cot\alpha-\operatorname{cosec}4\alpha.$

44. $\operatorname{cosec}\alpha\operatorname{cosec}2\alpha+\operatorname{cosec}2\alpha\operatorname{cosec}3\alpha\equiv2\cot\alpha\operatorname{cosec}3\alpha.$

45. $\sin2\alpha+\sin4\alpha+\sin6\alpha+\sin8\alpha\equiv4\sin5\alpha\cos2\alpha\cos\alpha$

46. $\cos10\alpha+\cos8\alpha+3\cos4\alpha+3\cos2\alpha\equiv8\cos\alpha\cos^3 3\alpha.$

47. $\sin3\alpha\sin^3\alpha+\cos3\alpha\cos^3\alpha\equiv\cos^3 2\alpha.$

48. $\cos^3\alpha\left(\frac{1}{3}\sin3\alpha\right)+\sin^3\alpha\left(\frac{1}{3}\cos3\alpha\right)\equiv\frac{1}{4}\sin4\alpha.$

49. $\cot\frac{\alpha}{2}-\cot\alpha\equiv\operatorname{cosec}\alpha.$

50. $\sec\alpha\equiv1+\tan\alpha\tan\frac{\alpha}{2}.$

51. $\tan4\alpha\equiv\frac{4\tan\alpha(1-\tan^2\alpha)}{1-6\tan^2\alpha+\tan^4\alpha}.$

52. $\frac{\sin19\alpha+\sin17\alpha}{\sin10\alpha+\sin8\alpha}\equiv2\cos9\alpha$

53. $\frac{\cos3\alpha+\sin3\alpha}{\cos\alpha-\sin\alpha}\equiv1+2\sin2\alpha.$

54. $\frac{\sin3\alpha\sin3\beta-\sin3\beta\sin2\alpha}{\sin2\alpha\sin\beta-\sin2\beta\sin\alpha}\equiv1+4\cos\alpha\cos\beta.$

55. $\frac{\sin\alpha+\sin3\alpha+\sin5\alpha+\sin7\alpha}{\cos\alpha+\cos3\alpha+\cos5\alpha+\cos7\alpha}\equiv\tan4\alpha.$

56. $\frac{\sin\alpha+2\sin3\alpha+\sin5\alpha}{\sin3\alpha+2\sin5\alpha+\sin7\alpha}\equiv\frac{\sin3\alpha}{\sin5\alpha}.$

例題解自 43. 至 56.

(43) $\frac{1}{\sin2\alpha}+\frac{\cos4\alpha}{\sin4\alpha}=\frac{2\cos2\alpha+2\cos^2 2\alpha-1}{2\sin2\alpha\cos2\alpha}=\frac{2\cos2\alpha\times2\cos^2\alpha-1}{2\sin2\alpha\cos2\alpha}$

$=\frac{2\cos^2\alpha}{\sin2\alpha}-\frac{1}{\sin4\alpha}=\frac{\cos\alpha}{\sin\alpha}-\frac{1}{\sin4\alpha}.$

(44) $\frac{1}{\sin\alpha\sin2\alpha}+\frac{1}{\sin2\alpha\sin3\alpha}=\frac{\sin3\alpha+\sin\alpha}{\sin\alpha\sin2\alpha\sin3\alpha}$

$=\frac{2\sin2\alpha\cos\alpha}{\sin\alpha\sin2\alpha\sin3\alpha}=\frac{2\cos\alpha}{\sin\alpha\sin3\alpha}=2\cot\alpha\operatorname{cosec}3\alpha.$

(45) $(\sin 2\alpha+\sin 8\alpha)+(\sin 4\alpha+\sin 6\alpha)=2\sin 5\alpha\cos 3\alpha+2\sin 5\alpha\cos\alpha$

$=2\sin 5\alpha(\cos 3\alpha+\cos\alpha)=4\sin 5\alpha\cos 2\alpha\cos\alpha.$

(46) $2\cos 9\alpha\cos\alpha+6\cos 3\alpha\cos\alpha=2\cos\alpha(4\cos^3 3\alpha-3\cos 3\alpha+3\cos 3\alpha).$

(47) $\frac{1}{2}\sin^2\alpha(2\sin 3\alpha\sin\alpha)+\frac{1}{2}\cos^2\alpha(2\cos 3\alpha\cos\alpha)$

$=\frac{1}{2}\sin^2\alpha(\cos 2\alpha-\cos 4\alpha)+\frac{1}{2}\cos^2\alpha(\cos 2\alpha+\cos 4\alpha)$

$=\frac{1}{2}\cos 2\alpha+\frac{1}{2}\cos 4\alpha(\cos^2\alpha-\sin^2\alpha)=\frac{1}{2}\{\cos 2\alpha+(2\cos^2 2\alpha-1)\cos 2\alpha\}.$

(48) 與前例同樣.

(49) $\frac{\cos\frac{\alpha}{2}}{\sin\frac{\alpha}{2}}-\frac{\cos\alpha}{\sin\alpha}=\frac{2\cos^2\frac{\alpha}{2}-\cos\alpha}{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}=\frac{1}{\sin\alpha}=\operatorname{cosec}\alpha.$

(50) $\frac{1}{\cos\alpha}=\frac{\cos\frac{\alpha}{2}\left(1-2\sin^2\frac{\alpha}{2}+2\sin^2\frac{\alpha}{2}\right)}{\cos\alpha\cos\frac{\alpha}{2}}=\frac{\cos\frac{\alpha}{2}\left(\cos\alpha+2\sin^2\frac{\alpha}{2}\right)}{\cos\alpha\cos\frac{\alpha}{2}}$

$=\frac{\cos\alpha\cos\frac{\alpha}{2}+\sin\alpha\sin\frac{\alpha}{2}}{\cos\alpha\cos\frac{\alpha}{2}}=1+\tan\alpha\tan\frac{\alpha}{2}.$

(51) $\tan 4\alpha=2\tan 2\alpha/(1-\tan^2 2\alpha),\ \tan 2\alpha=2\tan\alpha/(1-\tan^2\alpha);$

(52) $\frac{2\sin 18\alpha\cos\alpha}{2\sin 9\alpha\cos\alpha}=\frac{2\sin 9\alpha\cos 9\alpha}{\sin 9\alpha}=2\cos 9\alpha.$

(53) $\frac{4(\cos^3\alpha-\sin^3\alpha)-3(\cos\alpha-\sin\alpha)}{\cos\alpha-\sin\alpha}=4(1+\cos\alpha\sin\alpha)-3$

(54) $\frac{(3\sin\alpha-4\sin^3\alpha)2\sin\beta\cos\beta-(3\sin\beta-4\sin^3\beta)2\sin\alpha\cos\alpha}{2\sin\alpha\cos\alpha\sin\beta-2\sin\beta\cos\beta\sin\alpha}$

$=\frac{-3(\cos\alpha-\cos\beta)-4\{(1-\cos^2\alpha)\cos\beta-(1-\cos^2\beta)\cos\alpha\}}{\cos\alpha-\cos\beta}$

$=\frac{-3(\cos\alpha-\cos\beta)+4(\cos\alpha-\cos\beta)+4\cos\alpha\cos\beta(\cos\alpha-\cos\beta)}{\cos\alpha-\cos\beta}$

$=1+4\cos\alpha\cos\beta.$

(55) $\frac{(\sin 7\alpha+\sin\alpha)+(\sin 5\alpha+\sin 3\alpha)}{(\cos 7\alpha+\cos\alpha)+(\cos 5\alpha+\cos 3\alpha)}=\frac{2\sin 4\alpha\cos 3\alpha+2\sin 4\alpha\cos\alpha}{2\cos 4\alpha\cos 3\alpha+2\cos 4\alpha\cos\alpha}$

$=\frac{2\sin 4\alpha(\cos 3\alpha+\cos\alpha)}{2\cos 4\alpha(\cos 3\alpha+\cos\alpha)}=\tan 4\alpha.$

(56) $\frac{(\sin 5\alpha+\sin\alpha)+2\sin 3\alpha}{(\sin 7\alpha+\sin 3\alpha)+2\sin 5\alpha}=\frac{2\sin 3\alpha\cos 2\alpha+2\sin 3\alpha}{2\sin 5\alpha\cos 2\alpha+2\sin 5\alpha}$

$=\sin 3\alpha/\sin 5\alpha.$

57. $\dfrac{\cos\alpha-\cos3\alpha}{\sin3\alpha-\sin\alpha}\equiv\tan2\alpha.$

58. $\dfrac{\cos2\alpha-\cos4\alpha}{\sin4\alpha-\sin2\alpha}\equiv\tan3\alpha.$

59. $8(\cos^8\alpha-\sin^8\alpha)\equiv\cos6\alpha+7\cos2\alpha.$

60. $64(\cos^8\alpha+\sin^8\alpha)\equiv\cos8\alpha+28\cos4\alpha+35.$

61. $\sec^2\dfrac{\alpha}{2}\sec\alpha\left(\cot^2\dfrac{\alpha}{2}-\cot^2\dfrac{3}{2}\alpha\right)\equiv8\left(1+\cot^2\dfrac{3}{2}\alpha\right).$

62. $\{\sec\alpha+\operatorname{cosec}(1+\sec\alpha)\}\left(1-\tan^2\dfrac{\alpha}{2}\right)\left(1-\tan^2\dfrac{\alpha}{4}\right)$

$$\equiv\left(\sec\frac{\alpha}{2}+\operatorname{cosec}\frac{\alpha}{2}\right)\sec^2\frac{\alpha}{4}.$$

63. $\dfrac{(\sec\alpha\sec\beta+\tan\alpha\tan\beta)^2-(\tan\alpha\sec\beta+\sec\alpha\tan\beta)^2}{2(1+\tan^2\alpha\tan^2\beta)-\sec^2\alpha\sec^2\beta}$

$$\equiv\frac{\sec2\alpha\sec2\beta}{\sec^2\alpha\sec^2\beta}.$$

64. $\left(\tan^2\dfrac{\alpha}{2}\operatorname{cosec}^2\dfrac{\alpha}{6}-\sec^2\dfrac{\alpha}{2}\right)\cot\dfrac{2}{3}\alpha$

$$\equiv\left(\operatorname{cosec}^2\frac{\alpha}{6}-\sec^2\frac{\alpha}{2}\right)\tan\frac{\alpha}{3}.$$

65. $\cot\alpha+\cot2\alpha+\cot4\alpha\equiv\operatorname{cosec}4\alpha(2+2\cos2\alpha+3\cos4\alpha).$

66. $\operatorname{cosec}\alpha\equiv\dfrac{2\sin2\alpha+2\cos2\alpha}{\cos\alpha-\sin\alpha-\cos3\alpha+\sin3\alpha}.$

例題解自 57. 至 66.

(57) $\dfrac{2\sin2\alpha\sin\alpha}{2\cos2\alpha\sin\alpha}=\tan2\alpha.$ (58) 與前例同樣.

(59) $8\{(\cos^2\alpha+\sin^2\alpha)^2-2\cos^2\alpha\sin^2\alpha\}(\cos^2\alpha+\sin^2\alpha)(\cos^2\alpha-\sin^2\alpha)$

$=8(1-\frac{1}{2}\sin^22\alpha)\cos2\alpha=2(4-2\sin^22\alpha)\cos2\alpha$

$=2(3+\cos4\alpha)\cos2\alpha=6\cos2\alpha+2\cos4\alpha\cos2\alpha$

$=6\cos2\alpha+\cos6\alpha+\cos2\alpha.$

(60) $64\{(\cos^4\alpha+\sin^4\alpha)^2-2\sin^4\alpha\cos^4\alpha\}=64\{(1-2\sin^2\alpha\cos^2\alpha)^2-2\sin^4\alpha\cos^4\alpha\}$

$=8\{8(1-\frac{1}{2}\sin^22\alpha)^2-\sin^42\alpha)\}.$

$$=8(8-8\sin^2 2\alpha+\sin^4 2\alpha)=8\{8-4(1-\cos4\alpha)+\sin^4 2\alpha\}$$
$$=32+32\cos4\alpha+2(1-\cos4\alpha)^2=34+28\cos4\alpha+2\cos^2 4\alpha$$
$$=35+28\cos4\alpha+(2\cos^2 4\alpha-1)=35+28\cos4\alpha=\cos8\alpha.$$

(61) $$\frac{1}{\cos^2\frac{\alpha}{2}\cos\alpha}\left(\cot\frac{\alpha}{2}+\cot\frac{3}{2}\alpha\right)\left(\cot\frac{\alpha}{2}-\cot\frac{3}{2}\alpha\right)$$

$$\frac{1}{\cos^2\frac{\alpha}{2}\cos\alpha}\left(\frac{\sin2\alpha}{\sin\frac{\alpha}{2}\sin\frac{3}{2}\alpha}\right)\left(\frac{\sin\alpha}{\sin\frac{\alpha}{2}\sin\frac{3}{2}\alpha}\right)=\frac{8}{\sin^2\frac{3}{2}\alpha}.$$

(62) $$\left(\frac{1}{\cos\alpha}+\frac{\cos\alpha+1}{\sin\alpha\cos\alpha}\right)\frac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}}\cdot\frac{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{4}}{\cos^2\frac{\alpha}{4}}$$

$$=\left(\frac{\sin\alpha+2\cos^2\frac{\alpha}{2}}{\sin\alpha\cos\alpha}\right)\frac{\cos\alpha\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}\cos^2\frac{\alpha}{4}}=\frac{2\cos\frac{\alpha}{2}\left(\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}\right)}{2\sin\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\cos^2\frac{\alpha}{4}}$$

$$=\left(\sec\frac{\alpha}{2}+\operatorname{cosec}\frac{\alpha}{2}\right)\sec^2\frac{\alpha}{4}.$$

(63) $$\frac{(\sec^2\alpha-\tan^2\alpha)(\sec^2\beta-\tan^2\beta)}{2(1+\tan^2\alpha\tan^2\beta)-(1+\tan^2\alpha)(1+\tan^2\beta)}=\frac{1}{(1-\tan^2\alpha)(1-\tan^2\beta)}$$

$$=\frac{\cos^2\alpha\cos^2\beta}{(\cos^2\alpha-\sin^2\alpha)(\cos^2\beta-\sin^2\beta)}=\frac{\cos^2\alpha\cos^2\beta}{\cos2\alpha\cos2\beta}.$$

(64) $$\frac{\sin^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{6}}{\cos^2\frac{\alpha}{2}\sin^2\frac{\alpha}{6}}\cdot\frac{\cos\frac{2\alpha}{3}}{\sin\frac{2\alpha}{3}}=\frac{\left(\sin\frac{\alpha}{2}+\sin\frac{\alpha}{6}\right)\left(\sin\frac{\alpha}{2}-\sin\frac{\alpha}{6}\right)\cos\frac{2\alpha}{3}}{\cos^2\frac{\alpha}{2}\sin^2\frac{\alpha}{6}\sin\frac{2\alpha}{3}}$$

$$=\frac{4\sin\frac{\alpha}{3}\cos\frac{\alpha}{6}\cos\frac{\alpha}{3}\sin\frac{\alpha}{6}\cos\frac{2\alpha}{3}}{2\cos\frac{\alpha}{2}\sin^2\frac{\alpha}{6}\sin\frac{\alpha}{3}\cos\frac{\alpha}{3}}=\frac{\sin\frac{\alpha}{3}\cos\frac{2\alpha}{3}}{\cos^2\frac{\alpha}{2}\sin^2\frac{\alpha}{6}}$$

$$=\frac{\tan\frac{\alpha}{3}\cos\frac{2\alpha}{3}\cos\frac{\alpha}{3}}{\cos^2\frac{\alpha}{2}\sin^2\frac{\alpha}{6}}=\frac{\tan\frac{\alpha}{3}\left(\cos\alpha+\cos\frac{\alpha}{3}\right)}{2\cos\frac{\alpha}{2}\sin^2\frac{\alpha}{6}}$$

$$=\frac{\tan\frac{\alpha}{3}\left(2\cos^2\frac{\alpha}{2}-2\sin^2\frac{\alpha}{6}\right)}{2\cos^2\frac{\alpha}{2}\sin^2\frac{\alpha}{6}}=\tan\frac{\alpha}{3}\left(\operatorname{cosec}^2\frac{\alpha}{6}-\sec^2\frac{\alpha}{2}\right).$$

(65) $$\frac{\cos\alpha}{\sin\alpha}+\frac{\cos2\alpha}{\sin2\alpha}+\frac{\cos4\alpha}{\sin4\alpha}=\frac{2\cos2\alpha(2\cos^2\alpha+\cos2\alpha)+\cos4\alpha}{\sin4\alpha}$$
$$=\operatorname{cosec}4\alpha\{2\cos2\alpha(\cos2\alpha+1+\cos2\alpha)+\cos4\alpha\}.$$

(66) $$\operatorname{cosec}\alpha=\frac{1}{\sin\alpha}=\frac{2\sin2\alpha+2\cos2\alpha}{2\sin2\alpha\sin\alpha+2\cos2\alpha\sin\alpha}$$
$$=\frac{2\sin2\alpha+2\cos2\alpha}{(\cos\alpha-\cos3\alpha)+(\sin3\alpha-\sin\alpha)}.$$

67. $\sin 4\alpha \tan^4\alpha + 4\tan^3\alpha + 2\sin 4\alpha \tan^2\alpha$

$$-4\tan\alpha + 4\sin 4\alpha \equiv 0.$$

68. $\operatorname{cosec}\alpha \operatorname{cosec} 2\alpha + \operatorname{cosec} 2\alpha \operatorname{cosec} 3\alpha$

$$\equiv \operatorname{cosec}\alpha\,(\cot\alpha - \cot 3\alpha).$$

69. $\tan\alpha + 2\tan 2\alpha + 4\tan 4\alpha + 8\cot 8\alpha \equiv \cot\alpha.$

70. $\cos 2\alpha + \cos 2\alpha \cos\alpha \equiv \cos 2\alpha \sin\frac{3\alpha}{2} \operatorname{cosec}\frac{1}{2}\alpha.$

71. $2\sin 7\alpha \cos\alpha + 16\sin\alpha \cos^5\alpha$

$$\equiv \sin 6\alpha + 4\sin 2\alpha\,(1 + 2\cos^3 2\alpha).$$

72. $\{\sec\alpha + \operatorname{cosec}\alpha\,(1 + \sec\alpha)\}\left(1 - \tan^2\frac{\alpha}{2}\right)\left(1 + \tan^2\frac{\alpha}{4}\right)$

$$\equiv \left(\sec\frac{\alpha}{2} + \operatorname{cosec}\frac{\alpha}{2}\right)\sec^2\frac{\alpha}{4}.$$

73. $\dfrac{\sin 3\theta + 2\sin 5\theta + \sin 7\theta}{\sin 5\theta + 2\sin 7\theta + \sin 9\theta} \equiv \dfrac{\sin 5\theta}{\sin 7\theta}.$

74. $\dfrac{\sin\theta + 2\sin 3\theta + \sin 5\theta}{\cos\theta - 2\cos 3\theta + \cos 5\theta} \equiv \dfrac{4\sin\theta - 3\operatorname{cosec}\theta}{4\cos\theta - 3\sec\theta}.$

75. $\dfrac{\tan 5\theta + \tan 3\theta}{\tan 5\theta - \tan 3\theta} \equiv 4\cos 2\theta \cos 4\theta.$

76. $\dfrac{\cos 2\theta - \cos 4\theta}{\sin 4\theta - \sin 2\theta} + \dfrac{\cos\theta - \cos 3\theta}{\sin\theta - \sin 3\theta} \equiv \dfrac{\sin\theta}{\cos 2\theta \cos 3\theta}.$

例題解自 67. 至 76.

(67) $\sin 4\alpha\,(\tan^4\alpha + 2\tan^2\alpha + 1) + 4\tan\alpha\,(\tan^2\alpha - 1)$

$$= \sin 4\alpha\,(\tan^2\alpha + 1)^2 + 4\tan\alpha\left(\frac{\sin^2\alpha - \cos^2\alpha}{\cos^2\alpha}\right)$$

$$= \sin 4\alpha \sec^4\alpha + 4\tan\alpha\left(\frac{-\cos 2\alpha}{\cos^2\alpha}\right) = \frac{2\sin 2\alpha \cos 2\alpha}{\cos^4\alpha} - \frac{4\sin\alpha \cos 2\alpha}{\cos^3\alpha}$$

$$= \frac{2\cos 2\alpha\,(2\sin\alpha - 2\sin\alpha)}{\cos^3\alpha} = 0.$$

(68) $\frac{1}{\sin2\alpha}\left(\frac{1}{\sin\alpha}+\frac{1}{\sin3\alpha}\right)=\frac{1}{\sin2\alpha}\left(\frac{2\sin2\alpha\cos\alpha}{\sin\alpha\sin3\alpha}\right)$

$=\frac{2\cos\alpha}{\sin\alpha\sin3\alpha}=\frac{\sin2\alpha}{\sin^2\alpha\sin3\alpha}=\operatorname{cosec}\alpha\left\{\frac{\sin(3\alpha-\alpha)}{\sin\alpha\sin3\alpha}\right\}$

$=\operatorname{cosec}\alpha(\cot\alpha-\cot3\alpha).$

(69) $\tan\alpha+2\tan2\alpha+4\tan4\alpha+\frac{8}{\tan8\alpha}=\tan\alpha+2\tan2\alpha+4\tan4\alpha$

$+\frac{8(1-\tan^2 4\alpha)}{2\tan4\alpha}=\tan\alpha+2\tan2\alpha+\frac{4}{\tan4\alpha}=\tan\alpha+2\tan2\alpha+\frac{4(1-\tan^2 2\alpha)}{2\tan2\alpha}$

$=\tan\alpha+\frac{2}{\tan2\alpha}=\tan\alpha+\frac{2(1-\tan^2\alpha)}{2\tan\alpha}=\frac{1}{\tan\alpha}=\cot\alpha.$

(70) $2\cos2\alpha\cos\alpha+\cos2\alpha=\frac{\cos2\alpha\left(2\cos\alpha\sin\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)}{\sin\frac{\alpha}{2}}$

$=\cos2\alpha\left(\sin\frac{3}{2}\alpha-\sin\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)\operatorname{cosec}\frac{\alpha}{2}=\cos2\alpha\sin\frac{3}{2}\alpha\operatorname{cosec}\frac{\alpha}{2}.$

(71) $\sin8\alpha+\sin6\alpha+8\sin2\alpha\cos^2\alpha=\sin8\alpha+\sin6\alpha+4\sin2\alpha(\cos2\alpha+1)$

$=\sin6\alpha+2\sin4\alpha\cos4\alpha+4\sin2\alpha(\cos2\alpha+1)$

$=\sin6\alpha+4\sin2\alpha(\cos4\alpha\cos2\alpha+\cos2\alpha+1)$

$=\sin6\alpha+4\sin2\alpha\{\cos2\alpha(\cos4\alpha+1)+1\}=\sin6\alpha+4\sin2\alpha(2\cos^3 2\alpha+1).$

(72) $\left(\frac{1}{\cos\alpha}+\frac{\cos\alpha+1}{\sin\alpha\cos\alpha}\right)\frac{\cos\alpha}{\cos^2\frac{\alpha}{2}}\cdot\frac{\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{4}}=\frac{\sin\alpha+\cos\alpha+1}{\sin\alpha\cos^2\frac{\alpha}{2}\cos^2\frac{\alpha}{4}}$

$=\frac{2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+2\cos^2\frac{\alpha}{2}}{2\sin\frac{\alpha}{2}\cos^2\frac{\alpha}{2}\cos^2\frac{\alpha}{4}}=\left(\sec\frac{\alpha}{2}+\operatorname{cosec}\frac{\alpha}{2}\right)\sec^2\frac{\alpha}{4}.$

(73) $\frac{2\sin5\theta\cos2\theta+2\sin5\theta}{2\sin7\theta\cos2\theta+2\sin7\theta}=\frac{\sin5\theta}{\sin7\theta}.$

(74) $\frac{2\sin3\theta\cos2\theta+2\sin3\theta}{2\cos3\theta\cos2\theta-2\cos3\theta}=\frac{\sin3\theta(\cos2\theta+1)}{\cos3\theta(\cos2\theta-1)}=\frac{\sin3\theta\cos^2\theta}{-\cos3\theta\sin^2\theta}$

$=\frac{(3\sin\theta-4\sin^3\theta)\cos^2\theta}{-(4\cos^3\theta-3\cos\theta)\sin^2\theta}=\frac{4\sin\theta-3\operatorname{cosec}\theta}{4\cos\theta-3\sec\theta}.$

(75) $\frac{\sin(5\theta+3\theta)}{\cos5\theta\cos3\theta}\div\frac{\sin(5\theta-3\theta)}{\cos5\theta\cos3\theta}=\frac{\sin8\theta}{\sin2\theta}=\frac{2\sin4\theta\cos4\theta}{\sin2\theta}.$

(76) $\frac{2\sin3\theta\sin\theta}{2\cos3\theta\sin\theta}+\frac{2\sin2\theta\sin\theta}{-2\cos2\theta\sin\theta}=\frac{\sin(3\theta-2\theta)}{\cos3\theta\cos2\theta}.$

77. $\dfrac{\cos\theta}{1-\sin\theta}+\dfrac{\cos\phi}{1-\sin\phi}\equiv\dfrac{2(\sin\theta-\sin\phi)}{\sin(\theta-\phi)+\cos\theta-\cos\phi}.$

78. $\dfrac{\cos(n-2)\alpha-\cos n\alpha}{\sin n\alpha-\sin(n-2)\alpha}\equiv\tan(n-1)\alpha.$

79. $\dfrac{\sec 2^{2n+1}\alpha-1}{\sec 2^{2n}\alpha-1}\equiv\dfrac{\tan 2^{2n+1}\alpha}{\tan 2^{2n-1}\alpha}.$

80. $\dfrac{\sin n\alpha}{\cos 2n\alpha+\cos\alpha}\equiv\dfrac{\sec(n+\frac{1}{2})\alpha-\sec(n-\frac{1}{2})\alpha}{4\sin\frac{1}{2}\alpha}.$

81. $\dfrac{\sin\alpha\pm\sin n\alpha+\sin(2n-1)\alpha}{\cos\alpha\pm\cos n\alpha+\cos(2n-1)\alpha}\equiv\tan n\alpha.$

82. $\sin n\alpha\cos(n+2)\alpha-\cos^2(n+1)\alpha+\sin^2\alpha\equiv0.$

83. $\sin n\alpha\,\mathrm{cosec}^2\alpha\sec\alpha-\cos n\alpha\sec^2\alpha\,\mathrm{cosec}\,\alpha$

$$\equiv4\sin(n-1)\alpha\,\mathrm{cosec}^2 2\alpha.$$

84. $\dfrac{1+(\mathrm{cosec}\,\alpha\tan\chi)^2}{1+(\mathrm{cosec}\,\beta\tan\chi)^2}\equiv\dfrac{1+(\cot\alpha\sin\chi)^2}{1+(\cot\beta\sin\chi)^2}.$

85. $\sqrt{1+\sin\alpha}\equiv1+2\sin\dfrac{\alpha}{4}\sqrt{1-\sin\dfrac{\alpha}{2}}.$

86. $\cos^2(\alpha-\beta)+\cos^2\beta-2\cos(\alpha-\beta)\cos\alpha\cos\beta\equiv\sin^2\alpha.$

87. $\sin^2(\alpha-\beta)+\sin^2\beta+2\sin(\alpha-\beta)\sin\beta\cos\alpha\equiv\sin^2\alpha.$

88. $\sin(\beta-\alpha)\sin(\delta-\gamma)+\sin(\gamma-\beta)\sin(\delta-\alpha)$

$$\equiv\sin(\gamma-\alpha)\sin(\beta-\delta).$$

89. $\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)$

$$\equiv-4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}.$$

例題解自 77. 至 89.

(77) $$\frac{\cos^2\frac{\theta}{2}-\sin^2\frac{\theta}{2}}{\left(\cos\frac{\theta}{2}-\sin\frac{\theta}{2}\right)^2}+\frac{\cos^2\frac{\phi}{2}-\sin^2\frac{\phi}{2}}{\left(\cos\frac{\phi}{2}-\sin\frac{\phi}{2}\right)^2}=\frac{\cos\frac{\theta}{2}+\sin\frac{\theta}{2}}{\cos\frac{\theta}{2}-\sin\frac{\theta}{2}}+\frac{\cos\frac{\phi}{2}+\sin\frac{\phi}{2}}{\cos\frac{\phi}{2}-\sin\frac{\phi}{2}}$$

$$=\frac{2\cos\frac{\theta}{2}\cos\frac{\phi}{2}-2\sin\frac{\theta}{2}\sin\frac{\phi}{2}}{\cos\frac{\theta}{2}\cos\frac{\phi}{2}+\sin\frac{\theta}{2}\sin\frac{\phi}{2}-\sin\frac{\theta}{2}\cos\frac{\phi}{2}-\cos\frac{\theta}{2}\sin\frac{\phi}{2}}=\frac{2\cos\left(\frac{\theta}{2}+\frac{\phi}{2}\right)}{\cos\left(\frac{\theta}{2}-\frac{\phi}{2}\right)-\sin\left(\frac{\theta}{2}+\frac{\phi}{2}\right)}$$

$$=\frac{4\cos\left(\frac{\theta}{2}+\frac{\phi}{2}\right)\sin\left(\frac{\theta}{2}-\frac{\phi}{2}\right)}{2\sin\left(\frac{\theta}{2}-\frac{\phi}{2}\right)\cos\left(\frac{\theta}{2}-\frac{\phi}{2}\right)-2\sin\left(\frac{\theta}{2}+\frac{\phi}{2}\right)\sin\left(\frac{\theta}{2}-\frac{\phi}{2}\right)}=\frac{2(\sin\theta-\sin\phi)}{\sin(\theta-\phi)-(\cos\phi-\cos\theta)}$$

(78) $\frac{2\sin(n-1)\alpha\sin\alpha}{2\cos(n-1)\alpha\sin\alpha}=\tan(n-1)\alpha.$

(79) $\frac{1-\cos 2^{2n+1}\alpha}{\cos 2^{2n+1}\alpha}\Big/\frac{1-\cos 2^{2n}\alpha}{\cos 2^{2n}\alpha}=\frac{2\sin^2 2^{2n}\alpha}{\cos 2^{2n+1}\alpha}\Big/\frac{2\sin^2 2^{2n-1}\alpha}{\cos 2^{2n}\alpha}$

$=\frac{2\sin^2 2^{2n}\alpha\cos 2^{2n}\alpha}{2\sin^2 2^{2n-1}\alpha\cos 2^{2n+1}\alpha}=\frac{\sin 2^{2n+1}\alpha\sin 2^{2n}\alpha}{2\sin^2 2^{2n-1}\alpha\cos 2^{2n+1}\alpha}$

$=\frac{\tan 2^{2n+1}\alpha(2\sin 2^{2n-1}\alpha\cos 2^{2n-1}\alpha)}{2\sin^2 2^{2n-1}\alpha}=\frac{\tan 2^{2n+1}\alpha}{\tan 2^{2n-1}\alpha}.$

(80) $\frac{\sin n\alpha\sin\frac{1}{2}\alpha}{2\cos(n+\frac{1}{2})\alpha\cos(n-\frac{1}{2})\alpha\sin\frac{1}{2}\alpha}=\frac{\frac{1}{2}\{\cos(n-\frac{1}{2})\alpha-\cos(n+\frac{1}{2})\alpha\}}{2\cos(n+\frac{1}{2})\alpha\cos(n-\frac{1}{2})\alpha\sin\frac{1}{2}\alpha}.$

(81) $\frac{2\sin n\alpha\cos(n-1)\alpha\pm\sin n\alpha}{2\cos n\alpha\cos(n-1)\alpha\pm\cos n\alpha}=\frac{\sin n\alpha}{\cos n\alpha}.$

(82) $\frac{1}{2}\{\cos(2n+2)\alpha+\cos 2\alpha\}-\frac{1}{2}\{1+\cos(2n+2)\alpha\}+\frac{1}{2}(1-\cos 2\alpha)=0.$

(83) $\frac{\sin n\alpha}{\sin^2\alpha\cos\alpha}-\frac{\cos n\alpha}{\cos^2\alpha\sin\alpha}=\frac{\sin n\alpha\cos\alpha-\cos n\alpha\sin\alpha}{\sin^2\alpha\cos^2\alpha}=\frac{\sin(n-1)\alpha}{4\sin^2 2\alpha}.$

(84) $\frac{1+(1+\cot^2\alpha)\frac{\sin^2\chi}{\cos^2\chi}}{1+(1+\cot^2\beta)\frac{\sin^2\chi}{\cos^2\chi}}=\frac{\cos^2\chi+\sin^2\chi+\cot^2\alpha\sin^2\chi}{\cos^2\chi+\sin^2\chi+\cot^2\beta\sin^2\chi}.$

(85) $\sqrt{\left(\cos^2\frac{\alpha}{2}+\sin^2\frac{\alpha}{2}+2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\right)}=\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}$

$=1-2\sin^2\frac{\alpha}{4}+2\sin\frac{\alpha}{4}\cos\frac{\alpha}{4}=1+2\sin\frac{\alpha}{4}\sqrt{\left(\cos\frac{\alpha}{4}-\sin\frac{\alpha}{4}\right)^2}.$

(86) $\cos^2(\alpha-\beta)+\cos^2\beta-\cos(\alpha-\beta)\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}$

$=\cos^2\beta-\cos(\alpha+\beta)\cos(\alpha-\beta)=\cos^2\beta-(\cos^2\alpha-\sin^2\beta).$ (87) 與前例同樣.

(88) $\frac{1}{2}\{\cos(\beta-\alpha-\delta+\gamma)-\cos(\beta-\alpha+\delta-\gamma)\}+\frac{1}{2}\{\cos(\gamma-\beta-\delta+\alpha)$

$-\cos(\gamma-\beta+\delta-\alpha)\}=\frac{1}{2}\{\cos(\beta-\alpha-\delta+\gamma)-\cos(\gamma-\beta+\delta-\alpha)\}=\sin(\beta-\delta)\sin(\gamma-\alpha).$

(89) $2\sin\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\alpha-\beta)+2\sin\frac{1}{2}(\beta-\alpha)\cos\frac{1}{2}(\beta-2\gamma+\alpha)$

$=2\sin\frac{1}{2}(\alpha-\beta)\{\cos\frac{1}{2}(\alpha-\beta)-\cos\frac{1}{2}(\beta-2\gamma+\alpha)\}$

$=2\sin\frac{1}{2}(\alpha-\beta)\{2\sin\frac{1}{2}(\alpha-\gamma)\sin\frac{1}{2}(\beta-\gamma)\}$

$=-4\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\gamma-\alpha)\sin\frac{1}{2}(\beta-\gamma).$

90. $\{\sin(\alpha-\beta)+\sin(\alpha+3\beta)\}\sec 2\beta$

$$\equiv(\cos 2\beta-\cos 2\alpha)\operatorname{cosec}(\alpha-\beta).$$

91. $\sin(2\alpha+\beta)\operatorname{cosec}\alpha-2\cos(\alpha+\beta)\equiv\sin\beta\operatorname{cosec}\alpha.$

92. $\cos(\alpha+\beta)\sin(\alpha-\beta)+\cos(\beta+\gamma)\sin(\beta-\gamma)$

$$+\cos(\gamma+\delta)\sin(\gamma-\delta)+\cos(\delta+\alpha)\sin(\delta-\alpha)\equiv 0.$$

93. $\sin(\delta-\beta)\sin(\alpha-\gamma)+\sin(\beta-\gamma)\sin(\alpha-\delta)$

$$+\sin(\gamma-\delta)\sin(\alpha-\beta)\equiv 0.$$

94. $\sin\alpha\sin(\beta-\gamma)\cos(\beta+\gamma-\alpha)$

$$+\sin\beta\sin(\gamma-\alpha)\cos(\gamma+\alpha-\beta)$$

$$+\sin\gamma\sin(\alpha-\beta)\cos(\alpha+\beta-\gamma)\equiv 0.$$

95. $\sin\alpha\sin\beta\sin(\alpha-\beta)+\sin\beta\sin\gamma\sin(\beta-\gamma)$

$$+\sin\gamma\sin\alpha\sin(\gamma-\alpha)$$

$\equiv\frac{1}{4}\{\sin(2\alpha-2\beta)+\sin(2\beta-2\gamma)+\sin(2\gamma-2\alpha)\}.$

96. $\sin(\alpha-\beta)\cos(\gamma-\beta)\cos(\alpha-\gamma)$

$+\sin(\beta-\gamma)\cos(\alpha-\gamma)\cos(\beta-\alpha)+\sin(\gamma-\alpha)\cos(\beta-\alpha)\cos(\gamma-\beta)$

$\equiv\frac{1}{4}\{\sin(2\beta-2\alpha)+\sin(2\gamma-2\beta)+\sin(2\alpha-2\gamma)\}.$

97. $\cos\beta\cos\gamma\sin(\gamma-\beta)+\cos\gamma\cos\alpha\sin(\alpha-\gamma)$

$+\cos\alpha\cos\beta\sin(\beta-\alpha)\equiv\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha).$

98. $\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$

$\equiv 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}.$

99. $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$\equiv 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}.$

100. $\cos 2\alpha\cos^2(\beta+\gamma)+\cos 2\beta\cos^2(\gamma+\alpha)+\cos 2\gamma\cos^2(\alpha+\beta)$

$\equiv\cos 2\alpha\cos 2\beta\cos 2\gamma+2\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta).$

101. $\sin\theta-3\sin(\theta+\alpha)+3\sin(\theta+2\alpha)-\sin(\theta+3\alpha)$

$\equiv 8\sin^3\frac{\alpha}{2}\cos\left(\theta+\frac{3\alpha}{2}\right).$

例題解自 90. 至 101.

(90) $2\sin(\alpha+\beta)\cos 2\beta\frac{1}{\cos 2\beta}=2\sin(\alpha+\beta)\sin(\alpha-\beta)\frac{1}{\sin(\alpha-\beta)}$

$=(\cos 2\beta-\cos 2\alpha)\operatorname{cosec}(\alpha-\beta).$

(91) $\frac{\sin 2\alpha\cos\beta+\cos 2\alpha\sin\beta}{\sin\alpha}-2(\cos\alpha\cos\beta-\sin\alpha\sin\beta)$

$=2\cos\alpha\cos\beta+\left(\frac{1}{\sin\alpha}-2\sin\alpha\right)\sin\beta-2\cos\alpha\cos\beta+2\sin\alpha\sin\beta.$

(92) $\frac{1}{2}(\sin 2\alpha-\sin 2\beta)+\frac{1}{2}(\sin 2\beta-\sin 2\gamma)+\frac{1}{2}(\sin 2\gamma-\sin 2\delta)+\frac{1}{2}(\sin 2\delta-\sin 2\alpha)=0.$

(93) 與前同樣.

(94) $\frac{1}{2}\{\cos(\alpha-\beta+\gamma)-\cos(\alpha+\beta-\gamma)\}\cos(\beta+\gamma-\alpha)$

$+\frac{1}{2}\{\cos(\beta-\gamma+\alpha)-\cos(\beta+\gamma-\alpha)\}\cos(\gamma+\alpha-\beta)$

$+\frac{1}{2}\{\cos(\gamma-\alpha+\beta)-\cos(\gamma+\alpha-\beta)\}\cos(\alpha+\beta-\gamma)=0.$

(95) $\sin\alpha\sin\beta\sin(\alpha-\beta)=\frac{1}{2}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}\sin(\alpha-\beta)$

$=\frac{1}{2}\{\frac{1}{2}\sin(2\alpha-2\beta)-\cos(\alpha+\beta)\sin(\alpha-\beta)\}$

$=\frac{1}{4}\{\sin(2\alpha-2\beta)-(\sin 2\alpha-\sin 2\beta)\}.$

由是原式$=\frac{1}{4}\{\sin(2\alpha-2\beta)+\sin(2\beta-2\gamma)+\sin(2\gamma-2\alpha)$

$-(\sin 2\alpha-\sin 2\beta+\sin 2\beta-\sin 2\gamma+\sin 2\gamma-\sin 2\alpha)\}.$ (96) 與前同樣.

(97) 如前例得原式$=\frac{1}{4}\{\sin(2\beta-2\gamma)+\sin(2\gamma-2\alpha)+\sin(2\alpha-2\beta)\}.$

如例題 89. 卽可得其結果.

(98) $2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-2\cos\left(\frac{\alpha+\beta}{2}+\gamma\right)\sin\frac{\alpha+\beta}{2}$

$=2\sin\frac{\alpha+\beta}{2}\left\{\cos\frac{\alpha-\beta}{2}-\cos\left(\frac{\alpha+\beta}{2}+\gamma\right)\right\}=4\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha+\gamma}{2}\sin\frac{\beta+\gamma}{2}.$

(99) 與前同樣.

(100) $\cos 2\alpha\cos^2(\beta+\gamma)=\frac{1}{2}\cos 2\alpha\{1+\cos(2\beta+2\gamma)\}$

$=\frac{1}{2}\{\cos 2\alpha+\frac{1}{2}[\cos(2\alpha+2\beta+2\gamma)+\cos(2\beta+2\gamma-2\alpha)]\},$ 由是

原式$=\frac{1}{2}(\cos 2\alpha+\cos 2\beta+\cos 2\gamma)+\frac{3}{4}\cos(2\alpha+2\beta+2\gamma)$

$+\frac{1}{4}\{\cos(2\beta+2\gamma-2\alpha)+\cos(2\gamma+2\alpha-2\beta)+\cos(2\alpha+2\beta-2\gamma)\}.$

如前例得 $\cos(2\beta+2\gamma-2\alpha)+\cos(2\gamma+2\alpha-2\beta)+\cos(2\alpha+2\beta-2\gamma)$

$=4\cos 2\gamma\cos 2\alpha\cos 2\beta-\cos(2\alpha+2\beta+2\gamma)$ 及 $\cos(2\alpha+2\beta+2\gamma)$

$=4\cos(\alpha+\beta)\cos(\beta+\gamma)\cos(\gamma+\alpha)-\cos(2\alpha+2\beta+2\gamma).$ 故得其證.

(101) $6\cos\frac{2\theta+3\alpha}{2}\sin\frac{\alpha}{2}-2\cos\frac{2\theta+3\alpha}{2}\sin\frac{3\alpha}{2}$

$=2\cos\frac{2\theta+3\alpha}{2}\left(3\sin\frac{\alpha}{2}-\sin\frac{3\alpha}{2}\right)=2\cos\left(\theta+\frac{3\alpha}{2}\right)\sin^3\frac{\alpha}{2}.$

102. $vers(180^\circ-\alpha)\equiv 2vers\frac{180^\circ+\alpha}{2}vers\frac{180^\circ-\alpha}{2}$.

103. $\sqrt{vers\alpha\, vers\beta}\equiv vers\frac{\alpha+\beta}{2}-vers\frac{\alpha-\beta}{2}$.

104. $\frac{sin\beta}{sin\alpha}\equiv\frac{sin(2\alpha+\beta)}{sin\alpha}-2cos(\alpha+\beta)$.

105. $$\frac{sin\alpha}{sin(\alpha-\beta)sin(\alpha-\gamma)}+\frac{sin\beta}{sin(\beta-\gamma)sin(\beta-\alpha)}+\frac{sin\gamma}{sin(\gamma-\alpha)sin(\gamma-\beta)}\equiv 0.$$

106. $$\frac{sin(\theta-\alpha)}{sin(\alpha-\beta)sin(\alpha-\gamma)}+\frac{sin(\theta-\beta)}{sin(\beta-\gamma)sin(\beta-\alpha)}+\frac{sin(\theta-\gamma)}{sin(\gamma-\alpha)sin(\gamma-\beta)}\equiv 0.$$

107. $$\frac{sin(\theta-\beta)sin(\theta-\gamma)}{sin(\alpha-\beta)sin(\alpha-\gamma)}+\frac{sin(\theta-\gamma)sin(\theta-\alpha)}{sin(\beta-\gamma)sin(\beta-\alpha)}+\frac{sin(\theta-\alpha)sin(\theta-\beta)}{sin(\gamma-\alpha)sin(\gamma-\beta)}\equiv 1.$$

108. $$\frac{tan\alpha}{tan(\alpha-\beta)tan(\alpha-\gamma)}+\frac{tan\beta}{tan(\beta-\gamma)tan(\beta-\alpha)}+\frac{tan\gamma}{tan(\gamma-\alpha)tan(\gamma-\beta)}\equiv tan\alpha\, tan\beta\, tan\gamma.$$

109. $$\frac{sin\frac{1}{2}(\alpha+\beta)sin\frac{1}{2}(\alpha+\gamma)}{sin\frac{1}{2}(\alpha-\beta)sin\frac{1}{2}(\alpha-\gamma)}cos\alpha+\frac{sin\frac{1}{2}(\beta+\gamma)sin\frac{1}{2}(\beta+\alpha)}{sin\frac{1}{2}(\beta-\gamma)sin\frac{1}{2}(\beta-\alpha)}cos\beta+\frac{sin\frac{1}{2}(\gamma+\alpha)sin\frac{1}{2}(\gamma+\beta)}{sin\frac{1}{2}(\gamma-\alpha)sin\frac{1}{2}(\gamma-\beta)}cos\gamma\equiv cos(\alpha+\beta+\gamma).$$

例題解自 102. 至 109.

(102) $1-cos(180^\circ-\alpha)=1+cos\alpha=2cos^2\frac{\alpha}{2}=2\left(1-sin^2\frac{\alpha}{2}\right)$

$=2\left(1+sin\frac{\alpha}{2}\right)\left(1-sin\frac{\alpha}{2}\right)=2\left\{1-cos\left(90^\circ+\frac{\alpha}{2}\right)\right\}\left\{1-cos\left(90^\circ-\frac{\alpha}{2}\right)\right\}$.

(103) $\sqrt{2sin^2\frac{\alpha}{2}2sin^2\frac{\beta}{2}}=2sin\frac{\alpha}{2}sin\frac{\beta}{2}=cos\frac{\alpha-\beta}{2}-cos\frac{\alpha+\beta}{2}$

$=\left(1-cos\frac{\alpha+\beta}{2}\right)-\left(1-cos\frac{\alpha-\beta}{2}\right)=vers\frac{\alpha+\beta}{2}-vers\frac{\alpha-\beta}{2}$.

(104) $\frac{\sin\beta}{\sin\alpha}=\frac{\sin(2\alpha+\beta)-\{\sin(2\alpha+\beta)-\sin\beta\}}{\sin\alpha}=\frac{\sin(2\alpha+\beta)-2\cos(\alpha+\beta)\sin\alpha}{\sin\alpha}$.

(105) 原式$=\frac{\sin\alpha\sin(\beta-\gamma)+\sin\beta\sin(\gamma-\alpha)+\sin\gamma\sin(\alpha-\beta)}{-\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}$,

此分子$=\frac{1}{2}\{\cos(\alpha-\beta+\gamma)-\cos(\alpha+\beta-\gamma)+\cos(\beta-\gamma+\alpha)-\cos(\beta+\gamma-\alpha)$

$+\cos(\gamma-\alpha+\beta)-\cos(\gamma+\alpha-\beta)\}=0$,

(106) 與前例同樣.

(107) $\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}=\frac{\sin(\beta-\gamma)\sin(\theta-\beta)\sin(\theta-\gamma)}{-\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}$

$=\frac{\frac{1}{2}\sin(\beta-\gamma)\{\cos(\beta-\gamma)-\cos(2\theta-\beta-\gamma)\}}{-\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}$

$=\frac{\frac{1}{4}\{\sin(2\beta-2\gamma)-\sin(2\theta-2\gamma)+\sin(2\theta-2\beta)\}}{-\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}$, 由是

原式$=\frac{\frac{1}{4}\{\sin(2\beta-2\gamma)+\sin(2\gamma-2\alpha)+\sin(2\alpha-2\beta)\}}{-\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)}$. 以下與例題89.同樣.

(108) $\frac{\tan\alpha}{\tan(\alpha-\beta)\tan(\alpha-\gamma)}=\frac{\tan\alpha(1+\tan\alpha\tan\beta)(1+\tan\alpha\tan\gamma)}{(\tan\alpha-\tan\beta)(\tan\alpha-\tan\gamma)}$

$=\frac{\tan\alpha(\tan\alpha-\tan\gamma)+\tan^2\alpha(\tan^2\beta-\tan^2\gamma)+\tan^3\alpha\tan\beta\tan\gamma(\tan\beta-\tan\gamma)}{-(\tan\alpha-\tan\beta)(\tan\beta-\tan\gamma)(\tan\gamma-\tan\alpha)}$,

原式$=\frac{\tan\alpha\tan\beta\tan\gamma\{\tan^2\alpha(\tan\beta-\tan\gamma)+\tan^2\beta(\tan\gamma-\tan\alpha)+\tan^2\gamma(\tan\alpha-\tan\gamma)\}}{-(\tan\alpha-\tan\beta)(\tan\beta-\tan\gamma)(\tan\gamma-\tan\alpha)}$

$=\tan\alpha\tan\beta\tan\gamma$.

(109) $\frac{\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha+\gamma)}{\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\alpha-\gamma)}\cos\alpha=\frac{\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha+\gamma)}{-\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)}\cos\alpha$

$=\frac{\frac{1}{2}\{\cos\frac{1}{2}(\alpha+\gamma)-\cos\frac{1}{2}(2\beta+\alpha-\gamma)\}\sin\frac{1}{2}(\alpha+\gamma)\cos\alpha}{-\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)}$

$=\frac{\frac{1}{4}\{\sin(\alpha+\gamma)-\sin(\alpha+\beta)+\sin(\beta-\gamma)\}\cos\alpha}{-\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)}$

$=\frac{\frac{1}{8}\{\sin(2\alpha+\gamma)+\sin\gamma-\sin(2\alpha+\beta)-\sin\beta+\sin(\alpha+\beta-\gamma)-\sin(\alpha-\beta+\gamma)\}}{-\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)}$

原式$=\frac{1}{-8\sin\frac{1}{2}(\alpha-\beta)\sin\frac{1}{2}(\beta-\gamma)\sin\frac{1}{2}(\gamma-\alpha)}\{\sin(2\alpha+\gamma)-\sin(2\beta+\gamma)$

$+\sin(2\beta+\alpha)-\sin(2\gamma+\alpha)+\sin(2\gamma+\beta)-\sin(2\alpha+\beta)\}$.

110. $1+\sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\gamma\sin\alpha-\cos\alpha\cos\beta\cos\gamma$

$$\equiv 2\left\{\sin\frac{\alpha+\beta}{2}\cos\frac{\gamma}{2}+\cos\frac{\alpha-\beta}{2}\sin\frac{\gamma}{2}\right\}^2.$$

111. $4\sin(\theta-\alpha)\sin(m\theta-\alpha)\cos(\theta-m\theta)$

$\equiv 1+\cos(2\theta-2m\theta)-\cos(2\theta-2\alpha)-\cos(2m\theta-2\alpha).$

112. $(1+\sec2\theta)(1+\sec4\theta)(1+\sec8\theta)\cdots(1+\sec2^n\theta)$

$\equiv\tan2^n\theta\cot\theta.$

113. $\left(\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}\right)\left(\cos\frac{\alpha}{2^2}+\cos\frac{\beta}{2^2}\right)\cdots\left(\cos\frac{\alpha}{2^n}+\cos\frac{\beta}{2^n}\right)$

$$\equiv\frac{1}{2^n}\left(\frac{\cos\alpha-\cos\beta}{\cos\frac{\alpha}{2^n}-\cos\frac{\beta}{2^n}}\right).$$

114. $(2\cos\theta-1)(2\cos2\theta-1)(2\cos2^2\theta-1)\cdots(2\cos2^{n-1}\theta-1)$

$$\equiv\frac{2\cos2^n\theta+1}{2\cos\theta+1}$$

115. $\sin\theta\equiv2^n\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cos\frac{\theta}{2^{n-1}}\sin\frac{\theta}{2^n}.$

116. $2\cos\theta\equiv\sqrt{2+\sqrt{2+\sqrt{2+\cdots+\sqrt{2+2\cos2^n\theta}}}}.$

但 $\sqrt{\ }$ 爲 n 個.

117. $\cos^2(\alpha+\beta)+\cos^2(\alpha-\beta)-\cos2\alpha\cos2\beta$ 試化爲最簡之式.

118. 求 $\cos^2\alpha+\cos^2\beta+\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma-1$ 之因子.

119. 求 $\frac{1}{2}\{\cos(3\alpha+2\beta)+\cos(5\alpha-4\beta)\}$ 之因子.

例題解自 110. 至 119.

(110) $1+\frac{1}{2}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}+\sin\gamma(\sin\beta+\sin\alpha)$

$-\frac{1}{2}\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}\cos\gamma$

$=1+\frac{1}{2}\cos(\alpha-\beta)(1-\cos\gamma)-\frac{1}{2}\cos(\alpha+\beta)(1+\cos\gamma)+2\sin\gamma\sin\frac{\beta+\alpha}{2}\cos\frac{\alpha-\beta}{2}$

$=1+\left(2\cos^2\frac{\alpha-\beta}{2}-1\right)\sin^2\frac{\gamma}{2}-\left(1-2\sin^2\frac{\alpha+\beta}{2}\right)\cos^2\frac{\gamma}{2}+4\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$$=2\left\{\cos^2\frac{\alpha-\beta}{2}\sin^2\frac{\gamma}{2}+\sin^2\frac{\alpha+\beta}{2}\cos^2\frac{\gamma}{2}+2\sin\frac{\gamma}{2}\cos\frac{\gamma}{2}\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\right\}.$$

(111) $2\sin(\theta-\alpha)\{\sin(\theta-\alpha)+\sin(2m\theta-\theta-\alpha)\}$

$=2\sin^2(\theta-\alpha)+\cos(2m\theta-2\theta)-\cos(2m\theta-2\alpha).$

(112) $1+\sec 2\theta=\dfrac{\cos 2\theta+1}{\cos 2\theta}=\dfrac{\sin 2\theta}{\cos 2\theta}\cdot\dfrac{2\cos^2\theta}{\sin 2\theta}=\dfrac{\tan 2\theta}{\tan\theta}$，故

原式$=\dfrac{\tan 2\theta}{\tan\theta}\cdot\dfrac{\tan 2^2\theta}{\tan 2\theta}\cdot\dfrac{\tan 2^3\theta}{\tan 2^2\theta}\cdots\dfrac{\tan 2^n\theta}{\tan 2^{n-1}\theta}=\dfrac{\tan 2^n\theta}{\tan\theta}.$

(113) $\cos\dfrac{\alpha}{2}+\cos\dfrac{\beta}{2}=\dfrac{2\cos^2\frac{\alpha}{2}-2\cos^2\frac{\beta}{2}}{2\left(\cos\frac{\alpha}{2}-\cos\frac{\beta}{2}\right)}=\dfrac{\cos\alpha-\cos\beta}{2\left(\cos\frac{\alpha}{2}-\cos\frac{\beta}{2}\right)}$，故

原式$=\dfrac{\cos\alpha-\cos\beta}{2\left(\cos\frac{\alpha}{2}-\cos\frac{\beta}{2}\right)}\cdot\dfrac{\cos\frac{\alpha}{2}-\cos\frac{\beta}{2}}{2\left(\cos\frac{\alpha}{2^2}-\cos\frac{\beta}{2^2}\right)}\cdot\dfrac{\cos\frac{\alpha}{2^2}-\cos\frac{\beta}{2^2}}{2\left(\cos\frac{\alpha}{2^3}-\cos\frac{\beta}{2^3}\right)}\cdots\dfrac{\cos\frac{\alpha}{2^{n-1}}-\cos\frac{\beta}{2^{n-1}}}{2\left(\cos\frac{\alpha}{2^n}-\cos\frac{\beta}{2^n}\right)}.$

(114) $2\cos\theta-1=\dfrac{4\cos^2\theta-1}{2\cos\theta+1}=\dfrac{2\cos 2\theta+1}{2\cos\theta+1}$，故

原式$=\dfrac{2\cos 2\theta+1}{2\cos\theta+1}\cdot\dfrac{2\cos 2^2\theta+1}{2\cos 2\theta+1}\cdot\dfrac{2\cos 2^3\theta+1}{2\cos 2^2\theta+1}\cdots\dfrac{2\cos 2^n\theta+1}{2\cos 2^{n-1}\theta+1}.$

(115) $\sin\theta=2\cos\dfrac{\theta}{2}\sin\dfrac{\theta}{2}=2\cos\dfrac{\theta}{2}2\cos\dfrac{\theta}{2^2}\sin\dfrac{\theta}{2^2}$. 以下同樣.

(116) $2\cos^2\theta=1+\cos 2\theta$，故 $2\cos\theta=\sqrt{2+2\cos 2\theta}$

$=\sqrt{2+\sqrt{2+2\cos 2^2\theta}}=\sqrt{2+\sqrt{2+\sqrt{2+2\cos 2^3\theta}}}$. 以下同樣.

(117) $\frac{1}{2}\{\cos(2\alpha+2\beta)+1+\cos(2\alpha-2\beta)+1\}$

$-\frac{1}{2}\{\cos(2\alpha+2\beta)+1+\cos(2\alpha-2\beta)\}=1.$

(118) $\cos^2\alpha-2\cos\alpha\cos\beta\cos\gamma+\cos^2\beta\cos^2\gamma+\cos^2\beta+\cos^2\gamma-1-\cos^2\beta\cos^2\gamma$

$=(\cos\alpha-\cos\beta\cos\gamma)^2-(1-\cos^2\beta)(1-\cos^2\gamma)=(\cos\alpha-\cos\beta\cos\gamma)^2-\sin^2\beta\sin^2\gamma$

$=(\cos\alpha-\cos\beta\cos\gamma+\sin\beta\sin\gamma)(\cos\alpha-\cos\beta\cos\gamma-\sin\beta\sin\gamma)$

$=\{\cos\alpha-\cos(\beta+\gamma)\}\{\cos\alpha-\cos(\beta-\gamma)\}$

$=4\sin\dfrac{\alpha+\beta+\gamma}{2}\sin\dfrac{\beta+\gamma-\alpha}{2}\sin\dfrac{\alpha+\beta-\gamma}{2}\sin\dfrac{\beta-\gamma-\alpha}{2}.$

(119) $2\cos\dfrac{(3\alpha+2\beta)+(5\alpha-4\beta)}{2}\cos\dfrac{(3\alpha+2\beta)-(5\alpha-4\beta)}{2}$

$=2\cos(4\alpha-\beta)\cos(3\beta-\alpha).$

120. 次式化爲最簡之式.

$$(x\cos2\alpha+y\sin2\alpha-1)(x\cos2\beta+y\sin2\beta-1)$$
$$-\{x\cos(\alpha+\beta)+y\sin(\alpha+\beta)-\cos(\alpha-\beta)\}^3.$$

121. 化次式爲最簡之式.

$$\{x\cos(\alpha+\beta)+y\sin(\alpha+\beta)-\cos(\alpha-\beta\}\{x\cos(\gamma+\delta)+y\sin(\gamma+\delta)$$
$$-\cos(\gamma-\delta)\}-\{x\cos(\alpha+\gamma)+y\sin(\alpha+\gamma)-\cos(\alpha-\gamma)\}\{x\cos(\beta+\delta)$$
$$+y\sin(\beta+\delta)-\cos(\beta-\delta)\}.$$

122. 設 $\tan2\alpha=-\frac{3}{4}$, 則 $\cos3\alpha=\pm\frac{13}{50}\sqrt{10}$. 試證之.

123. 有 $\sin\theta=\frac{1}{3}$, 求 $\tan\theta$ 及 $\sin\frac{\theta}{2}$ 之值.

124. 由 $\tan\theta=\frac{7}{24}$ 以求 $\cos2\theta$, $\tan3\theta$, 及 $\tan\frac{\theta}{2}$.

125. 設 $\tan\alpha=\frac{1}{7}$, $\tan\beta=\frac{1}{2}$, 則 $\tan(\beta-2\alpha)=\frac{2}{11}$. 試證之.

126. 設 $\tan\alpha=\frac{5}{12}$, $\cos2\beta=\frac{527}{625}$, 則 $\operatorname{cosec}\frac{\alpha+\beta}{2}=5\sqrt{13}$.

127. 有 $\cos\theta=\frac{5}{11}$, 求 $\tan\frac{\theta}{2}$, $\tan\theta$, 及 $\tan2\theta$.

128. 有 $\sin\theta=\frac{120}{169}$ 求 $\tan\frac{\theta}{2}$ 及 $\cos\frac{3}{2}\theta$.

129. 設 $\kappa=\frac{\pi}{7}$, 則 $\cos3\kappa-\cos2\kappa+\cos\kappa=\frac{1}{2}$. 試證之.

例題解自 120 至 129.

(120) 原式 $=x^2\cos2\alpha\cos2\beta+y^2\sin2\alpha\sin2\beta+xy(\sin2\alpha\cos2\beta+\cos2\alpha\sin2\beta)$
$-x(\cos2\alpha+\cos2\beta)-y(\sin2\alpha+\sin2\beta)+1-x^2\cos^2(\alpha+\beta)-y^2\sin^2(\alpha+\beta)-\cos^2(\alpha-\beta)$
$-2xy\sin(\alpha+\beta)\cos(\alpha+\beta)+2x\cos(\alpha+\beta)\cos(\alpha-\beta)+2y\sin(\alpha+\beta)\cos(\alpha-\beta)$
$=-x^2\sin^2(\alpha-\beta)-y^2\sin^2(\alpha-\beta)+\sin^2(\alpha-\beta)=-(x^2+y^2-1)\sin^2(\alpha-\beta).$

(121) 如前例解之.卽可得
$-(x^2+y^2-1)\sin(\beta-\gamma)\sin(\alpha-\delta).$

(122) $\tan 2\alpha=-\frac{3}{4}=\frac{2\tan\alpha}{1-\tan^2\alpha}$. 從此得 $\tan\alpha=3$, 或 $-\frac{1}{3}$. 故

$\cos\alpha=\frac{1}{\sqrt{(1+\tan^2\alpha)}}=\frac{\sqrt{10}}{10}$, 或 $\frac{3\sqrt{10}}{10}$.

$\cos 3\alpha=4\cos^3\alpha-3\cos\alpha=\frac{-13}{50}\sqrt{10}$, 或 $\frac{13}{50}\sqrt{10}$.

(123) $\tan\theta=\frac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}=\frac{\frac{1}{3}}{\sqrt{(1-\frac{1}{9})}}=\frac{\sqrt{2}}{4}$, $\sin\frac{\theta}{2}=\sqrt{\frac{1-\sqrt{1-\sin^2\theta}}{2}}$

$=\sqrt{\frac{1-\sqrt{\frac{8}{9}}}{2}}=\sqrt{\frac{3-2\sqrt{2}}{6}}=\sqrt{\frac{2-2\sqrt{2}+1}{6}}=\frac{\sqrt{2}-1}{\sqrt{6}}$.

(124) $\cos 2\theta=2\cos^2\theta-1=\frac{2}{1+\tan^2\theta}-1=\frac{2}{1+(\frac{7}{24})^2}-1=\frac{527}{625}$,

$\tan 3\theta=\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}$ 〔公式 (5)〕$=\frac{3(\frac{7}{24})-(\frac{7}{24})^3}{1-3(\frac{7}{24})^2}=\frac{2\times1679}{24\times429}$

$\tan\theta=\frac{7}{24}=\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$, 故 $\tan\frac{\theta}{2}=\frac{1}{7}$ 或 -7.

(125) $\tan 2\alpha=\frac{2\tan\alpha}{1-\tan^2\alpha}=\frac{\frac{2}{7}}{1-\frac{1}{49}}=\frac{7}{24}$,

故 $\tan(\beta-2\alpha)=\frac{\tan\beta-\tan\alpha}{1+\tan\beta\tan 2\alpha}=\frac{\frac{1}{2}-\frac{7}{24}}{1+\frac{1}{2}\times\frac{7}{24}}=\frac{2}{11}$,

(126) $\cos\alpha=\frac{1}{\sqrt{(1+\tan^2\alpha)}}=\frac{12}{13}$, 故 $\sin\alpha=\frac{5}{13}$, $\cos\beta=\sqrt{\frac{1+\cos 2\beta}{2}}=\frac{24}{25}$,

故 $\sin\beta=\frac{7}{25}$, $\operatorname{cosec}\frac{\alpha-\beta}{2}=\frac{1}{\sin\frac{\alpha-\beta}{5}}=\frac{1}{\sqrt{\frac{1-\cos(\alpha-\beta)}{2}}}$

$=\sqrt{\frac{2}{1-\cos\alpha\cos\beta-\sin\alpha\sin\beta}}=5\sqrt{13}$.

(127) $\tan\frac{\theta}{2}=\frac{\sqrt{6}}{4}$, $\tan\theta=\frac{4\sqrt{6}}{5}$, $\tan 2\theta=-\frac{40}{71}\sqrt{6}$.

(128) $\tan\frac{\theta}{2}=\frac{5}{12}$ 或 $-\frac{12}{5}$, $\cos\frac{3\theta}{2}=\frac{828}{13^3}$.

(129) $\cos 3\chi-\cos 2\chi+\cos\chi=y$, $5\chi+2\chi=\pi$, 故

$y=\cos 3\chi+\cos 5\chi+\cos\chi$, $2y^2=2(\cos 3\chi+\cos 5\chi+\cos\chi)^2$, 即

$2y^2=3+\cos 10\chi+\cos 6\chi+\cos 2\chi+2(\cos 8\chi+\cos 2\chi+\cos 6\chi+\cos 4\chi+\cos 2\chi)$

$=3-5(\cos 3\chi-\cos 2\chi+\cos\chi)=3-5y$,

$\therefore\ (y+3)(2y-1)=0\quad\therefore\ y=\frac{1}{2}$.

130. $tan\theta+cot\theta=2\left(\frac{m^2+n^2}{m^2-n^2}\right)$, 試求 $cos2\theta$ 之值.

131. $(a-b)sec\theta=\sqrt{\left(a^4+\frac{a^2b^2}{a^2-1}\right)}$, $(a+b)sec\phi=\sqrt{\left(a^4+\frac{a^2b^2}{a^2-1}\right)}$, 則

$tan\frac{1}{2}(\theta-\phi)=\frac{b}{a\sqrt{(a^2-1)}}$. 試證之.

132. $tan\theta=\frac{sin\alpha cos\gamma-sin\beta sin\gamma}{cos\alpha cos\gamma-cos\beta sin\gamma}$,

$tan\phi=\frac{sin\alpha sin\gamma-sin\beta cos\gamma}{cos\alpha sin\gamma-cos\beta cos\gamma}$, 試求 $tan(\theta+\phi)$.

133. $sin\alpha=\frac{cos\alpha}{\sqrt{1-m^2sin^2\alpha}}$, 試求 $tan\alpha$.

134. $sin\theta+sin\phi=a$, $cos\theta+cos\phi=b$, 試表示下列各式爲 a, b 之項.

(1) $sin\theta sin\phi$, (2) $cos\theta cos\phi$, (3) $tan\theta+tan\phi$,

(4) $cos2\theta+cos2\phi$, (5) $tan\frac{\theta}{2}+tan\frac{\phi}{2}$ (6) $cos3\theta+cos3\phi$.

135. $x=rsin\frac{1}{2}(\theta-\alpha)$, $y=rsin\frac{1}{2}(\theta+\alpha)$, 則

$x^2-2xycos\alpha+y^2=r^2sin^2\alpha$. 試證之.

例題解自 130. 至 135.

(130) $\because$ $\frac{sin^2\theta+cos^2\theta}{cos\theta sin\theta}=\frac{2(m^2+n^2)}{m^2-n^2}$, $\frac{2}{sin2\theta}=\frac{2(m^2+n^2)}{m^2-n^2}$,

$sin2\theta=\frac{m^2-n^2}{m^2+n^2}$, $\therefore$ $cos2\theta=\sqrt{(1-sin^22\theta)}=\frac{2mn}{m^2+n^2}$.

(131) $\because$ $\frac{a-b}{cos\theta}=a\sqrt{\frac{a^4-a^2+b^2}{a^2-1}}$, $cos\theta=\frac{a-b}{a}\sqrt{\frac{a^2-1}{a^4-a^2+b^2}}$,

$cos\phi=\frac{a+b}{a}\sqrt{\frac{a^2-1}{a^4-a^2+b^2}}$, $\therefore$ $sin\theta=\frac{a^3-a+b}{a\sqrt{(a^4-a^2+b^2)}}$.

$sin\phi=\frac{a^3-a-b}{a\sqrt{(a^4-a^2+b^2)}}$. 由是得

$cos(\theta-\phi)=cos\theta cos\phi+sin\theta sin\phi=(a^4-a^2-b^2)/(a^4-a^2+b^2)$,

$$\tan\frac{1}{2}(\theta-\phi)=\sqrt{\frac{1-\cos(\theta-\phi)}{1+\cos(\theta-\phi)}}=\frac{b}{a\sqrt{(a^2-1)}}.$$

(132) 依題意得 $\tan\theta+\tan\phi=\dfrac{\sin(\alpha+\beta)\{\cos(\alpha-\beta)\sin 2\gamma-1\}}{(\cos\alpha\cos\gamma-\cos\beta\sin\gamma)(\cos\alpha\sin\gamma-\cos\beta\sin\gamma)}$,

$$1-\tan\theta\tan\phi=\frac{\cos(\alpha+\beta)\{\cos(\alpha-\beta)\sin 2\gamma-1)\}}{(\cos\alpha\cos\gamma-\cos\beta\sin\gamma)(\cos\alpha\sin\gamma-\cos\beta\cos\gamma)},$$

$$\therefore\ \tan(\theta+\phi)=\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}=\tan(\alpha+\beta).$$

(133) $\tan\alpha=1/\sqrt{(1-m^2)}$.

(134) $\because$ $(\sin\theta+\sin\phi)^2+(\cos\theta+\cos\phi)^2=a^2+b^2$ 卽 $2+2\cos(\theta-\phi)=a^2+b^2$,

$\therefore$ $\cos(\theta-\phi)=\dfrac{a^2+b^2}{2}-1$, 又 $(\cos\theta+\cos\phi)^2-(\sin\theta+\sin\phi)^2=b^2-a^2$, 卽

$\cos 2\theta+\cos 2\phi+2\cos(\theta+\phi)=b^2-a^2$, $2\cos(\theta+\phi)\{\cos(\theta-\phi)+1\}=b^2-a^2$,

$$\therefore\ \cos(\theta+\phi)=\frac{b^2-a^2}{a^2+b^2}.$$

(1) $\sin\theta\sin\phi=\dfrac{\cos(\theta-\phi)-\cos(\theta+\phi)}{2}=\dfrac{(a^2+b^2)^2-4b^2}{4(a^2+b^2)}$.

(2) $\cos\theta\cos\phi=\frac{1}{2}\{\cos(\theta+\phi)+\cos(\theta-\phi)\}=\{(a^2+b^2)^2-4a^2\}/4(a^2+b^2)$.

(3) $\tan\theta+\tan\phi=\dfrac{\sin(\theta+\phi)}{\cos\theta\cos\phi}=\dfrac{\sqrt{1-\cos^2(\theta+\phi)}}{\cos\theta\cos\phi}=\dfrac{8ab}{(a^2+b^2)^2-4a^2}$.

(4) $\cos 2\theta+\cos 2\phi=2\cos(\theta+\phi)\cos(\theta-\phi)=\dfrac{(b^2-a^2)(a^2+b^2-2)}{a^2+b^2}$.

(5) $\tan\dfrac{\theta}{2}+\tan\dfrac{\phi}{2}=\dfrac{\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{\theta}{2}\cos\frac{\phi}{2}}=\dfrac{2\sin\frac{1}{2}(\theta+\phi)}{\cos\frac{1}{2}(\theta+\phi)+\cos\frac{1}{2}(\theta-\phi)}$

$$=\frac{2\sqrt{1-\cos(\theta+\phi)}}{\sqrt{1+\cos(\theta+\phi)}+\sqrt{1+\cos(\theta-\phi)}}=\frac{2\sqrt{2a^2}}{\sqrt{2b^2}+\sqrt{\left(\frac{a^2+b^2}{2}\right)(a^2+b^2)}}=\frac{4a}{2b+a^2+b^2}$$

(6) $\cos 3\theta+\cos 3\phi=4(\cos^3\theta+\cos^3\phi)-3(\cos\theta+\cos\phi)$

$=(\cos\theta+\cos\phi)\{4(\cos\theta+\cos\phi)^2-12\cos\theta\cos\phi-3\}$

$=b\left\{4b^2-\dfrac{3(a^2+b^2)^2-12a^2}{a^2+b^2}-3\right\}$.

(135) $2xy\cos\alpha=2r^2\cos\alpha\sin\frac{1}{2}(\theta+\alpha)\sin\frac{1}{2}(\theta-\alpha)$

$=r^2\cos\alpha(\cos\alpha-\cos\theta)=r^2(\cos^2\alpha-\cos\alpha\cos\theta)$,

又 $y^2+y^2=r^2\{\sin^2\frac{1}{2}(\theta+\alpha)+\sin^2\frac{1}{2}(\theta-\alpha)\}$

$=\frac{1}{2}r^2\{2-\cos(\theta+\alpha)-\cos(\theta-\alpha)\}=r^2(1-\cos\theta\cos\alpha)$.

136. $a\sin\theta+b\cos\theta=a\,\mathrm{cosec}\,\theta+b\sec\theta=c$, 則

$\sin 2\theta=2ab/(c^2-a^2-b^2)$. 試證之.

137. $2\tan\alpha=3\tan\beta$, 則

$$\tan(\alpha-\beta)=\frac{\tan\beta}{2+3\tan^2\beta}=\frac{\sin 2\beta}{5-\cos 2\beta}.$$ 試證之.

138. $\sin\alpha=\tan\beta$, 則 $\sin(\alpha-\beta)\cos\beta=\sin 2\beta\sin^2\frac{\alpha}{2}$. 試證之.

139. $2\sec\theta=\sec(\theta+2\alpha)+\sec(\theta-2\alpha)$, 則 $\cos^2\theta=2\cos^2\alpha$. 試證之.

140. $\tan^2\theta-\sec^2\alpha=1$, 則

$\sec\theta+\tan^3\theta\,\mathrm{cosec}\,\theta=(3+\tan^2\alpha)^{\frac{3}{2}}$.

141. $\tan\frac{1}{2}(\theta-a)=\dfrac{3\sin\alpha}{5-3\cos\alpha}$, 則 $\tan\dfrac{\theta}{2}=4\tan\dfrac{\alpha}{2}$.

142. $\tan\theta=\dfrac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha}$, 則 $\tan\dfrac{\theta}{2}=\dfrac{(1-\sin\beta)(1-\cos\alpha)}{\sin\alpha\cos\beta}$.

143. $\tan\alpha+\sin\alpha=m$, $\tan\alpha-\sin\alpha=n$, 則

$\cos\alpha=(m-n)/(m+n)$.

144. $m\sin\alpha=n\cos\alpha$, 則 $\sin\alpha=\pm\dfrac{n}{\sqrt{(m^2+n^2)}}$.

145. $\tan^2\alpha=1+2\tan^2\beta$, 則 $\cos 2\beta=1+2\cos 2\alpha$.

146. $\left(\dfrac{\sin\alpha}{\sin\beta}\right)^2+(\cos\alpha\cos\gamma)^2=1$, 則 $\sin\gamma=\tan\alpha\cot\beta$.

例題解自 136 至 146.

(136) $(a\sin\theta+b\cos\theta)(a\cos\theta+b\sin\theta)=c^2\sin\theta\cos\theta$, 卽

$(a^2+b^2)\sin\theta\cos\theta+ab=c^2\sin\theta\cos\theta$.

(137) $\tan(\alpha-\beta)=\dfrac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\dfrac{\frac{3}{2}\tan\beta-\tan\beta}{1+\frac{3}{2}\tan^2\beta}=\dfrac{\tan\beta}{2+3\tan^2\beta}$

$=\dfrac{\sin\beta\cos\beta}{2\cos^2\beta+3\sin^2\beta}=\dfrac{\sin 2\beta}{2(2+\sin^2\beta)}=\dfrac{\sin 2\beta}{5-\cos 2\beta}.$

(138) $\sin(\alpha-\beta)\cos\beta=(\sin\alpha\cos\beta-\cos\alpha\sin\beta)\cos\beta$

$=\sin\alpha\dfrac{\cos\beta}{\sin\beta}\cdot\frac{1}{2}\sin 2\beta-\cos\alpha\frac{1}{2}\sin 2\beta=\frac{1}{2}\sin 2\beta\left(\dfrac{\sin\alpha}{\tan\beta}-\cos\alpha\right)$

$=\frac{1}{2}\sin 2\beta(1-\cos\alpha)=\sin 2\beta\sin^2\frac{1}{2}\alpha.$

(139) $\dfrac{2}{\cos\theta}=\dfrac{\cos(\theta-2\alpha)+\cos(\theta+2\alpha)}{\cos(\theta+2\alpha)\cos(\theta-2\alpha)}$ 卽 $\dfrac{2}{\cos\theta}=\dfrac{4\cos\theta\cos 2\alpha}{\cos 2\theta+\cos 4\alpha}$,

故 $2\cos^2\theta-1+\cos 4\alpha=2\cos^2\theta\cos 2\alpha$, 故 $\cos^2\theta=2\cos^2\alpha.$

(140) $\sec^2\theta-1-\sin^2\alpha=1$, 故 $\sec\theta=(3+\tan^2\alpha)^{\frac{1}{2}}$. 由是

$\sec\theta+\tan^3\theta\operatorname{cosec}\theta=\dfrac{\sin^2\theta+\cos^2\theta}{\cos^3\theta}=\sec^3\theta=(3+\tan^2\alpha)^{\frac{3}{2}}.$

(141) $\dfrac{\tan\frac{\theta}{2}-\tan\frac{\alpha}{2}}{1+\tan\frac{\theta}{2}\tan\frac{\alpha}{2}}=\dfrac{6\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{5-3\left(2\cos^2\frac{\alpha}{2}-1\right)}$, $\left(\tan\frac{\theta}{2}-\tan\frac{\alpha}{2}\right)\left(4-3\cos^2\frac{\alpha}{2}\right)$

$=3\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(1+\tan\frac{\theta}{2}\tan\frac{\alpha}{2}\right)$, $\tan\frac{\theta}{2}\left(4-3\cos^2\frac{\alpha}{2}-3\sin^2\frac{\alpha}{2}\right)$

$=3\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}+\tan\frac{\alpha}{2}\left(4-3\cos^2\frac{\alpha}{2}\right)$, $\tan\frac{\theta}{2}=4\tan\frac{\alpha}{2}.$

(142) $\dfrac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=\dfrac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha}$, $1-\tan^2\frac{\theta}{2}=2\tan\frac{\theta}{2}\left(\dfrac{\sin\beta+\cos\alpha}{\sin\alpha\cos\beta}\right)$,

由是 $\tan\frac{\theta}{2}=\dfrac{-(\sin\beta+\cos\alpha)+(\sin\beta\cos\alpha+1)}{\sin\alpha\cos\beta}.$

(143) $2\tan\alpha=m+n$, $2\sin\alpha=m-n$, $\therefore$ $\cos\alpha=(m-n)/(m+n).$

(144) $m^2\sin^2\alpha=n^2(1-\sin^2\alpha)$. 故如題言.

(145) $\sec^2\alpha-1=1+2(\sec^2\beta-1)$, 變 sec 爲 cos, 卽得.

(146) $\dfrac{\sin^2\alpha}{\sin^2\beta}+\cos^2\alpha(1-\sin^2\gamma)=1$,

$\sin^2\alpha-\sin^2\gamma\cos^2\alpha\sin^2\beta=\sin^2\alpha\sin^2\beta$, 從此可得其證.

147. $\cos\chi=\dfrac{\cos\alpha}{\sin\gamma}$, $\cos(90°-\chi)=\dfrac{\cos\beta}{\sin\gamma}$, 則

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1$. 試證之.

148. $\cos\theta=\cos\alpha\cos\beta$, $\cos\theta'=\cos\alpha'\cos\beta$, 及 $\tan\dfrac{\theta}{2}\tan\dfrac{\theta'}{2}=\tan\dfrac{\beta}{2}$,

則 $\sin^2\beta=(\sec\alpha-1)(\sec\alpha'-1)$. 試證之.

149. $\tan 2\alpha=\dfrac{2(ab+cd)}{a^2-b^2+c^2-d^2}$, $\tan 2\beta=\dfrac{2(ac+bd)}{a^2-c^2+b^2-d^2}$,

則 $\tan(\alpha-\beta)=\dfrac{b-c}{a+d}$. 試證之.

150. $m\sin(\theta+\phi)=\cos(\theta-\phi)$,

則 $\dfrac{1}{1-m\sin2\theta}+\dfrac{1}{1-m\sin2\phi}=\dfrac{2}{1-m^2}$. 試證之.

151. $(1+2\cos^{\frac{2}{3}}\alpha)(1+2\cos^{\frac{2}{3}}\beta)=3$, 則

$\dfrac{(1+8\cos^2\alpha)^{\frac{3}{2}}}{\sin^3\alpha\cos\alpha}=\dfrac{(1+8\cos^2\beta)^{\frac{3}{2}}}{\sin^3\beta\cos\beta}$. 試證之.

152. $\dfrac{\cos\theta\cos\frac{\phi}{2}}{\cos\left(\theta-\frac{\phi}{2}\right)}+\dfrac{\cos\phi\cos\frac{\theta}{2}}{\cos\left(\theta-\frac{\phi}{2}\right)}=1$, 則 $\cos\theta+\cos\phi=1$, 試證之.

例題解自 147. 至 152.

(147) $\left(\dfrac{\cos\alpha}{\sin\gamma}\right)^2+\left(\dfrac{\cos\beta}{\sin\gamma}\right)^2=\cos^2\chi+\sin^2\chi=1$. 從此可得其證.

(148) $\dfrac{\sin\frac{\theta}{2}\sin\frac{\theta'}{2}}{\cos\frac{\theta}{2}\cos\frac{\theta'}{2}}=\dfrac{\sin\frac{\beta}{2}}{\cos\frac{\beta}{2}}$, 即 $\dfrac{\sin\theta\sin\theta'}{4\cos^2\frac{\theta}{2}\cos^2\frac{\theta'}{2}}=\dfrac{\sqrt{(1-\cos\beta)}}{\sqrt{(1+\cos\beta)}}$.

即 $\dfrac{(1-\cos^2\theta)(1-\cos^2\theta')}{(1+\cos\theta)^2(1+\cos\theta')^2}=\dfrac{1-\cos\beta}{1+\cos\beta}$, 即 $\dfrac{1+\cos\theta\cos\theta'}{\cos\theta+\cos\theta'}=\dfrac{1}{\cos\beta}$,

故 $\dfrac{1+\cos\alpha\cos\alpha'\cos^2\beta}{\cos\beta(\cos\alpha+\cos\alpha')}=\dfrac{1}{\cos\beta}$, 故 $\cos^2\beta=\sec\alpha+\sec\alpha'-\sec\alpha\sec\alpha'$.

(149) $\sec 2\alpha=\sqrt{(1+\tan^2 2\alpha)}=\dfrac{\sqrt{\{(a-d)^2+(b+c)^2\}\{(a+d)^2+(b-c)^2\}}}{a^2-b^2+c^2-d^2}$,

同樣 $\sec 2\alpha \sec 2\beta=\dfrac{\{(a-d)^2+(b+c)^2\}\{(a+d)^2+(b-c)^2\}}{(a^2-d^2)^2-(b^2-c^2)^2}$.

$$1+\tan 2\alpha \tan 2\beta=1+\frac{4(ab+cd)(ac+bd)}{(a^2-d^2)^2-(b^2-c^2)^2}=\frac{\{(a+d)^2-(b-c)^2\}\{(a-d)^2+(b+c)^2\}}{(a^2-d^2)^2-(b^2-c^2)^2},$$

$$\tan(\alpha-\beta)=\frac{\sin(\alpha-\beta)}{\cos(\alpha-\beta)}=\sqrt{\frac{1-\cos(2\alpha-2\beta)}{1+\cos(2\alpha-2\beta)}}=\sqrt{\frac{\sec 2\alpha\sec 2\beta-(1+\tan 2\alpha\tan 2\beta)}{\sec 2\alpha\sec 2\beta+(1+\tan 2\alpha\tan 2\beta)}}$$

$$=\sqrt{\frac{2(b-c)^2\{(a-d)^2+(b+c)^2\}}{2(a+d)^2\{(a-d)^2+(b+c)^2\}}}=\frac{b-c}{a+d}.$$

(150) $\dfrac{1}{1-m\sin 2\theta}+\dfrac{1}{1-m\sin 2\phi}=\dfrac{2-m(\sin 2\theta+\sin 2\phi)}{1-m(\sin 2\theta+\sin 2\phi)+m^2\sin 2\theta\sin 2\phi}$

$$=\frac{2\{1-\sin(\theta+\phi)\cos(\theta-\phi)\}}{1-2m\sin(\theta+\phi)\cos(\theta-\phi)+\frac{1}{2}m^2\{\cos(2\theta-2\phi)-\cos(2\theta+2\phi)\}}$$

$$=\frac{2\{1-\cos^2(\theta-\phi)\}}{1-2\cos^2(\theta-\phi)+m^2\{\sin(\theta+\phi)-\sin^2(\theta-\phi)\}}=\frac{2\sin^2(\theta-\phi)}{\sin^2(\theta-\phi)-m^2\sin^2(\theta-\phi)}.$$

(151) 從既知關係式得 $\cos^{\frac{2}{3}}\alpha=\dfrac{1-\cos^{\frac{2}{3}}\beta}{1+2\cos^{\frac{2}{3}}\beta}$, 故

$$\frac{(1+8\cos^2\alpha)^{\frac{3}{2}}}{\sin^3\alpha\cos\alpha}=\frac{\left\{1+8\left(\dfrac{1-\cos^{\frac{2}{3}}\beta}{1+2\cos^{\frac{2}{3}}\beta}\right)^3\right\}^{\frac{3}{2}}}{\left\{1-\left(\dfrac{1-\cos^{\frac{2}{3}}\beta}{1+2\cos^{\frac{2}{3}}\beta}\right)^3\right\}^{\frac{3}{2}}\left(\dfrac{1-\cos^{\frac{2}{3}}\beta}{1+2\cos^{\frac{2}{3}}\beta}\right)^{\frac{3}{2}}}$$

$$=\frac{\{(1+2\cos^{\frac{2}{3}}\beta)^3+8(1-\cos^{\frac{2}{3}}\beta)^3\}^{\frac{3}{2}}(1+2\cos^{\frac{2}{3}}\beta)^{\frac{3}{2}}}{\{(1+2\cos^{\frac{2}{3}}\beta)^3-(1-\cos^{\frac{2}{3}}\beta)^3\}^{\frac{3}{2}}(1-\cos^{\frac{2}{3}}\beta)^{\frac{3}{2}}}=\frac{(1-2\cos^{\frac{2}{3}}\beta+4\cos^{\frac{4}{3}}\beta)^{\frac{3}{2}}(1+2\cos^{\frac{2}{3}}\beta)^{\frac{3}{2}}}{\cos\beta\,(1+\cos^{\frac{2}{3}}\beta+\cos^{\frac{4}{3}}\beta)^{\frac{3}{2}}(1-\cos^{\frac{2}{3}}\beta)^{\frac{3}{2}}}$$

$$=\frac{(1+8\cos^2\beta)^{\frac{3}{2}}}{\cos\beta\,(1-\cos^2\beta)^{\frac{3}{2}}}=\frac{(1+8\cos^2\beta)^{\frac{3}{2}}}{\cos\beta\sin^3\beta}.$$

(152) $\dfrac{\frac{1}{2}\left\{\cos\left(\theta+\frac{\phi}{2}\right)+\cos\left(\theta-\frac{\phi}{2}\right)\right\}}{\cos\left(\theta-\frac{\phi}{2}\right)}+\dfrac{\frac{1}{2}\left\{\cos\left(\phi+\frac{\theta}{2}\right)+\cos\left(\phi-\frac{\theta}{2}\right)\right\}}{\cos\left(\phi-\frac{\theta}{2}\right)}=1.$

從 $\dfrac{\cos\left(\theta+\frac{\phi}{2}\right)}{\cos\left(\theta-\frac{\phi}{2}\right)}=\dfrac{-\cos\left(\phi+\frac{\theta}{2}\right)}{\cos\left(\phi-\frac{\theta}{2}\right)}$, 得 $\dfrac{\cos\theta\cos\frac{\phi}{2}}{\sin\theta\sin\frac{\phi}{2}}=\dfrac{\sin\phi\sin\frac{\theta}{2}}{\cos\phi\cos\frac{\theta}{2}}$.

$\cos\theta\cos\phi=4\sin^2\frac{\theta}{2}\sin^2\frac{\phi}{2}=(1-\cos\theta)(1-\cos\phi)$. 從此卽得其證.

153. $\tan x=(2+\sqrt{3})\tan\frac{x}{3}$，試求 $\tan x$ 之值.

154. $\alpha=\frac{\pi}{17}$，試求 $\frac{\cos\alpha\cos13\alpha}{\cos3\alpha+\cos5\alpha}$ 之值.

155. $\sec(\phi+\alpha)+\sec(\phi-\alpha)=2\sec\phi$，則

$\cos\phi=\sqrt{2}\cos\frac{\alpha}{2}$.

156. $\tan\frac{\theta}{2}=\left(\frac{1+c}{1-c}\right)^{\frac{1}{2}}\tan\frac{\phi}{2}$，則 $\cos\theta=\frac{\cos\phi-c}{1-c\cos\phi}$.

157. $\tan\theta\tan\phi=\sqrt{\frac{a-b}{a+b}}$，則

$(a-b\cos2\theta)(a-b\cos2\phi)=a^2-b^2$.

158. $x=y\cos R+z\cos Q$, $y=z\cos P+x\cos R$,

$P+Q+R=(2n+1)\pi$，則 $z=x\cos Q+y\cos P$，及

$\cos P=\frac{y^2+z^2-x^2}{2yz}$，但 n 爲正整數.

159. $\sin\alpha=p\sin\beta$, $\cos\alpha=q\cos\beta$，及

$\sin\alpha+\cos\alpha=\gamma(\sin\beta+\cos\beta)$，則

$(p-r)^2(1-q^2)+(q-r)^2(1-p^2)=0$.

160. $\tan^2x+\sec2x=\frac{7\sqrt{3}-10}{\sqrt{3}}$，則 $\cos2x=\frac{-5-4\sqrt{3}}{23}$.

例題解自 153. 至 160.

(153) $\frac{3\tan\frac{x}{3}-\tan^3\frac{x}{3}}{1-3\tan^2\frac{x}{3}}=(2+\sqrt{3})\tan\frac{x}{3}$，即 $\tan\frac{x}{3}=2-\sqrt{3}$.

(154) $\frac{\cos\alpha\cos13\alpha}{2\cos4\alpha\cos\alpha}=\frac{\cos(\pi-4\alpha)}{2\cos4\alpha}=\frac{-\cos4\alpha}{2\cos4\alpha}=-\frac{1}{2}$.

(155) $\frac{\cos(\phi-\alpha)+\cos(\phi+\alpha)}{\cos(\phi+\alpha)\cos(\phi-\alpha)}=\frac{2}{\cos\phi}$ 即 $\frac{4\cos\phi\cos\alpha}{\cos2\phi+\cos2\alpha}=\frac{2}{\cos\phi}$,

即 $\frac{\cos\phi\cos\alpha}{\cos^2\phi+\cos^2\alpha-1}=\frac{1}{\cos\phi}$ 故 $\cos^2\phi=1+\cos\alpha=2\cos^2\frac{1}{2}\alpha$.

(156) $\dfrac{sec^2\frac{\theta}{2}-1}{sec^2\frac{\phi}{2}-1}=\dfrac{cos^2\frac{\phi}{2}\left(1-cos^2\frac{\theta}{2}\right)}{cos^2\frac{\theta}{2}\left(1-cos^2\frac{\phi}{2}\right)}=\dfrac{1+c}{1-c}$, 即

$\dfrac{(1+cos\phi)(1-cos\theta)}{(1+cos\theta)(1-cos\phi)}=\dfrac{1+c}{1-c}$, 即 $\dfrac{1-cos\theta\, cos\phi}{cos\phi-cos\theta}=\dfrac{1}{c}$.

(157) $\dfrac{sin^2\theta\, sin^2\phi}{cos^2\theta\, cos^2\phi}=\dfrac{a-b}{a+b}$, 即 $\dfrac{(1-cos2\theta)(1-cos2\phi)}{(1+cos2\theta)(1+cos2\phi)}=\dfrac{a-b}{a+b}$,

故 $\dfrac{1+cos2\theta\, cos2\phi}{cos2\theta+cos2\phi}=\dfrac{a}{b}$, 即 $-a(cos2\theta+cos2\phi)+b\, cos2\theta\, cos2\phi=-b$.

故 $a^2-ab(cos2\theta+cos2\phi)+b^2cos2\theta\, cos2\phi=a^2-b^2$.

(158) 從 $x-y\, cos R-z\, cos Q=0$, $x\, cos R-y+z\, cos P=0$, 得

$\dfrac{x}{cos Q+cos P\, cos R}=\dfrac{y}{cos P+cos Q\, cos R}=\dfrac{z}{1-cos^2R}$ 但 $P+Q+R=(2n+1)\pi$,

故令 $\dfrac{x}{sin P\, sin R}=\dfrac{y}{sin Q\, sin R}=\dfrac{z}{sin^2R}$, 即 $\dfrac{x}{sin P}=\dfrac{y}{sin Q}=\dfrac{z}{sin R}=K$,

則 $x\, cos Q+y\, cos P=K(sin P\, cos Q+sin Q\, cos P)=K\, sin(P+Q)=K\, sin R=z$.

又 $x(x-y\, cos R-z\, cos Q)+y(x\, cos R-y+z\, cos P)+z(x\, cos Q+y\, cos P-z)=0$,

即 $x^2-y^2-z^2+2yz\, cos P=0$.

(159) $p^2sin^2\beta+q^2cos^2\beta=sin^2\alpha+cos^2\alpha=1$, 即 $p^2tan^2\beta+q^2=1+tan^2\beta$,

故 $tan^2\beta=\dfrac{-(1-q^2)}{1-P^2}$, 又 $p\, sin\beta+q\, cos\beta=r(sin\beta+cos\beta)$,

故 $tan\beta=\dfrac{-(q-r)}{P-r}$, 故 $\dfrac{-(1-q^2)}{1-P^2}=\dfrac{(q-r)^2}{(P-r)^2}$.

(160) $\dfrac{sin^2\chi}{cos^2\chi}+sec2\chi=\dfrac{7\sqrt{3}-10}{\sqrt{3}}$, 即 $\dfrac{1-cos2\chi}{1+cos2\chi}+\dfrac{1}{cos2\chi}=\dfrac{21-10\sqrt{3}}{3}$,

故 $cos^22\chi(24-10\sqrt{3})-cos2\chi(10\sqrt{3}-15)=3$, 即

$cos^22\chi-\dfrac{5(3\sqrt{3}-2)cos2\chi}{46}=\dfrac{12+5\sqrt{3}}{46}$, 故

$cos2\chi=\dfrac{5(3\sqrt{3}-2)\pm\sqrt{(2983+620\sqrt{3})}}{92}$,

$\sqrt{(2983+620\sqrt{3})}=\sqrt{(2883+620\sqrt{3}+100)}=31\sqrt{3}+10$.

但 $\dfrac{7\sqrt{3}-10}{\sqrt{3}}$ 為負數, 而 $tan^2\chi$ 為正,

故 $sec2\chi$ 為負, 則 $cos2\chi$ 亦為負.

161. $\tan\frac{\theta}{2}=\sqrt{\frac{1+e}{1-e}}\tan\frac{\alpha}{2}$, $r=\frac{a(1-e^2)}{1+e\cos\theta}$, 則

$\sqrt{r}\cos\frac{\theta}{2}=\sqrt{a(1-e)}\cos\frac{\alpha}{2}$,

$\sqrt{r}\sin\frac{\theta}{2}=\sqrt{a(1+e)}\sin\frac{\alpha}{2}$.

162. $\frac{\tan(\alpha-\beta)}{\tan\alpha}+\frac{\sin^2\gamma}{\sin^2\alpha}=1$, 則 $\frac{\tan\alpha}{\tan\gamma}=\frac{\tan\gamma}{\tan\beta}$.

163. $\frac{\tan\theta}{\tan\alpha}=\frac{1+\cos^2\theta}{1+\sin^2\theta}$, 則 $\sin(3\theta+\alpha)=7\sin(\theta-\alpha)$.

164. $\tan^2\chi=\tan(\alpha+\chi)\tan(\alpha-\chi)$, 則 $\sin2\chi=\sqrt{2}\sin\alpha$.

165. $\frac{\sin(\chi+\alpha)}{\sin(\chi+\beta)}=\sqrt{\frac{\sin2\alpha}{\sin2\beta}}$ 則 $\tan^2\chi=\tan\alpha\tan\beta$.

166. $a\cos\phi=b\cos\theta$, 則 $\cot\frac{1}{2}(\phi+\theta)\cot\frac{1}{2}(\phi-\theta)=\frac{a+b}{a-b}$.

167. $\tan(\alpha+\beta)=3\tan\alpha$, 則

$\sin(2\alpha+2\beta)+\sin2\alpha=2\sin2\beta$.

168. $\frac{\tan^2\alpha}{\tan^2\beta}=\frac{\cos\beta(\cos\chi-\cos\alpha)}{\cos\alpha(\cos\chi-\cos\beta)}$, 則

$\tan^2\frac{\chi}{2}=\tan^2\frac{\alpha}{2}\tan^2\frac{\beta}{2}$.

例題解自 161. 至 168.

(161) $(1-e)\tan^2\frac{\theta}{2}=(1+e)\tan^2\frac{\alpha}{2}$,

$(1-e)\left(\frac{1}{\cos^2\frac{\theta}{2}}-1\right)=(1+e)\left(\frac{1}{\cos^2\frac{\alpha}{2}}-1\right)$, 故 $\cos^2\frac{\theta}{2}=\frac{(1-e)\cos^2\frac{\alpha}{2}}{1-e\cos\alpha}$,

$\cos\theta=2\cos^2\frac{\theta}{2}-1=\frac{\cos\alpha-e}{1-e\cos\alpha}$, 故從 $r(1+e\cos\theta)=a(1-e^2)$, 得

$r\left\{1+\frac{e(\cos\alpha-e)}{1-e\cos\alpha}\right\}=a(1-e^2)$, 即 $r=a(1-e\cos\alpha)=\frac{a(1-e)\cos^2\frac{\alpha}{2}}{\cos^2\frac{\theta}{2}}$

(162) $\frac{cosec^2\alpha}{cosec^2\gamma}=1-\frac{\tan\alpha-\tan\beta}{\tan\alpha(1+\tan\alpha\tan\beta)}=\frac{\tan\beta(1+\tan^2\alpha)}{\tan\alpha(1+\tan\alpha\tan\beta)}$,

$\frac{\tan^2\gamma(1+\tan^2\alpha)}{\tan^2\alpha(1+\tan^2\gamma)}=\frac{\tan\beta(1+\tan^2\alpha)}{\tan\alpha(1+\tan\alpha\tan\beta)}$, $\frac{\tan^2\gamma}{1+\tan^2\gamma}=\frac{\tan\alpha\tan\beta}{1+\tan\alpha\tan\beta}$.

(163) $\frac{\sin\theta\cos\alpha}{\cos\theta\sin\alpha}=\frac{1+\cos^2\theta}{1+\sin^2\theta}$, $\sin\theta\cos\alpha(1+\sin^2\theta)=\cos\theta\sin\alpha(1+\cos^2\theta)$,

$\sin(\theta-\alpha)=\sin\theta\cos\alpha(1-\cos^2\theta)=\cos\theta\sin\alpha(1-\sin^2\theta)$,

$\sin(\theta-\alpha)+\sin(\theta-\alpha)=\frac{1}{2}\sin 2\theta\cos(\theta+\alpha)$,

$4\sin(\theta-\alpha)=\frac{1}{2}\{\sin(3\theta+\alpha)+\sin(\theta-\alpha)\}$.

(164) $\tan^2\chi=\frac{\sin^2\alpha\cos^2\chi-\cos^2\alpha\sin^2\chi}{\cos^2\alpha\cos^2\chi-\sin^2\alpha\sin^2\chi}=\frac{\sin^2\alpha-\cos^2\alpha\tan^2\chi}{\cos^2\alpha-\sin^2\alpha\tan^2\chi}$,

$\tan^2\chi+\frac{1}{\tan^2\chi}=\frac{2\cos^2\alpha}{\sin^2\alpha}$ 卽 $\left(\tan\chi+\frac{1}{\tan\chi}\right)^2=\frac{2}{\sin^2\alpha}$, 故

$\frac{\tan^2\chi+1}{\tan\chi}=\frac{\sqrt{2}}{\sin\alpha}$, $\frac{1}{\sin\chi\cos\chi}=\frac{\sqrt{2}}{\sin\alpha}$,

(165) $\frac{(\sin\chi\cos\alpha+\cos\chi\sin\alpha)^2}{(\sin\chi\cos\beta+\cos\chi\sin\beta)^2}=\frac{\sin\alpha\cos\alpha}{\sin\beta\cos\beta}$,

$\left(\frac{\tan\chi+\tan\alpha}{\tan\chi+\tan\beta}\right)^2=\frac{\tan\alpha}{\tan\beta}$. 從此卽得其證.

(166) $\cot\frac{\phi+\theta}{2}\cot\frac{\phi-\theta}{2}=\frac{\cos\frac{1}{2}(\phi+\theta)\cos\frac{1}{2}(\phi-\theta)}{\sin\frac{1}{2}(\phi+\theta)\sin\frac{1}{2}(\phi-\theta)}=\frac{\cos\phi+\cos\theta}{\cos\theta-\cos\phi}$

$=\frac{\frac{b}{a}\cos\theta+\cos\theta}{\cos\theta-\frac{b}{a}\cos\theta}=\frac{a+b}{a-b}$.

(167) $\sin(\alpha+\beta)\cos\alpha=3\cos(\alpha+\beta)\sin\alpha$. 卽

$\sin(2\alpha+\beta)+\sin\beta=3\{\sin(2\alpha+\beta)-\sin\beta\}$, $\sin(2\alpha+\beta)=2\sin\beta$,

故 $\sin(2\alpha+\beta)\cos\beta=\sin 2\beta$, 卽 $\sin(2\alpha+2\beta)+\sin 2\alpha=2\sin 2\beta$.

(168) $\frac{\sin^2\alpha\cos\beta}{\sin^2\beta\cos\alpha}=\frac{2\cos^2\frac{\chi}{2}-(1+\cos\alpha)}{2\cos^2\frac{\chi}{2}-(1+\cos\beta)}$. 簡之,得

$2\cos^2\frac{\chi}{2}(1+\cos\alpha\cos\beta)=(1+\cos\alpha)(1+\cos\beta)$,

$\frac{2}{1+\tan^2\frac{\chi}{2}}=\frac{(1+\cos\alpha)(1+\cos\beta)}{1+\cos\alpha\cos\beta}$, 卽 $\tan^2\frac{\chi}{2}=\frac{(1-\cos\alpha)(1-\cos\beta)}{(1+\cos\alpha)(1+\cos\beta)}$.

169. $\left(\frac{\tan\alpha}{\sin\theta}-\frac{\tan\beta}{\tan\theta}\right)^2=\tan^2\alpha-\tan^2\beta$, 則 $\cos\theta=\frac{\tan\beta}{\tan\alpha}$.

170. $\frac{\sin(\alpha-\beta)}{\sin\beta}=\frac{\sin(\alpha+\theta)}{\sin\theta}$, 則

$$\cot\beta-\cot\theta=\cot(\alpha+\theta)+\cot(\alpha-\beta).$$

171. $\tan\theta=m$, $\tan\phi=n$, 則

$$\sin 2(\theta+\phi)=\frac{2(m+n)(1-mn)}{(1+m^2)(1+n^2)}.$$

172. $\cos\theta=\frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$, 則 $\tan\frac{\theta}{2}=\pm\frac{\tan\frac{\alpha}{2}}{\tan\frac{\beta}{2}}$.

173. $\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\cos(\alpha-\beta)=n^2(\sin\alpha-\sin\beta)^2$,

則 $\tan\frac{\alpha}{2}=\frac{1\pm n}{1\mp n}\cos\frac{\beta}{2}$.

174. $\operatorname{vers}\alpha=\chi$, $\operatorname{vers}\beta=m\chi$, $\operatorname{vers}\gamma=1-m$, $\alpha+\beta=\gamma$,

則 $\chi=1\pm\sqrt{\frac{2m}{1+m}}$.

175. $\sin x\cos y=\tan\alpha\cot\gamma$, $\sin y\cos\chi=\tan\beta\cot\gamma$,

$\cos^2 y-\cos^2\chi=\cos^2\gamma$, 則 $\sec^2\alpha-\sec^2\beta=\sin^2\gamma$.

176. $\sin\beta=m\sin(2\alpha+\beta)$, 則

$$\tan(\alpha+\beta)=\frac{1+m}{1-m}\tan\alpha.$$

177. $\sin\theta=\frac{a}{b}\sin\phi$, $\cos\theta=\frac{a'}{b'}\cos\phi$, 則

$$\sin(\theta\pm\phi)=\frac{\sqrt{(b^2-a^2)(a'^2-b'^2)}}{ab'\mp a'b},$$

及 $\cos(\theta\pm\phi)=\frac{aa'\mp bb'}{ab'\pm a'b}$.

例題解自 169 至 177.

(169) $\frac{\tan^2\alpha}{1-\cos^2\theta}-\frac{2\tan\alpha\tan\beta\cos\theta}{1-\cos^2\theta}+\frac{\tan^2\beta\cos^2\theta}{1-\cos^2\theta}=\tan^2\alpha-\tan^2\beta$

(170) $\sin\theta\sin(\alpha-\beta)=\sin\beta\sin(\alpha+\theta)$, 故

$\frac{\sin(\alpha-\beta+\theta)}{\sin\theta\sin(\alpha-\beta)}=\frac{\sin(\alpha+\theta-\beta)}{\sin\beta\sin(\alpha+\theta)}$, 卽 $\cot\theta+\cot(\alpha-\beta)=\cot\beta-\cot(\alpha+\theta)$.

(171) $\sin 2(\theta+\phi)=2\sin(\theta+\phi)\cos(\theta+\phi)$

$=2\cos^2\theta\cos^2\phi(\tan\theta+\tan\phi)(1-\tan\theta\tan\phi)=\frac{2(\tan\theta+\tan\phi)(1-\tan\theta\tan\phi)}{(1+\tan^2\theta)(1+\tan^2\phi)}$.

(172) $2\cos^2\frac{\theta}{2}-1=\frac{\cos\alpha-\cos\beta}{1-\cos\alpha\cos\beta}$, 卽 $2\cos^2\frac{\theta}{2}=\frac{(1+\cos\alpha)(1-\cos\beta)}{1-\cos\alpha\cos\beta}$,

故 $\tan^2\frac{\theta}{2}=\frac{(1-\cos\alpha)(1+\cos\beta)}{(1+\cos\alpha)(1-\cos\beta)}=\frac{\tan^2\frac{\alpha}{2}}{\tan^2\frac{\beta}{2}}$.

(173) $(\sin\alpha-\sin\beta)^2+2\sin\alpha\sin\beta\{1-\cos(\alpha-\beta)\}=n^2(\sin\alpha-\sin\beta)^2$,

$4(n^2-1)\cos^2\frac{\alpha+\beta}{2}\sin^2\frac{\alpha-\beta}{2}-2\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}\sin^2\frac{\alpha-\beta}{2}=0$,

$2(n^2-1)\cos^2\frac{\alpha+\beta}{2}-\left\{2\cos^2\frac{\alpha-\beta}{2}-2\cos^2\frac{\alpha+\beta}{2}\right\}=0$, 故

$n^2\cos^2\frac{\alpha+\beta}{2}=\cos^2\frac{\alpha-\beta}{2}$, 故 $\frac{1+\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}=\pm n$.

(174) $\cos\gamma=\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$, 卽

$m=(1-\chi)(1-m\chi)-\sqrt{\{1-(1-\chi)^2\}\{1-(1-m\chi)^2\}}$. 從此可得其證.

(175) $\sin(x+y)=(\tan\alpha+\tan\beta)\cos\gamma$, $\sin(x-y)=(\tan\alpha-\tan\beta)\cot\gamma$,

又 $4\cos\frac{x+y}{2}\cos\frac{x-y}{2}\sin\frac{x+y}{2}\sin\frac{x-y}{2}=\sin(x+y)\sin(x-y)=\cos^2\gamma$,

故 $(\tan^2\alpha-\tan^2\beta)\cot^2\gamma=\cos^2\gamma$, 故 $\sec^2\alpha-\sec^2\beta=\sin^2\gamma$.

(176) $\sin\{(\alpha+\beta)-\alpha\}=m\sin\{(\alpha+\beta)+\alpha\}$. 從此可得其證.

(177) 從 $\frac{a^2}{b^2}\sin^2\phi+\frac{a'^2}{b'^2}\cos^2\phi=\sin^2\theta+\cos^2\theta=1$, 得

$\sin\phi=\frac{b\sqrt{b'^2-a'^2}}{\sqrt{a^2b'^2-a'^2b^2}}$, $\cos\phi=\frac{b'\sqrt{a^2-b^2}}{\sqrt{a^2b'^2-a'^2b^2}}$, 故

$\sin(\theta\pm\phi)=\sin\theta\cos\phi\pm\cos\theta\sin\phi=\frac{ab'\pm a'b}{bb'}\sin\phi\cos\phi$.

178. 原缺

179. $b\sin(x+\theta)=c\sin(y-\theta)$, $b\cos x-c\cos y=0$, 則

$2\tan\theta=\tan y-\tan x$.

180. $\tan\theta=\dfrac{x\sin\alpha}{y-x\cos\alpha}$, $\tan\phi=\dfrac{y\sin\alpha}{x-y\cos\alpha}$, 則

$\tan(\theta+\phi)=-\tan\alpha$.

181. $\tan^2\theta=\tan(\theta-\alpha)\tan(\theta-\beta)$, 則

$\tan2\theta=\dfrac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$.

182. $a\cos\theta+b\sin\theta=a\cos\phi+b\sin\phi$, 則

$\tan(\theta+\phi)=\dfrac{2ab}{a^2-b^2}$.

183. $\tan\phi=\dfrac{\sin\alpha\sin\theta}{\cos\theta-\cos\alpha}$, 則 $\tan\theta=\dfrac{\sin\alpha\sin\phi}{\cos\phi\mp\cos\alpha}$.

184. $\sqrt{2}\cos\alpha=\cos\beta+\cos^3\beta$, $\sqrt{2}\sin\alpha=\sin\beta-\sin^3\beta$,

則 $\pm\sin(\beta-\alpha)=\cos2\beta=\dfrac{1}{3}$.

185. $\dfrac{\tan^2\theta}{\tan^2\alpha}+\dfrac{\tan^2\phi}{\tan^2\beta}=1$, $\dfrac{\sin\theta}{\sin\alpha}=\dfrac{\sin\phi}{\sin\beta}$, 則

$\sin\theta=\dfrac{\pm\sin\alpha}{\sqrt{(1\pm\cos\alpha\cos\beta)}}$.

186. $\tan\dfrac{z}{2}=\tan\dfrac{x}{2}\tan\dfrac{y}{2}$, 則

$\tan z=\dfrac{\sin x\sin y}{\cos x+\cos y}$.

187. $\dfrac{\sin(\theta-\alpha)}{\sin(\theta-\beta)}=\dfrac{a}{b}$, $\dfrac{\cos(\theta-\alpha)}{\cos(\theta-\beta)}=\dfrac{a'}{b'}$, 則

$\cos(\alpha-\beta)=(aa'+bb')/(ab'+a'b)$.

例題解自179.至187.

(179) $\dfrac{\sin(x+\theta)}{\cos x}=\dfrac{\sin(y-\theta)}{\cos y}$ 從此可得其證.

(180) $\tan(\theta+\phi)=\dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\dfrac{\sin a\left(\dfrac{x}{y-x\cos a}+\dfrac{y}{x-y\cos a}\right)}{1-\dfrac{xy\sin^2 a}{(y-x\cos a)(x-y\cos a)}}$.

(181) $\tan^2\theta=\dfrac{(\tan\theta-\tan a)(\tan\theta-\tan\beta)}{(1+\tan\theta\tan a)(1+\tan\theta\tan\beta)}$, 解之.則

$\tan\theta(\tan a+\tan\beta)(1+\tan^2\theta)=\tan a\tan\beta(1-\tan^4\theta)$. 故

$\tan 2\theta=\dfrac{2\tan\theta}{1-\tan^2\theta}=\dfrac{2\tan a\tan\beta}{\tan a+\tan\beta}=\dfrac{2\sin a\sin\beta}{\sin(a+\beta)}$.

(182) $b(\sin\theta-\sin\phi)=a(\cos\phi-\cos\theta)$, 故 $\tan\dfrac{\theta+\phi}{2}=\dfrac{b}{a}$,

$\tan(\theta+\phi)=\dfrac{2\tan\frac{1}{2}(\theta+\phi)}{1-\tan^2\frac{1}{2}(\theta+\phi)}=\dfrac{2b/a}{1-(b/a)^2}=\dfrac{2ab}{a^2-b^2}$.

(183) $\dfrac{\sin\phi}{\cos\phi}=\dfrac{\sin a\tan\theta}{1-\cos a\sqrt{(1+\tan^2\theta)}}$, 從此求得 $\tan\theta$ 之證.

(184) $(\cos\beta+\cos^3\beta)^2+(\sin\beta-\sin^3\beta)^2=2(\cos^2 a+\sin^2 a)=2$, 即

$3\cos^2 2\beta+8\cos 2\beta-3=0$, 故 $\cos 2\beta=\frac{1}{3}$.

又 $\dfrac{\sqrt{2}\cos a}{\cos\beta}-\dfrac{\sqrt{2}\sin a}{\sin\beta}=(1+\cos^2\beta)-(1-\sin^2\beta)=1$, 即

$\sqrt{2}\sin(\beta-a)=\dfrac{1}{2}\sin 2\beta=\pm\sqrt{1-\dfrac{1}{9}}=\pm\dfrac{\sqrt{2}}{3}$, 故 $\sin(\beta-a)=\pm\dfrac{1}{3}$

(185) $\sin\phi=\dfrac{\sin\beta\sin\theta}{\sin a}$ 故 $\tan^2\phi=\dfrac{\sin^2\beta\sin^2\theta}{\sin^2 a-\sin^2\beta\sin^2\theta}$, 故

$\dfrac{\cos^2 a\sin^2\theta}{\sin^2 a(1-\sin^2\theta)}+\dfrac{\cos^2\beta\sin^2\theta}{\sin^2 a-\sin^2\beta\sin^2\theta}=1$. 從此又得其證.

(186) 變 $\tan\frac{z}{2}$ 爲 $\tan z$, 即可得其證.

(187) $\dfrac{\tan\theta\cos a-\sin a}{\tan\theta\cos\beta-\sin\beta}=\dfrac{a}{b}$, $\dfrac{\cos a+\tan\theta\sin a}{\cos\beta+\tan\theta\sin\beta}=\dfrac{a'}{b'}$.

從此兩方程式消去 $\tan\theta$. 即得.

188. $\cos^2\theta=\dfrac{\cos\alpha}{\cos\beta},\ \cos^2\theta'=\dfrac{\cos\alpha'}{\cos\beta},\ \dfrac{\tan\theta}{\tan\theta'}=\dfrac{\tan\alpha}{\tan\alpha'}$，則

$$\tan^2\frac{\alpha}{2}\tan^2\frac{\alpha'}{2}=\tan^2\frac{\beta}{2}.$$

189. $\cos\alpha=\cos\beta\cos\phi=\cos\beta'\cos\phi',\ \sin\alpha=2\sin\dfrac{\phi}{2}\sin\dfrac{\phi'}{2}$，

則 $\tan^2\dfrac{\alpha}{2}=\tan^2\dfrac{\beta}{2}\tan^2\dfrac{\beta'}{2}.$

190. $\tan\phi=\cos\theta\tan\alpha,\ \tan\alpha'=\tan\theta\sin\phi$，則

$$\tan^2\frac{\phi}{2}=\tan\frac{\alpha+\alpha'}{2}\tan\frac{\alpha-\alpha'}{2}.$$

191. $\dfrac{\tan(\theta+\alpha)}{x}=\dfrac{\tan(\theta+\beta)}{y}=\dfrac{\tan(\theta+\gamma)}{z}$，則

$$\frac{x+y}{x-y}\sin^2(\alpha-\beta)+\frac{y+z}{y-z}\sin^2(\beta-\gamma)+\frac{z+x}{z-x}\sin^2(\gamma-\alpha)=0.$$

192. $\cos\theta=\cos\alpha\cos\beta,\ \cos\theta'=\cos\alpha'\cos\beta$，及

$\tan\dfrac{\theta}{2}\tan\dfrac{\theta'}{2}=\tan\dfrac{\beta}{2}$，則 $\sin^2\beta=(\sec\alpha-1)(\sec\alpha'-1).$

例題解自 188. 至 192.

(188) $\tan^2\theta=\dfrac{\cos\beta-\cos\alpha}{\cos\alpha},\ \tan^2\theta'=\dfrac{\cos\beta-\cos\alpha'}{\cos\alpha'}$，故

$\dfrac{\tan^2\theta}{\tan^2\theta'}=\dfrac{\cos\alpha'(\cos\beta-\cos\alpha)}{\cos\alpha(\cos\beta-\cos\alpha')}=\dfrac{\tan^2\alpha}{\tan^2\alpha'}$ 從此得 $\cos\beta=\dfrac{\cos\alpha+\cos\alpha'}{1+\cos\alpha\cos\alpha'}.$

故 $\tan^2\dfrac{\beta}{2}=\dfrac{(1-\cos\alpha)(1-\cos\alpha')}{(1+\cos\alpha)(1+\cos\alpha')}=\tan^2\dfrac{\alpha}{2}\tan^2\dfrac{\alpha'}{2}.$

(189) $2\sin^2\dfrac{\phi}{2}=1-\dfrac{\cos\alpha}{\cos\beta},\ 2\sin^2\dfrac{\phi'}{2}=1-\dfrac{\cos\alpha}{\cos\beta}$，故

$\cos^2\alpha=4\sin^2\dfrac{\phi}{2}\sin^2\dfrac{\phi'}{2}=\left(1-\dfrac{\cos\alpha}{\cos\beta}\right)\left(1-\dfrac{\cos\alpha}{\cos\beta}\right).$ 從此得

$\cos\alpha=\dfrac{\cos\beta+\cos\beta'}{1+\cos\beta\cos\beta'}$ 以下與前例同樣.

(190) $\cos\theta=\frac{\tan\phi}{\tan\alpha}$, $\sin\theta=\frac{\tan\alpha'\cos\theta}{\sin\phi}=\frac{\tan\alpha'}{\tan\alpha\cos\phi}$, 故

$$\frac{\tan^2\phi}{\tan^2\alpha}+\frac{\tan^2\alpha'}{\tan^2\alpha\cos^2\phi}=\cos^2\theta+\sin^2\theta=1,\ 卽\ \tan^2\phi=\frac{\tan^2\alpha-\tan^2\alpha'}{1+\tan^2\alpha'},$$

故 $\tan^2\phi=\frac{\tan^2\alpha-\tan^2\alpha'}{1+\tan^2\alpha'}=\left(\frac{2\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}}\right)^2$. 從此得

$$\tan^4\frac{\phi}{2}-\frac{2\tan^2\frac{\phi}{2}(\tan^2\alpha+\tan^2\alpha'+2)}{\tan^2\alpha-\tan^2\alpha'}=-1,\ 卽$$

$$\tan^2\frac{\phi}{2}=\frac{\tan^2\alpha+\tan^2\alpha'+2\pm2\sec\alpha\sec\alpha'}{\tan^2\alpha-\tan^2\alpha'}=\frac{(\sec\alpha\mp\sec\alpha')^2}{\sec^2\alpha-\sec^2\alpha'}$$

$$=\frac{\sec\alpha\mp\sec\alpha'}{\sec\alpha\pm\sec\alpha'}=\frac{\cos\alpha'\mp\cos\alpha}{\cos\alpha'\pm\cos\alpha}=\tan\frac{\alpha\pm\alpha'}{2}\tan\frac{\alpha\mp\alpha'}{a}.$$

(191) $\frac{\tan(\theta+\alpha)+\tan(\theta+\beta)}{x+y}=\frac{\tan(\theta+\alpha)-\tan(\theta+\beta)}{x-y}$, 卽

$$\frac{\sin(2\theta+\alpha+\beta)}{x+y}=\frac{\sin(\alpha-\beta)}{x-y},\ 故$$

$$\frac{x+y}{x-y}\sin^2(\alpha-\beta)=\sin(2\theta+\alpha+\beta)\sin(\alpha-\beta)$$

$$=\tfrac{1}{2}\{\cos(2\theta+2\beta)-\cos(2\theta+2\alpha)\},$$

$$\frac{y+z}{y-z}\sin^2(\beta-\gamma)=\tfrac{1}{2}\{\cos(2\theta+2\gamma)-\cos(2\theta+2\beta)\},$$

$$\frac{z+x}{z-x}\sin^2(\gamma-\alpha)=\tfrac{1}{2}\{\cos(2\theta+2\alpha)-\cos(2\theta+2\gamma)\},$$

從此三方程式相加.卽可得其證.

(192) $2\cos^2\frac{\theta}{2}-1=\cos\alpha\cos\beta$, 卽 $\frac{2}{1+\tan^2\frac{\theta}{2}}=1+\cos\alpha\cos\beta$, 卽

$$\tan^2\frac{\theta}{2}=\frac{1-\cos\alpha\cos\beta}{1+\cos\alpha\cos\beta},\ 及\ \tan^2\frac{\theta'}{2}=\frac{1-\cos\alpha'\cos\beta}{1+\cos\alpha'\cos\beta},\ 故$$

$$\tan^2\frac{\theta}{2}\tan^2\frac{\theta'}{2}=\frac{(1-\cos\alpha\cos\beta)(1-\cos\alpha'\cos\beta)}{(1+\cos\alpha\cos\beta)(1+\cos\alpha'\cos\beta)}=\tan^2\frac{\beta}{2},\ 故$$

$$\frac{1-(\cos\alpha+\cos\alpha')\cos\beta+\cos\alpha\cos\alpha'\cos^2\beta}{1+(\cos\alpha+\cos\alpha')\cos\beta+\cos\alpha\cos\alpha'\cos^2\beta}=\frac{1-\cos\beta}{1+\cos\beta},$$

從此可得其證.

193. $x\cos(\alpha+\beta)+\cos(\alpha-\beta)=x\cos(\beta+\gamma)+\cos(\beta-\gamma)$

$=x\cos(\gamma+\alpha)+\cos(\gamma-\alpha)$,

則 $\dfrac{\tan\alpha}{\tan\frac{1}{2}(\beta+\gamma)}=\dfrac{\tan\beta}{\tan\frac{1}{2}(\gamma+\alpha)}=\dfrac{\tan\gamma}{\tan\frac{1}{2}(\alpha+\beta)}$.

194. $a\sin\alpha+b\sin\beta+c\sin\gamma=0$,

$a\cos\alpha+b\cos\beta+c\cos\gamma=0$,

則 $\dfrac{\sin(\beta-\gamma)}{a}=\dfrac{\sin(\gamma-\alpha)}{b}=\dfrac{\sin(\alpha-\beta)}{c}$.

195. $x\cos\beta+y\cos\alpha=z$, $x\sin\beta-y\sin\alpha=0$

則 $\dfrac{x}{\sin\alpha}=\dfrac{y}{\sin\beta}=\dfrac{z}{\sin(\alpha+\beta)}$.

196. $\cos\alpha+\cos\beta+\cos\gamma=0$.

則 $\cos3\alpha+\cos3\beta+\cos3\gamma=12\cos\alpha\cos\beta\cos\gamma$.

197. $\cos(\alpha-\beta)\cos\alpha=a\cos\beta\sin\beta$,

$\sin(\alpha-\beta)\cos\alpha=\dfrac{a}{2}\cos^2\beta$, 則 $a=\dfrac{2\sin\beta}{1+3\sin^2\beta}$.

198. $x^2\cos\alpha\cos\beta+x(\sin\alpha+\sin\beta)+1=0$,

$x^2\cos\beta\cos\gamma+x(\sin\beta+\sin\gamma)+1=0$,

則 $x^2\cos\gamma\cos\alpha+x(\sin\gamma+\sin\alpha)+1=0$.

例題解自 193. 至 198.

(193) 從既知方程式消去x,則

$$\frac{\cos(\alpha-\beta)-\cos(\beta-\gamma)}{\cos(\alpha+\beta)-\cos(\beta+\gamma)}=\frac{\cos(\beta-\gamma)-\cos(\gamma-\alpha)}{\cos(\beta+\gamma)-\cos(\gamma+\alpha)},\text{ 即}$$

$$\frac{\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}}{\{\cos(\alpha-\beta)+\cos(\alpha+\beta)\}}-\frac{\{\cos(\beta-\gamma)-\cos(\beta+\gamma)\}}{\{\cos(\beta-\gamma)+\cos(\beta+\gamma)\}}$$

$$=\frac{\{\cos(\beta-\gamma)-\cos(\beta+\gamma)\}-\{\cos(\gamma-\alpha)-\cos(\gamma+\alpha)\}}{\{\cos(\beta-\gamma)+\cos(\beta+\gamma)\}-\{\cos(\gamma-\alpha)+\cos(\gamma+\alpha)\}},$$

即 $\dfrac{\sin\beta(\sin\alpha-\sin\gamma)}{\cos\beta(\cos\alpha-\cos\gamma)}=\dfrac{\sin\gamma(\sin\beta-\sin\alpha)}{\cos\gamma(\cos\beta-\cos\alpha)}$, 即 $\dfrac{\tan\beta}{\tan\frac{1}{2}(\gamma+\alpha)}=\dfrac{\tan\gamma}{\tan\frac{1}{2}(\alpha+\beta)}$.

(194) 從兩方程式,得 $\dfrac{a}{\sin\beta\cos\gamma-\cos\beta\sin\gamma}=\dfrac{b}{\sin\gamma\cos\alpha-\cos\gamma\sin\alpha}$

$=\dfrac{c}{\sin\alpha\cos\beta-\cos\alpha\sin\beta}$, 即 $\dfrac{a}{\sin(\beta-\gamma)}=\dfrac{b}{\sin(\gamma-\alpha)}=\dfrac{c}{\sin(\alpha-\beta)}$.

(195) 令 $\dfrac{x}{\sin\alpha}=\dfrac{y}{\sin\beta}=K$, 則 $K\sin\alpha\cos\beta+K\sin\beta\cos\alpha=z$,

$$\therefore\ K=\frac{z}{\sin(\alpha+\beta)}.$$

(196) 從原方程式,得 $\cos^3\alpha+\cos^3\beta+\cos^3\gamma=3\cos\alpha\cos\beta\cos\gamma$,

即 $4\cos^3\alpha+4\cos^3\beta+4\cos^3\gamma=12\cos\alpha\cos\beta\cos\gamma$,

又由原方程式,得 $3\cos\alpha+3\cos\beta+3\cos\gamma=0$, 由是

$$(4\cos^3\alpha-3\cos\alpha)+(4\cos^3\beta-3\cos\beta)+(4\cos^3\gamma-3\cos\gamma)=12\cos\alpha\cos\beta\cos\gamma.$$

(197) 以第一除第二,得 $\tan(\alpha-\beta)=\dfrac{1}{2\tan\beta}$. 從此 $\tan\alpha=\dfrac{1+\sin^2\beta}{\sin\beta\cos\beta}$,

又 $a^2\cos^2\beta\sin^2\beta+\dfrac{a^2}{4}\cos^4\beta=\cos^2\alpha\{\cos^2(\alpha-\beta)+\sin^2(\alpha-\beta)\}=\cos^2\alpha$,

故 $\tan^2\alpha+1=\dfrac{4}{a^2\cos^2\beta(1+3\sin^2\beta)}$, 即

$\dfrac{(1+\sin^2\beta)^2}{\sin^2\beta\cos^2\beta}+1=\dfrac{4}{a\cos^2\beta(1+3\sin^2\beta)}$. 故 $a^2=\dfrac{4\sin^2\beta}{(1+3\sin^2\beta)^2}$.

(198) 從既知兩方程式之第二式減去第一式,而以 x 除之,則

$$x\cos\beta(\cos\gamma-\cos\alpha)=-(\sin\gamma-\sin\alpha),$$

又第一式以 $\cos\gamma$ 乘之,第二式以 $\cos\alpha$ 乘之,相減.則

$$x\sin\beta(\cos\gamma-\cos\alpha)=x\sin(\gamma-\alpha)-(\cos\gamma-\cos\alpha),$$

由是 $(\sin\gamma-\sin\alpha)^2+\{x\sin(\gamma-\alpha)-(\cos\gamma-\cos\alpha)\}^2$

$$=x^2(\cos^2\beta+\sin^2\beta)(\cos\gamma-\cos\alpha)^2,$$

即 $x^2\{(\cos\gamma-\cos\alpha)^2-\sin^2(\gamma-\alpha)\}+2x\sin(\gamma-\alpha)(\cos\gamma-\cos\alpha)$

$$+2\cos(\gamma-\alpha)-2=0,$$

即 $4x^2\sin^2\frac{1}{2}(\gamma-\alpha)\left\{\sin^2\dfrac{\gamma+\alpha}{2}-\cos^2\dfrac{\gamma-\alpha}{2}\right\}$

$$-8x\sin^2\tfrac{1}{2}(\gamma-\alpha)\sin\frac{\gamma+\alpha}{2}\cos\frac{\gamma-\alpha}{2}-4\sin^2\tfrac{1}{2}(\gamma-\alpha)=0.$$

199. $\cos\theta+\cos\phi+\cos\psi+\cos\theta\cos\phi\cos\psi=0$, 則

$\operatorname{cosec}^2\theta+\operatorname{cosec}^2\phi+\operatorname{cosec}^2\psi\pm2\operatorname{cosec}\theta\operatorname{cosec}\phi\operatorname{cosec}\psi=1.$

200. $(1+\sin\theta)(1+\sin\phi)(1+\sin\psi)=\cos\theta\cos\phi\cos\psi$, 則

$\sec^2\theta+\sec^2\phi+\sec^2\psi=1+2\sec\theta\sec\phi\sec\psi.$

201. $x+y\cos\alpha+z\sin\alpha=\cos(\beta-\gamma)$,

$x+y\cos\beta+z\sin\beta=\cos(\gamma-\alpha)$,

$x+y\cos\gamma+z\sin\gamma=\cos(\alpha-\beta)$,

則 $x=4\cos\frac{1}{2}(\alpha-\beta)\cos\frac{1}{2}(\beta-\gamma)\cos\frac{1}{2}(\gamma-\alpha)$.

202. $\tan\frac{\alpha}{2}=\tan^3\frac{\beta}{2}$, $\tan\beta=2\tan\phi$, 則

$\alpha+\beta=2\phi$.

203. $\sin x\sin y=\sin(\alpha+\beta)\sin\gamma$,

$\cos x\cos y=\cos(\alpha+\beta)\cos\gamma$,

$\cos^2x+\cos^2y=1+\cos^2(\alpha+\beta+\gamma)$,

則 $\sin^2(\alpha+\beta)+\sin^2\gamma=\sin^2(\alpha+\beta+\gamma)$.

204. $\tan(\pi\cot\theta)=\cot(\pi\tan\theta)$, 則

$\tan\theta=\frac{2n+1}{4}\pm\frac{\sqrt{(4n^2+4n-15)}}{4}$, 但 n 爲任意整數.

例題解自 199. 至 204.

(199) 令 $\cos\theta=x$, $\cos\phi=y$, $\cos\psi=z$, $\sin\theta=m$, $\sin\phi=n$,

$\sin\psi=p$, 則 $x+y+z=-xyz$, $x^2+y^2+z^2+2(xy+yz+zx)=x^2y^2z^2$,

即 $(1-m^2)+(1-n^2)+(1-p^2)+2(xy+yz+zx)=(1-m^2)(1-n^2)(1-p^2)$,

即 $2(xy+yz+zx)=m^2n^2+n^2p^2+p^2m^2-m^2n^2p^2-2.$

故 $4\{x^2y^2+y^2z^2+z^2x^2+2xyz(x+y+z)\}=(m^2n^2+n^2p^2+p^2m^2-m^2n^2p^2-2)^2$,

即 $4\{(1-m^2)(1-n^2)+(1-n^2)(1-p^2)+(1-p^2)(1-m^2)-2x^2y^2z^2\}$

$=(m^2n^2+n^2p^2+p^2m^2-m^2n^2p^2-2)^2$

即 $4\{3-2(m^2+n^2+p^2)+m^2n^2+n^2p^2+p^2m^2-2(1-m^2)(1-n^2)(1-p^2)\}$

$=(m^2n^2+n^2p^2+p^2m^2-m^2n^2p^2-2\}^2$,

故 $m^2n^2+n^2p^2+p^2m^2-m^2n^2p^2=\pm 2mnp$. 由是

$\frac{1}{p^2}+\frac{1}{m^2}+\frac{1}{n^2}\pm\frac{1}{mnp}=1$, 即得其證.

(200) 用前例之記法.得 $(1+m)(1+n)(1+p)=xyz$. 而 m, n, p, 以 x, y, z 之項示之,則可得證明如前例.

(201) 第一以λ乘之.第二以μ乘之.與第三相加.則

$$x(\lambda+\mu+1)+y(\lambda\cos\alpha+\mu\cos\beta+\cos\gamma)+z(\lambda\sin\alpha+\mu\sin\beta+\sin\gamma)$$
$$=\lambda\cos(\beta-\gamma)+\mu\cos(\gamma-\alpha)+\cos(\alpha-\beta),$$

令 $\lambda\cos\alpha+\mu\cos\beta+\cos\gamma=0$. $\lambda\sin\alpha+\mu\sin\beta+\sin\gamma=0$, 則

$\frac{\lambda}{\sin(\beta-\gamma)}=\frac{\mu}{\sin(\gamma-\alpha)}=\frac{1}{\sin(\alpha-\beta)}$ 故代入下式.

$x(\lambda+\mu+1)-\{\lambda\cos(\beta-\gamma)+\mu\cos(\gamma-\alpha)+\cos(\alpha-\beta)\}=0$, 則

$$x\{\sin(\beta-\gamma)+\sin(\gamma-\alpha)+\sin(\alpha-\beta)\}$$
$$=\sin(\beta-\gamma)\cos(\beta-\gamma)+\sin(\gamma-\alpha)\cos(\gamma-\alpha)+\sin(\alpha-\beta)\cos(\alpha-\beta)$$
$$=\tfrac{1}{2}\{\sin 2(\beta-\gamma)+\sin 2(\gamma-\alpha)+\sin 2(\alpha-\beta)\}.$$

由例題88.得

$$-4x\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\beta}{2}=-2\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta).$$

(202) $2\tan\phi=\dfrac{2\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}$. 即 $\tan\phi=\dfrac{\tan\frac{\beta}{2}\left(1+\tan^2\frac{\beta}{2}\right)}{\left(2-\tan^2\frac{\beta}{2}\right)\left(1+\tan^2\frac{\beta}{2}\right)}$

$$=\frac{\tan\frac{\beta}{2}+\tan^3\frac{\beta}{2}}{1-\tan^4\frac{\beta}{2}}=\frac{\tan\frac{\beta}{2}+\tan\frac{\alpha}{2}}{1-\tan\frac{\alpha}{2}\tan\frac{\beta}{2}}=\tan\tfrac{1}{2}(\alpha+\beta),\ 故\ \tfrac{1}{2}(\alpha+\beta)=\phi.$$

(203) 從第一得 $(1-\cos^2x)(1-\cos^2y)=\sin^2(\alpha+\beta)\sin^2\gamma$, 即

$1-(\cos^2x+\cos^2y)+\cos^2x\cos^2y=\sin^2(\alpha+\beta)\sin^2\gamma$, 故由第二,第三得

$1-\{1+\cos^2(\alpha+\beta+\gamma)\}+\cos^2(\alpha+\beta)\cos^2\gamma=\sin^2(\alpha+\beta)\sin^2\gamma$, 即

$\sin^2(\alpha+\beta+\gamma)-1+\{1-\sin^2(\alpha+\beta)\}(1-\sin^2\gamma)=\sin^2(\alpha+\beta)\sin^2\gamma$.

(204) $\tan(\pi\cot\theta)=\tan\{n\pi+\frac{\pi}{2}-\pi\tan\theta\}$ (第三編15.節)

故 $\cot\theta=n+\frac{1}{2}-\tan\theta$, 即 $\tan^2\theta+\tan\theta\left(\frac{2n+1}{2}\right)+1=0$.

205. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1,\ \cos^2\theta+\cos^2\phi+\cos^2\psi=1,$ 及

$\cos\alpha\cos\theta+\cos\beta\cos\phi+\cos\gamma\cos\psi=0,$ 則

$$\frac{\sin\theta\sin2\theta}{\cos\alpha}+\frac{\sin\phi\sin2\phi}{\cos\beta}+\frac{\sin\psi\sin2\psi}{\cos\gamma}+\frac{2\cos\theta\cos\phi\cos\psi}{\cos\alpha\cos\beta\cos\gamma}=0.$$

206. $a\cot\theta+b\cot\phi+c\cot\psi$

$$=(a+b+c)\cot\theta\cot\phi\cot\psi,$$

$(b+c)\cot\phi\cot\psi+(c+a)\cot\psi\cot\theta+(a+b)\cot\theta\cot\phi=0,$ 則

$a\sin2\theta+b\sin2\phi+c\sin2\psi=0.$

207. $\cos\theta=\cos\alpha\cos\beta+\sin\alpha\sin\beta\sqrt{(1-c^2\sin^2\theta)},$

$\cos\phi=\cos\alpha\cos\beta+\sin\alpha\sin\beta\sqrt{(1-c^2\sin^2\phi)},$

則 $\cos\theta+\cos\phi=\dfrac{2\cos\alpha\cos\beta}{1-c^2\sin^2\alpha-\sin^2\beta},$

及 $1+\cos\theta\cos\phi=\dfrac{\cos^2\alpha+\cos^2\beta}{1-c^2\sin^2\alpha\sin^2\beta}.$

208. $\tan^2\alpha\tan\alpha'=\tan^2\beta\tan\beta'=\tan^2\gamma\tan\gamma'$

$=\tan\alpha\tan\beta\tan\gamma.$

$\operatorname{cosec}2\alpha+\operatorname{cosec}2\beta+\operatorname{cosec}2\gamma=0,$ 則

$\tan(\alpha-\alpha')=\tan(\beta-\beta')=\tan(\gamma-\gamma').$

例題解自 205. 至 208.

(205) 去第四方程式前節之分母，則

$$2\cos\beta\cos\gamma\cos\theta(1-\cos^2\theta)+2\cos\gamma\cos\alpha\cos\phi(1-\cos^2\phi)+2\cos\alpha\cos\beta\cos\psi(1-\cos^2\psi)$$

$$+2\cos\theta\cos\phi\cos\psi=2\{\cos\beta\cos\gamma\cos\theta(\cos^2\phi+\cos^2\psi)+\cos\gamma\cos\alpha\cos\phi(\cos^2\psi+\cos^2\theta)$$

$$+\cos\alpha\cos\beta\cos\psi(\cos^2\theta+\cos^2\phi)+\cos\theta\cos\phi\cos\psi\}$$

$$=2\{\cos\gamma\cos\theta\cos\phi(\cos\beta\cos\phi+\cos\alpha\cos\theta)+\cos\alpha\cos\phi\cos\psi(\cos\gamma\cos\psi+\cos\beta\cos\phi)$$

$$+\cos\beta\cos\psi\cos\theta(\cos\alpha\cos\theta+\cos\gamma\cos\psi)+\cos\theta\cos\phi\cos\psi\}$$

$=2\{-\cos^2\gamma\cos\theta\cos\psi-\cos^2\alpha\cos\theta\cos\phi\cos\psi-\cos^2\beta\cos\theta\cos\phi\cos\psi+\cos\theta\cos\phi\cos\psi\}$

$=2\cos\theta\cos\phi\cos\psi(1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma)=0.$

(206) 從第一得 $a\cos\theta\cos(\phi+\psi)+b\cos\phi\cos(\psi+\theta)+c\cos\psi\cos(\theta+\phi)=0,$

又從第二得 $a\cos\theta\sin(\phi+\psi)+b\cos\phi\sin(\psi+\theta)+c\cos\psi\sin(\theta+\phi)=0,$

故令 $\dfrac{a\cos\theta}{\sin(\phi-\psi)}=\dfrac{b\cos\phi}{\sin(\psi-\theta)}=\dfrac{c\cos\psi}{\sin(\theta-\phi)}=\lambda$, 則

$$2a\sin\theta\cos\theta+2b\sin\phi+2c\sin\psi\cos\psi$$

$$=\lambda\{2\sin\theta\sin(\phi-\psi)+2\sin\phi\sin(\psi-\theta)+2\sin\psi\sin(\theta-\phi)\}$$

$$=\lambda\{\cos(\theta+\psi-\phi)-\cos(\theta+\phi-\psi)+\cos(\phi+\theta-\psi)-\cos(\phi+\psi-\theta)$$

$$+\cos(\psi+\phi-\theta)-\cos(\psi+\theta-\phi)\}=0.$$

(207) 令 $\cos\theta$ 及 $\cos\phi$ 為 x. 由兩方程式.得

$x=\cos\alpha\cos\beta+\sin\alpha\sin\beta\sqrt{\{1-c^2(1-x^2)\}}$. 故

$x^2-2x\cos\alpha\cos\beta+\cos^2\alpha\cos^2\beta=\sin^2\alpha\sin^2\beta(1-c^2+c^2x^2)$, 即

$x^2(1-c^2\sin^2\alpha\sin^2\beta)-2x\cos\alpha\cos\beta+\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta(1-c^2)=0,$

故 $\cos\theta+\cos\phi=\dfrac{2\cos\alpha\cos\beta}{1-c^2\sin^2\alpha\sin^2\beta}$,

及 $\cos\theta\cos\phi=\dfrac{\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta(1-c^2)}{1-c^2\sin^2\alpha\sin^2\beta}$

即 $1+\cos\theta\cos\phi=\dfrac{1+\cos^2\alpha\cos^2\beta-\sin^2\alpha\sin^2\beta}{1-c^2\sin^2\alpha\sin^2\beta}=\dfrac{\cos^2\alpha+\cos^2\beta}{1-c^2\sin^2\alpha\sin^2\beta}$.

(208) 令 $\tan\alpha=x,\ \tan\beta=y,\ \tan\gamma=z,\ \tan\alpha'=x',$

$\tan\beta'=y',\ \tan\gamma'=z'$, 則 $x^2x'=y^2y'=z^2z'=xyz.$ (1)

$\dfrac{1+x^2}{2x}+\dfrac{1+y^2}{2y}+\dfrac{1+z^2}{2z}=0$, 即 $xyz(x+y+z)+xy+yz+zx=0.$ (2)

$\tan(\alpha-\alpha')=\dfrac{x-x'}{1+xx'}=\dfrac{x-yz/x}{1+yz}$ 〔見 (1)〕$=\dfrac{x^2-yz}{x(1+yz)}$, 但

$x^2-yz=x^2+xyz(x+y+z)+xy+zx$ 〔見 (2)〕 $=x(1+yz)(x+y+z),$

故 $\tan(\alpha-\alpha')=\dfrac{x^2-yz}{x(1+yz)}=x+y+z=\tan(\beta-\beta')=\tan(\gamma-\gamma').$

209. $\alpha+\beta=\frac{\pi}{2}$, 則

$$\frac{\left(1-\tan\frac{\alpha}{2}\right)\left(1-\tan\frac{\beta}{2}\right)}{\left(1+\tan\frac{\alpha}{2}\right)\left(1+\tan\frac{\beta}{2}\right)}=\frac{\sin\alpha+\sin\beta-1}{\sin\alpha+\sin\beta+1}.$$

210. $\beta\neq\gamma$, 則

$$\frac{\cos(\alpha+\beta+\theta)}{\sin(\alpha+\beta)\cos^2\gamma}=\frac{\cos(\gamma+\alpha+\theta)}{\sin(\gamma+\alpha)\cos^2\beta}=\frac{\cos(\beta+\gamma+\theta)}{\sin(\beta+\gamma)\cos^2\alpha}.$$

211. $\frac{\cos(\alpha+\beta+\theta)}{\sin(\alpha+\beta)\cos^2\gamma}=\frac{\cos(\gamma+\alpha+\theta)}{\sin(\gamma+\alpha)\cos^2\beta}$, 則

$$\frac{\sin(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)}{\cos(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta)+\sin^2(\alpha+\beta+\gamma)}=\cot\theta.$$

212. $\frac{x\cos3\theta+y\sin3\theta}{\cos^3\theta}=\frac{y\cos3\theta-x\sin3\theta}{\sin^3\theta}=x^2+y^2$,

則 $x^2+y^2+x=2$.

例題解自 209. 至 212.

(209) $$\frac{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\left(\cos\frac{\beta}{2}-\sin\frac{\beta}{2}\right)}{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)\left(\cos\frac{\beta}{2}+\sin\frac{\beta}{2}\right)}=\frac{\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right)\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)}{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2\left(\cos\frac{\beta}{2}+\sin\frac{\beta}{2}\right)^2}$$

$$=\frac{\cos\alpha\cos\beta}{(1+\sin\alpha)(1+\sin\beta)}=\frac{(1-\sin\alpha)(1-\sin\beta)}{\cos\alpha\cos\beta}$$

$$=\frac{\cos\alpha\cos\beta-(1-\sin\alpha)(1-\sin\beta)}{(1+\sin\alpha)(1+\sin\beta)-\cos\alpha\cos\beta}=\frac{\cos(\alpha+\beta)+\sin\alpha+\sin\beta-1}{\sin\alpha+\sin\beta+1-\cos(\alpha+\beta)}.$$

(210) $\frac{\cos(\alpha+\beta+\theta)}{\sin(\alpha+\beta)\cos^2\gamma}=\frac{\cos(\gamma+\alpha+\theta)}{\sin(\gamma+\alpha)\cos^2\beta}=\lambda$, 則

$$\frac{\cos(\alpha+\beta+\theta)}{\cos\gamma}=\lambda\sin(\alpha+\beta)\cos\gamma=\tfrac{1}{2}\lambda\{\sin(\alpha+\beta+\gamma)+\sin(\alpha+\beta+\gamma)\},$$

$$\frac{\cos(\gamma+\alpha+\theta)}{\cos\beta}=\frac{1}{2}\lambda\{\sin(\alpha+\beta+\gamma)+\sin(\gamma+\alpha-\beta)\},$$ 故

$$\frac{\cos(\alpha+\beta+\theta)\cos\beta-\cos(\gamma+\alpha+\theta)\cos\gamma}{\cos\gamma\cos\beta}=\frac{1}{2}\lambda\{\sin(\alpha+\beta-\gamma)-\sin(\gamma+\alpha-\beta)\},$$

即 $$\frac{\cos(\alpha+2\beta+\theta)-\cos(\alpha+2\gamma+\theta)}{2\cos\gamma\cos\beta}=-\lambda\cos\alpha\sin(\beta-\gamma),$$ 故

$\lambda=\sin(\alpha+\beta+\gamma+\theta)/2\cos\alpha\cos\beta\cos\gamma$. 輪換此式之 α,β,γ. 其值亦不變,故題云云.

(211) $$\frac{\cos(\alpha+\beta)\cos\theta-\sin(\alpha+\beta)\sin\theta}{\sin(\alpha+\beta)\cos^2\gamma}=\frac{\cos(\gamma+\alpha)\cos\theta-\sin(\gamma+\alpha)\sin\theta}{\sin(\gamma+\alpha)\cos^2\beta},$$

故 $$\cot\theta\left\{\frac{\cos(\alpha+\beta)\sin(\gamma+\alpha)}{\cos^2\gamma}-\frac{\sin(\alpha+\beta)\cos(\gamma+\alpha)}{\cos^2\beta}\right\}$$

$$=\cot\theta\left\{\frac{\sin(\beta+\gamma+2\alpha)-\sin(\beta-\gamma)}{2\cos^2\gamma}-\frac{\sin(\beta+\gamma+2\alpha)+\sin(\beta-\gamma)}{2\cos^2\beta}\right\}$$

$=\sin(\alpha+\beta)\sin(\gamma+\alpha)\left\{\frac{1}{\cos^2\gamma}-\frac{1}{\cos^2\beta}\right\}$ 故 $\frac{1}{2}\cot\theta\{\sin(\beta+\gamma+2\alpha)(\cos^2\beta-\cos^2\gamma)$

$-\sin(\beta-\gamma)(\cos^2\beta+\cos^2\gamma)\}=\sin(\alpha+\beta)\sin(\gamma+\alpha)(\cos^2\beta-\cos^2\gamma)$,

即 $\frac{1}{2}\cot\theta\{\sin(\beta+\gamma+2\alpha)\sin(\beta+\gamma)+\cos^2\beta+\cos^2\gamma\}$

$=\sin(\alpha+\beta)\sin(\gamma+\alpha)\sin(\beta+\gamma)$ 但 $\sin(\beta+\gamma+2\alpha)\sin(\beta+\gamma)+\cos^2\beta+\cos^2\gamma$

$=\sin^2(\alpha+\beta+\gamma)-\sin^2\alpha+\cos^2\beta+\cos^2\gamma=2\sin^2(\alpha+\beta+\gamma)$

$+\frac{1}{2}\{1-2\sin^2(\alpha+\beta+\gamma)+1-2\sin^2\alpha+2\cos^2\beta-1+2\cos^2\gamma-1\}$

$=2\sin^2(\alpha+\beta+\gamma)+\frac{1}{2}\{\cos2(\alpha+\beta+\gamma)+\cos2\alpha+\cos2\beta+\cos2\gamma\}$

$=2\sin^2(\alpha+\beta+\gamma)+\cos(2\alpha+\beta+\gamma)\cos(\beta+\gamma)+\cos(\beta+\gamma)\cos(\beta-\gamma)$

$=2\sin^2(\alpha+\beta+\gamma)+\cos(\beta+\gamma)\{\cos(2\alpha+\beta+\gamma)+\cos(\beta-\gamma)\}$

$=2\sin^2(\alpha+\beta+\gamma)+2\cos(\beta+\gamma)\cos(\alpha+\beta)\cos(\gamma+\alpha)$.

(212) $$(x^2+y^2)^2=\frac{(x\cos3\theta+y\sin3\theta)^2+(y\cos3\theta-x\sin3\theta)^2}{\cos^6\theta+\sin^6\theta}$$

$$=\frac{x^2+y^2}{1-3\sin^2\theta\cos^2\theta},$$ 故 $$1-3\sin^2\theta\cos^2\theta=\frac{1}{x^2+y^2},$$

$$x^2+y^2=\frac{(x\cos3\theta+y\sin3\theta)\cos3\theta-(y\cos3\theta-x\sin3\theta)\sin2\theta}{\cos^3\theta\cos3\theta-\sin^3\theta\sin3\theta}$$

$$=\frac{x}{1-6\sin^2\theta\cos^2\theta},$$ 故 $$1-6\sin^2\theta\cos^2\theta=\frac{x}{x^2+y^2}.$$

213. $\tan\alpha=\frac{n}{m}$ 則 $\sqrt{\frac{m-n}{m+n}}+\sqrt{\frac{m+n}{m-n}}=\frac{2\cos\alpha}{\sqrt{\cos 2\alpha}}$.

214. $p=2\cos\alpha-5\cos^3\alpha+4\cos^5\alpha$,

$q=2\sin\alpha-5\sin^3\alpha+4\sin^5\alpha$, 則

$p\cos 3\alpha+q\sin 3\alpha=\cos 2\alpha$,

$p\sin 3\alpha-q\cos 3\alpha=\frac{1}{2}\sin 2\alpha$.

215. $\sin(\alpha+\beta)\cos\gamma=\sin(\alpha+\gamma)\cos\beta$, 則

$\beta-\gamma$ 爲 π 之倍數,或 α 爲 $\frac{1}{2}\pi$ 之奇倍數.

216. $\frac{\sin(\alpha+\theta)}{\sin(\alpha+\phi)}=\frac{\sin(\beta+\theta)}{\sin(\beta+\phi)}$ 則 α, β 之差.或 θ, ϕ 之差

爲 π 之倍數.

217. $\frac{\sin(\alpha+\theta)}{\sin(\alpha+\phi)}+\frac{\sin(\beta+\theta)}{\sin(\beta+\phi)}=\frac{\cos(\alpha+\theta)}{\cos(\alpha+\phi)}+\frac{\cos(\beta+\theta)}{\cos(\beta+\phi)}$, 則

$\alpha\sim\beta=(2n+1)\frac{\pi}{2}$, 或 $\theta\sim\phi=n\pi$.

例題解自 213. 至 217.

(213) $\frac{n}{m}=\frac{\sin\alpha}{\cos\alpha}$ 即 $\frac{m-n}{m+n}=\frac{\cos\alpha-\sin\alpha}{\cos\alpha+\sin\alpha}=\sqrt{\frac{1-2\sin\alpha\cos\alpha}{1+2\sin\alpha\cos\alpha}}$

$=\sqrt{\frac{1-\sin 2\alpha}{1+\sin 2\alpha}}=\sqrt{\frac{(1-\sin 2\alpha)^2}{1-\sin^2 2\alpha}}=\frac{1-\sin 2\alpha}{\cos 2\alpha}=\frac{(\cos\alpha-\sin\alpha)^2}{\cos 2\alpha}$

故 $\sqrt{\frac{m-n}{m+n}}=\frac{\cos\alpha-\sin\alpha}{\sqrt{\cos 2\alpha}}$, 同樣 $\sqrt{\frac{m+n}{m-n}}=\frac{\cos\alpha+\sin\alpha}{\sqrt{\cos 2\alpha}}$.

(214) $p=2\cos\alpha+\cos^3\alpha(-5+4\cos^3\alpha)$

$=2\cos\alpha+\frac{1}{4}(\cos 3\alpha+3\cos\alpha)(-3+2\cos 2\alpha)$

$=2\cos\alpha-\frac{3}{4}\cos 3\alpha-\frac{9}{4}\cos\alpha+\frac{1}{2}\cos 3\alpha\cos 2\alpha+\frac{3}{2}\cos 2\alpha\cos\alpha$

$=-\frac{3}{4}\cos 3\alpha-\frac{1}{4}\cos\alpha+\frac{1}{4}(\cos 5\alpha+\cos\alpha)+\frac{3}{4}(\cos 3\alpha+\cos\alpha)$

$=\frac{1}{4}(\cos 5\alpha+3\cos\alpha)$,

同樣.得 $q=\frac{1}{4}(\sin 5a+3\sin a)$，故

$p\cos 3a+q\sin 3a=\frac{1}{4}(\cos 5a+3\cos a)\cos 3a+\frac{1}{4}(\sin 5a+3\sin a)\sin 3a$

$=\frac{1}{4}(\cos 5a\cos 3a+\sin 5a\sin 3a)+\frac{3}{4}(\cos 3a\cos a+\sin 3a\sin a)$

$=\frac{1}{4}\cos 2a+\frac{3}{4}\cos 2a=\cos 2a,$

又 $p\sin 3a-q\cos 3a=\frac{1}{4}(\cos 5a+3\cos a)\sin 3a-\frac{1}{4}(\sin 5a+3\sin a)\cos 3a$

$=\frac{3}{4}(\sin 3a\cos a-\cos 3a\sin a)-\frac{1}{4}(\sin 5a\cos 3a-\cos 5a\sin 3a)$

$=\frac{3}{4}\sin 2a-\frac{1}{4}\sin 2a=\frac{1}{2}\sin 2a.$

(215) $\frac{1}{2}\{\sin(\alpha+\beta+\gamma)+\sin(\alpha+\beta-\gamma)\}=\frac{1}{2}\{\sin(\alpha+\gamma+\beta)+\sin(\alpha+\gamma-\beta)\},$

故 $\sin(\alpha+\beta-\gamma)\sim\sin(\alpha+\gamma-\beta)=0,$ 卽 $\cos\alpha\sin(\beta\sim\gamma)=0,$

故 $\cos\alpha=0=\cos(2n+1)\frac{\pi}{2}$ 卽 $\alpha=(2n+1)\frac{\pi}{2},$

或 $\sin(\beta\sim\gamma)=0=\sin n\pi,$ 卽 $\beta\sim\gamma=n\pi.$

(216) $2\sin(\alpha+\theta)\sin(\beta+\phi)=2\sin(\beta+\theta)\sin(\alpha+\phi),$ 卽

$\cos(\alpha+\theta-\beta-\phi)-\cos(\alpha+\theta+\beta+\phi)=\cos(\beta+\theta-\alpha-\phi)-\cos(\beta+\theta+\alpha+\phi),$

故 $\cos(\alpha+\theta-\beta-\phi)\sim\cos(\beta+\theta-\alpha-\phi)=0,$ 卽

$\sin(\theta\sim\phi)\sin(\beta\sim\alpha)=0,$ 故 $\sin(\theta\sim\phi)=0=\sin n\pi,$

卽 $\theta\sim\phi=n\pi,$ 或 $\sin(\beta\sim\alpha)=0=\sin n\pi,$ 卽 $\beta\sim\alpha=n\pi,$

(217) $\frac{\sin\{(\alpha+\theta)\sim(\alpha+\phi)\}}{\sin(\alpha+\phi)\cos(\alpha+\phi)}+\frac{\sin\{(\beta+\theta)\sim(\beta+\phi)\}}{\sin(\beta+\phi)\cos(\beta+\phi)}=0,$ 卽

$\sin(\theta\sim\phi)\{\sin(\beta+\phi)\cos(\beta+\phi)+\sin(\alpha+\phi)\cos(\alpha+\phi)\}=0,$ 卽

$\frac{1}{2}\sin(\theta\sim\phi)\{\sin 2(\beta+\phi)+\sin 2(\alpha+\phi)\}=0,$ 卽

$\sin(\theta\sim\phi)\sin(\alpha+\beta+2\phi)\cos(\alpha\sim\beta)=0,$ 故

$\sin(\theta\sim\phi)=0=\sin n\pi,$ 卽 $\theta\sim\phi=n\pi.$

或 $\cos(\alpha\sim\beta)=0=\cos(2n+1)\frac{\pi}{2},$ 卽 $\alpha\sim\beta=(2n+1)\frac{\pi}{2}.$

但此方程式之因子 $\sin(\alpha+\beta+2\phi)$ 爲 0.

因 $\sin(\alpha+\phi)$, $\sin(\beta+\phi)$, $\cos(\alpha+\phi)$, $\cos(\beta+\phi)$ 不爲 0.

而 $\sin(\alpha+\beta+2\phi)=\sin(\alpha+\phi)\cos(\beta+\phi)+\cos(\alpha+\phi)\sin(\beta+\phi)=0.$

故得以此因子而除此方程式.

218. $\tan\beta, \tan 2\beta, \tan\alpha$ 爲等差級數,則

$\tan(\alpha-\beta)=\sin 2\beta.$

219. $\sin(\beta+\gamma-\alpha), \sin(\gamma+\alpha-\beta), \sin(\alpha+\beta-\gamma)$ 爲等差級數,則

$\tan\alpha, \tan\beta, \tan\gamma$ 亦爲等差級數.

220. $\cos(\phi-\alpha), \cos\phi, \cos(\phi+\alpha)$ 爲調和級數,則

$\cos\phi=\sqrt{2}\cos\frac{1}{2}\alpha.$

221. $\sin\alpha, \sin\beta, \sin\gamma$ 爲等差級數,則

$\tan\frac{\beta+\gamma}{2}, \tan\frac{\gamma+\alpha}{2}, \tan\frac{\alpha+\beta}{2}$ 亦爲等差級數.

222. 設 $\sin\alpha$ 及 $\sin\beta$ 爲 $\sin\theta$ 及 $\cos\theta$ 之等差中項及等比中項,則 $\cos 2\alpha=\frac{1}{2}\cos 2\beta=\cos^2\left(\frac{\pi}{4}+\theta\right).$

223. $\cot\alpha, \cot\beta, \cot\gamma$ 爲等差級數,則

$\cot(\beta-\alpha), \cot\beta, \cot(\beta-\gamma)$

及 $\frac{\sin(\beta+\gamma)}{\sin\alpha}, \frac{\sin(\gamma+\alpha)}{\sin\beta}, \frac{\sin(\alpha+\beta)}{\sin\gamma}$

亦爲等差級數.

例題解自 218. 至 223.

(218) $2\tan 2\beta=\tan\beta+\tan\alpha$, 卽 $\frac{4\tan\beta}{1-\tan^2\beta}=\tan\beta+\tan\alpha,$

故 $\tan\alpha=\frac{\tan\beta(3+\tan^2\beta)}{1-\tan^2\beta}$. 由是 $\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}$

$$=\frac{\tan\beta\left(\frac{3+\tan^2\beta}{1-\tan^2\beta}-1\right)}{1+\frac{\tan^2\beta(3+\tan^2\beta)}{1-\tan^2\beta}}=\frac{2\tan\beta(1+\tan^2\beta)}{(1+\tan^2\beta)^2}=\frac{2\tan\beta}{\sec^2\beta}=\sin 2\beta.$$

(219) $\sin(\beta+\gamma-\alpha)+\sin(\alpha+\beta-\gamma)=2\sin(\gamma+\alpha-\beta)$, 即

$2\sin\beta\cos(\gamma-\alpha)-2\{\sin(\gamma+\alpha)\cos\beta-\cos(\gamma+\alpha)\sin\beta\}$, 故

$$\tan\beta=\frac{\sin(\gamma+\alpha)}{\cos(\gamma-\alpha)+\cos(\gamma+\alpha)}=\frac{1}{2}(\tan\gamma+\tan\alpha).$$

(220) $\dfrac{2}{\cos\phi}=\dfrac{1}{\cos(\phi-\alpha)}+\dfrac{1}{\cos(\phi+\alpha)}=\dfrac{\cos(\phi+\alpha)+\cos(\phi-\alpha)}{\cos(\phi+\alpha)\cos(\phi-\alpha)}$, 即

$\dfrac{2}{\cos\phi}=\dfrac{2\cos\phi\cos\alpha}{\cos^2\phi-\sin^2\alpha}$, 故 $\cos^2\phi=\dfrac{\sin^2\alpha}{1-\cos\alpha}=1+\cos\alpha=2\cos^2\dfrac{\alpha}{2}$.

(221) $\sin\alpha-\sin\beta=\sin\beta-\sin\gamma$, 即 $\cos\dfrac{\alpha+\beta}{2}\sin\dfrac{\alpha-\beta}{2}=\cos\dfrac{\beta+\gamma}{2}\sin\dfrac{\beta-\gamma}{2}$,

即 $\cos\dfrac{\alpha+\beta}{2}\sin\left(\dfrac{\gamma+\alpha}{2}-\dfrac{\beta+\gamma}{2}\right)=\cos\dfrac{\beta+\gamma}{2}\sin\left(\dfrac{\alpha+\beta}{2}-\dfrac{\gamma+\alpha}{2}\right)$. 解此方程式兩邊之括弧,各以 $\cos\dfrac{\alpha+\beta}{2}\cos\dfrac{\beta+\gamma}{2}\cos\dfrac{\gamma+\alpha}{2}$ 除之,則

$$\tan\frac{\gamma+\alpha}{2}-\tan\frac{\beta+\gamma}{2}=\tan\frac{\alpha+\beta}{2}-\tan\frac{\gamma+\alpha}{2}.$$

(222) $2\sin\alpha=\sin\theta+\cos\theta$ 及 $\sin^2\beta=\sin\theta\cos\theta$, 故

$\cos2\alpha=1-2\sin^2\alpha=1-\frac{1}{2}(\sin\theta+\cos\theta)^2=\frac{1}{2}(1-2\sin\theta\cos\theta)$

$=\frac{1}{2}(1-2\sin^2\beta)=\frac{1}{2}\cos2\beta$, 又 $\cos2\alpha=\frac{1}{2}(1-2\sin\theta\cos\theta)$

$=\frac{1}{2}(1-\sin2\theta)=\frac{1}{2}\left\{1+\cos\left(\frac{\pi}{2}+2\theta\right)\right\}=\cos^2\frac{1}{2}\left(\frac{\pi}{2}+2\theta\right)$.

(223) $2\cot\beta=\cot\alpha+\cot\gamma$,

$$\cot(\beta-\alpha)+\cot(\beta-\gamma)=\frac{\cot\alpha\cot\beta+1}{\cot\alpha-\cot\beta}+\frac{\cot\beta\cot\gamma+1}{\cot\gamma-\cot\beta}$$

$$=\frac{2\cot\alpha\cot\beta\cot\gamma-(\cot\beta-1)(\cot\alpha+\cot\gamma)-2\cot\beta}{\cot\alpha\cot\gamma-\cot\beta(\cot\gamma+\cot\alpha)+\cot^2\beta}$$

$=\dfrac{2\cot\beta(\cot\alpha\cot\gamma-\cot^2\beta)}{\cot\alpha\cot\gamma-\cot^2\beta}=2\cot\beta$, 故如題言.

又 $\dfrac{\sin(\beta+\gamma)}{\sin\alpha}=\dfrac{\sin(\alpha+\beta+\gamma-\alpha)}{\sin\alpha}=\sin(\alpha+\beta+\gamma)\cot\alpha-\cos(\alpha+\beta+\gamma)$,

同樣, $\sin(\gamma+\alpha)/\sin\beta=\sin(\alpha+\beta+\gamma)\cot\beta-\cos(\alpha+\beta+\gamma)$,

及 $\sin(\alpha+\beta)/\sin\gamma=\sin(\alpha+\beta+\gamma)\cot\gamma-\cos(\alpha+\beta+\gamma)$, 由是

$$\frac{\sin(\beta+\gamma)}{\sin\alpha}+\frac{\sin(\alpha+\beta)}{\sin\gamma}=\sin(\alpha+\beta+\gamma)(\cot\alpha+\cot\gamma)-2\cos(\alpha+\beta+\gamma)$$

$=2\{\sin(\alpha+\beta+\gamma)\cot\beta-\cos(\alpha+\beta+\gamma)\}=2\sin(\gamma+\alpha)/\sin\beta$.

224. $\tan\alpha, \tan\beta, \tan\gamma$, 爲等差級數,

$\tan\alpha, \tan\beta, \tan\delta$ 爲調和級數,

則 $\dfrac{\tan\gamma}{\tan\delta}=1-\dfrac{8\sin^2(\alpha-\beta)}{\sin 2\alpha\sin 2\beta}$.

設 x, y, z 爲等差級數.證次例各式.

225. $\sin x-\sin z=2\sin(x-y)\sin y$.

226. $\dfrac{\tan y}{\tan(y-z)}=\dfrac{\sin x+\sin z}{\sin x-\sin z}=\dfrac{\tan\frac{1}{2}(x+z)}{\tan\frac{1}{2}(x-z)}$.

227. $1/(\tan x+\tan z)+\frac{1}{2}\tan y$

$=1/(\cot x+\cot z)+\frac{1}{2}\cot y$.

228. $\tan\dfrac{x}{2}=\tan^3\dfrac{z}{2}$, $\tan z=2\tan y$, 則

x, y, z 爲等差級數.

229. $\tan\phi=\dfrac{\sin\theta\cos\theta'}{\sin\theta'+\cos\theta}$, 則

$\tan\frac{1}{2}\phi=\tan\frac{1}{2}\theta\tan(\frac{1}{4}\pi-\frac{1}{2}\theta')$.

例題解自 224 至 229.

(224) $\tan\alpha+\tan\gamma=2\tan\beta$, $1/\tan\alpha+1/\tan\delta=2/\tan\beta$,

故 $\dfrac{\tan\gamma}{\tan\delta}=(2\tan\beta-\tan\alpha)(2/\tan\beta-1/\tan\alpha)$

$=\dfrac{5\tan\alpha\tan\beta-2\tan^2\alpha-2\tan^2\beta}{\tan\alpha\tan\beta}=1-\dfrac{2(\tan\alpha-\tan\beta)^2}{\tan\alpha\tan\beta}$

$=1-\dfrac{2\sin^2(\alpha-\beta)}{\cos^2\alpha\cos^2\beta\tan\alpha\tan\beta}=1-\dfrac{8\sin^2(\alpha-\beta)}{\sin 2\alpha\sin 2\beta}$.

(225) $x+z=2y$, $\therefore$ $2(x-y)=x-z$, 由是

$$\sin x-\sin z=2\cos\tfrac{1}{2}(x+z)\sin\tfrac{1}{2}(x-z)=2\cos y\sin(x-y).$$

(226) $\dfrac{\tan y}{\tan(y-z)}=\dfrac{2\sin y\cos(y-z)}{2\cos y\sin(y-z)}=\dfrac{\sin(2y-z)+\sin z}{\sin(2y-z)-\sin z}$ (見 3. 節公式)

$=\dfrac{\sin x+\sin z}{\sin x-\sin z}$ (依題意) 以下見例題28.

(227) $\dfrac{1}{\tan x+\tan z}+\dfrac{1}{2}\tan y=\dfrac{\cos x\cos z}{\sin(x+z)}+\dfrac{\sin y}{2\cos y}=\dfrac{\cos x\cos z}{\sin 2y}+\dfrac{\sin y}{2\cos y}$

$=\dfrac{\cos x\cos z+\sin^2 y}{2\sin y\cos y}=\dfrac{\cos(x+z)+\sin x\sin z+\sin^2 y}{\sin 2y}$

$=\dfrac{\cos 2y+\sin x\sin z+\sin^2 y}{\sin 2y}=\dfrac{\sin x\sin z+1-\sin^2 y}{\sin 2y}=\dfrac{\sin x\sin z}{\sin 2y}+\dfrac{\cos^2 y}{\sin 2y}$

$=\dfrac{\sin x\sin z}{\sin(x+z)}+\dfrac{\cos^2 y}{2\sin y\cos y}=\dfrac{1}{\cot x+\cot z}+\dfrac{\cot y}{2}.$

(228) 與例題202同. 今示其解如次.

$\tan\dfrac{x}{2}+\tan\dfrac{z}{2}=\tan\dfrac{z}{2}\left(1+\tan^2\dfrac{z}{2}\right)$, $1-\tan\dfrac{x}{2}\tan\dfrac{z}{2}=1-\tan^4\dfrac{z}{2}$,

故 $\tan\left(\dfrac{x}{2}+\dfrac{z}{2}\right)=\dfrac{\tan\frac{x}{2}+\tan\frac{z}{2}}{1-\tan\frac{x}{2}\tan\frac{z}{2}}=\dfrac{\tan\frac{z}{2}\left(1+\tan^2\frac{z}{2}\right)}{1-\tan^2\frac{z}{2}}=\dfrac{\tan\frac{z}{2}}{1-\tan^2\frac{z}{2}}$

$=\dfrac{1}{2}\tan z=\tan y$, 故 $y=\dfrac{x+z}{2}$,

(229) $\tan\phi=\dfrac{2\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}}$ (2. 節公式), 故 $\dfrac{2}{\tan\phi}=\dfrac{1}{\tan\frac{\phi}{2}}-\tan\dfrac{\phi}{2}$ (1),

又 $\dfrac{2}{\tan\phi}=\dfrac{2(\sin\theta'+\cos\theta)}{\sin\theta\cos\theta'}$ (題意) $=\dfrac{4\left\{\cos\left(\frac{\pi}{2}-\theta'\right)+\cos\theta\right\}}{2\sin\theta\sin\left(\frac{\pi}{2}-\theta'\right)}$

$=\dfrac{8\cos p\cos q}{\cos 2p-\cos 2q}$ $\left(\text{但 } p=\dfrac{\pi}{4}-\dfrac{\theta'}{2}-\dfrac{\theta}{2},\ q=\dfrac{\pi}{4}-\dfrac{\theta'}{2}+\dfrac{\theta}{2}\right)$,

$=\dfrac{2\{(\cos p+\cos q)^2-(\cos p-\cos q)^2\}}{2\cos^2 p-2\cos^2 q}=\dfrac{\cos p+\cos q}{\cos p-\cos q}-\dfrac{\cos p-\cos q}{\cos p+\cos q}$ (2),

比較 (1), (2). 得 $\tan\dfrac{\phi}{2}=\dfrac{\cos p-\cos q}{\cos p+\cos q}=\dfrac{\cos\left(\frac{\pi}{4}-\frac{\theta'}{2}-\frac{\theta}{2}\right)-\cos\left(\frac{\pi}{4}-\frac{\theta'}{2}+\frac{\theta}{2}\right)}{\cos\left(\frac{\pi}{4}-\frac{\theta'}{2}-\frac{\theta}{2}\right)+\cos\left(\frac{\pi}{4}-\frac{\theta'}{2}+\frac{\theta}{2}\right)}$

$=\left\{2\sin\left(\dfrac{\pi}{4}-\dfrac{\theta'}{2}\right)\sin\dfrac{\theta}{2}\right\}\Big/\left\{2\cos\left(\dfrac{\pi}{4}-\dfrac{\theta'}{2}\right)\cos\dfrac{\theta}{2}\right\}=\tan\left(\dfrac{\pi}{4}-\dfrac{\theta'}{2}\right)\tan\dfrac{\theta}{2}.$

證次之各恆同式.

230. $\tan\frac{1}{2}(\alpha+\beta)\tan\frac{1}{2}(\alpha-\beta)$

$$\equiv\frac{cosec2\alpha cosec\beta-cosec2\beta cosec\alpha}{cosec2\alpha cosec\beta+cosec2\beta cosec\alpha}.$$

231. $\left(\frac{\tan\alpha}{\tan\beta}-\frac{\tan\beta}{\tan\alpha}\right)+\left(\frac{\tan\beta}{\tan\gamma}-\frac{\tan\gamma}{\tan\beta}\right)+\left(\frac{\tan\gamma}{\tan\alpha}-\frac{\tan\alpha}{\tan\gamma}\right)\equiv$

$2\{\sin2(\beta-\alpha)+\sin2(\gamma-\beta)+\sin2(\alpha-\beta)\}/\sin2\alpha\sin2\beta\sin2\gamma.$

232. $\sin\alpha\sin\beta\{cosec\alpha cosec(\alpha+\beta)+cosec(\alpha+\beta)cosec(\alpha+2\beta)$

$+cosec(\alpha+2\beta)cosec(\alpha+3\beta)\}\equiv\sin3\beta cosec(\alpha+3\beta).$

233. $\frac{1}{a+b\cos\theta}\equiv\frac{\sec^2\frac{1}{2}\theta}{a+b+(a-b)\tan^2\frac{1}{2}\theta}.$

234. $\alpha+\beta=\omega$, 則

$\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta\cos\omega=\sin^2\omega.$

235. $\alpha+\beta=\omega$. 及 $\sin\frac{1}{2}\alpha=m\sin\frac{1}{2}\beta$, 則

$\tan\frac{1}{2}(\alpha-\beta)=(m-1)\tan\frac{1}{2}\omega/(m+1).$

236. $\alpha+\beta=\omega$, 及 $\tan\alpha=m\tan\beta$, 則

$\sin\omega=(m+1)\sin(\alpha-\beta)/(m-1).$

237. $\tan\theta+\frac{1}{2\tan\theta+}\frac{1}{2\tan\theta+}\frac{1}{2\tan\theta+}\cdots$至$\infty\equiv\sec\theta.$

238. $2\cot\theta+\frac{1}{2\cot\theta+}\frac{1}{2\cot\theta+}\frac{1}{2\cot\theta+}\cdots$至$\infty\equiv\cot\frac{\theta}{2}.$

例題解自 230. 至 238.

(230) $\tan\frac{\alpha+\beta}{2}\tan\frac{\alpha-\beta}{2}=\frac{2\sin\frac{1}{2}(\alpha+\beta)\sin\frac{1}{2}(\alpha-\beta)}{2\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}=\frac{\cos\beta-\cos\alpha}{\cos\beta+\cos\alpha}$

$$=\frac{2\sin\alpha\sin\beta(\cos\beta-\cos\alpha)}{2\sin\alpha\sin\beta(\cos\beta+\cos\alpha)}=\frac{\sin\alpha\sin2\beta-\sin2\alpha\sin\beta}{\sin\alpha\sin2\beta+\sin2\alpha\sin\beta}.$$

此分母子以 $\sin2\alpha\sin2\beta\sin\alpha\sin\beta$ 除之,卽得.

(231) $\frac{\tan\alpha}{\tan\beta}-\frac{\tan\beta}{\tan\alpha}=\frac{\tan^2\alpha-\tan^2\beta}{\tan\alpha\tan\beta}=\frac{\sec^2\alpha-\sec^2\beta}{\tan\alpha\tan\beta}=\frac{\cos^2\beta-\cos^2\alpha}{\sin\alpha\sin\beta\cos\alpha\cos\beta}$

$$=\frac{2(\cos2\beta-\cos2\alpha)}{\sin2\alpha\sin2\beta}=\frac{2\sin2\alpha\cos2\beta-2\sin2\gamma\cos2\alpha}{\sin2\alpha\sin2\beta\sin2\gamma}$$

$$=\frac{\sin(2\gamma+2\beta)+\sin(2\gamma-2\beta)+\sin(2\alpha-2\gamma)-\sin(2\alpha+2\gamma)}{\sin2\alpha\sin2\beta\sin2\gamma}$$

原式之他兩項.亦得同樣之結果.故相加卽得其證.

(232) 將原式左邊{}內最初之二項相加而簡單之,則

原式之左邊$=\sin\alpha\sin\beta\left\{\frac{2\cos\beta}{\sin\alpha\sin(\alpha+2\beta)}+\frac{1}{\sin(\alpha+2\beta)\sin(\alpha+3\beta)}\right\}$

$=\frac{\sin\beta\{2\sin(\alpha+3\beta)\cos\beta+\sin\alpha\}}{\sin(\alpha+2\beta)\sin(\alpha+3\beta)}$, 但此分子$=\sin\beta\{\sin(\alpha+2\beta)+\sin(\alpha+4\beta)+\sin\alpha\}$

$=\sin\beta\{\sin(\alpha+2\beta)+2\sin(\alpha+2\beta)\cos2\beta\}$

$=\sin\beta\sin(\alpha+2\beta)\{1+2\cos2\beta\}=\sin(\alpha+2\beta)\sin3\beta$, 故如題言.

(233) $\frac{1}{a+b\cos\theta}=\frac{1}{a+b(2\cos^2\frac{1}{2}\theta-1)}=\frac{\sec^2\frac{1}{2}\theta}{(a-b)\sec^2\frac{1}{2}\theta+2b}$

$=\sec^2\frac{1}{2}\theta/\{(a-b)(1+\tan^2\frac{1}{2}\theta)+2b\}$.

(234) $\cos^2\alpha+\cos^2\beta-2\cos\alpha\cos\beta\cos\omega=\cos^2\alpha+\cos^2\beta+\cos^2\omega$

$-2\cos\alpha\cos\beta\cos\omega-1+\sin^2\omega$

$=4\sin\frac{1}{2}(\alpha+\beta+\omega)\sin\frac{1}{2}(\beta+\omega-\alpha)\sin\frac{1}{2}(\alpha+\beta-\omega)\sin\frac{1}{2}(\beta-\omega-\alpha)+\sin^2\omega$ (例題118.)

但依題意得$\alpha+\beta-\omega=0$, ∴ 原式$=\sin^2\omega$.

(235) $\frac{\sin\frac{1}{2}\alpha}{\sin\frac{1}{2}\beta}=\frac{m}{1}$, 卽 $\frac{\sin\frac{1}{2}\alpha-\sin\frac{1}{2}\beta}{\sin\frac{1}{2}\alpha+\sin\frac{1}{2}\beta}=\frac{m-1}{m+1}$. 卽

$\frac{2\cos\frac{1}{4}(\alpha+\beta)\sin\frac{1}{4}(\alpha-\beta)}{2\sin\frac{1}{4}(\alpha+\beta)\cos\frac{1}{4}(\alpha-\beta)}=\frac{\tan\frac{1}{4}(\alpha-\beta)}{\tan\frac{1}{4}\omega}=\frac{m-1}{m+1}$.

(236) 與前例同樣.

(237) 令此無窮連分數爲x,則

$x-\tan\theta=\frac{1}{2\tan\theta+}\ \frac{1}{2\tan\theta+}\ \frac{1}{2\tan\theta}+\cdots$

卽 $x-\tan\theta=\frac{1}{2\tan\theta+(x-\tan\theta)}$. 從此可求得$x$.

(238) 與前例同樣,得$x=2\cot\theta+\frac{1}{x}$. 從此可求得$x$.

239. n 項之連分數爲 $\frac{1}{2sin\theta-}\ \frac{1}{2sin\theta-}\cdots\frac{1}{2sin\theta+x}$,

其 n 爲偶數,則等於 $\frac{sin n\theta+x cos(n-1)\theta}{x sin n\theta-cos(n+1)\theta}$.

其 n 爲奇數,則等於 $\frac{cos n\theta-x sin(n-1)\theta}{x cos n\theta+sin(n+1)\theta}$. 試證之.

例題解 239.

(239) $\frac{2}{2sin\theta-}\ \frac{1}{2sin\theta+x}=\frac{2sin\theta+x}{2sin\theta(2sin\theta+x)-1}$

$=\frac{2sin\theta cos\theta+x cos\theta}{2x sin\theta cos\theta-(1-4sin^2\theta)cos\theta}=\frac{sin2\theta+x cos\theta}{x sin2\theta-cos3\theta}$,

$\frac{1}{2sin\theta-}\ \frac{1}{2sin\theta-}\ \frac{1}{2sin\theta+x}=\frac{1}{2sin\theta-}\ \frac{sin2\theta+x cos\theta}{x sin2\theta-cos3\theta}$

$=\frac{x sin2\theta-cos3\theta}{2sin\theta(x sin2\theta-cos3\theta)-(sin2\theta+x cos\theta)}=\frac{cos3\theta-x sin2\theta}{x cos3\theta+sin4\theta}$, 同樣,

$\frac{1}{2sin\theta-}\ \frac{1}{2sin\theta-}\ \frac{1}{2sin\theta-}\ \frac{1}{2sin\theta+x}=\frac{1}{2sin\theta-}\ \frac{cos3\theta-x sin2\theta}{x cos3\theta+sin4\theta}$

$=\frac{x cos3\theta+sin4\theta}{2sin\theta(x cos3\theta+sin4\theta)-(cos3\theta-x sin2\theta)}=\frac{sin4\theta+x cos3\theta}{x sin4\theta-cos5\theta}$,

以此類推.假定 n 爲偶數.則此連分數 $=\frac{sin n\theta+x cos(n-1)\theta}{x sin n\theta-cos(n+1)\theta}$,

n 爲奇數,則此連分數 $=\frac{cos n\theta-x sin(n-1)\theta}{x cos n\theta+sin(n+1)\theta}$.

而 n 爲偶數,則得 $n+1$ 項之連分數 $=\frac{1}{2sin\theta-}\ \frac{sin n\theta+x cos(n-1)\theta}{x sin n\theta-cos(n+1)\theta}$

$=\frac{x sin n\theta-cos(n+1)\theta}{2sin\theta\{x sin n\theta-cos(n+1)\theta\}-\{sin n\theta+x cos(n-1)\theta\}}$

$=\frac{cos(n+1)\theta-x sin n\theta}{x cos(n+1)\theta+sin(n+2)\theta}$

故本題之結果.對於偶數 n 爲眞,則對於 $n+1$ 亦爲眞.然對 n 之 2, 4 爲眞,故對於 3, 5 亦眞.

同樣,可證其對於奇數 n 爲眞.

(注意) 連分數 $\cfrac{1}{a\pm\cfrac{1}{b\pm\cfrac{1}{c}}}$ 略爲 $\frac{1}{a\pm}\ \frac{1}{b\pm}\ \frac{1}{c}$

本題採用簡略之記法.

第伍編

三角之和

1. 三角之和 由前編1.節及2.節得次之恆同式

$$\sin(\alpha+\beta+\gamma)\equiv\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\gamma\cos\alpha$$
$$+\sin\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta\sin\gamma.$$

$$\cos(\alpha+\beta+\gamma)\equiv\cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\beta\cos\gamma$$
$$-\sin\beta\sin\gamma\cos\alpha-\sin\gamma\sin\alpha\cos\beta.$$

$$\tan(\alpha+\beta+\gamma)\equiv\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha}.$$

〔證〕 $\sin\{(\alpha+\beta)+\gamma\}=\sin(\alpha+\beta)\cos\gamma+\cos(\alpha+\beta)\sin\gamma$

$=(\sin\alpha\cos\beta+\cos\alpha\sin\beta)\cos\gamma+(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\sin\gamma$

$=\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\gamma\cos\alpha+\sin\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta\sin\gamma.$

又 $\cos\{(\alpha+\beta)+\gamma\}=\cos(\alpha+\beta)\cos\gamma-\sin(\alpha+\beta)\sin\gamma$

$=(\cos\alpha\cos\beta-\sin\alpha\sin\beta)\cos\gamma-(\sin\alpha\cos\beta+\cos\alpha\sin\beta)\sin\gamma$

$=\cos\alpha\cos\beta\cos\gamma-\sin\alpha\sin\beta\cos\gamma-\sin\alpha\sin\gamma\cos\beta-\sin\beta\sin\gamma\cos\alpha.$

又 $\tan\{(\alpha+\beta)+\gamma\}=\dfrac{\tan(\alpha+\beta)+\tan\gamma}{1-\tan(\alpha+\beta)\tan\gamma}$

$$=\frac{\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}+\tan\gamma}{1-\dfrac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}\tan\gamma}=\frac{\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma}{1-\tan\alpha\tan\beta-\tan\alpha\tan\gamma-\tan\beta\tan\gamma}.$$

2. 注意 於前節之公式令$\alpha=\beta=\gamma$則可得第四編5.節之公式.

例題五

證次列各恆同式

1. $\cot(\alpha+\beta+\gamma)\equiv\dfrac{\cot\alpha\cot\beta\cot\gamma-\cot\alpha-\cot\beta-\cot\gamma}{\cot\beta\cot\gamma+\cot\gamma\cot\alpha+\cot\alpha\cot\beta-1}$.

2. $\dfrac{\sin(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}\equiv\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma$.

3. $\dfrac{\cos(\alpha+\beta+\gamma)}{\cos\alpha\cos\beta\cos\gamma}\equiv1-\tan\beta\tan\gamma-\tan\gamma\tan\alpha-\tan\alpha\tan\beta$.

4. $\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)+\sin(\alpha+\beta-\gamma)$
$-\sin(\alpha+\beta+\gamma)\equiv4\sin\alpha\sin\beta\sin\gamma$.

5. $\cos(\beta+\gamma-\alpha)+\cos(\gamma+\alpha-\beta)+\cos(\alpha+\beta-\gamma)$
$+\cos(\alpha+\beta+\gamma)\equiv4\cos\alpha\cos\beta\cos\gamma$.

6. $\sin(\alpha+\beta+\gamma)+\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)$
$-\sin(\alpha+\beta-\gamma)\equiv4\cos\alpha\cos\beta\sin\gamma$.

7. $\cos(\beta+\gamma-\alpha)+\cos(\gamma+\alpha-\beta)-\cos(\alpha+\beta-\gamma)$
$-\cos(\alpha+\beta+\gamma)\equiv4\sin\alpha\sin\beta\cos\gamma$.

8. $\sin\alpha+\sin\beta+\sin\gamma-4\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta\cos\frac{1}{2}\gamma$
$\equiv2\sin\frac{1}{4}(\alpha+\beta+\gamma-\pi)\{\cos\frac{1}{4}(3\alpha-\beta-\gamma+\pi)+\cos\frac{1}{4}(3\beta-\gamma-\alpha+\pi)$
$+\cos\frac{1}{4}(3\gamma-\alpha-\beta+\pi)+\cos\frac{1}{4}(\alpha+\beta+\gamma-\pi)\}$.

9. $\sin(\alpha+\beta)\cos\beta-\sin(\alpha+\gamma)\cos\gamma$
$\equiv\sin(\beta-\gamma)\cos(\alpha+\beta+\gamma)$.

10. $\sin(\alpha+\beta)\sin(\beta+\gamma)-\sin\alpha\sin\gamma$
$\equiv\sin\beta\sin(\alpha+\beta+\gamma)$.

11. $\sin(\alpha+\beta-2\gamma)\cos\beta-\sin(\alpha+\gamma-2\beta)\cos\gamma$
$\equiv\sin(\beta-\gamma)\{\cos(\beta+\gamma-\alpha)+\cos(\gamma+\alpha-\beta)+\cos(\alpha+\beta-\gamma)\}$.

例題解自 1. 至 11.

(1) $\cot(\alpha+\beta+\gamma)=\dfrac{\cos(\alpha+\beta+\gamma)}{\sin(\alpha+\beta+\gamma)}$ 由 1. 節解其分數之括弧母子.各以

$\sin\alpha\sin\beta\sin\gamma$ 除之.卽得. (2) 及 (3) 均可由 1. 節得之.

(4) $\{\sin(\beta+\gamma-\alpha)+\sin(\gamma+\alpha-\beta)\}-\{\sin(\alpha+\beta+\gamma)-\sin(\alpha+\beta-\gamma)\}$

$=2\sin\gamma\cos(\beta-\alpha)-2\cos(\alpha+\beta)\sin\gamma=2\sin\gamma\{\cos(\beta-\alpha)-\cos(\alpha+\beta)\}$

$=4\sin\gamma\sin\beta\sin\alpha$ (5) 至 (7)、均與此同法.

(8) 依前例 5. 變化原式左邊之項 $4\cos\frac{1}{2}\alpha\cos\frac{1}{2}\beta\cos\frac{1}{2}\gamma$ 則

$$\sin\alpha+\sin\beta+\sin\gamma-\left(\cos\frac{\beta+\gamma-\alpha}{2}+\cos\frac{\gamma+\alpha-\beta}{2}+\cos\frac{\alpha+\beta-\gamma}{2}+\cos\frac{\alpha+\beta+\gamma}{2}\right),$$

但 $\sin\alpha-\cos\dfrac{\beta+\gamma-\alpha}{2}=\sin\alpha-\sin\left(\dfrac{\pi}{2}-\dfrac{\beta+\gamma-\alpha}{2}\right)$

$$=2\cos\left(\frac{\pi}{4}+\frac{3\alpha-\beta-\gamma}{4}\right)\sin\left(\frac{\alpha+\beta+\gamma}{4}-\frac{\pi}{4}\right),$$

同樣 $\sin\beta-\cos\dfrac{\gamma+\alpha-\beta}{2}=2\cos\left(\dfrac{\pi}{4}+\dfrac{3\beta-\gamma-\alpha}{4}\right)\sin\left(\dfrac{\alpha+\beta+\gamma}{4}-\dfrac{\pi}{4}\right),$

$\sin\gamma-\cos\dfrac{\alpha+\beta-\gamma}{2}=2\cos\left(\dfrac{\pi}{4}+\dfrac{3\gamma-\alpha-\beta}{4}\right)\sin\left(\dfrac{\alpha+\beta+\gamma}{4}-\dfrac{\pi}{4}\right),$

又 $-\cos\dfrac{\alpha+\beta+\gamma}{2}=-\sin\left(\dfrac{\pi}{2}-\dfrac{\alpha+\beta+\gamma}{2}\right)$

$$=2\sin\left(\frac{\alpha+\beta+\gamma}{4}-\frac{\pi}{4}\right)\cos\left(\frac{\alpha+\beta+\gamma}{4}-\frac{\pi}{4}\right),$$

加此四個恆同式.卽可得其結果.

(9) 原式之左邊$=\frac{1}{2}\{\sin(\alpha+2\beta)+\sin\alpha\}-\frac{1}{2}\{\sin(\alpha+2\gamma)+\sin\alpha\}$

$=\frac{1}{2}\{\sin(\alpha+2\beta)-\sin(\alpha+2\gamma)\}=\cos(\alpha+\beta+\gamma)\sin(\beta-\gamma).$

(10) 原式之左邊$=\frac{1}{2}\{\cos(\alpha-\gamma)-\cos(\alpha+2\beta+\gamma)\}$

$-\frac{1}{2}\{\cos(\alpha-\gamma)-\cos(\alpha+\gamma)\}=\frac{1}{2}\{\cos(\alpha+\gamma)-\cos(\alpha+2\beta+\gamma)\}.$

(11) 原式之左邊$=\frac{1}{2}\{\sin(\alpha+2\beta-2\gamma)+\sin(\alpha-2\gamma)\}$

$-\frac{1}{2}\{\sin(\alpha+2\gamma-2\beta)+\sin(\alpha-2\beta)\}$

$=\frac{1}{2}\{\sin(\alpha+2\beta-2\gamma)-\sin(\alpha+2\gamma-2\beta)\}+\frac{1}{2}\{\sin(\alpha-2\gamma)-\sin(\alpha-2\beta)\}$

$=\cos\alpha\sin(2\beta-2\gamma)+\cos(\alpha-\beta-\gamma)\sin(\beta-\gamma)$

$=\sin(\beta-\gamma)\{2\cos\alpha\cos(\beta-\gamma)+\cos(\alpha-\beta-\gamma)\}.$

12. $\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)$

$$\equiv 4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}.$$

13. $\cos\alpha+\cos\beta+\cos\gamma+\cos(\alpha+\beta+\gamma)$

$$\equiv 4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}.$$

14. $\sin(2x+\theta)+\sin(2y+\theta)+\sin(2z+\theta)-\sin(2x+2y+2z+3\theta)$

$\equiv 4\sin(x+y+\theta)\sin(y+z+\theta)\sin(z+x+\theta).$

15. $\sin(x-y)+\sin(y-z)+\sin(z-x)$

$\equiv -4\sin\frac{x-y}{2}\sin\frac{y-z}{2}\sin\frac{z-x}{2}.$

16. $\cos(x-y)+\cos(y-z)+\cos(z-x)+1$

$\equiv 4\cos\frac{x-y}{2}\cos\frac{y-z}{2}\cos\frac{z-x}{2}.$

17. $\sin3\alpha\sin(\beta-\gamma)+\sin3\beta\sin(\gamma-\alpha)+\sin3\gamma\sin(\alpha-\beta)$

$\equiv 4\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha+\beta+\gamma).$

18. $\cos3\alpha\sin(\beta-\gamma)+\cos3\beta\sin(\gamma-\alpha)+\cos3\gamma\sin(\alpha-\beta)$

$\equiv 4\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)\cos(\alpha+\beta+\gamma).$

19. $\tan(\alpha+\beta+\gamma)$

$\equiv \frac{\sin3\alpha\sin(\beta-\gamma)+\sin3\beta\sin(\gamma-\alpha)+\sin3\gamma\sin(\alpha-\beta)}{\cos3\alpha\sin(\beta-\gamma)+\cos3\beta\sin(\gamma-\alpha)+\cos3\gamma\sin(\alpha-\beta)}.$

20. $\cos(\alpha+\beta+\gamma)\cos(\alpha+\beta-\gamma)\cos(\beta+\gamma-\alpha)\cos(\gamma+\alpha-\beta)$

$+\sin(\alpha+\beta+\gamma)\sin(\alpha+\beta-\gamma)\sin(\beta+\gamma-\alpha)\sin(\gamma+\alpha-\beta)$

$\equiv \cos2\alpha\cos2\beta\cos2\gamma.$

21. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma-2\sin\alpha\sin\beta\sin\gamma-1$

$\equiv -4\cos\left(\frac{\alpha+\beta+\gamma}{2}-\frac{\pi}{4}\right)\cos\left(\frac{\alpha+\beta-\gamma}{2}+\frac{\pi}{4}\right)$

$\cos\left(\frac{\beta+\gamma-\alpha}{2}-\frac{\pi}{4}\right)\cos\left(\frac{\gamma+\alpha-\beta}{2}+\frac{\pi}{4}\right)$

例題解自12.至21.

(12) $\{\sin\alpha+\sin\beta\}-\{\sin(\alpha+\beta+\gamma)-\sin\gamma\}$

$$=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}-2\cos\frac{\alpha+\beta+2\gamma}{2}\sin\frac{\alpha+\beta}{2}$$

$$=2\sin\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2}-\cos\frac{\alpha+\beta+2\gamma}{2}\right)=4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}.$$

(13) 與前例同樣,可用3.節之公式

(14) 於前例(12)之$\alpha,\beta,$及γ.順次代以$2x+\theta, 2y+\theta,$及$2z+\theta$.即得其證.

(15) 於前例(12)之$\alpha,\beta,$及γ.順次代以$x-y, y-z,$及$z-x$.則

$\sin(\alpha+\beta+\gamma)=\sin(x-y+y-z+z-x)=\sin 0=0.$ 即得其證.

(16) 於前例(13)之α,β及γ.順次代以$x-y, y-z,$及$z-x$.則

$\cos(\alpha+\beta+\gamma)=\cos 0=1.$ 即可得其證.

(17) 原式之左邊$=\frac{1}{2}\{\cos(3\alpha-\beta+\gamma)-\cos(3\alpha+\beta-\gamma)\}$

$+\frac{1}{2}\{\cos(3\beta-\gamma+\alpha)-\cos(3\beta+\gamma-\alpha)\}+\frac{1}{2}\{\cos(3\gamma-\alpha+\beta)-\cos(3\gamma+\alpha-\beta)\}$

$=\frac{1}{2}\{\cos(3\alpha-\beta+\gamma)-\cos(3\beta+\gamma-\alpha)\}+\frac{1}{2}\{\cos(3\beta-\gamma+\alpha)-\cos(3\gamma+\alpha-\beta)\}$

$+\frac{1}{2}\{\cos(3\gamma-\alpha+\beta)-\cos(3\alpha+\beta-\gamma)\}=\sin(2\beta-2\alpha)\sin(\alpha+\beta+\gamma)$

$+\sin(2\gamma-2\beta)\sin(\alpha+\beta+\gamma)+\sin(2\alpha-2\gamma)\sin(\alpha+\beta+\gamma)$

$=\sin(\alpha+\beta+\gamma)\{\sin(2\beta-2\alpha)+\sin(2\gamma-2\beta)+\sin(2\alpha-2\gamma)\}$

$=\sin(\alpha+\beta+\gamma)\{4\sin(\alpha-\beta)\sin(\beta-\gamma)\sin(\gamma-\alpha)\}$　由例題15.

(18) 與前例同樣　(19) 從 17. 18. 兩例題.即可得其證.

(20) 原式之左邊$=\frac{1}{2}\{\cos2\gamma+\cos(2\alpha+2\beta)\}\frac{1}{2}\{\cos2\gamma+\cos(2\alpha-2\beta)\}$

$-\frac{1}{2}\{\cos2\gamma-\cos(2\alpha+2\beta)\}\frac{1}{2}\{\cos2\gamma-\cos(2\alpha-2\beta)\}$

$=\frac{1}{4}\{2\cos2\gamma\cos(2\alpha+2\beta)+2\cos2\gamma\cos(2\alpha-2\beta)\}$

$=\frac{1}{2}\cos2\gamma\{\cos(2\alpha+2\beta)+\cos(2\alpha-2\beta)\}=\cos2\gamma\cos2\beta\cos2\alpha.$

(21) $\frac{1}{2}(1-\cos2\alpha)+\frac{1}{2}(1-\cos2\beta)+\sin^2\gamma-\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}\sin\gamma-1$

$=\cos(\alpha+\beta)\sin\gamma-\frac{1}{2}(\cos2\alpha+\cos2\beta)+\sin\gamma\{\sin\gamma-\cos(\alpha-\beta)\}$

$=\cos(\alpha+\beta)\{\sin\gamma-\cos(\alpha-\beta)\}+\sin\gamma\{\sin\gamma-\cos(\alpha-\beta)\}$

$=\{\sin\gamma-\cos(\alpha-\beta)\}\{\cos(\alpha+\beta)+\sin\gamma\}$

$=-\left\{\cos\left(\frac{\pi}{2}+\gamma\right)+\cos(\alpha-\beta)\right\}\left\{\cos(\alpha+\beta)+\cos\left(\frac{\pi}{2}-\gamma\right)\right\}.$

用例題118.之解法.更爲便利.

22. $\cos\beta\sin\frac{\alpha+\beta}{2}\sin\frac{\gamma-\delta}{2}+\cos\gamma\sin\frac{\alpha+\gamma}{2}\sin\frac{\delta-\beta}{2}$

$+\cos\delta\sin\frac{\alpha+\delta}{2}\sin\frac{\beta-\gamma}{2}$

$\equiv 2\sin\frac{\gamma-\delta}{2}\sin\frac{\delta-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\alpha+\beta+\gamma+\delta}{2}.$

23. $\cos2(\alpha+\beta+\gamma)+\cos(2\alpha+\beta+\gamma)+\cos(2\beta+\gamma+\alpha)+\cos(2\gamma+\alpha+\beta)$

$+\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\beta)$

$\equiv 8\cos(\alpha+\beta+\gamma)\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\beta}{2}-1.$

24. $\{\sin\beta+\sin\gamma-\sin(\beta+\gamma)\}\{\sin\gamma+\sin\alpha-\sin(\gamma+\alpha)\}$

$\{\sin\alpha+\sin\beta-\sin(\gamma+\alpha)\}$

$\equiv 16\sin^2\frac{\alpha}{2}\sin^2\frac{\beta}{2}\sin^2\frac{\gamma}{2}\{\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)\}.$

25. $2\sin^2\alpha\sin\frac{\beta+\gamma}{2}\sin\frac{\beta-\gamma}{2}+2\sin^2\beta\sin\frac{\gamma+\alpha}{2}\sin\frac{\gamma-\alpha}{6}$

$+2\sin^2\gamma\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}$

$\equiv -(\cos\alpha-\cos\beta)(\cos\beta-\cos\gamma)(\cos\gamma-sos\alpha).$

26. $(\cos\alpha+\cos\beta+\cos\gamma)\{\cos2\alpha+\cos2\beta+\cos2\gamma-\cos(\beta+\gamma)-\cos(\gamma+\alpha)$
$-\cos(\alpha+\beta)\}-(\sin\alpha+\sin\beta+\sin\gamma)\{\sin2\alpha+\sin2\beta+\sin2\gamma-\sin(\beta+\gamma)$
$-\sin(\gamma+\alpha)-\sin(\alpha+\beta)\}\equiv\cos3\alpha+\cos3\beta+\cos3\gamma-3\cos(\alpha+\beta+\gamma).$

例題解自 22 至 26.

(22) $\cos\beta\sin\frac{\alpha+\beta}{2}\sin\frac{\gamma-\delta}{2}=\frac{1}{2}\cos\beta\left(\cos\frac{\alpha+\beta-\gamma+\delta}{2}-\cos\frac{\alpha+\beta+\gamma-\delta}{2}\right)$

$=\frac{1}{4}\left(\cos\frac{\alpha-\beta-\gamma+\delta}{2}-\cos\frac{\alpha+3\beta-\gamma+\delta}{2}-\cos\frac{\alpha-\beta+\gamma-\delta}{2}-\cos\frac{\alpha+3\beta+\gamma-\delta}{2}\right),$

他之兩項 $\cos\gamma\sin\frac{\alpha+\gamma}{2}\sin\frac{\delta-\beta}{2}$ 及 $\cos\delta\sin\frac{\alpha+\delta}{2}\sin\frac{\beta-\gamma}{2}$ 亦可同樣變化之.
相加.則

原式$=\frac{1}{4}\left(\cos\frac{\alpha+3\beta-\gamma+\delta}{2}-\cos\frac{\alpha+3\gamma+\delta-\beta}{2}\right),$

$$+\frac{1}{4}\left(\cos\frac{\alpha+3\gamma-\delta+\beta}{2}-\cos\frac{\alpha+3\delta+\beta-\gamma}{2}\right)$$

$$+\frac{1}{4}\left(\cos\frac{\alpha+3\delta-\beta+\gamma}{2}-\cos\frac{\alpha+3\beta+\gamma-\delta}{2}\right)$$

$$=\tfrac{1}{2}\sin(\gamma-\beta)\sin\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)+\tfrac{1}{2}\sin(\delta-\gamma)\sin\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)$$

$$+\tfrac{1}{2}\sin(\beta-\delta)\sin\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)$$

$$=\tfrac{1}{2}\sin\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)\{\sin(\gamma-\beta)+\sin(\delta-\gamma)+\sin(\beta-\delta)\}$$

$$=2\sin\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)\sin\tfrac{1}{2}(\beta-\gamma)\sin\tfrac{1}{2}(\gamma-\delta)\tfrac{1}{2}(\delta-\beta).$$ (例題 15.)

(23) $\cos 2(\alpha+\beta+\gamma)+\{\cos(2\alpha+\beta+\gamma)+\cos(\beta+\gamma)\}+\{\cos(2\beta+\gamma+\alpha)$
$+\cos(\gamma+\alpha)\}+\{\cos(2\gamma+\alpha+\beta)+\cos(\alpha+\beta)\}=2\cos^2(\alpha+\beta+\gamma)-1$
$+2\cos\alpha\cos(\alpha+\beta+\gamma)+2\cos\beta\cos(\alpha+\beta+\gamma)+2\cos\gamma\cos(\alpha+\beta+\gamma)$
$=2\cos(\alpha+\beta+\gamma)\{\cos(\alpha+\beta+\gamma)+\cos\alpha+\cos\beta+\cos\gamma\}-1$
$=8\cos(\alpha+\beta+\gamma)\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}-1.$ (例題 13.)

(24) $\sin\beta+\sin\gamma-\sin(\beta+\gamma)=2\sin\frac{\beta+\gamma}{2}\left(\cos\frac{\beta-\gamma}{2}-\cos\frac{\beta+\gamma}{2}\right)$
$=4\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma}{2}\sin\frac{\beta}{2}$, 由是原式之左邊如下.
$\left(4\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma}{2}\sin\frac{\beta}{2}\right)\left(4\sin\frac{\gamma+\alpha}{2}\sin\frac{\alpha}{2}\sin\frac{\gamma}{2}\right)\left(4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta}{2}\sin\frac{\alpha}{2}\right)$
$=16\sin^2\frac{\gamma}{2}\sin^2\frac{\beta}{2}\sin^2\frac{\alpha}{2}\left(4\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}\sin\frac{\beta+\alpha}{2}\right)$
$=16\sin^2\frac{\gamma}{2}\sin^2\frac{\beta}{2}\sin^2\frac{\alpha}{2}\{\sin\alpha+\sin\beta+\sin\gamma-\sin(\alpha+\beta+\gamma)\}$ (12).

(25) $(1-\cos^2\alpha)(\cos\gamma-\cos\beta)+(1-\cos^2\beta)(\cos\alpha-\cos\gamma)+(1-\cos^2\gamma)(\cos\beta-\cos\alpha)$
$=\cos^2\alpha(\cos\beta-\cos\gamma)+\cos^2\beta(\cos\gamma-\cos\alpha)=\cos^2\gamma(\cos\alpha-\cos\beta).$

(26) 令 $\cos\alpha=a, \cos\beta=b, \cos\gamma=c, \sin\alpha=x, \sin\beta=y, \sin\gamma=z,$
則 $a^2+x^2=b^2+y^2=c^2+z^2=1,$

原式$=(a+b+c)\{a^2-x^2+b^2-y^2+c^2-z^2-(bc-yz)-(ca-zx)-(ab-xy)\}$
$-(x+y+z)\{2ax+2by+2cz-(yc+bz)-(za+cx)-(xb+ay)\}$
$=(a+b+c)\{(a^2+b^2+c^2-bc-ca-ab)-(x^2+y^2+z^2-yz-zx-xy)\}$
$-(x+y+z)\{a(2x-y-z)+b(2y-z-x)+c(2z-x-y)\}$
$=a^3+b^3+c^3-3abc-3a(x^2-yz)-3b(y^2-zx)-3c(z^2-xy)$
$=a(a^2-3x^2)+b(b^2-3y^2)+c(c^2-3z^2)-3(abc-ayz-byx-czx)$
$=(4a^3-3a)+(4b^3-3b)+(4c^3-3c)-3(abc-ayz-byx-czx)$
$=\cos 3\alpha+\cos 3\beta+\cos 3\gamma-3\cos(\alpha+\beta+\gamma).$
但 $a(a^2-3x^2)=a\{a^2-3(1-a^2)\}=4a^3-3a.$

27. $(\sin\alpha+\sin\beta+\sin\gamma)\{\cos2\alpha+\cos2\beta+\cos2\gamma$

$-\cos(\beta+\gamma)-\cos(\gamma+\alpha)-\cos(\alpha+\beta)\}+(\cos\alpha+\cos\beta+\cos\gamma)$

$\{\sin2\alpha+\sin2\beta+\sin2\gamma-\sin(\beta+\gamma)-\sin(\gamma+\alpha)-\sin(\alpha+\beta)\}$

$\equiv\sin3\alpha+\sin3\beta+\sin3\gamma-3\sin(\alpha+\beta+\gamma).$

28. 設 $\alpha,\beta,\gamma,\delta$ 爲不等. 則

$$\frac{\cos2\alpha}{\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\alpha-\delta}{2}}+\frac{\cos2\beta}{\sin\frac{\beta-\alpha}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\beta-\delta}{2}}$$

$$+\frac{\cos2\gamma}{\sin\frac{\gamma-\alpha}{2}\sin\frac{\gamma-\beta}{2}\sin\frac{\gamma-\delta}{2}}+\frac{\cos2\delta}{\sin\frac{\delta-\alpha}{2}\sin\frac{\delta-\beta}{2}\sin\frac{\delta-\gamma}{2}}$$

$$=8\sin\frac{\alpha+\beta+\gamma+\delta}{2}.$$

29. 設 $\sin\alpha+\sin\beta+\sin\gamma=\cos\alpha+\cos\beta+\cos\gamma=0,$

則 $\cos3\alpha+\cos3\beta+\cos3\gamma=3\cos(\alpha+\beta+\gamma),$

及 $\sin3\alpha+\sin3\beta+\sin3\gamma=3\sin(\alpha+\beta+\gamma).$

例題解自 27. 至 29.

(27) 與前例同法.

(28) 令原式之通分母 $=\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\alpha-\delta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\beta-\delta}{2}\sin\frac{\gamma-\delta}{2}$

則原式之分子 $=\cos2\alpha\sin\frac{\beta-\gamma}{2}\sin\frac{\beta-\delta}{2}\sin\frac{\gamma-\delta}{2}-\cos2\beta\sin\frac{\alpha-\gamma}{2}\sin\frac{\alpha-\delta}{2}\sin\frac{\gamma-\delta}{2}$

$+\cos2\gamma\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\delta}{2}\sin\frac{\beta-\delta}{2}-\cos2\delta\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}$ 〔由例題15. 得次式〕

$=\frac{1}{4}\cos2\alpha\{\sin(\beta-\gamma)+\sin(\gamma-\delta)-\sin(\beta-\delta)\}-\frac{1}{4}\cos2\beta\{\sin(\alpha-\gamma)+\sin(\gamma-\delta)$

$-\sin(\alpha-\delta)\}+\frac{1}{4}\cos2\gamma\{\sin(\alpha-\beta)+\sin(\beta-\delta)-\sin(\alpha-\delta)\}-\frac{1}{4}\cos2\delta\{\sin(\alpha-\beta)$

$+\sin(\beta-\gamma)-\sin(\alpha-\gamma)\}=\frac{1}{4}\sin(\beta-\gamma)(\cos2\alpha-\cos2\delta)+\frac{1}{4}\sin(\gamma-\delta)(\cos2\alpha-\cos2\beta)$

$-\frac{1}{4}\sin(\beta-\delta)(\cos2\alpha-\cos2\gamma)-\frac{1}{4}\sin(\alpha-\gamma)(\cos2\beta-\cos2\delta)$

$+\frac{1}{4}\sin(\alpha-\delta)(\cos2\beta-\cos2\gamma)+\frac{1}{4}\sin(\alpha-\beta)(\cos2\gamma-\cos2\delta)$

$$=-\tfrac{1}{2}\sin(\beta-\gamma)\sin(\alpha+\delta)\sin(\alpha-\delta)-\tfrac{1}{2}\sin(\gamma-\delta)\sin(\alpha+\beta)\sin(\alpha-\beta)$$
$$+\tfrac{1}{2}\sin(\beta-\delta)\sin(\alpha+\gamma)\sin(\alpha-\gamma)+\tfrac{1}{2}\sin(\alpha-\gamma)\sin(\beta+\delta)\sin(\beta-\delta)$$
$$-\tfrac{1}{2}\sin(\alpha-\delta)\sin(\beta+\gamma)\sin(\beta-\gamma)-\tfrac{1}{2}\sin(\alpha-\beta)\sin(\gamma+\delta)\sin(\gamma-\delta)$$
$$=-\tfrac{1}{2}\sin(\beta-\gamma)\sin(\alpha-\delta)\{\sin(\alpha+\delta)+\sin(\beta+\gamma)\}$$
$$-\tfrac{1}{2}\sin(\gamma-\delta)\sin(\alpha-\beta)\{\sin(\alpha+\beta)+\sin(\gamma+\delta)\}$$
$$+\tfrac{1}{2}\sin(\beta-\delta)\sin(\alpha-\gamma)\{\sin(\alpha+\gamma)+\sin(\beta+\delta)\}$$
$$=-\sin(\beta-\gamma)\sin(\alpha-\delta)\sin\tfrac{1}{2}(\alpha+\delta+\beta+\gamma)\cos\tfrac{1}{2}(\alpha+\delta-\beta-\gamma)$$
$$-\sin(\gamma-\delta)\sin(\alpha-\beta)\sin\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)\cos\tfrac{1}{2}(\alpha+\beta-\gamma-\delta)$$
$$+\sin(\beta-\delta)\sin(\alpha-\gamma)\sin(\alpha+\gamma+\beta+\delta)\cos\tfrac{1}{2}(\alpha+\gamma-\beta-\delta)$$
$$=\tfrac{1}{2}\sin\tfrac{1}{2}(\alpha+\beta+\gamma+\delta)[-\{\cos(\alpha-\delta-\beta+\gamma)-\cos(\alpha-\delta+\beta-\gamma)\}\cos\tfrac{1}{2}(\alpha+\delta-\beta-\gamma)$$
$$-\{\cos(\alpha-\beta-\gamma+\delta)-\cos(\alpha-\beta+\gamma-\delta)\}\cos\tfrac{1}{2}(\alpha+\beta-\gamma-\delta)$$
$$+\{\cos(\alpha-\gamma-\beta+\delta)-\cos(\alpha-\gamma+\beta-\delta)\}\cos\tfrac{1}{2}(\alpha+\gamma-\beta-\delta)]$$
$$=\sin\frac{\alpha+\beta+\gamma+\delta}{2}\Big[-\Big(\cos^2\frac{\alpha-\beta+\gamma-\delta}{2}-\cos^2\frac{\alpha+\beta-\gamma-\delta}{2}\Big)\cos\frac{\alpha-\beta-\gamma+\delta}{2}$$
$$-\Big(\cos^2\frac{\alpha-\beta-\gamma+\delta}{2}-\cos^2\frac{\alpha-\beta+\gamma-\delta}{2}\Big)\cos\frac{\alpha+\beta-\gamma-\delta}{2}$$
$$+\Big(\cos^2\frac{\alpha-\beta-\gamma+\delta}{2}-\cos^2\frac{\alpha+\beta-\gamma-\delta}{2}\Big)\cos\frac{\alpha-\beta+\gamma-\delta}{2}\Big]$$
$$=\sin\frac{\alpha+\beta+\gamma+\delta}{2}\Big[-\Big(\cos^2\frac{\alpha-\beta+\gamma-\delta}{2}-\cos^2\frac{\alpha+\beta-\gamma-\delta}{2}\Big)\cos\frac{\alpha-\beta-\gamma+\delta}{2}$$
$$+\Big(\cos\frac{\alpha-\beta+\gamma-\delta}{2}-\cos\frac{\alpha+\beta-\gamma-\delta}{2}\Big)\cos\frac{\alpha+\beta-\gamma-\delta}{2}\cos\frac{\alpha-\beta+\gamma-\delta}{2}$$
$$+\Big(\cos\frac{\alpha-\beta+\gamma-\delta}{2}-\cos\frac{\alpha+\beta-\gamma-\delta}{2}\Big)\cos^2\frac{\alpha-\beta-\gamma+\delta}{2}\Big]$$
$$=\sin\frac{\alpha+\beta+\gamma+\delta}{2}\Big(\cos\frac{\alpha-\beta+\gamma-\delta}{2}-\cos\frac{\alpha+\beta-\gamma-\delta}{2}\Big)$$
$$\Big(\cos\frac{\alpha-\beta-\gamma+\delta}{2}-\cos\frac{\alpha+\beta-\gamma-\delta}{2}\Big)\Big(\cos\frac{\alpha-\beta-\gamma+\delta}{2}-\cos\frac{\alpha-\beta+\gamma-\delta}{2}\Big)$$
$$=8\sin\frac{\alpha+\beta+\gamma+\delta}{2}\sin\frac{\alpha-\delta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\delta}{2}\sin\frac{\alpha-\beta}{2}\sin\frac{\gamma-\delta}{2},$$

由是原式 $=8\sin\dfrac{\alpha+\beta+\gamma+\delta}{2}$.

(29) ∵ $\sin\alpha+\sin\beta+\sin\gamma=0,\ \cos\alpha+\cos\beta+\cos\gamma=0$

∴ 例題26.及27.之恆同式之左邊爲0

由是 $0=\cos3\alpha+\cos3\beta+\cos3\gamma-3\cos(\alpha+\beta+\gamma)$,

及 $0=\sin3\alpha+\sin3\beta+\sin3\gamma-3\sin(\alpha+\beta+\gamma)$.

設 $\alpha+\beta+\gamma=\pi$ 求自30.至69.各式之證

30. $\sin\alpha+\sin\beta+\sin\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$

31. $\cos\alpha+\cos\beta+\cos\gamma=1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$

32. $\cos\alpha+\cos\beta+\cos\gamma$

$=\sin\frac{\alpha}{2}\cos\frac{\beta-\gamma}{2}+\sin\frac{\beta}{2}\cos\frac{\gamma-\alpha}{2}+\sin\frac{\gamma}{2}\cos\frac{\alpha-\beta}{2}.$

33. $\sin\alpha+\sin\beta-\sin\gamma=4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$

34. $\cos\alpha+\cos\beta-\cos\gamma=4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\sin\frac{\gamma}{2}-1.$

35. $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma.$

36. $\cot\alpha+\cot\beta+\cot\gamma$

$=\cot\alpha\cot\beta\cot\gamma+\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma.$

37. $\sin2\alpha+\sin2\beta+\sin2\gamma=4\sin\alpha\sin\beta\sin\gamma.$

38. $\cos2\alpha+\cos2\beta+\cos2\gamma=-1-4\cos\alpha\cos\beta\cos\gamma.$

39. $\sin2\alpha+\sin2\beta+\sin2\gamma$

$=2(\sin\alpha+\sin\beta+\sin\gamma)(\cos\alpha+\cos\beta+\cos\gamma-1)$

40. $\sin\frac{\alpha}{2}+\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}=1+4\sin\frac{\beta+\gamma}{4}\sin\frac{\gamma+\alpha}{4}\sin\frac{\alpha+\beta}{4}.$

41. $\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}=4\cos\frac{\beta+\gamma}{4}\cos\frac{\gamma+\alpha}{4}\cos\frac{\alpha+\beta}{4}.$

42. $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}$

$=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}.$

43. $\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}.$

44. $\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\frac{4+4\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta\sin\frac{1}{2}\gamma}{\sin\alpha+\sin\beta+\sin\gamma}.$

45. $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}+\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\tan\frac{\gamma}{2}\tan\frac{\alpha}{2}=1.$

例題解自 30. 至 45.

(30) 於例題8.之右邊.令 $\alpha+\beta+\gamma=\pi$. 即得其證.

(31) 從例題13.得 $\cos\alpha+\cos\beta+\cos\gamma+\cos\pi$

$$=4\cos\frac{\pi-\alpha}{2}\cos\frac{\pi-\beta}{2}\cos\frac{\pi-\gamma}{2},$$

即 $\cos\alpha+\cos\beta+\cos\gamma-1=4\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta\sin\frac{1}{2}\gamma$.

(32) 原式之左邊 $=\frac{1}{2}(\cos\alpha+\cos\beta)+\frac{1}{2}(\cos\beta+\cos\gamma)+\frac{1}{2}(\cos\gamma+\cos\alpha)$

$=\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)+\cos\frac{1}{2}(\beta+\gamma)\cos\frac{1}{2}(\beta-\gamma)+\cos\frac{1}{2}(\gamma+\alpha)\cos\frac{1}{2}(\gamma-\alpha)$,

但 $\frac{1}{2}(\alpha+\beta)=\frac{1}{2}\pi-\frac{1}{2}\gamma,\ \frac{1}{2}(\beta+\gamma)=\frac{1}{2}\pi-\frac{1}{2}\alpha,\ \frac{1}{2}(\gamma+\alpha)=\frac{1}{2}\pi-\frac{1}{2}\beta$.

(33) 原式之右邊 $=2\sin\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)-\sin\{\pi-(\alpha+\beta)\}$

$=2\sin\frac{1}{2}(\alpha+\beta)\{\cos\frac{1}{2}(\alpha-\beta)-\cos\frac{1}{2}(\alpha+\beta)\}=4\sin\frac{1}{2}(\pi-\gamma)\sin\frac{1}{2}\alpha\sin\frac{1}{2}\beta$.

(34) 與前例同法.

(35) 於1.節之公式 $\tan(\alpha+\beta+\gamma)=\tan\pi=0$, 故右節之分子當爲0.即

$\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma=0$.

(36) 原式之左邊 $=\frac{\cos\alpha}{\sin\alpha}+\frac{\cos\beta}{\sin\beta}+\frac{\cos\gamma}{\sin\gamma}=\frac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}+\frac{\cos\gamma}{\sin\gamma}$

$$=\frac{\sin^2\gamma+\sin\alpha\sin\beta\cos\gamma}{\sin\alpha\sin\beta\sin\gamma}=\frac{1+\cos\gamma(\sin\alpha\sin\beta-\cos\gamma)}{\sin\alpha\sin\beta\sin\gamma}$$

$$=\frac{1+\cos\gamma\{\sin\alpha\sin\beta+\cos(\alpha+\beta)\}}{\sin\alpha\sin\beta\sin\gamma}=\frac{1+\cos\gamma\cos\alpha\cos\beta}{\sin\alpha\sin\beta\sin\gamma}.$$

(37) 於例題14.令 $x=\alpha,\ y=\beta,\ z=\gamma,\ \theta=0$, 則

$\sin2\alpha+\sin2\beta+\sin2\gamma=4\sin(\pi-\gamma)\sin(\pi-\alpha)\sin(\pi-\beta)$.

(38) 原式之左邊 $=2\cos(\alpha+\beta)\cos(\alpha-\beta)+\cos2\gamma=-2\cos\gamma\cos(\alpha-\beta)$

$+2\cos^2\gamma-1=-2\cos\gamma\{\cos(\alpha-\beta)+\cos(\alpha+\beta)\}-1=-4\cos\alpha\cos\beta\cos\gamma-1$.

(39) 由例題37.得

原式之左邊 $=32\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$.

$=2(\cos\alpha+\cos\beta+\cos\gamma-1)(\sin\alpha+\sin\beta+\sin\gamma)$ 〔例題30. 31.〕

(40) $2\sin\frac{\alpha+\beta}{4}\cos\frac{\alpha-\beta}{4}+\sin\frac{\pi-(\alpha+\beta)}{2}=2\sin\frac{\alpha+\beta}{4}\cos\frac{\alpha-\beta}{4}+1-2\sin^2\frac{\alpha+\beta}{4}$

$$=2\sin\frac{\alpha+\beta}{4}\left\{\cos\frac{\alpha-\beta}{4}-\cos\left(\frac{\pi}{2}-\frac{\alpha+\beta}{4}\right)\right\}+1=4\sin\frac{\alpha+\beta}{4}\sin\frac{\pi-\alpha}{4}\sin\frac{\pi-\beta}{4}+1.$$

(41) 與前例同樣. (42) 與36例題同樣.

(43) 於例題1之 α,β,γ. 代以 $\frac{1}{2}\alpha,\frac{1}{2}\beta,\frac{1}{2}\gamma$. 即得其證.

(44) 從例題42.容易導得.即用例題30.

(45) $\tan(\frac{1}{2}\alpha+\frac{1}{2}\beta+\frac{1}{2}\gamma)=\tan\frac{1}{2}\pi=\infty$ 由1.節之公式.即得其證.

46. $\cot\alpha\cot\beta+\cot\beta\cot\gamma+\cot\gamma\cot\alpha=1.$

47. $8(\sin\alpha+\sin\beta+\sin\gamma)\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

$=\sin2\alpha+\sin2\beta+\sin2\gamma.$

48. $(\tan\frac{1}{2}\alpha+\tan\frac{1}{2}\beta+\tan\frac{1}{2}\gamma)(\cot\frac{1}{2}\alpha+\cot\frac{1}{2}\beta+\cot\frac{1}{2}\gamma)$

$=1+\operatorname{cosec}\frac{1}{2}\alpha\operatorname{cosec}\frac{1}{2}\beta\operatorname{cosec}\frac{1}{2}\gamma.$

49. $\dfrac{\tan\alpha+\tan\beta+\tan\gamma}{(\sin\alpha+\sin\beta+\sin\gamma)^2}=\dfrac{\tan\frac{1}{2}\alpha\tan\frac{1}{2}\beta\tan\frac{1}{2}\gamma}{2\cos\alpha\cos\beta\cos\gamma}.$

50. $\sin\alpha(\cos\beta+\cos\gamma)+\sin\beta(\cos\gamma+\cos\alpha)+\sin\gamma(\cos\alpha+\cos\beta)$

$=\sin\alpha+\sin\beta+\sin\gamma.$

51. $\sin\alpha\sin(\alpha+2\gamma)+\sin\beta\sin(\beta+2\alpha)+\sin\gamma\sin(\gamma+2\beta)=0.$

52. $\sin(\alpha+2\beta-\gamma)+\sin(\beta+2\gamma-\alpha)+\sin(\gamma+2\alpha-\beta)$

$=4\cos\left(\alpha-\frac{\gamma}{2}\right)\cos\left(\beta-\frac{\alpha}{2}\right)\cos\left(\gamma-\frac{\beta}{2}\right).$

53. $\cos\left(\frac{3\alpha}{2}+\beta-2\gamma\right)+\cos\left(\frac{3\beta}{2}+\gamma-2\alpha\right)+\cos\left(\frac{3\gamma}{2}+\alpha-2\beta\right)$

$=4\cos\frac{5\alpha-2\beta-\gamma}{4}\cos\frac{5\beta-2\gamma-\alpha}{4}\cos\frac{5\gamma-2\alpha-\beta}{4}.$

54. $\sin\left(\alpha+\frac{\beta}{2}\right)+\sin\left(\beta+\frac{\gamma}{2}\right)+\sin\left(\gamma+\frac{\alpha}{2}\right)+1$

$=4\cos\frac{\alpha-\beta}{4}\cos\frac{\beta-\gamma}{4}\cos\frac{\gamma-\alpha}{4}.$

55. $\sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\gamma\sin\alpha$

$=2\left(\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right)$

56. $\cos\alpha\cos\beta+\cos\beta\cos\gamma+\cos\gamma\cos\alpha+1$

$=2\left(\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2}-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right).$

57. $\cos\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma+\cos\beta\operatorname{cosec}\gamma\operatorname{cosec}\alpha$

$+\cos\gamma\operatorname{cosec}\alpha\operatorname{cosec}\beta=2.$

例題解自 46. 至 57.

(46) 與前例同法. 而從例題 1. 卽可得其證.

(47) 原式之左邊 $=32\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$. (例題 30.)

$=4\sin\alpha\sin\beta\sin\gamma=\sin 2\alpha+\sin 2\beta+\sin 2\gamma$ (例題 37.)

(48) 原式之左邊 $=\left(\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}+\sec\frac{\alpha}{2}\sec\frac{\beta}{2}\sec\frac{\gamma}{2}\right)\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}$

(題例 42. 及 43.) $=1+\operatorname{cosec}\frac{\alpha}{2}\operatorname{cosec}\frac{\beta}{2}\operatorname{cosec}\frac{\gamma}{2}$.

(49) 例題 35. 之恆同式. 以 30. 之恆同式之平方約之. 卽得.

(50) 由例題 31. $\sin\alpha(\cos\beta+\cos\gamma)=\sin\alpha\left(1+4\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}-\cos\alpha\right)$

$=\sin\alpha+4\sin\alpha\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}-\frac{1}{2}\sin 2\alpha$,

故 $\sin\alpha(\cos\beta+\cos\gamma)+\sin\beta(\cos\gamma+\cos\alpha)+\sin\gamma(\cos\alpha+\cos\beta)$

$=\sin\alpha+\sin\beta+\sin\gamma+4(\sin\alpha+\sin\beta+\sin\gamma)\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

$-\frac{1}{2}(\sin 2\alpha+\sin 2\beta+\sin 2\gamma)$ 由例題 30. 及 37.

$=\sin\alpha+\sin\beta+\sin\gamma+16\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}$

$-2\sin\alpha\sin\beta\sin\gamma=\sin\alpha+\sin\beta+\sin\gamma+2\sin\alpha\sin\beta\sin\gamma$

$-2\sin\alpha\sin\beta\sin\gamma=\sin\alpha+\sin\beta+\sin\gamma$.

(51) $\sin\alpha\sin(\alpha+2\gamma)=\sin(\beta+\gamma)\sin(\beta-\gamma)=\sin^2\beta-\sin^2\gamma$ (例題四 1.)

故原式左邊 $=(\sin^2\beta-\sin^2\gamma)+(\sin^2\gamma-\sin^2\alpha)+(\sin^2\alpha-\sin^2\beta)=0$.

(52) 原式之左邊 $=\sin(2\gamma-\beta)+\sin(2\alpha-\gamma)+\sin(2\beta-\alpha)$,

但 $(2\gamma-\beta)+(2\alpha-\gamma)+(2\beta-\alpha)=\alpha+\beta+\gamma=\pi$, 故從例題 30. 卽得.

(53) $\left(\frac{3\alpha}{2}+\beta-2\gamma\right)+\left(\frac{3\beta}{2}+\gamma-2\alpha\right)+\left(\frac{3\gamma}{2}+\alpha-2\beta\right)=\frac{\pi}{2}$.

故從例題 41. 卽得.

(54) 左邊 $=\cos\frac{\gamma-\alpha}{2}+\cos\frac{\alpha-\beta}{2}+\cos\frac{\beta-\gamma}{2}+1$, 故從例題 16. 卽得.

(55) $\sin\alpha\sin\beta=\frac{1}{2}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}=\frac{1}{2}\{\cos(\alpha-\beta)+\cos\gamma\}$,

故原式左邊 $=\frac{1}{2}\{\cos(\alpha-\beta)+\cos(\beta-\gamma)+\cos(\gamma-\alpha)+\cos\gamma+\cos\beta+\cos\alpha\}$,

故從例題 16. 及 31. 卽得. (56) 與前例同法.

(57) 原式左邊 $=\dfrac{\cos\alpha\sin\alpha+\cos\beta\sin\beta+\cos\gamma\sin\gamma}{\sin\alpha\sin\beta\sin\gamma}$

$=\frac{1}{2}(\sin 2\alpha+\sin 2\beta+\sin 2\gamma)/\sin\alpha\sin\beta\sin\gamma$, 從例題 37. 卽得.

58. $(\sin\beta-\sin\gamma)\cot\frac{\alpha}{2}+(\sin\gamma-\sin\alpha)\cot\frac{\beta}{2}$

$+(\sin\alpha-\sin\beta)\cot\frac{\gamma}{2}=0.$

59. $\left(1-\sin\frac{\beta}{2}\right)\left(1-\sin\frac{\gamma}{2}\right)\cos\frac{\alpha}{2}+\left(1-\sin\frac{\gamma}{2}\right)\left(1-\sin\frac{\alpha}{2}\right)\cos\frac{\beta}{2}$

$+\left(1+\sin\frac{\alpha}{2}\right)\left(1-\sin\frac{\beta}{2}\right)\cos\frac{\gamma}{2}=\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$

60. $(\sin\alpha+\sin\beta)(\cos\beta+\cos\gamma)(\cos\gamma+\cos\alpha)+(\sin\beta+\sin\gamma)$
$(\cos\gamma+\cos\alpha)(\cos\alpha+\cos\beta)+(\sin\gamma+\sin\alpha)(\cos\alpha+\cos\beta)(\cos\beta+\cos\gamma)$

$=(\sin\alpha+\sin\beta)(\sin\beta+\sin\gamma)(\sin\gamma+\sin\alpha).$

61. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2\cos\alpha\cos\beta\cos\gamma+2.$

62. $\sin^2\alpha+\sin^2\beta-\sin^2\gamma=2\sin\alpha\sin\beta\cos\gamma.$

63. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1-2\cos\alpha\cos\beta\cos\gamma.$

64. $\sin^2\frac{\alpha}{2}+\sin^2\frac{\beta}{2}+\sin^2\frac{\gamma}{2}=1-2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$

65. $\cos^2\frac{\alpha}{2}+\cos^2\frac{\beta}{2}+\cos^2\frac{\gamma}{2}=2+2\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}.$

66. $\cos\alpha\sin\beta\sin\gamma+\cos\beta\sin\gamma\sin\alpha+\cos\gamma\sin\alpha\sin\beta$

$=1+\cos\alpha\cos\beta\cos\gamma.$

67. $\frac{\tan\alpha}{\tan\beta}+\frac{\tan\beta}{\tan\gamma}+\frac{\tan\gamma}{\tan\alpha}+\frac{\tan\alpha}{\tan\gamma}+\frac{\tan\beta}{\tan\alpha}+\frac{\tan\gamma}{\tan\beta}$

$=\sec\alpha\sec\beta\sec\gamma-2.$

68. $\sin3\alpha+\sin3\beta+\sin3\gamma=-4\cos\frac{3\alpha}{2}\cos\frac{3\beta}{2}\cos\frac{3\gamma}{2}.$

例題解自 58 至 68.

(58) $(\sin\beta-\sin\gamma)\cot\frac{\alpha}{2}=2\cos\frac{\beta+\gamma}{2}\sin\frac{\beta-\gamma}{2}\cot\left(\frac{\pi}{2}-\frac{\beta+\gamma}{2}\right)$

$=2\sin\frac{\beta+\gamma}{2}\sin\frac{\beta-\gamma}{2}=\cos\gamma-\cos\beta.$ 他之二項.亦可同樣得

$\cos\alpha-\cos\gamma$, $\cos\beta-\cos\alpha$,

(59) $\left(1-\sin\frac{\beta}{2}\right)\left(1-\sin\frac{\gamma}{2}\right)\cos\frac{\alpha}{2}=\cos\frac{\alpha}{2}-\cos\frac{\alpha}{2}\left(\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\right)$

$+\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\alpha}{2}=\cos\frac{\alpha}{2}-\frac{1}{2}\left(\sin\frac{\alpha+\beta}{2}-\sin\frac{\alpha-\beta}{2}+\sin\frac{\alpha+\gamma}{2}+\sin\frac{\gamma-\alpha}{2}\right)$

$+\frac{1}{4}(\sin\beta+\sin\gamma-\sin\alpha)$, (例題33.)

$=\frac{1}{2}\left(2\cos\frac{\alpha}{2}-\cos\frac{\beta}{2}-\cos\frac{\gamma}{2}\right)+\frac{1}{2}\left(\sin\frac{\alpha-\beta}{2}-\sin\frac{\gamma-\alpha}{2}\right)+\frac{1}{4}(\sin\beta+\sin\gamma-\sin\alpha)$,

故原式$=\frac{1}{4}(\sin\alpha+\sin\beta+\sin\gamma)$故由例題30.即得其證.

(60) $(\sin\alpha+\sin\beta)(\cos\beta+\cos\gamma)(\cos\gamma+\cos\alpha)=8\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$\cos\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma+\alpha}{2}\cos\frac{\gamma-\alpha}{2}=8\cos\frac{\alpha-\beta}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\cos\frac{\gamma}{2}$

$=2\cos\frac{\alpha-\beta}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}(\sin\alpha+\sin\beta-\sin\gamma)$, 〔例題33.〕

故原式$=2\cos\frac{\alpha-\beta}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}(\sin\alpha+\sin\beta-\sin\gamma)$

$=8\cos\frac{\alpha-\beta}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}$, 〔例題30.〕

$=8\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}\sin\frac{\beta+\gamma}{2}\cos\frac{\beta-\gamma}{2}\sin\frac{\gamma+\alpha}{2}\cos\frac{\gamma-\alpha}{2}$

$=(\sin\alpha+\sin\beta)(\sin\beta+\sin\gamma)(\sin\gamma+\sin\alpha)$.

(61) 原式$=\frac{1}{2}\{3-(\cos2\alpha+\cos2\beta+\cos2\gamma)\}=\frac{1}{2}(3+1+4\cos\alpha\cos\beta\cos\gamma)$ (見例題38)

(62) 原式$=\sin^2\alpha+\sin^2\beta+\sin^2\gamma-2\sin^2\gamma$ 故由前例

$=2\cos\alpha\cos\beta\cos\gamma+2(1-\sin^2\gamma)=2\cos\gamma\{\cos\alpha\cos\beta-\cos(\alpha+\beta)\}$.

(63) 從例題61.即可導得其結果.

(64) 於例題31.可變$\cos\alpha$爲$1-2\sin^2\frac{\alpha}{2}$. (65) 亦與上同樣.

(66) $\cos\alpha\sin\beta\sin\gamma=\cos\alpha\{\cos\beta\cos\gamma-\cos(\beta+\gamma)\}=\cos\alpha\cos\beta\cos\gamma+\cos^2\alpha$

可用例題63.以代原式之左邊.

(67) $\frac{\tan\alpha}{\tan\beta}+\frac{\tan\gamma}{\tan\beta}=\frac{\sin(\alpha+\gamma)}{\cos\alpha\cos\gamma\tan\beta}=\frac{\cos\beta}{\cos\alpha\cos\gamma}$ 故

原式$=\frac{\cos^2\beta+\cos^2\gamma+\cos^2\alpha}{\cos\alpha\cos\beta\cos\gamma}=\frac{1-2\cos\alpha\cos\beta\cos\gamma}{\cos\alpha\cos\beta\cos\gamma}$, 〔例題63.〕

(68) 原式$=\sin3\alpha+2\sin\frac{3\beta+3\gamma}{2}\cos\frac{3\beta-3\gamma}{2}$

$=\sin3\alpha-2\cos\left(\frac{3\pi}{2}-\frac{3\beta+3\gamma}{2}\right)\cos\frac{3\beta-3\gamma}{2}$

$=2\sin\frac{3\alpha}{2}\cos\frac{3\alpha}{2}-2\cos\frac{3\alpha}{3}\cos\frac{3\beta-3\gamma}{2}=2\cos\frac{3\alpha}{2}\left(\sin\frac{3\alpha}{2}-\cos\frac{3\beta-3\gamma}{2}\right)$

$=-2\cos\frac{3\alpha}{2}\left(\cos\frac{3\beta+3\gamma}{2}+\cos\frac{3\beta-3\gamma}{2}\right)=-4\cos\frac{3\alpha}{2}\cos\frac{3\beta}{2}\cos\frac{3\gamma}{2}$.

69. $\sin^3\alpha+\sin^3\beta+\sin^3\gamma$

$$=3\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}+\cos\frac{3}{2}\alpha\cos\frac{3}{2}\beta\cos\frac{3}{2}\gamma.$$

70. $\cos3\alpha\sin(\beta-\gamma)+\cos3\beta\sin(\gamma-\alpha)+\cos3\gamma\sin(\alpha-\beta)$

$$=-4\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta).$$

71. $\cos^3\alpha\sin(\beta-\gamma)+\cos^3\beta\sin(\gamma-\alpha)+\cos^3\gamma\sin(\alpha-\beta)$

$$=-\sin(\beta-\gamma)\sin(\gamma-\alpha)\sin(\alpha-\beta).$$

72. $\sin4\alpha+\sin4\beta+\sin4\gamma=-4\sin2\alpha\sin2\beta\sin2\gamma.$

73. $\cos4\alpha+\cos4\beta+\cos4\gamma=4\cos2\alpha\cos2\beta\cos2\gamma-1.$

74. $\sin^2 2\alpha+\sin^2 2\beta+\sin^2 2\gamma=2-2\cos2\alpha\cos2\beta\cos2\gamma.$

75. $\sin6\alpha+\sin6\beta+\sin6\gamma=4\sin3\alpha\sin3\beta\sin3\gamma.$

76. $\sin^4\alpha+\sin^4\beta+\sin^4\gamma$

$$=\tfrac{1}{2}(3+4\cos\alpha\cos\beta\cos\gamma+\cos2\alpha\cos2\beta\cos2\gamma).$$

77. $\cos^4\alpha+\cos^4\beta+\cos^4\gamma$

$$=\tfrac{1}{2}(1-4\cos\alpha\cos\beta\cos\gamma+\cos2\alpha\cos2\beta\cos2\gamma).$$

78. $\sin n\alpha+\sin n\beta+\sin n\gamma$

$$=4\sin\frac{n\pi}{2}\cos\frac{n\alpha}{2}\cos\frac{n\beta}{2}\cos\frac{n\gamma}{2}.$$

但 n 爲 $4m+1$ 或 $4m+3$. 則爲正整數.

79. $\sin n\alpha+\sin n\beta+\sin n\gamma$

$$=-4\cos\frac{n\pi}{2}\sin\frac{n\alpha}{2}\sin\frac{n\beta}{2}\sin\frac{n\gamma}{2}.$$

但 n 爲 $4m$ 或 $4m+2$. 則爲正整數.

80. $\cot m\alpha\cot m\beta+\cot m\beta\cot m\gamma+\cot m\gamma\cot m\alpha=1.$

例題解自 69. 至 80.

(69) 原式之左邊$=\frac{1}{4}\{3(\sin\alpha+\sin\beta+\sin\gamma)-(\sin3\alpha+\sin3\beta+\sin3\gamma)\}$

$=\frac{1}{4}\left\{12\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}+4\cos\frac{3}{2}\alpha\cos\frac{3}{2}\beta\cos\frac{3}{2}\gamma\right\}$

(70) 於例題18. 令$\alpha+\beta+\gamma=\pi$, 則可直得其證.

(71) $\cos^3\alpha\sin(\beta-\gamma)=\frac{1}{4}(\cos3\alpha+3\cos\alpha)\sin(\beta-\gamma)=\frac{1}{4}\cos3\alpha\sin(\beta-\gamma)$

$+\frac{3}{4}\cos\alpha\sin(\beta-\gamma)=\frac{1}{4}\cos3\alpha\sin(\beta-\gamma)-\frac{3}{4}\cos(\beta+\gamma)\sin(\beta-\gamma)$

$=\frac{1}{4}\cos3\alpha\sin(\beta-\gamma)-\frac{3}{8}(\sin2\beta-\sin2\gamma)$.

故原式$=\frac{1}{4}\{\cos3\alpha\sin(\beta-\gamma)+\cos3\beta\sin(\gamma-\alpha)+\cos3\gamma\sin(\alpha-\beta)\}$.

(72) 原式$=2\sin(2\alpha+2\beta)\cos(2\alpha-2\beta)+\sin4\gamma=-2\sin2\gamma\cos(2\alpha-2\beta)$

$+2\sin2\gamma\cos2\gamma=-2\sin2\gamma\{\cos(2\alpha-2\beta)-\cos(2\alpha+2\beta)\}=-4\sin2\gamma\sin2\alpha\sin2\beta$.

(73) 與前例同樣.

(74) 從前例 $1-2\sin^22\alpha+1-2\sin^22\beta+1-2\sin^22\gamma$

$=4\cos2\alpha\cos2\beta\cos2\gamma-1$.

(75) 原式$=2\sin(3\alpha+3\beta)\cos(3\alpha-3\beta)+\sin6\gamma=2\sin(3\pi-3\gamma)\cos(3\alpha-3\beta)$

$+2\sin3\gamma\cos3\gamma=2\sin3\gamma\{\cos(3\alpha-3\beta)-\cos(3\alpha+3\beta)\}=4\sin3\gamma\sin3\alpha\sin3\beta$.

(76) 從第四編5.節之公式 $\sin^4\alpha=\sin^3\alpha\sin\alpha=\frac{1}{4}(3\sin\alpha-\sin3\alpha)\sin\alpha=\frac{1}{4}\{3\sin^2\alpha$

$-\frac{1}{2}(\cos2\alpha-\cos4\alpha)\}=\frac{1}{8}\{6\sin^2\alpha-(1-2\sin^2\alpha)+\cos4\alpha\}=\frac{1}{8}(8\sin^2\alpha+\cos4\alpha-1)$.

故 $\sin^4\alpha+\sin^4\beta+\sin^4\gamma=\sin^2\alpha+\sin^2\beta+\sin^2\gamma+\frac{1}{8}(\cos4\alpha+\cos4\beta+\cos4\gamma)-\frac{3}{8}$,

以下用例題61, 73.

(77) 與前例同樣.

(78) 原式$=2\sin\frac{n\alpha+n\beta}{2}\cos\frac{n\alpha-n\beta}{2}+\sin n\gamma=2\sin\frac{n\pi-n\gamma}{2}\cos\frac{n\alpha-n\beta}{2}+\sin n\gamma$.

$n=4m+1$ 或 $4m+3$, 則

$\sin\frac{n\pi-n\gamma}{2}=\sin\left\{2m\pi\pm\left(\frac{\pi}{2}\mp\frac{n\gamma}{2}\right)\right\}=\pm\sin\left(\frac{\pi}{2}\mp\frac{n\gamma}{2}\right)=\pm\cos\frac{n\gamma}{2}$,

及 $\sin\frac{n\pi}{2}=\pm1$. 由是

原式$=\pm2\cos\frac{n\gamma}{2}\left(\cos\frac{n\alpha-n\beta}{2}+\cos\frac{n\alpha+n\beta}{2}\right)=2\sin\frac{n\pi}{2}\cos\frac{n\gamma}{2}\cos\frac{n\alpha}{2}\cos\frac{n\beta}{2}$

(79) 原式$=2\sin\frac{n\pi-n\gamma}{2}\cos\frac{n\alpha-n\beta}{2}+\sin n\gamma$, $n=4m$, 或 $4m+2$, 則

$\sin\frac{n\pi-n\gamma}{2}=\sin\left(2m\pi-\frac{n\gamma}{2}\right)=-\sin\frac{n\gamma}{2}$ 或 $\sin\left\{(2m+1)\pi-\frac{n\gamma}{2}\right\}=\sin\frac{n\gamma}{2}$.

及 $\cos\frac{n\pi}{2}=\cos2m\pi=1$ 或 $\cos(2m+1)\pi=-1$, 由是

原式$=\mp2\sin\frac{n\gamma}{2}\left(\cos\frac{n\alpha-n\beta}{2}-\cos\frac{n\alpha+n\beta}{2}\right)=-4\cos\frac{n\pi}{2}\sin\frac{n\gamma}{2}\sin\frac{n\alpha}{2}\sin\frac{n\beta}{2}$.

(80) $\cot m\pi=\infty$, 故從例題1. 容易得其證.

81. $\cot\alpha+\dfrac{\sin\alpha}{\sin\beta\sin\gamma}=\cot\gamma+\dfrac{\sin\gamma}{\sin\alpha\sin\beta}.$

82. $\dfrac{\sin\alpha-\sin\beta\cos\gamma}{\cos\beta}=\dfrac{\sin\beta-\sin\alpha\cos\gamma}{\cos\alpha}.$

83. $\tan\dfrac{3}{2}\alpha=\dfrac{\sin3\beta-\sin3\gamma}{\cos3\gamma-\cos3\beta}.$

84. $\cos\dfrac{\alpha}{2}=\sqrt{\dfrac{(\sin\alpha+\sin\beta+\sin\gamma)(\sin\beta+\sin\gamma-\sin\alpha)}{4\sin\beta\sin\gamma}}.$

85. $\tan^2\dfrac{\alpha}{2}=\dfrac{(\sin\alpha+\sin\gamma-\sin\beta)(\sin\alpha+\sin\beta-\sin\gamma)}{(\sin\alpha+\sin\beta+\sin\gamma)(\sin\beta+\sin\gamma-\sin\alpha)}$

86. $\sin^2\alpha\sin2\gamma+\sin^2\gamma\sin2\alpha=\sin^2\beta\sin2\alpha+\sin^2\alpha\sin2\beta.$

87. $\sin\alpha\sin\beta+\cos^2\left(\alpha+\dfrac{\gamma}{2}\right)=\cos^2\dfrac{\gamma}{2}.$

88. $\tan\dfrac{\alpha}{2}+\cos\dfrac{\alpha}{2}\sec\dfrac{\beta}{2}\sec\dfrac{\gamma}{2}$

$=\tan\dfrac{\beta}{2}+\cos\dfrac{\beta}{2}\sec\dfrac{\gamma}{2}\sec\dfrac{\alpha}{2}=\tan\dfrac{\gamma}{2}+\cos\dfrac{\gamma}{2}\sec\dfrac{\alpha}{2}\sec\dfrac{\beta}{2}.$

89. $(\sin\alpha+\sin\beta+\sin\gamma)^2+(\cos\alpha+\cos\beta+\cos\gamma-1)^2+4(\cos\alpha+\cos\beta$

$+\cos\gamma-1)=4(\sin\beta\sin\gamma+\sin\gamma\sin\alpha+\sin\alpha\sin\beta).$

90. $\sec^2\beta+\sec^2\gamma=2\sec\beta\sec\gamma\cos\alpha$

$=\sec\beta\sec\gamma\sin\alpha(\tan\beta+\tan\gamma).$

例題解自 81. 至 90.

(81) $\dfrac{\cos\alpha}{\sin\alpha}+\dfrac{\sin\alpha}{\sin\beta\sin\gamma}=\dfrac{\cos\alpha\sin\beta\sin\gamma+\sin^2\alpha}{\sin\alpha\sin\beta\sin\gamma}$

$=\dfrac{\{\sin(\alpha+\beta)-\sin\alpha\cos\beta\}\sin\gamma+\sin^2\alpha}{\sin\alpha\sin\beta\sin\gamma}=\dfrac{\sin^2\gamma+\sin\alpha(\sin\alpha-\cos\beta\sin\gamma)}{\sin\alpha\sin\beta\sin\gamma}$

$=\dfrac{\sin^2\gamma+\sin\alpha\{\sin(\beta+\gamma)-\cos\beta\sin\gamma\}}{\sin\alpha\sin\beta\sin\gamma}=\dfrac{\sin^2\gamma+\sin\alpha\sin\beta\cos\gamma}{\sin\alpha\sin\beta\sin\gamma}$

$=\dfrac{\sin\gamma}{\sin\alpha\sin\beta}+\cot\gamma.$

(82) $\dfrac{\sin(\beta+\gamma)-\sin\beta\sin\gamma}{\cos\beta}=\dfrac{\cos\beta\sin\gamma}{\cos\beta}=\dfrac{\cos\alpha\sin\gamma}{\cos\alpha}$

$=\{\sin(\gamma+\alpha)-\sin\alpha\cos\gamma\}/\cos\alpha=(\sin\beta-\sin\alpha\cos\gamma)/\cos\alpha.$

(83) $\tan\dfrac{3\alpha}{2}=\dfrac{\sin\frac{3\alpha}{2}}{\cos\frac{3\alpha}{2}}=\dfrac{\sin\left(\frac{3\pi}{2}-\frac{3\beta+3\gamma}{2}\right)}{\cos\left(\frac{3\pi}{2}-\frac{3\beta+3\gamma}{2}\right)}=\dfrac{-\cos\frac{3\beta+3\gamma}{2}}{-\sin\frac{3\beta+3\gamma}{2}}$

$=2\cos\dfrac{2\beta+3\gamma}{2}\sin\dfrac{3\beta-3\gamma}{2}\Big/2\sin\dfrac{3\beta+3\gamma}{2}\sin\dfrac{3\beta-3\gamma}{2}=\dfrac{\sin3\beta-\sin3\gamma}{\cos3\gamma-\cos3\beta}.$

(84) $\cos\dfrac{\alpha}{2}=(\sin\beta+\sin\gamma-\sin\alpha)/4\sin\dfrac{\beta}{2}\sin\dfrac{\gamma}{2}$, (例題 33) $\cos^2\dfrac{\alpha}{2}$

$=\dfrac{4\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}(\sin\beta+\sin\gamma-\sin\alpha)}{16\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}=\dfrac{(\sin\alpha+\sin\beta+\sin\gamma)(\sin\beta+\sin\gamma-\sin\alpha)}{4\sin\beta\sin\gamma}$

(85) 用例題 30. 及 33. 即可得其證.

(86) $\sin^2\alpha\sin2\gamma+\sin^2\gamma\sin2\alpha=2\sin\alpha\sin\gamma(\sin\alpha\cos\gamma+\sin\gamma\cos\alpha)$

$=2\sin\alpha\sin\gamma\sin(\gamma+\alpha)=2\sin\alpha\sin\gamma\sin\beta=2\sin\alpha\sin\beta\sin(\alpha+\beta)$

$=2\sin\alpha\sin\beta(\sin\alpha\cos\beta+\cos\alpha\sin\beta)=\sin^2\alpha\sin2\beta+\sin^2\beta\sin2\alpha.$

(87) $\frac{1}{2}\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}+\frac{1}{2}\{1+\cos(2\alpha+\gamma)\}$

$=\frac{1}{2}\{\cos(\alpha-\beta)+\cos\gamma+1+\cos(\alpha+\pi-\beta)\}$

$=\frac{1}{2}\{\cos(\alpha-\beta)+2\cos^2\dfrac{\gamma}{2}-\cos(\alpha-\beta)\}=\cos^2\dfrac{\gamma}{2}.$

(88) $\tan\dfrac{\alpha}{2}+\cos\dfrac{\alpha}{2}\sec\dfrac{\beta}{2}\sec\dfrac{\gamma}{2}=\dfrac{\sin\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}+\cos^2\frac{\alpha}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$

$=\dfrac{\sin\frac{\alpha}{2}\left\{\cos\frac{\beta+\gamma}{2}-\sin\frac{\beta}{2}\sin\frac{\gamma}{2}\right\}+\cos^2\frac{\alpha}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}=\dfrac{1-\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}$

他之兩節. 亦等於此分數. 故得其證.

(89) 解原式之括弧. 集其一個相應項而簡單之. 則

原式 $=2\{\cos(\alpha-\beta)+\cos\gamma+\cos(\beta-\gamma)+\cos\alpha+\cos(\gamma-\alpha)+\cos\beta\}$

$=2\{\cos(\alpha-\beta)-\cos(\alpha+\beta)+\cos(\beta-\gamma)-\cos(\beta+\gamma)+\cos(\gamma-\alpha)-\cos(\gamma+\alpha)\}$

$=4\{\sin\alpha\sin\beta+\sin\beta\sin\gamma+\sin\gamma\sin\alpha\}.$

(90) 原式之左邊 $=\dfrac{\cos^2\gamma+\cos^2\beta+2\cos\beta\cos\gamma\cos\alpha}{\cos^2\beta\cos^2\gamma}$, 故

原子 $=\cos^2\gamma+\cos\beta\{2\cos\gamma\cos\alpha-\cos(\gamma+\alpha)\}=\cos^2\gamma-\cos(\gamma+\alpha)\cos(\gamma-\alpha)$

$=\cos^2\gamma-(\cos^2\gamma-\sin^2\alpha)=\sin^2\alpha=\sin\alpha\sin(\beta+\gamma)$, 由是

原式 $=\sin\alpha\sin(\beta+\gamma)/\cos^2\beta\cos^2\gamma=\sin\alpha(\tan\beta+\tan\gamma)/\cos\beta\cos\gamma.$

91. 設 $\cos\alpha=\cos\beta\cos\gamma$ 則 $\cot\beta\cot\gamma=\frac{1}{2}$.

92. 設 $\sin\alpha=\cos\beta\cos\gamma$ 則 $\tan\beta+\tan\gamma=1$.

93. 設 $\sin2\alpha:\sin2\beta:\sin2\gamma::5:4:3$ 則

$\tan\alpha=\pm1,\ \tan\beta=\pm2,\ \tan\gamma=\pm3.$

94. 設 $\sin\left(\alpha+\frac{\gamma}{2}\right)=n\sin\frac{\gamma}{2}$ 則 $\tan\frac{\alpha}{2}\tan\frac{\beta}{2}=\frac{n-1}{n+1}$.

95. 設 $\sin\alpha:\sin\beta:\sin\gamma::x:y:z$ 則

$$(x-y)\cot\frac{\gamma}{2}+(y-z)\cot\frac{\alpha}{2}+(z-x)\cot\frac{\beta}{2}=0.$$

96. $\sin^3\omega=\sin(\alpha-\omega)\sin(\beta-\omega)\sin(\gamma-\omega)$ 則

$\cot\omega=\cot\alpha+\cot\beta+\cot\gamma.$

97. $x\sin\alpha+y\sin\beta+z\sin\gamma=0$ 則

$$(y+z\cos\alpha)(z+x\cos\beta)(x+y\cos\gamma)$$
$$+(y\cos\alpha+z)(z\cos\beta+x)(x\cos\gamma+y)=0$$

98. 設 $\sin\alpha,\sin\beta,\sin\gamma$ 爲等差級數,則

$\tan\frac{\alpha}{2}\tan\frac{\gamma}{2}=\frac{1}{3}$.

99. 設 $\tan\alpha,\tan\beta,\tan\gamma$ 爲等差級數,則

$$\cos(\beta+\gamma-\alpha)=\frac{4+5\cos2\gamma}{5+4\cos2\gamma}.$$

例題解自 91. 至 99.

(91) 從 $-\cos(\beta+\gamma)=\cos\beta\cos\gamma$ 卽得其證.

(92) 從 $\sin(\beta+\gamma)=\cos\beta\cos\gamma$ 卽得其證.

(93) $\because\ \sin2\beta=\frac{4}{5}\sin2\alpha,\ \sin2\gamma=\frac{3}{5}\sin2\alpha,$

$\therefore\ \sin2\beta+\sin2\gamma=\frac{7}{5}\sin2\alpha$, 卽 $2\sin(\beta+\gamma)\cos(\beta-\gamma)=\frac{14}{5}\sin\alpha\cos\alpha,$

故 $\cos(\beta-\gamma)=\frac{7}{5}\cos\alpha$. 又從 $\sin2\beta-\sin2\gamma=\frac{1}{5}\sin2\alpha$, 得

$\sin(\beta-\gamma)=-\frac{1}{5}\sin\alpha$. 故 $\frac{49}{25}\cos^2\alpha+\frac{1}{25}\sin^2\alpha=\cos^2(\beta-\gamma)+\sin^2(\beta-\gamma)=1$,

$49+\tan^2\alpha=25(1+\tan^2\alpha)$, 故 $\tan^2\alpha=1$.

(94) $\sin\left(\alpha+\frac{\pi-\alpha-\beta}{2}\right)=n\sin\left(\frac{\pi-\alpha-\beta}{2}\right)$, $\cos(\alpha-b)=n\cos(\alpha+\beta)$.

卽 $1+\tan\alpha\tan\beta=n(1-\tan\alpha\tan\beta)$, 故 $\tan\alpha\tan\beta=\frac{n-1}{n+1}$.

(95) 令 $\sin\alpha/x=\sin\beta/y=\sin\gamma/z=K$, 則

$$K=\frac{\sin\alpha-\sin\beta}{x-y}=\frac{2\sin\frac{\gamma}{2}\sin\frac{\alpha-\beta}{2}}{x-y}=\frac{2\cos\frac{\gamma}{2}\sin\frac{\alpha-\beta}{2}}{(x-y)\cot\frac{\gamma}{4}}$$

$$=\frac{\sin\frac{1}{2}(\gamma+\alpha-\beta)-\sin\frac{1}{2}(\gamma-\alpha+\beta)}{(x-y)\cot\frac{\gamma}{2}}=\frac{\cos\beta-\cos\alpha}{(x-y)\cot\frac{\gamma}{2}},$$

故 $(x-y)\cot\frac{\gamma}{2}=\frac{\cot\beta-\cot\alpha}{K}$ 其他項亦同樣.

(96) $\sin^3\omega=\frac{1}{2}\sin(\alpha-\omega)\{\cos(\beta-\gamma)-\cos(\beta+\gamma-2\omega)\}$

$=\frac{1}{4}\{\sin(\alpha+\beta-\gamma-\omega)+\sin(\alpha-\beta+\gamma-\omega)-\sin(\alpha+\beta+\gamma-3\omega)+\sin(\beta+\gamma-\alpha-\omega)\}$

卽 $4\sin^3\omega=\sin(2\gamma+\omega)+\sin(2\beta+\omega)-\sin3\omega+\sin(2\alpha+\omega)$,

故 $3\sin\omega=\sin\omega(\cos2\alpha+\cos2\beta+\cos2\gamma)+\cos\omega(\sin2\alpha+\sin2\beta+\sin2\gamma)$,

卽 $3\sin\omega=-\sin\omega(1+4\cos\alpha\cos\beta\cos\gamma)+4\cos\omega\sin\alpha\sin\beta\sin\gamma$,

故 $\cot\omega=\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma+\cot\alpha\cot\beta\cot\gamma$. 以下見例題36.

(97) 以 $z=-\frac{x\sin\alpha+y\sin\beta}{\sin\gamma}$ 代原式之第一項而變化之, 則

$(y+z\cos\alpha)(z+x\cos\beta)(x+y\cos\gamma)=-\frac{\sin\alpha\sin\beta}{\sin^2\gamma}(y\cos\beta-x\cos\alpha)(y+x\cos\gamma)(x+y\cos\gamma)$,

同樣 $(y\cos\alpha+z)(z\cos\beta+x)(x\cos\gamma+y)$

$=\frac{\sin\alpha\sin\beta}{\sin^2\gamma}(y\cos\gamma+x)(y\cos\beta-x\cos\alpha)(x\cos\gamma+y)$. 故如題言.

(98) 從 $\sin\alpha+\sin\gamma=2\sin\beta=2\sin(\alpha+\gamma)$, 得 $\cos\frac{\alpha-\gamma}{2}=2\cos\frac{\alpha+\gamma}{2}$,

兩式以 $\cos\frac{\alpha}{2}\cos\frac{\gamma}{2}$ 除之卽得.

(99) 從 $\tan\alpha+\tan\gamma=2\tan\beta$, 得 $\frac{\sin(\gamma+\alpha)}{\cos\alpha\cos\gamma}=\frac{2\sin(\gamma+\alpha)}{\cos\beta}$, 故

$\cos\beta=2\cos\alpha\cos\gamma$, 又令 $\cos(\beta+\gamma-\alpha)=-\cos2\alpha=x$, 則

$x=\cos\beta\cos(\gamma-\alpha)-\sin\beta\sin(\gamma-\alpha)=2\cos\alpha\cos\gamma\cos(\gamma-\alpha)-(\cos^2\alpha-\cos^2\gamma)$,

變化此方程式, 則

$x^2(5+4\cos2\gamma)-x(4+10\cos2\gamma+5\cos^22\gamma)+\cos2\gamma(4+5\cos2\gamma)=0$

$(x-\cos2\gamma)\{x(5+4\cos2\gamma)-(4+5\cos2\gamma)\}=0$.

設 $\alpha+\beta+\gamma=2\pi$, 試證次列各式.

100. $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2-2\cos\alpha\cos\beta\cos\gamma.$

101. $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1+2\cos\alpha\cos\beta\cos\gamma.$

102. $\sin\alpha(1+2\cos\beta)+\sin\beta(1+2\cos\gamma)+\sin\gamma(1+2\cos\alpha)$

$$=-4\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}.$$

103. 設 $\cos\alpha=\frac{(d-a)(b-c)}{(d+a)(b+c)}$, $\cos\beta=\frac{(d-b)(c-a)}{(d+b)(c+a)}$,

$\cos\gamma=\frac{(d-c)(a-b)}{(d+c)(a+b)}$, 則

$\tan\frac{1}{2}\alpha+\tan\frac{1}{2}\beta+\tan\frac{1}{2}\gamma=\pm1.$

例題解自 100. 至 103.

(100) 與例題 74. 同. (101) 由前例可得其證.

(102) 原式 $=\sin\alpha+\sin\beta+\sin\gamma+\sin(\alpha+\beta)+\sin(\alpha-\beta)+\sin(\beta+\gamma)+\sin(\beta-\gamma)$
$+\sin(\gamma+\alpha)+\sin(\gamma-\alpha)=\sin\alpha+\sin\beta+\sin\gamma-\sin\gamma+\sin(\alpha-\beta)-\sin\alpha$
$+\sin(\beta-\gamma)-\sin\beta+\sin(\gamma-\alpha)=\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)$

$$=\sin(\alpha-\beta)+2\sin\frac{\beta-\alpha}{2}\cos\frac{\beta-2\gamma+\alpha}{2}=3\sin\frac{\alpha-\beta}{2}\left(\cos\frac{\alpha-\beta}{2}-\cos\frac{\beta-2\gamma+\alpha}{2}\right)$$

$$=4\sin\frac{\alpha-\beta}{2}\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}=-4\sin\frac{\alpha-\beta}{2}\sin\frac{\gamma-\alpha}{2}\sin\frac{\beta-\gamma}{2}.$$

(103) $2\cos^2\frac{\alpha}{2}=\frac{(d-c)(b-c)}{(d+a)(b+c)}+1=\frac{2(db+ac)}{(d+a)(b+c)}$, 故

$$\tan^2\frac{\alpha}{2}=\sec^2\frac{\alpha}{2}-1=\frac{1}{\cos^2\frac{\alpha}{2}}-1=\frac{(d+a)(b+c)}{db+ac}-1=\frac{ab+cd}{db+ac},$$

又 $\frac{\alpha}{2}+\frac{\beta}{2}+\frac{\gamma}{2}=\pi$, 故由例題 35.

$$\tan\frac{\alpha}{2}+\tan\frac{\beta}{2}+\tan\frac{\gamma}{2}=\tan\frac{\alpha}{2}\tan\frac{\beta}{2}\tan\frac{\gamma}{2}$$

$$\pm\sqrt{\frac{ab+cd}{db+ac}}\times\pm\sqrt{\frac{bc+ad}{dc+ab}}\times\pm\sqrt{\frac{ac+bd}{da+bc}}=\pm1.$$

設 $\alpha+\beta+\gamma+\delta=2\pi$, 試證次列各式.

104. $\sin\alpha+\sin\beta+\sin\gamma+\sin\delta=4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$.

105. $\cos\alpha+\cos\beta+\cos\gamma+\cos\delta=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$.

106. $\sin\alpha-\sin\beta+\sin\gamma-\sin\delta=4\cos\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$.

107. $\cos\alpha-\cos\beta+\cos\gamma-\cos\delta=4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\cos\frac{\gamma+\alpha}{2}$.

108. $\sin(\alpha+\gamma)\sin(\alpha+\delta)=\sin(\beta+\gamma)\sin(\beta+\delta)$.

109. $\cos\frac{\alpha}{2}\cos\frac{\delta}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}-\cos\frac{\beta}{2}\cos\frac{\gamma}{2}\sin\frac{\alpha}{2}\sin\frac{\delta}{2}$

$=\sin\frac{\alpha+\beta}{2}\sin\frac{\gamma+\alpha}{2}\cos\frac{\alpha+\delta}{4}$.

110. $\cos\frac{\alpha}{2}\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\cos\frac{\delta}{2}=\sin\frac{\alpha}{2}\sin\frac{\beta}{2}+\sin\frac{\gamma}{2}\sin\frac{\delta}{2}$.

例題解自 104. 至 110.

(104) $2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\sin\frac{\gamma+\delta}{2}\cos\frac{\gamma-\delta}{2}=2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}$

$+2\sin\frac{\alpha+\beta}{2}\cos\frac{\gamma-\delta}{2}=2\sin\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2}+\cos\frac{\gamma-\delta}{2}\right)$

$=4\sin\frac{\alpha+\beta}{2}\cos\frac{\beta+\gamma-\alpha-\delta}{4}\cos\frac{\alpha+\gamma-\beta-\delta}{4}=4\sin\frac{\alpha+\beta}{2}\sin\frac{\beta+\gamma}{2}\sin\frac{\gamma+\alpha}{2}$.

(105), (106), 及 **(107).** 均與前例同法.

(108) $\sin\{2\pi-(\beta+\delta)\}\sin\{2\pi-(\beta+\gamma)\}=\sin(\beta+\delta)\sin(\beta+\gamma)$.

(109) $\frac{1}{4}\left(\sin\frac{\alpha+\beta}{2}-\sin\frac{\alpha-\beta}{2}\right)\left(\sin\frac{\gamma+\delta}{2}+\sin\frac{\gamma-\delta}{2}\right)$

$-\frac{1}{4}\left(\sin\frac{\alpha+\beta}{2}+\sin\frac{\alpha-\beta}{2}\right)\left(\sin\frac{\gamma+\delta}{2}-\sin\frac{\gamma-\delta}{2}\right)$

$=\frac{1}{2}\left(\sin\frac{\alpha+\beta}{2}\sin\frac{\gamma-\delta}{2}-\sin\frac{\alpha-\beta}{2}\sin\frac{\gamma+\delta}{2}\right)=\frac{1}{2}\sin\frac{\alpha+\beta}{2}\left(\sin\frac{\gamma-\delta}{2}-\sin\frac{\alpha-\beta}{2}\right)$

$=\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha+\gamma-\beta-\delta}{2}\sin\frac{\beta+\gamma-\alpha-\delta}{2}=\sin\frac{\alpha+\beta}{2}\sin\frac{\alpha+\gamma}{2}\cos\frac{\alpha+\delta}{2}$,

(110) $\cos\frac{\alpha+\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}+\cos\frac{\gamma+\delta}{2}+\sin\frac{\gamma}{2}\sin\frac{\delta}{2}$

$=\cos\frac{\alpha-\beta}{2}+\sin\frac{\alpha}{2}\sin\frac{\beta}{2}-\cos\frac{\alpha+\beta}{2}+\sin\frac{\gamma}{2}\sin\frac{\delta}{2}$.

設 $\alpha+\beta+\gamma=\frac{\pi}{2}$, 試求次列各式之證.

111. $\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha=1.$

112. $\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma+\sec\alpha\sec\beta\sec\gamma.$

113. $(\tan\alpha+\tan\beta+\tan\gamma)(\cot\alpha+\cot\beta+\cot\gamma)$

$=1+\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma.$

114. $\cot\alpha(\tan\beta+\tan\gamma)+\cot\beta(\tan\gamma+\tan\alpha)$

$+\cot\gamma(\tan\alpha+\tan\beta)+2=\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma.$

例題解自 111. 至 114.

(111) 於 1. 節公式. $\tan(\alpha+\beta+\gamma)=\tan\frac{\pi}{2}=\frac{1}{0}$,

故 $1-\tan\alpha\tan\beta-\tan\beta\tan\gamma-\tan\gamma\tan\alpha=0.$

(112) 於 1. 節公式. $\sin(\alpha+\beta+\gamma)=\sin\frac{\pi}{2}=1$,

故 $1=\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\gamma\cos\alpha+\sin\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta\sin\gamma.$

兩邊試以 $\cos\alpha\cos\beta\cos\gamma$ 除之. 則

$\sec\alpha\sec\beta\sec\gamma=\tan\alpha+\tan\beta+\tan\gamma-\tan\alpha\tan\beta\tan\gamma.$

(113) 於例題 111. 之兩邊. 以 $\tan\alpha\tan\beta\tan\gamma$ 約之. 則

$\cot\alpha+\cot\beta+\cot\gamma=\frac{1}{\tan\alpha\tan\beta\tan\gamma}$, 又由前例

$(\tan\alpha+\tan\beta+\tan\gamma)(\cot\alpha+\cot\beta+\cot\gamma)$

$=(\tan\alpha\tan\beta\tan\gamma+\sec\alpha\sec\beta\sec\gamma)\frac{1}{\tan\alpha\tan\beta\tan\gamma}$

$=1+\frac{\sec\alpha\sec\beta\sec\gamma}{\tan\alpha\tan\beta\tan\gamma}=1+\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma.$

(114) $\tan\alpha\cot\alpha=\tan\beta\cot\beta=\tan\gamma\cot\gamma=1$, 故

原式之左邊 $=\cot\alpha(\tan\alpha+\tan\beta+\tan\gamma)-1+\cot\beta(\tan\alpha+\tan\beta+\tan\gamma)-1$

$+\cot\gamma(\tan\alpha+\tan\beta+\tan\gamma)-1+2$

$=(\tan\alpha+\tan\beta+\tan\gamma)(\cot\alpha+\cot\beta+\cot\gamma)-1$

$=1+\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma-1=\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma.$

設 $\alpha+\beta+\gamma=(2n+1)\pi$，求次列各式之證.

115. $\sin^2\beta+\sin^2\gamma-\sin^2\alpha=2\cos\alpha\sin\beta\sin\gamma$.

116. $\cos^4\frac{\alpha}{2}+\cos^4\frac{\beta}{2}+\cos^4\frac{\gamma}{2}-2\left(\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}+\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}\right.$
$\left.+\cos^2\frac{\gamma}{2}\cos^2\frac{\alpha}{2}\right)+4\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}=0.$

117. $\cot\alpha+\cot\beta+\cot\gamma-2(\cot2\alpha+\cot2\beta+\cot2\gamma)$
$=\left(\cot\frac{\alpha}{2}+\cot\frac{\beta}{2}+\cot\frac{\gamma}{2}\right)(\sec\alpha-1)(\sec\beta-1)(\sec\gamma-1).$

118. $\sin^22\alpha+\sin^22\beta+\sin^22\gamma+2\cos2\alpha\cos2\beta\cos2\gamma=2.$

119. $\left(1-\sin\frac{\beta}{2}\right)\left(1-\sin\frac{\gamma}{2}\right)\cos\frac{\alpha}{2}+\left(1-\sin\frac{\gamma}{2}\right)\left(1-\sin\frac{\alpha}{2}\right)\cos\frac{\beta}{2}$
$+\left(1-\sin\frac{\alpha}{2}\right)\left(1-\sin\frac{\beta}{2}\right)\cos\frac{\gamma}{2}=\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}.$

例題解自 115. 至 119.

(115) 與例題 62. 同樣.

(116) 原式 $=-\left(\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\right)\left(\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}-\cos\frac{\gamma}{2}\right)$
$\left(\cos\frac{\alpha}{2}-\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\right)\left(-\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}\right)+4\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}$,
但與例題 41. 用同樣之法.
$\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}+\cos\frac{\gamma}{2}=4\cos\frac{\alpha+\beta}{4}\cos\frac{\beta+\gamma}{4}\cos\frac{\gamma+\alpha}{4}$，與此同法.得
$\cos\frac{\alpha}{2}+\cos\frac{\beta}{2}-\cos\frac{\gamma}{2}=4\cos\frac{\alpha+\beta}{4}\sin\frac{\beta+\gamma}{4}\sin\frac{\gamma+\alpha}{4}$，由是
原式 $=-256\sin^2\frac{\alpha+\beta}{4}\sin^2\frac{\beta+\gamma}{4}\sin^2\frac{\gamma+\alpha}{4}\cos^2\frac{\alpha+\beta}{4}\cos^2\frac{\beta+\gamma}{4}\cos^2\frac{\gamma+\alpha}{4}$
$+4\cos^2\frac{\alpha}{2}\cos^2\frac{\beta}{2}\cos^2\frac{\gamma}{2}=-4\sin^2\frac{\alpha+\beta}{2}\sin^2\frac{\beta+\gamma}{2}\sin^2\frac{\gamma+\alpha}{2}$
$+4\sin^2\frac{\beta+\gamma}{2}\sin^2\frac{\gamma+\alpha}{2}\sin^2\frac{\alpha+\beta}{2}=0.$

(117) $\cot\alpha-2\cot2\alpha=\frac{\cos\alpha}{\sin\alpha}-\frac{2\cos2\alpha}{\sin2\alpha}=\frac{\cos^2\alpha-(\cos^2\alpha-\sin^2\alpha)}{\sin\alpha\cos\alpha}=\tan\alpha,$

故原式之左邊 $=\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma$ 〔例題 35.〕

又原式之右邊 $=\cot\frac{\alpha}{2}\cot\frac{\beta}{2}\cot\frac{\gamma}{2}\cdot\frac{(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)}{\cos\alpha\cos\beta\cos\gamma}$，〔例題 43.〕

$=\frac{\cos\frac{\alpha}{2}\cos\frac{\beta}{2}\cos\frac{\gamma}{2}}{\sin\frac{\alpha}{2}\sin\frac{\beta}{2}\sin\frac{\gamma}{2}}\cdot\frac{(1-\cos\alpha)(1-\cos\beta)(1-\cos\gamma)}{\cos\alpha\cos\beta\cos\gamma}=\frac{\sin\alpha\sin\beta\sin\gamma}{\cos\alpha\cos\beta\cos\gamma}$

$=\tan\alpha\tan\beta\tan\gamma.$

(118) 從例題 74. (119) 從例題 59.

120. 設 $\alpha+\beta+\gamma=\frac{\pi}{4}$, 則

$$\cos(6\beta+4\gamma-8\alpha)+\cos(6\gamma+4\alpha-8\beta)+\cos(6\alpha+4\beta-8\gamma)$$
$$=4\cos(5\alpha-2\beta-\gamma)\cos(5\beta-2\gamma-\alpha)\cos(5\gamma-2\alpha-\beta).$$

121. $\alpha+\beta+\gamma=n\pi$, 則

$$\tan\alpha+\tan\beta+\tan\gamma=\tan\alpha\tan\beta\tan\gamma.$$

122. $\alpha+\beta+\gamma=(2n+1)\pi$ 或 $(2n+\frac{1}{2})\pi$, 則

$$(\sin\alpha+\cos\alpha)(\sin\beta+\cos\beta)(\sin\gamma+\cos\gamma)$$
$$=2\sin\alpha\sin\beta\sin\gamma+2\cos\alpha\cos\beta\cos\gamma+1.$$

123. $\alpha+\beta+\gamma=2n\pi$ 或 $(2n-\frac{1}{2})\pi$, 則

$$(\sin\alpha+\cos\alpha)(\sin\beta+\cos\beta)(\sin\gamma+\cos\gamma)$$
$$=2\sin\alpha\sin\beta\sin\gamma+2\cos\alpha\cos\beta\cos\gamma-1.$$

設 $\alpha+\beta+\gamma=2s$, 試求自 124. 至 128. 各式之證.

124. $1+2\cos\alpha\cos\beta\cos\gamma-\cos^2\alpha-\cos^2\beta-\cos^2\gamma$

$$=4\sin s\sin(s-\alpha)\sin(s-\beta)\sin(s-\gamma).$$

125. $1-2\cos\alpha\cos\beta\cos\gamma-\cos^2\alpha-\cos^2\beta-\cos^2\gamma$

$$=4\cos s\cos(s-\alpha)\cos(s-\beta)\cos(s-\gamma).$$

126. $4\cos\alpha\cos\beta\cos\gamma=\cos2(s-\alpha)+\cos2(s-\beta)+\cos2(s-\gamma)+\cos2s.$

127. $4\sin\alpha\sin\beta\sin\gamma=\sin2(s-\alpha)+\sin2(s-\beta)+\sin2(s-\gamma)-\sin2s.$

128. $\tan(s-\alpha)+\tan(s-\beta)+\tan(s-\gamma)-\tan s=$

$$4\sin\alpha\sin\beta\sin\gamma/(1-\cos^2\alpha-\cos^2\beta-coc^2\gamma-2\cos\alpha\cos\beta\cos\gamma).$$

129. 設 $s=\alpha+\beta+\gamma+\delta$ 及

$$\cos(s-2\alpha)+\cos(s-2\beta)=\cos(s-2\gamma)+\cos(s-2\delta)$$

則 $\tan\alpha\tan\beta=\tan\gamma\tan\delta.$

例題解自 120. 至 129.

(120) $(6\beta+4\gamma-8\alpha)+(6\gamma+4\alpha-8\beta)+(6\alpha+4\beta-8\gamma)=2\alpha+2\beta+2\gamma=\frac{\pi}{2}$,

故由例題 44. 可得其證.

(121) 於 1. 節公式 $\tan(\alpha+\beta+\gamma)=\alpha+\beta+\gamma$, 可代以 $n\pi$ 而得其證.

(122) 解原式左邊之括弧. 括之如 1. 節之公式. 則

$2\sin\alpha\sin\beta\sin\gamma+2\cos\alpha\cos\beta\cos\gamma+\sin(\alpha+\beta+\gamma)-\cos(\alpha+\beta+\gamma)$,

依意題 $\sin(\alpha+\beta+\gamma)=\sin(2n+1)\pi=0$, $\cos(\alpha+\beta+\gamma)=\cos(2n+1)\pi=-1$,

或 $\sin(\alpha+\beta+\gamma)=\sin\{(2n+\frac{1}{2})\pi\}=+1$, $\cos(\alpha+\beta+\gamma)=\cos\{(2n+\frac{1}{2})\pi\}=0$.

(123) 與前例同樣.

(124) 由例題四 113 而得.

$$原式=4\sin\frac{\alpha+\beta+\gamma}{2}\sin\frac{\alpha+\beta-\gamma}{2}\sin\frac{\beta+\gamma-\alpha}{2}\sin\frac{\gamma+\alpha-\beta}{2}$$

$$=4\sin s\sin(s-\gamma)\sin(s-\alpha)\sin(s-\beta).$$

(125) 與前例同法.

(126) $4\cos\alpha\cos\beta\cos\gamma=2\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}\cos\gamma$

$=\cos(\alpha+\beta+\gamma)+\cos(\alpha+\beta-\gamma)+\cos(\gamma+\alpha-\beta)+\cos(\gamma-\alpha+\beta)$

$=\cos2s+\cos(2s-2\gamma)+\cos(2s-2\beta)+\cos(2s-2\alpha)$.

(127) 與前例同樣.

(128) 原式之左邊 $=\dfrac{\sin(2s-\alpha-\beta)}{\cos(s-\alpha)\cos(s-\beta)}-\dfrac{\sin\gamma}{\cos(s-\gamma)\cos s}$

$$=\frac{\sin\gamma\{\cos(s-\gamma)\cos s-\cos(s-\alpha)\cos(s-\beta)\}}{\cos(s-\alpha)\cos(s-\beta)\cos(s-\gamma)\cos s}$$

$$=\frac{\sin\gamma\{\cos(2s-\gamma)+\cos\gamma-\cos(2s-\alpha-\beta)-\cos(\alpha-\beta)\}}{2\cos(s-\alpha)\cos(s-\beta)\cos(s-\gamma)\cos s}$$

$$=\frac{\sin\gamma\{\cos(\alpha+\beta)-\cos(\alpha-\beta)\}}{2\cos(s-\alpha)\cos(s-\beta)\cos(s-\gamma)\cos s}$$ 〔由例題 125.〕

$$=\frac{4\sin\gamma\sin\alpha\sin\beta}{1-\cos^2\alpha-\cos^2\beta-\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma}.$$

(129) $2\cos(s-\alpha-\beta)\cos(\alpha-\beta)=2\cos(s-\gamma-\delta)\cos(\gamma-\delta)$,

即 $\cos(\gamma+\delta)\cos(\alpha-\beta)=\cos(\alpha+\beta)\cos(\gamma-\delta)$,

解兩邊之括弧以 $\cos\alpha\cos\beta\cos\gamma\cos\delta$ 約之, 即得.

第陸編

特別角之三角函數值

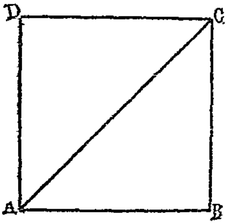
1. 四十五度之三角函數 求得45°之三角函數值.由是可求得135°, 225°及315°等之三角函數值.

ABCD爲正方形.其對角線AC與AB成角BAC爲45°.故

$\sin BAC=\frac{BC}{AC}$, 但 AB=BC, $AC=\sqrt{(AB^2+BC^2)}=BC\sqrt{2}$, 即 $\frac{BC}{AC}=\frac{1}{\sqrt{2}}$,

故 $\sin 45°=\frac{1}{\sqrt{2}}$, 又 $\cos 45°=\sin(90°-45°)=\sin 45°=\frac{1}{\sqrt{2}}$,

$\tan 45°=\frac{\sin 45°}{\cos 45°}=\frac{1}{\sqrt{2}}\div\frac{1}{\sqrt{2}}=1=\cot 45°$, $\sec 45°=\frac{1}{\cos 45°}=\sqrt{2}=\operatorname{cosec} 45°$,



又 $\sin 135°=\sin(180°-45°)=\sin 45°=\frac{1}{\sqrt{2}}$,

$\cos 135°=\cos(180°-45°)=-\cos 45°=-\frac{1}{\sqrt{2}}$,

$\tan 135°=\frac{\sin 135°}{\cos 135°}=\frac{1}{\sqrt{2}}\div-\frac{1}{\sqrt{2}}=-1.$

又 $\sin 225°=\sin(180°+45°)=-\sin 45°=-\frac{1}{\sqrt{2}}$,

$\cos 225°=\cos(180°+45°)=-\cos 45°=-\frac{1}{\sqrt{2}}$,

$\tan 225=\frac{\sin 225°}{\cos 225°}=-\frac{1}{\sqrt{2}}\div-\frac{1}{\sqrt{2}}=1.$

又 $\sin 315°=\sin(360°-45°)=-\sin 45°=-\frac{1}{\sqrt{2}}$,

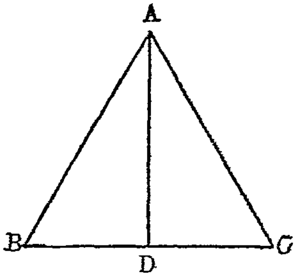
$\cos 315°=\cos 45°=\frac{1}{\sqrt{2}}$, $\tan 315°=\frac{\sin 315°}{\cos 315°}=-\frac{1}{\sqrt{2}}\div\frac{1}{\sqrt{2}}=-1.$

2. 三十度之三角函數值 求得30°之三角函數值由是可求得60°, 120°, 150°, 210°, 240°, 330°, 等之三角函數值.

ABC爲等邊三角形.令AD爲BC線上之垂線.角BAC爲60°.

故角BAD等於30°.

而 $AD^2=AB^2-BD^2=AB^2-\left(\frac{AB}{2}\right)^2$, 故 $AD=\frac{1}{2}AB\sqrt{3}$, $\frac{AD}{AB}=\frac{1}{2}\sqrt{3}$,



$\sin \mathrm{BAD}=\frac{\mathrm{BD}}{\mathrm{AB}}=\frac{\frac{1}{2}\mathrm{AB}}{\mathrm{AB}}=\frac{1}{2}$，即 $\sin 30^\circ=\frac{1}{2}$.

$\cos \mathrm{BAD}=\frac{\mathrm{AD}}{\mathrm{AB}}$ 即 $\cos 30^\circ=\frac{1}{2}\sqrt{3}$.

$\tan \mathrm{BAD}=\frac{\sin 30^\circ}{\cos 30^\circ}=\frac{1}{2}\div\frac{1}{2}\sqrt{3}=\frac{1}{\sqrt{3}}$.

$\sin 60^\circ=\sin(90^\circ-30)=\cos 30^\circ=\frac{1}{2}\sqrt{3}$.

$\cos 60^\circ=\cos(90^\circ-30^\circ)=\sin 30^\circ=\frac{1}{2}$.

$\tan 60^\circ=\frac{\sin 60^\circ}{\cos 60^\circ}=\frac{1}{2}\sqrt{3}\div\frac{1}{2}=\sqrt{3}$.

又 $150^\circ=180^\circ-30^\circ$, $210^\circ=180^\circ+30^\circ$, $240^\circ=180^\circ+60^\circ$, $300^\circ+360^\circ-60^\circ$, $330=360^\circ-30^\circ$. 知此關係. 可容易求得其三角函數值.

3. 三角函數值之表 前兩節所得之三角函數值.表示如次.

角	度	*sin*	*cos*	*tan*	*cot*	*sec*	*cosec*
30°	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$\frac{2}{\sqrt{3}}$	2
45°	$\frac{\pi}{4}$	$\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	1	1	$\sqrt{2}$	$\sqrt{2}$
60°	$\frac{\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	2	$\frac{2}{\sqrt{3}}$
120°	$\frac{2\pi}{3}$	$\frac{1}{2}\sqrt{3}$	$-\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	-2	$\frac{2}{\sqrt{3}}$
135°	$\frac{3\pi}{4}$	$\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	-1	-1	$-\sqrt{2}$	$\sqrt{2}$
150°	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	2
210°	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$	$\sqrt{3}$	$-\frac{2}{\sqrt{3}}$	-2
225°	$\frac{5\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$-\frac{1}{\sqrt{2}}$	1	1	$-\sqrt{2}$	$-\sqrt{2}$
240°	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$	$\frac{1}{\sqrt{3}}$	-2	$-\frac{2}{\sqrt{3}}$
300°	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$	$-\frac{1}{\sqrt{3}}$	2	$-\frac{2}{\sqrt{3}}$
315°	$\frac{7\pi}{4}$	$-\frac{1}{\sqrt{2}}$	$\frac{1}{\sqrt{2}}$	-1	-1	$\sqrt{2}$	$-\sqrt{2}$
330°	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$	$-\sqrt{3}$	$\frac{2}{\sqrt{3}}$	-2

例　題　六

求下列各式之證.

1. $\sin 15^\circ=\frac{1}{4}(\sqrt{6}-\sqrt{2})$, $\cos 15^\circ=\frac{1}{4}(\sqrt{6}+\sqrt{2})$, $\tan 15^\circ=2-\sqrt{3}$.

2. $\sin 75^\circ=\frac{1}{4}(\sqrt{6}+\sqrt{2})$, $\cos 75^\circ=\frac{1}{4}(\sqrt{6}-\sqrt{2})$, $\tan 75^\circ=2+\sqrt{3}$.

3. $\sin 105^\circ=\frac{1}{4}(\sqrt{6}+\sqrt{2})$, $\cos 105^\circ=-\frac{1}{4}(\sqrt{6}-\sqrt{2})$.

4. $\sin 22^\circ\frac{1}{2}=\frac{\sqrt{(2-\sqrt{2})}}{2}$, $\tan 22^\circ\frac{1}{2}=\sqrt{2}-1$.

5. $\sin 7^\circ\frac{1}{2}=\frac{1}{4}\sqrt{(8-2\sqrt{6}-2\sqrt{2})}$,

$\tan 7^\circ\frac{1}{2}=(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$.

6. $\tan 11^\circ\frac{1}{4}=\sqrt{2(2+\sqrt{2})}-\sqrt{2}-1$.

7. $\cot 11^\circ\frac{1}{4}=1+\sqrt{2}+\sqrt{2(2+\sqrt{2})}$.

8. $\sin 67^\circ\frac{1}{2}=\frac{1}{2}\sqrt{(2+\sqrt{2})}$.

9. $\cos 33^\circ 45'=\frac{1}{2}\sqrt{\{2+\sqrt{(2-\sqrt{2})}\}}$.

10. $\tan 37^\circ\frac{1}{2}=\sqrt{6}+\sqrt{3}-\sqrt{2}-2$.

11. $\tan 52^\circ\frac{1}{2}=\sqrt{6}-\sqrt{3}-\sqrt{2}+2$.

12. $\sin 18^\circ=\frac{\sqrt{5}-1}{4}$, $\cos 18^\circ=\frac{\sqrt{10+2\sqrt{5}}}{4}$.

13. $\tan 18^\circ=\sqrt{1-\frac{2}{5}\sqrt{5}}$.

14. $\sin 36^\circ=\frac{1}{4}\sqrt{(10-2\sqrt{5})}$, $\cos 36^\circ=\frac{1}{4}(\sqrt{5}+1)$.

15. $\tan 36^\circ=\sqrt{(5-2\sqrt{5})}$.

16. $\sin 9^\circ=\frac{1}{4}(\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}})$

17. $\cos 9^\circ=\frac{1}{4}(\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}})$.

例題解自 1. 至 17.

(1) $cos30°=1-2sin^2 15°$, 故 $sin15°=\sqrt{\frac{1-cos30°}{2}}=\sqrt{\frac{1-\frac{1}{2}\sqrt{3}}{2}}$

$=\sqrt{\frac{2-\sqrt{3}}{4}}=\frac{1}{2}\sqrt{\frac{4-2\sqrt{3}}{2}}=\frac{1}{2}\sqrt{\frac{(\sqrt{3}-1)^2}{2}}=\frac{\sqrt{3}-1}{2\sqrt{2}}=\frac{\sqrt{6}-\sqrt{2}}{4}$.

$cos15°=\sqrt{\frac{1+cos30°}{2}}$, $tan\,15°=\frac{sin15°}{cos15°}=\sqrt{\frac{1-cos30°}{1+cos30°}}$.

(2) $sin75°=sin(90°-15°)=cos15°=\frac{\sqrt{6}+\sqrt{2}}{4}$.

(3) $sin105°=sin(180°-75°)=sin75°$.

(4) $sin\,22°\frac{1}{2}=\sqrt{\frac{1-cos45°}{2}}=\sqrt{\frac{1-\frac{1}{\sqrt{2}}}{2}}=\frac{\sqrt{(2-\sqrt{2})}}{2}$.

(5) $sin7°\frac{1}{2}=\sqrt{\frac{1-cos15°}{2}}=\sqrt{\frac{1-\frac{1}{4}(\sqrt{6}+\sqrt{2})}{2}}=\frac{1}{4}\sqrt{(8-2\sqrt{6}-2\sqrt{2})}$,

$tan\,7°\frac{1}{2}=\sqrt{\frac{1-cos15°}{1+cos15°}}=\sqrt{\frac{4-(\sqrt{6}+\sqrt{2})}{4+(\sqrt{6}+\sqrt{2})}}=\frac{4-(\sqrt{6}+\sqrt{2})}{\sqrt{\{4^2-(\sqrt{6}+\sqrt{2})^2\}}}=\frac{4-\sqrt{6}-\sqrt{2}}{\sqrt{(8-4\sqrt{3})}}$

$=\frac{(4-\sqrt{6}-\sqrt{2})(\sqrt{3}+1)}{(\sqrt{3}-1)\sqrt{2}\times(\sqrt{3}+1)}=\sqrt{6}+\sqrt{2}-\sqrt{3}-2=(\sqrt{2}-1)(\sqrt{3}-\sqrt{2})$.

(6) 可由 $tan11°\frac{1}{4}=\sqrt{\frac{1-sin22°\frac{1}{2}}{1+cos22°\frac{1}{2}}}$ 求得. (7) 可由前例求得.

(8) $sin\,67°\frac{1}{2}=\sqrt{\frac{1-cos135°}{2}}=\sqrt{\frac{1+\frac{1}{\sqrt{2}}}{2}}=\frac{1}{2}\sqrt{(2+\sqrt{2})}$.

(9) $cos33°45'=cos\frac{135°}{4}=\sqrt{\frac{1+cos\frac{135°}{2}}{2}}=\sqrt{\frac{1}{2}\left(1+\sqrt{\frac{1+cos135°}{2}}\right)}$.

(10) $tan\,37°\frac{1}{2}=\sqrt{\frac{1-cos75°}{1+cos75°}}=\sqrt{\frac{1-\frac{1}{4}(\sqrt{6}-\sqrt{2})}{1+\frac{1}{4}(\sqrt{6}-\sqrt{2})}}=\sqrt{\frac{4-\sqrt{6}+\sqrt{2}}{4+\sqrt{6}-\sqrt{2}}}$

$=\frac{4-\sqrt{6}+\sqrt{2}}{\sqrt{(8+4\sqrt{3})}}=\frac{4-\sqrt{6}+\sqrt{2}}{(\sqrt{3}+1)\sqrt{2}}=\frac{(4-\sqrt{6}+\sqrt{2})(\sqrt{6}-\sqrt{2})}{4}=\sqrt{6}+\sqrt{3}-\sqrt{2}-2$.

(11) $tan\,52°\frac{1}{2}=\sqrt{\frac{1-cos105°}{1+cos105°}}$. (12) $cos54°=sin(90°-54°)=sin36°$,

即 $4cos^3 18°-3cos18°=2sin18°cos18°$, 故 $4(1-sin^2 18°)-3=2sin18°$,

由是 $sin18°=\frac{1}{2}(\sqrt{5}-1)$. (13), (14), (15) 可由例題 12. 求得.

(16) $sin9°=\sqrt{\frac{1-cos18°}{2}}=\sqrt{\frac{1+sin18°+1-sin18°-2\sqrt{1-sin^2 18°}}{4}}$

$=\frac{\sqrt{1+sin18°}-\sqrt{1-sin18°}}{2}=\frac{1}{2}\left(\sqrt{1+\frac{\sqrt{5}-1}{4}}-\sqrt{1-\frac{\sqrt{5}-1}{4}}\right)$.

(17) 與前例同樣.

18. $\tan 82°\frac{1}{2}=(\sqrt{3}+\sqrt{2})(\sqrt{2}+1)$.

19. $\tan 54°=\sqrt{1+\frac{2}{5}\sqrt{5}}$, $\sec 54°=\sqrt{\dfrac{2(\sqrt{5}+1)}{\sqrt{5}}}$.

20. $\tan 9°=\sqrt{5}+1-\sqrt{(5+2\sqrt{5})}$.

21. $\sin 3°=\frac{1}{8}\{(\sqrt{5}-1)\sqrt{(2+\sqrt{3})}-\sqrt{(10+2\sqrt{5})(2-\sqrt{3})}\}$.

22. $\cos 12°=\frac{1}{8}(\sqrt{5}-1+\sqrt{30+6\sqrt{5}})$.

23. $\cos 27°=\frac{1}{4}(\sqrt{5+\sqrt{5}}+\sqrt{3-\sqrt{5}})$.

24. $\tan 27°=\sqrt{5}-1-\sqrt{(5-2\sqrt{5})}$.

25. $\sin 63°=\dfrac{\sqrt{(10+2\sqrt{5})}+\sqrt{5}-1}{4\sqrt{2}}$.

26. $\sin 87°=\frac{1}{8}\{(\sqrt{5}-1)\sqrt{(2-\sqrt{3})}+\sqrt{(10+2\sqrt{5})(2+\sqrt{3})}\}$.

27. $\cos 42°=\frac{1}{8}(\sqrt{15}-\sqrt{3}+\sqrt{10+2\sqrt{5}})$.

28. $(\tan 7°\frac{1}{2}+\tan 37°\frac{1}{2}+\tan 67°\frac{1}{2})(\tan 22°\frac{1}{2}+\tan 52°\frac{1}{2}+\tan 82°\frac{1}{2})$

$=17+8\sqrt{3}$.

29. $\dfrac{(\tan 67°\frac{1}{2}-\tan 7°\frac{1}{2})(\tan 127°\frac{1}{2}+\tan 22°\frac{1}{2})}{(\tan 22°\frac{1}{2}+\tan 7°\frac{1}{2})(\tan 127°\frac{1}{2}-\tan 67°\frac{1}{2})}=1$.

30. $\sin^2 24°-\sin^2 6°=\dfrac{\sqrt{5}-1}{8}$.

31. $\tan\dfrac{\pi}{10}\tan\dfrac{3\pi}{10}=\dfrac{1}{\sqrt{5}}$.

32. $\cos 1°-\cos 59°=\sin 29°$

33. $\tan(45°\pm\alpha)=\dfrac{1\pm\tan\alpha}{1\mp\tan\alpha}$.

34. $\tan(45°+\alpha)-\tan(45°-\alpha)=2\tan 2\alpha$.

例題解自 18. 至 34.

(18) $\tan 82^\circ\frac{1}{2}=\cot(90^\circ-82^\circ\frac{1}{2})=1/\tan 7^\circ\frac{1}{2}=1/(\sqrt{3}-\sqrt{2})(\sqrt{2}-1)$.

(19) $\tan 54^\circ=\cot(90^\circ-54^\circ)=1/\tan 36^\circ=1/\sqrt{(5-2\sqrt{5})}$.

(20) 可從例題16.及17.求得.

(21) $\sin 3^\circ=\sin(18^\circ-15^\circ)=\sin 18^\circ\cos 15^\circ-\cos 18^\circ\sin 15^\circ$.

(22) $\cos 12^\circ=\cos(30^\circ-18^\circ)=\cos 30^\circ\cos 18^\circ+\sin 30^\circ\sin 18^\circ$.

(23) $\cos 27^\circ=\sqrt{\frac{1+\cos 54^\circ}{2}}=\sqrt{\frac{1+\sin 36^\circ}{2}}=\sqrt{\frac{1}{2}\left(1+\frac{1}{4}\sqrt{10-2\sqrt{5}}\right)}$

$=\frac{1}{4}\sqrt{\{5+\sqrt{5}+2\sqrt{(5+\sqrt{5})(3-\sqrt{5})}+3-\sqrt{5}\}}=\frac{1}{4}(\sqrt{5+\sqrt{5}}+\sqrt{3-\sqrt{5}})$.

(24) $\tan 27^\circ=\frac{\sin 27^\circ}{\cos 27^\circ}=\frac{\sqrt{\{1-\frac{1}{16}(\sqrt{5+\sqrt{5}}+\sqrt{3-\sqrt{5}})^2\}}}{\frac{1}{4}(\sqrt{5+\sqrt{5}}+\sqrt{3-\sqrt{5}})}=\frac{\sqrt{5+\sqrt{5}}-\sqrt{3-\sqrt{5}}}{\sqrt{5+\sqrt{5}}+\sqrt{3-\sqrt{5}}}$.

(25) $\sin 63^\circ=\cos(90^\circ-63^\circ)=\cos 27^\circ=\frac{1}{4}\{\sqrt{5+\sqrt{5}}-\sqrt{\frac{1}{2}(6-2\sqrt{5})}\}$.

(26) $\sin 87^\circ=\cos(90^\circ-87^\circ)=\cos 3^\circ=\sqrt{(1-\sin^2 3^\circ)}$.

(27) $\cos 42^\circ=\cos(60^\circ-18^\circ)=\cos 60^\circ\cos 18^\circ+\sin 60^\circ\sin 18^\circ$.

(28) 原式$=\left\{\frac{\sin(7^\circ\frac{1}{2}+67^\circ\frac{1}{2})}{\cos 7^\circ\frac{1}{2}\cos 67^\circ\frac{1}{2}}+\frac{\sin 37^\circ\frac{1}{2}}{\cos 37^\circ\frac{1}{2}}\right\}\left\{\frac{\sin(22^\circ\frac{1}{2}+82^\circ\frac{1}{2})}{\cos 22^\circ\frac{1}{2}\cos 82^\circ\frac{1}{2}}+\frac{\sin 52^\circ\frac{1}{2}}{\cos 52^\circ\frac{1}{2}}\right\}$

$=\left(\frac{2\sin 75^\circ}{\cos 60^\circ+\cos 75^\circ}+\frac{\sin 75^\circ}{\cos 75^\circ+1}\right)\left(\frac{2\sin 105^\circ}{\cos 60^\circ+\cos 105^\circ}+\frac{\sin 105^\circ}{\cos 105^\circ+1}\right)$

$=\left(\frac{4\cos 15^\circ}{1+2\sin 15^\circ}+\frac{\cos 15^\circ}{1+\sin 15^\circ}\right)\left(\frac{4\cos 15^\circ}{1-2\cos 15^\circ}+\frac{\cos 15^\circ}{1-\sin 15^\circ}\right)$.

(29) 原式$=\frac{\sin(67^\circ\frac{1}{2}-7^\circ\frac{1}{2})\cos 22^\circ\frac{1}{2}\cdot\sin(127^\circ\frac{1}{2}+22^\circ\frac{1}{2})\cos 67^\circ\frac{1}{2}}{\sin(22^\circ\frac{1}{2}+7^\circ\frac{1}{2})\cos 67^\circ\frac{1}{2}\cdot\sin(127^\circ\frac{1}{2}-67^\circ\frac{1}{2})\cos 22^\circ\frac{1}{2}}$

$=\frac{\sin 60^\circ\sin 150^\circ}{\sin 30^\circ\sin 60^\circ}=\frac{\sin 150^\circ}{\sin 30^\circ}=\frac{\sin 30^\circ}{\sin 30^\circ}=1$.

(30) 原式$=\sin(24^\circ+6^\circ)\sin(24^\circ-6^\circ)=\sin 30^\circ\sin 18^\circ$.

(31) $\tan 18^\circ\tan 54^\circ=\tan 18^\circ\cot 36^\circ=\tan 18^\circ/\tan 36^\circ$.

(32) $2\sin 30^\circ\sin 29^\circ$.　　(33) 用$\tan 45^\circ=1$於第四編2.節之公式.

(34) 用第四編2.節公式而$\tan 45^\circ=1$.又用第四編4.節之公式.

35. $\sin(36^\circ+\alpha)-\sin(36^\circ-\alpha)\equiv\frac{\sqrt{5}+1}{2}\sin\alpha.$

36. $\sin(72^\circ+\alpha)-\sin(72^\circ-\alpha)\equiv\frac{\sqrt{5}-1}{2}\sin\alpha.$

37. $4\sin\alpha\sin(60^\circ-\alpha)\sin(60^\circ+\alpha)\equiv\sin3\alpha.$

38. $4\cos\alpha\cos(60^\circ-\alpha)\cos(60^\circ+\alpha)\equiv\cos3\alpha.$

39. $\tan\alpha\tan(60^\circ+\alpha)\tan(120^\circ+\alpha)\equiv-\tan3\alpha.$

40. $\sin\alpha+\sin(72^\circ+\alpha)-\sin(72^\circ-\alpha)\equiv\sin(36^\circ+\alpha)-\sin(36^\circ-\alpha).$

41. $\cos(36^\circ+\alpha)+\cos(36^\circ-\alpha)=\frac{\sqrt{5}+1}{2}\cos\alpha.$

42. $\cos(72^\circ+\alpha)+\cos(72^\circ-\alpha)=\frac{\sqrt{5}-1}{2}\cos\alpha.$

43. $\cos\alpha+\cos(72^\circ+\alpha)+\cos(72^\circ-\alpha)$

$\equiv\cos(36^\circ+\alpha)+\cos(36^\circ-\alpha).$

44. $\cos\alpha+\cos(120^\circ-\alpha)+\cos(120^\circ+\alpha)\equiv0$

45. $\tan\alpha+\sec\alpha\equiv\tan\left(45^\circ+\frac{\alpha}{2}\right).$

46. $\frac{1\pm\sin\alpha}{\cos\alpha}\equiv\tan\left(45^\circ\pm\frac{\alpha}{2}\right).$

47. $\tan(45^\circ+\alpha)+\tan(45^\circ-\alpha)=2\sec2\alpha.$

48. $\sin(45^\circ+\alpha)-\sin(45^\circ-\alpha)\equiv\sqrt{2}\sin\alpha.$

49. $\tan\alpha+\tan(60^\circ+\alpha)+\tan(120^\circ+\alpha)\equiv3\tan3\alpha.$

50. $\cot\alpha+\cot(60^\circ+\alpha)+\cot(120^\circ+\alpha)\equiv3\cot3\alpha.$

51. $\cot\alpha\cot(60^\circ+\alpha)+\cot(60^\circ+\alpha)\cot(120^\circ+\alpha)$

$+\cot(120^\circ+\alpha)\cot\alpha=-3.$

52. $\sin2\alpha\equiv\frac{1-\cot^2(45^\circ+\alpha)}{1+\cot^2(45^\circ+\alpha)}.$

53. $\frac{\tan\alpha+\sec\alpha}{\cot\alpha+\operatorname{cosec}\alpha}\equiv\tan\frac{\alpha}{2}\tan\left(45^\circ+\frac{\alpha}{2}\right).$

54. $\frac{\sin3\alpha+\cos3\alpha}{\sin3\alpha-\cos3\alpha}\equiv\frac{2\sin2\alpha+1}{2\sin2\alpha-1}\tan(45^\circ-\alpha).$

例題解自 35 至 54.

(35) $2\cos 36^\circ \sin\alpha=\frac{\sqrt{5}+1}{2}\sin\alpha.$ (36) 同上.

(37) $4\sin\alpha(\sin^2 60^\circ-\sin^2\alpha)=3\sin\alpha-4\sin^3\alpha=\sin 3\alpha.$

(38) $4\cos\alpha(\cos^2\alpha-\sin^2 60^\circ)=4\cos^3\alpha-3\cos\alpha=\cos 3\alpha.$ (39) 從前例.

(40) $\sin\alpha+2\cos 72^\circ\sin\alpha=\sin\alpha+2\sin 18^\circ\sin\alpha=\sin\alpha+\frac{\sqrt{5}-1}{2}\sin\alpha$

$=2\times\frac{\sqrt{5}+1}{4}\sin\alpha=2\cos 36^\circ\sin\alpha=\sin(36^\circ+\alpha)-\sin(36^\circ-\alpha).$

(41), (42) 與例題 35 同樣.

(43) $\cos\alpha+2\cos 72^\circ\cos\alpha=\cos\alpha+\frac{\sqrt{5}-1}{2}\cos\alpha=2\cos 36^\circ\cos\alpha.$

(44) $\cos\alpha+2\cos 120^\circ\cos\alpha=\cos\alpha-\cos\alpha=0.$

(45) $$\frac{\sin\alpha+1}{\cos\alpha}=\frac{-\cos(90^\circ+\alpha)+1}{\sin(90^\circ+\alpha)}=\frac{2\sin^2\left(45^\circ+\frac{\alpha}{2}\right)}{2\sin\left(45^\circ+\frac{\alpha}{2}\right)\cos\left(45^\circ+\frac{\alpha}{2}\right)}.$$

(46) $$\frac{\sin^2\frac{\alpha}{2}+\cos^2\frac{\alpha}{2}\pm 2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}}{\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}}=\frac{\cos\frac{\alpha}{2}\pm\sin\frac{\alpha}{2}}{\cos\frac{\alpha}{2}\mp\sin\frac{\alpha}{2}}=\frac{1\pm\tan\frac{\alpha}{2}}{1\mp\tan\frac{\alpha}{2}}.$$

(47) $$\frac{1+\tan\alpha}{1-\tan\alpha}+\frac{1-\tan\alpha}{1+\tan\alpha}=\frac{2(1+\tan^2\alpha)}{1-\tan^2\alpha}=\frac{2}{\cos^2\alpha-\sin^2\alpha}=\frac{2}{\cos 2\alpha}.$$

(48) $2\cos 45^\circ\sin\alpha=\sqrt{2}\sin\alpha.$

(49) $$\tan\alpha+\frac{\tan 60^\circ+\tan\alpha}{1-\tan 60^\circ\tan\alpha}+\frac{\tan 120^\circ+\tan\alpha}{1-\tan 120^\circ\tan\alpha}$$

$$=\tan\alpha+\frac{\sqrt{3}+\tan\alpha}{1-\sqrt{3}\tan\alpha}-\frac{\sqrt{3}-\tan\alpha}{1+\sqrt{3}\tan\alpha}=\frac{3(3\tan\alpha-\tan^3\alpha)}{1-3\tan^2\alpha}=3\tan 3\alpha.$$

(50) 與前例同樣.

(51) $$\frac{\tan(120^\circ+\alpha)+\tan\alpha+\tan(60^\circ+\alpha)}{\tan\alpha\tan(60^\circ+\alpha)\tan(120^\circ+\alpha)}=\frac{3\tan 3\alpha}{-\tan 3\alpha}=-3.$$

(52) $$\sin 2\alpha=-\cos(90^\circ+2\alpha)=\frac{\cos^2(45^\circ+\alpha)-\sin^2(45^\circ+\alpha)}{\cos^2(45^\circ+\alpha)+\sin^2(45^\circ+\alpha)}.$$

(53) $$\frac{(\sin\alpha+1)\sin\alpha}{(\cos\alpha+1)\cos\alpha}=\frac{2\sin^2(45^\circ+\frac{1}{2}\alpha)2\sin\frac{1}{2}\alpha\cos\frac{1}{2}\alpha}{2\cos^2\frac{1}{2}\alpha 2\sin(45^\circ+\frac{1}{2}\alpha)\cos(45^\circ+\frac{1}{2}\alpha)}.$$

(54) $$\frac{4(\cos^3\alpha-\sin^3\alpha)-3(\cos\alpha-\sin\alpha)}{4(\cos^3\alpha+\sin^3\alpha)-3(\cos\alpha+\sin\alpha)}=\frac{(\cos\alpha-\cos\alpha)(2\sin 2\alpha+1)}{(\cos\alpha+\sin\alpha)(2\sin 2\alpha-1)}$$

$$=\frac{(\sin 45^\circ\cos\alpha-\cos 45^\circ\sin\alpha)(2\sin 2\alpha+1)}{(\cos 45^\circ\cos\alpha+\sin 15^\circ\sin\alpha)(2\sin 2\alpha-1)}.$$

55. $\cot(\alpha+15^\circ)-\tan(\alpha-15^\circ)\equiv\dfrac{4\cos2\alpha}{2\sin2\alpha+1}$.

56. $\sin3(\alpha-15^\circ)\equiv4\cos(\alpha-45^\circ)\cos(\alpha+15^\circ)\sin(\alpha-15^\circ)$.

57. $\sin^3+\sin^3(120^\circ+\alpha)+\sin^3(240^\circ+\alpha)\equiv-\frac{3}{4}\sin3\alpha$.

58. $\tan^2\left(\dfrac{\pi}{4}+\dfrac{\alpha}{2}\right)\equiv\dfrac{\sec\alpha+\tan\alpha}{\sec\alpha-\tan\alpha}$.

59. $\sin\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)+\cos\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)\equiv\dfrac{\sin\theta}{\sqrt{\operatorname{vers}\theta}}$.

60. $\sec\theta\equiv\frac{1}{2}\left\{\tan\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)+\cot\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)\right\}$.

61. $\tan(30^\circ+\alpha)\tan(30^\circ-\alpha)\equiv\dfrac{2\cos2\alpha-1}{2\cos2\alpha+1}$.

62. $\sin80^\circ\equiv\sin40^\circ+\sin20^\circ$.

63. $\sin70^\circ\equiv\sin50^\circ+\sin10^\circ$.

64. $\cos20^\circ+\cos100^\circ+\cos140^\circ\equiv0$.

65. $\cos12^\circ+\cos60^\circ+\cos84^\circ\equiv\cos24^\circ+\cos48^\circ$.

66. $\sin85^\circ\equiv\cos55^\circ+\sin25^\circ$.

67. $\cos55^\circ+\cos65^\circ+\cos175^\circ\equiv0$.

68 $\dfrac{\sin80^\circ+\sin10^\circ}{\cos80^\circ+\cos10^\circ}\equiv1$.

69. $\dfrac{\cos50^\circ-\cos70^\circ}{\sin70^\circ-\sin50^\circ}\equiv\sqrt{3}$.

70. $\sin\dfrac{2\pi}{7}+\sin\dfrac{4\pi}{7}-\sin\dfrac{6\pi}{7}\equiv4\sin\dfrac{\pi}{7}\sin\dfrac{2\pi}{7}\sin\dfrac{3\pi}{7}$.

71. $\cos60^\circ+2\cos70^\circ+\cos80^\circ\equiv4\cos^25^\circ\cos70^\circ$.

72. $\cos47^\circ-\cos61^\circ-\cos11^\circ+\cos25^\circ\equiv\sin7^\circ$.

73 $1+\tan65^\circ+\tan70^\circ\equiv\tan65^\circ\tan70^\circ$.

74 $4\sin110^\circ\sin70^\circ\sin40^\circ\equiv\sin80^\circ+2\sin40^\circ$.

例 題 解 自 55. 至 74.

(55) $\dfrac{\cos\{(a+15^\circ)+(a-15^\circ)\}}{\sin(a+15^\circ)\cos(a-15^\circ)}=\dfrac{\cos 2a}{\frac{1}{2}(\sin 2a+\sin 30^\circ)}=\dfrac{4\cos 2a}{2\sin 2a+1}.$

(56) $3\sin(a-15^\circ)-4\sin^3(a-15^\circ)=\sin(a-15^\circ)\{3-4\sin^2(a-15^\circ)\}$

$=\sin(a-15^\circ)\{1+2\cos(2a-30^\circ)\}=2\sin(a-15^\circ)\{\cos 60^\circ+\cos(2a-30^\circ)\}$

$=4\sin(a-15^\circ)\cos(a-45^\circ)\cos(a+15^\circ).$

(57) 原式$=\sin^3 a+(\sin 120^\circ\cos a+\cos 120^\circ\sin a)^3+(\sin 240^\circ\cos a+\cos 240^\circ\sin a)^3$

$=\sin^3 a+(\frac{1}{2}\sqrt{3}\cos a-\frac{1}{2}\sin a)^3+(-\frac{1}{2}\sqrt{3}\cos a-\frac{1}{2}\sin a)^3$

$=\sin^3 a+\frac{1}{8}(-18\cos^2 a\sin a-2\sin^3 a)=-\frac{3}{4}(4\sin^3 a-3\sin a)=-\frac{3}{4}\sin 3a.$

(58) $\tan^2\left(\dfrac{\pi}{4}+\dfrac{a}{2}\right)=\dfrac{\sin^2\left(\dfrac{\pi}{4}+\dfrac{a}{2}\right)}{\cos^2\left(\dfrac{\pi}{4}+\dfrac{a}{2}\right)}=\dfrac{1-\cos\left(\dfrac{\pi}{2}+a\right)}{1+\cos\left(\dfrac{\pi}{2}+a\right)}=\dfrac{1+\sin a}{1-\sin a}$

(59) 原式$=\sqrt{\left\{1+\sin\left(\dfrac{\pi}{2}-\theta\right)\right\}}$

$=\sqrt{\{(1-\cos^2\theta)/(1-\cos\theta)\}}=\sin\theta/\sqrt{\operatorname{vers}\theta}.$

(60) $\sec\theta=\dfrac{1}{\cos\theta}=\dfrac{1}{\sin\left(\dfrac{\pi}{2}-\theta\right)}=\dfrac{\sin^2\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)+\cos^2\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)}{2\sin\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)\cos\left(\dfrac{\pi}{4}-\dfrac{\theta}{2}\right)}.$

(61) $\dfrac{\sin(30^\circ+a)\sin(30^\circ-a)}{\cos(30^\circ+a)\cos(30^\circ-a)}=\dfrac{\sin^2 30^\circ-\sin^2 a}{\cos^2 a-\sin^2 30^\circ}=\dfrac{1-4\sin^2 a}{4\cos^2 a-1}=\dfrac{2\cos 2a-1}{2\cos 2a+1}.$

(62) $\sin 80^\circ=\cos(90^\circ-10^\circ)=\cos 10^\circ=2\sin 30^\circ\cos 10^\circ=\sin 40^\circ+\sin 20^\circ.$

(63) $\sin 70^\circ=\cos 20^\circ=2\sin 30^\circ\cos 20^\circ=\sin 50^\circ+\sin 10^\circ.$

(64) $\cos 20^\circ+\cos(180^\circ-80^\circ)+\cos(180^\circ-40^\circ)=\cos 20^\circ-\cos 80^\circ-\cos 40^\circ$

$=2\sin 50^\circ\sin 30^\circ-\sin(90^\circ-40^\circ)=\sin 50^\circ-\sin 50^\circ=0.$

(65) $\cos 12^\circ+2\cos 72^\circ\cos 12^\circ=\cos 12^\circ\left(1+\dfrac{\sqrt{5}-1}{2}\right)=\dfrac{\sqrt{5}+1}{2}\cos 12^\circ$

$=2\cos 36^\circ\cos 12^\circ=\cos 48^\circ+\cos 24^\circ.$

(66) $\sin 85^\circ=\cos 5^\circ=2\cos 60^\circ\cos 5^\circ=\cos 65^\circ+\cos 55^\circ=\sin 25^\circ+\cos 55^\circ.$

(67) $2\cos 60^\circ\cos 5^\circ-\cos(180^\circ-175^\circ)=\cos 5^\circ-\cos 5^\circ=0.$

(68) $\dfrac{2\sin 45^\circ\cos 35^\circ}{2\cos 45^\circ\cos 35^\circ}=1.$ (69) $\dfrac{2\sin 10^\circ\sin 60^\circ}{2\cos 60^\circ\sin 10^\circ}=\sqrt{3}.$

(70) $\sin\dfrac{2\pi}{7}-2\cos\dfrac{5\pi}{7}\sin\dfrac{\pi}{7}=2\sin\dfrac{\pi}{7}\left(\cos\dfrac{\pi}{7}-\cos\dfrac{5\pi}{7}\right).$

(71) $2\cos 70^\circ\cos 10^\circ+2\cos 70^\circ=2\cos 70^\circ(\cos 10^\circ+1).$

(72) $2\sin 54^\circ\sin 7^\circ-2\sin 18^\circ\sin 7^\circ=2\sin 7^\circ(\sin 54^\circ-\sin 18^\circ).$

(73) $1+\dfrac{\sin 135^\circ}{\cos 65^\circ\cos 70^\circ}=\dfrac{\cos 65^\circ\cos 70^\circ+\dfrac{1}{\sqrt{2}}}{\cos 65^\circ\cos 70^\circ}=\dfrac{\frac{1}{2}(\cos 5^\circ+\cos 135^\circ)+\dfrac{1}{\sqrt{2}}}{\cos 65^\circ\cos 70^\circ}$

$=\dfrac{\frac{1}{2}\left(\cos 5^\circ+\dfrac{1}{\sqrt{2}}\right)}{\cos 65^\circ\cos 70^\circ}=\dfrac{\frac{1}{2}(\cos 5^\circ-\cos 135^\circ)}{\cos 65^\circ\cos 70^\circ}=\dfrac{\sin 65^\circ\sin 70^\circ}{\cos 65^\circ\cos 70^\circ}.$ (74) 乃 (71) 之反法.

75. $\cos 40^\circ + \cos 80^\circ + \cos 160^\circ \equiv 0.$

76. $\cos 40^\circ \cos 80^\circ + \cos 80^\circ \cos 160^\circ + \cos 160^\circ \cos 40^\circ \equiv -\frac{3}{4}.$

77. $\cos 40^\circ \cos 80^\circ \cos 160^\circ \equiv -\frac{1}{8}.$

78. $\sin^2 10^\circ + \cos^2 20^\circ - \sin 10^\circ \cos 20^\circ \equiv \frac{3}{4}.$

79. $\sin 10^\circ + \cos^2 40^\circ + \sin 10^\circ \cos 40 \equiv \frac{3}{4}.$

80. $\cos 20^\circ \cos 40^\circ \cos 80^\circ \equiv (\frac{1}{2})^3.$

81. $\cos 55^\circ \cos 65^\circ + \cos 65^\circ \cos 175^\circ + \cos 175^\circ \cos 55^\circ \equiv -\frac{3}{4}.$

82. $\cos 55^\circ \cos 65^\circ \cos 175^\circ \equiv -\dfrac{1+\sqrt{3}}{8\sqrt{2}}.$

83. $\cos\dfrac{\pi}{15}\cos\dfrac{2\pi}{15}\cos\dfrac{3\pi}{15}\cos\dfrac{4\pi}{15}\cos\dfrac{5\pi}{15}\cos\dfrac{6\pi}{15}\cos\dfrac{7\pi}{15} \equiv \left(\dfrac{1}{2}\right)^7.$

84. $\cos^4\dfrac{\pi}{8} + \cos^4\dfrac{3\pi}{8} + \cos^4\dfrac{5\pi}{8} + \cos^4\dfrac{7\pi}{8} \equiv \dfrac{3}{2}.$

85. $\cos^4\dfrac{\pi}{9} + \cos^4\dfrac{2\pi}{9} + \cos^4\dfrac{3\pi}{9} + \cos^4\dfrac{4\pi}{9} \equiv \dfrac{19}{16}.$

86. $\cos^8\dfrac{\pi}{8} + \cos^8\dfrac{3\pi}{8} + \cos^8\dfrac{5\pi}{8} + \cos^8\dfrac{7\pi}{8} \equiv \dfrac{17}{16}.$

87. $\sin^2 24^\circ - \sin^2 6^\circ = \dfrac{\sqrt{5}-1}{8}.$

88. $\cos^2 18^\circ \sin^2 36^\circ - \cos 36^\circ \sin 18^\circ \equiv \dfrac{1}{16}.$

89. $\dfrac{\cos 27^\circ - \sin 27^\circ}{\cos 27^\circ + \sin 27^\circ} \equiv \dfrac{\sqrt{(25-10\sqrt{5})}}{5}.$

例題解自 75. 至 89.

(75) 與例題64同樣.

(76) $\frac{1}{2}(\cos 40^\circ + \cos 120^\circ + \cos 80^\circ + \cos 240^\circ + \cos 120^\circ + \cos 200^\circ)$ 由前例

$= \frac{1}{2}(2\cos 120^\circ + \cos 240^\circ - \cos 160^\circ + \cos 200^\circ)$

$= \frac{1}{2}(-1 + \cos 240^\circ) - \frac{1}{2}(\cos 60^\circ - \cos 200^\circ)$

$= -\sin^2 120^\circ - \sin 180^\circ \sin 20^\circ = -(\frac{1}{2}\sqrt{3})^2 - 0 \times \sin 20^\circ = -\frac{3}{4}.$

(77) $\frac{1}{2}\cos 80°(\cos 120°+\cos 200°)=-\frac{1}{4}\cos 80°+\frac{1}{2}\cos 80°\cos 200°$

$=-\frac{1}{4}\cos 80°+\frac{1}{4}(\cos 120°+\cos 280°)=-\frac{1}{8}-\frac{1}{4}(\cos 80°-\cos 280°)$

$=1-\frac{1}{8}-\frac{1}{2}\sin 180°\sin 100°=-\frac{1}{8}$.

(78) $\frac{1}{2}(1-\cos 20°)+\frac{1}{2}(1+\cos 40°)-\frac{1}{2}(\sin 30°-\sin 10°)$

$=1-\frac{1}{2}(\cos 20°-\cos 40°)-\frac{1}{2}(\frac{1}{2}-\sin 10°)=1-\sin 30°\sin 10°-\frac{1}{4}+\frac{1}{2}\sin 10°$.

(79) 與前例同樣. (80) 與例題77.同法.

(81) $\frac{1}{2}(\cos 10°+\cos 120°+\cos 110°+\cos 240°+\cos 120°+\cos 230°)$

$=\frac{1}{2}\{2\cos 120°+\cos 240°+(\cos 10°+\cos 110°)+\cos 230°\}$

$=\frac{1}{2}\{-1+\cos 240°+2\cos 60°\cos 50°+\cos 230°\}=\frac{1}{2}(-2\sin^2 120°+\cos 50°+\cos 230°)$

$=\frac{1}{2}(-\frac{3}{2}+2\cos 90°\cos 140°)=\frac{1}{2}(-\frac{3}{2}+2\times 0\times\cos 140°)=-\frac{3}{4}$.

(82) $\frac{1}{2}\cos 65°(\cos 120°+\cos 230°)=-\frac{1}{4}\cos 65°+\frac{1}{2}\cos 65°\cos 230°$

$=-\frac{1}{4}\cos 65°+\frac{1}{4}(\cos 165°+\cos 295°)=\frac{1}{4}\cos(180°-15°)-\frac{1}{4}(\cos 65°-\cos 295°)$

$=-\frac{1}{4}\cos 15°-\frac{1}{2}\sin 115°\sin 180°=-\frac{1}{4}\cos 15°$.

(83) $\left(\cos\frac{\pi}{15}\cos\frac{4\pi}{15}\right)\left(\cos\frac{2\pi}{15}\cos\frac{7\pi}{15}\right)\left(\cos\frac{3\pi}{15}\cos\frac{6\pi}{15}\right)\cos\frac{5\pi}{15}=\frac{1}{2}\left(\cos\frac{3\pi}{15}+\cos\frac{5\pi}{15}\right)$

$\frac{1}{2}\left(\cos\frac{5\pi}{15}+\cos\frac{9\pi}{15}\right)\frac{1}{2}\left(\cos\frac{3\pi}{15}+\cos\frac{9\pi}{15}\right)\cos\frac{5\pi}{15}=\frac{1}{16}\left(\frac{\sqrt{5}+1}{4}+\frac{1}{2}\right)\left(\frac{1}{2}-\frac{\sqrt{5}-1}{4}\right)$

$\left(\frac{\sqrt{5}+1}{4}-\frac{\sqrt{5}-1}{4}\right)=\frac{1}{16}\left(\frac{3+\sqrt{5}}{4}\right)\left(\frac{3-\sqrt{5}}{4}\right)\left(\frac{2}{4}\right)=\left(\frac{1}{2}\right)^7$.

(84) $\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}+\left\{\cos\left(\pi-\frac{3\pi}{8}\right)\right\}^4+\left\{\cos\left(\pi-\frac{\pi}{8}\right)\right\}^4=2\left(\cos^4\frac{\pi}{8}+\cos^4\frac{3\pi}{8}\right)$

$=2\left(\cos^4\frac{\pi}{8}+\sin^4\frac{\pi}{8}\right)=2\left(1-2\sin^2\frac{\pi}{8}\cos^2\frac{\pi}{8}\right)=2\left(1-\frac{1}{2}\sin^2\frac{\pi}{2}\right)$.

(85) 試改弧度爲六十分法.則

$\left(\frac{1+\cos 40°}{2}\right)^2+\left(\frac{1+\cos 80°}{2}\right)^2+\cos^4 60°+\left(\frac{1+\cos 160°}{2}\right)^2=\frac{3}{4}+\frac{1}{16}+\frac{1}{2}(\cos 40°$

$+\cos 80°+\cos 160°)+\frac{1}{4}(\cos^2 40°+\cos^2 80°+\cos^2 160°)$ 由例題75.得

$=\frac{13}{16}+\frac{1}{4}\left\{(\cos 40°+\cos 80°+\cos 160°)^2-2(\cos 40°\cos 80°+\cos 80°\cos 160°+\cos 160°\cos 40°)\right\}$

$=\frac{13}{16}+\frac{1}{4}\left\{-2\times-\frac{3}{4}\right\}$ 〔從例題75, 76.〕$=\frac{19}{16}$.

(86) $2\left(\cos^8\frac{\pi}{8}+\sin^8\frac{\pi}{8}\right)=2\left\{\left(\cos^4\frac{\pi}{8}-\sin^4\frac{\pi}{8}\right)^2+2\sin^4\frac{\pi}{8}\cos^4\frac{\pi}{8}\right\}$

$=2\left\{\cos^2\frac{\pi}{4}+\frac{1}{8}\sin^4\frac{\pi}{4}\right\}=2\left(\frac{1}{2}+\frac{1}{32}\right)=\frac{17}{16}$. (87), (88) 皆與前諸例同樣.

(89) $\frac{\cos 27°-\cos 63°}{\cos 27°+\cos 63°}=\frac{2\sin 45°\sin 18°}{2\cos 45°\cos 18°}=\tan 45°\tan 18°=1\times\sqrt{(1-\frac{2}{5}\sqrt{5})}$.

90. $\dfrac{\sin 60^\circ-\sin 30^\circ}{\sin 60^\circ+\sin 30^\circ}\equiv\dfrac{\tan 60^\circ-\tan 45^\circ}{\tan 60^\circ+\tan 45^\circ}.$

91. $\dfrac{1+\cot 60^\circ}{1-\cot 60^\circ}\equiv\left(\dfrac{1+\cos 30^\circ}{1-\cos 30^\circ}\right)^{\frac{1}{2}}.$

92. $\cos\alpha+\cos\left(\dfrac{2\pi}{3}+\alpha\right)+\cos\left(\dfrac{2\pi}{3}-\alpha\right)\equiv 0.$

93. $\cos\alpha\cos\left(\dfrac{2\pi}{3}+\alpha\right)+\cos\left(\dfrac{2\pi}{3}+\alpha\right)\cos\left(\dfrac{2\pi}{3}-\alpha\right)$

$$+\cos\left(\frac{2\pi}{3}-\alpha\right)\cos\alpha\equiv-\frac{3}{4}.$$

94. $\cos\alpha\cos\left(\dfrac{2\pi}{3}+\alpha\right)\cos\left(\dfrac{2\pi}{3}-\alpha\right)\equiv\dfrac{1}{4}\cos 3\alpha.$

95. $\sin^2\left(\dfrac{\pi}{8}+\dfrac{\theta}{2}\right)-\sin^2\left(\dfrac{\pi}{8}-\dfrac{\theta}{2}\right)\equiv\dfrac{1}{\sqrt{2}}\sin\theta.$

96. $\sec^2\dfrac{\pi-\alpha}{4}+\sec^2\dfrac{\pi+\alpha}{4}\equiv\sec^2\dfrac{\pi-\alpha}{4}\sec^2\dfrac{\pi+\alpha}{4}.$

97. $\operatorname{cosec}\alpha+\operatorname{cosec}\left(\alpha+\dfrac{2\pi}{3}\right)+\operatorname{cosec}\left(\alpha+\dfrac{4\pi}{3}\right)\equiv 3\operatorname{cosec}3\alpha.$

98. $\sin^3\alpha+\sin^3(120^\circ+\alpha)-\sin^3(120^\circ-\alpha)\equiv-\frac{3}{4}\sin 3\alpha.$

99. $\cos 3\left(\alpha+\dfrac{\pi}{4}\right)+\cos\left(\alpha+\dfrac{\pi}{4}\right)\left(1+2\sin 2\alpha\right)\equiv 0.$

100. $\cos^4\alpha+\cos^4\left(\dfrac{2\pi}{3}-\alpha\right)+\cos^4\left(\dfrac{4\pi}{3}-\alpha\right)\equiv\dfrac{9}{8}.$

101. $\sin(90^\circ-\alpha)+\sin(18^\circ+\alpha)+\sin(18^\circ-\alpha)$

$$\equiv\sin(54^\circ+\alpha)+\sin(54^\circ-\alpha).$$

例題解自 90. 至 101.

(90) $\dfrac{(\sin 60^\circ\cos 45^\circ-\cos 60^\circ\sin 45^\circ)\div\cos 60^\circ\cos 45^\circ}{(\sin 60^\circ\cos 45^\circ+\cos 60^\circ\sin 45^\circ)\div\cos 60^\circ\cos 45^\circ}=\dfrac{\tan 60^\circ-\tan 45^\circ}{\tan 60^\circ+\tan 45^\circ}.$

(91) $\dfrac{\sin 60^\circ+\cos 60^\circ}{\sin 60^\circ-\cos 60^\circ}=\dfrac{\cos 30^\circ+\frac{1}{2}}{\cos 30^\circ-\frac{1}{2}}=\sqrt{\left(\dfrac{\cos^2 30^\circ+\cos 30^\circ+\frac{1}{4}}{\cos^2 30^\circ-\cos 30^\circ+\frac{1}{4}}\right)}$

$$=\sqrt{\left(\frac{\frac{3}{4}+\cos 30^\circ+\frac{1}{4}}{\frac{3}{4}-\cos 30^\circ+\frac{1}{4}}\right)}\sqrt{\left(\frac{1+\cos 30^\circ}{1-\cos 30^\circ}\right)}.$$

(92) $\cos\alpha+2\cos\dfrac{2\pi}{3}\cos\alpha=\cos\alpha-\cos\alpha=0.$

(93) $\frac{1}{2}\left\{\cos\frac{2\pi}{3}+\cos\left(\frac{2\pi}{3}+2\alpha\right)+\cos2\alpha+\cos\frac{4\pi}{3}+\cos\left(\frac{2\pi}{3}-2\alpha\right)+\cos\frac{2\pi}{3}\right\}$.

$=\frac{1}{2}\left\{2\cos\frac{2\pi}{3}+\cos\frac{4\pi}{3}+\cos2\alpha+2\cos\frac{2\pi}{3}\cos2\alpha\right\}=\frac{1}{2}\left\{-1-\frac{1}{2}+\cos2\alpha-\cos2\alpha\right\}$.

(94) $\frac{1}{2}\cos\alpha\left\{\cos\frac{4\pi}{3}+\cos2\alpha\right\}=\frac{1}{2}\cos\alpha\{-\frac{1}{2}+\cos2\alpha\}=\frac{1}{4}\cos\alpha(4\cos^2\alpha-3)$.

(95) $\frac{1}{2}\left\{1-\cos\left(\frac{\pi}{4}+\theta\right)-1+\cos\left(\frac{\pi}{4}-\theta\right)\right\}=\frac{1}{2}\left\{\cos\left(\frac{\pi}{4}-\theta\right)-\cos\left(\frac{\pi}{4}+\theta\right)\right\}$

$=\sin\frac{\pi}{4}\sin\theta=\frac{1}{\sqrt{2}}\sin\theta$.

(96) $\dfrac{\cos^2\frac{\pi+\alpha}{4}+\cos^2\frac{\pi-\alpha}{4}}{\cos^2\frac{\pi+\alpha}{4}\cos^2\frac{\pi-\alpha}{4}}=\dfrac{2+\cos\frac{\pi+\alpha}{2}+\cos\frac{\pi-\alpha}{2}}{2\cos^2\frac{\pi+\alpha}{4}\cos^2\frac{\pi-\alpha}{4}}=\dfrac{2-\sin\frac{\alpha}{2}+\sin\frac{\alpha}{2}}{2\cos^2\frac{\pi+\alpha}{4}\cos^2\frac{\pi-\alpha}{4}}$

(97) $\dfrac{1}{\sin\alpha}+\dfrac{\sin\left(\alpha+\frac{4\pi}{3}\right)+\sin\left(\alpha+\frac{2\pi}{3}\right)}{\sin\left(\alpha+\frac{2\pi}{3}\right)\sin\left(\alpha+\frac{4\pi}{3}\right)}=\dfrac{1}{\sin\alpha}+\dfrac{2\sin(\alpha+\pi)\cos\frac{\pi}{3}}{-\sin\left(\frac{\pi}{3}-\alpha\right)\sin\left(\frac{\pi}{3}+\alpha\right)}$

$=\dfrac{1}{\sin\alpha}-\dfrac{-\sin\alpha}{\sin^2\frac{\pi}{3}-\sin^2\alpha}=\dfrac{1}{\sin\alpha}+\dfrac{4\sin\alpha}{3-4\sin^2\alpha}=\dfrac{3}{3\sin\alpha-4\sin^3\alpha}$.

(98) $\frac{1}{4}\{3\sin\alpha-\sin3\alpha+3\sin(120^\circ+\alpha)-\sin3(120^\circ+\alpha)$
$-3\sin(120^\circ-\alpha)+\sin3(120^\circ-\alpha)\}$
$=\frac{3}{4}\{\sin\alpha+\sin(120^\circ+\alpha)-\sin(120^\circ-\alpha)\}-\frac{1}{4}(\sin3\alpha+\sin3\alpha+\sin3\alpha)\}$
$=\frac{3}{4}\{\sin\alpha+2\cos120^\circ\sin\alpha\}-\frac{3}{4}\sin3\alpha=\frac{3}{4}\{\sin\alpha-\sin\alpha\}-\frac{3}{4}\sin3\alpha$.

(99) $\cos3\left(\alpha+\frac{\pi}{4}\right)+\cos\left(\alpha+\frac{\pi}{4}\right)\left\{1-2\cos\left(\frac{\pi}{2}+2\alpha\right)\right\}$

$=\cos3\left(\alpha+\frac{\pi}{4}\right)+\cos\left(\alpha+\frac{\pi}{4}\right)\left\{1-4\cos^2\left(\frac{\pi}{4}+\alpha\right)+2\right\}$

$=\cos3\left(\alpha+\frac{\pi}{4}\right)-\left\{4\cos^3\left(\alpha+\frac{\pi}{4}\right)-3\cos\left(\alpha+\frac{\pi}{4}\right)\right\}=\cos3\left(\alpha+\frac{\pi}{4}\right)-\cos3\left(\alpha+\frac{\pi}{4}\right)$.

(100) $\cos^4\alpha+\cos^4\left(\frac{\pi}{3}+\alpha\right)+\cos^4\left(\frac{\pi}{3}-\alpha\right)$

$=\left\{\frac{1+\cos2\alpha}{2}\right\}^2+\left\{\frac{1+\cos\left(\frac{2\pi}{3}+2\alpha\right)}{2}\right\}^2+\left\{\frac{1+\cos\left(\frac{2\pi}{3}-2\alpha\right)}{2}\right\}^2$

$=\frac{3}{4}+\frac{1}{2}\left\{\cos2\alpha+\cos\left(\frac{2\pi}{3}+2\alpha\right)+\cos\left(\frac{2\pi}{3}-2\alpha\right)\right\}$

$+\frac{1}{4}\left\{\cos^22\alpha+\cos^2\left(\frac{2\pi}{3}+2\alpha\right)+\cos^2\left(\frac{2\pi}{3}-2\alpha\right)\right\}=\frac{3}{4}+\frac{1}{2}\left\{\cos2\alpha+2\cos\frac{2\pi}{3}\cos2\alpha\right\}$

$+\frac{1}{8}\left\{1+\cos4\alpha+1+\cos\left(\frac{4\pi}{3}+4\alpha\right)+1+\cos\left(\frac{4\pi}{3}-4\alpha\right)\right\}$

$=\frac{3}{4}+\frac{1}{2}\left\{\cos2\alpha-\cos2\alpha\right\}+\frac{1}{8}\left\{3+\cos4\alpha+2\cos\frac{4\pi}{3}\cos4\alpha\right\}$

$=\frac{3}{4}+\frac{1}{8}\left\{3+\cos4\alpha-\cos4\alpha\right\}=\frac{9}{8}$.

(101) 與下題同.參照下題之解法卽得.

102. $\cos(90°-\alpha)+\cos(18°-\alpha)-\cos(18°+\alpha)$

$$\equiv\cos(54°-\alpha)-\cos(54°+\alpha).$$

103. $\sqrt{vers\,\alpha}\left\{\sin\left(45°-\frac{\alpha}{2}\right)+\cos\left(45°-\frac{\alpha}{2}\right)\right\}\equiv\sin\alpha.$

104. $\cos 11\alpha+3\cos 9\alpha+3\cos 7\alpha+\cos 5\alpha$

$\equiv 16\cos^3\alpha\cos\left(4\alpha+\frac{\pi}{4}\right)\cos\left(4\alpha-\frac{\pi}{4}\right).$

105. $2\cos\frac{45°}{2^n}\equiv\sqrt{2+\sqrt{2+\sqrt{2+\cdots\cdots}}}$ 至 $n+1$ 項.

106. $\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\equiv-\frac{1}{2}.$

107. $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}+\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}+\cos\frac{6\pi}{7}\cos\frac{2\pi}{7}\equiv-\frac{1}{2}.$

108. $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}\equiv\frac{1}{8}.$

109. $\cos\frac{11\pi}{36}+\cos\frac{13\pi}{36}+\cos\frac{35\pi}{36}\equiv 0.$

110. $\cos\frac{11\pi}{36}\cos\frac{13\pi}{36}+\cos\frac{13\pi}{36}\cos\frac{35\pi}{85}+\cos\frac{35\pi}{35\pi}\cos\frac{11\pi}{36}=-\frac{3}{4}.$

111. $\cos\frac{11\pi}{36}\cos\frac{13\pi}{36}\cos\frac{35\pi}{36}\equiv-\frac{\sqrt{3}+1}{8\sqrt{2}}.$

112. $\left(x-2\cos\frac{2\pi}{7}\right)\left(x-2\cos\frac{4\pi}{7}\right)\left(x-2\cos\frac{6\pi}{7}\right)$

$\equiv x^4+x^3-2x-1.$

113. $\tan\theta\tan\left(\frac{\pi}{3}+\theta\right)+\tan\left(\frac{\pi}{3}+\theta\right)\tan\left(\frac{2\pi}{3}+\theta\right)$

$+\tan\left(\frac{2\pi}{3}+\theta\right)\tan\theta=-3.$

114. $\tan\theta\tan\left(\frac{\pi}{3}+\theta\right)\tan\left(\frac{2\pi}{3}+\theta\right)=-\tan 3\theta.$

115. $x^3-3x^2\tan 3\theta-3x+\tan 3\theta$

$\equiv\left\{x-\tan\theta\right\}\left\{x-\tan\left(\frac{\pi}{3}+\theta\right)\right\}\left\{x-\tan\left(\frac{2\pi}{3}+\theta\right)\right\}.$

例題解自 102 至 115.

(102) $\sin\alpha+2\sin 18^\circ \sin\alpha=\sin\alpha(1+2\sin 18^\circ)=\sin\alpha\left(1+\frac{\sqrt{5}-1}{2}\right)$

$=2\sin\alpha\left(\frac{\sqrt{5}+1}{4}\right)=2\sin\alpha\sin 54^\circ=\cos(54^\circ-\alpha)-\cos(54^\circ+\alpha)$.

(103) $\sqrt{1-\cos\alpha}\left\{\sin 45^\circ\cos\frac{\alpha}{2}-\cos 45^\circ\sin\frac{\alpha}{2}+\cos 45^\circ\cos\frac{\alpha}{2}+\sin 45^\circ\sin\frac{\alpha}{2}\right\}$

$=\sqrt{2\sin^2\frac{\alpha}{2}}\left\{\sqrt{2}\cos\frac{\alpha}{2}\right\}=2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}=\sin\alpha$.

(104) $2\cos 8\alpha\cos 3\alpha+6\cos 8\alpha\cos\alpha=2\cos 8\alpha(\cos 3\alpha+3\cos\alpha)$

$=8\cos 8\alpha\cos^3\alpha=8\cos^3\alpha(\cos^2 4\alpha-\sin^2 4\alpha)$

$=8\left(\sqrt{2}\right)^2\cos^3\alpha\left(\cos^2 4\alpha\cos^2\frac{\pi}{3}-\sin^2 4\alpha\sin^2\frac{\pi}{4}\right)$

$=16\cos^2\alpha\cos\left(4\alpha-\frac{\pi}{4}\right)\cos\left(4\alpha+\frac{\pi}{4}\right)$.

(105) $2\cos 45^\circ=\sqrt{2}$, $\quad 2\cos\frac{45^\circ}{2}=\sqrt{2+2\cos 45^\circ}=\sqrt{2+\sqrt{2}}$,

$2\cos\frac{45^\circ}{2^2}=\sqrt{2+2\cos\frac{45^\circ}{2}}=\sqrt{2+\sqrt{2+\sqrt{2}}}$, 由是

$2\cos\frac{45^\circ}{2^n}=\sqrt{2+2\cos\frac{45^\circ}{2^{n-1}}}=\sqrt{2+\sqrt{2+2\cos\frac{45^\circ}{2^{n-2}}}}$, 以下同理.

(106) $2\cos\frac{4\pi}{7}\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}=\cos\frac{4\pi}{7}\left(2\cos\frac{2\pi}{7}+1\right)=\cos\frac{4\pi}{7}\left(3-4\sin^2\frac{\pi}{7}\right)$

$=\dfrac{\cos\frac{4\pi}{7}\left(3\sin\frac{\pi}{7}-4\sin^3\frac{\pi}{7}\right)}{\sin\frac{\pi}{7}}=\dfrac{\cos\frac{4\pi}{7}\sin\frac{3\pi}{7}}{\sin\frac{\pi}{7}}=\dfrac{\sin\pi-\sin\frac{\pi}{7}}{2\sin\frac{\pi}{7}}=-\frac{1}{2}$.

(107) $\frac{1}{2}\left(2\cos\frac{2\pi}{7}+\cos\frac{6\pi}{7}+\cos\frac{10\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}\right)$ 由前例

$=\frac{1}{2}\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right)+\frac{1}{2}\left(\cos\frac{2\pi}{7}+\cos\frac{8\pi}{7}+\cos\frac{10\pi}{7}\right)=\frac{1}{2}\left(-\frac{1}{2}\right)+\frac{1}{2}\left(\cos\frac{16\pi}{7}\right.$

$\left.+\cos\frac{8\pi}{7}+\cos\frac{24\pi}{7}\right)=-\frac{1}{4}+\frac{1}{2}\cos\frac{16\pi}{7}\left(1+2\cos\frac{8\pi}{7}\right)=-\frac{1}{4}+\dfrac{\sin 4\pi-\sin\frac{4\pi}{7}}{4\sin\frac{4\pi}{7}}=-\frac{1}{4}-\frac{1}{4}$.

(108) $\frac{1}{2}\cos\frac{6\pi}{7}\left(\cos\frac{2\pi}{7}+\cos\frac{6\pi}{7}\right)=\frac{1}{4}\left(\cos\frac{4\pi}{7}+\cos\frac{8\pi}{7}+\cos\frac{12\pi}{7}+1\right)$

$=\frac{1}{4}\cos\frac{8\pi}{7}\left(1+2\cos\frac{4\pi}{7}\right)+\frac{1}{4}=\dfrac{\cos\frac{8\pi}{7}\sin\frac{6\pi}{7}}{4\sin\frac{2\pi}{7}}+\frac{1}{4}=\dfrac{\sin 2\pi-\sin\frac{2\pi}{7}}{8\sin\frac{2\pi}{7}}+\frac{1}{4}=-\frac{1}{8}+\frac{1}{4}$.

(109), (110), (111) 均與前三例同法. **(112)** 又從前三例. **(113)** 與 **51.** 題同樣.

(114) $\dfrac{\sin\theta\sin\left(\frac{\pi}{3}+\theta\right)\sin\left(\frac{2\pi}{3}+\theta\right)}{\cos\theta\cos\left(\frac{\pi}{3}+\theta\right)\cos\left(\frac{2\pi}{3}+\theta\right)}=\dfrac{\sin\theta\left(\frac{3}{4}-\sin^2\theta\right)}{-\cos\theta\left(\cos^2\theta-\frac{3}{4}\right)}=-\dfrac{\sin 3\theta}{\cos 3\theta}$.

(115) 從例題 **49.**, **113.** 及 **114.**

116. $\dfrac{\sin 5\theta-\cos 5\theta}{\sin 5\theta+\cos 5\theta}\equiv\tan\left(\theta-\dfrac{\pi}{4}\right)\dfrac{1-2\sin 2\theta-4\sin^2 2\theta}{1+2\sin 2\theta-4\sin^2 2\theta}$.

117. $\tan\theta+\tan\left(\dfrac{\pi}{5}+\theta\right)+\tan\left(\dfrac{2\pi}{5}+\theta\right)+\tan\left(\dfrac{3\pi}{5}+\theta\right)$

$$+\tan\left(\frac{4\pi}{5}+\theta\right)\equiv 5\tan 5\theta.$$

118. $\dfrac{1}{\cos\dfrac{2\pi}{7}+\cos 2\phi}+\dfrac{1}{\cos\dfrac{4\pi}{7}+\cos 2\phi}+\dfrac{1}{\cos\dfrac{6\pi}{7}+\cos 2\phi}$

$$\equiv\frac{7\tan 7\phi-\tan\phi}{2\sin 2\phi}.$$

119. $\cos\dfrac{\pi}{13}+\cos\dfrac{3\pi}{13}+\cos\dfrac{9\pi}{13}\equiv\dfrac{1+\sqrt{13}}{4}$,

及 $\cos\dfrac{5\pi}{13}+\cos\dfrac{7\pi}{13}+\cos\dfrac{11\pi}{13}\equiv\dfrac{1-\sqrt{13}}{4}$.

例題解自 116. 至 119.

(116) 由例題四 35. 題. 原式 $=\dfrac{(\sin\theta-\cos\theta)(2\cos 4\theta-2\sin 2\theta-1)}{(\sin\theta+\cos\theta)(2\cos 4\theta+2\sin 2\theta-1)}$

$$=\frac{\left(\sin\theta\cos\frac{\pi}{4}-\cos\theta\sin\frac{\pi}{4}\right)\left(1-2\sin 2\theta-4\sin^2 2\theta\right)}{\left(\sin\theta\sin\frac{\pi}{4}+\cos\theta\cos\frac{\pi}{4}\right)\left(1+2\sin 2\theta-4\sin^2 2\theta\right)}=\frac{\sin\left(\theta-\frac{\pi}{4}\right)\left(1-2\sin 2\theta-4\sin^2 2\theta\right)}{\cos\left(\theta-\frac{\pi}{4}\right)\left(1+2\sin 2\theta-4\sin^2 2\theta\right)}.$$

(117) $\tan\theta+\tan\left(\dfrac{\pi}{5}+\theta\right)+\tan\left(\dfrac{2\pi}{5}+\theta\right)-\tan\left(\dfrac{2\pi}{5}-\theta\right)-\tan\left(\dfrac{\pi}{5}-\theta\right)$

$$=\tan\theta+\frac{\sin 2\theta}{\cos\left(\frac{\pi}{5}+\theta\right)\cos\left(\frac{\pi}{5}-\theta\right)}+\frac{\sin 2\theta}{\cos\left(\frac{2\pi}{5}+\theta\right)\cos\left(\frac{2\pi}{5}-\theta\right)}$$

$$=\frac{\sin\theta}{\cos\theta}+\frac{\sin 2\theta}{\cos^2\theta-\sin^2\frac{\pi}{5}}+\frac{\sin 2\theta}{\cos^2\theta-\sin^2\frac{2\pi}{5}}=\frac{\sin\theta}{\cos\theta}+\frac{\sin 2\theta}{\cos^2\theta-\frac{10-2\sqrt{5}}{16}}+\frac{\sin 2\theta}{\cos^2\theta-\frac{10+2\sqrt{5}}{16}}$$

$$=\frac{\sin 2\theta}{2\cos^2\theta}+\frac{8\sin 2\theta}{8\cos^2\theta-5+\sqrt{5}}+\frac{8\sin 2\theta}{8\cos^2\theta-5-\sqrt{5}}=\frac{\sin 2\theta}{2\cos^2\theta}+\frac{16\sin 2\theta(8\cos^2\theta-5)}{(8\cos\theta-5)^2-5}$$

$$=\frac{\sin 2\theta(80\cos^4\theta-60\cos^2\theta+5)}{2\cos^2\theta(16\cos^2\theta-20\cos^2\theta+5)}=\frac{5\sin\theta\{16(1-\sin^2\theta)^2-12(1-\sin^2\theta)+1\}}{16\cos^5\theta-20\cos^3\theta+5\cos\theta}$$

$=\dfrac{5(16\sin^5\theta-20\sin^3\theta+5\sin\theta)}{16\cos^5\theta-20\cos^3\theta+5\cos\theta}$. 由例題四 33. 及 34. 得

$$=\frac{5\sin 5\theta}{\cos 5\theta}=5\tan 5\theta.$$

(118) 將原式左邊之分母通分相加,則

$$\text{分子}=\left(\cos\frac{4\pi}{7}+\cos2\phi\right)\left(\cos\frac{6\pi}{7}+\cos2\phi\right)+\left(\cos\frac{4\pi}{7}+\cos2\phi\right)\left(\cos\frac{2\pi}{7}+2\phi\right)$$

$$+\left(\cos\frac{2\pi}{7}+\cos2\phi\right)\left(\cos\frac{6\pi}{7}+\cos2\phi\right)=\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}+\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}+\cos\frac{6\pi}{7}\cos\frac{2\pi}{7}$$

$$+2\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right)\cos2\phi+3\cos^2 2\phi=-\tfrac{1}{2}-\cos2\phi+3\cos^2 2\phi,$$

從例題105,及107.

$$\text{又分母}=\left(\cos\frac{2\pi}{7}+\cos2\phi\right)\left(\cos\frac{4\pi}{7}+\cos2\phi\right)\left(\cos\frac{6\pi}{7}+\cos2\phi\right)$$

$$=\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}+\left(\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}+\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}+\cos\frac{6\pi}{7}\cos\frac{2\pi}{7}\right)\cos2\phi$$

$$+\left(\cos\frac{2\pi}{7}+\cos\frac{4\pi}{7}+\cos\frac{6\pi}{7}\right)\cos^2 2\phi+\cos^3 2\phi=\tfrac{1}{8}-\tfrac{1}{2}\cos2\phi-\tfrac{1}{2}\cos^2 2\phi+\cos^3 2\phi,$$

$$\text{由是原式左邊}=\frac{4(6\cos^2 2\phi-2\cos2\phi-1)}{8\cos^3 2\phi-4\cos^2 2\phi-4\cos2\phi+1},$$

$$\text{又原式之右邊}=\frac{7\tan7\phi-\tan\phi}{2\sin2\phi}=\frac{7\sin7\phi\cos\phi-\sin\phi\cos7\phi}{2\sin2\phi\cos7\phi\cos\phi}$$

$$=\frac{3\sin8\phi+3\sin6\phi+\sin(7\phi-\phi)}{\sin2\phi(\cos8\phi+\cos6\phi)}=\frac{6\sin4\phi\cos4\phi+4(3\sin2\phi-4\sin^3 2\phi)}{\sin2\phi(2\cos^2 4\phi-1+4\cos^3 2\phi-3\cos2\phi)}$$

$$=\frac{12\cos2\phi(2\cos^2 2\phi-1)+12-16(1-\cos^2 2\phi)}{2(2\cos^2 2\phi-1)^2-1+4\cos^3 2\phi-3\cos2\phi}=\frac{4(6\cos^2 2\phi-2\cos2\phi-1)}{8\cos^3 2\phi-4\cos^2 2\phi-4\cos2\phi+1}.$$

(119) 令第一為x,第二為y,則

$$x+y=\cos\frac{\pi}{13}+\cos\frac{11\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{9\pi}{13}+\cos\frac{5\pi}{13}+\cos\frac{7\pi}{13}$$

$$=2\cos\frac{6\pi}{18}\left(\cos\frac{5\pi}{13}+\cos\frac{3\pi}{13}+\cos\frac{\pi}{13}\right)=2\cos\frac{6\pi}{13}\left\{\cos\frac{\pi}{13}\left(2\cos\frac{4\pi}{13}+1\right)\right\}$$

$$=\frac{2\cos\frac{6\pi}{13}\cos\frac{\pi}{13}\left(3\sin\frac{2\pi}{13}-4\sin^3\frac{2\pi}{13}\right)}{\sin\frac{2\pi}{13}}=\frac{2\cos\frac{6\pi}{13}\cos\frac{\pi}{13}\sin\frac{6\pi}{13}}{\sin\frac{2\pi}{14}}$$

$$=\frac{\sin\frac{12\pi}{13}\cos\frac{\pi}{13}}{\sin\frac{2\pi}{13}}=\frac{\sin\left(\pi-\frac{\pi}{13}\right)\cos\frac{\pi}{13}}{\sin\frac{2\pi}{13}}=\frac{\frac{1}{2}\sin\frac{2\pi}{13}}{\sin\frac{2\pi}{13}}=\frac{1}{2}.$$

又 $xy=\cos\frac{\pi}{13}\cos\frac{5\pi}{13}+$ 以下是二因子之積之項,此各二因子之積,為各項之和.

$$\text{則}\quad xy=\frac{1}{2}\left(3\cos\frac{2\pi}{13}+3\cos\frac{4\pi}{13}+2\cos\frac{6\pi}{13}+3\cos\frac{8\pi}{13}+2\cos\frac{10\pi}{13}+\cos\frac{12\pi}{13}+2\cos\frac{14\pi}{13}\right.$$

$$\left.+\cos\frac{16\pi}{13}+\cos\frac{20\pi}{13}\right)=\frac{3}{2}\left(\cos\frac{2\pi}{13}+\cos\frac{4\pi}{13}+\cos\frac{6\pi}{13}+\cos\frac{8\pi}{15}+\cos\frac{10\pi}{13}+\cos\frac{12\pi}{13}\right)$$

$$=\frac{3\cos\left(\frac{2\pi}{13}+\frac{5\pi}{13}\right)\sin\frac{6\pi}{13}}{2\sin\frac{\pi}{13}}=-\frac{3\cos\frac{6\pi}{13}\sin\frac{6\pi}{13}}{2\sin\frac{\pi}{13}}=-\frac{3\sin\frac{12\pi}{13}}{4\sin\frac{\pi}{13}}=-\frac{3}{4}.$$

由 $x+y=\frac{1}{2}$ 及 $xy=-\frac{3}{4}$, 得 $x=\frac{1+\sqrt{13}}{4}$ 及 $y=\frac{1-\sqrt{13}}{4}$.

120 設 $\tan\theta=\frac{1}{\sqrt{3}}$ 及 $\tan\phi=\frac{1}{\sqrt{15}}$，則 $\sin(\theta+\phi)=\sin 60^\circ\cos 36^\circ$.

121. 設 $\cos 60^\circ=\sin 36^\circ\cos\alpha$, $\cos 36^\circ=\sin 60^\circ\cos\beta$.
則 $\tan\alpha+\tan\beta=1$.

122. 設 $a=\frac{\pi}{17}$，則 $\frac{\cos\alpha\cos 13\alpha}{\cos 3\alpha+\cos 5\alpha}=-\frac{1}{2}$.

123. 設 $\tan\frac{\theta}{2}=(1-\tan^2\alpha)/(1+\tan^2\alpha)$,

則 $2\cot 2\alpha=\cot^{\frac{1}{2}}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-\tan^{\frac{1}{2}}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$.

124. 設 $\tan\theta=\sin\alpha\cos\beta/(\sin\beta+\cos\alpha)$，則

$\tan\frac{\theta}{2}=\tan\frac{\alpha}{2}\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)$ 或 $-\cot\frac{\alpha}{2}\cot\left(\frac{\pi}{4}-\frac{\beta}{2}\right)$.

125. 設 $\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\tan^5\left(\frac{\pi}{4}+\frac{\phi}{2}\right)$，則

$$\sin\theta=5\sin\phi\left\{\frac{\left(1+\sin^2\phi\cot\frac{\pi}{5}\right)\left(1+\sin^2\phi\cot\frac{2\pi}{5}\right)}{\left(1+\sin^2\phi\tan\frac{\pi}{5}\right)\left(1+\sin^2\phi\tan\frac{2\pi}{5}\right)}\right\}.$$

126. 設 $\tan\phi=\frac{1+2c^2}{1-c^2}\tan\theta$, $\tan\left(\frac{\pi}{4}+\frac{\phi}{2}\right)=\frac{1+c}{1-c}\tan\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$,
則 $\sin\theta=\frac{2}{c}$.

例題解自 120. 至 126.

(120) $\sin(\theta+\phi)=\sin\theta\cos\phi+\cos\theta\sin\phi=\cos\theta\cos\phi(\tan\theta+\tan\phi)$

$=\frac{\tan\theta+\tan\phi}{\sqrt{(1+\tan^2\theta)}\sqrt{(1+\tan^2\phi)}}=\frac{\frac{1}{\sqrt{3}}+\frac{1}{\sqrt{15}}}{\sqrt{\left(1+\frac{1}{3}\right)}\sqrt{\left(1+\frac{1}{15}\right)}}$

$=\frac{(\sqrt{5}+1)\sqrt{3}}{8}=\frac{\sqrt{3}}{2}\times\frac{\sqrt{5}+1}{4}=\sin 60^\circ\cos 36^\circ$.

(121) $\frac{1}{2}=\frac{1}{4}\sqrt{10-2\sqrt{5}}\cos\alpha$, $\sec\alpha=\frac{1}{2}\sqrt{10-2\sqrt{5}}$, $1+\tan^2\alpha=\frac{1}{4}(10-2\sqrt{5})$,
$\tan\alpha=\frac{1}{2}\sqrt{(6-2\sqrt{5})}=\frac{\sqrt{5}-1}{2}$, 又 $\sec\beta=\frac{2\sqrt{3}}{\sqrt{5}+1}=\frac{(\sqrt{5}-1)\sqrt{3}}{2}$
$1+\tan^2\beta=\frac{(6-2\sqrt{5})3}{4}$, $\tan\beta=\frac{1}{2}\sqrt{(14-6\sqrt{5})}=\frac{3-\sqrt{5}}{2}$.

由是 $\tan\alpha+\tan\beta=\frac{\sqrt{5}-1}{2}+\frac{3-\sqrt{5}}{2}=1.$

(122) $\frac{\cos\alpha\cos13\alpha}{2\cos4\alpha\cos\alpha}=\frac{\cos(17\alpha-4\alpha)}{2\cos4\alpha}=\frac{\cos17\alpha}{2}+\frac{\sin17\alpha\tan4\alpha}{2}$

$$=\frac{\cos\pi}{2}+\frac{\sin\pi\tan4\alpha}{2}=-\frac{1}{2}.$$

(123) $\tan^2\alpha=\frac{1-\tan\frac{\theta}{2}}{1+\tan\frac{\theta}{2}}=\frac{\tan\frac{\pi}{4}-\tan\frac{\theta}{2}}{1+\tan\frac{\pi}{4}\tan\frac{\theta}{2}}=\tan\left(\frac{\pi}{4}-\frac{\theta}{2}\right)\cot^2\alpha$

$=\cot\left(\frac{\pi}{4}-\frac{\theta}{2}\right)$, $2\cot2\alpha=\frac{1-\tan^2\alpha}{\tan\alpha}=\cot\alpha-\tan\alpha=\cot^{\frac{1}{2}}\left(\frac{\pi}{4}-\frac{\theta}{2}\right)-\tan^{\frac{1}{2}}\left(\frac{\pi}{4}-\frac{\theta}{2}\right).$

(124) 從 $\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}=\frac{\sin\alpha\cos\beta}{\sin\beta+\cos\alpha}$ 得 $\tan\frac{\theta}{2}=\pm\frac{(1-\cos\alpha)(1-\sin\beta)}{\sin\alpha\cos\beta}$,

$\tan\frac{\beta}{2}=\frac{(1-\cos\alpha)(1-\sin\beta)}{\sin\alpha\cos\beta}=\tan\left(\frac{\pi}{4}-\frac{\beta}{2}\right)\tan\frac{\alpha}{2}.$

(125) $\tan^2\left(\frac{\pi}{4}+\frac{\theta}{2}\right)=\left\{\tan^2\left(\frac{\pi}{4}+\frac{\phi}{2}\right)\right\}^5$, 卽 $\frac{1+\sin\theta}{1-\sin\theta}=\left(\frac{1+\sin\phi}{1-\sin\phi}\right)^5$,

$\therefore\ \sin\theta=\frac{(1+\sin\phi)^5-(1-\sin\phi)^5}{(1+\sin\phi)^5+(1-\sin\phi)^5}=\frac{\sin\phi(5+10\sin^2\phi+\sin^4\phi)}{1+10\sin^2\phi+5\sin^4\phi}$

$$=\sin\phi\left[\frac{5\left(1+\sin^2\phi\frac{5-2\sqrt{5}}{5}\right)\left(1+\sin^2\phi\frac{5+2\sqrt{5}}{5}\right)}{\left(1+\sin^2\phi\frac{5}{5-2\sqrt{5}}\right)\left(1+\sin^2\phi\frac{5}{5+2\sqrt{5}}\right)}\right]$$

$$=5\sin\phi\left[\frac{\left(1+\sin^2\phi\cot\frac{2\pi}{5}\right)\left(1+\sin^2\phi\cot\frac{\pi}{5}\right)}{\left(1+\sin^2\phi\tan\frac{2\pi}{5}\right)\left(1+\sin^2\phi\tan\frac{\pi}{5}\right)}\right].$$

(126) 由第二方程式 $\frac{1+\tan\frac{\phi}{2}}{1-\tan\frac{\phi}{2}}=\frac{1+c}{1-c}\times\frac{2+\tan\frac{\theta}{2}}{1-\tan\frac{\theta}{2}}$ 卽 $\tan\frac{\phi}{2}=\frac{c+\tan\frac{\theta}{2}}{1+c\tan\frac{\theta}{2}}$,

故從第一方程式 $\frac{2\tan\frac{\phi}{2}}{1-\tan^2\frac{\phi}{2}}=\frac{1+2c^2}{1-c^2}\times\frac{2\tan\frac{\theta}{2}}{1-\tan^2\frac{\theta}{2}}$, 卽

$$\frac{\left(1+c\tan\frac{\theta}{2}\right)\left(c+\tan\frac{\theta}{2}\right)}{\left(1+c\tan\frac{\theta}{2}\right)^2-\left(c+\tan\frac{\theta}{2}\right)^2}=\frac{(1+2c^2)\tan\frac{\theta}{2}}{\left(1-c^2\right)\left(1-\tan^2\frac{\theta}{2}\right)}.$$

卽 $\frac{c+(1+c^2)\tan\frac{\theta}{2}+c\tan^2\frac{\theta}{2}}{\left(1-c^2\right)\left(1-\tan^2\frac{\theta}{2}\right)}=\frac{(1+2c^2)\tan\frac{\theta}{2}}{\left(1-c^2\right)\left(1-\tan^2\frac{\theta}{2}\right)}$,

故 $1-c\tan\frac{\theta}{2}+\tan^2\frac{\theta}{2}=0.$ 故 $\sin\theta=\frac{2}{c}.$

127. $\alpha=(n+\frac{1}{4}\pm\frac{1}{8})\pi$, 則 $\tan\alpha+\cot\alpha=4$.

128. $\alpha+\beta+\gamma=90^\circ$, 則

$$\frac{\sin\alpha+\sin\beta+\sin\gamma-1}{\cos\alpha+\cos\gamma+\cos\beta}=\frac{\left(1-\tan\frac{\alpha}{2}\right)\left(1-\tan\frac{\beta}{2}\right)\left(1-\tan\frac{\gamma}{2}\right)}{\left(1+\tan\frac{\alpha}{2}\right)\left(1+\tan\frac{\beta}{2}\right)\left(1+\tan\frac{\gamma}{2}\right)}$$

129. $\alpha+\beta+\gamma=90^\circ$, 則

$$\frac{\cos\alpha+\sin\gamma-\sin\beta}{\cos\beta+\sin\gamma-\sin\alpha}=\frac{1+\tan\frac{1}{2}\alpha}{1+\tan\frac{1}{2}\beta}.$$

130. $\alpha+\beta+\gamma=\pi$, 則

$$\cos2\alpha+\cos2\beta+\sin2\gamma=4\cos\gamma\cos\left(\frac{\pi}{4}+\alpha\right)\sin\left(\beta-\frac{\pi}{4}\right),$$

$$\cos2\alpha-\cos2\beta+\cos2\gamma+1=-4\cos\gamma\sin\left(\frac{\pi}{4}+\alpha\right)\cos\left(\frac{\pi}{4}+\beta\right),$$

$$\sin\alpha+\sin\beta+\cos\gamma+1=4\cos\frac{\gamma}{2}\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right),$$

$$\sin\alpha-\sin\beta+\cos\gamma-1=-4\sin\frac{\gamma}{2}\sin\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\sin\left(\frac{\pi}{4}-\frac{\beta}{2}\right),$$

$$\sin\left(\alpha-\frac{\pi}{3}\right)+\sin\left(\beta-\frac{\pi}{3}\right)+\sin\left(\gamma-\frac{\pi}{3}\right)$$

$$=-4\sin\left(\frac{\alpha}{6}-\frac{\pi}{6}\right)\sin\left(\frac{\beta}{2}-\frac{\pi}{6}\right)\sin\left(\frac{\gamma}{2}-\frac{\pi}{6}\right).$$

例題解自 127. 至 130.

(127) $\tan\alpha=\tan\left(n+\frac{1}{4}\pm\frac{1}{8}\right)\pi=\tan\left(\frac{\pi}{4}\pm\frac{\pi}{6}\right)=\dfrac{1\pm\tan\frac{\pi}{6}}{1\mp\tan\frac{\pi}{6}}=\dfrac{\sqrt{3}\pm1}{\sqrt{3}\mp1}$,

故 $\tan\alpha+\cot\alpha=\dfrac{\sqrt{3}\pm1}{\sqrt{3}\mp1}+\dfrac{\sqrt{8}\mp1}{\sqrt{3}\pm1}=\dfrac{(\sqrt{3}\pm1)^2+(\sqrt{3}\mp1)^2}{3-1}=4.$

(128) $$\frac{2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\cos(\alpha+\beta)-1}{2\cos\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+\sin(\alpha+\beta)}=\frac{2\sin\frac{\alpha+\beta}{2}\left(\sin\frac{\alpha-\beta}{2}-\sin\frac{\alpha+\beta}{2}\right)}{2\cos\frac{\alpha+\beta}{2}\left(\cos\frac{\alpha-\beta}{2}+\sin\frac{\alpha+\beta}{2}\right)}$$

$$=\frac{\sin\frac{90^\circ-\gamma}{2}\left\{\cos\frac{\alpha-\beta}{2}-\cos\left(90^\circ-\frac{\alpha+\beta}{2}\right)\right\}}{\cos\frac{90^\circ-\gamma}{2}\left\{\cos\frac{\alpha-\beta}{2}+\cos\left(90^\circ-\frac{\alpha+\beta}{2}\right)\right\}}=\frac{2\sin\frac{90^\circ-\gamma}{2}\sin\frac{90^\circ-\alpha}{2}\sin\frac{90^\circ-\beta}{2}}{2\cos\frac{90^\circ-\gamma}{2}\cos\frac{90^\circ-\alpha}{2}\cos\frac{90^\circ-\beta}{2}}$$

$$=\tan\left(45^\circ-\frac{\alpha}{2}\right)\tan\left(45^\circ-\frac{\beta}{2}\right)\tan\left(45^\circ-\frac{\gamma}{2}\right)=\frac{1-\tan\frac{\alpha}{2}}{1+\tan\frac{\alpha}{2}}\cdot\frac{1-\tan\frac{\beta}{2}}{1+\tan\frac{\beta}{2}}\cdot\frac{1-\tan\frac{\gamma}{2}}{1+\tan\frac{\gamma}{2}}$$

(129) $\dfrac{\cos\alpha-\sin\beta+\cos(\alpha+\beta)}{\cos\beta-\sin\alpha+\cos(\alpha+\beta)}=\dfrac{\cos\alpha(1+\cos\beta)-\sin\beta(1+\sin\alpha)}{\cos\beta(1+\cos\alpha)-\sin\alpha(1+\sin\beta)}$

$$=\frac{2\left(\cos^2\frac{\alpha}{2}-\sin^2\frac{\alpha}{2}\right)\cos^2\frac{\beta}{2}-2\sin\frac{\beta}{2}\cos\frac{\beta}{2}\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)^2}{2\left(\cos^2\frac{\beta}{2}-\sin^2\frac{\beta}{2}\right)\cos^2\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\cos\frac{\alpha}{2}\left(\cos\frac{\beta}{2}+\sin\frac{\beta}{2}\right)^2}$$

$$=\frac{\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)\cos\frac{\beta}{2}\left\{\left(\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\right)\cos\frac{\beta}{2}-\sin\frac{\beta}{2}\left(\cos\frac{\alpha}{2}+\sin\frac{\alpha}{2}\right)\right\}}{\left(\cos\frac{\beta}{2}+\sin\frac{\beta}{2}\right)\cos\frac{\alpha}{2}\left\{\left(\cos\frac{\beta}{2}-\sin\frac{\beta}{2}\right)\cos\frac{\alpha}{2}-\sin\frac{\alpha}{2}\left(\cos\frac{\beta}{2}+\sin\frac{\beta}{2}\right)\right\}}=\frac{1+\tan\frac{\alpha}{2}}{1+\tan\frac{\beta}{2}}$$

(130) $2\cos(\alpha+\beta)\cos(\alpha-\beta)+2\sin\gamma\cos\gamma=-2\cos\gamma\cos(\alpha-\beta)+2\sin(\alpha+\beta)\cos\gamma$

$=2\cos\gamma(\sin\alpha\cos\beta+\cos\alpha\sin\beta-\cos\alpha\cos\beta-\sin\alpha\sin\beta)$

$=2\cos\gamma(\cos\alpha-\sin\alpha)(\sin\beta-\cos\beta)$

$=4\cos\gamma\left(\cos\alpha\cos\frac{\pi}{4}-\sin\alpha\sin\frac{\pi}{4}\right)\left(\sin\beta\cos\frac{\pi}{4}-\cos\beta\sin\frac{\pi}{4}\right)$

$=4\cos\gamma\cos\left(\frac{\pi}{4}+\alpha\right)\sin\left(\beta-\frac{\pi}{4}\right)$.

第二之證.亦與第一同樣.

$2\sin\frac{\alpha+\beta}{2}\cos\frac{\alpha-\beta}{2}+2\cos^2\frac{\gamma}{2}=2\cos\frac{\gamma}{2}\left\{\cos\frac{\alpha-\beta}{2}+\cos\left(\frac{\pi}{2}-\frac{\alpha+\beta}{2}\right)\right\}$

$=4\cos\frac{\gamma}{2}\cos\left(\frac{\pi}{4}-\frac{\alpha}{2}\right)\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)$.

第四之證.亦與第三同樣.

$2\sin\left(\frac{\alpha+\beta}{2}-\frac{\pi}{3}\right)\cos\frac{\alpha-\beta}{2}+\sin\left(\gamma-\frac{\pi}{3}\right)=-2\sin\left(\frac{\gamma}{2}-\frac{\pi}{6}\right)\cos\frac{\alpha-\beta}{2}+\sin\left(\gamma-\frac{\pi}{3}\right)$

$=2\sin\left(\frac{\gamma}{2}-\frac{\pi}{6}\right)\left\{\cos\left(\frac{\gamma}{2}-\frac{\pi}{6}\right)\cos\frac{\alpha-\beta}{2}\right\}$

$=2\sin\left(\frac{\gamma}{2}-\frac{\pi}{6}\right)\left\{\cos\left(\frac{\pi}{3}-\frac{\alpha+\beta}{2}\right)-\cos\frac{\alpha-\beta}{2}\right\}$

$=4\sin\left(\frac{\gamma}{2}-\frac{\pi}{6}\right)\sin\left(\frac{\pi}{6}-\frac{\alpha}{2}\right)\sin\left(\frac{\beta}{2}-\frac{\pi}{6}\right)$

$=-4\sin\left(\frac{\gamma}{2}-\frac{\pi}{6}\right)\sin\left(\frac{\alpha}{2}-\frac{\pi}{6}\right)\sin\left(\frac{\beta}{2}-\frac{\pi}{6}\right)$.

分角之公式

3. 分角之公式 於第四編4.節之公式

$\sin 2a=2\sin a\cos a$ 其 a 代以 $\frac{1}{2}a$,

則 $\sin a=2\sin\frac{1}{2}a\cos\frac{1}{2}a$,

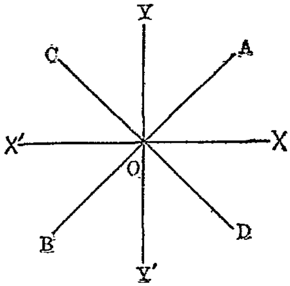
又 $1=\sin^2\frac{1}{2}a+\cos^2\frac{1}{2}a$.

由是 $\sin\frac{a}{2}+\cos\frac{a}{2}=\pm\sqrt{(1+\sin a)}$,

及 $\sin\frac{a}{2}-\cos\frac{a}{2}=\pm\sqrt{(1-\sin a)}$,

故 $2\sin\frac{a}{2}=\pm\sqrt{(1+\sin a)}\pm\sqrt{(1-\sin a)}$.

及 $2\cos\frac{a}{2}=\pm\sqrt{(1+\sin a)}\mp\sqrt{(1-\sin a)}$.



此公式中根號之正負.可由 $\frac{a}{2}$ 之値定之.如圖 XOX′, YOY′ 爲直交二軸. AB 及 CD. 爲角 XOY 及角 YOX′ 之二等分線. $\frac{a}{2}$ 爲在角 AOD, AOC, COB, BOD 之間.從而得决定其根號之正負.

$\frac{a}{2}$ 在 45° 與 −45° 之間.卽在角 AOD 之內.就其絕對値而論,則 $\sin\frac{a}{2}<\cos\frac{a}{2}$, 而 $\cos\frac{a}{2}$ 爲正,

故 $\sin\frac{a}{2}+\cos\frac{a}{2}=+\sqrt{(1+\sin a)}$, $\sin\frac{a}{2}-\cos\frac{a}{2}=-\sqrt{(1-\sin a)}$,

故 $2\sin\frac{a}{2}=\sqrt{(1+\sin a)}-\sqrt{(1-\sin a)}$, $2\cos\frac{a}{2}=\sqrt{(1+\sin a)}-\sqrt{(1-\sin a)}$.

$\frac{a}{2}$ 在 45° 與 135° 之間.卽在角 AOC 之內.就絕對値而論.

$\sin\frac{a}{2}>\cos\frac{a}{2}$, 而 $\sin\frac{a}{2}$ 爲正.

故 $\sin\frac{a}{2}+\cos\frac{a}{2}=+\sqrt{(1+\sin a)}$, $\sin\frac{a}{2}-\cos\frac{a}{2}=+\sqrt{(1-\sin a)}$,

故 $2\sin\frac{a}{2}=+(1+\sin a)+\sqrt{(1-\sin a)}$, $2\cos\frac{a}{2}=+\sqrt{(1+\sin a)}-\sqrt{(1-\sin a)}$.

$\frac{a}{2}$ 在 135° 與 225° 之間.卽在角 COB 之內.就絕對値而論.

$\sin\frac{a}{2}<\cos\frac{a}{2}$, 而 $\cos\frac{a}{2}$ 爲負.

故　$\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}=-\sqrt{(1+\sin\alpha)}$. $\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}=+\sqrt{(1-\sin\alpha)}$,

故　$2\sin\frac{\alpha}{2}=-\sqrt{(1+\sin\alpha)}+\sqrt{(1-\sin\alpha)}$, $2\cos\frac{\alpha}{2}=-\sqrt{(1+\sin\alpha)}-\sqrt{(1-\sin\alpha)}$.

$\frac{\alpha}{2}$ 在 225° 與 315° 之間. 卽在角 BOD 之內. 就絕對値而論.

$\sin\frac{\alpha}{2}>\cos\frac{\alpha}{2}$ 而 $\sin\frac{\alpha}{2}$ 爲負.

故　$\sin\frac{\alpha}{2}+\cos\frac{\alpha}{2}=-\sqrt{(1+\sin\alpha)}$, $\sin\frac{\alpha}{2}-\cos\frac{\alpha}{2}=-\sqrt{(1-\sin\alpha)}$,

故　$2\sin\frac{\alpha}{2}=-\sqrt{(1+\sin\alpha)}-\sqrt{(1-\sin\alpha)}$, $2\cos\frac{\alpha}{2}=-\sqrt{(1+\sin\alpha)}+\sqrt{(1-\sin\alpha)}$.

例　題　七

1. α 在 450° 與 630° 之間. 試表示

$2\sin\frac{\alpha}{2}=\pm\sqrt{(1+\sin\alpha)}\pm\sqrt{(1-\sin\alpha)}$ 之正負.

2. $2\sin\alpha=+\sqrt{(1+\sin2\alpha)}-\sqrt{(1-\sin2\alpha)}$, 則 α 在何者之間.
3. $2\cos\alpha=-\sqrt{(1+\sin2\alpha)}-\sqrt{(1-\sin2\alpha)}$, 則 α 在何者之間.
4. $2\sin\alpha=-\sqrt{(1+\sin2\alpha)}+\sqrt{(1-\sin2\alpha)}$, 則 α 在

$2n\pi+\frac{3\pi}{4}$ 與 $2n\pi+\frac{5\pi}{4}$ 之間.

例題解自 1 至 4.

(1) $\frac{\alpha}{2}$ 在 225° 與 315° 之間. 故如 3. 節之例解.

(2) $\sin\alpha+\cos\alpha=+\sqrt{(1+\sin2\alpha)}$, $\sin\alpha-\cos\alpha=-\sqrt{(1-\sin2\alpha)}$,

故　$\sin\alpha>\cos\alpha$. 而 $\cos\alpha$ 爲正. 故 α 在 $-\frac{\pi}{4}$ 與 $\frac{\pi}{4}$ 之間. 卽一般爲 $2n\pi\pm\frac{\pi}{4}$.

(3) $\sin\alpha+\cos\alpha=-\sqrt{(1+\sin2\alpha)}$, $\sin\alpha-\cos\alpha=+\sqrt{(1-\sin2\alpha)}$,

故　$\cos\alpha>\sin\alpha$, 而 $\cos\alpha$ 爲負. 故 α 在 $\frac{3\pi}{4}$ 與 $\frac{5\pi}{4}$ 之間. 卽一般在 $2n\pi+\frac{3\pi}{4}$ 與 $2n\pi+\frac{5\pi}{4}$ 之間.

(4) 與前例同樣.

第柒編

方程式

1. 方程式 三角函數之方程式.示其解法如次.

〔第一例〕 解 $\sin\theta=1$.

$\sin\theta=1=\sin\frac{\pi}{2}$, 故 $\theta=\frac{\pi}{2}$ 或 $n\pi+(-1)^n\frac{\pi}{2}$.

n 爲偶數. 卽 $2m$. 則 $\theta=2m\pi+\frac{\pi}{2}$.

n 爲奇數. 卽 $2m+1$. 則 $\theta=(2m+1)\pi-\frac{\pi}{2}=2m\pi+\frac{\pi}{2}$.

由是 θ 之根爲 $2m\pi+\frac{\pi}{2}$.

〔第二例〕 解 $\sin^2\theta=\sin^2\alpha$.

$\sin\theta=\pm\sin\alpha=\sin\alpha$ 或 $\sin(-\alpha)$, 故 $\theta=n\pi+(-1)^n\alpha$ 或 $n\pi-(-1)^n\alpha$, 由是 θ 之根爲 $n\pi\pm\alpha$.

例題八

解次列各方程式

1. $\cos\theta=0$.
2. $\sin\theta=\cos\alpha$.
3. $\cos\theta=1$.
4. $\tan\theta=1$.
5. $\cos\theta=\cos\alpha$.
6. $\tan\theta=\tan\alpha$.
7. $\sec\theta=\sec\alpha$.
8. $\operatorname{cosec}\theta=\operatorname{cosec}\alpha$.
9. $\sin\theta+\cos\theta=\frac{1}{\sqrt{2}}$.
10. $\cos\theta+\sin\theta=\sqrt{2}$.
11. $\cos\theta-\sin\theta=\sqrt{2}$.
12. $\sin^2\theta=\frac{3}{2}\cos\theta$.
13. $\sqrt{3}\sin\theta-\cos\theta=\sqrt{2}$.
14. $\sec^2\theta-\frac{5}{2}\sec\theta+1=0$.
15. $6\cot^2\theta=1+4\cos^2\theta$
16. $\sin3\theta+\cos3\theta=\frac{1}{\sqrt{2}}$.
17. $\sin\theta+\sin2\theta+\sin3\theta=0$

例題解自 1. 至 17.

(1) $\cos\theta=0=\cos\frac{\pi}{2}$, $\therefore\ \theta=2n\pi\pm\frac{\pi}{2}$.

(2) $\sin\theta=\sin\left(\frac{\pi}{2}-\alpha\right)=\sin\left\{n\pi+(-1)^n\left(\frac{\pi}{2}-\alpha\right)\right\}$ $\therefore\ \theta=n\pi+(-1)^n\left(\frac{\pi}{2}-\alpha\right)$,

n 爲偶數 $(2m)$, 則 $\theta=2m\pi+\frac{\pi}{2}-\alpha=(2m+\frac{1}{2})\pi-\alpha$,

n 爲奇數 $(2m+1)$, 則 $\theta=(2m+1)\pi-\frac{\pi}{2}+\alpha=(2m+\frac{1}{2})\pi+\alpha$,

由是 $\theta=(2m+\frac{1}{2})\pi\pm\alpha$.

(3) $\because\ \cos\theta=-1=\cos\pi$, $\therefore\ \theta=(2n+1)\pi$.

(4) $\because\ \tan\theta=1=\tan\frac{\pi}{4}$, $\therefore\ \theta=n\pi+\frac{\pi}{4}$.

(5) $\because\ \cos\theta=\cos(2n\pi\pm\alpha)$, $\therefore\ \theta=2n\pi\pm\alpha$ (6) $\theta=n\pi+\alpha$.

(7) $\theta=2n\pi\pm\alpha$. (8) $\theta=n\pi+(-1)^n\alpha$.

(9) $\sin\theta\cos\frac{\pi}{4}+\cos\theta\sin\frac{\pi}{4}=\frac{1}{2}$, 即 $\sin\left(\theta+\frac{\pi}{4}\right)=\frac{1}{2}=\sin\frac{\pi}{6}$,

故 $\theta+\frac{\pi}{4}=n\pi+(-1)^n\frac{\pi}{6}$ 即 $\theta=n\pi+(-1)^n\frac{\pi}{6}-\frac{\pi}{4}$,

n 爲偶數 $(2m)$, 則 $\theta=2m\pi+\frac{\pi}{6}-\frac{\pi}{4}=2m\pi-\frac{\pi}{12}$,

n 爲奇數 $(2m+1)$, 則 $\theta=(2m+1)\pi-\frac{\pi}{6}-\frac{\pi}{4}=2m\pi+\frac{7\pi}{12}$.

(10) $\sin\left(\theta+\frac{\pi}{4}\right)=1=\sin\frac{\pi}{2}$, $\therefore\ \theta=2m\pi+\frac{\pi}{4}$.

(11) $\cos\left(\theta+\frac{\pi}{4}\right)=1=\cos0$, 故 $\theta=2n\pi+\frac{\pi}{4}$ 或 $2n\pi-\frac{3\pi}{4}$.

(12) $1-\cos^2\theta=\frac{3}{2}\cos\theta$, 即 $\cos\theta=\frac{1}{2}$ 或 -2. 但就絕對値而論.

$\cos$ 比 1 小. 故 -2 可省略. 故 $\cos\theta=\frac{1}{2}=\cos\frac{\pi}{3}$,

故 $\theta=2n\pi\pm\frac{\pi}{3}$.

(13) $\frac{1}{2}\sqrt{3}\sin\theta-\frac{1}{2}\cos\theta=\frac{\sqrt{2}}{2}$, 即 $\cos\frac{\pi}{6}\sin\theta-\sin\frac{\pi}{6}\cos\theta=\frac{1}{\sqrt{2}}$,

即 $\sin\left(\theta-\frac{\pi}{6}\right)=\frac{1}{\sqrt{2}}=\sin\frac{\pi}{4}$, 故 $\theta=n\pi+\left(-1\right)^n\frac{\pi}{4}+\frac{\pi}{6}$.

(14) $\sec\theta=2$ 或 $\frac{1}{2}$, 即 $\sec\theta=2=\sec\frac{\pi}{3}$, 故 $\theta=2n\pi\pm\frac{\pi}{3}$.

又 $\sec$ 比 1 大. 故 $\frac{1}{2}$ 可省.

(15) $\frac{6\cos^2\theta}{1-\cos^2\theta}=1+4\cos^2\theta$, 故 $4\cos^4\theta+3\cos^2\theta=1$.

故 $\cos^2\theta=-1$, 或 $\frac{1}{4}$. 故 $\cos\theta=\pm\frac{1}{2}$. 故 $\theta=n\pi\pm\frac{\pi}{3}$.

(16) $3\theta=n\pi+(-1)^n\frac{\pi}{6}-\frac{\pi}{4}$ (例題 9.)

(17) $\sin\theta+2\sin\theta\cos\theta+3\sin\theta-4\sin^3\theta=0$.

故 $\sin\theta=0$, 或 $\cos\theta=0$, 或 $\cos\theta=-\frac{1}{2}$, 故 $\theta=\frac{n\pi}{2}$ 或 $2n\pi\pm\frac{2\pi}{3}$.

18. $\cos\theta-\cos 2\theta=\sin 3\theta.$

19. $\cos 2\theta-\cos 4\theta=\sin\theta.$

20. $\sin 4\theta-\sin 2\theta=\sin\theta.$

21. $\sin 6\theta=2\sin 4\theta-\sin 2\theta.$

22. $\sin 3\theta=2\sin\theta.$

23. $\tan\theta=2\sqrt{3}\cos\theta.$

24. $2\sin\theta=\tan\theta.$

25. $\tan\theta+\cot\theta=\frac{4}{\sqrt{3}}.$

26. $\cos 3\theta=\sin\theta.$

27. $2(\cos\theta+\sec\theta)=5.$

28. $\sin 3\theta+\sin\theta+\cos 4\theta=1.$

29. $\cos\theta+\tan\theta=\sec\theta.$

30. $\sin\theta+\sin 2\theta=\sin 3\theta+\sin 4\theta.$

31. $\sin\theta+\sin 2\theta=\cos\theta+\cos 2\theta.$

32. $\cos 2\theta=\cos\theta+\sin\theta.$

33. $4\sin\theta\cos\theta+1=2(\sin\theta+\cos\theta).$

34. $\cot\theta-\tan\theta=\cot\alpha-\tan\alpha.$

35. $\tan 5\theta=\tan\theta.$

36. $\cot\theta-\tan\theta=\cos\theta+\sin\theta.$

37. $\cos n\theta+\cos(n-2)\theta=\cos\theta$

38. $2\cot 2\theta-\tan 2\theta=3\cot 3\theta.$

39. $8\cot\theta=\sec^2\frac{\theta}{2}+\operatorname{cosec}^2\frac{\theta}{2}.$

40. $\sec^2\theta+\operatorname{cosec}^2\theta=3\sec^4\theta.$

41. $\operatorname{cosec}3\theta+\operatorname{cosec}2\theta=\sin\theta\operatorname{cosec}2\theta\operatorname{cosec}3\theta.$

例題解自 18. 至 41.

(18) $2\sin\frac{3\theta}{2}\sin\frac{\theta}{2}=2\sin\frac{3\theta}{2}\cos\frac{3\theta}{2}$, 故 $\sin\frac{3\theta}{2}=0$, 或 $\sin\frac{\theta}{2}=\cos\frac{3\theta}{2}$,

故 $\sin\frac{3\theta}{2}=\sin\pi$, 故 $\frac{3\theta}{2}=n\pi$, 卽 $\theta=\frac{2}{3}n\pi$,

或 $\cos\left(\frac{\pi}{2}-\frac{\theta}{2}\right)=\cos\left(2n\pi\pm\frac{3\theta}{2}\right)$, 故 $\frac{\pi}{2}-\frac{\theta}{2}=2n\pi\pm\frac{3\theta}{2}$.

(19) $2\sin 3\theta\sin\theta=\sin\theta$, 故 $\sin\theta=0$, 或 $\sin 3\theta=\frac{1}{2}$,

故 $\theta=n\pi$ 或 $\frac{1}{3}\left\{n\pi+(-1)^n\frac{\pi}{6}\right\}$. (20) $\theta=n\pi$, 或 $3\theta=2n\pi\pm\frac{\pi}{3}$.

(21) $2\sin 4\theta\cos 2\theta=2\sin 4\theta$, 故 $\sin 4\theta=0$, 或 $\cos 2\theta=1$,

由是 $\theta=\frac{1}{4}n\pi$, 或 $n\pi$.

(22) $3\sin\theta-4\sin^3\theta=2\sin\theta$, 故 $\sin\theta=0$, 或 $\pm\frac{1}{2}$. 卽 $\theta=n\pi$, $n\pi\pm\frac{\pi}{6}$.

(23) $\sin\theta=2\sqrt{3}\cos^2\theta$, 卽 $\sin\theta=2\sqrt{3}(1-\sin^2\theta)$, 故 $\sin\theta=\frac{\sqrt{3}}{2}$,

即 $\theta=n\pi+(-1)^n\frac{\pi}{3}$.　(24) $\theta=n\pi$ 或 $2n\pi\pm\frac{\pi}{3}$.

(25) $\frac{\cos^2\theta+\sin^2\theta}{\cos\theta\sin\theta}=\frac{4}{\sqrt{3}}$, 故 $\sin 2\theta=\frac{\sqrt{3}}{2}$, $2\theta=n\pi+(-1)^n\frac{\pi}{3}$.

(26) $\cos\left(\frac{\pi}{2}-\theta\right)-\cos 3\theta=0$, 即 $2\sin\left(\frac{\pi}{4}+\theta\right)\sin\left(2\theta-\frac{\pi}{4}\right)=0$.

故 $\frac{\pi}{4}+\theta=n\pi$, 或 $2\theta-\frac{\pi}{4}=n\pi$, 即 $\theta=(4n-1)\frac{\pi}{4}$, 或 $(4n+1)\frac{\pi}{8}$.

(27) $2\cos^2\theta+2=5\cos\theta$, $\cos\theta=\frac{1}{2}$, 故 $\theta=2n\pi\pm\frac{\pi}{3}$.

(28) $2\sin 2\theta\cos\theta+1-2\sin^2 2\theta=1$, 故 $\sin 2\theta=0$, 即 $\theta=\frac{n\pi}{2}$,

或 $\cos\theta=\cos\left(\frac{\pi}{2}-2\theta\right)$, 即 $\theta=2n\pi\pm\left(\frac{\pi}{2}-2\theta\right)$.

(29) $\cos^2\theta+\sin\theta=1$, 即 $\sin\theta=\sin^2\theta$. 故 $\theta=n\pi$ 或 $n\pi+\frac{\pi}{2}$.

(30) $\sin\theta-\sin 3\theta=\sin 4\theta-\sin 2\theta$, 即 $-2\cos 2\theta\sin\theta=2\cos 3\theta\sin\theta$,

故 $\sin\theta=0$, 或 $\cos 2\theta=-\cos 3\theta=\cos(\pi-3\theta)$. 故 $\theta=n\pi$, 或 $\theta=(2n+1)\frac{\pi}{5}$.

(31) $\theta=(4n\pm1)\pi$, 或 $\frac{n\pi}{3}+\frac{\pi}{6}$.

(32) $\cos^2\theta-\sin^2\theta=\cos\theta+\sin\theta$. 故 $\cos\theta+\sin\theta=0$, 或 $\cos\theta-\sin\theta=1$.

即 $\cos\left(\frac{\pi}{4}+\theta\right)=\frac{1}{\sqrt{2}}$. 故 $\theta=2n\pi$ 或 $2n\pi-\frac{\pi}{2}$.

(33) $(1-2\sin\theta)(1-2\cos\theta)=0$, 故 $\theta=2n\pi\pm\frac{\pi}{3}$ 或 $n\pi+(-1)^n\frac{\pi}{6}$.

(34) $\tan\theta=\tan\alpha$, 或 $\tan\theta=\tan\left(\frac{\pi}{2}+\alpha\right)$, 即 $\theta=n\pi+\alpha$ 或 $n\pi+\frac{\pi}{2}+\alpha$.

(35) $\theta=\frac{n\pi}{4}$.

(36) $\frac{\cos^2\theta\sin^2\theta}{\sin\theta\cos\theta}=\cos\theta+\sin\theta$, 故 $\cos\theta+\sin\theta=0$, 或 $\cos\theta-\sin\theta=\sin\theta\cos\theta$.

即 $1-\sin 2\theta=\frac{1}{4}\sin^2 2\theta$. 故 $\sin 2\theta=2(\sqrt{2}-1)$.

(37) $2\cos(n-1)\theta\cos\theta=\cos\theta$. 故 $\cos\theta=0$, 或 $\cos(n-1)\theta=\frac{1}{2}$.

即 $\theta=(4n+1)\frac{\pi}{2}$ 或 $\frac{6n\pm1}{n-1}\cdot\frac{\pi}{2}$.

(38) $2(\cot 2\theta-\cot 3\theta)=\cot 3\theta+\tan 2\theta$. 即 $\frac{2\sin\theta}{\sin 2\theta\sin 3\theta}=\frac{\cos\theta}{\sin 3\theta\cos 2\theta}$,

$2\sin\theta(2\cos^2\theta-1)=2\sin\theta\cos\theta$, 故 $\theta=n\pi$ 或 $(2n+1)\frac{\pi}{3}$.

(39) $\frac{8\cos\theta}{\sin\theta}=\frac{1}{\sin^2\frac{1}{2}\theta\cos^2\frac{1}{2}\theta}$. 即 $2\sin\theta\cos\theta=1$, $\theta=n\pi+\frac{\pi}{4}$.

(40) $n\pi+\frac{\pi}{6}$.

(41) $\sin 3\theta+\sin 2\theta=\sin\theta$. 即由 $3\sin\theta-4\sin^3\theta+2\sin\theta\cos\theta=\sin\theta$.

得 $\theta=n\pi$, $2n\pm\frac{\pi}{3}$, $(2n\pm1)\pi$.

42. $2\sin^2\theta=\cos^2\frac{3\theta}{2}$.　43. $2\sin^2\theta+\sin^2 2\theta=2$.

44. $\sin^2 2\theta-\sin^2\theta=\sin^2\frac{\pi}{6}$.　45. $\operatorname{cosec}\theta=\operatorname{cosec}\frac{\theta}{2}$.

46. $\sin\theta\sin 3\theta=\frac{1}{2}$.　47. $4\sin^2\theta+\sin^2 2\theta=3$.

48. $(1-\tan\theta)(1+\sin 2\theta)=1+\tan\theta$.

49. $\sin\theta+\sin 2\theta+\sin 3\theta+\sin 4\theta=0$.

50 $\sin\theta-\cos\theta=4\sin\theta\cos^2\theta$.　51. $\tan 2\theta=\tan\frac{2}{\theta}$.

52. $\cos\theta\cos 3\theta=\cos 5\theta\cos 7\theta$.　53. $\tan^2\theta+\cot^2\theta=2$.

54. $\sin 3\theta=8\sin^3\theta$.　55. $\sin 9\theta+\sin 5\theta+2\sin^2\theta=1$.

56. $\sin 7\theta-\sin\theta=\sin 3\theta$.　57. $\sin 5\theta=\cos 2\theta$.

58. $\sin 3\theta+\sin 5\theta=\sqrt{2}\cos\theta$.　59. $\tan 3\theta=3\tan\theta$.

60. $\cos\theta-\cos 5\theta=\sin 2\theta$.　61. $\cos\theta+\cos 2\theta=\sin 3\theta$.

62. $\sin 4\theta-\sin 2\theta=\sqrt{3}\sin\theta$.　63. $3\tan^2 2\theta=1$.

64. $\tan\theta+\tan 2\theta=\tan 3\theta$.　65. $\cos 2\theta+\sin\theta+\cos^2\theta=\frac{7}{4}$.

66. $2\sin^2\theta+3\cos\theta=3$.　67. $\sin\theta+\cos\theta=\sin 2\theta+\cos 2\theta$.

68. $\cos\theta+\cos 7\theta=\cos 4\theta$.

69. $\cos\theta+\cos 3\theta=\cos 2\theta+\cos 4\theta$.

70. $\sin\theta+\sin 3\theta+\sin 5\theta+\sin 7\theta=0$.

71. $\sin 5\theta=16\sin^5\theta$.　72. $\tan 2\theta=8\cos^2\theta-\cot\theta$.

73. $4\sin^2\theta-2(\sqrt{3}+1)\sin\theta+\sqrt{3}=0$.

74. $\sqrt{3}\tan^2\theta-(\sqrt{3}+1)\tan\theta+1=0$.

75. $\tan^2\theta-\tan\theta(\sqrt{3}+1)+\sqrt{3}=0$.

76. $8\sin^4\theta-6\sin^2\theta+1=0$.　77. $4\sin^2\theta+3\operatorname{cosec}^2\theta=7$.

78. $16\sin^4\theta-16\sin^2\theta+3=0$.

79. $(4-\sqrt{3})(\sec\theta+\operatorname{cosec}\theta)=4(\sin\theta\tan\theta+\cos\theta\cot\theta)$.

例題解自 42. 至 79.

(42) $8\sin^2\frac{\theta}{2}\cos^2\frac{\theta}{2}=\cos^2\frac{\theta}{2}\left(4\cos^2\frac{\theta}{2}-3\right)^2$，故 $\theta=(2n+1)\pi$ 或 $2n\pi\pm\frac{\pi}{12}$.
或 $2n\pi\pm\frac{5\pi}{12}$.

(43) $(2n+1)\frac{\pi}{2}$，或 $n\pi+(-1)^n\frac{\pi}{4}$.

(44) $n\pi\pm\frac{\pi}{10}$ 或 $n\pi\pm\frac{3\pi}{10}$. (45) $4n\pi$ (46) $(2n+1)\frac{\pi}{4}$, 或 $n\pi\pm\frac{\pi}{6}$.

(47) $\cos 2\theta(\cos 2\theta+2)=0$. 故 $\theta=(2n+1)\frac{\pi}{4}$.

(48) $n\pi$ 或 $(4n-1)\frac{\pi}{4}$. (49) $(4n\pm1)\frac{\pi}{2}$ 或 $\frac{2n\pi}{5}$ 或 $(2n+1)\pi$.

(50) $\cos\theta=-(3\sin\theta-4\sin^3\theta)$. 故 $\cos\theta=\cos\left(\frac{\pi}{2}+3\theta\right)$, $\theta=2n\pi\pm\left(\frac{\pi}{2}+3\theta\right)$.

(51) $2\theta=n\pi+\frac{2}{\theta}$, 故 $\theta=\frac{1}{4}(n\pi\pm\sqrt{n^2\pi^2+16})$.

(52) $\cos 2\theta+\cos 4\theta=\cos 2\theta+\cos 12\theta$, 故 $\theta=\frac{n\pi}{4}$ 或 $\frac{n\pi}{8}$.

(53) $\tan^4\theta-2\tan^2\theta+1=0$. 即 $\tan^2\theta-1=0$, 故 $\theta=n\pi\pm\frac{\pi}{4}$.

(54) $\theta=n\pi$ 或 $n\pi\pm\frac{\pi}{6}$.

(55) $\cos 2\theta(2\sin 7\theta-1)=0$, $\theta=(2n+1)\frac{\pi}{4}$, 或 $7\theta=n\pi+(-1)^n\frac{\pi}{6}$.

(56) $\theta=\frac{n\pi}{3}$, 或 $4\theta=2n\pi\pm\frac{\pi}{3}$. (57) $2\theta=2n\pi\pm\left(\frac{\pi}{2}-5\theta\right)$.

(58) $\theta=(4n\pm1)\frac{\pi}{2}$, 或 $4\theta=n\pi+(-1)^n\frac{\pi}{4}$. (59) $\theta=n\pi$.

(60) $\theta=\frac{n\pi}{2}$, 或 $3\theta=n\pi+(-1)^n\frac{\pi}{4}$. (61) $\theta=\frac{(2n+1)\pi}{3}$. 或 $2\theta=2n\pi\pm\left(\frac{\pi}{2}-\frac{3\theta}{2}\right)$. (62) $\theta=n\pi$ 或 $(12n\pm1)\frac{\pi}{18}$. (63) $2\theta=n\pi\pm\frac{\pi}{6}$.

(64) $\frac{\sin 3\theta}{\cos\theta\cos 2\theta}=\frac{\sin 3\theta}{\cos 3\theta}$. 故 $\sin 3\theta=0$, 或 $\cos\theta\cos 2\theta=\cos 3\theta$, 即 $\cos 3\theta+\cos\theta=2\cos 3\theta$. 故 $\cos 3\theta=\cos\theta$. 或 $\theta=\frac{n\pi}{3}$ 或 $\frac{n\pi}{2}$ 或 $n\pi$.

(65) $1-2\sin 2\theta+\sin\theta+1-\sin^2\theta=\frac{7}{4}$. 故 $\theta=n\pi+(-1)^n\frac{\pi}{6}$, 或 $\sin\theta=-\frac{1}{6}$.

(66) $\theta=(4n+1)\frac{\pi}{2}$ 或 $2n\pi\pm\frac{\pi}{3}$. (67) $\theta=2n\pi$, 或 $\frac{3\theta}{2}=2n\pi\pm\left(\frac{\pi}{2}-\frac{3\theta}{2}\right)$.

(68) $\theta=(2n+1)\frac{\pi}{8}$, 或 $3\theta=2n\pi\pm\frac{\pi}{3}$.

(69) $\theta=(2n+1)\frac{\pi}{2}$ 或 $2\theta=2n\pi\pm3\theta$.

(70) $4\sin 4\theta\cos 2\theta\cos\theta=0$. 故 $\theta=\frac{n\pi}{4}$ 或 $(2n+1)\frac{\theta}{4}$ 或 $(2n+1)\frac{\pi}{2}$.

(71) $16\sin^5\theta-20\sin^5\theta+5\sin\theta=16\sin^5\theta$. 即 $\sin\theta(4\sin^2\theta-1)=0$, 故 $\theta=n\pi$ 或 $n\pi\pm\frac{\pi}{6}$. (72) $\frac{\sin 2\theta}{\cos 2\theta}=8\cos^2\theta-\frac{\cos\theta}{\sin\theta}$, 故 $\cos\theta=0$, 或 $\frac{2\sin^2\theta}{\cos 2\theta}=8\sin\theta\cos\theta-1$, 即 $2\sin 4\theta=1$. 故 $\theta=(2n+1)\frac{\pi}{2}$, 或 $4\theta=n\pi+(-1)^n\frac{\pi}{6}$. (73) $n\pi+(-1)^n\frac{\pi}{6}$ 或 $n\pi+(-1)^n\frac{\pi}{3}$. (74) $n\pi+\frac{\pi}{4}$ 或 $n\pi+\frac{\pi}{6}$. (75) $n\pi+\frac{\pi}{4}$ 或 $n\pi+\frac{\pi}{3}$.

(76) $n\pi\pm\frac{\pi}{4}$ 或 $n\pi\pm\frac{\pi}{6}$. (77) $(2n+1)\frac{\pi}{2}$ 或 $n\pi\pm\frac{\pi}{3}$.

(78) $n\pi\pm\frac{\pi}{6}$ 或 $n\pi\pm\frac{\pi}{3}$. (79) $\theta=(4n-1)\frac{\pi}{4}$, 或 $2\theta=n\pi+(-1)^n\frac{\pi}{3}$.

80. $\sin(\theta+\alpha)=\cos(\theta-\alpha)$.

81. $\cos(\theta+\alpha)+\cos\theta=2\cos\frac{\alpha}{2}$.

82. $\sin(\theta+\alpha)+\cos(\theta+\alpha)=\sin(\theta-\alpha)+\cos(\theta-\alpha)$.

83. $\sin\alpha+\sin(\theta-\alpha)+\sin(2\theta+\alpha)=\sin(\theta+\alpha)+\sin(2\theta-\alpha)$.

84. $4\sin\theta\sin(\theta-\alpha)=2\cos\alpha-1$.

85. $\cos(x+\frac{3}{2})\alpha+\cos(x+\frac{1}{2})\alpha=\sin\alpha$.

86. $x^2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)+x\cos(\alpha-\beta)=2\cos\frac{\beta}{2}$.

87. $\cot 2^{x-1}\alpha-\cot 2^x\alpha=\operatorname{cosec}4\alpha$.

88. $\cos\beta\sqrt{(a^2-x^2)}+a\sin\alpha=x\sin\beta$.

89. $m\operatorname{vers}\theta=n\operatorname{vers}(\alpha-\theta)$.

90. $\cos n\theta+\cos(n-2)\theta=\cos\theta$.

91. $\cos3\theta+\cos5\theta+\sqrt{2}(\cos\theta+\sin\theta)\cos\theta=0$.

92. $8\sin\left(\theta-\frac{\pi}{3}\right)\cos^3\theta+8\cos\left(\theta-\frac{\pi}{3}\right)\sin^3\theta-6\sin\left(2\theta-\frac{\pi}{3}\right)=\sqrt{3}$.

93. $\cos^2\theta-\cos^2\alpha=2\cos^3\theta(\cos\theta-\cos\alpha)-2\sin^3\theta(\sin\theta-\sin\alpha)$.

94. $\tan\theta+2\cot2\theta=\sin\theta\left(1+\tan\theta\tan\frac{\theta}{2}\right)$.

95. $\cos^3\theta\sin3\theta+\sin^3\theta\cos3\theta=\frac{3}{4}$.

96. $\sec\left(\frac{\pi}{4}+\theta\right)+\sec\left(\frac{\pi}{4}-\theta\right)=2\sqrt{2}$.

97. $\sin\frac{p+1}{2}\theta+\sin\frac{p-1}{2}\theta=\cos\frac{\theta}{2}$.

98. $\cos2\theta-\cos120^\circ=\cos\theta-\cos60^\circ$.

99. $\sin(\theta+2\alpha)-\sin(2\theta+\alpha)=\sin\frac{\alpha-\theta}{2}$.

100. $\sin5\theta+\sin3\theta+\sqrt{2}(\sin\theta+\cos\theta)\cos\theta=0$.

101. $\sin(\theta-\alpha)=\sin\theta-\sin\alpha$.

例題解自 80. 至 101.

(80) $\cos(\theta-\alpha)=\cos\left(\frac{\pi}{2}-\theta-\alpha\right)$. 故 $\theta-\alpha=2n\pi\pm\left(\frac{\pi}{2}-\theta-\alpha\right)$, $\theta=(4n+1)\frac{\pi}{4}$.

(81) $\theta=2n\pi-\frac{\alpha}{2}$. (82) $\theta=(4n+1)\frac{\pi}{4}$.

(83) $\sin\alpha+2\sin\frac{3\theta}{2}\cos\frac{\theta+2\alpha}{2}=2\sin\frac{3\theta}{2}\cos\frac{\theta-2\alpha}{2}$. 卽

$4\sin\frac{3\theta}{2}\sin\frac{\theta}{2}\sin\alpha=\sin\alpha$, 卽 $2\cos^2\theta-\cos\theta=\frac{1}{2}$. 故 $\theta=2n\pi\pm\frac{\pi}{5}$. (84) $2n\pi\pm\frac{3\pi}{5}$.

(85) $\cos(x+1)\alpha=\cos\left(\frac{\pi}{2}-\frac{1}{2}\alpha\right)$. 故 $\alpha x=(4n\pm1)\frac{\pi}{2}\mp\frac{1}{2}\alpha-\alpha$.

(86) $x=\dfrac{-\cos(\alpha-\beta)\pm\sqrt{\left\{\cos^2(\alpha-\beta)+8\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)\cos\frac{\beta}{2}\right\}}}{2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)}$,

但根號內之式$=\cos^2(\alpha-\beta)+4\cos\alpha\{\cos(\alpha-\beta)+\cos\alpha\}=\{\cos(\alpha-\beta)+2\cos\alpha\}^2$,

故 $x=\dfrac{-\cos(\alpha-\beta)\pm\{\cos(\alpha-\beta)+2\cos\alpha\}}{2\cos\alpha\cos\left(\alpha-\frac{\beta}{2}\right)}=\dfrac{1}{\cos\left(\alpha-\frac{\beta}{2}\right)}$ 或 $\dfrac{-2\cos\frac{\beta}{2}}{\cos\alpha}$.

(87) $\dfrac{\sin(2^x-2^{x-1})\alpha}{\sin 2^{x-1}\alpha\sin 2^x\alpha}=\dfrac{1}{\sin 4\alpha}$. 卽 $\dfrac{1}{\sin 2^x\alpha}=\dfrac{1}{\sin 4\alpha}$. 故 $x=2$.

(88) $\pm\alpha\cos(\alpha\pm\beta)$. (89) $\cot\frac{\theta}{2}=\cot\frac{\alpha}{2}\pm\dfrac{1}{\sin\frac{\alpha}{2}}\sqrt{\dfrac{m}{n}}$.

(90) $\theta=(2n+1)\frac{\pi}{2}$, 或 $(n-1)\theta=2n\pi\pm\frac{\pi}{3}$.

(91) $\cos\theta\left\{\cos 4\theta+\cos\left(\frac{\pi}{4}+\theta\right)\right\}=0$, $\theta=(2n+1)\frac{\pi}{2}$, 或 $4\theta=2n\pi\pm\left(\frac{3\pi}{4}-\theta\right)$.

(92) $2\sin\left(\theta-\frac{\pi}{3}\right)(\cos 3\theta+3\cos\theta)+2\cos\left(\theta-\frac{\pi}{3}\right)(3\sin\theta-\sin 3\theta)$

$-6\sin\left(2\theta-\frac{\pi}{3}\right)=\sqrt{3}$, 卽 $-2\sin\left(2\theta+\frac{\pi}{3}\right)+6\sin\left(2\theta-\frac{\pi}{3}\right)$

$-6\sin\left(2\theta-\frac{\pi}{3}\right)=\sqrt{3}$ 故 $2\theta+\frac{\pi}{3}=n\pi+(-1)^n\frac{4\pi}{3}$.

(93) $2(\cos^2\theta-\cos^2\alpha)=(3\cos\theta+\cos 3\theta)(\cos\theta-\cos\alpha)-(3\sin\theta-\sin 3\theta)(\sin\theta-\sin\alpha)$,

故 $\cos 2\theta-\cos(3\theta-\alpha)-3\cos(\theta+\alpha)=3\sin^2\theta+\cos^2\theta-2\cos^2\alpha-1$,

卽 $\cos 2\theta-\cos(2\theta-\alpha)-3\cos(\theta+\alpha)=-2\cos 2\theta-\cos 2\alpha$,

故 $3\cos 2\theta-3\cos(\theta+\alpha)-\cos(3\theta-\alpha)+\cos 2\alpha=0$,

故 $4\sin\frac{3\theta+\alpha}{2}\sin^3\frac{\alpha-\theta}{2}=0$, 故 $\theta=\frac{2n\pi}{3}-\frac{\alpha}{3}$, 或 $\alpha-\theta=2n\pi$.

(94) $(2n+1)\frac{\pi}{4}$. (95) $(4n+1)\frac{\pi}{8}$. (96) $\theta=2n\pi$ 或 $2n\pi\pm\frac{2\pi}{3}$.

(97) $\theta=(4n\pm1)\pi$, $p\theta=2n\pi+(-1)^n\frac{\pi}{3}$. (98) $(2n+1)\frac{\pi}{2}$, 或 $2n\pi\pm\frac{\pi}{3}$.

(99) $\alpha-\theta=2n\pi$, 或 $3\theta+3\alpha=4n\pi\pm\frac{2\pi}{3}$.

(100) $\theta=(2n+1)\frac{\pi}{2}$, 或 $4\theta=n\pi+(-1)^n\left(\frac{5\pi}{4}+\theta\right)$. (101) $2n\pi+\alpha$.

102. $\sin\frac{n+1}{2}\theta+\sin\frac{n-1}{2}\theta=\sin\theta.$

103. $\tan(\alpha+\theta)\tan(\alpha-\theta)=\frac{1-2\cos2\alpha}{1+2\cos2\alpha}.$

104. $(\cot\theta-\tan\theta)^2(2-\sqrt{3})=4(2+\sqrt{3}).$

105. $2\sqrt{2}\cos\left(\frac{\pi}{4}+\theta\right)(1+\sin\theta)=1+\cos2\theta.$

106. $\cos3\theta-(\sqrt{3}+1)\cos2\theta+(\sqrt{3}+3)\cos\theta-\sqrt{3}-1=0.$

107. $\sec^2\theta\,\mathrm{cosec}^2\theta+2\,\mathrm{cosec}^2\theta=8.$

108. $\sin p\theta+\sin q\theta+\sin(p+q)\theta=0.$

109. $2\sin2\theta-4\sin\left(\theta+\frac{\pi}{6}\right)+\sqrt{3}=0.$

110. $\frac{\tan(\theta-15^\circ)}{\tan(\theta+15^\circ)}=\frac{1}{3}.$

111. $\frac{1}{\sin3\theta}+\frac{1}{\sin2\theta}=\frac{\sin2\theta}{\sin\theta\sin3\theta}.$

112. $\frac{\tan m\theta}{\tan n\theta}=1.$

113. $\frac{\sin^4\theta}{\sin^2\alpha}+\frac{\cos^4\theta}{\cos^2\alpha}=1.$

114. $\tan\theta+\tan2\theta+\tan3\theta=0.$

115. $\tan\theta+\tan n\theta+\tan m\theta=\tan\theta\tan n\theta\tan m\theta.$

116. $\tan\left(\frac{\pi}{2\sqrt{2}}\sin\theta\right)=\cot\left(\frac{\pi}{2\sqrt{2}}\cos\theta\right).$

117. $2(\sin^4\theta+\cos^4\theta)=1.$

118. $\sec4\theta-\sec2\theta=2.$

119. $3(\sin^4\theta-\cos^4\theta)+4\cos^6\theta=\cos^3 2\theta.$

120. $a\tan(\theta-\alpha)+b\tan(\theta+\alpha)=(a-b)\cot\alpha.$

121. $\mathrm{cosec}^2\frac{\theta}{2}-\sec^2\frac{\theta}{2}=2\sqrt{3}\,\mathrm{cosec}^2\theta.$

122. $3\cos^2\theta+2\sqrt{3}\cos\theta=5\frac{1}{4}.$

123. $\mathrm{cosec}4\alpha-\mathrm{cosec}4\theta=\cot4\alpha-\cot4\theta.$

124. $(1+\sin\theta)(1-2\sin\theta)^2=(1-\cos\theta)(1+2\cos\theta)^2.$

例 題 解 自 102. 至 124.

(102) $\theta=(2m+1)\pi$ 或 $\frac{4m\pi}{n-1}$ 或 $\frac{2(2m+1)\pi}{n+1}$.

(103) $\theta=n\pi\pm\frac{\pi}{6}$. (104) $(\cot\theta-\tan\theta)^2=4(2+\sqrt{3})^2$. 故

$\cot\theta-\tan\theta=\pm2(2+\sqrt{3})$, $\theta=n\pi\pm\frac{\pi}{12}$.

(105) $\theta=2n\pi-\frac{\pi}{2}$, 或 $2n\pi$. (106) $2n\pi\pm\frac{\pi}{3}$ 或 $2n\pi\pm\frac{\pi}{6}$.

(107) $\theta=(2n+1)\frac{\pi}{4}$, 或 $n\pi\pm\frac{2\pi}{3}$.

(108) $\frac{2n\pi}{p+q}$ 或 $\frac{(2n+1)\pi}{p}$, 或 $\frac{(2n+1)\pi}{q}$.

(109) $\theta=2n\pi\pm\frac{\pi}{6}$. (110) $\frac{\tan(\theta+15^\circ)+\tan(\theta-15^\circ)}{\tan(\theta+15^\circ)-\tan(\theta-15^\circ)}=\frac{3+1}{3-1}$, 卽 $\frac{\sin2\theta}{\sin30^\circ}=2$. 故 $\sin2\theta=1$. 故 $\theta=(4n+1)\frac{\pi}{4}$. (111) $2\theta=n\pi+(-1)^n\theta$.

(112) $\theta=\frac{p\pi}{m-n}$. (113) $\sin^2\theta\left(\frac{\sin^2\theta}{\sin^2a}-1\right)+\cos^2\theta\left(\frac{\cos^2\theta}{\cos^2a}-1\right)=0$. 故 $(\sin^2\theta-\sin^2a)\sin(\theta\pm a)=0$, 故 $\theta=n\pi\pm a, n\pi+(-1)^na, n\pi+(-1)^n(\pi+a)$,

(114) $\frac{\sin4\theta}{\cos\theta\cos3\theta}+\frac{\sin2\theta}{\cos2\theta}=0$. 故 $\sin2\theta=0$ 或

$\frac{2\cos2\theta}{\frac{1}{2}(\cos4\theta+\cos2\theta)}+\frac{1}{\cos2\theta}=0$, 卽 $\cos\theta=\frac{-1\pm5}{12}$.

故 $\theta=\frac{n\pi}{2}$ 或 $n\pi\pm\frac{2\pi}{3}$, $\cos2\theta=\frac{1}{3}$.

(115) $\tan\theta+\frac{\tan n\theta+\tan m\theta}{1-\tan n\theta\tan m\theta}=0$. 卽 $\tan(n+m)\theta=-\tan\theta$,

故 $(n+m)\theta=p\pi+(\pi-\theta)$.

(116) $\tan\left(\frac{\pi}{2\sqrt{2}}\sin\theta\right)=\tan\left(\frac{\pi}{2}-\frac{\pi}{2\sqrt{2}}\cos\theta\right)$, 故 $\frac{1}{\sqrt{2}}\sin\theta=1-\frac{1}{\sqrt{2}}\cos\theta$, 卽 $\cos\left(\theta-\frac{\pi}{4}\right)=1$, 故 $\theta=(4n+1)\frac{\pi}{2}+\frac{\pi}{4}$. (117) $\theta=(2n+1)\frac{\pi}{4}$.

(118) $\cos2\theta-\cos4\theta=2\cos2\theta\cos4\theta=\cos2\theta+\cos6\theta$, 故 $\cos6\theta=\cos(\pi-4\theta)$, 由是 $6\theta=2n\pi\pm(\pi-4\theta)$. (119) $n\pi$. (120) $\tan\theta=\frac{a-b}{a+b}\cot a$.

(121) $\theta=2n\pi\pm\frac{\pi}{6}$. (122) $\theta=2n\pi\pm\frac{\pi}{6}$.

(123) $\frac{\sin4\theta-\sin4a}{\sin4\theta\sin4a}=\frac{\sin(4\theta-4a)}{\sin4\theta\sin4a}$, 故 $\sin(2\theta-2a)=1$,

或 $\cos(2\theta+2a)=\cos\{2n\pi\pm(2\theta-2a)\}$, 故 $\theta=(4n+1)\frac{\pi}{4}+a$ 或 $\frac{n\pi}{2}$.

(124) $(1+\sin\theta)(1-4\sin\theta+4\sin^2\theta)=(1-\cos\theta)(1+4\cos\theta+4\cos^2\theta)$, 故 $3(\cos\theta+\sin\theta)-4(\cos^3\theta+\sin^3\theta)=0$. 故 $\theta=n\pi-\frac{\pi}{4}$, 或 $\frac{n\pi}{2}+(-1)^n\frac{\pi}{12}$.

125. $7\cos 3\theta=\sin^2\theta+\cos 2\theta$.　　126. $\tan 5\theta=5\tan\theta$.

127. $vers\,\theta=vers(\theta-\alpha)$.　　128. $\tan\theta+\cot\theta=4$.

129. $4(1+8\,cosec^2 4\theta)=(\tan^2\theta+\cot^2\theta)(\tan^2\alpha+\cot^2\alpha)$.

130. $\sin^2 2\theta-\sin^2\theta=\frac{1}{4}$.

131. $\tan\theta+\tan 2\theta+\tan 3\theta+\tan 4\theta=0$.

132. $\sin 3\theta=\sin\theta\cos 2\theta$.

133. $16^{\cos^2 x+2\sin^2 x}+4^{2\cos^2 x}=40$.

134. $7346\times 7^{\sec x}+7^{1+\sec x}-7010\times 7^{2\sec x}$
$-7^{3+2\sec x}+3\times 7^{2+3\sec x}=147$.

135. $\left.\begin{array}{l}\sin\theta+\sin\phi=p\\ \cos\theta+\cos\phi=q\end{array}\right\}$.　　136. $\left.\begin{array}{l}\cos\theta=\tan\phi\\ \cot\theta=\sin\phi\end{array}\right\}$.

137. $\left.\begin{array}{l}2\sin(\theta-\phi)=1\\ \sin(\theta+\phi)=1\end{array}\right\}$.　　138. $\left.\begin{array}{l}\cos(2\theta+3\phi)=\frac{1}{2}\\ \cos(3\theta+2\phi)=\frac{\sqrt{3}}{2}\end{array}\right\}$.

139. $\left.\begin{array}{l}a\sin\theta+b\cos\phi=c\\ a\,cosec\,\theta+b\sec\phi=c\end{array}\right\}$.　　140. $\left.\begin{array}{l}a\sec^2\theta-b\cos\phi=2a\\ b\cos^2\theta-a\sec\phi=2b\end{array}\right\}$.

例題解自 125 至 140.

(125) $\theta=(2n+1)\frac{\pi}{2}$，或 $\cos\theta=\frac{1\pm\sqrt{2353}}{56}$.

(126) $\frac{\sin 5\theta}{\cos 5\theta}=\frac{5\sin\theta}{\cos\theta}$. 由例題第四33.及34.而從

$\frac{16\sin^5\theta-20\sin^3\theta+5\sin\theta}{16\cos^5\theta-20\cos^3\theta+5\cos\theta}=\frac{5\sin\theta}{\cos\theta}$ 得 $\theta=n\pi$，或 $\cos 2\theta=-\frac{1}{4}$.

(127) $\theta=n\pi+\frac{\alpha}{2}$.　(128) $n\pi+\frac{\pi}{12}$，或 $n\pi+\frac{5\pi}{12}$.

(129) 試變原方程式爲 $\sin 2\theta$ 之項.則

$\sin^4 2\theta=\frac{2\sin^2 2\theta}{\sin^2 2\alpha}-\frac{4\cos^2 2\alpha}{\sin^2 2\alpha}$，故 $\sin 2\theta=\pm\sqrt{2}\cot 2\alpha$.

(130) $\theta=n\pi+\frac{\pi}{10}$，$\theta=n\pi+\frac{3\pi}{10}$.

(131) $\frac{\sin5\theta}{\cos\theta\cos4\theta}+\frac{\sin5\theta}{\cos2\theta\cos3\theta}=0$, 故 $\sin5\theta=0$, 或 $\cos2\theta\cos3\theta+\cos\theta\cos4\theta=0$,

卽 $\cos2\theta(4\cos^3\theta-3\cos\theta)+\cos\theta(2\cos^2 2\theta-1)=0$, 故 $\cos\theta=0$, 或

$\cos2\theta(2\cos2\theta-1)+2\cos^2 2\theta-1=0$, 故 $\theta=\frac{n\pi}{5}$ 或 $(2n+1)\frac{\pi}{2}$,

(132) $\theta=n\pi$, 或 $(2n+1)\frac{\pi}{2}$.

(133) $4^{2\cos^2x}+4^{\sin^2x}+4^{1+\cos2x}=40$. 卽 $4^3\cdot4^{-\cos2x}+4\cdot4^{\cos2x}=40$,

故 $4^{\cos2x}=4^{\frac{1}{2}}$ 或 $4^{\frac{3}{2}}$, 故 $\cos2x=\frac{1}{2}$. 故 $x\pi=n\pm\frac{\pi}{6}$.

(134) $7346\times7^{\sec x}+7\times7^{\sec x}-7010\times7^{2\sec x}-343\times7^{2\sec x}+147\times7^{3\sec x}=147$,

$(7^{\sec x}-1)\{147(7^{2\sec x}+7^{\sec x}+1)-7353\times7^{\sec x}\}=0$, 故

$7^{\sec x}=1$, 或 $147\times7^{2\sec x}-7206\times7^{\sec x}+147=0$, 卽 $7^{\sec x}=49$ 或 $\frac{1}{49}$.

由是 $7^{\sec x}=7^0$ 或 7^2 或 7^{-2}, 故 $x=2n\pi\pm\frac{\pi}{3}$ 或 $2n\pi\pm\frac{2\pi}{3}$.

(135) $(p-\sin\theta)^2+(q-\cos\theta)^2=\sin^2\phi+\cos^2\phi=1$. 卽 $p\sin\theta+q\cos\theta=\frac{1}{2}(p^2+q^2)$,

$\frac{q}{p}=\tan\alpha$, 則 $\sin\alpha=\frac{q}{\sqrt{(p^2+q^2)}}$, $\cos\alpha=\frac{p}{\sqrt{(p^2+q^2)}}$, 故

$\sqrt{p^2+q^2}(\sin\theta\cos\alpha+\cos\theta\sin\alpha)=\frac{1}{2}(p^2+q^2)$, 令 $\frac{1}{2}\sqrt{(p^2+q^2)}=\sin\beta$, 則

$\sin(\theta+\alpha)=\sin\beta$, 故 $\theta=n\pi+(-1)^n\beta-\alpha$.

(136) $\cos\theta=\frac{\sin\phi}{\cos\phi}=\frac{\cot\theta}{\cos\phi}$, 故 $\cos\phi=\frac{\cot\theta}{\cos\theta}=\frac{1}{\sin\theta}$,

故 $\cot^2\theta+\frac{1}{\sin^2\theta}=\sin^2\phi+\cos^2\phi=1$, 故 $\cos2\theta=-1$, $\theta=(2n+1)\frac{\pi}{2}$.

(137) $\theta-\phi=n\pi+(-1)^n\frac{\pi}{6}$, $\theta+\phi=(4m+1)\frac{\pi}{2}$, 故

$2\theta=n\pi+(4m+1)\frac{\pi}{2}+(-1)^n\frac{\pi}{6}$, 卽 $2\phi=(4m+1)\frac{\pi}{2}-n\pi-(-1)^n\frac{\pi}{6}$.

(138) $2\theta+3\phi=2n\pi\pm\frac{\pi}{3}$, $3\theta+2\phi=2m\pi\pm\frac{\pi}{6}$.

(139) $\cos\phi=\frac{c-a\sin\theta}{b}=\frac{b\sin\theta}{c\sin\theta-a}$,

故 $\sin\theta=\frac{c^2+a^2-b^2\pm\sqrt{(c^2-a^2)^2-2b^2(b^2+c^2)+b^4}}{2ac}$.

(140) $\cos\phi=\frac{a(1-2\cos^2\theta)}{b\cos^2\theta}$, $\sec\phi=\frac{b(\cos^2\theta-2)}{a}$. 由是

$\frac{(1-2\cos^2\theta)(\cos^2\theta-2)}{\cos^2\theta}=\sec\phi\cos\phi=1$, $\cos\theta=\pm1$, 故 $\theta=n\pi$.

141. $\left.\begin{aligned}\cos(\theta+3\phi)=\sin(2\theta+2\phi)\\ \sin(3\theta+\phi)=\cos(2\theta+2\phi)\end{aligned}\right\}.$

142. $\left.\begin{aligned}m\cos\theta+n\cos\phi=1\\ m\sin\theta-n\sin\phi=1\end{aligned}\right\}.$

143. $\left.\begin{aligned}p\sin^4\theta-q\sin^4\phi=p\\ p\cos^4\theta-q\cos^4\phi=q\end{aligned}\right\}.$

144. $\left.\begin{aligned}\sin\theta=\sqrt{2}\sin\phi\\ \tan\theta=\sqrt{3}\tan\phi\end{aligned}\right\}.$

145. $\frac{\sin\theta}{x}=\frac{\cos\theta}{y},\ \frac{\cos^2\theta}{x^2}+\frac{\sin^2\theta}{y^2}=\frac{10}{3(x^2+y^2)}.$

146. $\tan\left(\frac{\pi}{4}+\theta\right)+\tan\left(\frac{\pi}{4}-\theta\right)=\left(\frac{8\sqrt{2}}{1+\sqrt{2}}\right)^{\frac{1}{2}}$ 求適合於 θ 之最小值.

147. 由 $\sec\alpha\sec\theta+\tan\alpha\tan\theta=\sec\beta$ 求 $\tan\theta$.

148. 由 $\tan\frac{\theta}{2}=\frac{\tan\theta+c-1}{\tan\theta+c+1}$ 求 $\tan\frac{\theta}{2}$.

149. 由 $\cos\theta\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)=\sin\beta\cos\frac{\theta}{2}$ 求 $\cos\frac{\theta}{2}$.

150. 有 $\tan(\pi\sin\theta)=\cot(\pi\cos\theta)$ 求 $\sin\left(\theta+\frac{\pi}{4}\right)$.

151. 設 $\tan(\cot\theta)=\cot\theta(\tan\theta)$, 則

$\sin2\theta=\frac{4}{(2n+1)\pi}$. 但 n 爲 -1 或 0 以外之任意整數.

152. 試以 $\cos\theta=-\frac{1}{2}$ 表示爲 $\cos\theta+\cos3\theta=\frac{1}{2}$ 之一根.

153. 試示 $\tan\theta=\cos\theta$ 之各爲 $\left(\frac{\sqrt{5}-1}{2}\right)^{\frac{1}{2}}$.

例題解自 141. 至 153.

(141) $\theta+3\phi=2n\pi\pm\left(\frac{\pi}{2}-2\theta-2\phi\right)$, 故 $3\theta+5\phi=2n\pi+\frac{\pi}{2}$ (1),

或 $-\theta+\phi=2n\pi-\frac{\pi}{2}$ (2), 由 $3\theta+\phi=2m\pi\pm\left(\frac{\pi}{2}-2\theta-2\phi\right)$. 故

$5\theta+3\phi=2m\pi+\frac{\pi}{2}$ (3) 或 $\theta-\phi=2m\pi-\frac{\pi}{2}$ (4),

從 (1), (3) 得 $\theta=(5m-3n)\frac{\pi}{8}+\frac{\pi}{16}$, $\phi=(5n-3m)\frac{\pi}{8}+\frac{\pi}{16}$,

由 (2), (4) 得 $\theta=(5m+n-1)\frac{\pi}{4}$, $\phi=(n-3m+1)\frac{\pi}{4}$.

(142) $(1-m\cos\theta)^2+(m\sin\theta-1)^2=n^2(\cos^2\phi+\sin^2\phi)=n^2$, 卽

$\cos\theta+\sin\theta=\frac{2+m^2-n^2}{2m}$, 故 $\sin\left(\theta+\frac{\pi}{4}\right)=\frac{2+m^2-n^2}{2m\sqrt{2}}$,

同樣 $\cos\left(\phi+\frac{\pi}{4}\right)=\frac{2-m^2+n^2}{2n\sqrt{2}}$.

(143) 從第二 $p(1-\sin^2\theta)^2-q(1-\sin^2\phi)^2=q$. 卽 $q\cos^2\phi=p\cos^2\theta$,

由此與第一得 $\cos^4\theta=\frac{q}{p(q-p)}$, $\cos^4\phi=\frac{1}{q-p}$.

(144) $\theta=n\pi\pm\frac{\pi}{4}$, $\phi=n\pi\pm\frac{\pi}{6}$.

(145) 從第一得 $x=y\tan\theta$. 以此代入第二, 則

$\theta=n\pi\pm\frac{\pi}{3}$, 或 $n\pi\pm\frac{\pi}{6}$.

(146) $$\frac{\sin\frac{\pi}{2}}{\cos\left(\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}-\theta\right)}=\left(\frac{8\sqrt{2}}{1+\sqrt{2}}\right)^{\frac{1}{2}},$$ 卽

$$\frac{1}{\sin\left(\frac{\pi}{4}+\theta\right)\cos\left(\frac{\pi}{4}+\theta\right)}=\left(\frac{8\sqrt{2}}{1+\sqrt{2}}\right)^{\frac{1}{2}},\quad \sin\left(\frac{\pi}{2}+2\theta\right)=\left(\frac{1+\sqrt{2}}{2\sqrt{2}}\right)^{\frac{1}{2}},$$

故 $\cos 2\theta=\frac{\pi}{8}$. 卽 $\theta=\frac{\pi}{16}$.

(147) $(1+\tan^2\alpha)(1+\tan^2\theta)=(\sec\beta-\tan\alpha\tan\theta)^2$, $\tan\theta=\frac{-\sin\alpha\pm\sin\beta}{\cos\alpha\cos\beta}$.

(148) $\tan\frac{\theta}{2}=\pm\sqrt{\frac{c-1}{c+1}}$.

(149) $\cos\frac{\theta}{2}=\cos\left(\frac{\pi}{4}-\frac{\beta}{2}\right)$, $-\frac{1}{2}\sec\left(\frac{\pi}{4}-\frac{\beta}{2}\right)$.

(150) $\pi\sin\theta=n\pi+\frac{\pi}{2}-\pi\cos\theta$. 故 $\sin\left(\theta+\frac{\pi}{4}\right)=\frac{2n+1}{2\sqrt{2}}$.

(151) $\cot\theta=n\pi\frac{\pi}{2}-\tan\theta$, 故 $\sin 2\theta=\frac{4}{(2n+1)\pi}$.

(152) $\cos\theta+4\cos^3\theta-3\cos\theta=\frac{1}{2}$, 卽 $8\cos^3\theta-4\cos\theta-1=0$.

(153) 由 $\tan\theta=\cos\theta$, 得 $\sin\theta=\cos^2\theta=1-\sin^2\theta$, 故

$\sin\theta=\frac{1\pm\sqrt{5}}{2}$. 故 $\cos\theta=\sqrt{(1-\sin^2\theta)}=\sqrt{\left\{1-\left(\frac{1\pm\sqrt{5}}{2}\right)^2\right\}}=\sqrt{\frac{\pm\sqrt{5}-1}{2}}$.

154. 於 $\sin^2 x+2b\sin x+c=0$ 其 b 爲正數. 試求 $\sin x$ 當有兩實根之關係.

155. $\cos 2x+b\cos x+c=0$ 其 b 爲正數. 則 $\cos x$ 至少當有一實根. 試求其關係式.

156. $\sin^2(n+1)\theta=\sin^2 n\theta+\sin^2(n-1)\theta$, 則 $(n+1)\theta$, $n\theta$, $(n-1)\theta$ 之和等於 π. 試求 n 之值.

157. $\sin\theta+\sin\phi=\sin(\theta+\phi)$, 則 θ, ϕ, $\theta+\phi$ 均爲 2π 之倍數.

158. 有五個角成等差級數. 其中央一角之餘弦, 等於其餘四角餘弦之和. 試求其通差.

159. 解 $2\sec\theta\,\mathrm{cosec}\,\theta-\mathrm{cosec}\,\theta\cot\theta=\sec\theta$ 以求其 θ 之角度. 但 θ 爲三角形之最大角.

160. 解 $8\sin^3\theta-6\sin\theta+1=0$ 以求六個正角. 但均小於四直角.

161. $\sin\theta+\sin\phi=\sqrt{3}(\cos\phi-\cos\theta)$, 則

$\sin 3\theta+\sin 3\phi=0$.

162. $\sin\theta+\sin 3\theta=\sin 2\theta+\sin 4\theta$, 則 θ 在大於 0 小於 2π 之內. 有七個正數值. 試證之.

163. $\tan\theta=\tan\beta\tan(\alpha+\theta)$. 其 θ 爲實數. 則 $\tan\beta$ 當在 $(\sec\alpha-\tan\alpha)^2$ 與 $(\sec\alpha+\tan\alpha)^2$ 之間. 試證之.

164 $\dfrac{2\cos\theta}{\sqrt{2}}+\dfrac{\sqrt{3}}{2\cos\theta}$ 爲極小. 試求 θ 之值.

例題解自 154. 至 164.

(154) $\sin x=-b\pm\sqrt{(b^2-c)}$. 故 $b^2\not<c$ 及 $b+\sqrt{(b^2-c)}\not>1$.

(155) $\cos x=\frac{1}{4}\{-b\pm\sqrt{(b^2-8\,\overline{c-2}}\}$. 故 $b^2\not<(8c-1)$, $\sqrt{\{b^2-8(c-1)\}}-b\not>1$.

(156) $(n+1)\theta+n\theta+(n-1)\theta=\pi$, 即 $n\theta=\frac{\pi}{3}$. 由原方程式

$\sin^2(n+1)\theta-\sin^2(n-1)\theta=\sin^2 n\theta$, 即 $\sin\{(n+1)\theta+(n-1)\theta\}\sin\{(n+1)\theta-(n-1)\theta\}=\sin^2 n\theta$,

即 $\sin 2n\theta\sin 2\theta=\sin^2 n\theta$. 故 $\sin\frac{2\pi}{3}\sin\frac{2\pi}{3n}=\sin^2\frac{\pi}{3}$,

故 $\sin\frac{2\pi}{3n}=\sin\frac{\pi}{3}$ 即 $\frac{2\pi}{3n}=\frac{\pi}{3}$, 故 $n=2$.

(157) $2\sin\frac{\theta+\phi}{2}\cos\frac{\theta-\phi}{2}=\sin(\theta+\phi)$, 故 $\cos\frac{\theta-\phi}{2}=\cos\frac{\theta+\phi}{2}$, 或

$\sin\frac{\theta+\phi}{2}=0$, $\frac{\theta+\phi}{2}=2n\pi\pm\frac{\theta-\phi}{2}$, 即 $\theta=2n\pi$,

$\phi=2n\pi$, 或 $\theta+\phi=2n\pi$.

(158) $\cos(\theta+2\delta)=\cos\theta+\cos(\theta+2\delta)+\cos(\theta+3\delta)+\cos(\theta+4\delta)$, 從此得

$\cos\delta=-1$, 或 $\frac{1}{2}$, 故 $\delta=(2n+1)\pi$, 或 $2n\pi\pm\frac{\pi}{3}$.

(159) 從原方程式 $\sin\theta=\frac{1}{2}$, 故 $\theta=180°-30°=150°$.

(160) $-2(3\sin\theta-4\sin^3\theta)+1=0$, 即 $\sin 3\theta=\frac{1}{2}$, 故 $\theta=\frac{n\pi}{3}+(-1)^n\frac{\pi}{18}\not>2\pi$,

故 $n\not>6$. 即令 n 為 0, 1, 2, 3, 4, 5, 則 $\theta=\frac{\pi}{18}$, $\frac{5\pi}{18}$, $\frac{13\pi}{18}$, $\frac{17\pi}{18}$, $\frac{25\pi}{18}$, $\frac{29\pi}{18}$.

(161) 從原方程式 $\cos^2\frac{\theta-\phi}{2}=3\sin^2\frac{\theta-\phi}{2}$, 故 $\cos(\theta-\phi)=\frac{1}{2}$, 故

$\cos^2\theta\cos^2\phi=\frac{1}{4}-\sin\theta\sin\phi+\sin^2\theta\sin^2\phi$ 即 $4(\sin^2\theta-\sin\theta\sin\phi+\sin^2\phi)=3$,

故 $4(\sin^3\theta+\sin^3\phi)=3(\sin\theta+\sin\phi)$. 從此即得其證.

(162) $2\sin 2\theta\cos\theta=2\sin 3\theta\cos\theta$. 故 $\cos\theta=0$, 即 $\theta=\frac{\pi}{2}$, $\frac{3\pi}{2}$.

又 $3\theta=n\pi+(-1)^n 2\theta$, 故 $\theta=\frac{n\pi}{3-(-1)^n 2}$. 故令 $n=1, 3, 5, 7, 9$. 則

$\theta=\frac{\pi}{5}$, $\frac{3\pi}{5}$, π, $\frac{7\pi}{5}$, $\frac{9\pi}{5}$. 故都合 θ 之值有七個.

(163) $\tan\alpha\tan^2\theta+(\tan\beta-1)\tan\theta+\tan\alpha\tan\beta=0$, θ 為實數, 則

$(\tan\beta-1)^2-4\tan^2\alpha\tan\beta\not<0$, 即

$\{\tan\beta-(\sec\alpha+\tan\alpha)^2\}\{\tan\beta-(\sec\alpha-\tan\alpha)^2\}\not<0$, 故如題言.

(164) 令極小值為 y, 則 $4\cos^2\theta-2\sqrt{3}y\cos\theta+3=0$, 故

$(2\sqrt{3}y)^2-4(3\times3)\not<0$, 即 $(y+2)(y-2)\not<0$, 故 $y=2$,

故從 $\frac{2\cos\theta}{\sqrt{3}}+\frac{\sqrt{3}}{2\cos\theta}=2$ 得 $\cos\theta=\frac{\sqrt{3}}{2}$, 故 $\theta=2n\pi\pm\frac{\pi}{6}$.

165. 對於θ之任意實數值. $a\sin^2\theta+b\sin\theta\cos\theta+c\cos^2\theta$ 常在 $\frac{1}{2}(a+c)+\frac{1}{2}\sqrt{\{b^2+(a-c)^2\}}$ 與 $\frac{1}{2}(a+c)-\frac{1}{2}\sqrt{\{b^2-(a-c)^2\}}$ 之間.

166. $x+y$ 爲實數. 求 $\sin x+\sin y$ 之界限, 及其極大值.

167. 設θ爲實數值. 則 $\frac{\tan(\theta+\alpha)}{\tan(\theta-\alpha)}$ 在 $\frac{1-2\sin2\alpha}{1+2\sin2\alpha}$ 與 $\frac{1+2\sin2\alpha}{1-2\sin2\alpha}$ 之間.

168. $\tan\frac{\theta}{2}=\frac{\tan\theta+a-1}{\tan\theta+a+1}$ 爲實數. 則 $a^2>1$.

169. 設 $\tan(2\alpha-3\beta)=\cot(3\alpha-2\beta)$, $\tan(2\alpha+3\beta)$ $=\cot(3\alpha+2\beta)$, 則α及β爲$\frac{\pi}{10}$之倍數.

170. $\sin\theta\cos(\alpha-\theta)$ 爲極大. 則θ之一切值. 可於0及π之間求之. 但α又在0與$\frac{\pi}{2}$之間.

171. 設 $x\cos(\alpha+\beta)+\cos(\alpha-\beta)=x\cos(\beta+\gamma)+\cos(\beta-\gamma)$ $=x\cos(\gamma+\alpha)+\cos(\gamma-\alpha)$, 則

$$\frac{\tan\alpha}{\tan\frac{1}{2}(\beta+\gamma)}=\frac{\tan\beta}{\tan\frac{1}{2}(\gamma+\alpha)}=\frac{\tan\gamma}{\tan\frac{1}{2}(\alpha+\beta)}.$$

172. 設 $\cos\theta=\cos\alpha\cos\beta-\sin\alpha\sin\beta\sqrt{(1-c^2\sin^2\theta)}$,

及 $\cos\phi=\cos\alpha\cos\beta+\sin\alpha\sin\beta\sqrt{(1-c^2\sin^2\phi)}$,

則 $\cos\theta+\cos\phi=\frac{2\cos\alpha\cos\beta}{1-c^2\sin^2\alpha\sin^2\beta}$,

及 $1+\cos\theta\cos\phi=\frac{\cos^2\alpha+\cos^2\beta}{1+c^2\sin^2\alpha\sin^2\beta}$.

173. $\tan^2\theta=\tan(\alpha+\theta)\tan(\alpha-\theta)$, 則 $\sin2\theta=\sqrt{2}\sin\alpha$.

例題解自 165. 至 173.

(165) 令 $a\sin^2\theta+b\sin\theta\cos\theta+c\cos^2\theta=y$, 則

$(a-y)\tan^2\theta+b\tan\theta+c-y=0$, $\tan\theta$ 爲實根. 則 θ 當爲實數, 故

$b^2-4(a-y)(c-y)\not<0$. 卽 $-\left\{y-\frac{c+a}{2}+\frac{1}{2}\sqrt{b^2+(a-c)^2}\right\}$

$\left\{y-\frac{c+a}{2}-\frac{1}{2}\sqrt{b^2+(a-c)^2}\right\}\not<0$, 故如題言.

(166) 令 $x+y=a$, $\sin x+\sin y=z$, 則 $2\sin\frac{a}{2}\cos\frac{x-y}{2}=z$, 而

$\cos\frac{x-y}{2}<1$, 卽 $\frac{z}{2\sin\frac{a}{2}}<1$, 故 $z<2\sin\frac{a}{2}$, 由是 z 在 $2\sin\frac{a}{2}$ 與

$-2\sin\frac{a}{2}$ 之間, 而 z 之極大値爲 $2\sin\frac{a}{2}$.

(167) $\frac{\tan(\theta+a)}{\tan(\theta-a)}=y$, $\tan a(1+y)\tan^2\theta+(1+\tan^2 a)(1-y)\tan\theta+\tan a(1+y)=0$,

故 $(1+\tan^2 a)^2(1-y)^2-4\tan^2 a(1+y)^2\not<0$, 卽

$\left(\frac{1+2\tan a+\tan^2 a}{1-2\tan a+\tan^2 a}-y\right)\left(\frac{1-2\tan a+\tan^2 a}{1+2\tan a+\tan^2 a}-y\right)\not<0$, 卽

$\left(\frac{1+2\sin 2a}{1-2\sin 2a}-y\right)\left(\frac{1-2\sin 2a}{1+2\sin 2a}-y\right)\not<0$, 故如題言.

(168) 從原方程式 $\tan\frac{\theta}{2}=1$, 或 $\tan\frac{\theta}{2}=\frac{1}{a+1}\sqrt{(a^2-1)}$, 故 $a^2>1$.

(169) 從第一 $2a-3\beta=n\pi+\frac{\pi}{2}-(3a-2\beta)$,

從第二 $2a+3\beta=m\pi+\frac{\pi}{2}-(3a+2\beta)$,

故 $a=(m+n+1)\frac{\pi}{10}$, 及 $\beta=(5m-5n)\frac{\pi}{10}$.

(170) $\sin\theta\cos(a-\theta)=y$, $\sin a+\sin(2\theta-a)=2y$, 令 y 爲極大. 則

$\sin(2\theta-a)=1$, 故 $\theta=\frac{n\pi}{2}+(-1)^n\frac{\pi}{4}+\frac{a}{2}$; 故 $\theta=\frac{\pi}{4}+\frac{a}{2}$.

(171) $x=\frac{\cos(\beta-\gamma)-\cos(a-\beta)}{\cos(a+\beta)-\cos(\beta+\gamma)}=\frac{\cos(\gamma-a)-\cos(\beta-\gamma)}{\cos(\beta+\gamma)-\cos(\gamma+a)}$, 故

$\frac{\sin\left(\beta-\frac{\gamma+a}{2}\right)}{\sin\left(\beta+\frac{\gamma+a}{2}\right)}=\frac{\sin\left(\gamma-\frac{a+\beta}{2}\right)}{\sin\left(\gamma+\frac{a+\beta}{2}\right)}$, 卽 $\frac{\tan\beta-\tan\frac{1}{2}(\gamma+a)}{\tan\beta+\tan\frac{1}{2}(\gamma+a)}=\frac{\tan\gamma-\tan\frac{1}{2}(a+\beta)}{\tan\gamma+\tan\frac{1}{2}(a+\beta)}$,

故 $\frac{\tan\beta}{\tan\frac{1}{2}(\gamma+a)}=\frac{\tan\gamma}{\tan\frac{1}{2}(a+\beta)}$. (172) 由第一及第二

$$\cos\theta=\frac{\cos a\cos\beta\pm\sqrt{\{\cos^2 a\cos^2\beta+(1-c^2\sin^2 a\sin^2\beta-\cos^2 a-\cos^2\beta)(1-c^2\sin^2 a\sin^2\beta)\}}}{1-c^2\sin^2 a\sin^2\beta},$$

$$\cos\phi=\frac{\cos a\cos\beta\pm\sqrt{\{\cos^2 a\cos^2\beta+(1-c^2\sin^2 a\sin^2\beta-\cos^2 a-\cos^2\beta)(1-c^2\sin^2 a\sin^2\beta)\}}}{1-c^2\sin^2 a\sin^2\beta}.$$

(173) 解括弧. 可變其 $\tan\theta$, $\tan a$ 爲 $\sin 2\theta$, $\sin a$.

174. 從 $\sec\alpha\cos(\theta+\phi)=1+\tan\theta\tan\phi$,

$\sec\beta\cos(\theta-\phi)=1-\tan\theta\tan\phi$, 以求 $\cos(\theta+\phi)$ 及 $\cos(\theta-\phi)$.

175. β,γ 爲 $\sin(\alpha+x)=m\sin\alpha$ 之不等根, 則

$\cos\frac{1}{2}(\beta-\gamma)+m\sin\frac{1}{2}(\beta+\gamma)=0.$

176. $\sin^2x\sin^2y+\sin^2(x+y)=(\sin x+\sin y)^2$. 適合其 x, y 之實數. 爲 π 之倍數.

177. $\dfrac{\sin(\theta+\alpha)}{\sin(\theta+\beta)}=\sqrt{\dfrac{\sin2\alpha}{\sin2\beta}}$, 則 $\tan^2\theta=\tan\alpha\tan\beta$.

178. $a\cos\theta=b\cos\phi$, 則 $\cot\dfrac{\theta+\phi}{2}\cot\dfrac{\theta-\phi}{2}=\dfrac{a+b}{a-b}$.

179. β,γ 爲不等值. 而適合於 $\dfrac{\cos\alpha\cos x}{a}+\dfrac{\sin\alpha\sin x}{b}=\dfrac{1}{c}$,

則 $(b^2+c^2-a^2)\cos\beta\cos\gamma+(c^2+a^2-b^2)\sin\beta\sin\gamma=a^2+b^2-c^2$.

180. $\tan^2x\tan\dfrac{x}{2}=1$, 其一切實根. 均適合於

$\cos2x=2-\sqrt{5}$. 試證之.

181. $\cos3x=-\dfrac{3\sqrt{3}}{4\sqrt{2}}$ 其 $\cos x$ 之三值爲

$\sqrt{\dfrac{3}{2}}\sin\dfrac{\pi}{10}$, $\sqrt{\dfrac{3}{2}}\sin\dfrac{\pi}{6}$, 及 $-\sqrt{\dfrac{3}{2}}\sin\dfrac{3\pi}{10}$.

例題解自 174. 至 181.

(174) 從第一 $\dfrac{\cos(\theta+\phi)}{\cos\alpha}=\dfrac{2\cos(\theta-\phi)}{\cos(\theta+\phi)+\cos(\theta-\phi)}$, 從第二

$\dfrac{\cos(\theta-\phi)}{\cos\beta}=\dfrac{2\cos(\theta+\phi)}{\cos(\theta+\phi)+\cos(\theta-\phi)}$. 故 $\cos(\theta+\phi)=\dfrac{2\cos\alpha\sqrt{\cos\beta}}{\sqrt{\cos\alpha}+\sqrt{\cos\beta}}$.

(175) $\because$ $\sin(\alpha+\beta)=m\sin\alpha$, $\therefore$ $\cot\alpha\sin\beta=m-\cos\beta$,

又 $\sin(\alpha+\gamma)=m\sin\alpha$, $\therefore$ $\cot\alpha\sin\gamma=m-\cos\gamma$,

從此消去 $\cot\alpha$, 即得其證.

(176) 解括弧而簡單之.則 $\sin x\sin y(\sin x\sin y-2\cos x\cos y+2)=0$,

故 $\sin x=0$, 或 $\sin y=0$, 故 $x=m\pi, y=n\pi$, 或 $\sin x\sin y-2\cos x\cos y+2=0$,

即 $\frac{1}{2}\{\cos(x-y)-\cos(x+y)\}-\cos(x+y)-\cos(x-y)+2=0$, 即

$\sin^2\frac{x+y}{2}+3\sin^2\frac{x-y}{2}=0$, 故 $\sin\frac{x+y}{2}=0$, $\sin\frac{x-y}{2}=0$,

故 $x+y=2m\pi$, $x-y=2n\pi$.

(177) 從 $\frac{(\tan\theta\cos\alpha+\sin\alpha)^2}{(\tan\theta\cos\beta+\sin\beta)^2}=\frac{\sin\alpha\cos\alpha}{\sin\beta\cos\beta}$ 得 $\tan^2\theta=\tan\alpha\tan\beta$.

(178) $\because$ $\frac{\cos\theta}{\cos\phi}=\frac{b}{a}$. 故從 $\frac{\cos\theta+\cos\phi}{\cos\theta-\cos\phi}=\frac{a+b}{a-b}$ 即得其證.

(179) 於 $bc\cos x\cos\alpha+ac\sin x\sin\alpha=ab$, 令 x 爲 β,γ.

則 $bc\cos\beta\cos\alpha+ac\sin\beta\sin\alpha=ab$, $bc\cos\gamma\cos\alpha+ac\sin\gamma\sin\alpha=ab$,

故 $\frac{\cos\alpha}{a^2bc(\sin\beta-\sin\gamma)}=\frac{\sin\alpha}{b^2ac(\cos\gamma-\cos\beta)}=\frac{-1}{c^2ab(\cos\beta\sin\gamma-\sin\beta\cos\gamma)}$,

即 $\frac{\cos\alpha}{a\cos\frac{1}{2}(\beta+\gamma)}=\frac{\sin\alpha}{b\sin\frac{1}{2}(\beta+\gamma)}=\frac{1}{c\cos\frac{1}{2}(\beta-\gamma)}$, 故

$a^2\cos^2\frac{1}{2}(\beta+\gamma)+b^2\sin^2\frac{1}{2}(\beta+\gamma)=c^2\cos^2\frac{1}{2}(\beta-\gamma)$, 即

$a^2\{1+\cos(\beta+\gamma)\}+b^2\{1-\cos(\beta+\gamma)\}=c^2\{1+\cos(\beta-\gamma)\}$, 解括弧.則

$a^2+b^2-c^2+a^2(\cos\beta\cos\gamma-\sin\beta\sin\gamma)-b^2(\cos\beta\cos\gamma-\sin\beta\sin\gamma)$

$=c^2(\cos\beta\cos\gamma+\sin\beta\sin\gamma)$,

即 $a^2+b^2-c^2=(b^2+c^2-a^2)\cos\beta\cos\gamma+(c^2+a^2-b^2)\sin\beta\sin\gamma$.

(180) 從第一 $\frac{\sin^2x\sin\frac{x}{2}}{\cos^2x\cos\frac{x}{2}}=1$, 即 $\frac{2\sin^2x\sin^2\frac{x}{2}}{\cos^3x}=1$, 即

$\sin^2x(1-\cos x)=\cos^3x$, 故 $\sin^2x=\cos x$, 即 $\cos^2x+\cos x-1=0$.

故 $\cos x=\frac{\pm\sqrt{5}-1}{2}$, 又由第二 $2\cos^2x-1=2-\sqrt{5}$, 故 $\cos x=\pm\frac{\sqrt{5}-1}{2}$.

(181) $4\cos^3x-3\cos x=-\frac{3\sqrt{3}}{4\sqrt{2}}$, 即

$8(\sqrt{2}\cos x)^3-12(\sqrt{2}\cos x)+3\sqrt{3}=0$, 即

$2(\sqrt{2}\cos x)\{4(\sqrt{2}\cos x)^2-3\}-3\{2(\sqrt{2}\cos x)-\sqrt{3}\}=0$, 故

$2(\sqrt{2}\cos x)-\sqrt{3}=0$, 即 $\cos x=\frac{\sqrt{3}}{\sqrt{2}}\left(\frac{1}{2}\right)=\sqrt{\frac{3}{2}}\sin\frac{\pi}{6}$, 或

$2(\sqrt{2}\cos x)\{2(\sqrt{2}\cos x)+\sqrt{3}\}-3=0$, $\cos x=\sqrt{\frac{3}{2}}\left(\frac{-1\pm\sqrt{5}}{4}\right)$.

182. $\tan\frac{\alpha+x}{2}\tan\chi=m$ 之一切根.爲次式之根.

$$\sin\alpha\sin\chi\{1-\cos(\alpha+\chi))\}$$
$$+m^2\cos\alpha\cos\chi\{1+\cos(\alpha+\chi)\}=m\sin^2(\alpha+\chi).$$

183. α,β 爲比 π 小之不等值,適合於 $a\cos2\chi+b\sin2\chi=1$,
及 $(l\cos^2 2\alpha+m\sin^2 2\alpha)(l\cos^2 2\beta+m\sin^2 2\beta)$
$=\{l\cos^2(\alpha+\beta)+m\sin^2(\alpha+\beta)\}^2$. 則 $l=m$, 或
$a^2-b^2=\frac{m-l}{m+l}$.

184. $\alpha,\beta,\gamma,\delta$ 爲比 π 小之不等值.其
β,γ 適合於 $a\cos2\chi+b\sin2\chi=1$, 及
α,δ 適合於 $a'\cos2\chi+b'\sin2\chi=1$, 而
$\{l\cos^2(\alpha+\beta)+m\sin^2(\alpha+\beta)\}\{l\cos^2(\gamma+\delta)+m\sin^2(\gamma+\delta)\}$
$=\{l\cos^2(\alpha+\gamma)+m\sin^2(\alpha+\gamma)\}\{l\cos^2(\beta+\delta)+m\sin^2(\beta+\delta)\}$,
則 $l=m$, 或 $aa'-bb'=\frac{m-l}{m+l}$.

例題解自 182. 至 184.

(182) 從第一 $m=\frac{\sin\frac{1}{2}(\alpha+\chi)\sin\chi}{\cos\frac{1}{2}(\alpha+\chi)\cos\chi}$, 以 $2\sin\frac{\alpha+\chi}{2}$ 乘其分母子.
變化其角 $\frac{\alpha+\chi}{2}$ 爲角 $(\alpha+\chi)$.而後去其分母.

$$\sin\alpha\sin\chi\{1-\cos(\alpha+\chi)\}=m\sin\alpha\cos\chi\sin(\alpha+\chi), \quad (1)$$

又前之分母子.以 $2\cos\frac{1}{2}(\alpha+\chi)$ 乘之.與前同樣變化.去其分母.再以 m 乘其兩節.則

$$m^2\cos\alpha\cos\chi\{1+\cos(\alpha+\chi)\}=m\sin^2(\alpha+\chi)-m\sin\alpha\cos\chi\sin(\alpha+\chi), \quad (2)$$

將 (1), (2) 之各節.順次相加.即得其證.

(183) 由原方程式 $(a^2+b^2)\sin^2 2\chi-2b\sin2\chi+1-a^2=0$, 從此方程式求 $\sin2\chi$ 之兩值.則依題意.

$$\sin2\alpha=\frac{b\pm a\sqrt{a^2+b^2-1}}{a^2+b^2}, \quad 或 \quad \sin2\beta=\frac{b\mp a\sqrt{a^2+b^2-1}}{a^2+b^2},$$

故 $\sin 2\alpha \sin 2\beta = \frac{(b \pm a\sqrt{a^2+b^2-1})(b \mp a\sqrt{a^2+b^2-1})}{(a^2+b^2)^2} = \frac{1-a^2}{a^2+b^2}$,

同樣. 表原方程式爲 $\cos 2\chi$. 則 $\cos 2\alpha \cos 2\beta = (1-b^2)/(a^2+b^2)$,

故 $\cos 2(\alpha+\beta) = \cos 2\alpha \cos 2\beta - \sin 2\alpha \sin 2\beta$

$= (1-b^2)/(a^2+b^2) - (1-a^2)/(a^2+b^2) = (a^2-b^2)/(a^2+b^2)$.

由是得 $\cos^2(\alpha+\beta) = a^2/(a^2+b^2)$, $\sin^2(\alpha+\beta) = a^2/(a^2+b^2)$.

又解第二方程式之括弧.都集而括之.則

$l^2(\cos 2\alpha \cos 2\beta)^2 + lm\{1-\cos 2(\alpha+\beta) - 2\sin 2\alpha \sin 2\beta \cos 2\alpha \cos 2\beta\}$

$+m^2(\sin 2\alpha \sin 2\beta)^2 = \{l\cos^2(\alpha+\beta) + m\sin^2(\alpha+\beta)\}^2$,

此方程式代用前所得之各值.則

$l^2\left(\frac{1-b^2}{a^2+b^2}\right)^2 + lm\left\{1-\left(\frac{a^2-a^2}{a^2+b^2}\right)^2 - 2\left(\frac{1-b^2}{a^2+b^2}\right)\left(\frac{1-a^2}{a^2+b^2}\right)\right\} + m^2\left(\frac{1-a^2}{a^2+b^2}\right)^2$

$= \left(\frac{la^2}{a^2+b^2} + \frac{mb^2}{a^2+b^2}\right)^2$, 即 $(l-m)\{l(a^4-b^4+2b^2-1) - m(b^4-a^4+2a^2-1)\} = 0$.

故 $l=m$, 或 $\frac{a^4-b^4+2b^2-1}{b^4-a^4+2a^2-1} = \frac{m}{l}$, 即 $a^2-b^2 = \frac{m-l}{m+l}$.

(184) 由前例. $\cos(\beta+\gamma) = a/\sqrt{a^2+b^2}$, $\sin(\beta+\gamma) = b/\sqrt{a^2+b^2}$,

$\cos(\alpha+\delta) = a'/\sqrt{a'^2+b'^2}$, $\sin(\alpha+\delta) = b'/\sqrt{a'^2+b'^2}$. 又參照前例之解法.

$\cos 2(\beta-\gamma) = \cos 2\beta \cos 2\gamma + \sin 2\beta \sin 2\gamma = \frac{1-b^2}{a^2+b^2} + \frac{1-a^2}{a^2+b^2} = \frac{2-a^2-b^2}{a^2+b^2}$.

由是 $\cos(\beta-\gamma) = 1/\sqrt{a^2+b^2}$, 同樣, $\cos(\alpha-\delta) = 1/\sqrt{a'^2+b'^2}$.

括最後之方程式如次.然後由題意.可省略 $\sin(\beta-\gamma)$ 等因子.則

$\{l-(l-m)\sin^2(\alpha+\beta)\}\{l-(l-m)\sin^2(\gamma+\delta)\}$

$= \{l-(l-m)\sin^2(\alpha+\gamma)\}\{l-(l-m)\sin^2(\beta+\delta)\}$.

解兩節之括弧.則當含有 $l-m$ 之項.由是 $l=m$, 或

$(l-m)\{\sin^2(\alpha+\beta)\sin^2(\gamma+\delta) - \sin^2(\alpha+\gamma)\sin^2(\beta+\delta)\} = l\{\sin^2(\alpha+\beta) - \sin^2(\alpha+\gamma)$

$+\sin^2(\gamma+\delta) - \sin^2(\beta+\delta)\}$. 即 $(l-m)[\frac{1}{4}\{\cos(\alpha+\beta-\gamma-\delta) - \cos(\alpha+\beta+\gamma+\delta)\}^2$

$-\frac{1}{4}\{\cos(\alpha+\gamma-\beta-\delta) - \cos(\alpha+\beta+\gamma+\delta)\}^2] = l\{\sin(2\alpha+\beta+\gamma)\sin(\beta-\gamma)$

$+\sin(2\delta+\beta+\gamma)\sin(\gamma-\beta)\}$, 即 $(l-m)[\cos^2(\alpha+\beta-\gamma-\delta) - \cos^2(\alpha+\gamma-\beta-\delta)$

$-2\cos(\alpha+\beta+\gamma+\delta)\{\cos(\alpha+\beta-\gamma-\delta) - \cos(\alpha+\gamma-\beta-\delta)\}]$

$= 4l\sin(\gamma-\beta)\{\sin(2\delta+\beta+\gamma) - \sin(2\alpha+\beta+\gamma)\}$, 即

$(l-m)\{2\sin 2(\alpha-\delta)\cos(\gamma-\beta) - 4\cos(\alpha+\beta+\gamma+\delta)\sin(\alpha-\delta)\}$

$= 8l\cos(\alpha+\beta+\gamma+\delta)\sin(\delta-\alpha)$, 從此變化歸回之後

$(l-m)\cos(\alpha-\delta)\cos(\beta-\gamma) = (m+l)\{\cos(\alpha+\delta)\cos(\beta+\gamma) - \sin(\alpha+\delta)\sin(\beta+\gamma)\}$,

即 $(m-l)(1/\sqrt{a'^2+b'^2})(1/\sqrt{a^2+b^2}) = (m+l)\{(a/\sqrt{a^2+b^2})$

$(a'/\sqrt{a'^2+b'^2}) - (b/\sqrt{a^2+b^2})(b'/\sqrt{a'^2+b'^2})\}$, 故 $aa'-bb' = (m-l)/(m+l)$.

185. $\alpha,\beta,\gamma,\delta$. 為 $\sin 2\theta-m\cos\theta-n\sin\theta+\gamma=0$ 之四根

則 $\sin\alpha+\sin\beta+\sin\gamma+\sin\delta=m,$

$\cos\alpha+\cos\beta+\cos\gamma+\cos\delta=n,$

$\sin 2\alpha+\sin 2\beta+\sin 2\gamma+\sin 2\delta=2mn-4\gamma,$

$\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\cos 2\delta=n^2-m^2,$

及 $\alpha+\beta+\gamma+\delta=(2p+1)\pi$. 但 p 為整數.

186. $\dfrac{\cos(\alpha+\theta)}{\sin^3\alpha}=\dfrac{\cos(\beta+\theta)}{\sin^3\beta}=\dfrac{\cos(\gamma+\theta)}{\sin^3\gamma}$. 其 α,β,γ

為小於 π 之不等值. 則 $\alpha+\beta+\gamma=\pi,$

及 $\tan\theta=(1+\cos\alpha\cos\beta\cos\gamma)/(\sin\alpha\sin\beta\sin\gamma)$.

187. 有 $u=\dfrac{\sin x}{\sin\alpha}=\dfrac{\sin y}{\sin\beta}=\dfrac{\sin z}{\sin\gamma}$ 及 $x+y+z=2\pi,$

試求 x, y, z 之正弦及 u 之值.

例題解自 185. 至 187.

(185) 試變化原方程式為 $\sin\theta$ 之項. 則

$\sin^4\theta-m\sin^3\theta+\frac{1}{4}(m^2+n^2-4)\sin^2\theta+\frac{1}{2}(2m-n\gamma)\sin\theta-\frac{1}{4}(m^2-\gamma^2)=0,$

故由方程式之性質. $\sin\alpha+\sin\beta+\sin\gamma+\sin\delta=m.$

同樣. 試變化原方程式為 $\cos\theta$. 則 $\cos\alpha+\cos\beta+\cos\gamma+\cos\delta=n.$

又變化原方程式為 $\sin 2\theta$. 則 $\sin^4 2\theta-2(mn-2\gamma)\sin^3\theta+(\sin^2\theta$ 及其以下之項$)=0$. 故 $\sin 2\alpha+\sin 2\beta+\sin 2\gamma+\sin 2\delta=2mn-4\gamma.$

同樣. 試變化為 $\cos 2\theta$. 則 $\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\cos 2\delta=n^2-m^2.$

又 $\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\cos 2\delta=(\cos\alpha+\cos\beta+\cos\gamma+\cos\delta)^2$
$-(\sin\alpha+\sin\beta+\sin\gamma+\sin\delta)^2=\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\cos 2\delta-2\cos(\alpha+\beta)$
$-2\cos(\beta+\gamma)-2\cos(\gamma+\alpha)-2\cos(\alpha+\delta)-2\cos(\beta+\delta)-2\cos(\gamma+\delta)$, 卽
$\cos(\alpha+\beta)+\cos(\beta+\gamma)+\cos(\gamma+\alpha)+\cos(\alpha+\delta)+\cos(\beta+\delta)+\cos(\gamma+\delta)=0,$

但 $\alpha+\beta+\gamma+\delta=(2p+1)\pi$. 故上之關係式. 其前節可為 0.

(186) 試變原方程式爲 $\frac{\cot\alpha-\tan\theta}{\sin^2\alpha}=\frac{\cot\beta-\tan\theta}{\sin^2\beta}=\frac{\cot\gamma-\tan\theta}{\sin^2\gamma}=K$,

則 $K=\frac{\cot\alpha-\cot\beta}{\sin^2\alpha-\sin^2\beta}=\frac{-\sin(\alpha-\beta)}{\sin\alpha\sin\beta\sin(\alpha+\beta)\sin(\alpha-\beta)}=\frac{-1}{\sin\alpha\sin\beta\sin(\alpha+\beta)}$

同樣. $K=\frac{-1}{\sin\alpha\sin\beta\sin(\alpha+\beta)}=\frac{-1}{\sin\beta\sin\gamma\sin(\beta+\gamma)}$,

故 $\sin\alpha\sin(\alpha+\beta)=\sin\gamma\sin(\beta+\gamma)$, 解其括弧而再括之如次.

$2\cos\beta(\sin^2\alpha-\sin^2\gamma)+\sin\beta(\sin2\alpha-\sin2\gamma)=0$. 以此變化之而依題意去其通因子 $\sin(\alpha-\gamma)$. 則 $\cos\beta\sin(\alpha+\gamma)+\sin\beta\cos(\alpha+\gamma)=0$,

即 $\sin(\alpha+\beta+\gamma)=0$, 故 $\alpha+\beta+\gamma=\pi$ 或 2π.

又 $K=\frac{-1}{\sin\alpha\sin\beta\sin(\alpha+\beta)}=\frac{-1}{\sin\beta\sin\gamma\sin(\beta+\gamma)}=\frac{-1}{\sin\gamma\sin\alpha\sin(\gamma+\alpha)}$

$=\frac{-(\sin\alpha+\sin\beta+\sin\gamma)}{\sin\alpha\sin\beta\sin\gamma\{\sin(\alpha+\beta)+\sin(\beta+\gamma)+\sin(\gamma+\alpha)\}}$, (因 $\alpha+\beta+\gamma=\pi$ 故也),

$\therefore\ K=\frac{-(\sin\alpha+\sin\beta+\sin\gamma)}{\sin\alpha\sin\beta\sin\gamma(\sin\gamma+\sin\alpha+\sin\beta)}=\frac{-1}{\sin\alpha\sin\beta\sin\gamma}$, (1)

又 $K=\frac{\cot\alpha+\cot\beta+\cot\gamma-3\tan\theta}{\sin^2\alpha+\sin^2\beta+\sin^2\gamma}$, (由例題五 36. 及 61.).

$\therefore\ K=\frac{\cot\alpha\cot\beta\cot\gamma+\operatorname{cosec}\alpha\operatorname{cosec}\beta\operatorname{cosec}\gamma-3\tan\theta}{2+2\cos\alpha\cos\beta\cos\gamma}$, (2)

從 (1), (2) 即可得第二之結果.

(187) $u\sin\gamma=\sin z=-\sin(x+y)$

$=-\sin x\cos y-\cos x\sin y=-u\sin\alpha\cos y-u\sin\beta\cos x$,

故 $u=0$, 或 $\sin\gamma=-\sin\alpha\cos y-\sin\beta\cos x$,

即 $\sin\alpha\cos y=-\sin\gamma-\sin\beta\cos x$, 又 $\sin\alpha\sin y=\sin x\sin\beta$,

上之兩式. 平方之相加, 則 $\sin^2\alpha=\sin^2\gamma+\sin^2\beta+2\sin\gamma\sin\beta\cos x$,

故 $\cos x=\frac{\sin^2\alpha-\sin^2\gamma-\sin^2\beta}{2\sin\gamma\sin\beta}$.

同法. 由求得 $\cos y$, $\cos z$ 之值, 從而 u 之值亦可求得.

又令 $u=0$, 則從原式 $\sin x=0$, $\sin y=0$, $\sin z=0$,

故 $x=0, y=\pi, z=\pi$. $x=\pi, y=0. z=\pi$, $x=\pi, y=\pi, z=0$.

又 $x=0, y=0, z=2\pi$. $x=0, y=2\pi. z=0$. $x=2\pi, y=0, z=0$.

第捌編

消去法

1. 消去法 代數學上.從其若干組之聯立方程式.消去其未知數量.謂之消去法.而其消得之新方程式.爲消去其未知數量之結果.或稱其殘餘之關係式.今三角法之消去法.亦同此意義也.

2. 例解 今示數例如次.

〔第一〕 從 $sin\theta=a,\ cos\theta=b$ 消去 θ.

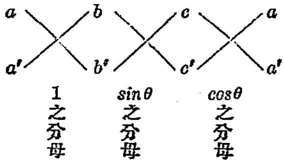
$sin^2\theta+sin^2\theta=a^2+b^2$ 但 $sin^2\theta+cos\theta=1$ 故 $a^2+b^2=1$.

〔第二〕 從 $sec\theta=a,\ tan\theta=b$, 求 a, b 之關係式.

$sec^2\theta-tan^2\theta=a^2-b^2$, 但 $sec^2\theta-tan^2\theta=1$, 故 $a^2-b^2=1$.

〔第三〕 從 $asin\theta+bcos\theta+c=0$, 及 $a'sin\theta+b'cos\theta+c'=0$. 消去 θ.

由代數學聯立方程式之解法.(十文字法).得其兩方程式之係數如次.



$$\frac{sin\theta}{bc'-b'c}=\frac{cos\theta}{ca'-c'a}=\frac{1}{ab'-a'b},$$

故 $\dfrac{sin^2\theta+cos^2\theta}{(bc'-b'c)^2+(ca'-c'a)^2}=\dfrac{1}{(ab'-ab')^2}$, 但 $sin^2\theta+cos^2\theta=1$,

故 $(bc'-b'c)^2+(ca'-c'a)^2=(ab'-ab')^2$, 是卽所求之結果.

〔第四〕 從 $asin\theta+bsin\phi+c=0,\ a'sin\theta+b'sin\phi+c'=0$, 及 $msin^2\theta+nsin^2\phi=p$. 消去 θ 及 ϕ.

由第三而得最初兩方程式爲 $\dfrac{sin\theta}{bc'-b'c}=\dfrac{sin\phi}{ca'-c'a}=\dfrac{1}{ab'-a'b}$, 故

$\frac{m\sin^2\theta+n\sin^2\phi}{m(bc'-b'c)^2+n(ca'-c'a)^2}=\frac{1}{(ab'-a'b)^2}$. 但 $m\sin^2\theta+n\sin^2\phi=p$.

由是 $p(ab'-a'b)^2=m(bc'-b'c)^2+n(ca'-c'a)^2$.

例題九

試消去次列各式之 θ.

1. $\tan\theta+\sin\theta=a,\quad \tan\theta-\sin\theta=b$.
2. $\sec\theta+\tan\theta=a,\quad \operatorname{cosec}\theta+\cot\theta=b$.
3. $\sin\theta+\sin 2\theta=a,\quad \cos\theta+\cos 2\theta=b$.
4. $a\sin\theta+b\cos\theta=a,\quad \operatorname{cosec}\theta+b\sec\theta=c$.
5. $a\sec^2\theta-b\cos\theta=2a,\quad b\cos^2\theta-a\sec\theta=2b$.

例題解自 1. 至 5.

(1) $\tan\theta=\frac{1}{2}(a+b)$, $\sin\theta=\frac{1}{2}(a-b)$. $\therefore$ $\cos\theta=\sin\theta/\tan\theta=(a-b)/(a+b)$,

$\therefore$ $\sin^2\theta+\cos^2\theta=\frac{(a-b)^2}{4}+\frac{(a-b)^2}{(a+b)^2}=1$, 卽 $(a^2-b^2)^2=16ab$.

(2) 從原兩方程式 $\frac{1+\sin\theta}{\cos\theta}=a$, $\frac{1+\cos\theta}{\sin\theta}=b$.

由加法. $\frac{\sin\theta+\cos\theta+1}{\sin\theta\cos\theta}=a+b$.

又由乘法. $\frac{1+\cos\theta+\sin\theta}{\sin\theta\cos\theta}+1=ab$. $\therefore$ $ab-a-b=1$.

(3) 將原方程式之兩節.平方之相加.則 $2+2\cos(2\theta-\theta)=a^2+b^2$,
卽 $1+2\cos\theta=a^2+b^2-1$, 又變原兩方程式爲 $\sin\theta(1+2\cos\theta)=a$,
及 $\cos\theta(1+2\cos\theta)=b+1$, $\therefore$ $(1+2\cos\theta)^2=a^2+(b+1)^2$. 由是
$a^2+(b+1)^2=(a^2+b^2-1)^2$. 卽 $(a^2+b^2)^2-3(a^2+b^2)-2b=0$.

(4) $a\sin\theta+b\cos\theta=c$, $a\cos\theta+b\sin\theta=c\sin\theta\cos\theta$, 此兩方程式之
各節.平方之相加.則 $a^2+b^2+4ab\sin\theta\cos\theta=c^2(1+\sin^2\theta\cos^2\theta)$, (1),
又兩方程式之各節相乘.則 $(a^2+b^2)\sin\theta\cos\theta+ab=c^2\sin\theta\cos\theta$, (2).
從 (1), (2) 消去 $\sin\theta\cos\theta$ 之項.則

$$(a^2+b^2-c^2)^3-4a^2b^2(a^2+b^2-c^2)=a^2b^2c^2.$$

(5) 由原兩方程式.得 $a-b\cos^3\theta=2a\cos^2\theta$, $b\cos^3\theta-a=2b\cos\theta$. 此二式相加.以 $\cos\theta$ 除之.則 $a\cos\theta+b=0$, 從此可得 $a^2-b^2=0$.

6. $a\sin^2\theta+b\cos^2\theta=c,\quad a\operatorname{cosec}^2\theta+b\sec^2\theta=d.$

7. $a\tan\theta+b\sec\theta=h,\quad a\cot\theta+b\cos\theta=k.$

8. $x\cos\theta+y\sin\theta=x'\cos\theta+y'\sin\theta=1.$

9. $x\sin\theta-y\cos\theta=\sqrt{x^2+y^2},\quad \frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}=\frac{1}{x^2+y^2}.$

10. $n\sin\phi-m\cos\phi=2m\sin\theta,\ n\sin2\phi-m\cos2\phi=n.$

11. $\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1,\quad \frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2.$

12. $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2,\quad \frac{ax\sin\theta}{\cos^2\theta}+\frac{by\cos\theta}{\sin^2\theta}=0.$

13. $\frac{x}{a}=\frac{\sec^2\theta-\cos^2\theta}{\sec^2\theta+\cos^2\theta},\quad \frac{2b}{y}=\sec^2\theta+\cos^2\theta.$

14. $\tan\theta=\frac{(1+m)\tan\theta}{1-m\tan^2\theta},\ \tan\beta=\frac{(1-m)\tan\theta}{1+m\tan^2\theta}.$

15. $\sin\theta\cos\theta(\cos\theta-\sin\theta)=a,\ \sin\theta\cos\theta(\cos\theta+\sin\theta)=b.$

16. $x\cos\theta-y\sin\theta=a\cos2\theta,\quad x\sin\theta+y\cos\theta=2a\sin2\theta.$

例題解自 6 至 16.

(6) $a\sin^2\theta+b\cos^2\theta=c,$ (1) $a\cos^2\theta+b\sin^2\theta=d\sin^2\theta\cos^2\theta,$ (2)

(1), (2) 相加. 則 $a+b=c+d\sin^2\theta\cos^2\theta$ (3). 又 (1), (2) 相乘. 則 $(a^2+b^2)\sin^2\theta\cos^2\theta+ab(\sin^4\theta+\cos^4\theta)=cd\sin^2\theta\cos^2\theta,$ 卽 $\{cd-(a-b)^2\}\sin^2\theta\cos^2\theta=ab,$ (4). 故從 (3), (4) 消去 $\sin^2\theta\cos^2\theta,$ 則 $(a+b-c)\{cd-(a-b)^2\}=abd.$ 卽 $d\{c(a+b-c)-ab\}+(a-b)\{-(a^2-b^2)+c(a-b)\}=0.$ 卽 $d(b-c)(c-a)+a(a-b)(c-a)+b(a-b)(b-c)=0.$

故 $\frac{d}{a-b}+\frac{a}{b-c}+\frac{b}{c-a}=0.$

(7) 從原兩方程式.得 $a\tan\theta+b\sec\theta=h$. (1), $a\sec\theta+b\tan\theta=k\tan\theta\sec\theta$ (2).

由 (1), (2) 相加.得 $(a+b)(\sec\theta+\tan\theta)=k\tan\theta\sec\theta+h$ (3).

又由 (1), (2) 相減.得 $(a-b)(\sec\theta-\tan\theta)=k\tan\theta\sec\theta-h$ (4).

又 (3), (4) 相乘.得 $\sec^2\theta-\tan^2\theta=1$, 故

$a^2-b^2=k^2\tan^2\theta\sec^2\theta-h^2$ (5). 又從 (3), (4) 得

$$(\sec\theta+\tan\theta)^2-(\sec\theta-\tan\theta)^2=\left(\frac{k\tan\theta\sec\theta+h}{a+b}\right)^2-\left(\frac{k\tan\theta\sec\theta-h}{a-b}\right)^2,$$

卽 $(a^2-b^2)^2\tan\theta\sec\theta=-ab(k^2\tan^2\theta\sec^2\theta+h^2)+kh(a^2+b^2)\tan\theta\sec\theta$, (6).

(5) 之 $k^2\tan^2\theta\sec^2\theta$ 代入 (6). 且轉項將各節平方之.則

$a^2b^2(a^2-b^2+2h^2)^2=\{kh(a^2+b^2)-(a^2-b^2)\}^2\tan^2\theta\sec^2\theta$. 由此方程式與 (5) 而得

$k^2a^2b^2(a^2-b^2+2h^2)^2=\{kh(a^2+b^2)-(a^2-b^2)\}^2(a^2-b^2+h^2)$.

(8) 由 2. 節第三得 $(y-y')^2+(x-x')^2=(xy'-x'y)^2$.

(9) 試將第一方程式之兩節.平方而變化之.則

$$x^2\cos^2\theta+2xy\sin\theta\cos\theta+y^2\sin^2\theta=0,$$

卽 $x\cos\theta+y\sin\theta=0$. 故 $\sin^2\theta=x^2/(x^2+y^2)$, $\cos^2\theta=y^2/(x^2+y^2)$.

以此代入第二方程式.則 $x^2/a^2+y^2/b^2=1$.

(10) 由第二方程式.得 $\sin^2\theta=(m+n-2n\sin\phi\cos\phi)/2m$. 以此與第一方程式消去 $\sin\theta$. 則 $(n\sin\phi+m\cos\phi)^2=2m(m+n)$.

(11) 去兩方程式之分母.求得 $x=a\cos\theta$, $y=b\sin\theta$,

故 $(x/a)^2+(y/b)^2=\cos^2\theta+\sin^2\theta=1$.

(12) 由兩方程式.得 $ax=(a^2-b^2)\cos^3\theta$, $by=-(a^2-b^2)\sin^3\theta$,

故 $\cos\theta=\dfrac{(ax)^{\frac{1}{3}}}{(a^2-b^2)^{\frac{1}{3}}}$, $\sin\theta=\dfrac{(by)^{\frac{1}{3}}}{-(a^2-b^2)^{\frac{1}{3}}}$, 故 $(ax)^{\frac{2}{3}}+(by)^{\frac{2}{3}}=(a^2-b^2)^{\frac{2}{3}}$.

(13) 從原兩方程式.得 $\sec^2\theta=b(a+x)/(ay)$, $\cos^2\theta=b(a-x)/(ay)$,

由是 $b^2(a^2-x^2)/(ay)^2=1$, 卽 $b^2x^2+a^2y^2=a^2b^2$,

(14) $(1+m\tan^2\theta)^2-(1-m\tan^2\theta)^2=\{(1-m)^2\cot^2\beta-(1+m)^2\cot^2\alpha\}\tan^2\theta$,

卽 $4m=(1-m)^2\cot^2\beta-(1+m)^2\cot^2\alpha$, 卽 $(1+m)^2-(1-m)^2$

$=(1-m)^2\cot^2\beta-(1+m)^2\cot^2\alpha$. 卽 $(1+m)^3\operatorname{cosec}^2\alpha=(1-m)^2\operatorname{cosec}^2\beta$.

(15) 原兩方程式.平方之相加.得 $2\sin^2\theta\cos^2\theta=a^2+b^2$, 又由減法得

$4\sin^3\theta\cos^3\theta=b^2-a^2$. 故 $2(a^2+b^2)^3=(a^2-b^2)^2$,

(16) 由原方程式.得 $x=a\cos\theta(1+2\sin^2\theta)$. $y=a\sin\theta(1+2\cos^2\theta)$.

故 $x+y=a(\sin\theta+\cos\theta)^3$, $x-y=a(\cos\theta-\sin\theta)^3$, 由是 $(x+y)^{\frac{2}{3}}+(x-y)^{\frac{2}{3}}=2a^{\frac{2}{3}}$.

17. $x+a=a(2\cos\theta-\cos 2\theta)$, $y=a(2\sin\theta-\sin 2\theta)$.

18. $x=a(\cos\theta+\cos 2\theta)$, $y=b(\sin\theta+\sin 2\theta)$.

19. $x=a(1+\sin^2\theta\cos 2\theta)$, $y=a\sin^2\theta\sin 2\theta$.

20. $x=2a\cos\theta\cos 2\theta-a\cos\theta$, $y=2\cos\theta\sin 2\theta-a\sin\theta$.

21. $(a+b)(x+y)=\cos\theta(1+2\sin^2\theta)$,

$(a-b)(x-y)=\sin\theta(1+2\cos^2\theta)$.

22. $\cos\alpha\cos\theta/a^2+\sin\alpha\sin\theta/b^2+1=0$,

$\cos\beta\cos\theta/a^2+\sin\beta\sin\theta/b^2+1=0$.

23. $\operatorname{cosec}\theta\tan^3\theta(\operatorname{cosec}^2\theta+1)+\sec\theta=a$,

$\tan^2\theta(\operatorname{cosec}^2\theta+1)-\tan\theta=b$.

24. $m\sin 2\theta=n\sin\theta$, $p\cos 2\theta=q\cos\theta$.

25. $a^3y\sin\theta+b^3x\cos\theta+ab(a^2\sin^2\theta+b^2\cos^2\theta)=0$,

$ax\sec-by\operatorname{cosec}\theta=a^2-b^2$.

26. $2\cos^2\theta+a\sec\theta=2\sin^2\theta+b\operatorname{cosec}\theta=3$.

27. $x\sin\theta-y\cos\theta=a\sqrt{(1+\sin 2\theta\cos\alpha)}$,

$x\cos\theta-y\sin\theta=a\cos 2\theta\cos\alpha/\sqrt{(1+\sin 2\theta\cos\alpha)}$.

例題解自 17. 至 27.

(17) $(x+a)^2+y^2=a^2\{(2\cos\theta-\cos 2\theta)^2+(2\sin\theta-\sin 2\theta)^2\}$. 即 $x^2+y^2+2ax=4a^2(1-\cos\theta)$, (1). 又變化兩原方程式,順次爲 $x=2a\cos\theta(1-\cos\theta)$, 及 $y=2a\sin\theta(1-\cos\theta)$, 此兩方程式.各平方之相加.則 $x^2+y^2=4a^2(1-\cos\theta)^2$, (2).

由 (1) 及 (2) 得 $(x^2+y^2+2ax)^2=4a^2(x^2+y^2)^2$.

(18) $(x/a)^2+(y/b)^2=(\cos\theta+\cos 2\theta)^2+(\sin\theta+\sin 2\theta)^2=1+(1+2\cos\theta)$, (1),

又由原兩方程式.得 $x/a+1=\cos\theta(1+2\cos\theta)$, $y/b=\sin\theta(1+2\cos\theta)$,

故 $\left(\frac{x}{a}+1\right)^2+\left(\frac{y}{b}\right)^2=(1+2\cos\theta)^2$. (2). 故從 (1), (2) 得

$\left\{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2-1\right\}^2=\left(\frac{x}{a}+1\right)^2+\left(\frac{y}{b}\right)^2$, 即 $\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)^2-3\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=\frac{2x}{a}$.

(19) 由原兩方程式.得 $(x-a)^2+y^2=a^2\sin^4\theta(\cos^2 2\theta+\sin^2 2\theta)=a^2\sin^4\theta$, (1).

又由原兩方程式.得 $2(x^2+y^2)=2a^2\{(1+\sin^2\theta\cos 2\theta)^2+\sin^4\theta\sin^2 2\theta\}$

$=a^2(2+4\sin^2\theta-6\sin^4\theta)$, 但由最初之原方程式.得 $3a(x-a)$

$=a^2(3\sin^2\theta-6\sin^4\theta)$, 故 $2(x^2+y^2)-3a(x-a)-2a^2=a^2\sin^2\theta$, (2).

故從 (1), (2) 得 $a^2\{(x-a)^2+y^2\}=\{2(x^2+y^2)-3ax+a^2\}^2$.

(20) $x^2+y^2=a^2\{(2\cos\theta\cos 2\theta-\cos\theta)^2+(2\cos\theta\sin 2\theta-\sin\theta)^2\}=a^2$

$=a^2\{4\cos^2\theta+1-4\cos\theta\cos(2\theta-\theta)\}$ 卽 $x^2+y^2=a^2$.

(21) 從原方程式.由加法.(但參照例題 16.)

$2(ax+by)=(\cos\theta+\sin\theta)^3$, 由減法. $2(bx+ay)=(\cos\theta-\sin\theta)^3$,

故 $(ax+by)^{\frac{2}{3}}+(bx+ay)^{\frac{2}{3}}=2^{\frac{1}{3}}$.

(22) 由 2. 節第三. $a^4(\sin\alpha-\sin\beta)^2+b^4(\cos\alpha-\cos\beta)^2=\sin^2(\alpha-\beta)$.

(23) 變化原兩方程式.順次得 $2\sec^3\theta=a$, 及 $2\tan^3\theta=b$.

由是 $a^{\frac{2}{3}}-b^{\frac{2}{3}}=2^{\frac{2}{3}}(\sec^2\theta-\tan^2\theta)=2^{\frac{2}{3}}$.

(24) 由第一得 $\cos\theta=\frac{n}{2m}$, 由第二得 $p(2\cos^2\theta-1)=q\cos\theta$,

故 $p(n^2-2m^2)=qmn$.

(25) 由第一得 $a^3\sin\theta(y+b\sin\theta)=-b^3\cos\theta(x+a\cos\theta)$. (1).

由第二得 $b\cos\theta(y-b\sin\theta)=a\sin\theta(x-a\cos\theta)$, (2).

(1) 及 (2) 相乘.得 $a^3b\sin\theta\cos\theta(y^2-b^2\sin^2\theta)=-ab^3\sin\theta\cos\theta(x^2-a^2\cos^2\theta)$.

故 $a^2y^2+b^2x^2=a^2b^2$.

(26) 變原兩方程式.順次爲 $a=3\cos\theta-2\cos^3\theta$.

$b=3\sin\theta-2\sin^3\theta$, 故由加法. $a+b=(\cos\theta+\sin\theta)(1+2\sin\theta\cos\theta)$

$=(\cos\theta+\sin\theta)^3$, 又由減法. $a-b=(\cos\theta-\sin\theta)^3$,

故由例題 16. 與同樣之方向.得 $(a+b)^{\frac{2}{3}}+(a-b)^{\frac{2}{3}}=2$.

(27) 試將原兩方程式相乘.得 $(x^2-y^2)\sin\theta\cos\theta+xy(\cos^2\theta-\sin^2\theta)$

$=a^2\cos 2\theta\cos\alpha$, 卽 $(x^2-y^2)\sin 2\theta+2xy\cos 2\theta=2a^2\cos 2\theta\cos\alpha$,

故 $\cos 2\theta=\frac{(x^2-y^2)\sin 2\theta}{2(a^2\cos\alpha-xy)}$, (1). 又將原兩方程式之各節.平方之相加.

得 $x^2+y^2=a^2(1+\sin 2\theta\cos\alpha)+\frac{a^2\cos^2 2\theta\cos^2\alpha}{1+\sin 2\theta\cos\alpha}$, 卽

$\sin 2\theta=\frac{a^2(1+\cos^2\alpha)-x^2-y^2}{(x^2+y^2-2a^2)\cos\alpha}$ (2). 由 (1) 及 $\sin^2\theta+\cos^2\theta=1$, 得

$\sin^2 2\theta=\frac{4(a^2\cos\alpha-xy)^2}{(x^2-y^2)^2+4(a^2\cos\alpha-xy)^2}$. 由此及 (2). 得

$$\frac{4(a^2\cos\alpha-xy)^2}{(x^2-y^2)^2+4(a^2\cos\alpha-xy)^2}=\frac{\{a^2(1+\cos^2\alpha)-x^2-y^2\}^2}{(x^2+y^2-2a^2)^2\cos^2\alpha}.$$

28. $sin^2\theta - psin\theta + 1 = 0,\ cos^2\theta - qcos\theta + 1 = 0.$

29. $sin^2\theta - psin\theta + m = 0,\ cos^2\theta - qcos\theta + n = 0.$

30. $x(1 + sin^2\theta - cos\theta) - ysin\theta(1 + cos\theta) = a(1 + cos\theta).$

$y(1 + cos^2\theta) - xsin\theta cos\theta = asin\theta.$

31. $acos\alpha cos\theta + bsin\alpha sin\theta = c,$

$acos\beta cos\theta + bsin\beta sin\theta = c.$

32. $p^2bxsec\theta - q^2aycosec\theta + ab(p^2 - q^2) = 0,$

$p^2bxsec^2\theta sin\theta + q^2aycosec^2\theta cos\theta = 0.$

33. $\frac{sin\alpha}{b^2-1} = \frac{cos\alpha}{2bsin2\theta} = \frac{1}{1+2bcos2\theta+b^2}.$

34. $asec\theta + bcosec\theta = c,\ asec\theta - bcosec\theta = cos2\theta.$

35. $acos2\theta + bsin2\theta = c,\ lcos3\theta + msin3\theta = 0.$

36. $acos\theta + bsin\theta = acos3\theta + bsin3\theta = c.$

37. $\frac{cos3\theta}{a} + \frac{cos2\theta}{b} + \frac{cos\theta}{c} = 0,$

$\frac{cos2\theta}{a} + \frac{cos\theta}{b} + \frac{1}{2c} = 0.$

例題解自 28 至 37.

(28) 由第一. $p^2sin^2\theta = (sin^2\theta + 1)^2$, 卽 $sin^4\theta - (p^2 - 2)sin^2\theta + 1 = 0$, (1).

由第二. $q^2(1 - sin^2\theta) = (2 - sin^2\theta)^2$, 卽 $sin^4\theta + (q^2 - 4)sin^2\theta - (q^2 - 4) = 0$, (2).

由 2. 節第三而從 (1), (2) 得

$$\frac{sin^4\theta}{(p^2-2)(q^2-4)-(q^2-4)} = \frac{sin^2\theta}{1+(q^2-4)} = \frac{1}{(q^2-4)+(p^2-2)},$$ 卽

$$\frac{sin^4\theta}{(q^2-3)^3} = \frac{sin^4\theta}{(q^2-4)(p^2-3)} \times \frac{1}{p^2+q^2-6}.$$ 故

$$(q^2-3)^2 = (q^2-4)(p^2-3)(p^2+q^2-6).$$

(29) 與前例同法. $\{q^2 + m^2 - (n+1)^2\}^2$

$= \{p^2q^2 - p^2(n+1)^3 - q^2(m^2+2m) + 2m(n+1)(m+n+1)\}\{p^2 + q^2 - 2(m+n+1)\}.$

(30) 由 2. 節第三. 得 $x/(1+\cos\theta)=y/\sin\theta=a/(1-\cos\theta)$,

即 $y^2/\sin^2\theta=ax/(1-\cos\theta)$, 故 $y^2=ax$.

(31) 與前例同法. 得 $c^2a^2(\cos\beta-\cos\alpha)^2+b^2c^2(\sin\alpha-\sin\beta)^2=a^2b^2\sin^2(\alpha-\beta)$.

(32) 從兩方程式求得 x, 及 y. 順次變化其所求之值. 爲

$$\cos\theta=\frac{p^{\frac{2}{3}}x^{\frac{1}{3}}}{a^{\frac{1}{3}}(p^2-q^2)^{\frac{1}{3}}} \quad 及 \quad \sin\theta=\frac{q^{\frac{2}{3}}y^{\frac{1}{3}}}{b^{\frac{1}{3}}(p^2-q^2)^{\frac{1}{3}}},$$

由是 $p^{\frac{4}{3}}x^{\frac{2}{3}}/a^{\frac{2}{3}}+q^{\frac{4}{3}}y^{\frac{2}{3}}/b^{\frac{2}{3}}=(p^2-q^2)^{\frac{2}{3}}$.

(33) $b^2=(1+\sin\alpha)/(1-\sin\alpha)$, 此分母子以 $1-\sin\alpha$ 乘之. 開平方. 得

$$b=\pm\cos\alpha/(1-\sin\alpha)=\pm\sin\left(\frac{\pi}{2}+\alpha\right)\Big/\left\{1+\cos\left(\frac{\pi}{2}+\alpha\right)\right\}=\pm\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

(34) 從原兩方程式行加減. 得 $2a/\cos\theta=c+\cos2\theta$, (1).

及 $2b/\sin\theta=c-\cos2\theta$, (2) 由 (1), (2) 得 $8ab/\sin2\theta=c^2-\cos^22\theta$,

即 $64a^2b^2=(1-\cos^22\theta)(c^2-\cos^22\theta)^2$, (3). 又 (1) 之兩節. 平方而變化之. 則 $\cos^32\theta+(2c+1)\cos^22\theta+(c^2+2c)\cos2\theta+c^2-8a^2=0$. 同樣. 由 (2) 得 $-\cos^32\theta+(2c+1)\cos^22\theta-(c^2+2c)\cos2\theta+c^2-8b^2=0$, 從此兩方程式. 求得 $\cos^22\theta=-\dfrac{c^2-4(a^2+b^2)}{2c+1}$, 以此代入 (3) 且去其分母. 則

$\{c^3+c^2-2(a^2+b^2)\}^2\{(c+1)^2-4(a^2+b^2)\}=16a^2b^2(2c+1)^3$.

(35) 變化原兩方程式. 順次得

$(a+c)\tan^2\theta-2b\tan\theta+c-a=0$, (1). 及 $m\tan^3\theta+3l\tan\theta-3m\tan^2\theta-l=0$, (2).

$m\tan\theta$ 乘 (1). $(a+c)$ 乘 (2), 消去 $\tan^3\theta$ 之項. 則

$\{3l(a+c)+2bm\}\tan^2\theta-2m(a+2c)\tan\theta-l(a+c)=0$, (3).

從 (1) 及 (3) 用 2. 節第三消去法消去 θ. 即得.

(36) 由 2. 節第三. 得 $\dfrac{a}{\sin3\theta-\sin\theta}=\dfrac{b}{\cos\theta-\cos3\theta}=\dfrac{c}{\sin2\theta}$. 即

$\dfrac{a}{\cos2\theta}=\dfrac{b}{\sin2\theta}=\dfrac{c}{\cos\theta}$. 故 $a^2+b^2=\dfrac{c^2}{\cos^2\theta}$ 及 $\sin\theta=b/(2c)$.

由是 $(a^2+b^2)b^2=4c^2(a^2+b^2-c^2)$.

(37) 由 2. 節第三. 得

$$\frac{1}{a}\Big/(\tfrac{1}{2}\cos2\theta-\cos^2\theta)=\frac{1}{b}\Big/(\cos\theta\cos2\theta-\tfrac{1}{2}\cos3\theta)=\frac{1}{c}\Big/(\cos3\theta\cos\theta-\cos^22\theta)$$

即 $-\dfrac{2}{a}=\dfrac{2}{b\cos\theta}=-\dfrac{1}{c\sin^2\theta}$, 故 $\sin^2\theta=\dfrac{a}{2c}$, $\cos\theta=-\dfrac{a}{b}$,

由是 $\dfrac{a}{2c}+\left(-\dfrac{a}{b}\right)^2=1$, 即 $ab^2+2c(a^2-b^2)=0$.

38. $x=a\cos(\theta+\alpha),\ y=b\cos(\theta-\beta).$

39. $\dfrac{\sin(\theta-\alpha)}{\sin(\theta-\beta)}=\dfrac{a}{b},\quad \dfrac{\cos(\theta-\alpha)}{\cos(\theta-\beta)}=\dfrac{a'}{b'}.$

40. $3(\cos(\theta+\alpha)-2\sin(\theta+\alpha)=\cos(\theta-\alpha),$

$3\cos(\theta+\beta)+4\sin(\theta+\beta)=\cos(\theta-\beta).$

41. $\sin(\theta+\alpha)=\sin(\theta+\beta)=a\sin2\theta.$

42. $x=a\cos(2\theta+\alpha),\ y=b\sin(3\theta+\beta).$

43. $\sin\theta\cot\alpha=\sin(\theta+\phi)\cot\beta=\sin(\theta+\phi+\psi)\cot\gamma.$

44. $(a+b)\tan(\theta-\phi)=(a-b)\tan(\theta+\phi),$

$a\cos2\phi+b\cos2\theta=c.$

45. $(a-b)\sin(\theta+\phi)=(a+b)\sin(\theta-\phi),$

$a\tan\frac{1}{2}\theta-b\tan\frac{1}{2}\phi=c.$

46. $6\tan(\theta+\alpha)=3\tan(\theta+\beta)=2\tan(\theta+\gamma).$

例題解自 38. 至 46.

(38) 將原兩方程式相加而變化之,則

$(x/a+y/b)/\cos\frac{1}{2}(\alpha-\beta)=2\cos\frac{1}{2}(2\theta-\alpha-\beta),$

同樣.由減法得 $(x/a-y/b)/\sin\frac{1}{2}(\alpha-\beta)=2\sin\frac{1}{2}(2\theta-\alpha-\beta)$, 平方此兩方程式之各節.相加而變化之.則

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{2xy}{ab}\cos(\alpha-\beta)=\sin^2(\alpha-\beta).$$

(39) 由原兩方程式得 $\tan\theta=\dfrac{b\sin\alpha-a\sin\beta}{b\cos\alpha-a\cos\beta}=\dfrac{-b'\cos\alpha+a'\cos\beta}{b'\sin\alpha-a'\sin\beta},$

故 $\cos(\alpha-\beta)=(aa'+bb')/(ab'+a'b).$

(40) 與前例同樣.得 $\tan\theta=\dfrac{\cos\alpha-\sin\alpha}{\cos\alpha+2\sin\alpha}=\dfrac{\cos\beta+2\sin\beta}{2(\sin\beta-\cos\beta)}.$

故 $\cos(\alpha-\beta)+\sin\alpha\sin\beta=0.$

(41) 變化原兩方程式.順次得 $\cos\alpha\tan\theta-2a\sin\theta+\sin\alpha=0$, 及 $\cos\beta\tan\theta-2a\sin\theta+\sin\beta=0$, 由 2. 節第三.得

$\dfrac{\tan\theta}{2a\cos\frac{1}{2}(\alpha+\beta)}=\dfrac{\sin\theta}{\cos\frac{1}{2}(\alpha-\beta)}=\dfrac{1}{2a\sin\frac{1}{2}(\alpha+\beta)}$, 故 $a^2\sin^2(\alpha+\beta)=\cos^2\frac{1}{2}(\alpha-\beta).$

(42) $\cos3(2\theta+\alpha)=4\cos^3(2\theta+\alpha)-3\cos(2\theta+\alpha)=4\left(\dfrac{x}{a}\right)^3-3\left(\dfrac{x}{a}\right)=p.$

即 $\cos 6\theta\cos 3\alpha-\sin 6\theta\sin 3\alpha=p$, (1), 又 $\sin 2(3\theta+\beta)=2\sin(3\theta+\beta)\cos(3\theta+\beta)$
$=2(y/b)\sqrt{1-(y/b)^2}=\sqrt{1-\{1-2(y/b)^2\}^2}=\sqrt{1-q^2}$, 但 $1-2(y/b)^2=q$. 即
$\cos 6\theta\sin 2\beta+\sin 6\theta\cos 2\beta=\sqrt{1-q^2}$, (2). 從 (1) 及 (2) 用 2. 節第三之法.得

$$\frac{\cos 6\theta}{p\cos 2\beta+\sqrt{1-q^2}\sin 3\alpha}=\frac{\sin 6\theta}{-p\sin 2\beta+\sqrt{1-q^2}\cos 3\alpha}=\frac{1}{\cos(3\alpha-2\beta)},$$

令 $3\alpha-2\beta=\omega$, 則 $\{p\cos 2\beta+\sqrt{1-q^2}\sin 3\alpha\}^2+\{-p\sin 2\beta+\sqrt{1-q^2}\cos 3\alpha\}^2=\cos^2\omega$,
即 $p^2+2p\sqrt{1-q^2}\sin\omega+\sin^2\omega-q^2=0$, $\therefore$ $2pq\cos\omega=p^2+q^2-\sin^2\omega$.

(43) $\tan\theta\cot\alpha=(\tan\theta\cos\phi+\sin\phi)\cot\beta=\{\tan\theta\cos(\phi+\psi)+\sin(\phi+\psi)\}\cot\gamma$,

故 $\dfrac{\sin\phi\cot\beta}{\cot\alpha-\cos\phi\cot\beta}=\dfrac{\sin(\phi+\psi)\cot\gamma}{\cos\alpha-\cos(\phi+\psi)\cot\gamma}$, 去此分母變化之,則
$\sin\phi\tan\gamma+\sin\psi\tan\alpha=\sin(\phi+\psi)\tan\beta$.

(44) $\dfrac{a+b}{a-b}=\dfrac{\sin(\theta+\phi)\cos(\theta-\phi)}{\cos(\theta+\phi)\sin(\theta-\phi)}=\dfrac{\sin 2\theta+\sin 2\phi}{\sin 2\theta-\sin 2\phi}$, 即 $\dfrac{a}{b}=\dfrac{\sin 2\theta}{\sin 2\phi}$,
由此及第二方程式.即可得 $c^2+a^2-2ca\cos 2\phi=b^2$.

(45) 從第一方程式 $\tan\theta=\dfrac{a}{b}\tan\phi=\dfrac{2\tan\dfrac{\theta}{2}}{1-\tan^2\dfrac{\theta}{2}}$, 又從第二方程式得

$\tan\frac{1}{2}\theta=\dfrac{c+b\tan\frac{1}{2}\phi}{a}$, 故 $a^2\tan\dfrac{\phi}{2}=\left(b+c\tan\dfrac{\phi}{2}\right)\left(c+b\tan\dfrac{\phi}{2}\right)$.

(46) 由第一方程式 $2\sin(\theta+\alpha)\cos(\theta+\beta)=\sin(\theta+\beta)\cos(\theta+\alpha)$. 用第四編 3.節之公式而變化之.則 $\sin(2\theta+\alpha+\beta)=-3\sin(\alpha-\beta)$, (1).
同樣.由第二方程式得 $\sin(2\theta+\beta+\gamma)=-5\sin(\beta-\gamma)$, (2).
由 (1), (2) 加減而變化之.則

$$2\sin\left(2\theta+\beta+\frac{\gamma+\alpha}{2}\right)=\frac{-3\sin(\alpha-\beta)-5\sin(\beta-\gamma)}{\cos\frac{1}{2}(\gamma-\alpha)},$$

$$2\cos\left(2\theta+\beta+\frac{\gamma+\alpha}{2}\right)=\frac{3\sin(\alpha-\beta)-5\sin(\beta-\gamma)}{\sin\frac{1}{2}(\gamma-\alpha)},$$

平方此兩方程式.相加而變化之.則
$\sin^2(\gamma-\alpha)=9\sin^2(\alpha-\beta)+25\sin^2(\beta-\gamma)-30\sin(\alpha-\beta)\sin(\beta-\gamma)\cos(\gamma-\alpha)$, (3).
又用例題五 62.同樣之方法.得
$\sin^2(\alpha-\beta)+\sin^2(\beta-\gamma)-\sin^2(\gamma-\alpha)=-2\sin(\alpha-\beta)\sin(\beta-\gamma)\cos(\gamma-\alpha)$, (4).
因 $\sin^2(\alpha-\beta)+\sin^2(\beta-\gamma)+\sin^2(\gamma-\alpha)-2\sin^2(\gamma-\alpha)$
$=\frac{1}{2}[3-\{\cos 2(\alpha-\beta)+\cos 2(\beta-\gamma)+\cos 2(\gamma-\alpha)\}]-2\sin^2(\gamma-\alpha)$
$=\frac{1}{2}\{3-4\cos(\alpha-\beta)\cos(\beta-\gamma)\cos(\gamma-\alpha)+1\}-2\sin^2(\gamma-\alpha)$ [例題五 16.].
$=-2\cos(\alpha-\beta)\cos(\beta-\gamma)\cos(\gamma-\alpha)+2\cos^2(\gamma-\alpha)$
$=2\cos(\gamma-\alpha)[-\cos(\alpha-\beta)\cos(\beta-\gamma)+\cos\{-(\alpha-\beta)-(\beta-\gamma)\}]$
$=-2\sin(\alpha-\beta)\sin(\beta-\gamma)\cos(\gamma-\alpha)$.
由是從 (3) 及 (4) 得 $3\sin^2(\alpha-\beta)+5\sin^2(\beta-\gamma)=2\sin^2(\gamma-\alpha)$.

47. $\dfrac{\cos^3\theta}{\cos(\alpha-3\theta)}=\dfrac{\sin^3\theta}{\sin(\alpha-3\theta)}=m.$

48. $x\cos(\theta+\alpha)+y\sin(\theta+\alpha)=a\sin 2\theta,$

$y\cos(\theta+\alpha)-x\sin(\theta+\alpha)=2a\cos 2\theta.$

49. $\cos(\theta-\phi)\cos\theta=a\sin\phi\cos\phi,$

$\sin(\theta-\phi)\cos\theta=\frac{1}{2}a\cos^3\phi.$

50. $a\sin\left(\theta+\dfrac{\pi}{4}\right)+b\sin\left(\theta-\dfrac{\pi}{4}\right)=\dfrac{c}{\sqrt{2}},$

$a\cos\left(\theta-\dfrac{\pi}{4}\right)+b\cos\left(\theta+\dfrac{\pi}{4}\right)=c\sin\left(2\theta+\dfrac{\pi}{4}\right).$

51. $a\cos 3\theta+b\sin 3\theta=c\cos\theta,$

$a\sin 3\theta+b\operatorname{coc} 3\theta=c\cos\left(\theta+\dfrac{\pi}{6}\right).$

52. $\sin 3\theta+m\sin(\alpha+\theta)+2\sin(\beta-\theta)\sin(\gamma-\theta)\sin(\beta+\gamma+\theta)=0,$

$\cos 3\theta+m\cos(\alpha+\theta)+2\cos(\beta-\theta)\cos(\gamma-\theta)\cos(\beta+\gamma+\theta)=0.$

例題解自 47. 至 52.

(47) 由第一 $3\cos\theta+\cos 3\theta=4m(\cos\alpha\cos 3\theta+\sin\alpha\sin 3\theta),$

及由第二 $3\sin\theta-\sin 3\theta=4m(\sin\alpha\cos 3\theta-\cos\alpha\sin 3\theta),$

即 $4m\sin\alpha\sin 3\theta+(4m\cos\alpha-1)\cos 3\theta-3\cos\theta=0,$

及 $(4m\cos\alpha-1)\sin 3\theta-4m\sin\alpha\cos 3\theta+3\sin\theta=0,$ 由 2. 節第三. 得

$$\frac{\sin 3\theta}{12m\sin(\theta-\alpha)-3\sin\theta}=\frac{\cos 3\theta}{-12m\cos(\theta-\alpha)+3\cos\theta}=\frac{1}{-16m^2+8m\cos\alpha-1},$$

故 $\{12m\sin(\theta-\alpha)-3\sin\theta\}^2+\{-12m\cos(\theta-\alpha)+3\cos\theta\}^2=(-16m^2+8m\cos\alpha-1)^2,$

即 $144m^2+9-72m\cos\alpha=(8m\cos\alpha-16m^2-1)^2,$

故 $(m\cos\alpha-2m^2+1)(8m\cos\alpha-16m^2-1)=0.$

(48) 由 2. 節第三 $\dfrac{x}{-2\sin(\theta+\alpha)\cos 2\theta+\sin 2\theta\cos(\theta+\alpha)}$

$$=\frac{y}{\sin 2\theta\sin(\theta+\alpha)+2\cos 2\theta\cos(\theta+\alpha)}=\frac{a}{\cos^2(\theta+\alpha)+\sin^2(\theta+\alpha)},$$

即 $\dfrac{x}{\sin(\theta-\alpha)-\cos 2\theta\sin(\theta+\alpha)}=\dfrac{y}{\cos(\theta-\alpha)+\cos 2\theta\cos(\theta+\alpha)}=a,$ 故

$x\cos\alpha+y\sin\alpha=a[\sin(\theta-\alpha)\cos\alpha+\cos(\theta-\alpha)\sin\alpha-\cos 2\theta\{\sin(\theta+\alpha)\cos\alpha$

$-\cos(\theta+\alpha)\sin\alpha\}]=a\sin\theta(1-\cos 2\theta)=2a\sin^3\theta.$

故 $(x\cos\alpha+y\sin\alpha)^{\frac{1}{3}}=(2a)^{\frac{1}{3}}\sin\theta$, 同樣 $(x\sin\alpha-y\cos\alpha)^{\frac{1}{3}}=(2a)^{\frac{1}{3}}\cos\theta$,

由是 $(x\cos\alpha+y\sin\alpha)^{\frac{2}{3}}+(x\sin\alpha-y\cos\alpha)^{\frac{2}{3}}=(2a)^{\frac{2}{3}}$.

(49) 由第一及第二得 $\cos(2\theta-\phi)+\cos\phi=2a\sin\phi\cos\phi$,

$\sin(2\theta-\phi)-\sin\phi=a\cos^2\phi$, 故 $(2a\sin\phi\cos\phi-\cos\phi)^2+(a\cos^2\phi+\sin\phi)^2$

$=\cos^2(2\theta-\phi)+\sin^2(2\theta-\phi)$, 由是 $a(1+3\sin^2\phi)=2\sin\phi$.

(50) 由加法 $a\left\{\sin\left(\theta+\frac{\pi}{4}\right)+\cos\left(\theta-\frac{\pi}{4}\right)\right\}+b\left\{\sin\left(\theta-\frac{\pi}{4}\right)+\cos\left(\theta+\frac{\pi}{4}\right)\right\}$

$=c\left\{\frac{1}{\sqrt{2}}+\sin\left(2\theta+\frac{\pi}{4}\right)\right\}$, 即 $a(\sin\theta+\cos\theta)=c\cos\theta(\sin\theta+\cos\theta)$,

故 $a=c\cos\theta$, 同樣,由減法 $b=c\sin\theta$, 故 $a^2+b^2=c^2$.

(51) 由原方程式相加.得

$$(a+b)(\sin3\theta+\cos3\theta)=c\left\{\cos\theta+\cos\left(\theta+\frac{\pi}{6}\right)\right\},$$

即 $(a+b)\sin\left(3\theta+\frac{\pi}{4}\right)=c\sqrt{2}\cos\left(\theta+\frac{\pi}{12}\right)\cos\frac{\pi}{12}=\frac{c(\sqrt{3}+1)}{2}\cos\left(\theta+\frac{\pi}{12}\right)$,

同樣.由減法得 $(a-b)\cos\left(3\theta+\frac{\pi}{4}\right)=\frac{c(\sqrt{3}-1)}{2}\sin\left(\theta+\frac{\pi}{12}\right)$,

故 $\theta+\frac{\pi}{12}=\phi$, 令 $\frac{c(\sqrt{3}+1)}{2(a+b)}=k$, $\frac{c(\sqrt{3}-1)}{2(a-b)}=k'$, 則

$$\sin3\phi=k\cos\phi,\ \cos3\phi=k'\sin\phi,\ 故\ 1=k^2\cos^2\phi+k'^2\sin^2\phi,$$

即 $\cos^2\phi=\frac{1-k'^2}{k^2-k'^2}$, $\sin^2\phi=\frac{k^2-1}{k^2-k'^2}$,

又 $3\sin\phi-4\sin^3\phi=k\cos\phi$, $4\cos^3\phi-3\cos\phi=k'\sin\phi$,

故 $k'\sin^2\phi(3-4\sin^2\phi)=k\cos^2\phi(4\cos^2\phi-3)$, 由是

$$\frac{k'(k^2-1)}{k^2-k'^2}\left\{3-\frac{4(k^2-1)}{k^2-k'^2}\right\}=\frac{k(1-k'^2)}{k^2-k'^2}\left\{\frac{4(1-k'^2)}{k^2-k'^2}-3\right\},$$

簡之.則 $(k-k')^2(3-kk')=4(1-kk')^2$.

(52) 以 $\sin\theta$ 乘第一, $\cos\theta$ 乘第二.相加.則

$\cos2\theta(1+\cos2\theta)+m\cos\alpha+\frac{1}{2}\{\cos2\beta+\cos2\gamma+\cos2(\beta+\gamma)-1\}=0$,

又以 $\cos\theta$ 乘第一, $\sin\theta$ 乘第二.相減.則

$\sin2\theta(1+\cos2\theta)+m\sin\alpha+\frac{1}{2}\{\sin2\beta+\sin2\gamma-\sin2(\beta+\gamma)\}=0$,

爲便利計.令 $\cos2\theta(1+\cos2\theta)+a=0$, 及 $\sin2\theta(1+\cos2\theta)+b=0$,

則 $(1+\cos2\theta)^2=a^2+b^2$, 故 $\cos^2 2\theta=\frac{a^2}{(1+\cos2\theta)^2}=\frac{a^2}{a^2+b^2}$,

故 $\left\{1+\frac{a}{\sqrt{(a^2+b^2)}}\right\}^2=a^2+b^2$, 即 $1+\frac{2a}{\sqrt{(a^2+b^2)}}+\frac{a^2}{a^2+b^2}=a^2+b^2$,

即 $4a^2(a^2+b^2)=\{(a^2+b^2)^2-2a^2-b^2\}^2$.

次之各聯立方程式消去其θ及ϕ.

53. $x=a\cos^m\theta\cos^m\phi,\ y=b\cos^m\theta\sin^m\phi,\ z=c\sin^m\theta.$

54. $\cos^2\theta=\dfrac{\cos\alpha}{\cos\beta},\ \cos^2\phi=\dfrac{\cos\gamma}{\cos\beta},\ \dfrac{\tan\theta}{\tan\phi}=\dfrac{\tan\alpha}{\tan\gamma}.$

55. $x\cos\theta/a+y\sin\theta/b=x\cos\phi/a+y\sin\phi/b=1,$

$$\theta-\phi=2\alpha,$$

56. $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=\dfrac{ax}{\cos\phi}-\dfrac{by}{\sin\phi}=a^2-b^2,\ \theta-\phi=\frac{1}{2}\pi.$

57. $x\cos\theta/a+y\sin\theta/b=x\cos\phi/a+y\sin\theta/b=1,$

$\cos\theta\cos\phi/a^2+\sin\theta\sin\phi/b^2=0.$

58. $a\cos^2\theta+b\sin^2\theta=m\cos^2\phi,$

$a\sin^2\theta+b\cos^2\theta=n\sin^2\phi,\ m\tan^2\theta=n\tan^2\phi.$

59. $a\cos\theta+b\sin\theta=a\cos\phi+b\sin\theta=c,\ \tan\theta\tan\phi=d.$

60. $x\cos\theta+y\sin\theta=x\cos\phi+y\sin\phi=2a,$

$2\cos\frac{1}{2}\theta\cos\frac{1}{2}\phi=1.$

61. $x=(a\sin^2\theta+b\cos^2\theta)\cos^2\phi+c\sin^2\phi,\ y=a\cos^2\theta+b\sin^2\theta,$

$z=(b-a)\sin\theta\cos\theta\cos\phi.$

例題解自53.至61.

(53) 由第三得 $\sin\theta=(z/c)^{\frac{1}{m}}$, 由第一,第二得

$\cos\phi=(x/a)^{\frac{1}{m}}/\cos\theta,\ \sin\phi=(y/b)^{\frac{1}{m}}/\cos\theta,$

故 $\left(\dfrac{x}{a}\right)^{\frac{2}{m}}+\left(\dfrac{y}{b}\right)^{\frac{2}{m}}=(\cos^2\phi+\sin^2\phi)\cos^2\theta=1-\sin^2\theta=1-\left(\dfrac{z}{c}\right)^{\frac{2}{m}}.$

(54) 由原三方程式得 $\sec^2\theta=\dfrac{\cos\beta}{\cos\alpha},$

$\sec^2\phi=\dfrac{\cos\beta}{\cos\gamma},\ \dfrac{\sec^2\theta-1}{\sec^2\phi-1}=\dfrac{\tan^2\alpha}{\tan^2\gamma},$

故 $\dfrac{\cos\gamma(\cos\beta-\cos\alpha)}{\cos\alpha(\cos\beta-\cos\gamma)}=\dfrac{\tan^2\alpha}{\tan^2\gamma}.$

(55) 由2節第三 $\frac{x}{a^2b(\sin\theta-\sin\phi)}=\frac{y}{ab^2(\cos\phi-\cos\theta)}=\frac{1}{ab\sin(\theta-\phi)}$,

即 $\frac{bx}{ab\cos\frac{1}{2}(\theta+\phi)}=\frac{ay}{ab\sin\frac{1}{2}(\theta+\phi)}=\frac{1}{\cos\alpha}$, 故 $\frac{b^2x^2+a^2y^2}{a^2b^2}=\frac{2}{1+\cos 2\alpha}$.

(56) $\theta=\frac{1}{2}\pi+\phi$, 故由第一 $\frac{ax}{-\sin\phi}-\frac{by}{\cos\phi}=a^2-b^2$. 由此與第二求

得 $\sin\phi$, 及 $\cos\phi$, 試消去 ϕ. 則 $2(a^2x^2+b^2y^2)^3=(a^2x^2-b^2y^2)^2(a^2-b^2)^2$.

(57) 表原方程式爲 $\cos\theta$ 及 $\cos\phi$ 之項以 λ 代之.則

$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)\lambda^2-\frac{2x}{a}\lambda+1-\frac{y^2}{b^2}=0$, 由二次方程式之性質, 從 $\lambda=\cos\theta$ 或

$\cos\phi$, 得 $\cos\theta\cos\phi=\left(1-\frac{y^2}{b^2}\right)\Big/\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$. 叉表兩方程式爲 $\sin\theta$ 及

$\sin\phi$ 之項則 $\sin\theta\sin\phi=\left(1-\frac{x^2}{b^2}\right)\Big/\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)$,

∴ 從第三方程式. $x^2+y^2=a^2+b^2$.

(58) 由最初兩方程式相加. 得 $a+b=m\cos^2\phi+n\sin^2\phi$, 從此

求得 $\sec^2\phi$, 叉由第一求得 $\sec^2\theta$. 代入由第三所得之式之內. 即

代入 $m(\sec^2\theta-1)=n(\sec^2\phi-1)$ 之內. 即得 $(a+b)(m+n)=2mn$.

(59) 從最初兩方程式依例題57. 得 $\cos\theta\cos\phi=\frac{c^2-b^2}{a^2+b^2}$,

$\sin\theta\sin\phi=\frac{c^2-a^2}{a^2+b^2}$. 故由第三. 得 $(c^2-a^2)/(c^2-b^2)=d$.

(60) 如例題57. 得 $\left(\frac{x^2}{4a^2}+\frac{y^2}{4a^2}\right)\lambda^2-\frac{x}{a}\lambda+1-\frac{y^2}{4a^2}=0$, 故

$\cos\theta+\cos\phi=\frac{4ax}{x^2+y^2}$, 及 $\cos\theta\cos\phi=\frac{4a^2-y^2}{x^2+y^2}$. 叉從第三方程式得

$4\cos^2\frac{1}{2}\theta\cos^2\frac{1}{2}\phi=1$, 即 $(1+\cos\theta)(1+\cos\phi)=1$. 即

$\cos\theta+\cos\phi+\cos\theta\cos\phi=0$, 故 $y^2=4a(a+x)$.

(61) 從第二得 $\cos^2\theta=\frac{y-b}{a-b}$, $\sin^2\theta=\frac{a-y}{a-b}$, 代入第一. 以求 $\cos^2\phi$

則 $\cos^2\phi=\frac{x-c}{a+b-c-y}$. 將第三平方之. 而代用 $\sin^2\theta$, $\cos^2\theta$, 及 $\cos^2\phi$ 之值

則 $z^2(y+c-a-b)=(x-c)(y-b)(y-a)$.

62. $\sec\theta-\sec\phi=a$, $\tan\frac{1}{2}\theta/\tan\frac{1}{2}\phi=b$, $\tan\theta\tan\phi=c$.

63. $a=\sin\theta\cos\phi\sin\alpha+\cos\theta\cos\alpha$,

$b=\sin\theta\cos\phi\cos\alpha-\cos\theta\sin\alpha$, $c=\sin\theta\sin\phi\sin\alpha$.

64. $a\sin^2\theta+c\cos^2\theta=b$, $c\sin^2\phi+a\cos^2\phi=d$,

$a\tan\theta=c\tan\phi$.

65. $\tan\theta+\tan\phi=a$, $\cot\theta+\cot\phi=b$, $\theta+\phi=c$.

66. $a^2\cos^2\theta-b^2\cos^2\phi=c^2$, $a\cos\theta+b\cos\phi=d$,

$a\tan\theta=b\tan\phi$.

67. $x\cos\theta+y\sin\theta=x\cos\phi+y\sin\phi=1$,

$a\cos\theta\cos\phi+b\sin\theta\sin\phi+c(\cos\theta+\cos\phi)+d(\sin\theta+\sin\phi)=0$.

68. $\frac{x}{a}\cos\theta+\frac{y}{b}\cos\phi=\frac{x}{a}\sin\theta+\frac{y}{b}\sin\phi=1$,

$\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2=m(\sin2\theta-\sin2\phi)$.

例題解自 62. 至 68.

(62) 由第一. $\cos\phi=\frac{\cos\theta}{1-a\cos\theta}$, (1). 由第二. $\frac{\sin\theta(1+\cos\phi)}{\sin\phi(1+\cos\theta)}=b$,

即 $b^2=\frac{(1-\cos\theta)(1+\cos\phi)}{(1+\cos\theta)(1-\cos\phi)}$, 故 $\frac{b^2+1}{b^2-1}=\frac{1-\cos\theta\cos\phi}{\cos\phi-\cos\theta}$, (2).

從 (1), (2) 得 $\{a(b^2+1)+b^2-1\}\cos^2\theta+a(b^2-1)\cos\theta-(b^2-1)=0$, (3).

又由第三. $1-(\cos^2\theta+\cos^2\phi)+(1-c^2)\cos^2\theta\cos^2\phi=0$, 故從此方程式及(1)消去$\phi$.則

$(1-c^2-a^2)\cos^2\theta+2a\cos^3\theta+(a^2-2)\cos^2\theta-2a\cos\theta+1=0$, (4)

以 b^2-1 乘 (4). 而後與 (3) 相加. 以 $\cos\theta$ 除之. 則

$$(1-c^2-a^2)(b^2-1)\cos^3\theta+2a(b^2-1)\cos^2\theta$$
$$+\{a^2(b^2-1)+a(b^2+1)\}\cos\theta-a(b^2-1)=0, \qquad (5).$$

以 a 乘 (3). 從 (5) 減之. 再以 $\cos\theta$ 除之. 則

$$(1-c^2-a^2)(b^2-1)\cos^2\theta-a\{a(b^2+1)-(b^2-1)\}\cos\theta$$
$$+a(b^2+1)-(b^2-1)=0, \qquad (6).$$

從 (3) 減 (6). 則

$$\{(c^2+a^2)(b^2-1)+a(b^2+1)\}\cos^2\theta+a^2(b^2+1)\cos\theta-a(b^2+1)=0, \qquad (7).$$

以 $a(b^2+1)$ 乘 (3), b^2-1 乘 (7). 相減. 得

$\{a^2(b^2+1)^2-(c^2+a^2)(b^2-1)^2\}\cos^2\theta+0$, 故 $4a^2b^2=c^2(b-1)^2$.

(63) $a^2+b^2=(\sin\theta\cos\phi\sin a+\cos\theta\cos a)^2+(\sin\theta\cos\phi\cos a-\cos\theta\sin a)^2$

$=\sin^2\theta\cos^2\phi+\cos^2\theta=1-\sin^2\theta\sin^2\phi=\dfrac{c^2}{\sin^2 a}-1,\ a^2+b^2-1+\dfrac{c^2}{\sin^2 a}=0.$

(64) 從第一. $\sin^2\theta=\dfrac{b-c}{a-c}$, $\cos^2\theta=\dfrac{a-b}{a-c}$, 故 $\tan^2\theta=\dfrac{b-c}{a-b}$,

同樣. 從第二. $\tan^2\phi=\dfrac{a-d}{d-c}$, 又從第三. $a^2\tan^2\theta=c^2\tan^2\phi$, 故

$\dfrac{a^2(b-c)}{a-b}=\dfrac{c^2(a-d)}{d-c}$, 卽 $a^2\{bd-c(b+d)+c^2\}=c^2\{a^2-a(b+d)+bd\}$,

故 $bd(a^2-c^2)=ac(b+d)(a-c)$, 卽 $bd(a+c)=ac(b+d)$.

(65) 從第二. $\tan\phi+\tan\theta=b\tan\theta\tan\phi$, 故從第一. $\tan\theta\tan\phi=a/b$,

故 $\tan(\theta+\phi)=\dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$, 故 $\tan c=\dfrac{ab}{b-a}$.

(66) 以第二除第一. 得 $a\cos\theta-b\cos\phi=c^2/d$, 由此與第二. 得

$\cos\theta=\dfrac{c^2+d^2}{2ad}$, $\cos\phi=\dfrac{d^2-c^2}{2bd}$, 從第三. $a^2\left(\dfrac{1}{\cos^2\theta}-1\right)=b^2\left(\dfrac{1}{\cos^2\phi}-1\right)$,

由是 $a^2\{4a^2d^2/(c^2+d^2)^2-1\}=b^2\{4b^2d^2/(d^2-c^2)^2-1\}$.

(67) 與例題 57. 及 60. 同法. 求得 $\cos\theta\cos\phi$, $\sin\theta\sin\phi$, $\cos\theta+\cos\phi$, 及 $\sin\theta+\sin\phi$, 從第三. $a+b+c+2cx+2dy=ay^2+bx^2$.

(68) 由 2. 節第三. $\dfrac{x/a}{\sin\phi-\cos\phi}=\dfrac{y/b}{\cos\theta-\sin\theta}=\dfrac{-1}{\sin(\theta-\phi)}$, 故

$\dfrac{(x/a)^2-(y/b)^2}{\sin 2\theta-\sin 2\phi}=\dfrac{1}{\sin^2(\theta-\phi)}$, 故從第三. $\sin^2(\theta-\phi)=\dfrac{1}{m}$,

又將第一, 第二平方之相加. 且變化之. 則

$\left(\dfrac{x}{a}\right)^2+\left(\dfrac{y}{b}\right)^2+\dfrac{2xy}{ab}\sqrt{\{1-\sin^2(\theta-\phi)\}}=2$, 故 $\left(\dfrac{x}{a}\right)^2+\left(\dfrac{y}{b}\right)^2+\dfrac{2xy}{abm}\sqrt{(m^2-1)}=2.$

69. $b\tan\phi - c\tan\theta = a$, $d\sin\phi - e\sin\theta = 0$,

$be\sec^3\phi - cd\sec^3\theta = 0$.

70. $a\tan 3\theta + b\tan\theta = a\tan 3\phi + b\tan\phi = c$, $\theta + \phi = \frac{1}{4}\pi$.

71. $a\cos\theta + b\cos\phi = a\sin\theta + b\sin\phi = c$, $\theta + \phi = n\pi$.

72. $b + c\cos\theta = m\cos(\theta - a)$,

$b + c\cos\phi = m\cos(\phi - a)$, $\theta - \phi = 2u$.

73. $ax/\cos\theta - by/\sin\theta = ax/\cos\phi - by/\sin\phi = c^2$,

$\cos\frac{1}{2}(\theta - \phi)/\cos\frac{1}{2}(\theta + \phi) = c/a$.

例 題 解 自 69. 至 73.

(69) 從第二. $\sin\phi = e\sin\theta/d$, 從第三. $\cos\phi = (be)^{\frac{1}{3}}\cos\theta/(cd)^{\frac{1}{3}}$,

而 $\sin^2\phi + \cos^2\phi = 1$, 從此求 $\sin^2\theta$, $\cos^2\theta$, 卽可得 $\tan^2\theta$, 如

$\tan^2\theta = \dfrac{d^2\{(be)^{\frac{2}{3}} - (cd)^{\frac{2}{3}}\}}{(cd)^{\frac{2}{3}}(d^2 - e^2)}$, 同樣. $\tan^2\phi = \dfrac{e^2\{(be)^{\frac{2}{3}} - (cd)^{\frac{2}{3}}\}}{(be)^{\frac{2}{3}}(d^2 - e^2)}$, 以此兩値代第一

方程式之平方. $b^2\tan^2\phi + c^2\tan^2\theta - 2bc\tan\phi\tan\theta = a^2$ 且簡單之.

則 $(be)^{\frac{2}{3}} - (cd)^{\frac{2}{3}} = a^{\frac{2}{3}}(d^2 - e^2)^{\frac{1}{3}}$.

(70) 表最初兩方程式爲 $\tan\theta$ 及 $\tan\phi$ 之項. 令爲 λ. 則得次之方程式.

$(a+3b)\lambda^3 - 3c\lambda^2 - (3a+b)\lambda + c = 0$, 令 x, y, z, 爲 λ 入之三根,

其 $x = \tan\theta$, $y = \tan\phi$, 則從第三方程式.

$(x+y)/(1-xy) = \tan\frac{\pi}{4}$, 卽 $x + y = 1 - xy$, (1).

又由三次方程式之性質. $x+y+z = 3c/(a+3b)$, (2).

$xy + yz + zx = -(3a+b)/(a+3b)$, (3). $xyz = -c/(a+3b)$, (4).

從 (1), (2) 得 $z = 3c/(a+3b) - 1 + xy$, (5). 又從 (1), (3), (5) 得

$xy + (x+y)z = xy + (1-xy)\left(\dfrac{3c}{a+3b} - 1 + xy\right) = -\dfrac{3a+b}{a+3b}$, 卽

$(a+3b)x^2y^2 - 3(a+3b-c)xy - 2a + 2b - 3c = 0$, (6). 同樣. 從 (1) 及 (5) 得

$(a+3b)x^2y^2 - (a+3b-3c)xy + c = 0$, (7).

從 (6) 及 (7) 相減得 $xy=-\frac{a-b+2c}{a+3b}$, 以此代入 (7). 則

$$(a-b+2c)^2+(a+3b-3c)(a-b+2c)+c(a+3b)=0,\ 即$$

$$(a-b+2c)(2a+2b-c)+c(a+3b)=0,\ 即$$

$$(a-b+c)(a+b-c)+2c(a+b)=0.$$

(71) ∵ $\phi=n\pi-\theta$. 故由最初兩方程式.得

$$(a\pm b)\cos\theta=(a\mp b)\sin\theta=c.$$

$$\therefore\ \frac{c^2}{(a\pm b)^2}+\frac{c^2}{(a\mp b)^2}=\cos^2\theta+\sin^2\theta.\quad \therefore\ 2c^2(a^2+b^2)=(a^2-b^2)^2.$$

(72) 從 2. 節第三. $\frac{b}{\cos\theta\cos(\phi-a)-\cos\phi\cos(\theta-a)}=\frac{c}{\cos(\theta-a)-\cos(\phi-a)}$

$=\frac{m}{\cos\theta-\cos\phi}$, 化此爲最簡式. 而令 $\theta-\phi=2u$. 則 $\frac{b}{2\cos u\sin a}$

$=c\Big/\sin\left(\frac{\theta+\phi}{2}-a\right)=m\Big/\sin\frac{\theta+\phi}{2}$, $\therefore\ \sin\frac{\theta+\phi}{2}=\frac{2m\cos u\sin a}{b}$,

又 $\sin\left(\frac{\theta+\phi}{2}-a\right)=\sin\frac{\theta+\phi}{2}\cos a-\sin a\cos\frac{\theta+\phi}{2}=\frac{2\cos u\sin a}{b}$. 以上式

$\sin\frac{\theta+\phi}{2}$ 代之, 則 $\cos\frac{\theta+\phi}{2}=\frac{2\cos u(m\cos x-c)}{b}$. 故由

$\sin^2\frac{\theta+\phi}{2}+\sin^2\frac{\theta+\phi}{2}=1$, 得 $4\cos^2u(m^2+c^2-2cm\cos a)=b^2$.

(73) 由最初兩方程式用2節第三. 化其分母爲最簡式. 則

$\frac{ax}{\cos\theta\cos\phi\cos\frac{1}{2}(\theta+\phi)}=\frac{-by}{\sin\theta\sin\phi\sin\frac{1}{2}(\theta+\phi)}=\frac{c^2}{\cos\frac{1}{2}(\theta-\phi)}$. 用第三方程式. 且

變化之. 則得 $x=c\cos\theta\cos\phi=c\left(\cos^2\frac{\theta+\phi}{2}+\cos^2\frac{\theta-\phi}{2}-1\right)$, 故用第三方程式.

而 $\cos^2\frac{\theta-\phi}{2}=\frac{c(x+c)}{a^2+c^2}$, (1), 同樣. 用第三方程式. 求得

$y=-\frac{c^2}{b}\left(\cos^2\frac{\theta-\phi}{2}-\cos^2\frac{\theta+\phi}{2}\right)\sqrt{\left\{\frac{1-\cos^2\frac{1}{2}(\theta+\phi)}{\cos^2\frac{1}{2}(\theta-\phi)}\right\}}$, 故從第三

$b^2y^2=c^4\left(\cos^2\frac{\theta-\phi}{2}-\frac{a^2}{c^2}\cos^2\frac{\theta-\phi}{2}\right)^2\left\{\frac{c^2-a^2\cos^2\frac{1}{2}(\theta-\phi)}{c^2\cos^2\frac{1}{2}(\theta-\phi)}\right\}$, (2).

由 (1) 及 (2) 消去 $\cos^2\frac{\theta-\phi}{2}$. 則

$$b^2y^2=(c^2-a^2)^2\left(\frac{x+c}{a^2+c^2}\right)\left\{c-\frac{a^2(x+c)}{a^2+c^2}\right\}.$$

即 $b^2(a^2+c^2)^2y^2=(c^2-a^2)^2(x+c)\{c(a^2+c^2)-a^2(x+c)\}$.

74. $a\sin\phi=b\sin\theta,\ a\sin(\theta+\phi)=c\sin\theta,$

$\cos\theta-\cos\phi=2m.$

75. $\sin\theta+\cos\phi=4mn,\ \cos\theta+\sin\phi=2(m^2-n^2),$

$\cos\frac{1}{2}(\theta+\phi)-\sin\frac{1}{2}(\theta+\phi)=(m^2-n^2)\sqrt{2}.$

76. $\dfrac{\sin(\theta-\phi)}{\sin(\theta+\phi)}=\dfrac{a}{b},\ \dfrac{\sin\theta}{\sin\phi}=\dfrac{b}{x},\ \cos(\theta-\phi)=\dfrac{c}{x}.$

77. $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=\dfrac{x\cos\phi}{a}+\dfrac{y\sin\phi}{b}=1,$

$4\cos\frac{1}{2}(\theta-\phi)\cos\frac{1}{2}(\alpha-\theta)\cos\frac{1}{2}(\alpha-\phi)=1.$

78. $x\cos(\theta+2\phi)-y\sin(\theta+2\phi)=a,$

$x\cos\theta+y\sin\theta=a,\ b\sin(\theta+\phi)=a\sin\phi.$

79. $\cos\theta=\dfrac{\sin\beta}{\sin\alpha},\ \cos\phi=\dfrac{\sin\gamma}{\sin\alpha},$

$\cos(\theta-\phi)=\sin\beta\sin\gamma.$

例題解自 74 至 79.

(74) 由第一. $a^2(\sin^2\theta-\sin^2\phi)=(a^2-b^2)\sin^2\theta.$ 即

$-a^2(\cos^2\theta-\cos^2\phi)=(a^2-b^2)\sin^2\theta.$ 即

$-a^2(\cos\theta+\cos\phi)(\cos\theta-\cos\phi)=(a^2-b^2)\sin^2\theta,$ 故由第三.

$-a^2(2\cos\theta-2m)2m=(a^2-b^2)\sin^2\theta.$ 即

$(a^2-b^2)\cos^2\theta-4ma^2\cos\theta-(a^2-b^2-4m^2a^2)=0,$ (A),

又解第二括弧.由第一 $\sin\phi$ 及第三 $\cos\phi$ 代之.則得

$\cos\theta=(2am+c)/(a+b).$ 以此代入 (A).則

$$(a-b)\{(a+b)^2-c^2\}+4abcm=0.$$

(75) $(\sin\theta+\cos\phi)^2+(\cos\theta+\sin\phi)^2=4\{4m^2n^2+(m^2-n^2)^2\},$ 即

$\sin(\theta+\phi)=2(m^2+n^2)^2-1.$ 又將第三平方之.則

$1-\sin(\theta+\phi)=2(m^2-n^2)^2.$ 由此 $m^4+n^4=\frac{1}{2}.$

(76) 由第二.得 $\dfrac{\sin(\theta-\phi)\{1+\cos(\theta+\phi)\}}{\sin(\theta+\phi)\{1+\cos(\theta-\phi)\}}=\dfrac{b-x}{b+x}$ 以此代第一第三之值而簡單之.則 $\cos(\theta+\phi)=b(c+x)(b-x)/\{ax(b+x)\}-1$, 又由第一第三得 $\sin^2(\theta+\phi)=\dfrac{b^2}{a^2}\left(1-\dfrac{c^2}{x^2}\right)$, 故 $\dfrac{b^2(x^2-c^2)}{a^2x^2}+\left\{\dfrac{b(c+x)(b-x)}{ax(b+x)}-1\right\}^2=1$, 即

$b(x-b)(b+x)^2+b(x+c)(b-x)^2-2ax(b^2-x^2)=0$, 即 $(a+b)x^2=b^2(a-b+2c)$

(77) 由例題55. $\cos^2\dfrac{\theta-\phi}{2}=1\Big/\left\{\left(\dfrac{x}{a}\right)^2+\left(\dfrac{y}{b}\right)^2\right\}$,

又 $\cos^2\dfrac{\theta+\phi}{2}=\left(\dfrac{x}{a}\right)^2\Big/\left\{\left(\dfrac{x}{a}\right)^2+\left(\dfrac{y}{b}\right)^2\right\}$, 又從第三.

$\cos\frac{1}{2}(\theta-\phi)\{\cos\alpha\cos\frac{1}{2}(\theta+\phi)+\sin\alpha\sin\frac{1}{2}(\theta+\phi)\}+\cos^2\frac{1}{2}(\theta-\phi)=\frac{1}{2}$, 即

$$\frac{1}{\sqrt{\left\{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\right\}}}\left[\cos\alpha\frac{x/a}{\sqrt{\left\{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\right\}}}+\sin\alpha\frac{y/b}{\sqrt{\left\{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2\right\}}}\right]$$

$+\dfrac{1}{\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2}=\dfrac{1}{2}$, 即 $2(x/a)\cos\alpha+2(y/b)\sin\alpha+2=(x/a)^2+(y/b)^2$,

由是 $(x/a-\cos\alpha)^2+(y/b-\sin\alpha)^2=3$.

(78) 由第二減第一而變化之.且以 $\sin(\theta+\phi)$ 除之.則

$x\sin\phi+y\cos\phi=0$, 由此與第二得 $\dfrac{x}{\cos\phi}=\dfrac{-y}{\sin\phi}=\dfrac{a}{\cos(\theta+\phi)}$, 故

$x^2+y^2=\dfrac{y^2}{\sin^2\phi}=\dfrac{a^2}{1-\sin^2(\theta+\phi)}$ (由第三) $=\dfrac{a^2b^2}{b^2-a^2\sin^2\phi}$,

即 $x^2+y^2=\dfrac{a^2y^2+a^2b^2}{a^2\sin^2\phi+(b^2-a^2\sin^2\phi)}=a^2\left(\dfrac{y^2}{b^2}+1\right)$.

(79) 由第三 $(\cos\theta\cos\phi-\sin\beta\sin\gamma)^2=\sin^2\theta\sin^2\phi=(1-\cos^2\theta)(1-\cos^2\phi)$,

即 $\cos^2\theta+\cos^2\phi-2\cos\theta\cos\phi\sin\beta\sin\gamma=1-\sin^2\beta\sin^2\gamma$, 以此代入第一及第二.則 $\dfrac{1}{\sin^2\alpha}(\sin^2\beta+\sin^2\gamma-2\sin^2\beta\sin^2\gamma)=1-\sin^2\beta\sin^2\gamma$,

故 $\sin^2\alpha=\dfrac{\sin^2\beta+\sin^2\gamma-2\sin^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}=\dfrac{\sin^2\beta\cos^2\gamma+\cos^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}$,

故 $\cos^2\alpha=1-\dfrac{\sin^2\beta\cos^2\gamma+\cos^2\beta\sin^2\gamma}{1-\sin^2\beta\sin^2\gamma}=\dfrac{\cos^2\beta\cos^2\gamma}{1-\sin^2\beta\sin^2\gamma}$,

由是 $\tan^2\alpha=\dfrac{\sin^2\beta\cos^2\gamma+\cos^2\beta\sin^2\gamma}{\cos^2\beta\cos^2\gamma}=\tan^2\beta+\tan^2\gamma$.

80. $\sin(\theta+\alpha)=\sin(\phi+\alpha)=\sin\beta,$

$a\sin(\theta+\phi)+b\sin(\theta-\phi)=c.$

81. $a\cos^3\theta+b\cos^2\theta+c\cos\theta+d=0,$

$a\cos^3\phi+b\cos^2\phi+c\cos\phi+d=0,\ \cos\theta+\cos\phi=m.$

82. $x\cos(\theta-\alpha)+y\cos\theta=2a\sin(\theta+\gamma)\cos\theta\cos(\theta-\alpha),$

$x\sin(\theta-\alpha)+y\sin\theta=2a\{\sin(\theta+\gamma)\sin\theta\cos(\theta-\alpha)-\cos\beta\cos\theta\},$

但 $\alpha+\beta+\gamma=\pi.$

83. $\gamma=\dfrac{ab\cos(\alpha-\theta)}{\sqrt{(a^2\sin^2\theta+b^2\cos^2\theta)}}=\dfrac{ab\cos(\alpha-\phi)}{\sqrt{(a^2\sin^2\phi+b^2\sin^2\phi)}},$

$\tan\theta\tan\phi=-b^2/a^2.$

84. $\sin\theta=p\sin\phi.\ \cos\theta=q\cos\phi,$

$\sin\theta+\cos\theta=r(\sin\phi+\cos\phi).$

例題解自 80. 至 84.

(80) 由最初兩方程式相減.得 $\sin(\theta+\alpha)-\sin(\phi+\alpha)=0,$ 卽

$2\cos\left(\dfrac{\theta+\phi}{2}+\alpha\right)\sin\dfrac{\theta-\phi}{2}=0,$ 故 $\cos\left(\dfrac{\theta+\phi}{2}+\alpha\right)=0,$ 或 $\sin\dfrac{\theta-\phi}{2}=0,$

故 $\dfrac{\theta+\phi}{2}+\alpha=\dfrac{\pi}{2},$ (1). 或 $\dfrac{\theta-\phi}{2}=0,$ (2),

又由最初兩方程式相加.得 $2\sin\left(\dfrac{\theta+\phi}{2}+\alpha\right)\cos\dfrac{\theta-\phi}{2}=2\sin\beta,$ (3).

故從 (1) $\cos\dfrac{\theta-\phi}{2}=\sin\beta.$ 卽 $\cos(\theta-\phi)=2\sin^2\beta-1,$ (4). 又第三代入

(1),及 (4).則 $a\sin2\alpha+b\sqrt{1-(2\sin^2\beta-1)^2}=c,$ 由是 $a\sin2\alpha\pm b\sin2\beta=c,$

或入 (2).則從 (3) 得 $\sin\left(\dfrac{\theta+\phi}{2}+\alpha\right)=\sin\beta$ 卽

$\cos(\theta+\phi+2\alpha)=1-2\sin^2\beta=\cos2\beta,$ 故 $\theta+\phi=-(2\alpha\pm2\beta).$ 由是從

(2) 及第三得 $a\sin(\theta+\phi)=c,$ 故 $a\sin(2\alpha\pm2\beta)=-c.$

(81) 令 $\cos\theta=x$, $\cos\phi=y$, 由第一減第二, 以 $x-y$ 除之. 則

$a(x^2+xy+y^2)+b(x+y)+c=0$, 但由第三. $x+y=m$,

故 $ax^2-amx+am^2+bm+c=0$, (1). 又 (1) 以 x 乘之. 由第一減之. 則

$(am+b)x^2-(am+b)mx+d=0$, (2).

又 (1) 以 $am+b$ 乘之. (2) 以 a 乘之. 相減得

$(am+b)(am^2+bm+c)-ad=0$.

(82) 以 $\sin\theta$ 乘第一, 以 $\cos\theta$ 乘第二, 由減法. 得

$x\sin\alpha=2a\cos\beta\cos^2\theta$, (1). 又以 $\sin(\theta-\alpha)$ 乘第一. 以 $\cos(\theta-\alpha)$ 乘第二,

由減法得 $y\sin\alpha=2a\cos(\theta-\alpha)\{\sin(\theta+\gamma)\sin\alpha-\cos\beta\cos\theta\}$

$=2a\cos(\theta-\alpha)\{\sin\theta\cos\gamma\sin\alpha+\cos\theta(\sin\gamma\sin\alpha-\cos\beta)\}$

$=2a\cos(\theta-\alpha)\{\sin\theta\cos\gamma\sin\alpha+\cos\theta[\sin\gamma\sin\alpha+\cos(\alpha+\gamma)]\}$

$=2a\cos(\theta-\alpha)\{\sin\theta\cos\gamma\sin\alpha+\cos\theta\cos\alpha\cos\gamma\}$

$=2a\cos\gamma\cos^2(\theta-\alpha)$, (2). 又從 (1) 得 $\tan^2\theta=\dfrac{2a\cos\beta}{x\sin\alpha}-1$, (3).

又從 (1), (2) 得 $\dfrac{\cos(\theta-\alpha)}{\cos\theta}=\sqrt{\dfrac{y\cos\beta}{x\cos\gamma}}$. 以此表示 $\tan\theta$ 之項. 則

$\tan\theta=\dfrac{1}{\sin\alpha}\left(\sqrt{\dfrac{y\cos\beta}{x\cos\gamma}}-\cos\alpha\right)$, (4). 故由 (3) 及 (4) 得

$$\frac{\sin\alpha(2a\cos\beta-x\sin\alpha)}{x}=\frac{y\cos\beta}{x\cos\gamma}-2\cos\alpha\sqrt{\frac{y\cos\beta}{x\cos\gamma}}+\cos^2\alpha,$$

由是 $\dfrac{x}{\cos\beta}+\dfrac{y}{\cos\alpha}-2\cos\alpha\sqrt{\dfrac{xy}{\cos\beta\cos\gamma}}=2a\sin\alpha$.

(83) $\gamma^2=\dfrac{a^2b^2\cos^2(\alpha-\theta)/\cos^2\theta}{(a^2\sin^2\theta+b^2\cos^2\theta)/\cos^2\theta}=\dfrac{a^2b^2(\cos\alpha+\sin\alpha\tan\theta)^2}{a^2\tan^2\theta+b^2}$, (1).

同樣. $\gamma^2=\dfrac{a^2b^2(\cos\alpha+\sin\alpha\tan\phi)^2}{a^2\tan^2\phi+b^2}$. 但 $\tan\phi=-\dfrac{b^2}{a^2\tan\theta}$, 故

$\gamma^2=\dfrac{(a^2\cos\alpha\tan\theta-b^2\sin\alpha)^2}{a^2\tan^2\theta+b^2}$, (2). 故由 (1), (2) 相加. 得

$$2\gamma^2=\frac{a^2b^2(\cos\alpha+\sin\alpha\tan\theta)^2+(a^2\cos\alpha\tan\theta-b^2\sin\alpha)^2}{a^2\tan^2\theta+b^2}.$$

即 $2\gamma^2=\dfrac{(a^2\tan^2\theta+b^2)(a^2\cos^2\alpha+b^2\sin^2\alpha)}{a^2\tan^2\theta+b^2}=a^2\cos^2\alpha+b^2\sin^2\alpha$.

(84) 本題容易消去 θ, ϕ. 故

$(p-r)^2(1-q^2)+(q-r)^2(1-p^2)=0$.

85. 設 $a\cos(\theta+\phi)+b\cos(\theta-\phi)+c=0,$

$a\cos(\phi+\psi)+b\cos(\phi-\psi)+c=0,$

$a\cos(\psi+\theta)+b\cos(\psi-\theta)+c=0,$

則 $a^2-b^2+2bc=0.$

86. 設 $a\cos\theta+b\cos\phi+c\cos\psi=0,$

$a\sin\theta+b\sin\phi+c\sin\psi=0,$

$a\sec\theta+b\sec\phi+c\sec\psi=0,$

則 $a^2+b^4+c^4-2a^2b^2-2b^2c^2-2c^2a^2=0.$

87. 設 $x=y\cos\gamma+z\cos\beta,\ y=z\cos\alpha+x\cos\gamma,$

$z=x\cos\beta+y\cos\alpha$. 試從此消去 x, y, z.

88. $\frac{x^2}{a^2}\cos\theta=\frac{y^2}{a^2}\cos\theta+\frac{z^2}{b^2}\cos\phi,$

$\frac{x}{\sin(\theta+\phi)}=\frac{y}{\sin(\theta-\phi)}=\frac{z}{\sin 2\theta}$, 試從此消去 x, y, z.

例題解自 85. 至 88.

(85) 於最初兩方程式.令 θ 及 ψ 代 λ.則可得次之一個新方程式.

$a\cos(\phi+\lambda)+b\cos(\phi-\lambda)+c=0,$ 即

$(a+b)\cos\phi\cos\lambda-(a-b)\sin\phi\sin\lambda+c=0,$ 以此示 $\cos\lambda$ 之項.則

$\{(a+b)^2\cos^2\phi+(a-b)^2\sin^2\phi\}\cos^2\lambda+2c(a+b)\cos\phi\ ccs\lambda+c^2-(a-b)^2\sin^2\phi=0,$

由是 $\cos\theta\cos\psi=\frac{c^2-(a-b)^2\sin^2\phi}{(a+b)^2\cos^2\phi+(a-b)^2\sin^2\phi}=\frac{c^2-(a-b)^2\sin^2\phi}{M},$

同樣 $\sin a\sin\psi=\frac{c^2-(a+b)^2\cos^2\phi}{(a+b)^2\cos^2\phi+(a-b)^2\sin^2\phi}=\frac{c^2-(a+b)^2\cos^2\phi}{M},$

故 $\cos(\theta+\psi)=\frac{(a+b)^2\cos^2\phi-(a-b)^2\sin^2\phi}{M}=\frac{M-2(a-b)^2\sin^2\phi}{M},$

及 $\cos(\theta-\psi)=\frac{2c^2-(a+b)^2\cos^2\phi-(a-b)^2\sin^2\phi}{M}=\frac{2c^2-M}{M},$

故由第三方程式 $a\left(\frac{M-2(a-b)^2\sin^2\phi}{M}\right)+b\left(\frac{2c^2-M}{M}\right)+c=0,$

即 $M(a-b+c)+2bc^2-2a(a-b)^2\sin^2\phi=0,$

但 $M=(a+b)^2\cos^2\phi+(a-b)^2\sin^2\phi=(a+b)^2-4ab\sin^2\phi,$

故 $\{(a+b)^2-4ab\sin^2\phi\}(a-b+c)+2bc^2-2a(a-b)^2\sin^2\phi=0,$

即 $(a+b)^2(a-b+c)+2bc^2-2a\{(a-b)^2+2b(a-b+c)\}\sin^2\phi=0,$

即 $(a+b)\{a^2-b^2+c(a-b)+2bc\}+2bc^2-2a(a^2-b^2+2bc)\sin^2\phi=0,$

即 $(a+b)(a^2-b^2+2bc)+c(a^2-b^2+2bc)-2a(a^2-b^2+2bc)\sin^2\phi=0,$

即 $(a^2-b^2+2bc)(a+b+c-2a\sin^2\phi)=0,$ 故省略 $a+b+c-2a\sin^2\phi,$

則 $a^2-b^2+2bc=0.$

(86) 由第一. $\cos\psi=-\frac{a\cos\theta+b\cos\phi}{c}$, 又由第三. $\cos\psi=-\frac{c\cos\theta\cos\phi}{a\cos\phi+b\cos\theta}$,

故 $(a\cos\theta+b\cos\phi)(a\cos\phi+b\cos\theta)=c^2\cos\theta\cos\phi,$ 即

$(a^2+b^2-c^2)\cos\theta\cos\phi+ab(\cos^2\theta+\cos^2\phi)=0,$ (A). 又第二轉項而平方之.則

$a^2\sin^2\theta+b^2\sin^2\phi+2ab\sin\theta\sin\phi=c^2\sin^2\psi,$ 又轉項而平方之.則

$(a^2+b^2-c^2-a^2\cos^2\theta-b^2\cos^2\phi+c^2\cos^2\psi)^2=4a^2b^2(1-\cos^2\theta)(1-\cos^2\phi),$ 故

$\{a^2+b^2-c^2-a^2\cos^2\theta-b^2\cos^2\phi+(a\cos\theta+b\cos\phi)^2\}^2=4a^2b^2(1-\cos^2\theta)(1-\cos^2\phi),$

即 $(a^2+b^2-c^2+2ab\cos\theta\cos\phi)^2=4a^2b^2(1-\cos^2\theta-\cos^2\phi+\cos^2\theta\cos^2\phi);$

即 $(a^2+b^2-c^2)^2-4a^2b^2+4ab\{(a^2+b^2-c^2)\cos\theta\cos\phi+ab(\cos^2\theta+\cos^2\phi)\}=0,$

故從 (A) 得 $(a^2+b^2-c^2)^2-4a^2b^2=0.$

(87) 由 2. 節第三 $\frac{x}{-\cos\gamma\cos\alpha-\cos\beta}=\frac{y}{-\cos\beta\cos\gamma-\cos\alpha}=\frac{z}{-1+\cos^2\gamma}$,

令此式等於K.則 $x=K(\cos\gamma\cos\alpha+\cos\beta),\ y=K(\cos\beta\cos\gamma+\cos\alpha),$

$z=K\sin^2\gamma$, 但此結果,由最初兩方程式得來,以此代入第三方程式.

則 $\sin^2\alpha+\sin^2\beta+\sin^2\gamma=2(1+\cos\alpha\cos\beta\cos\gamma).$

(88) 令最後方程式之各分數爲K.則

$x=K\sin(\theta+\phi),\ y=K\sin(\theta-\phi),\ z=K\sin2\theta,$ 以此代入最初方程式

$b^2\cos\theta(x^2-y^2)=a^2z^2\cos\phi$ 之內.則

$b^2\cos\theta\{\sin^2(\theta+\phi)-\sin^2(\theta-\phi)\}=a^2\sin^22\theta\cos\phi,$

即 $b^2\cos\theta\sin\theta\cos\phi\cos\theta\sin\phi=a^2\sin^2\theta\cos^2\theta\cos\phi,$

簡之.則 $b^2\sin\phi=a^2\sin\theta.$

89. 試從 $\frac{a\tan^2\theta-x}{\tan 2\alpha\tan 2\beta}=\frac{2a\tan\theta}{\tan 2\alpha+\tan 2\beta}=a-x$

消去 x.

90. 設 $m=\left(\frac{\sin^2\theta}{a^2}+\frac{\cos^2\theta}{b^2}\right)\cos^2\omega+\frac{\sin^2\omega}{c^2}$,

$n=\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}$, $p=\left(\frac{1}{b^2}-\frac{1}{a^2}\right)\sin\theta\cos\theta\cos\omega$.

例 $\left(m-\frac{1}{c^2}\right)\left(n-\frac{1}{a^2}\right)\left(n-\frac{1}{b^2}\right)+p\left(\frac{1}{a^2}+\frac{1}{b^2}-\frac{1}{c^2}\right)=np^2$.

例題解自 89. 至 90.

(89) $x=a\left(1-\frac{2\tan\theta}{\tan 2\alpha+\tan 2\beta}\right)$, $x=a\tan\theta\left(\tan\theta-\frac{2\tan 2\alpha\tan 2\beta}{\tan 2\alpha+\tan 2\beta}\right)$,

$\therefore\ 1-\tan^2\theta=\frac{2\tan\theta(1-\tan 2\alpha\tan 2\beta)}{\tan 2\alpha+\tan 2\beta}$.

$\frac{\tan 2\alpha+\tan 2\beta}{1-\tan 2\alpha\tan 2\beta}=\frac{2\tan\theta}{1-\tan^2\theta}$, 卽 $\tan(2\alpha+2\beta)=\tan 2\theta$.

(90) 由第二.得 $\sin^2\theta=\left(n-\frac{1}{a^2}\right)\Big/\left(\frac{1}{b^2}-\frac{1}{a^2}\right)$,

$\cos^2\theta=-\left(n-\frac{1}{b^2}\right)\Big/\left(\frac{1}{b^2}-\frac{1}{a^2}\right)$,

以此代入第一.而求 $\cos^2\omega$. 則 $\cos^2\omega=\frac{m-1/c^2}{1/b^2-1/a^2-n-1/c^2}$,

故由第三得 $p^2=\left(\frac{1}{b^2}-\frac{1}{a^2}\right)^2\sin^2\theta\cos^2\theta\cos^2\omega$,

卽 $p^2=\frac{-(m-1/c^2)(n-1/a^2)(n-1/b^2)}{1/b^2-1/a^2-n-1/c^2}$.

第玖編

三角反函數

1. 三角反函數

如 $x=\sin\alpha$. 其 x 已知其爲三角函數矣. 換言之, x 爲表示 α 角之正弦. (卽三角函數) 反之. α 爲 x 之三角反函數. 卽 α 爲 x 爲正弦所示之角也. 卽 α 等於 x 爲正弦之角.

其記法如次

$\alpha=\sin^{-1}x$ 卽 $\sin^{-1}x$ 示其角度.

但 $\sin^{-1}x$. 不含有代數學之負指數 $a^{-1}=\frac{1}{a}$ 之意義.

卽 $\sin^{-1}x$ 非 $\frac{1}{\sin x}$ 之意義.

然爲便利計. 於 $\sin\alpha=x$ 亦得記爲 $\alpha=\frac{x}{\sin}=\sin^{-1}x$,

但此式中 $\frac{x}{\sin}$ 之 $\sin$. 不表若何之數量. 故無意義.

因得如負指數之記法. 於記憶上便利. 故特述之.

其他皆得如上記之

如 $\sin(\sin^{-1}x)=\sin^{1-1}x=\sin^{\circ}x=x$,

因令 $\sin^{-1}x=\alpha$ 則 $x=\sin\alpha$, 故 $\sin(\sin^{-1}x)=\sin\alpha=x$.

又如 $\sin^{-1}(\sin x)=\sin^{-1+1}x=\sin^{\circ}x=x$,

因令 $\sin^{-1}(\sin x)=y$ 則 $\sin x=\sin y$, 故 $x=y$

〔譯註〕 由角而舉其函數. 謂之三角函數. 由函數而舉其角. 謂之三角反函數.

2. 注意

嚴格論之則 $\sin\alpha=\sin\{n\pi+(-1)^n\alpha\}$ (第三編 15. 節)

故 $\sin^{-1}(\sin\alpha)=n\pi+(-1)^n\alpha$,

同樣 $\cos^{-1}(\cos\alpha)=2n\pi\pm\alpha$,

$\tan^{-1}(\tan\alpha)=n\pi+\alpha$,

然通例但用簡單之角而已.

例題十

1. 求次各式之證.

$$\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{x}{\sqrt{1-x^2}}=\cot^{-1}\frac{\sqrt{1-x^2}}{x}=\sec^{-1}\frac{1}{\sqrt{1-x^2}}=\operatorname{cosec}^{-1}\frac{1}{x},$$

$$\cos^{-1}x=\sin^{-1}\sqrt{1-x^2}=\tan^{-1}\frac{\sqrt{1-x^2}}{x}=\cot^{-1}\frac{x}{\sqrt{1-x^2}}=\sec^{-1}\frac{1}{x}=\operatorname{cosec}^{-1}\frac{1}{\sqrt{1-x^2}},$$

$$\tan^{-1}x=\sin^{-1}\frac{x}{\sqrt{1+x^2}}=\cos^{-1}\frac{1}{\sqrt{1+x^2}}=\cot^{-1}\frac{1}{x}=\sec^{-1}\sqrt{1+x^2}=\operatorname{cosec}^{-1}\frac{\sqrt{1+x^2}}{x},$$

$$\cot^{-1}x=\sin^{-1}\frac{1}{\sqrt{x^2+1}}=\cos^{-1}\frac{1}{\sqrt{x^2+1}}=\tan^{-1}\frac{1}{x}=\sec^{-1}\frac{\sqrt{x^2+1}}{x}=\operatorname{cosec}^{-1}\sqrt{1+x^2},$$

$$\sec^{-1}x=\sin^{-1}\frac{\sqrt{x^2-1}}{x}=\cos^{-1}\frac{1}{x}=\tan^{-1}\sqrt{x^2-1}=\cot^{-1}\frac{1}{\sqrt{x^2-1}}=\operatorname{cosec}^{-1}\frac{x}{\sqrt{x^2-1}},$$

$$\operatorname{cosec}^{-1}x=\sin^{-1}\frac{1}{x}=\cos^{-1}\frac{\sqrt{x^2-1}}{x}=\tan^{-1}\frac{1}{\sqrt{x^2-1}}=\cot^{-1}\sqrt{x^2-1}=\sec^{-1}\frac{x}{\sqrt{x^2-1}},$$

例題解 1.

(1) 令 $\sin^{-1}x=\alpha$, 則 $x=\sin\alpha$, 又 $\cos\alpha=\sqrt{1-\sin^2\alpha}$,

故 $=\sqrt{1-x^2}$, 故 $\alpha=\cos^{-1}\sqrt{1-x^2}$, 由是 $\sin^{-1}x=\cos^{-1}\sqrt{1-x^2}$.

$\tan\alpha=\frac{\sin\alpha}{\sqrt{1-\sin^2\alpha}}=\frac{x}{\sqrt{1-x^2}}$ 故 $\alpha=\tan^{-1}\frac{x}{\sqrt{1-x^2}}$ 由是 $\sin^{-1}x=\tan^{-1}\frac{x}{\sqrt{1-x^2}}$,

$\sec\alpha=\frac{1}{\sqrt{1-\sin^2\alpha}}=\frac{1}{\sqrt{1-x^2}}$, 故 $\alpha=\sec^{-1}\frac{1}{\sqrt{1-x^2}}$, 由是 $\sin^{-1}x=\sec^{-1}\frac{1}{\sqrt{1-x^2}}$.

又令 $\tan^{-1}x=\alpha$, 則 $x=\tan\alpha$, $\sin\alpha=\frac{\tan\alpha}{\sec\alpha}=\frac{\tan\alpha}{\sqrt{1+\tan^2\alpha}}=\frac{x}{\sqrt{1+x^2}}$,

故 $\alpha=\sin^{-1}\frac{x}{\sqrt{1+x^2}}$ 由是 $\tan^{-1}x=\sin^{-1}\frac{x}{\sqrt{1+x^2}}$,

令 $\operatorname{cosec}^{-1}x=\alpha$ 則 $x=\operatorname{cosec}\alpha$, $\sin\alpha=\frac{1}{\operatorname{cosec}\alpha}=\frac{1}{x}$,

由是 $\operatorname{cosec}^{-1}x=\sin^{-1}\frac{1}{x}$ 餘仿此故略.

求次各式之證

2. $\sin^{-1}a+\cos^{-1}a=\frac{\pi}{2}$.

3. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$.

4. $\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=\frac{\pi}{4}$.

5. $\tan^{-1}\frac{4}{3}+\tan^{-1}7=\frac{3}{4}\pi$.

6. $\tan^{-1}\frac{1}{3}+\sin^{-1}\frac{1}{\sqrt{5}}=\frac{\pi}{4}$.

7. $\cot^{-1}\frac{3}{4}+\cot^{-1}\frac{1}{7}=\frac{3}{4}\pi$.

8. $\cos^{-1}\frac{9}{\sqrt{82}}+\sin^{-1}\frac{4}{\sqrt{41}}=\frac{\pi}{4}$.

9. $\sin^{-1}\frac{1}{\sqrt{5}}+\cot^{-1}3=\frac{\pi}{4}$.

10. $\cos^{-1}\frac{9}{\sqrt{82}}+\operatorname{cosec}^{-1}\frac{\sqrt{41}}{4}=\frac{\pi}{4}$.

11. $\tan^{-1}\frac{2a-b}{b\sqrt{3}}+\tan^{-1}\frac{2b-a}{a\sqrt{3}}=\frac{\pi}{3}$.

12. $4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{\pi}{4}$.

13. $\cot^{-1}\frac{2a-b}{b\sqrt{3}}+\cot^{-1}\frac{2b-a}{a\sqrt{3}}=\frac{\pi}{3}$.

例題解自 2. 至 13.

(2) 令 $\sin^{-1}a=\theta$, $\cos^{-1}a=\phi$, 則 $a=\sin\theta=\cos\phi$, 故

$\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right)=\cos\phi$, 故 $\theta+\phi=\frac{\pi}{2}$, 即 $\sin^{-1}a+\cos^{-1}a=\frac{\pi}{2}$.

(3) 令 $\tan^{-1}\frac{1}{2}=x$, $\tan^{-1}\frac{1}{3}=y$, 則 $\frac{1}{2}=\tan x$, $\frac{1}{3}=\tan y$, 故

$\tan(x+y)=\frac{\tan x+\tan y}{1-\tan x\tan y}=\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=1=\tan\frac{\pi}{4}$, 故

$x+y=\frac{\pi}{4}$, 即 $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\frac{\pi}{4}$, (4), (5) 與前二例同.

(6) 令 $\tan^{-1}\frac{1}{3}=x$. $\sin^{-1}\frac{1}{\sqrt{5}}=y$, 則 $\frac{1}{3}=\tan x$, $\frac{1}{\sqrt{5}}=\sin y$, 故

$\tan y=\frac{\sin y}{\sqrt{(1-\sin^2 y)}}=\frac{1}{2}$, 故與例題 3. 同, 自 (7) 迄 (11) 均與前例同.

(12) 令 $4\tan^{-1}\frac{1}{5}=x$, $\tan^{-1}\frac{1}{239}=y$, 則 $\frac{1}{5}=\tan\frac{x}{4}$, $\frac{1}{239}=\tan y$,

$\tan\frac{x}{2}=\frac{2\tan\frac{x}{4}}{1-\tan^2\frac{x}{4}}=\frac{\frac{2}{5}}{1-\frac{1}{25}}=\frac{5}{12}$, $\tan x=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}=\frac{\frac{10}{12}}{1-\frac{25}{144}}=\frac{120}{119}$,

故與例題 3. 同

(13) 證明原式之前節等於 $\frac{1}{\sqrt{3}}$ 即得

14. $\tan^{-1}\frac{3}{4}=2\tan^{-1}\frac{1}{3}$,

15. $\sin\left(\sin^{-1}\frac{1}{2}+\cos^{-1}\frac{1}{2}\right)=1$.

16. $\tan(\tan^{-1}x+\cot^{-1}x)=\infty$.

17. $3\sin^{-1}x=\sin^{-1}(3x-4x^3)$.

18. $2\cot^{-1}x=\operatorname{cosec}^{-1}\frac{1-x^2}{2x}$.

19. $\sin^{-1}\sqrt{\frac{\beta}{\alpha+\beta}}=\tan^{-1}\sqrt{\frac{\beta}{\alpha}}$.

20. $\sin^{-1}\left(\frac{x-a+b}{2b}\right)^{\frac{1}{2}}=\frac{1}{2}\cos^{-1}\left(\frac{a-x}{b}\right)$.

21. $\sin^{-1}x\pm\sin^{-1}y=\sin^{-1}(x\sqrt{1-y^2}\pm y\sqrt{1-x^2})$.

22. $\cos^{-1}x\pm\cos^{-1}y=\cos^{-1}\{xy\mp\sqrt{(1-x^2)(1-y^2)}\}$.

23. $\tan^{-1}x\pm\tan^{-1}y=\tan^{-1}\frac{x\pm y}{1\mp xy}$.

24. $\sin^{-1}\frac{3}{5}+\sin^{-1}\frac{8}{17}=\sin^{-1}\frac{77}{85}$.

25. $\sin^{-1}\frac{3}{5}-\sin^{-1}\frac{5}{13}=\sin^{-1}\frac{16}{65}$.

26. $\tan^{-1}\frac{3}{5}-\cot^{-1}\frac{7}{3}=\cot^{-1}\frac{22}{3}$.

27. $\tan^{-1}x+\tan^{-1}\frac{2x}{1-x^2}=\tan^{-1}\frac{3x+x^3}{1-3x^2}$.

28. $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}$.

29. $\cos^{-1}\frac{63}{65}+2\tan^{-1}\frac{1}{5}=\sin^{-1}\frac{3}{5}$.

30. $\sin^{-1}x+\cos^{-1}y=\sin^{-1}\{xy+\sqrt{(1-x^2)(1-y^2)}\}$.

31. $\cot^{-1}a+\cot^{-1}(4a^3+3a)=2\cot^{-1}2a$.

32. $\cot^{-1}(a^3+a^2+a)^{\frac{1}{2}}+\cot^{-1}\left(a+\frac{1}{a}+1\right)^{\frac{1}{2}}\equiv\tan^{-1}\left(\frac{1}{a^3}+\frac{1}{a^2}+\frac{1}{a}\right)^{\frac{1}{2}}$.

33. $2\tan^{-1}\frac{1}{408}-\tan^{-1}\frac{1}{1393}=\tan^{-1}\frac{1}{239}$.

34. $\cos^{-1}\frac{1-x^2}{1+x^2}-\cos^{-1}\frac{1-y^1}{1+y^2}=\tan^{-1}\frac{2(x-y)(1+xy)}{(1+xy)^2-(x-y)^2}$.

35. $\operatorname{vers}^{-1}a+\operatorname{vers}^{-1}b=\operatorname{vers}^{-1}\{a+b-ab+\sqrt{(2a-a^2)(2b-b^2)}\}$.

例題解自 14 至 35.

(14) 令 $\tan^{-1}\frac{3}{4}=x$, 則 $\frac{3}{4}=\tan x$, 及 $\frac{1}{2}\tan^{-1}\frac{3}{4}=\frac{x}{2}$, 故

$\tan x=\frac{3}{4}=\frac{2\tan\frac{x}{2}}{1-\tan^2\frac{x}{2}}$, 從此得 $\tan\frac{x}{2}=\frac{2}{3}$, 卽 $\frac{x}{2}=\tan^{-1}\frac{2}{3}$,

卽 $x=2\tan^{-1}\frac{1}{3}$, 卽 $\tan^{-1}\frac{3}{4}=2\tan^{-1}\frac{1}{3}$.

(15) 令 $\sin^{-1}\frac{1}{2}=x$, $\cos^{-1}\frac{1}{2}=y$. 則 $\frac{1}{2}=\sin x=\cos y$, 故 $x=30°$,

$y=60°$, 故 $\sin(x+y)=\sin 90°=1$. 卽如題言.

(16) 與前例同樣.

(17) 令 $3\sin^{-1}x=a$, 則 $x=\sin\frac{a}{3}$, $\sin a=3\sin\frac{a}{3}-4\sin^3\frac{a}{3}$, (第四編 5. 節)

故 $\sin a=3x-4x^3$, 故 $a=\sin^{-1}(3x-4x^3)$.

(18) 令 $2\cot^{-1}x=a$, 則 $x=\cot\frac{a}{2}$, 故 $\operatorname{cosec} a=1/\sin a=\frac{\sin^2\frac{a}{2}+\cos^2\frac{a}{2}}{2\sin\frac{a}{2}\cos\frac{a}{2}}$

$=\frac{\tan\frac{a}{2}+\cot\frac{a}{2}}{2}=\frac{1}{2}\left(\frac{1}{x}+x\right)=\frac{1+x^2}{2x}$.

(19), (20). 與前例同樣.

(21) $\sin(\sin^{-1}x\pm\sin^{-1}y)=\sin(a\pm\beta)=\sin a\cos\beta\pm\cos a\sin\beta$

$=\sin a\sqrt{1-\sin^2\beta}\pm\sin\beta\sqrt{1-\sin^2 a}=x\sqrt{1-y^2}\pm y\sqrt{1-x^2}$.

(22) 從 $\cos(\cos^{-1}x\pm\cos^{-1}y)=\cos(a\pm\beta)$, 卽得.

(23) 從 $\tan(\tan^{-1}x\pm\tan^{-1}y)=\tan(a\pm\beta)$, 卽得.

(24), (25) (27) 由前三例卽得. (26) 可化爲 $\cot(a-\beta)$ 以求之.

(28) 令 $\tan^{-1}\frac{1}{4}=a$, $\tan^{-1}\frac{2}{9}=\beta$, 則 $\frac{1}{4}=\tan a$, $\frac{2}{9}=\tan\beta$,

故 $\tan(a+\beta)=\frac{\tan a+\tan\beta}{1-\tan a\tan\beta}=\frac{\frac{1}{4}+\frac{2}{9}}{1-\frac{1}{4}\times\frac{2}{9}}=\frac{1}{2}$,

$\cos 2(a+\beta)=2\cos^2(a+\beta)-1=\frac{2}{1+\tan^2(a+\beta)}-1=\frac{2}{1+\frac{1}{4}}-1=\frac{3}{5}$,

故 $2(a+\beta)=\cos^{-1}\frac{3}{5}$ 卽 $\tan^{-1}\frac{1}{4}+\tan^{-1}\frac{2}{9}=\frac{1}{2}\cos^{-1}\frac{3}{5}$.

(29) 至 (34) 可參照前列諸例求之

(35) 可令 $\operatorname{vers}^{-1}a=x$, $\operatorname{vers}^{-1}b=y$, $1-a=\cos x$, $1-b=\cos y$,

$\operatorname{vers}(x+y)=1-\cos(x+y)$ 以求之

36. $sin^{-1}\frac{4}{5}+sin^{-1}\frac{5}{13}+sin^{-1}\frac{16}{65}=\frac{\pi}{2}$.

37. $tan^{-1}\frac{1}{2}+tan^{-1}\frac{1}{5}+tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

38. $tan^{-1}\frac{2}{3}+tan^{-1}\frac{1}{4}+tan^{-1}\frac{10}{11}=\frac{\pi}{2}$.

39. $tan^{-1}1+tan^{-1}2+tan^{-1}3=\pi$.

40. $tan^{-1}\alpha+tan^{-1}\beta+tan^{-1}\frac{1-\alpha-\beta-\alpha\beta}{1+\alpha+\beta-\alpha\beta}=\frac{\pi}{4}$.

41. $3tan^{-1}\frac{1}{4}+tan^{-1}\frac{1}{20}+tan^{-1}\frac{1}{1985}=\frac{\pi}{4}$.

42. $4tan^{-1}\frac{1}{5}-2tan^{-1}\frac{1}{408}+tan^{-1}\frac{1}{1393}=\frac{\pi}{4}$.

43. $tan^{-1}\frac{1}{3}+tan^{-1}\frac{1}{5}+tan^{-1}\frac{1}{7}+tan^{-1}\frac{1}{8}=\frac{\pi}{4}$.

44. $tan^{-1}\frac{1}{3}+tan^{-1}\frac{1}{4}+tan^{-1}\frac{1}{5}+tan^{-1}\frac{1}{47}=\frac{\pi}{4}$.

45. $tan\left(3tan^{-1}\frac{1}{7}+tan^{-1}\frac{1}{3}+tan^{-1}\frac{1}{26}-\frac{\pi}{4}\right)=\frac{1}{2057}$.

46. $tan^{-1}\left(1+\frac{2}{x}\right)+tan^{-1}\left(1-\frac{2}{x}\right)$

$$+tan^{-1}(1+x)+tan^{-1}(1-x)=(2n+1)\frac{\pi}{2}.$$

例題解自 36. 至 46.

(36) 令 $\frac{4}{5}=sin\alpha$, $\frac{5}{13}=sin\beta$, $\frac{16}{65}=sin\gamma$, 則

$cos\alpha=\frac{3}{5}$, $cos\beta=\frac{12}{13}$, $cos\gamma=\frac{63}{65}$, 由第五編1.節之公式得

$sin(\alpha+\beta+\gamma)=sin\alpha\,cos\beta\,cos\gamma+sin\beta\,cos\gamma\,cos\alpha+sin\gamma\,cos\alpha\,cos\beta-sin\alpha\,sin\beta\,sin\gamma$

$=\frac{4}{5}\times\frac{12}{13}\times\frac{63}{65}+\frac{5}{13}\times\frac{63}{65}\times\frac{3}{5}+\frac{16}{65}\times\frac{3}{5}\times\frac{12}{13}-\frac{4}{5}\times\frac{5}{13}\times\frac{16}{65}=1.$

故 $sin(\alpha+\beta+\gamma)=\frac{\pi}{2}$, 卽 $\alpha+\beta+\gamma=\frac{\pi}{2}$.

(37), (38), (39), 可如次例求得之

(40) $\alpha=tanx$, $\beta=tany$, $\frac{1-\alpha-\beta-\alpha\beta}{1+\alpha+\beta-\alpha\beta}=tanz$,

$tan(x+y)=\frac{tanx+tany}{1-tanx\,tany}=\frac{\alpha+\beta}{1-\alpha\beta}$, $tan(x+y+z)=tan\{(x+y)+z\}$

$$=\frac{\tan(x+y)+\tan z}{1-\tan(x+y)\tan z}=\frac{\frac{\alpha+\beta}{1-\alpha\beta}+\frac{1-\alpha-\beta-\alpha\beta}{1+\alpha+\beta-\alpha\beta}}{1-\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)\left(\frac{1-\alpha-\beta-\alpha\beta}{1+\alpha+\beta-\alpha\beta}\right)}$$

$$+\frac{(\alpha+\beta)(1-\alpha\beta)+(\alpha+\beta)^2+(1-\alpha\beta)^2-(\alpha+\beta)(1-\alpha\beta)}{(1-\alpha\beta)^2+(\alpha+\beta)(1-\alpha\beta)+(\alpha+\beta)^2-(\alpha+\beta)(1-\alpha\beta)}=\frac{(\alpha+\beta)^2+(1-\alpha\beta)^2}{(1-\alpha\beta)^2+(\alpha+\beta)^2}=1.$$

故 $x+y+z=\frac{\pi}{4}$. (41) 與次例同樣.

(42) 令 $4\tan^{-1}\frac{1}{5}=\alpha$, $2\tan^{-1}\frac{1}{408}=\beta$. $\tan^{-1}\frac{1}{1393}=\gamma$, 則 $\frac{1}{1393}=\tan\gamma$,

$$\frac{1}{408}=\tan\frac{\beta}{2},\quad \tan\beta=\frac{2\tan\frac{\beta}{2}}{1-\tan^2\frac{\beta}{2}}=\frac{2\times\frac{1}{408}}{1-\left(\frac{1}{408}\right)^2}=\frac{816}{166463},$$

$$\tan(\beta-\gamma)=\frac{\tan\beta-\tan\gamma}{1+\tan\beta\tan\gamma}=\frac{\frac{816}{166463}-\frac{1}{1393}}{1+\frac{816}{166463}\times\frac{1}{1393}}=\frac{1}{239},$$ 故

$\alpha-\beta+\gamma=\alpha-(\beta-\gamma)=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}=\frac{\pi}{4}$, (例題 12.).

(43) $\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{1/3+1/5}{1-\frac{1}{3}\times\frac{1}{5}}=\frac{4}{7}$, $\tan(\gamma+\delta)=\frac{\tan\gamma+\tan\delta}{1-\tan\gamma\tan\delta}$

$=\frac{1/7+1/8}{1-\frac{1}{7}\times\frac{1}{8}}=\frac{3}{11}$, $\tan(\alpha+\beta+\gamma+\delta)=\frac{\tan(\alpha+\beta)+\tan(\gamma+\delta)}{1-\tan(\alpha+\beta)\tan(\gamma+\delta)}$

$=\frac{4/7+3/11}{1-\frac{4}{7}\times\frac{3}{11}}=\frac{65}{65}=1=\tan\frac{\pi}{4}$, 故 $\alpha+\beta+\gamma+\delta=\frac{\pi}{4}$. (44) 與前例同樣

(45) 令 $3\tan^{-1}\frac{1}{7}=\alpha$, $\tan^{-1}\frac{1}{3}=\beta$, $\tan^{-1}\frac{1}{26}=\gamma$ 則 $\tan\alpha=\frac{3\tan\frac{\alpha}{3}-\tan^3\frac{\alpha}{3}}{1-3\tan^2\frac{\alpha}{3}}$

$$=\frac{3\left(\frac{1}{7}\right)-\left(\frac{1}{7}\right)^3}{1-3\times\left(\frac{1}{7}\right)^2}=\frac{73}{161},\ \tan(\alpha+\beta)=\frac{\frac{73}{161}+\frac{1}{3}}{1-\frac{72}{161}\times\frac{1}{3}}=\frac{38}{41},\ \tan\left(\gamma-\frac{\pi}{4}\right)=\frac{\tan\gamma-1}{\tan\gamma+1}$$

$$=\frac{\frac{1}{26}-1}{\frac{1}{26}+1}=-\frac{25}{27};\ \tan\left(\alpha+\beta+\gamma-\frac{\pi}{4}\right)=\frac{\tan(\alpha+\beta)+\tan\left(\gamma-\frac{\pi}{4}\right)}{1-\tan(\alpha+\beta)\tan\left(\gamma-\frac{\pi}{4}\right)}=\frac{\frac{38}{41}+\frac{-25}{27}}{1-\frac{38}{41}\times\frac{-25}{27}}$$

$$=\frac{1}{2057}.$$

(46) $$\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\frac{\left(1+\frac{2}{x}\right)+\left(1-\frac{2}{x}\right)}{1-\left(1+\frac{2}{x}\right)\left(1-\frac{2}{x}\right)}=\frac{x^2}{2},$$

$$\tan(\gamma+\delta)=\frac{(1+x)+(1-x)}{1-(1+x)(1-x)}=\frac{2}{x^2}\ \text{故}\ \tan(\alpha+\beta+\gamma+\delta)=\frac{\frac{x^2}{2}+\frac{2}{x^2}}{1-\frac{x^2}{2}\times\frac{2}{x^2}}=\infty=\frac{\pi}{2}.$$

47. $\cos^{-1}\frac{4}{5}-\sin^{-1}\frac{1}{\sqrt{10}}+\tan^{-1}\frac{1}{2}=\frac{\pi}{4}$.

48. $\tan^{-1}\frac{a-b}{1+ab}+\tan^{-1}\frac{b-c}{1+bc}+\tan^{-1}c=\tan^{-1}a$.

49. $6\tan^{-1}x=\cos^{-1}\frac{1-15x^2+15x^4-x^6}{(1+x^2)^3}$.

50. $\sin^{-1}\frac{4x(x^2-1)}{(x^2+1)^2}-\cos^{-1}\frac{x^2-1}{x^2+1}=2\cot^{-1}x$.

51. $\tan^{-1}\frac{2x}{2+x^2+x^4}+\tan^{-1}(x-1)+\tan^{-1}(x+1)=2\tan^{-1}x$.

52. $\sin(\sin^{-1}x)=\sin^{-1}(\sin x)=x$.

53. $\cos(\sin^{-1}x)=\sin(\cos^{-1}x)$.

54. $\tan^{-1}\{(\sqrt{2}+1)\tan\alpha\}-\tan^{-1}\{\sqrt{2}-1)\tan\alpha\}=\tan^{-1}(\sin2\alpha)$.

55. $2\tan(\tan^{-1}\alpha+\tan^{-1}\alpha^3)=\tan(2\tan^{-1}\alpha)$.

56. $\tan^{-1}(\frac{1}{2}\tan2\alpha)+\tan^{-1}(\cot\alpha)+\tan^{-1}(\cot^3\alpha)=0$.

57. $\tan\left(\frac{\pi}{4}+\frac{1}{2}\cos^{-1}\frac{a}{b}\right)+\tan\left(\frac{\pi}{4}-\frac{1}{2}\cos^{-1}\frac{a}{b}\right)=\frac{2b}{a}$.

58. $\frac{a^3}{2}\operatorname{cosec}^2\left(\frac{1}{2}\tan^{-1}\frac{a}{b}\right)+\frac{b^3}{2}\sec^2\left(\frac{1}{2}\tan^{-1}\frac{b}{a}\right)=(a+b)(a^2+b^2)$.

例題解自 47. 至 58.

(47) $\frac{4}{5}=\cos\alpha,\ \tan\alpha=\frac{\sqrt{1-\cos^2\alpha}}{\cos\alpha}=\frac{\sqrt{1-\left(\frac{4}{5}\right)^2}}{\frac{4}{5}}=\frac{3}{4},\ \frac{1}{\sqrt{10}}=\sin\beta,$

$\tan\beta=\frac{\sin\beta}{\sqrt{(1-\sin^2\beta)}}=\frac{\frac{1}{\sqrt{10}}}{\sqrt{\left(1-\frac{1}{10}\right)}}=\frac{1}{3},\ \frac{1}{2}=\tan\gamma,$ 餘與前舊例同樣.

(48) 可參照從前諸例.

(49) 令 $6\tan^{-1}x=\alpha$, 則 $x=\tan\frac{\alpha}{6}$ 故

$$\tan\frac{\alpha}{2}=\frac{3\tan\frac{\alpha}{6}-\tan^3\frac{\alpha}{6}}{1-3\tan^2\frac{\alpha}{6}}=\frac{3x-x^3}{1-3x^2},\ \sec\alpha=\sqrt{1+\tan^2\alpha}=\sqrt{1+\left\{\frac{2\tan\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}}\right\}^2}$$

$$=\frac{1+\tan^2\frac{\alpha}{2}}{1-\tan^2\frac{\alpha}{2}},\ \text{故}\ \cos\alpha=\frac{1-\tan^2\frac{\alpha}{2}}{1+\tan^2\frac{\alpha}{2}}=\frac{(1-3x^2)^2-(3x-x^3)^2}{(1-3x^2)^2+(3x-x^3)^2}=\frac{1-15x^2+15x^4-x^6}{(1+x^2)^3}.$$

(50) $\frac{4x(x^2-1)}{(x^2+1)^2}=\sin\alpha,\ \frac{x^2-1}{x^2+1}=\cos\beta$ 則 $\cos\alpha=\frac{x^4-6x^2+1}{(x^2+1)^2}$，故

$\cot\alpha=\frac{\cos\alpha}{\sin\alpha}=\frac{x^4-6x^2+1}{4x(x^2-1)}$，同樣 $\cot\beta=\frac{x^2-1}{2x}$，用第四編 2. 節之公式

$$\cot(\alpha-\beta)=\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}=\frac{x^2-1}{2x},\ \text{故}\ \cot\frac{\alpha-\beta}{2}=\frac{1+\cos(\alpha-\beta)}{\sin(\alpha-\beta)}$$

$=\sqrt{1+\cot^2(\alpha-\beta)}+\cot(\alpha-\beta)=x$，卽 $\alpha-\beta=2\cot^{-1}x$.

(51) 可從 $\tan\frac{1}{2}(\alpha+\beta+\gamma)$ 導得但應參照例題 40. 及其他之例.

(52) 令 $\sin^{-1}x=\alpha$，則 $x=\sin\alpha$，而 $\sin(\sin^{-1}x)=\sin\alpha=x$，

又令 $\sin^{-1}(\sin x)=\beta$ 則 $\sin x=\sin\beta$ 故 $x=\beta=\alpha$

(53) 令 $x=\sin\alpha,\ x=\cos\beta$ 則 $\beta=\frac{\pi}{2}-\alpha$，故 $\cos(\sin^{-1}x)=\cos\alpha$，

又 $\sin(\cos^{-1}x)=\sin\beta=\sin\left(\frac{\pi}{2}-\alpha\right)=\cos\alpha$，故如題言

(54) 令 $(\sqrt{2}+1)\tan\alpha=\tan x,\ (\sqrt{2}-1)\tan\alpha=\tan y$，則

$$\tan(x-y)=\frac{(\sqrt{2}+1)\tan\alpha-(\sqrt{2}-1)\tan\alpha}{1+(\sqrt{2}+1)(\sqrt{2}-1)\tan^2\alpha}=\frac{2\tan\alpha}{1+\tan^2\alpha}=\sin2\alpha.$$

(55) 令 $a=\tan x,\ a^3=\tan y$ 則 $2\tan(x+y)=\frac{2(a+a^3)}{1-a^4}$

$$=2a/(1-a^2)=2\tan x/(1-\tan^2x)=\tan2x=\tan(2\tan^{-1}a).$$

(56) $\frac{1}{2}\tan2\alpha=\tan x,\ \cot\alpha=\tan y,\ \cot^3\alpha=\tan z$ 故

$$\tan(x+y)=-\cot^3\alpha,\ \tan(x+y+z)=\frac{-\cot^3\alpha+\cot^3\alpha}{1-(-\cot^3\alpha)\cot^3\alpha}=0.$$

(57) 令 $\frac{1}{2}\cos^{-1}\frac{a}{b}=x$，則 $\frac{a}{b}=\cos2x=2\cos^2x-1$，故 $\tan x=\sqrt{\frac{b-a}{b+a}}$，

$$\tan\left(\frac{\pi}{4}+x\right)=\frac{1+\tan x}{1-\tan x}=\frac{\sqrt{(b+a)}+\sqrt{(b-a)}}{\sqrt{(b+a)}-\sqrt{(b-a)}},\ \tan\left(\frac{\pi}{4}-x\right)=\frac{\sqrt{(b+a)}-\sqrt{(b-a)}}{\sqrt{(b+a)}+\sqrt{(b-a)}}.$$

(58) 令 $\frac{a}{b}=\tan2x,\ \frac{b}{a}=\tan2y$，則 $\frac{a^3}{2}\operatorname{cosec}^2x+\frac{b^3}{2}\sec^2y$

$$=\frac{a^3}{1-\cos2x}+\frac{b^3}{1+\cos2x}=\frac{a^3\sqrt{(1+\tan^22x)}}{\sqrt{(1+\tan^22x)}+1}+\frac{b^3\sqrt{(1+\tan^22y)}}{\sqrt{(1+\tan^22y)}+1}$$

$$=\frac{a^3\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)}-b}+\frac{b^3\sqrt{(a^2+b^2)}}{\sqrt{(a^2+b^2)}-a}=(a^2+b^2)(a+b).$$

59. $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)=\cos^{-1}\left(\frac{b+a\cos x}{a+b\cos x}\right).$

60. $\tan^{-1}\left(\frac{a\cos x}{1-a\sin x}\right)-\tan^{-1}\left(\frac{a-\sin x}{\cos x}\right)=x.$

61. $\cos^{-1}\left(\frac{\tan\alpha+\tan\beta}{1+\tan\alpha\tan\beta}\right)=2\tan^{-1}\sqrt{\tan(45^\circ-\alpha)\tan(45^\circ-\beta)}.$

62. $\cos^{-1}\frac{a+b\cos x}{b+a\cos x}-\cos^{-1}\frac{a-b\cos x}{b-a\cos x}=\cos^{-1}\frac{a^2\sin^2x+a^2-b^2}{a^2\sin^2x-a^2+b^2}.$

63. $$\sin\left(\frac{2\pi}{3}+\cos^{-1}\frac{\alpha}{\beta}\right)\sin\left(\frac{2\pi}{3}-\cos^{-1}\frac{\alpha}{\beta}\right)$$

$$-\cos\left(\frac{2\pi}{3}+\cos^{-1}\frac{\alpha}{\beta}\right)\cos\left(\frac{2\pi}{3}-\cos^{-1}\frac{\alpha}{\beta}\right)=\frac{1}{2}.$$

64. $\frac{1}{2}\tan^{-1}[2\tan\{\alpha+\tan^{-1}(\tan^3\alpha)\}]=\alpha.$

65. $\operatorname{cosec}^{-1}\sin\tan^{-1}\cos\tan^{-1}\sin\cos^{-1}\tan\sin^{-1}x=\sqrt{\frac{3-4x^2}{1-x^2}}.$

66. $\sin 2\cos^{-1}\tan 3\cos^{-1}x=\frac{2(3x^2-1)\sqrt{x^6-15x^4+15x^2-1}}{x^2(x^2-3)^2}.$

例題解自 59. 至 66.

(59) 令 $2\tan^{-1}\left(\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}\right)=y$, 則 $\sqrt{\frac{a-b}{a+b}}\tan\frac{x}{2}=\tan\frac{y}{2}$,

以此值代入 $\cos y=2\cos^2\frac{y}{2}-1=\frac{1-\tan^2\frac{y}{2}}{1+\tan^2\frac{y}{2}}$ 之中. 則得 $\cos y=\frac{b+a\cos x}{a+b\cos x}$.

(60) $\frac{a\cos x}{1-a\sin x}=\tan y$, $\frac{a-\sin x}{\cos x}=\tan z$ 以此代入下式

$\tan(y-z)=\frac{\tan y-\tan z}{1+\tan y\tan z}$ 則得

$\tan(y-z)=\frac{\sin x(a^2-2a\sin x+1)}{\cos x(a^2-2a\sin x+1)}=\tan x.$

(61) $\frac{\tan\alpha+\tan\beta}{1+\tan\alpha\tan\beta}=\cos x$ 代入下式 $\tan\frac{x}{2}=\frac{\sin\frac{x}{2}}{\cos\frac{x}{2}}=\frac{\sqrt{1-\cos x}}{\sqrt{1+\cos x}}$ 則

$\tan\frac{x}{2}=\sqrt{\frac{(1-\tan\alpha)(1-\tan\beta)}{(1+\tan\alpha)(1+\tan\beta)}}=\sqrt{\tan(45^\circ-\alpha)\tan(45^\circ-\beta)}$.

(62) 可從 $\cos(\alpha+\beta)$ 導得.

(63) $-\cos\left\{\left(\frac{2\pi}{3}+\cos^{-1}\frac{\alpha}{\beta}\right)+\left(\frac{2\pi}{2}-\cos^{-1}\frac{\alpha}{\beta}\right)\right\}=-\cos\frac{4\pi}{3}=\cos\frac{\pi}{3}=\frac{1}{2}$.

(64) 令 $\tan^{-1}(\tan^3\alpha)=x$ 則 $\tan^3\alpha=\tan x$, 故

原式$=\frac{1}{2}\tan^{-1}\left[2\tan\{\alpha+x\}\right]=\frac{1}{2}\tan^{-1}\left[\frac{2(\tan\alpha+\tan x)}{1-\tan\alpha\tan\beta}\right]$

$=\frac{1}{2}\tan^{-1}\left[\frac{2\tan\alpha(1+\tan^2\alpha)}{1-\tan^4\alpha}\right]=\frac{1}{2}\tan^{-1}\left[\frac{2\tan\alpha}{1-\tan^2\alpha}\right]=\frac{1}{2}\tan^{-1}\left[\tan 2\alpha\right]=\alpha$.

(65) 令 $\sin^{-1}x=a$, 則 $x=\sin a$, 故 $\tan\sin^{-1}x=\tan a=\frac{x}{\sqrt{1-x^2}}$,

令 $\cos^{-1}\tan\sin^{-1}x=\cos^{-1}\frac{x}{\sqrt{1-x^2}}=b$, 則 $\frac{x}{\sqrt{1-x^2}}=\cos b$, $\sin b=\sqrt{1-\frac{x^2}{1-x^2}}=\sqrt{\frac{1-2x^2}{1-x^2}}$,

令 $\tan^{-1}\sin\cos^{-1}\tan\sin^{-1}x=\tan^{-1}\sqrt{\frac{1-2x^2}{1-x^2}}=c$ 則 $\sqrt{\frac{1-2x^2}{1-x^2}}=\tan c$, 故

$\cos c=\frac{1}{\sqrt{1+\tan^2 c}}=\sqrt{\frac{1-x^2}{2-3x^2}}$,

令 $\tan^{-1}\cos\tan^{-1}\sin\cos^{-1}\tan\sin^{-1}x=\tan^{-1}\sqrt{\frac{1=x^2}{2-3x^2}}=d$, 則

$\sqrt{\frac{1-x^2}{2-3x^2}}=\tan d$, 故 $\sin d=\sqrt{1-\cos^2 d}=\sqrt{1-\frac{1}{1+\tan^2 d}}=\frac{\tan d}{\sqrt{1+\tan^2 d}}$

$=\sqrt{\frac{1-x^2}{3-4x^2}}$, $\sec^{-1}\sin\tan^{-1}\cos\tan^{-1}\sin\cos^{-1}\tan\sin^{-1}x=e$, 故

$\sqrt{\frac{1-x^2}{3-4x^2}}=\sec e$ 由是原式$=\frac{1}{\sec e}=\sqrt{\frac{3-4x^2}{1-x^2}}$.

(66) 令 $3\cot^{-1}x=a$, 則 $x=\cot\frac{a}{3}$, 卽 $\tan\frac{a}{3}=\frac{1}{x}$, 故

$\tan a=\frac{3\tan\frac{a}{3}-\tan^3\frac{a}{3}}{1-3\tan^2\frac{a}{3}}=\frac{3/x-1/x^3}{1-3/x^2}=\frac{3x^2-1}{x^3-3x}$,

令 $2\cos^{-1}\tan a=2\cos^{-1}\frac{3x^2-1}{x^3-3x}=b$, 則 $\frac{3x^2-1}{x^3-3x}=\cos\frac{b}{2}$, 故

$\sin b=2\cos\frac{b}{2}\sqrt{1+\cos^2\frac{b}{2}}=\frac{2(3x^2-1)}{x^3-3x}\sqrt{1-\left(\frac{3x^2-1}{x^3-3x}\right)^2}$.

67　$\tan^{-1}(\cot x)+2x=\tan^{-1}(\tan x)+\frac{2n+1}{2}\pi.$

68.　$\tan^{-1}a+\tan^{-1}b+\tan^{-1}c=\pi$,　則　$a+b+c=abc.$

69.　$u=\cot^{-1}\sqrt{\cos a}-\tan^{-1}\sqrt{\cos a}$　則　$\sin u=\tan^2\frac{a}{2}.$

70.　$\tan^{-1}a+\tan^{-1}b+\tan^{-1}c=\frac{\pi}{4}$　則

$$\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}=\frac{(1+a)(1+b)(1+c)}{(1-a)(1-b)(1-c)}.$$

71.　$\sin^{-1}\frac{x}{a}+\sin^{-1}\frac{y}{b}=\sin^{-1}\frac{c^2}{ab}$,　則

$$b^2x^2+2(a^2b^2-c^4)^{\frac{1}{2}}xy+a^2y^2=c^4.$$

72.　$\tan(\theta-\alpha)\tan(\theta-\beta)=\tan^2\theta$　則

$$\theta=\frac{1}{2}\tan^{-1}\frac{2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}.$$

73.　$2\theta=\phi+\sin^{-1}(a\sin\phi)$,　則

$$\phi=\theta+\tan^{-1}\left(\frac{1-a}{1+a}\tan\theta\right).$$

74.　$\cos^{-1}\frac{x}{a}=2\sin^{-1}\frac{y}{a}$,　則　$a^2=ax+2y^2.$

例題解自 67. 至 74.

(67) 令 $\tan^{-1}(\cot x)=y$, 則 $\cot x=\tan y$, $\tan y=$ $\tan\left(\frac{2n+1}{2}\pi-x\right)$, 故 $y=\frac{2n+1}{2}\pi-x$, 即 $2x+y=x+\frac{2n+1}{2}\pi$.
由是 $\tan^{-1}(\cot x)+2x=y+2x=x+\frac{2n+1}{2}\pi=\tan^{-1}(\tan x)+\frac{2n+1}{2}\pi$.

(68) 令 $a=\tan\alpha$, $b=\tan\beta$, $c=\tan\gamma$, 則由第五編 1. 節之公式. 從 $\tan(\alpha+\beta+\gamma)=0$, 可得其證.

(69) 令 $\cot^{-1}\sqrt{\cos\alpha}=x,\ \tan^{-1}\sqrt{\cos\alpha}=y$, 則 $\sqrt{\cos\alpha}=\cot x=\tan y$, 卽

$x=\frac{\pi}{2}-y$, 由是 $\sin u=\sin(x-y)=\sin\left(\frac{\pi}{2}-2y\right)=\cos 2y$

$=\frac{1-\tan^2 y}{1+\tan^2 y}=\frac{1-\cos\alpha}{1+\cos\alpha}=\tan^2\frac{\alpha}{2}$.

(70) 令 $\tan^{-1}a=\alpha,\ \tan^{-1}b=\beta,\ \tan^{-1}c=\gamma$ 則

$\frac{1+a}{1-a}=\frac{1+\tan\alpha}{1-\tan\alpha}=\tan\left(\frac{\pi}{4}+\alpha\right),\ \frac{1+b}{1-b}=\tan\left(\frac{\pi}{4}+\beta\right),\ \frac{1+c}{1-c}=\tan\left(\frac{\pi}{4}+\gamma\right)$,

依題意 $\alpha+\beta+\gamma=\frac{\pi}{4}$, 故 $\left(\frac{\pi}{4}+\alpha\right)+\left(\frac{\pi}{4}+\beta\right)+\left(\frac{\pi}{4}+\gamma\right)=\pi$, 故由例題五 35.

得 $\tan\left(\frac{\pi}{4}+\alpha\right)+\tan\left(\frac{\pi}{4}+\beta\right)+\tan\left(\frac{\pi}{4}+\gamma\right)=\tan\left(\frac{\pi}{4}+\alpha\right)\tan\left(\frac{\pi}{4}+\beta\right)\tan\left(\frac{\pi}{4}+\gamma\right)$,

卽 $\frac{1+a}{1-a}+\frac{1+b}{1-b}+\frac{1+c}{1-c}=\frac{(1+a)(1+b)(1+c)}{(1-a)(1-b)(1-c)}$.

(71) 令 $\sin^{-1}\frac{x}{a}=\alpha,\ \sin^{-1}\frac{y}{b}=\beta,\ \sin^{-1}\frac{c^2}{ab}=\gamma$, 則

$\frac{x}{a}=\sin\alpha,\ \frac{y}{b}=\sin\beta,\ \frac{c^2}{ab}=\sin\gamma$, 故 $\alpha+\beta=\gamma$, 卽

$\cos(\alpha+\beta)=\cos\gamma$, 卽 $\cos\alpha\cos\beta=\cos\gamma+\sin\alpha\sin\beta$, 雙方平方之.則

$(1-\sin^2\alpha)(1-\sin^2\beta)=\cos^2\gamma+2\sin\alpha\sin\beta\cos\gamma+\sin^2\alpha\sin^2\beta$,

故 $\sin^2\alpha+\sin^2\beta-\sin^2\gamma+2\sin\alpha\sin\beta\cos\gamma=0$,

卽 $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{c^4}{a^2b^2}+\frac{2xy}{ab}\sqrt{\left(1-\frac{c^4}{a^2b^2}\right)}=0$.

(72) 從原式得 $\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\cos(\theta-\alpha)\cos(\theta-\beta)}=\frac{\sin^2\theta}{\cos^2\theta}$, 卽

$\frac{\cos(\alpha-\beta)-\cos(2\theta-\alpha-\beta)}{\cos(\alpha-\beta)+\cos(2\theta-\alpha-\beta)}=\frac{1-\cos 2\theta}{1+\cos 2\theta}$, 故 $\frac{\cos(2\theta-\alpha-\beta)}{\cos(\alpha-\beta)}=\cos 2\theta$.

由是 $\tan 2\theta=\{\cos(\alpha-\beta)-\cos(\alpha+\beta)\}/\sin(\alpha+\beta)$.

(73) 由原方程式 $\sin(2\theta-\phi)=a\sin\phi$, 則

$\sin 2\theta\cos\phi-\cos 2\theta\sin\phi=a\sin\phi$, 從此可得

$2\cos\theta\sin(\phi-\theta)=(1-a)\sin\phi$, 及 $2\sin\theta\cos(\phi-\theta)=(1+a)\sin\phi$.

(74) 令 $\cos^{-1}\frac{x}{a}=\alpha,\ \sin^{-1}\frac{y}{a}=\beta$, 則 $\alpha=2\beta$,

$\cos\alpha=\cos 2\beta=1-2\sin^2\beta$ 從此可得其證.

75. $vers^{-1}\frac{x}{a}-vers^{-1}\frac{bx}{a}=vers^{-1}(1-b)$ 則

$\frac{x}{a}=1\pm\sqrt{\frac{2b}{1+b}}$.

76. $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{y}{b}=\alpha$, 則

$\left(\frac{x}{a}\right)^2-\frac{2xy}{ab}\cos\alpha+\left(\frac{y}{b}\right)^2=\sin^2\alpha$.

77. $x^2=a^2+b^2+ab$, $\tan^2\phi=2\,cosec\left(2\tan^{-1}\frac{b}{a}\right)$,

則 $x=\sqrt{ab}\sec\phi$.

78. $\tan^3\theta=\tan(\theta-\alpha)$, 則 $\theta=\frac{\sin^{-1}(3\sin\alpha)+\alpha}{4}$.

79. $\tan^3\left\{\frac{\sin^{-1}(3\sin\alpha)+\alpha}{4}\right\}=\tan\left\{\frac{\sin^{-1}(3\sin\alpha)-3\alpha}{4}\right\}$.

80. $\tan^{-1}\frac{a_1-a_2}{1+a_1a_2}+\tan^{-1}\frac{a_2-a_3}{1+a_2a_3}+\cdots\cdots+\tan^{-1}\frac{a_{n-1}-a_n}{1+a_{n-1}a_n}$

$=\tan^{-1}a_1-\tan^{-1}a_n$.

81. $\tan^{-1}\frac{c_1x-y}{c_1y+x}+\tan^{-1}\frac{c_2-c_1}{c_2c_1+1}+\tan^{-1}\frac{c_3-c_2}{c_3c_2+1}+\cdots\cdots$

$+\tan^{-1}\frac{c_n-c_{n-1}}{c_nc_{n-1}+1}+\tan^{-1}\frac{1}{c}=\tan^{-1}\frac{x}{y}$.

82. $\theta=\tan^{-1}\frac{x\sqrt{3}}{2c-x}$, $\phi=\tan^{-1}\frac{2x-c}{c\sqrt{3}}$ 則 $\theta-\phi=\frac{\pi}{6}$.

例題解自 75. 至 82.

(75) 令 $\frac{x}{a}=1-\cos\alpha$, $\frac{bx}{a}=1-\cos\beta$, $b=\cos\gamma$, 則

$\cos(\alpha-\beta)=\cos\gamma$, 而如例題 70. 之解法. 則

$\cos^2\alpha+\cos^2\beta+\cos^2\gamma-2\cos\alpha\cos\beta\cos\gamma=1$ 卽

$\left(1-\frac{x}{a}\right)^2+\left(1-\frac{bx}{a}\right)^2+b^2-2\left(1-\frac{x}{a}\right)\left(1-\frac{bx}{a}\right)b=1$, 從此即得其證.

(76) 令 $\frac{x}{a}=\cos\theta,\ \frac{y}{b}=\cos\phi$ 則 $\cos(\theta+\phi)=\cos\alpha$, 如前證例.

$\cos^2\theta+\cos^2\phi+\cos^2\alpha-2\cos\theta\cos\phi\cos\alpha=1$, 從此可得其證.

(77) 令 $\tan^{-1}\frac{b}{a}=\theta$, 則 $\frac{b}{a}=\tan\theta,\ \tan^2\phi=2\operatorname{cosec}2\theta=\frac{\sin^2\theta+\cos^2\theta}{\sin\theta\cos\theta}$

$=\tan\theta+\frac{1}{\tan\theta}=\frac{b}{a}+\frac{a}{b}$, 即 $\sec^2\phi-1=\frac{b}{a}+\frac{a}{b}$, 故

$ab\sec^2\theta=a^2+ab+b^2=x^2$, 即 $x=\sqrt{ab}\sec\phi$.

(78) 從原式 $\frac{\sin^3\theta}{\cos^3\theta}=\frac{\sin(\theta-\alpha)}{\cos(\theta-\alpha)}$ 即 $\frac{3\sin\theta-\sin3\theta}{3\cos\theta+\cos3\theta}=\frac{\sin(\theta-\alpha)}{\cos(\theta-\alpha)}$,

去分母括之.則 $3\{\sin\theta\cos(\theta-\alpha)-\cos\theta\sin(\theta-\alpha)\}=\sin3\theta\cos(\theta-\alpha)$

$+\cos3\theta\sin(\theta-\alpha)$, 即 $3\sin\alpha=\sin(4\theta-\alpha)$, 故 $\theta=\frac{\sin^{-1}(3\sin\alpha)+\alpha}{4}$.

(79) 由前例令 $\frac{\sin^{-1}(3\sin\alpha)+\alpha}{4}=\theta$, 則 $\frac{\sin^{-1}(3\sin\theta)-3\alpha}{4}=\theta-\alpha$,

故如題言.

(80) 由例題23. 原式$=(\tan^{-1}a_1-\tan^{-1}a_2)+(\tan^{-1}a_2-\tan^{-1}a_3)$

$+\cdots\cdots+(\tan^{-1}a_{n-2}-\tan^{-1}a_{n-1})+(\tan^{-1}a_{n-1}-\tan^{-1}a_n)=\tan^{-1}a_1-\tan^{-1}a_n$.

(81) $\tan^{-1}\frac{\frac{x}{y}-\frac{1}{c_1}}{1+\frac{1}{c_1}\cdot\frac{x}{y}}=\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{1}{c_1}$, (例題23.) 故

原式$=\left(\tan^{-1}\frac{x}{y}-\tan^{-1}\frac{1}{c_1}\right)+\left(\tan^{-1}\frac{1}{c_1}-\tan^{-1}\frac{1}{c_2}\right)+\cdots\cdots+\left(\tan^{-1}\frac{1}{c_{n-1}}-\tan^{-1}\frac{1}{c_n}\right)$

$+\tan^{-1}\frac{1}{c_n}=\tan^{-1}\frac{x}{y}$.

(82) $\tan(\theta-\phi)=\frac{\tan\theta-\tan\phi}{1+\tan\theta\tan\phi}=\frac{\frac{x\sqrt{3}}{2c-x}-\frac{2x-c}{c\sqrt{3}}}{1+\left(\frac{x\sqrt{3}}{2c-x}\right)\left(\frac{2x-c}{c\sqrt{3}}\right)}$

$=\frac{3cx-(5cx-2x^2-2c^2)}{(2c^2-cx+2x^2-cx)\sqrt{3}}=\frac{1}{\sqrt{3}}=\tan\frac{\pi}{6}$, 故 $\theta-\phi=\frac{\pi}{6}$.

83. $sin^2\theta+sin^2\phi=\frac{1}{2}$, 則

$$sin^{-1}(sin\theta+sin\phi)-sin^{-1}(sin\theta-sin\phi)=\frac{\pi}{2}.$$

84. $sin^{-1}\frac{(-1)^m}{2}$, $cos^{-1}\frac{(-1)^m}{2}$, 及 $tan^{-1}(-1)^m$ 求其一般之值.

例題解自 83. 至 84.

(83) 令 $sin^{-1}(sin\theta+sin\phi)=x$, $sin^{-1}(sin\theta-sin\phi)=y$, 則

$sinx=sin\theta+sin\phi$, $siny=sin\theta-sin\phi$, 故 $sin^2x+sin^2y=1$,

即 $sinx=cosy$, $cosx=siny$. 由是

$sin(x+y)=sinx\,cosy+cosx\,siny=sin^2x+cos^2x=1$, 故 $x+y=\frac{\pi}{2}$,

(84) 令 $sin^{-1}\frac{(-1)^m}{2}=\theta$, 則 $sin\theta=\frac{(-1)^m}{2}$, 故 θ最簡單之值,

為 $(-1)^m\frac{\pi}{6}$, 其一般之值, 為 $n\pi+(-1)^n\theta=n\pi+(-1)^{m+n}\frac{\pi}{6}$. (第三編15.節).

令 $cos^{-1}\frac{(-1)^m}{2}=\phi$, 則 $cos\phi=(-1)^m\frac{1}{2}$, 故$m$為偶數, 則$\phi$最簡單之值為

$\frac{\pi}{3}$, 又m為奇數, 則ϕ最簡單之值為$\pi+\frac{\pi}{3}$. 故由第三編15.節一般之形為

$2n\pi\pm\frac{\pi}{3}$, 或 $2n\pi\pm\left(\pi+\frac{\pi}{3}\right)$. 括此二形狀為 $(2n+m)\pi\pm\frac{\pi}{3}$.

又令 $tan^{-1}(-1)^m=\psi$, 則 $tan\psi=(-1)^m$

故ψ一般之值為 $n\pi+(-1)^m\frac{\pi}{4}$.

方　程　式

3. 反函數之方程式

三角方程式.含有反函數或求其反函數者.謂之反函數之方程式.今各示一例如次.

例如 $\sin^{-1}x=a$, 解答爲 $x=\sin a$ (此方程式含有反函數)

又如 $\sin\theta=a$. 解答爲 $\theta=\sin^{-1}a$, (此方程式求其反函數者也)

例　題　十　一

解次列各方程式

1. $\sin^{-1}x+\sin^{-1}\frac{x}{2}=\frac{\pi}{4}$　　2. $\sin^{-1}x+\cos^{-1}2x=\frac{\pi}{3}$.

3. $\tan^{-1}x+\tan^{-1}\frac{2x}{1-x^2}=\frac{\pi}{3}$.

4. $\sin^{-1}\frac{2x}{1+x^2}+\cos^{-1}\frac{1-x^2}{1+x^2}=\frac{\pi}{4}$.

例題解自 1. 至 4.

(1) 令 $\sin^{-1}n=\theta$, $\sin^{-1}\frac{x}{2}=\phi$) 則 $x=\sin\theta$, $\frac{x}{2}=\sin\phi$, 故 $\theta+\phi=\frac{\pi}{4}$,

$\cos(\theta+\phi)=\cos\frac{\pi}{4}$, 卽 $\cos\theta\cos\phi-\sin\theta\sin\phi=\frac{1}{\sqrt{2}}$, 卽

$\sqrt{(1-x^2)(1-\frac{1}{4}x^2)}-\frac{1}{2}x^2=\frac{1}{\sqrt{2}}$, 從此 $x=\pm\frac{1}{\sqrt{17}}\sqrt{2(5-2\sqrt{2})}$.

(2) 令 $x=\sin\theta$, $2x=\cos\phi$, 則 $\sin(\theta+\phi)=\sin\frac{\pi}{3}$, 卽

$\sin\theta\cos\phi+\cos\theta\sin\phi=\frac{1}{2}\sqrt{3}$, 卽 $2x^2+\sqrt{(1-x^2)(1-4x^2)}=\frac{1}{2}\sqrt{3}$,

從此 $x=\pm\sqrt{\frac{5+2\sqrt{3}}{52}}$.

(3) 令 $x=\tan\theta$, 則 $\frac{2x}{1-x^2}=\frac{2\tan\theta}{1-\tan^2\theta}=\tan 2\theta$, 故 $3\theta=\frac{\pi}{3}$, 卽

$\theta=\frac{\pi}{9}$, 從此 $x=\tan\frac{\pi}{9}$.

(4) 令 $\frac{2x}{1+x^2}=\sin\theta$, 則 $\frac{1-x^2}{1+x^2}=\sqrt{1-\left(\frac{2x}{1+x^2}\right)^2}=\cos\theta$, 故 $\theta+\theta=\frac{\pi}{4}$,

卽 $\theta=\pi/8$, 由此 $2x/(1+x^2)=\sin\pi/8$, 從此 $x=\tan\pi/16$.

5. $\tan^{-1}(x+1)=3\tan^{-1}(x-1)$.

6. $\sin^{-1}\frac{2a}{1+a^2}+\sin^{-1}\frac{2b}{1+b^2}=2\tan^{-1}x$.

7. $\tan^{-1}(x+1)=2\tan^{-1}(x-1)+\tan^{-1}\frac{x}{2-x}$.

8. $\cos^{-1}\frac{x}{a}+\cos^{-1}\frac{b}{x}=\cos^{-1}\left(\frac{2b}{a}-1\right)$.

9. $\sin^{-1}\frac{a}{x}+\sin^{-1}\frac{b}{2}=\frac{\pi}{2}$.

10. $\sin^{-1}2x-\sin^{-1}x\sqrt{3}=\sin^{-1}x$.

11. $\tan^{-1}(x-1)+\tan^{-1}x+\tan^{-1}(x+1)=\tan^{-1}3x$.

12. $\tan^{-1}\frac{1}{x-2}+\tan^{-1}\frac{1}{x}+\tan^{-1}\frac{1}{x+2}+\tan^{-1}\frac{1}{x+3}=\frac{\pi}{4}$.

13. $\tan^{-1}\frac{1}{4}+2\tan^{-1}\frac{1}{5}+\tan^{-1}\frac{1}{6}+\tan^{-1}\frac{1}{x}=\frac{\pi}{4}$.

14. $\tan^{-1}\frac{1}{x}+\tan^{-1}\frac{1}{a^2-x+1}=\tan^{-1}\frac{1}{a-1}$.

15. $3\tan^{-1}\frac{1}{2+\sqrt{3}}-\tan^{-1}\frac{1}{x}=\tan^{-1}\frac{1}{3}$.

例題解自 5. 至 15.

(5) 令 $\tan^{-1}(x+1)=\theta, \tan^{-1}(x-1)=\phi$, 則 $\theta=3\phi$, 即

$\tan\theta=\tan 3\phi=\frac{3\tan\phi-\tan^3\phi}{1-3\tan^2\phi}$, 即 $x+1=\frac{3(x-1)-(x-1)^3}{1-3(x-1)^2}$,

簡之則 $x(x^2-2)=0$, 故 $x=0$, 或 $\pm\sqrt{2}$.

(6) 令 $\frac{2a}{1+a^2}=\sin\theta, \frac{2a}{1+b^2}=\sin\phi, x=\tan\psi$, 則

$\tan\theta=\frac{\sin\theta}{\sqrt{(1-\sin^2\theta)}}=\frac{2a}{1-a^2}, \tan\phi=\frac{2b}{1-b^2}$, 又 $\theta+\phi=2\psi$, 故

$\tan(\theta+\phi)=\tan 2\psi$, 即 $\frac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\frac{2\tan\psi}{1-\tan^2\psi}$ 以此代用上之各值

而變化之. 則 $\frac{a+b}{1-ab}\Big/\left\{1-\left(\frac{a+b}{1-ab}\right)^2\right\}=x\big/(1-x^2)$, 故 $x=\pm\frac{a+b}{1-ab}$.

(7) 令 $x+1=\tan\theta$, $x-1=\tan\phi$, $\frac{x}{2-x}=\tan\psi$ 則 $\tan 2\phi=\frac{2(x-1)}{x(2-x)}$, 故 $\tan 3\theta=\tan(2\phi+\psi)$, 卽 $\frac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=\frac{\tan 2\phi+\tan\psi}{1-\tan 2\phi\tan\psi}$, 以此代用上之各值而變化之. 則 $(x^2+2x-2)\{(x+1)(x^3-6x^2+6x)-(2-x)(3x^2+6x+2)\}=0$, 故從 $x^2+2x-2=0$ 得 $x=-1\pm\sqrt{3}$, 或從 { } 之因子得 $x=1\pm\sqrt{3}$ 或 $\pm\sqrt{-2}$.

(8) 令 $\frac{a}{x}=\cos\theta$, $\frac{b}{x}=\cos\phi$, $\frac{2b}{a}-1=\cos\psi$, 則 $\cos(\theta+\phi)=\cos\psi$, 解之. 代用上之各值. 則 $x=\pm\sqrt{ab}$.　　(9) 與前同樣. 得 $x=\pm\sqrt{a^2+b^2}$.

(10) 令 $2x=\sin\theta$, $x\sqrt{3}=\sin\phi$, $x=\sin\psi$ 而從 $\sin(\theta-\phi)=\sin\psi$ 求之. 則得 $x=0$, 或 $\pm\frac{1}{2}$,　　(11) 令 $x-1=\tan\alpha$, $x=\tan\beta$, $x+1=\tan\gamma$, $3x=\tan\delta$, 則 $\alpha+\beta+\gamma=\delta$, 卽 $\tan(\alpha+\beta)=\tan(\delta-\gamma)$ 解之. 代用上之各值. 則得 $x=\frac{1}{2}$ 或 $-\frac{1}{2}$ 或 0.

(12) 令 $\frac{1}{x-2}=\tan\alpha$, $\frac{1}{x}=\tan\beta$, $\frac{1}{x+2}=\tan\gamma$, $\frac{1}{x+3}=\tan\delta$, 則 $\tan(\alpha+\beta)=\frac{2x-2}{x^2-2x-1}$, $\tan(\gamma+\delta)=\frac{2x+5}{x^2+5x+5}$, $\alpha+\beta+\gamma+\delta=\frac{\pi}{4}$,

故 $\tan(\alpha+\beta)=\tan\left\{\frac{\pi}{4}-(\gamma+\delta)\right\}$ 解之. 代用上之各值而變化之. 則

$(x-5)(x^3+4x^2+x-4)=0$, 故 $x=5$.

(13) 令 $\frac{1}{4}=\tan\alpha$, $\frac{1}{5}=\tan\frac{\beta}{2}$, 故 $\tan\beta=\frac{5}{12}$, 又令 $\frac{1}{6}=\tan\gamma$, $\frac{1}{x}=\tan\theta$. 則可得 $\tan(\alpha+\beta+\gamma)=\frac{235}{226}$ 故從

$\alpha+\beta+\gamma=\frac{\pi}{4}$ 得 $\frac{235}{226}=\frac{1-\frac{1}{x}}{1+\frac{1}{x}}$, 故 $x=-\frac{461}{9}$.

(14) 令 $1/x=\tan\alpha$, $1/(a^2-x+1)=\tan\beta$, $1/(a-1)=\tan\gamma$. 則 $\tan(\alpha+\beta)=\tan\gamma$, 卽 $\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}=\tan\gamma$, 以此代用上之各值而簡單之. 則 $x^2-x(a^2+1)+a(a^2-a+1)=0$, 故 $x=a$, 或 a^2-a+1.

(15) 令 $\frac{1}{2+\sqrt{3}}=2-\sqrt{3}=\tan\frac{\alpha}{3}$, 則 $\tan\alpha=\frac{3\tan\frac{\alpha}{3}-\tan^3\frac{\alpha}{3}}{1-3\tan^2\frac{\alpha}{3}}=1$,

又 $\frac{1}{x}=\tan\theta$, $\frac{1}{3}=\tan\beta$, 故 $\alpha-\theta=\beta$, 卽 $\tan\theta=\tan(\alpha-\beta)$,

卽 $\frac{1}{x}=\left(1-\frac{1}{3}\right)\Big/\left(1+\frac{1}{3}\right)$. 故 $x=2$.

16. $sec^{-1}\frac{x}{a}+sec^{-1}a=sec^{-1}\frac{x}{b}+sec^{-1}b.$

17. $tan^{-1}x+\frac{1}{2}sec^{-1}5x=\frac{\pi}{4}.$

18. $cot^{-1}(x-1)-cot^{-1}(x+1)=\frac{\pi}{12}.$

19. $vers^{-1}x-vers^{-1}ax=vers^{-1}(1-a).$

20. $vers^{-1}(1+x)-vers^{-1}(1-x)=tan^{-1}(2\sqrt{1-x^2}).$

21. $cos^{-1}\frac{1-x^2}{1+x^2}+tan^{-1}\frac{2x}{1-x^2}=\frac{4}{3}\pi.$

22. $2sin^{-1}\frac{1}{\sqrt{(x^2-2x+2)}}=\frac{\pi}{6}+chord^{-1}\frac{1}{\sqrt{(x^2+2x+2)}}.$

但 *chord* 爲弧之弦.

23. $sin\,2cos^{-1}cot\,2tan^{-1}x=0.$

24. $2tan^{-1}(cos\,x)=tan^{-1}(3\,cosec\,x).$

25. $sin^{-1}x+sin^{-1}y=\frac{2}{3}\pi,\ cos^{-1}x-cos^{-1}y=\frac{\pi}{3}.$

例題解自 16. 至 25.

(16) 令 $sec^{-1}\frac{x}{a}=\theta,\ sec^{-1}a=\alpha,\ sec^{-1}\frac{x}{b}=\phi,\ sec^{-1}b=\beta$, 則

$\frac{a}{x}=cos\theta,\ \frac{1}{a}=cos\alpha,\ \frac{b}{x}=cos\phi,\ \frac{1}{b}=cos\beta$ 又 $\theta+\alpha=\phi+\beta$,

故 $cos\theta\,cos\alpha-sin\theta\,sin\alpha=cos\phi\,cos\beta-sin\phi\,sin\beta$, 即

$\frac{1}{x}-\sqrt{\left(1-\frac{a^2}{x^2}\right)\left(1-\frac{1}{a^2}\right)}=\frac{1}{x}-\sqrt{\left(1-\frac{b^2}{x^2}\right)\left(1-\frac{1}{b^2}\right)}$, 故 $x=\pm ab.$

(17) 令 $tan^{-1}x=\alpha,\ \frac{1}{2}sec^{-1}5x=\beta$, 則 $x=tan\,\alpha,\ 5x=sec\,2\beta$,

又 $\alpha+\beta=\frac{\pi}{4}$, 故 $5x=sec\left(\frac{\pi}{2}-2\alpha\right)=cosec\,2\alpha=\frac{sin^2\alpha+cos^2\alpha}{2sin\,\alpha\,cos\,\alpha}$,

即 $10x=tan\,\alpha+1/tan\,\alpha$, 即 $10x=x+1/x$, 故 $x=\pm\frac{1}{3}.$

(18) 令 $x-1=\cot\alpha, x+1=\cot\beta$, 則 $\alpha-\beta=\pi/12$, 故

$\cot(\alpha-\beta)=\cot\frac{\pi}{12}$, 即 $\frac{\cot\alpha\cot\beta+1}{\cot\beta-\cot\alpha}=2+\sqrt{3}$, 即 $\frac{x^2}{2}=2+\sqrt{3}$

故 $x=\pm\sqrt{(4+2\sqrt{3})}=\pm(\sqrt{3}+1)$.

(19) 令 $vers^{-1}x=\alpha, vers^{-1}ax=\beta, vers^{-1}(1-x)=\gamma$, 則

$\cos\alpha=1-x, \cos\beta=1-ax, \cos\gamma=a$, 又 $\alpha-\beta=\gamma$, 即 $\cos(\alpha-\gamma)=\cos\beta$,

即 $(1-x)a+\sqrt{(2x-x^2)(1-a^2)}=1-ax$, 即 $(2x-x^2)(1-a^2)=(1-a)^2$

由是 $x=1\pm\sqrt{2a/(1+a)}$。

(20) 令 $vers^{-1}(1+x)=\alpha, vers^{-1}(1-x)=\beta, \tan^{-1}(2\sqrt{1+x^2})=\gamma$, 則

$\cos\alpha=-x, \cos\beta=x, \tan\gamma=2\sqrt{1-x^2}$, 故 $\alpha+\beta=\pi$, 又 $\alpha-\beta=\gamma$, 故

$2\alpha=\pi+\gamma$, 即 $\cos2\alpha=-\cos\gamma$, 故 $(2\cos^2\gamma-1)^2=1/(1+\tan^2\gamma)$,

即 $(2x^2-1)^2=1/(5-4x^2)$, 即 $4x^6-(3x^2-1)^2=0, x=\pm1$, 或 $\pm\frac{1}{2}$.

(21) $\frac{1-x^2}{1+x^2}=\cos\alpha, \frac{2x}{1-x^2}=\tan\beta$, 則 $\cos\beta=\frac{1}{\sqrt{(1+\tan^2\beta)}}=\frac{1-x^2}{1+x^2}$,

故 $\alpha=\beta$, 又 $\alpha+\beta=\frac{4\pi}{3}$, 故 $\alpha=\frac{2\pi}{3}$, 故 $\frac{1-x^2}{1+x^2}=\cos\frac{2\pi}{3}=-\frac{1}{2}$,

故 $x=\pm\sqrt{3}$.

(22) $\frac{1}{\sqrt{(x^2-2x+2)}}=\sin\frac{\alpha}{2}, \frac{2}{\sqrt{(x^2+2x+2)}}=chord\,\beta=2\sin\frac{\beta}{2}$,

$\alpha=\frac{\pi}{6}+\beta$ 故 $\cos\left(\frac{\alpha}{2}-\frac{\beta}{2}\right)=\cos\frac{\pi}{12}$, 解此括弧.代入上之各值.且變化之.則

$x^2/\sqrt{(x^4+4)}=\frac{1}{2}\sqrt{(2+\sqrt{3})}$ 故 $x=\pm(\sqrt{3}+1)$.

(23) 令 $\tan^{-1}x=\theta$, 則 $x=\tan\theta, \cot2\theta=\frac{1-\tan^2\theta}{2\tan\theta}=\frac{1-x^2}{2x}$,

$\sin2\cos^{-1}\frac{1-x^2}{2x}=0$, 故 $2\cos^{-1}\frac{1-x^2}{2x}=n\pi, \frac{1-x^2}{2x}=\cos\frac{n\pi}{2}$,

故 $\frac{1-x^2}{2x}=0$, 或 ±1 故 $x=\pm1$, 或 $\pm1/\pm\sqrt{2}$.

(24) 令原式之各節爲θ,則 $\cos x=\tan\frac{\theta}{2}, 2\operatorname{cosec}x=\tan\theta$,

從此消去θ,即可得$x=\pi/4$.

(25) $x=\sin\theta=\cos\left(\frac{\pi}{2}-\theta\right), y=\sin\phi=\cos\left(\frac{\pi}{2}-\phi\right)$,

$\theta+\phi=\frac{2}{3}\pi, \left(\frac{\pi}{2}-\theta\right)-\left(\frac{\pi}{2}-\phi\right)=\frac{\pi}{3}$, 故 $\theta=\frac{\pi}{6}, \phi=\frac{\pi}{2}$.

26. $\sin\theta - \operatorname{cosec}\theta = \frac{4}{3}$.

27. $m\sec^2\theta\tan(a-\theta) = n\tan\theta\sec^2(a-\theta)$.

28. $\tan(\theta-a)\tan(\theta-\beta) = \tan^2\theta$.

29. $\cos^2\theta\cos^2 a + 4\cos\theta\sin(\theta-a)\sin^3 a = \sin^2 a\cos^2 a$.

30. $\operatorname{cosec}\theta + \operatorname{cosec}3\theta = 4\cos^2\frac{\theta}{2}\operatorname{cosec}2\theta$.

31. $\frac{\sin a\cos(\beta+\theta)}{\sin\beta\cos(a+\theta)} = \frac{\tan\beta}{\tan a}$.

32. $35\sin3\theta + 20\cos3\theta + 39\sin\theta - 20\cos\theta = 0$.

33. $(1+\cos a\cos\theta)\tan(\theta+a) + (\cos a+\cos\theta)\sin a = 0$.

34. $\cos7\theta + 7\cos\theta = 0$.

35. $\tan^2(a+\theta) - \tan^2(a-\theta) = \tan a\tan\theta$.

36. $\tan^2\theta = 2\tan a\tan\beta\sec\theta + \tan^2 a + \tan^2\beta$.

例題解自 26. 至 36.

(26) $\sin\theta - \cos\theta = \frac{4}{3}\sin\theta\cos\theta$, 雙方平方之, 則 $1-\sin2\theta = \frac{4}{9}\sin^2 2\theta$,

$\sin2\theta = \frac{3}{4}$, 故 $\theta = \frac{1}{2}\sin^{-1}\frac{3}{4}$,

(27) 試將原方程式變化之, 則 $m\sin(a-\theta)\cos(a-\theta) = n\sin\theta\cos\theta$, 即

$m\sin(2a-2\theta) = n\sin2\theta$, 故 $2\theta = n\pi + \tan^{-1}\left(\frac{m\sin2a}{m\cos2a+n}\right)$.

(28) $\frac{(\tan\theta-\tan a)(\tan\theta-\tan\beta)}{(1+\tan\theta\tan a)(1+\tan\theta\tan\beta)} = \tan^2\theta$, 去其分母而括之, 則

$\tan^2\theta - \tan\theta(\tan a+\tan\beta) + \tan a\tan\beta = \tan^2\theta + \tan^3\theta(\tan a+\tan\beta) + \tan^4\theta\tan a\tan\beta$,

即 $\tan a\tan\beta(1-\tan^4\theta) = \tan\theta(\tan a+\tan\beta)(1+\tan^2\theta)$,

故 $\frac{2\tan\theta}{1-\tan^2\theta} = \frac{2\tan a\tan\beta}{\tan a+\tan\beta}$, 即 $\tan2\theta = \frac{\sin a\sin\beta}{\sin(a+\beta)}$,

由是 $2\theta = n\pi + \tan^{-1}\frac{\sin a\sin\beta}{\sin(a+\beta)}$.

(29) $\cos^2\theta\cos^2 a + 4\cos\theta\sin^3 a(\sin\theta\cos a - \cos\theta\sin a) = \sin^2 a\cos^2 a$,

故 $\cos^2\theta(\cos^2 a - 4\sin^4 a) + 4\sin\theta\cos\theta\sin^3 a\cos a = \sin^2 a\cos^2 a$.

即 $\cos^2\alpha-4\sin^4\alpha+4\tan\theta\sin^3\alpha\cos\alpha=\sin^2\alpha\cos^2\alpha(1+\tan^2\theta)$,

即 $\tan^2\theta-4\tan\theta\tan\alpha+4\tan^2\alpha=\cot^2\alpha$, 故 $\tan\theta-2\tan\alpha=\pm\cot\alpha$,

由是 $\theta=n\pi+\tan^{-1}(2\tan\alpha\pm\cot\alpha)$.

(30) $\dfrac{\sin3\theta+\sin\theta}{\sin\theta\sin3\theta}=2(1+\cos\theta)\dfrac{1}{\sin2\theta}$, 即 $\dfrac{4\sin\theta-4\sin^3\theta}{\sin\theta(3\sin\theta-4\sin^3\theta)}=\dfrac{1+\cos\theta}{\sin\theta\cos\theta}$

即 $\dfrac{4\cos^2\theta}{3-4\sin^2\theta}=\dfrac{1+\cos\theta}{\cos\theta}$, 故 $4\cos^2\theta-\cos\theta=1$, 故 $\theta=2n\pi\pm\cos^{-1}\left(\dfrac{1\pm\sqrt{17}}{8}\right)$.

(31) 與例題28.同法, 得 $\theta=n\pi+\tan^{-1}\dfrac{\sin(\alpha+\beta)}{\sin\alpha\sin\beta}$.

(32) $35(3\sin\theta-4\sin^3\theta)+20(4\cos^3\theta-3\cos\theta)+39\sin\theta-20\cos\theta=0$, 即

$\sin\theta(144-140\sin^2\theta)-80\cos\theta(1-\cos^2\theta)=0$, 試化此爲最簡之式, 則

$1+35\cos^2\theta=20\sin\theta\cos\theta$, 雙方平方之而示 $\cos\theta$ 之項, 則

$1625\cos^4\theta-330\cos^2\theta+1=0$, 故 $\theta=2n\pi\pm\cos^{-1}\dfrac{1}{\sqrt{5}}$ 或 $2n\pi\pm\cos^{-1}\dfrac{1}{4\sqrt{13}}$.

(33) $\cos\theta\{\sin(\theta+\alpha)\cos\alpha+\cos(\theta+\alpha)\sin\alpha\}+\sin(\theta+\alpha)+\sin\alpha\cos\alpha\cos(\theta+\alpha)=0$,

即 $\cos\theta\sin(\theta+2\alpha)+\sin(\theta+\alpha)+\frac{1}{2}\sin2\alpha\cos(\theta+\alpha)=0$ 即

$\sin(2\theta+2\alpha)+\sin2\alpha+2\sin(\theta+\alpha)+\sin2\alpha\cos(\theta+\alpha)=0$ 即

$2\sin(\theta+\alpha)\{\cos(\theta+\alpha)+1\}+\sin2\alpha\{\cos(\theta+\alpha)+1\}=0$, 故

$2\sin(\theta+\alpha)=-\sin2\alpha$, 故 $\theta=n\pi-\alpha-(-1)^n\sin^{-1}(\sin2\alpha/2)$.

或 $\cos(\theta+\alpha)=-1$, 故 $\theta=(2n+1)\pi-\alpha$.

(34) $2\cos4\theta\cos3\theta+6\cos\theta=0$ 即 $(2\cos^22\theta-1)(4\cos^3\theta-3\cos\theta)+3\cos\theta=0$,

即 $(2\cos^22\theta-1)(2\cos2\theta-1)+3=0$, 故 $(1+\cos2\theta)(2\cos^22\theta-3\cos2\theta+2)=0$,

由是 $\cos2\theta=-1$, 即 $2\theta=(2n+1)\pi$.

或 $\cos2\theta=\dfrac{3\pm\sqrt{-7}}{4}$, 即 $2\theta=2n\pi\pm\cos^{-1}\dfrac{3\pm\sqrt{-7}}{4}$.

(35) 試變原方程式爲 $\dfrac{\sin2\theta\sin2\alpha}{\cos^2(\alpha+\theta)\cos^2(\alpha-\theta)}=\dfrac{\sin\theta\sin\alpha}{\cos\theta\cos\alpha}$, 故

$4\cos^2\theta\cos^2\alpha=(\cos^2\theta-\sin^2\alpha)^2$, 即 $\cos^4\theta-2\cos^2\theta(1+\cos^2\alpha)+\sin^4\alpha=0$,

故 $\cos\theta=1\pm\cos\alpha$, 故 $\theta=2n\pi\pm\cos^{-1}(1\pm\cos\alpha)$.

(36) 示原方程式爲 $\sec\theta$ 之項, 則

$\sec^2\theta-2\tan\alpha\tan\beta\sec\theta=\tan^2\alpha+\tan^2\beta+1$, 故

$\sec\theta=\tan\alpha\tan\beta\pm\sqrt{\tan^2\alpha\tan^2\beta+\tan^2\alpha+\tan^2\beta+1}$,

即 $\sec\theta=\tan\alpha\tan\beta\pm\sqrt{(1+\tan^2\alpha)(1+\tan^2\beta)}$,

由是 $\theta=2n\pi\pm\sec^{-1}(\tan\alpha\tan\beta\pm\sec\alpha\sec\beta)$,

37. $\sin(\pi\cos\theta)=\cos(\pi\sin\theta)$.

38. $\tan(n\cos\theta)=\cot(n\tan\theta)$.

39. $\sin(n\cot\theta)=\cos(n\sin\theta)$.

40. $p\sin^4\theta-q\sin^4\phi=p,\quad p\cos^4\theta-q\cos^4\phi=q$.

41. $\sin\theta+\cos\phi=a,\ \sin\phi+\cos\theta=b$.

42. $2m\cos\theta=a\sin(\theta+\phi)-b\sin(\theta-\phi)$,

$2n\cos\phi=a\sin(\theta+\phi)+b\sin(\theta-\phi)$.

43. $a\tan\phi=b\tan\theta,\ \sin(\theta+\phi)=a+b$.

44. $n\sin\theta-m\cos\theta=2n\sin\phi$.

$n\sin2\theta-m\cos2\phi=m$.

45. $\sin(\theta-\phi)=\frac{1}{10}$,

$$\frac{\sin^2\theta\cos^2\phi+\cos^2\theta\sin^2\theta}{\sin^2\theta\cos^2\phi-\cos^2\theta\sin^2\phi}=\frac{5}{3}.$$

例題解自 37 至 45.

(37) $\sin(\pi\cos\theta)=\sin\left(\frac{\pi}{2}-\pi\sin\theta\right)=\sin\left\{n\pi+(-1)^n\left(\frac{\pi}{2}-\pi\sin\theta\right)\right\}$,

故 $\cos\theta+(-1)^n\sin\theta=n+(-1)^n\frac{1}{2}$, 即 $\cos\left\{\theta-(-1)^n\frac{\pi}{4}\right\}=\frac{2n+(-1)^n}{2\sqrt2}$,

由是 $\theta=(-1)^n\frac{\pi}{4}+\cos^{-1}\frac{2n+(-1)^n}{2\sqrt2}$.

(38) $\tan(n\cot\theta)=\tan(p\pi+\frac{\pi}{2}-n\tan\theta)$, 故

$n(\tan\theta+\cot\theta)=\frac{(2p+1)\pi}{2}$, 即 $\sin2\theta=\frac{4n}{(2p+1)\pi}$,

由是 $\theta=\frac{m\pi}{2}+(-1)^m\frac{1}{2}\sin^{-1}\frac{4n}{(2p+1)\pi}$.

(39) 可參照前二例, $\theta=(-1)^p\frac{\pi}{4}+\cos^{-1}\frac{\{2p+(-1)^p\}\pi}{2n\sqrt2}$.

(40) 由第一及第二相減. $p(1-2\cos^2\theta)-q(1-2\cos^2\phi)=p-q$,

故 $\cos^2\phi=\frac{p}{q}\cos^2\theta$, 以此代入第二. 則 $p\cos^4\theta-q\left(\frac{p}{q}\cos^2\theta\right)^2=q$,

故 $\theta=\cos^{-1}\frac{\sqrt{q}}{\sqrt[4]{(pq-p^2)}}$, 又 $\phi=\cos^{-1}\sqrt[4]{\frac{p}{q-p}}$.

(41) 第一及第二各節.平方之相加.得 $2+2\sin(\theta+\phi)=a^2+b^2$,

$\theta+\phi=\sin^{-1}\frac{a^2+b^2-2}{2}$, (1). 又由減法 $\cos 2\phi-\cos 2\theta+2\sin(\theta-\phi)=a^2-b^2$,

即 $2\sin(\theta+\phi)\sin(\theta-\phi)+2\sin(\theta-\phi)=a^2-a^2$, 故

$\sin(\theta-\phi)=\frac{a^2-b^2}{2\sin(\theta+\phi)+2}=\frac{a^2-b^2}{a^2+b^2}$, 即 $\theta-\phi=\sin^{-2}\frac{a^2-b^2}{a^2+b^2}$, (2).

由 (1), (2) 即可得 θ, ϕ.

(42) 由兩方程式相加.得 $m\cos\theta+n\cos\phi=a\sin(\theta+\phi)$,

相減.得 $m\cos\theta-n\cos\phi=-b\sin(\theta-\phi)$,

上二式相乘. $m^2\cos^2\theta-n^2\cos^2\phi=-ab\sin(\theta+\phi)\sin(\theta-\phi)=-ab(\cos^2\phi-\cos^2\theta)$

故 $\frac{\cos\theta}{\cos\phi}=\sqrt{\frac{n^2-ab}{m^2-ab}}=k$ (1). 又由除法

$\frac{m\cos\theta+n\cos\phi}{m\cos\theta-n\cos\phi}=\frac{a\sin(\theta+\phi)}{-b\sin(\theta-\phi)}$, 即 $\frac{mk+n}{mk-n}=\frac{a(k\sin\phi+\sin\theta)}{-b(k\sin\phi-\sin\theta)}$,

故 $\frac{\sin\theta}{\sin\phi}=\frac{b(mk+n)+a(mk-n)}{b(mk+n)-a(mk-n)}k$, (2). 從 (1), (2) 即可得 θ, ϕ.

(43) 從第二 $\theta+\phi=n\pi+(-1)^n\sin^{-1}(a+b)$, (1). 又從第一

$\frac{\sin\theta\cos\phi}{\cos\theta\sin\phi}=\frac{a}{b}$, 即 $\frac{\sin(\theta+\phi)}{\sin(\theta-\phi)}=\frac{a+b}{\sin(\theta-\phi)}=\frac{a+b}{a-b}$. 故

$\theta-\phi=m\pi+(-1)^m\sin^{-1}(a-b)$, (2) 從 (1), (2) 即可得 θ, ϕ.

(44) 從第一 $\sin\phi=\frac{n\sin\theta-m\cos\theta}{2m}$, 從第二 $\cos^2\phi=\frac{n\sin\theta\cos\theta}{m}$,

故 $\frac{(n\sin\theta-m\cos\theta)^2}{4m^2}+\frac{n\sin\theta\cos\theta}{m}=1$, 由是

$\theta=\gamma\pi+\tan^{-1}\left\{\frac{2m}{n^2-4m^2}\left(\sqrt{n^2-3m^2}-\frac{n}{2}\right)\right\}$.

(45) 從第二 $\frac{\sin\theta\cos\phi}{\cos\theta\sin\phi}=\pm 2$, 故可如前例得

$2\theta=\sin^{-1}\frac{3}{10}+\sin^{-1}\frac{1}{10}$, 或 $\sin^{-1}\frac{1}{30}+\sin^{-1}\frac{1}{10}$.

46. 設 $\sin^2\theta+\sin^2\phi=\frac{1}{2}$, 則 $(2n+1)\frac{\pi}{2}$ 爲

$\sin^{-1}(\sin\theta+\sin\phi)+\sin^{-1}(\sin\theta-\sin\phi)$ 之一值.

47. $\tan^{-1}x+\cot^{-1}y=\tan^{-1}3$ 求其 x, y 之正整數.

48. 若 c 爲正整數, 則 $\tan^{-1}x+\tan^{-1}y=\tan^{-1}c$ 無正整數解答, 而同時 $\cot^{-1}x+\cot^{-1}y=\cot^{-1}c$, 則有 $1+c^2$ 整除個數之正整答數.

49. 試證, $2a$ 爲次式之一值

$$\cos^{-1}\frac{1-a^2\cos 2a-2a\sin a}{1+a^2-2a\sin a}-\cos^{-1}\frac{\cos 2a+2a\sin a-a^2}{1+a^2-2a\sin a}$$

50. $\alpha, \beta, \gamma, \cdots\cdots$ 爲 $x^n-p_1x^{n-1}+p_2x^{n-2}\cdots\cdots=0$ 之根, 則

$$\tan^{-1}\alpha+\tan^{-1}\beta+\tan^{-1}\gamma+\cdots\cdots=\tan^{-1}\frac{p_1-p_3+p_5-\cdots\cdots}{1-p_2+p_4-\cdots\cdots}$$

例題解自 46. 至 50.

(46) 令 $\sin\theta+\sin\phi=\sin x, \sin\theta-\sin\phi=\sin y$ 則

$\sin^2x+\sin^2y=2(\sin^2\theta+\sin^2\phi)=1$, 故 $x+y=(2n+1)\frac{\pi}{2}$.

(47) $\tan\left(\tan^{-1}x+\tan^{-1}\frac{1}{y}\right)=3$, 即 $\frac{x+1/y}{1-x/y}=3$, $xy+1=3y-3x$,

即 $(3-x)(3+y)=10$, 故 $3-x=1, 3+y=10$, 即 $x=2, y=7$,

或 $3-x=2, 3+y=5$, 即 $x=1, y=2$.

(48) $\tan(\tan^{-1}x+\tan^{-1}y)=c$, 即 $(x+y)/(1-xy)=c$,

即 $x=(c-y)/(1+cy)$, 但 c 爲正整數則令 y 爲正整數

$c-y$ 比 cy 小, 故 x 當爲分數. 故如題言.

又 $\cot(\cot^{-1}x+\cot^{-1}y)=c$, 卽 $\frac{xy-1}{x+y}=c$, 卽 $x=c+\frac{1+c^2}{y-c}$,

故 x正整數之數,爲$1+c^2$之整除數之數明甚.

(49) 令原式爲$x-y$, 則 $\cos x=\frac{1-a^2\cos 2\alpha-2a\sin\alpha}{1+a^2-2a\sin\alpha}$,

$\cos y=\frac{\cos 2\alpha+2a\sin\alpha-a^2}{1+a^2-2a\sin\alpha}$, 故 $\sin x=\frac{2a\cos\alpha(1-a\sin\alpha)}{1+a^2-2a\sin\alpha}$,

$\sin y=\frac{2\cos\alpha(a-\sin\alpha)}{1+a^2-2a\sin\alpha}$, 由是 $\cos(x-y)=\cos x\cos y+\sin x\sin y$

$$=\frac{1}{(1+a^2-2a\sin\alpha)^2}\Big\{(1-a^2\cos 2\alpha-2a\sin\alpha)(\cos 2\alpha+2a\sin\alpha-a^2)$$
$$+4a\cos^2\alpha(-a\sin\alpha)(a-\sin\alpha)\Big\}$$

$$=\frac{1}{(1+a^2-2a\sin\alpha)^2}\Big\{[(1-a\sin\alpha)^2-a^2\cos^2\alpha][\cos^2\alpha-(a-\sin\alpha)^2]$$
$$+4a\cos^2\alpha(1-a\sin\alpha)(a-\sin\alpha)\Big\}$$

$$=\frac{1}{(1+a^2-2a\sin\alpha)^2}\Big\{[\cos\alpha(1-a\sin\alpha)+a\cos\alpha(a-\sin\alpha)]^2$$
$$-[a\cos^2\alpha-(1-a\sin\alpha)(a-\sin\alpha)]^2\Big\}$$

$$=\frac{1}{(1+a^2-2a\sin\alpha)^2}\Big\{\cos^2\alpha(1+a^2-2a\sin\alpha)^2-\sin^2\alpha(1+a^2-2a\sin\alpha)^2\Big\}$$

$=\cos^2\alpha-\sin^2\alpha=\cos 2\alpha$, 故 $x-y=2\alpha$.

(50) 令 $\tan^{-1}\alpha=x$, $\tan^{-1}\beta=y$, $\tan^{-1}\gamma=z$,.........

又從方程式之性質, $\alpha+\beta+\gamma+\ldots\ldots=p_1$ $\alpha\beta+\alpha\gamma+\ldots\ldots+\beta\gamma+\ldots\ldots=p_2$,

$\alpha\beta\gamma+\ldots\ldots=p_3$,......... 故 $\tan(x+y)=\frac{\alpha+\beta}{1-\alpha\beta}=\frac{p_1}{1-p_2}$,

$$\tan(x+y+z)=\frac{\frac{\alpha+\beta}{1-\alpha\beta}+\gamma}{1-\left(\frac{\alpha+\beta}{1-\alpha\beta}\right)\gamma}=\frac{\alpha+\beta+\gamma-\alpha\beta\gamma}{1-(\alpha\beta+\alpha\gamma+\beta\gamma)}=\frac{p_1-p_3}{1-p_2},$$

同樣. $\tan(x+y+z+u)=\frac{p_1-p_3+p_5}{1-p_2+p_4}$,

故 $x+y+z+\ldots\ldots\ldots=\tan^{-1}\frac{p_1-p_3+p_5\ldots\ldots\ldots\ldots}{1-p_2+p_4\ldots\ldots\ldots}$.

由三角法解次列各方程式

51. $\sqrt{x(1-y)}+\sqrt{y(1-x)}=a,$

$\sqrt{xy}+\sqrt{(1-x)(1-y)}=b.$ 但 a, b 爲實數.

52. $\sqrt{x(1-y)}+\sqrt{y(1-x)}=a,$

$\sqrt{x(1-x)}+\sqrt{y(1-y)}=b,$ 但 a, b 爲實數.

53. $\frac{x+y}{1-xy}+\frac{1+xy}{x-y}=a,\quad \frac{x-y}{1+xy}+\frac{1-xy}{x+y}=b.$

54. $x=a\sqrt{1-z^2},\ y=b\sqrt{1-z^2},$

$z=\sqrt{(1-x^2)(1-y^2)}.$

55. $\frac{x+y}{\sqrt{(1+x^2)(1+y^2)}}=a,\quad \frac{1+xy}{\sqrt{(1+x^2)(1+y^2)}}=b.$

56. $x+2y-xy^2+(1-2xy-y^2)\sqrt{3}=0,$

$y+2x-x^2y+(1-2xy-x^2)(2+\sqrt{3})=0.$

例題解自 51. 至 56.

(51) 以 a, b 爲實數, x, y 比 1 小, 故得 $x=\sin^2\theta,$ $y=\sin^2\phi$, 故從原方程式 $\sin\theta\cos\phi+\cos\theta\sin\phi=a,$

卽 $\sin(\theta+\phi)=a,$ 及 $\sin\theta\sin\phi+\cos\theta\cos\phi=b.$

卽 $\cos(\theta-\phi)=b,$ 故 $\theta+\phi=\sin^{-1}a,\ \theta-\phi=\cos^{-1}b.$

由是 $\theta=\frac{1}{2}(\sin^{-1}a+\cos^{-1}b),\quad \phi=\frac{1}{2}(\sin^{-1}a-\cos^{-1}b),$

卽 $x=\sin^2\frac{1}{2}(\sin^{-1}a+\cos^{-1}b).\ y=\sin^2\frac{1}{2}(\sin^{-1}a-\cos^{-1}b).$

(52) 試如前例, 從第一, $\sin(\theta+\phi)=a,$ 從第二. $\sin\theta\cos\theta+\sin\phi\cos\phi=b,$ 卽 $\sin(\theta+\phi)\cos(\theta-\phi)=b,$

卽 $\cos(\theta-\phi)=b/a,$ 故 $\theta+\phi=\sin^{-1}a,\ \theta-\phi=\cos^{-1}b/a,$

故 $x=\sin^2\frac{1}{2}(\sin^{-1}a+\cos^{-1}b/a),\ y=\sin^2\frac{1}{2}(\sin^{-1}a-\cos^{-1}b/a).$

(53) 令 $x=\tan\theta, y=\tan\phi$, 則從第一, $\tan(\theta+\phi)+\cot(\theta-\phi)=a$

即 $\dfrac{\cos 2\phi}{\cos(\theta+\phi)\sin(\theta-\phi)}=a$, 即 $\sin 2\theta-\sin 2\phi=\dfrac{2\cos 2\phi}{a}$, 同樣.從第二,

$\sin 2\theta+\sin 2\phi=\dfrac{2\cos 2\phi}{b}$, 故 $\tan 2\phi=\dfrac{a-b}{ab}$,

又 $\sin 2\theta=\dfrac{a+b}{ab}\cos 2\phi=\dfrac{a+b}{ab\sqrt{(1+\tan^2 2\phi)}}=\dfrac{a+b}{\sqrt{a^2b^2+(a-b)^2}}$, 由是

$x=\tan\theta=\tan\left\{\dfrac{1}{2}\sin^{-1}\dfrac{a+b}{\sqrt{a^2b^2+(a-b)^2}}\right\}$, $y=\tan\left(\dfrac{1}{2}\tan^{-1}\dfrac{a-b}{ab}\right)$.

(54) 令 $x=\sin\theta, y=\sin\phi, z=\sin\psi$, 則原三方程式順次

爲 $\sin\theta=a\cos\psi, \sin\phi=b\cos\psi, \sin\psi=\cos\theta\cos\phi$,

故 $\cos^2\theta=1-a^2\cos^2\psi, \cos^2\phi=1-b^2\cos^2\psi, \cos^2\psi=1-\cos^2\theta\cos^2\phi$,

故 $\cos^2\psi=1-(1-a^2\cos^2\psi)(1-b^2\cos^2\psi)$, 故 $\cos^2\psi=\dfrac{a^2+b^2-1}{a^2b^2}$,

$\sin^2\psi=1-\dfrac{a^2+b^2-1}{a^2b^2}=\dfrac{(1-a^2)(1-b^2)}{a^2b^2}$, 由是 $z=\sin^{-1}\sqrt{\dfrac{(1-a^2)(1-b^2)}{a^2b^2}}$,

$x=\sin\theta=a\cos\psi=a\dfrac{\sqrt{(a^2+b^2-1)}}{a}$, $y=b\cos\psi=\dfrac{\sqrt{(a^2+b^2-1)}}{b}$.

(55) 令 $x=\tan\theta, y=\tan\phi$, 從原兩方程式,

$\sin(\theta+\phi)=a, \cos(\theta-\phi)=b$, 故 $2\theta=\sin^{-1}a+\sin^{-1}b$,

$2\phi=\sin^{-1}a-\sin^{-1}b$, 故 $x=\tan\dfrac{\sin^{-1}a+\sin^{-1}b}{2}, y=\tan\dfrac{\sin^{-1}a-\sin^{-1}b}{2}$.

(56) 令 $x=\tan\theta, y=\tan\phi$, 從第一,

$\tan\theta(1-\tan^2\phi)+2\tan\phi+(1-\tan^2\phi-2\tan\theta\tan\phi)\tan 60^\circ=0$,

即 $\tan\theta+\dfrac{2\tan\phi}{1-\tan^2\phi}+\left(1-\dfrac{2\tan\phi}{1-\tan^2\phi}\tan\theta\right)\tan 60^\circ=0$ 即

$\tan\theta+\tan 60^\circ+\tan 2\phi(1-\tan\theta\tan 60^\circ)=0$, 即

$\tan(\theta+60^\circ)=-\tan 2\phi$, (1), 又從第二,

$\tan\phi(1-\tan^2\theta)+2\tan\theta+(1-\tan^2\theta-2\tan\theta\tan\phi)\cot 15^\circ=0$,

即 $\cot(\phi-15^\circ)=\tan 2\theta$, (2)

從 (1) 及 (2), 得 $x=\dfrac{1}{\sqrt{3}}, y=1$, 或 $x=-\dfrac{1}{\sqrt{3}}, y=-2+\sqrt{3}$.

第拾編

極限

1. 極限 於代數式或三角函數式,其內之變數,以某一值代之,則其式有時爲 $\frac{0}{0}$,又變化其式,以前之某一值代入之,則得某有限值.如是者,稱此有限值,曰其式之極限.

例如 $\frac{x^2-1}{x-1}$, 令 $x=1$, 則 $\frac{1^2-1}{1-1}=\frac{0}{0}$, 而 $\frac{x^2-1}{x-1}=x+1$, 故令 $x=1$, 則可得 $x+1$ 爲 2.

故稱 $\frac{x^2-1}{x-1}$ 之極限爲 2. 記之如次, $lim_{x=1}\left(\frac{x^2-1}{x-1}\right)=2$

其 *lim* 爲 *limits* (極限) 之略字.而 $x=1$,則表此極限爲 2.

$\frac{0}{0}$ 之意義,本爲不定之形,以此形而等於 2,在初學者固不易解.然如下之說明,則直可得其解矣.

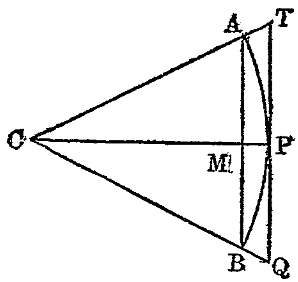
例如 $\frac{x^2-1}{x-1}$. 其 x,順次遞代以 3, 2, $1\frac{1}{2}$, $1\frac{1}{10}$, $1\frac{1}{100}$, $1\frac{1}{1000}$, …………, 而漸近於 1. 則其相應之 $\frac{x^2-1}{x-1}$ 之值,遞次等於 4, 3, $2\frac{1}{2}$, $2\frac{1}{10}$, $2\frac{1}{100}$, $2\frac{1}{1000}$, ……… 而漸近於 2.

故 $\frac{x^2-1}{x-1}=\frac{0}{0}$, 其等於 2, 非突然而得者. 乃由 $x=1\frac{1}{1000\ldots0}$ 至其極限等於 1 而得.

即 $\frac{x^2-1}{x-1}=2\frac{1}{1000\ldots0}$ 至其極限$=2$.

2. 定理壹 令比$\frac{\pi}{2}$更小之正角弧度為θ,則

$\sin\theta<\theta<\tan\theta$. 又$\theta$無限減小,則此各二量之比之極限等於1.



令單位圓之中心為C,平分弧AB於P,從P引切線與半徑CA, CB之引長線交於T, Q. 而AB與CP之交點為M. 則角ACB=2角ACM=2θ, 然AM=$\sin\theta$, 弧AP=θ, TP=$\tan\theta$, TQ即2TP為外切多角形之一邊. 而AB即2AM, 為與之相應之內切多角形之一邊. 由是與第一編8.節同理.

2AM<2弧AP<2TP, 即 $\sin\theta<\theta<\tan\theta$, (1)

因θ為比$\frac{\pi}{2}$小之正角. 故$\sin\theta$為正數. 故(1)以$\sin\theta$除之,則$1<\frac{\theta}{\sin\theta}<\frac{1}{\cos\theta}$, 於此不等式,其$\theta$為次第減小至於極限,令$\theta=0$, 則$\cos\theta=1$, 故

$1<\lim_{\theta=0}\left(\frac{\theta}{\sin\theta}\right)<1$, 即 $\lim_{\theta=0}\left(\frac{\sin\theta}{\theta}\right)=1$, (2)

又 $\frac{\theta}{\tan\theta}=\frac{\theta}{\sin\theta}\times\cos\theta$, 故 $\theta=0$, 則 $\cos\theta=1$,

故 $\lim_{\theta=0}\left(\frac{\theta}{\tan\theta}\right)=1$.

3. 推論壹 m無限增大,則$m\sin\frac{a}{m}$ 及 $m\tan\frac{a}{m}$之極限等於a.

$m\sin\frac{a}{m}=a\times\frac{\sin(a/m)}{a/m}$, 若 $m=\infty$, 則 $\frac{a}{m}=0$, 故 $\sin\frac{a}{m}=0$,

但由定理壹, $\frac{\sin(a/m)}{a/m}$ 之極限為1, 故

$\lim_{m=\infty}\left(m\sin\frac{a}{m}\right)=a$, 同樣, $\lim_{m=\infty}\left(m\tan\frac{a}{m}\right)=a$.

4. 推論貳 於定理一之角,以六十分法表之,令其角爲$x°$,則

$\frac{\sin x°}{x°}$ 及 $\frac{x°}{\tan x°}$ 之極限爲 $\frac{\pi}{180}$.

$\because\ \pi:\theta=180°:x°\quad\therefore\ x=180\theta/\pi$, 即 $\sin x°=\sin\theta$,

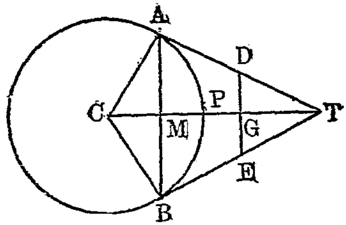
故 $\frac{\sin x°}{x°}=\frac{\sin\theta}{180\theta/\pi}=\frac{\pi}{180}\left(\frac{\sin\theta}{\theta}\right)$,

故由定理一, $\lim_{x=0}\left(\frac{\sin x°}{x°}\right)=\frac{\pi}{180}$.

5. 定理貳 令θ爲比$\frac{\pi}{2}$小之正角弧度,則

$$\tan\theta-\theta>\theta-\sin\theta,$$

弧 AP=弧 PB=θ, 平分 AT, BT 二切線於D及E,則DE與CT之交點G,必平分MT.



由是 AD+DE+BE>弧 APB,

即 2AD+2DG>2弧 AP,

即 AT+AM>2弧 AP,

即 $\tan\theta+\sin\theta>2\theta$, 故

$\tan\theta-\theta>\theta-\sin\theta$.

6. 定理三 θ爲比$\frac{\pi}{4}$小之正角弧度,則

$\sin\theta$ 在 θ 與 $\theta-\frac{\theta^2}{4}$ 之間, $\cos\theta$ 在 $1-\frac{\theta^2}{2}$ 與 $1-\frac{\theta^2}{2}+\frac{\theta^4}{16}$ 之間,

又 $\tan\theta$ 在 $\theta+\frac{\theta^3}{2}$ 與 $\theta+\frac{\theta^3}{4}$ 之間,

〔證〕 $\sin\theta=2\sin\frac{\theta}{2}\cos\frac{\theta}{2}=2\tan\frac{\theta}{2}\cos^2\frac{\theta}{2}$,

但 $\tan\frac{\theta}{2}>\frac{\theta}{2}$ (定理壹), 故 $\sin\theta>\theta\cos^2\frac{\theta}{2}$, 即 $>\theta\left(1-\sin^2\frac{\theta}{2}\right)$.

又 $\sin\frac{\theta}{2}<\frac{\theta}{2}$, 故 $\sin\theta>\theta\left(1-\frac{\theta^2}{4}\right)$, 即 $\sin\theta>\theta-\frac{\theta^3}{4}$.

由是 $\theta>\sin\theta>\theta-\frac{\theta^3}{4}$, (1)

又 $\cos\theta=1-2\sin^2\frac{\theta}{2}>1-2\left(\frac{\theta}{2}\right)^2$, 即 $\cos\theta>1-\frac{\theta^2}{2}$,

從 (1) $\sin\frac{\theta}{2}>\frac{\theta}{2}-\frac{1}{4}\left(\frac{\theta}{2}\right)^3$,

故 $\cos\theta=1-2\sin^2\frac{\theta}{2}<1-2\left\{\frac{\theta}{2}-\frac{1}{4}\left(\frac{\theta}{2}\right)^3\right\}^2<1-\frac{\theta^2}{2}+\frac{\theta^4}{16}$,

由是 $1-\frac{\theta^2}{2}+\frac{\theta^4}{16}>\cos\theta>1-\frac{\theta^2}{2}$, (2)

又 $\tan\theta=\frac{\sin\theta}{\cos\theta}<\frac{\theta}{1-\frac{\theta^2}{2}}$, 但 $\sin\theta<\theta$, (定理壹) 及從 (2),

知 $\tan\theta<\theta\left(1+\frac{\theta^2}{2}+\cdots\cdots\right)$, 即 $\tan\theta<\theta+\frac{\theta^3}{2}$,

從 (2) $\cos\theta<1-\frac{\theta^2}{2}+\frac{\theta^4}{16}$, 及從 (1) $\sin\theta>\theta-\frac{\theta^3}{4}$, 故

$\tan\theta>\frac{\theta-\frac{\theta^3}{4}}{1-\frac{\theta^2}{2}+\frac{\theta^4}{16}}$, 即 $\tan\theta>\theta+\frac{\theta^3}{4}$,

由是 $\theta+\frac{\theta^3}{2}>\tan\theta>\theta+\frac{\theta^3}{4}$, (3)

7. 定理四 θ 爲比 $\frac{\pi}{2}$ 小之正角弧度,則

$\sin\theta>\theta-\frac{\theta^3}{6}$, $\cos\theta<1-\frac{\theta^2}{2}+\frac{\theta^4}{24}$, $\tan\theta>\theta+\frac{\theta^3}{3}$.

〔證〕 $\sin\theta=3\sin\frac{\theta}{3}-4\sin^3\frac{\theta}{3}$,

故 $3\sin\frac{\theta}{3}=3^2\sin\frac{\theta}{3^2}-3\times4\sin^3\frac{\theta}{3^2}$,

$3^2\sin\frac{\theta}{3^2}=3^3\sin\frac{\theta}{3^3}-3^2\times4\sin^3\frac{\theta}{3^3}$,

....

$3^{n-1}\sin\frac{\theta}{3^{n-1}}=3^n\sin\frac{\theta}{3^n}-3^{n-1}\times4\sin^3\frac{\theta}{3^n}$.

此各相應邊相加,減去同類項,則

$\sin\theta=3^n\sin\frac{\theta}{3^n}-4\left\{\sin^3\frac{\theta}{3}+3\sin^3\frac{\theta}{3^2}+3^2\sin^3\frac{\theta}{3^3}+\cdots\cdots+3^{n-1}\sin^3\frac{\theta}{3^n}\right\}$,

但 $\sin\frac{\theta}{3}<\frac{\theta}{3}$, $\sin\frac{\theta}{3^2}<\sin\frac{\theta}{3^2}$, $\sin\frac{\theta}{3^3}<\frac{\theta}{3^3}$, ..., 故

$$\sin\theta > 3^n\sin\frac{\theta}{3^n} - 4\left\{\left(\frac{\theta}{3}\right)^3 + 3\left(\frac{\theta}{3^2}\right)^3 + 3^2\left(\frac{\theta}{3^3}\right)^3 + \cdots\cdots + 3^{n-1}\left(\frac{\theta}{3^n}\right)^3\right\},$$

故 $\sin\theta > \theta\left\{\dfrac{\sin(\theta/3^n)}{\theta/3^n}\right\} - \dfrac{4\theta^3}{27}\left(1 + \dfrac{1}{3^2} + \dfrac{1}{3^3} + \cdots\cdots + \dfrac{1}{3^{2n-2}}\right).$

$n=\infty$，則可達於極限，故

$\sin\theta > \theta - \dfrac{4\theta^3}{27}\left(\dfrac{1}{1-\frac{1}{9}}\right)$， 即 $\sin\theta > \theta - \dfrac{\theta^3}{6}$，

由是得 $\sin\dfrac{\theta}{2} > \dfrac{\theta}{2} - \dfrac{\theta^3}{48}$， 故

$\cos\theta = 1 - 2\sin^2\dfrac{\theta}{2} < 1 - 2\left(\dfrac{\theta}{2} - \dfrac{\theta^3}{48}\right)^2$， 即 $\cos\theta < 1 - \dfrac{\theta^2}{2} + \dfrac{\theta^4}{24}$，

由是 $\tan\theta > \dfrac{\theta - \theta^3/6}{1 - \theta^2/2 + \theta^4/24}$， 即 $\tan\theta > \theta + \dfrac{\theta^3}{4}$.

8. 定理五

θ 爲小於 $\dfrac{\pi}{2}$ 之正角弧度，則

$$\frac{3\sin\theta}{2+\cos\theta} < \theta < \frac{\tan\theta + 2\sin\theta}{3},$$

〔證〕 $\dfrac{\sin\theta}{2+\cos\theta} = \dfrac{2\sin\frac{1}{2}\theta\cos\frac{1}{2}\theta}{1+2\cos^2\frac{1}{2}\theta} = \dfrac{2\sin\frac{1}{2}\theta}{\sec\frac{1}{2}\theta + 2\cos\frac{1}{2}\theta}$， (1) 但於 2. 節之圖，CT=CA+AT，即 $\sec\theta = 1 + AT$，CM=CP−MP，即 $\cos\theta = 1 - MP$，

故 $\sec\theta + \cos\theta = 2 + AT - MP$，但 AT>MP，故 $\sec\theta + \cos\theta > 2$，

同樣. $\sec\dfrac{\theta}{2} + \cos\dfrac{\theta}{2} > 2$，及 $\sec\dfrac{\theta}{2} + 2\cos\dfrac{\theta}{2} > 2 + \cos\dfrac{\theta}{2}$， (2)

從 (1) 及 (2)，$\dfrac{\sin\theta}{2+\cos\theta} < 2\left(\dfrac{\sin\frac{1}{2}\theta}{2+\cos\frac{1}{2}\theta}\right) < 2^2\left(\dfrac{\sin\frac{1}{4}\theta}{2+\cos\frac{1}{4}\theta}\right) < \cdots\cdots$

$$< 2^n\left(\frac{\sin\frac{1}{2^n}\theta}{2+\cos\frac{1}{2^n}\theta}\right). \quad 即 \quad < \theta\left(\frac{\sin\frac{1}{2^n}\theta}{\frac{1}{2^n}\theta}\right)\left(\frac{1}{2+\cos\frac{1}{2^n}\theta}\right).$$

至於極限. 令 $n=\infty$，則由定理壹，即 $<\theta\,(1)\left(\dfrac{1}{2+1}\right)$， 即 $<\dfrac{\theta}{3}$

又 $\tan\theta + 2\sin\theta = \dfrac{\sin\theta(1+2\cos\theta)}{\cos\theta} = \dfrac{\sin\theta}{\cos\theta\cos^2\frac{1}{2}\theta}\left(4\cos^4\dfrac{\theta}{2} - \cos^2\dfrac{\theta}{2}\right),$

$$2\left(\tan\frac{\theta}{2}+2\sin\frac{\theta}{2}\right)=\frac{2\sin\frac{1}{2}\theta(1+2\cos\frac{1}{2}\theta)}{\cos\frac{1}{2}\theta}=\frac{\sin\theta}{\cos^2\frac{1}{2}\theta}\left(1+2\cos\frac{\theta}{2}\right)$$

$$=\frac{\sin\theta}{\cos\theta\cos^2\frac{1}{2}\theta}\left(4\cos^3\frac{\theta}{2}+2\cos^2\frac{\theta}{2}-2\cos\frac{\theta}{2}-1\right).$$

$$\left(4\cos^4\frac{\theta}{2}-\cos^2\frac{\theta}{2}\right)-\left(4\cos^3\frac{\theta}{2}+2\cos^2\frac{\theta}{2}-2\cos\frac{\theta}{2}-1\right)$$

$$=1+2\cos\frac{\theta}{2}-3\cos^2\frac{\theta}{2}-4\cos^3\frac{\theta}{2}+4\cos^4\frac{\theta}{2}$$

$$=\left(1-\cos\frac{\theta}{2}\right)\left(1+3\cos\frac{\theta}{2}\right)-4\cos^3\frac{\theta}{2}\left(1-\cos\frac{\theta}{2}\right)=\left(1-\cos\frac{\theta}{2}\right)\left(1+3\cos\frac{\theta}{2}-4\cos^3\frac{\theta}{2}\right)$$

$$=\left(1-\cos\frac{\theta}{2}\right)^2\left(1+2\cos\frac{\theta}{2}\right)^2=\text{正數}.$$

故 $4\cos^4\frac{\theta}{2}-\cos^2\frac{\theta}{2}>4\cos^3\frac{\theta}{2}+2\cos^2\frac{\theta}{2}-2\cos\frac{\theta}{2}-1,$

故 $\tan\theta+2\sin\theta>2\left(\tan\frac{\theta}{2}+2\sin\frac{\theta}{2}\right)>2^2\left(\tan\frac{\theta}{4}+2\sin\frac{\theta}{4}\right)>\cdots\cdots\cdots$

$$>2^n\left(\tan\frac{\theta}{2^n}+2\sin\frac{\theta}{2^n}\right),\ \text{即}\ >\theta\left\{\left(\frac{\tan\frac{\theta}{2^n}}{\frac{\theta}{2^n}}\right)+2\left(\frac{\sin\frac{\theta}{2^n}}{\frac{\theta}{2^n}}\right)\right\}$$

令 $n=\infty$, 則由定理叄, $>\theta(1+2)$ 即 $>3\theta$.

9. π 之界限 求法如次,

$\sin\frac{\pi}{20}=\frac{1}{4}(\sqrt{3+\sqrt{5}}-\sqrt{5-\sqrt{5}})=\cdot1564345,$ (例題六 16.)

$\cos\frac{\pi}{20}=\frac{1}{4}(\sqrt{3+\sqrt{5}}+\sqrt{5-\sqrt{5}})=\cdot9876883,$ (同 17.)

$\tan\frac{\pi}{20}=\sqrt{5}+1-\sqrt{(5+2\sqrt{5})}=\cdot1583844,$ (同 21.)

由前節定理五. $\frac{3\sin\theta}{2+\cos\theta}<\theta<\frac{\tan\theta+2\sin\theta}{3},$

即 $\frac{3(\cdot1564345)}{2+\cdot9876883}<\frac{\pi}{20}<\frac{\cdot1583844+2(\cdot1564345)}{3},$

即 $\frac{9\cdot3860700}{2\cdot9876883}<\pi<\frac{9\cdot425068}{3},$

即 $3\cdot14158<\pi<3\cdot14167.$

由是 π 之值. 可得至 3·141 之眞數,

而得 $\pi=3\cdot14159$, 或 $\pi=3\cdot1416$,

10. 正餘弦及正切之第壹略近值

θ 當極微小時，則 $\sin\theta$, θ, $\tan\theta$ 殆相等,(定理一)然 $\tan\theta-\theta>\theta-\sin\theta$, (定理二) 故僅令 $\sin\theta=\theta$, 其 $\tan\theta-\theta>0$. 惟令 $\tan\theta=\theta$, $\cos\theta=1$, 則能使 $\sin\theta=\theta$ 之略近值爲精密. 由是 $\sin\theta=\theta$ 不可不如次.

$\cos\theta=1-2\left(\sin\frac{\theta}{2}\right)^2$, 令 $\sin\theta=\theta$, 則 $\sin\frac{\theta}{2}=\frac{\theta}{2}$, 此較精密一層.

故 $\cos\theta=1-2\left(\frac{\theta}{2}\right)^2=1-\frac{\theta^2}{2}$, 故 $\tan\theta=\theta\Big/\left(1-\frac{\theta^2}{2}\right)=\theta\left(1+\frac{\theta^2}{2}+\frac{\theta^4}{4}+\cdots\cdots\right)$,

由是第一略近值 $\begin{cases}\sin\theta=\theta\\ \cos\theta=1-\frac{\theta^2}{2}\\ \tan\theta=\theta+\frac{\theta^3}{2}.\end{cases}$

11. 同第二略近值

由第一略近值,求其更密之略近值如次.

由定理一, $\sin\theta<\theta$, 卽 $\sin\frac{\theta}{2}<\frac{\theta}{2}$,

故 $\cos\theta=1-2\sin^2\frac{\theta}{2}<1-2\left(\frac{\theta}{2}\right)^2$, 卽 $\cos\theta<1-\frac{\theta^2}{2}$,

卽 $\left.\begin{matrix}\sin\theta<\theta\\ \cos\theta<1-\frac{\theta^2}{2}\end{matrix}\right\}$ (1). 由定理三, $\left.\begin{matrix}\sin\theta>\theta-\frac{\theta^3}{4}\\ \cos\theta<1-\frac{\theta^2}{2}+\frac{\theta^4}{16}\end{matrix}\right\}$ (2)

故從 (1) $\tan\theta<\dfrac{\theta}{1-\frac{\theta^2}{2}}<\theta\left(1+\frac{\theta^2}{2}+\frac{\theta^4}{4}+\cdots\cdots\right)$ 卽 $<\theta+\frac{\theta^3}{2}+\frac{\theta^5}{4}+\cdots\cdots\cdots$

又從 (2) $\tan\theta>\dfrac{\theta-\frac{\theta^3}{4}}{1-\frac{\theta^2}{2}+\frac{\theta^4}{16}}$, 卽 $>\theta+\frac{\theta^3}{4}+\cdots\cdots\cdots$

但可省略比 θ^3 更高之方乘,

故 $\tan\theta<\theta+\frac{\theta^3}{2}$, $\tan\theta>\theta+\frac{\theta^3}{4}$, 卽與定理三相符合.

由是第二略近值 $\begin{cases}\sin\theta=\theta-\frac{\theta^3}{4},\\ \cos\theta=1-\frac{\theta^2}{2},\\ \tan\theta=\theta-\frac{\theta^3}{3},\end{cases}$

12. 同第三略近值 由第二略近值,更求其精密之略近值如次.

由定理四, 第三略近值 $\begin{cases} \sin\theta=\theta-\dfrac{\theta^3}{6}, \\ \cos\theta=1-\dfrac{\theta^2}{2}, \\ \tan\theta=\theta+\dfrac{\theta^3}{3}, \end{cases}$

〔注意〕 單位圓之1°之弧 =·01745, 而 $\dfrac{(\cdot 01745)^4}{24}$ 所得之小數,其前六位爲0,故 $\theta \ngtr 1°$ 之弧,在定理四之證明中,於 $\cos\theta<1-\dfrac{\theta^2}{2}+\dfrac{\theta^4}{24}$,省略$\dfrac{\theta^4}{24}$亦可合於第三略近值至小數六位.

例題十二

$\theta=\dfrac{\pi}{2}$ 求下列各式之極限.

1. $\sec\theta-\tan\theta$. **2.** $\operatorname{cosec}\theta-\cot\theta$.

3. $(1+\sin\theta+\cos\theta)/(\sin\theta+\cos\theta-1)$.

例題解自1. 至 3.

(1) $\sec\theta-\tan\theta=\dfrac{1-\sin\theta}{\cos\theta}=\sqrt{\dfrac{11-\sin\theta}{1+\sin\theta}}$, 故令 $\theta=\dfrac{\pi}{2}$, 則

$$\frac{1-\sin\theta}{\cos\theta}=\frac{1-\sin\frac{\pi}{2}}{\cos\frac{\pi}{2}}=\frac{1-1}{0}=\frac{0}{0}, \text{ 又 } \sqrt{\frac{1-\sin\theta}{1+\sin\theta}}=\sqrt{\frac{1-1}{1+1}}=0.$$

由是 $\lim\limits_{\theta=\frac{\pi}{2}}(\sec\theta-\tan\theta)=0.$

(2) $\operatorname{cosec}\theta-\cot\theta=\dfrac{1-\cos\theta}{\sin\theta}=\sqrt{\dfrac{1-\cos\theta}{1+\cos\theta}}$, 故令 $\theta=\dfrac{\pi}{2}$, 則

$$\lim_{\theta=\frac{\pi}{2}}(\operatorname{cosec}\theta-\cot\theta)=\sqrt{\frac{1-0}{1+0}}=1.$$

(3) $\dfrac{1-\sin\theta+\cos\theta}{\sin\theta+\cos\theta-1}=\dfrac{\sqrt{(1-\sin^2\theta)}+(1-\sin\theta)}{\sqrt{(1-\sin^2\theta)}-(1-\sin\theta)}=\dfrac{\sqrt{(1+\sin\theta)}+\sqrt{(1-\sin\theta)}}{\sqrt{(1+\sin\theta)}-\sqrt{(1-\sin\theta)}}$,

令 $\theta=\dfrac{\pi}{2}$, 則 極限$=\dfrac{\sqrt{(1+1)}+\sqrt{(1-1)}}{\sqrt{(1+1)}-\sqrt{(1-1)}}=1.$

$x=0$, 求次各式之極限.

4. $\sin x''/x''$.

5. $(\tan x'+\tan x'')/x''$.

6. $\sin\alpha x^\circ/\sin\beta x^\circ$.

7. $\operatorname{vers}\alpha x^\circ/\operatorname{vers}\beta x^\circ$.

弧度 θ 爲 0, 求次各式之極限.

8. $(1-\cos\theta)/\theta^2$.

9. $(\theta-\sin\theta)/\theta^3$.

10. $(\tan\theta-\sin\theta)/\sin^3\theta$.

11. $\sin n\theta\cos(n-1)\theta/\sin\theta$.

12. $(\tan 2\theta-2\tan\theta)/\theta^3$.

13. $\sin 4\theta\cot\theta/(\operatorname{vers}2\theta\cot^2 2\theta)$.

14. $\dfrac{\cos\theta-\cos m\theta}{\cos\theta-\cos n\theta}$.

15. $\dfrac{\sin^{-1}\theta-\theta}{\sin^3\theta}$.

例題解自 4. 至 15.

(4) $\because\ \pi:\theta=180\times60\times60'':x''$, $\therefore\ x''=\dfrac{648000\cdot\theta}{\pi}$,

故由 4. 節推論二, 得 $\lim_{x=0}\left(\dfrac{\sin x''}{x''}\right)=\dfrac{\pi}{648000}$.

(5) 同樣. $\lim_{x=0}\left(\dfrac{\tan x'}{x'}+\dfrac{\tan x''}{x''}\right)=\dfrac{\pi}{180\times60}+\dfrac{\pi}{648000}=\dfrac{61\pi}{648000}$.

(6) $\dfrac{\sin\alpha x^\circ}{\sin\beta x^\circ}=\dfrac{\sin\dfrac{180\alpha\theta}{\pi}}{\sin\dfrac{180\beta\theta}{\pi}}=\dfrac{\alpha}{\beta}\cdot\left(\dfrac{\sin\dfrac{180\alpha\theta}{\pi}\Big/\dfrac{180\alpha\theta}{\pi}}{\sin\dfrac{180\beta\pi}{\pi}\Big/\dfrac{180\beta\pi}{\pi}}\right)$, 故令

$x=\theta=0$, 則 $\operatorname{Lim}_{x=0}\left(\dfrac{\sin\alpha x^\circ}{\sin\beta x^\circ}\right)=\dfrac{\alpha}{\beta}\left(\dfrac{1}{1}\right)=\dfrac{\alpha}{\beta}$.

(7) $\dfrac{\operatorname{vers}\alpha x^\circ}{\operatorname{vers}\beta x^\circ}=\dfrac{2\sin^2\dfrac{\alpha x^\circ}{2}}{2\sin^2\dfrac{\beta\alpha^\circ}{2}}=\left(\dfrac{\sin\dfrac{\alpha x^\circ}{2}}{\sin\dfrac{\beta x^\circ}{2}}\right)^2$, 故由前例, 得

$$\operatorname{Lim}_{x=0}\left(\frac{\operatorname{vers}\alpha x^\circ}{\operatorname{vers}\beta x^\circ}\right)=\left(\frac{\alpha}{\beta}\right)^2.$$

(8) $\dfrac{1-\cos\theta}{\theta^2}=\dfrac{2\sin^2\dfrac{\theta}{2}}{\theta^2}=\dfrac{1}{2}\left(\dfrac{\sin\dfrac{\theta}{2}}{\dfrac{\theta}{2}}\right)^2$, 故由定理一, 得

$$lim_{x=0}\left(\frac{1-\cos\theta}{\theta^2}\right)=\tfrac{1}{2}(1)^2=\tfrac{1}{2}.$$

(9) $\frac{\theta-\sin\theta}{\theta^3}=\left\{\theta-\left(\theta-\frac{\theta^3}{6}\right)\right\}\Big/\theta^3$ (略近值) 由 12. 節

故 $\frac{\theta-\sin\theta}{\theta^3}=\frac{1}{6}$, 即 $lim_{\theta=0}\left(\frac{\theta-\sin\theta}{\theta^3}\right)=\frac{1}{6}.$

(10) $\frac{\tan\theta-\sin\theta}{\sin^3\theta}=\frac{1-\cos\theta}{\cos\theta\sin^2\theta}=\frac{1}{\cos\theta(1+\cos\theta)},$

故 $lim_{\theta=0}\left(\frac{\tan\theta-\sin\theta}{\sin^3\theta}\right)=\frac{1}{\cos\theta(1+\cos\theta)}=\frac{1}{2}.$

(11) $\frac{\sin n\theta\cos(n-1)\theta}{\sin\theta}=n\left(\frac{\sin n\theta/(n\theta)}{\sin\theta/\theta}\right)\cos(n-1)\theta,$

故 $Lim_{\theta=0}\left(\frac{\sin n\theta\sin(n-1)\theta}{\sin\theta}\right)=n\left(\frac{1}{1}\right)\cos\theta=n.$

(12) $\frac{\tan 2\theta-2\tan\theta}{\theta^3}=\frac{2\theta+\frac{(2\theta)^3}{3}-2\left(\theta+\frac{\theta^3}{3}\right)}{\theta^3}$ (略近值) 由 12. 節

故令 $\theta=0$, 則 極限=2.

(13) $\frac{\sin 4\theta\cot\theta}{\text{vers}\,2\theta\cot^2 2\theta}=\frac{\sin 4\theta\tan^2 2\theta}{2\sin^2\theta\tan\theta}=\frac{8(\sin 4\theta/4\theta)(\tan 2\theta/2\theta)^2}{(\sin\theta/\theta)^2(\tan\theta/\theta)},$

令 $\theta=0$, 則 極限=8.

(14) $\frac{\cos\theta-\cos m\theta}{\cos\theta-\cos n\theta}=\frac{2\sin\frac{1}{2}(m+1)\theta\sin\frac{1}{2}(m-1)\theta}{2\sin\frac{1}{2}(n+1)\theta\sin\frac{1}{2}(n-1)\theta}$

$$=\frac{\left\{\frac{\sin\frac{1}{2}(m+1)\theta}{\frac{1}{2}(m+1)\theta}\right\}\left\{\frac{\sin\frac{1}{2}(m-1)\theta}{\frac{1}{2}(m-1)\theta}\right\}(m+1)(m-1)}{\left\{\frac{\sin\frac{1}{2}(n+1)\theta}{\frac{1}{2}(n+1)\theta}\right\}\left\{\frac{\sin\frac{1}{2}(n-1)\theta}{\frac{1}{2}(n-1)\theta}\right\}(n+1)(n-1)},$$ 故 極限$=\frac{m^2-1}{n^2-1},$

(15) 令 $\sin^{-1}\theta=x$, 則 $\theta=\sin x$, 故由 12. 節,

原式$=\frac{x-\sin x}{\sin^3\sin x}=\frac{x-\left(x-\frac{x^3}{6}\right)}{\left(\sin x-\frac{\sin^3 x}{6}\right)^3}=\frac{\frac{x^3}{6}}{\sin^3 x\left(1-\frac{\sin^2 x}{6}\right)^3}=\frac{\frac{1}{6}(x/\sin x)^3}{\left(1-\frac{\sin^2 x}{6}\right)^3},$

令 $\theta=0, x=0$, 故由定理一. 極限$=\frac{1}{6}.$

16. $\dfrac{\theta+\sin\theta-\sin2\theta}{2\theta+\tan\theta-\tan3\theta}$.

17. $\dfrac{\theta+\tan\theta-\tan2\theta}{2\theta+\tan\theta-\tan3\theta}$.

18. $\dfrac{(2\sin\theta-\sin2\theta)^2}{(\sec\theta-\cos2\theta)^3}$.

19. $\dfrac{\left(\theta+\sin2\theta-6\sin\frac{\theta}{2}\right)^2}{\left(4+\cos\theta-5\cos\frac{\theta}{2}\right)}$.

20. $\dfrac{m\sin\theta-\sin m\theta}{\theta(\cos\theta-\cos m\theta)}$.

21. $\dfrac{\tan\sin\theta-\sin\tan\theta}{\theta^7}$.

22. $\sin\frac{\theta}{2}\cos2\theta/(\mathrm{vers}\,\theta\cot\theta)$.

23. $\tan^3\theta/(\sec2\theta-1)$.

n 爲正整數,至無限增大,求次各式之極限.

24. $\cos\frac{a}{3}\cos\frac{a}{4}\cos\frac{a}{8}\cdots\cdots\cos\frac{a}{2^n}$

25. $\left(1-\tan^2\frac{a}{2}\right)\left(1-\tan^2\frac{a}{4}\right)\left(1-\tan^2\frac{a}{8}\right)\cdots\cdots\left(1-\tan^2\frac{a}{2^n}\right)$.

例題解自 16. 至 25.

(16) 由定理四,原式 $=\dfrac{\theta+\theta-\frac{\theta^3}{6}-\left(2\theta-\frac{8\theta^3}{6}\right)}{2\theta+\theta+\frac{\theta^3}{3}-\left(3\theta+\frac{27\theta^3}{3}\right)}=-\dfrac{7}{52}$, 故極限 $=-\dfrac{7}{52}$.

(17) 同上,得 $\dfrac{7}{26}$.

(18) $\dfrac{(2\sin\theta-\sin2\theta)^2}{(\sec\theta-\cos2\theta)^3}=\dfrac{4\sin^2\theta(1-\cos\theta)^2}{(1+\cos\theta-2\cos^3\theta)^3/\cos^3\theta}=\dfrac{4(1-\cos^2\theta)(1-\cos\theta)^2\cos^3\theta}{(1-\cos\theta)^3(1+2\cos\theta+2\cos^2\theta)^3}$

$=\dfrac{4(1+\cos\theta)\cos^3\theta}{(1+2\cos\theta+2\cos^2\theta)^3}$, 故 $\lim_{\theta=0}\dfrac{(2\sin\theta-\sin2\theta)^2}{(\sec\theta-\cos2\theta)^3}=\dfrac{4(1+1)}{(1+2+2)^3}=\dfrac{8}{125}$.

(19) 由定理四, 原式 $=\dfrac{\left\{\theta+2\theta-\frac{8\theta^3}{6}-6\left(\frac{\theta}{2}-\frac{1}{6}\cdot\frac{\theta^3}{8}\right)\right\}^2}{\left\{4+1-\frac{\theta^2}{2}-5\left(1-\frac{1}{2}\cdot\frac{\theta^2}{4}\right)\right\}}=8\left(\dfrac{92}{3}\right)^2=$ 極限.

(20) 同上. 原式 $=\dfrac{m\left(\theta-\frac{\theta^3}{6}\right)-\left(m\theta-\frac{m^3\theta^3}{6}\right)}{\theta\left\{1-\frac{\theta^2}{2}-\left(1-\frac{m^2\theta^2}{2}\right)\right\}}=\dfrac{m^3-m}{3(m^2-1)}=\dfrac{m}{3}=$ 極限.

(21) 同上, 原式 $=\left\{\tan\left(\theta-\frac{\theta^3}{6}\right)-\sin\left(\theta+\frac{\theta^3}{3}\right)\right\}\Big/\theta^7$

$=\dfrac{\theta-\frac{\theta^3}{6}+\frac{1}{3}\left(\theta-\frac{\theta^3}{6}\right)^3-\left\{\theta+\frac{\theta^3}{3}-\frac{1}{6}\left(\theta+\frac{\theta^3}{3}\right)^3\right\}}{\theta^7}=\dfrac{1}{12}+\dfrac{\theta^2}{216}$, ∴ 極限 $=\dfrac{1}{12}$.

(22) $\dfrac{\sin\frac{\theta}{2}\cos 2\theta}{vers\,\theta\cot\theta}=\dfrac{\sin\frac{\theta}{2}\cos 2\theta\sin\theta}{(1-\cos\theta)\cos\theta}=\dfrac{2\sin^2\frac{\theta}{2}\cos\frac{\theta}{2}\cos 2\theta}{2\sin^2\frac{\theta}{2}\cos\theta}=\dfrac{\cos\frac{\theta}{2}\cos 2\theta}{\cos\theta}$,

故 $\lim_{\theta=0}\left(\dfrac{\sin\frac{\theta}{2}\cos 2\theta}{vers\,\theta\cot\theta}\right)=\dfrac{\cos 0\cos 0}{\cos 0}=1$.

(23) $\dfrac{\tan^2\theta}{\sec 2\theta-1}=\dfrac{\sin^2\theta\cos 2\theta}{\cos^2\theta(1-\cos 2\theta)}=\dfrac{\sin^2\theta\cos^2\theta}{2\cos^2\theta\sin^2\theta}=\dfrac{\cos^2\theta}{2\cos^2\theta}$,

由是 極限$=\dfrac{1}{2}$

(24) 由例題四 115,

$\cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\cos\frac{\alpha}{8}\cdots\cdots\cos\frac{\alpha}{2^n}=\dfrac{\sin\alpha}{2^{n+1}\sin\frac{\alpha}{2^{n+1}}}=\left(\dfrac{\frac{\alpha}{2^{n+1}}}{\sin\frac{\alpha}{2^{n+1}}}\right)\dfrac{\sin\alpha}{\alpha}$,

故 $n=\infty$, 由定理一, 得極限$=\dfrac{\sin\alpha}{\alpha}$.

(25) $1-\tan^2\frac{\alpha}{2}=\dfrac{\cos\alpha}{\cos^2\frac{\alpha}{2}}$, 故

原式$=\dfrac{\cos\alpha\cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\cdots\cdots\cos\frac{\alpha}{2^{n+1}}}{\left(\cos\frac{\alpha}{2}\cos\frac{\alpha}{4}\cos\frac{\alpha}{8}\cdots\cdots\cos\frac{\alpha}{2^n}\right)^2}$ 故由前例 $n=\infty$, 則

原式之極限$=\dfrac{\sin 2\alpha/(2\alpha)}{(\sin\alpha/\alpha)^2}=\dfrac{\alpha}{\tan\alpha}$.

〔註〕 極限之問題,原式爲$\dfrac{0}{0}$,求與此相當之有限值,已於1.節言之矣,評 $\lim_{x=1}\left(\dfrac{x^2-1}{x-1}\right)=2$, 則 $\dfrac{0}{0}=2$, 而略言 $x=1$,究非正確之論,必云x漸近於1殆等於1時,則極限爲2.

以上例題.於此點,望初學者注意.

π之值 於例題24. 令 $\alpha=\dfrac{\pi}{2}$, 則

$\cos\frac{\pi}{4}\cos\frac{\pi}{8}\cos\frac{\pi}{16}\cdots\cdots$至$\infty=\dfrac{\sin(\pi/2)}{\pi/2}$,

即 $\dfrac{\sqrt{2}}{2}\cdot\dfrac{\sqrt{2+\sqrt{2}}}{2}\cdot\dfrac{\sqrt{2+\sqrt{2+\sqrt{2}}}}{2}\cdots\cdots=\dfrac{2}{\pi}$,

故 $\pi=2\cdot\dfrac{2}{\sqrt{2}}\cdot\dfrac{2}{\sqrt{2+\sqrt{2}}}\cdot\dfrac{2}{\sqrt{2+\sqrt{2+\sqrt{2}}}}\cdots\cdots\cdots$

26. θ 爲甚小之弧度,則

$$3\theta=2(\sin 2\theta-\sin\theta)+\tan 2\theta-\tan\theta.$$ (略近值)

27. $\cos\theta/\theta+\theta/\cos\theta$ 有最小正數值,則 $\theta>\sqrt{3}-1$.

28. θ 爲比 $\frac{\pi}{2}$ 小之正角,則 $\sin\theta>\tan\theta-\frac{\tan^3\theta}{2}$.

29. θ 爲甚小之弧度,省其立方乘及

$\cos(\alpha-\theta)=\cos\alpha\cos\beta$, 則

$$\theta=-2\cot\alpha\sin^2\frac{\beta}{2}\left(1-\cot^2\alpha\sin^2\frac{\beta}{2}\right).$$ (略近值)

30. θ 及 ϕ 爲甚小之弧度,省其立方乘及

$\cot\alpha\sin(\beta+\theta)=\cot\alpha\sin\beta\cos\phi-\cos\beta\sin\phi$, 則

$$\phi=-\theta\cot\alpha+\frac{\theta^2}{2}\cot\alpha\tan\beta(1-\cot^2\alpha).$$ (略近值)

例題解自 26. 至 30.

(26) 由定理二, $2\theta<\sin\theta+\tan\theta$, 由定理一, $\theta>\sin\theta$,

故 $3\theta=2\sin\theta+\tan\theta$ (略近值), 同樣, $6\theta=2\sin 2\theta+\tan 2\theta$, (略近值)

由是 $3\theta=2(\sin 2\theta-\sin\theta)+\tan 2\theta-\tan\theta$.

(27) $\because$ $\frac{\cos\theta}{\theta}+\frac{\theta}{\cos\theta}=2+\left(\sqrt{\frac{\cos\theta}{\theta}}-\sqrt{\frac{\theta}{\cos\theta}}\right)^2$, 故此式最小之値爲 2,

即 $\frac{\cos\theta}{\theta}+\frac{\theta}{\cos\theta}=2$, 即 $\theta=\cos\theta$.

即 $\theta=1-\frac{\theta^2}{2}$, (12. 節) 故 $\theta=\sqrt{3}-1$.

(28) $\sin\theta-(\tan\theta-\tan^3\theta/2)=\sin\theta-\frac{\sin\theta}{\cos\theta}+\frac{\sin^3\theta}{2\cos^3\theta}$

$=\frac{\sin\theta}{\cos\theta}\left(\cos\theta-1+\frac{\sin^2\theta}{2\cos^2\theta}\right)=\frac{\sin\theta}{\cos\theta}(1-\cos\theta)\left(-1+\frac{1+\cos\theta}{2\cos^2\theta}\right)$

$$=\frac{\sin\theta}{2\cos^3\theta}(1-\cos\theta)(1+\cos\theta-2\cos^2\theta)=\frac{\sin\theta}{2\cos^2\theta}(1-\cos\theta)^2(1+2\cos\theta),$$

但依題意, $\cos\theta$ 爲正, 故此式爲正, 故如題言.

〔別解〕 由 12 節 $\sin\theta=\theta-\frac{\theta^3}{6}$,

$$\tan\theta-\frac{\tan^3\theta}{2}=\theta+\frac{\theta^3}{2}-\frac{1}{2}\left(\theta+\frac{\theta^3}{3}\right)^3=\theta-\frac{\theta^3}{6}-\frac{\theta^5}{2}-\frac{\theta^7}{6}-\frac{\theta^9}{54},$$

$$\therefore\ \sin\theta>\tan\theta-\frac{\tan^3\theta}{2}.$$

(29) 由既知之關係式 $\cos\alpha\cos\theta+\sin\alpha\sin\theta=\cos\alpha\cos\beta$, 依題意, 用12節且去 θ^3, 則 $\cos\alpha\left(1-\frac{\theta^2}{2}\right)+\theta\sin\alpha=\cos\alpha\cos\beta$,

即 $2\theta\sin\alpha-\theta^2\cos\alpha=-4\cos\alpha\sin^2\frac{\beta}{2}$, 取 θ 之一次略近值 (省略 θ^2),

則 $2\theta\sin\alpha=-4\cos\alpha\sin^2\frac{\beta}{2}$, 即 $\theta=-2\cot\alpha\sin^2\frac{\beta}{2}$, 以此代入前方程式之 θ^2,

則 $2\theta\sin\alpha-4\cot^2\alpha\sin^4\frac{\beta}{2}\cos\alpha=-4\cos\alpha\sin^2\frac{\beta}{2}$,

由是 $\theta=-2\cot\alpha\sin^2\frac{\beta}{2}\left(1-\cot^2\alpha\sin^2\frac{\beta}{2}\right)$.

(30) 從既知關係式

$$\cot\alpha\left\{\sin\beta\left(1-\frac{\theta^2}{2}\right)+\theta\cos\beta\right\}=\cot\alpha\sin\beta\left(1-\frac{\phi^2}{2}\right)-\phi\cos\beta,$$

即 $\phi=-\theta\cot\alpha-\frac{1}{2}(\theta^2-\phi^2)\cot\alpha\tan\beta$, 取 θ, ϕ 之一次略近值, 省略 ϕ^2, 則 $\phi=-\theta\cot\alpha-\frac{\theta^2}{2}\cot\alpha\tan\beta$, 以此代入前之方程式之 ϕ^2. 略省 θ^3 以上之項, 則

$$\phi=-\theta\cot\alpha-\tfrac{1}{2}(\theta^2-\theta^2\cot^2\alpha)\cot\alpha\tan\beta$$

$$=-\theta\cot\alpha+\frac{\theta^2}{2}\cot\alpha\tan\beta(1-\cot^2\alpha).$$

一般角之三角函數值

13. 十秒之正餘弦 $10''$ 之弧度$=\frac{10\pi}{180\times60\times60}=\frac{\pi}{64800}$,

故 $\sin 10''<\frac{\pi}{64800}$, (定理一), 又 $\sin 10''>\frac{\pi}{64800}-\frac{1}{4}\left(\frac{\pi}{64800}\right)^3$, (11 節)

試用 π 之略近值 $=3\cdot1415926535 89793\ldots\ldots\ldots$, 則

$\frac{\pi}{64800}=\cdot000048481368110\ldots\ldots\ldots$ 故 $\sin 10''$ 比此數小,

而 $\frac{\pi}{64800}<\cdot00005$, 故

$\sin 10''>\cdot000048481368110\ldots\ldots\ldots-\frac{1}{4}(\cdot00005)^3$,

即 $\sin 10''>\cdot000048481368078\ldots\ldots\ldots$ 由是

$\cdot000048481368110\ldots\ldots\ldots>\sin 10''>\cdot000048481368078\ldots\ldots\ldots$

故知 $\sin 10''=\cdot000048481368\ldots\ldots\ldots$, 而此值正合眞值至小數十二位, 故其誤差在 $\frac{1}{10^{12}}$ 以下, 即在一兆分之一以下.

又 $\cos 10''=\sqrt{(1-\sin^2 10'')}=\cdot9999999988248\ldots\ldots\ldots$

14. 秒之弧度及正弦 由前節知 $\sin 10''$ 與 $10''$ 之弧度, 合小數十二位, 故 $\sin 1''=1''$ 之弧度 (略近值),

由是 $n \not> 10$, 得表之如次,

$\sin n''=n''$ 之弧度$=n\times1''$ 之弧度$=n\sin 1''$, (略近值)

$\therefore\ n=\frac{n'' \text{之弧度}}{\sin 1''}$, (略近值)

15. 若干秒之正弦值 有 $10''$ 爲通差之等差級數, 計算其所成角之正弦如次.

令 α 爲任意之角, 則

$$\sin(n+1)\alpha+\sin(n-1)\alpha=2\sin n\alpha\cos\alpha.$$

令 $2\cos\alpha=2-k$, 則 $\sin(n+1)\alpha+\sin(n-1)\alpha=(2-k)\sin n\alpha$.

故 $\sin(n+1)\alpha-\sin n\alpha=\sin n\alpha-\sin(n-1)\alpha-k\sin n\alpha$,

今令 $\alpha=10''$, 則由 13 節已知其 $\sin\alpha$ 及 $\cos\alpha$ 矣,

故 $k=2-2\cos 10''=\cdot 0000000023504\ldots\ldots\ldots\ldots$

而 $n=1$, 則 $\sin 20''-\sin 10''=\sin 10''-k\sin 10''$,

故得 $\sin 20''$, 又令 $n=2$, 則可得 $\sin 30''$, $n=3$ 以下, 可依次得知 $\sin 40''$ 等之值. 以 k 之值計算, 可得達其充分之目的.

16. 餘論 計算自 0° 至 45° 各角之正弦, 則 45° 以上之正弦, 由次之公式, 容易求得.

$$\sin(45°+d)-\sin(45°-d)=2\cos 45°\sin d=\sqrt{2}\cdot\sin d.$$

計算 $\sqrt{2}$ 之小數, 頗覺煩雜, 今爲避此煩雜起見, 特求得 0° 至 60° 各角之正弦, 則 60° 以上之正弦, 用次之公式計算, 更覺容易.

$$\sin(60°+d)-\sin(60°-d)=2\cos 60°\sin d=\sin d.$$

又求得正切如次.

$$\tan(45°+d)-\tan(45°-d)=2\tan 2d.$$

又餘割 $\operatorname{cosec}A=\frac{1}{2}\left(\tan\frac{A}{2}+\cot\frac{A}{2}\right)$.

例題十三

1. 求 $\sin 1'$ 及 $\cos 1'$ 之略近值, 至小數八位止.

2. 6° 之弧度爲 θ, $\cos 6°$ 之略近值爲 $1-\frac{\theta^2}{2}+\frac{\theta^4}{24}$, 試以 $1-\frac{\theta^2}{2}$ 代其略近值, 有若干位小數相合.

例題解自1至2.

(1) 1′ 之弧度 $=\frac{3\cdot 1416}{180\times 60}=\cdot 00029088=\theta$, 用 $\sin 1'=\theta-\frac{\theta^2}{2}$, 則

$\frac{\theta^3}{6}$ 至小數十位爲 0, 故 $\sin 1'=\theta=\cdot 00029088$.

又 $\cos 1'=1-\frac{1}{2}(\cdot 0002909)^2=\cdot 99999995$.

(2) 6° 之弧度 $=\frac{6\times 3\cdot 1415926}{180}=\cdot 10471975=\theta$, $1-\frac{\theta^2}{2}=\cdot 9945$, 即至四位相合.

3. 求 $\tan 46^\circ$ 之值.

4. $19\sin\theta=\frac{2165}{2166}$, 則 θ 殆等於半徑角十六分之一.

5. $\cos\theta=\theta$, 求 θ 之略近值.

6. $\cot\theta=\theta$, 則 θ 殆等於 $49^\circ\ 17'$.

例題解自 3. 至 6.

(3) 1° 之弧度 $=\frac{3\cdot1415926}{180}=\cdot0174533$, 由 12 節

$\tan 1^\circ=\cdot0174533-\frac{1}{3}(\cdot0174533)^3=\cdot0174551$,

$\tan 46^\circ=\tan(45^\circ+1^\circ)=\frac{\tan 45^\circ+\tan 1^\circ}{1-\tan 45^\circ\tan 1^\circ}=\frac{1+\tan 1^\circ}{1-\tan 1^\circ}$

$=\frac{1\cdot0174551}{\cdot9825449}=1\cdot0355303.$

(4) $\sin\theta=\frac{1}{19}\left(1-\frac{1}{2166}\right)=\frac{1}{19}\left(1-\frac{1}{6\times361}\right)=\frac{1}{19}-\frac{1}{6}\left(\frac{1}{19}\right)^3$,

由 12 節 $\sin\theta=\theta-\frac{\theta^3}{6}$, 故 $\theta=\frac{1}{19}$, 即爲 $\frac{\text{半徑角}}{19}$.

(5) 45° 之弧 $=\frac{\pi}{4}=\cdot7854$, $\cos\frac{\pi}{4}=\frac{1}{\sqrt{2}}=\cdot7071$, 故 $\theta=\frac{\pi}{4}-\delta$,

而 δ 甚微小, 故 $\cos\left(\frac{\pi}{4}-\delta\right)=\frac{\pi}{4}-\delta$, 即

$\cos\frac{\pi}{4}\cos\delta+\sin\frac{\pi}{4}\sin\delta=\frac{\pi}{4}-\delta$, 即 $\frac{\delta}{\sqrt{2}}+\frac{1}{\sqrt{2}}\left(1-\frac{\delta^2}{2}\right)=\cdot7854-\delta$,

即 $\delta^2=2(2\cdot4142)\delta-\cdot2214$, 故 $\delta=\cdot0463$.

由是 $\theta=\frac{\pi}{4}-\delta=\cdot7854-\cdot0463=\cdot7391$.

(6) $\cot\theta=\theta$, 即 $\theta\tan\theta=1$, 但 $\frac{\pi}{4}=\cdot7854$, $\cot 45^\circ=1$,

故 $\theta=\frac{\pi}{4}+\delta$, $\left(\frac{\pi}{4}+\delta\right)\tan\left(\frac{\pi}{4}+\delta\right)=1$, $\left(\frac{\pi}{4}+\delta\right)(1+\tan\delta)=1-\tan\delta$,

即 $(\cdot7854+\delta)(1+\delta)=1-\delta$, 由 δ 之二次式, 得 $\delta=\cdot0751$.

故 $\theta=\cdot7854+\cdot0751=\cdot8605=\frac{\cdot8605}{3\cdot1416}\times180^\circ=49^\circ17'$.

第拾壹編

對數及對數級數

1. 對數已於代數學中説明矣，此處再説，似嫌重複，今於用對數表之前，略示其簡單之原理.

設 $a^x=n$，其 x 稱爲以 a 爲底所成 n 之對數.

記之如次. $x=log_a n$.

例如 $3^4=81$，其 4 爲以 3 爲底所成 81 之對數.

即 $4=log_3 81$.

對數之性質如次.

〔第一〕 1之對數爲0.

$\because a^0=1$, $\therefore 0=log_a 1$.

〔第二〕 底數之對數等於1.

$\because a^1=a$, $\therefore 1=log_a a$.

〔第三〕 積之對數，等於其因子之對數之和.

$\because a^x=m$, $a^y=n$, 即 $x=log_a m$, $y=log_a n$.

又 $a^{x+y}=m\times n$, 即 $x+y=log_a(m\times n)$.

由是 $log_a(m\times n)=log_a m+log_a n$.

〔第四〕 商之對數，等於被除數之對數減除數之對數.

$\because a^{x-y}=m\div n$, 即 $x-y=log_a(m\div n)$

由是 $log_a(m\div n)=log_a m-log_a n$.

〔第五〕 某數方乘之對數，等於其數之對數乘其指數.

$\because a^x=m$, 而 $(a^x)^r=a^{rx}=m^r$,

故 $rx=log_a(m^r)$, 由是 $log_a(m^r)=r\times log_a m$.

〔第六〕 於相異底數,求其所成一數之對數之關係.

$a^x=m$, $b^y=m$, 卽 $x=log_a m$, $y=log_b m$,

於 $a^x=b^y$, 故 $a^{\frac{x}{y}}=b$, 及 $b^{\frac{y}{x}}=a$,

卽 $\frac{x}{y}=log_a b$, 及 $\frac{y}{x}=log_b a$,

由是 $log_a b\times log_b a=\frac{x}{y}\times\frac{y}{x}=1$.

注意 $log_a b\times log_b a=1$, 此乃最要切用之公式也.

2. 常用對數 以10爲底之對數,謂之常用對數,一般計算用之.

於常用對數,其任意數之對數,在整數部,謂之示標.在小數部,謂之假數.而示標可由視察得之.

例如有整數位之一數,比10^n大而比10^{n+1}小.故其一數之對數,爲比$log_{10}10^n=n$大而比$log_{10}10^{n+1}=n+1$小.

故其一數之對數$=n+$假數$=n+$小數.

例如 $345\cdot6>100$. 又 $345\cdot6<1000$.

故 $log_{10}345\cdot6>log_{10}10^2$ 卽 >2, 又 $log_{10}345\cdot6<log_{10}10^3$ 卽 <3.

由是 $log_{10}345\cdot6=2+$小數, 卽 $log_{10}345\cdot6$之示標爲2.

又比1小之一數,其小數底(卽單位與有效數字間之位數也)有n位爲0,則其一數比$\frac{1}{10^n}$大,而比$\frac{1}{10^{n+1}}$小,故其對數在$log10^{-n}$與$log10^{-n-1}$之間.故其一數之對數,在$-n$與$-n-1$之間而等於$-(n+1)+$小數.

例如 $\cdot03456$在$\cdot1$與$\cdot01$之間,卽在10^{-1}與10^{-2}之間,

故 $log_{10}\cdot03456$在-1與-2之間,而等於$-2+$小數,卽示標爲-2.

同樣. $log_{10}\cdot003456$之示標爲-3, 又$log_{10}\cdot3456$之示標爲-1.

由是整數之示標爲正,小數之示標爲負,而假數常爲正.

例如檢表知$log_{10}534=2\cdot7275413$,

然 $log_{10}53400=log_{10}(534\times10^2)=log_{10}534+log_{10}10^2=log_{10}534+2$,

故 $=2\cdot7275413+2=4\cdot7275413$.

同樣 $log_{10}5\cdot34=log_{10}(534\times10^{-2})=log_{10}534-2=0\cdot7275413$,

又 $log_{10}\cdot0534=loy_{10}(534\times10^{-4})=2\cdot7275413-4=-2+\cdot7275413$,

記之為 $log_{10}\cdot0534=\overline{2}\cdot727513$. 卽 2 為負, 小數為正.

對數級數

3. 指數級數 $a^x=\{1+(a-1)\}^x$ 明甚, 將此式由二項式之定理展開之, 則

$$a^x=1+x(a-1)+\frac{x(x-1)}{1.2}(a-1)^2+\frac{x(x-1)(x-2)}{1.2.3}(a-1)^3+\frac{x(x-1)(x-2)(x-3)}{1.2.3.4}(a-1)^4+\cdots\cdots$$

$$=1+x\{a-1-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\cdots\cdots\cdots\cdots\cdots\}+x^2\mathrm{K},$$

但 $x^2\mathrm{K}$ 為表示 x^2 及 x^2 以上之高次項.

其不含 x 之數量, 以 $c_1, c_2, c_3, c_4\cdots\cdots\cdots\cdots$ 代之, 則 a^x 之展開式如次,

故 $a^x=1+c_1x+c_2x^2+c_3x^3+c_4x^4+\cdots\cdots\cdots\cdots\cdots$ (1)

$c_1=a-1-\frac{1}{2}(a-1)^2+\frac{1}{3}(a-1)^3-\frac{1}{4}(a-1)^4+\cdots\cdots\cdots\cdots\cdots$

試以 $x+y$ 代其 x, 則

$a^{x+y}=1+c_1(x+y)+c_2(x+y)^2+c_3(x+y)^3+\cdots\cdots\cdots\cdots\cdots$

$=1+c_1y+c_2y^2+c_3y^3+\cdots\cdots\cdots+(c_1+2c_2y+3c_3y^2+\cdots\cdots\cdots)x+x^2\mathrm{H}$, (2)

但 $x^2\mathrm{H}$ 為表示 x^2 及其以上之 x 之高次項.

又 $a^{x+y}=a^xa^y=a^y(1+c_1x+c_2x^2+c_3x^3+\cdots\cdots\cdots\cdots)$

$=a^y+c_1a^yx+c_2a^yx^2+c_3a^yx^3+\cdots\cdots\cdots\cdots$ (3)

試將 (2), (3) 比較 x 之係數, 則

$c_1+2c_2y+3c_3y^2+4c_4y^3+\cdots\cdots\cdots=c_1a^y=c_1(1+c_1y+c_2y^2+c_3y^3+\cdots\cdots\cdots)$.

由此恆同式比較 $y, y^2, y^3,\cdots\cdots\cdots$ 之係數, 則

$2c_2=c_1^2,\ 3c_3=c_1c_2,\ 4c_4=c_1c_3,\cdots\cdots\cdots\cdots$

由是 $c_2=\frac{c_1^2}{2},\ c_3=\frac{c_1c_2}{3}=\frac{c_1^3}{1.2.3},\ c_4=\frac{c_1c_3}{4}=\frac{c_1^4}{1.2.3.4},\cdots\cdots\cdots\cdots\cdots$

以此代入 (1), 則

$$a^x=1+c_1x+\frac{c_1^2x^2}{\underline{|2}}+\frac{c_1^3x^3}{\underline{|3}}+\frac{c_1^4x^4}{\underline{|4}}+\cdots\cdots\cdots \quad (4)$$

此結果, 對於 x 之任何值皆合理, 故令 $x=\frac{1}{c_1}$, 則

$$a^{\frac{1}{c_1}}=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\frac{1}{\underline{|4}}+\cdots\cdots\cdots$$

此級數, 通例以 e 表之, 故 $a^{\frac{1}{c_1}}=e$, 卽 $a=e^{c_1}$, 卽 $c_1=\log_e a$,

由此得 (4) 如次

$$a^x=1+(\log_e a)x+\frac{(\log_e a)^2x^2}{\underline{|2}}+\frac{(\log_e a)^3x^3}{\underline{|3}}+\cdots\cdots\cdots \quad (5)$$

又以 e 代其 a, 則 $\log_e a=\log_e e=1$, (1 節第二)

由是從 (5) $e^x=1+x+\frac{x^2}{\underline{|2}}+\frac{x^3}{\underline{|3}}+\cdots\cdots\cdots$ (6)

令 $x=1$, 則 $e=1+1+\frac{1}{\underline{|2}}+\frac{1}{\underline{|3}}+\cdots\cdots\cdots=2\cdot718281828\cdots\cdots\cdots$.

e 為訥白爾 (*Napier*) 氏對數之底數, 卽訥白爾氏所發明之對數也.

4. 訥白爾 (Napier) 氏對數之級數 $\log_e(1+x)$

依 x 遞昇方乘展開之, 因前節 $c_1=\log_e a$, 故由前節

$$\log_e a=a-1-\tfrac{1}{2}(a-1)^2+\tfrac{1}{3}(a-1)^3-\tfrac{1}{4}(a-1)^4+\cdots\cdots\cdots$$

試以 $1+x$ 代其 a, 則

$$\log_e(1+x)=x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\cdots\cdots\cdots \quad (7)$$

又以 $-x$ 代其 x, 則

$$\log_e(1-x)=-x-\frac{x^2}{2}-\frac{x^3}{3}-\frac{x^4}{4}-\cdots\cdots\cdots\cdots \quad (7')$$

從 (7) 減去此式, 則 $\log_e(1+x)-\log_e(1-x)=\log_e\frac{1+x}{1-x}$,

卽 $\log_e\frac{1+x}{1-x}=2\left(x+\frac{x^3}{3}+\frac{x^5}{5}+\cdots\cdots\cdots\right)$.

令 $x=\frac{m-n}{m+n}$, 則得 $\frac{1+x}{1-x}=\frac{m}{n}$, 故

$$\log_e\frac{m}{n}=2\left\{\frac{m-n}{m+n}+\frac{1}{3}\left(\frac{m-n}{m+n}\right)^3+\frac{1}{5}\left(\frac{m-n}{m+n}\right)^5+\cdots\cdots\right\}\cdots\cdots\cdots(8).$$

令 $n=1$, 則 $log_e m=2\left\{\frac{m-1}{m+1}+\frac{1}{3}\left(\frac{m-1}{m+1}\right)^3+\frac{1}{5}\left(\frac{m-1}{m+1}\right)^5+\cdots\cdots\right\}$. (9)

於 (8) 令 $m=n+1$, 則

$$log_e(n+1)-log_e n=2\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\cdots\cdots\right\}. \quad (10)$$

此 (10) 乃計算上最簡便之形也.

〔例〕 於 (9) 令 $m=2$, 則

$$log_e 2=2\left\{\frac{1}{3}+\frac{1}{3}\left(\frac{1}{3}\right)^3+\frac{1}{5}\left(\frac{1}{3}\right)^5+\cdots\cdots\right\}=\cdot 6931471\cdots\cdots$$

於 (10) 令 $n=2$, 則

$$log_e 3-log_e 2=2\left\{\frac{1}{5}+\frac{1}{3}\left(\frac{1}{5}\right)^3+\frac{1}{5}\left(\frac{1}{5}\right)^5+\cdots\cdots\right\}.$$

由是 $log_e 3=1\cdot 0986122\cdots\cdots$

又 $log_e 9=log_e 3^2=2log_e 3=2\times 1\cdot 09816122\cdots\cdots$

故於 (10) 令 $n=9$, 則

$$log_e 10-log_e 9=2\left\{\frac{1}{19}+\frac{1}{3}\left(\frac{1}{19}\right)^3+\frac{1}{5}\left(\frac{1}{19}\right)^5+\cdots\cdots\right\},$$

故 $log_e 10=2\cdot 3025850\cdots\cdots$

5. 常用對數之級數 由1節第六

$log_e 10\times log_{10}e=1$, 故 $log_{10}e=\frac{1}{log_e 10}=\frac{1}{2\cdot 3025850\cdots}=\cdot 43429448\cdots\cdots$

此以 μ 代之, 卽名對數之**模數**, 卽 $log_{10}e=\cdot 4342948\cdots\cdots=\mu$.

某數之常用對數, 等於以 μ 乘其數之訥白爾氏對數.

令某數爲 n, $log_e n=x$, 則 $n=e^x$, 故

$log_{10}n=log_{10}e^x=x\,log_{10}e=log_e n\times\mu$.

由是 (10) 乘以 μ, 則

$$\mu log_e(n+1)-\mu log_e n=2\mu\left\{\frac{1}{2n+1}+\frac{1}{3(2n+1)^3}+\frac{1}{5(2n+1)^5}+\cdots\cdots\right\}.$$

卽 $log_{10}(n+1)-log_{10}n=2\mu\left\{\frac{1}{2n+1}+\frac{1}{2(2n+1)^3}+\frac{1}{5(2n+1)^5}+\cdots\right\}$, (11)

同樣. 從 (7) $log_{10}(1+x)=\mu\left\{x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+\cdots\cdots\right\}$, (12)

例題十四

1. 以$\sqrt[3]{4}$為底數，試求128之對數.

2. 以$\sqrt{3}$為底數，試求$243\sqrt[3]{9}$之對數.

3. 求$log_3 2187$, $log_{10}\cdot 0001$, $log_2 cos 45°$之各值.

4. 從$5^{6-4x}=2^{x+3}$求x之略近值，但$log_{10}2=\cdot 301030$.

5. 有$log_{10}\cdot 224=a$, $log_{10}125=b$, 求$log_{10}2$及$log_{10}7$之各值.

6. $log_6 725$及$log_6\sqrt[5]{(\cdot 0725)}$之示標各如何.

7. 已知$log_{10}\cdot 2=\cdot 301030$, $log_{10}405=2\cdot 607455$, 求$log_{10}\cdot 003$.

8. 知$log_{10}2=\cdot 301030$, $log_{10}7=\cdot 845098$,

   求 $log_{10}98$ 及 $log_{10}\left(\frac{4}{343}\right)^{\frac{1}{2}}$.

9. 知$log_{10}2=\cdot 301030$, $log_{10}3=\cdot 477121$, 求$log_{10}(\cdot 0020736)^{\frac{1}{5}}$

例題解自1至9.

(1)　$128=2^7=(2^2)^{\frac{7}{2}}=(4)^{\frac{7}{2}}=4^{\frac{21}{6}}=(4^{\frac{1}{3}})^{\frac{21}{2}}=(\sqrt[3]{4})^{10\cdot 5}$.

由是　$log_{\sqrt[3]{4}}128=10\cdot 5$.

(2)　$243\sqrt[3]{9}=3^5\cdot 3^{\frac{2}{3}}=3^{\frac{17}{3}}=(\sqrt{3})^{\frac{34}{3}}$, 故 $log_{\sqrt{3}}243\sqrt[3]{9}=\frac{34}{3}$.

或　$log_{\sqrt{3}}(243\sqrt[3]{9})=log_{\sqrt{3}}243+log_{\sqrt{3}}\sqrt[3]{9}=log_{\sqrt{3}}\sqrt{3}^{10}+log_{\sqrt{3}}\sqrt{3}^{\frac{4}{3}}=10+\frac{4}{3}$,

(3) $2187=3^7$, 故 $log_3 2187=7$.

$\cdot 0001=\frac{1}{10^4}=10^{-4}$, 故 $log_{10}\cdot 0001=-4$,

$cos 45°=\frac{1}{\sqrt{2}}=2^{-\frac{1}{2}}$, 故 $log_2 cos 45°=-\frac{1}{2}$.

(4) $(6-4x)log_{10}5=(x+3)log_{10}2$, 故 $x=\frac{6log_{10}5-3log_{10}2}{4log_{10}5+3log_{10}2}$, 即

$$x=\frac{6log_{10}(10\div 2)-3log_{10}2}{4log_{10}(10\div 2)+3log_{10}2}=\frac{3(2-3log_{10}2)}{4-log_{10}2}=\frac{3(2-3\times\cdot 301030)}{4-\cdot 301030}=\cdot 889\ldots\ldots$$

(5) $a=log_{10}\cdot 224=log_{10}\frac{7\times 2^5}{1000}=log_{10}7+5log_{10}2-3$, (1)

$b=log_{10}125=log_{10}\frac{1000}{8}=3-3log_{10}2$, (2)

從 (1), 及 (2), 得 $log_{10}2=1-\frac{b}{3}$ 及 $log_{10}7=a+\frac{5b}{3}-2$.

(6) $6^3<725<6^4$, 故 $log_6 725$ 之示標爲 6.

又 $log_6(\cdot 0725)^{\frac{1}{5}}=\frac{1}{5}log_6\cdot 0725$, $6^{-1}>\cdot 0725>6^{-2}$,

故 $-\frac{1}{5}>log_6(\cdot 0725)^{\frac{1}{5}}>-\frac{2}{5}$, 故 $log_6(\cdot 0725)^{\frac{1}{5}}=-1+$假數.

由是示標爲 -1.

(7) $log_{10}405=log_{10}(3^4\times 10\div 2)=4log_{10}3+1-log_{10}2$

$=4log_{10}(\cdot 003\times 10^3)+1-log_{10}2=4(log_{10}\cdot 003+3)+1-log_{10}2$, 故

$$log_{10}\cdot 003=\frac{log_{10}405+log_{10}2-13}{4}=\frac{2\cdot 607455+\cdot 301030-13}{4}$$

$$=\frac{-12+1\cdot 908485}{4}+\bar{3}\cdot 477121.$$

(8) $log_{10}98=log_{10}(2\times 7^2)=log_{10}2+2log_{10}7$

$=\cdot 301030+2\times\cdot 845098=1\cdot 991226$.

$$log_{10}\left(\frac{4}{343}\right)^{\frac{1}{2}}=\frac{1}{2}(log_{10}2^2-log_{10}7^3)=\frac{1}{2}(2log_{10}2-3log_{10}7)$$

$$=\frac{1}{2}(2\times\cdot 301030-3\times\cdot 845098)=-\cdot 966617=\bar{1}\cdot 033383.$$

(9) $log_{10}(\cdot 0020736)^{\frac{1}{3}}=\frac{1}{3}log_{10}(3^4\times 2^8\div 10^7)=\frac{1}{3}(4log_{10}3+8log_{10}2-7)$

$=\frac{1}{3}(4\times\cdot 477121+8\times\cdot 301030-7)=\frac{1}{3}(-3+\cdot 316724)$

$=\bar{1}\cdot 105575$.

10. 試證 $e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\cdots$爲不盡數.

11. 試決定無限級數 $\frac{2}{3!}+\frac{4}{5!}+\frac{6}{7!}$ 之值.

12. 試證 $\frac{e}{2}=\frac{1}{2!}+\frac{1+2}{3!}+\frac{1+2+3}{4!}+\cdots\cdots$.

次列各式.其 n 增至無限,求其極限.

13. $\left(\cos\frac{\alpha}{n}\right)^n$, 14. $\left(\frac{\sin\alpha/n}{\alpha/n}\right)^n$, 15. $\left(\cos\frac{\alpha}{n}\right)^{n^2}$.

16. $\left(\cos\frac{\alpha}{n}\right)^{n^3}$. 17. $\left(\frac{n-1}{n}\right)^n$.

例題解自 10. 至 17.

(10) $e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\cdots\cdots+\frac{1}{n!}+\frac{1}{(n+1)!}+\cdots\cdots=\frac{m}{n}$,

但 m, n 爲有理數,此以 $n!$ 乘之,則

$m\,(n-1)!=$整數$+\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\cdots\cdots$.

即 $\frac{1}{n+1}+\frac{1}{(n+1)(n+2)}+\frac{1}{(n+1)(n+2)(n+3)}+\cdots\cdots=m\,(n-1)!-$整數$=$整數

然此右式之分數,比 $\frac{1}{n+1}+\frac{1}{(n+1)^2}+\frac{1}{(n+1)^3}+\cdots\cdots$即$\frac{\frac{1}{n+1}}{1-\frac{1}{n+1}}=\frac{1}{n}$更小.即比 1 較小,而云等於整數,是不合理,故如題言.

(11) 原級數$=\left(\frac{1}{2!}-\frac{1}{3!}\right)+\left(\frac{1}{4!}-\frac{1}{5!}\right)+\left(\frac{1}{6!}-\frac{1}{7!}\right)+\cdots\cdots$ 又於 3. 節 (6)

令 $x=-1$, 則 $e^{-1}=\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\cdots\cdots$.

由是原級數$=e^{-1}$.

(12) $e=1+\frac{1}{1!}+\frac{1}{2!}+\frac{1}{3!}+\frac{1}{4!}+\cdots\cdots+\frac{1}{n!}+\cdots\cdots$

$$=\frac{1}{2!}+\frac{2\cdot3}{3!}+\frac{3\cdot4}{4!}+\frac{4\cdot5}{5!}+\frac{5\cdot6}{6!}\cdots\cdots+\frac{(n+1)(n+2)}{(n+2)!}+\cdots\cdots,$$

$$\frac{e}{2}=\frac{1}{\lfloor 2}+\frac{3}{\lfloor 3}+\frac{6}{\lfloor 4}+\frac{10}{\lfloor 5}+\cdots\cdots+\frac{\frac{1}{2}(n+1)(n+2)}{\lfloor n+2}+\cdots\cdots$$

$$=\frac{1}{\lfloor 2}+\frac{1+2}{\lfloor 3}+\frac{1+2+3}{\lfloor 4}+\frac{1+2+3+4}{\lfloor 5}+\cdots\cdots+\frac{1+2+3+\cdots\cdots+(n+1)}{\lfloor n+2}+\cdots\cdots$$

(13) $\cos\frac{a}{n}=\left(1-\sin^2\frac{a}{n}\right)^{\frac{1}{2}}$, 故 $\log_e\left(\cos\frac{a}{n}\right)^n=\log_e\left(1-\sin^2\frac{a}{n}\right)^{\frac{n}{2}}$

$=-\frac{n}{2}\left(\sin^2\frac{a}{n}+\frac{1}{2}\sin^4\frac{a}{n}+\frac{1}{3}\sin^6\frac{a}{n}+\cdots\cdots\right)$, 〔由 4 節 (7′)〕

但 $\sin^2\frac{a}{n}+\frac{1}{2}\sin^4\frac{a}{n}+\frac{1}{3}\sin^6\frac{a}{n}+\cdots\cdots<\sin^2\frac{a}{n}+\sin^4\frac{a}{n}+\sin^6\frac{a}{n}+\cdots\cdots$

即 $<\frac{\sin^2 a/n}{1-\sin^2 a/n}$, 即 $<\tan^2\frac{a}{n}$, 故 $\log_e\left(\cos\frac{a}{n}\right)^n<-\frac{n}{2}\tan^2\frac{a}{n}$.

即 $<-\frac{1}{2}\left(\frac{\tan^2 a/n}{a^2/n^2}\right)\frac{a^2}{n}$, $n=\infty$, 則 $\frac{\tan^2 a/n}{a^2/n^2}=1$ (2 節) 及 $\frac{a^2}{n}=0$,

故 $\lim_{n=\infty}\log_e\left(\cos\frac{a}{n}\right)^n=0$, 即 $\lim_{n=\infty}\left(\cos\frac{a}{n}\right)^n=1$.

(14) n 增大, 則 $\sin\frac{a}{n}=\frac{a}{n}-\frac{a^3}{6n^3}$, 故

$$\log\left(\frac{\sin a/n}{a/n}\right)^n=n\log_e\left(\frac{\frac{a}{n}-\frac{a^3}{6n^3}}{a/n}\right)=n\log_e\left(1-\frac{a^2}{6n^2}\right)$$

$=-n\left(\frac{a^2}{6n^2}+\frac{a^4}{72n^4}+\cdots\cdots\right)$, 故 $n=\infty$, 則 $\log_e\left(\frac{\sin a/n}{a/n}\right)^n=0$,

即 $\lim_{n=\infty}\left(\sin\frac{a}{n}\Big/\frac{a}{n}\right)^n=1$.

(15) $\log_e\left(\cos\frac{a}{n}\right)^{n^2}=n^2\log_e\left(1-\frac{a^2}{2n^2}\right)=-n^2\left(\frac{a^2}{2n^2}+\frac{a^4}{8n^4}+\cdots\cdots\right)$

$n=\infty$, 則 $\log_e\left(\cos\frac{a}{n}\right)^{n^2}=-\frac{a^2}{2}$, 故 $\lim_{n=\infty}\left(\cos\frac{a}{n}\right)^{n^2}=e^{-\frac{a^2}{2}}$

(16) $\log_e\left(\cos\frac{a}{n}\right)^{n^3}=-n^3\left(\frac{a^2}{2n^2}+\frac{a^4}{8n^4}+\cdots\cdots\right)$,

$n=\infty$, 則 $\log_e\left(\cos\frac{a}{n}\right)^{n^3}=-\infty$, 即 $\lim_{n=\infty}\left(\cos\frac{a}{n}\right)^{n^3}=0$.

(17) $\left(1-\frac{1}{n}\right)^n=1-n\left(\frac{1}{n}\right)+\frac{n(n-1)}{\lfloor 2}\left(\frac{1}{n}\right)^2-\frac{n(n-1)(n-2)}{\lfloor 3}\left(\frac{1}{n}\right)^3+\cdots\cdots$

$=1-1+\frac{1}{\lfloor 2}\left(1-\frac{1}{n}\right)-\frac{1}{\lfloor 3}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)+\cdots\cdots$

故 $n=\infty$, 則 $\lim_{n=\infty}\left(1-\frac{1}{n}\right)^n=1-1+\frac{1}{\lfloor 2}-\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}\cdots\cdots=e^{-1}$.

比例差之理論

6. 比例差 n 爲五位之數，即 $100000>n>10000$，而 $d<1$，

則 $\dfrac{\log(n+d)-\log n}{\log(n+1)-\log n}=\dfrac{d}{1}$，但 $\log$ 爲 $\log_{10}$ 之略號，以下仿此.

$$\log(n+d)=\log\left\{n\left(1+\frac{d}{n}\right)\right\}=\log n+\log\left(1+\frac{d}{n}\right)$$

$$=\log n+\mu\left\{\frac{d}{n}-\frac{1}{2}\ \frac{d^2}{n^2}+\frac{1}{3}\ \frac{d^3}{n^3}-\cdots\cdots\right\}\quad \text{〔5 節 (12)〕}$$

$n>10000$, $d<1$, 而 $\mu=\cdot43422945\cdots\cdots<\cdot5$, 故 $\dfrac{d}{n}<\cdot0001$.

故 $\dfrac{1}{2}\ \dfrac{\mu d^2}{n^2}\not>\dfrac{1}{4}(\cdot0001)^2$ 即 $\not>\cdot0000000025$，而 $\dfrac{\mu}{3}\ \dfrac{d^3}{n^3}$ 更小，

由是次之略近值，最少亦可合至小數七位.

$$\log(n+d)-\log n=\mu d/n.$$

同樣. $\log(n+1)-\log n=\mu/n.$

故
$$\frac{\log(n+d)-\log n}{\log(n+1)-\log n}=\frac{d}{1}. \qquad (1)$$

7. 三角函數之比例差 示其正弦他準此.

$$\sin(\alpha+\delta)=\sin\alpha\cos\delta+\cos\alpha\sin\delta=\sin\alpha(1-\tfrac{1}{2}\delta^2)+\cos\alpha(\delta-\tfrac{1}{3}\delta^3)$$

$$=\sin\alpha+\delta\cos\alpha-\tfrac{1}{2}\delta^2\sin\alpha-\tfrac{1}{3}\delta^3\cos\alpha,$$

δ 又大於 $1'$ 之弧，故 $\delta\not>\cdot0003$，(例題十三 1 之解)

故 $\tfrac{1}{2}\delta^2\not>\cdot00000005$，又 $\sin\alpha<1$，故次之略近值，正合小數至七位.

$$\sin(\alpha+\delta)-\sin\alpha=\delta\cos\alpha,$$

同樣. $\sin(\alpha+\delta')-\sin\alpha=\delta'\cos\alpha$，但 δ' 通例爲 $1'$ 之差.

故
$$\frac{\sin(\alpha+\delta)-\sin\alpha}{\sin(\alpha+\delta')-\sin\alpha}=\frac{\delta}{\delta'}. \qquad (2)$$

8. 三角函數值 由比例差之應用而求得之.

例 求 $\sin 18°1'$ 之值.

$\sin 18^\circ=\frac{\sqrt{5}-1}{4}$, (例題六 12.) $=\cdot 3090169$, $\cos 18^\circ=\frac{\sqrt{10+2\sqrt{5}}}{4}=\cdot 9510565$,

$\sin 1'=\cdot 00029088$, (例題十三. 1.) $\cos 1'=\cdot 99999995$. 故

$\sin 18^\circ 1'=\sin 18^\circ \cos 1'+\cos 18^\circ \sin 1'=$

$$=\cdot 3090169\times\cdot 99999995+\cdot 9510565\times\cdot 00029088=\cdot 3092936,$$

又可用 (2) 以求 $\sin 18^\circ 0'23''$,

$60''$ 之差 $=\sin 18^\circ 1'-\sin 18^\circ=\cdot 3092936-\cdot 3090169=\cdot 0002767$,

故 $\sin 18^\circ 0'23''=\sin 18^\circ+\frac{23}{60}\times\cdot 0002767$

$$=\cdot 3090169+\cdot 0001060=\cdot 3091229.$$

例 題 十 五

1. 試用表以求 $\sin 37^\circ 23' 47''$.
2. 設 $\cos a=\cdot 8241637$, 試用表以求其 a.
3. 試求 $\sin 30^\circ 1' 17''$. 但不用表.
4. $\cos a=\cdot 4996532$, $1'$ 之差 $=\cdot 0002519$, 試不用表以求其 a.

例 題 十 五 之 解 答

(1) 由表 $\sin 37^\circ 23'=\cdot 6071447$, $60''$ 之差 $=\cdot 0002311$,

故 $\sin 37^\circ 23' 47''=\cdot 6071447+\frac{47}{60}\times\cdot 0002311=\cdot 6073257$.

(2) 由表 $\cos 34^\circ 30'=\cdot 8241262$, $1'$ 之差 $=\cdot 0001648$,

又 $\cdot 8241637-\cdot 8241262=\cdot 0000375$, 故 $\frac{375}{1648}\times 60''=13''$,

故 $34^\circ 30'-13''=34^\circ 29' 47''=a$.

(3) 從 $\sin 30^\circ=\frac{1}{2}$, 求得 $\sin 30^\circ 1' 17''=\cdot 5003233$ (8 節).

(4) $\cos 60^\circ=\frac{1}{2}$, 故 $\cos 60^\circ-\cos a=\cdot 0003468$, 故

$\frac{3468}{2519}\times 1'=1' 22''\cdot 6$, 故 $a=60^\circ 1' 22''\cdot 6$.

對數表用法

9. 已知眞數求對數 由表求其對數.法加次.

如求5340234之對數.

由表求其兩數相近之對數,而取其差,如

$$\begin{array}{r} \log 5340300 = 6\cdot 7275657 \\ \log 5340200 = 6\cdot 7275575 \\ \hline \text{差 } 100 = \ \cdot 0000082 \end{array}$$

其示標6,乃由視察得之.而$5340234-5340200=34$, 故由

6節(1) $\frac{34}{100}\times\cdot 0000082=\cdot 0000028$　由是

$\log 5340234=6\cdot 7275575+\cdot 0000028=6\cdot 7275603.$

10. 旣知對數 由表求其相當之眞數,法如次.

如用表求$\bar{1}\cdot 3702074$相當之對數,則取其相近之兩數,如

$$\begin{array}{r} 4\cdot 3702169 = \log 23454 \\ 4\cdot 3701984 = \log 23453 \\ \hline \cdot 0000185 = \text{差}\quad 1 \end{array}$$

又 $\cdot 3702074-\cdot 3701984=\cdot 0000090$, 故由(1)

$\frac{90}{185}\times 1=\cdot 486$, 故 $4\cdot 3702074=\log(23453+\cdot 486)$.

即 $\bar{1}\cdot 43702074=\log\cdot 23453489$, 是即所求之數.

11. 三角函數之對數表 此表均以10加,故書log爲L以別之.

例如 $\sin 30^\circ=\frac{1}{2}=\cdot 5$, 故 $\log\sin 30^\circ=\bar{1}\cdot 6989700$.

表此爲 $L\sin 30^\circ=9\cdot 6989700$.

即 $\log\sin 30^\circ=10+\bar{1}\cdot 6989700-10=9\cdot 6989700-10$.

例題十六

1. 有 $\log 12440=4\cdot 0948204$, $\log 12441=4\cdot 0948553$,
   求 $\log 12440\cdot 35$.

2. 有 $log 1{\cdot}3325 = {\cdot}1246672$, $log 1{\cdot}3326 = {\cdot}1246998$,

求與 $-1{\cdot}8753145$ 各對數相當之眞數.

3. 有 $log 7 = {\cdot}8450980$, $log 58751 = 4{\cdot}7690153$ 及

$log 58752 = 4{\cdot}7690227$, 求 $({\cdot}07)^{\frac{1}{5}}$.

4. 有 $L sin 17°1' = 9{\cdot}4663483$, $L sin 17° = 9{\cdot}4659353$,

求 $L sin 17°0'12''$.

5. 有 $L cot 72°15' = 9{\cdot}5052891$, $L cot 72°16' = 9{\cdot}5048538$,

求 $L cot 72°15'35''$.

例題解自 1. 至 5.

(1) 1 之差 $=4{\cdot}0948553-4{\cdot}0948204={\cdot}0000349$, 故

$\frac{{\cdot}35}{1}\times{\cdot}0000349={\cdot}0000122$, 故 $log 12440{\cdot}35=4{\cdot}0948204+{\cdot}0000122$

$=4{\cdot}0948326$.

(2) ${\cdot}0001$ 之差 $={\cdot}0000326$, $-1{\cdot}8753145=\overline{2}{\cdot}1246855$,

故 ${\cdot}1246855-{\cdot}1246672={\cdot}0000183$, 故 $\frac{183}{326}\times{\cdot}001={\cdot}0000561$,

故 $log(1{\cdot}3325+{\cdot}0000561)={\cdot}1246855$, 故所求之數 $={\cdot}013325561$.

(3) $log({\cdot}07)^{\frac{1}{5}}=\frac{1}{5}log{\cdot}07=\frac{1}{5}(\overline{2}{\cdot}8450980)=\frac{1}{5}(-5+3{\cdot}8450980)$

$=\overline{1}{\cdot}7690196$, 1 之差 $=4{\cdot}7690227-4{\cdot}7690153={\cdot}0000074$.

故 $\frac{7690196-7690153}{74}\times1={\cdot}581$, 故 $log 58751{\cdot}581=4{\cdot}7690196$,

由是 $({\cdot}07)^{\frac{1}{5}}={\cdot}58751581$.

(4) 1′ 之差 $=9{\cdot}4663483-9{\cdot}4659353={\cdot}0004130$,

$\frac{12}{60}\times{\cdot}0004130={\cdot}0000826$, 故

$L sin 17°0'12''=9{\cdot}4659353+{\cdot}0000826=9{\cdot}4660179$.

(5) 1′ 之差 $={\cdot}0004353$, $\frac{35}{60}\times{\cdot}0004353={\cdot}0002539$,

故 $L cot 72°15'35''=9{\cdot}5052891-{\cdot}0002539=9{\cdot}5050352$.

6. 有 $L\cot 81^\circ 46' = 9\cdot 1604569$, $10''$ 之差 $= \cdot 0001486$ 與 $L\cot a = 9\cdot 1603493$, 由此求其 a.

〔解〕 $9\cdot 1604569 - 9\cdot 1603493 = \cdot 0001076$, 故 $\frac{1076}{1486} \times 10'' = 7''$,

故 $L\cot a = L\cot(81^\circ 46' + 7'') = 9\cdot 1603493$, 故 $a = 81^\circ 46' 7''$.

7. 有 $L\tan 37^\circ 19' = 9\cdot 8821007$, $1'$ 之差 $= \cdot 0002621$ 與 $L\tan a = 9\cdot 8823059$, 從此求其 a.

〔解〕 $9\cdot 8823059 - 9\cdot 8821007 = \cdot 0002052$,

故 $\frac{2052}{2621} \times 60'' = 47''$,

故 $L\tan a = L\tan(37^\circ 19' + 47'') = 9\cdot 8820359$.

故 $a = 37^\circ 19' 47''$.

第拾貳編

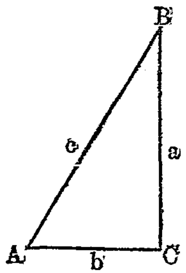
三角形邊及角之關係

1. 三角形之邊及角 三角形 ABC, 對於角 A, B, C 之邊 BC, CA, AB. 順次以 a, b, c 代之.

三角形三邊 a, b, c 與三角 A, B, C 共有六項, 知其三項, 卽可求其餘三項.

〔譯注〕 有三邊, 卽可求得其三角, 有三角, 則不能求得其三邊, 故所知之三項, 最少必有一邊在內.

2. 直角三角形 C 爲直角, 則有次之關係.



$a=\sqrt{c^2-b^2},\quad a=c\sin A.$

$c=\sqrt{a^2+b^2},\quad b=c\cos A.$

$B=90°-A,\quad a=b\tan A.$

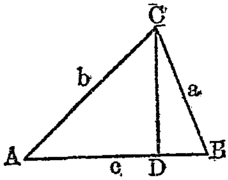
〔證〕 依幾何學 $c^2=a^2+b^2$,

又 $A+B+C=180°$, 但 $C=90°$, 故

$A+B=90°$, 由第二編 1 節 $\frac{a}{c}=\sin A$,

$\frac{b}{c}=\cos A$, $\frac{a}{b}=\tan A$.

3. 三角形之公式

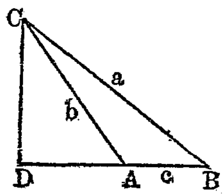


$$\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin C}{c},\quad (1)$$

〔證〕 引 AB 之垂線 CD, 則

於一圖, $CD=b\sin A$, $CD=a\sin B$,

故 $\frac{\sin A}{a}=\frac{\sin B}{b}$.



又於二圖, $CD=b\sin CAD=b\sin(180°-A)$

$=b\sin A$. $CD=a\sin B$.

故與前同.

$$\left.\begin{aligned}2bc\cos A&=b^2+c^2-a^2\\2ca\cos B&=c^2+a^2-b^2\\2ab\cos C&=a^2+b^2-c^2\end{aligned}\right\}\quad(2)$$

〔證〕 $CD^2=b^2-AD^2=a^2-BD^2$,

於一圖, $b^2-AD^2=a^2-(c-AD)^2$, $2c.AD=b^2+c^2-a^2$,

$AD=b\cos A$, ∴ $2bc\cos A=b^2+c^2-a^2$.

又於二圖, $b^2-AD^2=a^2-(c+AD)^2$, $-2c.AD=b^2+c^2-a^2$,

$AD=b\cos CAD=b\cos(180°-A)=-b\cos A$. ∴ $2bc\cos A=b^2+c^2-a^2$.

〔譯補〕 餘二式仿此.

$$\left.\begin{aligned}\sin\frac{A}{2}&=\sqrt{\frac{(s-b)(s-c)}{bc}}\\\cos\frac{A}{2}&=\sqrt{\frac{s(s-a)}{bc}}\\\tan\frac{A}{2}&=\sqrt{\frac{(s-b)(s-c)}{s-a}}\end{aligned}\right\}\quad(3)\quad 但\ a+b+c=2s.$$

〔證〕 從 (2) $\cos A=1-2\sin^2\frac{A}{2}=\frac{b^2+c^2-a^2}{2bc}$,

故 $\sin^2\frac{A}{2}=\frac{a^2-(b-c)^2}{4bc}=\frac{(a+b-c)(a-b+c)}{4bc}=\frac{(2s-2c)(2s-2b)}{4bc}=\frac{(s-c)(s-b)}{bc}$,

$$\cos^2\frac{A}{2}=\frac{1+\cos A}{2}=\frac{2bc+b^2+c^2-a^2}{4bc}=\frac{(b+c+a)(b+c-a)}{4bc}=\frac{2s(2s-2a)}{4bc}=\frac{s(s-a)}{bc},$$

又 $\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\sqrt{\frac{(s-b)(s-c)}{bc}}\Big/\sqrt{\frac{s(s-a)}{bc}}=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}$.

$$\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2},\quad(4)$$

〔證〕 從 (1) $\frac{b}{c}=\frac{\sin B}{\sin C}$, 故 $\frac{b+c}{b-c}=\frac{\sin B+\sin C}{\sin B-\sin C}$

$$=\frac{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}{2\cos\frac{1}{2}(B+C)\sin\frac{1}{2}(B-C)}=\frac{\sin\frac{1}{2}(180°-A)\cos\frac{1}{2}(B-C)}{\cos\frac{1}{2}(180°-A)\sin\frac{1}{2}(B-C)}=\frac{\cot\frac{A}{2}}{\tan\frac{1}{2}(B-C)}.$$

$$a=(b-c)\sec\tan^{-1}\left(\frac{2\sin\frac{A}{2}}{b-c}\sqrt{bc}\right),\quad (5)$$

〔證〕 從 (2) $a^2=b^2+c^2-2bc\cos A=b^2+c^2-2bc\left(1-2\sin^2\frac{A}{2}\right)$

$$=(b-c)^2\left\{1+\frac{4bc\sin^2\frac{A}{2}}{(b-c)^2}\right\},$$

$\tan\theta=\frac{2\sin\frac{A}{2}}{b-c}\sqrt{bc}$ · 卽 $\theta=\tan^{-1}\left(\frac{2\sin\frac{A}{2}}{b-c}\sqrt{bc}\right)$,

故 $a^2=(b-c)^2(1+\tan^2\theta)=(b-c)^2\sec^2\theta.$

$S=\frac{1}{2}bc\sin A=\sqrt{s(s-a)(s-b)(s-c)}.$ (6) 但S爲面積.

〔證〕 $CD=b\cdot\sin A$, $S=\frac{1}{2}AB.CD=\frac{1}{2}cb\sin A$,

$$\sin^2A=1-\cos^2A=1-\left(\frac{b^2+c^2-a^2}{2bc}\right)^2=\frac{(2bc+b^2+c^2-a^2)(2bc-b^2-c^2+a^2)}{4b^2c^2},$$

故 $2bc\sin A=\sqrt{(b+c+a)(b+c-a)(a+b-c)(a-b+c)}$

$$=\sqrt{2s(2s-2a)(2s-2c)(2s-2b)}=4\sqrt{s(s-a)(s-b)(s-c)}.$$

由是 $S=\frac{1}{2}bc\sin A=\sqrt{s(s-a)(s-b)(s-c)}.$

例題十七

$c=90^\circ$, 求次各式之證.

1. $\tan^2(45^\circ-\frac{1}{2}A)=(c-a)/(c+a).$

2. $\tan^2(45^\circ+\frac{1}{2}A)=(c+a)/(c-a).$

例題解自 1. 至 2.

(1) $\tan^2(45^\circ-\frac{1}{2}A)=\tan^2\frac{1}{2}(90^\circ-A)=\frac{\sin^2\frac{1}{2}(90^\circ-A)}{\cos^2\frac{1}{2}(90^\circ-A)}=\frac{1-\cos(90^\circ-A)}{1+\cos(90^\circ-A)}$

$=\frac{1-\sin A}{1+\sin A}=\frac{1-a/c}{1+a/c}=\frac{c-a}{c+a}$, (2) 從前例卽得.

3. $\cos 2A=\frac{b^2-a^2}{c^2}$.　　4. $\tan 2A=\frac{b^2-a^2}{c^2}$.

5. $\sin 3A=\frac{3ab^2-a^3}{c^3}$.　　6. $\cos 3A=\frac{b^3-3a^2b}{c^3}$.

7. $\tan\frac{A}{2}=\frac{a}{b+c}$.　　8. $\sin^2\frac{A}{2}=\frac{c-b}{2c}$.

求次列各任意三角形之證.

9. $\frac{\sin A+2\sin B}{\sin C}=\frac{a+2b}{c}$.

10. $\frac{\sin^2 A-m\sin^2 B}{\sin^2 C}=\frac{a^2-mb^2}{c^2}$.

11. $(a+b)\sin\frac{C}{2}=c\cos\frac{A-B}{2}$.

12. $(a-b)\cos\frac{C}{2}=c\sin\frac{A-B}{2}$.

13. $c=a\cos B+b\cos C$.

14. $c\cos(A-B)=a\cos A+b\cos B$.

15. $c\sin(A-B)=a\sin A-b\sin B$.

16. $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0$.

17. $c(a\cos B-b\cos A)=a^2-b^2$.

18. $\frac{a\cos B-b\cos A}{\sin(A-B)}=\frac{c}{\sin C}$.

例題解自 3. 至 18.

(3) $\cos 2A=\cos^2 A-\sin^2 A=\left(\frac{b}{c}\right)^2-\left(\frac{a}{c}\right)^2=\frac{b^2-a^2}{c^2}$.

(4) $\tan 2A=\frac{\sin 2A}{\cos 2A}=\frac{2\sin A\cos A}{\cos 2A}=\frac{2(a/c)(b/c)}{(b^2-a^2)/c^2}=\frac{2ab}{b^2-a^2}$.

(5) $\sin 3A=3\sin A-4\sin^3 A=3\left(\frac{a}{c}\right)-4\left(\frac{a}{c}\right)^3=\frac{3ac^2-4a^3}{c^3}=\frac{3a(a^2+b^2)-4a^3}{c^3}$.

(6) $\cos 3A=4\cos^3 A-3\cos A=4\left(\frac{b}{c}\right)^3-3\left(\frac{b}{c}\right)=\frac{4b^3-3bc^2}{c^3}=\frac{4b^3-3b(a^2+b^2)}{c^3}$.

(7) $\tan\frac{A}{2}=\frac{\sin\frac{A}{2}}{\cos\frac{A}{2}}=\frac{2\sin\frac{A}{2}\cos\frac{A}{2}}{2\cos^2\frac{A}{2}}=\frac{\sin A}{1+\cos A}=\frac{a/c}{1+b/c}=\frac{a}{c+b}.$

(8) $\sin^2\frac{A}{2}=\frac{1-\cos A}{2}=\frac{1-b/c}{2}=\frac{c-b}{2c}.$

(9) $\frac{\sin A}{\sin C}+\frac{2\sin B}{\sin C}=\frac{a}{c}+\frac{2b}{c},$ (10) $\frac{\sin^2 A}{\sin^2 C}-\frac{m\sin^2 B}{\sin^2 C}=\frac{a^2}{c^2}-\frac{mb^2}{c^2}.$

(11) 從公式 (1), $a=\frac{c\sin A}{\sin C}$, $b=\frac{c\sin B}{\sin C}$, 故 $(a+b)\cos\frac{C}{2}$

$=\frac{c(\sin A+\sin B)}{\sin C}\sin\frac{C}{2}=\frac{2c\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}C}=\frac{c\cos\frac{1}{2}C\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}.$

(12) 同上.

(13) 從 3 節一圖, $c=BD+AD=a\cos B+b\cos A$. 又從二圖, $c=BD-AD$

$=a\cos B-b\cos CAD=a\cos B-b\cos(180°-A)=a\cos B+b\cos A.$

(14) $c\cos(A-B)=\frac{a\sin C}{\sin A}\cos(A-B)=\frac{a}{\sin A}\sin(A+B)\cos(A-B)$

$=\frac{a}{\sin A}(\sin A\cos B+\cos A\sin B)(\cos A\cos B+\sin A\sin B)$

$=\frac{a}{\sin A}\left\{\sin A\cos A(\cos^2 B+\sin^2 B)+\sin B\cos B(\cos^2 A+\sin^2 A)\right\}$

$=\frac{a}{\sin A}(\sin A\cos A+\sin B\cos B)=a\cos A+\frac{a}{\sin A}(\sin B\cos B)$

$=a\cos A+\frac{b}{\sin B}(\sin B\cos B).$

(15) $c\sin(A-B)=\frac{a\sin C}{\sin A}\sin(A-B)=\frac{a}{\sin A}\sin(A+B)\sin(A-B)$

$=\frac{a}{\sin A}(\sin^2 A-\sin^2 B)=a\sin A-\frac{a}{\sin A}\sin^2 B=a\sin A-b\sin B.$

(16) 從前例, $c\sin(A-B)=a\sin A-b\sin B,$

$a\sin(B-C)=b\sin B-c\sin C,\ b\sin(C-A)=c\sin C-a\sin A,$

相加, 則 $a\sin(B-C)+b\sin(C-A)+c\sin(A-B)=0.$

(17) 由公式 (2), $ca\cos B=\frac{c^2+a^2-b^2}{2}$, $bc\cos A=\frac{b^2+c^2-a^2}{2}.$

故 $c(a\cos B-b\cos A)=\frac{c^2+a^2-b^2}{2}-\frac{b^2+c^2-a^2}{2}=a^2-b^2.$

(18) 由前例, $c(a\cos B-b\cos A)=\frac{c^2\sin^2 A}{\sin^2 C}-\frac{c^2\sin^2 B}{\sin^2 C}=\frac{c^2(\sin^2 A-\sin^2 B)}{\sin^2 C}$

$=\frac{c^2\sin(A+B)\sin(A-B)}{\sin^2(A+B)}=\frac{c^2\sin(A-B)}{\sin(A+B)}=\frac{c^2\sin(A-B)}{\sin C}.$

故 $(a\cos B-b\cos A)/\sin(A-B)=c/\sin C.$

19. $2(a\cos A-b\cos B)\sin C=c(\sin 2A-\sin 2B)$.

20. $\dfrac{a-c\cos B}{b-c\cos A}=\dfrac{\sin B}{\sin A}$.

21. $\dfrac{b+c}{a}=\dfrac{\cos B+\cos C}{1-\cos A}$.

22. $\dfrac{a}{\sin A}=\dfrac{b+c}{\sin B+\sin C}=\dfrac{b-c}{\sin B-\sin C}$.

23. $\dfrac{a}{b}-\dfrac{b}{a}=c\left(\dfrac{\cos B}{b}-\dfrac{\cos A}{a}\right)$.

24. $\dfrac{a}{\cos B}-\dfrac{b}{\cos A}=\cos C\left(\dfrac{b}{\cos B}-\dfrac{a}{\cos A}\right)$.

25. $\dfrac{\cos A}{b}-\dfrac{\cos B}{a}=\dfrac{\cos C}{c}\left(\dfrac{\sin B}{\sin A}-\dfrac{\sin A}{\sin B}\right)$.

26. $s(s-a)-(s-b)(s-c)=bc\cos A$.

27. $a\cos A+b\cos B+c\cos C=2a\sin B\sin C$.

28. $\cos A+\cos B+\cos C=1+\dfrac{2a\sin B\sin C}{a+b+c}$.

29. $a+b+c=(b+c)\cos A+(c+a)\cos B+(a+b)\cos C$.

30. $c^2=(a+b)^2\sin^2\dfrac{C}{2}+(a-b)^2\cos^2\dfrac{C}{2}$.

例 題 解 自 19. 至 30.

(19) $2(a\cos A-b\cos B)\sin C=2\left(\dfrac{c\sin A}{\sin C}\cos A-\dfrac{c\sin B}{\sin C}\cos B\right)\sin C$

$=c(2\sin A\cos A-2\sin B\cos B)=c(\sin 2A-\sin 2B)$.

(20) $\dfrac{a-c\cos B}{b-c\cos A}=\dfrac{(2a^2-2ca\cos B)b}{(2b^2-2bc\cos A)a}=\dfrac{2a^2-(c^2-a^2-b^2)}{2b^2-(b^2+c^2-a^2)}\times\dfrac{b}{a}$

$$=\frac{a^2+b^2-c^2}{a^2+b^2-c^2}\times\frac{b}{a}=\frac{b}{a}=\frac{\sin B}{\sin A}.$$

(21) $\frac{b+c}{a}=\frac{b}{a}+\frac{c}{a}=\frac{\sin B}{\sin A}+\frac{\sin C}{\sin A}=\frac{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}{\sin A}$

$=\frac{2\cos\frac{1}{2}A\cos\frac{1}{2}(B-C)}{2\sin\frac{1}{2}A\cos\frac{1}{2}A}=\frac{\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A}=\frac{2\sin\frac{1}{2}A\cos\frac{1}{2}(B-C)}{2\sin^2\frac{1}{2}A}=\frac{2\cos\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}{1-\cos A}$

$=(\cos B+\cos C)/(1-\cos A)$.

(22) $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{b+c}{\sin B+\sin C}=\frac{b-c}{\sin B-\sin C}$.

(23) $\frac{a}{b}=\frac{b}{a}=\frac{a^2-b^2}{ab}=\frac{c(a\cos B-b\cos A)}{ab}$. (例題 17.)

(24) $a^2(2bc\cos A)-b^2(2ca\cos B)=a^2(b^2+c^2-a^2)-b^2(c^2+a^2-b^2)$,

卽 $2abc(a\cos A-b\cos B)=-(a^2-b^2)(a^2+b^2-c^2)$, 由例題 17,

$=-c(a\cos B-b\cos A)2ab\cos C$, 故 $a\cos A-b\cos B=\cos C(b\cos A-a\cos B)$.

此可以 $\cos A$, $\cos B$ 除之.

(25) $\frac{\cos A}{b}-\frac{\cos B}{a}=\frac{b^2+c^2-a^2}{2b^2c}-\frac{c^2+a^2-b^2}{2a^2c}=\frac{a^2(b^2+c^2-a^2)-b^2(c^2+a^2-b^2)}{2a^2b^2c}$

$=\frac{-(a^2-b^2)(a^2+b^2-c^2)}{2a^2b^2c}=\frac{-(a^2-b^2)2ab\cos C}{2a^2b^2c}=\left(\frac{b}{a}-\frac{a}{b}\right)\frac{\cos C}{c}$

$=(\cos C/c)(\sin B/\sin A+\sin A/\sin B)$.

(26) $s(s-a)-(s-b)(s-c)=s(b+c-a)-bc=\frac{1}{2}(b^2+c^2-a^2)=bc\cos A$.

(27) $a\cos A+b\cos B+c\cos C=\frac{c\sin A}{\sin C}\cos A+\frac{c\sin B}{\sin C}\cos B+\frac{c\sin C}{\sin C}\cos C$

$=\frac{c}{2\sin C}(\sin 2A+\sin 2B+\sin 2C)=\frac{2c\sin A\sin B\sin C}{\sin C}$, (例題五 37.)

$=2c\sin A\sin B=2a\sin C\sin B$.

(28) $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, (例題五 31.)

$=1+4\left\{\frac{s(s-a)(s-b)(s-c)}{abcs}\right\}$. [公式 (3)] $=1+\frac{4s^2}{abcs}$ [公式 (6)]

$=1+\frac{ac\sin B\,ba\sin C}{\frac{1}{2}abc(a+b+c)}=1+\frac{2a\sin B\sin C}{a+b+c}$, (29) 應用前例.

(30) $c^2=a^2+b^2-2ab\cos C=a^2+b^2-2ab\left(2\cos^2\frac{C}{2}-1\right)=(a+b)^2-4ab\cos^2\frac{C}{2}$,

故 $c^2\sin^2\frac{C}{2}=(a+b)^2\sin^2\frac{C}{2}-ab\sin^2C$.

同樣. $c^2\cos^2\frac{C}{2}=(a-b)^2\cos^2\frac{C}{2}-ab\cos^2C$. 此二式相加卽得.

31. $a\sin^2 C = c(\cos C\cos A + \cos B)$.

32. $a(\cos B\cos C + \cos A) = b(\cos C\cos A + \cos B)$.

$$= c(\cos A\cos B + \cos C).$$

33. $(a+b+c)(\cos A + \cos B + \cos C)$

$$= 2a\cos^2\frac{A}{2} + 2b\cos^2\frac{B}{2} + 2c\cos^2\frac{C}{2}.$$

34. $\dfrac{\cos^2\frac{A}{2}}{a} + \dfrac{\cos^2\frac{B}{2}}{b} + \dfrac{\cos^2\frac{C}{2}}{c} = \dfrac{s^2}{abc}$.

35. $\dfrac{\cos A}{a} + \dfrac{\cos B}{b} + \dfrac{\cos C}{c} = \dfrac{a^2+b^2+c^2}{2abc}$.

36. $\dfrac{b^2\cos A}{a} + \dfrac{c^2\cos B}{b} + \dfrac{a^2\cos C}{c} = \dfrac{a^4+b^4+c^4}{2abc}$.

37. $\dfrac{\cos A\cos B}{ab} + \dfrac{\cos B\cos C}{bc} + \dfrac{\cos C\cos A}{ca} = \dfrac{\sin^2 A}{2^2}$.

38. $\tan A = \dfrac{a\sin C}{b - a\cos C}$. **39.** $\dfrac{\tan A}{\tan B} = \dfrac{c^2+a^2-b^2}{b^2+c^2-a^2}$.

40. $\cot B - \cot A = \dfrac{a^2-b^2}{ab}\operatorname{cosec} A$.

41. $(s-b)\tan\dfrac{B}{2} = (s-c)\tan\dfrac{C}{2} = (s-a)\tan\dfrac{A}{2}$.

例題解自 31 至 41.

(31) $a\sin^2 C = a\sin C\sin C = c\sin A\sin C = \frac{1}{2}c\{\cos(C-A) - \cos(C+A)\}$

$= \frac{1}{2}c\{\cos C\cos A + \sin C\sin A - \cos(180^\circ - B)\}$.

即 $a\sin^2 C = \frac{1}{2}c(\cos C\cos A + \cos B) + \frac{1}{2}c\sin C\sin A$

$= \frac{1}{2}c(\cos C\cos A + \cos B) + \frac{1}{2}a\sin^2 C$.

(32) $a(\cos B\cos C + \cos A) = a\{\cos(B+C) + \sin B\sin C + \cos A\}$

$= a\{\cos(180^\circ - A) + \sin B\sin C + \cos A\} = a\sin B\sin C$

$=b\sin A\sin C=b\{\cos A\cos C-\cos(A+C)\}=b(\cos A\cos C+\cos B)$.

(33) $(a+b+c)(\cos A+\cos B+\cos C)=a+b+c+2a\sin B\sin C$ (例題 28.)

$=a+b+c+a\cos A+b\cos B+c\cos C$ (例題 27.)

$=a(1+\cos A)+b(1+\cos B)+c(1+\cos C)=2a\cos^2\frac{A}{2}+2b\cos^2\frac{B}{2}+2c\cos^2\frac{C}{2}$.

(34) 將原式之左邊代入公式(3),則

$$\frac{s(s-a)}{abc}+\frac{s(s-b)}{abc}+\frac{s(s-c)}{abc}=\frac{s(3s-a-b-c)}{abc}=\frac{s^2}{abc}.$$

(35) 代入公式(2),則 $\frac{b^2+c^2-a^2}{2abc}=\frac{c^2+a^2-b^2}{2abc}=\frac{a^2+b^2-c^2}{2abc}=\frac{a^2+b^2+c^2}{2abc}$.

(36) 代入公式(2),卽可得其結果.

(37) 代入公式(2),則

$$\frac{(b^2+c^2-a^2)(c^2+a^2-b^2)}{4a^2b^2c^2}+\frac{(c^2+a^2-b^2)(a^2+b^2-c^2)}{4a^2b^2c^2}+\frac{(a^2+b^2-c^2)(b^2+c^2-a^2)}{4a^2b^2c^2}$$

$$=\frac{2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4}{4a^2b^2c^2}=\frac{16s(s-a)(s-b)(s-c)}{4a^2b^2c^2}$$

$$=\frac{16S^2}{4a^2b^2c^2}=\frac{(bc\sin A)^2}{a^2b^2c^2},\ [公式(6)]=\frac{\sin^2 A}{a^2}.$$

(38) $\tan A=\frac{c\sin A}{c\cos A}=\frac{a\sin C}{b-a\cos C}$, [由例題 13.]

(39) $\frac{\tan A}{\tan B}=\frac{a\sin C}{b-a\cos C}\div\frac{b\sin A}{c-b\cos A}$, [依前例] $\frac{a\sin C(c-b\cos A)}{b\sin A(b-a\cos C)}$

$$=\frac{ac(c-b\cos A)}{ba(b-a\cos C)}=\frac{2c^2-2bc\cos A}{2b^2-2ab\cos C}=\frac{2c^2-(b^2+c^2-a^2)}{2b^2-(a^2+b^2-c^2)}.$$

(40) $\cot B-\cot A=\frac{1}{\tan B}-\frac{1}{\tan A}=\frac{c-b\cos A}{b\sin A}-\frac{b-a\cos C}{a\sin C}$, (例題 38.)

$$=\frac{c-b\cos A}{b\sin A}-\frac{b-a\cos C}{c\sin A}=\frac{c^2-b^2-bc\cos A+ab\cos C}{bc\sin A}$$

$$=\frac{2c^2-2b^2-(b^2+c^2-a^2)+(a^2+b^2-c^2)}{2bc\sin A}=\frac{a^2-b^2}{bc\sin A}=\frac{a^2-b^2}{bc}\operatorname{cosec}A,$$

(41) $(s-b)\tan\frac{B}{2}=(s-b)\sqrt{\frac{(s-a)(s-c)}{s(s-b)}}=\sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$.

42. $(a^2-b^2)\cot C+(b^2-c^2)\cot A+(c^2-a^2)\cot B=0.$

43 $(a-b)\cot\frac{C}{2}+(b-c)\cot\frac{A}{2}+(c-a)\cot\frac{B}{2}=0.$

44. $1-\tan\frac{A}{2}\tan\frac{B}{2}=c/s.$

45. $a^2(s-a)\sec^2\frac{A}{2}+b^2(s-b)\sec^2\frac{B}{2}+c^2(s-c)\sec^2\frac{C}{2}=2abc.$

46. $(\cot\frac{1}{2}B+\cot\frac{1}{2}C)/\cot\frac{1}{2}A=\frac{a}{s-a}.$

47. $\frac{\sin 2A}{a^2(b^2+c^2-a^2)}=\frac{\sin 2B}{b^2(c^2+a^2-b^2)}=\frac{\sin 2C}{c^2(a^2+b^2-c^2)}.$

48. $(\cot\frac{1}{2}A-\operatorname{cosec}\frac{1}{2}A)/(\cot\frac{1}{2}B+\cot\frac{1}{2}C)=(b+c-a)/(2a).$

49. $b^2\sin 2C-c^2\sin 2B=2bc\sin(B-C).$

50. $b^2\sin 2C+c^2\sin 2B=2bc\sin A.$

51. $\tan A(\sin^2 B+\sin^2 C-\sin^2 A)=\tan B(\sin^2 C+\sin^2 A-\sin^2 B).$

52. $bc\cos^2\frac{A}{2}+ca\cos^2\frac{B}{2}+ab\cos^2\frac{C}{2}=s^2.$

53. $c(\sin^2 A+\sin^2 B)=\sin C(a\sin A+b\sin B).$

54. $4(a\sin A+b\sin B+c\sin C)\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$

$$=(a+b+c)(\sin^2 A+\sin^2 B+\sin^2 C).$$

例題解自 42. 至 54.

(42) 令 $a/\sin A=b/\sin B=c/\sin C=k$, 則

原式$=k^2\{(\sin^2 A-\sin^2 B)\cot C+(\sin^2 B-\sin^2 C)\cot A+(\sin^2 C-\sin^2 A)\cot B\}$

$=k^2\{\sin(A+B)\sin(A-B)\frac{\cos C}{\sin C}+\sin(B+C)\sin(B-C)\frac{\cos A}{\sin A}+\sin(C+A)\sin(C-A)\frac{\cos B}{\sin B}\}$

$=-k^2\{\cos(A+B)\sin(A-B)+\cos(B+C)\sin(B-C)+\cos(C+A)\sin(C-A)\}$

$=-\frac{1}{2}k^2(\sin 2A-\sin 2B+\sin 2B-\sin 2C+\sin 2C-\sin 2A)=0.$

(43) 試如前例定k,則

原式 $=k\left\{(\sin A-\sin B)\cot\frac{C}{2}+(\sin B-\sin C)\cot\frac{A}{2}+(\sin C-\sin A)\cot\frac{B}{2}\right\}$

$=2k\left\{\cos\frac{A+B}{2}\sin\frac{A-B}{2}\frac{\cos\frac{1}{2}C}{\sin\frac{1}{2}C}+\cos\frac{B+C}{2}\sin\frac{B-C}{2}\frac{\cos\frac{1}{2}A}{\sin\frac{1}{2}A}+\cos\frac{C+A}{2}\sin\frac{C-A}{2}\frac{\cos\frac{1}{2}B}{\sin\frac{1}{2}B}\right\}$

$=-2k\left(\sin^2\frac{A}{2}-\sin^2\frac{B}{2}+\sin^2\frac{B}{2}-\sin^2\frac{C}{2}+\sin^2\frac{C}{2}-\sin^2\frac{A}{2}\right)=0.$

(44) 原式$=1-\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\times\frac{(s-c)(s-a)}{s(s-b)}\right\}}=1-\frac{s-c}{s}=\frac{c}{s}.$

(45) $\frac{(s-a)a^2}{\cos^2\frac{1}{2}A}+\frac{(s-b)b^2}{\cos^2\frac{1}{2}B}+\frac{(s-c)c^2}{\cos^2\frac{1}{2}C}=\frac{a^2bc}{s}+\frac{b^2ca}{s}+\frac{c^2ab}{s}=2abc.$

(46) 原式$=\frac{\sin\frac{1}{2}(B+C)\sin\frac{1}{2}A}{\sin\frac{1}{2}B\sin\frac{1}{2}C\cos\frac{1}{2}A}=\frac{\sin\frac{1}{2}A}{\sin\frac{1}{2}B\sin\frac{1}{2}C}$, 以公式(3)之 $\sin\frac{A}{2}$, $\sin\frac{B}{2}$, $\sin\frac{C}{2}$ 之值代入.卽可得其證.

(47) $\frac{\sin 2A}{a^2(b^2+c^2-a^2)}=\frac{2\sin A\cos A}{a^2(2bc\cos A)}=\frac{\sin A}{a}\left(\frac{2}{2abc}\right)=\frac{\sin B}{b}\left(\frac{2}{2abc}\right)$

$=2\sin B\cos B/(2b^2ca\cos B)=\sin 2B/\{b^2(c^2+a^2-b^2)\}.$

(48) 原式$=\frac{\sin\frac{1}{2}B\sin\frac{1}{2}C(2\cos^2\frac{1}{2}A-1)}{\sin\frac{1}{2}A\sin\frac{1}{2}(B+C)}=\frac{\sin\frac{1}{2}B\sin\frac{1}{2}C\cos\frac{1}{2}A}{\sin\frac{1}{2}A\cos\frac{1}{2}A}$

$=\frac{\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin\frac{1}{2}A}=\sqrt{\frac{(s-c)(s-a)}{ca}}\sqrt{\frac{(s-a)(s-b)}{ab}}\sqrt{\frac{bc}{(s-b)(s-c)}}=\frac{s-a}{a}.$

(49) 原式 $=2b^2\sin C\cos C-2c^2\sin B\cos B=2bc\sin B\cos C-2cb\sin C\cos B$

$=2bc(\sin B\cos C-\cos B\sin C)=2bc\sin(B-C).$

(50) 試如前例,則原式$=2bc\sin(B+C)=2bc\sin A.$

(51) $\tan A(\sin^2B+\sin^2C-\sin^2A)=\tan A(2\sin B\sin C\cos A)$ 〔例題五62.〕

$=2\sin A\sin B\sin C=\tan B(2\sin C\sin A\cos B).$

(52) 原式 $=bc\left\{\frac{s(s-a)}{bc}\right\}+ca\left\{\frac{s(s-b)}{ca}\right\}+ab\left\{\frac{s(s-c)}{ab}\right\}.$

(53) $c\sin^2A+c\sin^2B=a\sin C\sin A+b\sin C\sin B.$

(54) $4\left(\frac{a}{\sin A}\sin^2B+\frac{b}{\sin B}\sin^2B+\frac{c}{\sin C}\sin^2C\right)\sqrt{\left\{\frac{s(s-a)}{bc}\times\frac{s(s-b)}{ca}\times\frac{s(s-c)}{ab}\right\}}$

$=\frac{4a}{\sin A}(\sin^2A+\sin^2B+\sin^2C)\frac{s\sqrt{s\{(s-a)(s-b)(s-c)\}}}{abc}$, 由公式(6)

$=(4a/\sin A)(\sin^2A+\sin^2B+\sin^2C)\frac{1}{2}sbc\sin A/(abc)=2s(\sin^2A+\sin^2B+\sin^2C).$

55. $a\cos\frac{B}{2}\cos\frac{C}{2}\operatorname{cosec}\frac{A}{2}=s.$

56. $a\sec A+b\sec B+c\sec C=a\sec A\tan B\tan C.$

57. $a^{\frac{1}{2}}=\sqrt{\dfrac{(a+2b-3c)\sin A}{\sin A+2\sin B-3\sin C}}.$

58. $(b+c)\sqrt{bc\sin B\sin C}=b^2\sin C+c^2\sin B.$

59. $\sqrt{c+(a-b)\cos\frac{C}{2}}+\sqrt{c-(a-b)\cos\frac{C}{2}}=2\sqrt{c}\cos\dfrac{A-B}{4}.$

60. $a^{\frac{1}{2}}(b^{\frac{3}{2}}+c^{\frac{3}{2}})\cos A+b^{\frac{1}{2}}(c^{\frac{3}{2}}+a^{\frac{3}{2}})\cos B+c^{\frac{1}{2}}(a^{\frac{3}{2}}+b^{\frac{3}{2}})\cos C$

$$=a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}(a^{\frac{1}{2}}+b^{\frac{1}{2}}+c^{\frac{1}{2}}).$$

61. $\left\{\dfrac{2(a^2-b^2)}{\cos 2B-\cos 2A}\right\}^{\frac{3}{2}}=\dfrac{abc}{\sin A\sin B\sin C}.$

62. $\dfrac{a}{\sin\frac{1}{2}A}=\dfrac{b+c}{\cos\frac{1}{2}(B-C)}.$　　**63.** $\dfrac{a}{\cos\frac{1}{2}A}=\dfrac{b-c}{\sin\frac{1}{2}(B-C)}.$

64. $\dfrac{\operatorname{vers}A}{\operatorname{vers}B}=\dfrac{a(c+a-b)}{b(b+c-a)}.$

例題解自55.至64.

(55) $\dfrac{a\cos\frac{1}{2}B\cos\frac{1}{2}C}{\sin\frac{1}{2}A}=\dfrac{a\sqrt{s(s-b)/(ac)}\sqrt{s(s-c)/(ab)}}{\sqrt{(s-b)(s-c)/(bc)}}=s.$

(56) $\dfrac{a}{\cos A}+\dfrac{b}{\cos B}+\dfrac{c}{\cos C}=\dfrac{2abc}{c^2+a^2-b^2}+\dfrac{2abc}{a^2+b^2-c^2}+\dfrac{2abc}{b^2+c^2-a^2}$

$$=\frac{2abc(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)}{(c^2+a^2-b^2)(a^2+b^2-c^2)(b^2+c^2-a^2)}=\frac{32abcs(s-a)(s-b)(s-c)}{8a^2b^2c^2\cos A\cos B\cos C}$$

$$=\frac{4(\frac{1}{2}ac\sin B)(\frac{1}{2}ab\sin C)}{abc\cos A\cos B\cos C}=a\sec A\tan B\tan C.$$

(57) $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=\dfrac{a+2b-3c}{\sin A+2\sin B-3\sin C}.$

故 $a^{\frac{1}{2}}=\sqrt{\dfrac{(a+2b-3c)\sin A}{\sin A+2\sin B-3\sin C}}$.

(58) $(b+c)\sqrt{\dfrac{ac^2\sin B\,ab^2\sin C}{a^2}}=\dfrac{(b+c)2S}{a}=\dfrac{2Sb}{a}+\dfrac{2Sc}{a}$

$=\dfrac{ab^2\sin C}{a}+\dfrac{ac^2\sin B}{a}=b^2\sin C+c^2\sin B.$

(59) (原式之左邊)$^2=2c+2\sqrt{c^2-(a-b)^2\cos^2\dfrac{C}{2}}$

$=2c+2\sqrt{c^2-(a-b)^2\dfrac{s(s-c)}{ab}}=2c+\sqrt{4c^2-\dfrac{(a-b)^2(a+b+c)(a+b-c)}{ab}}$

$=2c+(a+b)\sqrt{\dfrac{4(s-a)(s-b)}{ab}}=2c+\left(\dfrac{c\sin A}{\sin C}+\dfrac{c\sin B}{\sin C}\right)2\sin\dfrac{C}{2}$

$=2c\left\{1+\dfrac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos\frac{1}{2}C}\right\}=2c\left(1+\cos\dfrac{A-B}{2}\right)=4c\cos^2\dfrac{A-B}{4}$,

由是原式左邊$=2\sqrt{c}\cos\dfrac{A-B}{4}$.

(60) 試代用公式於原式左邊,則

$\dfrac{a^{\frac{1}{2}}(b^{\frac{3}{2}}+c^{\frac{3}{2}})(b^2+c^2-a^2)}{2bc}+\dfrac{b^{\frac{1}{2}}(c^{\frac{3}{2}}+a^{\frac{3}{2}})(c^2+a^2-b^2)}{2ca}+\dfrac{c^{\frac{1}{2}}(a^{\frac{3}{2}}+b^{\frac{3}{2}})(a^2+b^2-c^2)}{2ab}$

$=\dfrac{1}{2abc}\left\{a^{\frac{3}{2}}(b^{\frac{3}{2}}+c^{\frac{3}{2}})(b^2+c^2-a^2)+b^{\frac{3}{2}}(c^{\frac{3}{2}}+a^{\frac{3}{2}})(c^2+a^2-b^2)+c^{\frac{3}{2}}(a^{\frac{3}{2}}+b^{\frac{3}{2}})(a^2+b^2-c^2)\right\}$

$=\dfrac{1}{2abc}\left\{a^{\frac{3}{2}}b^{\frac{3}{2}}(b^2+c^2-a^2+c^2+a^2-b^2)+\text{以下等勢}\right\}$

$=\dfrac{1}{2abc}\left\{2a^{\frac{3}{2}}b^{\frac{3}{2}}c^2+2b^{\frac{3}{2}}c^{\frac{3}{2}}a^2+2c^{\frac{3}{2}}a^{\frac{3}{2}}b^2\right\}=a^{\frac{1}{2}}b^{\frac{1}{2}}c^{\frac{1}{2}}(c^{\frac{1}{2}}+a^{\frac{1}{2}}+b^{\frac{1}{2}})$.

(61) 令 $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k$, 則

$k^2=\dfrac{a^2-b^2}{\sin^2A-\sin^2B}=\dfrac{2(a^2-b^2)}{\cos 2B-\cos 2A}$, 又 $k^3=\dfrac{abc}{\sin A\sin B\sin C}$.

(62) 從前例 $k=\dfrac{a}{\sin A}=\dfrac{b+c}{\sin B+\sin C}=\dfrac{b+c}{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}$,

即 $\dfrac{a}{\sin\frac{1}{2}A}=\dfrac{b+c}{\cos\frac{1}{2}(B-C)}$. (63) 同上.

(64) 原式$=\sin^2\dfrac{A}{2}\quad\sin^2\dfrac{B}{2}=\dfrac{(s-b)(s-c)}{bc}\Big/\dfrac{(s-c)(s-a)}{ca}=\dfrac{a(s-b)}{b(s-a)}$.

65. $a\left(b\cos^2\frac{C}{2}-c\cos^2\frac{B}{2}\right)^2=(b-c)\left(b^2\cos^2\frac{C}{2}-c^2\cos^2\frac{B}{2}\right).$

66. $a\cos(B-C)+b\cos(C-A)+c\cos(A-B)$

$$+a\cos A+b\cos B+c\cos C=6a\sin B\sin C.$$

67. $\frac{\sin A}{\cos B}+\frac{\sin B}{\cos C}+\frac{\sin C}{\cos A}+\frac{\sin A}{\cos C}+\frac{\sin B}{\cos A}+\frac{\sin C}{\cos B}$

$=\sin A+\sin B+\sin C+(\cos A+\cos B+\cos C)\tan A\tan B\tan C.$

68. $\frac{bc}{(b-a)(c-a)}\tan^2\frac{B}{2}\tan^2\frac{C}{2}+\frac{ca}{(c-b)(a-b)}\tan^2\frac{C}{2}\tan^2\frac{A}{2}$

$$+\frac{ab}{(a-c)(b-c)}\tan^2\frac{A}{2}\tan^2\frac{B}{2}=1.$$

69. $\frac{\sin(A-B)}{ab}+\frac{\sin(B-C)}{bc}+\frac{\sin(C-A)}{ca}=0.$

70. $\frac{a^2\sin(B-C)}{\sin A}+\frac{b^2\sin(C-A)}{\sin B}+\frac{c^2\sin(A-B)}{\sin C}=0.$

71. $a\sin(B-C)\cos(B+C-A)+b\sin(C-A)\cos(C+A-B)$

$$+c\sin(A-B)\cos(A+B-C)=0.$$

72. $\frac{a^2-b^2}{ab\sin^2\frac{1}{2}C}+\frac{b^2-c^2}{bc\sin^2\frac{1}{2}A}+\frac{c^2-a^2}{ca\sin^2\frac{1}{2}B}$

$=-8\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(B-C)\sin\frac{1}{2}(C-A)/(\sin A\sin B\sin C).$

例題解自 65 至 72.

(65) $a\left(b\cos^2\frac{C}{2}-c\cos^2\frac{B}{2}\right)^2=a\left\{b\frac{s(s-c)}{ab}-c\frac{s(s-b)}{ca}\right\}^2$

$=\frac{s^2(b-c)^2}{a}=(b-c)\left\{\frac{s(bs-cs)}{a}\right\}=(b-c)\left\{\frac{bs(s-c)-cs(s-b)}{a}\right\}$

$=(b-c)\left\{b^2\frac{s(s-c)}{ab}-c^2\frac{s(s-b)}{ca}\right\}=(b-c)\left(b^2\cos^2\frac{C}{2}-c^2\cos^2\frac{B}{2}\right).$

(66) 原式$=a\{\cos A+\cos(B-C)\}+b\{\cos B+\cos(C-A)\}+c\{\cos C+\cos(A-B)\}$

$=2a\cos\frac{1}{2}(A+B-C)\cos\frac{1}{2}(A+C-B)+2b\cos\frac{1}{2}(B+C-A)\cos\frac{1}{2}(B+A-C)$

$+2c\cos\frac{1}{2}(C+A-B)\cos\frac{1}{2}(C+B-A)=2a\sin C\sin B+2b\sin A\sin C+2c\sin B\sin A$

$=2a\sin B\sin C+2a\sin B\sin C+2a\sin C\sin B=6a\sin B\sin C.$

(67) $\frac{\sin(B+C)}{\cos B}+\frac{\sin(C+A)}{\cos C}+\frac{\sin(A+B)}{\cos A}+\frac{\sin(B+C)}{\cos C}+\frac{\sin(C+A)}{\cos A}+\frac{\sin(A+B)}{\cos B}$

$=\tan B\cos C+\sin C+\tan C\cos A+\sin A+\tan A\cos B+\sin B+\sin B+\cos B\tan C$

$+\sin C+\cos C\tan A+\sin A+\cos A\tan B=\sin A+\sin B+\sin C$

$+\tan A(\cos A+\cos B+\cos C)+\tan B(\cos A+\cos B+\cos C)+\tan C(\cos A+\cos B+\cos C)$

$=\sin A+\sin B+\sin C+(\cos A+\cos B+\cos C)(\tan A+\tan B+\tan C)$

但由例題五 35. $\tan A+\tan B+\tan C=\tan A\tan B\tan C.$

(68) $\frac{bc}{(b-a)(c-a)}\tan^2\frac{B}{2}\tan^2\frac{C}{2}=\frac{bc}{(b-a)(c-a)}\times\frac{(s-a)(s-c)}{s(s-b)}\times\frac{(s-a)(s-b)}{s(s-c)}$

$=\frac{bc(s-c)^2}{s^2(b-a)(c-a)}$, $\frac{bc(s-c)^2}{s^2(b-a)(c-a)}+\frac{ca(s-b)^2}{s^2(c-b)(a-b)}+\frac{ab(s-c)^2}{s^2(a-c)(b-c)}=1.$

(69) 令 $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$, 則原式$=\frac{\sin(A-B)}{k^2\sin A\sin B}+\frac{\sin(B-C)}{k^2\sin B\sin C}$

$+\frac{\sin(C-A)}{k^2\sin C\sin A}=\frac{\sin C\sin(A-B)+\sin A\sin(B-C)+\sin B\sin(C-A)}{k^2\sin A\sin B\sin C}=0.$

因分子$=\sin(A+B)\sin(A-B)+$以下等勢$=\sin^2A-\sin^2B+$以下等勢$=0.$

(70) 如前例定k, 則原式等於前例之分子.

(71) 由例題 69. $c\sin(A-B)=-b\sin(C-A)-a\sin(B-C)$, 故

原式$=a\sin(B-C)\cos(B+C-A)+b\sin(C-A)\cos(C+A-B)-\{b\sin(C-A)$

$+a\sin(B-C)\}\cos(A+B-C)=a\sin(B-C)\{\cos(B+C-A)-\cos(A+B-C)\}$

$+b\sin(C-A)\{\cos(C+A-B)-\cos(A+B-C)\}=-2a\sin(B-C)\sin B\sin(C-A)$

$+2b\sin(C-A)\sin A\sin(B-C)=-2a\sin B\sin(B-C)\sin(C-A)$

$+2a\sin B\sin(B-C)\sin(C-A)=0.$

(72) 如 69. 定k, 則

原式$=\frac{k^2(\sin^2A-\sin^2B)}{k^2\sin A\sin B\sin^2\frac{1}{2}C}+\frac{k^2(\sin^2B-\sin^2C)}{k^2\sin B\sin C\sin^2\frac{1}{2}A}+\frac{k^2(\sin^2C-\sin^2A)}{k^2\sin C\sin A\sin^2\frac{1}{2}B}$

$=\frac{2(1+\cos C)\sin(A-B)}{\sin A\sin B\sin C}+\frac{2(1+\cos A)\sin(B-C)}{\sin A\sin B\sin C}+\frac{2(1+\cos B)\sin(C-A)}{\sin A\sin B\sin C}$

分子$=2\{\sin(A-B)+\sin(B-C)+\sin(C-A)\}-2\{\cos(A+B)\sin(A-B)+$ 以下等勢$\}$

$=-8\sin\frac{1}{2}(A-B)\sin\frac{1}{2}(B-C)\sin\frac{1}{2}(C-A)-\{\sin2A-\sin2B+$ 以下等勢$\}$. 〔例題五 15.〕

73. $\dfrac{\sin^2\frac{1}{2}A}{a}+\dfrac{\sin^2\frac{1}{2}B}{b}+\dfrac{\sin^2\frac{1}{2}C}{c}=\dfrac{2ab+2bc+2ca-a^2-b^2-c^2}{4abc}$.

74. $\dfrac{a\cos\frac{1}{2}(B-C)}{bc\cos\frac{1}{2}(B+C)}+\dfrac{b\cos\frac{1}{2}(C-A)}{ca\cos\frac{1}{2}(C+A)}+\dfrac{c\cos\frac{1}{2}(A-B)}{ab\cos\frac{1}{2}(A+B)}$

$$=\frac{2(ab+bc+ca)}{abc}.$$

75. $\dfrac{a\sin\frac{1}{2}(B-C)}{\sin\frac{1}{2}A}+\dfrac{b\sin\frac{1}{2}(C-A)}{\sin\frac{1}{2}B}+\dfrac{c\sin\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}=0.$

76. $\dfrac{a^2\sin(B-C)}{\sin B+\sin C}+\dfrac{b^2\sin(C-A)}{\sin C+\sin A}+\dfrac{c^2\sin(A-B)}{\sin A+\sin B}=0.$

77. $(\sin A+\sin B+\sin C)\left(\tan\dfrac{A}{2}+\tan\dfrac{B}{2}+\tan\dfrac{C}{2}\right)$

$$=4+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}.$$

78. $\left(\dfrac{\sin A+\sin B+\sin C}{a+b+c}\right)^2=\dfrac{a\cos A+b\cos B+c\cos C}{2abc}.$

79. $\cot\dfrac{A}{2}+\cot\dfrac{B}{2}+\cot\dfrac{C}{2}=\dfrac{a+b+c}{b+c-a}\cot\dfrac{A}{2}.$

例題解自 73 至 79.

(73) $\dfrac{\sin^2\frac{1}{2}A}{a}+\dfrac{\sin^2\frac{1}{2}B}{b}+\dfrac{\sin^2\frac{1}{2}C}{c}$

$=\dfrac{(s-b)(s-c)}{abc}+\dfrac{(s-c)(s-a)}{abc}+\dfrac{(s-a)(s-b)}{abc}=\dfrac{3s^2-2s(a+b+c)+ab+bc+ca}{abc}$

$=\dfrac{-s^2+ab+bc+ca}{abc}=\dfrac{-(a+b+c)^2+4(ab+bc+ca)}{4abc}.$

(74) 由例題62. $\dfrac{a\cos\frac{1}{2}(B-C)}{\sin\frac{1}{2}A}=b+c,$

即 $\dfrac{a\cos\frac{1}{2}(B-C)}{bc\cos\frac{1}{2}(B+C)}=\dfrac{b+c}{bc}$, 故

原式$=\frac{b+c}{bc}+\frac{c+a}{ca}+\frac{a+b}{ab}=\frac{a(b+c)+b(c+a)+c(a+b)}{abc}$

$=\frac{2(ab+bc+ca)}{abc}$.

(75)　由例題63.　$a\sin\frac{1}{2}(B-C)\operatorname{cosec}\frac{1}{2}A=\frac{(b-c)\cos\frac{1}{2}A}{\sin\frac{1}{2}A}=(b-c)\cot\frac{1}{2}A$,

故原式$=(b-c)\cot\frac{1}{2}A+(c-a)\cot\frac{1}{2}B+(a-b)\cot\frac{1}{2}C=0$.　例題43.

(76)　令　$a/\sin A=b/\sin B=c/\sin C=k$.　則

$$\frac{a^2\sin(B-C)}{\sin B+\sin C}=\frac{k^2\sin^2 A\sin(B-C)}{2\sin\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}=\frac{k^2\sin^2 A\sin\frac{1}{2}(B-C)}{\cos\frac{1}{2}A}$$

$$=2k^2\sin A\sin\frac{A}{2}\sin\frac{B-C}{2}=k^2\sin A\left(\cos\frac{A-B+C}{2}-\cos\frac{A+B-C}{2}\right)$$

$=k^2\sin A\{\cos(90^\circ-B)-\cos(90^\circ-C)\}=k^2\sin A(\sin B-\sin C)$,　故

原式$=k^2\{\sin A(\sin B-\sin C)+\sin B(\sin C-\sin A)+\sin C(\sin A-\sin B)\}$

$=k^2\{0\}=0$.

(77)　原式$=(\sin A+\sin B+\sin C)\frac{4+4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}{\sin A+\sin B+\sin C}$,　(例題五44.)

$=4+4\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$.

(78)　$\left(\frac{\sin A+\sin B+\sin C}{a+b+c}\right)^2=\frac{\sin^2 A}{a^2}=\frac{b^2c^2\sin^2 A}{a^2b^2c^2}=\frac{4S^2}{a^2b^2c^2}$

$$=\frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{4a^2b^2c^2}=\frac{a^2(b^2+c^2-a^2)+b^2(c^2+a^2-b^2)+c^2(a^2+b^2-c^2)}{4a^2b^2c^2}$$

$$=\frac{\frac{a}{2bc}(b^2+c^2-a^2)+\frac{b}{2ca}(c^2+a^2-b^2)+\frac{c}{2ab}(a^2+b^2-c^2)}{2abc}$$

$$=\frac{a\cos A+b\cos B+c\cos C}{2abc}.$$

(79)　$\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}+\sqrt{\frac{s(s-b)}{(s-c)(s-a)}}+\sqrt{\frac{s(s-c)}{(s-a)(s-b)}}$

$$=\frac{(s-a+s-b+s-c)\sqrt{s}}{\sqrt{\{(s-a)(s-b)(s-c)\}}}=\frac{(3s-a-b-c)\sqrt{s}}{\sqrt{\{(s-a)(s-b)(s-c)\}}}$$

$$=\frac{s}{(s-a)}\sqrt{\frac{s(s-a)}{(s-b)(s-c)}}=\frac{a+b+c}{b+c-a}\cot\frac{A}{2}.$$

80. $\dfrac{a\cos 2(B-C)}{\cos B\cos C}+\dfrac{b\cos 2(C-A)}{\cos C\cos A}+\dfrac{c\cos 2(A-B)}{\cos A\cos B}$

$=8(a\cos A+b\cos B+c\cos C)$.

81. $a^3\cos 3B+b^3\cos 3A=c^3-3abc\cos(A-B)$.

82. $a^2-2ab\cos(60^\circ+C)=c^2-2bc\cos(60^\circ+A)$.

83. $b^2\cos 2C+2bc\cos(B-C)+c^2\cos 2B=a^2$.

84. $\dfrac{a^2\cos\frac{1}{2}(B-C)}{\cos\frac{1}{2}(B+C)}+\dfrac{b^2\cos\frac{1}{2}(C-A)}{\cos\frac{1}{2}(C+A)}+\dfrac{c^2\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}$

$=2(ab+bc+ca)$.

85. $a^2\{\cos(B-C)+\cos A\}/4\sin A=S$.

86. $\dfrac{s^2}{\cot\frac{1}{2}A+\cot\frac{1}{2}B+\cot\frac{1}{2}C}=S$.

例題解自 80 至 86.

(80) 令 $\dfrac{a}{\sin A}=\dfrac{b}{\sin B}=\dfrac{c}{\sin C}=k$, 則

$\dfrac{a\cos 2(B-C)}{\cos B\cos C}=\dfrac{k\sin 2A\cos 2(B-C)}{2\cos A\cos B\cos C}=\dfrac{k\{\sin(2A+2B-2C)+\sin(2A+2C-2B)\}}{4\cos A\cos B\cos C}$

$=\dfrac{-k(\sin 4C+\sin 4B)}{4\cos A\cos B\cos C}$, 故原式$=\dfrac{-k(\sin 4A+\sin 4B+\sin 4C)}{2\cos A\cos B\cos C}$

$=\dfrac{2k\sin 2A\sin 2B\sin 2C}{\cos A\cos B\cos C}=16k\sin A\sin B\sin C=16a\sin B\sin C$

$=8a\{\cos(B-C)-\cos(B+C)\}=8a\{\cos(B-C)+\cos A\}=8k\sin A\cos(B-C)+8a\cos A$

$=8k\sin(B+C)\cos(B-C)+8a\cos A=4k(\sin 2B+\sin 2C)+8a\cos A$

$=8k\sin B\cos B+8k\sin C\cos C+8a\cos A$

$=8b\cos B+8c\cos C+8a\cos A$.

(81) 令 $a/\sin A=b/\sin B=c/\sin C=k$, 則

$a^3\cos 3B+b^3\cos 3A=k^3(\sin^3 A\cos 3B+\sin^3 B\cos 3A)=k^3\{4(\sin^3 A\cos^3 B+\sin^3 B\cos^3 A)$

$-3(\sin^3 A\cos B+\sin^3 B\cos A)\}=k^3\{4\sin^3 C-12\sin A\sin B\cos A\cos B\sin C$

$-3\sin A\cos B(1-\cos^2 A)-3\sin B\cos A(1-\cos^2 B)\}$

$=k^3[4\sin^3 C-6\cos A\cos B\sin C\{\cos(A-B)-\cos(A+B)\}-3\sin C$

$+3\cos A\cos B(\sin A\cos A+\sin B\cos B)]=k^3[\sin^3 C-3\sin C\cos^2 C$

$-6\cos A\cos B\sin C\{\cos(A-B)+\cos C\}+3\cos A\cos B\sin C\cos(A-B)]$

$=k^3[\sin^3 C-3\sin C\{\cos^2 C+2\cos A\cos B\cos(A-B)+2\cos A\cos B\cos C$

$-\cos A\cos B\cos(A-B)\}]=k^3[\sin^3 C-3\sin C\{\cos C\cos(A-B)+\cos A\cos B\cos(A-B)\}]$

$=k^3[\sin^3 C-3\sin C\cos(A-B)\{-\cos(A+B)+\cos A\cos B\}]$

$=k^3\sin^3 C-3k^3\sin C\sin A\sin B\cos(A-B)$

$=c^3-3abc\cos(A-B).$

(82) $a^2-2ab(\cos 60°\cos C-\sin 60°\sin C)=a^2-\frac{1}{2}(a^2+b^2-C^2)+2ab\sin 60°\sin C$

$=\frac{1}{2}(a^2+c^2-b^2)+2bc\sin 60°\sin A=\frac{1}{2}(a^2+c^2-b^2)+2bc\cos 60°\cos A-2bc\cos(60°+A)$

$=\frac{1}{2}(a^2+c^2-b^2)+\frac{1}{2}(b^2+c^2-a^2)-2bc\cos(60°+A)$

$=c^2-2bc\cos(60°+A).$

(83) $b^2(\cos^2 C-\sin^2 C)+2bc\cos(B-C)+c^2(\cos^2 B-\sin^2 B)$

$=(b\cos C+c\cos B)^2-(b\sin C-c\sin B)^2-2bc(\cos B\cos C+\sin B\sin C)+2bc\cos(B-C)$

$=(b\cos C+c\cos B)^2-(c\sin B-c\sin B)^2$

$=\left\{\frac{b(a^2+b^2-c^2)}{2ab}+\frac{c(a^2+c^2-b^2)}{2ac}\right\}^2=\left(\frac{2a^2}{2a}\right)^2=a^2.$

(84) $\frac{a^2\cos\frac{1}{2}(B-C)}{\cos\frac{1}{2}(B+C)}=\frac{2a^2\cos\frac{1}{2}(B+C)\cos\frac{1}{2}(B-C)}{2\cos^2\frac{1}{2}(B+C)}=\frac{a^2(\cos B+\cos C)}{1+\cos(B+C)}$

$=\frac{a^2(\cos B+\cos C)}{1-\cos A}$，於此代用公式 (2) 之 $\cos A$, $\cos B$, $\cos C$ 之值而簡約之，

則 $=ab+ac$. 故 原式$=2(ab+bc+ca)$.

(85) $\frac{a^2\{\cos(B-C)-\cos(B+C)\}}{4\sin A}=\frac{2a^2\sin B\sin C}{4\sin A}=\frac{ab\sin B\sin C}{2\sin B}=\frac{1}{2}ab\sin C=S.$

(86) $\frac{s^2}{\cot\frac{1}{2}A+\cot\frac{1}{2}B+\cot\frac{1}{2}C}=\frac{s^2}{\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C}$，〔例題五 43.〕

$=\frac{s^2\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$

$=s^2\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\times\frac{(s-c)(s-a)}{ca}\times\frac{(s-a)(s-b)}{ab}\right\}}\Big/\sqrt{\left\{\frac{s(s-a)}{bc}\times\frac{s(s-b)}{ca}\times\frac{s(s-c)}{ab}\right\}}$

$=\sqrt{s(s-a)(s-b)(s-c)}=S.$

87. $\frac{1}{12}\{(a^2+b^2)\sin 2C+(b^2+c^2)\sin 2A+(c^2+a^2)\sin 2B\}=S.$

88. $\frac{(a^2-b^2)\sin A\sin B}{2\sin(A-B)}=S$, 及 $\frac{a^2}{2(\cot B+\cot C)}=S.$

89. $\left(\frac{a^2}{\sin A}+\frac{b^2}{\sin B}+\frac{c^2}{\sin C}\right)\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=S.$

90. $8S^2(\cot^2 A+\cot^2 B+\cot^2 C+1)=a^4+b^4+c^4.$

91. $(b^2-c^2)\cot^2\frac{A}{2}+(c^2-a^2)\cot^2\frac{B}{2}+(a^2-b^2)\cot^2\frac{C}{2}$

$$+2s^2(a-b)(b-c)(c-a)/S^2=0.$$

自A, B, 及C引各對邊之垂線, 順次令爲h_1, h_2及h_3, 示次各式之證.

92. $\frac{h_1^2}{h_2h_3}+\frac{h_2^2}{h_3h_1}+\frac{h_3^2}{h_1h_2}=\frac{ab}{c^2}+\frac{bc}{a^2}+\frac{ca}{b^2}.$

93. $2h_3(ab\cos C+bc\cos A+ca\cos B)=ab(a\sin A+b\sin B+c\sin C).$

例題解自87至93.

(87) 令 $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$, 則

$$\frac{k^2}{12}\left\{(\sin^2 A+\sin^2 B)\sin 2C+(\sin^2 B+\sin^2 C)\sin 2A+(\sin^2 C+\sin^2 A)\sin 2B\right\}$$

$$=\frac{k^2}{24}\left\{(2-\cos 2A-\cos 2B)\sin 2C+(2-\cos 2B-\cos 2C)\sin 2A\right.$$
$$\left.+(2-\cos 2C-\cos 2A)\sin 2B\right\}$$

$$=\frac{k^2}{24}\left\{2(\sin 2A+\sin 2B+\sin 2C)-\sin(2A+2B)-\sin(2B+2C)-\sin(2C+2A)\right\}$$

$$=\frac{k^2}{24}\left\{8\sin A\sin B\sin C+\sin 2C+\sin 2A+\sin 2B\right\}$$

$$=\frac{k^2}{2}\sin A\sin B\sin C=\tfrac{1}{2}bc\sin A=S.$$

(88) 定k如前例. 則原式$=k^2(\sin^2 A-\sin^2 B)\sin A\sin B/2\sin(A-B)$

$=k^2\sin(A+B)\sin(A-B)\sin A\sin B/2\sin(A-B)=\frac{1}{2}k^2\sin C\sin A\sin B$

$=\frac{1}{2}bc\sin A=S.$

及原式$=\dfrac{a^2\sin B\sin C}{2\sin(B+C)}=\dfrac{a^2\sin B\sin C}{2\sin A}=\dfrac{ab\sin A\sin C}{2\sin A}=\frac{1}{2}ab\sin C=S.$

(89) 如 80. 定 k, 則

原式$=k(a+b+c)\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}$

$=k^2(\sin A+\sin B+\sin C)\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}.$ 由例題 30.

$=4k^2\cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}=\dfrac{1}{2}k^2\sin A\sin B\sin C=S.$

(90) 令 $\sin A/a=\lambda$, 則

$\cot^2A=\dfrac{\cos^2A}{\sin^2A}=\dfrac{(b^2+c^2-a^2)^2/(2bc)^2}{a^2\lambda}=\dfrac{(b^2+c^2-a^2)^2}{4a^2b^2c^2\lambda^2}=\dfrac{(b^2+c^2-a^2)^2}{16s^2}.$

故 原式$=8S^2\left\{\dfrac{(b^2+c^2-a^2)^2+(c^2+a^2-b^2)^2+(a^2+b^2-c^2)^2}{16s^2}+1\right\}$

$=\frac{1}{2}\{3(a^4+b^4+c^4)-2(a^2b^2+b^2c^2+c^2a^2)+16S^2\}$

$=\frac{1}{2}\{3(a^4+b^4+c^4)-2(a^2b^2+b^2c^2+c^2a^2)+2(a^2b^2+b^2c^2+c^2a^2)-a^4-b^4-c^4\}$

$=a^4+b^4+c^4.$

(91) $(b^2-c^2)\cot^2\dfrac{A}{2}=(b^2-c^2)\dfrac{s(s-a)}{(s-b)(s-c)}$, 故

原式$=\dfrac{s(b^2-c^2)(b+c-a)^2+s(c^2-a^2)(c+a-b)^2+s(a^2-b^2)(a+b-c)^2}{4(s-a)(s-b)(s-c)}$

$+\dfrac{2s^2(a-b)(b-c)(c-a)}{S^2}=\dfrac{-2s^2(a-b)(b-c)(c-a)}{s(s-a)(s-b)(s-c)}$

$+\dfrac{2s^2(a-b)(b-c)(c-a)}{S^2}=0.$

(92) $ah_1=bh_2=ch_3$, 故 $\dfrac{a^2h_1^2}{bh_2ch_3}=1$, 即從 $\dfrac{h_1^2}{h_2h_3}=\dfrac{bc}{a^2}$ 得其證.

(93) $h_3(2ab\cos C+2bc\cos A+2ca\cos B)$

$=h_3(a^2+b^2-c^2+b^2+c^2-a^2+c^2+a^2-b^2)=a^2h_3+b^2h_3+c^2h_3$, 但 $h_3=\dfrac{2s}{a}$,

$=\dfrac{2Sa^2}{c}+\dfrac{2Sb^2}{c}+2Sc=\dfrac{bc\sin A\,a^2}{c}+\dfrac{ca\sin B\,b^2}{c}+abc\sin C$

$=ab(a\sin A+b\sin B+c\sin C).$

94. $(h_1 \sin A + h_2 \sin B + h_3 \sin C)^2 = 18\,S \sin A \sin B \sin C$.

95. $\frac{\cos A}{b} = \frac{\cos B}{a}$，則其三角形爲等脚三角形或直角三角形.

96. $b = c$, $A = 60°$，則其三角形爲等邊.

97. $\sin^2 A = \sin^2 B + \sin^2 C$，則其三角形爲直角三角形.

98. $\sin A = 2 \sin B \cos C$，則其三角形爲等脚三角形.

99. $\frac{\tan A}{\tan B} = \frac{\sin A}{\sin B}$，則其三角形爲等脚三角形.

100. $\sin B \sin C = \cos^2 \frac{A}{2}$，則 $b = c$.

101. $a \tan A + b \tan B = (a+b) \tan \frac{A+B}{2}$，則 $A = B$.

102. 三角爲30°, 60°及90°，則其對邊順次爲$1:\sqrt{3}:2$.

103. $a:b:c = 2:3:4$，則

$\cos A = 7/8$, $\cos B = 11/16$, $\cos C = -1/4$.

104. $a = 35$, $b = 84$, $c = 91$，則

$\tan \frac{A}{2} = \frac{1}{5}$, $\tan \frac{B}{2} = \frac{2}{3}$, $\tan \frac{C}{2} = 1$, $S = 1470$.

105. $a = 125$, $b = 123$, $c = 62$，則

$\tan \frac{A}{2} = \frac{4}{5}$, $\tan \frac{B}{2} = \frac{3}{4}$, $\tan \frac{C}{2} = \frac{8}{31}$, $S = 3720$.

例題解自 94 至 105.

(94) $\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = k$，又 $a h_1 = b h_2 = c h_3 = 2s$.

即 $k h_1 \sin A = k h_2 \sin B = k h_3 \sin C = 2s$，故

$k^2(h_1 \sin A+h_2 \sin B+h_3 \sin C)^2=36s^2$. 又 $k=\frac{abc}{bc\sin A}=\frac{abc}{2s}$,

即 $2sk=abc=k^3 \sin A \sin B \sin C$, 故 $k^2=\frac{2s}{\sin A \sin B \sin C}$,

由是 $(h_1 \sin A+h_2 \sin B+h_3 \sin C)^2=\frac{36s^2}{k^2}=18s \sin A \sin B \sin C$.

(95) 由 $a\cos A=b\cos B$, 得 $\frac{a(b^2+c^2-a^2)}{2bc}=\frac{b(c^2+a^2-b^2)}{2ca}$, 試將此式簡單之, 則 $(a^2-b^2)(c^2-a^2-b^2)=0$, $\therefore$ $a=b$, 或 $c^2=a^2+b^2$.

(96) $B=C=\frac{1}{2}(180°-A)=\frac{1}{2}(180°-60°)=60°$.

(97) 令 $\sin A/a=\sin B/b=\sin C/c=\lambda$, 則從 $\sin^2 A=\sin^2 B+\sin^2 C$,

得 $a^2\lambda^2=b^2\lambda^2+c^2\lambda^2$, 即 $a^2=b^2+c^2$

(98) $\sin(B+C)=2\sin B\cos C$, 故 $\sin(B-C)=0$, $B=C$.

(99) $\frac{\sin A\cos B}{\sin B\cos A}=\frac{\sin A}{\sin B}$, 故 $\cos B=\cos A$, 故 $B=A$.

(100) $2\sin B\sin C=2\cos^2\frac{A}{2}=1+\cos A=1-\cos(B+C)$.

故 $\cos(B-C)=1$, $B-C=0$, 故 $B=C$.

(101) 令 $a/\sin A=b/\sin B=c/\sin C=k$, 則

$k\sin A\tan A+k\sin B\tan B=k(\sin A+\sin B)\tan\frac{A+B}{2}$,

$\frac{\sin^2 A}{\cos A}+\frac{\sin^2 B}{\cos B}=\frac{2\sin^2\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{\cos\frac{1}{2}(A+B)}=\frac{4\sin^2\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{2\cos^2\frac{1}{2}(A+B)}$

$=\frac{\{1-\cos(A+B)\}(\cos A+\cos B)}{1+\cos(A+B)}$, 則 $\frac{\sin^2 A\cos B+\sin^2 B\cos A}{\cos A\cos B(\cos A+\cos B)}=\frac{1-\cos(A+B)}{1+\cos(A+B)}$,

故 $\frac{\cos B+\cos A}{\cos A\cos B(\cos A+\cos B)}=\frac{2}{1+\cos(A+B)}$, 故 $1+\cos(A+B)=2\cos A\cos B$,

故 $\cos(A-B)=1$, $A-B=0$, $A=B$.

(102) $A=30°$, $B=60°$, 故 $c=2a$, $b=\sqrt{(c^2-a^2)}=a\sqrt{3}$.

(103) $\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{3^2+4^2-2^2}{2\times3\times4}=\frac{7}{8}$.

(104) $\tan\frac{1}{2}A=\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}=\sqrt{\frac{21\times14}{105\times70}}=\frac{1}{5}$,　(105) 同上.

106. $b=6,\ c=4,\ \cos A=\frac{1}{3}$, 則 $a=6$.

107. $B=45^\circ,\ C=60^\circ,\ a=2(\sqrt{3}\ \ 1)$, 則 $s=6+2\sqrt{3}$.

108. $a=2c,\ b=3c$, 則 $\cos B=-1$.

109. $a=\frac{\sqrt{6}-\sqrt{2}}{4},\ b=\frac{1}{\sqrt{2}},\ c=\frac{\sqrt{3}}{2}$, 則

$A=15^\circ,\ B=45^\circ,\ C=120^\circ$.

110. 三邊爲 $m,\ n,\ \sqrt{m^2+mn+n^2}$, 則其最大角爲 120°.

111. 三邊爲 $x^2+x+1,\ x^2-1,\ 2x+1$, 則其最大角爲 120°.

112. $A=3B$, 則 $\sin B=\frac{1}{2}\sqrt{\frac{3b-a}{b}}$.

113 三角形之周邊等於 $2c\cos\frac{A}{2}\cos\frac{B}{2}\sec\frac{A+B}{2}$.

114. $b-a=mc$ 則 $\cos\left(A+\frac{C}{2}\right)=m\cos\frac{C}{2}$.

115. $b-a=mc$ 則 $\cot\frac{B-A}{2}=\frac{1+m\cos B}{m\sin B}$.

116. $\frac{\tan A-\tan B}{\tan A+\tan B}=\frac{c-b}{c}$, 則 $A=60^\circ$

117. 三邊爲 $\frac{x}{y}+\frac{y}{z},\ \frac{y}{z}+\frac{z}{x},\ \frac{z}{x}+\frac{x}{y}$,

則 $s=\sqrt{\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)}$.

例題解自 106 至 117.

(106) $a^2=b^2+c^2-2bc\cos A=6^2+4^2-2\times6\times4\times\frac{1}{3}=36,\ \therefore\ a=6$.

(107) $s=\frac{1}{2}bc\sin A=\frac{1}{2}\times\frac{a\sin B}{\sin A}\times\frac{a\sin C}{\sin A}\sin A=\frac{a^2\sin B\sin C}{2\sin A}$

$=\frac{4(\sqrt{3}+1)^2\sin45^\circ\sin60^\circ}{2\sin75^\circ}=\frac{8(2+\sqrt{3})\times\frac{1}{2}\sqrt{2}\times\frac{1}{2}\sqrt{3}}{2\times\frac{1}{4}(\sqrt{6}+\sqrt{2})}=6+2\sqrt{3}$.

(108) $\cos B=\frac{a^2+c^2-b^2}{2ac}=\frac{4c^2+c^2-9c^2}{2\times 2c^2}=\frac{-4}{4}=-1.$

(109) $\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{\frac{1}{2}+\frac{3}{4}-\frac{8-4\sqrt{3}}{16}}{2\left(\sqrt{\frac{1}{2}}\right)\left(\frac{\sqrt{3}}{2}\right)}=\frac{\sqrt{6}+\sqrt{2}}{4}$, $\therefore$ $A=15°$.

(110) $\cos A=\frac{m^2+n^2-(m^2+mn+n^2)}{2mn}=-\frac{1}{2}$, 故 $\cos(180°-A)=\frac{1}{2}$.

故 $180°-A=60°$, 卽 $A=120°$.

(111) $\cos A=\frac{(x^2-1)^2+(2x+1)^2-(x^2+x+1)^2}{2(x^2-1)(2x+1)}=\frac{-(2x+1)(x^2-1)}{2(x^2-1)(2x+1)}=-\frac{1}{2}$, $A=120°$.

(112) $\frac{\sin A}{\sin B}=\frac{\sin 3B}{\sin B}=\frac{3\sin B-4\sin^3 B}{\sin B}=3-4\sin^2 B=\frac{a}{b}$.

故 $\sin^2 B=\frac{3b-a}{4b}$.

(113) $a+b+c=\frac{c\sin A}{\sin C}+\frac{c\sin B}{\sin C}+c=\frac{c}{\sin C}(\sin A+\sin B+\sin C)$

$=\frac{c}{\sin C}4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{2c\cos\frac{1}{2}A\cos\frac{1}{2}B}{\sin\frac{1}{2}C}$.

(114) 由 $b-a=mc$, 得 $\frac{c\sin B}{\sin C}-\frac{c\sin A}{\sin C}=mc$, $\sin B-\sin A=m\sin C$

$2\cos\frac{B+A}{2}\sin\frac{B-A}{2}=m\sin C$, $2\sin\frac{C}{2}\sin\frac{B-A}{2}=2m\sin\frac{C}{2}\cos\frac{C}{2}$, 卽

$\sin\frac{180°-A-C-A}{2}=m\cos\frac{C}{2}$, 卽 $\cos\left(A+\frac{C}{2}\right)=m\cos\frac{C}{2}$.

(115) 由前例. $\sin\frac{B-A}{2}=m\cos\frac{C}{2}=m\cot\frac{180°-A-B}{2}=m\sin\frac{A+B}{2}$

$=m\sin\left(B-\frac{B-A}{2}\right)=m\left(\sin B\cos\frac{B-A}{2}-\cos B\sin\frac{B-A}{2}\right)$, 故

$1=m\left(\sin B\cot\frac{B-A}{2}-\cos B\right)$. 故 $\cot\frac{B-A}{2}=\frac{1+m\cos B}{m\sin B}$.

(116) $\frac{\sin(A-B)}{\sin(A+B)}=1-\frac{b}{c}$, 卽 $\frac{\sin(A+B)-\sin(A-B)}{\sin C}=\frac{b}{c}=\frac{\sin B}{\sin C}$

$2\cos A\sin B=\sin B$, 故 $\cos A=\frac{1}{2}$, 故 $A=60°$.

(117) $s=\frac{x}{y}+\frac{y}{z}+\frac{z}{x}$, $s-a=\frac{z}{x}$, $s-b=\frac{x}{y}$, $s-c=\frac{y}{z}$,

故 $S=\sqrt{\left\{\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)\frac{z}{x}\times\frac{x}{y}\times\frac{y}{z}\right\}}=\sqrt{\left(\frac{x}{y}+\frac{y}{z}+\frac{z}{x}\right)}$.

118. $\dfrac{\sin A}{\sin B}=\dfrac{m}{n}$, $\dfrac{\cos A}{\cos B}=\dfrac{p}{q}$, 則 $\cos C=\dfrac{mp-nq}{np-mq}$.

119. $c^2y+b^2z=a^2z+c^2x=b^2x+a^2y$, 則

$x/\sin 2A=y/\sin 2B=z/\sin 2C$.

120. $C=2B$, $A\neq B$, 則 $c^2=(a+b)b$.

121. 三邊爲 $m+n$, $m-n$, $\sqrt{2(m^2+n^2)}$, 其一角之正弦爲

$\frac{1}{4}(\sqrt{5}-1)$, 求他二角.

122. $C=60°$, 則 $\dfrac{1}{a+c}+\dfrac{1}{b+c}=\dfrac{3}{a+b+c}$.

123. $2\cos A+\cos B+\cos C=2$, 則 $2a=b+c$.

124. 三邊爲 a, b, c, 其對角爲 $2\theta, 3\theta, 4\theta$, 則

$\tan^2\theta=\left(\dfrac{2b}{a+c}\right)^2-1$.

125. A', B', C', 爲外角, 則

$2bc\,\mathrm{vers}\,A'+2ca\,\mathrm{vers}\,B'+2ab\,\mathrm{vers}\,C'=4s^2$.

126. $y\sin^2 A+x\sin^2 B=z\sin^2 B+y\sin^2 C=x\sin^2 C+z\sin^2 A$,

則 $x:y:z=\sin 2A:\sin 2B:\sin 2C$.

例題解自 118 至 126.

(118) $\dfrac{\sin(B+C)}{\sin B}=\cos C+\cot B\sin C=\dfrac{m}{n}$, $\cot B=\dfrac{m-n\cos C}{n\sin C}$

$=\dfrac{-\cos(B+C)}{\cos B}=-\cos C+\tan B\sin C=\dfrac{p}{q}$, $\tan B=\dfrac{p+p\cos C}{q\sin C}$

故 $\dfrac{(m-n\cos C)(p+q\cos C)}{nq\sin^2 C}=\cot B\tan B=1$, 即

$mp+(mq-np)\cos C-nq\cos^2 C=nq\sin^2 C,\ \cos C=\dfrac{nq-mp}{mq-np}.$

(119) 令 $a/\sin A=b/\sin B=c/\sin C=k$, 則

$k^2\sin^2 Cy+k^2\sin^2 Bz=k^2\sin^2 Az+k^2\sin^2 Cx=k^2\sin^2 Bx+k^2\sin^2 Ay,$

即 $x\sin^2 B+y(\sin^2 A-\sin^2 C)-z\sin^2 B=0,$

$x\sin^2 C-y\sin^2 C+z(\sin^2 A-\sin^2 B)=0.$

從此兩方程式,準第八編2節第三,即可得其證.

(120) $\dfrac{\sin C}{\sin B}=\dfrac{\sin 2B}{\sin B}=2\cos B=\dfrac{c}{b},\ \dfrac{a^2+c^2-b^2}{2ac}=\dfrac{c}{b},$

$(a-b)\{b(a+b)-c^2\}=0$, 但 $a-b\neq 0$, 故 $b(a+b)-c^2=0.$

(121) $\cos A=\dfrac{(m+n)^2+(m-n)^2-2(m^2+n^2)}{2(m+n)(m-n)}=0$, 故 $A=90°,$

又 $\sin B=\frac{1}{4}(\sqrt{5}-1)$, 故 $B=15°,\ C=180°-(15°+90°)=75°.$

(122) $2ab\cos C=a^2+b^2-c^2,\ 2ab\cos 60°=a^2+b^2-c^2,\ ab=a^2+b^2-c^2,$ 即

$(a+b)^2-c^2=3ab,\ (a+b+c)(a+b-c)=3ab,$

$(a+b+c)(a+b+2c)=3ab+3c(a+b+c),$

$(a+b+c)\{(a+c)+(b+c)\}=3(a+c)(b+c)$, 故 $\dfrac{1}{b+c}+\dfrac{1}{a+c}=\dfrac{3}{a+b+c}.$

(123) $\dfrac{b^2+c^2-a^2}{bc}+\dfrac{c^2+a^2-b^2}{2ca}+\dfrac{a^2+b^2-c^2}{2ab}=2$, 故

$2a(b^2+c^2-a^2)+b(c^2+a^2-b^2)+c(a^2+b^2-c^2)=4abc$, 即

$bc(b+c)+a^2(b+c)-(b^3+c^3)-2a(a^2-b^2-c^2+2bc)=0$, 即

$(b+c)(bc+a^2-b^2+bc-c^2)-2a(a^2-b^2-c^2+2bc)=0$, 即

$(a+b-c)(a-b+c)(b+c-2a)=0$, 故 $b+c=2a.$

(124) $\dfrac{\sin 2\theta+\sin 4\theta}{\sin 3\theta}=\dfrac{a+c}{b}$, 即 $2\cos\theta=\dfrac{a+c}{b},$

$\tan^2\theta=1/\cos^2\theta-1=\{2b/(a+c)\}^2-1.$

(125) $2bc\,\mathrm{vers}A'=2bc(1-\cos A')=2bc(1+\cos A)=4bc\cos^2\dfrac{A}{2}=4s(s-a),$

故 原式$=4s(s-a+s-b+s-c)=4s^2.$

(126) 由原兩方程式消去z, 則

$y\sin^2 A(\sin^2 B+\sin^2 C-\sin^2 A)=x\sin^2 B(\sin^2 C+\sin^2 A-\sin^2 B)$, 由例題五62.

$y\sin^2 A\sin B\sin C\cos A=x\sin^2 B\sin C\sin A\cos B,$

即 $y\sin 2A=x\sin 2B$, 故 $x:y=\sin 2A:\sin 2B.$

127. $C=90^\circ$, 則 $\cos(2A-B)=a(3c^2-4a^2)/c^3$.

128. $\cos\theta=\dfrac{a}{b+c}$, $\cos\phi=\dfrac{b}{c+a}$, $\cos\psi=\dfrac{c}{a+b}$, 則

$$\tan^2\frac{\theta}{2}+\tan^2\frac{\phi}{2}+\tan^2\frac{\psi}{2}=1,\ \text{及}$$

$$\tan\frac{\theta}{2}\tan\frac{\phi}{2}\tan\frac{\psi}{2}=\pm\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}.$$

129. a, b, c 爲等差級數, 則 $\tan\dfrac{A}{2}\tan\dfrac{C}{2}=\dfrac{1}{3}$.

130. 同上. $\cos\frac{1}{2}(A-C)=2\cos\frac{1}{2}(A+C)$.

131. 同上. $a\cos^2\dfrac{C}{2}+c\cos^2\dfrac{A}{2}=\dfrac{3}{2}b$.

132. 同上. 最大角與最小角之差爲 90°, 則三邊之比爲 $\sqrt{7}+1:\sqrt{7}:\sqrt{7}-1$.

133. A, B, C 爲等差級數, 則

$$2\cos\frac{A-C}{2}=\frac{a^3+c^3}{b^3}=\frac{a+c}{\sqrt{(a^2-ac+c^2)}}.$$

例題解自 127 至 133.

(127) $\cos(2A-B)=\cos\{2(90^\circ-B)-B\}=-\cos3B=3\cos B-4\cos^3B$

$$=\frac{3a}{c}-\frac{4a^3}{c^3}=\frac{3ac^2-4a^3}{c^3}.$$

(128) $2\cos^2\dfrac{\theta}{2}-1=\dfrac{a}{b+c}$, 即 $\cos^2\dfrac{\theta}{2}=\dfrac{a+b+c}{2(b+c)}$,

$\tan^2\dfrac{\theta}{2}=\dfrac{1}{\cos^2\frac{1}{2}\theta}-1=\dfrac{2(b+c)}{a+b+c}-1=\dfrac{b+c-a}{a+b+c}$, 同樣

$\tan^2\dfrac{\phi}{2}=\dfrac{c+a-b}{a+b+c}$, $\tan^2\dfrac{\psi}{2}=\dfrac{a+b-c}{a+b+c}$, 由是

$$\tan^2\frac{\theta}{2}+\tan^2\frac{\phi}{2}+\tan^2\frac{\psi}{2}=\frac{1}{a+b+c}(b+c-a+c+a-b+a+b-c)=1.$$

$$\tan^2\frac{\theta}{2}\tan^2\frac{\phi}{2}\tan^2\frac{\psi}{2}=\frac{(b+c-a)(c+a-b)(a+b-c)}{(a+b+c)^3}=\frac{(s-a)(s-b)(s-c)}{s^3}$$

$$=\frac{(s-b)(s-c)}{s(s-a)}\times\frac{(s-c)(s-a)}{s(s-b)}\times\frac{(s-a)(s-b)}{s(s-c)}=\tan^2\frac{A}{2}\tan^2\frac{B}{2}\tan^2\frac{C}{2}.$$

(129) $\tan\frac{A}{2}\tan\frac{C}{2}=\sqrt{\left\{\frac{(s-b)(s-c)}{s(s-a)}\times\frac{(s-a)(s-b)}{s(s-c)}\right\}}=\frac{c+a-b}{a+b+c}=\frac{b}{3b}=\frac{1}{3}.$

(130) $\cos\frac{A-C}{2}=\cos\frac{A}{2}\cos\frac{C}{2}+\sin\frac{A}{2}\sin\frac{C}{2}=\sqrt{\frac{s(s-a)s(s-c)}{bc\cdot ab}}$

$+\sqrt{\frac{(s-b)(s-c)(s-a)(s-b)}{bc\cdot ab}}=\frac{(s+s-b)\sqrt{(s-c)(s-a)}}{b\sqrt{ca}}=\frac{(c+a)\sqrt{(s-c)(s-a)}}{b\sqrt{ca}}$

$=2\sqrt{\frac{(s-c)(s-a)}{ca}}=2\sin\frac{B}{2}=2\cos\frac{C+A}{2}.$

(131) $a\cos^2\frac{C}{2}+c\cos^2\frac{A}{2}=\frac{as(s-c)}{ab}+\frac{cs(s-a)}{bc}=\frac{s(2s-c-a)}{b}=s$

$=\frac{1}{2}(a+b+c)=\frac{3}{2}b.$

(132) $A-C=90°$, $\frac{a}{\sin A}=\frac{a}{\sin(90°+C)}=\frac{a}{\cos C}=\frac{c}{\sin C}=\frac{\sqrt{(a^2+c^2)}}{\sqrt{(\sin^2C+\cos^2C)}}$,

故 $\cos C=\frac{a}{\sqrt{(a^2+c^2)}}$, $2ab\cos C=a^2+b^2-c^2$, 即

$\frac{a^2(c+a)}{\sqrt{(a^2+c^2)}}=\frac{1}{4}(c+a)^2-(c^2-a^2)$, 即 $\frac{a^4}{a^2+c^2}=\frac{1}{16}\{c+a-4(c-a)\}^2$, 故

$9\left(\frac{a^4}{c^4}\right)-30\left(\frac{a^3}{c^3}\right)+34\left(\frac{a^2}{c^2}\right)-30\left(\frac{a}{c}\right)+9=0$, 令 $\frac{a}{c}=\lambda$, 則

$9\lambda^4-30\lambda^3+34\lambda^2-30\lambda+9=0$, 故 $\lambda=\frac{a}{c}=\frac{4+\sqrt{7}}{3}=\frac{\sqrt{7}+1}{\sqrt{7}-1}$,

$\frac{a+c}{c}=\frac{2\sqrt{7}}{\sqrt{7}-1}=\frac{2b}{c}$, ∴ $a:b:c=\sqrt{7}+1:\sqrt{7}:\sqrt{7}-1.$

(133) $A+C=2B$, 故 $A+B+C=3B=180°$ ∴ $B=60°$.

$2ac\cos B=c^2+a^2-b^2$, 即 $b^2=c^2-ca+a^2$.

$\frac{\sin C+\sin A}{c+a}=\frac{\sin B}{b}$, $\frac{2\sin\frac{1}{2}(C+A)\cos\frac{1}{2}(C-A)}{c+a}=\frac{\sin B}{b}$, 即

$2\cos\frac{1}{2}(C-A)=\frac{c+a}{b}=\frac{c+a}{\sqrt{(c^2-ca+a^2)}}.$

又 $2\cos\frac{1}{2}(C-A)=\frac{(c+a)b^2}{b^3}=\frac{(c+a)(c^2-ca+a^2)}{b^3}=\frac{c^3+a^3}{b^3}.$

134. 同上. 其通差爲 D, 則 $\tan D=\frac{4b^2-(c+a)^2}{(c+a)^2}$.

135. A, B, C 成等比級數, 其通比爲 $\frac{1}{2}$, 則其最大邊與周邊之比. 等於 $2\sin\frac{90°}{7}:1$.

136. a, b, c 爲調和級數, 則

$$\cos\frac{B}{2}=\sqrt{\frac{\sin C\sin A}{\cos C+\cos A}}.$$

137. a^2, b^2, c^2 成等差級數, 則

$\cot A, \cot B, \cot C$ 亦成等差級數.

138. 同上. $a\sec A, b\sec B, c\sec C$ 爲調和級數.

139. $\sin A, \sin B, \sin C$ 爲調和級數, 則

$1-\cos A, 1-\cos B, 1-\cos C$ 亦爲調和級數.

140. $\cot\frac{A}{2}, \cot\frac{B}{2}, \cot\frac{C}{2}$ 爲等差級數, 則

$$\cot\frac{A}{2}\cot\frac{C}{2}=3.$$

例題解自 134 至 140.

(134) 由前例, $B=60°$, 故 $ac=a^2+c^2-b^2$.

$$\tan A=\frac{\sin A}{\cos A}=\frac{2bc\sin A}{2bc\cos A}=\frac{4S}{b^2+c^2-a^2}=\frac{\sqrt{\{4c^2a^2-(c^2+a^2-b^2)^2\}}}{2c^2-ac}$$

$$=\frac{\sqrt{\{4c^2a^2-a^2c^2\}}}{c(2c-a)}=\frac{a\sqrt{3}}{2c-a},$$

$$\tan D=\tan(A-60°)=\frac{\tan A-\sqrt{3}}{1+\tan A\sqrt{3}}=\frac{\frac{a\sqrt{3}}{2c-a}-\sqrt{3}}{1+\frac{3a}{2c-a}}=\frac{(a-c)\sqrt{3}}{a+c},$$

$$\tan^2 D=\frac{3a^2-6ac+3c^2}{(a+c)^2}=\frac{4(a^2+c^2-ac)-(a+c)^2}{(a+c)^2}=\frac{4b^2-(a+c)^2}{(a+c)^2}.$$

(135) 令 $A=8\theta, B=4\theta, C=2\theta$, 则 $14\theta=180^\circ, 7\theta=90^\circ$,

$$\frac{a}{a+b+c}=\frac{\sin 8\theta}{\sin 8\theta+\sin 4\theta+\sin 2\theta}=\frac{\sin 8\theta}{\sin(180^\circ-6\theta)+\sin 4\theta+\sin 2\theta}=\frac{\sin 8\theta}{\sin 6\theta+\sin 2\theta+\sin 4\theta}$$

$$=\frac{2\sin 4\theta\cos 4\theta}{2\sin 4\theta\cos 2\theta+\sin 4\theta}=\frac{2\cos(90^\circ-3\theta)}{2\cos 2\theta+1}=\frac{2\sin 3\theta}{2(1-2\sin^2\theta)+1}=\frac{2(3\sin\theta-4\sin^3\theta)}{3-4\sin^2\theta}$$

$$=2\sin\theta=2\sin\frac{90^\circ}{7}.$$

(136) $b=\dfrac{2ac}{a+c}$, $\cos\dfrac{B}{2}=\sqrt{\dfrac{s(s-b)}{ca}}=\sqrt{\dfrac{S^2}{ca(s-c)(s-a)}}=\sqrt{\dfrac{\frac{1}{2}bc\sin A\,\frac{1}{2}ab\sin C}{ca(s-c)(s-a)}}$

$$=\sqrt{\frac{2ab^2c\sin A\sin C}{b(a+c)\{b^2-(c-a)^2\}}}=\sqrt{\frac{2abc\sin A\sin C}{ab^2+a^2c+b^2c+ac^2-c^3-a^3}}$$

$$=\sqrt{\frac{2abc\sin A\sin C}{a(b^2+c^2-a^2)+c(a^2+b^2-c^2)}}=\sqrt{\frac{\sin A\sin C}{\frac{1}{2bc}(b^2+c^2-a^2)+\frac{1}{2ab}(a^2+b^2-c^2)}}=\sqrt{\frac{\sin A\sin C}{\cos A+\cos C}}$$

(137) $2b^2=a^2+c^2$, 故 $2\sin^2 B=\sin^2 A+\sin^2 C$,

$\sin^2 B+\sin B\sin(C+A)=\sin A\sin(B+C)+\sin C\sin(A+B)$,

$\sin^2 B+\sin B\sin C\cos A+\sin B\cos C\sin A=\sin A\sin B\cos C+\sin A\cos B\sin C$

$+\sin C\sin A\cos B+\sin C\cos A\sin B$, 即 $\sin^2 B=2\sin A\sin C\cos B$,

故 $\dfrac{\sin(A+C)}{\sin A\sin C}=2\cot B$, 即 $\cot A+\cot C=2\cot B$.

(138) $2b^2=a^2+c^2$, $b\sec B=\dfrac{2abc}{2ac\cos B}=\dfrac{2abc}{a^2+c^2-b^2}=\dfrac{2ac}{b}$,

又 $\dfrac{2ac\sec A\sec C}{a\sec A+c\sec C}=\dfrac{2ac}{a\cos C+c\cos A}=\dfrac{4abc}{a^2+b^2-c^2+b^2+c^2-a^2}=\dfrac{2ac}{b}$.

(139) $\sin B=\dfrac{2\sin A\sin C}{\sin A+\sin C}$, $2\sin A\sin B+2\sin B\sin C=4\sin A\sin C$,

即 $\cos(A-B)-\cos(A+B)+\cos(B-C)-\cos(B+C)=2\{\cos(C-A)-\cos(C+A)\}$

即 $\cos A\cos B+\cos C+\cos B\cos C+\cos A+\sin B(\sin A+\sin C)$

$=2\cos C\cos A+2\cos B+2\sin C\sin A$, 故

$\cos B=\dfrac{\cos A+\cos C-2\cos A\cos C}{2-\cos A-\cos C}$, 即 $1-\cos B=\dfrac{2(1-\cos A)(1-\cos C)}{(1-\cos A)+(1-\cos C)}$.

(140) $2\cot\dfrac{B}{2}=2\cot\frac{1}{2}(180^\circ-A-C)=2\tan\frac{1}{2}(A+C)=\cot\dfrac{A}{2}+\cot\dfrac{C}{2}$, 故

$\dfrac{2(\cot\frac{1}{2}C+\cot\frac{1}{2}A)}{\cot\frac{1}{2}A\cot\frac{1}{2}C-1}=\cot\frac{1}{2}A+\cot\frac{1}{2}C$, 故 $\cot\frac{1}{2}A\cot\frac{1}{2}C=3$.

141. $C>90^\circ$, 則 $\tan A\tan B<1$.

142. $A=B=C$, 則 $8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}<1$.

143. $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\not>\frac{3\sqrt{3}}{8}$.

144. $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\not>\frac{3}{4}$.

145. $\sin A+\sin B+\sin C\not>\sin 2A+\sin 2B+\sin 2C$.

例題解自 141 至 145.

(141) $1-\tan A\tan B=\frac{\cos(A+B)}{\cos A\cos B}=\frac{-\cos C}{\cos A\cos B}$, $C>90^\circ$, 則 $-\cos C$ 爲正, 又 A, B 爲銳角, 故 $1>\tan A\tan B$.

(142) $1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=1-8\sqrt{\left\{\frac{(s-b)(s-c)}{bc}\times\frac{(s-c)(s-a)}{ca}\times\frac{(s-a)(s-b)}{ab}\right\}}$

$=\frac{abc-(a+b-c)(b+c-a)(c+a-b)}{abc}=\frac{1}{abc}\left\{(a-b)^2(b+c-a)+(b-c)^2(c+a-b)\right.$

$\left.+(c-a)^2(c+a-b)\right\}$=正數, 故 $1>8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

(143) 由例題五 30. $\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\frac{1}{4}(\sin A+\sin B+\sin C)$,

因 $A+B+C=180^\circ$, 故此式極大爲 $\sin A=\sin B=\sin C$

$=\sin 60^\circ=\frac{1}{2}\sqrt{3}$, 故此式之極大值$=\frac{3}{8}\sqrt{3}$, 故如題言.

(144) $\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}$ 之極大爲 $\sin\frac{A}{2}=\sin\frac{B}{2}=\sin\frac{C}{2}$

$=\sin 30^\circ=\frac{1}{2}$, 故此式之極大值$=\frac{3}{4}$, 故如題言.

(145) $\sin A+\sin B+\sin C-(\sin 2A+\sin 2B+\sin 2C)$

$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-4\sin A\sin B\sin C$

$=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}\left(1-8\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$, 由例題 142. 此式爲正, 故如題言.

第拾叁編

三角形之解法

1. 三角形之解法

三角形有三邊與三角,合計六項.知其內三項,即可求得他三項.但所知三項均屬三角,則只能求得三邊之比,不能求其原邊之長.

三角形之解法,有二方法如次.

〔第一〕 三角函數之値及三邊之長,各以其本然之値入算,是之謂**眞數計算法**.

〔第二〕 眞數之計算,即用乘除或平方根等之計算.故除簡易已知數量之外,其他實際上用之甚少.

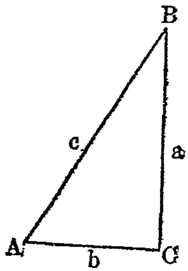
故用第二法.此法名**對數之計算**.即使用對數表,以加減代乘除等等之計算法也.

使用對數以求三角函數之眞數,其方法及原理,已於第十編及第十一編言之.讀者可取而觀之.

直角三角形之眞數計算

2. 直角三角形之眞數計算

已於第十二編示其公式及圖形矣.茲再詳示如次.



$a=\sqrt{c^2-b^2}$,　$a=c\sin A$.

$c=\sqrt{a^2+b^2}$,　$b=c\cos A$.

$B=90^\circ-A$,　$a=b\tan A$.

直角三角形有一角爲90°,此即爲已知項之一項,故於其他二角及三邊之內,任知其二,即可求得其餘各項.

例題十八

已知C=90°及次列各值，求其餘各項.

1. $b=355,\ c=923.$　　2. $a=36,\ b=27.$

3. $c=125,\ A=37°30'.$　　4. $a=75,\ A=39°15'.$

5. $b=100,\ A=27°30'25''.$

已知C=90°及次列各值，試毋用三角函數表，以求其餘各項.

6. $a=3\sqrt{7},\ b=\sqrt{21}.$　　7. $a=b=4.$

8. $c=12,\ a=6.$　　9. $c=4,\ a=2\sqrt{3}.$

10. $c=2,\ a=\sqrt{2}.$　　11. $c=10,\ A=30°.$

12. $c=12,\ A=15°.$　　13. $a=5\sqrt{3},\ A=60°.$

14. $a=10,\ A=45°.$

例題解自1至14.

(1) 由公式 $a=\sqrt{(c^2-b^2)}=\sqrt{(c+b)(c-b)}=\sqrt{(923+355)(923-355)}$

$=\sqrt{1278\times568}=\sqrt{(71\times18\times71\times8)}=\sqrt{(71^2\times144)}=71\times12=852,$

又 $\cos A=\frac{b}{c}=\frac{355}{923}=\frac{15}{13}=\cdot384615=\cos67°22'49''$ (從表)

故 $A=67°22'49''$ $B=90°-67°22'49''=22°37'11''.$

(2) $c=\sqrt{(a^2+b^2)}=\sqrt{(36^2+27^2)}=\sqrt{\{9^2(4^2+3^2)\}}=9\sqrt{25}=45,$

又 $\tan A=\frac{a}{b}=\frac{36}{27}=\frac{4}{3}=1\cdot333333=\tan53°7'48'',$

故 $A=53°7'48''$, $B=90°-53°7'48''=36°52'12''.$

(3) $b=c\cos A=125\cos37°30'=125\times\cdot793353$ (從表)

$=\frac{1000}{8}\times\cdot793353=793\cdot353\div8=99\cdot169,$

又 $a=c\sin A=125\sin37°30'=125\times\cdot608761=76\cdot095,$

$B=90°-A=90°-37°30'=52°30'.$

(4) $c=\frac{a}{\sin A}=\frac{75}{\sin 39°15'}=\frac{75}{\cdot 632705}$, (從表)$=118\cdot 53$,

此可用餘剩之眞數表,如下式求之. $c=a\,cosec\,A=75\times cosec\,39°15'$

$b=a/\tan A=a\cot A=75\cot 39°15'=75\times 1\cdot 223939$ (從表)$=\frac{300}{4}\times 1\cdot 223939$

$=367\cdot 1817\div 4=91\cdot 7955$, $B=90°-A=90°-39°15'=50°45'$.

(5) $a=b\tan A=100\tan 27°30'25''=100\times\cdot 520721$ (從表)$=52\cdot 0721$

$c=\frac{b}{\cos A}=\frac{100}{\cos 27°30'25''}=\frac{100}{\cdot 886955}$ (從表)$=112\cdot 74$,

$B=90°-A=90°-27°30'25''=62°29'35''$.

(6) $c=\sqrt{(a^2+b^2)}=\sqrt{\{(3\sqrt{7})^2+\sqrt{(21)^2}\}}=\sqrt{(63+21)}=2\sqrt{21}$

$\tan A=\frac{a}{b}=\frac{3\sqrt{7}}{\sqrt{21}}=\sqrt{3}=\tan 60°$ (第六編3節) 故 $A=60°$, $B=30°$.

(7) $c=\sqrt{(a^2+b^2)}=\sqrt{(4^2+4^2)}=4\sqrt{2}$,

$\tan A=\frac{a}{b}=\frac{4}{4}=1=\tan 45°$ (第六編3節) 故 $A=45°=B$.

(8) $b=\sqrt{(c^2-a^2)}=\sqrt{(12^2-6^2)}=\sqrt{6^2(2^2-1)\}}=6\sqrt{3}$,

$\sin A=\frac{a}{c}=\frac{6}{12}=\frac{1}{2}=\sin 30°$, (第六編3節) 故 $A=30°$, $B=60°$.

(9) $b=\sqrt{(c^2-a^2)}=\sqrt{\{4^2-(2\sqrt{3})^2\}}=2$

$\sin A=\frac{a}{c}=\frac{2\sqrt{3}}{4}=\frac{\sqrt{3}}{2}=\sin 60°$, (第六編3節) 故 $A=60°$, $B=30°$.

(10) $b=\sqrt{(c^2-a^2)}=\sqrt{(2^2-\sqrt{2}^2)}=\sqrt{2}$, 由前法 $A=45°=B$.

(11) $a=c\sin A=10\sin 30°=10\times\frac{1}{2}$ (第六編3節)$=5$

$b=c\cos A=10\cos 30°=10\times\frac{\sqrt{3}}{2}$ (第六編3節)$=5\sqrt{3}$

$B=90°-A=90°-30°=60°$.

(12) $a=c\sin A=12\sin 15°=12\times\frac{1}{4}(\sqrt{6}-\sqrt{2})$ 〔例題六1〕$=3(\sqrt{6}-\sqrt{2})$,

$b=c\cos A=12\cos 15°=12\times\frac{1}{4}(\sqrt{6}+\sqrt{2})$ 〔例題六1〕$=3(\sqrt{6}+\sqrt{2})$,

$B=90°-A=90°-15°=75°$.

(13) $B=30°$, $c=10$, $b=5$. (14) $B=45°$, $b=10$, $c=10\sqrt{2}$.

直角三角形之對數計算

3. 直角三角形之對數計算

從2節之公式如次.

$a=\sqrt{c^2-b^2}=\sqrt{(c+b)(c-b)}$ 由第十一編1節

$$\log a=\tfrac{1}{2}\{\log(c+b)+\log(c-b)\}$$

從 $a=c\sin A$ 得 $\log a=\log c+\log\sin A$

但由第十一編11節 $\log\sin A=L\sin A-10$

由此 $\log a=\log c+L\sin A-10,$

同樣 $\log b=\log c+L\cos A-10,$

$$\log c=\log b+L\tan A-10.$$

例題十九

於次列各式,計算其未知項之值.

1. $c=365,\quad A=33°12'.$
2. $a=33\cdot33,\quad A=83°33'.$
3. $a=105\cdot5,\ B=46°3'$
4. $c=111\cdot1,\quad b=37\ 5.$
5. $a=29\cdot37,\ b=37\cdot29.$

例題解自1至5.

(1) $\log a=\log c+L\sin A-10=\log365+L\sin33°12'-10$

$=2\cdot562293+9\cdot738434-10$ (從表)$=2\cdot300727=\log199\cdot86,$

故 $a=199\cdot86,$ 同樣. 從 $\log b=\log c+L\sin A-10$ 得 $b,$

卽 $b=305\cdot42$ 又 $B=90°-A=90°-33°12'=56°48'.$

(2) $\log c=\log a-(L\sin A-10)=10+\log a-L\sin A$

$=10+\log33\cdot33-L\sin83°33'=10+1\cdot522835-9\cdot997242$

$=1\cdot525593=\log33\cdot542$ (從表) 故 $c=33\cdot542,$

從 $\log b=\log a-(L\tan A-10)=10+\log a-L\tan A$ 得 $b=3\cdot768$

又 $B=90°-A=90°-83°33'=6°27'.$

(3) $\log c=10+\log a-\mathrm{L}\sin\mathrm{A}=10+\log a-\mathrm{L}\sin(90°-\mathrm{B})$

$=10+\log a-\mathrm{L}\cos\mathrm{B}=10+\log 105\cdot5-\mathrm{L}\cos 46°3'=10+2\cdot023253-9\cdot841379$

$=2\cdot181874=\log 152\cdot01$, 故 $c=152\cdot01$.

同樣. $\log b=10+\log a-\mathrm{L}\cot\mathrm{A}$ 從此得 $b=109\cdot44$.

(4) $\log a=\frac{1}{2}\{\log(c+b)+\log(c-b)\}=\frac{1}{2}\{\log(111\cdot1+37\cdot5)+\log(111\cdot1-37\cdot5)\}$

$=\frac{1}{2}(\log 148\cdot6+\log 73\cdot6)=\frac{1}{2}(2\cdot172019+1\cdot866878)=2\cdot019448$

$=\log 104\cdot58$, 故 $a=104\cdot58$.

$\mathrm{L}\cos\mathrm{A}=10+\log b-\log c=10+\log 37\cdot5-\log 111\cdot1$

$=10+1\cdot574031-2\cdot045714=9\cdot528317=\mathrm{L}\cos 70°16'24''$.

故 $\mathrm{A}=70°16'24''$, $\mathrm{B}=90°-70°16'24''=19°43'36''$.

(5) $\mathrm{L}\tan\mathrm{A}=10+\log a-\log b=10+1\cdot467904-1\cdot571592=9\cdot896312$

$=\mathrm{L}\tan 38°13'28''$ 故 $\mathrm{A}=38°13'28''$, $\mathrm{B}=90°-\mathrm{A}=90°-38°13'28''=51°46'32''$.

從 $\log c=10+\log a-\mathrm{L}\sin\mathrm{A}$ 得 $c=47\cdot467$.

三角形之眞數計算

4. 三角形之眞數計算 第十二編3.節之公式及圖形.

再示如次.

$$\frac{\sin\mathrm{A}}{a}=\frac{\sin\mathrm{B}}{b}=\frac{\sin\mathrm{C}}{c}, \qquad (1)$$

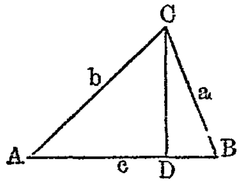
$$2bc\cos\mathrm{A}=b^2+c^2-a^2, \qquad (2)$$

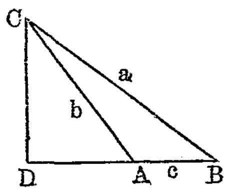
$$\cos\frac{\mathrm{A}}{2}=\sqrt{\frac{s(s-a)}{bc}}, \qquad (3)$$

$$\tan\frac{\mathrm{B}-\mathrm{C}}{2}=\frac{b-c}{b+c}\cot\frac{\mathrm{A}}{2}, \qquad (4)$$

$$\mathrm{S}=\tfrac{1}{2}bc\sin\mathrm{A}$$

$$=\sqrt{s(s-a)(s-b)(s-c)}, \qquad (5)$$





5. 三角函數之正負 於三角形ABC. 其 $A+B+C=180°$,

故 $\frac{1}{2}A+\frac{1}{2}B+\frac{1}{2}C=90°$, 故 $\frac{1}{2}A$, $\frac{1}{2}B$, $\frac{1}{2}C$ 均爲銳角,由是得決定三角函數之正負.如次.

〔第一〕 $\sin A$, $\sin B$, $\sin C$, 常爲正. 〔公式 (1)〕

〔第二〕 $\cos A$, $\cos B$, $\cos C$, 正或負. 〔公式 (2)〕

〔第三〕 $\cos\frac{A}{2}$, $\cos\frac{B}{2}$, $\cos\frac{C}{2}$ 常爲正. 〔公式 (3)〕

〔第四〕 $B>C$, 則 $\tan\frac{B-C}{2}$ 常爲正. 〔公式 (4)〕

6. 兩意之式 知三角形之二邊及一對角,以求他項,則可得二答數. 今知 a, b 及 A, 則從公式 (1). $\sin B=b\sin A/a$.

由前節第一. $\sin B$ 常爲正. 而 $\sin B=\sin(180°-B)$.

故B之値,爲B或 $180°-B$,分叙如次.

〔第一〕 $b=a$, 則 $B=A$. 故所求之B,等於已知數A.由是B只有一値.

〔第二〕 $b<a$, 則 $B<A$. 故B比已知數之A小.

此B之値.爲B,或 $180°-B$. 則 $180°-B<A$, 卽 $180°<A+B$. 然三角形二角之和大於180°爲不合理,由是B之値亦只有一個.

〔第三〕 $b>a$, 則 $B>A$. 故B比旣知數大,

此B之値.令爲B,或 $180°-B$. 則 $B>A$, $180°-B>A$,

卽 $180°>A+B$, 卽與三角形任二角之和小於180°之理相合.

由是B之値.於B,或 $180°-B$ 均適合.

此第三之式.稱爲兩意之式.

7. 二次方程式之解說 兩意之式.已於平面幾何學講義第一編第二節47.及48.說明矣,玆更用代數學解釋之,

先令 a, b, A 爲已知數.且 $b>a$,

然 $\sin B=\frac{b\sin A}{a}$, 由前節所得B之二値.令爲 B_1, B.

卽於前節 B 之二值 B 及 $180°-B$, 以 B_1 及 B_2 代之.

由是 $b>a$. 其方程式爲 $\sin B=b\sin A/a$. 而 B 之二根爲 B_1 及 B_2. 此二根之關係式爲 $B_1+B_2=180°$.

又從公式 (2) $2bc\cos A=b^2+c^2-a^2$, 卽 $c^2-2bc\cos A+(b^2-a^2)=0$,

但 $b>a$, 故此 c 之二次方程式其第三項 b^2-a^2 爲正.

解此二次方程式 $c=b\cos A\pm\sqrt{b^2\cos^2 A-b^2+a^2}$

卽 $c=b\cos A\pm\sqrt{a^2-b^2\sin^2 A}$

令 c 之二根爲 c_1 及 c_2, 則

$c_1=b\cos A+\sqrt{(a^2-b^2\sin^2 A)}$, $c_2=b\cos A-\sqrt{(a^2-b^2\sin^2 A)}$.

由是 c 之二根之關係式爲 $c_1+c_2=2b\cos A$, $c_1c_2=b^2-a^2$.

以此圖示之. 已知 a, b, A, 而

$b>a$, 則得兩三角形 AB_1C, AB_2C,

卽 $AB_1=c_1$, $AB_2=c_2$

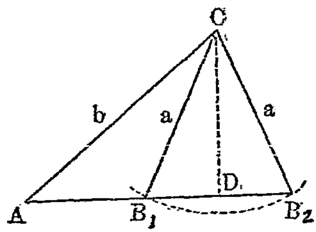
角 $AB_1C=B_1$, 角 $AB_2C=B_2$,

從 C 引 AB_1 之垂線 CD, 則

$CD=AC\sin A=b\sin A$,

故前二次方程式之根, 爲

$c=b\cos A\pm\sqrt{a^2-CD^2}$.



c 爲實根, 則 $a>CD$. 故 $a<CD$ 則爲虛根, 是不合理, 卽三角形不能成立.

若 c 爲等根, (卽一個值) 則 $a=CD$.

是爲直角三角形.

令兩三角形 AB_1C, AB_2C 爲 S_1 及 S_2, 則

公式 (5) $S=\frac{1}{2}bc\sin A$, S 之二根爲 S_1 及 S_2.

卽 $S_1=\frac{1}{2}bc_1\sin A$, 及 $S_2=\frac{1}{2}bc_2\sin A$.

此以前所得 c_1 及 c_2 之值代入之, 則

$$S_1=\tfrac{1}{2}b\sin A(b\cos A+\sqrt{a^2-b^2\sin^2 A})$$

$$=\tfrac{1}{2}b\sin A(b\cos A-\sqrt{a^2-b^2\sin^2 A})$$

由是此二根之關係式, 爲 $S_1+S_2=\frac{1}{4}b^2\sin 2A$

$$S_1S_2=\tfrac{1}{4}b^2\sin^2 A(b^2-a^2).$$

例題二十

1. $a:b:c=4:7:9$ 其 A, B, C 之值如何.

2. $a=15,\ b=16,\ c=29$, 求 C 之值.

3. $a=7,\ b=8,\ c=9$, 其最大角之值如何.

4. $b:c=4:5,\ \ a=1000,\ A=37°19'$, 求 b 之值.

5. $b:c=9:7,\ \ A=64°12'$, 求 B 及 C 之值.

6. $b=35,\ c=21,\ A=50°$, 求 a 之值.

7. $a=1,\ b=9,\ C=65°$, 求 A 及 B 之值.

8. $A=20°41'20'',\ B=51°38'55''$ 及 $a=24\frac{1}{2}$, 求 c 之值如何.

9. $A=49°,\ B=51°,\ a=1$, 求 b 及 c 之值.

10. $a=19,\ b=1,\ A-B=90°$, 求 C 之值.

11. $c:a-b=9:2,\ C=60°$, 求 A 及 B 之值.

例題解自 1. 至 11.

(1) 令 $a/4=b/7=c/9=k$, 則 $a=4k,\ b=7k,\ c=9k$,

從公式 (2) $2bc\cos A=b^2+c^2-a^2$,

即 $2(7k)(9k)\cos A=(7k)^2+(9k)^2-(4k)^2$,

故 $\cos A=\dfrac{7^2+9^2-4^2}{2\times7\times9}=\dfrac{114}{2\times7\times9}=\dfrac{19}{21}=.904762=\cos 25°12'32''$.

由是 $A=25°12'32''$, 同樣. $B=48°11'23''$, $C=106°36'5''$.

(2) $\cos C=\dfrac{a^2+b^2-c^2}{2ab}=\dfrac{15^2+16^2-29^2}{2\times15\times16}=\dfrac{-360}{30\times16}=-\dfrac{3}{4}=-.75$.

但從表 $.75=\cos 41°24'35''$, 故 $-.75=-\cos 41°24'35''$,

即 $-.75=\cos(180°-41°24'35'')$, 故 $C=180°-41°24'35''=138°35'25''$.

(3) c爲最大邊.故與此相對之角C.亦爲最大角.與前例同法求之.卽可得 $C=73°23'51''$.

(4) $2bc\cos A=b^2+c^2-a^2$, 但 $c=\frac{5b}{4}$, 故 $\frac{10b^2}{4}\cos A=b^2+\frac{25b^2}{16}-a^2$,

故 $b=\frac{4a}{\sqrt{(41-40\cos A)}}=\frac{4\times1000}{\sqrt{(44-40\cos 37°19')}}=\frac{4000}{\sqrt{(41-40\times\cdot795297)}}$

$=\frac{4000}{\sqrt{(9\cdot18812)}}=\frac{4000}{3\cdot031}=1319\cdot6.$

(5) 從 $b:c=9:7$ 得 $b+c:b-c=9+7:9-7$ 故 $\frac{b-c}{b+c}=\frac{1}{8}$

從公式(4) $\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{1}{8}\cot\frac{64°12'}{2}=\frac{\cot 32°6'}{8}=\frac{1\cdot594137}{8}$

$=\cdot199267=\tan 11°16'10''$, 故 $\frac{1}{2}(B-C)=11°16'10''$,

又 $\frac{1}{2}(B+C)=\frac{1}{2}(180°-A)=\frac{1}{2}(180°-64°12')=57°54'$

由是 $B=69°10'10''$, $C=46°37'50''$.

(6) 可從 $2bc\cos A=b^2+c-a^2$ 得 $a=26\cdot86$.

(7) 從 $\tan\frac{1}{2}(B-A)=\frac{b-a}{b+a}\cot\frac{A}{2}=\frac{9-1}{9+1}\cot\frac{65°}{2}$. 如例題5.得

$A=6°1'54''$, $B=108°58'6''$.

(8) $C=180°-(20°41'20''+51°38'55'')=180°-72°20'15''$

$c=\frac{a\sin C}{\sin A}=\frac{24\frac{1}{2}\sin(180°-72°20'15'')}{\sin 20°41'20''}=\frac{49\sin 72°20'15''}{2\sin 20°41'20''}=66\cdot078.$

(9) $C=180°-(B+A)=180°-(49°+51°)=80°$.

$b=\frac{a\sin B}{\sin A}=\frac{1\times\sin 51°}{\sin 49°}=\frac{\cdot777146}{\cdot754710}=1\cdot02973$, $c=\frac{a\times\sin C}{\sin A}=1\cdot30488.$

(10) 從公式(4) $\cot\frac{C}{2}=\frac{a+b}{a-b}\tan\frac{A-B}{2}=\frac{19+1}{19-1}\tan\frac{90}{2}=\frac{10}{9}\times1=1\cdot111111$

$=\cot 41°50'14''$, 故 $C=83°58'28''$.

(11) $\frac{c}{\sin C}=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{a-b}{\sin A-\sin B}=\frac{a-b}{2\cos\frac{1}{2}(A+B)\sin\frac{1}{2}(A-B)}$, 卽

$\sin\frac{1}{2}(A-B)=\frac{a-b}{c}\times\frac{\sin C}{2\cos(90°-\frac{1}{2}C)}=\frac{a-b}{c}\cos\frac{C}{2}=\frac{2}{9}\cos 30°=\cdot192450$

$=\sin 11°5'45''$ 由是 $A=71°5'45''$, $B=48°54'15''$.

12. $A=76°2'30''$, $a=2000$, $b=1000$ 求 B, C 之值.

13. $B=52°32'15''$, $b=32$, $c=40$ 求 A, C 之值.

14. 三角形之三角成等差級數,其最大邊與最小邊之比,若 5:4, 求各角之值.

15. $b=8$, $c=5$, $A=60°$, 求 a 之值.

16. 此邊爲他一邊之半,其夾角爲60°, 求兩角之值.

17. $a=2$, $b=\sqrt{6}$, $c=\sqrt{3}+1$, 求各角.

18. $a=m$, $b=n$, $c=\sqrt{(m^2+mn+n^2)}$, 求最大角.

19. $a=17$, $b=25$, $c=26$, 求其面積.

20. $a=2(\sqrt{3}+1)$, $B=45°$, $C=60°$, 求其面積.

21. $b=14$, $c=7$, $C=30°$, 求 A, B, 及 a.

22. $a=\sqrt{2}$, $b=2$, $A=45°$, 求 B 角.

23. $a=2$, $c=\sqrt{3}+1$, $A=45°$, 求 B, C, 及 b.

例題解自 12 至 23.

(12) $\sin B=\frac{b\sin A}{a}=\frac{1000\sin 76°2'30''}{2000}=\frac{.970471}{2}=.485236$
$=\sin 29°1'40''$. 依題意. $a>b$, 故 $A>B$, 故 $B=29°1'40''$,
$C=180°-(A+B)=180°-(76°2'30''+29°1'40'')=74°55'50''$.

(13) $\sin C=\frac{c\sin B}{b}=\frac{40\sin 52°32'15''}{32}=\frac{5\times .793751}{4}=.992189$
$=\sin 82°50'3''$, $c>b$, 故 $C>B$. 依 $C=82°50'3''$ 或 $C=180°-82°50'3''$
$=97°9'57''$, 由是 $A=44°37'42''$ 或 $A=30°17'48''$.

(14) 令最大角$=A$, 最小角$=C$, 依題意. $B=\frac{1}{2}(A+C)$,

卽 $B=\frac{1}{2}(180^\circ-B)$, 故 $B=60^\circ$, 又 $a:c=5:4$, 故 $\frac{a-c}{a+c}=\frac{5-4}{5+4}=\frac{1}{9}$,

$\tan\frac{1}{2}(A-C)=\frac{a-c}{a+c}\cot\frac{B}{2}=\frac{1}{9}\cot 30^\circ=\frac{1\cdot 732051}{9}=\cdot 192450=\tan 10^\circ 53'36''$,

故 $\frac{1}{2}(A-C)=10^\circ 53'30''$, 又 $\frac{1}{2}(A+C)=\frac{1}{2}(180^\circ-B)=\frac{1}{2}(180^\circ-60^\circ)=60^\circ$,

由是 $A=70^\circ 58'36''$, $C=49^\circ 6'24''$.

(15) $2bc\cos A=b^2+c^2-a^2$, 卽 $2\times 8\times 5\cos 60^\circ=8^2+5^2-a^2$ 故 $a=7$.

(16) 令 $c=\frac{1}{2}b$, $A=60^\circ$ 則 $\frac{1}{2}(B+C)=\frac{1}{2}(180^\circ-A)=60^\circ$,

$\tan\frac{1}{2}(B-C)=\frac{b-c}{b+c}\cot\frac{1}{2}A=\frac{\frac{1}{2}b}{\frac{3}{2}b}\cot 30^\circ=\frac{1}{3}\times\sqrt{3}=\frac{1}{\sqrt{3}}=\tan 30^\circ$,

故 $\frac{1}{2}(B-C)=30^\circ$ 由是 $B=90^\circ$, $C=30^\circ$.

(17) $\cos A=\frac{b^2+c^2-a^2}{2bc}=\frac{6+(\sqrt{3}+1)^2-4}{2\sqrt{6}(\sqrt{3}+1)}=\frac{(\sqrt{3}+1)\sqrt{3}}{(\sqrt{3}+1)\sqrt{6}}=\frac{1}{\sqrt{2}}=\cos 45^\circ$,

故 $A=45^\circ$, 同法.. $B=60^\circ$, $C=75^\circ$.

(18) $\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{m^2+n^2-(m^2+mn+n^2)}{2mn}=-\frac{1}{2}$,

但 $\frac{1}{2}=\cos 60^\circ$, 故 $-\frac{1}{2}=\cos(180^\circ-60^\circ)=\cos 120^\circ$ 故 $C=120^\circ$.

(19) $s=\frac{1}{2}(a+b+c)=\frac{1}{2}(17+25+26)=34$, $s-a=17$, $s-b=9$, $s-c=8$,

從公式(5). $S=\sqrt{s(s-a)(s-b)(s-c)}=\sqrt{34\times 17\times 9\times 8}=17\times 3\times 4=204$.

(20) $A=180^\circ-(B+C)=180^\circ-(45^\circ+60^\circ)=75^\circ$, $c=\frac{a\sin C}{\sin A}$,

從公式(5). $S=\frac{1}{2}ac\sin B=\frac{1}{2}a\times\frac{a\sin C}{\sin A}\sin B=\frac{a^2\sin B\sin C}{2\sin A}$

$=\frac{a^2\sin 45^\circ\sin 60^\circ}{2\sin 75^\circ}=\frac{4(\sqrt{3}+1)^2\times\frac{1}{\sqrt{2}}\times\frac{\sqrt{3}}{2}}{\frac{1}{2}(\sqrt{6}+\sqrt{2})}$, (例題六 2.) $=6+2\sqrt{3}$.

(21) $\sin B=\frac{b\sin C}{c}=\frac{14\sin 30^\circ}{7}=2\times\frac{1}{2}=1$, 故 $B=90^\circ$, $A=60^\circ$.

(22) $\sin B=\frac{b\sin A}{a}=\frac{2\sin 45^\circ}{\sqrt{2}}=\sqrt{2}\times\frac{1}{\sqrt{2}}=1$, 故 $B=90^\circ$.

(23) $\sin C=\frac{c\sin A}{a}=\frac{(\sqrt{3}+1)\sin 45^\circ}{2}=\frac{\sqrt{3}+1}{2\sqrt{2}}=\frac{\sqrt{6}+\sqrt{2}}{4}=\sin 75^\circ=\sin(180^\circ-75^\circ)$,

$c>a$, 故 $C>A$, 故 $C=75^\circ$ 或 $180^\circ-75^\circ=105^\circ$

由是 $C=75^\circ$, $B=60^\circ$, $b=\sqrt{6}$, 或 $C=105^\circ$, $B=30^\circ$, $b=\sqrt{2}$.

24. $A=15^\circ$, $a=4$, $b=4+\sqrt{48}$, 求 B.

25. $A=18^\circ$, $a=4$, $b=4+\sqrt{80}$, 求 B, C, 及 c.

26. $A=30^\circ$, $b=100$, $a=40$, 求 B.

27. $B=45^\circ$, $b=2$, $c=\sqrt{3}-1$, 求 A, C 及 a.

28. $C=120^\circ$, 又 c 之中央線爲 m, 則 a 或 b 之值爲

$$\frac{1}{2\sqrt{2}}(\sqrt{3c^2-4m^2}\pm\sqrt{12m^2-c^2}).$$

29. 三角形之一角爲60°, 其夾邊之比爲5:3, 則其他之二角, 爲 $tan^{-1}\frac{3}{7}\sqrt{3}$ 及 $tan^{-1}5\sqrt{3}$.

30. 已知 $a+b$, c 及 C, 則

$$a=m\sin^2\tfrac{1}{2}\phi;\ b=m\cos^2\tfrac{1}{2}\phi.$$

但 $m=a+b$, $\sin\phi=\pm\frac{1}{m}\sqrt{(m+c)(m-c)}\sec^2\frac{1}{2}C.$

31. $\cos\theta=(a-b)/c$, 則

$$\cos\frac{A+B}{2}=\frac{c\sin\theta}{2\sqrt{ab}},\quad \cos\frac{A-B}{2}=\frac{(a+b)\sin\theta}{2\sqrt{ab}}.$$

例題解自 24 至 31.

(24) $\sin B=\frac{b\sin A}{a}=\frac{(4+\sqrt{48})\sin 15^\circ}{4}=(1+\sqrt{3})\frac{\sqrt{3}-1}{2\sqrt{2}}=\frac{1}{\sqrt{2}}.$

$b>a$, 故 $B>A$, 由是 $B=45^\circ$ 或 $180^\circ-45^\circ=135^\circ$.

(25) $\sin B=\frac{b\sin A}{a}=\frac{(4+\sqrt{80})\sin 18^\circ}{4}=(1+\sqrt{5})\frac{\sqrt{5}-1}{4}$, [例題六 12]

$=1$. 故 $B=90^\circ$, $C=90^\circ-18^\circ=72^\circ$.

$c=b\cos A=(4+\sqrt{80})\cos 18^\circ=4(1+\sqrt{5})\frac{\sqrt{(10+2\sqrt{5})}}{4}=\sqrt{\{(1+\sqrt{5})^2(10+2\sqrt{5})\}}$

$=2\sqrt{(3+\sqrt{5})(5+\sqrt{5})}=4\sqrt{(5+2\sqrt{5})}.$

(26) $\sin B=\frac{b\sin A}{a}=\frac{100\sin 30°}{40}=\frac{5}{4}>1$, 是不合理. 故不成三角形.

(27) $\sin C=\frac{c\sin B}{b}=\frac{(\sqrt{3}-1)\sin 45°}{2}=\frac{\sqrt{3}-1}{2\sqrt{2}}=\sin 15°=\sin(180°-15°)$

然 $b>c$, 故 $B>C$. 故 $C=15°$, $A=120°$, $a=\sqrt{6}$.

(28) 由平面幾何學講義三編172.節之定理, $a^2+b^2=2m^2+2(\frac{1}{2}C)^2$.

又 $2ab\cos 120°=a^2+b^2-c^2$, 故 $-ab=2m^2+\frac{1}{2}c^2-c^2$, $2ab=c^2-4m^2$.

故 $a+b=\sqrt{(a^2+b^2+2ab)}=\sqrt{(2m^2+\frac{1}{2}c^2+c^2-4m^2)}=\frac{1}{\sqrt{2}}\sqrt{(3c^2-4m^2)}$,

$a-b=\sqrt{(a^2+b^2-2ab)}=\sqrt{\{2m^2+\frac{1}{2}c^2-(c^2-4m^2)\}}=\frac{1}{\sqrt{2}}\sqrt{(12m^2-c^2)}$.

由此可得其證.

(29) 用公式(4). $\tan\frac{B-C}{2}=\frac{b-c}{b+c}\cot\frac{A}{2}=\frac{5-3}{5+3}\cot 30°=\frac{\sqrt{3}}{4}$, 故

$\tan\frac{1}{2}\{180°-(60°+C)-C\}=\frac{1}{4}\sqrt{3}$, 即 $\tan(60°-C)=\frac{1}{4}\sqrt{3}$, 即

$\frac{\tan 60°-\tan C}{1+\tan 60°\tan C}=\frac{\sqrt{3}}{4}$, $\frac{\sqrt{3}-\tan C}{1+\tan C\sqrt{3}}=\frac{\sqrt{3}}{4}$, 從此 $\tan C=\frac{3\sqrt{3}}{7}$,

又 $\tan\frac{1}{2}\{B-(180°-60°-B)\}=\frac{1}{4}\sqrt{3}$, 從此 $\tan B=5\sqrt{3}$.

(30) $2ab\cos C=a^2+b^2-c^2$, 即 $2ab(2\cos^2\frac{1}{2}C-1)=a^2+b^2-c^2$.

$4ab\cos^2\frac{1}{2}C=(a+b)^2-c^2=m^2-c^2$, $4ab=(m^2-c^2)\sec^2\frac{1}{2}C=m^2\sin^2\phi$,

即 $\sin\phi=\pm\frac{1}{m}\sqrt{(m+c)(m-c)}\sec\frac{1}{2}C$, 故 $(a+b)^2-4ab=m^2-m^2\sin^2\phi$,

即 $(a-b)^2=m^2\cos^2\phi$, 故 $a-b=m\cos\phi$, 但 $a+b=m$,

由是 $a=\frac{1}{2}m(1+\cos\phi)=m\cos^2\frac{1}{2}\phi$.

(31) $\sin^2\theta=1-\cos^2\theta=\frac{c^2-(a-b)^2}{c^2}=\frac{2ab-(a^2+b^2-c^2)}{c^2}=\frac{2ab-2ab\cos C}{c^2}$,

故 $c^2\sin^2\theta=2ab(1-\cos C)=4ab\sin^2\frac{1}{2}C=4ab\cos^2\frac{1}{2}(A+B)$,

故 $\cos\frac{A+B}{2}=\frac{c\sin\theta}{2\sqrt{ab}}$, 又 $\frac{\sin A}{a}=\frac{\sin B}{b}=\frac{\sin A+\sin B}{a+b}=\frac{\sin C}{c}$,

故 $\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A-B)}{a+b}=\frac{\sin(A+B)}{c}=\frac{2\sin\frac{1}{2}(A+B)\cos\frac{1}{2}(A+B)}{c}$,

故 $\cos\frac{1}{2}(A-B)=\frac{a+b}{c}\cos\frac{1}{2}(A+B)=\frac{a+b}{c}\times\frac{c\sin\theta}{2\sqrt{ab}}=\frac{(a+b)\sin\theta}{2\sqrt{ab}}$.

32. 已知底邊 a, 高 h, 及兩底角之差, 求其頂角.

33. 三邊, 高, 及面積, 均爲有理數, 則其一般之式, 順次爲 $u^2x^2+v^2y^2$, $u^2y^2+v^2x^2$, $(u^2-v^2)(x^2+y^2)$, $2uvxy$, $uvxy(u^2-v^2)(x^2+y^2)$. 試證之.

已知 a, b, A 兩意之式, 求次各式之證.

34. $c_1=2c_2$ 則 $3a=b\sqrt{(1+8\sin^2 A)}$.

35. 三角形 AB_1C, 爲三角形 AB_2C 之 n 倍, 則

$$\frac{a}{b}=\frac{1}{n+1}\sqrt{(n^2+1-2n\cos 2A)}.$$

36. $\frac{\sin C_1}{c_1}=\frac{\sin C_2}{c_2}$,

37. $\sin\frac{C_1-C_2}{2}=\frac{c_1-c_2}{2a}$.

38. $\frac{\sin C_1}{\sin B_1}+\frac{\sin C_2}{\sin B_2}=2\cos A$.

39. $\cot\frac{C_1+C_2}{2}=\tan A$.

40. $c_1^2-2c_1c_2\cos 2A+c_2^2=4a^2\cos^2 A$.

例題解自 32 至 40.

(32) $2S=ah=bc\sin A$, 但 A 爲頂角,

又從 $\frac{b}{\sin B}=\frac{c}{\sin C}=\frac{a}{\sin A}$ 得 $bc=\frac{a^2\sin B\sin C}{\sin^2 A}$, 故 $\sin B\sin C$

$=\frac{h}{a}\sin A$, 卽 $\cos(B-C)-\cos(B+C)=\frac{2h}{a}\sin A$, 卽 $\cos d+\cos A=\frac{2h}{a}\sin A$,

故 $\cos d\operatorname{cosec} A+\cot A=\frac{2h}{a}$, $\cos^2 d\,(1+\cot^2 A)=\left(\frac{2h}{a}-\cot A\right)^2$

由是 $\cot A=\frac{1}{a\sin^2 d}\{2h\pm\cos d\sqrt{(4h^2+a^2\sin^2 d)}\}$

(33) 於三角形 ABC, 令其高爲 AD, 則

$(m^2+n^2)=(m^2-n^2)^2+4m^2n^2$, 故於直角三角形ABD, 得

$AB=m^2+n^2$, $BD=m^2-n^2$, $AD=2mn$,

同樣. 於直角三角形ACD, 得 $AC=m'^2+n'^2$, $CD=m'^2-n'^2$, $AD=2m'n'$.

故 $mn=m'n'$, 由是 $m=ux$. $n=vy$, $m'=uy$. $n'=vx$, 則

$mn=m'n'=uvxy$, 故 $AB=u^2x^2+v^2y^2$, $AC=u^2y^2+v^2x^2$.

$BC=BD+CD=m^2-n^2+m'^2-n'^2=u^2x^2-v^2y^2+u^2y^2-v^2x^2=(u^2-v^2)(x^2+y^2)$.

$AD=2mn=2uvxy$,

又 $S=\frac{1}{2}AD\times BC=\frac{1}{2}(2uvxy)\times(u^2-v^2)(x^2+y^2)$.

(34) 於6.節之圖. $2bc_1\cos A=b^2+c_1^2-a^2$, 又由幾何學定理,

$a^2+b^2=2a^2+2c_2^2=2a^2+2(\frac{1}{2}c_1)^2$, 故 $c_1^2=2(b^2-a^2)$,

故 $2b\sqrt{2(b^2-a^2)}\cos A=b^2+2(b^2-a^2)-a^2=3(b^2-a^2)$, 從此可得其證.

(35) $S_1=nS_2$, 故 $c_1=nc_2$, 由平面幾何學講義第三編例題932.

$(n-1)b^2+a^2=(n-1+1)a^2+\frac{n-1}{n-1+1}c_1^2$, 即 $c_1^2=n(b^2-a^2)$,

又 $2bc_1\cos A=b^2+c_1^2-a^2$, 故 $2b\cos A\sqrt{n(b^2-a^2)}=b^2+n^2(b^2-a^2)-a^2$,

從此可得其證. (36) 此例甚容易, 故略.

(37) 由6.節之圖. $B_2D=BC\sin B_2CD$, 即 $\frac{1}{2}(c_1-c_2)=a\sin\frac{1}{2}(C_1-C_2)$.

(38) 同上. $\frac{\sin C_1}{\sin B_1}=\frac{c_1}{b}$, $\frac{\sin C_2}{\sin B_2}=\frac{c_2}{b}$, 故 $\frac{\sin C_1}{\sin B_1}+\frac{\sin C_2}{\sin B_2}=\frac{c_1+c_2}{b}$,

又 $2bc_1\cos A=b^2+c_1^2-a^2$, $2bc_2\cos A=b^2+c_2^2-a^2$, 由減法.

$2b\cos A(c_1-c_2)=c_1^2-c_2^2$, 即 $\frac{c_1-c_2}{b}=2\cos A$, 故得其證.

(39) 角 ACB_2+角 A=角 CB_1B_2=角 B_1, 即 $C_2=B_1-A$, $B_1=C_2+A$,

又 角 $ACB_1=C_1=180°-A-B_1=180°-A-(C_2+A)$, 故 $\frac{C_1+C_2}{2}=90°-A$,

故 $\cot\frac{C_1+C_2}{2}=\cot(90°-A)=\tan A$.

(40) $2bc_1\cos A=b^2+c_1^2-a^2$, 從例題37. $2\cos A=\frac{c_1+c_2}{b}$,

故 $bc_1\times\frac{c_1+c_2}{b}=\frac{(c_1+c_2)^2}{4\cos^2 A}+c_1^2-a^2$,

即 $4c_1c_2\cos^2 A=c_1^2+c_2^2+2c_1c_2-4a^2\cos^2 A$,

故 $4a^2\cos^2 A=c_1^2+c_2^2-2c_1c_2(2\cos^2 A-1)=c_1^2+c_2^2-2c_1c_2\cos 2A$.

41. $\dfrac{2b+c_1+c_2}{1+\cos A}+\dfrac{2b-c_1-c_2}{1-\cos A}=4b.$

42. $\dfrac{S_1}{S_2}+\dfrac{S_2}{S_1}=\dfrac{2(a^2+b^2\cos 2A)}{b^2-a^2}$

43. $S_1^2+S_2^2-2\xi_1\xi_2\cos 2A=\dfrac{a^2}{b^2}(S_1+S_2)^2.$

44. $S_1=KS_2$, 則 $\dfrac{b}{a}$ 在 1 與 $\dfrac{K+1}{K-1}$ 之間.

例題解自 41. 至 44.

(41) 從前例, $2b\cos A=c_1+c_2$, 卽 $2b(1+\cos A)=2b+c_1+c_2$,

故 $\dfrac{2b+c_1+c_2}{1+\cos A}=2b$, 同樣. $\dfrac{2b-c_1-c_2}{1-\cos A}=2b.$

(42) 從 6. 節之關係式. $S_1+S_2=\frac{1}{2}b^2\sin 2A$, $S_1S_2=\frac{1}{4}b^2\sin^2 A(b^2-a^2)$,

$$\frac{S_1}{S_2}+\frac{S_2}{S_1}=\frac{(S_1+S_2)^2-2S_1S_2}{S_1S_2}=\frac{\frac{1}{4}b^4\sin^2 2A-\frac{1}{2}b^2\sin^2 A(b^2-a^2)}{\frac{1}{4}b^2\sin^2 A(b^2-a^2)}$$

$$=\frac{4b^2\cos^2 A-2(b^2-a^2)}{b^2-a^2}=\frac{2\{a^2+b^2(2\cos^2 A-1\}}{b^2-a^2}.$$

(43) $S_1^2+S_2^2-2S_1S_2\cos 2A=(S_1+S_2)^2-2S_1S_2(1+\cos 2A)$

$=\frac{1}{4}b^4\sin^2 2A-\frac{1}{2}b^2\sin^2 A(b^2-a^2)2\cos^2 A=\frac{1}{4}b^4\sin^2 2A-\frac{1}{4}b^2\sin^2 2A(b^2-a^2)$

$=\frac{1}{4}a^2b^2\sin^2 2A=\dfrac{a^2}{b^2}(\frac{1}{4}b^4\sin^2 2A)=\dfrac{a^2}{b^2}(S_1+S_2)^2.$

(44) 於前例, 令 $S_1=KS_2$, 則

$K^2S_2^2+S_2^2-2KS_2^2\cos 2A=\dfrac{a^2}{b^2}(KS_2+S_2)^2$, 卽 $K^2+1-2K\cos 2A=\dfrac{a^2}{b^2}(K+1)^2$,

$\dfrac{b^2}{a^2}=\dfrac{(K+1)^2}{K^2+1-2K\cos^2 A}=\dfrac{(K+1)^2}{(K+1)^2+4K\sin^2 A}$. K 為正數, 故

$\dfrac{b^2}{a^2}<\dfrac{(K+1)^2}{(K+1)^2}$, 卽 $\dfrac{b}{a}<1$, 又 $\dfrac{b^2}{a^2}=\dfrac{(K+1)^2}{(K-1)^2-4K\cos^2 A}$, 故 $\dfrac{b}{a}>\dfrac{K+1}{K-1}$.

三角形之對數計算

7. 三角形之對數計算 用公式如次.

(已知二角及一對邊) $\log b=\log a+\mathrm{L}\sin\mathrm{B}-\mathrm{L}\sin\mathrm{A}$ (1)

〔證〕 從3.節之公式, $b=\dfrac{a\sin\mathrm{B}}{\sin\mathrm{A}}$, 故

$\log b=\log a+\mathrm{L}\sin\mathrm{B}-10-(\mathrm{L}\sin\mathrm{A}-10)$.

(已知二邊及一對角) $\mathrm{L}\sin\mathrm{B}=\log b+\mathrm{L}\sin\mathrm{A}-\log a$ (2)

〔證〕 以(1)轉項, 即得.

(已知三邊)

$$\mathrm{L}\cos\frac{\mathrm{A}}{2}=10+\tfrac{1}{2}\{\log s+\log(s-a)-\log b-\log c\} \quad (3)$$

〔證〕 從3.節公式(3). $\cos\dfrac{\mathrm{A}}{2}=\sqrt{\dfrac{s(s-a)}{bc}}$,

$\mathrm{L}\cos\dfrac{\mathrm{A}}{2}-10=\tfrac{1}{2}\{\log s+\log(s-a)-\log b-\log c\}$,

(已知二邊及夾角) $\mathrm{L}\tan\dfrac{\mathrm{B}-\mathrm{C}}{2}=\log(b-c)+\mathrm{L}\cot\dfrac{\mathrm{A}}{2}-\log(b+c)$, (4)

〔證〕 從3.節公式(4). $\tan\dfrac{\mathrm{B}-\mathrm{C}}{2}=\dfrac{b-c}{b+c}\cot\dfrac{\mathrm{A}}{2}$,

$\mathrm{L}\tan\dfrac{\mathrm{B}-\mathrm{C}}{2}-10=\log(b-c)+\log\cot\dfrac{\mathrm{A}}{2}-10-\log(b+c)$,

(求面積) $\log\mathrm{S}=\log b+\log c+\mathrm{L}\sin\mathrm{A}-\log 2-10$

$=\tfrac{1}{2}\{\log s+\log(s-a)+\log(s-b)+\log(s-c)\}$ (5)

〔證〕 從3.節公式(5). $\mathrm{S}=\tfrac{1}{2}bc\sin\mathrm{A}$.

$\log s=\log b+\log c+\mathrm{L}\sin\mathrm{A}-10-\log 2$.

又從 $\mathrm{S}=\sqrt{\{s(s-a)(s-b)(s-c)\}}$ 可得第二之公式.

例題二十一

1. $a=55$, $A=41°13'2''$, $B=71°19'5''$, 求 b 及 c.
2. $b=643$, $c=872$, $C=52°10'$, 求 B.
3. $b=873{\cdot}4$, $c=752{\cdot}8$ $C=54°23'$, 求 B.
4. $a=374{\cdot}5$, $c=576{\cdot}2$, $c=759{\cdot}3$, 求 A.
5. $a=4001$, $b=9760$, $c=7942$, 求 A.
6. $a=764{\cdot}2$, $b=1873{\cdot}5$, $B=32°58'$, 求 C.
7. $b=1234$, $c=567$. $A=8°9'10''$, 求 S.

例題解自1至7.

(1) 由公式(1). $\log b=\log 55+L\sin 71°19'5''-L\sin 41°13'22''$

$=1{\cdot}740363+9{\cdot}976494-9{\cdot}818878=1{\cdot}897979=\log 79$,

∴ $b=79$, $c=77$.

(2) 由公式(2). $L\sin B=\log 643+L\sin 52°10'-\log 872$

$=2{\cdot}808211+9{\cdot}897516-2{\cdot}940517=9{\cdot}765210=L\sin 35°37'7''$

∴ $B=35°37'7''$, 但 $b<c$, ∴ $B<C$, 故本題之B只有一値.

(3) 由公式(2). $L\sin B=\log 873{\cdot}4+L\sin 54°23°-\log 752{\cdot}8$

$=9{\cdot}974587=L\sin 70°33'24''$, 但 $b>c$. 故 $B>C$.

由是 $B=70°35'24''$, 或 $B=180°-70°35'24''=109°24'36''$.

(4) 從公式(3). $A=28°35'39''$, (5) $A=23°21'18''$.

(6) 從公式(4). $L\tan\frac{1}{2}(C-A)=\log(1873{\cdot}5-764{\cdot}2)+L\cot 16°29'$

$-\log(1873{\cdot}5+764{\cdot}2)=10{\cdot}152684=L\tan 54°52'13''$,

故 $C=\frac{1}{2}(C+A)+\frac{1}{2}(C-A)=128°23'13''$.

(7) 從公式(5). $S=49612$.

第拾肆編

高及距離之測量

1. 高及距離

應用三角形之解法.可以測量之高及距離.今示其方法如次.

〔第一〕 求視線(測者之目與物體之連結線)與水平線(平行於海面之線)所成之傾角,而物體高於測者之目時.此角謂之仰角.(或云高度)低於測者之目時,此角謂之俯角.(或云下臨角)

〔第二〕 求兩視線之角.

求此二種角所用之器械,爲六分儀(*Sex-tant*)及經緯儀(*Theodolite*)。

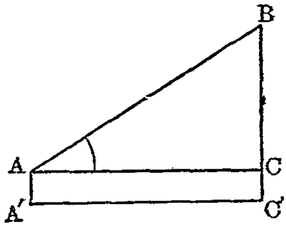
2. 物體高度之測法

〔例〕 有高塔BC,自其底c距地平上(卽水平)153尺之A點,測得塔頂B點之仰角爲33°10′,求塔高若干.

AC=153, 角BAC=33°10′

$BC=AC\,tan\,BAC=153\times tan\,33°10'$

$=153\times \cdot 653551=100$ (略近値).



(注意) 此例測者之眼高未算入,卽測者之眼,設在地平上A點也.

若算入測者之眼高,例如A′C′爲地平上之直線,則AA″=CC′=眼高.然塔高=BC′,故塔高=100尺+眼高.

3. 山高之測法

〔例〕 於地平上A點,測得山之頂點C之仰角CAD爲α度.向山之方向一直進行至a尺之B點,再測得CBD之仰角爲β度.求山之高.

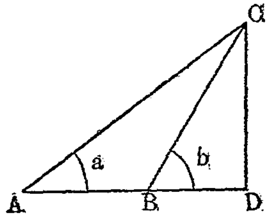
$AB=a$, 角 $CAD=\alpha$, 角 $CBD=\beta$,

故 角 $ACB=$ 角 $CBD-$角 $CAD=\beta-\alpha$.

由是 $BC=\dfrac{AB\sin CAB}{\sin ACB}=\dfrac{a\sin\alpha}{\sin(\beta-\alpha)}$,

故 $CD=BC\sin CBD$.

即 $CD=\dfrac{a\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$.



4. 不能接近物體之高之測法

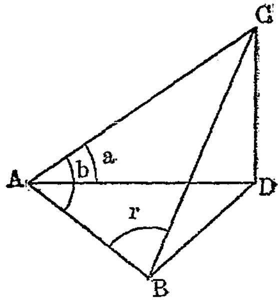
〔例〕 有塔高 CD, 於地平上 A 點測之, 得角 $CAD=\alpha$, 角 $CAB=\beta$, 但 AB 之距離爲 a. 又於 B 點測之, 得角 $CBA=\gamma$, 求塔高 CD 若干.

$AB=a$, 角 $CAD=\alpha$, 角 $CAB=\beta$,

角 $CBA=\gamma$,

故 $AC=\dfrac{AB\sin CBA}{\sin ACB}=\dfrac{a\sin\gamma}{\sin(\beta+\gamma)}$.

故 $CD=AC\sin CAD=\dfrac{a\sin\gamma\sin\alpha}{\sin(\beta+\gamma)}$.



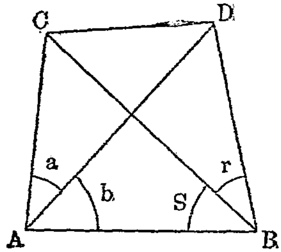
5 難以互視之二點之距離測定法

〔例〕 有兩點 C.D. 於 A 及 B 二處測之, 測得 角 $CAD=\alpha$, 角 $DAB=\beta$, 角 $DBC=\gamma$, 角 $CBA=\delta$, $AB=a$, 則 CD 之距離如何.

$AC=\dfrac{AB\sin ACB}{\sin CBA}=\dfrac{a\sin(\alpha+\beta+\delta)}{\sin\delta}$,

又 $AD=\dfrac{AB\sin ADB}{\sin ABD}=\dfrac{a\sin(\beta+\gamma+\delta)}{\sin(\gamma+\delta)}$,

於三角形 ACD. 知其二邊 AC, AD, 及夾角 α, 故由前編. 可求得 CD 之長.

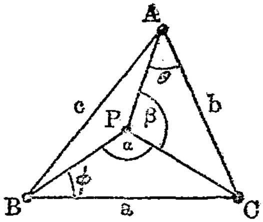


6. 三點問題

知三點A, B, C之位置,卽知其距離.

BC=a, CA=b, AB=c, 而於一點P, 測得與此三定點之夾角 BPC=α, CPA=β, 求自P點至各點之距離.

P點在三角形ABC之內或外,均可同法解之.



令角PAC=θ, 角PBC=ϕ, 則

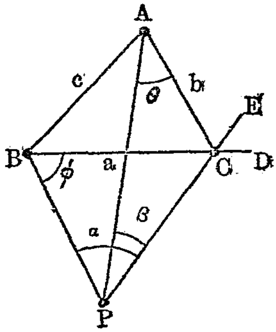
$$PC=\frac{a\sin\phi}{\sin\alpha},\ PC=\frac{b\sin\theta}{\sin\beta}$$

故 $$\frac{\sin\theta}{\sin\phi}=\frac{a\sin\beta}{b\sin\alpha}=\tan\omega.$$

故 $\tan\omega$ 爲已知數,

而 $$\frac{\sin\theta-\sin\phi}{\sin\theta+\sin\phi}=\frac{1-\tan\omega}{1+\tan\omega},$$

故 $$\frac{\tan\frac{1}{2}(\theta-\phi)}{\tan\frac{1}{2}(\theta+\phi)}=\tan(45^\circ-\omega)\ \dots\dots\dots\dots\ (A)$$



P在三角形ABC之內,則於四角形APBC, $\theta+\phi+\alpha+\beta+C=360^\circ$.

故 $\theta+\phi$ 爲已知數.

故從(A)得知 $\theta-\phi$,

由是得知 θ 及 ϕ,

P在三角形ABC之外,而隔A與BC,在BAC之內,則將BC延長至D, PC延長至E,則 角ACE=$\theta+\beta$, 角PCD=$\phi+\alpha$, 角ECD=角BCP=$180^\circ-\phi-\alpha$,

而 角ACE+角ACB+角PCD+角BCP+角ECD=360°,

卽 $\theta+\beta+C+\phi+\alpha+180^\circ-\phi-\alpha+180^\circ-\phi-\alpha=360^\circ$,

故 $\theta-\phi=\alpha-\beta-C$. 故 $\theta-\phi$ 爲已知數.

由是從(A)可知 $\theta+\phi$,

由是 P爲任意之位置,均可知其爲 θ 及 ϕ,

故 $$PA=\frac{b\sin(\theta+\beta)}{\sin\beta},\ PB=\frac{a\sin(\phi+\alpha)}{\sin\alpha},\ PC=\frac{a\sin\phi}{\sin\alpha}.$$

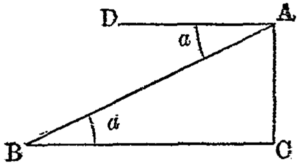
〔注意〕 本題乃測地圖或海圖,遠隔三定點之距離之法, 故港灣測量多用之.

又本題特別之式爲P對向於AB, BC, CA之角,若相等時,則爲幾何學上之題.(平面幾何學講義例題336, 669, 710.)

又 A, B,C, P同在圓周上,則其特別之式爲$\theta=\phi$, $\alpha+A=180°$, 而 (A) 爲0 (同上講義 126. 節定理)

7. 測物體俯角之事件

〔第一例〕 有人立於丘上,已知眼高AC=a,在地平上有一物體B,測得其俯角爲α度,問此人至B之距離如何.



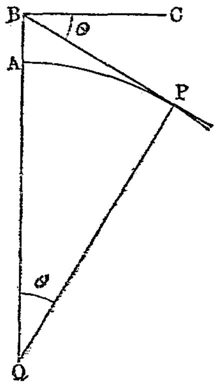
BC爲水平線, AD平行於BC,

然 角BAD=α=角ABC,

故 $AB=\dfrac{AC}{\sin ABC}=\dfrac{a}{\sin\alpha}$.

〔第二例〕 於高h哩之點B所得見地平上之點P,求PA地平距離,但地球爲半徑R哩之球體.

(本題爲自高處,求得望地平上之眼界)



AB=h, 俯角CBP=θ (弧度)

AO=PO=R, AP=x.

BP爲切線,故

$BP^2=AB(AB+2R)=h(h+2R)$

即 $BP=\sqrt{2Rh}\left(1+\dfrac{h}{2R}\right)^{\frac{1}{2}}$

$=\sqrt{2Rh}\left(1+\dfrac{h}{4R}-\dfrac{h^2}{32R^2}\cdots\cdots\cdots\cdots\right)$

又 $BP=BO\sin BOP=(h+R)\sin\theta$

$AP=x=R\theta$ (第一編 10. 節)

R殆近於400,000哩,而地上之高處在5哩以下, 即 $h\ngtr 5$, 故 h/R爲極微小.

由是省略 $\frac{h}{4R}$ 以下之項. $BP=\sqrt{2Rh}$, 其誤差甚小. 從而 θ 亦微小. 故 $\sin\theta=\theta$,

由是 $BP=(h+R)\theta$.

由是 $x=R\theta=\frac{R\sqrt{2Rh}}{h+R}$. 是卽所求眼界之距離也.

若高甚小, 則 $\frac{R}{R+h}=1$. (略近數) 則 $x=\sqrt{2Rh}$.

8. 羅針盤

航海測量, 及測風之方位等, 用羅針盤. (*Compass*) 其針常指北, 故 N(北), S(南), E(東), W(西). 卽以直交之 NS (南北直徑). 與 EW(東西直徑). 分圓周爲四等分, 每一等分爲一象限 (90°), 又分各象限爲八等分, 自一點至八點之間, 以各象限之角度示之, 例如北微東 (N *by* E) 爲第一象限之第一點. 卽示自北向東之11°15′也. 北北東 (NNE) 爲第二點, 自北向東22°30′. 北東北 (NEN) 爲第三點, 北東 (NE) 爲第四點, 北東東 (NEE) 爲第五點, 東北東 (ENE) 爲第六點, 東微北 (E *by* N) 爲第七點, 東 (E) 爲第八點. 順次表示其自北移向東 11°15′.

第四象限, 卽自東至南. 亦同樣分爲東微南 (E *by* S), 東南東 (ESE), 南東東 (SEE), 南東 (SE), 南東南 (SES), 南南東 (SSE), 南微東 (S *by* E), 南 (S) 之八點, 其他自北至西, 及自西至南, 均同樣也.

例題二十二

1. 立於地平上之塔高爲120尺, 於同地平上測得此塔頂之仰角爲36°25′, 求測點距塔底若干

2. 於同地平上. 離塔底 $369\frac{1}{2}$ 尺, 測得塔頂之仰角爲33°12′, 問塔高若干, 但眼高5尺.

例題解自1至2.

(1) 令塔頂爲A, 塔底爲C, 測者之處爲B, 則 $AC=120$, 角 $B=36°25'$,
$BC=AC\cot B=120\times\cot 36°25'=162\cdot7$. 是卽所求之尺度也.

(2) 同上. $BC=369\frac{1}{2}$ 尺, 角 $B=33°12'$,
$AC=BC\tan B+$ 眼高 $=369\frac{1}{2}\times\tan 33°12'+5=246\cdot7$ 尺, 卽塔之高.

3. 塔高150尺,其影75尺,求太陽之高度.

4. 於地平上之塔影爲176·2尺,太陽之高度爲33°12′,求塔之高.

5. 有風船在高於地平上2500尺之處.人自地平上之一點望之,得仰角40°35′,設此風船直下墜於地上,其墜點與測者所在之點,距離若干.

6. 有繩梯196尺,一端掛於100尺高之塔頂上,他端着地上,求其與地平線所成之角度.

7. 某海濱測某山(高12370尺)之仰角得8°22′,問此海濱與該山頂之水平距離如何.

8. 立於平行兩河岸上,於此岸上與人家相近處望對岸人家,沿河岸行1町,再望從前人家,則視線與河岸成30°角,問河寬.

9. 自海船上望見A,B兩島,東西對峙.而船適在A島之正南,於船上望B島,在北偏東爲35°,問測者之地位,距B島爲距A島之幾倍.

10. 同時測太陽與月,其兩視線成88°42′之交角.斯時太陽正照月之半面,(上弦或下弦).則自月至地與自月至太陽之兩直線,應成直角,問地距太陽爲距月若干倍.

11. 在風船之正反對,取地上二點A及B,(相距400碼).測得其高度爲64°15′及48°20′.求風船之高若干.

例題解自 3. 至 11.

(3) 令塔頂為 A, 底為 C, 影為 CB, 則 AC=150, BC=75, 角 ACB=直角

故 $\cot ABC=\frac{BC}{AC}=\frac{75}{150}=\cdot 5=\cot 63°26'6''$=高度.

(4) 同上. $AC=BC\cdot\tan ABC=176\cdot 2\tan 33°12'=115$=塔高.

(5) 風船為 A, 其直下之點為 B, 測者之處為 C, AB=2500, 角 ACB=40°35′,

故 $BC=AB\cot ACB=2500\cot 40°35'=2918\cdot 5$ 尺=所求之距離.

(6) 令塔頂為 A, 底為 C, 繩為 AB, 則 AC=100, AB=196,

故 $\sin ABC=\frac{AC}{AB}=\frac{100}{196}=\sin 30°40'37''$=傾度.

(7) 令山頂為 A, 測者之地點為 B, 水平距離為 BC, 則 AC=12370,

角 ABC=8°22′, $BC=AC\cot ABC=12370\cot 8°20'=6\frac{1}{2}$ 里.

(8) 令人家為 A, 其所望最近之點為 B, 則 AB 為河寬, 又 C 為第二點,

而 BC=1 町, 角 ACB=30°.

故 $AB=BC\tan ACB=1$ 町 $\times\tan 30°=1$ 町 $\times\frac{1}{\sqrt{3}}=20\sqrt{3}$ 間.

(9) 令測者之地位為 C, 則角 ACB=35°,

故 $\frac{AC}{BC}=\cos 35°$, $BC=\frac{AC}{\cos 35°}=1\cdot 222AC$, 即 1·222 倍.

(10) 令 S 為太陽, M 為月, E 為為地球上測者之位置, 則 SM 與 EM 成直角.

而角 MES=80°42′, 故 $EM=ES\cos MES$.

即 $ES=\frac{EM}{\cos MES}=\frac{EM}{\cos 88°42'}=EM\times 44$. 即 44 倍.

(11) 令 C 為風船, D 為風船直下之地點, 依題意, D 在直線 AB 之中間,

而 CD 為 AB 之垂線, 故 CD 為風船之高.

而 AB=400, 角 A=64°15′, 角 B=48°20′,

故 $AD=CD\cot A$, $BD=CD\cot B$,

故 $AB=AD+BD=CD(\cot A+\cot B)=\frac{CD\sin(A+B)}{\sin A\sin B}$,

故 $CD=\frac{AB\sin A\sin B}{\sin(A+B)}=\frac{400\sin 64°15'\sin 48°20'}{\sin 112°35'}=291\cdot 49$ 碼.

12. 某人於某地望一山頂，得仰角13′，依所持之地圖，(五萬分之一)測得其山頂與此處之水平距離爲5寸，又其山高出海面6400尺，問此處高出海面若干．

13. 谷川架直線之橋，其長爲a尺，自橋之兩端至橋下谷底一點，測得其傾度爲α及β，問谷深若干尺．

14. 自一市之A點，至B, C兩驛站，各通直街道，而AB爲3里，AC爲$1\frac{1}{2}$里，AB, AC兩街道爲60°交角，求BC之距離．

15. 於一塔之正南A，測其塔頂，得仰角30°．又於A之正西B測之，得仰角18°，則塔高爲$\frac{1}{4}AB\sqrt{(2\sqrt{5}-2)}$．

16. 於塔之正北A，測其仰角得45°．又於其地之正東B，測其仰角得15°，則塔高爲$\frac{1}{2}AB\,(3^{\frac{1}{4}}-3^{-\frac{1}{4}})$．

17. 有甲乙二船，同時自某港出發．甲向北東行，每時$7\frac{1}{2}$浬．乙向正北行，每時10浬．問船行1時30分之後，二船相離若干浬．

18. 於川之岸望對岸之木．(最近)得仰角60°．退後川岸40尺，再望前木，得仰角30°．問木之高及川之闊．

19. 自山麓B，測得山頂A之仰角爲60°，自B行上1哩至C再測之，得BCA爲135°，問山高若干碼，但BA與水平面成角30°．

例題解自 12. 至 19.

(12) 令山頂爲A,其底爲B,測者之地點爲C,則BC爲水平距離.
$\therefore$ BC$=\cdot5$尺$\times50000=25000$尺, 角 $ACB=13^\circ$ 故 $AB=BC\tan ACB$
$=25000$ 尺$\times\tan 13^\circ=5771\cdot7$ 尺. 故 C 之高$=6400$ 尺$-5771\cdot7$ 尺$=628\cdot3$ 尺.

(13) 令 $AB=a$ 爲橋之長, C 爲谷底, CD 卽 AB 之垂線, 爲谷之深.
故 $CD\cot\alpha+CD\cot\beta=AD+BD=AB=a$, 故 $CD=\dfrac{a\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$.

(14) $BC^2=AB^2+AC^2-2AB, AC\cos BAC=3^2+1\cdot5^2-2\times3\times1\cdot5\cos60^\circ=6\frac{3}{4}$
故 BC 爲 $\frac{3}{2}\sqrt{3}$ 里

(15) 令塔頂爲C,其底爲D,依題意, DA 與 BA 成直角,
而 $BD=CD\cot18^\circ=\dfrac{CD}{\sqrt{(1-\frac{2}{5}\sqrt{5})}}$ (例題六 13.) $=CD\sqrt{(2\sqrt{5}+5)}$;
又 $AD=CD\cot30^\circ=CD\sqrt{3}$, 故從 $AB^2=BD^2-AD^2$ 得
$CD^2\{(2\sqrt{5}+5)-3\}=AB^2$, 故 $CD=\frac{1}{4}AB\sqrt{(2\sqrt{5}-2)}$.

(16) 如前例, 得 $AD=CD\cot45^\circ=CD$,
$BD=CD\cot15^\circ=CD(2+\sqrt{3})$, 〔例題六 1.〕
故 $AB^2=CD^2\{(2+\sqrt{3})^2-1\}$, $CD^2=\dfrac{AB^2}{2(3+2\sqrt{3})}=\dfrac{AB^2(2\sqrt{3}-3)}{6}$
$CD=AB\sqrt{\dfrac{\sqrt{3}(2-\sqrt{3})}{6}}=AB\sqrt[4]{3}\left(\dfrac{\sqrt{3}-1}{\sqrt{6\times2}}\right)=AB\sqrt[4]{3}\left(\dfrac{1-\sqrt{3}-1}{2}\right)=\dfrac{AB}{2}\left(3^{\frac{1}{4}}-3^{-\frac{1}{4}}\right)$.

(17) 令 A 爲出發點, B, C 爲甲, 乙之位置, 則 $AB=7\frac{1}{2}$浬$\times1\frac{1}{2}=\frac{45}{4}$浬,
$AC=10$ 浬 $\times1\frac{1}{2}=15$ 浬, 又北東, 由北而東 45°, 故角 $BAC=45^\circ$,
故 $BC^2=AB^2+AC^2-2AB\times AC\cos BAC=\left(\frac{45}{4}\right)^2+15^2-2\left(\frac{45}{4}\right)\times15\cos45^\circ$,
故 $BC=10\cdot6$浬.

(18) AB 爲木高, BC 爲川廣, 令 BC 引長線上之一點爲 D, 則
角 $ACB=60^\circ$, 角 $ADB=30^\circ$, $CD=40$, 故角 $CAD=60^\circ-30^\circ=30^\circ$,
故 $AC=CD=40$, 由此 $AB=20\sqrt{3}$尺, $BC=20$尺.

(19) 從 B 引水平線, 與 A 之直下線會於 D, 則角 $CBD=30^\circ$,
角 $ABC=60^\circ-30^\circ=30^\circ$, 角 $ACB=135^\circ$, 故角 $BAC=15^\circ$, 故 $AB=\dfrac{BC\sin135^\circ}{\sin15^\circ}$
卽 $AB=\dfrac{1}{\sqrt{2}}\div\dfrac{\sqrt{6}-\sqrt{2}}{4}=\sqrt{3}+1$, $AD=AB\sin60^\circ=\frac{1}{2}(3+\sqrt{3})$哩$=880(3+\sqrt{3})$碼.

20. 向正北行之船,望見二燈臺在北東及北北東,再走20浬,望前二燈臺在正東成一直線,問二燈距離若干.

21. 立於同地平上之二塔,其高爲180尺及80尺,於各塔底互測其仰角,其一仰角等於他之仰角之二倍,問二塔距離若干.

22. 初測立木之仰角,得θ度,向此進行a尺,再測仰角得2θ度,又進行b尺,測得仰角爲4θ,則自三測處中之第二至立木之距離爲$a^2/(2b)$.

23. 有A, B, C三點,與同一平面上之塔DE之底D成一直線.其塔之仰角,在A爲θ,在B爲2θ,在C爲3θ,而$AB=a$, $BC=b$,則塔高爲$\{a\sqrt{(a+b)(3b-a)}\}/(2b)$.

24. 同上. AB=50尺, BC=20尺,求塔高及CD之距離各若干.

25. 同上.塔之仰角,在A爲θ,在B爲$\frac{\pi}{2}-\theta$,在C爲2θ,而$AB=a$, $BC=b$,則塔高爲$\frac{1}{2}\sqrt{(a+2b)(3a+2b)}$.

例題解自20.至25.

(20) 令A及B爲二燈臺,船在最高之處爲C,在次處爲D,則DBA成一直線 角ADC=90°,而CD=20浬,角ACD=45°,角BCD=22°30′ 角DAC=90°−角ACD=45°=角ACD,故AD=CD=20浬.

又 $BD=CD\tan BCD=20\tan 22°30'$,故$AB=AD-BD=11{\cdot}7158$浬.

(21) 二塔爲AB, CD. B, D爲其二底, AB=180尺, CD=80尺, BD=x,

角CBD=θ, 角ADB=2θ, 則 $\frac{80}{x}=tan\theta$, $\frac{180}{x}=tan2\theta=\frac{2tan\theta}{1-tan^2\theta}$,

故從 $\frac{180}{x}=2\left(\frac{80}{x}\right)\Big/\left\{1-\left(\frac{80}{x}\right)^2\right\}$, 得 $x=240$尺.

(22) 令立木DE之底爲D, 三測處順次爲A, B, C, 則AB=a, BC=b,

又令 BD=x, DE=h, 則 $\frac{DE}{AD}=\frac{h}{x+a}=tan\,\theta$, $\frac{DE}{BD}=\frac{h}{x}=tan2\theta$,

故從 $\frac{h}{x}=\frac{2tan\,\theta}{1-tan^2\theta}=2\left(\frac{h}{x+a}\right)\Big/\left\{1-\left(\frac{h}{x+a}\right)^2\right\}$ 得 $a^2-x^2=h^2$,

又從 $\frac{DE}{CD}=\frac{h}{x-b}=tan4\theta=\frac{2tan2\theta}{1-tan^22\theta}=2\left(\frac{h}{x}\right)\Big/\left\{1-\left(\frac{h}{x}\right)^2\right\}$ 得

$2bx-x^2=h^2$, 故 $2bx-x^2=a^2-x^2$.

(23) 同上. $\frac{h}{x+a}=tan\theta$, $\frac{h}{x}=tan2\theta$, $\frac{h}{x-b}=tan3\theta$,

從最初兩方程式, 依前例得 $a^2-x^2=h^2$, 又 $\frac{h}{x-b}=\frac{3tan\,\theta-tan^3\theta}{1-3\,tan^2\theta}$,

故從 $\frac{h}{x-b}=\left(\frac{3h}{x+a}-\frac{h^3}{(x+a)^3}\right)\Big/\left(1-\frac{3h^2}{(x+a)^2}\right)$ 得 $(a+x)^2(a+3b-2x)=h^2(3a+b+2x)$,

故 $(a+x)^2(a+3b-2x)=(a^2-x^2)(3a+b+2x)$, 此以$a-x$約之, 且化爲最簡之式, 則

$x=\frac{a(a-b)}{2b}$, 又 $h^2=a^2-x^2=a^2-\frac{a^2(a-b)^2}{4b^2}$.

(24) 前例之結果, x及h之值, 以$a=50$, $b=20$代之, 則

$h=\frac{25}{2}\sqrt{7}$尺, (塔高) 又CD$=17\frac{1}{2}$尺.

(25) 角之外, 均用前例所假定者, 則

$\frac{h}{x+a}=tan\theta$, $\frac{h}{x}=tan\left(\frac{\pi}{2}-\theta\right)=cot\,\theta$, 故 $\frac{h^2}{x(x+a)}=1$,

即 $h^2=x(x+a)$, 又 $\frac{h}{x-b}=tan2\theta=\frac{2tan\theta}{1-tan^2\theta}$

故從 $\frac{h}{x-b}=\frac{2h}{x+a}\Big/\left\{1-\left(\frac{h}{x+a}\right)^2\right\}$ 得 $h^2=(x+a)(a+2b-x)$, 故

$x(x+a)=(x+a)(a+2b-x)$, 故 $x=\frac{1}{2}a+b$, 此可代入下式, $h^2=x(x+a)$.

26. 有塔高h尺，於其正南測之，得仰角α，轉向正西步至A地測之，仰角β，同方向再進至B地測之，仰角γ，則AB之距離為$h(\sqrt{\cot^2\gamma-\cot^2\alpha}-\sqrt{\cot^2\beta-\cot^2\alpha})$尺.

27. 有高a尺之旗，立於塔頂上；隔塔底b尺之處望之，旗所對向之角為γ，求塔高若干。

28. 有塔在正北地點望之，得仰角15°，依前離塔底之距離，繞塔行100碼，至北東地點望之，仍得仰角15°，求塔高若干.

29. 立旗於塔頂上，於平面上之一點，測其對向之角，在塔為β度，在旗為α度，向塔行動a尺，測旗所對向之角，仍為α度，則塔高為$a\tan\beta/\{1-\tan\beta\tan(\alpha+\beta)\}$.

30. 有塔PQ，在α度傾斜之山道上，於此山道之同側，取A，B二點，與塔成一直線，從此二點測得塔頂之仰角為β，γ，求塔高若干. 但$AB=a$.

31. 有塔PQ，在山道中間，於此山道上取其正反對之二點A，B. 自塔底Q至A及B之距離為a尺及b尺，由A及B，測得塔所對向之角為β度及γ度，而塔高為h，山道之傾度為α，則

$$a+b=h(\cot\beta+\cot\gamma)\cos\alpha,$$

及 $$2\sin\alpha=(\cot\gamma-\cot\beta)\cos\alpha+(a-b)/h$$

例題解自26至31.

(26) 塔底為C，最初望塔之處為D，則角$BCD=90°$，故

$(AB+AC)^2=BD^2-CD^2=(h\cot\gamma)^2-(h\cot\alpha)^2$,

$AC^2=AD^2-CD^2=(h\cot\beta)^2-(h\cot\alpha)^2$,

由此可消去AC.

(27) 塔爲AB,旗爲AC,地平上之一點爲D, 則AC$=a$, BD$=b$, 角ADC$=\gamma$

又令 角ADB$=\theta$, AB$=x$, 則 $tan\theta=\frac{AB}{BD}=\frac{x}{b}$ $tan(\theta+\gamma)=\frac{BC}{BD}=\frac{x+a}{b}$,

即 $\frac{tan\theta+tan\gamma}{1-tan\theta\ tan\gamma}=\frac{x+a}{b}$, 故 $\frac{x+b\,tan\gamma}{b-x\,tan\gamma}=\frac{x+a}{b}$,

故 $x=\frac{1}{2}\{\sqrt{(a^2+4ab\ cot\gamma-4b^2)}-a\}$.

(28) 塔爲AB,二測處爲C,D. BC$=$BD$=y$ 爲半徑,並弧CD$=100$碼,

又 AB$=x$, 依題意, 角CBD$=45°$, 故 $2\pi y:100=360°:45°$,

故 $y=\frac{400}{\pi}$, $x=y\,tan15°=\frac{400}{\pi}(2-\sqrt{3})$碼.

(29) 塔爲AB,旗爲AC,平面上之二點爲D, E. AB$=x$, DE$=a$,

角ADB$=\beta$, 又角CDA$=$角CEA$=\alpha$, 故ACDE爲圓之內切四角形,

由是 角AEB$=$角ACD$=90°-$角CDE$=90°-(\alpha+\beta)$. 〔平面幾何學之定理〕,

$\frac{x}{BE}=tan\,AEB=tan\{90°-(\alpha+\beta)\}=cot(\alpha+\beta)$. 故 BE$=x\,tan(\alpha+\beta)$.

又 $\frac{x}{BD}=\frac{x}{a+BE}=tan\,\beta$, 故 BE$=\frac{x}{tan\,\beta}=a$, 由是消去BE.

(30) 自A及B,向塔引水平線AC, BD. 則角QAC$=$角QBD$=\alpha$,

角PAQ$=\beta$, 角PBQ$=\gamma$, 故角APB$=$角PAQ$-$角PBQ$=\beta-\gamma$,

角AQP$=90°+\alpha$, 故 $AP=\frac{AB\,sin PBA}{sin APB}=\frac{a\,sin\,\gamma}{sin\,(\beta-\gamma)}$, 又

$PQ=\frac{AP\,sin\,PAQ}{sin\,AQP}=\frac{AP\,sin\,\beta}{sin(90°+\alpha)}=\frac{a\,sin\,\beta\,sin\,\gamma}{cos\,\alpha\,sin(\beta-\gamma)}$.

(31) 自A,及B,向塔引水平線AD及BE,則角QAD$=$角QBE$=\alpha$,

角PAD$=$角PAQ$-$角QAD$=\beta-\alpha$, 角PBE$=$角PBQ$+$角QBE$=\gamma+\alpha$,

故 角APQ$=90°-$角PAD$=90°-(\beta-\alpha)$, 角BPQ$=90°-$角PBE$=90°-(\gamma+\alpha)$.

$AQ=\frac{PQ\,sin\,APQ}{sin\,PAQ}$, 即 $a=\frac{h\,cos\,(\beta-\alpha)}{sin\,\beta}$, 又 $BQ=\frac{PQ\,sin\,BPQ}{sin\,PBQ}$,

即 $b=\frac{h\,cos(\gamma+\alpha)}{sin\,\gamma}$, 故 $a+b=h\left\{\frac{cos(\beta-\alpha)}{sin\,\beta}\pm\frac{cos\,(\gamma+\alpha)}{sin\,\gamma}\right\}$.

又 $a-b=h\left\{\frac{cos\,(\beta-\alpha)}{sin\,\beta}-\frac{cos(\gamma+\alpha)}{sin\gamma}\right\}$

32. 河岸上有圓柱高200尺,上立30尺之銅像,下立6尺之人,自對岸望之,人與銅像之對角相等,求河寬.

33. 立於地上之桿,傾向南方,於正北及正南,取等距離之點,測得其仰角爲α及β,求桿之傾度.

34. 以高x尺之桿,立於y尺高之塔頂上,於塔底與同平面上一點,測得桿對向之極大角爲α,則

$$x : y=2\sin\alpha : 1-\sin\alpha.$$

35. 以高a尺之旗,立於b尺高之塔頂上,眼高h尺之人,測得塔與旗之對向角相等,則塔底至眼簾之距離,

爲$b\left(\dfrac{a+b-2h}{a-b}\right)^{\frac{1}{2}}$尺.

36. 以高h尺之銅像,立於圓柱石上,於地平上之一點,測得此像之對角爲α自此一直進行至k尺,再測得其對角爲β,求自第二測點至石之距離若干.

例題解自 32. 至 36.

(32) 令AB爲圓柱,AC爲銅像,BD爲圓柱下所立之人,E爲測者之地點,則河寬=BE=x, 角AEC=角BED=θ, 又角AEB=ϕ,

故 $\tan\theta=\dfrac{6}{x}$, $\tan\phi=\dfrac{200}{x}$, $\tan(\theta+\phi)=\dfrac{230}{x}$,

故 $\dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}=\dfrac{230}{x}$ 即 $\dfrac{\dfrac{6}{x}+\dfrac{200}{x}}{1-\dfrac{6}{x}\times\dfrac{200}{x}}=\dfrac{230}{x}$,

$206x^2=230(x^2-1200)$, 故 $x=10\sqrt{115}$ 尺.

(33) CD為桿長, A為北點, B為南點, $AD=BD=a$, 角$CDB=\theta$, 角$CAD=\alpha$

故 角$ACD=\theta-\alpha$, 又角$CBD=\beta$, 故角$BCD=180^\circ-(\theta+\beta)$,

由是 $\frac{CD}{a}=\frac{\sin\alpha}{\sin(\theta-\alpha)}=\frac{\sin\beta}{\sin(\theta+\beta)}$, 即 $\tan\theta=\frac{2\sin\beta\sin\alpha}{\sin(\beta-\alpha)}$.

(34) 令$AB=y=$塔高, $AC=x=$桿長, D為地平上之一點, 依題意, D為通過A,C圓周之切點, 故依幾何學圓之定理,

角$ACD=$角$ADB=\theta$,

又角$ADC=\alpha$, 故 $AD=\frac{AB}{\sin ADB}=\frac{y}{\sin\theta}$, 又 $AD=\frac{AC\sin ACD}{\sin ADC}=\frac{x\sin\theta}{\sin\alpha}$,

故 $y\sin\alpha=x\sin^2\theta=\frac{1}{2}x(1-\cos 2\theta)$, 又角BDC+角$BCD=90^\circ$, 即

$\theta+\alpha+\theta=90^\circ$, 即 $2\theta=90^\circ-\alpha$, 故 $y\sin\alpha=\frac{1}{2}x\{1-\cos(90^\circ-\alpha)\}$.

(35) AB為塔高, AC為旗, D為眼高, DE為地平線之垂線, 則

角BDA=角CDA, 故由幾何學比例之定理, $\frac{AB}{BD}=\frac{AC}{CD}$.

又 $BD^2=BE^2+DE^2=BE^2+h^2$, $CD^2=BE^2+(BC-DE)^2=BE^2+(a+b-h)^2$,

故 $\frac{b}{\sqrt{(BE^2+b^2)}}=\frac{a}{\sqrt{BE^2+(a+b-h)^2}}$, 即 $\frac{b^2}{BE^2+h^2}=\frac{a^2}{BE^2+(a+b-h)^2}$,

即 $\frac{a^2-b^2}{(a+b)^2-2(a+b)h}=\frac{b^2}{BE^2+h^2}$, 即 $\frac{a-b}{a+b-2h}=\frac{b^2}{BE^2+h^2}$,

故 $BE^2=\frac{b^2(a+b-2h)}{a-b}-h^2$, 由是

$BD^2=BE^2+h^2=\frac{b^2(a+b-2h)}{a-b}$, 故 $BD=b\left(\frac{a+b-2h}{a-b}\right)^{\frac{1}{2}}$.

(36) 令AB為圓柱, AC為銅像, D, E為測者之地點, 則

$\tan AEB=\frac{AB}{BE}$, 即 $\tan\phi=\frac{y}{x}$, $\tan BEC=\frac{BC}{BE}$, $\tan(\beta+\phi)=\frac{y+h}{x}$,

故 $\frac{\tan\beta+\frac{y}{x}}{1-\tan\beta\left(\frac{y}{x}\right)}=\frac{y+h}{x}$, 從此 $x^2+y^2+yh=hx\cot\beta$, (1)

又 $\tan ADB=\frac{AB}{BD}$, 即 $\tan\theta=\frac{y}{x+k}$, $\tan BDC=\frac{BC}{BD}$, 即 $\tan(\alpha+\theta)=\frac{y+h}{x+k}$,

與前同樣, 得 $x^2+y^2+yh+2xk+k^2=h(x+k)\cot\alpha$, (2)

由(1)及(2)得 $x=-\frac{-k^2+kh\cot\beta}{2k+h\cos\beta-h\cos\alpha}$.

37. 有物體直立於ABCD水平線上之D點，於物體之頂E，對向於AB及BC之角，均爲α度，而AB$=a$，BC$=b$，則

$$DE=\frac{2ab(a+b)\tan\alpha}{(a-b)^2+(a+b)^2\tan^2\alpha}.$$

38. 有人立於等高二煙筒間之直線上，測得其接近一煙筒之仰角爲60°，與此直線成直角，行80尺，測得二煙筒之仰角爲45°及30°，求煙筒之高及其距離.

39. 於此岸A，B二點，各與彼岸P，Q兩點(此四點同在平面上)聯結直線，均得α度，又QAB，PBA爲β，γ度，QBA比PBA更大，則

$$\frac{PQ}{AB}=\frac{\sin\alpha}{\sin(\alpha+\beta+\gamma)}=\frac{\sin(\alpha+\beta)\sin(\alpha+\gamma)-\sin\beta\sin\gamma}{\sin^2(\alpha+\beta+\gamma)}.$$

40. 有二桿AB，CD，其高爲a，c，立於河岸，在A正反對之岸P點，測得AB$=$AC，及B，D之仰角相等，求河寬.又

$$\cos CPD=\cos CPB=\frac{a^2}{c^2}.$$

例題解自37.至40.

(37) ∵ 角AEB$=$角BEC$=\alpha$，故 $\frac{AB}{AE}=\frac{BC}{CE}$，即 $\frac{a^2}{AD^2+DE^2}=\frac{b^2}{CD^2+DE^2}$

CD$=y$，DE$=x$ 則 $\frac{a^2}{(a+b+y)^2+x^2}=\frac{b^2}{y^2+x^2}=\frac{a^2-b^2}{(a+b)^2+2y(a+b)}=\frac{a-b}{a+b+2y}$，

故 $b^2(a+b+2y)=(a-b)(x^2+y^2)$， (1) 又 $\frac{CD}{DE}=\tan CED$，

即 $\frac{y}{x}=\tan\theta$， 及 $\frac{AD}{DE}=\tan AED$， 即 $\frac{a+b+y}{x}=\tan(2\alpha+\theta)$，

故 $\frac{a+b+y}{x}=\frac{\tan 2\alpha+\tan\theta}{1-\tan 2\alpha\tan\theta}=\frac{\tan 2\alpha+y/x}{1-\tan 2\alpha(y/x)}=\frac{x\tan 2\alpha+y}{x-y\tan 2\alpha}.$

故 $(a+b)(x-y\tan 2\alpha)=(x^2+y^2)\tan 2\alpha,$ (2)

以 (1) 約 (2), 則 $\dfrac{(a+b)(x-y\tan 2\alpha)}{b^2(a+b+2y)}=\dfrac{\tan 2\alpha}{a-b},$

故 $y=\dfrac{(a+b)\{x(a-b)-b^2\tan 2\alpha\}}{(a^2+b^2)\tan 2\alpha}$, 以此代入 (1) 而簡單之,再求兩邊之平方根,

則 $ab(a+b)\tan 2\alpha+2abx=x(a^2+b^2)\sec 2\alpha,$

故 $x=\dfrac{ab(a+b)\sin 2\alpha}{(a^2+b^2)-2ab\cos 2\alpha}=\dfrac{2ab(a+b)\sin\alpha\cos\alpha}{a^2+b^2-2ab(2\cos^2\alpha-1)}=\dfrac{2ab(a+b)\tan\alpha}{(a+b)^2\sec^2\alpha-4ab}$

$=\dfrac{2ab(a+b)\tan\alpha}{(a+b)^2(1+\tan^2\alpha)-4ab}=\dfrac{2ab(a+b)\tan\alpha}{(a-b)^2+(a+b)^2\tan^2\alpha}.$

(38) AB, CD 為二煙筒, 從直線 BD 上之一點, 引 EF 垂線, 令

$EF=80$, $AB=CD=h$, $BD=x$, 則 $BE=AB\cot AEB=h\cot 60^\circ=\dfrac{h}{\sqrt{3}}$,

$BF=AB\cot AFB=h\cot 45^\circ=h$, $BF^2=BE^2+EF^2$ 即

$h^2=\dfrac{h^2}{3}+80^2$, 故 $h=40\sqrt{6}$, 又 $DF=CD\cot CFD=h\cot 30^\circ=h\sqrt{3}$, $DF^2=DE^2+EF^2$,

即 $3h^2=\left(x-\dfrac{h}{\sqrt{3}}\right)^2+80^2$, 故 $x=40(\sqrt{14}+\sqrt{2})$.

(39) 角 PAQ=角 PBQ=α, 故 A,B,P,Q 在一圓周上, 故

$\dfrac{PQ}{PB}=\dfrac{\sin PBQ}{\sin PQB}=\dfrac{\sin\alpha}{\sin(180^\circ-PAB)}=\dfrac{\sin\alpha}{\sin PAB}$, $\dfrac{PB}{AB}=\dfrac{\sin PAB}{\sin APB}=\dfrac{\sin PAB}{\sin(\alpha+\beta+\gamma)}$,

故 $\dfrac{PQ}{AB}=\dfrac{\sin\alpha}{\sin(\alpha+\beta+\gamma)}=\dfrac{\sin\alpha\sin(\alpha+\beta+\gamma)}{\sin^2(\alpha+\beta+\gamma)}=\dfrac{\sin\{(\alpha+\gamma)-\gamma\}\sin\{(\alpha+\beta)+\gamma\}}{\sin^2(\alpha+\beta+\gamma)}.$

(40) $AB=AC=a$, $AP=x$, 又角 APB=角 CPD, 故

$\dfrac{AP}{AB}=\dfrac{CP}{CD}$, 即 $\dfrac{x}{a}=\dfrac{CP}{c}$, 故 $CP=\dfrac{cx}{a}$, 故 $CP^2=AP^2+AC^2$, 即

$\dfrac{c^2x^2}{a^2}=x^2+a^2$, 故 $x=\dfrac{a^2}{\sqrt{(c^2-a^2)}}$, PA 直立於縱平面 BACD 上, 故

角 DAP=90°, 故 $\cos APD=\dfrac{AP}{PD}=\dfrac{x}{\sqrt{\left(c^2+\dfrac{c^2x^2}{a^2}\right)}}=\dfrac{ax}{c\sqrt{(a^2+x^2)}}=\dfrac{ax}{c\left(\dfrac{cx}{a}\right)}=\dfrac{a^2}{c^2}$,

又 $2CP.PB\cos CPB=CP^2+PB^2-BC^2$, 即 $2\left(\dfrac{cx}{a}\right)\sqrt{(a^2+x^2)}\cos CPB=\dfrac{c^2x^2}{a^2}+a^2+x^2-2a^2$,

從此 $\cos CPB=a^2/c^2$.

41 有二燈桿，立於河之南北二岸，從其一桿向正西行a町測之，又同方向行b町測之，則初次所測之角，爲二次所測之角之三倍，求河寬若干.

42. A,B兩停車場，A在B之正西，今汽車自B出發，向北西走6里，適見A在其南西方，求AB之距離.

43. 有人行於直街道，此街道距a町之處有一塔，測得仰角爲α度，又在此街道，測得家屋與塔頂之交角爲β度，求塔高若干.

44. 有矩形之室爲ABCD，於C點測得A點屋高之仰角爲18°，測得B點屋高之仰角爲30°，AB爲48尺，求此屋之高.

45. 有壁高20尺，位於自南至59°5′之東方，太陽在其正南，仰角爲30°，求壁影之長.

46. 地平上有A,B,C三點，同在一直線上，當其對於圓柱塔之幅之角頂，$AB=m$, $BC=n$, 及A,B,C所對之角爲$2\alpha, 2\beta, 2\gamma$，則塔之直徑爲$2\sqrt{\left\{\dfrac{mn(m+n)}{m\csc^2\gamma+n\csc^2\alpha-(m+n)\csc^2\beta}\right\}}$.

例題解自41.至46.

(41) 令兩桿爲A,B，A之正西之二點爲C,D. $AC=a$, $CD=b$, 則角$ACB=3\theta$, 角$CDB=\theta$, 又$AB=x$, 由是

$\tan 3\theta=\dfrac{x}{a}$, $\tan\theta=\dfrac{x}{a+b}$, 故$=\dfrac{3\tan\theta-\tan^3\theta}{1-3\tan^2\theta}=\dfrac{x}{a}$.

卽 $\dfrac{\dfrac{3x}{a+b}-\left(\dfrac{x}{a+b}\right)^3}{1-3\left(\dfrac{x}{a+b}\right)^2}=\dfrac{x}{a}$, 卽 $\dfrac{3x(a+b)^2-x^3}{(a+b)\{(a+b)^2-3x^2\}}=\dfrac{x}{a}$,

由是 $x=(a+b)\sqrt{\{(b-2a)/(2a+3b)\}}$.

(42) 令 c 爲汽車, A 爲測點, 則 AC=6, 依題意, 角 ABC=45°, 角 BAC=45°, 故角 $c=90°$, 故 $AB=\sqrt{(BC^2+AC^2)}=\sqrt{(6^2+6^2)}=6\sqrt{2}$ 里.

(43) 自塔 CD 之底 D, 引 DA 直線, 垂於街道, 則 $DA=a$, 令測者之點爲 B, 則角 $DBC=\alpha$, 角 $ABC=\beta$, 由立體幾何學之定理, CD 爲街道平面上之垂綫, 而 DA 爲 AB 垂綫, 故角 CAB=90°, 故 $AB=AC\cot ABC=\sqrt{(a^2+x^2)}\cot\beta$,

又 $BD=CD\cot DBC=x\cot\alpha$, 但 $AD^2+AB^2=BD^2$,

故 $a^2+(a^2+x^2)\cot^2\beta=x^2\cot^2\alpha$, 故 $x^2=a^2(1+\cot^2\beta)/(\cot^2\alpha-\cot^2\beta)$,

$$x=a\sqrt{\frac{\sin^2\alpha}{\sin(\alpha+\beta)\sin(\beta-\alpha)}}=\frac{a\sin\alpha}{\sqrt{\{\sin(\alpha+\beta)\sin(\beta-\alpha)\}}}.$$

(44) 令 AB=48, BC=a, 高=x, 則

$\dfrac{x}{AC}=\dfrac{x}{\sqrt{(48^2+a^2)}}=\tan 18°=\sqrt{\left(1-\dfrac{2}{5}\sqrt{5}\right)}$, $\therefore 5x^2=(48^2+a^2)(5-2\sqrt{5})$,

又 $\dfrac{x}{BC}=\dfrac{x}{a}=\tan 30°=\dfrac{1}{\sqrt{3}}$, $\therefore a=x\sqrt{3}$, $\therefore 5x^2=(48^2+3x^2)(5-2\sqrt{5})$,

$$x=48\sqrt{\frac{5-2\sqrt{5}}{2(3\sqrt{5}-5)}}=48\sqrt{\frac{\sqrt{5}-1}{5}}=12\sqrt{(2\sqrt{5}-2)}=18{\cdot}84 \text{ 尺}.$$

(45) AB 爲壁高, BC 爲其影, CD 爲壁底之垂綫, 則 CD 可表壁全體之影之長, 而 AB=20, CD=x, 角 ACB=30°, 角 CBD=59°5′,

$\therefore \dfrac{AB}{BC}=\dfrac{20}{BC}=\tan 30°=\dfrac{1}{\sqrt{3}}$, $\therefore BC=20\sqrt{3}$, $\dfrac{CD}{BC}=\dfrac{x}{20\sqrt{3}}=\sin 59°5'$, $\therefore x=29{\cdot}719$ 尺.

(46) 令 O 爲圓柱之中心, R 爲半徑, 則

$AO=\dfrac{R}{\sin\alpha}=R\csc\alpha$, $DO=R\csc\beta$, $CO=R\csc\gamma$,

令角 $ABO=\theta$, 則 $2AB{\cdot}BO\cos\theta=AB^2+BO^2-AO^2$, 卽

$2mR\csc\beta\cos\theta=m^2+R^2\csc^2\beta-R^2\csc^2\alpha$, 同樣,

$-2nR\csc\beta\cos\theta=n^2+R^2\csc^2\beta-R^2\csc^2\gamma$,

從此兩方程式, 可消去 θ.

47. 風船之半徑爲R尺,其對於地平上一點之角度爲α,中心之仰角爲β,求中心之高.

48. 風船之仰角,於地平一直線上之三點A,B,C測之,得$cot^{-1}\alpha$, $cot^{-1}\beta$, $cot^{-1}\alpha$, 而AB=a, BC=b, 則風船之高爲 $\sqrt{\frac{ab}{\alpha^2-\beta^2}}$.

49. 風船以定方向及定速上昇,今於一定時刻,在同點測其仰角二次,試將各次測得之角度,以求計算風船昇高之式.

50. 一岩在船之北北西,其船向東北東行10浬,覩前岩在船之正西,求覩處至岩之距離.

51. 有船向南西行時,望見二物體,在北北西及西北西,行5浬後,再望前二物體,在北微西及北西,求此二物體相距若干浬.

52. 船向正北行,望正西二燈臺,恰成一直線,行一時間後,見前二燈臺在南西及南南西,此二燈臺相距8浬,求船一時間之速度.

例題解自47.至52.

(47) 令風船之中心爲O,地平上之一點爲A,則

$$AO=R\,csc\tfrac{1}{2}\alpha,$$

又 中心之高爲x, 則.

$$AO=x,\,csc\,\beta.$$

$$\therefore\ x=AO\,sin\,\beta=R\,csc\tfrac{1}{2}\alpha\,sin\,\beta.$$

(48) 令 $\alpha=\cot\theta$, $\beta=\cot\phi$, 風船之高爲 x, 風船爲O, 則

$OA=x\csc\theta=x\sqrt{(1+\alpha^2)}=OC$, $OB=x\sqrt{(1+\beta^2)}$,

故由例題46. $x^2=\dfrac{ab(a+b)}{a\csc^2\theta+b\csc^2\theta-(a+b)\csc^2\phi}$

$$=\frac{ab(a+b)}{(a+b)(1+\alpha^2)-(a+b)(1+\beta^2)}=\frac{ab}{\alpha^2-\beta^2}.$$

(49) 風船出立地上之一點爲O, 其方向爲OAB, 二回所測之高爲AE, BF, 卽 $AE=x$, $BF=y$, 而角 $AOE=\alpha$, 測者之地點爲C, 角 $ACE=\beta$, 角 $BCF=\gamma$, 又 $AB=a$, 則

$\dfrac{AE}{OA}=\dfrac{BF}{OB}=\dfrac{y-x}{a}=\sin\alpha$, $\therefore y=a\sin\alpha+x$, 又 $\dfrac{AE}{OE}=\dfrac{BF}{OF}=\dfrac{y-x}{EF}=\tan\alpha$,

$\therefore$ $EF=(y-x)\cot\alpha$, 又 $EC=AE\cot\beta=x\cot\beta$,

$FC=BF\cot\gamma=y\cot\gamma$, $CF^2=EF^2+CE^2$, 卽

$y^2\cot^2\gamma=(y-x)^2\cot^2\alpha+x^2\cot^2\beta$, 卽

$x^2(\cot^2\beta-\cot^2\gamma)-2ax\sin\alpha\cot^2\gamma=a^2\sin^2\alpha(\cot^2\gamma-\cot^2\alpha)$. 從此可求得 x.

(50) 令船開初之位置爲A, 其次之位置爲B, 岩爲C, 則角 $BAC=22°30'+67°30'=90°$, 又角 $ABC=90°-67°30'=22°30'$,

故 $BC=\dfrac{AB}{\cos ABC}=\dfrac{10}{\cos 22°30'}$ 卽所求距離之答爲4·142, 10·824 浬.

(51) 船之兩位置爲A,B, 二物體爲C, D, 則 $AB=5$,

角 $BAD=90°-67°30'+45°=67°30'$, 角 $ABD=45°+45°=90°$, 故

$BD=AB\tan BAD=5\tan 67°30'$, 又角 $ABC=45°+11°15'=56°15'$,

角 $BAC=90°-22°30'+45°=112°30'$, 角 $ACB=180°-(56°15'+112°30')$

$\sin 11°15' : \sin 112°30'=5 : BC$, $\therefore BC=5\sin 112°30'/\sin 11°15'$, 於三角形BCD, 已知其BC, BD, 及角 $CBD=45°-11°15'$, 故 $CD=15\cdot2025$ 浬.

(52) 令船之位置爲A,B, 二燈臺爲C,D, 則 $CD=8$, 角 $ABC=45°$,

角 $ABD=22°30'$, 角 $BDC=90°+22°30'=112°30'$ 角 $CBD=45°-22°30'=22°30'$,

$\therefore$ $BC=\dfrac{8\sin 112°30'}{\sin 22°30'}$, 又 $AB=AD$, $\therefore$ $AB=\sqrt{\dfrac{BD^2}{2}}=\frac{1}{2}BD\sqrt{2}$,

由是得 $AB=8+4\sqrt{2}$ 浬.

53. 有人立船上,望砲臺在船之東北東,船走正東4浬再望之,在北北東;求船之兩處至砲臺之距離.

54. 有二船平行而走,其初測得此船在他船北偏東爲α度,走一時間,望見北偏東爲β度,又走一時間,望見北偏東爲γ度,求此船所走之方向.

55. 於此屋窗下向彼屋測之,則對於彼屋頂及底之二視線,互相垂直,而頂上之仰角爲60°,中間道闊爲30尺,求彼屋之高.

56. 有a尺周圍之輪,走於圓弧道上,輪展n次,圓弧之弦爲c尺,則圓弧之半徑,殆等於$\dfrac{(na\sqrt{na})}{\{2\sqrt{6(na-c)}\}}$.

57. 有正方體立於地平面上,其三隅之仰角爲α,β,γ,則 $\sin^2 2\beta=(1-\sin^2\beta\cot^2\alpha)^2+(1-\sin^2\beta\cot^2\gamma)^2$.

58. 有正方柱之塔,於一直線上取三點測之,得仰角40°, 60°, 及45°,求此塔闊與高之比.

例題解自53至58.

(53) 令船開初之位置爲A,其次之位置爲B,砲台爲C,則
$AB=4$, 角$BAC=90°-67°30'=22°30'$ 角$ABC=90°+22°30'=112°30'$,
角$ACB=180°-(22°30'+112°30')=45°$, 由是
$\sin 45° : 4=\sin 112°30' : AC=\sin 22°30' : BC$ 故
$AC=\dfrac{4\cos 22°30'}{\sin 45°}=\dfrac{2}{\sin 22°30'}=\dfrac{2}{\frac{1}{2}\sqrt{(2-\sqrt{2})}}=2\sqrt{(2+\sqrt{2})}$ 浬, $BC=2\sqrt{(2-\sqrt{2})}$ 浬.

(54) 令船之初中後三位置爲A, B, C,他船之三位置爲P, Q, R,船走之方向自北而東θ度,而ABC與PQR平行, $AB=BC$, $PQ=QR$, 從B引BM與AP平行,截PQR於M點,則

$\frac{QM}{BM}=\frac{\sin QBM}{\sin BQM}=\frac{\sin(\beta-\alpha)}{\sin(\theta-\beta)}$, 又從C引CN與AP平行,截PQR於N點,

則 $\frac{RN}{CN}=\frac{\sin RCN}{\sin CRN}=\frac{\sin(\gamma-\alpha)}{\sin(\theta-\gamma)}$,

但 BM=CN, 又 QM 爲二船一時間之差, RN 爲二時間之差, 故

$RN=2QM$. $\therefore \frac{2\sin(\beta-\alpha)}{\sin(\theta-\beta)}=\frac{\sin(\gamma-\alpha)}{\sin(\theta-\gamma)}$,

$\therefore \tan\theta=\frac{2\sin(\beta-\alpha)\sin\gamma-\sin(\gamma-\alpha)\sin\beta}{2\sin(\beta-\alpha)\cos\gamma-\sin(\gamma-\alpha)\cos\beta}$.

(55) 令在窗下之測點爲A,彼屋之頂點爲B,底爲C,AD爲BC之垂線,則角 $BAD=60°$, 故角 $CAD=90°-60°=30°$, 角 $ABD=90°-60°=30°$,

角 $ACD=90°-30°=60°$, $\therefore BD=\sqrt{(AB^2-AD^2)}=\sqrt{(60^2-30^2)}=30\sqrt{3}$.

$AC^2-CD^2=30^2$, 即 $4CD^2-CD^2=30^2$, $\therefore CD=10\sqrt{3}$ $\therefore BC=40\sqrt{3}$ 尺.

(56) 令AB爲弦,O爲弧之中心,弧 $AB=na$, 角 $AOB=2\theta$,

由是 $\sin\theta=\frac{\frac{1}{2}c}{R}$, 即 $\theta-\frac{1}{6}\theta^3=\frac{c}{2R}$, 但 $\theta=\frac{na}{2R}$, 由是可得R.

(57) 令正方體之上隅爲A, B, C, 下隅爲D, E, F, 測者之點爲O, 從O引OM爲EF引長線之垂線, 又令正方體之一稜爲 a,

則 $OD^2=(a-OM)^2+EM^2=a^2-2a\cdot OM+OE^2$, $\therefore OM=\frac{a^2+OE^2-OD^2}{2a}$

$OF^2=(EM+a)^2+OM^2=a^2-2a\cdot EM+OE^2$ $\therefore EM=\frac{a^2+OE^2-OF^2}{2a}$,

$OE^2=EM^2+OM^2=\frac{(a^2+OE^2-OD^2)^2}{4a^2}+\frac{(a^2+OE^2-OF^2)^2}{4a^2}$

$OD=a\cot\alpha$, $OE=a\cot\beta$, $OF=a\cot\gamma$.

(58) 假定如前例,令 $AB=BC=a$, $AD=BE=h$, 則

$OD^2=a^2-2a\cdot OM+OE^2$, $OF^2=a^2-2a\cdot EM+OE^2$,

$OD=h\cot 45°=h$, $OE=h\cot 60°=\frac{h}{\sqrt{3}}$, $OF=h\cot 45°=h$

$\therefore OM=\frac{2h^2-3a^2}{6a}=EM$, $OM^2+EM^2=OE^2$, 從此 $\frac{h}{a}=\frac{1}{4}(\sqrt{6}+\sqrt{30})$.

59. 在山上測正方砲臺之幅,自其一隅之正南覘之,得臺面所對之角爲α度,向正西行a里又覘之,其對向之角亦與前同,又行b里,至他一隅之正南,則臺之幅爲

$(a+b)\sec\phi$, 但$\tan\phi=\dfrac{b\tan\alpha}{(a+b)}$.

60. 有船行到A處,初望見山頂S,由此向山之方向進行到B處,測S之仰角爲α度,而地球之中心爲O,以OS截地球表面之點C, 又弧AC$=a$, 弧BC$=b$, 則地球之半徑及山之高度,殆等於$\dfrac{a^2-b^2}{2b}\cot\alpha$ 及 $\dfrac{a^2b\tan\alpha}{a^2-b^2}$.

61. 有二岩A,B之距離d,測得其頂點之俯角α,β,而A頂點之水平線,通過B之頂點,則地球半徑爲$d/\cos^{-1}(\sec\alpha\cos\beta)$.

62. 初在船檣上(高出海面64呎)望見燈臺在地平上,後向燈臺進行30分,在船檣望之,即見燈臺在甲板上,(高出海面16呎)而地球半徑爲4000哩,求此船一時間之速度.

例題解自59.至62.

(59) AB爲正方形之一邊,C爲A之正南,測者之第一點,D爲其第二點,E爲B之正南,爲第三點, 則CD$=a$, DE$=b$.

角ACB=角ADB$=\alpha$, 又 AB, CE 之引長線之交角爲ϕ,

則 CE=AB $\cos\phi$, 即 AB=CE $\sec\phi=(a+b)\sec\phi$,

又角ACE=角BEC$=90°$, 又ABDC爲圓之內接四角形,

故角BDE=角BAC$=90°-\phi$ 角BCE$=90°-\alpha$, BE=DE$\tan$BDE$=b\tan(90°-\phi)$

$=b\cot\phi$ 及 BE=CE$\tan$BCE$=(a+b)\tan(90°-\alpha)=(a+b)\cot\alpha$,

故 $b\cot\phi=(a+b)\cot a$, 故 $\tan\phi=b\tan a/(a+b)$.

(60) 令 AS 爲 A 點之切線, BE 爲 B 點之切線, 則

角 $SBE=a$, 於直角三角形 SOA, $OA=OS\cos AOC$, 卽

$R=(R+x)\cos\frac{a}{R}=(R+x)\left(1-\frac{a^2}{2R^2}\right)$, (第十編 12 節) 故

$R+x=\frac{2R^3}{2R^2-a^2}$, 又角 $OBS=\frac{\pi}{2}+a$, 角 $BSO=\pi-$(角 OBS+角 BOC)

$=\pi\ \left(\frac{\pi}{2}+a+\frac{b}{R}\right)=\frac{\pi}{2}-\left(a+\frac{b}{R}\right)$, $\frac{OS}{OB}=\frac{\sin OBS}{\sin BSO}=\sin\left(\frac{\pi}{2}+a\right)\Big/\sin\left(\frac{\pi}{2}-a-\frac{b}{R}\right)$.

卽 $\frac{R+x}{R}=\frac{\cos a}{\cos\left(a+\frac{b}{R}\right)}=\frac{\cos a}{\cos a\cos\frac{b}{R}-\sin a\sin\frac{b}{R}}$ 故

$\frac{2R^2}{2R^2-a^2}=\frac{\cos a}{\cos a\left(1-\frac{b^2}{2R^2}\right)-\sin a\left(\frac{b}{R}-\frac{b^3}{6R^3}\right)}$, 簡之, 得 $R=\frac{(a^2-b^2)\cot a}{b}+\frac{b^3}{5R}$,

山高 b 與地球之半徑 R, 其比極小, 故省略 $\frac{b^3}{6R}$ 而得

$R=\frac{(a^2-b^2)\cot a}{b}$ 又從此可得 $x=R\left(\frac{2R^2}{2R^2-a^2}-1\right)$.

(61) A, B 爲二岩之頂點, O 爲地球之中心, 自 A 及 B 引地球之

切線 AC 及 BD, 則角 $CAO=\frac{\pi}{2}-a$, 角 $DBO=\frac{\pi}{2}-\beta$,

故 $AO=\frac{CO}{\sin CAO}=\frac{R}{\cos a}$, $BO=\frac{R}{\cos\beta}$, 又依題意, 角 $BAO=\frac{\pi}{2}$,

故 $\frac{AO}{BO}=\cos AOB=\cos\frac{d}{R}$, 由此 $\frac{\cos\beta}{\cos a}=\cos\frac{d}{R}$, 卽 $\cos^{-1}(\sec a\cos\beta)=\frac{d}{R}$.

(62) A 爲測者起初之位置, B 爲其次之位置, 依題意, AB 切於球面而通過燈臺, O 爲地球之中心, C 爲切點, 則

$AC=\sqrt{\{(R+64)^2-R^2\}}=\sqrt{(2R\times64)}$, (略近值), 又 $BC=\sqrt{(2R\times16)}$,

三十分之速度爲 $\sqrt{(2R\times64)}-\sqrt{(2R\times16)}$, 故一時間之速度 $=8\sqrt{2R}$

$=8\sqrt{2\times4000\times528}$ 呎 $=\frac{40}{9}\sqrt{2}$ 浬, (略近值)

63. 在塔之正南及正東有二點,對向塔頂上之旗之角相等,又旗頂之仰角爲$tan^{-1}\alpha$, $tan^{-1}\beta$, 此二點之距離爲a尺,則旗高爲 $\frac{a(\alpha\beta-1)}{\sqrt{(\alpha^2+\beta^2)}}$尺.

64. 雲高於水平1哩,其一端之仰角爲α度,則至雲一端之距離爲$\frac{\{(2R+1)\csc\alpha\}}{\{R+\cot\alpha\sqrt{(2R+1)}\}}$哩,但R爲地球半徑.

65. 人在山上,視地平上三塔,其對角均相等,而其俯角爲α,α',α'',三塔之高爲c,c',c'',則

$$\frac{\sin(\alpha'-\alpha'')}{c\cos\alpha}+\frac{\sin(\alpha''-\alpha)}{c'\cos\alpha'}+\frac{\sin(\alpha-\alpha')}{c''\cos\alpha''}=0.$$

66. 有塔高h,傾向正北θ度,在距其正南a, b之二地點測之,得仰角α, β, 則

$$\tan\theta=\frac{b-a}{b\cot\alpha-a\cot\beta},\quad h=\frac{b-a}{\cot\beta-\cot\alpha}.$$

例題解自 63. 至 66.

(63) 令塔爲AB,旗爲AE,二點爲C,D, AB=h, AE=x, 角AEC=角ADE=θ, 則 $\frac{BE}{BC}=\frac{h+x}{BC}=\tan ECB=\alpha$, $BC=\frac{h+x}{\alpha}$, 同樣. $BD=\frac{h+x}{\beta}$,

$BC^2+BD^2=CD^2$, 卽 $(h+x)^2\left(\frac{1}{\alpha^2}+\frac{1}{\beta^2}\right)=a^2$, $\therefore x=\frac{a\alpha\beta}{\sqrt{(\alpha^2+\beta^2)}}-h$,

又 $\frac{AB}{BC}=\frac{h}{BC}=\tan ACB=\tan(BCE-\theta)=\frac{\tan BCE-\tan\theta}{1+\tan BCE\tan\theta}=\frac{\alpha\tan\theta}{1+\alpha\tan\theta}$,

卽 $\frac{h\alpha}{h+x}=\frac{\alpha-\tan\theta}{1+\alpha\tan\theta}$, 卽 $\frac{h\sqrt{(\alpha^2+\beta^2)}}{a\beta}=\frac{\alpha-\tan\theta}{1+\alpha\tan\theta}$, 故 $\tan\theta=\frac{a\alpha\beta-h\sqrt{(\alpha^2+\beta^2)}}{a\beta+h\alpha\sqrt{(\alpha^2+\beta^2)}}$.

同樣，$\frac{h}{BD}=\tan(BDE-\theta)$，從此 $\tan\theta=\frac{a\alpha\beta-h\sqrt{(\alpha^2+\beta^2)}}{a\alpha+h\beta\sqrt{(\alpha^2+\beta^2)}}$，故

$a\alpha+h\beta\sqrt{(\alpha^2+\beta^2)}=a\beta+h\alpha\sqrt{(\alpha^2+\beta^2)}$，$\therefore h=\frac{a}{\sqrt{(\alpha^2+\beta^2)}}$，從此卽得其證.

(64) A爲雲之頂點，B爲其一端，C爲測者之位置，O爲地球之中心，

則AC爲切線，而角 $ACB=\alpha$，$AO=R+1$，$BC=x$，

角 $AOC=$角 $BAC=\theta$，$\cos AOC=\frac{CO}{AO}$，卽 $\cos\theta=\frac{R}{R+1}$，

$AC=\sqrt{(AO^2-CO^2)}=\sqrt{(R+1)^2-R^2}=\sqrt{(2R+1)}$，$\cot\theta=\frac{CO}{AC}=\frac{R}{\sqrt{(2R+1)}}$.

又 $x=\frac{AC\sin BAC}{\sin ABC}=\frac{\sqrt{2R+1}\sin\theta}{\sin(\theta+\alpha)}$，

故 $x=\frac{\sqrt{2R+1}}{\cos\alpha+\cot\theta\sin\alpha}=\frac{\sqrt{2R+1}}{\cos\alpha+\frac{R}{\sqrt{(2R+1)}}\sin\alpha}=\frac{(2R+1)\csc\alpha}{R+\sqrt{(2R+1)}\cot\alpha}$.

(65) h爲山之高，x,x',x''爲三塔至水平之距離，θ爲三塔之對角，

則 $\frac{h}{x}=\tan(\alpha+\theta)$，$\frac{h-c}{x}=\tan\alpha$，$\therefore \frac{h}{h-c}=\frac{\tan(\alpha+\theta)}{\tan\alpha}$，從此

$\tan\theta=\frac{c\sin\alpha\cos\alpha}{h-c\cos^2\alpha}$，同樣，$\tan\theta=\frac{c'\sin\alpha'\cos\alpha'}{h-c'\cos^2\alpha'}$，故 $\frac{c\sin\alpha\cos\alpha}{h-c\cos^2\alpha}=\frac{c'\sin\alpha'\cos\alpha'}{h-c'\cos^2\alpha'}$，

故 $\frac{\sin(\alpha-\alpha')}{c''\cos\alpha''}=\frac{h}{cc'c''}\left(\frac{c\sin\alpha}{\cos\alpha'\cos\alpha''}-\frac{c'\sin\alpha'}{\cos\alpha\cos\alpha''}\right)$，

$\frac{\sin(\alpha'-\alpha'')}{c\cos\alpha}=\frac{h}{cc'c''}\left(\frac{c'\sin\alpha'}{\cos\alpha''\cos\alpha}-\frac{c''\sin\alpha''}{\cos\alpha'\cos\alpha}\right)$，

$\frac{\sin(\alpha''-\alpha)}{c'\cos\alpha'}=\frac{h}{cc'c''}\left(\frac{c''\sin\alpha''}{\cos\alpha\cos\alpha'}-\frac{c\sin\alpha}{\cos\alpha''\cos\alpha'}\right)$，以此相加，卽得其證.

(66) 自塔之頂點，引至地平垂線之足，與塔底之距離爲x，

則 $h+x=h\cot\alpha$，故 $h=\frac{b-a}{\cot\beta-\cot\alpha}$，

又 $\tan\theta=\frac{h}{x}=\frac{h}{h\cot\alpha-a}=\frac{\frac{a-\beta}{\cot\alpha-\cot\beta}}{\frac{(a-b)\cot\alpha}{\cot\alpha-\cot\beta}-a}=\frac{b-a}{b\cot\alpha-a\cot\beta}$.

67. 傾於平面之塔，其底爲A，自A至平地C，引AC直線，令對向塔之角爲α度，再自AC直線上向塔進行至D點，令CD=AD，測其仰度爲β度，又ϕ爲AC與塔之交角，則$\cot\phi=\cot\alpha-2\cot\beta$.

68. 自某點測得二山之仰角爲60°，及40°，自此點至第二山頂764尺，測得第一山頂之仰角爲40°，該山頂上之旗之仰角爲70°，求旗之高.

69. 有高h_1及h_2尺之兩壁，互爲直交，當太陽正南時，其影長爲a_1及a_2，若太陽之仰角爲θ，第一壁與子午線平面之傾角爲ϕ，則 $\cot\theta=\sqrt{\left(\frac{a_1}{h_2}\right)^2+\left(\frac{a_2}{h_2}\right)^2}$，$\cot\phi=\frac{h_1a_2}{h_2a_1}$.

70. 有人步行直道上，測得同地平上二物體之最大角爲α，再行c里，望二物體成一直線，再測之，二物體與直道成β度，則兩物體之距離爲 $\frac{2c\sin\alpha\sin\beta}{\cos\alpha+\cos\beta}$.

71. 有二人同在平面上一點P，測得同平面上二物體A，B之對角爲α，後二人分道行至Q，R各測之，得AB之對角亦爲α，而 $PQ\perp PA$，$PR\perp PB$，$PQ=a$，$PR=b$，則
AB之長爲$\sqrt{(a^2+2ab\cos\beta+b^2)}$，但$\beta=\alpha$或$\pi-\alpha$.

例題解自67.至71.

(67) 令B爲塔頂，則$\frac{\sin CBD}{\sin BCD}=\frac{CD}{BD}$，即$\frac{\sin(\beta-\alpha)}{\sin\alpha}=\frac{\frac{1}{2}AC}{BD}$，

又$\frac{\sin ABD}{\sin BAD}=\frac{AD}{BD}$，即$\frac{\sin(\beta+\phi)}{\sin\phi}=\frac{\frac{1}{2}AC}{BD}$.

由是 $\frac{\sin(\beta-\alpha)}{\sin\alpha}=\frac{\sin(\beta+\phi)}{\sin\phi}$, 故 $\sin\beta\cot\alpha-\cos\beta=\sin\beta\cot\phi+\cos\beta$.

(68) 二山頂爲A, B, 一點爲C. 旗爲AD,

角 $ACB=180°-(60°+40°)=80°$, 角 $ABC=40°+40°=80°$, 故

$AB=AC=\frac{\frac{1}{2}BC}{\cos ACB}=\frac{382}{\cos 80°}$, 角 $ABD=70°-40°=30°$, 角 $ADB=90°-70°=20°$,

$AD=\frac{AB\sin ABD}{\sin ADB}=\frac{382\sin 30°}{\sin 20°\cos 80°}=\frac{191}{\sin 20°\cos 80°}=3211\cdot 2$ 尺.

(69) 高 h_1 之影爲 $h_1\cot\theta$, 而從此影之一端, 引第一壁之垂線爲 a_1,

卽第一壁之影, 故 $\sin\phi=\frac{a_1}{h_1\cot\theta}$, 第二壁與第一壁成直角, 故

$\cos\phi=\frac{a_2}{h_2\cot\theta}$, 由此 $\cot\phi=\frac{a_2h_1}{a_1h_2}$,

又 $\sin^2\phi+\cos^2\phi=1=\left\{\left(\frac{a_1}{h_1}\right)^2+\left(\frac{a_2}{h_2}\right)^2\right\}\frac{1}{\cot^2\theta}$.

(70) 令二物體爲A, B, 最大角點爲C, 則A, B, 通過圓周, 可切於C點之切線爲直道, 又令他點爲D, $CD=c$, 由幾何學之定理, 角 $ACD=$ 角 $ABC=\theta$,

故 $2\theta+\alpha+\beta=180°$, $\theta=90°-\frac{1}{2}(\alpha+\beta)$,

又 角 $BCD=\theta+\alpha=90°+\frac{1}{2}(\alpha-\beta)$, $BD=\frac{CD\sin BCD}{\sin DBC}=\frac{c\sin(\alpha+\theta)}{\sin\theta}=\frac{c\cos\frac{1}{2}(\alpha-\beta)}{\cos\frac{1}{2}(\alpha+\beta)}$,

又 $AD\cdot BD=DC^2$, 故 $AD=\frac{c^2}{BD}=\frac{c\cos\frac{1}{2}(\alpha+\beta)}{\cos\frac{1}{2}(\alpha-\beta)}$,

故 $AB=BD-AD=\frac{c\{\cos^2\frac{1}{2}(\alpha-\beta)-\cos^2\frac{1}{2}(\alpha+\beta)\}}{\cos\frac{1}{2}(\alpha+\beta)\cos\frac{1}{2}(\alpha-\beta)}=\frac{2c\sin\alpha\sin\beta}{\cos\alpha+\cos\beta}$.

(71) A, B, P, Q, R 在一圓周上, 而角APQ. 角BPR均爲直角,

故 AQ, BR 爲其圓之直徑, $AB=x$, $PA=\sqrt{(4R^2-a^2)}$, $PB=\sqrt{(4R^2-b^2)}$,

$x=2R\sin\alpha$, $2PA.PB\cos\alpha=PA^2+PB^2-x^2$, 解之, 則

$x^2=a^2+b^2\pm 2ab\cos\alpha=a^2+b^2+2ab\cos\beta$.

72. 有人行於直道上，測得二山之仰角爲 α, α', 其時高山在低山背後，順道行 c 里，再視二山，高山全隱沒於低山，又行 1 里，低山之仰角爲 β, 而低山之高爲 h, 高山之高爲 h', 則

$$h=\frac{(c+1)\sin\alpha\sin\beta}{\sin(\beta-\alpha)},\quad \frac{h'}{h}=\frac{h'\cot\alpha'-c}{h\cot\alpha-c}.$$

73. 上山取最短距離，初道與水平成 α 角，後道增加 β 度，達到山頂，由晴雨計測知其高爲 n 尺，而最初出立點之俯角爲 γ 度，則所上之路程爲 $n\cos\left(\frac{\alpha+\beta}{2}-\gamma\right)\left(\cos\frac{\beta-\alpha}{2}\sin\gamma\right)$.

74. 於水平面上取二點，測塔之仰角爲 α, β, 又取第三點，與此二點在同一直線上，且與此二點之距離爲 a, b, 再測塔之仰角爲 γ, 則此塔之高爲

$$\frac{\sin\alpha\sin\beta\sin\gamma(a^2b+ab^2)^{\frac{1}{2}}}{\{a\sin^2\alpha(\sin^2\gamma-\sin^2\beta)+b\sin^2\beta(\sin^2\gamma-\sin^2\alpha)\}^{\frac{1}{2}}}.$$

75. 有二物體相距 C, 在川之彼岸，於川之此岸，取相距 c 之二點測之，測得其對角爲 α, β, 求川之闊.

例題解自 72. 至 75.

(72) 令測者開初之點爲 A, 次點爲 B, 終點爲 C, 低山之底爲 D, 高山之低爲 E, 則 $AD=h\cot\alpha$, $AE=h'\cot\alpha'$, $CD=h\cot\beta$,

又 $AD=AB+BC+CD$, 即 $h\cot\alpha=c+1+h\cot\beta$,

故 $h=\dfrac{c+1}{\cot\alpha-\cot\beta}=\dfrac{(c+1)\sin\alpha\sin\beta}{\sin(\beta-\alpha)}$, $\dfrac{h'}{h}=\dfrac{BE}{BD}=\dfrac{h'\cot\alpha'-c}{h\cot\alpha-c}$.

(73) 令最初之點爲A, 次點爲B, 山頂爲C, B之底爲D, C之底爲E, 則 $AE=n\cot\gamma$, $AD=BD\cot\alpha$, $DE=(n-BD)\cot\beta$, $AE=AD+DE$,

卽 $n\cot\gamma=BD\cot\alpha+(n-BD)\cot\beta$, 故 $BD=\frac{n(\cot\gamma-\cot\beta)}{\cot\alpha-\cot\beta}=\frac{n\sin(\beta-\gamma)\sin\alpha}{\sin(\beta-\alpha)\sin\gamma}$,

$$AB+BC=\frac{BD}{\sin\alpha}+\frac{n-BD}{\sin\beta}=\frac{n\sin\alpha+BD(\sin\beta-\sin\alpha)}{\sin\alpha\sin\beta}$$

$$=\frac{n\sin\alpha+\frac{n\sin(\beta-\gamma)\sin\alpha}{\sin(\beta-\alpha)\sin\gamma}(\sin\beta-\sin\alpha)}{\sin\alpha\sin\beta}$$

$$=\frac{n\{\sin(\beta-\alpha)\sin\gamma+\sin(\beta-\gamma)(\sin\beta-\sin\alpha)\}}{\sin(\beta-\alpha)\sin\beta\sin\gamma}$$

$$=\frac{n\{\sin\beta\sin(\gamma-\alpha)+\sin(\beta-\gamma)\sin\beta\}}{\sin(\beta-\alpha)\sin\beta\sin\gamma}=\frac{n\{\sin(\gamma-\alpha)+\sin(\beta-\gamma)\}}{\sin(\beta-\alpha)\sin\gamma}.$$

從此可得其證.

(74) 初二點爲A, B, 中間一點爲C, 塔底爲O, 高爲x, 則

$OA=x\cot\alpha$, $OB=x\cot\beta$, $OC=x\cot\gamma$,

$2AC.CO\cos ACO=AC^2+CO^2-AO^2$, 卽 $2ax\cot\gamma\cos ACO=a^2+x^2\cot^2\gamma-x^2\cot^2\alpha$,

$2BC.CO\cos BCO=BC^2+CO^2-BO^2$, 卽 $-2bx\cot\gamma\cos ACO=b^2+x^2\cot^2\gamma-x^2\cot^2\beta$,

消去 $\cos ACO$以求x^2, 得

$$x^2=\frac{-ab(a+b)}{a^2(\cot^2\gamma-\cot^2\beta)+b^2(\cot^2\gamma-\cot^2\alpha)}.$$ 從此可得其證.

(75) 令彼岸之二點爲A, B, 此岸之二點爲C, D, 則ABCD爲平行四邊形, 而令AD, BC之交點爲O, 則

$\frac{AC}{\sin AOC}=\frac{AO}{\sin ACO}=\frac{CO}{\sin CAO}$, 卽 $\frac{AC}{\sin(\alpha+\beta)}=\frac{AD}{2\sin\alpha}=\frac{BC}{2\sin\beta}$, 又

$AD^2+BC^2=2AC^2+2c^2$, 卽 $\frac{4AC^2\sin^2\alpha}{\sin^2(\alpha+\beta)}+\frac{4AC^2\sin^2\beta}{\sin^2(\alpha+\beta)}=2AC^2+2c^2$,

$AC^2=\frac{c^2\sin^2(\alpha+\beta)}{2\sin^2\beta+2\sin^2\alpha-\sin^2(\alpha+\beta)}$, 又令川闊爲$x$, 則

$$cx=CA.AD\sin\beta=\frac{2CA^2\sin\alpha\sin\beta}{\sin(\alpha+\beta)}=\frac{2c^2\sin(\alpha+\beta)\sin\alpha\sin\beta}{2\sin^2\beta+2\sin^2\alpha-\sin^2(\alpha+\beta)}.$$

76. 塔之四周,繞以圓渠,在正午時,塔影越圓渠45尺,在太陽正西之影,則越渠120尺,此二影兩端之距離為375尺,又於渠之邊測塔之仰角為60°, 求塔高若干.

〔解〕 令圓渠之半徑為γ, 塔高為x, 則 $(\gamma+120)^2+(\gamma+45)^2=(375)^2$,

$\therefore\ \gamma=180,\ x=\gamma\tan 60°=\gamma\sqrt{3}=180\sqrt{3}$ 尺.

77 有A, B, C三點,同在平面上, AC=CB, 且AC⊥CB, 於O點測得AC, BC之對角為α, β, 從O引OC之垂線OO′, 令OO′$=d$, 從O′再測得AC, CB之對角為α', β', 求AB之距若干.

〔解〕 $AC=BC=\sqrt{\frac{1}{2}\overline{AB}^2}=\sqrt{x/2}$. 角ACO$=\theta$, 則角BCO$=90°-\theta$,

故 $OC=\dfrac{AC\sin(\alpha+\theta)}{\sin\alpha}=\dfrac{x\sin(\alpha+\theta)}{\sqrt{2}\sin\alpha}$, $OC=\dfrac{BC\sin(90°-\theta+\beta)}{\sin\beta}=\dfrac{x\cos(\theta-\beta)}{\sqrt{2}\sin\beta}$,

即 $OC\sqrt{2}\sin\alpha=x(\sin\alpha\cos\theta+\cos\alpha\sin\theta)$,

$OC\sqrt{2}\sin\beta=x(\cos\beta\cos\theta+\sin\beta\sin\theta)$, 從此兩方程式, 得

$x\sin\theta=\dfrac{OC\sqrt{2}\sin\alpha(\cos\beta-\sin\beta)}{\cos(\alpha+\beta)}$, $x\cos\theta=\dfrac{OC\sqrt{2}\sin\beta(\cos\alpha-\sin\alpha)}{\cos(\alpha+\beta)}$,

由此 $x^2\cos^2(\alpha+\beta)=2OC^2\{\sin^2\alpha+\sin^2\beta-2\sin\alpha\sin\beta\sin(\alpha+\beta)\}$,

同樣, $x^2\cos^2(\alpha'+\beta')=2O'C^2\{\sin^2\alpha'+\sin^2\beta'-2\sin\alpha'\sin\beta'\sin(\alpha'+\beta')\}$,

但 $O'C^2-OC^2=d^2$, 從此可得x.

第拾伍編

三角形之性質

1. 三角形之性質 此編說明三角形之外切圓,內切圓,傍切圓,及其他幾何學之問題,於三角形之性質有關係者.

此編之例題,於平面幾何學講義,列舉甚多,茲則就其所餘三角函數之關係者示之.

2. 外切圓 有三角形ABC,令其外切圓之半徑爲R,則

$$2R=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C},\quad S=\frac{abc}{4R}.$$

〔證〕 從B引外切圓之直徑BD, 則角BDC=角A, 且角BCD=90°,

故 $BD\sin BDC=BC$, 卽 $2R\sin A=a$,

又由第十二編3節公式(6), 得 $S=\frac{1}{2}bc\sin A=\frac{abc}{\frac{2a}{\sin A}}=\frac{abc}{4R}$.

3. 內切圓 有三角形ABC,令其內切圓之半徑爲r, 則

$$r=(s-a)\tan\frac{A}{2}=(s-b)\tan\frac{B}{2}=(s-c)\tan\frac{C}{2},\quad S=rs.$$

〔證〕 內切圓之中心爲O, 切於AC之點爲E, 則於直角三角形AEO,

得 $OE=AE\tan OAE$, 卽 $r=(s-a)\tan\frac{1}{2}A$.

又 S=三角形OAB+三角形OBC+三角形OCA$=\frac{1}{2}rc+\frac{1}{2}ra+\frac{1}{2}rb=\frac{1}{2}(a+b+c)r=sr$.

4. 傍切圓 有三角形ABC,令其外切於BC, CA, AB傍切圓之半徑爲r_1, r_2, r_3, 則

$$r_1=s\tan\frac{A}{2},\quad r_2=s\tan\frac{B}{2},\quad r_3=s\tan\frac{C}{2},$$

$$S=(s-a)r_1=(s-b)r_2=(s-c)r_3.$$

〔證〕令外切BC之傍切圓之中心爲O_1, AC引長線上之切點爲E,

則於直角三角形AEO_1, 得$O_1E=AE\tan O_1AE$, 卽$r_1=s\tan\frac{1}{2}A$,

又 S=三角形O_1AB+三角形O_1CA−三角形$O_1CB=\frac{1}{2}r_1c+\frac{1}{2}r_1b-\frac{1}{2}r_1a$

$=\frac{1}{2}(b+c-a)r_1=(s-a)r_1.$

例題二十三

求次各式之證. 但 s_1, s_2, s_3 爲 $(s-a)$, $(s-b)$, $(s-c)$.

1. $r=\sqrt{\dfrac{s_1s_2s_3}{s}}.$

2. $r_1=\sqrt{\dfrac{ss_2s_3}{s_1}}.$

3. $r_1r_2+r_2r_3+r_3r_1=s^2.$

4. $S=\sqrt{rr_1r_2r_3}.$

5. $\dfrac{r}{4R}=\sin\dfrac{A}{2}\sin\dfrac{B}{2}\sin\dfrac{C}{2}.$

6. $\dfrac{r_1}{4R}=\sin\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2}.$

7. $r_1-r=4R\sin^2\dfrac{A}{2}.$

8. $r_2+r_3=4R\cos^2\dfrac{A}{2}.$

9. $r_1+r_2+r_3-r=4R.$

10. $\dfrac{1}{r_1}+\dfrac{1}{r_2}+\dfrac{1}{r_3}=\dfrac{1}{r}.$

11. $\dfrac{1}{r_1-r}+\dfrac{1}{r_2+r_3}=\dfrac{4R}{a^2}.$

12. $\dfrac{1}{Rr}=\dfrac{2}{ab}+\dfrac{2}{bc}+\dfrac{2}{ca}.$

13. $\dfrac{r}{R}=4\left(\dfrac{s}{a}-1\right)\left(\dfrac{s}{b}-1\right)\left(\dfrac{s}{c}-1\right).$

例題解自 1. 至 13.

(1) $r=\frac{S}{s}$(3. 節)$=\frac{1}{s}\sqrt{s s_1 s_2 s_3}$〔第十二編公式 (6)〕$=\sqrt{\frac{s_1 s_2 s_3}{s}}$.

(2) $r_1=\frac{S}{s_1}$(4. 節)$=\frac{1}{s_1}\sqrt{s_1 s s_2 s_3}=\sqrt{\frac{s s_2 s_3}{s_1}}$.

(3) $r_1 r_2+r_2 r_3+r_3 r_1=\frac{S^2}{s_1 s_2}+\frac{S^2}{s_2 s_3}+\frac{S^2}{s_3 s_1}=\frac{S^2(s_1+s_2+s_3)}{s_1 s_2 s_3}$

$$=s(s-a+s-b+s-c)=2s^2.$$

(4) $S=\sqrt{s s_1 s_2 s_3}=\sqrt{\left(\frac{s}{r}\times\frac{S}{r_1}\times\frac{S}{r_2}\times\frac{S}{r_3}\right)}=\frac{S^2}{\sqrt{r r_1 r_2 r_3}}$.

(5) $\frac{r}{4R}=\frac{S}{s}\div\frac{abc}{S}=\frac{S^2}{sabc}=\sqrt{\left(\frac{s_2 s_3}{bc}\cdot\frac{s_3 s_1}{ca}\cdot\frac{s_1 s_2}{ab}\right)}$

$=\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$〔第十二編公式 (3)〕.

(6) $\frac{r_1}{4R}=\frac{S}{s_1}\div\frac{abc}{S}=\frac{S^2}{s_1 abc}=\sqrt{\left(\frac{s_2 s_3}{bc}\cdot\frac{s_2 s}{ca}\cdot\frac{s s_3}{ab}\right)}=\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}$.

(7) $r_1-r=4R\sin\frac{A}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2}-\sin\frac{B}{2}\sin\frac{C}{2}\right)$　〔由前二例〕

$=4R\sin\frac{A}{2}\cos\left(\frac{B+C}{2}\right)=4R\sin\frac{A}{2}\cos\left(90^\circ-\frac{A}{2}\right)$.

(8) $r_2+r_3=4R\cos\frac{A}{2}\left(\sin\frac{B}{2}\cos\frac{C}{2}+\cos\frac{B}{2}\sin\frac{C}{2}\right)=4R\cos\frac{A}{2}\sin\frac{B+C}{2}$.

(9) $r_1+r_2+r_3-r=4R\left(\sin^2\frac{A}{2}+\cos^2\frac{A}{2}\right)=4R$.

(10) $\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}=\frac{s_1}{S}+\frac{s_2}{S}+\frac{s_3}{S}=\frac{s}{S}=\frac{1}{r}$.

(11) $\frac{1}{r_1-r}+\frac{1}{r_2+r_3}=\frac{1}{4R\sin^2\frac{1}{2}A}+\frac{1}{4R\cos^2\frac{1}{2}A}=\frac{\cos^2\frac{1}{2}A+\sin^2\frac{1}{2}A}{4R\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A}$

$$=\frac{1}{R\sin^2 A}=\frac{1}{R(a/2R)^2}=\frac{4R}{a^2}.$$

(12) $\frac{1}{Rr}=\frac{4S}{abc}\times\frac{s}{S}=\frac{2(a+b+c)}{abc}=\frac{2}{bc}+\frac{2}{ca}+\frac{2}{ab}$.

(13) $\frac{r}{R}=\frac{S}{s}\times\frac{4S}{abc}=\frac{4(s-a)(s-b)(s-c)}{abc}=4\left(\frac{s}{a}-1\right)\left(\frac{s}{b}-1\right)\left(\frac{s}{c}-1\right)$.

14. $\dfrac{ab-r_1r_2}{r_3}=\dfrac{bc-r_2r_3}{r_1}=\dfrac{ca-r_3r_1}{r_2}=r.$

15. $r(r_1+r_2+r_3)+r_1r_2+r_2r_3+r_3r_1=ab+bc+ca.$

16. $(4R+r)(4R+r+s\sqrt{3})(4R+r-s\sqrt{3})=r_1^3+r_2^3+r_3^3-3r_1r_2r_3.$

17. $S=\dfrac{r_1r_2r_3}{\sqrt{(r_1r_2+r_2r_3+r_3r_1)}}.$

18. $R=\dfrac{(r_1+r_2)(r_2+r_3)(r_3+r_1)}{4(r_1r_2+r_2r_3+r_3r_1)}.$

19. $a=\dfrac{r_1(r_2+r_3)}{\sqrt{(r_1r_2+r_2r_3+r_3r_1)}}.$

20. $\dfrac{1}{r^2}+\dfrac{1}{r_1^2}+\dfrac{1}{r_2^2}+\dfrac{1}{r_3^2}=\dfrac{a^2+b^2+c^2}{S^2}.$

21. $\dfrac{r_1}{bc}+\dfrac{r_2}{ca}+\dfrac{r_3}{ab}=\dfrac{1}{r}-\dfrac{1}{2R}.$

22. $3\sqrt{\dfrac{r_1r_2r_3}{r}}-\sqrt{\dfrac{rr_2r_3}{r_1}}-\sqrt{\dfrac{rr_3r_1}{r_2}}-\sqrt{\dfrac{rr_1r_2}{r_3}}=2s.$

例題解自 14. 至 22.

(14) $\dfrac{ab-r_1r_2}{r_3}=\left(ab-\dfrac{S}{s_1}\cdot\dfrac{S}{s_2}\right)\Big/\dfrac{S}{s_3}=(ab-ss_3)\dfrac{s_3}{S}$

$=\dfrac{\{ab-\frac{1}{2}(a+b+c)\frac{1}{2}(a+b-c)\}s_3}{S}=\dfrac{\{c^2-(a-b)^2\}s_3}{4S}=\dfrac{(c+a-b)(c-a+b)s_3}{4S}$

$=\dfrac{s_1s_2s_3}{S}=\dfrac{S^2}{sS}=\dfrac{S}{s}=r.$

(15) 從前例 $\dfrac{(ab-r_1r_2)+(bc-r_2r_3)+(ca-r_3r_1)}{r_1+r_2+r_3}=r$, 卽

$ab+bc+ca-(r_1r_2+r_2r_3+r_3r_1)=r(r_1+r_2+r_3).$

(16) 原式之左邊$=(4R+r)\{(4R+r)^2-3s^2\}$. 從例題 3. 及 9.

$=(r_1+r_2+r_3)\{(r_1+r_2+r_3)^2-3(r_1r_2+r_2r_3+r_3r_1)\}$

$=(r_1+r_2+r_3)(r_1^2+r_2^2+r_3^2-r_1r_2-r_2r_3-r_3r_1)=r_1^3+r_2^3+r_3^3-3r_1r_2r_3.$

(17)　$S=\sqrt{ss_1s_2s_3}=\sqrt{\frac{s^3s_1s_2s_3}{s^2}}=\sqrt{\left(\frac{ss_2s_3}{s_1}\times\frac{ss_1s_3}{s_2}\times\frac{ss_1s_2}{s_3}\right)}\Big/\sqrt{s^2}$

$=\sqrt{(r_1r_2r_3)}/\sqrt{(r_1r_2+r_2r_3+r_3r_1)}$, 由例題2. 3. 及4.

(18)　$r_2+r_3=\frac{S}{s_2}+\frac{S}{s_3}=\frac{aS}{s_2s_3}$, 故 $(r_1+r_2)(r_2+r_3)(r_3+r_1)=\frac{abcS^3}{(s_1s_2s_3)^2}$

$=\frac{4RS^4}{(s_1s_2s_3)^2}$, (2.節) $=\frac{4R(ss_1s_2s_3)^2}{(s_1s_2s_3)^2}=4Rs^2=4R(r_1r_2+r_2r_3+r_3r_1).$

(19)　從前之解式　$r_2+r_3=\frac{aS}{s_2s_3}=\frac{a\sqrt{(ss_1s_2s_3)}}{s_2s_3}=a\sqrt{\frac{ss_1}{s_2s_3}}$

$=a\sqrt{\frac{s^2s_1}{ss_2s_3}}=\frac{as}{r_1}$(例題2.) $=\frac{a}{r_1}\sqrt{(r_1r_2+r_2r_3+r_3r_1)}.$

(20)　$\frac{1}{r^2}+\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}=\frac{s^2+s_1^2+s_2^2+s_3^2}{S^2}$, 以下容易求得, 故畧.

(21)　$\frac{r_1}{bc}=\frac{s\tan\frac{1}{2}A}{bc}$, (3.節) $=\frac{sa\tan\frac{1}{2}A}{abc}=\frac{s(2R\sin A)\tan\frac{1}{2}A}{abc}$, (2.節)

$=\frac{4sR\sin^2\frac{1}{2}A}{abc}$, 故 $\frac{r_1}{bc}+\frac{r_2}{ca}+\frac{r_3}{ab}=\frac{4sR}{abc}\left(\sin^2\frac{A}{2}+\sin^2\frac{B}{2}+\sin^2\frac{C}{2}\right)$

$=\frac{4sR}{abc}\left(1-2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$, [例題五64.] [用第十二編公式(3)]

$=\frac{4sR}{abc}\left(1-2\sqrt{\frac{s_2s_3}{bc}}\sqrt{\frac{s_3s_1}{ca}}\sqrt{\frac{s_1s_2}{ab}}\right)=\frac{4sR}{abc}\left(1-\frac{2s_1s_2s_3}{abc}\right)$

$=\frac{4sR}{abc}\left(1-\frac{2S^2}{abcs}\right)=\frac{4sR}{abc}-\frac{8RS^2}{a^2b^2c^2}=\frac{4sR}{4SR}-\frac{8RS^2}{(4SR)^2}$, (2.節)

$=\frac{s}{S}-\frac{1}{2R}=\frac{s}{sr}-\frac{1}{2R}$, (3.節) $=\frac{1}{r}-\frac{1}{2R}.$

(22)　原式之左邊 $=\left(\frac{3}{r}-\frac{1}{r_1}-\frac{1}{r_2}-\frac{1}{r_3}\right)\sqrt{r\,r_1\,r_2\,r_3}$

$=\left(\frac{3}{r}-\frac{1}{r_1}-\frac{1}{r_2}-\frac{1}{r_3}\right)S=\frac{3S}{r}-\frac{S}{r_1}-\frac{S}{r_2}-\frac{S}{r_3}=3s-s_1-s_2-s_3$

$=3s-(s-a)-(s-b)-(s-c)=a+b+c=2s.$

23. $\frac{r}{R}=\frac{a\cos A+b\cos B+c\cos C}{a+b+c}$.

24. $\frac{s}{r}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$.

25. $\tan^2\frac{A}{2}=\frac{rr_1}{r_2r_3}$.

26. $r=s\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

27. $r^2=S\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

28. $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=4R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{s}{r}$.

29. $\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}=\frac{r(r_1^2+r_2^2+r_3^2)}{r_1r_2r_3}$.

30. $a\cos A+b\cos B+c\cos C=4R\sin A\sin B\sin C$.

31. $r_1r_2r_3=r_3\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}$.

32. $a\cot A+b\cot B+c\cot C=2(R+r)$.

例題解自 23. 至 32.

(23) $\frac{r}{R}=\frac{S}{s}\Big/\frac{abc}{4S}=\frac{4S^2}{abcs}=\frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{4abcs}$

$=\frac{1}{2s}\left(a.\frac{b^2+c^2-a^2}{2bc}+b.\frac{c^2+a^2-b^2}{2ca}+c.\frac{a^2+b^2-c^2}{2ab}\right)=\frac{a\cos A+b\cos B+c\cos C}{a+b+c}$.

(24) $\frac{s}{r}=\frac{s^2}{S}=\sqrt{\frac{s^3}{s_1s_2s_3}}=\sqrt{\frac{ss_3}{s_1s_2}}\sqrt{\frac{ss_2}{s_1s_3}}\sqrt{\frac{ss_1}{s_2s_3}}$

$=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}$. (例題五 43.)

(25) $\tan^2\frac{A}{2}=\frac{s_1s_2}{ss_3}=\frac{S}{r_1}\times\frac{S}{r_2}\Big/\left(\frac{S}{r}\times\frac{S}{r_3}\right)=\frac{rr_3}{r_1r_2}$.

(26) $r^3=s_1s_2s_3\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=\frac{S^2}{s}\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}=r^2s\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}$.

(27) 可以r乘前例之式, 令$rs=S$以求之.

(28) $\tan\frac{A}{2}+\tan\frac{B}{2}+\tan\frac{C}{2}=\sqrt{\frac{s_2s_3}{ss_1}}+\sqrt{\frac{s_3s_1}{ss_2}}+\sqrt{\frac{s_1s_2}{ss_3}}=\frac{s_2s_3+s_3s_1+s_1s_2}{\sqrt{(s\,s_1\,s_2\,s_3)}}$

$=\frac{s_2s_3+s_3s_1+s_1s_2+s^2}{S}-\frac{s^2}{S}=\frac{bc+ca+ab}{S}-\frac{s^2}{rs}=\frac{bc+ca+ab}{abc/(4R)}-\frac{s}{r}=4R\left(\frac{1}{a}+\frac{1}{b}+\frac{1}{c}\right)-\frac{s}{r}$

(29) $\tan^2\frac{A}{2}=\frac{s_2s_3}{ss_1}=\frac{S}{r_2}\times\frac{S}{r_3}\Big/\left(s\times\frac{S}{r_1}\right)=\frac{Sr_1}{r_2r_3s}=\frac{rr_1}{r_2r_3}$,

$\tan^2\frac{A}{2}+\tan^2\frac{B}{2}+\tan^2\frac{C}{2}=\frac{rr_1}{r_2r_3}+\frac{rr_2}{r_3r_1}+\frac{rr_3}{r_1r_2}=\frac{r(r_1^2+r_2^2+r_3^2)}{r_1r_2r_3}$.

(30) $a\cos A+b\cos B+c\cos C=\frac{a(b^2+c^2-a^2)}{2bc}+\frac{b(c^2+a^2-b^2)}{2ca}+\frac{c(a^2+b^2-c^2)}{2ab}$

$=\frac{2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4}{2abc}=\frac{16S^2}{2abc}=\frac{8}{abc}\times\frac{a^2b^2c^2}{16R^2}=\frac{abc}{2R^2}$

$=(2R\sin A)(2R\sin B)(2R\sin C)/(2R^2)=4R^2\sin A\sin B\sin C$.

(31) $r_1=\frac{S}{s_1}=\frac{rs}{s_1}$, 故 $r_1r_2r_3=\frac{r^3s^3}{s_1s_2s_3}$, 又 $\tan^2\frac{A}{2}=\frac{s_2s_3}{ss_1}$,

故 $s=\frac{s_2s_3}{s_1}\cot^2\frac{A}{2}$, 故 $s^3=\left(\frac{s_2s_3}{s_1}\cot^2\frac{A}{2}\right)\left(\frac{s_3s_1}{s_2}\cot^2\frac{B}{2}\right)\left(\frac{s_1s_2}{s_3}\cot^2\frac{C}{2}\right)$

$=s_1s_2s_3\cot^2\frac{A}{2}\cot^2\frac{B}{2}\cot^2\frac{C}{2}$, 從此可得其證.

(32) $a\cot A=\frac{a\cos A}{\sin A}=2R\cos A$,(2.節) 由此

$a\cot A+b\cot B+c\cot C=2R(\cos A+\cos B+\cos C)$

$=2R\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$, [例題五 31.]

$=2R\left(1+4\sqrt{\frac{s_2s_3\cdot s_3s_1\cdot s_1s_2}{bc\cdot ca\cdot ab}}\right)=2R\left(1+\frac{4S^2}{abcs}\right)=2R\left(1+\frac{1}{R}\cdot r\right)$.

33. $r=2R\left(\cos^2\frac{A}{2}-\sin^2\frac{B}{2}-\sin^2\frac{C}{2}\right)$.

34. $\frac{a\tan\frac{A}{2}}{r_1-r}=\frac{b\tan\frac{B}{2}}{r_2-r}=\frac{c\tan\frac{C}{2}}{r_3-r}$.

35. $\sqrt{abc}\left(\sqrt{\frac{a}{r_1}}+\sqrt{\frac{b}{r_2}}+\sqrt{\frac{c}{r_3}}\right)$

$=16R\sqrt{r}\cos\frac{1}{4}(180°-A)\cos\frac{1}{4}(180°-B)\cos\frac{1}{4}(180°-C)$.

36. $\frac{a^2\cot\frac{A}{2}+b^2\cot\frac{B}{2}+c^2\cot\frac{C}{2}}{a^2\tan\frac{A}{2}+b^2\tan\frac{B}{2}+c^2\tan\frac{C}{2}}=\frac{R+r}{R-r}$.

37. $r\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=r_1\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)=a$.

38. $a^2+b^2+c^2=8R^2(1+\cos A\cos B\cos C)$.

39. $S=r_2r_3\tan\frac{A}{2}=r_2r_3\sqrt{\left(\frac{4R}{r_2+r_3}-1\right)}=rr_1\sqrt{\left(\frac{4R}{r_1-r}-1\right)}$.

例題解自 33 至 39.

(33) $r=(s-b)\tan\frac{B}{2}$, (3.節) $=\frac{1}{2}(c+a-b)\tan\frac{B}{2}$

$=\frac{1}{2}(2R\sin C+2R\sin A-2R\sin B)\tan\frac{B}{2}$, (2.節) $=R(\sin C+\sin A-\sin B)\tan\frac{B}{2}$

$=2R\left(\sin\frac{C+A}{2}\cos\frac{C-A}{2}-\sin\frac{B}{2}\cos\frac{B}{2}\right)\frac{\sin\frac{B}{2}}{\cos\frac{B}{2}}=2R\left(\cos\frac{C+A}{2}\cos\frac{C-A}{2}-\sin^2\frac{B}{2}\right)$

$=2R\left(\cos^2\frac{A}{2}-\sin^2\frac{C}{2}-\sin^2\frac{B}{2}\right)$.

(34) $\frac{a\tan\frac{A}{2}}{r_1-r}=\frac{2R\sin A\tan\frac{A}{2}}{2R\sin^2\frac{A}{2}}$, (2.節及例題 7.) $=\frac{4R\sin^2\frac{A}{2}}{4R\sin^2\frac{A}{2}}=1$

(35) $\sqrt{\frac{a}{r_1}}=\sqrt{\frac{2R\sin A}{s\tan\frac{A}{2}}}=\sqrt{\frac{4R\cos^2\frac{A}{2}}{s}}=2\cos\frac{A}{2}\sqrt{\frac{R}{s}}$, 故

$$\sqrt{abc}\left(\sqrt{\frac{a}{r_1}}+\sqrt{\frac{b}{r_2}}+\sqrt{\frac{c}{r_3}}\right)=\sqrt{4RS}\left(2\cos\frac{A}{2}\sqrt{\frac{R}{s}}+2\cos\frac{B}{2}\sqrt{\frac{R}{s}}+2\cos\frac{C}{2}\sqrt{\frac{R}{s}}\right)$$

$$=4R\sqrt{\frac{S}{s}}\left(\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}\right)=4R\sqrt{r}\left(4\cos\frac{B+C}{4}\cos\frac{C+A}{4}\cos\frac{A+B}{4}\right).$$

〔見例題五 41〕.

(36)　$a^2\cot\frac{A}{2}=4R^2\sin^2A\cot\frac{A}{2}=2R^2(2\sin A+\sin 2A).$

又　$a^2\tan\frac{A}{2}=4R^2\sin^2A\tan\frac{A}{2}=2R^2(2\sin A-\sin 2A).$

故原式之左邊 $=\frac{2R^2\{2(\sin A+\sin B+\sin C)+(\sin 2A+\sin 2B+\sin 2C)\}}{2R^2\{2(\sin A+\sin B+\sin C)-(\sin 2A+\sin 2B+\sin 2C)\}}$

$=\frac{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}+4\sin A\sin B\sin C}{8\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}-4\sin A\sin B\sin C}$, (例題五 30. 及 37.)

$=\frac{1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}{1-4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}}=\frac{1+\frac{r}{R}}{1-\frac{r}{R}}$,　(例題 5.)

(37)　三角形為 ABC, 其內切圓及傍切圓之中心為 O 及 O_1, 其切圓周 BC 之點為 D 及 E, 則 $BD=OD\cot DBO=r\cot\frac{1}{2}B$, $CD=r\cot\frac{1}{2}C$,

$BE=O_1E\cot EBO_1=r_1\cot\frac{1}{2}(180°-B)=r_1\tan\frac{1}{2}B$, $CE=r_1\tan\frac{1}{2}C$

而　$r(\cot\frac{1}{2}B+\cot\frac{1}{2}C)=BD+CD=BC=a$,

$r_1(\tan\frac{1}{2}B+\tan\frac{1}{2}D)=BE+CE=BC=a.$

(38)　$a^2+b^2+c^2=(2R\sin A)^2+(2R\sin B)^2+(2R\sin C)^2$, 〔2. 節〕

$=4R^2(\sin^2A+\sin^2B+\sin^2C)=4R^2(2\cos A\cos B\cos C+2)$, 〔例題五 61.〕

(39)　$S=\sqrt{rr_1r_2r_3}$, (例題 4.) $=r_2r_3\sqrt{\frac{rr_1}{r_2r_3}}=r_2r_3\tan\frac{A}{2}$, (例題 25.)

又由例題 10. $\frac{rr_1}{r_2r_3}=\frac{r_1-r}{r_2+r_3}=\frac{4R-(r_2+r_3)}{r_2+r_3}$, (例題 9.) $=\frac{4R}{r_2+r_3}-1.$

故　$S=r_2r_3\sqrt{\frac{rr_1}{r_2r_3}}=r_2r_3\sqrt{\left(\frac{4R}{r_2+r_3}-1\right)}$,

又　$S=rr_1\sqrt{\frac{r_2r_3}{rr_1}}=rr_1\sqrt{\frac{r_2+r_3}{r_1-r}}=rr_1\sqrt{\frac{4R-(r_1-r)}{r_1-r}}=rr_1\sqrt{\left(\frac{4R}{r_1-r}-1\right)}.$

幾何學上之應用

5. 幾何學上之應用

本編自1.節至4.節及其例題,已示三角形之內切圓,外切圓及傍切圓之單純者,茲更將是等圓所附隨之線及點之關係者示之,又三角形之諸線及諸點,其關於幾何學上之圖形者,亦依次略述之.

予講述平面幾何學講義,自直線至比例,其各卷末所揭之例題,於是等重要之線及點,均分類記載,茲編所說,與該講義錄互相連絡,且補其缺憾,但彼以幾何學解之,此以三角法解之,其方法各異,然其歸着點一也,初學者苟能彼此對照,互相研究,較之獨學一書,便益實多.

此處所用之圖形,其所附之文字記號,均與該幾何學講義錄同,學者宜注意.

6. 三角形之垂線

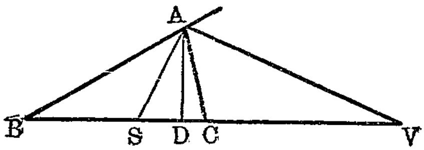
三角形之垂線AD=h,則

$$h=\frac{2S}{a}=c\sin B=b\sin C$$

因　$S=\frac{1}{2}AD.BC=\frac{1}{2}ah$,

又直角三角形ABD,

$AD=AB\sin B$,　即　$h=c\sin B$.



7. 三角形之頂角及其外角之二等分線

三角形之頂角A及其外角之二等分線,爲AS=p, AV=q,則

$$p=\frac{2bc\cos\frac{A}{2}}{b+c}=\frac{2\sqrt{bcss_1}}{b+c}$$

$$q=\frac{2bc\sin\frac{A}{2}}{c-b}=\frac{2\sqrt{bcs_2s_3}}{c-b},\quad 但\ c>b.$$

三角形ASB+三角形ASC=三角形ABC,即

$\frac{1}{2}cp\sin\frac{1}{2}A+\frac{1}{2}bp\sin\frac{1}{2}A=\frac{1}{2}bc\sin A$,　由是

$$p=\frac{bc\sin A}{\sin\frac{1}{2}A(c+b)}=\frac{2bc\cos\frac{A}{2}}{c+b}=\frac{2bc}{c+b}\sqrt{\frac{s(s-a)}{bc}}=\frac{2\sqrt{bcss_1}}{c+b}.$$

又 三角形ABV－三角形ACV＝三角形ABC，即

$\frac{1}{2}cq\sin(\mathrm{SAV+BAS})-\frac{1}{2}bq\sin(\mathrm{SAV-SAC})=\frac{1}{2}bc\sin A$，即

$\frac{1}{2}cq\sin\left(90°+\frac{A}{2}\right)-\frac{1}{2}bq\sin\left(90°-\frac{A}{2}\right)=\frac{1}{2}bc\sin A$，由是

$$q=\frac{bc\sin A}{\cos\frac{A}{2}(c-b)}=\frac{2bc\sin\frac{A}{2}}{c-b}=\frac{2bc}{c-b}\sqrt{\frac{(s-b)(s-c)}{bc}}=\frac{2\sqrt{bcs_2s_3}}{c-b}.$$

8. 三角形之中央線 從三角形之角頂A，至對邊中央點P所作之直線，即中央線AP＝m，則

$$m=\frac{1}{2}\sqrt{b^2+c^2+2bc\cos A}=\frac{1}{2}\sqrt{2(b^2+c^2)-a^2}.$$

BP＝CP，故由幾何學之定理.

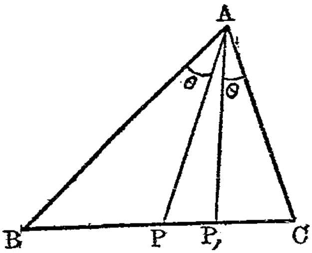
$2AP^2+2BP^2=AB^2+AC^2$，即

$2m^2+2(\frac{1}{2}a)^2=b^2+c^2$，故

$m=\frac{1}{2}\sqrt{2(b^2+c^2)-a^2}$，

又 $a^2=b^2+c^2-2bc\cos A$，由是

$m=\frac{1}{2}\sqrt{b^2+c^2+2bc\cos A}.$



9. 類似中央線 有三角形ABC，從其一角頂A，至對邊引 AP，AP_1 二直線，AP 爲中央線，而角BAP＝角CAP_1，則 AP_1 稱類似中央線.

設 $AP_1=m_1$，則 $m_1=\frac{2bcm}{b^2+c^2}$.

令角 BAP＝角 $CAP_1=\theta$，則 $2AB.AP\cos\theta=AB^2+AP^2-BP^2$，即

$2cm\cos\theta=c^2+m^2-\frac{1}{4}a^2=c^2+\frac{1}{4}\{2(b^2+c^2)-a^2\}-\frac{1}{4}a^2=\frac{1}{2}(b^2+3c^2-a^2)$，

故 $\cos\theta=\frac{b^2+3c^2-a^2}{4cm}$，又 $AB.AP\sin\theta=2$三角形BAP＝S，即

$cm\sin\theta=S=\frac{1}{2}bc\sin A$，故 $\sin\theta=\frac{b\sin A}{2m}=\frac{a\sin B}{2m}$，

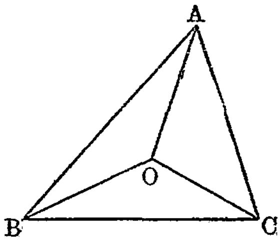
又從三角形 APP_1 $\frac{AP_1}{AP}=\frac{\sin APP_1}{\sin AP_1P}=\frac{\sin(\theta+B)}{\sin(\theta+C)}$.

即 $\frac{m_1}{m}=\frac{\sin\theta\cos B+\cos\theta\sin B}{\sin\theta\cos C+\cos\theta\sin C}$

$$=\frac{\frac{a\sin B}{2m}\times\frac{a^2+c^2-b^2}{2ac}+\frac{b^2+3c^2-a^2}{4cm}\times\sin B}{\frac{a\sin B}{2m}\times\frac{a^2+b^2-c^2}{2ab}+\frac{b^2+3c^2-a^2}{4cm}\times\frac{c\sin B}{b}}=\frac{b\sin B(a^2+c^2-b^2+b^2+3c^2-a^2)}{c\sin B(a^2+b^2-c^2+b^2+3c^2-a^2)}$$

$=\frac{2bc}{b^2+c^2}$, 故 $m_1=\frac{2bcm}{b^2+c^2}$.

10. 補題一 從三角形ABC之各角頂,引任意之直線,會於一點O,則 $\frac{\sin BAO.\sin CBO.\sin ACO}{\sin CAO.\sin ABO.\sin BCO}=1$.



因 $\frac{\sin BAO}{\sin ABO}=\frac{BO}{AO}$, $\frac{\sin CBO}{\sin BCO}=\frac{CO}{BO}$

$\frac{\sin ACO}{\sin CAO}=\frac{AO}{CO}$, 而

$\frac{\sin BAO}{\sin ABO}\times\frac{\sin CBO}{\sin BCO}\times\frac{\sin ACO}{\sin CAO}=\frac{BO}{AO}\times\frac{CO}{BO}\times\frac{AO}{CO}$.

故 $\frac{\sin BAO.\sin CBO.\sin ACO}{\sin CAO.\sin ABO.\sin BCO}=1$.

11. 補題二 三角形之三邊BC, CA, AB,於其上各取一點,順次命爲X, Y, Z,若 $\frac{\sin BAX\sin CBY\sin ACZ}{\sin CAX\sin ABY\sin BCZ}=1$,

則AX, BY, CZ會於一點.

命 AX, BY之交點爲O,則O爲直線AX及BY上之點,故

角BAX=角BAO, 角CAX=角CAO, 角ABY=角ABO, 角CBY=角CBO,

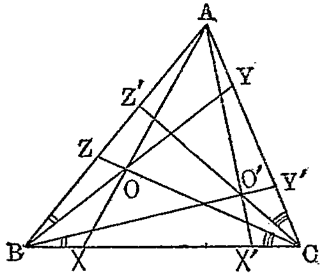
故 $\frac{\sin BAO\ \sin CBO.\sin ACZ}{\sin CAO.\sin ABO.\sin BCZ}=1$, 又連結CO,則由前節

$\frac{\sin BAO.\sin CBO.\sin ACO}{\sin CAO.\sin ABO.\sin BCO}=1$,

故角ACZ=角ACO, 角BCZ=角BCO,

由是CZ通過O.

12. 補題三 自三角形之各角頂,二引直線,此各二直線交角之二等分線,與原三角形各角之二等分線同.其各角頂所引之一直線之三個直線,會於一點,則其餘各角之一直線之三個直線,亦會於一點.



從A引二直線AX, AX',其交角之二等分線,與A之二等分線同,則

角BAX=角CAX', 角CAX=角BAX', 同樣,

角CBY=角ABY', 角ACZ=角BCZ',

令AX, BY, CZ會於一點O,

則由補題一,

$$\frac{\sin BAX.\sin CBY.\sin ACZ}{\sin CAX.\sin ABY.\sin BCZ}=1,\quad 即\quad \frac{\sin CAX'.\sin ABY'.\sin BCZ'}{\sin BAX'.\sin CBY'.\sin ACZ'}=1.$$

故由補題二, AX', BY', CZ'會於一點.

13. 三角形之類似重心 三角形之三個類似中央線,會於一點,此會點,名類似重心.

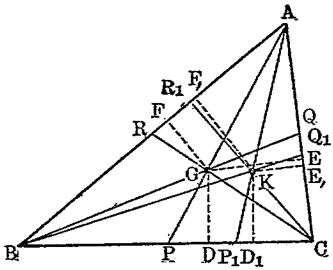
如次15.節之圖,自A引中央線AP,及類似中央線AP_1,而自B及C,亦各用中央線BQ及CR,與類似中央線線BQ_1及CR_1,由幾何學,AP, BQ, CR當會於一點,此會點名重心,以G代之,然由前節AP_1, BQ_1, CR_1亦當會於一點,此會點,名類似重心,以K代之.

14. 定理一 從三角形之重心G,引BC, CA, AB之垂線.令爲x, y, z, 則 $x:y:z=\frac{1}{a}:\frac{1}{b}:\frac{1}{c}$.

由幾何學,三角形GBC=三角形GCA=三角形GAB,

$\frac{1}{2}xa=\frac{1}{2}yb=\frac{1}{2}zc$, 即 $x\Big/\frac{1}{a}=y\Big/\frac{1}{b}=z\Big/\frac{1}{c}$.

15. 定理二 從三角形之類似重心K, 引BC, CA, AB之垂線, 命爲x_1, y_1, z_1, 則 $x_1 : y_1 : z_1 = a : b : c$.



由前節之假定, $GD=x$, $GE=y$, $GF=z$, 又 $KD_1=x_1$, $KE_1=y_1$. $KF_1=z_1$. 兩三角形 CKD_1, CGE 爲相似, 故 $KD_1 : GE=CK : CG$, 即 $x_1 : y=CK : CG$, 又兩三角形 CKE_1, CGD 爲相似, 故 $KE_1 : GD=CK : CG$, 即 $y_1 : x=CK : CG$, 故 $x_1 : y=y_1 : x$, 即 $x_1 : y_1=y : x$, 故由前節 $x_1 : y_1=\frac{1}{b} : \frac{1}{a}=a : b$, 同樣, $y_1 : z=b : c$, 由是 $x : y : z=a : b : c$.

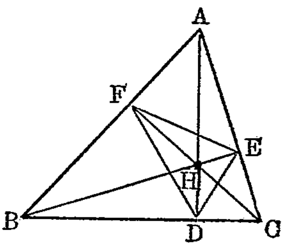
16. 垂足三角形 三角形三垂線之足D, E, F, 連結成一DEF三角形, 是名垂足三角形, 而 $EF=d$, 角 $EDF=D$,

則 $D=180°-2A$,

$A=90°-\frac{1}{2}D$,

$d=a\cos A$,

$a=d\csc\frac{1}{2}D$.



由幾何學之定理, 角 FDH=角 ABH=90°−A=角 HDE,

故　$D=$角 FDH+角 HDE$=180^\circ-2A$，及　$A=90^\circ-\frac{1}{2}D$，

又　角 FDB$=$角 EDC$=90^\circ-\frac{1}{2}D=A$，同樣，角 AEF$=$B，

又　$AF=AC\cos A=b\cos A$，　$\sin AEF:\sin A=AF:EF=b\cos A:d$，

故　$d=\dfrac{b\sin A\cos A}{\sin AEF}=\dfrac{b\sin A\cos A}{\sin B}=\dfrac{a\sin B\cos A}{\sin B}=a\cos A$，

又　$a=\dfrac{d}{\cos A}=\dfrac{d}{\cos(90^\circ-\frac{1}{2}D)}=d\csc\frac{1}{2}D$.

17. 四角形之面積　四角形 ABCD，其面積$=$P，$AB=a$，$BC=b$，$CD=c$，$DA=d$，又 $a+b+c+d=2p$，則

$$P=\sqrt{\left\{(p-a)(p-b)(p-c)(p-d)-abcd\cos^2\frac{A+C}{2}\right\}}.$$

四角形 ABCD，引對角線 BD，則得兩三角形 ABD 及 CBD，而

$BD^2=a^2+d^2-2ad\cos A$，及 $BD^2=b^2+c^2-2bc\cos C$，

由是　$\frac{1}{2}(a^2+d^2-b^2-c^2)=ad\cos A-bc\cos C$，

又　四角形 ABCD$=$三角形 ABD，+三角形 CBD，卽 $2P=ad\sin A+bc\sin C$，

故　$\frac{1}{4}(a^2+d^2-b^2-c^2)^2+4P^2=(ad\cos A-bc\cos C)^2+(ad\sin A+bc\sin C)^2$

$$=a^2d^2+b^2c^2-2abcd(\cos A\cos C-\sin A\sin C)$$

$$=a^2d^2+b^2c^2-2abcd\cos(A+C)$$

$$=a^2d^2+b^2c^2-2abcd\left(2\cos^2\frac{A+C}{2}-1\right)$$

$$=(ad+bc)^2-4abcd\cos^2\frac{A+C}{2},$$

故　$16P^2=4(ad+bc)^2-(a^2+d^2-b^2-c^2)^2-16abcd\cos^2\dfrac{A+C}{2}$

$$=\{2(ad+bc)+(a^2+d^2-b^2-c^2)\}\{2(ad+bc)-(a^2+d^2-b^2-c^2)\}-16abcd\cos^2\frac{A+C}{2}$$

$$=\{(a+d)^2-(b-c)^2\}\{(b+c)^2-(a-d)^2\}-16abcd\cos^2\frac{A+C}{2}$$

$$=(a+d+b-c)(a+d-b+c)(b+c+a-d)(b+c-a+d)-16abcd\cos^2\frac{A+C}{2}$$

$$=16(p-c)(p-b)(p-d)(p-a)-16abcd\cos^2\frac{A+C}{2}$$

由是　$P=\sqrt{\{(p-a)(p-b)(p-c)(p-d)-abcd\cos^2\frac{1}{2}(A+C)\}}$.

18. 圓之內切四角形 其面積如次,

$$P=\sqrt{\{(p-a)(p-b)(p-c)(p-d)\}}.$$

由幾何學之定理,圓之內切四角形,其兩對角之和爲180°,即

$$A+C=180°,\quad 故\quad \cos\frac{A+C}{2}=\cos 90°=0,$$

由是依前節可得其證.

19. 正多角形 正n多角形,其內切圓之半徑爲r,

外切圓之半徑爲R,一邊爲a,面積爲A,則

$$a=2R\sin\frac{\pi}{n}=2r\tan\frac{\pi}{n},$$

$$A=\tfrac{1}{2}nR^2\sin\frac{2\pi}{n}=nr^2\tan\frac{\pi}{n}.$$

令兩圓之共通中心爲O, $AB=a$,

$OA=OB=R$, $OD=r$, 然

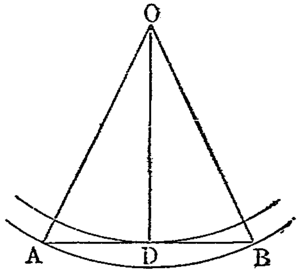
角$AOB=\frac{2\pi}{n}=2$角AOD,

故 $AD=AO\sin AOD$, 即$\frac{1}{2}a=R\sin\frac{\pi}{n}$.

又 $AD=OD\tan AOD$, 即 $\frac{1}{2}a=r\tan\frac{\pi}{n}$,

故 $A=n$三角形$AOB=n\times\frac{1}{2}AB.OD$,

$=\frac{1}{2}nar=\frac{1}{2}nr^2\tan\frac{\pi}{n}$,



又 $A=n$ 三角形$AOB=n\times\frac{1}{2}AO.BO\sin AOB=\frac{1}{2}nR^2\sin\frac{2\pi}{n}$.

20. 圓之面積 圓之半徑爲R,則其面積等於$R^2\pi$,

R爲半徑之圓,作其內切正n多角形,其邊數爲n,若n漸漸增大,則其正n多角形次第近於圓.$n=\infty$,則式如次,

$$A=\tfrac{1}{2}nR^2\sin\frac{2\pi}{n}=R^2\pi\times\frac{\sin\frac{2\pi}{n}}{\frac{2\pi}{n}}=R^2\pi\times 1,\quad (第十編2.節)$$

例　題　二　十　四

1. 自三角形之各角頂，至對邊 a, b, c 引垂線 h_1, h_2, h_3. 則

$$\frac{1}{S}=\sqrt{\left\{\left(\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}\right)\left(\frac{1}{h_1}+\frac{1}{h_2}-\frac{1}{h_3}\right)\left(\frac{1}{h_2}+\frac{1}{h_3}-\frac{1}{h_1}\right)\left(\frac{1}{h_3}+\frac{1}{h_1}-\frac{1}{h_2}\right)\right\}}.$$

2. $\sin A$

$$=\frac{\sqrt{\left\{\left(\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}\right)\left(\frac{1}{h_1}+\frac{1}{h_2}-\frac{1}{h_3}\right)\left(\frac{1}{h_2}+\frac{1}{h_3}-\frac{1}{h_1}\right)\left(\frac{1}{h_3}+\frac{1}{h_1}-\frac{1}{h_2}\right)\right\}}}{2.\frac{1}{h_2}.\ \frac{1}{h_3}}$$

3. $$\tan^2\frac{A}{2}=\frac{\left\{\left(\frac{1}{h_3}+\frac{1}{h_1}-\frac{1}{h_2}\right)\left(\frac{1}{h_1}+\frac{1}{h_2}-\frac{1}{h_3}\right)\right\}}{\left\{\left(\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}\right)\left(\frac{1}{h_2}+\frac{1}{h_3}-\frac{1}{h_1}\right)\right\}}$$

4. $h_1=2s\,/\,(\cot\frac{1}{2}B+\cot\frac{1}{2}C)$.

例　題　解　自　1　至　4.

(1) $a=\frac{2S}{h_1}$, $b=\frac{2S}{h_2}$, $c=\frac{2S}{h_3}$, (6. 節) 故由第十二編 3. 節公式 (6), 得 $S=\frac{1}{4}\sqrt{\{(a+b+c)(b+c-a)(c+a-b)(a+b-c)\}}$, 以前之 a, b, c 之值代入而變化之, 則 $S=S^2\sqrt{\left\{\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}\right)\left(\frac{1}{h_2}+\frac{1}{h_3}-\frac{1}{h_1}\right)\left(\frac{1}{h_3}+\frac{1}{h_1}-\frac{1}{h_2}\right)\left(\frac{1}{h_1}+\frac{1}{h_2}-\frac{1}{h_3}\right)\right\}}$.

(2) 由第十二編 3. 節公式 (6). $\sin A=\frac{2S}{bc}=\frac{2S}{\frac{2S}{h_2}\cdot\frac{2S}{h_3}}=\frac{1/S}{\left(2.\frac{1}{h_2}\cdot\frac{1}{h_3}\right)}$.

(3) 由第十二編 3. 節公式 (6), $\tan^2\frac{A}{2}=\frac{(c+a-b)(a+b-c)}{(a+b+c)(b+c-a)}$, 代入 a, b, c 之值，卽得其證.

(4) $b=\frac{h_1}{\sin C}$, $c=\frac{h_1}{\sin B}$, $a=\frac{2S}{h_1}=\frac{bc\sin A}{h_1}=\frac{h_1^2}{\sin B\sin C}\times\frac{\sin A}{h_1}$,

故 $2S=a+b+c=\left(\frac{\sin A}{\sin B\sin C}+\frac{1}{\sin C}+\frac{1}{\sin B}\right)h_1$

$=\left\{\frac{\sin(B+C)+\sin B+\sin C}{\sin B\sin C}\right\}h_1=\left(\frac{1+\cos C}{\sin C}+\frac{1+\cos B}{\sin B}\right)h_1=\left(\cot\frac{C}{2}+\cot\frac{B}{2}\right)h_1$,

5. $h_1=\dfrac{b^2 \sin 2C + c^2 \sin 2B}{2a}$.

6. $\sin^2\dfrac{A}{2}=\dfrac{h_2 h_3}{4 r_2 r_3}$. 7. $h_1=2R\sin B\sin C$.

8. 三角形之垂線AD,垂心H. 則

$AH=2R\cos A$, $HD=2R\cos B\cos C$.

9. 於銳角三角形,其 $AH=x$, $BH=y$, $CH=z$, 則

$S=\frac{1}{4}(ax+by+cz)$,

$a^2x\csc A+b^2y\csc B+c^2z\csc C=2abc$.

10. 同上, AH, BH, CH 三邊作三角形,其各角爲 α, β, γ, 則

$$1+\frac{\cos\alpha}{\cos A}+\frac{\cos\beta}{\cos B}+\frac{\cos\gamma}{\cos C}=\tfrac{1}{2}\sec A\sec B\sec C.$$

11. 自 B, C 至對邊引垂線 BE, CF, 再引長 EF 及 CB 交於Q, 則 $2(QE^2-QF^2)=(CQ^2-BQ^2)(\cos 2B+\cos 2C)$.

例題解自 5. 至 11.

(5) S=三角形ABD+三角形ACD, 但 $S=\frac{1}{2}h_1a$, 三角形 $ABD=\frac{1}{2}h_1BD$ $=\frac{1}{2}c\sin B.c\cos B=\frac{1}{4}c^2\sin 2B$, 同樣, 三角形 $ACD=\frac{1}{4}b^2\sin 2C$,

由是 $2h_1a=c^2\sin 2B+b^2\sin 2C$.

(6) $b_2=\dfrac{S}{r_2}$, $S_3=\dfrac{S}{r_3}$, (4.節), $b=\dfrac{2S}{h_2}$, $c=\dfrac{2S}{h_3}$, 故由

第十二編3.節之公式(3). $\sin^2\dfrac{A}{2}=\dfrac{S_2S_3}{bc}=\dfrac{\dfrac{S}{r_2}\times\dfrac{S}{r_3}}{\dfrac{2S}{h_2}\times\dfrac{2S}{h_3}}=\dfrac{h_2h_3}{4r_2r_3}$.

(7) $h_1=c\sin B=2R\sin C\sin B$, (2. 節)

(8) $BD=AB\cos ABD=c\cos B$, $HD=BD\tan HBD=c\cos B\tan(90^\circ-C)$

$=c\cos B\cot C=\frac{c}{\sin C}\cos B\cos C=2R\cos B\cos C$, (2. 節)

$AH=AD-HD=2R\sin C\sin B-2R\cos B\cos C$, (前例) $=-2R\cos(B+C)$.

(9) $S=\frac{1}{2}bc\sin A=\frac{1}{2}(2R\sin B)(2R\sin C)\sin A$, (2. 節)

$=\frac{1}{2}R^2(4\sin A\sin B\sin C)=\frac{1}{2}R^2(\sin 2A+\sin 2B+\sin 2C)$, (例題五 37.)

但 $\frac{1}{2}R^2\sin 2A=R^2\sin A\cos A=\frac{1}{4}(2R\sin A)(2R\cos A)=\frac{1}{4}ax$, (2. 節及前例)

由是 $S=\frac{1}{4}(ax+by+cz)$, 由平面幾何講義第二編例題 613.

令 AH=2MP 則得最簡之解式.

又 $a^2x\csc A=\frac{a^2}{\sin^2 A}x\sin A=4R^2(2R\cos A)\sin A$, (前例及 2. 節) $=4R^3\sin 2A$,

故 $a^2x\csc A+b^2y\csc B+c^2z\csc C=4R^3(\sin 2A+\sin 2B+\sin 2C)$

$=16R^3\sin A\sin B\sin C$, (例題五 37.) $=2(2R\sin A)(2R\sin B)(2R\sin C)=2abc$.

(10) 由第十二編公式 (2). $\cos a=\frac{y^2+z^2-x^2}{2yz}=\frac{4R^2(\cos^2 B+\cos^2 C-\cos^2 A)}{2(2R\cos B)(2R\cos C)}$,

(例題 8.) $=\frac{\cos^2 B+\cos^2 C-\cos^2 A}{2\cos B\cos C}$, 即 $\frac{\cos a}{\cos A}=\frac{\cos^2 B+\cos^2 C-\cos^2 A}{2\cos A\cos B\cos C}$, 故

原式之左邊 $=1+\frac{\cos^2 A+\cos^2 B+\cos^2 C}{2\cos A\cos B\cos C}=1+\frac{1-2\cos A\cos B\cos C}{2\cos A\cos B\cos C}$, (例題五 63.)

(11) EFQ 為三角形 ABC 之橫截線, 故由平面幾何學講義第四編例題 1510. BQ. CE. AF=CQ. AE. BF, 即 $BQ(a\cos C)(b\cos A)=CQ(c\cos A)(a\cos B)$,

故 $CQ=\frac{BQ\sin B\cos C}{\sin C\cos B}$, 故 $BQ^2-CQ^2=\frac{BQ^2\sin(B+C)\sin(B-C)}{\sin^2 C\cos^2 B}$

$=BQ^2(\cos 2C-\cos 2B)/(2\sin^2 C\cos^2 B)$

又 $QE=BQ\frac{\sin EBQ}{\sin BEQ}=BQ\frac{\sin(90^\circ+C)}{\sin(90^\circ-B)}=\frac{BQ\cos C}{\cos B}$,

同樣 $QF=\frac{CQ\cos B}{\cos C}=\frac{BQ\sin B\cos C}{\sin C\cos B}\times\frac{\cos B}{\cos C}=\frac{BQ\sin B}{\sin C}$,

故 $QE^2-QF^2=BQ^2\left(\frac{\cos^2 C}{\cos^2 B}-\frac{\sin^2 B}{\sin^2 C}\right)=\frac{BQ^2(\cos^2 2B-\cos^2 2C)}{4\cos^2 B\sin^2 C}$,

由是 $2(QE^2-QF^2)=(CQ^2-BQ^2)(\cos 2C+\cos 2B)$.

12. 知三角形之A角及h_2, h_3二垂線,求其餘邊角.

13. 證明三角形A角之二等分線爲$\frac{2bc\cos\frac{1}{2}A}{b+c}$.

14. 同上.A之外角之二等分線爲$\frac{2bc\sin\frac{1}{2}A}{b-c}$.

15. 令A角之二等分線爲AS,則

$$BS:CS=\sin C:\sin B.$$

16. 令A角外角之二等分線爲AV,則

$$BV:CV=\sin C:\sin B.$$

17. 同上　$\frac{a-SC}{SC}=\frac{2c\cos\frac{1}{2}A\cos\frac{1}{2}(B+C)}{a\sin B}$.

18. 同上　$\frac{1}{CV}=\frac{2\sin\frac{1}{2}A\sin\frac{1}{2}(B-C)}{a\sin B}$.

19. 三角形各角之二等分線爲AS, BT, CU.則

$$\text{三角形STU}=\frac{2S\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}(B-C)\cos\frac{1}{2}(C-A)\cos\frac{1}{2}(A-B)}.$$

20. 三角形ABC,其角A, B, C之二等分線爲p, q, r.

$$\frac{\cos\frac{1}{2}A}{p}+\frac{\cos\frac{1}{2}B}{q}+\frac{\cos\frac{1}{2}C}{r}=\frac{1}{a}+\frac{1}{b}+\frac{1}{c}.$$

21. 同上　$\frac{pqr(b+c)(c+a)(a+b)}{abc}=8sS$

22. 同上　$\frac{p^2(b+c)^2}{bc}+\frac{q^2(c+a)^2}{ca}+\frac{r^2(a+b)^2}{ab}=4s^2$.

例題解自 12. 至 22.

(12) $b=\frac{h_3}{\sin A}$, $c=\frac{h_2}{\sin A}$, $a^2=b^2+c^2-2bc\cos A=\frac{h^2{}_2+h^2{}_3-2h_2h_3\cos A}{\sin^2 A}$

又知 a 及 b, 故由 $\sin B=\frac{b\sin A}{a}$ 可得 B.

(13) 命 A 之二等分線為 AS, 則三角形 ASB＋三角形 ASC＝三角形 ABC,

即 $\frac{1}{2}c.AS\sin\frac{A}{2}+\frac{1}{2}b.AS\sin\frac{A}{2}=\frac{1}{2}bc\sin A$. 從此可得其證.

(14) 與前同樣. (15), (16) 亦甚容易,故略.

(17) 從例題15. $\frac{BS}{SC}=\frac{a-BC}{BC}=\frac{\sin C}{\sin B}=\frac{a\sin C}{a\sin B}=\frac{c\sin A}{a\sin B}=\frac{2c\sin\frac{1}{2}A\cos\frac{1}{2}A}{a\sin B}$

$=\frac{2c\cos\frac{1}{2}(B+C)\cos\frac{1}{2}A}{a\sin B}$.

(18) 由例題16. $\frac{CV-a}{CV}=\frac{\sin C}{\sin B}$, $\frac{a}{CV}=1-\frac{\sin C}{\sin B}=\frac{\sin B-\sin C}{\sin B}$.

(19) $AS=\frac{2bc\cos\frac{1}{2}A}{b+c}$, (例題13.) $AS=\frac{bc}{c+a}$, $AU=\frac{bc}{a+b}$, (平面幾何學講義錄第四編例題1273. 之解). 故三角形 $ASU=\frac{1}{2}AS.AU\sin\frac{1}{2}A=\frac{b^2c^2\sin A}{2(a+b)(b+c)}$,

同樣. 三角形 $AST=\frac{b^2c^2\sin A}{2(c+a)(b+c)}$, 故

三角形 STU＝三角形 ASU＋三角形 AST－三角形 ATU

$$=\frac{1}{2}bc\sin A\left\{\frac{bc}{(a+b)(b+c)}+\frac{bc}{(c+a)(b+c)}-\frac{bc}{(c+b)(c+b)}\right\}$$

$$=S\left\{\frac{2abc}{(a+b)(b+c)(c+a)}\right\}=\frac{2S}{(a/c+b/c)(b/a+c/a)(c/b+a/b)}$$

$$=\frac{2S\sin A\sin B\sin C}{(\sin A+\sin B)(\sin B+\sin C)(\sin C+\sin A)}$$

(20) 由例題13. $\frac{\cos\frac{1}{2}A}{p}=\frac{b+c}{2bc}=\frac{1}{2}\left(\frac{1}{c}+\frac{1}{b}\right)$, 從此可得其證.

(21) 見平面幾何學講義錄第四編例題1292.

(22) 由 13. $\frac{p^2(b+c)^2}{bc}=4bc\cos^2\frac{A}{2}=2bc(1+\cos A)=(b+c)^2-a^2$.

23. 於BC上取S之兩側之P, Q二點,角PAS=角QAS=θ,

則 $\frac{1}{AP}+\frac{1}{AQ}:\frac{1}{b}+\frac{1}{c}=\cos\theta:\cos\frac{A}{2}$.

24. 以A及A之外角之二等分線爲二邊,其夾角爲角,順次於A, B, C.得三個三角形爲S_1, S_2, S_3.則

$$1/S_1+1/S_3=1/S_2.$$

25. $BS=m$, $CS=n$, 則

$$2ap\,R\sin\tfrac{1}{2}A=bm^2+cn^2.$$

26. $h_1=\dfrac{mn(m+n)\sin A}{m^2+n^2-2mn\cos A}$.

27. p與BC之銳角爲ϑ. $c>b$, 則

$$\tan\vartheta=\frac{c+b}{c-b}\tan\frac{A}{2}.$$

28. 以三角形三外角之二等分線作第二三角形.又從第二三角形同法作第三三角形,以下仿此,作得第n三角形,求第n三角形之各角.

29. 同上.第二三角形之各邊爲

$4R\cos\frac{1}{2}A$,　$4R\cos\frac{1}{2}B$,　$4R\cos\frac{1}{2}C$.

例題解自23.至29.

(23) 於三角形APB, $\dfrac{c}{AP}=\dfrac{\sin APB}{\sin B}=\dfrac{\sin(B+\frac{1}{2}A-\theta)}{\sin B}$,

$$\frac{1}{AP}=\frac{\sin(B+\frac{1}{2}A-\theta)}{c\sin B},\quad \frac{1}{AQ}=\frac{\sin(C+\frac{1}{2}A-\theta)}{b\sin C}=\frac{\sin(C+\frac{1}{2}A-\theta)}{c\sin B},$$

$$\frac{1}{AP}+\frac{1}{AQ}=\frac{\sin(B+\frac{1}{2}A-\theta)+\sin(C+\frac{1}{2}A-\theta)}{c\sin B}=\frac{2\cos\theta\cos\frac{1}{2}(B-C)}{c\sin B},$$

$\frac{1}{c}\Big/\left(\frac{1}{AP}+\frac{1}{AQ}\right)=\frac{\sin B}{\cos\theta\cos\frac{1}{2}(B-C)}$，　$\frac{1}{b}\Big/\left(\frac{1}{AP}+\frac{1}{AQ}\right)=\frac{\sin C}{\cos\theta\cos\frac{1}{2}(B-C)}$，

故 $\left(\frac{1}{b}+\frac{1}{c}\right)\Big/\left(\frac{1}{AP}+\frac{1}{AQ}\right)=\frac{\sin B+\sin C}{\cos\theta\cos\frac{1}{2}(B-C)}=\frac{2\cos\frac{1}{2}A}{\cos\theta}$.

(24) 由 13. 及 14. $AS=\frac{2bc\cos\frac{1}{2}A}{b+c}$, $AV=\frac{2bc\sin\frac{1}{2}A}{b-c}$, 故

$S_1=\frac{1}{2}AS, AV\sin A=\frac{b^2c^2\sin^2 A}{b^2-c^2}$， 卽 $\frac{1}{S_1}=\frac{1}{c^2\sin^2 A}-\frac{1}{b^2\sin^2 A}$

$=\frac{1}{a^2\sin^2 C}-\frac{1}{b^2\sin^2 A}$， 同法 $\frac{1}{S_2}=\frac{1}{c^2\sin^2 B}-\frac{1}{b^2\sin^2 A}$

$\frac{1}{S_3}=\frac{1}{c^2\sin^2 B}-\frac{1}{a^2\sin^2 C}$. 從此卽得其證.

(25) $2ap R\sin\frac{A}{2}=a\times\frac{2bc\cos\frac{1}{2}A}{b+c}\times\frac{a}{\sin A}\times\sin\frac{A}{2}$, (2. 節及例題 13.)

$=\frac{a^2bc}{b+c}=\frac{a^2bc(b+c)}{(b+c)^2}=c\left(\frac{ab}{b+c}\right)^2+b\left(\frac{ac}{b+c}\right)^2=cn^2+bm^2$.

(26) $h_1=\frac{2S}{a}=\frac{bc\sin A}{a}=\frac{bc(m+n)\sin A}{a^2}=\frac{bc(m+n)\sin A}{b^2+c^2-2bc\cos A}$.

可令 $m/c=n/b=k$, 而代入上之最後分數式.

(27) $\phi=B+\frac{1}{2}A=B+\frac{1}{2}(180^\circ-C-B)$, 故 $\frac{1}{2}(C-B)=90^\circ-\phi$,

故用第十二編公式 (4) 得 $\tan\frac{C-B}{2}=\frac{c-b}{c+b}\cot\frac{A}{2}$. 從此可得其證.

(28) 第一角 $C_1=\pi-\frac{\pi-A}{2}-\frac{\pi-B}{2}=\frac{A+B}{2}=\frac{\pi}{2}-\frac{C}{2}$, 同樣.

$C_2=\frac{\pi}{2}-\frac{C_1}{2}=\frac{\pi}{2}-\frac{\pi}{2^2}+\frac{C}{2^2}$, $C_3=\frac{\pi}{2}-\frac{\pi}{2^2}+\frac{\pi}{2^3}-\frac{C}{2^3}$

$$C_{n-1}=\frac{\pi}{2}-\frac{\pi}{2^2}+\frac{\pi}{2^3}-\cdots\cdots-\left(-\frac{1}{2}\right)^{n-1}\pi+\left(-\frac{1}{2}\right)^{n-1}C=\frac{\frac{\pi}{2}\left\{1-\left(-\frac{1}{2}\right)^{n-1}\right\}}{1-\left(-\frac{1}{2}\right)}+\left(-\frac{1}{2}\right)^{n-1}C$$

$=\frac{\pi}{3}+\left(C-\frac{\pi}{3}\right)\left(-\frac{1}{2}\right)^{n-1}$.

(29) $AC_1=\frac{c\sin ABC_1}{\sin C_1}=\frac{c\sin(90^\circ-\frac{1}{2}B)}{\sin(90^\circ-\frac{1}{2}C)}=\frac{c\cos\frac{1}{2}B}{\cos\frac{1}{2}C}=\frac{2c\sin\frac{1}{2}C\cos\frac{1}{2}B}{\sin C}$

$=4R\sin\frac{1}{2}C\cos\frac{1}{2}B$, $AB_1=4R\sin\frac{1}{2}B\cos\frac{1}{2}C_1$, $B_1C_1=AB_1+AC_1=4R\sin\frac{1}{2}(B+C)$.

30. 同上.第二三角形.其各邊平方之和為$8R(4R+r)$.

31. 於ABC三角形,從其底邊BC上取一點P.令

$BP/CP=m/n$, 又 角$BAP=\alpha$, 角$CAP=\beta$, 角$APC=\theta$.

則 $(m+n)\cot\theta=m\cot\alpha-n\cot\beta=n\cot B-m\cot C$.

32. 同上.

$$(\cot B-\cot\theta)\sin\theta=BP/AP,$$

$$(\cot C+\cot\theta)\sin\theta=CP/AP.$$

33. 同上. AP為中央線,則

$$2\cot\theta=\cot C-\cot B.$$

34. 同上. $\tan\frac{\alpha-\beta}{2}=\tan\frac{B-C}{2}\tan^2\frac{A}{2}$.

35. 同上. $\cot\beta-\cot\alpha=\cot C-\cot B$.

36. C為直角,ϕ為A角二等分線與自A所引中央線之交角:

則$\tan\phi=\tan^3\frac{1}{2}A$.

37. 中央線$AP=b=\frac{1}{2}c$,

則 $\sin^2C=\frac{5}{8}$, $\sin^2B=\frac{5}{32}$.

例題解自30.至37.

(30) 由前例.各邊平方之和$=16R^2\left(\cos^2\frac{A}{2}+\cos^2\frac{B}{2}+\cos^2\frac{C}{2}\right)$

$=16R^2\left(2+2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$, 〔例題五65〕

$=8R\left(4R+4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)=8R(4R+r)$, 〔例題二十三5〕

(31) $\frac{AP}{BP}=\frac{\sin B}{\sin\alpha}=\frac{\sin(\theta-\alpha)}{\sin\alpha}$, $\frac{AP}{CP}=\frac{\sin C}{\sin\beta}=\frac{\sin(\theta+\beta)}{\sin\beta}$, 故

$\frac{BP}{CP}=\frac{m}{n}=\frac{\sin(\theta+\beta)\sin\alpha}{\sin(\theta-\alpha)\sin\beta}=\frac{\cot\beta+\cot\theta}{\cot\alpha-\cot\theta}$, 從此

$(m+n)\cot\theta=m\cot\alpha-n\cot\beta$, 又

$\frac{AP}{BP}=\frac{\sin B}{\sin\alpha}=\frac{\sin B}{\sin(\theta-B)}$, $\frac{AP}{CP}=\frac{\sin C}{\sin(\theta+C)}$. 從此用前法,即得第二式.

(32) 可從前解第二法得之.

(33) BP=CP, 故 $m=n$, 由此 $2\cot\theta=\cot\alpha-\cot\beta$.

(34) $\sin\alpha=\frac{BP\sin B}{AP}$, $\sin\beta=\frac{CP\sin C}{AP}$, $\sin\alpha-\sin\beta=\frac{a(\sin B-\sin C)}{2AP}$

$2\cos\frac{\alpha+\beta}{2}\sin\frac{\alpha-\beta}{2}=\frac{a\cos\frac{1}{2}(B+C)\sin\frac{1}{2}(B-C)}{AP}$, 即

$2\cos\frac{A}{2}\sin\frac{\alpha-\beta}{2}=\frac{a\sin\frac{1}{2}A\sin\frac{1}{2}(B-C)}{AP}$, $\sin\frac{\alpha-\beta}{2}=\frac{a\sin\frac{1}{2}(B-C)\tan\frac{1}{2}A}{AP}$,

又 $\sin\alpha+\sin\beta=\frac{a(\sin B+\sin C)}{2AP}$, 故 $\cos\frac{\alpha-\beta}{2}=\frac{a\cos\frac{1}{2}(B-C)\cot\frac{1}{2}A}{AP}$,

由是 $\tan\frac{1}{2}(\alpha-\beta)=\tan\frac{1}{2}(B-C)\tan^2\frac{1}{2}A$.

(35) 於 31. 令 $m=n$, 則可直得其證.

(36) 令 A 之二等分線爲 AS, 中央線爲 AP, 則

$\tan CAS=\tan\left(\frac{A}{2}+\phi\right)=\frac{CP}{AC}=\frac{a}{2b}=\frac{\tan A}{2}$, 即

$\frac{\tan\frac{A}{2}+\tan\phi}{1-\tan\frac{A}{2}\tan\phi}=\frac{\tan\frac{1}{2}A}{1-\tan^2\frac{1}{2}A}$. 從此得其證.

(37) 由幾何學之定理. $2AP^2+\frac{a^2}{2}=b^2+c^2$, 即 $2b^2+\frac{a^2}{2}=b^2+4b^2$,

故 $a^2=6b^2$, 又 $\cos C=\frac{a^2+b^2-c^2}{2ab}=\frac{6b^2+b^2-4b^2}{2b^2\sqrt{6}}=\frac{\sqrt{3}}{2\sqrt{2}}$

$\sin^2C=1-\cos^2C=1-\frac{3}{8}=\frac{5}{8}$, 同樣. $\cos B=\frac{3\sqrt{3}}{4\sqrt{2}}$

$\sin^2B=1-\cos^2B=1-\frac{27}{32}=\frac{5}{32}$.

38. $\tan \mathrm{APB}=\dfrac{2bc\sin \mathrm{A}}{b^2-c^2}$. **39.** $\dfrac{\sin \mathrm{BAP}}{\sin \mathrm{CAP}}=\dfrac{b}{c}$.

40. $\cot \mathrm{BAP}+\cot \mathrm{PAC}=4\cot \mathrm{A}+\cot \mathrm{B}+\cot \mathrm{C}$.

41. 從A, B, C引中央線 l, m, n, 則

$(2l^2-a^2)\tan \mathrm{A}=(2m^2-b^2)\tan \mathrm{B}=(2n^2-c^2)\tan \mathrm{C}=4\mathrm{S}$.

42. 垂足三角形, 其三邊爲 $a\cos \mathrm{A}$, $b\cos \mathrm{B}$, $c\cos \mathrm{C}$.

43. 同上. 其周邊爲 $4\mathrm{R}\sin \mathrm{A}\sin \mathrm{B}\sin \mathrm{C}$.

44. 三角之二等分線爲AS, BT, CU. 從A, B, C引TU, US, ST之垂線, 順次令爲 x, y, z, 則

$$\left(\frac{h_1}{x}\right)^2+\left(\frac{h_2}{y}\right)^2+\left(\frac{h_3}{z}\right)^2=11+8\sin\frac{\mathrm{A}}{2}\sin\frac{\mathrm{B}}{2}\sin\frac{\mathrm{C}}{2}.$$

例題解自38. 至44.

(38) 令垂線爲AD, 則 $\mathrm{AD}=h_1=\dfrac{bc\sin \mathrm{A}}{a}$,

$$\mathrm{PD}=\mathrm{PB}-\mathrm{BD}=\frac{a}{2}-c\cos \mathrm{B}=\frac{a}{2}-\frac{a^2+c^2-b^2}{2a}=\frac{b^2-c^2}{2a},$$

$$\tan \mathrm{APB}=\frac{\mathrm{AD}}{\mathrm{PD}}=\frac{2bc\sin \mathrm{A}}{b^2-c^2}.$$

(39) $\sin \mathrm{BAP}=\dfrac{\mathrm{BP}\sin \mathrm{B}}{\mathrm{AP}}=\dfrac{a\sin \mathrm{B}}{2\mathrm{AP}}$, $\sin \mathrm{CAP}=\dfrac{a\sin \mathrm{C}}{2\mathrm{AP}}$,

由是 $\dfrac{\sin \mathrm{BAP}}{\sin \mathrm{CAP}}=\dfrac{\sin \mathrm{B}}{\sin \mathrm{C}}=\dfrac{b}{c}$.

(40) 從C及P,引AB之垂線CF及PX, 則 $\mathrm{PX}=\frac{1}{2}h_3=\frac{1}{2}b\sin \mathrm{A}$,

$$\mathrm{AX}=\mathrm{AF}+\mathrm{FX}=\mathrm{AF}+\tfrac{1}{2}\mathrm{BF}=\tfrac{1}{2}(c+\mathrm{AF})=\tfrac{1}{2}(c+b\cos \mathrm{A}),$$ 故

$\cot \mathrm{BAP}=\dfrac{\mathrm{AX}}{\mathrm{PX}}=\dfrac{c+b\cos \mathrm{A}}{b\sin \mathrm{A}}$, 同樣. $\cot \mathrm{CAP}=\dfrac{b+c\cos \mathrm{A}}{c\sin \mathrm{A}}$, 故

$$\cot \mathrm{BAP}+\cot \mathrm{CAP}=\frac{b^2+c^2+2bc\cos \mathrm{A}}{bc\sin \mathrm{A}}=\frac{a^2+4bc\cos \mathrm{A}}{bc\sin \mathrm{A}}=4\cot \mathrm{A}+\frac{a^2}{bc\sin \mathrm{A}}.$$

又 $\cot \mathrm{B}+\cot \mathrm{C}=\dfrac{\sin(\mathrm{B}+\mathrm{C})}{\sin \mathrm{B}\sin \mathrm{C}}=\dfrac{\sin \mathrm{A}}{bc\sin^2 \mathrm{A}/a^2}=\dfrac{a^2}{bc\sin \mathrm{A}}$.

(41) $l^2=\frac{2b^2+2c^2-a^2}{2}$ 故 $2l^2-a^2=2(b^2+c^2-a^2)=2bc\cos A$

故 $(2b^2-a^2)\tan A=2bc\cos A\tan A=2bc\sin A=4S.$

(42) 令垂心為 H, 則角 AEF=角 AHF=B, (見幾何學) 故於三角形 AEF,

$$\sin A:\sin B=EF:AF=EF:b\cos A$$

$$EF=\frac{b\sin A\cos A}{\sin B}=a\cos A.$$

(43) $2R=\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}$, 故從前例,

$a\cos A+b\cos B+c\cos C=R(2\sin A\cos A+2\sin B\cos B+2\sin C\cos C)$

$=R(\sin 2A+\sin 2B+\sin 2C)=4R\sin A\sin B\sin C$, (例題五 37.).

(44) 由幾何學. $AT=\frac{bc}{a+c}$, $AU=\frac{bc}{a+b}$, 又 $UT=\frac{AU.AT\sin A}{x}$

$UT=\frac{b^2c^2\sin A}{x(a+b)(a+c)}=\frac{bch_1a}{x(a+b)(a+c)}$, $UT^2=AT^2+AU^2-2AT.AU\cos A$

故 $\frac{b^2c^2h_1{}^2a^2}{x^2(a+b)^2(a+c)^2}=\frac{b^2c^2}{(a+c)^2}+\frac{b^2c^2}{(a+b)^2}-\frac{2b^2c^2\cos A}{(a+c)(a+b)}$, 即

$$\left(\frac{h_1}{x}\right)^2=\frac{1}{a^2}\left\{(a+b)^2+(a+c)^2-2(a+b)(a+c)\cos A\right\}$$

$$=\frac{1}{a^2}\left\{(b-c)^2+4(a+b)(a+c)\sin^2\frac{A}{2}\right\}$$

$$=\left(\frac{b-c}{a}\right)^2+4\left(\frac{a+b}{a}\right)\left(\frac{a+c}{a}\right)\sin^2\frac{A}{2}$$

$$=\left(\frac{\sin B-\sin C}{\sin A}\right)^2+4\left(\frac{\sin A+\sin B}{\sin A}\right)\left(\frac{\sin A+\sin C}{\sin A}\right)\sin^2\frac{A}{2}$$

$$=\frac{\sin^2\frac{1}{2}(B-C)}{\cos^2\frac{1}{2}A}+\frac{\sin A(\sin A+\sin B+\sin C)+\sin B\sin C}{\cos^2\frac{1}{2}A}$$

$$=\frac{\frac{1}{2}\{1-\cos(B-C)\}+4\sin A\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C+\frac{1}{2}\{\cos(B-C)-\cos(B+C)\}}{\cos^2\frac{1}{2}A}$$

$$=\frac{\frac{1}{2}(1+\cos A)+8\sin\frac{1}{2}A\cos^2\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}{\cos^2\frac{1}{2}A}$$

$=1+8\sin\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=1+2(1+\cos B+\cos C-\cos A)$, [例題五 34.]

$=3+2(\cos B+\cos C-\cos A)$. 由是

$\left(\frac{h_1}{x}\right)^2+\left(\frac{h_2}{y}\right)^2+\left(\frac{h_3}{z}\right)^2=9+2(\cos A+\cos B+\cos C)$. 由例題五 31. 即得.

45. $\tan \mathrm{TSU}=\tan\left\{2\tan^{-1}\left(\frac{\sec\frac{1}{2}A}{2}\right)\right\}\cos\frac{B-C}{2}.$

46. 從任意一點P, 引三邊a, b, c之垂線x, y, z,

則 $x\sin A+y\sin B+z\sin C=2R\sin A\sin B\sin C.$

47. 同上. P至A, B, C之距離爲h, k, l,

則 $h^2x\sin A+k^2y\sin B+l^2z\sin C=b\,zx+c\,xy+a\,yz.$

48. 同上. 垂線之足爲H, K, L,

則 $b\,zx+c\,xy+a\,yz=4R\times$三角形HKL.

49. 垂足三角形之面積爲$\frac{1}{2}R^2\sin 2A\sin 2B\sin 2C.$

50. 垂心爲H. 則

$$\frac{AH}{\cos BHC}=\frac{BH}{\cos CHA}=\frac{CH}{\cos AHB}.$$

例題解自45.至50.

(45) 角$ASU=\theta$, 角$AST=\phi$, $\sin\theta:\sin AUS=AU:AS.$ 故

$$\frac{\sin\left(\theta+\frac{A}{2}\right)}{\sin\theta}=\frac{AS}{AU}=\frac{2bc\cos\frac{1}{2}A}{b+c}\div\frac{bc}{a+b},$$

卽 $\cos\frac{A}{2}+\cot\theta\sin\frac{A}{2}=\frac{2(a+b)\cos\frac{1}{2}A}{b+c}$, 故 $\tan\theta=\frac{(b+c)\tan\frac{1}{2}A}{2a+b-c}$

$$=\frac{(b+c)/a}{2+(b-c)/a}\tan\frac{A}{2}=\frac{(\sin B+\sin C)\tan\frac{1}{2}A}{2\sin A+\sin B-\sin C}=\frac{\cos\frac{1}{2}(B-C)}{2\cos\frac{1}{2}A+\sin\frac{1}{2}(B-C)}$$

同樣. $\tan\phi=\frac{\cos\frac{1}{2}(B-C)}{2\cos\frac{1}{2}A-\sin\frac{1}{2}(B-C)}$,

$\tan \text{TSU}=\tan(\theta+\phi)=\dfrac{\tan\theta+\tan\phi}{1-\tan\theta\tan\phi}$, 以前所得之值代入而簡單之,

則 $\tan \text{TSU}=\dfrac{4\cos\frac{1}{2}\text{A}\cos\frac{1}{2}(\text{B}-\text{C})}{4\cos^2\frac{1}{2}\text{A}-1}$, 令 $2\tan\dfrac{\chi}{2}=\sec\dfrac{\text{A}}{2}$,

則 $\tan \text{TSU}=\dfrac{8\tan\frac{1}{2}\chi\cos\frac{1}{2}(\text{B}-\text{C})}{4-4\tan^2\frac{1}{2}\chi}=\tan\chi\cos\dfrac{\text{B}-\text{C}}{2}$.

(46) $\dfrac{a}{\sin \text{A}}=\dfrac{b}{\sin \text{B}}=\dfrac{c}{\sin \text{C}}=2\text{R}$,

$ax+by+cz=bc\sin \text{A}$, 即 $2\text{R}(x\sin \text{A}+y\sin \text{B}+z\sin \text{C})$

$=4\text{R}^2\sin \text{B}\sin \text{C}\sin \text{A}$.

(47) 用48.之假定.AKPL在一圓周上, h 爲其直徑,故由2.節 $\text{KL}=h\sin \text{A}$, 由幾何學之定理. $h\cdot\text{KL}=y\cdot\text{AL}+z\cdot\text{AK}$, 即

$h^2\sin \text{A}=y\,\text{AL}+z\,\text{AK}$, $k^2\sin \text{B}=z\,\text{BH}+x\,\text{BL}$, $l^2\sin \text{C}=x\,\text{CK}+y\,\text{CH}$,

由是 $h^2x\sin \text{A}+k^2y\sin \text{B}+l^2z\sin \text{C}$

$=zx(\text{AK}+\text{CK})+xy(\text{AL}+\text{BL})+yz(\text{BH}+\text{CH})=zxb+xyc+yza$.

(48) $bzx=2\text{R}\sin \text{B}zx=2\text{R}zx\sin(180^\circ-\text{B})=2\text{R}zx\sin \text{HPL}$

$=4\text{R}$ 三角形 PHL, $cxy=4\text{R}$ 三角形 PKH, $ayz=4\text{R}$ 三角形 PLK.

以此三式相加,即得.

(49) 令垂足爲D,E,F,垂心爲H,則 $\text{HD}=\text{BD}\cot \text{BHD}=c\cos \text{B}\cot \text{C}$

$=\dfrac{c}{\sin \text{C}}\cos \text{B}\cos \text{C}=2\text{R}\cos \text{B}\cos \text{C}$. $\text{HE}=2\text{R}\cos \text{C}\cos \text{A}$.

$\text{HF}=2\text{R}\cos \text{A}\cos \text{B}$. 令前例之P爲H,則

三角形 $\text{DEF}=\dfrac{b\,\text{HF}.\text{HD}+c\,\text{HD}.\text{HE}+a\,\text{HE}.\text{HF}}{4\text{R}}$

$=\dfrac{4\text{R}^3\cos \text{A}\cos \text{B}\cos \text{C}(\cos \text{B}+\cos \text{C}+\cos \text{A})}{4\text{R}}$

$=4\text{R}^2\sin \text{A}\sin \text{B}\sin \text{C}\cos \text{A}\cos \text{B}\cos \text{C}$.

(50) 角 $\text{BHC}=180^\circ-\text{A}$, $\text{AH}=\dfrac{\text{AE}}{\cos \text{HAE}}=\dfrac{c\cos \text{A}}{\cos(90^\circ-\text{C})}=\dfrac{c\cos \text{A}}{\sin \text{C}}$, 故

$\dfrac{\text{AH}}{\cos \text{A}}=\dfrac{c}{\sin \text{C}}$, 即 $\dfrac{\text{AH}}{\cos \text{BHC}}=\dfrac{c}{\sin \text{C}}$, $\dfrac{\text{BH}}{\cos \text{CHA}}=\dfrac{a}{\sin \text{A}}$.

51. 自垂足D, E, F引各二鄰邊之垂線之六足,在一圓周上,則其圓之半徑爲

$$R(\cos^2 A\cos^2 B\cos^2 C+\sin^2 A\sin^2 B\sin^2 C)^{\frac{1}{2}}$$

52. 三角形之外心爲M,從A通過M引底BC之直線AD,

則 $MD=R\cos A/\cos(B-C)$.

53. 同上. $AD=\dfrac{2R\sin B\sin C}{\cos(B-C)}$.

54. 引長AM, BM, CM,順次截外切圓於D′, E′, F′,截對邊於D, E, F,則

$$\frac{DD'}{AD}+\frac{EE'}{BE}+\frac{FF'}{CF}=1.$$

55. 自M引三邊a, b, c之垂線MP, MQ, MR,

則 $4(MP^2+MQ^2+MR^2)=a^2\cot^2 A+b^2\cot^2 B+c^2\cot^2 C$.

56. $MH=R^2(1-8\cos A\cos B\cos C)$.

例題解自51.至56.

(51) 從D引AC, AB之垂線DG, DK,從E及F引BC之垂線EL及FN,

則 $BN=BF\cos B=a\cos^2 B$, $BK=BD\cos B=c\cos^2 B$,

又 $KN^2=BN^2+BK^2-2BN.BK\cos B=\cos B(a^2+c^2-2ac\cos B)=b^2\cos^4 B$,

故 $KN=b\cos^2 B$. 由此 $BN:KN:BK=a:b:c$,

故三角形BKN與三角形BCA爲相似形,而角BNK=C,

同樣. 三角形CGL, AGK亦與ABC相似,而角CGL=A,

角AGK=B, 故 角$LGK=180°-A-B=C$,

由是 角LGK=角BNK; 故由幾何學. N,L,G,K同在圓周上.

同樣. 從E引AB及從F引AC之垂線之足,亦在圓周上.

又從前之方法. 角 NKG=180°－角 BKN－角 AKG=180°－A－C=B,

$CN=a-BN=a-BF\cos B=a-a\cos^2 B=a\sin^2 B$, $CG=CD\cos C=b\cos^2 C$,

$$NG^2=CN^2+CG^2-2\,CN.CG\cos C=a^2\sin^4 B+b^2\cos^4 C-2ab\sin^2 B\cos^3 C$$

$$=4R^2\sin^2 B(\sin^2 A\sin^2 B+\cos^4 C-2\sin A\sin B\sin C)$$

$$=4R^2\sin^2 B\{\sin^2 A\sin^2 B+\cos^3 C(-\cos(A+B)-2\sin A\sin B)\}$$

$$=4R^2\sin^2 B\{\sin^2 A\sin^2 B-\cos^3 C\cos(A-B)\}$$

$$=4R^2\sin^2 B\{\sin^2 A\sin^2 B+\cos^2 C\cos(A+B)\cos(A-B)\}$$

$$=4R^2\sin^2 B\{\sin^2 A\sin^2 B+\cos^2 A\cos^2 B\cos^2 C-\sin^2 A\sin^2 B\cos^2 C\}$$

$$=4R^2\sin^2 B(\sin^2 A\sin^2 B\sin^2 C+\cos^2 A\cos^2 B\cos^2 C).$$

令所求圓之半徑爲 x, 則三角形 NKG 之外切圓半徑爲 x, 而由 2. 節

$$2x=\frac{NG}{\sin NKG}=\frac{NG}{\sin B}=2R(\sin^2 A\sin^2 B\sin^2 C+\cos^2 A\cos^2 B\cos^2 C)^{\frac{1}{2}}.$$

(52) 角 BMC=2A, 故 角 MCD=$\frac{1}{2}$(180°－角 BMC)=90°－A,

角 AMC=2B, 角 MDC=角 AMC－角 MCD=2B－(90°－A)=90°－(B－C),

由三角形 MDC $\dfrac{MD}{\sin MCD}=\dfrac{MC}{\sin MDC}$, 即 $MD=\dfrac{R\cos A}{\cos(B-C)}$.

(53) $AD=R+MD=R\left\{1+\dfrac{\cos A}{\cos(B-C)}\right\}=\dfrac{2R\sin B\sin C}{\cos(B-C)}$.

(54) $DD'=R-MD=\dfrac{R\{\cos(B-C)+\cos(B+C)\}}{\cos(B-C)}=\dfrac{2R\cos B\cos C}{\cos(B-C)}$,

$\dfrac{DD'}{AD}=\dfrac{\cos B\cos C}{\sin B\sin C}$, 同樣

$$\frac{DD'}{AD}+\frac{EE'}{BE}+\frac{FF'}{CF}=\frac{\sin A\cos B\cos C+\sin B\cos C\cos A+\sin C\cos A\cos B}{\sin A\sin B\sin C}$$

$$=\frac{\cos C\sin(A+B)+\sin C\cos A\cos B}{\sin A\sin B\sin C}=\frac{-\cos(A+B)+\cos A\cos B}{\sin A\sin B}=1.$$

(55) $MP=MC\sin MCP=R\sin(90°-A)=\dfrac{a\cos A}{2\sin A}=\dfrac{1}{2}a\cot A$.

(56) 由例題 8. $AH=2R\cos A$, 角 MAH＝角 CAM－角 CAH

$=90°-B-(90°-C)=C-B$, $MH^2=AM^2+AH^2-2AM.AH\cos MAH$

$=R^2+4R^2\cos^2 A-4R^2\cos A\cos(C-B)$

$=R^2[1-4\cos A\{\cos(C+B)+\cos(C-B)\}]=R^2(1-8\cos A\cos B\cos C)$.

57. 自各頂點引外切圓三切線,成PQR三角形與DEF垂足三角形,則

三角形DEF:三角形PQR$=4\cos^2A\cos^2B\cos^2C:1$.

58. 各角之二等分線,試引長之,令交外切圓,連結此交點之三角形,必比原形小.

59. A,C二角之二等分線,引長之令交外切圓.其交點爲A′,C′,而截BC, BA於P, Q,則

$$A'P:PQ:QC'=\sin^2\tfrac{1}{2}A:2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C:\sin^2\tfrac{1}{2}C.$$

60. 三角形內之一點爲P,P對於BC之角爲θ,PM截外切圓於Q,則

$MQ^2-MP^2=2(\cot A-\cot\theta)$ 三角形PBC.

61. 同上,三角形PBC, PCA, PAB之外切圓,其半徑爲r', r'', r''',則 $\left(\frac{abc}{r'r''r'''}\right)^2=$

$$\left(\frac{a}{r'}+\frac{b}{r''}+\frac{c}{r'''}\right)\left(-\frac{a}{r'}+\frac{b}{r'}+\frac{c}{r'''}\right)\left(\frac{a}{r'}-\frac{b}{r''}+\frac{c}{r'''}\right)\left(\frac{a}{r'}+\frac{b}{r''}-\frac{c}{r'''}\right).$$

例題解自57.至61.

(57) 三角形RAB爲等脚三角形,由幾何學.角RAB=角RBA=C

$AR=\frac{\frac{1}{2}AB}{\cos RAB}=\frac{c}{2\cos C}=\frac{R\sin C}{\cos C}$, $AQ=\frac{R\sin B}{\cos B}$,

$QR=AR+AQ=\frac{R\sin B}{\cos B\cos C}$, $PR=\frac{R\sin B}{\cos C\cos A}$,

三角形PQR $=\frac{1}{2}PR.QR\sin ARB=\frac{R^2\sin A\sin B}{2\cos A\cos B\cos^2C}\sin 2C=\frac{R^2\sin A\sin B\sin C}{\cos A\cos B\cos C}$.

由例題49. 三角形DEF$=\frac{1}{2}R^2\sin 2A\sin 2B\sin 2C$. 由此即得其證.

(58) 由平面幾何學講義錄第四編例題1484.第二之三角形,其面積爲 $Rs/2$, 又 $S=rs=2rs/2$.

但 r 在等邊三角形,則極大爲 $2r=R$,

故 $2r \not> R$, 故第二之三角形比S小.

(59) 從 A' 引 BC 之垂線 $A'D$, 而由幾何學

角 $A'PC=\frac{1}{2}A+\frac{1}{2}C=90^\circ-\frac{1}{2}B$, $A'D=BD\,tan\,A'BD=\frac{1}{2}a\,tan\frac{1}{2}A$.

$$A'P=\frac{A'D}{sin\,A'PD}=\frac{a\,tan\frac{1}{2}A}{2sin(90^\circ-\frac{1}{2}B)}=\frac{a\,tan\frac{1}{2}A}{2cos\frac{1}{2}B}\quad PD=A'P\,cos\,A'PD=\frac{a\,tan\frac{1}{2}A\,sin\frac{1}{2}B}{2cos\frac{1}{2}B},$$

$$BP=BD-PD=\frac{a}{2}\left(1-\frac{tan\frac{1}{2}A\,sin\frac{1}{2}B}{cos\frac{1}{2}B}\right)=\frac{a\,sin\frac{1}{2}C}{2cos\frac{1}{2}A\,cos\frac{1}{2}B},$$

$$\frac{1}{2}PQ=BP\,cos\,BPQ=\frac{a\,sin\frac{1}{2}C\,sin\frac{1}{2}B}{2cos\frac{1}{2}A\,cos\frac{1}{2}B},\quad 又\quad QC'=\frac{c\,tan\frac{1}{2}C}{2cos\frac{1}{2}B},$$

由是 $A'P:PQ:QC'=\frac{a\,tan\frac{1}{2}A}{2cos\frac{1}{2}B}:\frac{a\,sin\frac{1}{2}C\,sin\frac{1}{2}B}{cos\frac{1}{2}A\,cos\frac{1}{2}B}:\frac{c\,tan\frac{1}{2}C}{2cos\frac{1}{2}B}$

$$=sin\tfrac{1}{2}A:2sin\tfrac{1}{2}B\,sin\tfrac{1}{2}C:\frac{c}{a}\,tan\tfrac{1}{2}C\,cos\tfrac{1}{2}A,$$

$$=sin^2\tfrac{1}{2}A:2sin\tfrac{1}{2}A\,sin\tfrac{1}{2}B\,sin\tfrac{1}{2}C:\frac{sin\,C}{sin\,A}\,tan\tfrac{1}{2}C\,sin\tfrac{1}{2}A\,cos\tfrac{1}{2}A.$$

(60) 引長BP,截外切圓於 A', 則

角 $PA'C=A$, 角 $PCA'=\theta-A$, 故於三角形 $PA'C$. $PA'=\frac{PC\,sin(\theta-A)}{sin\,A}$,

又引長QP截外切圓於 Q', 則

$$MQ^2-MP^2=(MQ+MP)(MQ-MP)=PQ.PQ'=PB.PA'$$

$$=\frac{PB.PC\,sin(\theta-A)}{sin\,A}=\frac{PB.PC\,sin\,\theta\,sin(\theta-A)}{sin\,A\,sin\,\theta}=\frac{2\,\text{三角形}\,PBC\times sin(\theta-A)}{sin\,A\,sin\,\theta}.$$

(61) 角 $BPC=\theta$, 角 $CPA=\phi$, 角 $APB=\psi$, 但 $\theta+\phi+\psi=2\pi$,

由2.節 $\frac{a}{r'}=2sin\,\theta$, $\frac{b}{r''}=2sin\,\phi$, $\frac{c}{r'''}=2sin\,\psi$,

$$\frac{abc}{r'r''r'''}=8sin\,\theta\,sin\,\phi\,sin\,\psi,\quad \frac{a}{r'}+\frac{b}{r''}+\frac{c}{r'''}=2(sin\,\theta+sin\,\phi+sin\,\psi),$$

$$=8sin\frac{\theta}{2}sin\frac{\phi}{2}sin\frac{\psi}{2},\quad (\text{例題五}37.)\ \frac{a}{r'}+\frac{b}{r''}-\frac{c}{r'''}=8sin\frac{\theta}{2}cos\frac{\phi}{2}cos\frac{\psi}{2},$$

故同用此法,即可得

$$\left(\frac{a}{r'}+\frac{b}{r''}+\frac{c}{r'''}\right)\left(\frac{a}{r'}+\frac{b}{r''}-\frac{c}{r'''}\right)\left(\frac{a}{r'}-\frac{b}{r''}+\frac{c}{r'''}\right)\left(\frac{b}{r''}+\frac{c}{r'''}-\frac{a}{r'}\right)=\left(\frac{abc}{r'r''r'''}\right)^2.$$

62. 內心爲O,則

$$OA \sin BOC + OB \sin COA + OC \sin AOB = s.$$

63. $OA \cos BOC = OB \cos COA = OC \cos AOB.$

64. PA', PB', PC' 爲 BC, CA, AB 上之垂線,則

三角形 $A'B'C' = \frac{1}{2}(R^2 - PM^2) \sin A \sin B \sin C.$

65. 內切圓三邊之切點爲X, Y, Z,則

三角形 $XYZ = Sr/(2R)$

$$= 2R^2 \sin A \sin B \sin C \sin \frac{A}{2} \sin \frac{B}{2} \sin \frac{C}{2}.$$

66. 內切圓之切點,連結之作一三角形,又於其三角形之內切圓之切點,連結爲三角形,順是以下皆如是,則其三角形,次第近於等邊三角形.

67. 三角形之面積與內切圓之面積之比,若

$$\cot \frac{A}{2} \cot \frac{B}{2} \cot \frac{C}{2} : \pi.$$

68. 從A, B, C引直線.通過內心,而截外切圓於A′, B′, C′,則三角形 BOA′, COB′, AOC′ 之外切圓半徑之積,與三角形 COA′, AOB′, BOC′ 之外切圓半徑之積,均等於 $a^3b^3c^3/\{4r^2(a+b+c)^4\}$.

例題解自62至68.

(62) 令內切圓切於AC之點爲Y,則 $AO \cos OAY = AY$,即

$$AO \cos \frac{A}{2} = AO \cos\left(90° - \frac{B+C}{2}\right) = AO \sin \frac{B+C}{2} = AO \sin\left(180° - \frac{B+C}{2}\right) = AO \sin BOC$$

$= AY = s - a$,同樣. $BO \sin COA = s - b$, $CO \sin AOB = s - c$. 以此加之,即得.

(63) $OA\sin OAY=OY$, 即 $OA\cos BOC=r$, $OB\cos COA=r$.

(64) 引長BP,令截外切圓於Q,則PA′BC′及PA′CB′爲圓之內切四角形,故 角$PA'C'$=角PBA, 角$PA'B'$=角PCA, 故

角$B'A'C'$=角PBA+角PCA, 又於凹四角形ABPC(P在三角形ABC之內)

角A+角PBA+角PCA+360°−角BPC=360°,

即 角A+角$B'A'C'$+360°−角BPC=360°, 故

角$B'A'C'$=角BPC−A=角BPC−角BQC=角PCQ.

2三角形$A'B'C'=A'B'.A'C'\sin B'A'C'=A'B'.A'C'\sin PCQ$,

但 $PA'CB'$在一圓周上, 故 $A'B'=PC\sin C$, 同樣.

$A'C'=PB\sin B$, 故 2三角形$A'B'C'=PB.PC\sin B\sin C\sin PCQ$,

又 三角形PCQ, $PC\sin PCQ=PQ\sin A$,

故 2三角形$A'B'C'=PB.PQ\sin A\sin B\sin C$,

又於等脚三角形MQB,由平面幾何學講義第三編例題880.

$PB.PQ=MQ^2-MP^2=R^2-MP^2$, 故題云云.

(65) 可令前例之P點爲O點,即

三角形$XYZ=\frac{1}{2}(R^2-MO^2)\sin A\sin B\sin C$.

但 $MO^2=R^2-2Rr$, (平面幾何學講義第四編例題1471.)

故 三角形$XYZ=Rr\sin A\sin B\sin C=\frac{a}{2\sin A}\times r\times\frac{ab}{4R^2}\times\sin C$,

但 $2R=\frac{a}{\sin A}=\frac{b}{\sin B}$, 故

三角形$XYZ=\frac{ar}{2}\times\frac{b}{2R}\sin C=\frac{rS}{2R}$, 以下省畧.

(66) X, Y, Z之各角,順次命爲$90°-\frac{A}{2}$, $90°-\frac{B}{2}$, $90°-\frac{C}{2}$, 又第二三角形之各角爲$90°-\frac{X}{2}$, $90°-\frac{Y}{2}$, $90°-\frac{Z}{2}$,

此可從次第相等之極限證之.

(67) 三角形ABC:內切圓$=rs:r^2\pi=s:r\pi=s/r:\pi$,

但 $\frac{s}{r}=\frac{(s-a)+(s-b)+(s-c)}{r}=\frac{s-a}{r}+\frac{s-b}{r}+\frac{s-c}{r}$

$=\cot\frac{A}{2}+\cot\frac{B}{2}+\cot\frac{C}{2}=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}$.

(68) 可從平面幾何學之講義第四編1481.而得其證.

但用三角函數而求其證,更爲簡便.

69. 內切圓切AC之點為Y，則

$$AY = R(\sin B + \sin C - \sin A).$$

70. 三角形 $OMH = 2R^2 \sin\frac{1}{2}(B-C)\sin\frac{1}{2}(C-A)\sin\frac{1}{2}(A-B)$.

但H為垂心.

71. $OH = 2R\sqrt{\left(8\sin^2\frac{A}{2}\sin^2\frac{B}{2}\sin^2\frac{C}{2} - \cos A\cos B\cos C\right)}$.

72. p_1, p_2, p_3 為三角形MBC, MCA, MAB內切圓之半徑，則

$4R^2(p_1+p_2+p_3)+2R(ap_1+bp_2+cp_3)=abc$.

73. 自O至A, B, C之距離，順次令為 d, e, f，則

$$d^2\left(\frac{1}{b}-\frac{1}{c}\right)+e^2\left(\frac{1}{c}-\frac{1}{a}\right)+f^2\left(\frac{1}{a}-\frac{1}{b}\right)=0.$$

74. $\dfrac{\cos\frac{1}{2}A}{d}+\dfrac{\cos\frac{1}{2}B}{e}+\dfrac{\cos\frac{1}{2}C}{f}=s\left(\dfrac{1}{bc}+\dfrac{1}{ca}+\dfrac{1}{ab}\right)$.

例題解自69.至74.

(69) $AY=\frac{1}{2}(b+c-a)=\frac{1}{2}(2R\sin B+2R\sin C-2R\sin A)$.

(70) 從O, M, H, 引BC上之垂線OX, MP, HD, 則

$MP=R\cos A,\ OX=r=\frac{1}{2}(b+c-a)\tan\frac{1}{2}A=R(\sin B+\sin C-\sin A)\tan\frac{1}{2}A$

$=4R\sin\frac{1}{2}A\sin\frac{1}{2}B\sin\frac{1}{2}C$ (例題五33.) $=R(\cos A+\cos B+\cos C-1)$, [例題五31.]

$HD=BD\tan DBH=c\cos B\tan(90^\circ-C)=2R\sin C\cos B\cot C=2R\cos B\cos C$,

$PX=BX-BP=\frac{1}{2}(a+c-b)-\frac{1}{2}a=\frac{1}{2}(c-b)=R(\sin C-\sin B)$,

$XD=CX-CD=\frac{1}{2}(a+b-c)-b\cos C=R(\sin A+\sin B-\sin C-2\sin B\cos C)$,

三角形OMH=四角形OMPX+四角形OXDH−四角形MPDH

$=\frac{1}{2}\{PX(MP+OX)+XD(OX+HD)-(PX+XD)(MP+HD)\}$

$=\frac{1}{2}\{PX(OX-HD)+XD(OX-MP)\}$

$=\frac{1}{2}R^2\{(\sin C-\sin B)(\cos A+\cos B+\cos C-1-2\cos B\cos C)$

$+(\sin A+\sin B-\sin C-2\sin B\cos C)(\cos A+\cos B+\cos C-1-\cos A)\}$.

$$=\tfrac{1}{2}R^2[(\sin C-\sin B)\{2\cos A-\cos(C-B)\}+\sin(C-B)(\cos B+\cos C-1)]$$

$$=R^2\sin\frac{C-B}{2}\left[\sin\frac{A}{2}\{2\cos A-\cos(C-B)\}+\cos\frac{C-B}{2}(\cos B+\cos C-1)\right]$$

$$=R^2\sin\frac{C-B}{2}\left[2\sin\frac{A}{2}\cos A+\sin\frac{A}{2}-\cos\frac{C-B}{2}\right]$$

$$=R^2\sin\frac{C-B}{2}\left[\sin\frac{3A}{2}-\cos\frac{C-B}{2}\right]=R^2\sin\frac{C-B}{2}\left[\cos\frac{B+C-2A}{2}-\cos\frac{C-B}{2}\right]$$

$$=2R^2\sin\frac{C-B}{2}\sin\frac{A-B}{2}\sin\frac{C-A}{2}.$$

(71) 角 OAH = 角 BAH − 角 BAO $=90°-B-\tfrac{1}{2}A=\tfrac{1}{2}(C-B)$, $AH=2R\cos A$,

$OA=\dfrac{r}{\sin\frac{1}{2}A}=4R\sin\dfrac{B}{2}\sin\dfrac{C}{2}$, $OH^2=OA^2+AH^2-2AH.OA\cos OAH$

$$=4R^2\left\{4\sin^2\frac{B}{2}\sin^2\frac{C}{2}+\cos^2 A-4\sin\frac{B}{2}\sin\frac{C}{2}\cos A\cos\frac{C-B}{2}\right\}$$

$$=4R^2\left\{4\sin^2\frac{B}{2}\sin^2\frac{C}{2}(1-\cos A)+\cos^2 A-\cos A\sin B\sin C\right\}$$

$$=4R^2\left(8\sin^2\frac{B}{2}\sin^2\frac{C}{2}\sin^2\frac{A}{2}-\cos A\cos B\cos C\right).$$

(72) $p_1(BC+MB+MC)=MP.BC$, 即 $p_1(a+2R)=R\cos A.a$,

同樣. $p_2(b+2R)=R\cos B.b$, $p_3(c+2R)=R\cos C.c$, 故

$4R^2(p_1+p_2+p_3)+2R(ap_1+bp_2+cp_3)=2R^2(a\cos A+b\cos B+c\cos C)$

$$=\frac{2R^2\{a^2(b^2+c^2-a^2)+b^2(c^2+a^2-b^2)+c^2(a^2+b^2-c^2)\}}{2abc}$$

$$=\frac{16R^2S^2}{abc}=\frac{16R^2}{abc}\times\left(\frac{abc}{4R}\right)^2=abc.$$

(73) $OA=\dfrac{r}{\sin\frac{1}{2}A}$. 即 $d=\dfrac{r\sqrt{bc}}{\sqrt{s_2s_3}}$, $d^2\left(\dfrac{1}{b}-\dfrac{1}{c}\right)=\dfrac{r^2(c-b)}{s_2s_3}$,

故 $d^2\left(\dfrac{1}{b}-\dfrac{1}{c}\right)+e^2\left(\dfrac{1}{c}-\dfrac{1}{a}\right)+f^2\left(\dfrac{1}{a}-\dfrac{1}{b}\right)=\dfrac{r^2}{s_1s_2s_3}\left\{s_1(c-b)+s_2(a-c)+s_3(b-a)\right\}$

$$=\frac{r^2}{2s_1s_2s_3}\left\{(c^2-b^2)-a(c-b)+(a^2-c^2)-b(a-c)+(b^2-a^2)-c(b-a)\right\}=0.$$

(74) $\dfrac{\cos\frac{1}{2}A}{d}=\dfrac{\sin\frac{1}{2}A\cos\frac{1}{2}A}{r}=\dfrac{\sin A}{2r}$, 故

$$\frac{\cos\frac{1}{2}A}{d}+\frac{\cos\frac{1}{2}B}{e}+\frac{\cos\frac{1}{2}C}{f}=\frac{1}{2r}(\sin A+\sin B+\sin C)=\frac{2\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}}{r}$$

$$=\frac{2s\sqrt{s\,s_1s_2s_3}}{abcr}=\frac{2s^2S}{abc\,sr}=\frac{2s^2}{abc}=s\left(\frac{a+b+c}{abc}\right).$$

75. 三角形OBC, OCA, OBA.其內切圓之半徑,順次爲r_a, r_b, r_c,

則 $\frac{a}{r_a}+\frac{b}{r_b}+\frac{c}{r_c}=2\left(\cot\frac{A}{4}+\cot\frac{B}{4}+\cot\frac{C}{4}\right)$.

76. PA′, PB′, PC′ 爲三邊上之垂線,則連結三角形 PA′B′, PB′C′, PC′A′ 外切圓之中心所成三角形之面積,等於原形四分之一.

77. 引長銳角三角形之三垂線,交於外切圓.令其引長之部分爲α, β, γ,則 $\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=2(\tan A+\tan B+\tan C)$.

78. 從O對向於BC, CA, AB之角爲θ, ϕ, ψ,則

$$4\sin\theta\sin\phi\sin\psi=\sin A+\sin B+\sin C.$$

79. 內切圓與外切圓直徑之和,爲$a\cot A+b\cot B+c\cot C$.

80. G爲重心,則$OG^2=\frac{4}{9}R^2(1+\cos A\cos B\cos C)-\frac{4}{3}Rr+\frac{2}{3}r^2$.

例題解自 75. 至 80.

(75) 令r_a圓切BC之點爲K,則角OBC=$\frac{1}{2}$B, 角OCB=$\frac{1}{2}$C, 故

BK=$r_a\cot\frac{1}{4}B$, CK=$r_a\cot\frac{1}{4}C$, 故 $a=r_a(\cot\frac{1}{4}B+\cot\frac{1}{4}C)$, $\frac{a}{r_a}=\cot\frac{B}{4}+\cot\frac{C}{4}$,

$\frac{b}{r_b}=\cot\frac{C}{4}+\cot\frac{A}{4}$, $\frac{c}{r_c}=\cot\frac{A}{4}+\cot\frac{B}{4}$, 從此可得其證.

(76) B′, C′爲直角,故PB′C′之外切圓通過A, PA爲直徑,故令其中心爲O_1,則O_1爲PA之中央點, 同樣. PB′A′外切圓之中心O_2爲PC之中央點,又PA′C′外切圓之中心O_3爲PB之中央點,故三角形$O_1O_2O_3$之三邊,等於三角形ABC之三邊之半,由是其面積等於ABC四分之一.

(77) AD.a=BD.CD, 卽 $\frac{bc\sin A}{a}a=c\cos B.b\cos C$, 故

$\frac{a}{\alpha}=\frac{\sin A}{\cos B\cos C}$, 故 $\frac{a}{\alpha}+\frac{b}{\beta}+\frac{c}{\gamma}=\frac{\sin A}{\cos B\cos C}+\frac{\sin B}{\cos C\cos A}+\frac{\sin C}{\cos A\cos B}$

$=\frac{\sin A\cos A+\sin B\cos B+\sin C\cos C}{\cos A\cos B\cos C}=\frac{\sin 2A+\sin 2B+\sin 2C}{2\cos A\cos B\cos C}$

$=\frac{4\sin A\sin B\sin C}{2\cos A\cos B\cos C}$, 〔例題五 37.〕$=2\tan A\tan B\tan C$

$=2(\tan A+\tan B+\tan C)$, 〔例題五 35.〕.

(78) $\theta=90^\circ+\frac{A}{2}$, $\phi=90^\circ+\frac{B}{2}$, $\psi=90^\circ+\frac{C}{2}$, 故

$4\sin\theta\sin\phi\sin\psi=4\cos\frac{A}{2}\cos\frac{B}{2}\cos\frac{C}{2}=\sin A+\sin B+\sin C.$

(79) $2R=\frac{a}{\sin A}$, $\gamma=\frac{S}{s}$, 由此

$2R+2r=\frac{a}{\sin A}+\frac{2S}{s}=\frac{a}{\sin A}\left(1+\frac{2S\sin A}{as}\right)=\frac{a}{\sin A}\left(1+\frac{2Sbc\sin A}{abcs}\right)$

$=\frac{a}{\sin A}\left(1+\frac{4S^2}{abcs}\right)=\frac{a}{\sin A}\left(1+\frac{4ss_1s_2s_3}{abcs}\right)$

$=\frac{a}{\sin A}\left(1+4\sqrt{\frac{s_2s_3}{bc}}\sqrt{\frac{s_3s_1}{ca}}\sqrt{\frac{s_1s_2}{ab}}\right)=\frac{a}{\sin A}\left(1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\right)$

$=\frac{a}{\sin A}(\cos A+\cos B+\cos C)=a\cot A+\frac{a}{\sin A}\cos B+\frac{a}{\sin A}\cos C$

$=a\cot A+\frac{b}{\sin B}\cos B+\frac{c}{\sin C}\cos C=a\cot A+b\cot B+c\cot C.$

(80) AP 爲中央線, AD 爲垂線, 從 O, G 引 BC 之垂線 OX, GL.

則 $GL=\frac{AD}{3}=\frac{2rs}{3a}$, $PD=CP-CD=\frac{a}{2}-b\cos C=\frac{a}{2}-\frac{a^2+b^2-c^2}{2a}=\frac{c^2-b^2}{2a}$

$PL=\frac{PD}{3}=\frac{c^2-b^2}{6a}$, $BX=\frac{a+c-b}{2}$, $PX=BX-\frac{a}{2}=\frac{c-b}{2}$

$LX=PX-PL=\frac{(c-b)(3a-b-c)}{6a}$, 故 $OG^2=(OX-GL)^2+LX^2$

$=\left(r-\frac{2rs}{3a}\right)^2+\frac{(c-b)^2(3a-b-c)^2}{36a^2}=\frac{r^2}{9a^2}(2a-b-c)^2+\frac{1}{36a^2}(c-b)^2(3a-b-c)^2$

$=\frac{a^2+b^2+c^2}{18}+\frac{4s_1s_2s_3}{3s}-\frac{abc}{3s}=\frac{4R^2}{18}(\sin^2A+\sin^2B+\sin^2C)+\frac{4S^2}{3s^2}-\frac{4RS}{3S/r}$

$=\frac{4RS^2}{9}(1+\cos A\cos B\cos C)+\frac{4}{3}r^2-\frac{4Rr}{3}$, (例題五 61.)

81. BC, CA, AB 之傍切圓. 設其中心爲 O_1, O_2, O_3, 則

$$\frac{OO_1}{\sin\frac{1}{2}A}=\frac{OO_2}{\sin\frac{1}{2}B}=\frac{OO_3}{\sin\frac{1}{2}C}.$$

82. 傍心至內心之距離, 爲

$$a\sec\frac{A}{2},\quad b\sec\frac{B}{2},\quad c\sec\frac{C}{2}.$$

83. $OO_1=2\sqrt{R(r_1-r)}.$

84. $AO.AO_1=bc.$

85. $r.OO_1.OO_2.OO_3=\left(\frac{abc}{s}\right)^2.$

86. $\frac{a^2OO_1^2(b^2-c^2)}{\sin^2\frac{1}{2}A}+\frac{b^2OO_2^2(c^2-a^2)}{\sin^2\frac{1}{2}B}+\frac{c^2OO_3^2(a^2-b^2)}{\sin^2\frac{1}{2}C}=0.$

87. $O_1O_2.O_2O_3.O_3O_1=8r_1r_2r_3\csc A\csc B\csc C.$

88. $\tan OMO_1=\frac{2(\sin B-\sin C)}{2\cos A-1}.$

例題解自 81. 至 88.

(81) O 及 O_1 爲中心之圓, 切 AC 及所引長之邊上之點爲 Y 及 Y_1, 則

$AO_1=AY_1/\cos\frac{A}{2}=s/\cos\frac{A}{2},\quad AO=AY/\cos\frac{A}{2}=(s-a)/\cos\frac{A}{2},$

故 $OO_1=AO_1-AO=a/\cos\frac{A}{2}$, 卽 $OO_1/\sin\frac{A}{2}=\frac{2a}{\sin A}$,

同樣. $OO_2/\sin\frac{B}{2}=2b/\sin B,\quad OO_3/\sin\frac{C}{2}=2c/\sin C.$

(82) 從前例, 得 $OO_1=a\sec\frac{A}{2}$.

(83) $OO_1=\frac{O_1Y_1-OY}{\sin\frac{A}{2}}=\frac{r_1-r}{\sin\frac{A}{2}}$, 由前例 $OO_1^2=\frac{r_1-r}{\sin\frac{A}{2}}\times\frac{A}{\cos\frac{A}{2}}$

$=\frac{2a(r_1-r)}{\sin A}=\frac{4R\sin A(r_1-r)}{\sin A}=4R(r_1-r).$

(84) $AO.AO_1=\frac{s-a}{\cos\frac{A}{2}}\times\frac{s}{\cos\frac{A}{2}}=\frac{s(s-a)}{s(s-a)/(bc)}=bc.$

(85) $r\,OO_1.OO_2.OO_3=\frac{S}{s}\times\frac{abc}{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$

$$=\frac{abcS}{s\sqrt{\left\{\frac{ss_1}{bc}\times\frac{ss_2}{ca}\times\frac{ss_3}{ab}\right\}}}=\frac{a^2b^2c^2S}{s^2\sqrt{(ss_1s_2s_3)}}=(abc/s)^2.$$

(86) 令 $\frac{a}{\sin A}=\frac{b}{\sin B}=\frac{c}{\sin C}=k$, 則

$$\frac{a^2OO_1^2(b^2-c^2)}{\sin^2\frac{1}{2}A}=\frac{a^4(b^2-c^2)}{\sin^2\frac{1}{2}A\cos^2\frac{1}{2}A}=\frac{4a^4(b^2-c^2)}{\sin^2 A}=4k^2a^2(b^2-c^2),$$ 故

$$\frac{a^2OO_1^2(b^2-c^2)}{\sin^2\frac{1}{2}A}+\frac{b^2OO_2^2(c^2-a^2)}{\sin^2\frac{1}{2}B}+\frac{c^2OO_3^2(a^2-b^2)}{\sin^2\frac{1}{2}C}$$

$$=4k^2\{a^2(b^2-c^2)+b^2(c^2-a^2)+c^2(a^2-b^2)\}=0.$$

(87) 角 $O_1AO_2=90°$, 故 $O_1O_2=\frac{AO_1}{\sin O_2}=\frac{s/\cos\frac{1}{2}A}{\sin\frac{1}{2}(A+C)}=\frac{s}{\cos\frac{1}{2}A\cos\frac{1}{2}B}$,

故 $O_1O_2.O_2O_3.O_3O_1=\frac{s^3}{(\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C)^2}=\frac{s\,a^2b^2c^2}{s\,s_1s_2s_3}=\frac{s\,a^2b^2c^2}{S^2}$,

但 $rr_1r_2r_3=S^2$,〔例題二十三 4.〕 故 $r_1r_2r_3=S\left(\frac{S}{r}\right)=Ss$,

由是 $O_1O_2.O_2O_3.O_3O_1=\frac{Ss\,a^2b^2c^2}{S^3}=\frac{r_1r_2r_3a^2b^2c^2}{S.S.S}$

$$=\frac{r_1r_2r_3a^2b^2c^2}{\frac{1}{2}bc\sin A.\frac{1}{2}ca\sin B.\frac{1}{2}ab\sin C}=\frac{8r_1r_2r_3}{\sin A\sin B\sin C}.$$

(88) 由平面幾何學講義第四編例題 1471. 及 1472.

$OM^2=R^2-2Rr$, $O_1M^2=R^2+2Rr_1$, $OO_1^2=4R(r_1-r)$, 〔例題 83.〕

$$\cos OMO_1=\frac{OM^2+O_1M^2-OO_1^2}{2OM.O_1M}=\frac{(R^2-2Rr)+(R^2+2Rr_1)-4R(r_2-r)}{2\sqrt{\{(R^2-2Rr)(R^2+2Rr_1)\}}}$$

$$=\frac{R-(r_1-r)}{\sqrt{\{(R-2r)(R+2r_1)\}}},\quad \sin OMO_1=\frac{\sqrt{4R(r_1-r)-(r+r_1)^2}}{\sqrt{\{(R-2r)(R+2r_1)\}}},$$

$\tan OMO_1=\frac{\sqrt{4R(r_1-r)-(r_1+r)^2}}{R-(r_1-r)}$, $R=\frac{abc}{4S}$, $r_1=\frac{S}{s_1}$, $r=\frac{S}{s}$,

及 $S=\sqrt{ss_1s_2s_3}$, 代入而變化之, 則

$$\tan OMO_1=\frac{4(b-c)S}{a(b^2+c^2-a^2-bc)}=\frac{4(c\sin B/\sin C-c)\frac{1}{2}ab\sin C}{a(2bc\cos A-bc)}.$$

89. $a(s^2-AX_1^2)=4ss_2s_3$. 但 X_1 爲 r_1 傍切圓切於BC之點.

90. 三角形 $O_1O_2O_3=\dfrac{s^2}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$

$$=\frac{2RS}{r}=\frac{8sR^2S}{abc}$$

$$=\frac{abc}{2}\left\{\left(\frac{1}{a}+\frac{1}{b}\right)\tan\frac{C}{2}+\left(\frac{1}{b}+\frac{1}{c}\right)\tan\frac{A}{2}+\left(\frac{1}{c}+\frac{1}{a}\right)\tan\frac{B}{2}\right\}.$$

91. 於銳角三角形AM之引長線,交BC於D,則

$$DM\cos(B-C)=AM\cos A.$$

92. $\dfrac{\text{三角形 BOC}}{\text{三角形 }BO_1C}+\dfrac{\text{三角形 COA}}{\text{三角形 }CO_2A}+\dfrac{\text{三角形 AOB}}{\text{三角形 }AO_3B}=1.$

93. $\dfrac{a}{\text{三角形 }BO_1C}+\dfrac{b}{\text{三角形 }CO_2A}+\dfrac{c}{\text{三角形 }AO_3B}=\dfrac{2}{r}.$

94. 三角形 $OO_1O_2=2Rs_3=\dfrac{abc}{2s}\cot\dfrac{C}{2}.$

例 題 解 自 89. 至 94.

(89) $BX_1=s-c$, 故 $AX_1^2=AB^2+BX_1^2-2AB.BX_1\cos B$

$$=c^2+(s-c)^2-2(s-c)c\times\frac{c^2+a^2-b^2}{2ca}=s^2-2c(s-c)-\frac{(s-c)(a^2+c^2-b^2)}{a},$$

故 $a(s^2-AX_1^2)=(s-c)\{2ca+a^2+c^2-b^2\}=4(s-c)s(s-b)$.

(90) $O_1A=\dfrac{s}{\cos\frac{1}{2}A}$, (例題81.解), $O_2O_3=\dfrac{s}{\cos\frac{1}{2}B\cos\frac{1}{2}C}$, (例題87.)

三角形 $O_1O_2O_3=\dfrac{O_1A\times O_2O_3}{2}=\dfrac{s^2}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$

$$=\frac{s^2}{2\sqrt{\left(\frac{ss_1}{bc}\times\frac{ss_2}{ca}\times\frac{ss_3}{ab}\right)}}=\frac{abcs}{2S}=\frac{4RSs}{2S}=2Rs=\frac{2RS}{r}$$

$$=\frac{abc}{2}\left(\frac{s}{S}\right)=\frac{abc}{2}\left(\frac{s_1+s_2+s_3}{S}\right)=\frac{abc}{2}\left(\frac{s_1}{S}+\frac{s_2}{S}+\frac{s_3}{S}\right).$$

但 $\frac{s_1}{S}=\frac{s_1}{\sqrt{ss_1s_2s_3}}=\frac{\sqrt{ss_1}}{\sqrt{ss_2}\sqrt{ss_3}}=\frac{\sqrt{ss_1/(bc)}}{a\sqrt{ss_2/(ca)}\sqrt{ss_3/(ab)}}=\frac{\cos\frac{1}{2}A}{a\cos\frac{1}{2}B\cos\frac{1}{2}C}$

$=\frac{\sin(\frac{1}{2}B+\frac{1}{2}C)}{a\cos\frac{1}{2}B\cos\frac{1}{2}C}=\frac{1}{a}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)$, 由此

$$\frac{s_1}{S}+\frac{s_2}{S}+\frac{s_3}{S}=\frac{1}{a}\left(\tan\frac{B}{2}+\tan\frac{C}{2}\right)+\frac{1}{b}\left(\tan\frac{C}{2}+\tan\frac{A}{2}\right)+\frac{1}{c}\left(\tan\frac{A}{2}+\tan\frac{B}{2}\right)$$

$$=\left(\frac{1}{a}+\frac{1}{b}\right)\tan\frac{C}{2}+\left(\frac{1}{b}+\frac{1}{c}\right)\tan\frac{A}{2}+\left(\frac{1}{c}+\frac{1}{a}\right)\tan\frac{B}{2}.$$

(91) 角 MBD = 角 MCD = $\frac{1}{2}$(180°−角 BMC) = $\frac{1}{2}$(180°−2A) = 90°−A,

角 MDB = 角 C + 角 CAM = C+$\frac{1}{2}$(180°−角 AMC) = C+$\frac{1}{2}$(180°−2B)

= 90°−(B−C), $\sin$ MDB : $\sin$ MBD = BM : DM, 即

$\sin\{90°-(B-C)\}:\sin(90°-A)=AM:DM.$

(92) $\frac{\text{三角形 BOC}}{\text{三角形 BO}_1\text{C}}=\frac{r}{r_1}=\frac{s_1r}{S}$, 由此

$$\frac{\text{三角形 BOC}}{\text{三角形 BO}_1\text{C}}+\frac{\text{三角形 COA}}{\text{三角形 CO}_2\text{A}}+\frac{\text{三角形 AOB}}{\text{三角形 AO}_3\text{B}}=\frac{r(s_1+s_2+s_3)}{S}=\frac{rs}{S}=\frac{S}{S}.$$

(93) 三角形 $BO_1C=\frac{1}{2}ar_1=\frac{1}{2}a\times\frac{S}{s_1}$, 故 $\frac{a}{\text{三角形 BO}_1\text{C}}=\frac{2s_1}{S}$,

故 $\frac{a}{\text{三角形 BO}_1\text{C}}+\frac{b}{\text{三角形 CO}_2\text{A}}+\frac{c}{\text{三角形 AO}_3\text{B}}=\frac{2(s_1+s_2+s_3)}{S}=\frac{2s}{S}=\frac{2}{r}.$

(94) 三角形 $OO_1O_2=\frac{1}{2}CO\times O_1O_2=\frac{1}{2}\times\frac{s_3}{\cos\frac{1}{2}C}\times\frac{s}{\cos\frac{1}{2}A\cos\frac{1}{2}B}$

$$=\frac{ss_3}{2\sqrt{\frac{ss_3}{ab}}\sqrt{\frac{ss_1}{bc}}\sqrt{\frac{ss_2}{ca}}}=\frac{abcs_3}{2\sqrt{ss_1s_2s_3}}=\frac{abcs_3}{2S}=\frac{4RSs_3}{2S}=2Rs_3$$

$$=\frac{abc}{2s}\left(\frac{ss_3}{S}\right)=\frac{abc}{2s}\sqrt{\frac{ss_3}{s_1s_2}}=\frac{abc}{2s}\cot\frac{C}{2}.$$

95. $HO_1^2=4R^2\left(8\sin^2\frac{A}{2}\cos^2\frac{B}{2}\cos^2\frac{C}{2}-\cos A\cos B\cos C\right).$

96. r_1, r_2, r_3, 爲調和級數, r_1, r_2, R爲等比級數,

則 $\cos B=8/9,\quad R=9r,\quad \tan\frac{A}{2}\tan\frac{C}{2}=\frac{1}{3}.$

97. 三邊爲等差級數,其中項之邊之傍切圓之半徑,等於內切圓半徑之三倍.

98. 有兩個相似之三角形,其一個三角形之三邊,爲 a, b, c. 其他與此相應之三邊,順次爲 a_1, b_1, c_1. 設 a 及 b 之傍切圓相等,

則 $$\frac{a}{a_1}=\frac{\sin B+\sin C-\sin A}{\sin A+\sin C-\sin B}.$$

99. 三邊爲等比級數,其三垂線所成之三角形,與原三角形,相似.

100. 三角形 $O_1O_2O_3$. 其內切圓之半徑爲 r', 則

$$r':r=\cot\frac{A}{2}\cot\frac{B}{2}\cot\frac{C}{2}:\cos\frac{A}{2}+\cos\frac{B}{2}+\cos\frac{C}{2}.$$

例題解自 95. 至 100.

(95) $AO_1=\frac{s}{\cos\frac{1}{2}A}=\frac{a+b+c}{2\cos\frac{1}{2}A}=\frac{2R(\sin A+\sin B+\sin C)}{2\cos\frac{1}{2}A}=4R\cos\frac{B}{2}\cos\frac{C}{2}$

$AH=2R\cos A$, 角 $HAO_1=\frac{C-B}{2}$, $O_1H^2=AO_1^2+AH^2-2AO_1.AH\cos\frac{C-B}{2}$

$=4R^2\left(4\cos^2\frac{B}{2}\cos^2\frac{C}{2}+\cos^2 A-4\cos\frac{B}{2}\cos\frac{C}{2}\cos A\cos\frac{C-B}{2}\right)$

$=4R^2\left\{4\cos\frac{B}{2}\cos\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2}-\cos A\cos\frac{C-B}{2}\right)-\cos A\cos(B+C)\right\}$

$=4R^2\left\{4\cos\frac{B}{2}\cos\frac{C}{2}\left(\cos\frac{B}{2}\cos\frac{C}{2}-\cos A\cos\frac{C-B}{2}+\cos A\sin\frac{B}{2}\sin\frac{C}{2}\right)-\cos A\cos B\cos C\right\}$

$=4R^2\left\{4\cos\frac{B}{2}\cos\frac{C}{2}(1-\cos A)-\cos A\cos B\cos C\right\}.$

(96) $\frac{2}{r_2}=\frac{1}{r_1}+\frac{1}{r_3}$, 即 $\frac{2s_2}{S}=\frac{s_1}{S}+\frac{s_3}{S}$, 即 $2s_2=s_1+s_3$; 故 $b=\frac{a+c}{2}$.

故 $s=\frac{3b}{2}$, 又由 $b=\frac{a+c}{2}$. $\sin B=\frac{\sin A+\sin C}{2}$. 試將此變化之.

$\cos\frac{A-C}{2}=2\sin\frac{B}{2}$, 又 $r_2^2=rR$, 卽 $\frac{S^2}{s_2^2}=\frac{S}{s}\times\frac{abc}{4S}$, 卽

$\frac{a^2c^2\sin^2B}{4s_2^2}=\frac{abc}{4s}$, $\frac{ac\sin^2B}{b^2}=\frac{b}{6b}$, 即 $ac\times\frac{\sin A}{a}\times\frac{\sin C}{c}=\frac{1}{6}$.

故 $\sin A\sin C=\frac{1}{6}$, 卽 $\cos(A-C)-\cos(A+C)=\frac{1}{3}$, 卽

$\cos(A-C)+\cos B=\frac{1}{6}$, 卽 $2\cos^2\frac{A-C}{2}-1+\cos B=\frac{1}{3}$, 故

$2\left(2\sin\frac{B}{2}\right)^2-1+\cos B=\frac{1}{3}$, 從此得 $\cos B=\frac{8}{9}$,

$2R=\frac{b}{\sin B}$, 故 $r=\frac{S}{s}=\frac{2S}{3b}=\frac{ac\sin B}{3b}=\frac{b}{18\sin B}=\frac{2R}{18}$,

又 $\cos\frac{A-C}{2}=2\sin\frac{B}{2}=2\sqrt{\frac{1-\cos B}{2}}=\frac{2}{3\sqrt{2}}$, 卽

$\cos\frac{A}{2}\cos\frac{C}{2}+\sin\frac{A}{2}\sin\frac{C}{2}=\frac{2}{3\sqrt{2}}$, 又因 $\sin A\sin C=1/6$.

$\sin\frac{A}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{C}{2}=\frac{1}{24}$, 從此得 $\cos\frac{A}{2}\cos\frac{C}{2}-\sin\frac{A}{2}\sin\frac{C}{2}=\frac{1}{3\sqrt{2}}$.

由是 $\cos\frac{A}{2}\cos\frac{C}{2}=\frac{1}{2\sqrt{2}}$, 故 $\frac{\sin\frac{A}{2}\sin\frac{C}{2}\cos\frac{A}{2}\cos\frac{C}{2}}{\left(\cos\frac{A}{2}\cos\frac{C}{2}\right)^2}=\frac{\frac{1}{24}}{\left(\frac{1}{2\sqrt{2}}\right)^2}=\frac{1}{3}$.

(97) 由 $b=\frac{a+c}{2}$ 得 $s=3b/2$, $s_1=b/2$, 故 $r_1=\frac{S}{s_1}=\frac{2S}{b}=\frac{3S}{s}=3r$.

(98) 依題意. $\frac{S}{S_1}=\frac{b+c-a}{c_1+a_1-b_1}=\frac{2R(\sin B+\sin C-\sin A)}{2R_1(\sin C+\sin A-\sin B)}$,

又 $\frac{S}{S_1}=\frac{a^2}{a_1^2}=\frac{R^2}{R_1^2}$, 故 $\frac{a^2}{a_1^2}=\frac{a(\sin B+\sin C-\sin A)}{a_1(\sin C+\sin A-\sin B)}$.

(99) $ar^2=br=c$, 但 r 爲通比, 又 $ah_1=bh_2=ch_3$, 故

$ah_1=arh_2=ar^2h_3$. 卽 $h_3r^2=h_2r=h_1$, 由此

$\frac{a}{h_3}=\frac{b}{h_2}=\frac{c}{h_1}$. 其兩三角形之各邊有比例, 故爲相似形.

(100) $\frac{1}{2}r'(O_1O_2+O_2O_3+O_3O_1)=\frac{s^2}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$, (例題 90.)

卽 $r\left(\frac{s}{\cos\frac{1}{2}A\cos\frac{1}{2}B}+\frac{s}{\cos\frac{1}{2}B\cos\frac{1}{2}C}+\frac{s}{\cos\frac{1}{2}C\cos\frac{1}{2}A}\right)=\frac{s^2}{\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$, [例題 87.]

即 $r'(\cos\frac{1}{2}A+\cos\frac{1}{2}B+\cos\frac{1}{2}C)=s$, 但

$r\cot\frac{1}{2}A\cot\frac{1}{2}B\cot\frac{1}{2}C=r\sqrt{\left(\frac{ss_3}{s_1s_2}\times\frac{ss_1}{s_2s_3}\times\frac{ss_2}{s_3s_1}\right)}=\frac{rs^2}{S}=\frac{rs^2}{rs}=s$.

101. 同上. 三角形 $O_1O_2O_3$ 之周邊爲 $2s'$, 則

$$rs : r's' = 2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2} : 1.$$

102. 連結各傍切圓之切點, 成三個三角形. 由此三個三角形之和, 減連結內切圓切點之三角形, 則等於原形二倍.

103. 三角形之面積, 爲切於三邊四圓中之一圓, 連結其切點所成之三角形, 與連結其餘三圓中心之三角形之比例中項.

104. 三角形 $O_1O_2O_3$ 之中心至傍心之距離, 爲

$$8R\sin\tfrac{1}{4}(B+C),\quad 8R\sin\tfrac{1}{4}(C+A),\quad 8R\sin\tfrac{1}{4}(A+B).$$

105. 以 r_1, r_2, r_3, 爲三邊三角形, 其各角爲 α, β, γ, 則

$$a(s-a)\cos^2\frac{\alpha}{2}+b(s-b)\cos^2\frac{\beta}{2}+c(s-c)\cos^2\frac{\gamma}{2}$$

$$=\frac{3}{2}(bc+ca+ab-s^2).$$

例題解自 101. 至 105.

(101) $rs:r's'=S:S'=\sqrt{ss_1s_2s_3}:\dfrac{s^2}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$, 〔例題 90.〕

$$=\sqrt{\frac{s_2s_3}{ss_1}}\sqrt{\frac{s_3s_1}{ss_2}}\sqrt{\frac{s_1s_2}{ss_3}}:\frac{1}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}$$

$$=\tan\frac{A}{2}\tan\frac{B}{2}\tan\frac{C}{2}:\frac{1}{2\cos\frac{1}{2}A\cos\frac{1}{2}B\cos\frac{1}{2}C}=2\sin\tfrac{1}{2}A\sin\tfrac{1}{2}B\sin\tfrac{1}{2}C:1.$$

(102) 切於內切圓及 a, b, c 三傍切圓之點, 順次命爲 $X, Y, Z; X_1, Y_1, Z_1;$ $X_2, Y_2, Z_2; X_3, Y_3, Z_3$. 而角 $X=$角 $AYZ=$角 $AZY=90°-\frac{1}{2}A$, $XY=2CX\sin\frac{1}{2}C=2s_3\sin\frac{1}{2}C$, $XZ=2BX\sin\frac{1}{2}B=2s_2\sin\frac{1}{2}B$, 三角形 $XYZ=\frac{1}{2}XY.YZ\sin X=2s_2s_3\sin\frac{1}{2}B\sin\frac{1}{2}C\cos\frac{1}{2}A$

$$=2s_2s_3\sqrt{\frac{s_3s_1}{ca}}\sqrt{\frac{s_1s_2}{ab}}\sqrt{\frac{ss_1}{bc}}=\frac{2s_1s_2s_3S}{abc},$$

又 角 $X_1=180°-($角 BX_1Z_1+角 $CX_1Y_1)=180°-\dfrac{B+C}{2}=90°+\dfrac{A}{2}$,

$X_1Z_1=2BZ_1 sin\frac{180°-B}{2}=2s_3 cos\frac{B}{2}$, $X_1Y_1=2s_2 cos\frac{C}{2}$,

三角形 $X_1Y_1Z_1=\frac{1}{2}X_1Z_1.X_1Y_1 sin X_1=2s_2s_3 cos\frac{B}{2} cos\frac{C}{2} cos\frac{A}{2}$

$=2s_2s_3\sqrt{\frac{ss_2}{ca}}\sqrt{\frac{ss_3}{ab}}\sqrt{\frac{ss_1}{bc}}=\frac{2s_1s_2s_3 S}{abc}$, 由是

三角形 $X_1Y_1Z_1$＋三角形 $X_2Y_2Z_2$＋三角形 $X_3Y_3Z_3$－三角形 XYZ

$=\frac{2S}{abc}(ss_2s_3+ss_3s_1+ss_1s_2-s_1s_2s_3)$

$=\frac{2S}{abc}\{ss_3(s_2+s_1)+s_1s_2(s-s_3)\}=\frac{2S}{abc}\{ss_3c+s_1s_2c\}=\frac{2S}{ab}(ss_3+s_1s_2)$

$=\frac{2S}{ab}\left\{\frac{(a+b+c)(a+b-c)+(b+c-a)(c+a-b)}{4}\right\}=2S.$

(103) 三角形 $X_3Y_3Z_3=\frac{2ss_1s_2S}{abc}$, (前例)$=\frac{2ss_1s_2S}{4RS}=\frac{ss_1s_2}{2R}$,

又 三角形 $OO_1O_2=2Rs_3$(例題94.) 由是

三角形 $X_3Y_3Z_3\times$三角形 $OO_1O_2=\frac{ss_1s_2}{2R}\times 2Rs_3=ss_1s_2s_3=S^2$.

(104) 三角形 $O_1O_2O_3$ 之內心爲 O', 傍心爲 O_1', 則

$O'O'_1=\frac{O_2O_3}{cos\frac{1}{2}O_1}$, (例題 82.)$=s/\{cos\frac{1}{2}(B+C)cos\frac{1}{2}B\,cos\frac{1}{2}C\}$ (例題 87.)

$=\frac{4R\,cos\frac{1}{2}A\,cos\frac{1}{2}B\,cos\frac{1}{2}C}{cos\frac{1}{2}(B+C)cos\frac{1}{2}B\,cos\frac{1}{2}C}=\frac{4R\,sin\frac{1}{2}(B+C)}{cos\frac{1}{2}(B+C)}=8R\,sin\frac{1}{2}(B+C)$.

(105) $r_1+r_2+r_3=\frac{S}{s_1}+\frac{S}{s_2}+\frac{S}{s_3}=\frac{S(bc+ca+ab-s^2)}{s_1s_2s_3}$,

$r_2+r_3-r_1=\frac{S}{s_2}+\frac{S}{s_3}-\frac{S}{s_1}=\frac{S(s^2-a^2-bc)}{s_1s_2s_3}$, $r_2r_3=\frac{S^2}{s_2s_3}$,

故 $cos^2\frac{a}{2}=\frac{(r_2+r_3+r_1)(r_2+r_3-r_1)}{4r_2r_3}=\frac{(bc+ca+ab-s^2)(s^2-a^2-bc)}{4s_1^2s_2s_3}$,

故 $a(s-a)cos^2\frac{a}{2}=\frac{(bc+ca+ab-s^2)(as^2-a^3-abc)}{4s_1s_2s_3}$,

原式之左邊$=\frac{(bc+ca+ab-s^2)\{s^2(a+b+c)-(a^3+b^3+c^3+3abc)\}}{4s_1s_2s_3}$,

但 $s^2(a+b+c)-(a^3+b^3+c^3+3abc)=6s_1s_2s_3$.

106. 自九點圓之中心N,至BC作垂線Nn,則

$$Nn=\frac{1}{2}R\cos(C-B).$$

107. $Bn=\frac{1}{4}R(2\cos B\sin C+\sin A)$.

108. 引長AN,截BC於T,則

$$AN:NT=\cos(C-B)+2\cos A:\cos(C-B).$$

109. $BT:TC=\sin 2A+\sin 2B:\sin 2A+\sin 2C$.

110. $AN=\frac{1}{2}R\sqrt{(3+2\cos 2A-2\cos 2B-2\cos 2C)}$.

111. $ON=r\sim\frac{1}{2}R$. 112. $ON_1=r+\frac{1}{2}R$.

例題解自 106. 至 112.

(106) MN=NH, (平面幾何學講義第四編例題669.) 故

$$Nn=\frac{1}{2}(MP+HD)=\frac{1}{2}(R\cos A+2R\cos C\cos B)$$

$$=\frac{1}{2}R\{-\cos(C+B)+2\cos C\cos B\}=\frac{1}{2}R\cos(C-B).$$

(107) 如前例. $Pn=nD=\frac{1}{2}PD=\frac{1}{2}(CP-CD)=\frac{1}{2}\left(\frac{a}{2}-b\cos C\right)$,

$$Bn=BP+Pn=\frac{a}{2}+\frac{1}{2}\left(\frac{a}{2}-b\cos C\right)=\frac{1}{4}(3a-2b\cos C)$$

$$=\frac{1}{4}R(3\sin A-2\sin B\cos C)=\frac{1}{4}R\{\sin A+2[\sin(B+C)-\sin B\cos C]\}.$$

(108) 從N引AD垂線NL,則MN=NH,故

$$LH=\frac{1}{2}(MP-HD)=\frac{1}{2}(R\cos A-2R\cos B\cos C)$$

$$AL=AH-LH=2R\cos A-\frac{1}{2}(R\cos A-2R\cos B\cos C)$$

$$=\frac{R}{2}\{(\cos A+2\cos B\cos C)+2\cos A\}=\frac{R}{2}\{\cos(C-B)+2\cos A\}.$$

$$AN:NT=AL:Nn=\frac{R}{2}\{\cos(C-B)+2\cos A\}:\frac{1}{2}R\cos(C-B).$$

但參照平面幾何學講義第四編例題669.

(109) 由前例. $nD:Tn=AN:NP$, $Tn=nD\times\frac{NT}{AN}=\frac{PD}{2}\times\frac{NT}{AN}$,

但 $PD=\frac{a}{2}-b\cos C=R(\sin A-2\sin B\cos C)=R\sin(C-B)$, 故由前例.

$Tn=\frac{R\sin(C-B)}{2}\times\frac{\cos(C-B)}{2\cos A+\cos(C-B)}$, $BT=BP+Pn-Tn=\frac{a}{2}+\frac{PD}{2}-Tn$

$=R\sin A+\frac{R\sin(C-B)}{2}-\frac{R\sin(C-B)\cos(C-B)}{2\{2\cos A+\cos(C-B)\}}$

$=\frac{R[\{2\sin A+\sin(C-B)\}\{2\cos A+\cos(C-B)\}-\sin(C-B)\cos(C-B)]}{2\{2\cos A+\cos(C-B)\}}$

$=\frac{R\{2\sin A\cos(C-B)+2\cos A\sin(C-B)+4\sin A\cos A\}}{2\{\cos A+\cos(C-B)\}}=\frac{R(\sin 2B+\sin 2A)}{\cos A+\cos(C-B)}$

同樣. $CT=CP-Pn+Tn=\frac{R(\sin 2C+\sin 2A)}{\cos A+\cos(C-B)}$.

(110) $MN=NH$, 故 $AN^2=\frac{2AM^2+2AH^2-MH^2}{4}=\frac{2R^2+4R^2\cos^2 A-MH^2}{4}$,

但 $MH^2=R^2(1-8\cos A\cos B\cos C)$, 〔例題56.〕, 故

$AN^2=\frac{R^2}{4}(2\cos^2 A+8\cos A\cos B\cos C+1)$, 從此由例題38.卽可得其證.

(111) 從N引OX垂線NW, (但X爲內切圓切於BC之點)

然 $OW=OX-Nn=r-\frac{1}{2}R\cos(C-B)$, 〔例題106.〕

又 $NW=BX-Bn=\frac{1}{2}(a+c-b)-Bn$

$=R(\sin A+\sin C-\sin B)-\frac{1}{2}R(2\cos B\sin C+\sin A)$, 〔例題107.〕

$=\frac{1}{2}R\{2\sin C-2\sin B-\sin(C-B)\}=R\{(\sin C-\sin B)-\frac{1}{2}\sin(C-B)\}$,

$ON^2=OW^2+NW^2=r^2-Rr\cos(C-B)+\frac{1}{4}R^2\cos^2(C-B)$

$+R^2\{(\sin C-\sin B)^2-(\sin C-\sin B)\sin(C-B)+\frac{1}{4}\sin^2(C-B)\}$

$=r^2+\frac{1}{4}R^2-Rr\cos(C-B)+R^2(\sin C-\sin B)\{\sin C-\sin B-\sin(C-B)\}$

$=r^2+\frac{1}{4}R^2-Rr\cos(C-B)+4R^2\sin\frac{A}{2}\sin^2\frac{1}{2}(C-B)\left\{\cos\frac{C+B}{2}-\cos\frac{C-B}{2}\right\}$

$=r^2+\frac{1}{4}R^2-Rr\cos(C-B)-8R^2\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}\sin^2\frac{1}{2}(C-B)$.

但 $r=4R\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$, (例題二十三5.)

故 $ON^2=r^2+\frac{1}{4}R^2-Rr\{1-2\sin^2\frac{1}{2}(C-B)\}-2Rr\sin^2\frac{1}{2}(C-B)=(r-\frac{1}{2}R)^2$,

故 $ON=r\backsim\frac{1}{2}R$.

(112) 從N引O_1X_1之垂線NW_1,如前例.即得.

〔注意〕 例題111.及112.九點圓切於內切圓及傍切圓,爲幾何學上之定理,已於平面幾何學講義第四編例題1507及1508證明矣.

113. A=60°, 則A之二等分線, 必通過N.

114. 相似兩三角形.其第一形之外切圓,等於第二形之外切圓時,則其相應邊之比,爲 $4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}:1$.

115. AP爲中央線, AP_1爲類似中央線, 而角 $AP_1C=\phi$, 角 $APC=\psi$, 則 $\frac{\cos\phi}{\cos\psi}=\cos A$. $\tan\frac{\phi-\psi}{2}=\frac{c-b}{c+b}\tan\frac{A}{2}$,

$$\tan\phi=\frac{c^2+b^2}{c^2-b^2}\tan A.$$

例題解自113.至115.

(113) 由例題109. 得 $BT:TC=\sin 2A+\sin 2B:\sin 2A+\sin 2C$

$=2\sin(A+B)\cos(A-B):2\sin(A+C)\cos(A-C)$

$=\sin C\cos(60°-B):\sin B\cos(60°-C)$, 但 $B=180°-(60°+C)=120°-C$,

故 $\cos(60°-B)=\cos\{60°-(120°-C)\}=\cos\{-(60°-C)\}=\cos(60°-C)$.

由是 $BT:TC=\sin C\cos(60°-B):\sin B\cos(60°-C)=\sin C:\sin B$,

即 $BT:TC=c:b$, 故由幾何學之定理. AT爲A之二等分線.

(114) 第一形爲ABC, 第二形爲A'B'C', 其相應之邊爲 a 及 a'.

然 $a=2R\sin A$, $a'=R\left(\cot\frac{B}{2}+\cot\frac{C}{2}\right)=\frac{R\cos\frac{1}{2}A}{\sin\frac{1}{2}B\sin\frac{1}{2}C}$,

故 $\frac{a}{a'}=\frac{2\sin A\sin\frac{1}{2}B\sin\frac{1}{2}C}{\cos\frac{1}{2}A}=4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$.

(115) 於9.節之圖. $\cos\theta=\frac{b^2+3c^2-a^2}{4cm}$, $\sin\theta=\frac{S}{cm}$,

$$\frac{\cos\phi}{\cos\psi}=\frac{\cos\{180°-(C+\theta)\}}{\cos(B+\theta)}=\frac{-\cos C\cos\theta+\sin C\sin\theta}{\cos B\cos\theta-\sin B\sin\theta}$$

$$=\frac{-\cos C\left(\frac{b^2+3c^2-a^2}{4cm}\right)+\sin C\times\frac{S}{cm}}{\cos B\left(\frac{b^2+3c^2-a^2}{4cm}\right)-\sin B\times\frac{S}{cm}}=\frac{-\cos C(b^2+3c^2-a^2)+4S\sin C}{\cos B(b^2+3c^2-a^2)-4S\sin B}$$

$$=\frac{c\{-2ab\cos C(b^2+3c^2-a^2)+8S(ab\sin C)\}}{b\{2ac\cos B(b^2+3c^2-a^2)-8S(ac\sin B)\}}=\frac{c\{-(a^2+b^2-c^2)(b^2+3c^2-a^2)+16S^2\}}{b\{(a^2+c^2-b^2)(b^2+3c^2-a^2)-16S^2\}},$$

$$=\frac{c\{(a^4-4a^2c^2-b^4-2b^2c^2+3c^4)+(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)\}}{b\{(-a^4+2a^2b^2+2a^2c^2-b^4-2b^2c^2+3c^4)-(2b^2c^2+2c^2a^2+2a^2b^2-a^4-b^4-c^4)\}}$$

$$=\frac{c(-2b^4+2c^4+2a^2b^2-2a^2c^2)}{b(4c^4-4b^2c^2)}=\frac{(c^4-b^4)-a^2(c^2-b^2)}{2bc(c^2-b^2)}=\frac{c^2+b^2-a^2}{2bc}=\cos A.$$

$$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{S/(cm)}{(b^2+3c^2-a^2)/(4cm)}=\frac{4S}{b^2+3c^2-a^2}.$$

$$\tan\frac{\phi-\psi}{2}=\tan\frac{\{180^\circ-(C+\theta)\}-(B+\theta)}{2}=\tan\left\{\frac{180^\circ-(B+C)}{2}-\theta\right\}$$

$$=\tan\left(\frac{A}{2}-\theta\right)=\frac{\tan\frac{1}{2}A-\tan\theta}{1+\tan\frac{1}{2}A\tan\theta}=\frac{\tan\frac{1}{2}A-\tan\theta}{\tan\frac{1}{2}A+\tan^2\frac{1}{2}A\tan\theta}\times\tan\frac{A}{2}$$

$$=\frac{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}-\frac{4S}{b^2+3c^2-a^2}}{\sqrt{\frac{(s-b)(s-c)}{s(s-a)}}+\frac{(s-b)(s-c)}{s(s-a)}\times\frac{4S}{b^2+3c^2-a^2}}\times\tan\frac{A}{2}$$

$$=\frac{\frac{S}{s(s-a)}-\frac{4S}{b^2+3c^2-a^2}}{\frac{S}{s(s-a)}+\frac{4S(s-b)(s-c)}{s(s-a)(b^2+3c^2-a^2)}}\tan\frac{A}{2}=\frac{b^2+3c^2-a^2-4s(s-a)}{b^2+3c^2-a^2+4(s-b)(s-c)}\tan\frac{A}{2}$$

$$=\frac{b^2+3c^2-a^2-(a+b+c)(b+c-a)}{b^2+3c^2-a^2+(c+a-b)(a+b-c)}\tan\frac{A}{2}$$

$$=\frac{b^2+3c^2-a^2+a^2-b^2-2bc-c^2}{b^2+3c^2-a^2+a^2-b^2+2bc-c^2}\tan\frac{A}{2}=\frac{2c(c-b)}{2c(c+b)}\tan\frac{A}{2}.$$

$$\tan\phi=\tan\{180^\circ-(C+\theta)\}=-\tan(C+\theta)=\frac{\sin C\cos\theta+\cos C\sin\theta}{-\cos C\cos\theta+\sin C\sin\theta}$$

$$=\frac{\sin C\times\frac{b^2+3c^2-a^2}{4cm}+\cos C\times\frac{S}{cm}}{-\cos C\times\frac{b^2+3c^2-a^2}{4cm}+\sin C\times\frac{S}{cm}}=\frac{\sin C(b^2+3c^2-a^2)+4S\cos C}{-\cos C(b^2+3c^2-a^2)+4S\sin C}.$$

以 ab 乘其分母子而變化之,則

$$=\frac{2S(a^2+b^2-c^2)+2S(b^2+3c^2-a^2)}{8S^2-\frac{1}{2}(a^2+b^2-c^2)(b^2+3c^2-a^2)}=\frac{4S(b^2+c^2)}{c^4-b^4-a^2(c^2-b^2)},$$

$$=\frac{(c^2+b^2)2bc\sin A}{(c^2-b^2)(c^2+b^2-a^2)}=\frac{c^2+b^2}{c^2-b^2}\tan A.$$

116. P爲等邊三角形內之一點,則

$\cos(\mathrm{BPC}-60^\circ)=(\mathrm{BP}^2+\mathrm{CP}^2-\mathrm{AP}^2)/(2\mathrm{BP}.\mathrm{PC})$.

117. 一直線通過等邊三角形之中心,其兩端以二邊爲界,該直線被中心分爲x, y之二部分,而其等邊爲a,

則　$1/x^2-1/(xy)+1/y^2=9/a^2$.

118. 於直角等脚三角形內,作等邊三角形,其角頂置於三邊上,一邊與斜邊平行,則等邊三角形之面積,爲$2a^2\sin^2 15^\circ \sin 60^\circ$. 但$a$爲原形之一等邊.

119. 有直角等脚三角形,於其二等邊之引長線上,設自角頂等距離之二點,連結此二點之直線,爲底邊n倍. 由是所成梯形之內對角線之交角,爲

$$2\tan^{-1}\frac{n-1}{n+1},\quad 或\quad 2\tan^{-1}\frac{n+1}{n-1}.$$

120. 以銳角三角形之各邊爲底邊,以外切圓之半徑爲等邊,於其外方作等脚三角形,連結其頂角點之三角形,與原形爲全等形.

121. 平行於三角形之BC邊,引XY,截AB, AC於X,Y. 而XY $=$BX$+$CY,　則　$\mathrm{XY}=\frac{1}{2}a\cos\frac{1}{2}(\mathrm{B}-\mathrm{C})\sec\frac{1}{2}\mathrm{B}\sec\frac{1}{2}\mathrm{C}$.

例題解自 116. 至 121.

(116) ABC爲等邊三角形,連結AP, BP, CP. 於角BPC之間引PE,作等邊三角形CPE,連結EB,則於兩三角形APC, CEB.

CP$=$CE, AC$=$BC, 又 角ACP$=60^\circ-$角PCB$=$角BCE. 由是

此兩三角形爲全等形. 而 $AP=BE$, 故於三角形 BPE,

$\cos BPE=(BP^2+EP^2-BE^2)/(2BP.EP)$

$=(BP^2+CP^2-AP^2)/(2BP.CP)=\cos(BPC-60^\circ)$.

(117) 令等邊三角形 ABC 之中心爲 O, 一直線爲 XY. 則 $OX=x$, $OY=y$, 又角 $AXO=\theta$, 角 $AYO=180^\circ-(60^\circ+\theta)$,

故於兩三角形 AOX 及 AOY,

$\sin\theta=\dfrac{AO.\sin XAO}{x}=\dfrac{a\sqrt{3}\sin 30^\circ}{3x}=\dfrac{a\sqrt{3}}{6x}$, 及 $\sin(60^\circ+\theta)=\dfrac{a\sqrt{3}}{6y}$, 即

$\dfrac{\sqrt{3}}{2}\cos\theta+\frac{1}{2}\sin\theta=\dfrac{a\sqrt{3}}{6y}$, 從此兩方程式, 可消去 θ 之三角函數.

(118) 於直角等脚三角形 ABC, $A=90^\circ$, $AB=AC$,

又 令 D, E, F, 爲 BC, CA, AB 上之點. DEF 爲等邊三角形, 則 $B=45^\circ$, 角 BDF $=$角 $CDE=\frac{1}{2}(180^\circ-D)=60^\circ$, $BC=\sqrt{(AB^2+AC^2)}=\sqrt{(a^2+a^2)}=a\sqrt{2}$,

角 $BFD=180^\circ-(B+$角 $BDF)=180^\circ-(45^\circ+60^\circ)=75^\circ$, 於三角形 BDF,

$DF=\dfrac{BD\sin B}{\sin BFD}=\dfrac{a\sqrt{2}\sin 45^\circ}{2\sin 75^\circ}=\dfrac{a}{2\cos 15^\circ}=\dfrac{a\sin 15^\circ}{\sin 30^\circ}=2a\sin 15^\circ$,

故所求之面積$=\frac{1}{2}DF.DE\sin 60^\circ=\frac{1}{2}DF^2\sin 60^\circ=\frac{1}{2}(2a\sin 15^\circ)^2\sin 60^\circ$.

(119) 令 D, E 爲 AB, AC 引長線上之點. $AD=AE$, 則 $BD=CE$, $DE=nBC$,

$AB:AD=BC:DE=BC:nBC$, 故 $AD=nAB$.

$\tan ADC=\dfrac{AC}{AD}=\dfrac{AB}{nAB}=\dfrac{1}{n}$, 又令兩對角線 BE, CD 之交點爲 x, 則

角 $DXE=180^\circ-($角 $XDE+$角 $XED)=180^\circ-2$角 XDE

$=180^\circ-2(45^\circ-$角 $ADC)=90^\circ+2ADC$, 故 $\tan\frac{1}{2}DXE=\tan(45^\circ+ADC)$

$=\dfrac{1+\tan ADC}{1-\tan ADC}=\dfrac{n+1}{n-1}$, 又可得 $\tan\frac{1}{2}DXB=\dfrac{n-1}{n+1}$.

(120) 作 BC, CA, AB 上之等脚三角形之頂角爲 A,′ B,′ C,′ 外心爲 M, 兩等脚三角形 MBC, A′BC, R 爲等邊, 有同底 BC, 故爲全等形.

而角 $BA'C=$角 $BMC=2A$, 角 $A'CB=$角 $BCM=\frac{1}{2}(180^\circ-2A)=90^\circ-A$,

同樣. 角 $B'CA=90^\circ-B$, 故 角 $A'CB'=C+(90^\circ-A)+(90^\circ-B)=2C$,

$A'B'^2=A'C^2+B'C^2-2A'C.B'C\cos A'CB'=R^2+R^2-2R^2\cos 2C=4R^2\sin^2 C$,

故 $A'B'=2R\sin C=AB$, 同樣. $A'C'=AC$, $B'C'=BC$.

(121) $\dfrac{XY}{a}=\dfrac{AX}{c}=\dfrac{AY}{b}=\dfrac{AX+AY}{b+c}=\dfrac{b+c-XY}{b+c}$, $XY=\dfrac{a(b+c)}{a+b+c}$,

由是 b, c 可以 $\sin B$, $\sin C$ 之項表之.

122. 一點D爲直角,二直線CA, CB之對角爲γ,而CA=a, CB=b, 則

$$CD=\frac{ab\cos 2\gamma}{\sin\gamma\sqrt{(a^2+b^2-2ab\sin 2\gamma)}}.$$

123. 自正方形內切圓周上之一點P,向兩對角線之角爲α, β, 則 $\tan^2\alpha+\tan^2\beta=8$.

124. 有平行四邊形ABCD. AB=a, AD=b, 則兩對角線之長,爲$\sqrt{(a^2+b^2+2ab\cos A)}$, $\sqrt{(a^2+b^2-2ab\cos A)}$.

125. ABCD四角形之兩對角線AC, BD爲d, d',其交角爲θ,則其面積爲$\frac{1}{2}dd'\sin\theta$.

126. 已知梯形之平行二邊, AB=a, CD=b, 及二角C, D. 求其他之二邊.

127. 於ABCD四角形. 其角ABC=β, 角ADB=角CDB=$\frac{1}{2}\delta$, AB=a, BC=b, 則 $BD=\dfrac{ab\csc\frac{1}{2}\delta\sin(\beta+\delta)}{\sqrt{\{2a^2+b^2+2ab\cos(\beta+\delta)\}}}$.

例題解自 122. 至 127.

(122) 令角BCD=θ, 則 $\sin(\theta+\gamma)=\dfrac{CD\sin\gamma}{b}$, (1) 又

角ACD=$90°-\theta$, $\sin CAD=\sin(ACD+\gamma)=\cos(\theta-\gamma)=\dfrac{CD\sin\gamma}{a}$, (2)

(1), (2) 兩式相加減而變化之,則

$\sin\theta+\cos\theta=\dfrac{CD\sin\gamma(a+b)}{ab(\cos\gamma+\sin\gamma)}$, (3), 及

$\sin\theta-\cos\theta=\dfrac{CD\sin\gamma(a-b)}{ab(\cos\gamma-\sin\gamma)}$, (4)

而 $(\sin\theta+\cos\theta)^2+(\sin\theta-\cos\theta)^2=2$, 明甚. 從 (3), (4) 式代入之, 則

$$\frac{CD^2\sin^2\gamma}{a^2b^2}\left\{\frac{(a+b)^2}{(\cos\gamma+\sin\gamma)^2}+\frac{(a-b)^2}{(\cos\gamma-\sin\gamma)^2}\right\}=2.$$ 即

$CD^2\sin^2\gamma\{2(a^2+b^2)(\sin^2\gamma+\cos^2\gamma)-8ab\sin\gamma\cos\gamma\}=2a^2b^2(\cos^2\gamma-\sin^2\gamma)^2$,

即 $CD^2\sin^2\gamma(a^2+b^2-2ab\sin2\gamma)=a^2b^2\cos^22\gamma$.

(123) 於正方形ABCD, 其一邊爲 a, 從P引AC, BD之垂線PE, PF. 如是令內切圓之中心爲O, 則PO爲三角形PAC及PBD之中央線, 故由例題41.

$\{(\frac{1}{2}AC)^2-PO^2\}\tan\alpha=2\triangle PAC=PE\times AC$, 即

$(\frac{1}{2}a^2-\frac{1}{4}a^2)\tan\alpha=aPE\sqrt{2}$, 故 $\tan\alpha=\frac{4\sqrt{2}}{a}PE$, 同樣. $\tan\beta=\frac{4\sqrt{2}}{a}PF$,

由是 $\tan^2\alpha+\tan^2\beta=\frac{32}{a^2}(PE^2+PF^2)=\frac{32}{a^2}\left(\frac{a}{2}\right)^2=8$.

(124) 於三角形ABD,

$BD^2=AB^2+AD^2-2AB.AD\cos A=a^2+b^2-2ab\cos A$,

又於三角形ABC, $AC^2=a^2+b^2-2ab\cos(180°-A)$.

(125) 平行BD引LAM及PCN, 平行AC引LBP及MDN,

由是成LMNP之平行四邊形, 爲原形之二倍.

故四角形ABCD$=\frac{1}{2}$平行四邊形LMNP$=\frac{1}{2}LM.LP\sin LMP=\frac{1}{2}dd'\sin\theta$.

(126) 平行BC引AE, 截CD於E, 則於三角形ADE,

$AD=\frac{DE\sin AED}{\sin DAE}=\frac{(b-a)\sin C}{\sin(C+D)}$, $BC=AE=\frac{(b-a)\sin D}{\sin(C+D)}$.

(127) 令角BAD$=\theta$, 則於三角形ABD, $\sin\theta=\frac{BD\sin\frac{1}{2}\delta}{a}$, (1)

又於三角形BCD, $\sin BCD=-\sin(\theta+\beta+\delta)=\frac{BD\sin\frac{1}{2}\delta}{b}$, (2)

(1) (2), 加減而變化之, 則

$2\sin\left(\theta+\frac{\beta+\delta}{2}\right)=\frac{BD\sin\frac{1}{2}\delta(b-a)}{ab\cos\frac{1}{2}(\beta+\delta)}$, 及 $2\cos\left(\theta+\frac{\beta+\delta}{2}\right)=\frac{BD\sin\frac{1}{2}\delta(b+a)}{ab\sin(\beta+\delta)}$

此雙方之平方相加, 則 $4=\frac{BD^2\sin^2\frac{1}{2}\delta}{a^2b^2}\left\{\frac{(b-a)^2}{\cos^2\frac{1}{2}(\beta+\delta)}+\frac{(b+a)^2}{\sin^2\frac{1}{2}(\beta+\delta)}\right\}$

即 $a^2b^2\sin^2(\beta+\delta)=BD^2\sin^2\frac{1}{2}\delta\{a^2+b^2+2ab\cos(\beta+\delta)\}$.

128. 於圓之內切四角形ABCD,(18.節)

$$AC=\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}},\quad BD=\sqrt{\frac{(ac+bd)(ab+cd)}{ad+bc}}.$$

129. 同上. $\sin\frac{B}{2}=\sqrt{\frac{(p-a)(p-b)}{ab+cd}}$, $\cos\frac{B}{2}=\sqrt{\frac{(p-c)(p-d)}{ab+cd}}$.

130. 同上. 外切圓之半徑為R,則

$$4R=\sqrt{\frac{(ac+bd)(ab+cd)(ad+bc)}{(p-a)(p-b)(p-c)(p-d)}}.$$

131. 同上. 各兩對邊引長線之交點,連結成一直線,為

$$\sqrt{\frac{(ad+bc)(ab+cd)\{bd(c^2-a^2)^2+ca(b^2-d^2)^2\}}{(b^2-d^2)^2(c^2-a^2)^2}}.$$

例題解自128.至131.

(128) 於ABC三角形,

$AC^2=AB^2+BC^2-2AB.BC\cos B=a^2+b^2-2ab\cos B$,

於ADC三角形, $AC^2=AD^2+CD^2-2AD.CD\cos D=d^2+c^2-2dc\cos(180°-B)$,

可由此兩方程式消去 $\cos B$,

參照平面幾何學講義第四編例題1348.

(129) 由前例之兩方程式消去 AC^2,則

$a^2+b^2-2ab\cos B=c^2+d^2+2cd\cos B$, $\cos B=\frac{a^2+b^2-c^2-d^2}{2(ab+cd)}$,

$$\sin\frac{B}{2}=\sqrt{\frac{1-\cos B}{2}}=\sqrt{\frac{2(ab+cd)-(a^2+b^2-c^2-d^2)}{4(ab+cd)}}=\sqrt{\frac{(c+d)^2-(a-b)^2}{4(ab+cd)}}$$

但 $a+b+c+d=2p$, 故 $\sin\frac{B}{2}=\sqrt{\frac{(2p-2a)(2p-2b)}{4(ab+cd)}}$.

(130) 於ABC三角形, $AC=2R\sin B=4R\sin\frac{B}{2}\cos\frac{B}{2}$, 由前二例.

$$\sqrt{\frac{(ac+bd)(ad+bc)}{ab+cd}}=4R\sqrt{\frac{(p-a)(p-b)}{ab-cd}}\sqrt{\frac{(p-c)(p-d)}{ab+cd}}.$$

卽 $\sqrt{(ab+cd)(ac+bd)(ad+bc)}=4R\sqrt{(p-a)(p-b)(p-c)(p-d)}$

故 $4R=\sqrt{\dfrac{(ab+cd)(ac+bd)(ad+bc)}{(p-a)(p-b)(p-c)(p-d)}}$.

(131) 於內切四角形ABCD,其AB, DC引長線之交點爲Q, AD, BC引長線之交點爲R,則角Q=B−A,角R=C−B,

由例題129. $\sin B=2\sin\frac{B}{2}\cos\frac{B}{2}=2\sqrt{\dfrac{(p-a)(p-b)(p-c)(p-d)}{(ab+cd)^2}}$

卽 $\sin B=\dfrac{2P}{ab+cd}$, (18.節) 同樣. $\sin A=\dfrac{2P}{ad+bc}$,

又 $\cos C=1-2\sin^2\frac{C}{2}=1-\dfrac{2(p-a)(p-d)}{ad+bc}$, (例題129.)

$$=\frac{b^2+c^2-a^2-d^2}{2(ad+bc)},$$

又 $CQ=\dfrac{b\sin(180^\circ-B)}{\sin(B-A)}=\dfrac{b\sin B}{\sin(B-A)}$,

$QD=CQ+c=\dfrac{d\sin A}{\sin(B-A)}$, 故 $\dfrac{b\sin B}{\sin(B-A)}+c=\dfrac{d\sin A}{\sin(B-A)}$,

故 $\sin(B-A)=\dfrac{1}{c}(d\sin A-b\sin B)=\dfrac{1}{c}\left(\dfrac{2dP}{ad+bc}-\dfrac{2bP}{ab+cd}\right)$

$$=\frac{2P(d^2-b^2)}{(ad+bc)(ab+cd)},$$

由是 $CQ=\dfrac{b\sin B}{\sin(B-A)}=\dfrac{b(ad+bc)}{d^2-b^2}$, 同樣. $CR=\dfrac{c(ad+bc)}{a^2-c^2}$,

$QR^2=CQ^2+CR^2-2CQ.CR\cos C$

$$=\frac{b^2(ad+bc)^2}{(b^2-d^2)^2}+\frac{c^2(ad+bc)^2}{(c^2-a^2)^2}-\frac{2bc(ad+bc)^2}{(b^2-d^2)(c^2-a^2)}\times\frac{b^2+c^2-a^2-d^2}{2(ad+bc)}$$

$=\dfrac{ad+bc}{(b^2-d^2)^2(c^2-a^2)^2}\times M$, 但 M如次式

$$(ad+bc)\{b^2(c^2-a^2)^2+c^2(b^2-d^2)^2\}-bc(b^2-d^2)(c^2-a^2)\{(c^2-a^2)+(b^2-d^2)\}$$

$$=b(c^2-a^2)^2\{b(ad+bc)-c(b^2-d^2)\}+c(b^2-d^2)^2\{c(ad+bc)-b(c^2-a^2)\}$$

$$=b(c^2-a^2)^2d(ab+cd)+c(b^2-d^2)^2a(cd+ab)$$

$$=(ab+cd)\{bd(c^2-a^2)^2+ca(b^2-d^2)^2\}=M.$$

132. 自四角形ABCD內之一點O,至AB, BC, CD, DA上作OP, OQ, OR, OS,垂線,則四角形PQRS之面積,爲

$\frac{1}{2}P-\frac{1}{8}(OA^2 sin2A+OB^2 sin2B+OC^2 sin2C+OD^2 sin2D)$.

但 P爲ABCD四角形之面積.

133. 於四角形ABCD各邊上,向外方作直角等腳三角形,連結其相對頂角點之二直線相等,其各平方之二倍,等於次值.

$a^2+b^2+c^2+d^2+da(sin A-cos A)+ab(sin B-cos B)$

$+bc(sin C-cos C)+cd(sin D-cos D)$.

134. 三角形ABC之內心爲O,內切圓切於BC, CA, AB之點爲X, Y, Z.而三個四角形AYOZ, BZOX, CXOY內切圓之半徑爲p_1, p_2, p_3則

$$\frac{p_1}{r-p_1}+\frac{p_2}{r-p_2}+\frac{p_3}{r-p_3}=\frac{p_1}{r-p_1}\times\frac{p_2}{r-p_2}\times\frac{p_3}{r-p_3}.$$

例題解自132. 至134.

(132) $AP=OA cos OAP$, $AS=OA cos OAS$,

$OP=OA sin OAP$, $OS=OA sin OAS$,

三角形$APS=\frac{1}{2}AP.AS sin A=\frac{1}{2}OA^2 cos OAP cos OAS sin A$, (1),

三角形$OPS=\frac{1}{2}OP.OS sin POS=\frac{1}{2}OA^2 sin OAP sin OAS sin(180°-A)$, (2),

由(1)減(2),

三角形APS−三角形$OPS=\frac{1}{2}OA^2 cos(OAP+OAS) sin A$.

$=\frac{1}{2}OA^2 cos A sin A=\frac{1}{4}OA^2 sin 2A$.

三角形BPQ−三角形$OPQ=\frac{1}{4}OB^2 sin2B$,

三角形CQR−三角形$OQR=\frac{1}{4}OC^2 sin2C$,

三角形DSR−三角形$OSR=\frac{1}{4}OD^2 sin2D$. 由是

三角形APS+BPQ+CQR+DSR−(三角形OPS+OPQ+OQR+OSR)

$=\frac{1}{4}(OA^2 sin2A+OB^2 sin2B+OC^2 sin2C+OD^2 sin2D)$, (3),

2(三角形 OPS+OPQ+OQR+OSR)=2四角形 PQRS, (4),

(3), (4) 相加, 則

$$P=\frac{1}{4}(OA^2 sin2A+OB^2 sin2B+OC^2 sin2C+OD^2 sin2D)+2\text{四角形 PQRS}.$$

(133) AB, BC, CD, DA 上之直角等脚三角形, 其頂角點順次爲 P, Q, R, S 引長 PA, RD 令交於 X, 則

$PR^2=XP^2+XR^2-2XP.XR\,cosX$

$=\left(XA+\frac{a}{\sqrt{2}}\right)^2+\left(XD+\frac{c}{\sqrt{2}}\right)^2-2\left(XA+\frac{a}{\sqrt{2}}\right)\left(XD+\frac{c}{\sqrt{2}}\right)cos\,X$

$=XA^2+XD^2-2XA.XD\,cos\,X+\frac{a^2}{2}+\frac{c^2}{2}-ac\,cosX$

$+\sqrt{2}a(XA-XD\,cosX)+\sqrt{2}c(XD-XA\,cosX\}$,

但 角 XAD=180°-(角 PAB+A)=180°-(45°+A), 角 XDA=180°-(45°+D)

X=180°-(角 XAD+角 XDA)=-{90°-(A+D)}

又 $XA^2+XD^2-2XA.XD cosX=d^2$, $XA^2+d^2-2XA.d cosXAD=XD^2$

此兩方程式相加, 則 $2XA^2-2XA.XD cosX=2XA.d cosXAD$,

即 $XA-XD cosX=d\,cosXAD=-d\,cos(45°+A)$, 即

$XA-XD cosX=\frac{d}{\sqrt{2}}(sinA-cosA)$, 同樣. $XD-XA cosX=\frac{d}{\sqrt{2}}(sinD-cosD)$ 由是

$PR^2=d^2+\frac{a^2}{2}+\frac{c^2}{2}-ac\,sin(A+D)+ad(sinA-cosA)+cd(sinD-cosD)$

引長 PB, RC 令相交. 施用前法, 則 B+C=360°-(A+D),

故 $PR^2=b^2+\frac{a^2}{2}+\frac{c^2}{2}+ac\,sin(A+D)+bc(sinC-cosC)+ab(sinB-cosB)$,

將此二結果相加, 卽可得其證, 同樣. QS 亦可得同値.

(134) $AY=AZ=r\,cot\frac{1}{2}A$, 四角形 AYOZ= 三角形 AYZ+OYZ,

卽 $\frac{1}{2}p_1(AY+AZ+OY+OZ)=\frac{1}{2}AY.AZ sinA+\frac{1}{2}OY.OZ sinYOZ$,

卽 $p_1(r\,cot\frac{1}{2}A+r)=\frac{1}{2}r^2cot^2\frac{1}{2}A sinA+\frac{1}{2}r^2 sinA$,

卽 $2p_1(1+cot\frac{1}{2}A)=r\,sinA(cot^2\frac{1}{2}A+1)=2r\,cot\frac{A}{2}$, $\frac{p_1}{r-p_1}=cot\frac{A}{2}$,

同樣. $\frac{p_2}{r-p_2}=cot\frac{B}{2}$, $\frac{p_3}{r-p_3}=cot\frac{C}{2}$, 以此代入 (例題五 43.) 卽得

$cot\frac{1}{2}A+cot\frac{1}{2}B+cot\frac{1}{2}C=cot\frac{1}{2}A\,cot\frac{1}{2}B\,cot\frac{1}{2}C$.

135. 互交外切之三圓,其中心爲O_1, O_2, O_3, 其半徑爲p_1, p_2, p_3, 則 三角形$O_1O_2O_3=\sqrt{p_1p_2p_3(p_1+p_2+p_3)}$.

136. 同上. 通過三中心之圓周之半徑爲p,則

$$p=\frac{(p_1+p_2)(p_2+p_3)(p_3+p_1)}{4\sqrt{\{p_1p_2p_3(p_1+p_2+p_3)\}}}.$$

137. 同上. 外切三圓之圓之半徑爲p,則

$$\sqrt{\frac{p_2+p_3+p}{p_1}}+\sqrt{\frac{p_3+p_1+p}{p_2}}+\sqrt{\frac{p_1+p_2+p}{p_3}}=\sqrt{\frac{p_1+p_2+p_3}{p}}.$$

138. 同上. $\frac{1}{p}=\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\pm 2\sqrt{\left(\frac{1}{p_1p_2}+\frac{1}{p_2p_3}+\frac{1}{p_3p_1}\right)}$.

139. 兩等圓(其中心爲B, C)及一圓(其中心爲A)互相外切.切此三圓之二圓之中心距離,爲

$$\frac{4(1-\cos B)(1-2\cos B)}{4-5\cos B}.$$

例題解自 135. 至 139.

(135) 三角形之三邊,爲$O_1O_2=p_1+p_2$, $O_2O_3=p_2+p_3$, $O_3O_1=p_3+p_1$, 則其面積,可由第十二編3.節(6)之公式求之.

(136) p爲三角形$O_1O_2O_3$之外切圓之半徑.故由2.節.

$4p=\frac{(p_1+p_2)(p_2+p_3)(p_3+p_1)}{三角形O_1O_2O_3}$, 由前例可得證明.

(137) 有一圓外切於三圓,由題意.知此一圓切於三圓外切之間隙內.今此一圓之中心爲O,則

三角形$O_1O_2O_3$=三角形OO_1O_2+三角形OO_2O_3+三角形OO_3O_1,

由例題135.

$$\sqrt{p_1p_2p_3(p_1+p_2+p_3)}=\sqrt{pp_1p_2(p+p_1+p_2)}$$
$$+\sqrt{pp_2p_3(p+p_2+p_3)}+\sqrt{pp_3p_1(p+p_3+p_1)}$$

兩邊以$\sqrt{pp_1p_2p_3}$除之,卽得.

(138) 角$O_2OO_3=\alpha$, 角$O_3OO_1=\beta$, 角$O_1OO_2=\gamma$, $\alpha+\beta+\gamma=2\pi$,

故 $\cos^2\alpha+\cos^2\beta+\cos^2\gamma=1+2\cos\alpha\cos\beta\cos\gamma$, (例題五 101.)

$$\cos\alpha=\frac{(p+p_2)^2+(p+p_3)^2-(p_2+p_3)^2}{2(p+p_2)(p+p_3)}=\frac{p^2+p(p_2+p_3)-p_2p_3}{(p+p_2)(p+p_3)}$$

$$=1-\frac{2p_2p_3}{(p+p_2)(p+p_3)}=1-\frac{2(p+p_1)\lambda}{p_1},\quad 但\quad \frac{p_1p_2p_3}{(p+p_1)(p+p_2)(p+p_3)}=\lambda$$

同樣. $\cos\beta=1-\frac{2(p+p_2)\lambda}{p_2}$, $\cos\gamma=1-\frac{2(p+p_3)\lambda}{p_3}$, 由此

$$\left\{1-2\left(\frac{p}{p_1}+1\right)\lambda\right\}^2+\left\{1-2\left(\frac{p}{p_2}+1\right)\lambda\right\}^2+\left\{1-2\left(\frac{p}{p_3}+1\right)\lambda\right\}^2$$

$=1+2\left\{1-2\left(\frac{p}{p_1}+1\right)\lambda\right\}\left\{1-2\left(\frac{p}{p_2}+1\right)\lambda\right\}\left\{1-2\left(\frac{p}{p_3}+1\right)\lambda\right\}$, 卽解雙方之括弧,加減同類項,以$4\lambda^2$通除之,且代入$\lambda$之相等值,又以$p^2$通除之,則得次式

$$\frac{1}{p^2}-\frac{2}{p}\left(\frac{1}{p_1}+\frac{1}{p_2}+\frac{1}{p_3}\right)+\frac{1}{p_1^2}+\frac{1}{p_2^2}+\frac{1}{p_3^2}-\frac{2}{p_1p_2}-\frac{2}{p_2p_3}-\frac{2}{p_3p_1}=0.$$

(139) 兩等圓之切點爲D,其半徑爲b,一圓之半徑爲a,此三圓內切之大圓及外切之小圓之中心爲O, O′,其半徑爲R, r,則

$AB\cos ABC=BD$, 卽 $(a+b)\cos B=b$, 故 $a=\frac{b(1-\cos B)}{\cos B}$, $a+b=\frac{b}{\cos B}$,

$a-b=\frac{b(1-2\cos B)}{\cos B}$, 於三角形ABO, $AB^2+AO^2-BO^2=2AB.AO\cos BAO$

卽 $(a+b)^2+(R-a)^2-(R-b)^2=2(a+b)(R-a)\cos(90°-B)$, 故

$R=\frac{a(a+b)(1+\sin B)}{(a+b)\sin B+(a-b)}$, 於三角形ABO′用同樣之公式,則

$(a+b)^2+(r+a)^2-(r+b)^2=2(a+b)(r+a)\sin B$, $r=\frac{a(a+b)(1-\sin B)}{(a+b)\sin B-(a-b)}$,

又 $OO'=r+2a-R=2a-(R-r)$,

故 $OO'=2a-\frac{2a(a+b)\{(a+b)\sin^2B-(a-b)\}}{(a+b)^2\sin^2B-(a-b)^2}=\frac{4ab(a-b)}{(a+b)^2\sin^2B-(a-b)^2}$,

代入a, $a+b$, $a-b$之相等值,則

$$OO'=\frac{4b(1-\cos B)(1-2\cos B)}{\cos B(4-5\cos B)}=\frac{4AB(1-\cos B)(1-2\cos B)}{4-5\cos B}.$$

140. 有半圓ABC. 其中心爲O, 半徑爲R, 兩半徑OB, OC之直交角爲2α, 則兩三角形OAB, OCB之內切圓之兩中心距離, 爲
$R\sqrt{(2-\sin 2\alpha)}/\sqrt{(1+\sin\alpha)(1+\cos\alpha)}$.

141. 同心三圓. 與直線之交點A, B, C. 其直線與中心之距離爲p則其交點之切線所成三角形之面積, 爲

$$BC.CA.AB/(2p).$$

142. 於任意一直線上, 從A, B, C三點, 引x, y, z三垂線, 則三角形ABC之平方之四倍, 爲
$a^2x^2+b^2y^2+c^2z^2-2(bc\,yz\cos A+ca\,zx\cos B+ab\,xy\cos C)$.

例題解自140. 至142.

(140) $AB=AC\cos A=2R\cos\frac{1}{2}BOC=2R\cos\alpha$, $BC=AC\sin A=2R\sin\alpha$,
令三角形AOB, BOC之內切圓之中心爲M, N. 半徑爲a, b. 則
$a(AO+BO+AB)=AO.BO\sin AOB$, 卽 $a(2R+2R\cos\alpha)=R^2\sin 2\alpha$
$a=R\sin 2\alpha/2(1+\cos\alpha)$, 同樣. 由$b(BO+CO+BC)=BO.CO\sin BOC$, 得
$b=R\sin 2\alpha/2(1+\sin\alpha)$, 又 $OM=a/\sin\frac{1}{2}AOB=a/\cos\alpha$
$ON=b/\sin\alpha$, 故 $MN^2=a^2/\cos^2\alpha+b^2/\sin^2\alpha$

$$=\frac{R^2\sin^2\alpha}{(1+\cos\alpha)^2}+\frac{R^2\cos^2\alpha}{(1+\sin\alpha)^2}=\frac{R^2(1-\cos\alpha)}{1+\cos\alpha}=\frac{R^2(1-\sin\alpha)}{1+\sin\alpha}=\frac{R^2(2-\sin 2\alpha)}{(1+\cos\alpha)(1+\sin\alpha)}.$$

(141) A及C爲最大圓及最小圓之周上之點. 於直線ABC之引長線上. 引垂線OM. 於A及B之切線之交點爲F, 於C之切線AF, 及BF截E及D. 然$OM=p$, $OA=a$, $OB=b$, $OC=c$, $DE=f$, $EF=d$, $FD=e$, 角$COM=\theta$, 角$BOM=\phi$, 角$AOM=\psi$, F, A, B, O爲一圓周上之四點. 故 角F=角$AOB=\psi-\phi$, 同樣. 角$D=\phi-\theta$, 角$E=\pi-(\psi-\theta)$, 於三角形DEF, (此以S代之) 令

$$d/\sin(\phi-\theta)=e/\sin(\psi-\theta)=f/\sin(\psi-\phi)=\lambda,$$ 則

$d=\lambda sin(\phi-\frac{1}{2})$, $e=\lambda sin(\psi-\theta)$, $f=\lambda sin(\psi-\phi)$,

$2S=de sin(\psi-\phi)=\lambda^2 sin(\phi-\theta) sin(\psi-\theta) sin(\psi-\phi)$, (1)

又 $c=p/cos\theta$, $b=p/cos\phi$, $a=p/cos\psi$,

三角形 $ABO=\frac{1}{2}AB.p=\frac{1}{2}ab sin AOB$, 故 $AB.p=p^2 sin(\psi-\phi)/(cos\phi cos\psi)$,

卽 $AB=\frac{p sin(\psi-\phi)}{cos\phi cos\psi}$, 同樣 $BC=\frac{p sin(\phi-\theta)}{cos\phi cos\theta}$, $CA=\frac{p sin(\psi-\theta)}{cos\psi cos\theta}$,

$AB.BC.CA=p^3 sin(\psi-\theta) sin(\phi-\theta) sin(\phi-\psi)/(cos\theta cos\phi cos\psi)^2$. (2)

又 $2S=ad+cf-bc=p\lambda\left\{\frac{sin(\phi-\theta)}{cos\psi}+\frac{sin(\psi-\phi)}{cos\theta}-\frac{sin(\psi-\theta)}{cos\phi}\right\}$,

$$2S=\frac{p\lambda\{cos\theta cos\phi sin(\phi-\theta)+cos\phi cos\psi sin(\psi-\phi)+cos\psi cos\theta sin(\theta-\psi)\}}{cos\theta cos\phi cos\psi}$$

$$=\frac{p\lambda sin(\phi-\theta) sin(\psi-\phi) sin(\theta-\psi)}{cos\theta cos\phi cos\psi},\quad (3)$$ 〔由例題四 97.〕

將 (3) 平方之.以除 (1) 式.則

$2S=p^2 sin(\phi-\theta) sin(\psi-\phi) sin(\psi-\theta)/(cos\theta cos\phi cos\psi)^2$,

此右邊.可以 (2) 之右邊除 p 而代入之,卽得.

(142) 於三垂線 AD, BE, CF, 令 AD 爲中間之垂線,則

$\sqrt{AB^2-(AD-BE)^2}+\sqrt{AC^2-(AD-CF)^2}=\sqrt{BC^2-(CF-BE)^2}$, 卽

$\sqrt{c^2-(x-y)^2}+\sqrt{b^2-(x-z)^2}=\sqrt{a^2-(y-z)^2}$, 令 $y-z=m$, $z-x=n$, $x-y=p$

則 $\sqrt{c^2-p^2}+\sqrt{b^2-n^2}=\sqrt{a^2-m^2}$. 由方程式之解法,兩邊平方之,省其根號如次,

$(b^2+c^2-a^2)^2+(n^2+p^2-m^2)^2-2(b^2+c^2-a^2)(n^2+p^2-m^2)=4(b^2c^2+n^2p^2-b^2p^2-c^2n^2)$,

卽 $2a^2(n^2+p^2-m^2)+2b^2(p^2+m^2-n^2)+2c^2(m^2+n^2-p^2)$

$$=4b^2c^2-(b^2+c^2-a^2)^2+4n^2p^2-(n^2+p^2-m^2)^2,$$

但 $4b^2c^2-(b^2+c^2-a^2)^2=(4\text{三角形}ABC)^2$, $4n^2p^2-(n^2+p^2-m^2)$ 之因子爲 $m+n+p$, $m+n-p$, 等,其內必有一因子爲 0, 故此式爲 0. 又於上之方程式,其 m, n, p, 代入 $y-z, z-x, x-y$, 卽

$4a^2(x^2-xy-xz+yz)+4b^2(y^2-yz-yx+zx)+4c^2(z^2-zx-zy+xy)=16(\triangle ABC)^2$,

卽 $4(\triangle ABC)^2=a^2x^2+b^2y^2+c^2z^2-xy(a^2+b^2-c^2)-yz(b^2+c^2-a^2)-zx(c^2+a^2-b^2)$

$$=a^2x^2+b^2y^2+c^2z^2-xy(2ab\cos C)-yz(2bc\cos A)-zx(2ca\cos B).$$

143. 以三角形之角頂A, B, C爲中心, p_1, p_2, p_3爲半徑之三圓. 有共通一切線, 則

$$\frac{p_1^2}{h_1^2}+\frac{p_2^2}{h_2^2}+\frac{p_3^2}{h_3^2}-2\left(\frac{p_2p_3\cos A}{h_2h_3}+\frac{p_3p_1\cos B}{h_3h_1}+\frac{p_1p_2\cos C}{h_1h_2}\right)=1.$$

144. 三角形之內切圓. 其切於二邊之三圓之半徑, 爲 r_a, r_b, r_c. 則 $r_a=r\tan^2\frac{1}{4}(\pi-A)$, $r_b=r\tan^2\frac{1}{4}(\pi-B)$, $r_c=r\tan^2\frac{1}{4}(\pi-C)$, 但 r_a爲角A內之圓之半徑, r_b, r_c準此.

145. 同上. $\sqrt{r_a r_b}+\sqrt{r_b r_c}+\sqrt{r_c r_a}=r.$

146. 三角形之內切圓, 切於各二邊畫圓. 又於其圓與切其二邊畫圓. 以下如此, 畫至無限之圓. 則其總畫圓之面積之和, 爲

$$r^2\pi\left(\sin^4\frac{B+C}{4}\csc\frac{A}{2}+\sin^4\frac{C+A}{4}\csc\frac{B}{2}+\sin^4\frac{A+B}{4}\csc\frac{A+B}{2}\right).$$

147. 三角形之三傍切圓, 於各角之間畫六圓, 切各一邊及各一邊之引長線上, 則此六圓之一, 等於各三圓半徑之積.

例題解自 143. 至 147.

(143) p_1, p_2, p_3, 爲自A, B, C引至一直線之三垂線. 又由6.節

$a=2S/h_1$, $b=2S/h_2$, $c=2S/h_3$, 故 $x=p_1$, $y=p_2$, $z=p_3$,

三角形ABC=S, 代入前例之結果, 卽可得其證.

(144) r.及r_a爲半徑之二圓, 其中心爲O, O′, 令此二圓切於AB之點爲Z及Z′, 則OZ−O′Z′=OO′$\sin\frac{1}{2}A$, 卽

$$r-r_a=(r+r_a)\sin\tfrac{1}{2}A,\ 故$$

$$r_a=\frac{r(1-\sin\frac{1}{2}A)}{1+\sin\frac{1}{2}A}=\frac{r\{1-\cos(\frac{1}{2}\pi-\frac{1}{2}A)\}}{1+\cos(\frac{1}{2}\pi-\frac{1}{2}A)}=\frac{2r\sin^2(\frac{1}{4}\pi-\frac{1}{4}A)}{2\cos^2(\frac{1}{4}\pi-\frac{1}{4}A)}.$$

(145) 由前例. $\sqrt{r_a r_b}=\sqrt{r\tan^2\frac{1}{4}(\pi-A)r\tan^2\frac{1}{4}(\pi-B)}$, 由此

$$\sqrt{r_a r_b}+\sqrt{r_b r_c}+\sqrt{r_c r_a}=r\{\tan\tfrac{1}{4}(\pi-A)\tan\tfrac{1}{4}(\pi-B)$$

$$+\tan\tfrac{1}{4}(\pi-B)\tan\tfrac{1}{4}(\pi-C)+\tan\tfrac{1}{4}(\pi-C)\tan\tfrac{1}{4}(\pi-A)\}.$$

但 $\frac{1}{2}(\pi-A)+\frac{1}{2}(\pi-B)+\frac{1}{2}(\pi-C)=\pi$, 由例題五45. 上之相等式右邊{}之式等於1. 故如題言.

(146) 令 $\tan^2\frac{1}{4}(\pi-A)=\lambda$, 則由例題144. $r_a=\lambda r$, 又r_a與切於二邊所畫圓之半徑,與此同樣.得λr_a. 即 $\lambda^2 r$, 順次以下皆如是.畫得多圓之半徑爲$\lambda^3 r$, $\lambda^4 r$,... 故由等比無限級數,角A之間畫得一切圓之面積之和,爲

$$\pi(\lambda^2r^2+\lambda^4r^2+\lambda^6r^2+\lambda^8r^2+\ldots\ldots\ldots)=\pi r^2\left(\frac{\lambda^2}{1-\lambda^2}\right)$$

$$=\pi r^2\left\{\frac{\lambda^2}{(1+\lambda)(1-\lambda)}\right\}=\pi r^2\left[\frac{\tan^4\frac{1}{4}(\pi-A)}{\{1+\tan^2\frac{1}{4}(\pi-A)\}\{1-\tan^2\frac{1}{4}(\pi-A)\}}\right]$$

$$=\pi r^2\left[\frac{\tan^4\frac{1}{4}(\pi-A)\cos^2\frac{1}{4}(\pi-A)}{\sec^2\frac{1}{4}(\pi-A)\cos\frac{1}{2}(\pi-A)}\right]=\pi r^2\left\{\frac{\sin^4\frac{1}{4}(\pi-A)}{\cos\frac{1}{2}(\pi-A)}\right\}=\pi r^2\left\{\frac{\sin^4\frac{1}{4}(B+C)}{\sin\frac{1}{2}A}\right\},$$

同樣. 於B及C之間,切於內切圓,所畫一切圓之面積之和,爲

$$\pi r^2\left\{\frac{\sin^4\frac{1}{4}(C+A)}{\sin\frac{1}{2}B}\right\},\quad \pi r^2\left\{\frac{\sin^4\frac{1}{4}(A+B)}{\sin\frac{1}{2}C}\right\}.$$

(147) 令切於BC之傍切圓.(半徑爲r_1)而B之間所畫圓之半徑爲l,則由例題144. $l=r_1\tan^2\frac{1}{4}\{\pi-(\pi-B)\}=r_1\tan^2\frac{1}{4}B$, 順是以下,令BC之傍切圓與角C, AC之傍切圓與角C,同圓與角A, AB之傍切圓與角A,同圓與角B之間所畫圓之半徑,爲m, n, p, q, t,則 $m=r_1\tan^2\frac{1}{4}C$, $n=r_2\tan^2\frac{1}{4}C$, $p=r_2\tan^2\frac{1}{4}A$, $q=a_3\tan\frac{1}{4}A$,

$$t=r_3\tan^2\tfrac{1}{4}B,$$

由是 $lnq=(r_1\tan^2\frac{1}{4}B)(r_2\tan^2\frac{1}{4}C)(r_3\tan^2\frac{1}{4}A)$

$$=r_1r_2r_3\tan^2\tfrac{1}{4}B\tan^2\tfrac{1}{4}C\tan^2\tfrac{1}{4}A.$$

又 $mpt=(r_1\tan^2\frac{1}{4}C)(r_2\tan^2\frac{1}{4}A)(r_3\tan^2\frac{1}{4}B)$

$$=r_1r_2r_3\tan^2\tfrac{1}{4}C\tan^2\tfrac{1}{4}A\tan^2\tfrac{1}{4}B$$

148. 得畫外切圓及內切圓之四角形,雙方引長其各邊,畫四傍切圓,切於BC, CD, DA, AB之四邊.其傍切圓之半徑,順次爲r_a, r_b, r_c, r_d,內切圓之半徑爲r則

$$r_a r_b r_c r_d = r^4.$$

149. 正方錐之底邊與傍稜之比,爲$m:n$.則傍面傾斜底面之角,爲 $tan^{-1}\left(\frac{\sqrt{4n^2-2m^2}}{m}\right)$.

150. 正五角形.與其各對角線連結所生之正五角形.其面積之比,爲$3+\sqrt{5}:3-\sqrt{5}$.

151. a爲正五角形之一邊,R爲外切圓之半徑,則R/a殆等於17/20.

152. 正六角形之內切圓.作第二之內切正六角形.此正六角形之內切圓.作第三之正六角形.順是以下皆如是.至作第n正六角形.此一切六角形面積之和,爲等於$6a^2\sqrt{3}\left\{1-\left(\frac{3}{4}\right)^n\right\}$.

但a爲原形之一邊.

153. 於正六角形.連結其越一角之兩角頂.作第二正六角形.又從第二如此作第三正六角形.以下皆如此.作至無限.則其面積之和等於原形之半.

例題解自148.至153.

(148) AB, CD之引長線上,於BC之外方交於P. AD, BC,之引長線上,於CD之外方交於Q.由例題二十三10.

於三角形PBC.　$BC=r_a\left(cot\frac{\pi-B}{2}+cot\frac{\pi-C}{2}\right)=r_a\left(cot\frac{D}{2}+cot\frac{A}{2}\right)$.

因ABCD爲圓之內切四角形.故由幾何學定理. $A+B+C+D=\pi$,

於三角形PAD. $AD=r\left(cot\frac{A}{2}+cot\frac{D}{2}\right)$,故 $\frac{BC}{AD}=\frac{r_a}{r}$.

又於三角形PAD, $AD=r_c\left(cot\frac{\pi-A}{2}+cot\frac{\pi-D}{2}\right)=r_c(cot\frac{1}{2}C+cot\frac{1}{2}B)$.

於三角形PBC. $BC=r(cot\frac{1}{2}B+cot\frac{1}{2}C)$,

故 $\frac{AD}{BC}=\frac{r_c}{r}$, 故 $\frac{r_a}{r}\times\frac{r_c}{r}=\frac{BC}{AD}\times\frac{AD}{BC}=1$, 同樣. $\frac{r_b}{r}\times\frac{r_a}{r}=1$.

(149) A 爲錐體之頂點, BC 爲底面正方形之一邊, AO 爲高, 引 BC 之垂線 AD, 則 OD 爲 BC 之垂線, 故

$$tan\,ADO=\frac{AO}{DO}=\frac{\sqrt{AD^2-DO^2}}{\frac{1}{2}BC}=\frac{2\sqrt{AB^2-BD^2-\frac{1}{4}BC^2}}{BC}=2\sqrt{\frac{AB^2-\frac{1}{4}BC^2-\frac{1}{4}BC^2}{BC^2}}$$

$$=2\sqrt{\left(\frac{AB^2}{BC^2}-\frac{1}{2}\right)}=2\sqrt{\left(\frac{n^2}{m^2}-\frac{1}{2}\right)},\ 故\ ADO=tan^{-1}\left(\frac{\sqrt{4n^2-2m^2}}{m}\right).$$

(150) 正五角形ABCDE之對角線AC,截他之二對角線BE, BD於M, N點. 則MN爲第二正五角形之一邊, $AB^2=AC.AM$, (平面幾何學講義第四編例題1410.) $=(AM+AB)AM$, 故 $AM=\frac{1}{2}AB(\sqrt{5}-1)$, $MN=AC-2AM=AM+AB-2AM$ $=AB-AM^2=\frac{1}{2}AB(3-\sqrt{5})$,

故原形與第二之面積之比 $=AB^2:MN^2=AB^2:\frac{1}{4}AB^2(3-\sqrt{5})^2$

$$=(9-5):(3-\sqrt{5})^2=3+\sqrt{5}:3-\sqrt{5}.$$

(151) $a=2Rsin\frac{1}{5}\pi$, (19. 節) $=R\times\frac{1}{2}\sqrt{(10-2\sqrt{5})}$, 〔例題六 14.〕

$$=R\times\frac{1\cdot176}{2},\ (略近數)=R\times\frac{20}{17},\ (略近數)$$

(152) 令第一之面積 $=6\times\frac{1}{4}a^2\sqrt{3}=A$, 第一內切圓之半徑爲 r,

則 $r=\frac{1}{2}cot\frac{1}{6}\pi$, (19. 節) $=\frac{1}{2}a\sqrt{3}=$ 第二之一邊, 故

第二之面積 $=6\times\frac{1}{4}(\frac{1}{2}a\sqrt{3})^2\sqrt{3}=A\times\frac{3}{4}$, 同樣. 第三之面積 $=A\times\frac{3}{4}\times\frac{3}{4}$,

故用等比級數之公式.所求之和 $=A\left\{1+\frac{3}{4}+\left(\frac{3}{4}\right)^2+\cdots\cdots+\left(\frac{3}{4}\right)^{n-1}\right\}$.

(153) 於正六角形ABCDEF, 引對角線BF, BD, 截對角線AC於M, N, 則容易知 $AM=MN=CN=\frac{1}{3}AC$, 而

$AC^2=AB^2+BC^2-2AB.BC\,cos120^\circ=a^2+a^2-2a^2(-\frac{1}{2})=3a^2$,

故 $MN=\frac{1}{3}a\sqrt{3}$, 如前例. $A=4a^2\sqrt{3}/6$,

第二之面積 $=\frac{4}{6}MN^2\sqrt{3}=\frac{4}{6}(\frac{1}{3}a\sqrt{3})^2\sqrt{3}=A\times\frac{1}{3}$, 同樣

第三之面積 $=A\times\frac{1}{3}\times\frac{1}{3}$, 故由等比無限級數.

第二, 第三, 第四, .. 之和 $=\frac{1}{3}A\left(1+\frac{1}{3}+\frac{1}{3^2}+\cdots\cdots\right)=\frac{1}{3}A\times\frac{1}{1-\frac{1}{3}}=\frac{A}{2}$.

154. 如前例之正六角形爲正n多角形.則其和爲

$$A\cos^2\frac{2\pi}{n}\left(\sin\frac{3\pi}{n}\sin\frac{\pi}{n}\right).$$

155. 圓之外切正多角形ABCDE………之各邊上.從其圓之中心引垂線.連結其垂線之足所成多角形之面積,等於原積之半.則$\sin 2A+\sin 2B+\sin 2C+\sin 2D+\cdots\cdots=0$.

156. 圓之內切$3n$多角形ABCDE……之鄰接三邊,連續引至n回而爲a, b, c.又R爲外切圓半徑,則

$$AC^2=\left(ac+2bR\sin\frac{\pi}{n}\right)\left(bc+2aR\sin\frac{\pi}{n}\right)\Big/\left(ab+2cR\sin\frac{\pi}{n}\right).$$

例題解自154.至156.

(154) 正n多角形爲ABCD……如前例.從B至越一角之角頂,引兩對角線,截AC於M, N,令BK爲AC之垂線,則

$BK=AB\sin BAC=a\sin\frac{\pi}{n}$,

$MK=\frac{1}{2}MN=BK\cot MBK=a\sin\frac{\pi}{n}\cot\left\{\pi-\frac{(n-2)\pi}{n}\right\}=a\sin\frac{\pi}{n}\cot\frac{2\pi}{n}$,

卽 $MN=a\cos\frac{2\pi}{n}\Big/\cos\frac{\pi}{n}$, 令第二,第三,…多角形爲$A'$, A'',…

則 $A'=A\times\frac{MN^2}{a^2}=A\left(\cos\frac{2\pi}{n}\Big/\cos\frac{\pi}{n}\right)^2$, 同樣. $A''=A\left(\cos\frac{2\pi}{n}\Big/\cos\frac{\pi}{n}\right)^4$,

故所求之和$=A\left(\dfrac{\cos\frac{2\pi}{n}}{\cos\frac{\pi}{n}}\right)^2\left\{1+\left(\dfrac{\cos\frac{2\pi}{n}}{\cos\frac{\pi}{n}}\right)^2+\left(\dfrac{\cos\frac{2\pi}{n}}{\cos\frac{\pi}{n}}\right)^4+\cdots\cdots\right\}$

$=A\left(\dfrac{\cos\frac{2\pi}{n}}{\cos\frac{\pi}{n}}\right)^2\left\{1\Big/\left(1-\dfrac{\cos^2\frac{2\pi}{n}}{\cos^2\frac{\pi}{n}}\right)\right\}=\dfrac{A\cos^2\frac{2\pi}{n}}{\cos^2\frac{\pi}{n}-\cos^2\frac{2\pi}{n}}$.

$$=\frac{A\cos^2\frac{2\pi}{n}}{1-\sin^2\frac{\pi}{n}-\left(1-2\sin^2\frac{\pi}{n}\right)^2}=\frac{A\cos^2\frac{2\pi}{n}}{\sin\frac{\pi}{n}\left(3\sin\frac{\pi}{n}-4\sin^3\frac{\pi}{n}\right)}=\frac{A\cos^2\frac{2\pi}{n}}{\left(\sin\frac{\pi}{n}\sin\frac{3\pi}{n}\right)}$$

(155) 與例題131.同法.用同例之四角形ABCD=P爲多角形.則多角形

PQRS……$=\frac{1}{2}P-\frac{1}{8}(OA^2\sin2A+OB^2\sin2B+OC^2\sin2C+\ldots\ldots)$.

但依題意.多角形PQRS……$=\frac{1}{2}P$, $OA=OB=OC=\ldots\ldots$,

故 $\sin2A+\sin2B+\sin2C+\ldots\ldots=0$.

(156) 對向於弦a, b, c之中心角之和,爲$\frac{2\pi}{n}$.

又四角形ABCD,爲圓之內切四角形.故由例題128.

$$AC^2=\frac{(ac+b.AD)(bc+a.AD)}{ab+c.CD}.$$

但 $AD=2R\sin\frac{AOD}{2}=2R\sin\frac{\pi}{n}$, 由是

$$AC^2=\frac{\left(ac+2bR\sin\frac{\pi}{n}\right)\left(bc+2aR\sin\frac{\pi}{n}\right)}{ab+2cR\sin\frac{\pi}{n}}.$$

初等平面三角法

初等平面三角法講義.至本編終.自次編起.述高等平面三角法之一部分.以爲本講義之結束.

三角法原爲論三角形法解之學科.至於近世論三角函數,涉及代數學幾何學等.而高等三角法.則論及高等之函數.其範圍甚大.

此講義錄.以初等三角法之諸法,及蒐集例題而講述之爲主目的.故初等之部.蒐集材料足成四册.均便於學者之搜索.至高等之部.簡單說明.僅一册耳.只爲初學者示其大綱.開專攻之先路.

雖然.尙有望於初學者.第一.本編例題.大槪與幾何學例題互相關係.故初學者取予曩日所講述之平面及立體幾何學講義.彼此參照.便利甚多.如平面幾何學講義第四編.與本編例題二十三及二十四.有密接之關係.第二.本書及幾何學講義.予蒐集例題.每每隨意附以解義.於不識不知之間.故解不完全之處甚多.讀者能於例題解義外.再加考究.而施簡明之新法以解之,是予之所厚望也.

第拾陸編

棣美弗 (De Moivere) 氏之定理

1. 棣美弗 (De Moivere) 氏之定理 對於 n 任意之值 $\cos n\theta+\sqrt{-1}\sin n\theta$: 恆爲 $(\cos\theta+\sqrt{-1}\sin\theta)^n$ 之一值. 今證明如次.

〔第一〕 n 爲正整數, 則

$$(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\beta+\sqrt{-1}\sin\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta$$

$$+\sqrt{-1}(\sin\alpha\cos\beta+\cos\alpha\sin\beta)=\cos(\alpha+\beta)+\sqrt{-1}\sin(\alpha+\beta)$$

又 $(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\beta+\sqrt{-1}\sin\beta)(\cos\gamma+\sqrt{-1}\sin\gamma)$

$=\{\cos(\alpha+\beta)+\sqrt{-1}\sin(\alpha+\beta)\}(\cos\gamma+\sqrt{-1}\sin\gamma)$, 故如前例.

$=\cos(\alpha+\beta+\gamma)+\sqrt{-1}\sin(\alpha+\beta+\gamma)$. 由是

$(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\beta+\sqrt{-1}\sin\beta)(\cos\gamma+\sqrt{-1}\sin\gamma)$……至 n 因子

$=\cos(\alpha+\beta+\gamma+$……至 n 項$)+\sqrt{-1}\sin(\alpha+\beta+\gamma+$……至 n 項$)$.

今令 $\alpha=\beta=\gamma=\ldots$ 則上之恆同式如次.

$$(\cos\alpha+\sqrt{-1}\sin\alpha)^n=\cos n\alpha+\sqrt{-1}\sin n\alpha.$$

〔第二〕 n 爲負整數, 則

$$(\cos\alpha+\sqrt{-1}\sin\alpha)^n=(\cos\alpha+\sqrt{-1}\sin\alpha)^{-m}=\frac{1}{(\cos\alpha+\sqrt{-1}\sin\alpha)^m}$$

$$=\frac{1}{(\cos\alpha+\sqrt{-1}\sin\alpha)^m}\times\frac{(\cos\alpha-\sqrt{-1}\sin\alpha)^m}{(\cos\alpha-\sqrt{-1}\sin\alpha)^m}=\frac{(\cos\alpha+\sqrt{-1}\sin\alpha)^m}{(\cos^2\alpha+\sin^2\alpha)^m}$$

$=(\cos\alpha-\sqrt{-1}\sin\alpha)^m=\cos m\alpha-\sqrt{-1}\sin m\alpha$, 由第一.

$=\cos(-m\alpha)+\sqrt{-1}\sin(-m\alpha)=\cos n\alpha+\sqrt{-1}\sin n\alpha$.

〔第三〕 n 爲正或負之分數, 卽令 $n=p/q$,

由代數學之定理.可得某數之平方根二個.立方根三個.n 方根有 n 個之值.故 $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}$ 可得 q 個之值.而 $\cos\frac{p}{q}\theta+\sqrt{-1}\sin\frac{p}{q}\theta$ 等於其一值.故本此定理,得證本題如次.

$(\cos\theta+\sqrt{-1}\sin\theta)^n=(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}$, 由第一及第二,

$=(\cos p\theta+\sqrt{-1}\sin p\theta)^{\frac{p}{q}}$, 又由第一及第二.

$\left(\cos\frac{p}{q}\theta+\sqrt{-1}\sin\frac{p}{q}\theta\right)=\cos p\theta+\sqrt{-1}\sin p\theta$,

故 $(\cos p\theta+\sqrt{-1}\sin p\theta)^{\frac{1}{q}}$ 之一值.等於 $\cos\frac{p}{q}\theta+\sqrt{-1}\sin\frac{p}{q}\theta$,

即 $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}$ 之一值.等於 $\cos\frac{p}{q}\theta+\sqrt{-1}\sin\frac{p}{q}\theta$,

〔餘論〕 $\cos\theta$ 及 $\sin\theta$ 之 θ.可加 2π 及 2π 之倍數以變其值.故於第三 $(\cos\theta+\sqrt{-1}\sin\theta)^{\frac{p}{q}}$ 之 q 個值.可於 $\cos\frac{p}{q}\theta+\sqrt{-1}\sin\frac{p}{q}\theta$ 之 θ.代用次列 q 個之值.

即 $\frac{p}{q}\theta$, $\frac{p}{q}(\theta+2\pi)$, $\frac{p}{q}(\theta+4\pi)$,...... $\frac{p}{q}(\theta+2\pi q-2\pi)$.

例題二十五

1. $A=\cos2\alpha+\sqrt{-1}\sin2\alpha$, $B=\cos2\beta+\sqrt{-1}\sin2\beta$,

$C=\cos2\gamma+\sqrt{-1}\sin2\gamma$, $D=\cos2\delta+\sqrt{-1}\sin2\delta$,

則 $(A+B)(C+D)=4\cos(\alpha+\beta)\cos(\gamma-\delta)\{\cos(\alpha+\beta+\gamma+\delta)$
$+\sqrt{-1}\sin(\alpha+\beta+\gamma+\delta)\}$.

2. 同上. $AB+CD=2\cos(\alpha+\beta-\gamma-\delta)\{\cos(\alpha+\beta+\gamma+\delta)$
$+\sqrt{-1}\sin(\alpha+\beta+\gamma+\delta)\}$.

3. 同上. $\dfrac{1}{AB-CD}=\dfrac{\sin(\alpha+\beta+\gamma+\delta)-\sqrt{-1}\cos(\alpha+\beta+\gamma+\delta)}{2\sin(\alpha+\beta-\gamma-\delta)}$.

4. 同上. $(A-B)(C-D)=-4\sin(\alpha-\beta)\sin(\gamma-\delta)\times$
$\{\cos(\alpha+\beta+\gamma+\delta)-\sqrt{-1}\sin(\alpha+\beta+\gamma+\delta)\}$

5. 同上. $\frac{1}{(A+B)(C+D)}$

$$=\frac{\cos(\alpha+\beta+\gamma+\delta)-\sqrt{-1}\sin(\alpha+\beta+\gamma+\delta)}{4\cos(\alpha-\beta)\cos(\gamma-\delta)}.$$

6. 同上. $\frac{1}{(A-B)(C-D)}$

$$=\frac{\cos(\alpha+\beta+\gamma+\delta)+\sqrt{-1}\sin(\alpha+\beta+\gamma+\delta)}{4\sin(\beta-\alpha)\sin(\gamma-\delta)}.$$

例題解自 1. 至 6.

(1) $A+B=\cos 2\alpha+\cos 2\beta+\sqrt{-1}(\sin 2\alpha+\sin 2\beta)$

$=2\cos(\alpha-\beta)\{\cos(\alpha+\beta)+\sqrt{-1}\sin(\alpha+\beta)\}$, 同樣.

$C+D=2\cos(\gamma-\delta)\{\cos(\gamma+\delta)+\sqrt{-1}\sin(\alpha+\delta)\}$, 故由 1. 節.

$(A+B)(C+D)=4\cos(\alpha-\beta)\cos(\gamma-\delta)\{\cos(\alpha+\beta)$

$+\sqrt{-1}\sin(\alpha+\beta)\}\{\cos(\gamma+\delta)+\sqrt{-1}\sin(\gamma+\delta)\}$

$=4\cos(\alpha-\beta)\cos(\gamma-\delta)\{\cos(\alpha+\beta+\gamma+\delta)+\sqrt{-1}\sin(\alpha+\beta+\gamma+\delta)$.

(2) $AB=(\cos 2\alpha+\sqrt{-1}\sin 2\alpha)(\cos 2\beta+\sqrt{-1}\sin 2\beta)$

$=\cos(2\alpha+2\beta)+\sqrt{-1}\sin(2\alpha+2\beta)$, 同樣.

$CD=\cos(2\gamma+2\delta)+\sqrt{-1}\sin(2\gamma+2\delta)$, 故

$AB+CD=\cos(2\alpha+2\beta)+\cos(2\gamma+2\delta)+\sqrt{-1}\{\sin(2\alpha+2\beta)+\sin(2\gamma+2\delta)\}$

$=2\cos(\alpha+\beta-\gamma-\delta)\{\cos(\alpha+\beta+\gamma+\delta)+\sqrt{-1}\sin(\alpha+\beta+\gamma+\delta)\}$.

(3) 令如前例,則

$AB-CD=2\sin(\alpha+\beta-\gamma-\delta)\{\sin(\alpha+\beta+\gamma+\delta)+\sqrt{-1}\cos(\alpha+\beta+\gamma+\delta)\}$,

$1/(AB-CD)=1/[2\sin(\alpha+\beta-\gamma-\delta)\{\sin(\alpha+\beta+\gamma+\delta)+\sqrt{-1}\cos(\alpha+\beta+\gamma+\delta)\}]$,

以 $\sin(\alpha+\beta+\gamma+\delta)-\sqrt{-1}\cos(\alpha+\beta+\gamma+\delta)$ 乘其分母子,則

$$\frac{1}{AB-CD}=\frac{\sin(\alpha+\beta+\gamma+\delta)-\sqrt{-1}\cos(\alpha+\beta+\gamma+\delta)}{2\sin(\alpha+\beta-\gamma-\delta)\{\sin^2(\alpha+\beta+\gamma+\delta)+\cos^2(\alpha+\beta+\gamma+\delta)\}}$$

$$=\frac{\sin(\alpha+\beta+\gamma+\delta)-\sqrt{-1}\cos(\alpha+\beta+\gamma+\delta)}{2\sin(\alpha+\beta-\gamma-\delta)}.$$

(4), (5), (6). 亦與此方法同樣.

7. 同上. $\frac{BC}{(A+B)(A+C)}$

$$=\frac{\cos(\beta+\gamma-2\alpha)+\sqrt{-1}\sin(\beta+\gamma-2\alpha)}{4\cos(\alpha-\beta)\cos(\alpha-\gamma)}.$$

8. $\{\cos\theta+\cos\phi+\sqrt{-1}(\sin\theta+\sin\phi)\}^n$

$$+\{\cos\theta+\cos\phi-\sqrt{-1}(\sin\theta+\sin\phi)\}^n$$

$$=2^{n+1}\cos^n\tfrac{1}{2}(\theta-\phi)\cos\tfrac{1}{2}n(\theta+\phi).$$

9. 求 $\cos 4A+\sqrt{-1}\sin 4A$ 之平方根.

10. 有 $\cos A+\sqrt{-1}\sin B=\tan B+\sqrt{-1}\cot A$ 試求其A, B之度.

11. $\alpha=15°$, 試求次式之值.

$$\frac{(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos 2\alpha+\sqrt{-1}\sin 2\alpha)}{\cos 3\alpha-\sqrt{-1}\sin 3\alpha}.$$

12. $x_n=\cos\frac{\pi}{2^n}+\sqrt{-1}\sin\frac{\pi}{2^n}$, 則 $x_1x_2x_3x_4\cdots$至無限$=\cos\pi$

13. 於三角形ABC, 求次式一定之值.

$$(\cos A+\sqrt{-1}\sin A)(\cos B+\sqrt{-1}\sin B)(\cos C+\sqrt{-1}\sin C).$$

例題解自7. 至13.

(7) $BC=(\cos 2\beta+\sqrt{-1}\sin 2\beta)(\cos 2\gamma+\sqrt{-1}\sin 2\gamma)=\cos(2\beta+2\gamma)$ $+\sqrt{-1}\sin(2\beta+2\gamma)$, 又 $A+B=2\cos(\alpha-\beta)\{\cos(\alpha+\beta)+\sqrt{-1}\sin(\alpha+\beta)\}$,

及 $A+C=2\cos(\alpha-\gamma)\{\cos(\alpha+\gamma)+\sqrt{-1}\sin(\alpha+\gamma)\}$

$(A+B)(A+C)=4\cos(\alpha-\beta)\cos(\alpha-\gamma)\{\cos(2\alpha+\beta+\gamma)+\sqrt{-1}\sin(2\alpha+\beta+\gamma)\}$,

$$\frac{BC}{(A+B)(A+C)}=\frac{\cos(2\beta+2\gamma)+\sqrt{-1}\sin(2\beta+2\gamma)}{4\cos(\alpha-\beta)\cos(\alpha-\gamma)\{\cos(2\alpha+\beta+\gamma)+\sqrt{-1}\sin(2\alpha+\beta+\gamma)\}}.$$

此分母子以 $\cos(2\alpha+\beta+\gamma)-\sqrt{-1}\sin(2\alpha+\beta+\gamma)$ 乘之則

分母$=4\cos(\alpha-\beta)\cos(\alpha-\gamma)\{\cos^2(2\alpha+\beta+\gamma)+\sin^2(2\alpha+\beta+\gamma)\}$

$=4\cos(\alpha-\beta)\cos(\alpha-\gamma)$,

分子$=\{\cos(2\beta+2\gamma)+\sqrt{-1}\sin(2\beta+2\gamma)\}\{\cos(2\alpha+\beta+\gamma)-\sqrt{-1}\sin(2\alpha+\beta+\gamma)\}$

$=\cos\{(2\beta+2\gamma)-(2\alpha+\beta+\gamma)\}+\sqrt{-1}\sin\{(2\beta+2\gamma)-(2\alpha+\beta+\gamma)\}$

$=\cos(\beta+\gamma-2\alpha)+\sqrt{-1}\sin(\beta+\gamma-2\alpha)$.

(8)　原式$=2^n\cos^n\frac{\theta-\phi}{2}\left[\left\{\cos\frac{\theta+\phi}{2}+\sqrt{-1}\sin\frac{\theta+\phi}{2}\right\}^n+\left\{\cos\frac{\theta+\phi}{2}-\sqrt{-1}\sin\frac{\theta+\phi}{2}\right\}^n\right]$

$=2^n\cos^n\frac{\theta-\phi}{2}\left[\cos^n\frac{\theta+\phi}{2}+\sqrt{-1}\sin^n\frac{\theta+\phi}{2}+\cos^n\frac{\theta+\phi}{2}-\sqrt{-1}\sin^n\frac{\theta+\phi}{2}\right]$

$=2^{n+1}\cos^n\frac{1}{2}(\theta-\phi)\cos\frac{1}{2}n(\theta+\phi)$.

(9)　$(\cos4A+\sqrt{-1}\sin4A)^{\frac{1}{2}}=\pm(\cos2A+\sqrt{-1}\sin2A)$.

(10)　實數部與實數部比較.虛數部與虛數部比較.則

$\cos A=\tan B$,　$\sin B=\cot A$,　由此兩方程式,得 $\sin A\cos B=1$,

而 $\sin A$ 及 $\cos B$,不能比 1 大,故 $A=90°$, $B=0$.

(11)　原式之分母子,以 $\cos3\alpha+\sqrt{-1}\sin3\alpha$ 乘之,則

分母$=\cos^2 3\alpha+\sin^2 3\alpha=1$,　故

原式$=(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos2\alpha+\sqrt{-1}\sin2\alpha)(\cos3\alpha+\sqrt{-1}\sin3\alpha)$

$=\cos(\alpha+2\alpha+3\alpha)+\sqrt{-1}\sin(\alpha+2\alpha+3\alpha)$

$=\cos6\alpha+\sqrt{-1}\sin6\alpha$,　但　$\alpha=15°$, 故

$=\cos90°+\sqrt{-1}\sin90°=\sqrt{-1}$.

(12)　$x_1x_2x_3x_4$………至無窮

$=\left(\cos\frac{\pi}{2}+\sqrt{-1}\sin\frac{\pi}{2}\right)\left(\cos\frac{\pi}{2^2}+\sqrt{-1}\sin\frac{\pi}{2^2}\right)\left(\cos\frac{\pi}{2^3}+\sqrt{-1}\sin\frac{\pi}{2^3}\right)$………

$=\cos\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\cdots\right)\frac{\pi}{2}+\sqrt{-1}\sin\left(1+\frac{1}{2}+\frac{1}{2^2}+\frac{1}{2^3}+\cdots\cdots\right)\frac{\pi}{2}$

$=\cos\left(\frac{1}{1-\frac{1}{2}}\right)\frac{\pi}{2}+\sqrt{-1}\sin\left(\frac{1}{1-\frac{1}{2}}\right)\frac{\pi}{2}=\cos\pi+\sqrt{-1}\sin\pi=\cos\pi$.

(13)　$A+B+C=\pi$,　故

原式$=\cos(A+B+C)-\sqrt{-1}\sin(A+B+C)$

$=\cos\pi+\sqrt{-1}\sin\pi=\cos\pi=-1$.

14. 於恆同式 $\frac{(x-b)(x-c)}{(a-b)(a-c)}+\frac{(x-c)(x-a)}{(b-c)(b-a)}+\frac{(x-a)(x-b)}{(c-a)(c-b)}=1.$

設 $x=\cos 2\theta+\sqrt{-1}\sin 2\theta$, 而 a, b, c 之形亦與之相應. 則得次之恆同式.

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}\sin 2(\theta-\alpha)+\frac{\sin(\theta-\gamma)\sin(\theta-\alpha)}{\sin(\beta-\gamma)\sin(\beta-\alpha)}\sin(\theta-\beta)$$
$$+\frac{\sin(\theta-\alpha)\sin(\theta-\beta)}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}\sin 2(\theta-\gamma)=0.$$

15. 用此恆同式 $\frac{1}{(x-a)(x-b)}=\frac{1}{(a-b)(x-a)}-\frac{1}{(a-b)(x-b)}$,

證明次之恆同式.

$$\cos(2\theta+\alpha+\beta)\sin(\alpha-\beta)$$
$$=\cos(2\alpha+\theta+\beta)\sin(\theta-\beta)-\cos(2\beta+\theta+\alpha)\sin(\theta-\alpha).$$

例題解自 14. 至 15.

(14) 假定 $b=\cos 2\beta+\sqrt{-1}\sin 2\beta$, 則

$$x-b=\cos 2\theta-\cos 2\beta+\sqrt{-1}(\sin 2\theta-\sin 2\beta)$$
$$=-2\sin(\theta-\beta)\{\sin(\theta+\beta)-\sqrt{-1}\cos(\theta+\beta)\}, \text{ 同樣.}$$
$$x-c=-2\sin(\theta-\gamma)\{\sin(\theta+\gamma)-\sqrt{-1}\cos(\theta+\gamma)\}, \text{ 故}$$
$$(x-b)(x-c)=4\sin(\theta-\beta)\sin(\theta-\gamma)\{\sin(\theta+\beta)-\sqrt{-1}\cos(\theta+\beta)\}$$
$$\{\sin(\theta+\gamma)-\sqrt{-1}\cos(\theta+\beta)\}$$
$$=-4\sin(\theta-\beta)\sin(\theta-\gamma)\{\cos(2\theta+\beta+\gamma)+\sqrt{-1}\sin(2\theta+\beta+\gamma)\},$$

於此結果, x 代以 a, 則 θ 爲 α 而得 $(a-b)(a-c)$ 之相等式. 由是

$$\frac{(x-b)(x-c)}{(a-b)(a-c)}=\frac{\sin(\theta-\beta)\sin(\theta-\gamma)\{\cos(2\theta+\beta+\gamma)+\sqrt{-1}\sin(2\theta+\beta+\gamma)\}}{\sin(\alpha-\beta)\sin(\alpha-\gamma)\{\cos(2\alpha+\beta+\gamma)+\sqrt{-1}\sin(2\alpha+\beta+\gamma)\}}.$$

以 $\cos(2\alpha+\beta+\gamma)-\sqrt{-1}\sin(2\alpha+\beta+\gamma)$ 乘其分母子而變化之,則

$\frac{(x-b)(x-c)}{(a-b)(a-c)}=\frac{\sin(\theta-\beta)\sin(\theta-\gamma)\{\cos2(\theta-\alpha)+\sqrt{-1}\sin2(\theta-\alpha)\}}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}$, 同樣求得

$\frac{(x-c)(x-a)}{(b-c)(b-a)}$, $\frac{(x-a)(x-b)}{(c-a)(c-b)}$, 與之相加,則由恆同式此和等於1,即

$$\frac{\sin(\theta-\beta)\sin(\theta-\gamma)\{\cos2(\theta-\alpha)+\sqrt{-1}\sin2(\theta-\alpha)\}}{\sin(\alpha-\beta)\sin(\alpha-\gamma)}$$

$$+\frac{\sin(\theta-\gamma)\sin(\theta-\alpha)\{\cos2(\theta-\beta)+\sqrt{-1}\sin2(\theta-\beta)\}}{\sin(\beta-\gamma)\sin(\beta-\alpha)}$$

$$+\frac{\sin(\theta-\alpha)\sin(\theta-\beta)\{\cos2(\theta-\gamma)+\sqrt{-1}\sin2(\theta-\gamma)\}}{\sin(\gamma-\alpha)\sin(\gamma-\beta)}=1,$$

去此恆同式之分母子,括左邊爲實數部與虛數部.則右邊僅有實數部,故左邊之虛數部爲0.即得所求之結果.

(15) 令 $x=\cos2\theta+\sqrt{-1}\sin2\theta$, $a=\cos2\alpha+\sqrt{-1}\sin2\alpha$, 即

$x-a=-2\sin(\theta-\alpha)\{\sin(\theta+\alpha)+\sqrt{-1}\cos(\theta+\alpha)\}$, 同樣.

$x-b=-2\sin(\theta-\beta)\{\sin(\theta+\beta)+\sqrt{-1}\cos(\theta+\beta)\}$, 故

$$\frac{1}{(x-a)(x-b)}=\frac{1}{-4\sin(\theta-\alpha)\sin(\theta-\beta)\{\cos(2\theta+\alpha+\beta)-\sqrt{-1}\sin(2\theta+\alpha+\beta)\}}$$

$$=\frac{\cos(2\theta+\alpha+\beta)+\sqrt{-1}\sin(2\theta+\alpha+\beta)}{-4\sin(\theta-\alpha)\sin(\theta-\beta)}.$$

同樣求得 $\frac{1}{(a-b)(x-a)}$, $\frac{1}{(a-b)(x-b)}$, 之值,代入已知恆同式內.則

$$\frac{\cos(2\theta+\alpha+\beta)+\sqrt{-1}\sin(2\theta+\alpha+\beta)}{\sin(\theta-\alpha)\sin(\theta-\beta)}$$

$$=\frac{\cos(2\alpha+\theta+\beta)+\sqrt{-1}\sin(2\alpha+\theta+\beta)}{\sin(\theta-\alpha)\sin(\alpha-\beta)}$$

$$=\frac{\cos(2\beta+\theta+\alpha)+\sqrt{-1}\sin(2\beta+\theta+\alpha)}{\sin(\theta-\beta)\sin(\alpha-\beta)}$$ 去分母,則

$$=\cos(2\theta+\alpha+\beta)\sin(\alpha-\beta)+\sqrt{-1}\sin(2\theta+\alpha+\beta)\sin(\alpha-\beta)$$

$$=\cos(2\alpha+\theta+\beta)\sin(\theta-\beta)-\cos(2\beta+\theta+\alpha)\sin(\theta-\alpha)$$

$$+\sqrt{-1}\{\sin(2\alpha+\theta+\beta)\sin(\theta-\beta)-\sin(2\beta+\theta+\alpha)\sin(\theta-\alpha)\}.$$

於上之恆同式,可比較其兩邊之實數部而得.

16. $x^3+1=0$之兩虛根爲u, u^2. 則得恆同式.

$$\frac{3}{1+x^3}=\frac{1}{1+x}+\frac{1}{1+ux}+\frac{1}{1+u^2x},$$ 由此可證次之恆同式.

$$3\tan 3\theta=\tan\theta+\tan\left(\theta+\frac{\pi}{6}\right)+\tan\left(\theta-\frac{\pi}{6}\right).$$

17. 如前例定u, u^2. 則

$$\left\{\cos\alpha+u\cos\left(\alpha+\frac{\pi}{3}\right)+u^2\cos\left(\alpha+\frac{2\pi}{3}\right)\right\}$$

$$\left\{\cos\beta+u\cos\left(\beta+\frac{\pi}{3}\right)+u^2\cos\left(\beta+\frac{2\pi}{3}\right)\right\}$$

$$=\frac{3}{2}\left\{\cos(\alpha+\beta)+u\cos\left(\alpha+\beta+\frac{\pi}{3}\right)+u^2\cos\left(\alpha+\beta+\frac{2\pi}{3}\right)\right\}.$$

18. 同上. n爲正整數. 試化下式爲最簡之式.

$$\left\{\cos\alpha+u\cos\left(\alpha+\frac{\pi}{3}\right)+u^2\cos\left(\alpha+\frac{2\pi}{3}\right)\right\}^n.$$

19. 解次之方程式.

$$(\cos\theta+\sqrt{-1}\sin\theta)(\cos 2\theta+\sqrt{-1}\sin 2\theta)(\cos 3\theta+\sqrt{-1}\sin 3\theta)$$
$$\cdots\cdots\cdots(\cos n\theta+\sqrt{-1}\sin n\theta)=1.$$

例題解自 16. 至 19.

(16) $x^3+1=0$之三根爲1, $u=\frac{1+\sqrt{-3}}{2}$, $u^2=\frac{1-\sqrt{-3}}{2}$, 故

得 $u=\frac{1}{2}+\frac{\sqrt{3}}{2}\sqrt{-1}=\cos\frac{\pi}{3}+\sqrt{-1}\sin\frac{\pi}{3}$, $u^2=\cos\frac{\pi}{3}-\sqrt{-1}\sin\frac{\pi}{3}$,

而令 $x=\cos 2\theta+\sqrt{-1}\sin 2\theta$, 則

$$\frac{3}{1+x^3}=\frac{3}{1+\cos 6\theta+\sqrt{-1}\sin 6\theta}=\frac{3}{2\cos 3\theta(\cos 3\theta+\sqrt{-1}\sin 3\theta)}$$

$$=\frac{3(\cos 3\theta-\sqrt{-1}\sin 3\theta)}{2\cos 3\theta}=\frac{3}{2}(1-\sqrt{-1}\tan 3\theta),$$

$$\frac{1}{1+x}=\frac{1}{1+\cos 2\theta+\sqrt{-1}\sin 2\theta)}=\frac{1}{2}(1-\sqrt{-1}\tan\theta),$$

$$\frac{1}{1+ux}=\frac{1}{1+\left(\cos\frac{\pi}{3}+\sqrt{-1}\sin\frac{\pi}{3}\right)\left(\cos 2\theta+\sqrt{-1}\sin 2\theta\right)}$$

$$=\frac{1}{1+\cos\left(2\theta+\frac{\pi}{3}\right)+\sqrt{-1}\sin\left(2\theta+\frac{\pi}{3}\right)}=\frac{1}{2}\left\{1-\sqrt{-1}\tan\left(\theta+\frac{\pi}{6}\right)\right\},$$

$$\frac{1}{1+u^2x}=\frac{1}{1+\left(\cos\frac{\pi}{3}-\sqrt{-1}\sin\frac{\pi}{3}\right)\left(\cos2\theta+\sqrt{-1}\sin2\theta\right)}$$

$$=\frac{1}{1+\cos\left(2\theta-\frac{\pi}{3}\right)+\sqrt{-1}\sin\left(2\theta-\frac{\pi}{3}\right)}=\frac{1}{2}\left\{1-\sqrt{-1}\tan\left(\theta-\frac{\pi}{6}\right)\right\},$$

以此代入恆同式而括之,則

$$3(1-\sqrt{-1}\tan3\theta)=3-\sqrt{-1}\left\{\tan\theta+\tan\left(\theta+\frac{\pi}{6}\right)+\tan\left(\theta-\frac{\pi}{6}\right)\right\}.$$

(17) 依題意, $u^3=-1$, 故 $u^4=-u$, 由是

$$\text{原式}=\cos\alpha\cos\beta-\cos\left(\alpha+\frac{\pi}{3}\right)\cos\left(\beta+\frac{2\pi}{3}\right)-\cos\left(\alpha+\frac{2\pi}{3}\right)\cos\left(\beta+\frac{\pi}{3}\right)$$

$$+u\left\{\cos\left(\alpha+\frac{\pi}{3}\right)\cos\beta+\cos\left(\beta+\frac{\pi}{3}\right)\cos\alpha-\cos\left(\alpha+\frac{2\pi}{3}\right)\cos\left(\beta+\frac{2\pi}{3}\right)\right\}$$

$$+u^2\left\{\cos\left(\alpha+\frac{\pi}{3}\right)\cos\left(\beta+\frac{\pi}{3}\right)+\cos\alpha\cos\left(\beta+\frac{2\pi}{3}\right)+\cos\beta\cos\left(\alpha+\frac{2\pi}{3}\right)\right\}$$

$$=\frac{1}{2}\cos(\alpha+\beta)+\cos(\alpha-\beta)-\cos(\alpha+\beta+\pi)-\cos\left(\alpha-\beta-\frac{\pi}{3}\right)-\cos(\alpha+\beta+\pi)$$

$$-\cos\left(\alpha-\beta+\frac{\pi}{3}\right)\Big\}+\frac{1}{2}u\left\{\cos\left(\alpha+\beta+\frac{\pi}{3}\right)+\cos\left(\alpha-\beta+\frac{\pi}{3}\right)+\cos\left(\alpha+\beta+\frac{\pi}{3}\right)\right.$$

$$\left.+\cos\left(\alpha-\beta-\frac{\pi}{3}\right)-\cos\left(\alpha+\beta+\frac{\pi}{3}+\pi\right)-\cos(\alpha-\beta)\right\}+\frac{1}{2}u^2\left\{\cos\left(\alpha+\beta+\frac{2\pi}{3}\right)\right.$$

$$\left.+\cos(\alpha-\beta)+\cos\left(\alpha+\beta+\frac{2\pi}{3}\right)+\cos\left(\alpha-\beta-\frac{2\pi}{3}\right)+\cos\left(\alpha+\beta+\frac{2\pi}{3}\right)+\cos\left(\alpha-\beta+\frac{2\pi}{3}\right)\right\}$$

$$=\frac{1}{2}\left\{3\cos(\alpha+\beta)+\cos(\alpha-\beta)-\left[\cos\left(\alpha-\beta+\frac{\pi}{3}\right)+\cos\left(\alpha-\beta+\frac{\pi}{3}\right)\right]\right\}$$

$$+\frac{1}{2}u\left\{3\cos\left(\alpha+\beta+\frac{\pi}{3}\right)+\left[\cos\left(\alpha-\beta+\frac{\pi}{3}\right)+\cos\left(\alpha-\beta-\frac{\pi}{3}\right)\right]-\cos(\alpha-\beta)\right\}$$

$$+\frac{1}{2}u^2\left\{3\cos\left(\alpha+\beta+\frac{2\pi}{3}\right)+\cos(\alpha-\beta)+\left[\cos\left(\alpha-\beta+\frac{2\pi}{3}\right)+\cos\left(\alpha-\beta-\frac{2\pi}{3}\right)\right]\right\}$$

$$=\frac{1}{2}\left\{3\cos(\alpha+\beta)+\cos(\alpha-\beta)-2\cos(\alpha-\beta)\cos\frac{\pi}{3}\right\}+\frac{1}{2}u\left\{3\cos\left(\alpha+\beta+\frac{2\pi}{3}\right)\right.$$

$$\left.+2\cos(\alpha-\beta)\cos\frac{\pi}{3}-\cos(\alpha-\beta)\right\}+\frac{1}{2}u^2\left\{3\cos\left(\alpha+\beta+\frac{2\pi}{3}\right)+\cos(\alpha-\beta)+2\cos(\alpha-\beta)\cos\frac{2\pi}{3}\right\}.$$

(18) 由前例. $\left\{\cos\alpha+u\cos\left(\alpha+\frac{\pi}{3}\right)+u^2\cos\left(\alpha+\frac{2\pi}{3}\right)\right\}^2$

$$=\frac{3}{2}\left\{\cos2\alpha+u\cos\left(2\alpha+\frac{\pi}{3}\right)+u^2\cos\left(2\alpha+\frac{2\pi}{3}\right)\right\},\text{ 又}$$

$$\left\{\cos\alpha+u\cos\left(\alpha+\frac{\pi}{3}\right)+u^2\cos\left(\alpha+\frac{2\pi}{3}\right)\right\}^3=\frac{3}{2}\left\{\cos2\alpha+u\cos\left(2\alpha+\frac{\pi}{3}\right)+u^2\cos\left(2\alpha+\frac{2\pi}{3}\right)\right\},$$

$$\left\{\cos\alpha+u\cos\left(\alpha+\frac{\pi}{3}\right)+u^2\cos\left(\alpha+\frac{2\pi}{3}\right)\right\}=\left(\frac{3}{2}\right)^2\left\{\cos3\alpha+u\cos\left(3\alpha+\frac{\pi}{3}\right)+u^2\cos\left(3\alpha+\frac{2\pi}{3}\right)\right\},$$

以此類推. 故 $\text{原式}=\left(\frac{3}{2}\right)^{n-1}\left\{\cos n\alpha+u\cos\left(n\alpha+\frac{\pi}{3}\right)+u^2\cos\left(n\alpha+\frac{2\pi}{3}\right)\right\}.$

(19) 原方程式 $\cos(1+2+3+\dots+n)\theta+\sqrt{-1}\sin(1+2+3+\dots+n)\theta=1,$

卽 $\sqrt{-1}\sin\frac{1}{2}n(n+1)\theta=1-\cos\frac{1}{2}n(n+1)\theta$, 兩邊平方而變化之,則

$\cos\frac{1}{2}n(n+1)\theta=\cos\left(2m\pi\pm\frac{\pi}{3}\right)$, 故 $\theta=\frac{2(6m\pm1)\pi}{3n(n+1)}.$

諸角之三角函數

2. 和及積之記號

於 a, b, c 等文字.其 $a+b+c$ 記之爲 Σa, abc 記之爲 Πa, 又 $ab+bc+ca=\Sigma ab$,

$a+b+c+d=\Sigma a$, $abcd=\Pi a$, 但此文字之數不易知.故記之如次.

$\Sigma_{abc}a=a+b+c$, $\Sigma_{abcd}a=a+b+c+d$, $\Sigma_{abc}ab=ab+bc+ca$,

$\Sigma_{abcd}ab=ab+bc+ca+ad+bd+cd$, $\Pi_{abc}a=abc$, $\Pi_{abc}(b+c)=(b+c)(c+a)(a+b)$.

3. 諸角之和之正餘弦及正切

$$\sin(\alpha+\beta+\gamma+\cdots)=\cos\alpha\cos\beta\cos\gamma\cdots(t_1-t_3+t_5-t_7+\cdots),$$

$$\cos(\alpha+\beta+\gamma+\cdots)=\cos\alpha\cos\beta\cos\gamma\cdots(1-t_2+t_4-t_6+\cdots),$$

$$\tan(\alpha+\beta+\gamma+\cdots)=(t_1-t_3+t_5-t_7+\cdots)/(1-t_2+t_4-t_6+\cdots).$$

但 $\Sigma\tan\alpha=t_1$,　$\Sigma\tan\alpha\tan\beta=t_2$,

$\Sigma\tan\alpha\tan\beta\tan\gamma=t_3$,…………以下準此,

〔證〕 由 1. 節 $\cos(\alpha+\beta+\gamma+\ldots\ldots)+\sqrt{-1}\sin(\alpha+\beta+\gamma+\ldots\ldots)$

$=(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\beta+\sqrt{-1}\sin\beta)(\cos\gamma+\sqrt{-1}\sin\gamma)\ldots\ldots$

$=\cos\alpha(1+\sqrt{-1}\tan\alpha)\cos\beta(1+\sqrt{-1}\tan\beta)\cos\gamma(1+\sqrt{-1}\tan\gamma)\ldots\ldots$

$=\cos\alpha\cos\beta\cos\gamma\ldots(1+\sqrt{-1}\tan\alpha)(1+\sqrt{-1}\tan\beta)(1+\sqrt{-1}\tan\gamma)\ldots\ldots$

連乘 $1+\sqrt{-1}\tan\alpha$, $1+\sqrt{-1}\tan\beta$, $1+\sqrt{-1}\tan\gamma$,…… 分實數部與虛數部之二項如次.

$\cos(\alpha+\beta+\gamma+\ldots)+\sqrt{-1}\sin(\alpha+\beta+\gamma+\ldots)$

$=\cos\alpha\cos\beta\cos\gamma\ldots\{(1-t_2+t_4-t_6+\ldots)+\sqrt{-1}(t_1-t_3+t_5-t_7+\ldots)\}$,

試將兩邊之虛數部與實數部比較,則

$\sin(\alpha+\beta+\gamma+\ldots)=\cos\alpha\cos\beta\cos\gamma\ldots(t_1-t_3+t_5-t_7+\ldots)$,

$\cos(\alpha+\beta+\gamma+\ldots)=\cos\alpha\cos\beta\cos\gamma\ldots(1-t_2+t_4-t_6+\ldots)$,

以第二除第一,得 $\tan(\alpha+\beta+\gamma+\ldots)$.

例 題 二 十 六

證次列各恆同式.

1. $\sin(\alpha+\beta+\gamma)=\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\gamma\cos\alpha$

$$+\sin\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta\sin\gamma.$$

2. $\cos(\alpha+\beta+\gamma)=\cos\alpha\cos\beta\cos\gamma-\cos\alpha\sin\beta\sin\gamma$

$$-\cos\beta\sin\gamma\sin\alpha-\cos\gamma\sin\alpha\sin\beta.$$

3. $\sin(\alpha_1+\alpha_2+\alpha_3+\alpha_4+\cdots\cdots+\alpha_n)$

$$=s_1c_{n-1}-s_2c_{n-2}+s_3c_{n-5}-\cdots\cdots\cdots\cdots$$

但 $s_r\,c_{n-r}$ 爲自一切角內取 r 個正弦各積,與取其餘 $n-r$ 個餘弦各積相乘之和.

4. 同上. $\cos(\alpha_1+\alpha_2+\alpha_3+\cdots\cdots+\alpha_n)$

$$=c_n-c_{n-2}s_2+c_{n-4}s_4-\cdots\cdots\cdots$$

5. 由前二例表示次式.

$$\sin(\alpha+\beta+\gamma+\delta+\epsilon) \text{ 及 } \cos(\alpha+\beta+\gamma+\delta+\epsilon).$$

例 題 解 自 1. 至 5.

(1) $\cos(\alpha+\beta+\gamma)+\sqrt{-1}\sin(\alpha+\beta+\gamma)$

$=(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos\beta+\sqrt{-1}\sin\beta)(\cos\gamma+\sqrt{-1}\sin\gamma)$

$=\cos\alpha\cos\beta\cos\gamma-\cos\alpha\sin\beta\sin\gamma-\cos\beta\sin\gamma\sin\alpha$

$-\cos\gamma\sin\alpha\sin\beta+\sqrt{-1}(\sin\alpha\cos\beta\cos\gamma+\sin\beta\cos\gamma\cos\alpha$

$+\sin\gamma\cos\alpha\cos\beta-\sin\alpha\sin\beta\sin\gamma),$

故比較左邊與右邊之虛數項,卽得其證.

(2) 可從前例.比較兩邊之實數項而得其證.

(3) 及 (4) 可用3.節之公式求之.

(5) 由前二例. $\sin(\alpha+\beta+\gamma+\delta+\epsilon)=s_1c_4-s_3c_2+s_5,$

$\cos(\alpha+\beta+\gamma+\delta+\epsilon)=c_5-s_2c_3+s_4c_1.$

倍角之三角函數

4. 倍角之正餘弦第一　n 爲正整數,則

$$\sin n\theta=n\cos^{n-1}\theta\sin\theta-\frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\theta\sin^{3}\theta$$

$$+\frac{n(n-1)(n-2)(n-3)(n-4)}{\underline{|5}}\cos^{n-5}\theta\sin^{5}\theta+\cdots\cdots$$

$$\cos n\theta=\cos^{n}\theta-\frac{n(n-1)}{\underline{|2}}\cos^{n-2}\theta\sin^{2}\theta$$

$$+\frac{n(n-1)(n-2)(n-3)}{\underline{|4}}\cos^{n-4}\theta\sin^{4}\theta-\cdots\cdots$$

〔證〕 由 1. 節. $\cos n\theta+\sqrt{-1}\sin n\theta=(\cos\theta+\sqrt{-1}\sin\theta)^{n}$, 由二項式之定理,展開 $(\cos\theta+\sqrt{-1}\sin\theta)^{n}$. 得第二項之偶數方乘,爲 $(\sqrt{-1})^{\circ}=1$, $(\sqrt{-1})^{2}=-1$, $(\sqrt{-1})^{4}=-1$...及奇數方乘, $(\sqrt{-1})^{1}=\sqrt{-1}$, $(\sqrt{-1})^{3}=-\sqrt{-1}$, $(\sqrt{-1})^{5}=\sqrt{-1}$,... 括其實數及虛數之項,則

$$\cos n\theta+\sqrt{-1}\sin n\theta=\cos^{n}\theta-\frac{n(n-1)}{\underline{|2}}\cos^{n-4}\theta\sin^{4}\theta+\cdots\cdots$$

$$+\sqrt{-1}\left\{n\cos^{n-1}\theta\sin\theta-\frac{n(n-1)(n-2)}{\underline{|3}}\cos^{n-3}\theta\sin^{3}\theta+\cdots\cdots\right\}$$

比較此兩邊之實數及虛數之項,卽得其證.

5. 餘論　若 n 爲偶數,則 $(\cos\theta+\sqrt{-1}\sin\theta)^{n}$ 展開式之末項爲實數.

卽 $(\sqrt{-1})^{n}\sin^{n}\theta=(\sqrt{-1})^{2m}\sin^{n}\theta=(-1)^{m}\sin^{n}\theta=(-1)^{\frac{n}{2}}\sin^{n}\theta$,

又末項之次一項爲虛數,卽 $n(\sqrt{-1})^{n-1}\cos\theta\sin^{n-1}\theta$

$=n(\sqrt{-1})^{2m-1}\cos\theta\sin^{n-1}\theta=n(\sqrt{-1})^{2m-2}(\sqrt{-1})\cos\theta\sin^{n-1}\theta$

$=n(-1)^{m-1}(\sqrt{-1})\cos\theta\sin^{n-1}\theta=n(-1)^{\frac{n-1}{2}}\sqrt{-1}\cos\theta\sin^{n-1}\theta$, 由是

$\sin n\theta$ 展開式之末項爲 $n(-1)^{\frac{n-1}{2}}\sqrt{-1}\cos\theta\sin^{n-1}\theta$,

$\cos n\theta$ 展開式之末項爲 $(-1)^{\frac{n}{2}}\sin n\theta$,

若n爲奇數,則$(\cos\theta+\sqrt{-1}\sin\theta)^n$展開式之末項爲虛數.即

$(\sqrt{-1})^n\sin^n\theta=(\sqrt{-1})^{2m+1}\sin^n\theta=(\sqrt{-1})^{2m}(\sqrt{-1})\sin^n\theta$

$=(-1)^m(\sqrt{-1})\sin^n\theta=(-1)^{\frac{2n-1}{2}}(\sqrt{-1})\sin^n\theta,$

又末項之次一項爲實數,即

$n(\sqrt{-1})^{n-1}\cos\theta\sin^{n-1}\theta=n(\sqrt{-1})^{2m}\cos\theta\sin^{n-1}\theta=n(-1)^{\frac{n}{2}}\cos\theta\sin^{n-1}\theta$

由是$\sin n\theta$展開式之末項爲$n(-1)^{\frac{n}{2}}\cos\theta\sin^{n-1}\theta,$

$\cos n\theta$展開式之末項爲$(-1)^{\frac{2n-1}{2}}(\sqrt{-1})\sin^n\theta.$

6. 正餘弦之級數展開式

α爲弧度.則

$$\sin\alpha=\alpha-\frac{\alpha^3}{\lfloor 3}+\frac{\alpha^5}{\lfloor 5}-\frac{\alpha^7}{\lfloor 7}+\cdots\cdots\cdots$$

$$\cos\alpha=1-\frac{\alpha^2}{\lfloor 2}+\frac{\alpha^4}{\lfloor 4}-\frac{\alpha^6}{\lfloor 6}+\cdots\cdots\cdots$$

〔證〕於4.節之公式.令 $n\theta=\alpha$, 則 $n=\alpha/\theta$,

$$\sin n\theta=n\cos^{n-1}\theta\sin\theta-\frac{n(n-1)(n-2)}{\lfloor 3}\cos^{n-3}\theta\sin^3\theta+\cdots\cdots\cdots$$

即 $$\sin\alpha=\frac{\alpha}{\theta}\cos^{n-1}\theta\sin\theta-\frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{\theta^3\lfloor 3}\cos^{n-3}\theta\sin^3\theta+\cdots\cdots\cdots$$

$$=\alpha\cos^{n-1}\theta\left(\frac{\sin\theta}{\theta}\right)-\frac{\alpha(\alpha-\theta)(\alpha-2\theta)}{\lfloor 3}\cos^{n-3}\theta\left(\frac{\sin\theta}{\theta}\right)^3+\cdots\cdots$$

θ次第減小至於極限$\frac{\sin\theta}{\theta}=1$,(第十編2.節)而$\theta$殆等於0.故上之級數,可爲本題之結果.

又由 $\cos n\theta=\cos^n\theta-\frac{n(n-1)}{\lfloor 2}\cos^{n-2}\theta\sin^2\theta+\cdots\cdots\cdots$得

$$\cos\alpha=\cos^n\theta-\frac{n(n-1)}{\lfloor 2}\cos^{n-2}\theta\left(\frac{\sin\theta}{\theta}\right)^2+\cdots\cdots\cdots$$

又θ殆等於0.故$\cos^n\theta=1,\ \frac{\sin\theta}{\theta}=1.$

7. 餘論

又α爲$n°$(即n度)之角之弧度,則

$\sin n°=\sin\alpha$, 而 $\alpha=\frac{n\pi}{180}$, (從第十編4.節之解說)

由是前節之公式如次.

$$\sin n^\circ = \frac{n\pi}{180} - \frac{1}{\lfloor 3}\left(\frac{n\pi}{180}\right)^3 + \frac{1}{\lfloor 5}\left(\frac{n\pi}{180}\right)^5 - \cdots\cdots$$

$$\cos n^\circ = 1 - \frac{1}{2}\left(\frac{n\pi}{180}\right)^2 + \frac{1}{\lfloor 4}\left(\frac{n\pi}{180}\right)^4 - \cdots\cdots$$

8. 歛級數 正餘之級數(6.節)為歛級數.

〔證〕 $\sin\alpha$ 之級數,其第 n 項 $=\frac{(-1)^{n-1}\alpha^{2n-1}}{\lfloor 2n-1}$, 第 $(n+1)$ 項 $=\frac{(-1)^n\alpha^{2n+1}}{\lfloor 2n+1}$

故 $\frac{第(n+1)項}{第 n 項}=\frac{\alpha^2}{2n(3n+1)}$, 故 α 為任何值. n 為充分增大之值.

則 $\frac{\alpha^2}{2n(2n+1)}$ 比某有限值小明甚.

(見代數學無限級數之歛級數論)

而 $\cos\alpha$ 之級數,亦可同樣證之.

9. 6.節之證明尙未充分. 卽 $\cos\alpha$ 之級數. 其第 γ 項嚴密記之.為

$$(-1)^r\frac{n(n-1)(n-2)\cdots\cdots(n-2r+1)}{\lfloor 2r}\cos^{n-2r}\theta\sin^{2r}\theta,$$

令 $n\theta=\alpha$, 則 $(-1)^r\frac{\alpha(\alpha-\theta)(\alpha-2\theta)\cdots\cdots(\alpha-2r\theta+\theta)}{\lfloor 2r}\cos^{n-2r}\theta\left(\frac{\sin\theta}{\theta}\right)^{2r}$,

而於第十編2.節之證明中 $\cos^{n-2r}\theta$ 之極限為1, 而 $\left(\frac{\sin\theta}{\theta}\right)^{2r}$ 之極限亦為1.然其證明之缺點.為 $\frac{\alpha(\alpha-\theta)(\alpha-2\theta)\cdots\cdots(\theta-2r\theta+\theta)}{\lfloor 2r}$ 之極限.對於 r 之一切值.均為 $\frac{\alpha^{2r}}{\lfloor 2r}$,此未證明.今特證之.

$r=1$. 則 $\frac{\alpha(\alpha-\theta)}{2}$ 之極限為 $\frac{\alpha^2}{2}$ 明甚,而求其結果常為眞.由歸納法可得其證.今假定如次.

$$\frac{\alpha(\alpha-\theta)(\alpha-2\theta)\cdots\cdots(\alpha-2r\theta+\theta)}{\lfloor 2r} = \frac{\alpha^{2r}}{\lfloor 2r} + R.$$

但 θ 無限減小,右邊之極限為 $\frac{\alpha^{2r}}{\lfloor 2r}$, 則R消失.

此兩邊以 $\frac{\alpha-2r\theta}{2r+1}$ 乘之,則 $\frac{\alpha(\alpha-\theta)(\alpha-2\theta)\ldots\ldots(\alpha-2r\theta)}{\lfloor 2r+1}$

$$=\left(\frac{\alpha^{2r}}{\lfloor 2r}+\mathrm{R}\right)\left(\frac{\alpha}{2r+1}-\frac{2r\theta}{2r+1}\right)=\frac{\alpha^{2r+1}}{\lfloor 2r+1}+\frac{\mathrm{R}\alpha}{2r+1}-\frac{2r\theta}{2r+1}\left(\frac{\alpha^{2r}}{\lfloor 2r}+\mathrm{R}\right).$$

而 θ 無限減小.則右邊各項.除 $\frac{\alpha^{2r+1}}{\lfloor 2r+1}$ 外均消失,因由假定 θ 殆近於0,則R消失故也.

故 $\cos\alpha$ 之級數.其第 r 項之極限為 $\frac{\alpha^{2r}}{\lfloor 2r}$, 第 $r+1$ 項為 $\frac{\alpha^{2r+1}}{\lfloor 2r+1}$

然第2項之極限為 $\frac{\alpha^2}{\lfloor 2}$, 故第3項之極限為 $\frac{\alpha^3}{\lfloor 3}$ 以下類推.

10. 假設 令 $x=\cos\theta+\sqrt{-1}\sin\theta$, 則

$$\frac{1}{x}=\frac{1}{\cos\theta+\sqrt{-1}\sin\theta}=\cos\theta-\sqrt{-1}\sin\theta,$$

故 $x+\frac{1}{x}=2\cos\theta$, 及 $x-\frac{1}{x}=2\sqrt{-1}\sin\theta$, (1)

但 $x+\frac{1}{x}$ 對於 x 之任意值,常比2更大.

又 $\cos\theta$ 不能大於1, 故 $2\cos\theta$ 不能大於2.

故 $x+\frac{1}{x}=2\cos\theta$, 為不合理.

故此處之 x.只可認為 θ 之任何數量之一記號.

即 θ 為三角函數之角之量.故可認 x 為一記號.

又 $x^n=(\cos\theta+\sqrt{-1}\sin\theta)^n=\cos n\theta+\sqrt{-1}\sin n\theta$,

$$\frac{1}{x^n}=(\cos\theta-\sqrt{-1}\sin\theta)^n=\cos n\theta-\sqrt{-1}\sin n\theta,$$

故 $x^n+\frac{1}{x^n}=2\cos n\theta$, 及 $x^n-\frac{1}{x^n}=2\sqrt{-1}\sin n\theta$. (2)

由(1)及(2)之關係,得如次節以 $\sin\theta$ 及 $\cos\theta$ 表倍角正餘弦之項.而其最初比2大之 $x+\frac{1}{x}$,可假設其不大於2而等於 $2\cos\theta$ 以推考之.其結果亦合理.

11. 含倍角正餘弦之級數 n為正整數.

$$(2\cos\theta)^n=2\cos n\theta+n2\cos(n-2)\theta+\frac{n(n-1)}{\lfloor 2}2\cos(n-4)\theta$$

$$+\cdots\cdots+\frac{n(n-1)\cdots\cdots(n-r+1)}{\lfloor r}2\cos(n-2r)\theta+\cdots\cdots\cdots$$

n為偶數.

$$(2\sqrt{-1}\sin\theta)^n=2\cos n\theta-n2\cos(n-2)\theta+\frac{n(n-1)}{\lfloor 2}2\cos(n-4)\theta$$

$$+\cdots\cdots\cdots+(-1)^{n-r}\frac{n(n-1)\cdots\cdots(n-r+1)}{\lfloor r}2\cos(n-2r)\theta+\cdots\cdots$$

n為奇數.

$$(2\sqrt{-1}\sin\theta)^n=2\sqrt{-1}\sin n\theta-n2\sqrt{-1}\sin(n-2)\theta$$

$$+\frac{n(n-1)}{\lfloor 2}2\sqrt{-1}\sin(n-4)\theta$$

$$+\cdots\cdots+(-1)^{n-r}\frac{n(n-1)\cdots\cdots(n-r+1)}{\lfloor r}2\sqrt{-1}\sin(n-2\gamma)\theta+\cdots\cdots$$

〔證〕 由10.節. $(2\cos\theta)^n=(x+x^{-1})^n$

$$=x^n+nx^{n-2}+\frac{n(n-1)}{\lfloor 2}x^{n-4}+\cdots+\frac{n(n-1)\cdots(n-r+1)}{\lfloor r}x^{n-2r}+\cdots+nx^{-(n-2)}+x^{-n}$$

$$=(x^n+x^{-n})+n(x^{n-2}+x^{-(n-2)})+\frac{n(n-1)}{\lfloor 2}(x^{n-4}+x^{-(n-4)})+\cdots\cdots\cdots\cdots$$

$$=2\cos n\theta+n2\cos(n-2)\theta+\frac{n(n-1)}{\lfloor 2}2\cos(n-4)\theta+\cdots\cdots\cdots\cdots\cdots\cdots$$

又由10.節. n為偶數,則 $(2\sqrt{-1}\sin\theta)^n=(x-x^{-1})^n$

$$=x^n-nx^{n-2}+\frac{n(n-1)}{\lfloor 2}x^{n-4}-\cdots\cdots-nx^{-(n-2)}+x^{-n}$$

$$=(x^n+x^{-n})-n(x^{n-2}+x^{-(n-2)})+\frac{n(n-1)}{\lfloor 2}(x^{n-4}+x^{-(n-4)})-\cdots\cdots\cdots$$

$$=2\cos n\theta-n2\cos(n-2)\theta+\frac{n(n-1)}{\lfloor 2}2\cos(n-4)\theta-\cdots\cdots\cdots\cdots$$

又 n 爲奇數,則 $(2\sqrt{-1}\sin\theta)^n=(x-x^{-1})^n$

$$=x^n-nx^{n-2}+\frac{n(n-1)}{\lfloor 2}x^{n-4}-\cdots\cdots+nx^{-(n-2)}-x^{-n}$$

$$=(x^n-x^{-n})-n(x^{n-2}-x^{-(n-2)})+\frac{n(n-1)}{\lfloor 2}(x^{n-4}-x^{-(n-4)})\cdots\cdots\cdots$$

$$=2\sqrt{-1}\sin n\theta-n2\sqrt{-1}\sin(n-2)\theta+\frac{n(n-1)}{\lfloor 2}2\sqrt{-1}\sin(n-4)\theta-\cdots\cdots\cdots$$

12. 餘論

$(x+x^{-1})^n$ 之展開式有 $n+1$ 項.故 n 爲偶數.則有一中央項,卽第 $(\frac{1}{2}n+1)$ 項爲中央項.其項爲

$$+\frac{n(n-1)\cdots\cdots(\frac{1}{2}n+1)}{\lfloor \frac{1}{2}n}x^{\frac{n}{2}}(x^{-1})^{\frac{n}{2}}\quad 即\quad +\frac{n(n-1)\cdots\cdots(\frac{1}{2}n+1)}{\lfloor \frac{1}{2}n},$$

故 x 卽含有 θ,由是 n 爲偶數.則

$$(2\cos\theta)^n\ 之末項爲\ +\frac{n(n-1)\cdots\cdots(\frac{1}{2}n+1)}{\lfloor \frac{1}{2}n},$$

$$(2\sqrt{-1}\sin\theta)^n\ 之末項爲+(-1)^{\frac{n}{2}}\frac{n(n-1)\cdots\cdots(\frac{1}{2}n+1)}{\lfloor \frac{1}{2}n},$$

又 n 爲奇數,則 $(x+x^{-1})^n$ 有兩中央項,卽第 $\frac{1}{2}(n-1)+1$ 項與第 $\frac{1}{2}(n+1)+1$ 項.

故此兩項爲 $\frac{n(n-1)\cdots\cdots\frac{1}{2}(n+3)}{\lfloor \frac{1}{2}(n-1)}x$ 及

$\frac{n(n-1)\cdots\cdots\frac{1}{2}(n+3)}{\lfloor \frac{1}{2}(n-1)}x^{-1}$, 由是 n 爲奇數,則 $(2\cos\theta)^n$ 之末項爲

$$+\frac{n(n-1)\cdots\cdots\frac{1}{2}(n+3)}{\lfloor \frac{1}{2}(n-1)}(x+x^{-1}),\quad 卽\quad +\frac{n(n-1)\cdots\cdots\frac{1}{2}(n+3)}{\lfloor \frac{1}{2}(n-1)}2\cos\theta,$$

$$(2\sqrt{-1}\sin\theta)^n 之末項爲+(-1)^{\frac{n-1}{2}}\frac{n(n-1)\cdots\cdots\frac{1}{2}(n+3)}{\lfloor \frac{1}{2}(n-1)}2\sqrt{-1}\sin\theta.$$

13. 倍角之餘弦別式

於 4. 節. $\cos n\theta=\cos^n\theta-\frac{n(n-1)}{\lfloor 2}\cos^{n-2}\theta\sin^2\theta$

$$+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}\cos^{n-4}\theta\sin^4\theta+\cdots\cdots\cdots\cdots$$

示此級數爲 $\cos\theta$ 之項如次.

$$2\cos n\theta=(2\cos\theta)^n-n(2\cos\theta)^{n-2}+\frac{n(n-3)}{\underline{|2}}(2\cos\theta)^{n-4}-\cdots\cdots$$

〔證〕 由 10. 節 $x+\frac{1}{x}=2\cos\theta,\quad x^n+\frac{1}{x^n}=2\cos n\theta,$

令 $\quad(1-zx)\left(1-\frac{z}{x}\right)=1-z\left(x+\frac{1}{x}\right)+z^2=1-zc+z^2,$

但 $\quad c=2\cos\theta \qquad\qquad =1-z(c-z).$

此以對數式示之,則

$$\log(1-zx)+\log\left(1-\frac{z}{x}\right)=\log\{1-z(c-z)\},$$

由第十一編 4. 節. (但 $\log$ 爲 $\log_e$ 之略)

$$\left(zx+\frac{1}{2}z^2x^2+\frac{1}{3}z^3x^3+\cdots\cdots\right)+\left(\frac{z}{x}+\frac{1}{2}\cdot\frac{z^2}{x^2}+\frac{1}{3}\cdot\frac{z^3}{x^3}+\cdots\cdots\right)$$

$$=z(c-z)+\frac{1}{2}z^2(c-z)^2+\frac{1}{3}z^3(c-z)^3+\cdots\cdots+\frac{1}{n}z^n(c-z)^n+\cdots\cdots$$

於此恆同式之兩邊比較 z^n 之係數.則

左邊爲 $\frac{1}{n}x^n+\frac{1}{n}\cdot\frac{1}{x^n}$, 即 $\frac{1}{n}\left(x^n+\frac{1}{x^n}\right)$.

右邊於 $\frac{1}{n}z^n(c-z)^n$ 得 $\frac{1}{n}c^n$

於 $\frac{1}{n-1}z^{n-1}(c-z)^{n-1}$ 得 $-\frac{1}{n-1}(n-1)c^{n-2},$

於 $\frac{1}{n-2}z^{n-2}(c-z)^{n-2}$ 得 $\frac{1}{n-2}\frac{(n-2)(n-3)}{\underline{|2}}c^{n-4},$

於一般 $\frac{1}{n-r}z^{n-r}(c-z)^{n-r}$ 得

$$\frac{(-1)_r}{n-r}\frac{(n-r)(n-r-1)\cdots\cdots(n-2r+1)}{\underline{|r}}c^{n-2r},$$

故 $\frac{1}{n}\left(x^n+\frac{1}{x^n}\right)=\frac{1}{n}c^n-\frac{1}{n-1}(n-1)c^{n-2}+\frac{1}{n-2}\frac{(n-2)(n-3)}{\underline{|2}}c^{n-4}$

$$+\frac{(-1)^r}{n-r}\frac{(n-r)(n-r-1)\cdots\cdots(n-2r+1)}{\underline{|r}}c^{n-2r}+\cdots\cdots$$

卽 $x^n+\frac{1}{x^n}=c^n-nc^{n-2}+\frac{n(n-3)}{\underline{|2}}c^{n-4}-\cdots\cdots$

$$+(-1)^r\frac{n(n-r-1)(n-r-2)\cdots\cdots(n-2r+1)}{\underline{|r}}c^{n-2r}+\cdots\cdots$$

卽 $2\cos n\theta=(2\cos\theta)^n-n(2\cos\theta)^{n-2}+\frac{n(n-3)}{\underline{|2}}(2\cos\theta)^{n-4}-\cdots\cdots$

$$+(-1)^r\frac{n(n-r-1)(n-r-2)\cdots\cdots(n-2r+1)}{\underline{|r}}(2\cos\theta)^{n-2r}+\cdots\cdots$$

14. 倍角之正餘弦第二 n爲正整數.

n爲偶數.

$$\left.\begin{array}{l}\cos n\theta=1-\frac{n^2}{\underline{|2}}\sin^2\theta+\frac{n^2(n^2-2^2)}{\underline{|4}}\sin^4\theta-\frac{n^2(n^2-2^2)(n^2-4^2)}{\underline{|6}}\sin^6\theta+\cdots\cdots \\ \sin n\theta=n\cos\theta\left\{\sin^2\theta-\frac{n^2-2^2}{\underline{|3}}\sin^3\theta+\frac{(n^2-2^2)(n^2-4^2)}{\underline{|5}}\sin^5\theta-\cdots\cdots\right\}\end{array}\right\}$$

n爲奇數.

$$\sin n\theta=n\left\{\sin\theta-\frac{n^2-1}{\underline{|3}}\sin^3\theta+\frac{(n^2-1)(n^2-3^2)}{\underline{|5}}\sin^5\theta-\cdots\cdots\right\}$$

$$\cos n\theta=\cos\theta\left\{1-\frac{n^2-1}{\underline{|2}}\sin^2\theta+\frac{(n^2-1)(n^2-3^2)}{\underline{|4}}\sin^4\theta-\cdots\cdots\right\}$$

〔證〕 n爲偶數.則前節之級數之各項.凡爲$\cos\theta$項者爲偶數方乘,故$\cos^2\theta$可以$1-\sin^2\theta$代之.得表其各項爲$\sin\theta$之偶數方乘之項.故此得假定爲$\sin\theta$之遞昇方乘如次.

$$\cos n\theta=1+A_2\sin^2\theta+A_4\sin^4\theta+A_6\sin^6\theta+\cdots\cdots+A_n\sin^n\theta.$$

〔注意〕 於此假定式.其右邊之級數爲$n+1$項.則n爲正整數.故由二項式之定理.其第$n+1$項以下之係數爲0,卽此級數爲有限項.

於上所假定之恒同式.以$\theta+h$代其θ.則左邊如次.

$\cos n(\theta+h)=\cos n\theta\cos nh-\sin n\theta\sin nh,$ 由6.節.

$$=\cos n\theta\left(1-\frac{n^2h^2}{\lfloor 2}+\cdots\cdots\right)-\sin n\theta(nh-\cdots\cdots)$$

$$=\cos n\theta-nh\sin n\theta-\frac{n^2h^2}{2}\cos n\theta+\cdots\cdots\cdots \quad (1)$$

又於右邊 $A_{2r}\sin^{2r}\theta$ 之項.令 θ 爲 $\theta+h$, 則

$$A_{2r}\sin^{2r}(\theta+h)=A_{2r}(\sin\theta\cosh+\cos\theta\sinh)^{2r},\text{ 由 6. 節.}$$

$$=A_{2r}\left\{\sin\theta\left(1-\frac{h^2}{\lfloor 2}+\cdots\cdots\right)+\cos\theta\left(h-\cdots\cdots\cdots\right)\right\}^{2r}$$

$$=A_{2r}\left\{\sin\theta+\left(h\cos\theta-\frac{h^2}{\lfloor 2}\sin\theta-\cdots\cdots\cdots\right)\right\}^{2r}$$

$$=A_{2r}\left\{\sin^{2r}\theta+2r\sin^{2r-1}\theta\left(h\cos\theta-\frac{h^2}{\lfloor 2}\sin\theta-\cdots\cdots\cdots\right)\right.$$

$$\left.+\frac{2r(2r-1)}{\lfloor 2}\sin^{2r-2}\theta\left(h\cos\theta-\frac{h^2}{\lfloor 2}\sin\theta-\cdots\cdots\cdots\right)^2+\cdots\cdots\right\}, \quad (2)$$

故於 $A_{2r}\sin^{2r}(\theta+h)$, h^2 之係數爲 $A_{2r}\left(\frac{2r(2r-1)}{\lfloor 2}\sin^{2r-2}\theta\cos^2\theta-r\sin^{2r}\theta\right)$.

此式之 r, 順次令爲 1, 2, 3,……則恆同式之右邊.其第一,第二,第三,第四,……項 h^2 之係數.順次爲

$$A_2(\cos^2\theta-\sin^2\theta),\quad A_4(2.3\sin^2\theta\cos^2\theta-2\sin^4\theta),\cdots\cdots\cdots$$

由是比較右邊與左邊 h^2 之係數,則

$$-\frac{n^2}{2}\cos n\theta=A_2(\cos^2\theta-\sin^2\theta)+A_4(2.3\sin^2\theta\cos^2\theta-2\sin^4\theta)+\cdots\cdots\cdots$$

$$+A_{2r}\left\{\frac{2r(2r-1)}{\lfloor 2}\sin^{2r-2}\theta\cos^2\theta-r\sin^{2r}\theta\right\}+\cdots\cdots\cdots$$

$\cos n\theta$ 用最初假定 $\cos n\theta$ 之級數代之.右邊之 $\cos^2\theta$, 以 $1-\sin^2\theta$ 代之.則上之恆同式如次.

$$-\frac{n^2}{2}(1+A_2\sin^2\theta+\cdots\cdots+A_{2r}\sin^{2r}\theta+\cdots\cdots)=A_2(1-2\sin^2\theta)+\cdots\cdots\cdots$$

$$+A_{2r}\{r(2r-1)\sin^{2r-2}\theta-2r^2\sin^{2r}\theta\}$$

$$+A_{2r+2}\{(r+1)(2r+1)\sin^{2r}\theta-2(r+1)^2\sin^{2r+2}\theta\}+\cdots\cdots\cdots$$

比較此兩邊 $\sin^{2r}\theta$ 之係數,則

$$-\frac{n^2}{2}A_{2r}=-2r^2A_{2r}+(r+1)(2r+1)A_{2r+2}.$$

故 $A_{2r+2}=-\frac{n^2-(2r)^2}{(2r+2)(2r+1)}A_{2r}$, 但 $A_0=1$,

故 $A_2=-\frac{n^2}{1.2}A_0=-\frac{n^2}{1.2}$, $A_4=-\frac{n^2-2^2}{3.4}A_2=\frac{n^2(n^2-2^2)}{1.2.3.4}$,

以下類推.由是最初之恆同式如次.

$$\cos n\theta=1-\frac{n^2}{\underline{|2}}\sin^2\theta+\frac{n^2(n^2-2^2)}{\underline{|4}}\sin^4\theta-\frac{n^2(n^2-2^2)(n^2-4^2)}{\underline{|6}}\sin^6\theta+\cdots\cdots$$

次又於 (1)h 之係數爲 $-n\sin n\theta$,

又於 (2)$A_{2r}\sin^{2r}(\theta+h)$, 其 h 之係數爲 $A_{2r}(2r\sin^{2r-1}\theta\cos\theta)$.

此式之 r, 順次命爲 $1,2,3,\cdots\cdots$ 則恆同式之第二, 第三, 第四,$\cdots\cdots$

項, 其 h 之係數, 順次爲 $A_2(2\sin\theta\cos\theta)$, $A_4(4\sin^3\theta\cos\theta,)\cdots\cdots$

由是恆同式之左右兩邊比較 h 之係數, 則

$$-n\sin n\theta=A_2(2\sin\theta\cos\theta)+A_4(4\sin^3\theta\cos\theta)+\cdots\cdots$$ 如前得

$$=-n^2\sin\theta\cos\theta+\frac{n^2(n^2-2^2)}{\underline{|3}}\sin^3\theta\cos\theta\cdots\cdots$$

故 $$\sin n\theta=n\cos\theta\left\{\sin\theta-\frac{n^2-2^2}{\underline{|3}}\sin^3\theta+\cdots\cdots\right\}$$

n 爲奇數. 4. 節 $\sin n\theta$ 右邊之數. 爲 $\cos\theta$ 之偶數方乘. 故得以 $\sin\theta$ 表之. 故此爲 $\sin\theta$ 之遞昇方乘, 得假定爲次之恆同式.

$$\sin n\theta=A_1\sin\theta+A_3\sin^3\theta+\cdots\cdots+A_n\sin^n\theta.$$

試如前法推之. 可得 $\sin n\theta$ 及 $\cos n\theta$.

15. 倍角之正餘弦第三　n 爲正整數.

n 爲偶數.

$$(\sqrt{-1})^n\cos n\theta=1-\frac{n^2}{\underline{|2}}\cos^2\theta+\frac{n^2(n^2-2^2)}{\underline{|4}}\cos^4\theta-\cdots\cdots$$

$$(\sqrt{-1})^{n+2}\sin n\theta=n\sin\theta\left\{\cos\theta-\frac{n^2-2^2}{\underline{|3}}\cos^3\theta+\frac{(n^2-2^2)(n^2-4^2)}{\underline{|5}}\cos^5\theta\cdots\cdots\right\}$$

n 爲奇數, 則

$$(\sqrt{-1})^{n-1}\cos n\theta=n\left\{\cos\theta-\frac{n^2-1}{\underline{|3}}\cos^3\theta+\frac{(n^2-1)(n^2-3^2)}{\underline{|5}}\cos^5\theta-\cdots\cdots\right\}$$

$$(\sqrt{-1})^{n-1}\sin n\theta=\sin\theta\left\{1-\frac{n^2-1}{\underline{|2}}\cos^2\theta+\frac{(n^2-1)(n^2-3^2)}{\underline{|4}}\cos^4\theta-\cdots\cdots\right\}$$

〔證〕於前節之公式.其θ代以 $\frac{\pi}{2}-\theta$. 則於第一公式

n為偶數($2m$). 故 $\cos n\left(\frac{\pi}{2}-\theta\right)=\cos 2m\left(\frac{\pi}{2}-\theta\right)$

$$=\cos(m\pi-2m\theta)=(-1)^m\cos 2m\theta=(\sqrt{-1})^n\cos n\theta,$$

而其右邊為 $1-\frac{n^2}{|\underline{2}}\cos^2\theta+\frac{n^2(n^2-4^2)}{|\underline{4}}\cos^4\theta-\cdots\cdots$

以下同法.順次可得其證,

例題二十七

求次各式之證.

1. $\sin 4\theta=4\cos^3\theta\sin\theta-4\cos\theta\sin^3\theta.$

2. $\cos 4\theta=\cos^4\theta-6\cos^2\theta\sin^2\theta+\sin^4\theta.$

3. $\cos 6\theta=\cos^6\theta-15\cos^4\theta\sin^2\theta+15\cos^2\theta\sin^4\theta-\sin^6\theta.$

4. $\sin 9\theta=9\cos^8\theta\sin\theta-84\cos^6\theta\sin^3\theta$

$$+126\cos^4\theta\sin^5\theta-36\cos^2\theta\sin^7\theta+\sin^9\theta.$$

5. $$\tan n\theta=\frac{n\tan\theta-\frac{n(n-1)(n-2)}{|\underline{3}}\tan^3\theta+\cdots\cdots}{1-\frac{n(n-1)}{|\underline{2}}\tan^2\theta+\frac{n(n-1)(n-2)(n-3)}{|\underline{4}}\tan^4\theta-\cdots\cdots}$$

6. $$\cot n\theta=\frac{\cot^n\theta-\frac{n(n-1)}{|\underline{2}}\cot^{n-2}\theta+\frac{n(n-1)(n-2)(n-3)}{|\underline{4}}\cot^{n-4}\theta+\cdots\cdots}{n\cot^{n-1}\theta-\frac{n(n-1)(n-2)}{|\underline{3}}\cot^{n-3}\theta+\cdots\cdots}$$

7. $8\cos^4\theta=\cos 4\theta+4\cos 2\theta+3.$

8. $64\cos^7\theta=\cos 7\theta+7\cos 5\theta+21\cos 3\theta+35\cos\theta.$

9. $-64\sin^7\theta=\sin 7\theta-7\sin 5\theta+21\sin 3\theta-35\sin\theta.$

10. $128\cos^8\theta=\cos 8\theta+8\cos 6\theta+28\cos 4\theta+56\cos 2\theta+35.$

11. $128\sin^8\theta=\cos 8\theta-8\cos 6\theta+28\cos 4\theta-56\cos 2\theta+35.$

12. $512\sin^{10}\theta=\cos 10\theta-10\cos 8\theta+45\cos 6\theta$

$$-120\cos 4\theta+210\cos 2\theta-126.$$

13. $\cos 4\theta=1-8\cos^2\theta+8\cos^4\theta.$

14. $-\cos 6\theta=1-18\cos^2\theta+48\cos^4\theta-32\cos^6\theta.$

15. $\cos 6\theta=1-18\sin^2\theta+48\sin^4\theta-32\sin^6\theta.$

16. $\cos 9\theta=9\cos\theta-120\cos^3\theta+432\cos^5\theta$

$$-576\cos^7\theta+256\cos^9\theta.$$

例題解自 1 至 16.

(1), (2), 可於 4. 節之公式, 令 $n=4$ 以求之.

(3), (4), 亦與上同樣. 可令 $n=6$, $n=9$.

(5) $\tan n\theta=\frac{\sin n\theta}{\cos n\theta}$, 用 4. 節之公式. 以 $\cos^n\theta$ 除其分母子.

(6) 與前同樣.

(7) 於 11. 節之公式. 令 $n=4$, 而由 12. 節.

末項爲 $+\frac{4.3}{\lfloor\frac{4}{2}}=\frac{4.3}{\lfloor 2}$, 故由公式.

$(2\cos\theta)^4=2\cos 4\theta+4.2\cos 2\theta+\frac{4.3}{\lfloor 2}$, 從此可得其證.

(8) 由 12. 節. 末項 $=\frac{7.6.5.}{\lfloor\frac{1}{2}(7-1)}2\cos\theta$, 故從 11. 節公式.

$(2\cos\theta)^7=2\cos 7\theta+7.2\cos 5\theta+\frac{7.6}{\lfloor 2}2\cos 3\theta+\frac{7.6.5}{\lfloor 3}2\cos\theta.$

(9) 同樣. 末項 $=+(-1)^3\frac{7.6.5}{\lfloor\frac{1}{2}(7-1)}2\sqrt{-1}\sin\theta=-\frac{7.6.5}{\lfloor 3}2\sqrt{-1}\sin\theta$

$(2\sqrt{-1}\sin\theta)^7=2\sqrt{-1}\sin 7\theta-7.2\sqrt{-1}\sin 5\theta$

$$+\frac{7.6}{\lfloor 2}2\sqrt{-1}\sin 3\theta-\frac{7.6.5}{\lfloor 3}2\sqrt{-1}\sin\theta.$$

(10), (11), (12), 亦從同節之公式.

(13), 15 節公式. $(\sqrt{-1})^4\cos 4\theta=1-\frac{4^2}{\lfloor 2}\cos^2\theta+\frac{4^2(4^2-2^2)}{\lfloor 4}\cos^4\theta,$

卽 $\cos 4\theta=1-8\cos^2\theta+8\cos^4\theta.$

(14), (16), 同節之公式. (15) 可用 14. 節之公式.

17. 示 $\sin^{4n+2}\theta$ 爲 θ 倍角之餘弦之項.

18. 示 $\sin^{4n+1}\theta$ 爲 θ 倍角正弦之項.

19. 示 $\cos^{2n}\theta$ 爲 θ 倍角餘弦之項.

20. 求次各式之證.

$$\sin(x+h)=\sin x+h\cos x-\frac{h^2}{\lfloor 2}\sin x-\frac{h^3}{\lfloor 3}\cos x+\cdots\cdots\cdots\cdots$$

$$\cos(x+h)=\cos x-h\sin x-\frac{h^2}{\lfloor 2}\cos x+\frac{h^3}{\lfloor 3}\sin x+\cdots\cdots\cdots\cdots$$

21. 大於 α 之正數. 其 $\alpha-\frac{1}{6}\alpha^3+\frac{1}{120}\alpha^5$ 與 $\sin\alpha$ 之差小於 $\frac{1}{5040}$.

22. 同上. $1-\frac{\alpha^2}{2}+\frac{\alpha^4}{24}$ 與 $\cos\alpha$ 之差, 小於 $\frac{\alpha^6}{720}$.

23. $\frac{\sin\theta}{\theta}=\frac{2165}{2166}$, 則 θ 殆等於 $3°$ 之弧度.

24. $\frac{\sin\theta}{\theta}=\frac{1013}{1014}$, 則 θ 殆等於 $4°,24'$ 之弧度.

25. $\frac{\sin\theta}{\theta}=\frac{863}{864}$, 則 θ 殆等於 $5°$ 之弧度.

26. $\sin\left(\frac{\pi}{6}+\theta\right)=\cdot 51$, 省略 θ 二次以上之項. 求 θ 之略近值.

27. $\sin\left(\frac{\pi}{4}+\theta\right)=\cdot 71$, 其 θ^3 及 θ 之高次項得以省略. 求 θ 之略近值.

28. $\theta=\cos\theta$, 若只有一根. 則 θ 比 $\frac{\pi}{4}$ 小.

例題解自 17. 至 28.

(17) 由 11. 節第二公式與 12. 節.

$$-(2\sin\theta)^{4n+2}=2\cos(4n+2)\theta-(4n+2)2\cos 4n\theta+(2n+1)(4n+1)2\cos(4n-2)\theta-\cdots\cdots$$

$$-\frac{(4n+2)(4n+1)\cdots\cdots(2n+2)}{\lfloor 2n+1}.$$

(18) 由 11. 節第三公式與 12. 節

$$(2\sin\theta)^{4n+1}=2\sin(4n+1)\theta-(4n+1)2\sin(4n-1)\theta+(4n+1)2n.2\sin(4n-2)\theta-$$

$$+\frac{(4n+1)4n\cdots\cdots(2n+2)}{\lfloor 2n}2\sin\theta.$$

(19) 由 11. 節第一公式與 12. 節. $(2\cos\theta)^{2n}=2\cos 2n\theta+2n2\cos(2n-2)\theta$

$$+n(2n-1)2\cos(2n-4)\theta+\cdots\cdots+2n(2n-1)\cdots\cdots(n+1)/\lfloor n.$$

(20) $\sin(x+h)=\sin x\cosh+\cos x\sin h$ 由 6. 節.

$$=\sin x\left(1-\frac{h^2}{\lfloor 2}+\cdots\cdots\right)+\cos x\left(h-\frac{h^3}{\lfloor 3}+\cdots\cdots\right)=\sin x+h\cos x-\frac{h^2}{\lfloor 2}\sin x-\cdots\cdots$$

第二亦用同様之方法.

(21) 由 6. 節. $\sin a=a-\frac{a^3}{\lfloor 3}+\frac{a^5}{\lfloor 5}-\frac{a^7}{\lfloor 7}+\frac{a^9}{\lfloor 9}-\cdots\cdots$

故 $a-\frac{a^3}{\lfloor 3}+\frac{a^5}{\lfloor 5}-\sin a=\frac{a^7}{\lfloor 7}-\frac{a^9}{\lfloor 9}+\frac{a^{11}}{\lfloor 11}-\frac{a^{13}}{\lfloor 13}+\frac{a^{15}}{\lfloor 15}-\cdots\cdots$

即 $a-\frac{a^3}{6}+\frac{a^5}{120}-\sin a=\frac{a^7}{5040}-\frac{a^9}{\lfloor 9}\left(\frac{110-a^2}{110}\right)-\frac{a^{13}}{\lfloor 13}\left(\frac{210-a^2}{210}\right)-\cdots\cdots$

$a \ngtr \sqrt{110}$, 故上式右邊之負項皆爲負. 故如題言.

(22) 同上. (23) 由 6. 節 $\left(\theta-\frac{\theta^3}{\lfloor 3}\right)/\theta=-\frac{2165}{2166}$, 即

$1-\frac{\theta^2}{6}=\frac{2165}{2166}$, $\theta=\frac{1}{19}$, 故 θ 之弧度 $=\frac{180°}{3.1416}\times\frac{1}{19}=3°$ (略近數)

(24), (25) 同上. (26) $\theta=.01159$.

(27) $\theta^2-2\theta+(.0029\cdots\cdots)2\sqrt{2}=0$.

(28) 令 θ 爲負數 $(-a)$, 則 $-a=\cos(-a)=+\cos a$, 是不合理. 故 θ 爲正數. 而 $\cos\theta$ 之 θ 爲自 0 至 $\frac{1}{2}\pi$ 漸漸減少. 而 $\frac{1}{2}\pi$ 比 $\cos\frac{1}{2}\pi$ 大. 故 θ 比 $\frac{1}{2}\pi$ 小. 只有一値.

29. $\tan x = kx$, 則有無限之根.

30. β爲$\cos\theta=\theta$之近似根.而比眞根大.則

$\beta-\dfrac{\beta-\cos\beta}{1+\sin\beta}$尙爲近似根,亦比眞根大.

31. 於ABC三角形.其C之弧度爲$\pi-\theta$,當θ甚小時,則c殆等

於$(a+b)\left\{1-\dfrac{ab\theta^2}{2(a+b)^2}\right\}$.

32. 同上. B之弧度爲$\pi-\phi$,當ϕ幾乎甚小時.則

$$c=(b-a)\left\{1+\frac{a}{b}\frac{\phi^2}{\underline{|2}}-\left(\frac{a}{b}-\frac{3a^2}{b^2}-\frac{3a^3}{b^3}\right)\frac{\phi^4}{\underline{|4}}\right\}$$

33. 同上. $C=30°$, $a=1$, $c=250$求A之角度.

34. 設　$\tan x=a_1x+\dfrac{a_3x^3}{\underline{|3}}+\dfrac{a_5x^5}{\underline{|5}}+\cdots\cdots\cdots\cdots$　則

$$a_{2n+1}=\frac{(2n+1)2n}{\underline{|2}}a_{2n-1}-\frac{(2n+1)2n(2n-1)(2n-2)}{\underline{|4}}a_{2n-3}$$

$$+\cdots\cdots\cdots+(-1)^{n+1}(2n+1)a_1+(-1)^n.$$

例題解自29.至34.

(29) $\tan x$其x自0至$\frac{\pi}{2}$之間.卽自0至∞,而

$\tan\alpha=\tan(n\pi+\alpha)$.故$n$之値可取至無限.於其各取之値,而與$\tan(n\pi+\alpha)=k(n\pi+\alpha)$適合者之根必有一.故如題言.

(30) 依題意.令其第二之近似根爲$\beta-\alpha$,由原方程式.

$$\cos(\beta-\alpha)=\beta-\alpha,\quad 卽\quad \cos\beta\cos\alpha+\sin\beta\sin\alpha=\beta-\alpha,$$

α又爲甚小.故　$\cos\alpha=1$.(略近數)故

$\cos\beta+\sin\beta\sin\alpha=\beta-\alpha$, 即 $\alpha=\dfrac{\beta-\cos\beta}{1+\sin\beta\dfrac{\sin\alpha}{\alpha}}=\dfrac{\beta-\cos\beta}{1+\sin\beta}$,

由是 $\beta-\alpha=\beta-\dfrac{\beta-\cos\beta}{1+\sin\beta}$.

(31) $c^2=a^2+b^2-2ab\cos(\pi-\theta)=a^2+b^2+2ab\cos\theta$

$=a^2+b^2+2ab\left\{1-\dfrac{\theta^2}{2}\right\}^2$, (6.節) $=(a+b)^2-ab\theta^2=(a+b)^2\left\{1-\dfrac{ab\theta^2}{(a+b)^2}\right\}$

$=(a+b)^2\left\{1-\dfrac{ab\theta^2}{2(a+b)^2}\right\}^2$, (略近數).

(32) $c=a\cos B+b\cos A$, (例題十七 13.) $=a\cos(\pi-\phi)+b\sqrt{(1-\sin^2 A)}$

$=-a\cos\phi+b\sqrt{\left(1-\dfrac{a^2\sin^2 B}{b^2}\right)}=-a\cos\phi+b\sqrt{\left(1-\dfrac{a^2}{b^2}\sin^2\phi\right)}$, B為鈍角,故$b$為最大邊.故$a/b<1$.故上之平方根.由二項式之定理如次.

$c=-a\cos\phi+b\left(1-\dfrac{a^2}{2b^2}\sin\phi-\dfrac{a^4}{8b^4}\sin^4\phi+\dots\dots\right)$

$=-a\left(1-\dfrac{\phi^2}{\underline{|2}}+\dfrac{\phi^4}{\underline{|4}}-\dots\dots\right)+b\left\{1-\dfrac{a^2}{2b^2}\left(\phi-\dfrac{\phi^3}{\underline{|3}}+\dots\dots\right)^2-\dfrac{a^4}{8b^4}\left(\phi-\dots\dots\right)^4\right\}$

$=b-a+\dfrac{(b-a)a\phi^2}{b\underline{|2}}-\dfrac{b\phi^4}{\underline{|4}}\left(\dfrac{a}{b}-\dfrac{4a^2}{b^2}+\dfrac{3a^4}{b^4}\right)+\dots\dots$

(33) $\sin A=\dfrac{a\sin C}{c}=\dfrac{1\times\sin 30^\circ}{250}=\dfrac{1}{500}=$ A之弧度 (略近數)

故 $A=\dfrac{180^\circ}{3.1416}\times\dfrac{1}{500}=\dfrac{3}{26.18}\times 60'=7'$. (略近數)

(34) $\tan(-x)=-\tan x$, 故作$\tan x$之級數,為x之奇數方乘.

故 $\tan x=a_1x+\dfrac{a_3x^3}{\underline{|3}}+\dfrac{a_5x^5}{\underline{|5}}+\dots\dots$ 即

$\sin x=\cos x\left(a_1x+\dfrac{a_3x^3}{\underline{|3}}+\dfrac{a_5x^5}{\underline{|5}}+\dots\dots\right)$由6.節.

$$x-\frac{x^3}{\underline{|3}}+\dots\dots+(-1)^n\frac{x^{2n+1}}{\underline{|2n+1}}+\dots\dots=\left\{1-\frac{x^2}{\underline{|2}}+\dots\dots+(-1)^n\frac{x^{2n}}{\underline{|2n}}+\dots\dots\right\}$$

$$\times\left\{a_1x+\frac{a_3x^3}{\underline{|3}}+\dots\dots+\frac{a_{2n-1}x^{2n+1}}{\underline{|2n+1}}+\dots\dots\right\},$$

比較兩邊x^{2n+1}之係數則

$$\frac{(-1)^n}{\underline{|2n+1}}=\frac{a_{2n+1}}{\underline{|2n+1}}-\frac{a_{2n-1}}{\underline{|2}\,\underline{|2n-1}}+\frac{a_{2n-3}}{\underline{|4}\,\underline{|2n-3}}+\dots\dots+(-1)^n\frac{a_1}{\underline{|2n}}.$$

35. 設 $\sec\theta=a_0+a_2\theta^2+a_4\theta^4+\cdots\cdots+a_{2n}\theta^{2n}+\cdots\cdots$　則

$$a_{2n}=\frac{a_{2n-2}}{|\underline{2}}-\frac{a_{2n-4}}{|\underline{4}}+\cdots\cdots+\frac{(-1)^{n+1}a}{|\underline{2n}}.$$

求次各式之證.

36. $\tan x=x+\frac{x^3}{3}+\frac{2x^5}{15}+\frac{17x^7}{315}+\cdots\cdots\cdots$

37. $\sin^3 x=\frac{3}{4}\left\{\frac{3^2-1}{|\underline{3}}x^3-\frac{3^4-1}{|\underline{5}}x^5+\cdots\cdots+(-1)^{n-1}\frac{3^{2n}-1}{|\underline{2n+1}}x^{2n+1}+\cdots\right\}$

38. $\tan 6\theta=\frac{6\tan\theta-20\tan^3\theta+6\tan^5\theta}{1-15\tan^2\theta+15\tan^4\theta-\tan^6\theta}.$

39. $\sin^5\theta\cos^6\theta=\frac{1}{2^{10}}\{\sin 11\theta+\sin 9\theta$

$$-5(\sin 7\theta+\sin 5\theta)+10(\sin 3\theta+\sin\theta)\}.$$

40. 次之兩級數.n 爲奇數,則其數相等.n 爲偶數,則其一等於0.

$$1-\frac{n(n-1)}{|\underline{2}}+\frac{n(n-1)(n-2)(n-3)}{|\underline{4}}-\cdots\cdots\cdots\cdots$$

$$n-\frac{n(n-1)(n-2)}{|\underline{3}}+\frac{n(n-1)(n-2)(n-3)(n-4)}{|\underline{5}}-\cdots\cdots\cdots\cdots$$

41. 次之級數至 $n+1$ 項之和等於0.

$$1-(2n-1)+\frac{2n(2n-3)}{|\underline{2}}-\frac{2n(2n-3)(2n-5)}{|\underline{3}}+\cdots\cdots\cdots\cdots$$

例題解自35.至41.

(35)　$1=\cos\theta(a_0+a_2\theta^2+a_4\theta^4\cdots\cdots+a_{2n}\theta^{2n}+\cdots\cdots)$, 由6.節.

$$1=\left(1-\frac{\theta^2}{|\underline{2}}+\cdots\cdots+(-1)^{2n}\frac{\theta^{2n}}{|\underline{2n}}+\cdots\cdots\right)\left(a_0+a_2\theta^2+\cdots\cdots+a_{2n}\theta^{2n}+\cdots\cdots\right),$$

比較兩邊 θ^{2n} 之係數,則 $0=\frac{a_{2n-2}}{|\underline{2}}-\frac{a_{2n-4}}{|\underline{4}}+\cdots\cdots+\frac{(-1)^{2n}a_0}{|\underline{2n}}.$

但 $sec\theta$ 可如 $cos\theta$ 表 θ 之偶數方乘之項.

(36) 於例題34. 令 $n=o$, 則 $a_1=(-1)^\circ=1$,

令 $n=1$, 則 $a_{n+1}=(-1)^2(2+1)a_1+(-1)^1=3-1$, 故 $a_3=2$,

令 $n=2$, 則 $a_5=\frac{(4+1)4}{\lfloor 2}a_3-(4+1)a_1+(-1)^2=10\times2-5\times1+1=16$,

故 $tan x=a_1x+\frac{a_3x^3}{\lfloor 3}+\frac{a_5x^5}{\lfloor 5}+\ldots\ldots=x+\frac{2x^3}{\lfloor 3}+\frac{16x^5}{\lfloor 5}+\ldots\ldots$

(37) $sin^3x=\frac{1}{4}(3sin x-sin 3x)$, 由6.節.

$$=\frac{3}{4}\left\{x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}-\ldots\ldots+(-1)^n\frac{x^{2n+1}}{\lfloor 2n+1}+\ldots\ldots\right\}$$

$$-\frac{1}{4}\left\{3x-\frac{(3x)^3}{\lfloor 3}+\frac{(3x)^5}{\lfloor 5}-\ldots\ldots+(-1)^n\frac{(3x)^{2n+1}}{\lfloor 2n+1}+\ldots\ldots\right\}$$

$$=\frac{3}{4}\left\{\frac{3^2-1}{\lfloor 3}x^3-\frac{3^4-1}{\lfloor 5}x^5+\ldots\ldots+(-1)^{n-1}\frac{3^{2n}-1}{\lfloor 2n+1}x^{2n+1}+\ldots\ldots\right\}.$$

(38) 於例題5. 可令 $n=6$ 以求之.

(39) $sin^5\theta cos^6\theta=\frac{1}{2^5}cos\theta(2sin\theta cos\theta)^5=\frac{cos\theta}{2^5}(sin 2\theta)^5$, 由11.節.

$$=\frac{cos\theta}{2^4}\times\frac{1}{2^4}(sin 10\theta-5sin 6\theta+10sin 2\theta)$$

$$=\frac{1}{2^{10}}\{(sin 11\theta+sin 9\theta)-5(sin 7\theta+sin 5\theta)+10(sin 3\theta+sin\theta)\}.$$

(40) 於例題5. 令 $\theta=\frac{\pi}{4}$, 則

$$tan n\left(\frac{\pi}{4}\right)=\frac{n-\frac{n(n-1)(n-2)}{\lfloor 3}+\frac{n(n-1)(n-2)(n-3)(n-4)}{\lfloor 5}+\ldots\ldots}{1-\frac{n(n-1)}{\lfloor 2}+\frac{n(n-1)(n-2)(n-3)}{\lfloor 4}-\ldots\ldots}$$

而 n 爲奇數, 則 $tan n\left(\frac{\pi}{4}\right)=1$, 故分母相等.

又 n 爲偶數, 則 $tan n\left(\frac{\pi}{4}\right)=\infty$, 故分母等於0.

(41) 於11.節之公式, n 爲 $2n+1$, 以 $2\sqrt{-1}$ 除之, 則

$$2^{2n}sin^{2n+1}\theta=sin(2n+1)\theta-(2n+1)sin(2n-1)\theta+\frac{(2n+1)2n}{\lfloor 2}sin(2n-3)\theta-\ldots\ldots$$

$$2^{2n}\{\theta-\ldots\ldots\}^{2n+1}=\{(2n+1)\theta-\ldots\ldots\}-\{(2n+1)(2n-1)\theta+\ldots\ldots\}$$

$$+\frac{(2n+1)2n}{\lfloor 2}\{(2n-3)\theta-\ldots\ldots\}-\ldots\ldots$$

比較兩邊 θ 之係數, 則

$$0=(2n-1)\left\{1-(2n-1)+\frac{2n(2n-3)}{\lfloor 2}-\ldots\ldots\right\}.$$

42. 某數與其反商之和等於 $\cos A$. 其數如何.

43. $\cos(\sqrt{-1}\theta)$ 爲實數. 而 $\sin(\sqrt{-1}\theta)$ 爲虛數.

44. 示次式之證.

$$\frac{1-h^2}{(1-h)^2\cos^2\frac{1}{2}x+(1+h)^2\sin^2\frac{1}{2}x}=1+2h\cos x+2h^2\cos 2x+\cdots\cdots$$

45. $x=\cos\theta+\sqrt{-1}\sin\theta$, 及 $\sqrt{(1-c^2)}=nc-1$. 則

$$1+c\cos\theta=\frac{c}{2n}(1+nx)\left(1+\frac{n}{x}\right).$$

46. 求小弧度之長. 可用次列之規則. 試證之.

從半弧之弦之8倍, 減全弧之弦. 所餘之三分之一. 殆等於其弧之長.

47. 同上. 用次示之規則. 更爲精密.

四分弧之一之弦. 以256倍之. 再加此弧之弦減半弧之弦之40倍. 其餘以45除之.

例題解自 42. 至 47.

(42) 令某數爲 x, 則 $x+\frac{1}{x}=\cos A$, 卽 $x^2-x\cos A+1=0$,

故 $x=\frac{\cos A}{2}\pm\frac{\sin A\sqrt{-1}}{2}$.

(43) 試用6. 節之公式, 則

$$\cos(\sqrt{-1}\theta)=1-\frac{1}{\underline{|2}}(\sqrt{-1}\theta)^2+\frac{1}{\underline{|4}}(\sqrt{-1}\theta)^4-\cdots\cdots\cdots$$

$$=1+\frac{\theta^2}{\underline{|2}}+\frac{\theta^4}{\underline{|4}}+\cdots\cdots=\text{實數}.$$

$$\sin(\sqrt{-1}\theta)=\sqrt{-1}\theta-\frac{1}{\underline{|3}}(\sqrt{-1}\theta)^3+\cdots\cdots$$

$$=\sqrt{-1}\left(\theta+\frac{1}{\underline{|3}}\theta^3+\cdots\cdots\right)=\text{虛數}.$$

(44) $$\frac{1-h^2}{(1+h^2-2h)\cos^2\frac{1}{2}x+(1+h^2+2h)\sin^2\frac{1}{2}x}=\frac{1-h^2}{1-2h\cos x+h^2},$$

令 $=A_0+A_1h+A_2h^2+\cdots\cdots+A_nh^n+\cdots\cdots$ 則

$1-h^2=(1-2h\cos x+h^2)(A_0+A_1h+A_2h^2+\cdots\cdots+A_nh^n+\cdots\cdots)$,

於此恆同式之右邊,其 h^n 之係數爲 $A_{n-2}-2A_{n-1}\cos x+A_n$.

但於此恆同式,令 $h=0$, 則 $A_0=1$,

故此右邊之係數 $=-2A_0\cos x+A_1$, 左邊 h 之係數爲 O.

故 $-2A_0\cos x+A_1=0$, 卽 $A_1=2A_0\cos x=2\cos x$,

又於右邊 h^2 之係數 $=A_0-2A_1\cos x+A_2$, 左邊 h^2 之係數爲 -1,

故 $A_0-2A_1\cos x+A_2=-1$, 卽

$A_2=-1-A_0+2A_2\cos x=-1-1+4\cos^2x=2\cos 2x$, 由此可得其證.

(45) $x+\dfrac{1}{x}=2\cos\theta$, $\dfrac{c}{2n}(1+nx)\left(1+\dfrac{n}{x}\right)=\dfrac{c}{2n}\left\{1+n^2+n\left(x+\dfrac{1}{x}\right)\right\}$

$=\dfrac{c}{2n}(1+n^2)+\dfrac{c}{2}\left(x+\dfrac{1}{x}\right)=\dfrac{1}{2nc}\left\{(nc-1)^2+2nc+c^2-1\right\}+c\cos\theta$

$=\dfrac{1}{2nc}\{1-c^2+2nc+c^2-1\}+c\cos\theta=1+c\cos\theta$.

(46) 令半徑爲 r, 弧度爲 θ, 則其弧之長爲 $r\theta$, 然弧之弦 $=2r\sin\frac{1}{2}\theta$, 半弧之弦 $=2r\sin\frac{1}{4}\theta$, 弦與半弦之若干倍爲 l, m, 則 $r\theta=l(2r\sin\frac{1}{2}\theta)+m(2r\sin\frac{1}{4}\theta)$, 卽

$\theta=2l\left\{\dfrac{\theta}{2}-\dfrac{1}{\lfloor 3}\left(\dfrac{\theta}{2}\right)^3+\cdots\cdots\right\}+2m\left\{\dfrac{\theta}{4}-\dfrac{1}{\lfloor 3}\left(\dfrac{\theta}{4}\right)^3+\cdots\cdots\right\}$, θ^3 以下省略不計,

比較兩邊之 θ 及 θ^3 之係數, 則 $l+\dfrac{1}{2}m=1$, $\dfrac{1}{8}l+\dfrac{1}{64}m=0$,

故 $l=-\dfrac{1}{3}$, $m=\dfrac{8}{3}$, 由此得證.

(47) 令四分弧之弦之倍數爲 n, 則由前例.

$r\theta=l(2r\sin\frac{1}{2}\theta)+m(2r\sin\frac{1}{4}\theta)+n(2r\sin\frac{1}{8}\theta)$, 卽

$\theta=2l\left\{\dfrac{\theta}{2}-\dfrac{1}{\lfloor 3}\left(\dfrac{\theta}{2}\right)^3+\dfrac{1}{\lfloor 5}\left(\dfrac{\theta}{2}\right)^5\right\}+2m\left\{\dfrac{\theta}{4}-\dfrac{1}{\lfloor 3}\left(\dfrac{\theta}{4}\right)^3+\dfrac{1}{\lfloor 5}\left(\dfrac{\theta}{4}\right)^5\right\}$

$+2n\left\{\dfrac{\theta}{8}-\dfrac{1}{\lfloor 3}\left(\dfrac{\theta}{8}\right)^3+\dfrac{1}{\lfloor 5}\left(\dfrac{\theta}{8}\right)^5-\cdots\cdots\right\}$, 於此式如前例,求得 l, m, n 之值, 而用至 θ^5, 故比前更精密.

卽 $l+\dfrac{1}{2}m+\dfrac{1}{4}n=1$, $\dfrac{1}{8}l+\dfrac{1}{64}m+\dfrac{1}{512}n=0$, $\dfrac{1}{32}l+\dfrac{1}{1024}m+\dfrac{1}{32768}n=0$.

由是 $l=\dfrac{1}{45}$, $m=-\dfrac{40}{45}$, $n=\dfrac{256}{45}$,

方　根

16. 方根 方程式 $x^q=a$, 當有 q 個根.

即 $\sqrt[q]{a}$ 可得 q 個根. 但 q 爲正整數.

而是等之根. 得表爲 $A+\sqrt{-1}\,B$ 之形. 今順次證明如次.

但 A, B 爲實數.

17. 定理一 有 $A+\sqrt{-1}\,B$ 之形之式. 常得表爲 $r(\cos\theta+\sqrt{-1}\sin\theta)$ 之形.

〔證〕 令 $A=r\cos\theta$, $B=r\sin\theta$, 則

$A^2+B^2=r^2\cos^2\theta+r^2\sin^2\theta=r^2$, $B/A=r\sin\theta/(r\cos\theta)=\tan\theta$,

由此兩方程式, 得表 $\cos\theta$, $\sin\theta$, 及 r 爲 A, B 之項.

18. 定理二 $\sqrt[q]{a}$ 之値, 等於次之 q 個値.

$$a^{\frac{1}{q}},\quad a^{\frac{1}{q}}\left(\cos\frac{2\pi}{q}+\sqrt{-1}\sin\frac{2\pi}{q}\right),\quad a^{\frac{1}{q}}\left(\cos\frac{4\pi}{q}+\sqrt{-1}\sin\frac{4\pi}{q}\right),$$

$$\cdots\cdots,\quad a^{\frac{1}{q}}\left\{\cos\frac{(2q-2)\pi}{q}+\sqrt{-1}\sin\frac{(2q-2)\pi}{q}\right\}.$$

〔證〕 由前節令 $a=a+\sqrt{-1}.0=r(\cos\theta+\sqrt{-1}\sin\theta)$, 則

$r^2=a^2+0^2$, 故 $r=a$, 又 $\tan\theta=\dfrac{0}{a}=0$, 故 $\theta=2n\pi$.

故 $a^{\frac{1}{q}}=a^{\frac{1}{q}}(\cos 2n\pi+\sqrt{-1}\,2n\pi)^{\frac{1}{q}}$,

又由 1. 節餘論. 得 $(\cos 2n\pi+\sqrt{-1}\sin 2n\pi)^{\frac{1}{q}}$ 之値如次.

n 順次令爲 $0, 1, 2, 3, \ldots\ldots (q-1)$, 可得如次.

即 $\cos 0+\sqrt{-1}\sin 0$, $\cos\dfrac{2\pi}{q}+\sqrt{-1}\sin\dfrac{2\pi}{q}$, $\cos\dfrac{4\pi}{q}+\sqrt{-1}\sin\dfrac{4\pi}{q}$, $\ldots\ldots$,

$\cos\dfrac{(2q-2)\pi}{q}+\sqrt{-1}\sin\dfrac{(2q-2)\pi}{q}$,

故如題言.

例題二十八

示次各式爲 $r(\cos\theta+\sqrt{-1}\sin\theta)$ 之形.

1. $1-\sqrt{-1}$.　　2. $\sqrt{3}+\sqrt{-1}$.　　3. $1+\sqrt{-3}$.

求次各式所有之値.

4. $(4\sqrt{2}+4\sqrt{-2})^{\frac{1}{3}}$.　5. $(4\sqrt{3}+4\sqrt{-1})^{\frac{1}{3}}$.　6. $(\sqrt{3}+\sqrt{-1})^{\frac{1}{5}}$.

7. $\sqrt[4]{1}$,　　8. $\sqrt[5]{32}$.　　9. $\sqrt[3]{27}$.

10. $\sqrt[3]{-1}$.　　11. $\sqrt[6]{-1}$.

例題解自 1. 至 11.

(1) 由 17. 節. $\tan\theta=-1/1=-1$, 故 $\theta=-\pi/4$, $r^2=1^2+(-1)^2=2$,
故 $1-\sqrt{-1}=\sqrt{2}\left\{\cos\left(-\frac{1}{4}\pi\right)+\sqrt{-1}\sin\left(-\frac{\pi}{4}\right)\right\}$.

(2) $2\left(\cos\frac{1}{6}\pi+\sqrt{-1}\sin\frac{1}{6}\pi\right)$,　　(3) $2\left(\cos\frac{1}{3}\pi+\sqrt{-1}\sin\frac{1}{3}\pi\right)$.

(4) $4\sqrt{2}+4\sqrt{-2}=r(\cos\theta+\sqrt{-1}\sin\theta)$, 由 17. 節. $r=8$,
$\theta=2n\pi+\frac{1}{4}\pi$, 故所求之方根 $=2\left(\cos\frac{1}{3}\theta+\sqrt{-1}\sin\frac{1}{3}\theta\right)$, 令 n 爲 0, 1, 2, 則所求之方根爲 $2\left(\cos\frac{1}{12}\pi+\sqrt{-1}\sin\frac{1}{12}\pi\right)$, $2\left(\cos\frac{3}{4}\pi+\sqrt{-1}\sin\frac{3}{4}\pi\right)$
$2\left(\cos\frac{17}{12}\pi+\sqrt{-1}\sin\frac{17}{12}\pi\right)$.

(5) $2\left(\cos\frac{1}{18}\pi+\sqrt{-1}\sin\frac{1}{18}\pi\right)$, $2\left(\cos\frac{13}{18}\pi+\sqrt{-1}\sin\frac{13}{18}\pi\right)$,
$2\left(\cos\frac{25}{18}\pi+\sqrt{-1}\sin\frac{25}{18}\pi\right)$.　　(6) $\sqrt[5]{2}\left(\cos\frac{1}{30}\pi+\sqrt{-1}\sin\frac{1}{30}\pi\right)$,
$\sqrt[5]{2}\left(\cos\frac{13}{30}\pi+\sqrt{-1}\sin\frac{13}{30}\pi\right)$, $\sqrt[5]{2}\left(\cos\frac{25}{30}\pi+\sqrt{-1}\sin\frac{25}{30}\pi\right)$,
$\sqrt[5]{2}\left(\cos\frac{37}{30}\pi+\sqrt{-1}\sin\frac{37}{30}\pi\right)$, $\sqrt[5]{2}\left(\cos\frac{49}{30}\pi+\sqrt{-1}\sin\frac{49}{30}\pi\right)$.

(7) $1=\cos 2n\pi+\sqrt{-1}\sin 2n\pi$, 故 $\sqrt[4]{1}=\cos\frac{n\pi}{2}+\sqrt{-1}\sin\frac{n\pi}{2}$,
令 n 爲 0, 1, 2, 3, 則 $\sqrt[4]{1}=1, \sqrt{-1}, -1, -\sqrt{-1}$.

(8) $2\left\{\cos\frac{1}{5}\left(2n\pi\right)+\sqrt{-1}\sin\frac{1}{5}\left(2n\pi\right)\right\}$, 可令 n 爲 0, 1, 2, 3, 4.

(9) $3, \frac{3}{2}\left(-1+\sqrt{-3}\right)$, $\frac{3}{2}\left(-1-\sqrt{-3}\right)$.

(10) $\cos\frac{1}{6}\left(2n+1\right)\pi+\sqrt{-1}\sin\frac{1}{6}\left(2n+1\right)\pi$, 可令 n 爲 0, 1, 2.

(11) $\cos\frac{1}{12}\left(2n+1\right)\pi+\sqrt{-1}\sin\frac{1}{12}\left(2n+1\right)\pi$, 可令 0, 1, 2, 3, 4, 5.

12. $(-1+\sqrt{-3})^5+(-1-\sqrt{-3})^5=-2^5$

13. $\{(\sqrt{-3}-1)/(\sqrt{-3}+1)\}^6=1.$

14. $(\sqrt{-1})^{n+4}+(-\sqrt{-1})^n=2\cos\frac{n\pi}{2}.$

15. $A+B+C=\pi$, 則 $x+\frac{1}{x}=2\cos A$,

$y+\frac{1}{y}=2\cos B$, $z+\frac{1}{z}=2\cos C$. 而 $x^2+y^2+z^2=\alpha+\beta\sqrt{-1}$,

但α在$-\frac{3}{2}$與3之間.

16. $A+B+C=180°$, r為弧度單位. 而

$x+\frac{1}{x}=\cos A$, $y+\frac{1}{y}=\cos B$, $z+\frac{1}{z}=\cos C$, $u+\frac{1}{u}=\cos r$,

則$\log_u(xyz)$之值為π或$-\pi$.

例題解自 12. 至 16.

(12) 令 $-1+\sqrt{-3}=r(\cos\theta+\sqrt{-1}\sin\theta)$, 則 $-1-\sqrt{-3}=r(\cos\theta-\sqrt{-1}\sin\theta)$,

故 $r^2=(-1)^2+(\sqrt{3})^2=4$, 故 $r=2$, $\tan\theta=\frac{\sqrt{3}}{-1}=-\sqrt{3}$,

故 $\theta=\frac{2}{3}\pi$, 由是原式$=r^5(\cos\theta+\sqrt{-1}\sin\theta)^5+r^5(\cos\theta-\sqrt{-1}\sin\theta)^5$

$=r^5(\cos5\theta+\sqrt{-1}\sin5\theta)+r^5(\cos5\theta-\sqrt{-1}\sin5\theta)=2r^5\cos5\theta$

$=2(2^5)\cos\frac{10\pi}{3}=2^6\cos\left(3\pi+\frac{\pi}{3}\right)=-2^6\cos\frac{\pi}{3}=-2^6\times\frac{1}{2}=-2^5.$

(13) $\left(\frac{-1+\sqrt{-3}}{1+\sqrt{-3}}\right)^6=\left\{\frac{r(\cos\theta-\sqrt{-3}\sin\theta)}{r(\cos\theta+\sqrt{-3}\sin\theta)}\right\}^6=\left\{\frac{(\cos\theta-\sqrt{-1}\sin\theta)^2}{\cos^2\theta+\sin^2\theta}\right\}^6$

$=(\cos\theta-\sqrt{-1}\sin\theta)^{12}=\cos12\theta-\sqrt{-1}\sin12\theta$, 但從前例 $\theta=\frac{2}{3}\pi$,

$=\cos8\pi-\sqrt{-1}\sin8\pi=1.$

(14) 原式$=(\sqrt{-1})^4(\sqrt{-1})^n+(-\sqrt{-1})^n=(\sqrt{-1})^n+(-\sqrt{-1})^n.$

$=\left(\cos\frac{\pi}{2}+\sqrt{-1}\sin\frac{\pi}{2}\right)^n+\left(\cos\frac{\pi}{2}-\sqrt{-1}\sin\frac{\pi}{2}\right)^n$

$=\cos\frac{n\pi}{2}+\sqrt{-1}\sin\frac{n\pi}{2}+\cos\frac{n\pi}{2}-\sqrt{-1}\sin\frac{n\pi}{2}=2\cos\frac{n\pi}{2}.$

(15) 由 10. 節. $x=\cos A+\sqrt{-1}\sin A$, $y=\cos B+\sqrt{-1}\sin B$,

$z=\cos C+\sqrt{-1}\sin C$, 由是.

$x^2+y^2+z^2=(\cos A+\sqrt{-1}\sin A)^2+(\cos B+\sqrt{-1}\sin B)^2+(\cos C+\sqrt{-1}\sin C)^2$

$=\cos 2A+\cos 2B+\cos 2C+(\sin 2A+\sin 2B+\sin 2C)\sqrt{-1}$,

由是 $\alpha=\cos 2A+\cos 2B+\cos 2C$ (見例題五 38.)

$=-1-4\cos A\cos B\cos C$,

但 $\cos A\cos B\cos C$ 為極小之數. $\cos A=\cos B=\cos C$, 卽 $A=B=C=\frac{\pi}{3}$.

故 $\cos A\cos B\cos C=\left(\cos\frac{\pi}{3}\right)^3=\frac{1}{8}$, 由是 $\alpha=-1-4\times\frac{1}{8}=-\frac{3}{2}$,

又 $\cos A\cos B\cos C$ 為極大之數, 則 A, B, C 當有二個為 0, 其所餘之一個為 π,

故 $\cos A\cos B\cos C=\cos\pi\cos 0\cos 0=-1$,

由是 $\alpha=-1-4(-1)=3$,

又 $\beta=\sin 2A+\sin 2B+\sin 2C$.

(16) 令 $\log_u(xyz)=X$, 則 $_ux=xyz$, 卽

$(\cos\gamma+\sqrt{-1}\sin\gamma)X=(\cos A+\sqrt{-1}\sin A)(\cos B+\sqrt{-1}\sin B)(\cos C+\sqrt{-1}\sin C)$,

卽 $\cos X\gamma+\sqrt{-1}\sin X\gamma=\cos(A+B+C)+\sqrt{-1}\sin(A+B+C)$,

但 γ 為弧度單位, 卽半徑角. 故 $\gamma=\frac{180^\circ}{\pi}$, (第一編 9. 節)

故 $\cos\frac{180^\circ X}{\pi}+\sqrt{-1}\sin\frac{180^\circ X}{\pi}=\cos 180^\circ+\sqrt{-1}\sin 180^\circ$,

或 $=\cos(-180^\circ)+\sqrt{-1}\sin(-180^\circ)$,

由是 $\frac{180^\circ X}{\pi}180^\circ$ 或 $\frac{180^\circ X}{\pi}=-180^\circ$,

卽 $X=\pi$, 或 $-\pi$,

卽 $\log_u(xyz)=\pi$, 或 $-\pi$.

指　　數

19. 三角函數之指數值

e 爲納白爾 (*Napier*) 氏對數之底數, (第十一編3.節) 則

$$\sin\theta=\frac{e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}}{2\sqrt{-1}},\qquad \cos\theta=\frac{e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}}{2},$$

$$\tan\theta=\frac{e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}}{\sqrt{-1}\,(e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}})},$$

〔證〕 於第十一編3.節之公式(6). 令 $x=\theta\sqrt{-1}$, 則

$$e^{\theta\sqrt{-1}}=1+\theta\sqrt{-1}+\frac{(\theta\sqrt{-1})^2}{\lfloor 2}+\frac{(\theta\sqrt{-1})^3}{\lfloor 3}+\cdots\cdots=1+\theta\sqrt{-1}-\frac{\theta^2}{\lfloor 2}-\frac{\theta^3\sqrt{-1}}{\lfloor 3}+\cdots\cdots$$

又令 $x=-\theta\sqrt{-1}$, 則　$e^{-\theta\sqrt{-1}}=1+\theta\sqrt{-1}-\frac{\theta^2}{\lfloor 2}+\frac{\theta^3\sqrt{-1}}{\lfloor 3}-\cdots\cdots$

故　$\frac{e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}}{2}=\sqrt{-1}\left(\theta-\frac{\theta^3}{\lfloor 3}+\frac{\theta^5}{\lfloor 5}-\cdots\cdots\right)=\sqrt{-1}\sin\theta,$　(6. 節)

又　$\frac{e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}}{2}=1-\frac{\theta^2}{\lfloor 2}+\frac{\theta^4}{\lfloor 4}-\cdots\cdots=\cos\theta,$　(6. 節)

又　$\tan\theta=\frac{\sin\theta}{\cos\theta}=\frac{e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}}{\sqrt{-1}(e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}})}.$

20. 虛數式及指數式之比較

$$\cos\theta+\sqrt{-1}\sin\theta=e^{\theta\sqrt{-1}},\qquad \cos\theta-\sqrt{-1}\sin\theta=e^{-\theta\sqrt{-1}},$$

〔證〕 由前節.

$$\cos\theta+\sqrt{-1}\sin\theta=\frac{e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}}{2}+\sqrt{-1}\times\frac{e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}}{2\sqrt{-1}}=e^{\theta\sqrt{-1}},\ 及$$

$$\cos\theta-\sqrt{-1}\sin\theta=\frac{e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}}{2}-\sqrt{-1}\times\frac{e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}}{2\sqrt{-1}}=e^{-\theta\sqrt{-1}}.$$

21. 週期函數 $e^{\theta\sqrt{-1}}$爲週期函數.

$\cos\theta+\sqrt{-1}\sin\theta=e^{\theta\sqrt{-1}}$, (前節) 而令 n 爲正整數, 則
$\cos(2n\pi+\theta)+\sqrt{-1}\sin(2n\pi+\theta)=\cos\theta+\sqrt{-1}\sin\theta$, 故
$e^{(2n\pi+\theta)\sqrt{-1}}=e^{\theta\sqrt{-1}}$, 卽 $2n\pi$ 爲週期之增加數.

〔餘論〕 $e^{(2n\pi+\theta)\sqrt{-1}}=e^{2n\pi\sqrt{-1}}.e^{\theta\sqrt{-1}}=e^{\theta\sqrt{-1}}$,
故 $e^{2n\pi\sqrt{-1}}=1$, 卽 $\cos 2n\pi+\sqrt{-1}\sin 2n\pi=1$,
又令 $a=e^x$, 則 $\log_e a=x$, 卽 $a=e^{\log_e a}$, 故
$a=a\times 1=e^{\log_e a}\times e^{2n\pi\sqrt{-1}}=e^{\log_e a+2n\pi\sqrt{-1}}$.

例題二十九

由指數證次各式.

1. $\cos^2 a+\sin^2 a=1$.　　2. $\cos 2a=\cos^2 a-\sin^2 a$.
3. $\sin\theta=-\sin(-\theta)$.　　4. $\cos\theta=\cos(-\theta)$.
5. $\cos(a+\beta)\cos(a-\beta)=\cos^2 a-\sin^2\beta$.
6. $\cos 3\theta=4\cos^3\theta-3\cos\theta$.　　7. $\sin 3\theta=3\sin\theta-4\sin^3\theta$.

例題解自 1. 至 7.

(1) $\cos^2 a+\sin^2 a=\dfrac{(e^{a\sqrt{-1}}+e^{-a\sqrt{-1}})^2}{4}+\dfrac{(e^{a\sqrt{-1}}-e^{-a\sqrt{-1}})^2}{4(-1)}$

$$=\frac{4e^{a\sqrt{-1}}e^{-a\sqrt{-1}}}{4}=1.$$

(2) $\cos 2a=\dfrac{e^{2a\sqrt{-1}}+e^{-2a\sqrt{-1}}}{2}=\dfrac{(e^{a\sqrt{-1}}+e^{-a\sqrt{-1}})^2-2}{2}=\dfrac{4\cos^2 a-2}{2}$.

(3) $\sin\theta=\dfrac{e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}}{2\sqrt{-1}}=-\dfrac{e^{(-\theta)\sqrt{-1}}-e^{-(-\theta)\sqrt{-1}}}{2\sqrt{-1}}=-\sin(-\theta)$.

(4) 同上.

(5) $\cos(a+\beta)\cos(a-\beta)=\dfrac{e^{(a+\beta)\sqrt{-1}}+e^{-(a+\beta)\sqrt{-1}}}{2}$

$$\times\frac{e^{(a-\beta)\sqrt{-1}}+e^{-(a-\beta)\sqrt{-1}}}{2}$$

$$=\frac{e^{2a\sqrt{-1}}+e^{-2a\sqrt{-1}}+e^{2\beta\sqrt{-1}}+e^{-2\beta\sqrt{-1}}}{4}$$

$$=\frac{1}{2}\left(\frac{e^{2a\sqrt{-1}}+e^{-2a\sqrt{-1}}}{2}+\frac{e^{2\beta\sqrt{-1}}+e^{-2\beta\sqrt{-1}}}{2}\right)$$

$=\frac{1}{2}(\cos 2a+\cos 2\beta)$, (6) 及 (7) 亦同法.

8. $2\cos n\alpha\cos\alpha=\cos(n+1)\alpha+\cos(n-1)\alpha.$

9. $\dfrac{\sin 2\theta}{1-\cos 2\theta}=\cot\theta.$

10. $\cos(\alpha+\beta\sqrt{-1})+\sqrt{-1}\sin(\alpha+\beta\sqrt{-1})=e^{-\beta}(\cos\alpha+\sqrt{-1}\sin\alpha).$

11. $\cos(\alpha+\theta\sqrt{-1})-\sqrt{-1}\sin(\alpha+\beta\sqrt{-1})=e^{\beta}(\cos\alpha-\sqrt{-1}\sin\alpha).$

12. $2\sin(\alpha+\beta\sqrt{-1})=(e^{\beta}+e^{-\beta})\sin\alpha+\sqrt{-1}(e^{\beta}-e^{-\beta})\cos\alpha.$

13. $4\cos(\alpha+\beta\sqrt{-1})\cos(\alpha-\beta\sqrt{-1})=2^{2\beta}+e^{-2\beta}+2\cos 2\alpha.$

14. $4\sin(\alpha+\beta\sqrt{-1})\cos(\alpha-\beta\sqrt{-1})=2\sin 2\alpha+\sqrt{-1}(e^{2\beta}-e^{-2\beta}).$

15. $\sec(\alpha+\beta\sqrt{-1})=\dfrac{2(e^{\beta}+e^{-\beta})\cos\alpha+2\sqrt{-1}(e^{\beta}-e^{-\beta})\sin\alpha}{e^{2\beta}+2\cos 2\alpha+e^{-2\beta}}.$

16. $\tan(\alpha+\beta\sqrt{-1})=\dfrac{2\sin 2\alpha+\sqrt{-1}(e^{2\beta}-e^{-2\beta})}{e^{2\beta}+2\cos 2\alpha+e^{-2\beta}}.$

17. $(\sqrt{-1})^{\sqrt{-1}}=e^{-\frac{\pi}{2}}.$

18. 設 $\cos(\alpha+\beta\sqrt{-1})=\cos\phi+\sqrt{-1}\sin\phi$, 則

$\sin\phi=\pm\sin^2\alpha$, 及 $e^{\beta}-e^{-\beta}=\pm 2\sin\alpha.$

例題解自 8. 至 18.

(8) $2\cos n\alpha\cos\alpha=2\left(\dfrac{e^{n\theta\sqrt{-1}}+e^{-n\alpha\sqrt{-1}}}{2}\right)\left(\dfrac{e^{\alpha\sqrt{-1}}+e^{-\alpha\sqrt{-1}}}{2}\right)$

$=\frac{1}{2}\{e^{(n+1)\alpha\sqrt{-1}}+e^{-(n+1)\alpha\sqrt{-1}}+e^{(n-1)\alpha\sqrt{-1}}+e^{-(n-1)\alpha\sqrt{-1}}\}$

$=\cos(n+1)\alpha+\cos(n-1)\alpha.$

(9) $\dfrac{\sin 2\theta}{1-\cos 2\theta}=\dfrac{\frac{1}{2\sqrt{-1}}(e^{2\theta\sqrt{-1}}-e^{-2\theta\sqrt{-1}})}{1-\frac{1}{2}(e^{2\theta\sqrt{-1}}+e^{-2\theta\sqrt{-1}})}=\dfrac{e^{2\theta\sqrt{-1}}-e^{-2\theta\sqrt{-1}}}{-\sqrt{-1}(e^{2\theta\sqrt{-1}}-2+e^{-2\theta\sqrt{-1}})}$

$=\dfrac{e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}}{-\sqrt{-1}(e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}})}=\dfrac{2\cos\theta}{-(\sqrt{-1})2\sqrt{-1}\sin\theta}=\cot\theta.$

(10) $\cos(\alpha+\beta\sqrt{-1})+\sqrt{-1}\sin(\alpha+\beta\sqrt{-1})$

$=\frac{1}{2}\{e^{(\alpha+\beta\sqrt{-1})\sqrt{-1}}+e^{-(\alpha+\beta\sqrt{-1})\sqrt{-1}}\}$

$+\frac{1}{2}\{e^{(\alpha+\beta\sqrt{-1})\sqrt{-1}}-e^{(\alpha+\beta\sqrt{-1})\sqrt{-1}}\}=e^{(\alpha+\beta\sqrt{-1})\sqrt{-1}}=e^{-\beta}e^{\alpha\sqrt{-1}}$

$=e^{-\beta}(\cos\alpha+\sqrt{-1}\sin\alpha)$, 〔見 19. 節〕　(11) 同上.

(12) $2\sin(\alpha+\beta\sqrt{-1})=\frac{1}{\sqrt{-1}}\{e^{(\alpha+\beta\sqrt{-1})\sqrt{-1}}-e^{-(\alpha+\beta\sqrt{-1})\sqrt{-1}}\}$

$=-\sqrt{-1}\{e^{\alpha\sqrt{-1}}e^{-\beta}-e^{-\alpha\sqrt{-1}}e^{\beta}\}$, 由 19. 節.

$=-\sqrt{-1}\{(\cos\alpha+\sqrt{-1}\sin\alpha)e^{-\beta}-(\cos\alpha-\sqrt{-1}\sin\alpha)e^{\beta}\}$

$=(e^{\beta}+e^{-\beta})\sin\alpha+\sqrt{-1}(e^{\beta}-e^{-\beta})\cos\alpha.$

(13) 原式

$=\{e^{(\alpha+\beta\sqrt{-1})\sqrt{-1}}+e^{-(\alpha-\beta\sqrt{-1})\sqrt{-1}}e^{(\alpha-\beta\sqrt{-1})\sqrt{-1}}+e^{-(\alpha-\beta\sqrt{-1})\sqrt{-1}}$

$=e^{2\alpha\sqrt{-1}}+e^{-2\alpha\sqrt{-1}}+e^{-2\beta}+e^{2\beta}=2\cos2\alpha+e^{-2\beta}+c^{2\beta}$,　(14) 同上.

(15) $\sec(\alpha+\beta\sqrt{-1})=\frac{1}{\cos(\alpha+\beta\sqrt{-1})}=\frac{4\cos(\alpha-\beta\sqrt{-1})}{4\cos(\alpha+\beta\sqrt{-1})\cos(\alpha-\beta\sqrt{-1})}$

$=\frac{2\{e^{(\alpha-\beta\sqrt{-1})\sqrt{-1}}+e^{-(\alpha-\beta\sqrt{-1}\sqrt{-1}}}{e^{2\beta}+e^{-2\beta}+2\cos2\alpha}$, 〔例題 13.〕 $=\frac{2(e^{\alpha\sqrt{-1}}e^{\beta}+e^{-\alpha\sqrt{-1}}e^{-\beta})}{e^{2\beta}+e^{-2\beta}+2\cos2\alpha}$

但分子$=2\{(\cos\alpha+\sqrt{-1}\sin\alpha)e^{\beta}+(\cos\alpha-\sqrt{-1}\sin\alpha)e^{-\beta}\}$

$=2(e^{\beta}+e^{-\beta})\cos\alpha+2\sqrt{-1}(e^{\beta}-e^{-\beta})\sin\alpha.$

(16) 與前例同樣. 但由例題 13. 及 14.

(17) $\sqrt{-1}=\cos\frac{\pi}{2}+\sqrt{-1}\sin\frac{\pi}{2}=e^{\frac{\pi}{2}\sqrt{-1}}$

故 $(\sqrt{-1})^{\sqrt{-1}}=(e^{\frac{\pi}{2}\sqrt{-1}})^{\sqrt{-1}}=e^{-\frac{\pi}{2}}.$

(18) 由 $\cos(\alpha+\beta\sqrt{-1})=\cos\phi+\sqrt{-1}\sin\phi$,

得 $\frac{1}{2}\{e^{(\alpha+\beta\sqrt{-1})\sqrt{-1}}+e^{-(\alpha+\beta\sqrt{-1})\sqrt{-1}}\}$

$=\cos\phi+\sqrt{-1}\sin\phi$, 即 $e^{-\beta}(\cos\alpha+\sqrt{-1}\sin\alpha)+e^{\beta}(\cos\alpha-\sqrt{-1}\sin\alpha)$

$=2\cos\phi+2\sqrt{-1}\sin\phi$, 即 $(e^{\beta}+e^{-\beta})\cos\alpha-(e^{\beta}-e^{-\beta})\sin\alpha\sqrt{-1}$

$=2\cos\phi+2\sqrt{-1}\sin\phi$, 故 $(e^{\beta}+e^{-\beta})\cos\alpha=2\cos\phi$, 及

$(e^{\beta}-e^{-\beta})\sin\alpha=2\sin\phi$, 故 $(e^{\beta}+e^{-\beta})^2-(e^{\beta}-e^{-\beta})^2=\frac{4\cos^2\phi}{\cos^2\alpha}-\frac{4\sin^2\phi}{\sin^2\alpha}$,

由是得 $\sin\phi=\pm\sin^2\alpha$, 又從 $(e^{\beta}+e^{-\beta})^2\cos^2\alpha+(e^{\beta}-e^{-\beta})^2\sin^2\alpha$

$=4(\cos^2\phi+\sin^2\phi)$, 得 $e^{\beta}-e^{-\beta}=\pm2\sin\alpha.$

示次各式為$A+B\sqrt{-1}$之形.

19. $\cos(\alpha+\beta\sqrt{-1})$. **20.** $\sin(\alpha+\beta\sqrt{-1})$.

21. $\log_e(\alpha+\beta\sqrt{-1})$. **22.** $(\alpha+\beta\sqrt{-1})^{p+q\sqrt{-1}}$.

23. $a^{p+q\sqrt{-1}}$. **24.** $(\alpha+\beta\sqrt{-1})^{\sqrt{-1}}$.

25. $a^{\sqrt[6]{-1}}$. **26.** $(a+b\sqrt{-1}+c^{\sqrt{-1}})^{p+q\sqrt{-1}}$

27. 設 $A+B\sqrt{-1}=\log_e(m+n\sqrt{-1})$, 則

$\tan B=n/m$, $2A=\log_e(m^2+n^2)$.

28. 設 $u=(\pi+\beta\sqrt{-1})^{p+q\sqrt{-1}}$, 試示 $\log u$ 為

$A+B\sqrt{-1}$之形.

例題解自 19. 至 28.

(19) $\cos(\alpha+\beta\sqrt{-1})=\frac{1}{2}\{e^{(\alpha+\beta\sqrt{-1})\sqrt{-1}}+e^{-(\alpha+\beta\sqrt{-1})\sqrt{-1}}\}$

$=\frac{1}{2}\{e^{-\beta}e^{\alpha\sqrt{-1}}+e^{\beta}e^{-\alpha\sqrt{-1}}\}=\frac{1}{2}\{e^{-\beta}(\cos\alpha+\sqrt{-1}\sin\alpha)+e^{\beta}(\cos\alpha-\sqrt{-1}\sin\alpha)\}$

$=\frac{1}{2}(e^{\beta}+e^{-\beta})\cos\alpha+\frac{1}{2}(e^{-\beta}-e^{\beta})\sin\alpha\sqrt{-1}$.

(20) $\sin(\alpha+\beta\sqrt{-1})=\frac{1}{2}(e^{\beta}+e^{-\beta})\sin\alpha+\frac{1}{2}(e^{\beta}-e^{-\beta})\cos\alpha\sqrt{-1}$.

(21) $\alpha+\beta\sqrt{-1}=r(\cos\theta+\sqrt{-1}\sin\theta)$, 由16.節 $r^2=\alpha^2+\beta^2$, 及 $\tan\theta=\beta/\alpha$,

故 $\log_e(\alpha+\beta\sqrt{-1})=\log_e\{r(\cos\theta+\sqrt{-1}\sin\theta)\}=\log_e r+\log_e r^{\theta\sqrt{-1}}$

$=\log_e r+\theta\sqrt{-1}=\frac{1}{2}\log_e(\alpha^2+\beta^2)+\tan^{-1}(\beta/\alpha)\sqrt{-1}$.

(22) $(\alpha+\beta\sqrt{-1})^{p+q\sqrt{-1}}=\{r(\cos\theta+\sqrt{-1}\sin\theta)\}^{p+q\sqrt{-1}}$

$=r^{p+q\sqrt{-1}}e^{\theta\sqrt{-1}(p+q\sqrt{-1})}=r^p.e^{-\theta q}.r^{q\sqrt{-1}}e^{\theta p\sqrt{-1}}$, 但 $r=e^{\log_e r}$(21.節餘論)

$=r^p.e^{-\theta q}e^{(q\log_e r+\theta p)\sqrt{-1}}$.

$=r^p e^{-\theta q}\{\cos(q\log_e r+\theta p)+\sqrt{-1}\sin(q\log_e r+\theta p)\}$.

但由前例 $r=(\alpha^2+\beta^2)^{\frac{1}{2}}$,　$\theta=\tan^{-1}(\beta/\alpha)$.

(23) 於前例,令 $\alpha=a$, $\beta=0$, 則 $r=a$, $\theta=\tan^{-1}0$,

卽 $\tan\theta=0$, 故 $\theta=n\pi$, 由是

$a^{p+q\sqrt{-1}}=a^pe^{-qn\pi}\{\cos(q\log_e a+pn\pi)+\sqrt{-1}\sin(q\log_e a+pn\pi)\}$.

(24) 於例題22.令 $p=0$, $q=1$, 則 $r=1$, $\theta=\tan^{-1}\infty$

$(\alpha+\beta\sqrt{-1})^{\sqrt{-1}}=e^{-\theta}\{\cos(\log_e r)+\sqrt{-1}\sin(\log_e r)\}$

但 $r=(\alpha^2+\beta^2)^{\frac{1}{2}}$,　$\theta=\tan^{-1}(\beta/\alpha)$.

(25) 令 $\sqrt[6]{-1}=r(\cos\theta+\sqrt{-1}\sin\theta)$, 則 $\sqrt{-1}=r^3(\cos3\theta+\sqrt{-1}\sin3\theta)$

故 $r^3\cos3\theta=0$, $1=r^3\sin3\theta$, 故 $r=1$, $\theta=(4n+1)\pi/6$,

$a^{\sqrt[6]{-1}}=(e^{\log_e a})^{\sqrt[6]{-1}}$ [21.簡條論] $=(e^{\log_e a})^{(\cos\theta+\sqrt{-1}\sin\theta)}$

$=e^{\log_e a\cos\theta}.e^{\log_e a\sin\theta\sqrt{-1}}=e^{\log_e a\cos\theta}\{\cos(\log_e a\sin\theta)+\sqrt{-1}\sin(\log_e a\sin\theta)\}$,

但 $\theta=(4n+1)\pi/6$.

(26) $c^{\sqrt{-1}}=e^{\log_e c\sqrt{-1}}=\cos(\log_e c)+\sqrt{-1}\sin(\log_e c)$, 故

$(a+b\sqrt{-1}+c^{\sqrt{-1}})^{p+q\sqrt{-1}}=[\{(a+\cos(\log_e c)\}+\sqrt{-1}\{b+\sin(\log_e c)\}]^{p+q\sqrt{-1}}$

以下與例題22.同法.

(27) 由原恆同式. $e^{A+B\sqrt{-1}}=m+n\sqrt{-1}$, 卽 $e^A.e^{B\sqrt{-1}}=m+n\sqrt{-1}$

卽 $e^A(\cos B+\sqrt{-1}\sin B)=m+n\sqrt{-1}$, 故 $e^A\cos B=m$, 及

$e^A\sin B=n$, 故 $e^{2A}=m^2+n^2$, 卽 $2A=\log_e(m^2+n^2)$.

(28) 由例題22. $u=r^p.e^{-\theta q}e^{(q\log_e r+\theta p)\sqrt{-1}}$

$=e^{p\log_e r}.e^{-\theta q}.e^{(q\log_e r+\theta p)\sqrt{-1}}=e^{p\log_e r-\theta q+(q\log_e r+\theta p)\sqrt{-1}}$

由是 $\log_e u=p\log_e r-\theta q+(q\log_e r+\theta p)\sqrt{-1}$,

但由例題22. $r=(\alpha^2+\beta^2)^{\frac{1}{2}}$,　$\theta=\tan^{-1}(\beta/\alpha)$.

求次各式之證.

29. $\log_e \frac{a+b\sqrt{-1}}{a-b\sqrt{-1}} = 2\sqrt{-1}\tan^{-1}\frac{b}{a}$.

30. $\log_e \frac{\sin(x+y\sqrt{-1})}{\sin(x-y\sqrt{-1})} = 2\sqrt{-1}\tan^{-1}\left(\cot x\frac{e^y-e^{-y}}{e^y+e^{-y}}\right)$.

31. $\log_e \frac{\cos(x-y\sqrt{-1})}{\cos(x+y\sqrt{-1})} = 2\sqrt{-1}\tan^{-1}\left(\tan x\frac{e^y-e^{-y}}{e^y+e^{-y}}\right)$.

32. $\log_e 4\sin^2(x+y\sqrt{-1})$

$$=\log_e(e^{2y}-2\cos 2x+e^{-2y})+2\sqrt{-1}\tan^{-1}\left(\cot x\frac{e^y-e^{-y}}{e^y+e^{-y}}\right).$$

33. $\log_e 4\cos^2(x+y\sqrt{-1})$

$$=\log_e(e^{2y}+2\cos 2x+e^{-2y})-2\sqrt{-1}\tan^{-1}\left(\tan x\frac{e^y-e^{-y}}{e^y+e^{-y}}\right).$$

34. $\{\sin(\alpha-\theta)+e^{\pm\alpha\sqrt{-1}}\sin\theta\}^n$

$$=\sin^{n-1}\alpha\{\sin(\alpha-n\theta)+e^{\pm\alpha\sqrt{-1}}\sin n\theta\}.$$

例題解自29.至34.

(29) $\log_e\frac{r(\cos\theta+\sqrt{-1}\sin\theta)}{r(\cos\theta+\sqrt{-1}\sin\theta)} = \log_e(\cos\theta+\sqrt{-1}\sin\theta)^2$

$=\log_e(\cos 2\theta+\sqrt{-1}\sin 2\theta)=\log_e e^{2\theta\sqrt{-1}}=2\theta\sqrt{-1}$, 但 $\theta=\tan^{-1}\frac{b}{a}$, (17.節)

(30) 由例題20.

$$\log_e\frac{\frac{1}{2}(e^y+e^{-y})\sin x+\frac{1}{2}(e^y-e^{-y})\cos x\sqrt{-1}}{\frac{1}{2}(e^y+e^{-y})\sin x-\frac{1}{2}(e^y-e^{-y})\cos x\sqrt{-1}}$$

$=\log_e\frac{1+\sqrt{-1}\cot x(e^y-e^{-y})/(e^y+e^{-y})}{1-\sqrt{-1}\cot x(e^y-e^{-y})/(e^y+e^{-y})}$, 但 $\theta=\tan^{-1}\left(\cot x\frac{e^y-e^{-y}}{e^y+e^{-y}}\right)$

$$=\log_e\frac{1+\sqrt{-1}\tan\theta}{1-\sqrt{-1}\tan\theta}=\log_e\frac{\cos\theta+\sqrt{-1}\sin\theta}{\cos\theta-\sqrt{-1}\sin\theta}=\log_e(\cos2\theta+\sqrt{-1}\sin2\theta)$$

$$=\log_e e^{2\theta\sqrt{-1}}=2\theta\sqrt{-1}=2\sqrt{-1}\tan^{-1}\left(\cot x\frac{e^y-e^{-y}}{e^y+e^{-y}}\right).$$

(31) 同上.

(32) $\log_e 4\sin^2(x+y\sqrt{-1})=\log_e\{(e^y+e^{-y})\sin x+(e^y-e^{-y})\cos x\sqrt{-1}\}^2$,

[但由例題 20.] $=\log_e\{(e^y+e^{-y})^2\sin^2x(1+\sqrt{-1}\tan\theta)^2$,

$\left[但\quad \tan\theta=\cot x\frac{e^y-e^{-y}}{e^y+e^{-y}}\right]=\log_e(e^y+e^{-y})^2\frac{\sin^2x}{\sin^2x}(\cos\theta+\sqrt{-1}\sin\theta)^2$

$$=\log_e(e^y+e^{-y})^2\sin^2x(1+\tan^2\theta)e^{2\theta\sqrt{-1}}$$

$$=\log_e(e^y+e^{-y})^2\sin^2x\left\{1+\cot^2x\left(\frac{e^y-e^{-y}}{e^y+e^{-y}}\right)^2\right\}+\log_e e^{2\theta\sqrt{-1}}$$

$$=\log_e\{(e^y+e^{-y})^2\sin^2x+(e^y-e^{-y})^2\cos^2x\}+2\theta\sqrt{-1}$$

$$=\log_e(e^{2y}+e^{-2y}-2\cos2x)+2\sqrt{-1}\tan^{-1}\left(\cot x\frac{e^y-e^{-y}}{e^y+e^{-y}}\right).$$

(33) 同上.

(34) $\{\sin(a-\theta)+e^{\pm a\sqrt{-1}}\sin\theta\}^n$

$$=\{\sin(a-\theta)+(\cos a\pm\sqrt{-1}\sin a)\sin\theta\}^2.$$

$$=\{\sin a\cos\theta\pm\sqrt{-1}\sin a\sin\theta\}^n=\sin^n a(\cos\theta\pm\sqrt{-1}\sin\theta)^n$$

$$=\sin^n a(\cos\theta n\pm\sqrt{-1}\sin\theta n)$$

$$=\sin^{n-1}a(\sin a\sin\theta n\pm\sqrt{-1}\sin a\sin\theta n)$$

$$=\sin^{n-1}a\{\sin(a-\theta n)+\cos a\sin\theta n\pm\sqrt{-1}\sin a\sin\theta n\}$$

$$=\sin^{n-1}a\{\sin(a-\theta n)+(\cos a\pm\sqrt{-1}\sin a)\sin\theta n\}$$

$$=\sin^{n-1}a\{\sin(a-\theta n)+e^{\pm a\sqrt{-1}}\sin\theta n\}$$

35. $a^{\sqrt[4]{-1}}=a^{\cos\frac{1}{4}\pi}\left\{\cos\left(\sin\frac{\pi}{4}\log_e a\right)+\sqrt{-1}\sin\left(\sin\frac{\pi}{4}\log_e a\right)\right\}$.

36. $\log_e(\cos A+\sqrt{-1}\sin A)=-\log_e(\cos A-\sqrt{-1}\sin A)$.

37. 設 $k=\cos 1^\circ+\sqrt{-1}\sin 1^\circ$, 則

$$\frac{\sqrt{-1}\sin\theta^\circ-(1-\cos\theta^\circ)}{\theta}=\log_e k+\frac{\theta}{\underline{|2}}(\log_e k)^2+\frac{\theta^2}{\underline{|3}}(\log_e k)^3+\cdots\cdots$$

38. $\sin\log_e(a+b\sqrt{-1})=\alpha+\beta\sqrt{-1}$, 則

$$\log_e\sin\{\log_e\sqrt{(a^2+b^2)}+\sqrt{-1}\tan^{-1}(b/a)\}$$

$$=\log_e\sqrt{(\alpha^2+\beta^2)}+\sqrt{-1}\tan^{-1}(\beta/\alpha)\}$$

39. $(1+\sqrt{-1}\tan\theta)^{-\sqrt{-1}}$ 之實數部，為 $e^\theta\cos(\log_e\cos\theta)$.

40. $\frac{q}{2}\log_e(\alpha^2+\beta^2)+p\tan^{-1}\frac{\beta}{\alpha}$ 為 $\frac{\pi}{2}$ 之偶數倍，或奇數倍。

從而 $(\alpha+\beta\sqrt{-1})^{p+q\sqrt{-1}}$ 全為實數或全為虛數.

例題解自 35. 至 40.

(35) $-1=\cos\pi+\sqrt{-1}\sin\pi$, 故 $\sqrt[4]{-1}=(\cos\pi+\sqrt{-1}\sin\pi)^{\frac{1}{4}}$

$=\cos\frac{\pi}{4}+\sqrt{-1}\sin\frac{\pi}{4}$, 又 $a=e^{\log_e a}$, (21. 節餘論), 故

$a^{\sqrt[4]{-1}}=e^{\log_e a\sqrt[4]{-1}}=e^{\log_e a(\cos\frac{1}{4}\pi+\sqrt{-1}\sin\frac{1}{4}\pi)}$

$=e^{\log_e a\cos\frac{1}{4}\pi}\cdot e^{\log_e a\sin\frac{1}{4}\pi\sqrt{-1}}$

$=e^{\log_e a\cos\frac{1}{4}\pi}\left\{\cos\left(\log_e a\sin\frac{1}{4}\pi\right)+\sqrt{-1}\sin\left(\log_e a\sin\frac{1}{4}\pi\right)\right\}$.

(36) $\log_e(\cos A+\sqrt{-1}\sin A)=\log_e e^{A\sqrt{-1}}=A\sqrt{-1}$

又 $-\log_e(\cos A-\sqrt{-1}\sin A)=-\log_e e^{-A\sqrt{-1}}=-(-A\sqrt{-1})=A\sqrt{-1}$.

(37) $\frac{\sqrt{-1}\sin\theta^\circ-(1-\cos\theta^\circ)}{\theta}=\frac{\cos\theta^\circ+\sqrt{-1}\sin\theta^\circ-1}{\theta}=\frac{(\cos1^\circ+\sqrt{-1}\sin1^\circ)\theta-1}{\theta}$

$=\frac{k^\theta-1}{\theta}=\frac{(e^{\log_e k})\theta-1}{\theta}=\frac{e^{\theta\log_e k}-1}{\theta}$

$=\frac{1+\frac{\theta\log_e k}{\lfloor 1}+\frac{\theta^2(\log_e k)^2}{\lfloor 2}+\frac{\theta^3(\log_e k)^3}{\lfloor 3}+\cdots\cdots\cdots-1}{\theta}$.

(38) 由例題21.變 $\sin\log_e(a+b\sqrt{-1})=\alpha+\beta\sqrt{-1}$. 爲

$\sin\left\{\log_e\sqrt{(a^2+b^2)}+\sqrt{-1}\tan^{-1}\frac{b}{a}\right\}=\alpha+\beta\sqrt{-1}$, 卽

$\log_e\sin\left\{\log_e\sqrt{(a^2+b^2)}+\sqrt{-1}\tan^{-1}\frac{b}{a}\right\}=\log_e(\alpha+\beta\sqrt{-1})$, 又由例題21.

$=\log_e\sqrt{(\alpha^2+\beta^2)}+\sqrt{-1}\tan^{-1}\frac{\beta}{\alpha}$.

(39) $(1+\sqrt{-1}\tan\theta)^{-\sqrt{-1}}=\left(\frac{\cos\theta+\sqrt{-1}\sin\theta}{\cos\theta}\right)^{-\sqrt{-1}}=\frac{(e^{\theta\sqrt{-1}})^{-\sqrt{-1}}}{(\cos\theta)^{-\sqrt{-1}}}$

$=e^\theta(\cos\theta)^{\sqrt{-1}}=e^\theta(e^{\log_e\cos\theta})^{\sqrt{-1}}=e^\theta.e^{\log_e\cos\theta\sqrt{-1}}$

$=e^\theta\{\cos(\log_e\cos\theta)+\sqrt{-1}\sin(\log_e\cos\theta)\}$

故實數部爲 $e^\theta\cos(\log_e\cos\theta)$.

(40) 由例題22. $(\alpha+\beta\sqrt{-1})^{p+q\sqrt{-1}}=r^p e^{-\theta p}\cos\{(q\log_e r+\theta p)$

$+\sqrt{-1}\sin(q\log_e r+\theta p)\}$, 但 $r=\sqrt{(\alpha^2+\beta^2)}$, $\theta=\tan^{-1}\frac{\beta}{\alpha}$,

故 $q\log_e r+\theta p$ 卽 $\frac{1}{2}q\log_e(\alpha^2+\beta^2)+p\tan^{-1}\frac{\beta}{\alpha}$,爲 $\frac{\pi}{2}$ 之偶數倍數.(卽 $n\pi$) 則可消去正弦之項.全爲實數.

又爲 $\frac{\pi}{2}$ 之奇數倍數,$\left[\text{卽}\ (2n+1)\frac{\pi}{2}\right]$ 則得消去餘弦之項.全爲虛數.

41. $\cos^{-1}(\alpha+\beta\sqrt{-1})=\theta+\phi\sqrt{-1}$, 例

$$\frac{\alpha^2}{\cos^2\theta}-\frac{\beta^2}{\sin^2\theta}=1,\quad 及\quad \frac{\alpha^2}{(e^\phi+e^{-\phi})^2}+\frac{\beta^2}{(e^\phi-e^{-\phi})^2}=\frac{1}{4}.$$

42. 設 $\tan(\theta+\phi\sqrt{-1})=\cos\alpha+\sqrt{-1}\sin\alpha$, 則

$$\theta=n\pi\pm\frac{\pi}{4},\quad 及\quad e^{2\phi}=\pm\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$$

43. $\tan(\theta+\phi\sqrt{-1})=\tan\alpha+\sqrt{-1}\sec\alpha$, 則

$$e^{2\phi}=\pm\cot\frac{\alpha}{2}.$$

例題解自41.至43.

(41) $\alpha+\beta\sqrt{-1}=\cos(\theta+\phi\sqrt{-1})$, 由例題16.

$$=\tfrac{1}{2}(e^{\phi}+e^{-\phi})\cos\theta+\tfrac{1}{2}(e^{-\phi}-e^{\phi})\sin\beta$$

故 $\alpha=\frac{1}{2}(\phi+e^{-\phi})\cos\theta$, 及 $\beta=-\frac{1}{2}(e^{\phi}-e^{-\phi})\sin\theta$

故 $\dfrac{\alpha^2}{\cos^2\theta}-\dfrac{\beta^2}{\sin^2\theta}=\frac{1}{4}(e^{\phi}+e^{-\phi})^2-\frac{1}{4}(e^{\phi}-e^{-\phi})^2=1$

及 $\dfrac{\alpha^2}{(e^{\phi}+e^{-\phi})^2}+\dfrac{\beta^2}{(e^{\phi}-e^{-\phi})^2}=\frac{1}{4}\cos^2\theta+\frac{1}{4}\sin^2\theta=\frac{1}{4}$.

(42) 由例題19.

$$\frac{2\sin2\theta+\sqrt{-1}(e^{2\phi}-e^{-2\phi})}{e^{2\phi}+2\cos2\theta+e^{-2\phi}}=\cos\alpha+\sqrt{-1}\sin\alpha$$

故 $\dfrac{2\sin2\theta}{e^{2\phi}+2\cos2\theta+e^{-2\phi}}=\cos\alpha,\ \dfrac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+2\cos2\theta+e^{-2\phi}}=\sin\alpha,$ (A)

由此兩方程式(A), 得

$$\frac{4\sin^2 2\theta}{(e^{2\phi}+2\cos2\theta+e^{-2\phi})^2}+\frac{(e^{2\phi}-e^{-2\phi})^2}{(e^{2\phi}+2\cos2\theta+e^{-2\phi})^2}=1,\ 即$$

$$4\sin^2 2\theta+(e^{2\phi}-e^{-2\phi})^2=(e^{2\phi}+e^{-2\phi})^2+4(e^{2\phi}+e^{-2\phi})\cos 2\theta+4\cos^2 2\theta$$

即 $4(1-\cos^2 2\theta)=4+4(e^{2\phi}+e^{-2\phi})\cos 2\theta+4\cos^2 2\theta,$

由是 $\cos 2\theta=0$, 或 $\cos 2\theta=\frac{e^{2\phi}+e^{-2\phi}}{2},$

但 $\cos 2\theta=\frac{e^{\phi}+e^{-\phi}}{2}=\frac{(e^{\phi}-e^{-\phi})^2+2}{2}=\frac{(e^{\phi}-e^{-\phi})^2}{2}+1>1$

故此第二答不合理. 而 $\cos 2\theta=0=\cos\left(2n\pi\pm\frac{\pi}{2}\right),$

故 $\theta=n\pi\pm\frac{\pi}{4}.$

又於前兩方程式(A)之內之第二, 代入 $\cos 2\theta=0$, 則

$\frac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+e^{-2\phi}}=\sin\alpha$, 即 $\frac{2e^{2\phi}}{2e^{-2\phi}}=\frac{1+\sin\alpha}{1-\sin\alpha}=\left(\frac{\cos\frac{1}{2}\alpha+\sin\frac{1}{2}\alpha}{\cos\frac{1}{2}\alpha-\sin\frac{1}{2}\alpha}\right)^2$

即 $e^{4\phi}=\left(\frac{1+\tan\frac{1}{2}\alpha}{1-\tan\frac{1}{2}\alpha}\right)^2=\tan^2\left(\frac{\pi}{4}+\frac{\alpha}{2}\right),$

由是 $e^{2\phi}=\pm\tan\left(\frac{\pi}{4}+\frac{\alpha}{2}\right).$

(43) 如前例分 $\tan(\theta+\phi\sqrt{-1})$ 爲實數部與虛數部. 以此與右邊比較.

$$\frac{2\sin 2\theta}{e^{2\phi}+2\cos 2\theta+e^{-2\phi}}=\tan\alpha,\quad \frac{e^{2\phi}-e^{-2\phi}}{e^{2\phi}+2\cos 2\theta+e^{-2\phi}}=\sec\alpha, \qquad \text{(A)}$$

由(A)之兩方程式, $\frac{(e^{2\phi}-e^{-2\phi})^2-4\sin^2 2\phi}{(e^{2\phi}+2\cos 2\theta+e^{-2\phi})^2}=\sec^2\alpha-\tan^2\alpha=1,$

即 $(e^{2\phi}-e^{-2\phi})^2-4\sin^2 2\theta=(e^{2\phi}+e^{-2\phi})^2+4(e^{2\phi}+e^{-2\phi})\cos 2\theta+4\cos^2 2\theta,$

故 $\cos 2\theta=-\frac{2}{e^{2\phi}+e^{-2\phi}}$, 以此代入(A)之第二式, 則

$\frac{(e^{2\phi}-e^{-2\phi})(e^{2\phi}+e^{-2\phi})}{(e^{2\phi}+e^{-2\phi})^2-4}=\sec\alpha$, 即 $\frac{e^{2\phi}+e^{-2\phi}}{e^{2\phi}-e^{-2\phi}}=\frac{1}{\cos\alpha}$

即 $\frac{2e^{2\phi}}{2e^{-2\phi}}=e^{4\phi}=\frac{1+\cos\alpha}{1-\cos\alpha}=\cot^2\frac{\alpha}{2}.$

44. $\tan\theta = a\sin cx/(1-a\cos cx)$, 及

$r^2=1-2a\cos cx+a^2$ 則

$$1-\frac{n}{1}a\cos cx+\frac{n(n-1)}{\lfloor 2}a^2\cos 2cx+\cdots\cdots$$

$$+(-1)^n\cos ncx = r^n\cos n\theta.$$

45. $C=1+z\cos\theta+\frac{z^2}{\lfloor 2}\cos 2\theta+\frac{z^3}{\lfloor 3}\cos 3\theta+\cdots\cdots$

及 $S=z\sin\theta+\frac{z^2}{\lfloor 2}\sin 2\theta+\frac{z^2}{\lfloor 3}\sin 3\theta+\cdots\cdots$

則 $z\sin\theta=\tan^{-1}\frac{S}{C}$, $z\cos\theta=\frac{1}{2}\log_e(C^2+S^2)$.

46. $c=\frac{a-b}{a+b}$, 則 $\log_e\frac{a^2}{a^2\cos^2\theta+b^2\sin^2\theta}$

$$=4\{c\sin^2\theta-\tfrac{1}{2}c^2\sin^2 2\theta+\tfrac{1}{3}c^3\sin^2 3\theta-\cdots\cdots\}.$$

例題解自 44. 至 46.

(44) $1-a\cos cx=a\sin cx\cot\theta$, $r=\{(1-a\cos cx)^2+a^2\sin^2 cx\}^{\frac{1}{2}}$

$=(a^2\sin^2 cx\cot^2\theta+a^2\sin^2 cx)^{\frac{1}{2}}=a\sin cx\csc\theta$, 故原級數

$$=1-\frac{n}{1}a\frac{e^{cx\sqrt{-1}}+e^{-cx\sqrt{-1}}}{2}+\frac{n(n-1)}{\lfloor 2}a^2\frac{e^{2cx\sqrt{-1}}+e^{-2cx\sqrt{-1}}}{2}-\cdots\cdots$$

$$=\frac{1}{2}\left\{1-\frac{n}{1}ae^{cx\sqrt{-1}}+\frac{n(n-1)}{\lfloor 2}a^2e^{2x\sqrt{-1}}-\cdots\cdots\right\}$$

$$+\frac{1}{2}\left\{1-\frac{n}{1}ae^{-cx\sqrt{-1}}+\frac{n(n-1)}{\lfloor 2}a^2e^{-2\sqrt{-1}}-\cdots\cdots\right\}$$

$=\frac{1}{2}(1-ae^{cx\sqrt{-1}})^n+\frac{1}{2}(1-ae^{-cx\sqrt{-1}})^n$,

$1-ae^{cx\sqrt{-1}}=1-a(\cos cx+\sqrt{-1}\sin cx)$

$=a\sin cx\cot\theta-\sqrt{-1}a\sin cx=\frac{a\sin cx}{\sin\theta}\left(\cos\theta-\sqrt{-1}\sin\theta\right)$.

同樣. $1-ae^{-cx\sqrt{-1}}=\frac{a\sin cx}{\sin\theta}(\cos\theta+\sqrt{-1}\sin\theta)$, 由是

原級數$=\frac{a^n\sin^n cx}{2\sin^n\theta}\{(\cos\theta-\sqrt{-1}\sin\theta)^n+(\cos\theta+\sqrt{-1}\sin n\theta)\}$

$=\frac{a^n\sin^n cx}{2\sin^n\theta}\{\cos n\theta-\sqrt{-1}\sin n\theta+\cos n\theta+\sqrt{-1}\sin\theta)^n\}$

$=a^n\sin^n cx\csc^n\theta\cos n\theta=r^n\cos n\theta.$

(45) $C=1+\frac{1}{2}z(e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}})+\frac{1}{2\lfloor 2}z^2(e^{2\theta\sqrt{-1}}+e^{-2\theta\sqrt{-1}})+\ldots\ldots$

$=\frac{1}{2}(1+ze^{\theta\sqrt{-1}}\frac{1}{\lfloor 2}z^2e^{2\theta\sqrt{-1}}+\ldots\ldots)+\frac{1}{2}(1+ze^{-\theta\sqrt{-1}}+\frac{1}{\lfloor 2}z^2e^{-2\theta\sqrt{-1}}+\ldots\ldots)$

$=\frac{1}{2}e^x+\frac{1}{2}e^y$, 但 $ze^{\theta\sqrt{-1}}=x$, $ze^{-\theta\sqrt{-1}}=y$.

卽 $x=z(\cos\theta+\sqrt{-1}\sin\theta)$, $y=z(\cos\theta-\sqrt{-1}\sin\theta)$, 故

$C=\frac{1}{2}\{e^{z(\cos\theta+\sqrt{-1}\sin\theta)}+e^{z(\cos n\theta-\sqrt{-1}\sin\theta}\}=\frac{1}{2}e^{z\cos\theta}(e^{z\sin\theta\sqrt{-1}}+e^{-z\sin\theta\sqrt{-1}})$

$=\frac{1}{2}e^{z\cos\theta}\{\cos(z\sin\theta)+\sqrt{-1}\sin(z\sin\theta)+\cos(z\sin\theta)-\sqrt{-1}\sin(z\sin\theta)\}$

$=e^{z\cos\theta}\cos(z\sin\theta)$, 同樣. $S=e^{z\cos\theta}\sin(z\sin\theta)$,

故 $S/C=\tan(z\sin\theta)$, $C^2+S^2=e^{2z\cos\theta}$

(46) $\frac{a^2}{a^2\cos^2\theta+b^2\sin^2\theta}=\frac{2a^2}{(a^2+b^2)+(a^2-b^2)\cos 2\theta}$

$$=\frac{4a^2}{2(a^2+b^2)+(a^2-b^2)(e^{2\theta\sqrt{-1}}+e^{-2\theta\sqrt{-1}})}$$

$$=\frac{4a^2}{(a+b)^2+(a-b)^2+(a^2-b^2)(e^{2\theta\sqrt{-1}}+e^{-2\theta\sqrt{-1}})}$$

$$=\frac{4a^2}{(a+b)^2\{1+c^2+c(e^{2\theta\sqrt{-1}}+e^{-2\theta\sqrt{-1}})\}}$$

$=\frac{(1+c)^2}{(ce^{2\theta\sqrt{-1}}+1)(ce^{-2\theta\sqrt{-1}}+1)}$, 故 $\log_e\frac{a^2}{a^2\cos^2\theta+b^2\sin^2\theta}$

$=2\log_e(1+c)-\log_e(ce^{2\theta\sqrt{-1}}+1)-\log_e(ce^{-2\theta\sqrt{-1}}+1)=2\left(c-\frac{c^2}{2}+\frac{c^3}{3}-\ldots\ldots\right)$

$-\left(ce^{2\theta\sqrt{-1}}-\frac{c^2e^{4\theta\sqrt{-1}}}{2}+\frac{c^3e^{6\theta\sqrt{-1}}}{3}-\ldots\right)$

$-\left(ce^{-2\theta\sqrt{-1}}-\frac{c^2e^{-4\theta\sqrt{-1}}}{2}+\frac{c^3e^{-6\theta\sqrt{-1}}}{3}-\ldots\right)$

$=c\{-(e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}})\}^2-\frac{1}{2}c^2\{-(e^{2\theta\sqrt{-1}}-e^{-2\theta\sqrt{-1}})\}^2$

$=c\{4\sin^2\theta\}-\frac{1}{2}c^2\{4\sin^2 2\theta\}+\ldots\ldots\ldots$

47. $\sin x = n\sin(x+\alpha)$，則

$$x = n\sin\alpha + \frac{n^2}{2}\sin 2\alpha + \frac{n^3}{3}\sin 3\alpha + \cdots\cdots\cdots$$

48. 同上. $\alpha = \pi - 2x$，則

$$x = \sin 2x - \frac{1}{2}\sin 4x + \frac{1}{3}\sin 6x - \frac{1}{4}\sin 8x + \cdots\cdots\cdots$$

49. $\tan x = n\tan y$，則

$$x = y - m\sin 2y + \frac{m^2}{2}\sin 4y - \frac{m^2}{3}\sin 6y + \cdots\cdots\cdots$$

但 $m = \frac{1-n}{1+n}$.

50. 於 x 之方乘展開 $e^{ax}\cos bx$，其展開式 x^n 之係數，等於 $\frac{(a^2+b^2)^2}{\lfloor n}\cos n\theta$. 但 $\tan\theta = \frac{b}{a}$.

例題解自 47. 至 50.

(47) 由 19. 節. 變 $\sin x = n\sin(x+\alpha)$，則

$$e^{x\sqrt{-1}} - e^{-x\sqrt{-1}} = n\{e^{(x+\alpha)\sqrt{-1}} - e^{-(x+\alpha)\sqrt{-1}}\}$$，故

$$e^{2x\sqrt{-1}} - 1 = n\{e^{(2x+\alpha)\sqrt{-1}} - e^{-\alpha\sqrt{-1}}\}$$，故

$$e^{2x\sqrt{-1}}(1-ne^{\alpha\sqrt{-1}}) = 1-ne^{-\alpha\sqrt{-1}}$$，即 $$e^{2x\sqrt{-1}} = \frac{1-ne^{-\alpha\sqrt{-1}}}{1-ne^{\alpha\sqrt{-1}}}$$，

故 $$2x\sqrt{-1} = \log_e(1-ne^{-\alpha\sqrt{-1}}) - \log_e(1-ne^{\alpha\sqrt{-1}})$$

$$= n(e^{\alpha\sqrt{-1}} - e^{-\alpha\sqrt{-1}}) + \frac{n^2}{2}(e^{2\alpha\sqrt{-1}} - e^{-2\alpha\sqrt{-1}}) + \frac{n^3}{3}(e^{3\alpha\sqrt{-1}} - e^{-3\alpha\sqrt{-1}}) + \cdots\cdots$$

故 $x=n\dfrac{e^{a\sqrt{-1}}-e^{-a\sqrt{-1}}}{2\sqrt{-1}}+\dfrac{n}{2}\cdot\dfrac{e^{2a\sqrt{-1}}-e^{-2a\sqrt{-1}}}{2\sqrt{-1}}+\dfrac{n^3}{3}\cdot\dfrac{e^{3a\sqrt{-1}}-e^{-3a\sqrt{-1}}}{2\sqrt{-1}}+\cdots\cdots$

$=n\sin a+\dfrac{n^2}{2}\sin 2a+\dfrac{n^2}{3}\sin 3a+\cdots\cdots\cdots$

(48) 於前例之a,以$\pi-2x$代之,卽得其證.

(49) $\tan x=n\tan y$, 卽 $\dfrac{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}{e^{x\sqrt{-1}}+e^{-x\sqrt{-1}}}=\dfrac{n(e^{y\sqrt{-1}}-e^{-y\sqrt{-1}})}{e^{y\sqrt{-1}}+e^{-y\sqrt{-1}}}$

卽 $\dfrac{e^{2x\sqrt{-1}}-1}{e^{2x\sqrt{-1}}+1}=\dfrac{n(e^{2y\sqrt{-1}}-1)}{e^{2y\sqrt{-1}}+1}$, 故 $e^{2x\sqrt{-1}}=\dfrac{(1+n)e^{2y\sqrt{-1}}+1-n}{(1-n)e^{2y\sqrt{-1}}+1-n}$

令 $m=\dfrac{1-n}{1+n}$, 則 $e^{2x\sqrt{-1}}=e^{2y\sqrt{-1}}\left(\dfrac{1+me^{-2y\sqrt{-1}}}{1+me^{2y\sqrt{-1}}}\right)$, 故

$2x\sqrt{-1}=2y\sqrt{-1}+\log_e(1+me^{-2y\sqrt{-1}}-\log_e(1+me^{2y\sqrt{-1}})$

$=2y\sqrt{-1}-m(e^{2y\sqrt{-1}}+e^{-2y\sqrt{-1}})+\dfrac{m^2}{2}(e^{4y\sqrt{-1}}-e^{-4y\sqrt{-1}})-\cdots\cdots$

故 $x=y-m\left(\dfrac{e^{2y\sqrt{-1}}-e^{-2y\sqrt{-1}}}{2\sqrt{-1}}\right)+\dfrac{m^2}{2}\left(\dfrac{e^{4y\sqrt{-1}}-e^{-4y\sqrt{-1}}}{2\sqrt{-1}}\right)-\cdots\cdots$

$=y-m\sin 2y+\dfrac{m^2}{2}\sin 4y-\cdots\cdots\cdots$

(50) $e^{ax}\cos bx=\frac{1}{2}e^{ax}(e^{bx\sqrt{-1}}+e^{-bx\sqrt{-1}})=\frac{1}{2}e^{(a+b\sqrt{-1})x}+\frac{1}{2}e^{(a-b\sqrt{-1})x}$

展開此最後之結果,爲第十一編3.節指數級數,則x^n之係數爲

$\dfrac{1}{2\underline{|n}}\{(a+b\sqrt{-1})^n+(a-b\sqrt{-1})^n\}=\dfrac{r^n}{2\underline{|n}}\left\{\left(\dfrac{a}{r}+\dfrac{b}{r}\sqrt{-1}\right)^n+\left(\dfrac{a}{r}-\dfrac{b}{r}\sqrt{-1}\right)^n\right\}$

但令 $\dfrac{a}{r}=\cos\theta,\dfrac{b}{r}=\sin\theta$, 則 $r^2=a^2+b^2$, 由是 x^n之係數爲

$\dfrac{(a^2+b^2)^{\frac{n}{2}}}{2\underline{|n}}\{(\cos\theta+\sqrt{-1}\sin\theta)^n+(\cos\theta-\sqrt{-1}\sin\theta)^n\}$

$=\dfrac{(a^2+b^2)^{\frac{n}{2}}}{2\underline{|n}}\{\cos n\theta+\sqrt{-1}\sin n\theta+\cos n\theta-\sqrt{-1}\sin n\theta\}=\dfrac{(a^2+b^2)^{\frac{n}{2}}}{\underline{|n}}\cos n\theta$

51. 於三角形ABC.

$$B=\frac{b}{a}\sin C+\frac{b^2}{2a^2}\sin 2C+\frac{b^3}{3a^3}\sin 3C+\cdots\cdots\cdots$$

52. 同上.

$$\log_e c=\log_e a-\frac{b}{a}\cos C-\frac{b^2}{2a^2}\cos 2C-\frac{b^3}{3a^3}\cos 3C-\cdots\cdots\cdots$$

53. 同上. $\log_e b-\log_e a$

$$=\cos 2A-\cos 2B+\tfrac{1}{2}(\cos 4A-\cos 4B)+\tfrac{1}{3}(\cos 6A-\cos 6B)+\cdots\cdots\cdots$$

54. 同上. $\dfrac{b\cos A-a\cos B}{c}+\dfrac{b\cos 2A-a\cos 2B}{2c^2}$

$$+\frac{b\cos 3A-a\cos 3B}{3c^3}+\cdots\cdots\cdots\cdots=\log_e b-\log_e a.$$

例題解自51.至54.

(51) $\sin B=\frac{b}{a}\sin A=\frac{b}{a}\sin(B+C)$, 於例題47. 令$x$為B, n為$\frac{b}{a}$, a為C. 即得其證.

(52) $c^2=a^2+b^2-2ab\cos C=a^2+b^2-ab(e^{C\sqrt{-1}}+e^{-C\sqrt{-1}})$

$$=(a-be^{C\sqrt{-1}})(a-be^{-C\sqrt{-1}})=a^2\left(1-\frac{b}{a}e^{C\sqrt{-1}}\right)\left(1-\frac{b}{a}e^{-C\sqrt{-1}}\right),$$

$$2\log_e c=\log_e a+\log_e\left(1-\frac{b}{a}e^{C\sqrt{-1}}\right)+\log_e\left(1-\frac{b}{a}e^{-C\sqrt{-1}}\right)$$

$$=2\log_e a-\frac{b}{a}(e^{C\sqrt{-1}}+e^{-C\sqrt{-1}})-\frac{b^2}{2a^2}(e^{2C\sqrt{-1}}+e^{-2C\sqrt{-1}})\cdots$$ 〔第十一編4.節〕

故 $\log_e c=\log_e a-\frac{b}{a}\cos C-\frac{b^2}{2a^2}\cos 2C-\cdots\cdots\cdots\cdots$

本題 $b<a$, 則 $\frac{b}{a}\cos C+\frac{b^2}{2a^2}\cos 2C+\cdots\cdots\cdots$為斂級數.

(53) $\frac{b}{a}=\frac{\sin B}{\sin A}=\frac{e^{B\sqrt{-1}}-e^{-B\sqrt{-1}}}{e^{A\sqrt{-1}}-e^{-A\sqrt{-1}}}=\frac{e^{B\sqrt{-1}}(1-e^{-2B\sqrt{-1}})}{e^{A\sqrt{-1}}(1-e^{-2A\sqrt{-1}})}$,

$\log_e b-\log_e a=\log_e e^{B\sqrt{-1}}-\log_e e^{A\sqrt{-1}}+\log_e(1-e^{-B\sqrt{-1}})-\log_e(1-e^{-2A\sqrt{-1}})$

$=B\sqrt{-1}-A\sqrt{-1}-(e^{-2B\sqrt{-1}}+\frac{1}{2}e^{-4B\sqrt{-1}}+\frac{1}{3}e^{-6B\sqrt{-1}}+\cdots\cdots\cdots)$

$+(e^{-2A\sqrt{-1}}+\frac{1}{2}e^{-4A\sqrt{-1}}+\frac{1}{3}e^{-6A\sqrt{-1}}+\cdots\cdots\cdots)$, 〔第十一編4.節〕

$e^{-2B\sqrt{-1}}=\cos 2B-\sqrt{-1}\sin 2B,\quad e^{-4B\sqrt{-1}}=\cos 4B-\sqrt{-1}\sin 4B\cdots\cdots\cdots$

$e^{-2A\sqrt{-1}}=\cos 2A-\sqrt{-1}\sin 2A,\quad e^{-4A\sqrt{-1}}=\cos 4A-\sqrt{-1}\sin 4A,\cdots\cdots\cdots$

以此代入上之恆同式之左邊,則虛數部爲0.

(54) 令 $\alpha=\frac{b}{c}\cos A+\frac{b^2}{2c^2}\cos 2A+\frac{b^3}{3c^3}\cos 3A+\cdots\cdots\cdots$

及 $\beta=\frac{b}{c}\sin A+\frac{b^2}{2c^2}\sin 2A+\frac{b^3}{3c^3}\sin 3A+\cdots\cdots\cdots$

故 $\alpha+\sqrt{-1}\beta=\frac{b}{c}e^{A\sqrt{-1}}+\frac{b^2}{2c^2}e^{2A\sqrt{-1}}+\frac{b^3}{3c^3}e^{3A\sqrt{-1}}+\cdots\cdots\cdots$

$$=-\log_e\left(1-\frac{b}{c}e^{A\sqrt{-1}}\right)=-\log_e\left\{1-\frac{b}{c}(\cos A+\sqrt{-1}\sin A)\right\}$$

故 $1-\frac{b}{c}(\cos A+\sqrt{-1}\sin A)=e^{-\alpha-\beta\sqrt{-1}}=e^{-\alpha}\cdot e^{-\beta\sqrt{-1}}=e^{-\alpha}(\cos\beta-\sqrt{-1}\sin\beta)$

試比較此兩邊之實數部及虛數部,則

$e^{-\alpha}\cos\beta=\frac{c-b\cos A}{c}=\frac{a\cos B}{c},\quad e^{-\alpha}\sin\beta=\frac{b\sin A}{c}=\frac{a\sin B}{c}$, 平方之相加,則

$e^{-2\alpha}=\frac{a^2}{c^2}$, 故 $\alpha=\log_e c-\log_e a$.

同樣 令 $\alpha'=\frac{a}{c}\cos B+\frac{a^2}{2c^2}\cos 2B+\frac{a^3}{3c^3}\cos 3B+\cdots\cdots\cdots$

$\beta'=\frac{a}{c}\sin B+\frac{a^2}{2c^2}\sin 2B+\frac{a^3}{3c^3}\sin 3B+\cdots\cdots\cdots$

$\alpha'=\log_e c-\log_e b$,

故原式右邊 $=\alpha-\alpha'=(\log_e c-\log_e a)-(\log_e c-\log_e b)$

$=\log_e b-\log_e a$.

55. $x=\dfrac{2m+1}{2^n+1}$, 則

$$\cos x\cos 2x\cos 2^2x\cdots\cdots\cos 2^{n-1}x=1/2^n.$$

56. 設 $\cos\alpha+\cos\beta+\cos\gamma=\sin\alpha+\sin\beta+\sin\gamma=0$, 則

$$\frac{\cos 5\alpha+\cos 5\beta+\cos 5\gamma}{5}$$

$$=\frac{\cos 3\alpha+\cos 3\beta+\cos 3\gamma}{3}\cdot\frac{\cos 2\alpha+\cos 2\beta+\cos 2\gamma}{2}$$

$$-\frac{\sin 3\alpha+\sin 3\beta+\sin 3\gamma}{3}\cdot\frac{\sin 2\alpha+\sin 2\beta+\sin 2\gamma}{2}$$

及 $$\frac{\sin 5\alpha+\sin 5\beta+\sin 5\gamma}{5}$$

$$=\frac{\cos 3\alpha+\cos 3\beta+\cos\gamma}{3}\cdot\frac{\sin 2\alpha+\sin 2\beta+\sin 2\gamma}{2}$$

$$+\frac{\sin 3\alpha\sin 3\beta+\sin 3\gamma}{3}\cdot\frac{\cos 2\alpha+\cos 2\beta+\cos 2\gamma}{2}.$$

57. $$\frac{\cos 7\alpha+\cos 7\beta+\cos 7\gamma}{7}$$

$$=\frac{\cos 5\alpha+\cos 5\beta+\cos 5\gamma}{5}\cdot\frac{\cos 2\alpha+\cos 2\beta+\cos 2\gamma}{2}$$

$$-\frac{\sin 5\alpha+\sin 5\beta+\sin 5\gamma}{5}\cdot\frac{\sin 2\alpha+\sin 2\beta+\sin 2\gamma}{2}.$$

例題解自 55. 至 57.

(55) $\cos x\cos 2x\cos 2^2x\cdots\cdots\cos 2^{n-1}x$

$$=\tfrac{1}{2}(e^{x\sqrt{-1}}+e^{-x\sqrt{-1}})\tfrac{1}{2}(e^{2x\sqrt{-1}}+e^{-2x\sqrt{-1}})\cdots\cdots\tfrac{1}{2}(e^{2n-1x\sqrt{-1}}+e^{-2n-1x\sqrt{-1}}).$$

$$=\frac{1}{2^n}\cdot\frac{1}{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}(e^{2x\sqrt{-1}}-e^{-2x\sqrt{-1}})\times$$

$$(e^{2x\sqrt{-1}}+e^{-2x\sqrt{-1}})\ldots(e^{2^{n-1}x}+e^{-2^{n-1}x})$$

$$=\frac{1}{2^n}\cdot\frac{e^{2^nx\sqrt{-1}}-e^{-2^n\sqrt{-1}}}{e^{x\sqrt{-1}}-e^{-x\sqrt{-1}}}=\frac{1}{2^n}\cdot\frac{\sin 2^n x}{\sin x}$$

但 $\sin 2^n x=\sin 2^n\left(\frac{2m+1}{2^n+1}\right)\pi=\sin 2^n\left\{(2m+1)\pi-\frac{2m+1}{2^n+1}\pi\right\}$

$=\sin\frac{2m+1}{2^n+1}\pi=\sin x$, 故 $\frac{\sin 2^n x}{\sin x}=1$.

(56) $a+b+c=0$, 則 $\frac{a^5+b^5+c^5}{5}=\frac{a^3+b^3+c^3}{3}\cdot\frac{a^2+b^2+c^2}{2}$,

(大代數學講義129.章) 又依題意.

$(\cos\alpha+\sqrt{-1}\sin\alpha)+(\cos\beta+\sqrt{-1}\sin\beta)+(\cos\gamma+\sqrt{-1}\sin\gamma)=0$,

即 $e^{\alpha\sqrt{-1}}+e^{\beta\sqrt{-1}}+e^{\gamma\sqrt{-1}}=0$, 由是

$$\frac{e^{5\alpha\sqrt{-1}}+e^{5\beta\sqrt{-1}}+e^{5\gamma\sqrt{-1}}}{5}=\frac{e^{3\alpha\sqrt{-1}}+e^{3\beta\sqrt{-1}}+e^{3\gamma\sqrt{-1}}}{3}\times$$

$$\frac{e^{2\alpha\sqrt{-1}}+e^{2\beta\sqrt{-1}}+e^{2\gamma\sqrt{-1}}}{2},$$

即 $$\frac{\cos 5\alpha+\cos 5\beta+\cos 5\gamma+\sqrt{-1}(\sin 5\alpha+\sin 5\beta+\sin 5\gamma)}{5}$$

$$=\frac{\cos 3\alpha+\cos 3\beta+\cos 3\gamma+\sqrt{-1}(\sin 3\alpha+\sin 3\beta+\sin 3\gamma)}{3}$$

$$\times\frac{\cos 2\alpha+\cos 2\beta+\cos 2\gamma+\sqrt{-1}(\sin 2\alpha+\sin 2\beta+\sin 2\gamma)}{2}$$

$$=\left\{\frac{\cos 3\alpha+\cos 3\beta+\cos 3\gamma}{3}\cdot\frac{\cos 2\alpha+\cos 2\beta+\cos 2\gamma}{2}\right.$$

$$\left.-\frac{\sin 3\alpha+\sin 3\beta+\sin 3\gamma}{3}\cdot\frac{\sin 2\alpha+\sin 2\beta+\sin 2\gamma}{2}\right\}$$

$$+\sqrt{-1}\left\{\frac{\sin 3\alpha+\sin 3\beta+\sin 3\gamma}{3}\cdot\frac{\cos 2\alpha+\cos 2\beta+\cos 2\gamma}{2}\right.$$

$$\left.+\frac{\cos 3\alpha+\cos 3\beta+\cos 3\gamma}{3}\cdot\frac{\sin 2\alpha+\sin 2\beta+\sin 2\gamma}{2}\right\}$$

於兩邊之實數部及虛數部各比較之, 即得.

(57) $\frac{a^7+b^7+c^7}{7}=\frac{a^5+b^5+c^5}{5}\cdot\frac{a^2+b^2+c^2}{2}$, 從此可得其證.

反函數

22. 反函數

前數節所示諸法則.就其內有關乎反函數者.茲更為說明之.

23. Gregory 氏之級數

$$\theta=\tan\theta-\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta-\cdots\cdots\cdots\cdots$$

〔證〕 由19.節. $\sqrt{-1}\tan\theta=\dfrac{e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}}}{e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}}}$,

故 $\dfrac{1+\sqrt{-1}\tan\theta}{1-\sqrt{-1}\tan\theta}=\dfrac{2e^{\theta\sqrt{-1}}}{2e^{-\theta\sqrt{-1}}}=e^{2\theta\sqrt{-1}}$, 變此為對數式.則

$2\theta\sqrt{-1}=\log_e(1+\sqrt{-1}\tan\theta)-\log_e(1-\sqrt{-1}\tan\theta)$, 由第十一編4.節.

$=2\sqrt{-1}(\tan\theta-\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta-\cdots\cdots\cdots\cdots)$,

由是 $\theta=\tan\theta-\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta-\cdots\cdots\cdots\cdots$

24. 同上反函數之級數

$$\tan^{-1}x=x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\cdots\cdots\cdots\cdots$$

〔證〕 於前節之級數.令 $\tan\theta=x$, 即 $\theta=\tan^{-1}x$, 則

$\tan^{-1}x=x-\tfrac{1}{3}x^3+\tfrac{1}{5}x^5-\cdots\cdots\cdots\cdots$

25. 餘論

前節之級數.其第 n 項為 $\dfrac{(-1)^{n-1}}{2n-1}x^{2n-1}$, 第 $(n+1)$ 項為

$\dfrac{(-1)^n}{2n+1}x^{2n+1}$, 而其比為 $\dfrac{(-1)^n}{2n+1}x^{2n+1}\div\dfrac{(-1)^{n-1}}{2n-1}x^{2n-1}=-\dfrac{2n-1}{2n+1}x^2$.

故 $x<1$ 及 $x=1$, 由代數學知此級數為斂級數.

即 x 之絕對值要不比1大.

故 θ 要在 $\dfrac{\pi}{4}$ 與 $-\dfrac{\pi}{4}$ 之間.

然令 $\theta=n\pi+\phi$, 則 ϕ 在 $-\frac{\pi}{4}$ 與 $\frac{\pi}{4}$ 之間可也.

卽以此代入 23. 節之級數. 則

$$\phi=\tan\phi-\tfrac{1}{3}\tan^3\phi+\tfrac{1}{5}\tan^5\phi-\cdots\cdots$$

卽 $$\theta-n\pi=\tan\theta-\tfrac{1}{3}\tan^3\theta+\tfrac{1}{5}\tan^5\theta-\cdots\cdots$$

26. Gregory 氏級數之應用

$$\frac{\pi}{4}=1-\frac{1}{3}+\frac{1}{5}-\frac{1}{7}+\frac{1}{9}-\cdots\cdots$$

於 23. 節之級數, 令 $\theta=\frac{\pi}{4}$ 則 $\tan\frac{\pi}{4}=1$, 故得其證.

27. 尤拉 [Euler] 氏之級數

$$\frac{\pi}{4}=\frac{1}{2}-\frac{1}{3.2^3}+\frac{1}{5.2^5}-\frac{1}{7.2^7}-\cdots\cdots+\frac{1}{3}-\frac{1}{3.3^3}+\frac{1}{5.3^5}-\frac{1}{7.3^7}+\cdots\cdots$$

〔證〕 $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}=\tan^{-1}\frac{\frac{1}{2}+\frac{1}{3}}{1-\frac{1}{2}\cdot\frac{1}{3}}=\tan^{-1}1=\frac{\pi}{4}$,

卽 $\frac{\pi}{4}=\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{3}$, 由 24. 節.

$$=\frac{1}{2}-\frac{1}{3}\left(\frac{1}{2}\right)^3+\frac{1}{5}\left(\frac{1}{2}\right)^5-\cdots\cdots+\frac{1}{3}-\frac{1}{3}\left(\frac{1}{3}\right)^3+\frac{1}{5}\left(\frac{1}{3}\right)^5-\cdots\cdots$$

28. Maehin 氏之級數

$$\frac{\pi}{4}=4\left\{\frac{1}{5}-\frac{1}{3.5^3}+\frac{1}{5.5^5}-\frac{1}{7.5^7}+\cdots\cdots\right\}$$

$$-\left\{\frac{1}{239}-\frac{1}{3.239^3}+\frac{1}{5.239^5}-\frac{1}{7.239^7}+\cdots\cdots\right\}.$$

〔證〕 $\frac{\pi}{4}=4\tan^{-1}\frac{1}{5}-\tan^{-1}\frac{1}{239}$, (第九編例題拾 12.)

$$=4\left\{\frac{1}{5}-\frac{1}{3}\left(\frac{1}{5}\right)^3+\frac{1}{5}\left(\frac{1}{5}\right)^5-\frac{1}{7}\left(\frac{1}{5}\right)^7+\cdots\cdots\right\}$$

$$-\left\{\frac{1}{239}-\frac{1}{3}\left(\frac{1}{239}\right)^3+\frac{1}{5}\left(\frac{1}{239}\right)^5-\frac{1}{7}\left(\frac{1}{239}\right)^7+\cdots\cdots\right\}.$$

例題三十

求次各式之證.

1. $\frac{\pi}{8}=\frac{1}{1.3}+\frac{1}{5.7}+\frac{1}{9.11}+\cdots\cdots\cdots$

2. $\frac{\pi}{6}=\frac{1}{\sqrt{3}}\left(1-\frac{1}{3.3^2}+\frac{1}{5.3^2}-\frac{1}{7.3^3}+\cdots\cdots\cdots\right)$

3. $tan^{-1}\frac{m-nx}{n-mx}=tan^{-1}\frac{m}{n}-x+\frac{x^3}{3}-\frac{x^5}{5}+\frac{x^7}{7}+\cdots\cdots\cdots$

4. $sin^{-1}x=x+\frac{1}{.2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\frac{1.3.5}{2.4.6}\cdot\frac{x^7}{7}+\cdots\cdots\cdots$

5. $(sin^{-1}x)^2=2\left\{\frac{x^2}{2}+\frac{2}{3}\cdot\frac{x^4}{4}+\frac{2.4}{3.5}\cdot\frac{x^6}{6}+\frac{2.4.6}{3.5.7}\cdot\frac{x^8}{8}+\cdots\cdots\cdots\right\}$

6. $tan^{-1}\frac{x}{\sqrt{1-x^2}}=x+\frac{x^3}{6}+\frac{3x^5}{40}+\cdots\cdots\cdots$

$$+\frac{1.3\ldots\ldots(3r-1)}{2.4\ldots\ldots\ldots2r}\cdot\frac{x^{2r+1}}{2r+1}+\cdots\cdots\cdots$$

例題解自 1. 至 6.

(1) 由 26. 節 $\frac{\pi}{4}=\left(1-\frac{1}{3}\right)+\left(\frac{1}{5}-\frac{1}{7}\right)+\left(\frac{1}{9}-\frac{1}{11}\right)+\cdots\cdots\cdots$

$$=\frac{2}{1.3}+\frac{2}{5.7}+\frac{2}{9.11}+\cdots\cdots\cdots$$

(2) $tan\frac{\pi}{6}=\frac{1}{\sqrt{3}}$, 故 $\frac{\pi}{6}=tan^{-1}\frac{1}{\sqrt{3}}$, 由 24. 節

$$=\frac{1}{\sqrt{3}}-\frac{1}{3}\left(\frac{1}{\sqrt{3}}\right)^3+\frac{1}{5}\left(\frac{1}{\sqrt{3}}\right)^5-\cdots\cdots\cdots=\frac{1}{\sqrt{3}}\left(1-\frac{1}{3.3}+\frac{1}{5.3^2}-\cdots\cdots\cdots\right),$$

(3) $tan^{-1}\frac{m-nx}{n+mx}=\theta$, 則 $\frac{m-nx}{n+mx}=tan\,\theta=\frac{\frac{m}{n}-x}{1+\frac{m}{n}x}$,

$\tan\alpha=\frac{n}{m}$, $\tan\beta=x$, 卽 $\alpha=\tan^{-1}\frac{m}{n}$, $\beta=\tan^{-1}x$,

故 $\frac{m-nx}{n+mx}=\tan\theta=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}=\tan(\alpha-\beta)$,

故 $\tan^{-1}\frac{m-nx}{n+mx}=\theta=\alpha-\beta=\tan^{-1}\frac{m}{n}-\tan^{-1}x$

$$=\tan^{-1}\frac{m}{n}-\left(x-\frac{1}{3}x^3+\frac{1}{5}x^5-\cdots\cdots\right)$$

(4) 令 $\sin^{-1}x=\theta$, 則 $x=\sin\theta=\theta-\frac{\theta^3}{\underline{|3}}+\frac{\theta^5}{\underline{|5}}-\cdots\cdots\cdots$(6. 節)

又 $\sin^{-1}x=a_1x+a_3x^3+a_5x^5+\cdots\cdots\cdots$ 卽

$$\theta=a_1\left(\theta-\frac{\theta^3}{\underline{|3}}+\frac{\theta^5}{\underline{|5}}-\cdots\right)+a_3\left(\theta-\frac{\theta^3}{\underline{|3}}+\frac{\theta^5}{\underline{|5}}-\cdots\right)^3+a_5\left(\theta-\frac{\theta^3}{\underline{|3}}+\frac{\theta^5}{\underline{|5}}-\cdots\right)^5+\cdots\cdots$$

$$=a_1\theta+\left(a_3-\frac{a_1}{\underline{|3}}\right)\theta^3+\left(a_5-\frac{3}{\underline{|3}}a_3+\frac{a_1}{\underline{|5}}\right)\theta^5+\cdots\cdots\cdots$$

比較兩邊 θ 之同方乘, 則 $\theta=a_1\theta$, 故 $a_1=1$,

$a_3-\frac{a_1}{\underline{|3}}=0$, 故 $a_3=\frac{a_1}{\underline{|3}}=\frac{1}{2.3}$, $a_5-\frac{3}{\underline{|3}}a_3+\frac{a_1}{\underline{|5}}=0$,

故 $a_5=\frac{3}{\underline{|3}}\cdot\frac{1}{2.3}-\frac{1}{\underline{|5}}=\frac{1.3}{2.4.5}$,

(5) $(\sin^{-1}x)^2=\left\{x+\frac{1}{2}\cdot\frac{x^3}{3}+\frac{1.3}{2.4}\cdot\frac{x^5}{5}+\cdots\cdots\cdots\right\}^2$, (前例)

$$=x^2+\frac{x^4}{3}+x^6\left(\frac{1}{36}+\frac{3}{20}\right)+\cdots\cdots=x^2+\frac{4}{3}\cdot\frac{x^4}{4}+\frac{16}{3.5}\cdot\frac{x^6}{6}+\cdots\cdots$$

$$=2\left(\frac{x^2}{2}+\frac{2}{3}\cdot\frac{x^4}{4}+\frac{2.4}{3.5}\cdot\frac{x^6}{6}+\cdots\cdots\right)$$

(6) $\tan^{-1}\frac{x}{\sqrt{1-x^2}}=\tan^{-1}\frac{\sin\theta}{\sqrt{1-\sin^2\theta}}$, 但

$\theta=\sin^{-1}x=\tan^{-1}\tan\theta=\theta=\sin^{-1}x$, 見例題4.

但 $\frac{1.3.5\cdots\cdots(2r-1)}{2.4.6\cdots\cdots\cdots\cdots 2r}\cdot\frac{x^{2r+1}}{2r+1}$ 爲4. 及本例一般之項.

7. $\sin^{-1}(\cos\theta+\sqrt{-1}\sin\theta)$ 之一值,爲

$\cos^{-1}\sqrt{\sin\theta}+\sqrt{-1}\log_e(\sqrt{\sin\theta}+\sqrt{1+\sin\theta})$

但 θ 在 0 與 $\frac{\pi}{2}$ 之間.

8. $\log_e(1+\cos 2\theta+\sqrt{-1}\sin 2\theta)$ 之一值,爲

$\log_e(2\cos\theta)+\theta\sqrt{-1}$. 由此示 Gregory 氏之級數 $\frac{\pi}{4}$ 之值.

例題解自 7. 至 8.

(7) $\sin^{-1}(\cos\theta+\sqrt{-1}\sin\theta)=\alpha+\beta\sqrt{-1}$, 則由例題二十九 20.

$\cos\theta+\sqrt{-1}\sin\theta=\sin(\alpha+\beta\sqrt{-1})=\frac{1}{2}(e^{\beta}+e^{-\beta})\sin\alpha+\frac{1}{2}(e^{\beta}-e^{-\beta})\cos\alpha\sqrt{-1}$,

故 $\cos\theta=\frac{1}{2}(e^{\beta}+e^{-\beta})\sin\alpha,\quad \sin\theta=\frac{1}{2}(e^{\beta}-e^{-\beta})\cos\alpha$,

故 $\frac{4\cos^2\theta}{\sin^2\alpha}-\frac{4\sin^2\theta}{\cos^2\alpha}=(e^{\beta}+e^{-\beta})^2-(e^{\beta}-e^{-\beta})^2=4$,

從此得 $\cos\alpha=\sqrt{\sin\theta}$, 即 $\alpha=\cos^{-1}\sqrt{\sin\theta}$,

又 $\sin\theta=\frac{1}{2}(e^{\beta}-e^{-\beta})\cos\alpha=\frac{1}{2}(e^{\beta}-e^{-\beta})\sqrt{\sin\theta}$, 以此平方之. 以 $\sin\theta$

約其兩邊以求 e^{β}, 則 $e^{\beta}=\sqrt{\sin\theta}+\sqrt{1+\sin\theta}$,

故 $\beta=\log_e(\sqrt{\sin\theta}+\sqrt{1+\sin\theta})$

(8) $\log_e(1+\cos 2\theta+\sqrt{-1}\sin 2\theta)=\log_e\{2\cos\theta(\cos\theta+\sqrt{-1}\sin\theta)\}$

$=\log_e(2\cos\theta)+\log_e(\cos\theta+\sqrt{-1}\sin\theta)=\log_e(2\cos\theta)+\theta\sqrt{-1}$,

故 $\theta\sqrt{-1}=\log_e(1+\cos 2\theta+\sqrt{-1}\sin 2\theta)-\log_e(2\cos\theta)$,

令 $\theta=\frac{\pi}{4}$, 則 $\frac{\pi}{4}\sqrt{-1}=\log_e(1+\sqrt{-1})-\log_e(\sqrt{2})$, 由第十一編 4. 節

之公式 (7) 及 (9), 而展開右邊. 可與左邊虛數部比較.

第拾柒編

級數之和

1. 級數之和 此編示求三角函數之級數之和之法.其n項之和令爲S_n.無限項之和令爲S.

2. 等差級數之諸角 成等差級數之諸角.求其正餘弦之和之法.茲示如次.

$$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\cdots\cdots+\sin\{\alpha+(n-1)\beta\}$$
$$=\frac{\sin\{\alpha+\frac{1}{2}(n-1)\beta\}\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta},\cdots\cdots\cdots\cdots (1)$$

$$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+2\beta)+\cdots\cdots+\cos\{\alpha+(n-1)\beta\}$$
$$=\frac{\cos\{\alpha+\frac{1}{2}(n-1)\beta\}\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta},\cdots\cdots\cdots\cdots (2)$$

〔證〕
$$\cos(\alpha-\tfrac{1}{2}\beta)-\cos(\alpha+\tfrac{1}{2}\beta)=2\sin\tfrac{1}{2}\beta\sin\alpha$$
$$\cos(\alpha+\tfrac{1}{2}\beta)-\cos(\alpha+\tfrac{3}{2}\beta)=2\sin\tfrac{1}{2}\beta\sin(\alpha+\beta)$$
$$\cos(\alpha+\tfrac{3}{2}\beta)-\cos(\alpha+\tfrac{5}{2}\beta)=2\sin\tfrac{1}{2}\beta\sin(\alpha+2\beta)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cos\left(\alpha+\frac{2n-3}{2}\beta\right)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)=2\sin\tfrac{1}{2}\beta\sin\{\alpha+(n-1)\beta\}$$

以此相加,則 $\cos(\alpha-\frac{1}{2}\beta)-\cos\left(\alpha+\frac{2n-1}{2}\beta\right)=2\sin\frac{1}{2}\beta.S_n$

$$S_n=\frac{\cos(\alpha-\frac{1}{2}\beta)-\cos\{\alpha+\frac{1}{2}(2n-1)\beta\}}{2\sin\frac{1}{2}\beta}=\frac{\sin\{\alpha+\frac{1}{2}(n-1)\beta\}\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}.$$

又 $\sin(\alpha+\frac{1}{2}\beta)-\sin(\alpha-\frac{1}{2}\beta)=2\sin\frac{1}{2}\beta\cos\alpha,$

$\sin(\alpha+\frac{3}{2}\beta)-\sin(\alpha-\frac{1}{2}\beta)=2\sin\frac{1}{2}\beta\cos(\alpha+\beta),$

$\sin(\alpha+\frac{5}{2}\beta)-\sin(\alpha+\frac{3}{2}\beta)=2\sin\frac{1}{2}\beta\cos(\alpha+2\beta),$

....

$$\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin\left(\alpha+\frac{2n-3}{2}\beta\right)=2\sin\tfrac{1}{2}\beta\cos\{\alpha+(n-1)\beta\}$$

以此相加,則 $\sin\left(\alpha+\frac{2n-1}{2}\beta\right)-\sin(\alpha-\frac{1}{2}\beta)=2\sin\frac{1}{2}\beta.S_n$

$$S_n=\frac{\sin\{\alpha+\frac{1}{2}(2n-1)\beta\}-\sin(\alpha-\frac{1}{2}\beta)}{2\sin\frac{1}{2}\beta}=\frac{\cos\{\alpha+\frac{1}{2}(n-1)\beta\}\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}.$$

3. 方乘級數 求次式之和.但 m 爲正整數.

$\sin^m\alpha+\sin^m(\alpha+\beta)+\sin^m(\alpha+2\beta)+\cdots\cdots+\sin^m\{\alpha+(n-1)\beta\},\cdots\cdots(1)$

$\cos^m\alpha+\sin^m(\alpha+\beta)+\cos^m(\alpha+2\beta)+\cdots\cdots+\cos^m\{\alpha+(n-1)\beta\},\cdots\cdots(2)$

由第十六編11.節及12.節.而 m 爲偶數,則

$$(2\sqrt{-1}\sin\alpha)^m=2\cos m\alpha-m.2\cos(m-2)\alpha+\frac{m(m-1)}{\underline{|2}}2\cos(m-4)\alpha$$
$$-\cdots\cdots+\frac{m(m-1)(m-2)\cdots\cdots(\frac{1}{2}m+1)}{\underline{|\frac{1}{2}m}}$$

卽 $$2^{m-1}(\sqrt{-1})^m\sin^m\alpha=\cos m.\alpha-m\cos(m-2)\alpha+\frac{m(m-1)}{\underline{|2}}\cos(m-4)\alpha$$
$$-\cdots\cdots+\frac{m(m-1)(m-2)\cdots\cdots(\frac{1}{2}m-1)}{2\underline{|\frac{1}{2}m}},$$

同樣. $2^{m-1}(\sqrt{-1})^m\sin^m(\alpha+\beta)$, $2^{m-1}(\sqrt{-1})^m\sin(\alpha+2\beta)$, 求其相應項相加爲(1)之和.其 m 爲偶數,則 $2^{m-1}(\sqrt{-1})^mS_n$

$$=[\cos m\alpha+\cos(m\alpha+m\beta)+\cos(m\alpha+2m\beta)+\cdots\cdots+\cos\{m\alpha+(n-1)m\beta\}]$$
$$-m[\cos(m-2)\alpha+\cos\{(m-2)\alpha+(m-2)\beta\}+\cdots\cdots+\cos\{(m-2)\alpha+(n-1)(m-2)\beta\}]$$
$$+\frac{m(m-1)}{\underline{|2}}[\cos(m-4)\alpha+\cos\{(m-4)\alpha+(m-4)\beta\}+\cdots\cdots+\cos\{(m-4)\alpha+(n-1)(m-4)\beta\}]$$
$$-\cdots\cdots\cdots\cdots+\frac{nm(m-1)(m-2)\cdots\cdots(\frac{1}{2}m+1)}{2\underline{|\frac{1}{2}m}},$$

故由1.節(2).

$$S_n=\frac{1}{2^{m-1}(\sqrt{-1})^m}\left[\frac{\cos\{m\alpha+\frac{1}{2}(n-1)m\beta\}\sin\frac{1}{2}nm\beta}{\sin\frac{1}{2}m\beta}\right.$$

$$-\frac{m\cos\{(m-2)\alpha+\frac{1}{2}(n-1)(m-2)\beta\}\sin\frac{1}{2}n(m-2)\beta}{\sin\frac{1}{2}(m-2)\beta}$$

$$+\frac{m(m-1)\cos\{(m-4)\alpha+\frac{1}{2}(n-1)(m-4)\alpha\}\sin\frac{1}{2}n(m-4)\beta}{\sin\frac{1}{2}(m-4)\beta}$$

$$\left.-\cdots\cdots\cdots+\frac{nm(m-1)(m-2)\cdots\cdots(\frac{1}{2}m+1)}{2\,\underline{|\frac{1}{2}m}}\right]\cdots\cdots\cdots\cdots\cdots(A)$$

又m為奇數.由11.節及12.節得(1)之級數之和,為

$$(2\sqrt{-1}\sin\alpha)^m=2\sqrt{-1}\sin m\alpha-m2\sqrt{-1}\sin(m-2)\alpha+\frac{m(m-1)}{\underline{|2}}2\sqrt{-1}\sin(m-4)\alpha$$

$$-\cdots\cdots\cdots+\frac{m(m-1)(m-2)\cdots\cdots\frac{1}{2}(m+3)}{\underline{|\frac{1}{2}(m-1)}}\sin\alpha.$$

與前同法.可得所求.

(2)之級數.亦可同樣由11.節及12.節求得其和.而從$(2\cos\theta)^n$即可導得.但是等級數,若m之值甚大.則求其和甚為困難.

4. 指數之應用

已知$A_0+A_1xe^{\theta\sqrt{-1}}+A_2x^2e^{2\theta\sqrt{-1}}+A_3x^3e^{3\theta\sqrt{-1}}+\cdots\cdots\cdots$級數之和.求次兩級數和之和.

$$A_0+A_1x\cos\theta+A_2x^2\cos2\theta+A_3x^3\cos3\theta+\cdots\cdots\cdots(1)$$

及

$$A_1x\sin\theta+A_2x^2\sin2\theta+A_3x^3\sin3\theta+\cdots\cdots\cdots(2)$$

令(1)之和為C,(2)之和為S,則

$$C+S\sqrt{-1}=A_0+A_1x(\cos\theta+\sqrt{-1}\sin\theta)+A_2x^2(\cos2\theta+\sqrt{-1}\sin2\theta)$$

$$+A_3x^3(\cos3\theta+\sqrt{-1}\sin3\theta)+\cdots\cdots\cdots\cdots$$

$$=A_0+A_1xe^{\theta\sqrt{-1}}+A_2x^2e^{2\theta\sqrt{-1}}+A_3x^3e^{3\theta\sqrt{-1}}+\cdots\cdots=\alpha+\beta\sqrt{-1},$$

但依題意已知α及β,故$C=\alpha$, $S=\beta$.

例題三十一

求次各式之證.

1. $\sin\alpha+\sin\left(\alpha+\frac{2\pi}{n}\right)+\sin\left(\alpha+\frac{4\pi}{n}\right)+\ldots\ldots+\sin\left\{\alpha+\frac{2(n-1)\pi}{n}\right\}=0.$

2. $\cos\alpha+\cos\left(\alpha+\frac{2\pi}{n}\right)+\cos\left(\alpha+\frac{4n}{n}\right)+\ldots\ldots+\cos\left\{\alpha+\frac{2(n-1)\pi}{n}\right\}=0.$

3. $\sin\alpha+\sin 2\alpha+\sin 3\alpha+\ldots\ldots\ldots+\sin n\alpha$

$$=\sin\tfrac{1}{2}(n+1)\alpha\sin\tfrac{1}{2}n\alpha/\sin\tfrac{1}{2}\alpha.$$

4. $\cos\alpha+\cos 2\alpha+\cos 3\alpha+\ldots\ldots\ldots+\cos n\alpha$

$$=\cos\tfrac{1}{2}(n+1)\alpha\sin\tfrac{1}{2}n\alpha/\sin\tfrac{1}{2}\alpha.$$

5. $\cos^3\alpha+\cos^3 2\alpha+\cos^3 3\alpha+\ldots\ldots\ldots+\cos^3 n\alpha$

$$=\frac{\cos\frac{3}{2}(n+1)\alpha\sin\frac{3}{2}n\alpha}{4\sin\frac{3}{2}\alpha}+\frac{3\cos\frac{1}{2}(n+1)\alpha\sin\frac{1}{2}n\alpha}{4\sin\frac{1}{2}\alpha}.$$

6. $\sin^3\alpha+\sin^3 2\alpha+\sin^3 3\alpha+\ldots\ldots\ldots+\sin^3 n\alpha$

$$=\frac{3\sin\frac{1}{2}(n+1)\alpha\sin\frac{1}{2}n\alpha}{4\sin\frac{1}{2}\alpha}-\frac{\sin\frac{3}{2}(n+1)\alpha\sin\frac{3}{2}n\alpha}{4\sin\frac{3}{2}\alpha}.$$

7. $\sin^4\alpha+\sin^4(\alpha+\beta)+\sin^4(\alpha+2\beta)+\ldots\ldots+\sin^4\{\alpha+(n-1)\beta\}$

$$=\frac{\cos\{4\alpha+2(n-1)\beta\}\sin 2n\beta}{8\sin 2\beta}-\frac{\cos\{2\alpha+(n-1)\beta\}\sin n\theta}{2\sin\beta}+\frac{3n}{8}.$$

8. $\sin^4\alpha+\sin^4\left(\alpha+\frac{\pi}{n}\right)+\sin^4\left(\alpha+\frac{2\pi}{n}\right)+\ldots\ldots+\sin^4\left\{\alpha+\frac{(n-1)\pi}{n}\right\}$

$$=\frac{3}{8}n.$$

例題解自1.至8.

(1) 於2.節公式(1),令$\beta=\frac{2\pi}{n}$,則左邊爲本題之級數.又右邊爲

$\sin\frac{1}{2}n\beta=\sin\pi=0$,故其和爲0.

(2) 於2.節公式(2),令$\beta=\frac{2\pi}{n}$, 與前同樣求之卽得.

(3) 於2.節(1)令$\beta=\alpha$,則

$$S_n=\frac{\sin\{\alpha+\frac{1}{2}(n-1)\alpha\}\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}.$$

(4) 於2.節(2),可令$\beta=\alpha$以求之.

(5) $4\cos^3\alpha=\cos3\alpha+3\cos\alpha$, 同樣. $4\cos^3 2\alpha=\cos6\alpha+3\cos2\alpha$,

$4\cos^3 3\alpha=\cos9\alpha+3\cos3\alpha,\ldots\ldots\ldots4\cos^3 n\alpha=\cos3n\alpha+3\cos n\alpha$,

相加.則

$$4S_n=\cos3\alpha+\cos6\alpha+\cos9\alpha+\ldots\ldots\ldots+\cos3n\alpha$$

$$+3(\cos\alpha+\cos2\alpha+\cos3\alpha+\ldots\ldots\ldots+\cos n\alpha),\text{由前例.}$$

$$=\frac{\cos\frac{1}{2}(n+1)3\alpha\sin\frac{1}{2}n3\alpha}{\sin\frac{3}{2}\alpha}+\frac{3\cos\frac{1}{2}(n+1)\alpha\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}.$$

(6) 如前例用$4\sin^3\alpha=3\sin\alpha-\sin3\alpha$,則由例題3.卽可得其和.

(7) 於3.節(A),令$m=4$,則

$$S_n=\frac{1}{2^3(\sqrt{-1})^4}\Big[\frac{\cos\{4\alpha+2(n-1)\beta\}\sin2n\beta}{\sin2\beta}$$

$$-\frac{4\cos\{2\alpha+(n-1)\beta\}\sin n\beta}{\sin\beta}+\frac{n4.3}{2\ 2}\Big]$$

$$=\frac{\cos\{4\alpha+2(n-1)\beta\}\sin2n\beta}{8\sin2\beta}-\frac{\cos\{2\alpha+(n-1)\beta\}\sin n\beta}{2\sin\beta}+\frac{3n}{8}.$$

(8) 於前例,令$\beta=\frac{\pi}{n}$, 則 $\sin2n\beta=\sin2\pi=0$, $\sin n\beta=\sin\pi=0$.

故 $S_n=3n/8$.

9. $sin\alpha - sin(\alpha+\beta) + sin(\alpha+2\beta) - \cdots\cdots\cdots\cdots$

$$+(-1)^{n-1}sin\{\alpha+(n-1)\beta\} = \frac{sin\left\{\alpha+\frac{n-1}{2}(\beta+\pi)\right\}sin\frac{n}{2}(\beta+\pi)}{sin\frac{1}{2}(\beta+\pi)}.$$

10. $cos\alpha - cos(\alpha+\beta) + cos(\alpha+2\beta) - \cdots\cdots\cdots\cdots$

$$+(-1)^{n-1}cos\{\alpha+(n-1)\beta\} = \frac{cos\left\{\alpha+\frac{n-1}{2}(\beta+\pi)\right\}sin\frac{n}{2}(\beta+\pi)}{sin\frac{1}{2}(\beta+\pi)}.$$

11. $csc\,x + csc\,2x + csc\,4x + \cdots\cdots\cdots + csc\,2^{n-1}x$

$$= cot\tfrac{1}{2}x - cot\,2^{n-1}x.$$

12. $tan\,x + \frac{1}{2}tan\frac{x}{2} + \frac{1}{2^2}tan\frac{x}{2^2} + \cdots\cdots\cdots + \frac{1}{2^{n-1}}tan\frac{x}{2^{n-1}}$

$$= \frac{1}{2^{n-1}}cot\frac{x}{2^{n-1}} - 2cot2x.$$

13. $tan\,x + \frac{1}{2}tan\frac{x}{2} + \frac{x}{2^2}tan\frac{x}{2^2} + \cdots\cdots\cdots$ 至∞

$$= \frac{1}{x} - 2cot2x$$

14. $sin\alpha + x\,sin(\alpha+\beta) + x^2 sin(\alpha+2\beta)$

$$+\cdots\cdots\cdots + x^{n-1}sin\{\alpha+(n-1)\beta\}$$

$$= \frac{sin\alpha - x\,sin(\alpha-\beta) - x^n sin(\alpha+n\beta) + x^{n-1}sin\{\alpha+(n-1)\beta\}}{1-2x\,cos\beta+x^2}.$$

例題解自 9. 至 14.

(9) 於 2. 節 (1) 令 β 爲 $\beta+\pi$, 則可得本題之級數.

$S_n = sin\alpha + sin(\alpha+\beta+\pi) + sin(\alpha+2\beta+2\pi) + \cdots\cdots + sin\{\alpha+(n-1)\beta+(n-1)\pi\}$

$$=\frac{\sin\{\alpha+\frac{1}{2}(n-1)(\beta+\pi)\}\sin\frac{1}{2}n(\beta+\pi)}{\sin\frac{1}{2}(\beta+\pi)}.$$

(10) 於2.節(2),可用$\beta+\pi$以代其β以求之.

(11) $\csc x=\cot\frac{x}{2}-\cot x,\quad \csc 2x=\cot x-\cot 2x,\ldots\ldots\ldots\ldots$

$\csc 2^{n-1}x=\cot 2^{n-2}x-\cot 2^{n-1}x$, 相加.得 $S_n=\cot\frac{x}{2}-\cot 2^{n-1}x$.

(12) $\tan x=\cot x-2\cot 2x,\quad \frac{1}{2}\tan\frac{x}{2}=\frac{1}{2}\cot\frac{x}{2}-\cot x,$

$\frac{1}{2^2}\tan\frac{x}{2^2}=\frac{1}{2^2}\cot\frac{x}{2^2}-\frac{1}{2}\cot\frac{x}{2},\ldots\ldots\ldots\ldots\frac{1}{2^{n-1}}\tan\frac{x}{2^{n-1}}=\frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}}-\frac{1}{2^{n-2}}\cot\frac{x}{2^{n-2}}$

相加.得 $S_n=\frac{1}{2^{n-1}}\cot\frac{x}{2^{n-1}}-2\cot 2x$.

(13) 依前例. $S_n=\frac{1}{x}\left(\frac{x}{2^{n-1}}\Big/\tan\frac{x}{2^{n-1}}\right)-2\cot 2x$, 若 $n=\infty$, 則

$\frac{x}{2^{n-1}}$等0,故n增大.則 $\frac{x}{2^{n-1}}\Big/\tan\frac{x}{2^{n-1}}$ 之極限等於1.(第十編2.節)

由是 $S=\frac{1}{x}-2\cot 2x$.

(14) $$S_n=\frac{1}{2\sqrt{-1}}\{e^{\alpha\sqrt{-1}}-e^{-\alpha\sqrt{-1}}\}+\frac{x}{2\sqrt{-1}}\{e^{(\alpha+\beta)\sqrt{-1}}-e^{-(\alpha+\beta)\sqrt{-1}}\}$$

$$+\frac{x^2}{2\sqrt{-1}}\{e^{(\alpha+2\beta)\sqrt{-1}}-e^{-(\alpha+2\beta)\sqrt{-1}}\}$$

$$+\ldots\ldots\ldots+\frac{x^{n-1}}{2\sqrt{-1}}\{e^{(\alpha+\overline{n-1}\beta)\sqrt{-1}}-e^{-(\alpha+\overline{n-1}\beta)\sqrt{-1}}\}$$

$$=\frac{1}{2\sqrt{-1}}[e^{\alpha\sqrt{-1}}\{1+xe^{\beta\sqrt{-1}}+x^2e^{2\beta\sqrt{-1}}+\ldots\ldots\ldots+x^{n-1}e^{(n-1)\beta\sqrt{-1}}\}$$

$$-e^{-\alpha\sqrt{-1}}\{1+xe^{-\beta\sqrt{-1}}+x^2e^{-2B\sqrt{-1}}+\ldots\ldots\ldots+x^{n-1}e^{-(n-1)\beta\sqrt{-1}}\}]$$

$$=\frac{1}{2\sqrt{-1}}\left\{e^{\alpha\sqrt{-1}}\left(\frac{1-x^ne^{\beta\sqrt{-1}}}{1-xe^{\beta\sqrt{-1}}}\right)-e^{-\alpha\sqrt{-1}}\left(\frac{1-x^{-n}e^{-\beta\sqrt{-1}}}{1-xe^{-\beta\sqrt{-1}}}\right)\right\}$$

$$=\frac{1}{1-x(e^{\beta\sqrt{-1}}\div e^{-\beta\sqrt{-1}})+x^2}$$

$$\times\frac{1}{2\sqrt{-1}}[e^{\alpha\sqrt{-1}}-e^{-\alpha\sqrt{-1}}-e^{-(\alpha-\beta)\sqrt{-1}}-e^{-(\alpha-\beta)\sqrt{-1}}\}$$

$$-x^n\{e^{(\alpha+n\beta)\sqrt{-1}}-e^{-\beta\sqrt{-1}}\}+x^{n+1}\{e^{(\alpha+n\beta-\beta)\sqrt{-1}}-e^{-(\alpha-n\beta-\beta)\sqrt{-1}}\}$$

$$=\frac{1}{1-2x\cos\beta+x^2}[\sin\alpha-x\sin(\alpha-\beta)-x^n\sin(\alpha+n\beta)+x^{n+1}\sin\{\alpha+(n-1)\beta\}].$$

15. $x<1$, 則 $\sin\alpha+x\sin(\alpha+\beta)+x^2\sin(\alpha+2\beta)+\cdots\cdots$至$\infty$

$$=\{\sin\alpha-x\sin(\alpha-\beta)\}/(1-2x\cos\beta+x^2).$$

16. $\cos\alpha+x\cos(\alpha+\beta)+x^2\cos(\alpha+2\beta)+\cdots\cdots$

$$+x^{n-1}\cos\{\alpha+(n-1)\beta\}$$

$$=\frac{\cos\alpha-x\cos(\alpha-\beta)-x^n\cos(\alpha+n\beta)+x^{n+1}\cos\{\alpha+(n-1)\beta\}}{1-2x\cos\beta+x^2}.$$

17. $x<1$, 則

$\cos\alpha+x\cos(\alpha+\beta)+x^2\cos(\alpha+2\beta)+\cdots\cdots$至$\infty$

$$=\{\cos\alpha-x\cos(\alpha-\beta)\}/(1-2x\cos\beta+x^2).$$

18. $1+x\cos\theta+x^2\cos 2\theta+\cdots\cdots+x^{n-1}\cos(n-1)\theta$

$$=\frac{1-x\cos\theta-x^n\cos n\theta+x^{n+1}\cos(n-1)\theta}{1-2x\cos\theta+x^2}.$$

19. $1+x\sin\theta+x^2\sin 2\theta+\cdots\cdots+x^{n-1}\sin(n-1)\theta$

$$=1+\frac{x\sin\theta-x^n\sin n\theta+x^{n+1}\sin(n-1)\theta}{1-2x\cos\theta+x^2}.$$

例題解自 15. 至 19.

(15) 於前例, 令 $n=\infty$, 則 $x^n=0$, 及 $x^{n+1}=0$, 故得其證.

(16) 如例題14.導級數之全項爲指數而括之, 則

$$S_n=\tfrac{1}{2}[e^{\alpha\sqrt{-1}}\{1+xe^{2\beta\sqrt{-1}}+xe^{2\beta\sqrt{-1}}+\cdots\cdots+x^{n-1}e^{(n-1)\beta\sqrt{-1}}\}$$

$$+e^{-\alpha\sqrt{-1}}\{1+xe^{-\beta\sqrt{-1}}+x^2e^{-2\beta\sqrt{-1}}+\cdots\cdots+x^{n-1}e^{-(n-1)\beta\sqrt{-1}}\}$$

$$=\frac{1}{2}\left[e^{\alpha\sqrt{-1}}\left(\frac{1-x_n e^{n\beta\sqrt{-1}}}{1-xe^{\beta\sqrt{-1}}}\right)+e^{-\alpha\sqrt{-1}}\left(\frac{1-x^{-n}e^{-n\beta\sqrt{-1}}}{1-xe^{-\beta\sqrt{-1}}}\right)\right].$$

以此簡單之, 而餘弦之項可表以指數.

(17) 於前例, 令 $n=\infty$, 則 $x^n=0$ $x^{n+1}=0$ 故得其證.

(18) 由 4. 節. $C_n=1+x\cos\theta+x^2\cos2\theta+\dots\dots+x^{n-1}\cos(n-1)\theta$

$$S^{n-1}=x\sin\theta+x^2\sin3\theta+\dots\dots+x^{n-1}\sin(n-1)\theta$$

故 $C_n+S_{n-1}\sqrt{-1}=1+x(\cos\theta+\sqrt{-1}\sin\theta)+x^2(\cos2\theta+\sqrt{-1}\sin2\theta)$

$$+\dots\dots+x^{n-1}\{\cos(n-1)\theta+\sqrt{-1}\sin(n-1)\theta\}$$

$$=1+xe^{\theta\sqrt{-1}}+x^2e^{2\theta\sqrt{-1}}+\dots\dots+x^{n-1}e^{(n-1)\theta\sqrt{-1}}$$

$=\dfrac{1-x^ne^{n\theta\sqrt{-1}}}{1-xe^{\theta\sqrt{-1}}}$, 此分母子以 $1-xe^{-\theta\sqrt{-1}}$ 乘之, 則

$$=\frac{1-xe^{-\theta\sqrt{-1}}-x^ne^{n\theta\sqrt{-1}}+x^{n+1}e^{(n+1)\theta\sqrt{-1}}}{1-x(e^{\theta\sqrt{-1}}+e^{-\theta\sqrt{-1}})+x^2}$$

$$=\frac{1-x(\cos\theta-\sqrt{-1}\sin\theta)-x^n(\cos n\theta+\sqrt{-1}\sin n\theta)+x^{n+1}\{\cos(n-1)\theta+\sqrt{-1}\sin(n-1)\theta\}}{1-2x\cos\theta+x^2}$$

$$=\frac{1-x\cos\theta-x^n\cos n\theta+x^{n+1}\cos(n-1)\theta+\sqrt{-1}\{x\sin\theta-x^n\sin n\theta+x^{n+1}\sin(n-1)\theta\}}{1-2x\cos\theta+x^2}$$

故 $C_n=\dfrac{1-x\cos\theta-x^n\cos n\theta+x^{n+1}\cos(n-1)\theta}{1-2x\cos\theta+x^2}$, 即所求之和.

本題 $x<1$, 亦可得無限之和 $=\dfrac{1-x\cos\theta}{1-2x\cos\theta+x^2}$.

(19) 前例之 $C_n+S_{n-1}\sqrt{-1}$, 其右邊之虛數部與 S_{n-1} 比較, 則

$$S_{n-1}=\frac{x\sin\theta-x^n\sin n\theta+x^{n+1}\sin(n-1)\theta}{1-2x\cos\theta+x^2}$$

由是所求之和 $=1+S_{n-1}=1+\dfrac{x\sin\theta-x^n\sin n\theta+x^{n+1}\sin(n-1)\theta}{1-2x\cos\theta+x^2}$.

本題 $x<1$, 亦可得無限項之和, 爲

$$1+\frac{x\sin\theta}{1-2x\cos\theta+x^2}=\frac{1-x(2\cos\theta-\sin\theta)+x^2}{1-2x\cos\theta+x^2}.$$

20. $\sin\alpha+x\sin 2\alpha+x^2\sin 3\alpha+\cdots\cdots$ 至 ∞

$$=\sin\alpha/(1-2x\cos\alpha+x^2).$$

21. $\cos\alpha+x\cos 2\alpha+x^2\cos 3\alpha+\cdots\cdots$ 至 ∞

$$=\cos\alpha/(1-2x\cos\alpha+x^3).$$

22. $\sin\alpha+x\sin(\alpha+\beta)+\frac{x^2}{\underline{|2}}\sin(\alpha+2\beta)$

$$+\frac{x^3}{\underline{|3}}\sin(\alpha+3\beta)+\cdots\cdots \text{至 } \infty = e^{x\cos\beta}\sin(\alpha+x\sin\beta).$$

23. $\cos\alpha+x\cos(\alpha+\beta)+\frac{x^2}{\underline{|2}}\cos(\alpha+2\beta)$

$$+\frac{x^3}{\underline{|3}}\cos(\alpha+3\beta)+\cdots\cdots \text{至 } \infty = e^{x\cos\beta}\cos(\alpha+x\sin\beta).$$

24. $\sec\alpha\sec(\alpha+\beta)+\sec(\alpha+\beta)\sec(\alpha+2\beta)+\cdots\cdots$

$$+\sec(\alpha+\overline{n-1}\beta)\sec(\alpha+n\beta)$$

$$=\csc\beta\{\tan(\alpha+n\beta)-\tan\alpha\}.$$

25. $\sec\alpha\sec 2\alpha+\sec 2\alpha\sec 3\alpha+\sec 3\alpha\sec 4\alpha$

$$+\cdots\cdots+\sec n\alpha\sec(n+1)\alpha=\csc\alpha\{\tan(n+1)\alpha-\tan\alpha\}.$$

例題解自 20. 至 25.

(20) 於例題 15. 可令 $\alpha=\beta$.

(21) 於例題 17. 可令 $\alpha=\beta$.

(22) $S=\sin\alpha+x\sin(\alpha+\beta)+\frac{x^2}{\underline{|2}}\sin(\alpha+2\beta)+\frac{x^3}{\underline{|3}}\sin(\alpha+3\beta)+\cdots\cdots$

及 $C=\cos\alpha+x\cos(\alpha+\beta)+\frac{x^2}{\underline{|2}}\cos(\alpha+2\beta)+\frac{x^3}{\underline{|3}}\cos(\alpha+3\beta)+\cdots\cdots$

故 $C+S\sqrt{-1}=\cos\alpha+\sqrt{-1}\sin\alpha+x\{\cos(\alpha+\beta)+\sqrt{-1}\sin(\alpha+\beta)\}$

$$+\frac{x^2}{\underline{|2}}\{\cos(\alpha+2\beta)+\sqrt{-1}\sin(\alpha+2\beta)\}+\frac{x^3}{\underline{|3}}\{\cos(\alpha+3\beta)+\sqrt{-1}\sin(\alpha+3\beta)\}+\cdots\cdots$$

$$=e^{\alpha\sqrt{-1}}+xe^{(\alpha+\beta)\sqrt{-1}}+\frac{x^2}{\underline{|2}}e^{(\alpha+2\beta)\sqrt{-1}}+\frac{x^3}{\underline{|3}}e^{(\alpha+3\beta)\sqrt{-1}}+\cdots\cdots$$

$$=e^{\alpha\sqrt{-1}}\{1+xe^{\beta\sqrt{-1}}+\frac{1}{\underline{|2}}x^2e^{2\beta\sqrt{-1}}+\frac{1}{\underline{|3}}x^3e^{3\beta\sqrt{-1}}+\cdots\cdots\}$$

$$=e^{\alpha\sqrt{-1}}\{e\}^{xe^{\beta\sqrt{-1}}}=e^{\alpha\sqrt{-1}}\{e\}^{x(\cos\beta+\sqrt{-1}\sin\beta)}=e^{x\cos\beta+(\alpha+x\sin\beta)\sqrt{-1}}$$

$$=e^{x\cos\beta}e^{(\alpha+x\sin\beta)\sqrt{-1}}=e^{x\cos\beta}\{\cos(\alpha+x\sin\beta)+\sqrt{-1}\sin(\alpha+x\sin\beta)\}$$

由是 $S=e^{x\cos\beta}\sin(\alpha+x\sin\beta)$.

(23) 於前例而得 $C=e^{x\cos\beta}\cos(\alpha+x\sin\beta)$.

(24) $\sec(\alpha+\overline{n-1}\beta)\sec(\alpha+n\beta)=\dfrac{1}{\cos(\alpha+\overline{n-1}\beta)\cos(\alpha+n\beta)}$

$$=\frac{\sin\beta}{\sin\beta\cos(\alpha+\overline{n-1}\beta)\cos(\alpha+n\beta)}=\frac{\sin\{(\alpha+n\beta)-(\alpha+\overline{n-1}\beta)\}}{\sin\beta\cos(\alpha+\overline{n-1}\beta)\cos(\alpha+n\beta)}$$

$$=\frac{\sin(\alpha+n\beta)\cos(\alpha+\overline{n-1}\beta)-\cos(\alpha+n\beta)\sin(\alpha+\overline{n-1}\beta)}{\sin\beta\cos(\alpha+\overline{n-1}\beta)\cos(\alpha+n\beta)}$$

$$=\csc\beta\{\tan(\alpha+n\beta)-\tan(\alpha+\overline{n-1}\beta)\}.$$

令 n 爲 1, 2, 3, ……… 順次得下式.

$$\sec\alpha\sec(\alpha+\beta)=\csc\beta\{\tan(\alpha+\beta)-\tan\alpha\},$$

$$\sec(\alpha+\beta)\sec(\alpha+2\beta)=\csc\beta\{\tan(\alpha+2\beta)-\tan(\alpha+\beta)\}$$

$$\sec(\alpha+2\beta)\sec(\alpha+3\beta)=\csc\beta\{\tan(\alpha+3\beta)-\tan(\alpha+2\beta)\}$$

………………………………………………

但 $\sec(\alpha+\overline{n-1}\beta)\sec(\alpha+n\beta)=\csc\beta\{\tan(\alpha+n\beta)-\tan(\alpha+\overline{n-1}\beta)\}$,

此多式相加, 則

$$S_n=\csc\beta\{\tan(\alpha+n\beta)-\tan\alpha\}.$$

(25) 於前例, 令 $\alpha=\beta$, 則所求之和爲

$$\csc\alpha\{\tan(\alpha+n\alpha)-\tan(\overline{n-1}\alpha)\}.$$

求次各式n項之和.

26. $\sin^2\alpha+\sin^2(\alpha+\beta)+\sin^2(\alpha+2\beta)+\cdots\cdots\cdots$

27. $\sin^3\alpha+\sin^3(\alpha+\beta)+\sin^3(\alpha+2\beta)+\cdots\cdots\cdots$

28. $\cos^4\alpha+\cos^4(\alpha+\beta)+\cos^4(\alpha+2\beta)+\cdots\cdots\cdots$

29. $\csc\theta\sec2\theta-\csc2\theta\sec3\theta+\csc3\theta\sec4\theta-\cdots\cdots\cdots$

30. $\cos\theta\cos(\theta+\alpha)+\cos(\theta+\alpha)\cos(\theta+2\alpha)$

$$+\cos(\theta+2\alpha)\cos(\theta+3\alpha)+\cdots\cdots\cdots$$

31. $\sin(p+1)\theta\cos\theta+\sin(p+2)\theta\cos2\theta$

$$+\sin(p+3)\theta\cos3\theta+\cdots\cdots\cdots$$

32. $\sin\alpha\sin2\alpha+\sin2\alpha\sin3\alpha+\sin3\alpha\sin4\alpha+\cdots\cdots\cdots$

33. $\sin3\alpha\sin\alpha+\sin6\alpha\sin2\alpha+\sin12\alpha\sin4\alpha+\cdots\cdots\cdots$

34. $\tan\alpha+\cot\alpha+\tan2\alpha+\cot2\alpha+\tan4\alpha+\cot4\alpha+\cdots\cdots$但$n$為偶數.

例題解自 26. 至 34.

(26) $2S_n=2\sin^2\alpha+2\sin^2(\alpha+\beta)+2\sin^2(\alpha+2\beta)+\cdots\cdots$

$=(1-\cos2\alpha)+\{1-\cos(2\alpha+2\beta)\}+\{1-\cos(2\alpha+4\beta)\}+\cdots\cdots$

$=n-\{\cos2\alpha+\cos(2\alpha+2\beta)+\cos(2\alpha+4\beta)+\cdots\cdots\}$

但由2.節(2) $S_n=\dfrac{n}{2}-\dfrac{\cos\{2\alpha+(n-1)\beta\}\sin n\beta}{2\sin\beta}$.

(27) $\sin^3\alpha=\frac{1}{4}(3\sin\alpha-\sin3\alpha)$,

$\sin^3(\alpha+\beta)=\frac{1}{4}\{3\sin(\alpha+\beta)-\sin(3\alpha+3\beta)\}$,

$\sin^3(\alpha+2\beta)=\frac{1}{4}\{3\sin(\alpha+3\beta)-\sin(3\alpha+6\beta)\}$,

$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$

相加,則$S_n=\frac{1}{4}[3\{\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\cdots\cdots$至$n$項$\}$.

$-\{\sin(3\alpha+3\beta)+\sin(3\alpha+6\beta)+\sin(3\alpha+9\beta)+\cdots\cdots$至 n 項$\}]$

$$=\frac{1}{4}\left[\frac{3\sin\{\alpha+\frac{1}{2}(n-1)\beta\}\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}-\frac{\sin\{3\alpha+\frac{1}{2}(n-1)3\beta\}\sin\frac{1}{2}n\beta}{\sin\frac{1}{2}\beta}\right].$$ 〔2. 節 (1)〕

(28) $S_n=\sin^4\left(\frac{\pi}{2}+\alpha\right)+\sin^4\left(\frac{\pi}{2}+\alpha+\beta\right)+\sin^4\left(\frac{\pi}{2}+\alpha+2\beta\right)+\cdots\cdots$

$$=\frac{\cos\{2\pi+4\alpha+2(n-2)\beta\}\sin2n\beta}{8\sin2\beta}-\frac{\cos\{\pi+2\alpha(n-1)\beta\}\sin n\beta}{2\sin\beta}+\frac{3n}{8}.$$

見例題7.

(29) $S_n=\csc\left(\theta-\frac{\pi}{2}\right)\left\{\tan(n+1)\left(\theta-\frac{\pi}{2}\right)-\tan\left(\theta-\frac{\pi}{2}\right)\right\}$, (例題25.)

(30) $S_n=\frac{1}{2}\{\cos\alpha+\cos(2\theta+\alpha)\}+\frac{1}{2}(\cos\alpha+\cos(2\theta+3\alpha)\}$

$+\frac{1}{2}\{\cos\alpha+\cos(2\theta+5\alpha)\}+\cdots\cdots=\frac{1}{2}\{n\cos\alpha+\cos(2\theta+\alpha)+\cos(2\theta+3\alpha)+\cos(2\theta+5\alpha)+\cdots\cdots\}$

$$=\frac{1}{2}\left\{n\cos\alpha+\frac{\cos(2\theta+n\alpha)\sin n\alpha}{\sin\alpha}\right\},$$ 1. 節 (2)

(31) $S_n=\frac{1}{2}\{\sin(p+2)\theta+\sin p\theta\}+\frac{1}{2}\{\sin(p+4)\theta+\sin p\theta\}+\cdots\cdots$

$=\frac{1}{2}\{\sin(p+2)\theta+\sin(p+4)\theta+\cdots\cdots+n\sin p\theta\}$

$$=\frac{1}{2}\left\{\frac{\cos(p+n+1)\theta\sin n\theta}{\sin\theta}+n\sin p\theta\right\},$$ 2. 節 (1)

(32) $S_n=\frac{1}{2}(\cos\alpha-\cos3\alpha)+\frac{1}{2}(\cos\alpha-\cos5\alpha)+\frac{1}{2}(\cos\alpha-\cos7\alpha)+\cdots\cdots$

$$=\frac{1}{2}\left\{n\cos\alpha-\frac{\cos(2\alpha+n\alpha)\sin n\alpha}{\sin\alpha}\right\},$$ 2. 節 (2)

(33) $S_n=\frac{1}{2}(\cos2\alpha-\cos4\alpha)+\frac{1}{2}(\cos4\alpha-\cos8\alpha)+\cdots\cdots$

$+\frac{1}{2}(\cos2^n\alpha-\cos2^{n+1}\alpha)=\frac{1}{2}(\cos2\alpha-\cos2^{n+1}\alpha)$.

(34) 令 $\tan\alpha+\cot\alpha=2\csc2\alpha$, 以下同樣, 則

$S_n=2\csc2\alpha+2\csc4\alpha+\cdots\cdots+2\csc2^{\frac{1}{2}n}\alpha$

$=2(\cot\alpha-\cot2^{\frac{1}{2}n}\alpha)$. 見例題11.

35. $\tan\alpha+2\tan 2\alpha+4\tan 4\alpha+\cdots\cdots\cdots$

36. $\sin\theta\sin 3\theta+\sin\frac{\theta}{2}\sin\frac{3\theta}{2}+\sin\frac{\theta}{2^2}\sin\frac{3\theta}{2^2}+\cdots\cdots\cdots$

37. $\cot\theta\csc\theta+2\cot 2\theta\csc 2\theta+2^2\cot 2^2\theta\csc 2^2\theta+\cdots\cdots\cdots$

38. $\cot\theta\csc\theta+\frac{1}{2}\cot\frac{\theta}{2}\csc\frac{\theta}{2}+\frac{1}{2^2}\cot\frac{\theta}{2^2}\csc\frac{\theta}{2^2}+\cdots\cdots\cdots$

39. $\tan\theta\sec\theta+\tan\frac{\theta}{4}\sec\frac{\theta}{2}+\tan\frac{\theta}{8}\sec\frac{\theta}{4}+\cdots\cdots\cdots$

40. $\csc\alpha+\csc\frac{\alpha}{2}+\csc\frac{\alpha}{2^2}+\cdots\cdots\cdots$

41. $\frac{1}{2}\sec\alpha+\frac{1}{2^2}\sec\alpha\sec 2\alpha+\frac{1}{2^3}\sec\alpha\sec 2\alpha\sec 2^2\alpha+\cdots\cdots\cdots$

42. $\sin\alpha\sin^2\frac{\alpha}{2}-2\sin\frac{\alpha}{2}\sin^2\frac{\alpha}{4}+4\sin\frac{\alpha}{4}\sin^2\frac{\alpha}{8}+\cdots\cdots\cdots$

例題解自 35. 至 42.

(35) $\cot\alpha-\tan\alpha=\frac{\cos\alpha}{\sin\alpha}-\frac{\sin\alpha}{\cos\alpha}=\frac{\cos^2\alpha-\sin^2\alpha}{\sin\alpha\cos\alpha}=2\cot 2\alpha,$

故 $S_n=(\cot\alpha-2\cot 2\alpha)+2(\cot 2\alpha-2\cot 4\alpha)+2^2(\cot 4\alpha-2\cot 8\alpha)$

$+\cdots\cdots\cdots+2^{n-1}(\cot 2^{n-1}\alpha-2\cot 2^n\alpha)=\cot\alpha-2^n\cot 2^n\alpha.$

(36) $S_n=\frac{1}{2}(\cos 2\theta-\cos 4\theta)+\frac{1}{2}(\cos\theta-\cos 2\theta)+\frac{1}{2}\left(\cos\frac{\theta}{2}-\cos\theta\right)$

$+\cdots\cdots\cdots+\frac{1}{2}\left(\cos\frac{\theta}{2^{n-2}}-\cos\frac{\theta}{2^{n-3}}\right)=\frac{1}{2}\left(\cos\frac{\theta}{2^{n-2}}-\cos 4\theta\right).$

(37) $\cot\theta\csc\theta=\frac{\cos\theta}{\sin^2\theta}=\frac{2\cos^2\frac{1}{2}\theta-1}{\sin^2\theta}=\frac{2\cos^2\frac{1}{2}\theta}{\sin^2\theta}-\frac{1}{\sin^2\theta}$

$=\frac{1}{2}\csc^2\frac{1}{2}\theta-\csc^2\theta.$ 由是

$S_n=\left(\frac{1}{2}csc^2\frac{\theta}{2}-csc^2\theta\right)+2\left(\frac{1}{2}csc^2\theta-csc^2 2\theta\right)+\cdots\cdots\cdots$

$+2^{n-1}\left(\frac{1}{2}csc^2 2^{n-2}\theta-csc^2 2^{n-1}\theta\right)=\frac{1}{2}csc^2\frac{\theta}{2}-2^{n-1}csc^2 2^{n-1}\theta.$

(38) 如前例. $S_n=\left(\frac{1}{2}csc^2\frac{\theta}{2}-csc^2\theta\right)+\frac{1}{2}\left(\frac{1}{2}csc^2\frac{\theta}{2^2}-csc^2\frac{\theta}{2}\right)$

$+\cdots\cdots\cdots+\frac{1}{2^{n-1}}\left(\frac{1}{2}csc^2\frac{\theta}{2^n}-csc^2\frac{\theta}{2^{n-1}}\right)=\frac{1}{2^n}csc^2\frac{\theta}{2^n}-csc^2\theta.$

(39) $tan\frac{\theta}{2}sec\theta=\frac{sin\frac{1}{2}\theta}{cos\frac{1}{2}\theta cos\theta}=\frac{sin(\theta-\frac{1}{2}\theta)}{cos\frac{1}{2}\theta cos\theta}=\frac{sin\theta cos\frac{1}{2}\theta-cos\theta sin\frac{1}{2}\theta}{cos\frac{1}{2}\theta cos\theta}$

$=tan\theta-tan\frac{1}{2}\theta$, 故 $S_n=\left(tan\theta-tan\frac{\theta}{2}\right)+\left(tan\frac{\theta}{2}-tan\frac{\theta}{4}\right)+\cdots\cdots\cdots$

$+\left(tan\frac{\theta}{2^{n-1}}-tan\frac{\theta}{2^n}\right)=tan\theta-tan\frac{\theta}{2^n}.$

(40) $csc\alpha=\frac{1}{sin\alpha}=\frac{2cos^2\frac{1}{2}\alpha-cos\alpha}{sin\alpha}=\frac{2cos^2\frac{1}{2}\alpha}{sin\alpha}-\frac{cos\alpha}{sin\alpha}=cot\frac{\alpha}{2}-cot\alpha,$

故 $S_n=\left(cot\frac{\alpha}{2}-cot\alpha\right)+\left(cot\frac{\alpha}{2^2}-cot\frac{\alpha}{2}\right)+\left(cot\frac{\alpha}{2^3}-cot\frac{\alpha}{2^2}\right)$

$+\cdots\cdots\cdots+\left(cot\frac{\alpha}{2^n}-cot\frac{\alpha}{2^{n-1}}\right)=cot\frac{\alpha}{2^n}-cot\alpha.$

(41) $\frac{1}{2}sec\alpha=\frac{1}{2cos\alpha}=\frac{sin\alpha}{2sin\alpha cos\alpha}=\frac{sin(2\alpha-\alpha)}{sin2\alpha}=cos\alpha-cot2\alpha sin\alpha$

$=sin\alpha(cot\alpha-cot2\alpha),$

$\frac{1}{2^2}sec\alpha sec2\alpha=\frac{1}{2}sec\alpha sin2\alpha(cot2\alpha-cot4\alpha)=sin\alpha(cot2\alpha-cot4\alpha),$

$\frac{1}{2^3}sec\alpha sec2\alpha sec2^2\alpha=\frac{1}{2^2}sec\alpha sec2\alpha sin2^2\alpha(cot4\alpha-cot8\alpha)=sin\alpha(cot4\alpha-cot8\alpha).$

以下順次皆如此,相加,則 $S_n=sin\alpha(cot\alpha-cot2^n\alpha)$

(42) $sin\alpha sin^2\frac{\alpha}{2}=\frac{1}{2}sin\alpha(1-cos\alpha)=\frac{1}{2}sin\alpha-\frac{1}{2^2}sin2\alpha,$

故 $S_n=\left(\frac{1}{2}sin\alpha-\frac{1}{2^2}sin2\alpha\right)+2\left(\frac{1}{2}sin\frac{\alpha}{2}-\frac{1}{2^2}sin\alpha\right)+\cdots\cdots\cdots$

$+2^{n-1}\left(\frac{1}{2}sin\frac{\alpha}{2^{n-1}}-\frac{1}{2^2}sin\frac{\alpha}{2^{n-2}}\right)=2^{n-2}sin\frac{\alpha}{2^{n-1}}-\frac{1}{4}sin2\alpha.$

43. $\cos\frac{\theta}{2}+2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}+2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}+\cdots\cdots\cdots$

44. $\cos\theta\cos^2\frac{\theta}{2}\csc^2\frac{3\theta}{2}+\cos3\theta\cos^2\frac{3\theta}{2}\csc^2\frac{3^2\theta}{2}$

$$+\cos3^2\theta\cos^2\frac{3^2\theta}{2}\csc^2\frac{3^3\theta}{2}+\cdots\cdots\cdots$$

45. $\frac{1}{2}\log\tan2\theta+\frac{1}{2^2}\log\tan2^2\theta+\frac{1}{2^3}\log\tan2^3\theta+\cdots\cdots\cdots$

但此對數爲任意之底數.

46. $\cos\theta+\cos3\theta+\cos5\theta+\cdots\cdots+\cos(2n+1)\theta.$

47. $\sin\alpha+2\sin2\alpha+3\sin3\alpha+\cdots\cdots\cdots$

例題解自43.至47.

(43)　$\cos\frac{\theta}{2}=\frac{\sin\theta}{2\sin\frac{1}{2}\theta}=\frac{\sin\theta}{2}\left(\frac{2\cos^2\frac{1}{4}\theta-\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta}\right)=\frac{\sin\theta}{2}\left(\cot\frac{\theta}{4}-\cot\frac{\theta}{2}\right)$

$2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}=2\cos\frac{\theta}{2}\times\frac{\sin\frac{1}{2}\theta}{2}\left(\cot\frac{\theta}{8}-\cot\frac{\theta}{4}\right)=\frac{\sin\theta}{2}\left(\cot\frac{\theta}{8}-\cot\frac{\theta}{4}\right)$

$2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}=2^2\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\times\frac{\sin\frac{1}{4}\theta}{2}\left(\cot\frac{\theta}{16}-\cot\frac{\theta}{8}\right)$

$=2\cos\frac{\theta}{2}\times\frac{\sin\frac{1}{2}\theta}{2}\left(\cot\frac{\theta}{16}-\cot\frac{\theta}{8}\right)=\frac{\sin\theta}{2}\left(\cot\frac{\theta}{16}-\cot\frac{\theta}{8}\right)$

同樣得 $2^{n-1}\cos\frac{\theta}{2}\cos\frac{\theta}{2^2}\cos\frac{\theta}{2^3}\cdots\cdots\cos\frac{\theta}{2^n}=\frac{\sin\theta}{2}\left(\cot\frac{\theta}{2^{n+1}}-\cot\frac{\theta}{2^n}\right)$

此各式相加,則 $S_n=\frac{\sin\theta}{2}\left(\cot\frac{\theta}{2^{n+1}}-\cot\frac{\theta}{2}\right).$

(44)　$\cos\theta\cos^2\frac{\theta}{2}\csc^2\frac{3\theta}{2}=\frac{\cos\theta\cos^2\frac{1}{2}\theta}{\sin^2\frac{3}{2}\theta}=\frac{\cos\theta(1+\cos\theta)}{1-\cos3\theta}$

$=\frac{\cos\theta+\cos^2\theta}{(1-\cos\theta)(1+2\cos\theta)^2}=\frac{(1+2\cos\theta)^2-1}{4(1-\cos\theta)(1+2\cos\theta)^2}$

$$=\frac{1}{4}\left\{\frac{1}{1-\cos\theta}-\frac{1}{(1-\cos\theta)(1+2\cos\theta)^2}\right\}=\frac{1}{4}\left(\frac{1}{1-\cos\theta}-\frac{1}{1-\cos 3\theta}\right)$$

故 $S_n=\frac{1}{4}\left\{\left(\frac{1}{1-\cos\theta}-\frac{1}{1-\cos 3\theta}\right)+\left(\frac{1}{1-\cos 3\theta}-\frac{1}{1-\cos 3^2\theta}\right)+\cdots\cdots\right.$

$\left.+\left(\frac{1}{1-\cos 3^{n-1}\theta}-\frac{1}{1-\cos 3^n\theta}\right)\right\}=\frac{1}{4}\left(\frac{1}{1-\cos\theta}-\frac{1}{1-\cos 3^n\theta}\right).$

(45) $\frac{1}{2}\log\tan 2\theta=\frac{1}{2}\log\frac{\sin 2\theta}{\cos 2\theta}=\frac{1}{2}\log\frac{2\sin 2\theta^2}{\sin 4\theta}=\frac{1}{2}\log\frac{(2\sin 2\theta)^2}{2\sin 4\theta}$

$=\frac{1}{2}\{2\log(2\sin 2\theta)-\log(2\sin 4\theta)\}=\log(2\sin 2\theta)-\frac{1}{2}\log(2\sin 4\theta)$

$\frac{1}{2^2}\log\tan 2^2\theta=\frac{1}{2}\log(2\sin 4\theta)-\frac{1}{2^2}\log(2\sin 8\theta),$

$\frac{1}{2^n}\log\tan 2^n\theta=\frac{1}{2^{n-1}}\log(2\sin 2^n\theta)-\frac{1}{2^n}\log(2\sin 2^{n+1}\theta)$

故 $S_n=\log(2\sin 2\theta)-\frac{1}{2^n}\log(2\sin 2^{n+1}\theta).$

(46) $2\cos\theta\sin\theta=\sin 2\theta,\quad 2\cos 3\theta\sin\theta=\sin 4\theta-\sin 2\theta,$

$2\cos 5\theta\sin\theta=\sin 6\theta-\sin 4\theta,\cdots\cdots\cdots\cdots 2\cos(2n+1)\theta\sin\theta=\sin(2n+2)\theta-\sin 2n\theta,$

故 $2S_n\sin\theta=\sin(2n+2)\theta,$ 故 $S_n=\frac{\sin(2n+2)\theta}{2\sin\theta}.$

(47) $\sin\alpha+\sin 2\alpha+\sin 3\alpha+\cdots\cdots=\frac{\sin\frac{1}{2}(n+1)\alpha\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha},$ 〔例題 3.〕

$=\frac{1}{2\sin\frac{1}{2}\alpha}\left(\cos\frac{\alpha}{2}-\cos\frac{2n+1}{2}\alpha\right),$ 同樣. $\sin 2\alpha+\sin 3\alpha+\cdots\cdots\cdots+\sin n\alpha$

$=\frac{1}{2\sin\frac{1}{2}\alpha}\left(\cos\frac{3\alpha}{2}-\cos\frac{2n+1}{2}\alpha\right),\quad \sin 3\alpha+\sin 4\alpha+\cdots\cdots\cdots+\sin n\alpha$

$=\frac{1}{2\sin\frac{1}{2}\alpha}\left(\cos\frac{5\alpha}{2}-\cos\frac{2n+1}{2}\alpha\right)\cdots\cdots\cdots,\sin n\alpha=\frac{1}{2\sin\frac{1}{2}\alpha}\left(\cos\frac{2n-1}{2}\alpha-\cos\frac{2n+1}{2}\alpha\right)$

$S_n=\frac{1}{2\sin\frac{1}{2}\alpha}\left[\frac{\cos\{\frac{1}{2}\alpha+\frac{1}{2}(n-1)\alpha\}\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}-n\cos\frac{2n+1}{2}\alpha\right],$ 2. 節 (2).

$=\{(n+1)\sin n\alpha-n\sin(n+1)\alpha\}/\{2(1-\cos\alpha)\}.$

48. $1+\cos\alpha\cos\beta+\cos^2\alpha\cos2\beta+\cos^2\alpha\cos3\beta+\cdots\cdots\cdots$

49. $\sin\alpha+\sin\alpha\sin(\alpha+\beta)+\sin^2\alpha\sin(\alpha+2\beta)$

$$+\sin^3\alpha\sin(\alpha+3\beta)+\cdots\cdots\cdots$$

50. $1^2\cos\alpha+2^2\cos2\alpha+3^2\cos3\alpha+\cdots\cdots\cdots$

例題解自 48. 至 50.

(48) 於例題 18. 令 $x=\cos\alpha$, $\theta=\beta$, 則

$$S_n=\frac{1-\cos\alpha\cos\beta-\cos^n\alpha\cos n\alpha+\cos^{n+1}\alpha\cos(n-1)\alpha}{1-2\cos\alpha\cos\beta+\cos^2\alpha}.$$

(49) 於例題 14. 令 $x=\sin\alpha$, 則

$$S_n=\frac{\sin\alpha-\sin\alpha\sin(\alpha-\beta)-\sin^n\alpha\sin(\alpha+n\beta)+\sin^{n+1}\alpha\sin\{\alpha+(n-1)\beta\}}{1-2\sin\alpha\cos\beta+\sin^2\alpha}.$$

(50) 由第十六編 19. 節. $2S_n=1^2(e^{\alpha\sqrt{-1}}+e^{-\alpha\sqrt{-1}})+2^2(e^{2\alpha\sqrt{-1}}+e^{-2\alpha\sqrt{-1}})$
$+3^2(e^{3\alpha\sqrt{-1}}+e^{-3\alpha\sqrt{-1}})+\cdots\cdots+n^2(e^{n\alpha\sqrt{-1}}+e^{-n\alpha\sqrt{-1}})$, 令 $e^{\alpha\sqrt{-1}}=x$, 則

$$2S_n=x(1^2+2^2x+3^2x^2+\cdots\cdots+n^2x^{n-1})+x^{-1}(1^2+2^2x^{-1}+3^2x^{-2}+\cdots\cdots+n^2x^{-n+1})$$

令 $M=1^2+2^2x+3^2x^2+4^2x^3+\cdots\cdots+n^2x^{n-1}$

以 $(1-x)^3=1-3x+3x^2-x^3$ 乘之, 則

$$M(1-x)^3=1+(2^2-3)x+(3^2-3.2^2+3.1^2)x^2+(4^2-3.3^2+3.2^2-1^3)x^3$$
$$+\cdots\cdots+\{n^2-3(n-1)^2+3(n-2)^2-(n-3)^2\}x^{n-1}+\{-3n^2+3(n-1)^2-(n-2)^2\}x^n$$
$$+\{3n^2-(n-1)^2\}x^{n+1}-n^2x^{n+2},$$ 但 $3^2-3.2^2+3.1^2=0$, 又

恆同式, $k^2-3(k-1)^2+3(k-2)^2-(k-3)^2\equiv0$, 故

$$M(1-x)^3=1+x-(n^2+2x+1)x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}$$

$$2S_n=x\left\{\frac{1+x-(n^2+2n+1)x^n+(2n^2+2n-1)x^{n+1}-n^2x^{n+2}}{(1-x)^3}\right\}$$
$$+x^{-1}\left\{\frac{1+x^{-1}-(n^2+2n+1)x^{-n}+(2n^2+2n-1)x^{-n-1}-n^2x^{-n-2}}{(1-x^{-1})^3}\right\},$$

卽 $2S_n\{2-(x+x^{-1})\}^3=x(1-x^{-1})^3+x^{-1}(1-x)^3+x^2(1-x^{-1})^3+x^{-2}(1-x^{-3})$

$-(n^2+2n+1)\{x^{n+1}(1-x^{-1})^3+x^{-n-1}(1-x)^3\}$

$+(2n^2+2n-1)\{x^{n+2}(1-x^{-1})^3+x^{-n-2}(1-x)^3\}-n^2\{x^{n+3}(1-x^{-1})^3+x^{-n-3}(1-x)^3\}$

$=-(n^2+2n+1)\{(x^{n+1}+x^{-n-1})-3(x^n+x^{-n})+3(x^{n-1}+x^{-n+1})-(x^{n-2}+x^{-n+2})\}$

$+(2n^2+2n-1)\{(x^{n+2}+x^{-n-2})-3(x^{n+1}+x^{-n-1})+3(x^n+x^{-n})-(x^{n-1}+x^{-n+1})\}$

$-n^2\{(x^{n+3}+x^{-n-2})-3(x^{n+2}+x^{-n-2})+3(x^{n+1}+x^{-n-1})-(x^n+x^{-n})\}$

$8S_n(1-\cos\alpha)^3=-(n^2+2n+1)\{\cos(n+1)\alpha-3\cos n\alpha+3\cos(n-1)\alpha$

$-\cos(n-2)\alpha\}+(2n^2+2n-1)\{\cos(n+2)\alpha-3\cos(n+1)\alpha+3\cos n\alpha-\cos(n-1)\alpha\}$

$-n^2\{\cos(n+3)\alpha-3\cos(n+2)\alpha+3\cos(n+1)\alpha-\cos n\alpha\}$

$=n^2\{-\cos(n+3)\alpha+5\cos(n+2)\alpha-10\cos(n+1)\alpha+10\cos n\alpha-5\cos(n-1)\alpha$

$+\cos(n-2)\alpha\}+2n\{\cos(n+2)\alpha-4\cos(n+1)\alpha+6\cos n\alpha-4\cos(n-1)\alpha$

$+\cos(n-2)\alpha\}-\{\cos(n+2)\alpha-2\cos(n+1)\alpha+2\cos(n-1)\alpha-\cos(n-2)\alpha\}$

$=n^2\left\{2\sin\frac{(2n+1)\alpha}{2}\sin\frac{5\alpha}{2}-10\sin\frac{(2n+1)\alpha}{2}\sin\frac{3\alpha}{2}+20\sin\frac{(2n+1)\alpha}{2}\sin\frac{\alpha}{2}\right\}$

$+2n\{2\cos n\alpha\cos 2\alpha-8\cos n\alpha\cos\alpha+6\cos n\alpha\}$

$-\{-2\sin n\alpha\sin 2\alpha+4\sin n\alpha\sin\alpha\}$

$4S_n(1-\cos\alpha)^3=n^2\sin\frac{(2n+1)\alpha}{2}\left(\sin\frac{5\alpha}{2}-5\sin\frac{3\alpha}{2}+10\sin\frac{\alpha}{2}\right)$

$+2n\cos n\alpha(\cos 2\alpha-4\cos\alpha+3)-\sin n\alpha(-\sin 2\alpha+2\sin\alpha)$

$=n^2\sin\frac{(2n+1)\alpha}{2}\left\{2\cos 2\alpha\sin\frac{\alpha}{2}-8\cos\alpha\sin\frac{\alpha}{2}+6\sin\frac{\alpha}{2}\right\}+2n\cos n\alpha(2\cos^2\alpha-4\cos\alpha+2)$

$-2\sin n\alpha\sin\alpha(1-\cos\alpha)=4n^2\sin\frac{(2n+1)\alpha}{2}\sin\frac{\alpha}{2}(1-\cos\alpha)^2+4n\cos n\alpha(1-\cos\alpha)^2$

$-2\sin n\alpha\sin\alpha(1-\cos\alpha)$，由是

$$S_n=\frac{2n^2\sin\frac{1}{2}(2n+1)\alpha\sin\frac{\alpha}{2}+2n\cos n\alpha}{2(1-\cos\alpha)}-\frac{\sin n\alpha\sin\alpha}{2(1-\cos\alpha)^2}$$

$$=\frac{n^2\{\cos n\alpha-\cos(n+1)\alpha\}}{2(1-\cos\alpha)}-\frac{\sin n\alpha\sin\alpha}{2(1-\cos\alpha)^2},$$

51. $x\sin\alpha-x^2\sin(\alpha+\beta)+x^3\sin(\alpha+2\beta)-\cdots\cdots$

52. $1+n\cos\alpha+\dfrac{n(n-1)}{\underline{|2}}\cos2\alpha+\dfrac{n(n-1)(n-2)}{\underline{|3}}\cos3\alpha+\cdots\cdots$

53. $1+n\sin\alpha+\dfrac{n(n-1)}{\underline{|2}}\sin2\alpha+\dfrac{n(n-1)(n-2)}{\underline{|3}}\sin3\alpha+\cdots\cdots$

54. $\sin\alpha+nx\sin(\alpha+\beta)+\dfrac{n(n-1)}{\underline{|2}}x^2\sin(\alpha+2\beta)+\cdots\cdots$

55. $\cos x\sin(x+\tfrac{1}{2}n\pi)+n\cos(x+\tfrac{1}{2}\pi)\sin\{x+\tfrac{1}{2}(n-1)\pi\}$

$$+\frac{n(n-1)}{\underline{|2}}\cos(x+\pi)\sin\{x+\tfrac{1}{2}(n-2)\pi\}+\cdots\cdots$$

例題解自 51. 至 55.

(51) 令 $\beta=\pi+\gamma$, 則

$$S_n=x\{\sin\alpha+x\sin(\alpha+\gamma)+x^2\sin(\alpha+2\gamma)+\cdots\cdots\}$$

$$=x\left[\frac{\sin\alpha-x\sin(\alpha-\gamma)-x^n\sin(\alpha+n\gamma)+x^{n+1}\sin\{\alpha+(n-1)\gamma\}}{1-2x\cos\gamma+x^2}\right],\ (\text{例題 }14.)$$

$$=\frac{x\sin\alpha+x^2\sin(\alpha-\beta)-(-1)^nx^{n+1}\sin(\alpha+n\beta)+(-1)^{n+1}x^{n+2}\sin\{\alpha+(n-1)\beta\}}{1+2x\cos\beta+x^2}.$$

(52) $C_n=1+n\cos\alpha+\dfrac{n(n-1)}{\underline{|2}}\sin2\alpha+\cdots\cdots+n\cos(n-1)\alpha,$

$$S_{n-1}=n\sin\alpha+\frac{n(n-1)}{\underline{|2}}\sin2\alpha+\cdots\cdots+n\sin(n-1)\alpha,\ \text{故}$$

$$C_n+\sqrt{-1}S_{n-1}=1+n(\cos\alpha+\sqrt{-1}\sin\alpha)+\frac{n(n-1)}{\underline{|2}}(\cos2\alpha+\sqrt{-1}\sin2\alpha)$$

$$+\cdots\cdots+n\{\cos(n-1\alpha)+\sqrt{-1}\sin(n-1)\alpha\}=1+ne^{\alpha\sqrt{-1}}+\frac{n(n-1)}{\underline{|2}}e^{2\alpha\sqrt{-1}}+\cdots\cdots$$

$$+ne^{(n-1)\alpha\sqrt{-1}}=(1+e^{\alpha\sqrt{-1}})^n-e^{n\alpha\sqrt{-1}}=(1+\cos\alpha+\sqrt{-1}\sin\alpha)^n-(\cos n\alpha+\sqrt{-1}\sin n\alpha)$$

$$=2^n\cos^n\frac{\alpha}{2}\left(\cos\frac{\alpha}{2}+\sqrt{-1}\sin\frac{\alpha}{2}\right)^n-\cos n\alpha-\sqrt{-1}\sin n\alpha$$

$$=2^n\cos^n\tfrac{1}{2}\alpha\cos\tfrac{1}{2}n\alpha-\cos n\alpha+\sqrt{-1}\left(2^n\cos^n\frac{\alpha}{2}\sin\tfrac{1}{2}n\alpha-\sin n\alpha\right),$$

故　$C_n=2^n\cos^n\frac{1}{2}\alpha\cos\frac{1}{2}n\alpha-\cos n\alpha$.

(53)　依前例. $S_{n-1}=2^n\cos^n\frac{1}{2}\alpha\sin\frac{1}{2}n\alpha-\sin n\alpha$, 故

$S_n=1+S_{n-1}=1+2^n\cos^n\frac{1}{2}\alpha\sin\frac{1}{2}n\alpha-\sin n\alpha$.

(54)　$S_n=\sin\alpha+nx\sin(\alpha+\beta)+\frac{n(n-1)}{\lfloor 2}x^2\sin(\alpha+2\beta)+\ldots\ldots+nx^{n-1}\sin\{\alpha+(n-1)\beta\}$

$C_n=\cos\alpha+nx\cos(\alpha+\beta)+\frac{n(n-1)}{\lfloor 2}x^2\cos(\alpha+2\beta)+\ldots\ldots+nx^{n-1}\cos\{\alpha+(n-1)\beta\}$

故　$C_n+\sqrt{-1}S_n=e^{\alpha\sqrt{-1}}+nxe^{(\alpha+\beta)\sqrt{-1}}+\frac{n(n-1)}{\lfloor 2}x^2e^{(\alpha+2\beta)\sqrt{-1}}+\ldots\ldots$

$+nx^{n-1}e^{\{\alpha+(n-1)\}\beta\sqrt{-1}}=e^{\alpha\sqrt{-1}}(1+xe^{\beta\sqrt{-1}})^n-e^{\alpha\sqrt{-1}}x^ne^{n\beta\sqrt{-1}}$,

$=(\cos\alpha+\sqrt{-1}\sin\alpha)\{1+x(\cos\beta+\sqrt{-1}\sin\alpha)\}^n-x^n\{\cos(\alpha+n\beta)+\sqrt{-1}\sin(\alpha+n\beta)\}$

但令 $1+x\cos\beta+\sqrt{-1}\sin\beta=r(\cos\theta+\sqrt{-1}\sin\theta)$, 則

$1+x\cos\beta=r\cos\theta$,　$x\sin\beta=r\sin\theta$,　故　$\tan\theta=\frac{\sin\beta}{1+x\cos\beta}$,

及　$r^2=1+2x\cos\beta+x^2$,　由是

$C_n+\sqrt{-1}S_n=(\cos\alpha+\sqrt{-1}\sin\alpha)r^n(\cos\theta+\sqrt{-1}\sin\theta)^n$

$-x^n\{\cos(\alpha+n\beta)+\sqrt{-1}\cos(\alpha+n\beta)\}$

$=r^n(\cos\alpha+\sqrt{-1}\sin\alpha)(\cos n\theta+\sqrt{-1}\sin n\theta)$

$-x^n\{\cos(\alpha+n\beta)+\sqrt{-1}\sin(\alpha+n\beta)\}$, 比較實數部, 則

$C_n=r^n(\cos n\theta\cos\alpha-\sin n\theta\sin\alpha)-x^n\cos(\alpha+n\beta)$

$=r^n\cos(n\theta+\alpha)-x^n\cos(\alpha+n\beta)$.

(55)　$C_n=\cos x\sin(x+\frac{1}{2}n\pi)+n\cos(x+\frac{1}{2}\pi)\sin\{x+\frac{1}{2}(n-1)\alpha\}+\ldots\ldots\ldots\ldots$

$S_n=\sin x\cos(x+\frac{1}{2}n\pi)+n\sin(x+\frac{1}{2}\pi)\cos\{x+\frac{1}{2}(n-1)\pi\}+\ldots\ldots\ldots\ldots$

故　$C_n+S_n=\sin(2x+\frac{1}{2}n\pi)+n\sin(2x+\frac{1}{2}n\pi)+\frac{n(n-1)}{\lfloor 2}\sin(2x+\frac{1}{2}n\pi)+\ldots\ldots\ldots\ldots$

$=\sin(2x+\frac{1}{2}n\pi)\left\{1+n+\frac{n(n-1)}{\lfloor 2}+\ldots\ldots+n\right\}=\sin(2x+\frac{1}{2}nx)\{(1+1)^n-1\}$

$=2^n\sin(2x+\frac{1}{2}n\pi)-\sin(2x+\frac{1}{2}n\pi)$,　次取其差. 亦用同樣之方法.

$C_n-S_n=\sin\frac{1}{2}n\pi[(1-1)^n-(-1)^n]=-(-1)^n\sin\frac{1}{2}n\pi$

由是　$C_n=\frac{1}{2}[2^n\sin(2x+\frac{1}{2}n\pi)-\sin(2x+\frac{1}{2}n\pi)-(-1)^n\sin\frac{1}{2}n\pi]$

本題之項數, 令爲 $n+1$, 則可得簡便之答.

即　$C_{n+1}=2^{n-1}\sin(2x+\frac{1}{2}nx)$.

56. $n\sin\theta+(n-1)\sin 2\theta+(n-2)\sin 3\theta+\cdots\cdots\cdots$

57. $(n+1)n\sin\theta+n(n-1)\sin 2\theta+(n-1)(n-2)\sin 3\theta+\cdots\cdots\cdots$

58. $\sin^n\phi\cos n\theta+n\sin^{n-1}\phi\cos(n-1)\theta\sin(\theta-\phi)$

$$+\frac{n(n-1)}{|2}\sin^{n-2}\phi\cos(n-2)\phi\sin^2(\theta-\phi)+\cdots\cdots\cdots$$

例題解自 56. 至 58.

(56) $S_n=n\sin\theta+(n-1)\sin 2\theta+(n-2)\sin 3\theta+\cdots\cdots+1.\sin n\theta,$

$S_{n-1}=(n-1)\sin\theta+(n-2)\sin 2\theta+(n-3)\sin 3\theta+\cdots\cdots+1.\sin(n-1)\theta,$

由是 $S_n-S_{n-1}=\sin\theta+\sin 2\theta+\sin 3\theta+\cdots\cdots+\sin(n-1)\theta+\sin n\theta,$

依例題 3. $=\dfrac{\sin\frac{1}{2}(n+1)\theta\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}=\dfrac{1}{2\sin\frac{1}{2}\theta}\left\{\cos\dfrac{\theta}{2}-\cos\dfrac{(2n+1)\theta}{2}\right\}$

同樣 $S_{n-1}-S_{n-2}=\dfrac{1}{2\sin\frac{1}{2}\theta}\left\{\cos\dfrac{\theta}{2}-\cos\dfrac{(2n-1)\theta}{2}\right\},$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$S_3-S_2=\frac{1}{2\sin\frac{1}{2}\theta}\left\{\cos\frac{\theta}{2}-\cos\frac{7\theta}{2}\right\},$$

$$S_2-S_1=\frac{1}{2\sin\frac{1}{2}\theta}\left\{\cos\frac{\theta}{2}-\cos\frac{6\theta}{2}\right\},$$

$$S_1=\frac{1}{2\sin\frac{1}{2}\theta}\left\{\cos\frac{\theta}{2}-\cos\frac{3\theta}{2}\right\},$$

相加,則 $S_n=\dfrac{1}{2\sin\frac{1}{2}\theta}\left[n\cos\dfrac{\theta}{2}-\left\{\cos\dfrac{3\theta}{2}+\cos\dfrac{5\theta}{2}+\cdots\cdots+\cos\dfrac{(2n+1)\theta}{2}\right\}\right]$

$$=\frac{1}{2\sin\frac{1}{2}\theta}\left[n\cos\frac{\theta}{2}-\frac{\cos\{\frac{3}{2}\theta+\frac{1}{2}(n-1)\theta\}\sin\frac{1}{2}n\theta}{\sin\frac{1}{2}\theta}\right]\quad 2.\text{節 }(2)$$

$$=\frac{n}{2}\cot\frac{\theta}{2}-\frac{2\cos\{\frac{3}{2}\theta+\frac{1}{2}(n-1)\theta\}\sin\frac{1}{2}n\theta}{4\sin^2\frac{1}{2}\theta}$$

$$=\frac{n}{2}\cot\frac{\theta}{2}-\frac{\sin(n+1)\theta-\sin\theta}{4\sin^2\frac{1}{2}\theta}=\frac{n+1}{2}\cot\frac{\theta}{2}-\frac{\sin(n+1)\theta}{4\sin^2\frac{1}{2}\theta}.$$

(57) $S_n=(n+1)n\sin\theta+n(n-1)\sin2\theta+\dots\dots+2.1\sin n\theta,$

$$S_{n-1}=n(n-1)\sin\theta+(n-1)(n-2)\sin2\theta+\dots\dots+2.1\sin(n-1)\theta$$

故 $S_n-S_{n-1}=2\{n\sin\theta+(n-1)\sin2\theta+\dots\dots+\sin n\theta\}$

$$=(n+1)\cot\frac{\theta}{2}-\frac{\sin(n+1)\theta}{2\sin^2\frac{1}{2}\theta},$$ 見前例.

故 $S_{n-1}-S_{n-2}=n\cot\frac{\theta}{2}-\frac{\sin n\theta}{2\sin^2\frac{1}{2}\theta},$

$$\dots\dots\dots\dots\dots\dots\dots\dots$$

$$S_2-S_1=3\cot\frac{\theta}{2}-\frac{\sin3\theta}{2\sin^2\frac{1}{2}\theta},$$

$$S_1=2\cot\frac{\theta}{2}-\frac{\sin2\theta}{2\sin^2\frac{1}{2}\theta},$$ 相加,則

$$S_n=\{2+3+\dots\dots\dots+(n+1)\}\cot\frac{\theta}{2}-\frac{1}{2\sin^2\frac{1}{2}\theta}\{\sin2\theta+\sin3\theta+\dots\dots+\sin(n+1)\theta\}$$

$$=\frac{n(n+3)}{2}\cot\frac{\theta}{2}-\frac{\sin\{2\theta+\frac{1}{2}(n-1)\theta\}\sin\frac{1}{2}n\theta}{2\sin^2\frac{1}{2}\theta\sin\frac{1}{2}\theta},$$ 由 2. 節 (1)

$$=\frac{n(n+3)}{2}\cot\frac{\theta}{2}-\frac{\cos\frac{3}{2}\theta-\cos\frac{1}{2}(2n+3)\theta}{4\sin^3\frac{1}{2}\theta}.$$

(58) $C_n=\sin^n\phi\cos n\theta+n\sin^{n-1}\phi\cos(n-1)\theta\sin(\theta-\phi)+\dots\dots\dots\dots$

$$+n\sin\phi\cos\theta\sin^{n-1}(\theta-\phi),$$

$$S_n=\sin^n\phi\sin n\theta+n\sin^{n-1}\phi\sin(n-1)\theta\sin(\theta-\phi)+\dots\dots\dots\dots$$

$$+n\sin\phi\sin\theta\sin^{n-1}(\theta-\phi),$$

故 $C_n+\sqrt{-1}S_n=\sin^n\phi e^{n\theta\sqrt{-1}}+n\sin^{n-1}\phi\sin(n-\phi)e^{(n-1)\theta\sqrt{-1}}+\dots\dots\dots\dots$

$$+n\sin\phi\sin^{n-1}(\theta-\phi)e^{\theta\sqrt{-1}}$$

$$=\{\sin\phi e^{\theta\sqrt{-1}}+\sin(\theta-\phi)\}^n-\sin^n(\theta-\phi)$$

$$=\{\sin\phi(\cos\theta+\sqrt{-1}\sin\theta)+\sin\theta\cos\phi-\cos\theta\sin\phi\}^n-\sin^n(\theta-\phi)$$

$$=\sin^n\theta(\cos\phi+\sqrt{-1}\sin\phi)^n-\sin^n(\theta-\phi)$$

$$=\sin^n\theta\cos n\phi-\sin^n(\theta-\phi)+\sqrt{-1}\sin^n\theta\sin n\phi,$$

由是 $C_n=\sin^n\theta\cos n\phi-\sin^n(\theta-\phi).$

59. $\frac{1}{\sin\theta\sin 2\theta}+\frac{1}{\sin 2\theta\sin 3\theta}+\frac{1}{\sin 3\theta\sin 4\theta}+\cdots\cdots$

60. $\frac{1}{\sin\theta\cos 2\theta}+\frac{1}{\cos 2\theta\sin 3\theta}+\frac{1}{\sin 3\theta\cos 4\theta}+\cdots\cdots$

61. $\frac{1}{\cos\theta+\cos 3\theta}+\frac{1}{\cos\theta+\cos 5\theta}+\frac{1}{\cos\theta+\cos 7\theta}+\cdots\cdots$

62. $\frac{\sin\theta}{\cos\theta+\cos 2\theta}+\frac{\sin 2\theta}{\cos\theta+\cos 4\theta}+\frac{\sin 3\theta}{\cos\theta+\cos 6\theta}+\cdots\cdots$

63. $\frac{\cos\theta}{\cos\theta-\cos 2\theta}+\frac{\cos 2\theta}{\cos\theta-\cos 4\theta}+\frac{\cos 3\theta}{\cos\theta-\cos 6\theta}+\cdots\cdots$

64. $\frac{\sin}{1+2\cos\theta}+\frac{3\cos 3\theta}{1+2\cos 3\theta}+\frac{3^2\sin 3^2\theta}{1+2\cos 3^2\theta}+\cdots\cdots$

例題解自 59. 至 64.

(59) $\frac{1}{\sin\theta\sin 2\theta}=\frac{1}{\sin\theta}\left\{\frac{\sin(2\theta-\theta)}{\sin\theta\sin 2\theta}\right\}=\frac{1}{\sin\theta}(\cot\theta-\cot 2\theta)$, 故

$$S_n=\frac{1}{\sin\theta}[(\cot\theta-\cot 2\theta)+(\cot 2\theta-\cot 3\theta)+\cdots\cdots+\{\cot n\theta-\cot(n+1)\theta\}]$$

$$=\frac{1}{\sin\theta}\{\cot\theta-\cot(n+1)\theta\}.$$

(60) 令 $\theta=\phi-\frac{\pi}{2}$, 則 $S_n=\frac{1}{\cos\phi\cos 2\phi}+\frac{1}{\cos 2\phi\cos 3\phi}+\frac{1}{\cos 3\phi\cos 4\phi}+\cdots$

$\frac{1}{\cos\phi\cos 2\phi}=\frac{1}{\sin\phi}\left\{\frac{\sin(2\phi-\phi)}{\cos\phi\cos 2\phi}\right\}=\frac{1}{\sin\phi}(\tan 2\phi-\tan\phi)$, 故

$$S_n=\frac{1}{\sin\phi}[(\tan 2\phi-\tan\phi)+(\tan 3\phi-\tan 2\phi)+\cdots\cdots+\{\tan(n+1)\phi-\tan n\phi\}]$$

$$=\frac{1}{\sin\phi}\{\tan(n+1)\phi-\tan\phi\}=\frac{1}{\sin(\theta+\frac{1}{2}\pi)}\left\{\tan(n+1)\left(\theta+\frac{\pi}{2}\right)-\tan\left(\theta+\frac{\pi}{2}\right)\right\}.$$

(61) $\frac{1}{\cos\theta+\cos3\theta}=\frac{1}{2\cos2\theta\cos\theta}=\frac{1}{2\sin\theta}(\tan2\theta-\tan\theta)$，見前例

故如前例，得 $S_n=\frac{1}{2\sin\theta}\{\tan(n+1)\theta-\tan\theta\}$.

(62) $\frac{\sin\theta}{\cos\theta+\cos2\theta}=\frac{\sin\theta}{2\cos\frac{3}{2}\theta\cos\frac{1}{2}\theta}=\frac{1}{4\sin\frac{1}{2}\theta}\left(\frac{2\sin\theta\sin\frac{1}{2}\theta}{\cos\frac{3}{2}\theta\cos\frac{1}{2}\theta}\right)$

$=\frac{1}{4\sin\frac{1}{2}\theta}\left(\frac{\cos\frac{1}{2}\theta-\cos\frac{3}{2}\theta}{\cos\frac{3}{2}\theta\cos\frac{1}{2}\theta}\right)=\frac{1}{4\sin\frac{1}{2}\theta}\left(\frac{1}{\cos\frac{3}{2}\theta}-\frac{1}{\cos\frac{1}{2}\theta}\right)$

$S_n=\frac{1}{4\sin\frac{1}{2}\theta}\Big[\left(\frac{1}{\cos\frac{3}{2}\theta}-\frac{1}{\cos\frac{1}{2}\theta}\right)+\left(\frac{1}{\cos\frac{5}{2}\theta}-\frac{1}{\cos\frac{3}{2}\theta}\right)+\cdots\cdots$

$+\left\{\frac{1}{\cos\frac{1}{2}(2n+1)\theta}-\frac{1}{\cos\frac{1}{2}(2n-1)\theta}\right\}\Big]$

$=\frac{1}{4\sin\frac{1}{2}\theta}\left\{\frac{1}{\cos\frac{1}{2}(2n+1)\theta}-\frac{1}{\cos\frac{1}{2}\theta}\right\}$.

(63) $\frac{\cos\theta}{\cos\theta-\cos2\theta}=\frac{\cos\theta}{2\sin\frac{3}{2}\theta\sin\frac{1}{2}\theta}=\frac{1}{4\sin\frac{1}{2}\theta}\left(\frac{2\cos\theta\sin\frac{1}{2}\theta}{\sin\frac{3}{2}\theta\sin\frac{1}{2}\theta}\right)$

$=\frac{1}{4\sin\frac{1}{2}\theta}\left(\frac{\sin\frac{3}{2}\theta-\sin\frac{1}{2}\theta}{\sin\frac{3}{2}\theta\sin\frac{1}{2}\theta}\right)=\frac{1}{4\sin\frac{1}{2}\theta}\left(\frac{1}{\sin\frac{1}{2}\theta}-\frac{1}{\sin\frac{3}{2}\theta}\right)$，故

$S_n=\frac{1}{4\sin\frac{1}{2}\theta}\Big[\left(\frac{1}{\sin\frac{1}{2}\theta}-\frac{1}{\sin\frac{3}{2}\theta}\right)+\left(\frac{1}{\sin\frac{3}{2}\theta}-\frac{1}{\sin\frac{5}{2}\theta}\right)+\cdots\cdots$

$+\left\{\frac{1}{\sin\frac{1}{2}(2n-1)\theta}-\frac{1}{\sin\frac{1}{2}(2n+1)\theta}\right\}\Big]$

$=\frac{1}{4\sin\frac{1}{2}\theta}\left\{\frac{1}{\sin\frac{1}{2}\theta}-\frac{1}{\sin\frac{1}{2}(2n+1)\theta}\right\}$.

(64) $\frac{\sin\theta}{1+2\cos\theta}=\frac{\sin\theta}{3-4\sin^2\frac{1}{2}\theta}=\frac{2\sin\theta\sin\frac{1}{2}\theta}{2(3\sin\frac{1}{2}\theta-4\sin^3\frac{1}{2}\theta)}$

$=\frac{\cos\frac{1}{2}\theta-\cos\frac{3}{2}\theta}{2\sin\frac{3}{2}\theta}=\frac{\sin\theta-2\cos\frac{3}{2}\theta\sin\frac{1}{2}\theta}{4\sin\frac{1}{2}\theta\sin\frac{3}{2}\theta}$

$=\frac{\sin(\frac{3}{2}\theta-\frac{1}{2}\theta)-2\cos\frac{3}{2}\theta\sin\frac{1}{2}\theta}{4\sin\frac{1}{2}\theta\sin\frac{3}{2}\theta}=\frac{\sin\frac{3}{2}\theta\cos\frac{1}{2}\theta-3\cos\frac{3}{2}\theta\sin\frac{1}{2}\theta}{4\sin\frac{1}{2}\theta\sin\frac{3}{2}\theta}$

$=\frac{1}{4}\cot\frac{\theta}{2}-\frac{3}{4}\cot\frac{3}{2}\theta$, $\frac{3\sin3\theta}{1+2\cos3\theta}=\frac{3}{4}\cot\frac{3\theta}{2}-\frac{3^2}{4}\cot\frac{3^2\theta}{2}$,

此各式相加得 $S_n=\frac{1}{4}\cot\frac{1}{2}\theta-\frac{3^n}{4}\cot\frac{3^n}{2}\theta$.

65. $\dfrac{\sin 2\alpha}{\sin\alpha\sin 3\alpha}-\dfrac{\sin 4\alpha}{\sin 3\alpha\sin 5\alpha}+\dfrac{\sin 6\alpha}{\sin 5\alpha\sin 7\alpha}-\cdots\cdots\cdots\cdots$

66. $\dfrac{\sin\theta}{\cos^2\theta}+\dfrac{\sin 3\theta}{\cos^2 2\theta\cos^2\theta}+\dfrac{\sin 5\theta}{\cos^2 2\theta\cos^2 3\theta}+\cdots\cdots\cdots\cdots$

例題解自 65. 至 66.

(65) $\dfrac{\sin 2n\alpha}{\sin(2n-1)\alpha\sin(2n+1)\alpha}=\dfrac{1}{2\cos\theta}\left\{\dfrac{1}{\sin(2n-1)\theta}+\dfrac{1}{\sin(2n+1)\theta}\right\},$

令 n 順次爲 $n-1, n-2, \cdots\cdots\cdots, 3, 2, 1.$ 所得各式相加, 則

$$S_n=\frac{1}{2\cos\theta}\left[\left\{\frac{1}{\sin\theta}+\frac{1}{\sin 3\theta}\right\}-\left\{\frac{1}{\sin 3\theta}+\frac{1}{\sin 5\theta}\right\}+\left\{\frac{1}{\sin 5\theta}+\frac{1}{\sin 7\theta}\right\}\right.$$

$$\left.-\cdots\cdots+(-1)^{n-2}\left\{\frac{1}{\sin(2n-3)\theta}+\frac{1}{\sin(2n-1)\theta}\right\}+(-1)^{n-1}\left\{\frac{1}{\sin(2n-1)\theta}+\frac{1}{\sin(2n+1)\theta}\right\}\right]$$

$=\dfrac{1}{2\cos\theta}\left\{\dfrac{1}{\sin\theta}+(-1)^{n-1}\dfrac{1}{\sin(2n+1)\theta}\right\},$ 故 n 爲偶數或奇數, 從而

$S_n=\dfrac{1}{2\cos\theta}\left\{\dfrac{1}{\sin\theta}+\dfrac{1}{\sin(2n+1)\theta}\right\}$ 或 $\dfrac{1}{2\cos\theta}\left\{\dfrac{1}{2\cos\theta}-\dfrac{1}{\sin(2n+1)\theta}\right\}.$

(66) $\dfrac{\sin(2n-1)\theta}{\cos^2 n\theta\cos^2(n-1)\theta}=\dfrac{1}{\sin\theta}\left[\dfrac{\sin\{n\theta+(n-1)\theta\}\sin\{n\theta-(n-1)\theta\}}{\cos^2 n\theta\cos^2(n-1)\theta}\right]$

$=\dfrac{1}{\sin\theta}\left\{\dfrac{\sin^2 n\theta-\sin^2(n-1)\theta}{\cos^2 n\theta\cos^2(n-1)\theta}\right\}=\dfrac{1}{\sin\theta}\left\{\dfrac{\cos^2(n-1)\theta-\cos^2 n\theta}{\cos^2 n\theta\cos^2(n-1)\theta}\right\}$

$=\dfrac{1}{\sin\theta}\left\{\dfrac{1}{\cos^2 n\theta}-\dfrac{1}{\cos^2(n-1)\theta}\right\},$ 令 n 順次爲 $1, 2, 3, \cdots\cdots\cdots n-1,$ 所得各式相加, 則

$$S_n=\frac{1}{\sin\theta}\left[\frac{1}{\cos^2\theta}+\left\{\frac{1}{\cos^2 2\theta}-\frac{1}{\cos^2\theta}\right\}+\left\{\frac{1}{\cos^2 3\theta}-\frac{1}{\cos^2 2\theta}\right\}\right.$$

$$\left.+\cdots\cdots\cdots+\left\{\frac{1}{\cos^2 n\theta}-\frac{1}{\cos^2(n-1)\theta}\right\}\right]$$

$=\dfrac{1}{\sin\theta}\left[\dfrac{1}{\cos^2 n\theta}\right]=\csc\theta\sec^2 n\theta.$

67. $\tan^{-1}\frac{1}{1+1+1^2}+\tan^{-1}\frac{6}{1+2+2^2}+\tan^{-1}\frac{1}{1+3+3^2}+\cdots\cdots$

68. $\tan^{-1}x+\tan^{-1}\frac{x}{1+1.2x^2}+\tan^{-1}\frac{x}{1+2.3.x^2}+\cdots\cdots$

69. $\tan^{-1}\frac{3x^2}{1+1^2.2^2x^4}+\tan^{-1}\frac{3x^2}{1+2.^23^2x}+\cdots\cdots$

70. $\tan^{-1}\frac{1}{2}+\tan^{-1}\frac{1}{8}+\tan^{-1}\frac{1}{16}+\cdots\cdots+\tan^{-1}\frac{1}{2n^2}$

例題解自 67. 至 70.

(67) $\tan^{-1}\frac{1}{1+n+n^2}=\tan^{-1}\frac{\frac{1}{n}-\frac{1}{n+1}}{1+\frac{1}{n(n+1)}}=\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{n+1},$

故 $S_n=\left(\tan^{-1}\frac{1}{1}-\tan^{-1}\frac{1}{1+1}\right)+\left(\tan^{-1}\frac{1}{2}-\tan^{-1}\frac{1}{1+2}\right)+\left(\tan^{-1}\frac{1}{3}-\tan^{-1}\frac{1}{1+3}\right)$

$+\cdots\cdots+\left(\tan^{-1}\frac{1}{n}-\tan^{-1}\frac{1}{1+n}\right)=\tan^{-1}1-\tan^{-1}\frac{1}{1+n}=\frac{\pi}{4}-\tan^{-1}\frac{1}{1+n}.$

(68) $\tan^{-1}\frac{x}{1+(n-1)nx^2}=\tan^{-1}\frac{nx-(n-1)x}{1+nx.(n-1)x}=\tan^{-1}nx-\tan^{-1}(n-1)x$

故 $S_n=\tan^{-1}x+(\tan^{-1}2x-\tan^{-1}x)+(\tan^{-1}3x-\tan^{-1}2x)+\cdots\cdots$

$+\{\tan^{-1}nx-\tan^{-1}(n-1)x\}=\tan^{-1}nx.$

(69) $\tan^{-1}\frac{(2n+1)x^2}{1+n^2(n+1)^2x^4}=\tan^{-1}\frac{(n+1)^2x^2-n^2x^2}{1+(n+1)^2x^2.n^2x^2}=\tan^{-1}(n+1)^2x^2-\tan^{-1}n^2x^2$

試如前例．得 $S_n=\tan^{-1}(n+1)^2x^2-\tan^{-1}x^2.$

(70) $\tan^{-1}\frac{1}{2n^2}=\tan^{-1}\frac{(2n+1)-(2n-1)}{1+(2n+1)(2n-1)}=\tan^{-1}(2n+1)-\tan^{-1}(2n-1),$

$S_n=(\tan^{-1}3-\tan^{-1}1)+(\tan^{-1}5-\tan^{-1}3)+(\tan^{-1}7-\tan^{-1}5)+\cdots\cdots$

$+\{\tan^{-1}(2n+1)-\tan^{-1}(2n-1)\}=\tan^{-1}(2n+1)-\tan^{-1}1=\tan^{-1}(2n+1)-\frac{1}{4}\pi.$

71. $\cot^{-1}(2a^{-1}+a)+\cot^{-1}(2a^{-1}+3a)+\cot^{-1}(2a^{-1}+6a)$

$+\cot^{-1}(2a^{-1}+10a)+\cdots\cdots\cdots$

72. $1+2+3+\cdots\cdots\cdots$ 73. $1.2+2.3+3.4+\cdots\cdots\cdots$

74. $1^3+2^3+3^3+\cdots\cdots\cdots$

75. $1.3+2^2.1.3,+2^4.1.3+\cdots\cdots\cdots$

例題解自 71. 至 75.

(71) $\cot^{-1}(2a^{-1}+a)=\cot^{-1}\dfrac{2+a^2}{a}=\cot^{-1}\dfrac{1+\frac{1}{2}a.a}{a-\frac{1}{2}a}=\cot^{-1}\dfrac{a}{2}-\cot^{-1}a$

故 $S_n=\left(\cot^{-1}\dfrac{a}{2}-\cot^{-1}a\right)+(\cot^{-1}a-\cot^{-1}2a)+\cdots\cdots+\left(\cot^{-1}\dfrac{na}{2}-\cot^{-1}na\right)$

$=(\cot^{-1}\frac{1}{2}a-\cot^{-1}na)$.

(72) 依例題 3. $\sin\alpha+\sin 2\alpha+\sin 3\alpha+\cdots\cdots+\sin n\alpha=\dfrac{\sin\frac{1}{2}(n+1)\alpha\sin\frac{1}{2}n\alpha}{\sin\frac{1}{2}\alpha}$

即 $\left(\alpha-\dfrac{\alpha^3}{\underline{|3}}+\cdots\cdots\right)+\left\{2x-\dfrac{(2\alpha)^3}{\underline{|3}}+\cdots\cdots\right\}+\left\{3\alpha-\dfrac{(3\alpha)^3}{\underline{|3}}+\cdots\cdots\right\}$

$+\cdots\cdots+\left\{n\alpha-\dfrac{(n\alpha)^3}{\underline{|3}}+\cdots\cdots\right\}$

$$=\frac{\left[\frac{1}{2}(n+1)\alpha-\frac{1}{\underline{|3}}\{\frac{1}{2}(n+1)\alpha\}^3+\cdots\cdots\right]\left\{\frac{1}{2}n\alpha-\frac{1}{\underline{|3}}(\frac{1}{2}n\alpha)^2+\cdots\cdots\right\}}{\frac{1}{2}\alpha-\frac{1}{\underline{|3}}(\frac{1}{2}\alpha)^3+\cdots\cdots}$$

$=\dfrac{\frac{1}{4}n(n+1)\alpha^2-\alpha^2\text{ 以上之項}}{1\alpha-\frac{1}{\underline{|3}}(\frac{1}{2}\alpha)^3+\cdots\cdots}=\frac{1}{2}n(n+1)\alpha-\alpha$ 以上之項,

比較兩邊 α 之係數,得 $1+2+3+\cdots\cdots+n=\frac{1}{2}n(n+1)$.

(73) 依例題 33. $\sin\alpha\sin 2\alpha+\sin 2\alpha\sin 3\alpha+\cdots\cdots+\sin n\alpha\sin(n+1)\alpha$

$=\frac{1}{2}\left\{n\cos\alpha-\dfrac{\cos(2\alpha+n\alpha)\sin n\alpha}{\sin\alpha}\right\}=\dfrac{n\cos\alpha}{2}-\dfrac{\sin 2(n+1)\alpha-\sin 2\alpha}{4\sin\alpha}$

$=\dfrac{n\cos\alpha}{2}+\dfrac{\cos\alpha}{2}-\dfrac{\sin 2(n+1)\alpha}{4\sin\alpha}$,

即 $\left(\alpha-\dfrac{1}{\underline{|3}}\alpha^3+\cdots\cdots\right)\left(2\alpha-\dfrac{1}{\underline{|3}}8\alpha^3+\cdots\cdots\right)+\left(2\alpha-\dfrac{1}{\underline{|3}}8\alpha^3+\cdots\cdots\right)\left(3\alpha-\dfrac{1}{\underline{|3}}27\alpha^3+\cdots\cdots\right)$

$+\cdots\cdots+\left(n\alpha-\dfrac{1}{\underline{|3}}n^3\alpha^3+\cdots\cdots\right)\left\{(n+1)\alpha-\dfrac{1}{\underline{|3}}(n+1)^3\alpha^3+\cdots\cdots\right\}$

$$=\frac{n}{2}\left(1-\frac{1}{\lfloor 2}\alpha^2+\cdots\cdots\right)+\frac{1}{2}\left(1-\frac{1}{\lfloor 2}\alpha^2+\cdots\cdots\right)-\frac{2(n+1)\alpha-\frac{1}{\lfloor 3}8(n+1)^3\alpha^3+\cdots\cdots}{4\left(\alpha-\frac{1}{\lfloor 3}\alpha^3+\cdots\cdots\right)},$$

比較兩邊 α^2 之係數.得

$$1.2+2.3+\cdots\cdots+n(n+1)=-\frac{n}{4}-\frac{1}{4}-\frac{1}{4}\left\{-\frac{(n+1)(4n^2+8n+3)}{3}\right\}$$

$$=\frac{n+1}{4}\left(-1+\frac{4n^2+8n+3}{3}\right)=\frac{n(n+1)(n+2)}{3}.$$

(74) 依例題6. $\sin^3\alpha+\sin^3 2\alpha+\sin^3 3\alpha+\cdots\cdots+\sin^3 n\alpha$

$$=\frac{3\sin\frac{1}{2}(n+1)\alpha\sin\frac{1}{2}n\alpha}{4\sin\frac{1}{2}\alpha}-\frac{\sin\frac{3}{2}(n+1)\alpha\sin\frac{3}{2}n\alpha}{4\sin\frac{3}{2}\alpha}$$

$$=\frac{3\{\cos\frac{1}{2}\alpha-\cos\frac{1}{2}(2n+1)\alpha\}}{8\sin\frac{1}{2}\alpha}-\frac{\cos\frac{3}{2}\alpha-\cos\frac{3}{2}(2n+1)\alpha}{8\sin\frac{3}{2}\alpha},$$ 即

$$\left(\alpha-\frac{\alpha^3}{\lfloor 3}+\cdots\cdots\right)^3+\left(2\alpha-\frac{8\alpha^3}{\lfloor 3}+\cdots\cdots\right)^3+\cdots\cdots+\left(n\alpha-\frac{n^3\alpha^3}{\lfloor 3}+\cdots\cdots\right)^3$$

$$=\frac{3\left\{\frac{1}{2}n(n+1)\alpha^2-\frac{1}{48}n(n+1)(2n^2+2n+1)\alpha^4+\cdots\cdots\right\}}{8\left(\frac{1}{2}\alpha-\frac{1}{\lfloor 3}\cdot\frac{1}{8}\alpha^3+\cdots\cdots\right)}$$

$$-\frac{\left\{\frac{9}{2}n(n+1)\alpha^2-\frac{9}{48}n(n+1)(2n^2+2n+1)\alpha^4+\cdots\cdots\right\}}{8\left(\frac{3}{2}\alpha-\frac{1}{\lfloor 3}\cdot\frac{27}{8}\alpha^3+\cdots\cdots\right)},$$ 比較兩邊 α^3 之係數,

得 $1^3+2^3+\cdots\cdots+n^3=\frac{3}{8}\left\{-\frac{1}{12}n^2(n+1)^2\right\}-\frac{1}{8}\left\{-\frac{9}{4}n^2(n+1)^2\right\}=\frac{1}{4}n^2(n+1)^2.$

(75) 依例題33. $\sin 3\alpha\sin\alpha+\sin 2.3\alpha\sin 2\alpha+\sin 2^2.3\alpha\sin 2^2\alpha$

$+\cdots\cdots+\sin 2^{n-1}.3\alpha\sin 2^{n-1}\alpha=\frac{1}{2}(\cos 2\alpha-\cos 2^{n+1}\alpha)$, 即

$$\left(3\alpha-\frac{3^3\alpha^3}{\lfloor 3}+\cdots\cdots\right)\left(\alpha-\frac{\alpha^3}{\lfloor 3}+\cdots\cdots\right)+\left(2.3\alpha-\frac{2^3.3^3\alpha^3}{\lfloor 3}+\cdots\cdots\right)\left(2\alpha-\frac{2^3\alpha^3}{\lfloor 3}+\cdots\cdots\right)$$

$$+\cdots\cdots+\left(2^{n-1}.3\alpha-\frac{1}{\lfloor 3}2^{3n-3}.3^3\alpha^3+\cdots\cdots\right)\left(2^{n-1}\alpha-\frac{1}{\lfloor 3}2^{3n-3}\alpha^3+\cdots\cdots\right)$$

$$=\frac{1}{2}\left\{\left(1-\frac{1}{\lfloor 2}2^2\alpha^2+\cdots\cdots\right)-\left(1-\frac{1}{\lfloor 2}2^{2n+2}\alpha^2+\cdots\cdots\right)\right\},$$ 比較兩邊 α^2 之係數,

得 $1.3+2^2.1.3+\cdots\cdots+2^{2(n-1)}=2^{2n}-1.$

求次各式之證.

76. $\dfrac{\sin\theta+\sin3\theta+\sin5\theta+\cdots\cdots\text{至 }n\text{ 項}}{\cos\theta+\cos3\theta+\cos5\theta+\cdots\cdots\text{至 }n\text{ 項}}=\tan n\theta.$

77. $\dfrac{\sin\theta-\sin2\theta+\sin3\theta-\cdots\cdots\text{至 }n\text{ 項}}{\cos\theta-\cos2\theta+\cos3\theta-\cdots\cdots\text{至 }n\text{ 項}}=\tan\dfrac{(n+1)(\pi+\theta)}{2}.$

78. $\dfrac{\cos\alpha+\cos(\alpha+2\beta)+\cdots\cdots+\cos\{\alpha+2(n-1)\beta\}}{2\cos\{\alpha+(n-1)\beta\}}$

$-[\cos(n-1)\beta+\cos(n-3)\beta+\cdots\cdots+\cos\{\frac{3}{2}+\frac{1}{2}(-1)^{n-1}\}\beta]$

$=\frac{1}{4}\{1+(-1)^{n-2}\}.$

79. $\dfrac{\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+2\beta)+\cdots\cdots\text{至 }(2n-1)\text{ 項}}{\sin\alpha+\sin(\alpha+2\beta)+\sin(\alpha+4\beta)+\sin(\alpha+6\beta)+\cdots\cdots\text{至 }n\text{ 項}}$

之值,對於α任意之值常相同.

80. $\phi=\dfrac{\pi}{13}$, 則

$$\cos\phi+\cos3\phi+\cos9\phi=\frac{1+\sqrt{13}}{4},\quad \cos5\phi-\cos7\phi+\cos11\phi=\frac{1-\sqrt{13}}{4}.$$

例題解自 76. 至 80.

(76) 依 2. 節 $\dfrac{\sin n\theta\sin n\beta/\sin\beta}{\cos n\theta\sin n\beta/\sin\beta}=\dfrac{\sin n\theta}{\cos n\theta}=\tan n\theta.$

(77) 令 $\theta+\pi=\phi$, 則 $\theta=-(\pi-\phi)$, 故依例題 3. 及 4.

$$\frac{-\sin\phi-\sin2\phi-\sin3\phi-\cdots\cdots\text{至}n\text{項}}{-\cos\phi-\cos2\phi-\cos3\phi-\cdots\cdots\text{至}n\text{項}}=\frac{\sin\frac{1}{2}(n+1)\phi\sin\frac{1}{2}n\phi/\sin\frac{1}{2}\phi}{\cos\frac{1}{2}(n+1)\phi\sin\frac{1}{2}n\phi/\sin\frac{1}{2}\phi}$$

$=\tan\frac{1}{2}(n+1)\phi=\frac{1}{2}\tan\frac{1}{2}(n+1)(\theta+\pi).$

(78) 令左邊爲X及Y,則依2.節(2). $X=\frac{\cos\{\alpha+(n-1)\beta\}\sin n\beta/\sin\beta}{2\cos\{\alpha+(n-1)\beta\}}$

$$=\frac{\sin n\beta}{2\sin\beta}.$$

第一 n 爲奇數,則 $Y=\cos(\overline{n-1}-1)\beta+\cos(\overline{n-1}-2)\beta+\ldots\ldots+\cos(\overline{n-1}-\overline{n-3})\beta$,

故 Y爲 $\frac{n-3}{2}+1$, 卽 $\frac{n-1}{2}$ 項. 依2.節(2).

$Y=\frac{\cos[(n-1)]\beta+\frac{1}{2}\{\frac{1}{2}(n-1)-1\}(-2\beta)]\sin\frac{1}{2}(n-1)\beta}{\sin\beta}=\frac{\cos\frac{1}{2}(n+1)\beta\sin\frac{1}{2}(n-1)\beta}{\sin\beta}$

$=\frac{\sin n\beta-\sin\beta}{2\sin\beta}$, 故 $X-Y=\frac{1}{2}=\frac{1}{4}\{1+(-1)^{n-1}\}$.

第二 n 爲偶數,則 $Y=\cos(n-1)\beta+\cos(\overline{n-1}-2)\beta+\ldots\ldots+\cos(\overline{n-1}-\overline{n-2})\beta$,

故 Y爲 $\frac{n-2}{2}+1$, 卽 $\frac{n}{2}$ 項. 又依2.節(2).

$Y=\frac{\cos[(n-1)\beta+\frac{1}{2}\{\frac{1}{2}(n-1)(-2\beta)]\sin\frac{1}{2}\beta}{\sin\beta}=\frac{\sin n\beta}{2\sin\beta}$,

故 $X-Y=0=\frac{1}{4}\{1+(-1)^{n-1}\}$.

(79) 依2.節(1). $\frac{\sin\{\alpha+\frac{1}{2}(2n-2)\beta\}\sin\frac{1}{2}(2n-1)\beta/\sin\frac{1}{2}\beta}{\sin\{\alpha+(n-1)\beta\}\sin n\beta/\sin\beta}=\frac{\sin\frac{1}{2}(2n-1)\beta\sin\beta}{\sin n\beta\sin\frac{1}{2}\beta}$.

故此式關係 α.

(80) $x=\cos\phi+\cos3\phi+\cos9\phi$, $y=\cos5\phi+\cos7\phi+\cos11\phi$, 故

$x+y=\cos\phi+\cos3\phi+\cos5\phi+\cos7\phi+\cos9\phi+\cos11\phi$, 依2.節(2).

$$=\frac{\cos6\phi\sin6\phi}{\sin\phi}=\frac{\sin12\phi}{2\sin\phi}=\frac{\sin(\pi-\phi)}{2\sin\phi}=\frac{\sin\phi}{2\sin\phi}=\frac{1}{2},$$

又 $x=\cos\phi+\cos3\phi+\cos(\pi-4\phi)=\cos\phi+\cos\phi-\cos4\phi$,

$y=\cos(\pi-8\phi)+\cos(\pi-6\phi)+\cos(\pi-2\phi)=-\cos2\phi-\cos6\phi-\cos8\phi$,

卽 $-\frac{1}{2}y=\cos^2\phi+\cos^2 3\phi+\cos^2 4\phi-\frac{3}{2}$, 故

$x^2+\frac{1}{2}y=(\cos\phi+\cos3\phi-\cos4\phi)^2-\cos^2\phi-\cos^2 3\phi-\cos^2 4\phi+\frac{3}{2}$

$=\frac{3}{2}+2\cos\phi\cos3\phi-2\cos\phi\cos4\phi-2\cos3\phi\cos4\phi$

$=\frac{3}{2}-2\cos\phi\cos10\phi-2\cos3\phi\cos4\phi-2\cos\phi\cos4\phi$

$=\frac{3}{2}-(\cos11\phi+\cos9\phi+\cos7\phi+\cos\phi+\cos5\phi+\cos3\phi)=\frac{3}{2}-\frac{1}{2}=1$,

$x^2+\frac{1}{2}y=1$, 及 $x+y=\frac{1}{2}$, 得 $x=\frac{1}{4}(1+\sqrt{13})$, $y=\frac{1}{4}(1-\sqrt{13})$.

求次無限級數之和.

81. $\sin\alpha+\cos\alpha\sin(\alpha+\beta)+\cos^2\alpha\sin(\alpha+2\beta)+\cdots\cdots$

82. $\cos\alpha+\sin\alpha\cos(\alpha+\beta)+\sin^2\alpha\cos(\alpha+2\beta)+\cdots\cdots$

83. $\sin\theta+\frac{1}{\underline{|2}}\sin2\theta+\frac{1}{\underline{|3}}\sin3\theta+\cdots\cdots$

84. $\sin\theta-\frac{1}{\underline{|2}}\sin2\theta+\frac{1}{\underline{|3}}\sin3\theta-\cdots\cdots$

85. $\cos\theta-\frac{\cos\theta}{\underline{|1}}\cos2\theta+\frac{\cos^2\theta}{\underline{|2}}\cos3\theta+\cdots\cdots$

86. $1-\frac{\cos\theta}{\underline{|1}}\cos\phi+\frac{\cos^2\theta}{\underline{|2}}\cos2\phi-\frac{\cos^2\theta}{\underline{|3}}\cos3\phi+\cdots\cdots$

87. $x\sin\alpha+\frac{1}{\underline{|2}}x^2\sin2\alpha+\frac{1}{\underline{|3}}x^2\sin3\alpha+\cdots\cdots$

88. $\sin\alpha-\frac{1}{2}\sin2\alpha+\frac{1}{3}\sin3\alpha-\cdots\cdots$

89. $\cos\alpha+\frac{1}{2}\cos2\alpha+\frac{1}{3}\cos3\alpha+\cdots\cdots$

90 $\cos\alpha-\frac{1}{2}\cos2\alpha+\frac{1}{3}\cos3\alpha-\cdots\cdots$

91. $\frac{\cos\theta}{1}\cos\phi+\frac{\cos^2\theta}{2}\cos2\phi\frac{\cos^3\theta}{3}\cos3\phi+\cdots\cdots$

例題解自 81. 至 91.

(81) 於例題 15. 令 $x=\cos\alpha$, 則 $S=\frac{\sin\alpha-\cos\alpha\sin(\alpha-\beta)}{1-2\cos\alpha\cos\beta+\cos^2\alpha}$.

(82) 於例題 17. 令 $x=\sin\alpha$, 則 $S=\frac{\cos\alpha-\sin\alpha\cos(\alpha-\beta)}{1-2\sin\alpha\cos\beta+\sin^2\alpha}$.

(83) 於例題 22. 令 $\alpha=0$, $x=1$, $\beta=\theta$, 則

$$\sin\theta+\frac{1}{\underline{|2}}\sin2\theta+\frac{1}{\underline{|3}}\sin3\theta+\cdots\cdots=e^{\cos\theta}\sin(\sin\theta).$$

(84) 於例題 22. 令 $\alpha=0$, $x=-1$, $\beta=\theta$, 則

$$-\sin\theta+\frac{1}{\underline{|2}}\sin2\theta-\frac{1}{\underline{|3}}\sin3\theta+\cdots\cdots=e^{-\cos}\sin(-\sin\theta),$$

即 $\sin\theta-\frac{1}{\underline{|2}}\sin2\theta+\frac{1}{\underline{|3}}\sin3\theta-\cdots\cdots=e^{-\cos\theta}\sin(\sin\theta)$,

(85) 於例題23. 令 $x=\cos\theta,\ \alpha=\beta=\theta$, 則

$$\cos\theta+\cos\theta\cos2\theta+\frac{1}{\underline{|2}}\cos^2\theta\cos3\theta+\cdots\cdots=e^{\cos2\theta}\cos(\theta+\cos\theta\sin\theta).$$

(86) 於例題23. 令 $\alpha=0,\ x=-\cos\theta,\ \beta=\phi$, 則

$$1-\cos\theta\cos\phi+\frac{1}{\underline{|2}}\cos^2\theta\cos2\phi-\cdots\cdots=e^{-\cos\theta\cos\phi}\cos(-\cos\theta\sin\phi)$$

$$=e^{-\cos\theta\cos\phi}\cos(\cos\theta\sin\phi).$$

(87) 於例題22. 令 $\alpha=0,\ \beta=\alpha$, 則

$$x\sin\alpha+\frac{1}{\underline{|2}}x^2\sin2\alpha+\frac{1}{\underline{|3}}x^2\sin3\alpha+\cdots\cdots=e^{x\cos\alpha}\sin(x\sin\alpha).$$

(88) $2\sqrt{-1}S=e^{\alpha\sqrt{-1}}-e^{-\alpha\sqrt{-1}}-\frac{1}{2}e^{2\alpha\sqrt{-1}}-e^{-2\alpha\sqrt{-1}})$

$$+\frac{1}{3}(e^{3\alpha\sqrt{-1}}-e^{-3\alpha\sqrt{-1}})-\cdots\cdots$$

$$=(e^{\alpha\sqrt{-1}}-\frac{1}{2}e^{2\alpha\sqrt{-1}}+\frac{1}{3}e^{3\alpha\sqrt{-1}}-\cdots\cdots)-(e^{-\alpha\sqrt{-1}}-\frac{1}{2}e^{-2\alpha\sqrt{-1}}+\frac{1}{3}e^{-3\alpha\sqrt{-1}}-\cdots\cdots)$$

$=\log_e(1+e^{\alpha\sqrt{-1}})-\log_e(1+e^{-\alpha\sqrt{-1}})$, 見第十一編4.節.

$=\log_e\left(\dfrac{1+e^{\alpha\sqrt{-1}}}{1+e^{-\alpha\sqrt{-1}}}\right)=\log_e e^{\alpha\sqrt{-1}}=\alpha\sqrt{-1}$, 故 $S=\dfrac{\alpha}{2}$.

(89) 如前例. $2S=(e^{\alpha\sqrt{-1}}e^{\alpha\sqrt{-1}}+\frac{1}{2}e^{2\alpha\sqrt{-1}}+\frac{1}{3}e^{3\alpha\sqrt{-1}}+\cdots\cdots)$

$+(e^{-\alpha\sqrt{-1}}+\frac{1}{2}e^{-\alpha\sqrt{-1}}+\frac{1}{3}e^{-3\alpha\sqrt{-1}}+\cdots\cdots)=-\log_e(1-e^{\alpha\sqrt{-1}})-\log_e(1-e^{-\alpha\sqrt{-1}})$,

由第十一編4.節.

$$=-\log_e\{(1-e^{\alpha\sqrt{-1}})(1-e^{-\alpha\sqrt{-1}})\}=-\log_e\{2-(e^{\alpha\sqrt{-1}}+e^{-\alpha\sqrt{-1}})\}$$

$$=-\log_e(2-2\cos\alpha)=-\log_e(2\sin\tfrac{1}{2}\alpha)^2=-2\log_e(2\sin\tfrac{1}{2}\alpha),$$

故 $S=-\log_e(2\sin\frac{1}{2}\alpha)$.

(90) 如前例. $2S=\log_e(1+e^{\alpha\sqrt{-1}})+\log_e(1+e^{-\alpha\sqrt{-1}})$

$$=\log_e(2+e^{\alpha\sqrt{-1}}+e^{-\alpha\sqrt{-1}})=\log_e(2+2\cos\alpha)=\log_e(2\cos\tfrac{1}{2}\alpha)^2,$$

故 $S=\log_e(2\cos\frac{1}{2}\alpha)$.

(91) $2S=\dfrac{\cos\theta}{1}\left(e^{\phi\sqrt{-1}}+e^{-\phi\sqrt{-1}}\right)+\dfrac{\cos^2\theta}{2}(e^{2\phi\sqrt{-1}}+e^{-2\phi\sqrt{-1}})$

$+\dfrac{\cos^3\theta}{3}(e^{3\phi\sqrt{-1}}+e^{-3\phi})+\cdots\cdots=-\log_e(1+\cos\theta e^{\phi\sqrt{-1}})-\log_e(1+\cos\theta e^{-\phi\sqrt{-1}})$,

由第十一編4.節.

$$=-\log_e\{1+\cos\theta(e^{\phi\sqrt{-1}}+e^{-\phi\sqrt{-1}})+\cos^2\theta\}=-\log_e(1+2\cos\theta\cos\phi+\cos^2\theta)$$

故 $S=-\frac{1}{2}\log_e(1+2\cos\theta\cos\phi+\cos^2\theta)$.

92. $\frac{\sin\theta}{1}\sin\theta-\frac{\sin^{2}\theta}{2}\sin 2\theta+\frac{\sin^{3}\theta}{3}\sin 3\theta-\cdots\cdots$

93. $\frac{\cos\theta}{\underline{\lvert 1}}\sin\theta+\frac{\cos^{2}\theta}{\underline{\lvert 2}}\sin 2\theta+\frac{\cos^{3}\theta}{\underline{\lvert 3}}\sin 3\theta+\cdots\cdots$

94. $x\sin\theta-\frac{x^{2}\sin 2\theta}{2}+\frac{x^{3}\sin 3\theta}{3}-\cdots\cdots$

95. $\cos 2a+\frac{1}{2}\cos 6a+\frac{1}{3}\cos 10a+\cdots\cdots$

96. $\cos\theta\sin\theta+\cos^{2}\theta\sin 2\theta+\cos^{3}\theta\sin 3\theta+\cdots\cdots$

97. $1-\frac{1}{\underline{\lvert 2}}\cos 2a+\frac{1}{\underline{\lvert 4}}\cos 4a-\frac{1}{\underline{\lvert 6}}\cos 6a+\cdots\cdots$

98. $2\cos a+\frac{3}{2}\cos^{2}a+\frac{4}{3}\cos^{3}a+\frac{5}{4}\cos^{4}a+\cdots\cdots$

99. $\sin a-\frac{\cos a}{\underline{\lvert 1}}\sin(a+\beta)+\frac{\cos^{2}a}{\underline{\lvert 2}}\sin(a+2\beta)-\cdots\cdots$

例題解自 92. 至 99.

(92) $S=\frac{\sin\theta}{1}\sin\theta-\frac{\sin^{2}\theta}{2}\sin 2\theta+\frac{\sin^{3}\theta}{3}\sin 3\theta-\cdots\cdots$

$C=\frac{\sin\theta}{1}\cos\theta-\frac{\sin^{2}\theta}{2}\cos 2\theta+\frac{\sin^{2}\theta}{3}\cos 3\theta-\cdots\cdots$

$C+S\sqrt{-1}=\sin\theta e^{\theta\sqrt{-1}}-\frac{1}{2}\sin^{2}\theta e^{2\theta\sqrt{-1}}+\frac{1}{3}\sin^{3}\theta e^{3\theta\sqrt{-1}}-\cdots\cdots$

$=\log_{e}(1+\sin\theta e^{\theta\sqrt{-1}})=\log_{e}(1+\sin\theta\cos\theta+\sqrt{-1}\sin^{2}\theta)$,

故 $e^{C+S\sqrt{-1}}=1+\sin\theta\cos\theta+\sqrt{-1}\sin^{2}\theta$, 即

$e^{C}e^{S\sqrt{-1}}=e^{C}(\cos S+\sqrt{-1}\sin S)=1+\sin\theta\cos\theta+\sqrt{-1}\sin^{2}\theta$, 故

$e^{C}\cos S=1+\sin\theta\cos\theta$, 及 $e^{C}\sin S=\sin^{2}\theta$,

故 $\cot S=\frac{1+\sin\theta\cos\theta}{\sin^{2}\theta}=1+\cot^{2}\theta+\cot\theta$,

由是 $S=\cot^{-1}(1+\cot\theta+\cot^{2}\theta)$.

(93) 於例題 88. 令 $x=\cos\theta$, $a=0$, $\beta=\theta$, $S=e^{\cos 2\theta}\sin(\cos\theta\sin\theta)$.

(94) 令原級數爲 $C=x\cos\theta-\frac{1}{2}x^2\cos2\theta+\frac{1}{3}x^3\cos3\theta-\cdots\cdots\cdots$ 則

$S+C\sqrt{-1}=xe^{\theta\sqrt{-1}}-\frac{1}{2}x^2e^{2\theta\sqrt{-1}}+\frac{1}{3}x^3e^{3\theta\sqrt{-1}}-\cdots\cdots\cdots=\log_e(1+xe^{\theta\sqrt{-1}})$,

故 $1+xe^{\theta\sqrt{-1}}=e^{C+S\sqrt{-1}}$, 卽 $1+x\cos\theta+\sqrt{-1}x\sin\theta=e^C(\cos S+\sqrt{-1}\sin S)$

$e^C\cos S=1+x\cos\theta$, $e^C\sin S=x\sin\theta$, 由是

$\cot S=\dfrac{1+x\cos\theta}{x\sin\theta}$, 故 $S=\cot^{-1}\left(\dfrac{\csc\theta}{x}+\cot\theta\right)$.

(95) 令原級數爲 C, $\cos2\alpha+\frac{1}{3}\cos6\alpha+\frac{1}{5}\cos10\alpha+\cdots\cdots\cdots=S$, 則

$C+S\sqrt{-1}=e^{2\alpha\sqrt{-1}}+\frac{1}{3}e^{6\alpha\sqrt{-1}}\frac{1}{5}e^{10\alpha\sqrt{-1}}+\cdots\cdots\cdots=\frac{1}{2}\log e\,\dfrac{m}{n}$, 但 $e^{2\alpha\sqrt{-1}}=\dfrac{m-n}{m+n}$,

故 $\dfrac{1+e^{2\alpha\sqrt{-1}}}{1-e^{2\alpha\sqrt{-1}}}=\dfrac{m}{n}$, (見第十一編4.節) $2(C+S\sqrt{-1})=\log e\left(\dfrac{1+e^{2\alpha\sqrt{-1}}}{1-e^{2\alpha\sqrt{-1}}}\right)$

$=\log_e\left\{\dfrac{e^{\alpha\sqrt{-1}}+e^{-\alpha\sqrt{-1}}}{-(e^{\alpha\sqrt{-1}}-e^{-\alpha\sqrt{-1}})}\right\}=\log_e\left(\dfrac{2\cos\alpha}{-2\sqrt{-1}\sin\alpha}\right)$

$=\log_e(\sqrt{-1}\cot\alpha)$, 卽 $e^{2C+2S\sqrt{-1}}=\sqrt{-1}\cot\alpha$, 卽

$e^{2C}(\cos2S+\sqrt{-1}\sin2S)=\sqrt{-1}\cot\alpha$, 故 $e^{2C}\cos2S=0$, $e^{2C}\sin2S=\cot\alpha$,

故 $e^{2C}(\cos^2 2S+\sin^2 2S)=\cot^2\alpha$, $e^{2C}=\cot\alpha$, 故 $C=\frac{1}{2}\log_e(\cot\alpha)$.

(96) 令原級數爲 S, $C=\cos\theta\cos\theta+\cos^2\theta\cos2\theta+\cos^3\theta\cos3\theta$, 則

$C+S\sqrt{-1}=\cos\theta e^{\theta\sqrt{-1}}+\cos^2\theta e^{2\theta\sqrt{-1}}+\cos^3\theta e^{3\theta\sqrt{-1}}+\cdots\cdots\cdots$

$=\dfrac{\cos\theta e^{\theta\sqrt{-1}}}{1-\cos\theta e^{\theta\sqrt{-1}}}=\dfrac{\cos\theta(\cos\theta+\sqrt{-1}\sin\theta)}{1-\cos^2\theta-\cos\theta\sin\theta\sqrt{-1}}=\dfrac{\sqrt{-1}\cos\theta(\sin\theta-\sqrt{-1}\cos\theta)}{\sin\theta(\sin\theta-\sqrt{-1}\cos\theta)}=\sqrt{-1}\cot\theta$,

故 $S=\cot\theta$.

(97) $S=1-\dfrac{1}{2\lfloor 2}(e^{2\alpha\sqrt{-1}}+e^{-2\alpha\sqrt{-1}})+\dfrac{1}{2\lfloor 4}(e^{4\alpha\sqrt{-1}}+e^{-4\alpha\sqrt{-1}})\cdots\cdots\cdots$

$=\dfrac{1}{2}\left\{\left(1-\dfrac{1}{\lfloor 2}e^{2\alpha\sqrt{-1}}+\dfrac{1}{\lfloor 4}e^{4\alpha\sqrt{-1}}-\cdots\cdots\right)+\left(1-\dfrac{1}{\lfloor 2}e^{-2\alpha\sqrt{-1}}+\dfrac{1}{\lfloor 4}e^{-4\alpha\sqrt{-1}}-\cdots\cdots\right)\right\}$

$=\frac{1}{2}\{\cos(e^{\alpha\sqrt{-1}})+\cos(-e^{-\alpha\sqrt{-1}})\}=\frac{1}{2}\{\cos(\cos\alpha+\sqrt{-1}\sin\alpha)+\cos(\cos\alpha-\sqrt{-1}\sin\alpha)\}$

$=\cos\{\cos\alpha\cos(\sqrt{-1}\sin\alpha)\}$.

(98) $S=(\cos\alpha+\cos^2\alpha+\cos^3\alpha+\cdots\cdots\cdots)+(\cos\alpha+\frac{1}{2}\cos^2\alpha+\frac{1}{3}\cos^3\alpha+\cdots\cdots\cdots)$

$=\cos\alpha/(1-\cos\alpha)-\log_e(1-\cos\alpha)$.

(99) 於例題22. 令 $x=-\cos\alpha$, 則 $S=e^{-\cos\alpha\cos\beta}\sin(\alpha-\cos\alpha\sin\beta)$.

100. $e^{-x}\cos a-\frac{1}{3}e^{-3x}\cos 3a+\frac{1}{5}e^{-5x}\cos 5a-\cdots\cdots$

101. $\cos a-\frac{1}{3}\cos 3a+\frac{1}{5}\cos 5a-\cdots\cdots$

102. $\frac{p}{q}\cos a\sin\beta+\frac{p(p-q)}{\underline{|2}\ q^2}\cos^2 a\sin 2\beta$

$$+\frac{p(p-q)(p-2q)}{\underline{|3}\ q^3}\cos^3 a\sin 3\beta+\cdots\cdots$$

103. $e^{x}\sin x-\frac{1}{2}e^{2x}\sin 2x+\frac{1}{3}e^{2x}\sin 3x-\cdots\cdots$

104. $1+\frac{1}{\underline{|1}}e^{\cos a}\cos(\sin a)+\frac{1}{\underline{|2}}e^{2\cos a}\cos(2\sin a)$

$$+\frac{1}{\underline{|3}}e^{3\cos a}\cos(3\sin a)+\cdots\cdots$$

105. $\frac{1}{4}\tan\frac{1}{4}\pi+\frac{1}{8}\tan\frac{1}{8}\pi+\frac{1}{16}\tan\frac{1}{16}\pi+\cdots\cdots=1/\pi.$

例題解自 100. 至 105.

(100) $2S=e^{-x}(e^{a\sqrt{-1}}+e^{-a\sqrt{-1}})-\frac{1}{3}e^{-3x}(e^{2a\sqrt{-1}}+e^{-2a\sqrt{-1}})+\cdots\cdots$

$=e^{-x+a\sqrt{-1}}-\frac{1}{3}e^{3(-x+a\sqrt{-1})}+\frac{1}{5}e^{5(-x+a\sqrt{-1})}-\cdots\cdots$

$+e^{-x-a\sqrt{-1}}-\frac{1}{3}e^{3(-x-a\sqrt{-1})}+\frac{1}{5}e^{5(-x-a\sqrt{-1})}-\cdots\cdots$

$=\tan^{-1}e^{-x+a\sqrt{-1}}+\tan^{-1}e^{-x-a\sqrt{-1}}=\tan^{-1}\frac{e^{-x+a\sqrt{-1}}+e^{-x-a\sqrt{-1}}}{1-e^{-x+a\sqrt{-1}}\times e^{-x-a\sqrt{-1}}}$

$=\tan^{-1}\frac{e^{-x}(e^{a\sqrt{-1}}+e^{-a\sqrt{-1}})}{1-e^{-2x}}=\tan^{-1}\left(\frac{2\cos a}{e^{x}-e^{-x}}\right),$

由是 $S=\frac{1}{2}\tan^{-1}\{2\cos a/(e^{x}-e^{-x})\}.$

(101) 於前題,令 $x=0$, 則 $S=\frac{1}{2}\tan^{-1}\frac{2\cos a}{1-1}=\frac{1}{2}\tan^{-1}\infty,$

由是 $S=\frac{1}{2}(n+\frac{1}{2})\pi.$

(102) 令S為原級數, $C=1+\frac{p}{q}\cos a\cos\beta+\frac{p(p-q)}{\underline{|2}\ q^2}\cos^2 a\cos 2\beta,$

則 $C+S\sqrt{-1}=1+\frac{p}{q}\cos\alpha e^{\alpha\sqrt{-1}}+\frac{p(p-q)}{\lfloor 2\, q^2}\cos^2\alpha e^{2\beta\sqrt{-1}}+\cdots\cdots$

$=(1+\cos\alpha e^{\beta\sqrt{-1}})^{\frac{p}{q}}=(1+\cos\alpha\cos\beta+\sqrt{-1}\cos\alpha\sin\beta)^{\frac{p}{q}}$

$=\{r(\cos\theta+\sqrt{-1}\sin\theta)\}^{\frac{p}{q}}=r^{\frac{p}{q}}\left(\cos\frac{p\theta}{q}+\sqrt{-1}\sin\frac{p\theta}{q}\right)$,

由是　$S=r^{\frac{p}{q}}\sin\frac{p\theta}{q}$,　但　$r\cos\theta=1+\cos\alpha\cos\beta$,

$r\sin\theta=\cos\alpha\sin\beta$,　故　$\tan\theta=\frac{\cos\alpha\sin\beta}{1+\cos\alpha\cos\beta}$,

及　$r^2=(1+\cos\alpha\cos\beta)^2+\cos^2\alpha\sin^2\beta=1+2\cos\alpha\cos\beta+\cos^2\alpha$.

(103)　令S為原級數, $C=e^x\cos x-\frac{1}{2}e^{2x}\cos 2x+\frac{1}{3}e^{3x}\cos 3x-\cdots\cdots\cdots$

則　$C+S\sqrt{-1}=e^x e^{x\sqrt{-1}}-\frac{1}{2}e^{2x}e^{2x\sqrt{-1}}+\frac{1}{3}e^{3x}e^{3x\sqrt{-1}}-\cdots\cdots\cdots$

$=\log_e(1+e^{x+x\sqrt{-1}})$,　故　$e^{C+S\sqrt{-1}}=1+e^{x+x\sqrt{-1}}$,　即

$e^C(\cos S+\sqrt{-1}\sin S)=1+e^x(\cos x+\sqrt{-1}\sin x)$,

$e^C\cos S=1+e^x\cos x$,　$e^C\sin S=e^x\sin x$,

由是　$\tan S=\frac{e^x\sin x}{1+e^x\cos x}$,　故　$S=\tan^{-1}\left(\frac{e^x\sin x}{1+e^x\cos x}\right)$.

(104)　令C為原級數, $S=\frac{1}{\lfloor 1}e^{\cos\alpha}\sin(\sin\alpha)+\frac{1}{\lfloor 2}e^{2\cos\alpha}\sin(2\sin\alpha)+\cdots\cdots\cdots$

則　$C+S\sqrt{-1}=1+\frac{1}{\lfloor 1}e^{\cos\alpha+\sqrt{-1}\sin\alpha}+\frac{1}{\lfloor 2}e^{2(\cos\alpha+\sqrt{-1}\sin\alpha)}+\cdots\cdots\cdots$

$=e^{e^{\cos\alpha+\sqrt{-1}\sin\alpha}}=e^{e^{\cos\alpha}e^{\sqrt{-1}\sin\alpha}}=e^{e^{\cos\alpha}[\cos(\sin\alpha)+\sqrt{-1}\sin(\sin\alpha)]}$

$=e^{e^{\cos\alpha}\cos(\sin\alpha)}e^{\sqrt{-1}e^{\cos\alpha}\sin(\sin\alpha)}$

$=e^{e^{\cos\alpha}\cos(\sin\alpha)}[\cos\{e^{\cos\alpha}\sin(\sin\alpha)\}+\sqrt{-1}\sin\{e^{\cos\alpha}\sin(\sin\alpha)\}]$,

由是　$C=e^{e^{\cos\alpha}\cos(\sin\alpha)}\cos\{e^{\cos\alpha}\sin(\sin\alpha)\}$.

(105)　由例題13. $\tan\frac{\pi}{4}+\frac{1}{2}\tan\frac{\pi}{8}+\frac{1}{4}\tan\frac{\pi}{16}+\cdots\cdots=\frac{4}{\pi}-2\cot\frac{\pi}{2}=\frac{4}{\pi}$.

106. $\log\cos\theta+\log\cos\frac{\theta}{2}+\log\cos\frac{\theta}{2^2}+\log\cos\frac{\theta}{2^3}+\cdots\cdots\cdots$

107. $\frac{1+2}{\lfloor 1}\log_e 2+\frac{1+2^2}{\lfloor 2}(\log_e 2)^2+\frac{1+2^3}{\lfloor 3}(\log_e 2)^3+\cdots\cdots\cdots$

108. $1+\frac{1+2}{\lfloor 2}+\frac{1+2+2^2}{\lfloor 3}+\frac{1+2+2^2+2^3}{\lfloor 4}+\cdots\cdots\cdots$

109. $\frac{1}{\lfloor 2}+\frac{4}{\lfloor 3}+\frac{9}{\lfloor 4}+\frac{16}{\lfloor 5}+\cdots\cdots\cdots$

例題解自 106. 至 109.

(106) 由例題十二24. $\cos\frac{1}{2}\theta\cos\frac{1}{4}\theta\cos\frac{1}{8}\theta\cdots\cdots$之極限$=\frac{\sin\theta}{\theta}$,

故 $\cos\theta\cos\frac{\theta}{2}\cos\frac{\theta}{4}\cos\frac{\theta}{8}\cdots\cdots$之極限$=\cos\theta\times\frac{\sin\theta}{\theta}=\frac{\sin 2\theta}{2\theta}$

其對數式 $\log\cos\theta+\log\cos\frac{\theta}{2}+\log\cos\frac{\theta}{4}+\cdots\cdots=\log\left(\frac{\sin 2\theta}{2\theta}\right)$.

(107) $S=\log_e 2+\frac{1}{\lfloor 2}(\log_e 2)^3+\frac{1}{\lfloor 3}(\log_e 2)^3+\cdots\cdots+(2\log_e 2)+\frac{1}{\lfloor 2}(2\log_e 2)^2$

$+\frac{1}{\lfloor 3}(2\log_e 2)^3+\cdots\cdots=e^{\log_e 2}-1+e^{2\log_e 2}-1=2-1+2^2-1=4,$

但令 $e^{\log_e 2}=y$, 則 $\log_e 2=\log_e 2=\log_e y$, 故 $2=y$.

(108) $1+2+2^2+\cdots\cdots+2^n=2^{n+1}-1$, 由是

$S=2-1+\frac{2^2-1}{\lfloor 2}+\frac{2^2-1}{\lfloor 3}+\cdots\cdots=2+\frac{2^2}{\lfloor 2}+\frac{2^3}{\lfloor 3}+\cdots\cdots-\left(1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots\cdots\right)$

$=e^2-1-(e-1)=e(e-1).$

(109) $\frac{n^2}{\lfloor n+1}=\frac{n(n+1)-(n+1)+1}{\lfloor n+1}=\frac{1}{\lfloor n-1}-\frac{1}{\lfloor n}+\frac{1}{\lfloor n+1}$此式之$n$,順次

令爲1, 2, 3,……所得之式相加,則 $S=\frac{1}{\lfloor 2}+\left(1+\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots\cdots\right)-\left(\frac{1}{\lfloor 2}+\frac{1}{\lfloor 3}+\cdots\cdots\right)$

$+\left(\frac{1}{\lfloor 3}+\frac{1}{\lfloor 4}+\cdots\cdots\right)=\frac{1}{2}+(e-1)-(e-2)+(e-2-\frac{1}{2})=e-1.$

110. 自圓周上之任一點,引其圓內切正n多角形之各角頂之弦,求其各平方之和及四方乘之和.但圓之半徑爲R.

111. 同上.諸弦之長,從其近起,順次爲$c_1, c_2, c_3, \cdots\cdots$,則

$$c_1c_2+c_2c_3+c_3c_4+\cdots\cdots+c_{n-1}c_n-c_nc_1 \text{ 常爲一定.}$$

例題解自110.至111.

(110) 正n多角形爲$A_1A_2A_3\cdots\cdots\cdots A_n$.任一點P,在弧$A_1A_n$上.而令PA之弧度爲$2\alpha$, A_1A_2, A_2A_3,………, $A_{n-1}A_n$之弧度爲$2\pi/n$,則

三角形PA_1A_2, PA_2A_3,………, $PA_{n-1}A_n$由第十五編2.節.

$$PA_1=2R\sin\alpha,\quad PA_2=2R\sin\left(\alpha+\frac{\pi}{n}\right),\cdots\cdots\cdots PA_n=2R\sin\left\{\alpha+(n-1)\frac{\pi}{n}\right\}$$

故 $PA_1^2+PA_2^2+\cdots\cdots+PA_n^2=4R^2\left[\sin^2\alpha+\sin^2\left(\alpha+\frac{\pi}{n}\right)+\cdots\cdots+\sin^2\left\{\alpha+(n-1)\frac{\pi}{n}\right\}\right]$

$=2R^2\left[n-\dfrac{\cos\left\{2\alpha+\frac{1}{2}(n-1)\frac{\pi}{n}\right\}\sin\frac{n\pi}{n}}{\sin\frac{\pi}{n}}\right]$, (例題27.) 但 $\sin\frac{n\pi}{n}=\sin\pi=0$

$=2R^2[n-0]=2nR^2$.

又 $PA_1^4+PA_2^4+\cdots\cdots+PA_n^4=16R^4\left[\sin^4\alpha+\sin^4\left(\alpha+\frac{\pi}{n}\right)+\cdots\cdots+\sin^4\left\{\alpha+(n-1)\frac{\pi}{n}\right\}\right]$

由例題7. $\quad =16R^4\left[\frac{3n}{8}\right]=6nR^4$.

(111) 令 $PA_1=c_1$, $PA_2=c_2$,…………$PA_n=c_n$ 則

$\frac{1}{2}c_1c_2\sin\frac{\pi}{n}$=三角形$PA_1A_2$, 故 $c_1c_2=2\csc\frac{\pi}{n}$(三角形$PA_1A_2$).

同樣. $c_2c_3=2\csc\frac{\pi}{n}$(三角形$PA_2A_3$),………$c_{n-1}c_n=2\csc\frac{\pi}{n}$(三角形$PA_{n-1}A_n$),

$c_nc_1=2\csc A_nPA_1$(三角形PA_nA_1)$=2\csc\frac{(n-1)\pi}{n}$(三角形PA_nA_1),

由是 $c_1c_2+c_2c_3+\cdots\cdots\cdots+c_nc_{n-1}-c_nc_1=2\csc\frac{\pi}{n}$(三角形$PA_1A_2$),

$+PA_2A_3+\cdots\cdots\cdots+PA_{n-1}A_n-PA_n-PA_nA_1)=2\csc\frac{\pi}{n}$(正多角形)=一定.

112. 直角三角形ABC, 作CP_1, CP_2,......CP_n分C直角爲$n+1$等分，而P_1, P_2,......, P_n在AB之上，則

$$\frac{1}{CP_1^2}+\frac{1}{CP_2^2}+\cdots\cdots+\frac{1}{CP_n^2}=\frac{2}{AP^2 sin^2 2A}\left\{n+2 sin 2A\, cot\frac{\pi}{2(n+1)}\right\}.$$

113. 圓之內切正多角形.其各角頂各置一石.求其一個一個運送中心之勞力，與其一個一個運送其內一角頂之勞力之比.

114. 同上. n無限大，則其勞力之比爲$\pi:4$.

115. 以正n多角形之一角頂爲共通頂角，各邊爲底邊，則諸三角形之內切圓.其半徑之和爲$2R\{1-n sin^2\pi/(2n)\}^2$.

116. 同上. 內切圓面積之和.爲

$$16\pi R^2 sin^2\frac{\pi}{2^n}\left(\frac{n}{4}sin^2\frac{\pi}{2^n}+\frac{n-4}{8}\right).$$

例題解自112.至116.

(112) 令角ACP_1=角P_1CP_2=角P_2CP_3=.........=角$P_nCB=\frac{\pi}{n+1}=\theta$, 則於三角形$ACP_1$ $\frac{sin A}{CP_1}=\frac{sin(A+\theta)}{AC}=\frac{sin(A+\theta)}{AB cos A}$, 即 $\frac{1}{CP_1}=\frac{2 sin(A+\theta)}{AB sin 2A}$,

故 $\frac{1}{CP_1^2}+\frac{1}{CP_2^2}+\cdots\cdots+\frac{1}{CP_n^2}=\frac{4}{AB^2 sin^2 2A}\{sin^2(A+\theta)+sin^2(A+2\theta)+\cdots\cdots$

$+sin^2(A+n\theta)\}=\frac{2}{AB^2 sin^2 2A}\left[n-\frac{cos\{2A+2\theta+(n-1)\theta\} sin n\theta}{sin\theta}\right]$, 例題27.

$$=\frac{2}{AB^2 sin 2A}\left\{n-\frac{cos(2A+\frac{1}{2}\pi) sin(\frac{1}{2}\pi-\theta)}{sin\theta}\right\}.$$

(113) 運送中心勞力之比$=nR$, 由例題110.之記法，運送一角頂A_1之勞力之比$=A_1A_2+A_1A_3+A_1A_4+\cdots\cdots+A_1A_n$,

由第十五編2.節. $=2R\{sin(\pi/n)+sin(2\pi/n)+sin(3\pi/n)+\cdots\cdots+sin(n\pi/n)\}$

$$=\frac{2R\sin\frac{1}{2}(n+1)\pi/n\sin\frac{1}{2}n\pi/n}{\sin\frac{1}{2}\pi/n}$$，〔例題3.〕$=2R\cot\frac{\pi}{2n}$，

由是所求之比$=nR:2R\cot\frac{\pi}{2n}=n:2\cot\frac{\pi}{2n}$.

(114) 由前例. $\pi:4\left(\frac{\pi}{2n}\right)\quad\tan\frac{\pi}{2n}$，$n=\infty$，則 $\frac{\pi}{2n}\Big/\tan\frac{\pi}{2n}=1$.

(115) 用前例之記法，一邊$A_{m+1}A_{m+2}$爲底邊，A_1爲共通之頂角，其三角形內切圓之半徑爲r_m(但正多角形之一邊爲a.)

則由例題二十三37.

$$a=r_m(\cot\tfrac{1}{2}A_1A_{m+1}A_{m+2}+\cot\tfrac{1}{2}AA_{m+2}A_{m+1})=r_m\left\{\cot\frac{\pi-(m+1)\pi/n}{2}+\cot\frac{m\pi}{2n}\right\}$$

$$=r_m\left\{\tan\frac{(m+1)\pi}{2n}+\cot\frac{m\pi}{2n}\right\}，\text{故}\quad r_m=\frac{a}{2}\sec\frac{\pi}{2n}\left\{\sin\frac{(2m+1)\pi}{2n}-\sin\frac{\pi}{2n}\right\}.$$

三角形之數爲$n-2$個，故於上之方程式.其m順次爲$1,2,3,\dots\dots,n-2$相加，

則 $S=r_1+r_2+r_3+\dots\dots+r_{n-2}$

$$=\frac{a}{2}\sec\frac{\pi}{2n}\left\{\sin\frac{3\pi}{2n}+\sin\frac{5\pi}{2n}+\dots\dots+\sin\frac{(4n-1)\pi}{2n}-(n-2)\sin\frac{\pi}{2n}\right\}$$

$$=\frac{a}{2}\sec\frac{\pi}{2n}\left\{\frac{\sin(n\pi/2\pi)\sin(n-2)\pi/2n}{\sin(\pi/2n)}-(n-2)\sin\frac{\pi}{2n}\right\}，\quad 2.\text{節}.(1)$$

$$=\frac{2R\sin\pi/n}{2}\sec\frac{\pi}{2n}\left\{\frac{\cos\pi/n}{\sin\pi/2n}-(n-2)\sin\frac{\pi}{2n}\right\}=2R\left(1-n\sin^2\frac{\pi}{2n}\right)^2.$$

(116) 由前例. $\pi r_m{}^2=\frac{\pi a^2}{4}\sec^2\frac{\pi}{2n}\left\{\sin\frac{(2m+1)\pi}{2n}-\sin\frac{\pi}{2n}\right\}^2$

$$=\frac{\pi a^2}{4}\sec^2\frac{\pi}{2n}\left\{\sin^2\frac{(2m+1)\pi}{2n}-2\sin\frac{(2m+1)\pi}{2n}\sin\frac{\pi}{2n}+\sin^2\frac{\pi}{2n}\right\}$$

$$=\frac{\pi a^2}{8}\sec^2\frac{\pi}{2n}\left\{1-\cos\frac{(2m+1)\pi}{n}-4\sin\frac{(2m+1)\pi}{2n}\sin\frac{\pi}{2n}+2\sin^2\frac{\pi}{2n}\right\}.$$

其m爲$1,2,3,\dots\dots,n-2$所得之式相加，則面積之和爲

$$\frac{\pi a^2}{8}\sec^2\frac{\pi}{2n}\left[n-\frac{\cos\{3\pi/n+(n-3)\pi/n\}\sin(n-2)\pi/n}{\sin\pi/n}\right.$$

$$\left.-\frac{\sin\{3\pi/2n+\frac{1}{2}(n-3)\pi/n\}\sin\frac{1}{2}(n-2)\pi/n}{\sin\pi/2n}\times4\sin\frac{\pi}{2n}+n\sin^2\frac{\pi}{2n}\right]$$

〔但 $a=2R\sin\pi/n$〕 $=16\pi R^2\sin^2\pi/2n\{\frac{1}{4}n\sin^2\pi/2n+\frac{1}{8}(n-4)\}$.

117. 通過一定點,引平行正n多角形之各邊之諸直線.自他一定點引其諸直線之垂線.求各垂線之平方之和.

118. 正多角形之一邊,置於定直線上.於同平面上回轉.迨回轉一周後.正多角形之任一角頂之路之長,等於$\frac{4\pi R}{n}cot\frac{\pi}{2n}$.

例題解自 117. 至 118.

(117) 已知二點爲P, Q,則令PQ=d,其第一平行線與PQ之交角爲a,垂線之長,順次爲 $d sin\,a, d\,sin\left(a+\frac{\pi}{n}\right),\ d\,sin\left(a+\frac{2\pi}{n}\right),\ldots\ldots d\,sin\left(a+\frac{n-1}{n}\pi\right)$.

故垂線之平方之和,爲

$$d^2\left\{sin^2 a+sin^2\left(a+\frac{\pi}{n}\right)+sin^2\left(a+\frac{2\pi}{n}\right)+\cdots\cdots+sin^2\left(a+\frac{n-1}{n}\pi\right)\right\}$$

$$=\frac{d^2}{2}\left[n-\left\{cos 2a+cos\left(2a+\frac{2\pi}{n}\right)+cos\left(2a+\frac{4\pi}{n}\right)+\cdots\cdots+cos\left(2a+\frac{2n-2}{2}\right)\right\}\right]$$

$$=\frac{d^2}{2}\left[n-0\right],\ (\text{例題 } 2.)=\tfrac{1}{2}nd^2.$$

(118) 其邊在定直線上,其次之一邊,在其直線上之動點與某一角之動路,其時附着於直線之一角頂爲中心,其角頂與前之某一角頂連結之對角線爲半徑.其所畫弧之長爲其中心角$\frac{\pi}{n}$,

故某一角頂旋動,每面之半徑均如次.

$$2R\,sin\frac{\pi}{n},\quad 2R\,sin\frac{2\pi}{n},\quad 2R\,sin\frac{3\pi}{n},\ldots\ldots,2R\,sin\frac{(n-1)\pi}{n}.$$

由是所求之和,爲 $2R\left\{sin\frac{\pi}{n}+sin\frac{2\pi}{n}+sin\frac{3\pi}{n}+\cdots\cdots+sin\frac{(n-1)\pi}{n}\right\}\frac{2\pi}{n}$

$$=\frac{4R\pi}{n}\left\{\frac{sin\{\pi/n+\frac{1}{2}(n-1)\pi/n\}sin\,\pi/2n}{sin\,\pi/2n}\right\}$$

$$=\frac{4R\pi}{n}\left\{\frac{sin(n+1)\pi/2n}{sin\,\pi/2n}\right\}=\frac{4R\pi}{n}cot\frac{\pi}{2n}.$$

第拾捌編

三角函數之因子

1. 第一 n 爲偶數.

$$x^n-1=(x-1)(x+1)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\left(x^2-2x\cos\frac{4\pi}{n}+1\right)\cdots\cdots$$

$$\cdots\cdots\cdots\left(x^2-2x\cos\frac{n-4}{n}\pi+1\right)\left(x^2-2x\cos\frac{n-2}{n}+1\right)$$

n 爲奇數.

$$x_n-1=(x-1)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\left(x^2-2x\cos\frac{4\pi}{n}+1\right)\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\left(x^2-2x\cos\frac{n-3}{n}\pi+1\right)\left(x^2-2x\cos\frac{n-1}{n}\pi+1\right)$$

〔證〕 $\left(\cos\frac{2r\pi}{n}\pm\sqrt{-1}\sin\frac{2r\pi}{n}\right)^n=\cos 2r\pi\pm\sqrt{-1}\sin 3r\pi$, (第十六編1.節).

r 爲正整數, $=1\pm\sqrt{-1}\times 0=1.$

故 $\cos\frac{2r\pi}{n}\pm\sqrt{-1}\sin\frac{2r\pi}{n}$ 爲 $x_n=1$ 之一根明甚.

第一 n 爲偶數. 然令 $r=0$, 則 $\cos 0\pm\sqrt{-1}\sin 0=1.$

故 1 爲 $x^n=1$ 之一實根, 卽 $x-1$ 爲 x^n-1 之一因子.

令 $r=\frac{n}{2}$, 則 $\cos\pi\pm\sqrt{-1}\sin\pi=-1$, 故 -1 爲 $x^n=1$ 之一實根,

卽 $x+1$ 爲 x^n-1 之一因子.

令 r 爲 $1,2,3,\cdots\cdots\cdots\frac{n}{2}-1$, 則 $\cos\frac{2r\pi}{n}\pm\sqrt{-1}\sin\frac{2r\pi}{n}$ 之二因子, 有 $\frac{n}{2}-1$ 個. 合計得

$n-2$ 因子, 與前二因子共得 n 因子.

由是 $x^n-1=(x-1)(x+1)\left(x-\cos\frac{4\pi}{n}-\sqrt{-1}\sin\frac{4\pi}{n}\right)\left(x-\cos\frac{2\pi}{n}+\sqrt{-1}\sin\frac{2\pi}{n}\right)$

$\left(x-\cos\frac{4\pi}{n}-\sqrt{-1}\sin\frac{4\pi}{n}\right)\left(x-\cos\frac{4\pi}{n}+\sqrt{-1}\sin\frac{4\pi}{n}\right)\cdots\cdots\left(x-\cos\frac{n-1}{n}\pi-\sqrt{-1}\sin\frac{n\pi}{n}\right)$

$\left(x-\cos\frac{n-1}{n}\pi+\sqrt{-1}\sin\frac{n-1}{n}\pi\right)=(x-1)(x+1)\left(x^2-2x\cos\frac{2\pi}{n}+1\right)$

$\left(x^2-2x\cos\frac{2\pi}{n}+1\right)\cdots\cdots\left(x^2-2x\cos\frac{n-1}{n}\pi+1\right).$

〔譯註〕 n 爲奇數,亦可仿此證之.

2. 第二 n 爲偶數,則

$$x^n+1=\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{3\pi}{n}+1\right)\left(x^2-2x\cos\frac{5\pi}{n}+1\right)\cdots\cdots$$

$$\cdots\cdots\left(x^2-2x\cos\frac{n-3}{n}+1\right)\left(x^2-2x\cos\frac{n-1}{n}\pi+1\right)$$

n 爲奇數,則

$$x^n+1=(x+1)\left(x^2-2x\cos\frac{\pi}{n}\pi+1\right)\left(x^2-2x\cos\frac{3\pi}{n}+1\right)\cdots\cdots$$

$$\cdots\cdots\left(x^2-2x\cos\frac{n-4}{n}\pi+1\right)\left(x^2-2x\cos\frac{n-2}{n}\pi+1\right).$$

〔證〕 n 爲偶數,則 $x^n=-1$ 之根,均爲虛數.而令 r 爲正整數,

則 $\left(\cos\frac{2r+1}{n}\pi\pm\sqrt{-1}\sin\frac{2r+1}{n}\pi\right)^n=\cos(2r+1)\pi\pm\sqrt{-1}\sin(2r+1)\pi=-1\pm\sqrt{-1}0=-1,$

故 $\cos\frac{2r+1}{n}\pi\pm\sqrt{-1}\sin\frac{2r+1}{n}\pi$ 爲 $x^n=-1$ 之一對根.故於此式之 r, 令爲 $0, 1, 2, 3,$

$\cdots\cdots, \frac{n}{2}-1$, 則 $\frac{n}{2}$ 對之一次根皆合.即得 n 個根.

由是 $x^n+1=\left(x-\cos\frac{\pi}{n}-\sqrt{-1}\sin\frac{\pi}{n}\right)\left(x-\cos\frac{\pi}{n}+\sqrt{-1}\sin\frac{\pi}{n}\right)\left(x-\cos\frac{3\pi}{n}-\sqrt{-1}\sin\frac{3\pi}{n}\right)$

$\left(x-\cos\frac{3\pi}{n}+\sqrt{-1}\cos\frac{3\pi}{n}\right)\cdots\cdots\left(x-\cos\frac{n-1}{n}\pi-\sqrt{-1}\sin\frac{n-1}{n}\pi\right)$

$\left(x-\cos\frac{n-1}{n}\pi+\sqrt{-1}\sin\frac{n-1}{n}\pi\right)$

$=\left(x^2-2x\cos\frac{\pi}{n}+1\right)\left(x^2-2x\cos\frac{3\pi}{n}+1\right)\cdots\cdots\left(x^2-2x\cos\frac{n-1}{n}\pi+1\right).$

第二 n 爲奇數, $x^n=-1$, 則 -1 爲一實根,其餘 $n-1$ 個爲虛根.

故 $\cos\frac{2r+1}{n}\pm\sqrt{-1}\sin\frac{2r+1}{n}\pi$ 之 r, 以 $0,1,2,\dots\dots\frac{n-3}{2}$ 代之, 得次式.

$$x^n+1=(x+1)\left(x+\cos\frac{\pi}{n}+\sqrt{-1}\sin\frac{\pi}{n}\right)\left(x+\cos\frac{\pi}{n}-\sqrt{-1}\sin\frac{\pi}{n}\right)$$

$$\left(x+\cos\frac{2\pi}{n}+\sqrt{-1}\sin\frac{2\pi}{n}\right)\left(x+\cos\frac{2\pi}{n}-\sqrt{-1}\sin\frac{2\pi}{n}\right)\dots\dots\dots$$

$$\left(x+\cos\frac{n-2}{n}\pi+\sqrt{-1}\sin\frac{n-2}{n}\pi\right)\left(x+\cos\frac{n-2}{n}\pi-\sqrt{-1}\sin\frac{n-2}{n}\pi\right)$$

$$=(x+1)\left(x^2+2x\cos\frac{\pi}{n}+1\right)\left(x^2+2x\cos\frac{2\pi}{n}+1\right)\dots\dots\left(x^2+2x\cos\frac{n-2}{n}\right).$$

3. 第三 n 爲正整數.

$$x^{2n}-2x^n\cos\theta+1=\left(x^2-2x\cos\frac{\theta}{n}+1\right)\left(x^2-2x\cos\frac{2\pi+\theta}{n}+1\right)$$

$$\left(x^2-2x\cos\frac{4\pi+\theta}{n}+1\right)\left(x^2-2x\cos\frac{6\pi+\theta}{n}+1\right)\dots\dots\dots$$

$$\left\{x^2-2x\cos\frac{(2n-2)\pi+\theta}{n}+1\right\}.$$

〔證〕 $(x^n-\cos\theta-\sqrt{-1}\sin\theta)(x^n-\cos\theta+\sqrt{-1}\sin\theta)$

$=(x^n-\cos\theta)^2-(\sqrt{-1}\sin\theta)^2=x^{2n}-2x^n\cos\theta+1$, 故

$x^n\cos\theta\mp\sqrt{-1}\sin\theta$ 爲 $x^{2n}-2x^n\cos\theta+1$ 之因子, 而

$x^n=\cos\theta\pm\sqrt{-1}\sin\theta=\cos(2r\pi+\theta)\pm\sqrt{-1}\sin(2r\pi+\theta)$, 即

$x=\cos\frac{2r\pi+\theta}{n}\pm\sqrt{-1}\sin\frac{2r\pi+\theta}{n}$, 故

$x-\cos\frac{2r\pi+\theta}{n}\mp\sqrt{-1}\sin\frac{2r\pi+\theta}{n}$ 之 r, 代以 $0,1,2,3,\dots\dots, n$ 得次式,

$$x^{2n}-2x^n\cos\theta+1=\left(x-\cos\frac{\theta}{n}-\sqrt{-1}\sin\frac{\theta}{n}\right)\left(x-\cos\frac{\theta}{n}+\sqrt{-1}\sin\frac{\theta}{n}\right)$$

$$\left(x-\cos\frac{2\pi+\theta}{n}-\sqrt{-1}\sin\frac{2\pi+\theta}{n}\right)\left(x-\cos\frac{2\pi+\theta}{n}+\sqrt{-1}\sin\frac{2\pi+\theta}{n}\right)\dots\dots\dots$$

$$\dots\dots\dots\left\{x-\cos\frac{(2n-2)\pi+\theta}{n}-\sqrt{-1}\sin\frac{(2n-2)\pi+\theta}{n}\right\}$$

$$\left\{x-\cos\frac{(2n-2)\pi+\theta}{n}+\sqrt{-1}\sin\frac{(2n-2)\pi+\theta}{n}\right\}$$

$$=\left(x^2-2x\cos\frac{\theta}{n}+1\right)\left(x^2-2x\cos\frac{2\pi+\theta}{n}+1\right)\dots\dots\dots\left\{x^2-2x\cos\frac{(2n-2)\pi+\theta}{n}+1\right\}.$$

4. 第四 分 $\sin\theta, \cos\theta$ 爲無限數之因子.

$$\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots\cdots$$

$$\cos\theta=\left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\cdots\cdots$$

〔證〕 $\sin\theta=\theta-\frac{\theta^3}{\lfloor\!\underline{3}}+\frac{\theta^5}{\lfloor\!\underline{5}}-\cdots\cdots=\theta\left(1-\frac{\theta^2}{\lfloor\!\underline{3}}+\frac{\theta^4}{\lfloor\!\underline{5}}-\cdots\cdots\right)$,

次令 θ 爲 $n\pi\mp\theta$, 但 n 爲正整數, 則

$$\sin\theta=\pm\sin(n\pi\mp\theta)=(n\pi\mp\theta)\left\{1-\frac{(n\pi\mp\theta)^2}{\lfloor\!\underline{3}}+\frac{(n\pi\mp\theta)^4}{\lfloor\!\underline{5}}\cdots\cdots\right\}$$

故 $\theta, n\pi\mp\theta$ 爲 $\sin\theta$ 之因子, 而 n 得自 1 增大至無限.

即令 n 爲 1,2,3,………… 則

$$\sin\theta=A\theta(\pi+\theta)(\pi-\theta)(2\pi-\theta)(2\pi+\theta)(3\pi-\theta)(3\pi+\theta)\cdots\cdots$$

$$=A\theta(\pi^2-\theta^2)(2^2\pi^2-\theta^2)(3^2\pi^2-\theta^2)\cdots\cdots \qquad (1)$$

即 $\sin\theta/\theta=A(\pi^2-\theta^2)(2^2\pi^2-\theta^2)(3^2\pi^2-\theta^2)\cdots\cdots$

其 A 不含 θ, 故 θ 任如何變化, A 常爲一定.

而 θ 變小殆等於 0, 則上之恆同式爲

$$1=A\pi^2\cdot 2^2\pi^2\cdot 3^2\pi^2\cdots\cdots \qquad (2)$$

以 (2) 除 (1). 則 $\sin\theta=\theta\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)\cdots\cdots$

又 $\cos\theta=1-\frac{\theta^2}{\lfloor\!\underline{2}}+\frac{\theta^4}{\lfloor\!\underline{4}}-\frac{\theta^6}{\lfloor\!\underline{6}}+\cdots\cdots$ 令 $\theta=\pm\frac{(2n+1)\pi}{2}$,

則 $\theta=1-\frac{(2n+1)^2\pi^2}{\lfloor\!\underline{2}\ 2^2}+\frac{(2n+1)^4\pi^4}{\lfloor\!\underline{4}\ 2^4}-\frac{(2n+1)^6\pi^6}{\lfloor\!\underline{6}\ 2^6}+\cdots\cdots$

由是

$$\cos\theta=\frac{1}{\lfloor\!\underline{2}}\left\{\frac{(2n+1)^2\pi^2}{2^2}-\theta^2\right\}-\frac{1}{\lfloor\!\underline{4}}\left\{\frac{(2n+1)^4\pi^4}{2^4}-\theta^4\right\}+\frac{1}{\lfloor\!\underline{6}}\left\{\frac{(2n+1)^6\pi^6}{2^6}-\theta^6\right\}-\cdots\cdots$$

故 $\frac{(2n+1)^2\pi^2}{2^2}-\theta^2$ 爲 $\cos\theta$ 之因子, 如前例, 令 n 爲 0,1,2,3,………則

$cos\theta = A\left(\frac{\pi^2}{4}-\theta^2\right)\left(\frac{3^2\pi^2}{4}-\theta^2\right)\left(\frac{5^2\pi^2}{4}-\theta^2\right)\cdots\cdots\cdots$ 如前例求A,則

$cos\theta = \left(1-\frac{4\theta^2}{\pi^2}\right)\left(1-\frac{4\theta^2}{3^2\pi^2}\right)\left(1-\frac{4\theta^2}{5^2\pi^2}\right)\cdots\cdots\cdots$

例題三十二

證次列各式.

1. n為偶數,則

$$\sqrt{n}=2^{\frac{n-1}{2}}sin\frac{2\pi}{2n}sin\frac{4\pi}{2n}sin\frac{6\pi}{2n}\cdots\cdots sin\frac{(n-2)\pi}{2n}$$

$$=2^{\frac{n-1}{2}}cos\frac{2\pi}{2n}cos\frac{4\pi}{2n}cos\frac{6\pi}{2n}\cdots\cdots cos\frac{(n-2)\pi}{2n}.$$

2. n為奇數,則

$$\sqrt{n}=2^{\frac{n-2}{2}}sin\frac{2\pi}{2n}sin\frac{4\pi}{2n}sin\frac{6\pi}{2n}\cdots\cdots sin\frac{(n-1)\pi}{2n}.$$

例題解自1.至2.

(1) 1.節第一公式之兩邊,以$x-1$除之.得

$$x^{n-1}+x^{n-2}+x^{n-3}+\cdots\cdots\cdots+1=(x+1)\left(x^2-2x\,cos\frac{2\pi}{n}+1\right)\left(x^2-2x\,cos\frac{4\pi}{n}+1\right)\cdots\cdots\cdots$$

$$\cdots\cdots\cdots\left(x^2-2x\,cos\frac{n-2}{n}\pi+1\right),$$ 令$x=1$, 則

$$n=2^{\frac{n}{2}}\left(1-cos\frac{2\pi}{n}\right)\left(1-cos\frac{4\pi}{n}\right)\cdots\cdots\left(1-cos\frac{n-2}{n}\right)$$

$$=2^{\frac{2n-2}{2}}sin^2\frac{\pi}{n}\,sin^2\frac{2\pi}{n}\,sin^2\frac{3\pi}{n}\cdots\cdots\cdots sin^2\frac{n-2}{2n}\pi,$$

故 $\sqrt{n}=2^{\frac{n-1}{2}}sin\frac{\pi}{n}\,sin\frac{2\pi}{n}\,sin\frac{3\pi}{n}\cdots\cdots\cdots sin\frac{n-2}{2n}\pi,$

又1.節第一公式之兩邊,以$x+1$除之之後.如前法,令$x=-1$,即可得次式之證.

(2) 1.節第二公式之兩邊,以$x-1$除之之後.如前例,令$x=1$,即可得其證.

3. n 爲偶數,則

$$1=2^{\frac{n-1}{2}}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\sin\frac{5\pi}{2n}\cdots\cdots\sin\frac{(n-1)\pi}{2n}$$

$$=2^{\frac{n-1}{2}}\cos\frac{2\pi}{2n}\cos\frac{4\pi}{2n}\cos\frac{6\pi}{2n}\cdots\cdots\cos\frac{(n-2)\pi}{2n}.$$

4. n 爲奇數,則

$$1=2^{\frac{n-1}{2}}\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\sin\frac{5\pi}{2n}\cdots\cdots\sin\frac{(n-2)\pi}{2n}.$$

5. $$\sin n\alpha=2^{n-1}\sin\alpha\sin\left(\alpha+\frac{\pi}{n}\right)\sin\left(\alpha+\frac{2\pi}{2n}\right)\sin\left(\alpha+\frac{3\pi}{n}\right)$$

$$\cdots\cdots\cdots\sin\left(\alpha+\frac{n-1}{n}\pi\right).$$

6. $$\cos n\alpha=2^{n-1}\sin\left(\alpha+\frac{\pi}{2n}\right)\sin\left(\alpha+\frac{3\pi}{2n}\right)\sin\left(\alpha+\frac{2n-1}{2n}\pi\right)$$

$$\cdots\cdots\cdots\sin\left(\alpha+\frac{2n-1}{2n}\pi\right).$$

7. $$\sin\frac{\pi}{4n}\sin\frac{5\pi}{4n}\sin\frac{9\pi}{4n}\cdots\cdots\sin\frac{4n-3}{4n}\pi=2^{-n+\frac{1}{2}}.$$

8. $$\tan\alpha\tan\left(\alpha+\frac{\pi}{n}\right)\tan\left(\alpha+\frac{2\pi}{n}\right)\cdots\cdots\tan\left(\alpha+\frac{n-n}{n}\pi\right)$$

$$=(-1)^{\frac{n}{2}}.$$

例題解自 3. 至 8.

(3) 於 2. 節第一公式,令 $x=1$,則

$$2=2^{\frac{n}{2}}\left(1-\cos\frac{\pi}{n}\right)\left(1-\cos\frac{3\pi}{n}\right)\left(1-\cos\frac{4\pi}{n}\right)\cdots\cdots\left(1-\cos\frac{n-1}{n}\right)$$

$$=2^{n}\sin^2\frac{\pi}{2n}\sin^2\frac{3\pi}{2n}\sin^2\frac{5\pi}{2n}\cdots\cdots\cdots\sin^2\frac{2n-1}{2n}\pi,$$

故 $1=2^{n-1}\sin^2\frac{\pi}{2n}\sin^2\frac{3\pi}{2n}\sin^2\frac{5\pi}{2n}\sin^2\frac{7\pi}{2n}\cdots\cdots\sin^2\frac{n-1}{2n}\pi.$

(4) 於2.節第二之公式,令$x=1$,試如前例,卽得其證.

(5) 於3.節之公式,令$x=1$, $\theta=2n\alpha$, 則

$$2(1-\cos 2n\alpha)=2^n(1-\cos 2\alpha)\left\{1-\cos 2\left(\alpha+\frac{\pi}{n}\right)\right\}\left\{1-\cos 2\left(\alpha+\frac{2\pi}{n}\right)\right\}\cdots\cdots\cdots$$

$$\left\{1-\cos 2\left(\alpha+\frac{n-1}{n}\right)\right\},$$

卽 $2^2\sin^2 n\alpha=2^{2n}\sin\alpha\sin^2\left(\alpha+\frac{\pi}{n}\right)\sin^2\left(\alpha+\frac{2\pi}{n}\right)\cdots\cdots\sin^2\left(\alpha+\frac{n-1}{n}\pi\right),$

故 $\sin n\alpha=2^{n-1}\sin\alpha\sin\left(\alpha+\frac{\pi}{n}\right)\sin\left(\alpha+\frac{2\pi}{n}\right)\cdots\cdots\sin\left(\alpha+\frac{n-1}{n}\pi\right).$

(6) 於前例,令α爲$\alpha+\frac{\pi}{2n}$, 則

$$\sin\left(n\alpha+\frac{\pi}{2}\right)=2^{n-1}\sin\left(\alpha+\frac{3\pi}{2n}\right)\sin\left(\frac{\pi}{4n}+\frac{\pi}{n}\right)\cdots\cdots\sin\left(\alpha+\frac{\pi}{2n}+\frac{n-1}{n}\pi\right),$$

卽 $\cos n\alpha=2^{n-1}\sin\left(\alpha+\frac{\pi}{2n}\right)\sin\left(\alpha+\frac{2\pi}{2n}\right)\cdots\cdots\sin\left(\alpha+\frac{2n-1}{2n}\pi\right).$

(7) 於例題5.令$\alpha=\frac{\pi}{4n}$, 則

$$\sin\frac{\pi}{4}=2^{n-1}\sin\frac{\pi}{4n}\sin\left(\frac{\pi}{4n}+\frac{\pi}{n}\right)\sin\left(\frac{\pi}{4n}+\frac{2\pi}{n}\right)\sin\left(\frac{\pi}{4n}+\frac{3\pi}{n}\right)\cdots\cdots\cdots$$

$$\cdots\cdots\sin\left(\frac{\pi}{4n}+\frac{2n-1}{2n}\pi\right),$$

卽 $\frac{1}{\sqrt{2}}=2^{n-1}\sin\frac{\pi}{4n}\sin\frac{5\pi}{4n}\sin\frac{9\pi}{4n}\sin\frac{13\pi}{4n}\cdots\cdots\sin\frac{4n-1}{4n}\pi,$

由是 $\sin\frac{\pi}{4n}\sin\frac{5\pi}{4n}\sin\frac{9\pi}{4n}\cdots\cdots\sin\frac{4n-1}{4n}\pi=2^{-\frac{1}{2}-(n-1)}.$

(8) 由例題5. $2^{n-1}\sin\alpha\sin\left(\alpha+\frac{\pi}{n}\right)\sin\left(\alpha+\frac{2\pi}{n}\right)\cdots\cdots\sin\left(\alpha+\frac{n-1}{n}\pi\right)=\sin n\alpha,$

令α爲$\alpha+\frac{\pi}{2}$, 則 $2^{n-1}\cos\alpha\cos\left(\alpha+\frac{\pi}{n}\right)\cos\left(\alpha+\frac{2\pi}{n}\right)\cdots\cdots\cos\left(\alpha+\frac{n-1}{n}\pi\right)$

$=\sin n\left(\alpha+\frac{\pi}{2}\right)=\sin n\alpha\cos\frac{n\alpha}{2}+\cos n\alpha\sin\frac{n\alpha}{2}$, n爲偶數,故

$=\sin n\alpha(-1)^{\frac{n}{2}}$, 此以前式除之,得

$$\tan\alpha\tan\left(\alpha+\frac{\pi}{n}\right)\tan\left(\alpha+\frac{2\pi}{n}\right)\cdots\cdots\tan\left(\alpha+\frac{n-1}{n}\pi\right)=(-1)^{\frac{n}{2}}.$$

9. $\cos 5A = 16\cos A\cos(72^\circ - A)\cos(72^\circ + A)\cos(144^\circ - A)\cos(144^\circ + A)$.

10. $\sin 5A - \cos 5A$

$= 16\cos(A - 27^\circ)\cos(A + 9^\circ)\sin(A + 27^\circ)\sin(A - 9^\circ)(\cos A - \sin A)$.

11. $(1+x)\left(1+\frac{x}{2^2}\right)\left(1+\frac{x}{3^2}\right)$………之無限因子之連乘積,等於 x^n 之係數 $\frac{\pi}{\lfloor 2n+1}$.

例題解自 9. 至 11.

(9) 於前例,令 $n=5$, α 為 $90^\circ - A$, π 為 180°. 故

$\sin 5(90^\circ - A) = 2^4\sin(90^\circ - A)\sin(90^\circ - A + 72^\circ)\sin(90^\circ - A + 108^\circ)\sin(90^\circ - A + 144^\circ)$,

故 $\cos 5A = 16\cos A\cos(A - 36^\circ)\cos(A - 72^\circ)\cos(A - 108^\circ)\cos(A - 144^\circ)$

$= 16\cos A\cos\{180^\circ + (A - 36^\circ)\}\cos(72^\circ - A)$

$\cos\{180^\circ + (A - 108^\circ)\}\cos(144^\circ - A)$

$= 16\cos A\cos(144^\circ + A)\cos(72^\circ - A)\cos(72^\circ + A)\cos(144^\circ - A)$.

(10) $\sin 5A - \cos 5A = \sqrt{2}\sin(5A - 45^\circ) = \sqrt{2}\sin 5(A - 9^\circ)$

$= \sqrt{2}.2^4\sin(A - 9^\circ)\sin(A - 9^\circ + 36^\circ)\sin(A - 9^\circ + 72^\circ)\sin(A - 9^\circ + 108^\circ)$

$\sin(A - 9^\circ + 144^\circ)$

$= 16\sqrt{2}\sin(A - 9^\circ)\sin(A + 27^\circ)\sin(A + 63^\circ)\sin(A + 99^\circ)\sin(A + 135^\circ)$

$= 16\sqrt{2}\sin(A - 9^\circ)\sin(A + 27^\circ)\cos(27^\circ - A)\cos(A + 9^\circ)\cos(A + 45^\circ)$

$= 16\sqrt{2}\sin(A - 9^\circ)\sin(A + 27^\circ)\cos(A - 27^\circ)\cos(A + 9^\circ)\cos(A - 9^\circ)(\cos A - \sin A)\frac{1}{\sqrt{2}}$.

(11) 令 $x = -\frac{\theta^2}{\pi^2}$, 則 原式 $= \left(1 - \frac{\theta^2}{\pi^2}\right)\left(1 - \frac{\theta^2}{2^2\pi^2}\right)\left(1 - \frac{\theta^2}{3^2\pi^2}\right)$…………

$= \frac{\sin\theta}{\theta}$, (4. 節) $= \frac{\theta^2}{\lfloor 3} + \frac{\theta^4}{\lfloor 5} -$………$+ (-1)^n\frac{\theta^{2n}}{\lfloor 2n+1}$…………

而 $(-1)^n\frac{\theta^{2n}}{\lfloor 2n+1} = \frac{\theta^{2n}}{\lfloor 2n+1}\left(-\frac{\theta^2}{\pi^2}\right)^n = \frac{\pi^{2n}}{\lfloor 2n+1}x^n$, 故 x^{2n} 之係數 $= \frac{\pi^{2n}}{2n+1}$.

求次各式無限項之和

12. $\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\cdots\cdots$　**13.** $\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\cdots\cdots$

14. $\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\frac{1}{7^2}+\cdots\cdots\cdots$　**15.** $\frac{1}{1^4}+\frac{1}{3^4}+\frac{1}{5^4}+\frac{1}{7^4}+\cdots\cdots\cdots$

16. $\frac{1}{1^2}-\frac{1}{2^2}+\frac{1}{3^2}-\frac{1}{4^2}+\frac{1}{5^2}-\frac{1}{6^2}+\cdots\cdots\cdots$

例題解自12.至16.

(12) $\frac{\sin\theta}{\theta}=1-\frac{\theta^2}{\underline{|3}}+\frac{\theta^4}{\underline{|5}}-\cdots\cdots$(第十六編6.節)$=\left(1-\frac{\theta^2}{\pi^2}\right)\left(1-\frac{\theta^2}{2^2\pi^2}\right)\left(1-\frac{\theta^2}{3^2\pi^2}\right)$

$\cdots\cdots\cdots\cdots$(4.節), 卽

$$\log_e\left\{1-\left(\frac{\theta^2}{\underline{|3}}-\frac{\theta^4}{\underline{|5}}+\cdots\cdots\cdots\right)\right\}=\log_e\left(1-\frac{\theta^2}{\pi^2}\right)+\log_e\left(1-\frac{\theta^2}{2^2\pi^2}\right)+\log_e\left(1-\frac{\theta^2}{3^2\pi^2}\right)+\cdots\cdots$$

由第十一編4.節(7′) $-\left(\frac{\theta^2}{\underline{|3}}-\frac{\theta^4}{\underline{|5}}+\cdots\cdots\cdots\right)-\frac{1}{2}\left(\frac{\theta^2}{\underline{|3}}-\frac{\theta^4}{\underline{|5}}+\cdots\cdots\cdots\right)^2-\cdots\cdots\cdots$

$$=-\frac{\theta^2}{\pi^2}-\frac{\theta^4}{2\pi^4}-\cdots\cdots\cdots-\frac{\theta^2}{2^2\pi^2}-\frac{\theta^4}{2\cdot 2^4\pi^4}-\cdots\cdots\cdots-\frac{\theta^2}{3^2\pi^2}-\frac{\theta^4}{2\cdot 3^4\pi^4}-\cdots\cdots\cdots$$

比較 θ^2 之係數,得 $-\frac{1}{\underline{|3}}=-\frac{1}{\pi^2}-\frac{1}{2^2\pi^2}-\frac{1}{3\pi^2}-\cdots\cdots\cdots$　故 $S=\frac{\pi^2}{6}$.

(13) 於前例之展開式,比較 θ^4 之係數,得 $S=\pi^4/90$.

(14) $S=\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\cdots\cdots-\frac{1}{2^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}\right)+\cdots\cdots\cdots$

依例題12. $=\frac{\pi^2}{6}-\frac{1}{4}\times\frac{\pi^2}{6}=\frac{\pi^4}{8}$.

(15) $S=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\cdots\cdots-\frac{1}{2^4}\left(\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\cdots\cdots\cdots\right)=\frac{\pi^4}{96}$.

(16) $S=\frac{1}{1^2}+\frac{1}{3^2}+\frac{1}{5^2}+\cdots\cdots\cdots-\frac{1}{2^2}\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\cdots\cdots\cdots\right)=\frac{\pi^2}{12}$.

17. $\frac{1}{1^2}, \frac{1}{2^2}, \frac{1}{3^2}, \frac{1}{4^2}, \cdots\cdots$求其各二項之積之和.

18. 證 $\pi=3\cdot\frac{36}{35}\cdot\frac{144}{143}\cdot\frac{324}{323}\cdot\frac{576}{575}\cdots\cdots$

19. $\pi=2\cdot\frac{2}{1}\cdot\frac{2}{3}\cdot\frac{4}{3}\cdot\frac{4}{5}\cdot\frac{6}{5}\cdot\frac{6}{7}\cdot\frac{8}{7}\cdot\frac{8}{9}\cdots\cdots$同上.

20. $\sqrt{2}=\frac{4}{3}\cdot\frac{36}{35}\cdot\frac{100}{99}\cdot\frac{196}{195}\cdot\frac{324}{323}\cdots\cdots$同上.

21. $\frac{\sqrt{3}}{2}=\frac{8}{9}\cdot\frac{80}{81}\cdot\frac{224}{225}\cdot\frac{440}{441}\cdots\cdots$同上.

例題解自 17. 至 21.

(17) $\left(\frac{1}{1^2}+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\cdots\cdots\right)^2=\frac{1}{1^4}+\frac{1}{2^4}+\frac{1}{3^4}+\frac{1}{4^4}+\cdots\cdots+2S,$

由例題12.及13.得 $\left(\frac{\pi^4}{6}\right)^2=\frac{\pi^2}{90}+2S$, 故 $S=\frac{\pi^4}{120}$.

(18) 於4.節之第一公式, 令 $\theta=\pi/6$, 則

$$\frac{1}{2}=\frac{\pi}{6}\left(1-\frac{1}{6^2}\right)\left(1-\frac{1}{2^2\cdot 6^2}\right)\left(1-\frac{1}{3^2\cdot 6^2}\right)\cdots\cdots=\frac{\pi}{6}\cdot\frac{35}{36}\cdot\frac{143}{144}\cdot\frac{323}{324}\cdots\cdots$$

(19) 於同公式, 令 $\theta=\pi/2$, 則

$$\frac{1}{2}=\frac{\pi}{2}\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{4^2}\right)\left(1-\frac{1}{6^2}\right)\cdots\cdots=\frac{\pi}{2}\cdot\frac{1\cdot 3}{2\cdot 2}\cdot\frac{3\cdot 5}{4\cdot 4}\cdot\frac{5\cdot 7}{6\cdot 6}\cdots\cdots$$

(20) 於4.節之第二公式, 令 $\theta=\pi/4$, 則

$$\frac{1}{\sqrt{2}}=\left(1-\frac{1}{2^2}\right)\left(1-\frac{1}{2^2\cdot 3^2}\right)\left(1-\frac{1}{2^2\cdot 5^2}\right)\cdots\cdots=\frac{3}{4}\cdot\frac{35}{36}\cdot\frac{99}{100}\cdots\cdots$$

(21) 於同公式, 令 $\theta=\pi/6$, 可得其證.

22. $e^x-2\cos\theta+e^{-x}=4\sin^2\frac{\theta}{2}\left(1+\frac{x^2}{\theta^2}\right)\left\{1+\frac{x^2}{(2\pi-\theta)^2}\right\}\left\{1+\frac{x^2}{(2\pi+\theta)^2}\right\}$

$\left\{1+\frac{x^2}{(4\pi-\theta)^2}\right\}\left\{1+\frac{x^2}{(4\pi+\theta)^2}\right\}\left\{1+\frac{x^2}{(6\pi-\theta)^2}\right\}\left\{1+\frac{x^2}{(6\pi+\theta)^2}\right\}\cdots\cdots\cdots$

23. $e^x+e^{-x}=2\left(1+\frac{4x^2}{\pi^2}\right)\left(1+\frac{4x^2}{3^2\pi^2}\right)\left(1+\frac{4x^2}{5^2\pi^2}\right)\cdots\cdots\cdots$

24. $e^x-e^{-x}=2x\left(1+\frac{x^2}{\pi^2}\right)\left(1+\frac{x^2}{2^2\pi^2}\right)\left(1+\frac{x^2}{3^2\pi^2}\right)\cdots\cdots\cdots$

例題解自 22. 至 24.

(22) 由 3. 節. $x^{2n}-2x^ny^n\cos\theta+y^{2n}$ 之因子, 爲 $x^2-2xy\cos(2r\pi+\theta)+y^2$,

而 r 可取 $0,1,2,\cdots\cdots\cdots\cdots n-1$,

但 $\cos\frac{2(n-1)\pi+\theta}{n}=\cos\frac{2\pi-\theta}{n}$, $\cos\frac{2(n-2)\pi+\theta}{n}=\cos\frac{4\pi-\theta}{n}$ 等

故得此因子爲 $x^2-2xy\cos\frac{2r\pi\pm\theta}{n}+y^2$, 令 $x=1+\frac{x}{2n}$, $y=1-\frac{x}{2n}$,

則因子$=\left(1+\frac{x}{2n}\right)^2-2\left(1-\frac{x^2}{4n^2}\right)\cos\theta+\left(1-\frac{x}{2n}\right)^2=4\sin^2\frac{2r\pi\pm\theta}{2n}\left(1+\frac{x^2}{4n^2}\cos^2\frac{2r\pi\pm\theta}{2n}\right)$,

n 爲無限大, 則 $\frac{x^2}{4n^2}\cos^2\frac{2r\pi\pm\theta}{2n}$ 之極限$=\frac{x^2}{(2r\pi\pm\theta)}$, 故

因子$=4\sin^2\frac{2r\pi\pm\theta}{2n}\left\{1+\frac{x^2}{(2r\pi\pm\theta)}\right\}$, 而令 r 爲 $0,1,2,3,\cdots\cdots\cdots\cdots$, 則

$e^x-2\cos\theta+e^{-x}=4\sin^2\frac{\theta}{2n}4\sin^2\frac{2\pi\pm\theta}{2n}4\sin^2\frac{4\pi\pm\theta}{2n}\cdots\cdots\cdots\cdots$

$\left(1+\frac{x^2}{\theta^2}\right)\left\{1+\frac{x^2}{(2\pi\pm\theta)^2}\right\}\left\{1+\frac{x^2}{(4\pi\pm\theta)^2}\right\}\cdots\cdots\cdots$ 於例題 5. 令 $\alpha=\frac{\theta}{2}$, 則

$=4\sin^2\frac{\theta}{2}\left(1+\frac{x^2}{\theta^2}\right)\left\{1+\frac{x^2}{(2\pi\pm\theta)^2}\right\}\left\{1+\frac{x^2}{(4\pi\pm\theta)^2}\right\}\cdots\cdots\cdots\cdots$

(23) 可於前例, 令 $\theta=\frac{1}{2}\pi$ 以求之.

(24) 於例題 22. $4\sin^2\frac{\theta}{2}\left(1+\frac{x^2}{\theta^2}\right)=4\sin^2\frac{\theta}{2}+x^2\left\{\frac{4\sin^2\frac{1}{2}\theta}{(\frac{1}{2}\theta)^2}\right\}$,

可令其 $\theta=0$ 以求之.

25. $$\frac{\sin(\alpha-\theta)}{\sin\alpha}=\left(1-\frac{\theta}{\alpha}\right)\left(1+\frac{\theta}{\pi-\alpha}\right)\left(1-\frac{\theta}{\pi+\alpha}\right)\left(1+\frac{\theta}{2\pi-\alpha}\right)$$
$$\left(1-\frac{\theta}{2\pi+\alpha}\right)\left(1+\frac{\theta}{3\pi-\alpha}\right)\left(1-\frac{\theta}{3\pi+\alpha}\right)\cdots\cdots\cdots$$

26. $$\frac{\sin(\alpha+\theta)}{\sin\alpha}=\left(1+\frac{\theta}{\alpha}\right)\left(1+\frac{\theta}{\pi-\alpha}\right)\left(1+\frac{\theta}{\pi+\alpha}\right)\left(1-\frac{\theta}{2\pi-\alpha}\right)\cdots\cdots\cdots$$

27. $$\frac{\cos(\alpha-\theta)}{\cos\alpha}=\left(1+\frac{2\theta}{\pi-2\alpha}\right)\left(1-\frac{2\theta}{\pi+2\alpha}\right)\left(1+\frac{2\theta}{3\pi-2\alpha}\right)$$
$$\left(1-\frac{2\theta}{3\pi+2\alpha}\right)\left(1+\frac{2\theta}{5\pi-2\alpha}\right)\left(1-\frac{2\theta}{5\pi+2\alpha}\right)\cdots\cdots\cdots$$

28. $$\frac{\cos(\alpha+\theta)}{\cos\alpha}=\left(1-\frac{2\theta}{\pi+2\alpha}\right)\left(1+\frac{2\theta}{\pi+2\alpha}\right)\left(1-\frac{2\theta}{3\pi-2\alpha}\right)\cdots\cdots\cdots$$

29. $$\frac{\cos\theta+\cos\alpha}{1+\cos\alpha}=\left\{1-\frac{\theta^2}{(\pi\pm\alpha)^2}\right\}\left\{1-\frac{\theta^2}{(3\pi\pm\alpha)^2}\right\}\left\{1-\frac{\theta^2}{(5\pi\pm\alpha)^2}\right\}\cdots\cdots\cdots$$

30. $$\frac{\cos\theta-\cos\alpha}{1-\cos\alpha}=\left(1-\frac{\theta^2}{\alpha^2}\right)\left\{1-\frac{\theta^2}{(2\pi\pm\alpha)^2}\right\}\left\{1-\frac{\theta^2}{(4\pi\pm\alpha)^2}\right\}\cdots\cdots\cdots$$

31. $$\frac{\sin\theta+\sin\alpha}{\sin\alpha}=\left(1+\frac{\theta}{\alpha}\right)\left(1+\frac{\theta}{\pi-\alpha}\right)\left(1-\frac{\theta}{\pi+\alpha}\right)$$
$$\left(1+\frac{\theta}{2\pi+\alpha}\right)\left(1-\frac{\theta}{2\pi-\alpha}\right)\cdots\cdots\cdots$$

32. $$\frac{\sin\alpha-\sin\theta}{\sin\alpha}=\left(1-\frac{\theta}{\alpha}\right)\left(1-\frac{\theta}{\pi-\alpha}\right)\left(1+\frac{\theta}{\pi+\alpha}\right)$$
$$\left(1-\frac{\theta}{2\pi+\alpha}\right)\left(1+\frac{\theta}{2\pi-\alpha}\right)\cdots\cdots\cdots$$

例題解自 25. 至 32.

(25) 由 4. 節. $sin(\alpha-\theta)=(\alpha-\theta)\left\{1-\frac{(\alpha-\theta)^2}{\pi^2}\right\}\left\{1-\frac{(\alpha-\theta)^2}{2^2\pi^2}\right\}\left\{1-\frac{(\alpha-\theta)^2}{3^2\pi^2}\right\}\cdots\cdots$

又由 4. 節. $sin\alpha=\left(1-\frac{\alpha^2}{\pi^2}\right)\left(1-\frac{\alpha^2}{2^2\pi^2}\right)\left(1-\frac{\alpha^2}{3^2\pi^2}\right)\cdots\cdots$

但 $\frac{\alpha-\theta}{\alpha}=1-\frac{\theta}{\alpha}$, $\frac{1-\frac{(\alpha-\theta)^2}{\pi^2}}{1-\frac{\alpha^2}{\pi^2}}=\frac{\pi^2-\alpha^2+2\alpha\theta-\theta^2}{\pi^2-\alpha^2}=1+\frac{2\alpha\theta}{\pi^2-\alpha^2}-\frac{\theta^2}{\pi^2-\alpha^2}$

$=\left(1-\frac{\theta}{\pi-\alpha}\right)\left(1-\frac{\theta}{\pi+\alpha}\right)$, 同樣. $\frac{1-\frac{(\alpha-\theta)^2}{2^2\pi^2}}{1-\frac{\alpha^2}{2^2\pi^2}}=\left(1-\frac{\theta}{2\pi-\alpha}\right)\left(1-\frac{\theta}{2\pi+\alpha}\right)$.

以下同樣. 故得 $\frac{sin(\alpha-\theta)}{sin\alpha}$ 之結果.

(26) 至 (28) 同樣.

(29) $\frac{cos\theta+cos\alpha}{1+cos\alpha}=\frac{cos\frac{1}{2}(\theta+\alpha)cos\frac{1}{2}(\theta-\alpha)}{cos\frac{1}{2}\alpha cos\frac{1}{2}\alpha}$,

由 4. 節. $cos\frac{1}{2}(\theta+\alpha)=\left\{1-\frac{(\theta+\alpha)^2}{\pi^2}\right\}\left\{1-\frac{(\theta+\alpha)^2}{3^2\pi^2}\right\}\left\{1-\frac{(\theta+\alpha)^2}{5^2\pi^2}\right\}\cdots\cdots$

$cos\frac{1}{2}\alpha=\left(1-\frac{\alpha^2}{\pi^2}\right)\left(1-\frac{\alpha^2}{3^2\pi^2}\right)\left(1-\frac{\alpha^2}{5^2\pi^2}\right)\cdots\cdots$

$\frac{1-\frac{(\theta+\alpha)^2}{\pi^2}}{1-\frac{\alpha^2}{\pi^2}}=\frac{\pi^2-\alpha^2-2\alpha\theta-\theta^2}{\pi^2-\alpha^2}=1-\frac{2\alpha\theta}{\pi^2-\alpha^2}-\frac{\theta}{\pi^2-\alpha^2}=\left(1+\frac{\theta}{\pi+\alpha}\right)\left(1-\frac{\theta}{\pi-\alpha}\right)$,

以下同樣. 故 $\frac{cos\frac{1}{2}(\theta+\alpha)}{cos\frac{1}{2}\alpha}=\left(1+\frac{\theta}{\pi-\alpha}\right)\left(1-\frac{\theta}{\pi+\alpha}\right)\left(1-\frac{\theta}{3\pi+\alpha}\right)\left(1-\frac{\theta}{3\pi-\alpha}\right)\cdots\cdots$

同樣. $\frac{cos\frac{1}{2}(\theta+\alpha)}{cos\frac{1}{2}\alpha}=\left(1+\frac{\theta}{\pi+\alpha}\right)\left(1-\frac{\theta}{\pi-\alpha}\right)\left(1-\frac{\theta}{3\pi+\alpha}\right)\left(1-\frac{\theta}{3\pi-\alpha}\right)\cdots\cdots$

故原式之因子, 爲 $\left\{1-\frac{\theta^2}{(\pi+\alpha)^2}\right\}\left\{1-\frac{\theta^2}{(\pi-\alpha)^2}\right\}\left\{1-\frac{\theta^2}{(3\pi+\alpha)^2}\right\}\left\{1-\frac{\theta^2}{(3\pi-\alpha)^2}\right\}\cdots\cdots$

卽 $\left\{1-\frac{\theta^2}{(\pi\pm\alpha)^2}\right\}\left\{1-\frac{\theta^2}{(3\pi\pm\alpha)^2}\right\}\cdots\cdots$

(30) 至 (32) 同樣.

33. $\tan\alpha=$

$$\frac{2}{\pi-2\alpha}-\frac{2}{\pi+2\alpha}+\frac{2}{3\pi-2\alpha}-\frac{2}{3\pi+2\alpha}+\frac{2}{5\pi-2\alpha}-\frac{2}{5\pi+2\alpha}+\cdots\cdots$$

34. $\cot\alpha=\frac{1}{\alpha}-\frac{1}{\pi-\alpha}+\frac{1}{\pi+\alpha}-\frac{1}{2\pi-\alpha}+\frac{1}{2\pi+\alpha}-\cdots\cdots$

35. $1/\sin 2\alpha=$

$$\frac{1}{2\alpha}+\frac{1}{\pi-2\alpha}-\frac{1}{2\pi+2\alpha}-\frac{1}{\pi+2\alpha}+\frac{1}{2\pi-2\alpha}+\frac{1}{3\pi-2\alpha}-\frac{1}{4\pi-2\alpha}\cdots\cdots$$

36. $\frac{\pi}{3\sqrt{3}}=1-\frac{1}{2}+\frac{1}{4}-\frac{1}{5}+\frac{1}{7}-\frac{1}{8}+\frac{1}{10}-\cdots\cdots$

37. $\frac{\pi}{2\sqrt{3}}=1-\frac{1}{5}+\frac{1}{7}-\frac{1}{11}+\frac{1}{13}-\frac{1}{17}-\frac{1}{19}-\cdots\cdots$

例題解自33.至37.

(33) $\frac{\cos(\alpha-\theta)}{\cos\alpha}=\cos\theta+\tan\alpha\sin\theta=1-\frac{\theta^2}{\lfloor 2}+\cdots\cdots+\tan\alpha\left(\theta-\frac{\theta^3}{\lfloor 3}+\cdots\cdots\right)$

此右邊 θ 之係數 $=\tan\pi$, 又於例題27.公式之右邊

θ 之係數 $=\frac{2}{\pi-2\alpha}-\frac{2}{\pi+2\alpha}+\frac{2}{3\pi-2\alpha}-\frac{2}{3\pi+2\alpha}+\frac{2}{5\pi-2\alpha}-\cdots\cdots$

(34) 可如前例用例題26.以求之.

(35) $\tan\alpha+\cot\alpha=\frac{1}{\sin\alpha\cos\alpha}=\frac{2}{\sin 2\alpha}$, 故

$\frac{1}{\sin 2\alpha}=\frac{1}{2}(\tan\alpha+\cot\alpha)$, 此可代入前上例右邊之值以求之.

(36) 於例題33.令 $\alpha=\frac{\pi}{6}$, 卽得其證.

(37) 於例題34.令 $\alpha=\frac{\pi}{6}$, 卽得其證.

38. 自圓之內切正n多角形之一角頂,至他各角頂引直線.其各直線之積爲nR^{n-1}.

39. 自圓周上之一點,至其圓外切正$2n$多角形之各邊引垂線.令爲$p_1, p_2, p_3, \cdots\cdots p_{2n-1}, p_{2n}$. 則

$$p_1 p_3 p_5 \cdots\cdots p_{2n-1} + p_2 p_4 p_6 \cdots\cdots p_{2n} = r^n/2^{n-2}.$$

例題解自 38. 至 39.

(38) 自正n多角形$A_1A_2A_3A_4\cdots\cdots A_n$之一角頂$A_1$引對角線.其第$m$之對角線爲$A_1 A_{m-1}$,對向中心之角爲$m\frac{2\pi}{n}$.

故 $A_1A_{m+1}=2R\sin\frac{m\pi}{n}$, 令$m$爲$1,2,3,\cdots\cdots, n-1$,其各對角線之連乘積,爲$(2R)^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\cdots\cdots\sin\frac{(n-1)\pi}{n}$, 但由例題5. $n\left(\frac{\sin n\alpha}{n\alpha}\right)=2^{n-1}\left(\frac{\sin\alpha}{\alpha}\right)\sin\left(\alpha+\frac{\pi}{n}\right)\sin\left(\alpha+\frac{2\pi}{n}\right)\sin\left(\alpha+\frac{3\pi}{n}\right)\cdots\cdots\sin\left(\alpha+\frac{n-1}{n}\pi\right)$, 令$\alpha$殆等於0,則$n=2^{n-1}\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\sin\frac{3\pi}{n}\cdots\cdots\sin\frac{n-1}{n}\pi$,故所求之面積$=nR^{n-1}$.

(39) 外切多角形之各邊.切於圓周之點.順次爲A,B,C,D,$\cdots\cdots$圓周上之任意一點P.順次達離圓之中心爲O.

而令角$POA=2\theta$,則PA與A點切線之交角等於θ,

又 角$POB=2\theta+\frac{2\pi}{n}$, 故PB與B點之切線之交角爲$\theta+\frac{\pi}{n}$,又BC與C點之切線之交角爲$\theta+\frac{2\pi}{n}$.自P引A點之切線之垂線,令爲$p_1$.則 $p_1=PA\sin\theta=2r\sin^2\theta$,

$p_2=2r\sin^2\left(\theta+\frac{\pi}{n}\right)$, $p_3=2r\sin^2\left(\theta+\frac{2\pi}{n}\right)$,

故 $p_1p_3p_5\cdots\cdots P_{2n-1}=(2r)^n\sin^2\theta\sin^2\left(\theta+\frac{\pi}{n}\right)\sin^2\left(\theta+\frac{2\pi}{n}\right)\cdots\cdots\sin\left(\theta+\frac{n-1}{n}\pi\right)$

$=\frac{1}{2^{n-2}}r^n\sin^2 n\alpha$(例題5.) 又 $p_2p_4p_6\cdots\cdots p_{2n}=(2r)^n\sin^2\left(\theta+\frac{\pi}{2n}\right)\sin^2\left(\theta+\frac{3\pi}{2n}\right)\cdots\cdots\sin^2\left(\theta+\frac{2n-1}{2n}\pi\right)=\frac{1}{2^{n-2}}r^n\cos^2 n\alpha$,(例題6.) 相加卽得.

40. 以圓內切多角形之各角頂爲切點,作外切多角形.則自圓周上任意一點,引內切多角形各邊之垂線.其垂線之積,等於由其點至外切多角形之各邊所引垂線之積.

41. 分圓周爲$2n$等分,引半徑.自圓周上任一點,引其連次n半徑上之垂線.則其垂線之積,等於$\frac{R^n}{2^{n-1}}\sin\phi$.但$\phi$爲一端之半徑與前一點所引半徑之交角.

例題解自 40. 至 41.

(40) ABCD………爲內切多角形.P爲圓周上之任一點.對向PA之中心角爲2α,對向PB之中心角爲2β,對向PC之中心角爲2γ.順是以下皆如是.而從P引外切多角形各邊之垂線.順次令爲$p_1,p_2,p_3,\ldots\ldots$則 $p_1=\mathrm{PA}\sin\alpha$,

$p_2=\mathrm{PB}\sin\beta$, $p_3=\mathrm{PC}\sin\gamma\ldots\ldots\ldots$

故 $p_1p_2p_3\ldots\ldots=\mathrm{PA.PB.PC}\ldots\ldots\sin\alpha\sin\beta\sin\gamma,\ldots\ldots$

又從P引內切多角形之各邊AB,BC,CD,………之垂線.順次令爲$q_1,q_2,q_3,\ldots\ldots$則 $q_1=\mathrm{PA}\sin\beta$, $q_2=\mathrm{PB}\sin\gamma\ldots\ldots$

而最後所引一邊之垂線,爲從P至最後角頂之距離乘以$\sin\alpha$,由是

$$q_1q_2q_3\ldots\ldots=\mathrm{PA.PB.PC}\ldots\ldots\sin\beta\sin\gamma\ldots\ldots\sin\alpha,$$

故如題言.

(41) 第一垂線爲$r\sin\phi$,第二垂線爲$r\sin\left(\phi+\frac{\pi}{n}\right)$,第三垂線爲$r\sin\left(\phi+\frac{2\pi}{n}\right),\ldots\ldots$故各垂線之積,爲

$$r^n\sin\phi\sin\left(\phi+\frac{\pi}{n}\right)\sin\left(\phi+\frac{2\pi}{n}\right)\ldots\ldots\sin\left(\phi+\frac{n-1}{n}\pi\right)=\frac{r^n}{2^{n-1}}\sin n\phi.$$

第拾玖編

代數函數之方程式

1. 方程式解法　代數學之方程式,以三角函數解之.已於第九編例題十一 51.至 56.示其梗概矣.而其方法爲加減式所示之根,變爲乘除式,以便於對數計算.茲編亦用此方法解二次以上之方程式.但僅就用三角函數之便利者示之.

2. 二次方程式　次所示之四種.其p, q爲正數.

〔第一〕 解 $x^2-2px+q=0$,

由二次方程式之解法, $x=p\pm\sqrt{(p^2-q)}=p\left\{1\pm\sqrt{\left(1-\frac{q}{p^2}\right)}\right\}$.

設$p^2>q$,則爲實根.而$\frac{q}{p^2}<1$, 故 $\frac{q}{p^2}=sin^2\alpha$,

故 $x=p(1\pm cos\alpha)=2p\,cos^2\frac{1}{2}\alpha$, 或 $2p\,sin^2\frac{1}{2}\alpha$,

若 $p^2<q$,則爲虛根.而$q/p^2>1$, 故 $q/p^2=sec^2\alpha$,

故 $x=p\{1\pm\sqrt{(-1)}\,tan\,\alpha\}$.

〔第二〕 解 $x^2+2px+q=0$,

與第一同樣. $p^2>q$, 則 $q/p^2=sin^2\alpha$,

$x=p\{-1\pm cos\,\alpha\}=-2p\,cos^2\frac{1}{2}\alpha$, 或 $-2p\,sin^2\frac{1}{2}\alpha$,

又 $p^2<q$, 則 $q/p^2=sec^2\alpha$, $x=p\{-1\pm(-1)\,tan\,\alpha\}$.

〔第三〕 解 $x^2-2px-q=0$,

$$x=p\pm\sqrt{(p^2+q)}=p\left\{1\pm\sqrt{\left(1+\frac{q}{p^2}\right)}\right\},$$

此方程之根常爲實數.而q/p^2比1大或小,均無妨.

故令 $\tan^2\alpha=q/p^2$, 則 $x=p\{1\pm\sec\alpha\}$.

〔第四〕 解 $x^2+2px-q=0$,

與第三同様.得 $x=p\{-1\pm\sec\alpha\}=\frac{2p\sin^2\frac{1}{2}\alpha}{\cos\alpha}$ 或 $-\frac{2p\cos^2\frac{1}{2}\alpha}{\cos\alpha}$.

3. 三次方程式 $x^3+px^2+qx+r=0$, 以 $y-\frac{1}{3}p$ 代 x, 則得消去 x^2 之項,即如次式.

$$(y-\tfrac{1}{3}p)^3+p(y-\tfrac{1}{3}p)^2+q(y-\tfrac{1}{3}p)+r=0,$$

即 $$y^3-\left(\frac{1}{3}p^2-q\right)y+\left(\frac{2}{27}p^3-\frac{1}{3}pq+r\right)=0.$$

於代數學三次方程式之解法,先消去 x^2 之項,然後解之,用三角函數解之亦同.故將無 x^2 項之三次方程式解之如次,即一般三次方程式,皆可應用也.

〔第一〕 解 $x^3-ax-b=0$, 但 $4a^3>27b^2$,

令 $x=n\cos\alpha$, 則 $n^3\cos^3\alpha-an\cos\alpha-b=0$,

即 $$\cos^3\alpha-\frac{a\cos\alpha}{n^2}-\frac{b}{n^3}=0,$$ (1) 又由第四編5.節之公式,

$$\cos^3\alpha-\frac{3\cos\alpha}{4}-\frac{\cos3\alpha}{4}=0,$$ (2) 將 (1), (2) 比較,得

$$\frac{a}{n^2}=\frac{3}{4},\quad \frac{b}{n^3}=\frac{\cos3\alpha}{4},$$

故 $n=2\sqrt{\frac{a}{3}}$, $\cos3\alpha=\frac{4b}{n^3}=\sqrt{\frac{27b^2}{4a^3}}$, 但 $\frac{27b^2}{4a^3}<1$,

即 $\cos3\alpha<1$, 故 (2) 合理而 $\cos3\alpha=\cos(2\pi\pm3\alpha)$

由是 $x=n\cos\alpha=2\sqrt{\frac{a}{3}}\cos\alpha$, $2\sqrt{\frac{a}{3}}\cos\left(\frac{2\pi}{3}\pm\alpha\right)$, 爲所求之根.

〔餘論〕 於 $x^3\pm ax-b=0$, 或 $x^3\pm ax+b=0$, 而 $4a^3>27b^2$; $\cos3\alpha<1$. 故得與前同法解之.

〔第二〕 解 $x^3-ax+b=0$, 但 $4a^3<27b^2$.

原方程式爲 $x^3-ax+b=0$, (1) 此式假定如次.

$$x^3-3\alpha\beta x+\alpha^3+\beta^3=0, \ (2)$$

比較 (1), (2). 得 $\alpha^3+\beta^3=b$. $\alpha\beta=a/3$, 即 $\alpha^3\beta^3=a^3/27$,

由代數學二次方程式之性質, α^3, β^3 爲二根之方程式, 爲

$y^2-by+\frac{a^3}{27}=0$, 故 $y=\frac{b}{2}\pm\sqrt{\left(\frac{b^2}{4}-\frac{a^3}{27}\right)}$, 但 $4a^3<27b^2$, 故

$\frac{4a^3}{27b^2}=sin^2\theta$, 故 $y=\frac{b}{2}\pm\frac{b}{2}\sqrt{(1-sin^2\theta)}=\frac{b}{2}(1\pm cos\theta)$,

故 $\alpha=\sqrt[3]{\left\{\frac{b}{2}(1+cos\theta)\right\}}=b^{\frac{1}{3}}cos^{\frac{2}{3}}\theta$, $\beta=\sqrt[3]{\left\{\frac{b}{2}(1-cos\theta)\right\}}=b^{\frac{1}{3}}sin^{\frac{2}{3}}\theta$,

又由 (2), $(x+\alpha+\beta)(x+\omega\alpha+\omega^2\beta)(x+\omega^2\alpha+\omega\beta)=0$,

但 $\omega=\frac{1}{2}(1\pm\sqrt{-3})$, 見代數學.

故 $x=-(\alpha+\beta)=-b^{\frac{1}{3}}(sin^{\frac{2}{3}}\theta+cos^{\frac{2}{3}}\theta)$.

〔餘論〕 於 $x^3-ax-b=0$, 而 $4a^3<27b^2$,

其 $x=b^{\frac{1}{3}}(sin^{\frac{2}{3}}\theta+cos^{\frac{2}{3}}\theta)$.

又於 $x^3+ax+b=0$, 而 $4a^3<27b^2$, 其

$y^2-by-\frac{a^2}{27}=0$, 故 $y=\frac{b}{2}\pm\sqrt{\left(\frac{b^2}{4}+\frac{a^3}{27}\right)}$,

故 $\frac{4a^3}{27q^2}=tan^2\theta$, 故 $y=\frac{b}{2}(1\pm sec\theta)=\frac{1}{2}b\,sec\theta(cos\theta\pm1)$,

故 $\alpha=\sqrt[3]{\left\{\frac{b}{2}sec\theta(cos\theta+1)\right\}}=b^{\frac{1}{3}}sec^{\frac{1}{3}}\theta\,cos^{\frac{2}{3}}\theta$,

$\beta=\sqrt[3]{\left\{\frac{b}{2}sec\theta(cos\theta-1)\right\}}=b^{\frac{1}{3}}sec^{\frac{1}{3}}\theta\,cos^{\frac{2}{3}}\theta$.

由是 $x=-(\alpha+\beta)=-b^{\frac{1}{3}}sec^{\frac{1}{3}}\theta(cos^{\frac{2}{3}}\theta-sin^{\frac{2}{3}}\theta)$,

〔注意〕 三次方程式之解法, 係代加德(Descaretes)氏所發明. 而用通常之三角函數以推究之. 取其內之實根.

又根之整數用上兩法解之, 多爲不盡數. 例如 $x^3-7x-6=0$ 之根. 本爲 3, −2, −1. 然用第一法解之, 則爲不盡數. 元來高次方程式之解法. 爲計算不盡根而設. 是等之不便. 無足告也.

例題三十三

1. 解 $x^3-6x+4=0$.

2. $x^3-3x-1=0$ 之三根，爲

$2\cos 20°$, $-2\sin 10°$, 及 $-2\cos 40°$.

3. $x^5-px^3+qx-r=0$ 之根爲

$2\sqrt{\frac{p}{5}}\cos\alpha$, $2\sqrt{\frac{p}{5}}\cos\left(\frac{2\pi}{5}\pm\alpha\right)$, $2\sqrt{\frac{p}{5}}\cos\left(\frac{4\pi}{5}\pm\alpha\right)$,

但 $\cos 5\alpha=\frac{r}{2}\left(\frac{5}{p}\right)^{\frac{5}{2}}$, 又 $p^2=5q$, $\left(\frac{r}{2}\right)^2 \not> \left(\frac{p}{5}\right)^2$.

4. $x^5-10x^3+20x-8=0$.

5. $x^5+5x^4-20x^2-5x+3=0$.

6. 半圓之弦 AB, BC, CD 之長爲 1,2,3. 其直徑 AD 之長爲何.

7. 半圓之直徑爲 AB, 中心爲 O, 從 AB 之 B 之方引長至 C, 又自 C 引割線 CDE. 其圓外之部分 CD, 等於半徑. 而半徑爲 1. OC $=x$, 角 AOE $=\alpha$, 則 x 之方程式，爲

$$x^3-3x-2\cos\alpha=0$$

若 $\alpha=60°$, 求 x 之值若干.

例題解自 1. 至 7.

(1) 於 $x^3-6x+4=0$, 而 $4\times 6^3>27\times 4^2$, 故由 3. 節第一,

$\cos 3\alpha=\sqrt{\frac{27(-4)^2}{4\times 6^3}}=-\sqrt{\frac{1}{2}}=\cos(\pi-\frac{1}{4}\pi)$, 故 $\alpha=\frac{\pi}{4}$, 故所求之根爲

$$2\sqrt{\frac{6}{3}}\cos\frac{\pi}{4},\quad 2\sqrt{\frac{6}{3}}\cos\left(\frac{2\pi}{3}+\frac{\pi}{4}\right),\quad 2\sqrt{\frac{6}{3}}\cos\left(\frac{2\pi}{3}-\frac{\pi}{4}\right),$$

即　2, $-\sqrt{3}-1$, $\sqrt{3}-1$, 但由例題六 1. 及 2.

(2) 與前例同樣 $\cos 3\alpha=\sqrt{\dfrac{27\times 1^2}{4\times 3^3}}=\frac{1}{2}=\cos 60°$, 故 $\alpha=20°$,

由是 x 之根爲 $2\sqrt{\dfrac{3}{3}}\cos 20°$, $2\sqrt{\dfrac{3}{3}}\cos(120°+20°)$, $2\sqrt{\dfrac{3}{3}}\cos(120°-20°)$,

即　$2\cos 20°$, $2\cos(180°-40°)$, $2\cos(90°+10°)$.

(3) 由例題四 33. $\cos^5\alpha-\dfrac{5}{4}\cos^3\alpha\,\dfrac{5}{16}\cos\alpha=\dfrac{\cos 5\alpha}{16}$,

又於原方程式,令 $x=n\cos\alpha$, 則

$$\cos^5\alpha=\frac{p}{n^2}\cos^3\alpha+\frac{q}{n^4}\cos\alpha-\frac{r}{n^5}=0,$$ 比較此兩方程式,得

$n^2=4p/5$, $n^4=10q/5$, $\cos 5\alpha=16r/n^4=\dfrac{r}{2}\Big/\left(\dfrac{p}{5}\right)^{\frac{5}{2}}$,

由此關係式, $p^2=5q$, $\left(\dfrac{r}{2}\right)^2 \not> \left(\dfrac{p}{5}\right)^{\frac{5}{2}}$.

而 $\cos 5\alpha=\cos(2\pi\pm 5\alpha)$, $n=2\sqrt{\dfrac{p}{5}}$, 從此即得其證.

(4) 由前例. -2, $2\sqrt{2}\cos 9°$, $2\sqrt{2}\cos 63°$, $2\sqrt{2}\cos 81°$, $2\sqrt{2}\cos 153°$.

(5) 同上. 但令 $x=y-1$, 則得 -3, 1.7936, .2841, $-.55754$, -3.5201.

(6) 由幾何學,角 ABD=角 ACD=90°, 故 $BD^2=AD^2-AB^2=x^2-1$,

$AC^2=AD^2=CD^2=9$, $AC\times BD=AB\times CD+BC\times AD$,

即 $\sqrt{(x^2-9)}\sqrt{(x^2-1)}=3+2x$, 即 $x^3-14x-12=0$, 從此求得正數之根,爲 $x=4.1133$.

(7) 由平面幾何學講義第二編例題 492. 角 ACE=$\frac{1}{4}$角 AOE=$\frac{1}{2}\alpha$,

DO=CD=1, CO=x, 於三角形 CDO,

$2CO\cdot CD\cos ACE=CO^2+CD^2-DO^2$, 即 $2x\cos\dfrac{\alpha}{2}=x^2+1-1$,

故 $\cos\dfrac{\alpha}{2}=\dfrac{x}{2}$, 又 $\cos\alpha=4\cos^3\dfrac{\alpha}{3}-3\cos\dfrac{\alpha}{3}$,

故 $\cos\alpha=4\left(\dfrac{x}{2}\right)^3-3\left(\dfrac{x}{2}\right)$, 即 $x^3-3x-2\cos\alpha=0$,

令 $\alpha=60°$, 則 $x^3-3x-2\times\frac{1}{2}=0$,

由是得 $x=1.8749$.

因子之應用

4. 因子之應用 於第十八編所得解之式,今示例如次.

卽次之方程式,用第十八編解之.

$x^n \pm a^n=0$, $x^{2n}+2x^n\cos\alpha+1=0$.

〔第一〕 解 $x^n-a^n=0$,

由第十八編,得 $x=\cos\frac{2r\pi}{n}\pm\sqrt{-1}\sin\frac{2r\pi}{n}$.

n 爲偶數,則 r 順次可令爲 $0,1,2,3,\ldots\ldots\ldots\ldots,\frac{n}{2}-1$,

n 爲奇數,則 r 順次可令爲 $0,1,2,3,\ldots\ldots\ldots\ldots,\frac{n-1}{2}$.

〔第二〕 解 $x^n+a^n=0$,

由第十八編,得 $x=\cos\frac{2r+1}{n}\pi+\sqrt{-1}\sin\frac{2r+1}{n}$

n 爲偶數,則 r 可順次令爲 $0,1,2,3,\ldots\ldots\ldots\ldots,\frac{n}{2}-1$,

n 爲奇數,則 r 可順次令爲 $0,1,2,3,\ldots\ldots\ldots\ldots,\frac{n-1}{2}$.

〔第三〕 解 $x^{2n}-2x^n\cos\alpha+1=0$,

由第十八編 3. 節. $x=\cos\frac{3r\pi+\alpha}{n}\pm\sqrt{-1}\sin\frac{2r\pi+\alpha}{n}$.

其 n 順次可令爲 $0,1,2,3,\ldots\ldots\ldots\ldots,n$.

例題三十四

解次各方程式.

1. $x^5-1=0$.　　　**2.** $x^6-1=0$.

3. $x^4+1=0.$　　4. $x^{20}+1=0.$

5. $x^8-\sqrt{3}+\sqrt{-1}=0.$　　6. $x^5-32=0.$

7. $x^8-2x^4\cos 60^\circ+1=0.$　　8. $x^{10}-2x^5\cos 10^\circ+1=0.$

例題解自 1. 至 8.

(1) 由 4. 節第一, $x=\cos\frac{2r\pi}{5}+\sqrt{-1}\cos\frac{2r\pi}{5}$, 但 $r=0,1,2,3,4.$

(2) 同上. $x=\pm 1,\ \frac{1}{2}(1\pm\sqrt{-3}),\ \frac{1}{2}(-1\pm\sqrt{-3}).$

(3) 由 4. 節第二, $x=\pm\frac{1}{\sqrt{2}}\pm\sqrt{-1}.$

(4) $x=\cos\frac{(2r+1)\pi}{20}\pm\sqrt{-1}\sin\frac{(2r+1)\pi}{20},$

但 $r=0, 1, 2, 3, 4, 5, 6, 7, 8, 9.$

(5) 令 $\sqrt{3}+\sqrt{-1}=r(\cos\theta+\sqrt{-1}\sin\theta)$, 則

$\sqrt{3}=r\cos\theta,\ 1=r\sin\theta,\ \therefore\ r^2=4$, 卽 $r=2,$

$\sin\theta=\frac{1}{2}\ \therefore\ \theta=\frac{\pi}{6}$, 由是

$x^5=2\left\{\cos\left(2r\pi+\frac{\pi}{6}\right)+\sqrt{-1}\sin\left(2r\pi+\frac{\pi}{6}\right)\right\}$

故 $x=2^{\frac{1}{5}}\left(\cos\frac{12r+1}{30}\pi+\sqrt{-1}\sin\frac{12r+1}{30}\pi\right),$

但 r 可令爲 $0, 1, 2, 3, 4, 5.$

(6) $x=2\left(\cos\frac{2r\pi}{5}\pm\sqrt{-1}\sin\frac{2r\pi}{5}\right)$, 但 $r=0, 1, 2, 3, 4.$

(7) 由 4. 節第三, 得 $x=\cos 15^\circ\pm\sqrt{-1}\sin 15^\circ,\ \cos 105^\circ\pm\sqrt{-2}\sin 105^\circ$

$\cos 195^\circ\pm\sqrt{-1}\sin 195^\circ,\ \cos 285^\circ\pm\sqrt{-1}\sin 285^\circ.$

(8) $x=\cos 2^\circ\pm\sqrt{-1}\sin 2^\circ,\ \cos 74^\circ\pm\sqrt{-1}\sin 74^\circ$

$\cos 146^\circ\pm\sqrt{-1}\sin 146^\circ,\ \cos 218^\circ\pm\sqrt{-1}\sin 218^\circ,\ \cos 290^\circ\pm\sqrt{-1}\sin 290^\circ,$

9. $x^{12}-2x^{6}\cos\frac{2}{3}\pi+1=0.$ **10.** $x^{10}+\sqrt{3}\,x^{5}+1=0.$

11. $x^{n}\sin n\alpha-\frac{n}{\underline{|1}}x^{n-1}\sin(n\alpha+\beta)+\frac{n(n-1)}{\underline{|2}}x^{n-2}\sin(n\alpha+2\beta)$

$$+\cdots\cdots\cdots+(-1)^{n}\sin\{n\alpha+(n-1)\beta\}=0.$$

例題解自 9. 至 11.

(9) $x=\cos\frac{3r+1}{9}\pi\pm\sqrt{-1}\sin\frac{3r+1}{9}\pi$, 但 r 可取自 0 至 11.

(10) $x^{10}-2\left(-\frac{\sqrt{3}}{2}\right)x^{5}+1=0$, 故

$x^{10}=\cos\frac{18r+1}{60}\pi+\sqrt{-1}\sin\frac{18r+1}{60}\pi$, 但取 r 可自 0 至 9.

(11) $\sin\theta=\frac{1}{2\sqrt{-1}}(e^{\theta\sqrt{-1}}-e^{-\theta\sqrt{-1}})$, 故

$$x^{n}(e^{n\alpha\sqrt{-1}}-e^{-n\alpha\sqrt{-1}})-nx^{n-1}\{e^{(n\alpha+\beta)\sqrt{-1}}-e^{-(n\alpha+\beta)\sqrt{-1}}\}$$

$$+\frac{n(n-1)}{\underline{|2}}x^{n-2}\{e^{(n\alpha+\beta)\sqrt{-1}}-e^{-(n\alpha+\beta)\sqrt{-1}}\}=0,$$

即 $e^{n\alpha\sqrt{-1}}\{x^{n}-nx^{n-1}e^{\beta\sqrt{-1}}+\frac{n(n-1)}{\underline{|2}}x^{n-1}e^{2\beta\sqrt{-1}}-\cdots\cdots\cdots\}$

$$=e^{-n\alpha\sqrt{-1}}\{x^{n}-nx^{n-1}e^{-\beta\sqrt{-1}}+\frac{n(n-1)}{\underline{|2}}x^{n-2}e^{-2\beta\sqrt{-1}}-\cdots\cdots\cdots\},$$

即 $e^{n\alpha\sqrt{-1}}(n-e^{\beta\sqrt{-1}})^{n}x-e^{-\beta\sqrt{-1}}(x-e^{-\beta\sqrt{-1}})^{n}$,

但 $1=\left(\cos\frac{2r+2}{n}\pi+\sqrt{-1}\sin\frac{2r+2}{n}\pi\right)=\left(e^{\frac{2r+2}{n}\pi\sqrt{-1}}\right)^{n}$,

$e^{\alpha\sqrt{-1}}(x-e^{\beta\sqrt{-1}})=e^{-\alpha}(x-e^{-\beta\sqrt{-1}}e^{\frac{2r+2}{n}\pi\sqrt{-1}}$ 但 $\frac{\pi}{n}=\phi$, $r+1=m$,

故 $x=\dfrac{e^{(\alpha+\beta-m\phi)\sqrt{-1}}-e^{-(\alpha+\beta-m\alpha)\sqrt{-1}}}{e^{(\alpha-m\phi)\sqrt{-1}}-e^{-(\alpha-m\phi)\sqrt{-1}}}=\dfrac{\sin(\alpha+\beta-m\phi)}{\sin(\beta-m\phi)}$

平面三角法講義勘誤表

頁數	列數	原文	訂正
1	11	乃用法國	乃法國所用
		今不用	今亦不用
2	21	及 P2n	及 P_{2n}
	25	P4	P_4
	26	P8	P_8
3	1	P16	P_{16}
	2	P32	P_{32}
	3	P64	P_{64}
	4	P128	P_{128}
	5	P256	P_{256}
	6	P512	P_{512}
	7	P1024	P_{1024}
	8	P2048	P_{2048}
	9	P4096	P_{4096}
5	8	由下節定理	由7節定理
11	11	$sin\,a-$	$sin\,a=$
67	15	又得其證	可得其證
112	13	試以	各以
128	16	$a\,tan(\theta-\alpha)$	$a\,tan(\theta-\alpha)$
172	3	$3(cos(\theta-\alpha)$	$3\,cos(\theta-\alpha)$
188	5	例	則
205	12	$sin^{-1}n=\theta,\ sin^{-1}\frac{x}{2}=\phi)$	$sin^{-1}x=\theta\ sin^{-1}\frac{x}{2}=\phi$
211	8	試化此	試化

平面三角法講義勘誤表

頁數	列數	原文	訂正
213	6	$=a^2-a^2$	$=a^2-b^2$
	7	$sin^{-2}\frac{a^2-b^2}{a^2+b^2}$	$sin^{-1}\frac{a^2-b^2}{a^2+b^2}$
215	15	$=p1 \quad =p2$	$=p_1 \quad =p_2$
	16	$=p3$	$=p_3$
	17	$=\frac{p1-p3}{1-p2}$	$=\frac{p_1-p_3}{1-p_2}$
	18	$\frac{p1-p3+p5}{1-p2+p4}$	$\frac{p_1-p_3+p_5}{1-p_2+p_4}$
	19	$\frac{p1-p3+p5\ldots\ldots}{1-p2+p4\ldots\ldots}$	$\frac{p_1-p_3+p_5}{1-p_2+p_4}$
228	6	$cos\frac{a}{3}\ cos\frac{a}{4}\ldots\ldots$	$cos\frac{a}{2}\ cos\frac{a}{4}\ldots\ldots$
286	23	令爲 B_1, B	令爲 B_1, B_2
299	3	測量之高及……	測量高及……
336	13	$\frac{1}{2s}\left(a.\frac{b^2+c^2-a^2}{2bc}\right)+$	$\frac{1}{2s}\left(a.\frac{b^2+c^2-a^2}{2bc}+\right.$
347	11	$=S^2\sqrt{\left\{\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}\right)\left(\right.}$	$=S^2\sqrt{\left\{\left(\frac{1}{h_1}+\frac{1}{h_2}+\frac{1}{h_3}\right)\left(\right.\right.}$
365	7	=角 BPC−A	=角 BPC−角 A
400	4	三角形法解之學科	三角形解法之學科
404	5	$cos\frac{1}{2}n(\theta+\phi)$	$cos^n\frac{1}{2}(\theta+\phi)$
405	8	$cos\frac{1}{2}n(\theta+\phi)$	$cos^n\frac{1}{2}(\theta+\phi)$
412	9	$(\sqrt{-1})^4=-1$	$(\sqrt{-1})^4=1$
431	5	右邊之係數	右邊 h 之係數
	9	$=-1-A_0+2A_2\,cos\,x$	$=-1-A_0+2A_1\,cos\,x$
484	1	$+tan^{-1}\frac{6}{1+2+2^2}$	$+tan^{-1}\frac{1}{1+2+2}$
序1	11	三角之之形	三角之形

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