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A System of Equations Having No Nontrivial Solutions

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The object of this note is to prove the THEOREM: *The system of equations*

$$a_1^r + a_2^r + \ldots + a_{n-1}^r = b_1^r + b_2^r + \ldots + b_{n-1}^r, \quad r = 2, 3, \ldots, n;$$

has no nontrivial solutions in positive integers.

Key Words: Diophantine equations, Prouhet-Terry-Escott problem, symmetric functions.

1. The object of this note is to prove the THEOREM: *The system of equations*

$$a_1^r + a_2^r + \ldots + a_{n-1}^r = b_1^r + b_2^r + \ldots + b_{n-1}^r, \quad r = 2, 3, \ldots, n; \quad (1)$$

has no nontrivial solutions in positive integers.

In what follows, we write

A_r for $a_1^r + a_2^r + \ldots + a_{n-1}^r$, $\quad \cdot \geq 1$;

B_r for $b_1^r + b_2^r + \ldots + b_{n-1}^r$, $\quad r \geq 1$;

and all small letters denote integers ≥ 0 unless stated otherwise.

2. PROOF OF THE THEOREM: Let $a_1, a_2, \ldots, a_{n-1}$ be roots of the equation

$$x^{n-1} - \lambda_1 x^{n-2} + \lambda_2 x^{n-3} - \ldots + (-1)^{n-1}\lambda_{n-1} = 0; \quad (2)$$

so also let $b_1, b_2, \ldots, b_{n-1}$ be roots of the equation

$$y^{n-1} - \mu_1 y^{n-2} + \mu_2 y^{n-3} - \ldots + (-1)^{n-1}\mu_{n-1} = 0. \quad (3)$$

Then, we have

$$A_r - \lambda_1 A_{r-1} + \lambda_2 A_{r-2} - \ldots + (-1)^r r\lambda_r = 0, \quad r = 1, 2, \ldots, n, \ldots \quad (4)$$

with $\lambda_r = 0 \quad$ for $r \geq n$.

From (4) we have

$$r!\lambda_r = \begin{vmatrix} A_1 & 1 & 0 & 0 & 0 & \cdot & 0 \\ A_2 & A_1 & 2 & 0 & 0 & \cdot & 0 \\ A_3 & A_2 & A_1 & 3 & 0 & \cdot & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ A_{r-1} & A_{r-2} & A_{r-3} & A_{r-4} & A_{r-5} & \cdot & r-1 \\ A_r & A_{r-1} & A_{r-2} & A_{r-3} & A_{r-4} & \cdot & A_1 \end{vmatrix}, \quad (5)$$

with a similar expression for $r!\mu_r$.

Without loss of generality, we can take

$$A_1 = B_1 + h, \qquad h \geq 0. \quad (6)$$

Writing c_r for $r!\lambda_r$, and d_r for $r!\mu_r$, we then get

$$c_1 = d_1 + h;$$

$$c_2 = A_1^2 - A_2 = B_1^2 + 2hB_1 + h^2 - B_2 = d_2 + 2hd_1 + h^2,$$

$$c_3 = A_1^3 - 3A_1A_2 + 2A_3,$$

$$= (B_1 + h)^3 - 3(B_1 + h)B_2 + 2B_3,$$

$$= d_3 + 3hd_2 + 3h^2d_1 + h^3.$$

Assuming that

$$c_k = d_k + \binom{k}{1} hd_{k-1} + \binom{k}{2} h^2 d_{k-2} + \ldots + \binom{k}{k} h^k,$$

for $k = 1, 2, 3, \ldots, r-1$, (4) gives

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$$B_r-\frac{1}{1!}(d_1+h)B_{r-1}+\frac{1}{2!}(d_2+2hd_1+h^2)B_{r-2}-\ldots$$

$$+\frac{(-1)^{r-1}}{(r-1)!}\left\{d_{r-1}+\binom{r-1}{1}hd_{r-2}\right.$$

$$\left.+\ldots+\binom{r-1}{r-1}h^{r-1}\right\}(B_1+h)$$

$$+\frac{(-1)^r}{r!}rc_r=0, \qquad n\geqslant 2.$$

Since

$$B_k-\frac{d_1}{1!}B_{k-1}+\frac{d_2}{2!}B_{k-2}-\ldots+\frac{(-1)^{k-1}}{(k-1)!}d_{k-1}B_1$$

$$+\frac{(-1)^k}{k!}kd_k\equiv 0, \qquad k>1;$$

we get

$$c_r=d_r+\binom{r}{1}hd_{r-1}+\ldots+\binom{r}{r}h^r. \tag{7}$$

Symbolically, this result can be written in the form

$$c_r\simeq(d+h)_r, \qquad r=1, 2, \ldots, n-1, n, \ldots$$

In particular

$$c_n=d_n+\binom{n}{1}hd_{n-1}+\ldots+\binom{n}{n}h^n.$$

Since $c_n=0=d_n$, we must have

$$h\left\{\binom{n}{1}d_{n-1}+\ldots+\binom{n}{n}h^{n-1}\right\}=0. \tag{8}$$

As the expression in the curly brackets is positive, (8) can hold only with $h=0$. This shows that $c_i=d_i$, i.e., $\lambda_i=\mu_i$ for $i=1, 2, 3, \ldots, n-1$. Hence eqs (2) and (3) have the same roots and a's are b's in some order.

Alternatively

$$A_n=\lambda_1A_{n-1}-\lambda_2A_{n-2}+\ldots+(-1)^{n-2}\lambda_{n-1}A_1,$$

$$=\frac{c_1}{1!}A_{n-1}-\frac{c_2}{2!}A_{n-2}+\ldots+(-1)^{n-2}\frac{c_{n-1}}{(n-1)!}A_1,$$

$$=\frac{1}{1!}(d_1+h)B_{n-1}-\frac{1}{2!}(d_2+2hd_1+h^2)B_{n-2}+\ldots$$

$$+\frac{(-1)^{n-2}}{(n-1)!}\left\{d_{n-1}+\binom{n-1}{1}hd_{n-2}+\ldots\right.$$

$$\left.+\binom{n-1}{n-1}h^{n-1}\right\}(B_1+h),$$

$$=B_n+\frac{h}{(n-1)!}\left\{\binom{n}{1}d_{n-1}+\binom{n}{2}hd_{n-2}+\ldots\right.$$

$$\left.+\binom{n}{n}h^{n-1}\right\}.$$

Since $A_n=B_n$, the result follows as before.

3. H. Schmidt[1] has proved a special case of our theorem when $n=3$.

Proceeding on the same lines as we have done in section 2, it can be proved that any $(n-1)$ of the n equations

$$A_r=B_r, 1\leqslant r\leqslant n,$$

have only nontrivial solutions.

[1] H. Schmidt. Über einige diophantische Aufgaben 3 und 4 Grades. Jber. Deutsch Math. Verein **67** (1964/65) Abt. 1, 2–13.

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