DE

FUNCTIONE TRANSCENDENTE

QUAE LITTERA $\Gamma()$ OBSIGNATUR:

SIVE

DE INTEGRALI EULERIANO

SECUNDAE SPECIEI.

DISSERTATIO

INAUGURALIS MATHEMATICA

QUAM

CONSENSU ET AUCTORITATE

ORDINIS AMPLISSIMI PHILOSOPHORUM

IN

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Prooemium.

Conscripturus dissertationem cum diu dubitavissem, quam potissimum rem tractandam eligerem, operae pretium fore visum est, quae de integrali Euleriano secundae speciei, a clarissimo Legendre littera $\Gamma(\;)$ designato hucusque innotuere, ea summa, qua opus est, cum strenuitate discutere. Neglecta enim intuitione geometrica, in integralibus trium infimorum ordinum adhibenda, quae quamquam necessaria non est, tamen ad viam rectam tenendam permultum iuvat, cum in omni integralium definitorum theoria tum in theorematibus quibusdam ad functionem $\Gamma(\;)$ pertinentibus haud leves plerumque errores orti sunt, qui ut emendentur, cum studio agendum. Accedit, quod quae de functione nostra ab aliis alia ratione inventa et docta sunt, ea omnia iacent diffusa huc illuc quasi prostrata, quae est res sane molesta, quum integralis illius usus in tot tam diversis discussionum generibus amplissimus et in dies frequentior futurus sit, exquo tempore ill. Lejeune-Dirichlet *) et volumina et gravitatis centra et momenta inertiae corporum permultorum

*) Ueber eine neue Methode zur Bestimmung vielfacher Integrale von Herrn Lejeune-Dirichlet; vorgelesen in der Academie der Wissenschaften am 14. Februar 1839.

in integrale nostrum reduci posse docuit. Hic igitur mihi finis propositus erat, ut quae de functione $\Gamma(\ )$ extarent, colligerentur, distribuerentur, a vitiis sibi inhaerentibus purgarentur. Quae egomet ipse addidi, pauca sunt, spectantque cum ad integralia definita nonnulla in physice mathematica saepe occurrentia, ab integrali Euleriano nostro facillimis substitutionibus derivanda, tum vero ad connexum, quo singula huc pertinentia theoremata inter se continentur, quam maxime dilucidandum. Quod etiamsi in hoc meo labore multa sint, quae lectoribus doctis parum probentur, rogatos eos esse voluerim et iuvenilem aetatem meam respiciant et rem esse ad exponendum difficillimam.

Valorem integralis definiti

$$\int_0^\infty e^{-x} x^{a-1} dx$$

quem a sola constante quantitate (a) pendere pateat, Legendri exemplum secuti per Γ(a) obsignamus, etiamsi forsitan signum Π(a-1) ab illmo. Gauss introductum praeferendum esse videatur. In hoc intregrali dilucidando tria potissimum evolutionum genera distingui oportet. Primum enim sermo futurus est de proprietatibus functionis nostrae maximam partem notissimis; deinde exponendum, qua facillime ratione calculus illius numericus instituatur; disseremus denique de integralibus multis spectatissimis, quae expeditissimis plerumque substitutionibus ex integrali nostro derivari possunt.

I.

Hac in prima disquisitionis nostrae parte statim nobis occurrit quaestio, quae sint postulandae conditiones, ut functionis valor pro aliquo elemento (a) finitus sit.

Jam statuendo $e^{-x} = \frac{1}{z}$, erit:

$$\Gamma(a) = \int_1^\infty \frac{(\log\, z)^{a-1}}{z^2}\, dz$$

ex quo facillime, intuitione geometrica adiutrice, intelligitur, nisi forte numerus constans (a) ipse infinitus evadat, valorem ipsius Γ(a) finitum esse futurum. Neque vero constantem (a) negativum, ne nullitati quidem aequalem esse licere ex ista transformatione elucet. Quod clarius etiam hoc modo perspicitur. Per partes, ut aiunt, integrando fit:

$$\int e^{-x} x^{a} dx = - e^{-x} x^{a} + a \int e^{-x} x^{a-1} dx$$

ac perinde introducendo limites 0 et ∞

$$\int_0^\infty e^{-x} x^{a} dx = a \int_0^\infty e^{-x} x^{a-1} dx$$

siquidem valor ipsius (a) positivus nullitateque maior est; nam si numerus (a) negativus esset, producti fractionisve $e^{-x} x^{a}$ valor pro datis limitibus omnem limitem excederet. Hinc imposterum sermo non erit nisi de valoribus positivis nullitatemque superantibus ipsius (a), pro quibus secundum equationemi praecedentem hacc insignis habetur relatio:

$$\Gamma(a+1) = a\,\Gamma(a) \tag{1}$$

Quum autem pateat esse et

$$\Gamma(1) = 1$$

et pro quolibet numero positivo integro n

$$\Gamma(a+n) = a(a+1)(a+2)\ldots.\ (a+n-1).\ \Gamma(a) = [a,-1]^{n}\ \Gamma(a) \tag{2}$$

pro a = 1 prodit:

$$\Gamma(n+1) = n! \tag{3}$$

Hinc intimum esse functionem nostram inter facultatesque analyticas connexum ultro perspicitur, quam tamen rem, nuperrime a doctissimo matheseos candidato Weierstrass [1]) diligenter et accurate expositam hoc loco silentio praetermittimus.

Antequam vero ad alias functionis $\Gamma(\ )$ proprietates investigandas accedamus, animadvertendum est, et hanc aliam integralis nostri formam:

$$\Gamma(a) = \int_0^1 \left(\log \frac{1}{x}\right)^{a-1} dx \text{ sive } \Gamma(a) = \int_0^1 \left(\log \frac{1}{(1-x)}\right)^{a-1} dx$$

n scriptis Euleri praesertim et Krampii nobis occurrere, quarum altera ex primitivo integrali impetratur, statuendo log $\frac{1}{x}$ loco ipsius x, altera autem ex hac ipsa com-

[1]) Weierstrass, Ueber die Facultäten. Abhandlung im Programm des Gymnasium zu Deutsch-Krone, Schuljahr $18^{42}/_{43}$.

mutando x in 1—x sponte exit: et ab illmo. Liouville[1]) integrale $\Gamma(a)$ ita esse definitum, ut esset

$$\Gamma(a) = \frac{\Gamma(a+1)}{a}$$

quo facto functio $\Gamma(\ )$ extendi posset et in valores negativos elementi (a), dummodo exciperentur numeri integri,

$$0, -1, -2, -3, -4 \ldots.$$

Sed talis definitio, quamquam apta fuit ad autoris clarissimi quaestiones, continuitatis retinendae causa generaliter nobis non probatur.

Jam igitur reverti placet ad integrale $\Gamma(\ )$ magis etiam discutiendum, quod mutato valore ipsius x in kx transit in hoc:

$$\int_0^\infty e^{-kx} x^{a-1} dx = \frac{\Gamma(a)}{k^a} \tag{4}$$

litera k designante numerum positivum.

Illinc statutis

$$a = b + c\ ;\ k = 1 + n$$

multiplicando per $n^{b-1} dn$ alteraque integratione intra eosdem limites facienda nactum iri videmus:

$$\int_0^\infty\int_0^\infty e^{-(1+n)x}\, n^{b-1}\, x^{b+c-1}\, dx\, dn = \Gamma(b+c) \int_0^\infty \frac{n^{b-1}\, dn}{(1+n)^{b+c}}$$

Quum autem illud secundi ordinis integrale etiam ita possit exprimi:

$$\int_0^\infty n^{b-1} e^{-nx} dn \int_0^\infty e^{-x} x^{b+c-1} dx = \Gamma(b) \int_0^\infty \frac{e^{-x} x^{b+c-1}}{x^b} dx = \Gamma(b)\,\Gamma(c)$$

„integrale Eulerianum primae speciei" a Legendro nominatum

$$\int_0^\infty \frac{x^{b-1}\, dx}{(1+x)^{b+c}} = \frac{\Gamma(b)\,\Gamma(c)}{\Gamma(b+c)} \tag{5}$$

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[1]) Liouville, Mémoire sur le théorème des fonctions complémentaires. Crelle's Journal für Mathematik, Band XI, pag. 1 et sqq.

impetratur, quod immortalis Eulerus ipse nonnullique analytici post eum per

$$\left(\frac{b}{c}\right)$$

obsignaverunt, ita ut $\left(\frac{b}{c}\right) = \left(\frac{c}{b}\right)$ sit, alii autem maiore cum securitate per

(b,c).

Illi theorematí (5) maxime memorabili, per quod functio transcendens a duobus elementis pendens qua maxima potest cum simplicitate atque elegantia exprimitur, vel quo integralium Eulerianorum altera species per alteram exhibetur, alia etiam magis symmetrica forma dari potest. Fit enim statuendo $\frac{1}{x} - 1$ loco variabilis x

$$\int_0^1 (1-x)^{b-1} x^{c-1} dx = \frac{\Gamma(b)\ \Gamma(c)}{\Gamma(b+c)} \tag{6}$$

b et c designantibus numeros positivos.

Ill. Jacobi [1]) in commentatiuncula idem theorema, ad clmo. Poisson [2]) primo strenue demonstratum ita deduxit, ut duobus integralibus:

$$\Gamma(b) = \int_0^\infty e^{-x} x^{b-1} dx\ ,\ \Gamma(c) = \int_0^\infty e^{-y} y^{c-1} dy$$

inter se multiplicatis statueret:

$$x + y = r\ ,\ x = rw.$$

Quam nos secuti sumus demonstrandi rationem, ea maxime accommodatur ratiociniis illmi Dirichlet [3]) in fine commentationis egregiae: „Sur les intégrales Eulériennes."

Integralia (5) et (6) forma tantum inter se differenti, si $c = 1 - b$, ideoque $b < 1$ statueris, ultro transeunt in haec:

$$\int_0^\infty \frac{x^{b-1}}{1+x} dx = \Gamma(b)\ \Gamma(1-b) \tag{7}$$

$$\int_0^1 \frac{(1-x)^{b-1}}{x^b} dx = \int_0^1 \frac{x^{b-1}}{(1-x)^b} dx = \Gamma(b)\ \Gamma(1-b) \tag{8}$$

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1) Crelle's Journal f. M. Band XI, pag. 307.

2) Journal de l'Ecole polytechnique, 19 ième cahier pag 477.

3) Crelle's Iournal f. M. Band XV, pag. 258 et sqq.

Producti autem $\Gamma(b)\ \Gamma(1-b)$ valor, quae est res nota tudignissima, notis aliis functionibus exprimi potest. Etenim cum habeatur:

$$\int_0^\infty \frac{x^{b-1}}{1+x}dx = \int_0^1 \frac{x^{b-1}}{1+x}dx + \int_1^\infty \frac{x^{b-1}}{1+x}dx$$

quorum in dextera aequationis parte integralium quod est posterius substituto $\frac{1}{x}$ loco ipsius x transit in hoc:

$$\int_0^1 \frac{x^{-b}}{1+x}dx$$

cumque sit:

$$\frac{x^{b-1}}{1+x} = x^{b-1} - x^{b} + x^{b+1} - x^{b+2} + - \text{.. in inf.}$$

$$\frac{x^{-b}}{1+x} = x^{-b} - x^{-b+1} + x^{-b+2} - x^{-b+3} + - \text{.. in inf.}$$

has series per dx multiplicatas intra dasos limites integrando fit:

$$\Gamma(b)\ \Gamma(1-b) = \left\{\frac{1}{b} - \frac{1}{b+1} + \frac{1}{b+2} - \frac{1}{b+3} + - \ldots\right\} + \left\{\frac{1}{-b+1} - \frac{1}{-b+2} + \frac{1}{-b+3} - \frac{1}{-b+4} + - \ldots\right\}$$

$$= \frac{1}{b} + \frac{2b}{1^2-b^2} - \frac{2b}{2^2-b^2} + \frac{2b}{3^2-b^2} - + \ldots$$

$$= \frac{1}{b} + \sum_{\alpha\,0}^{\alpha\,\infty}(-1)^{\alpha}\frac{2b}{(\alpha+1)^2-b^2}$$

Notissimum autem est, ab illmo. Gudermann [1], hanc esse datam tangentis cyclicae in productum ex innumeris factoribus compositum evolutionem:

$$\text{tang}\,\frac{b\pi}{2} = b\prod_{\alpha:0}^{\alpha:\infty}\left\{\frac{(2\alpha+2)^2-b^2}{(2\alpha+1)^2-b^2}\right\}$$

unde sequitur esse:

$$\log\,\text{tang}\,\frac{b\pi}{2} = \log\,b + \sum_{\alpha:0}^{\alpha:\infty}\log\left\{(2\alpha+2)^2-b^2\right\} - \sum_{\alpha:0}^{\alpha:\infty}\log\left\{(2\alpha+1)^2-b^2\right\}$$

quae aequatio secundum quantitatem b differentiata fit:

$$\frac{\pi}{\sin b\pi} = \frac{1}{b} - \sum_{\alpha:0}^{\alpha:\infty}\frac{2b}{(2\alpha+2)^2-b^2} + \sum_{\alpha:0}^{\alpha:\infty}\frac{2b}{(2\alpha+1)^2-b^2}$$

[1]) Gudermann Theorie der Potentialfunctionen, pag. 69.

vel

$$\frac{\pi}{\sin b\pi} = \frac{1}{b} + \sum_{\alpha=0}^{\alpha=\infty} (-1)^{\alpha} \frac{2b}{(\alpha+1)^2-b^2}$$

ita ut sit

$$(9) \qquad \Gamma(b)\,\Gamma(1-b) = \frac{\pi}{\sin b\pi}$$

Quae relatio praeter ceteras insignis, iam Eulero olim probata, cum sit

$$b\,\Gamma(b) = \Gamma(b+1)$$

per hanc etiam magis symmetricam aequationem exhiberi potest:

$$\Gamma(1+b)\,\Gamma(1-b) = \frac{b\pi}{\sin b\pi}$$

quam ipsam valere constat pro omnibus valoribus numeri b inter limites $(+1)$ et (-1) sitis, limites hos ipsos modo excludas.

Ex ista relatione (9), quam magis infra, cum de Gaussiana functionis nostrae in-interpretatione disseretur, alia ratione demonstrabimus, pro $b=\frac{1}{2}$ exire patet:

$$(10) \qquad \Gamma(\tfrac{1}{2}) = +\sqrt{\pi}$$

qui valor particularis cum in omni nostra discussione maximi momenti sit, ut alio etiam modo a nobis eruatur liceat velimus. Jam igitur in aequatione (6) posito $b=c=\frac{1}{2}$ substitutoque x^2 loco ipsius x continetur haec:

$$\int_0^1 \frac{dx}{\sqrt{(1-x^2)}} = \tfrac{1}{2}[\Gamma(\tfrac{1}{2})]^2$$

cuius integralis cum sit valor

$$\text{arc sin }(1) - \text{arc sin }(0) = \frac{\pi}{2}$$

— neque enim aliorum arcuum admitti debere delectum apparet — relationem (10) impetrari videmus.

Jam cum sciamus, esse

$$\Gamma\left(\frac{a}{n}\right)\Gamma\left(\frac{n-a}{n}\right) = \frac{\pi}{\sin\frac{a}{n}\pi}$$

qiquidem n et a numeros designant positivos integros, quorum maior est n; quaeritur, suis tandem valor futurus sit producti

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\Gamma\left(\frac{3}{n}\right)\cdots\Gamma\left(\frac{n-1}{n}\right).$$

Esto ergo

$$\prod_{\alpha=1}^{\alpha=n-1} \Gamma\left(\frac{\alpha}{n}\right) = N$$

Si hanc aequationem factorum ordine inverso per se ipsam multiplicaveris, respectu aequationis (9) nancisceris:

$$N^2 = \frac{\pi^{n-1}}{\sin\frac{\pi}{n}.\sin\frac{2\pi}{n}.\sin\frac{3\pi}{n}\ldots\sin\frac{n-1}{n}\pi}$$

vel, quia $\sin.\frac{m\pi}{n} = 2\sin\frac{m\pi}{2n}\cos\frac{m\pi}{2n}$ est,

$$N^2 = \frac{\pi^{n-1}}{2^{n-1}\prod_1^{n-1}\sin\frac{\alpha\pi}{2n}\prod_1^{n-1}\cos\frac{\alpha\pi}{2n}}$$

relatis indicibus (n-1) et (1) ad variabilem numerum α.

At cum

$$\cos\frac{\alpha\pi}{2n} = \sin\left(\frac{\pi}{2} - \frac{\alpha\pi}{2n}\right) = \sin\left(\frac{n-\alpha}{2n}\right)\pi$$

sit cumque tribuendo ipsi α omnes deinceps valores integros inde ab $\alpha = 1$ usque ad $\alpha = n - 1$, $\sin\left(\frac{n-\alpha}{2n}\right)\pi$ transeat in $\sin.\frac{\alpha\pi}{2n}$, erit:

$$N^2 = \frac{\pi^{n-1}}{2^{n-1}\prod_1^{n-1}\sin.^2\frac{\alpha\pi}{2n}} = \frac{\pi^{n-1}}{\prod_1^{n-1}\left(1-\cos\frac{\alpha\pi}{n}\right)}$$

Quod ad productum numerice determinandum proficiscimur ab aequatione

$$x^{2n} - 1 = o$$

cuius 2n radices ad unam omnes contentae sunt in

$$x - e^{\frac{\alpha}{n}\pi i}$$

siquidem

$$i = \sqrt{-1}$$

ita ut sit:

$$\prod_1^{2-n}\left(x - e^{\frac{\alpha}{n}\pi i}\right) = o$$

aut separatis radicibus duabus realibus, quas respondere constat valoribus n et 2n ipsius α:

$$x^{2n-2}+x^{2n-4}+x^{2n-6}+\dots+x^2+1=\prod_{1}^{n-1}\left(x-e^{\frac{\alpha}{n}\pi i}\right)\prod_{n+1}^{2n-1}\left(x-e^{\frac{\alpha}{n}\pi i}\right)$$

Sed quia alterum ex his productis statuto $2n-\alpha$ loco ipsius α ita quoque potest exhiberi:

$$\prod_{n+1}^{2n-1}\left(x-e^{\frac{a}{n}\pi i}\right)=\prod_{1}^{n-1}\left(x-e^{2\pi i}\,.\,e^{-\frac{\alpha}{n}\pi i}\right)=\prod_{1}^{1}\left(x-e^{-\frac{\alpha}{n}\pi i}\right)$$

erit:

$$x^{2n-2}+x^{2n-4}+\dots+x^2+1=\prod_{1}^{n-1}\left(x^2-2x\cos\frac{\alpha}{n}\pi+1\right)$$

ponendoque $x=1$

$$n=2^{n-1}\prod_{1}^{n-1}\left(1-\cos\frac{\alpha}{n}\pi\right)$$

Hinc fit:

$$N^2=\frac{(2\pi)^{n-1}}{n}$$

vel

$$\Gamma\left(\frac{1}{n}\right)\Gamma\left(\frac{2}{n}\right)\dots\Gamma\left(\frac{n-1}{n}\right)=\frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$$

Quae functionis nostrae proprietas elegantissima contenta haberi potest in hac multo generaliori aequatione:

$$\Gamma(a)\,\Gamma\left(a+\frac{1}{n}\right)\dots\dots\Gamma\left(a+\frac{n-1}{n}\right)=(2\pi)^{\frac{n-1}{2}}\,.\,n^{\frac{1}{2}-na}\,\Gamma(na)$$

pro quovis positivo numero (a) iusta, dummodo n sit integer positivus. Illam demonstraturi initium capimus ab integrali definito

$$\int_0^\infty e^{-tx}dx=\frac{1}{t}$$

quod si multiplicaveris per dt, integratione intra limites 1 et t facta accipitur:

$$\int_1^t\int_0^\infty e^{-tx}dxdt=\log t$$

Quum autem sit

$$\int_1^t e^{-tx}dt=\frac{1}{x}\left(e^{-x}-e^{-tx}\right)$$

erit:

$$\log t = \int_0^\infty \frac{e^{-x} - e^{-tx}}{x}\, dx$$

Ex aequatione autem

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx,$$

theoremate adhibito uberrimo illo, quod Leibnitius, immortalis eius auctor „differentiationis de curva in curvam“ appellavit, significando $\frac{d\Gamma(a)}{da}$ per $\overset{1}{\Gamma}(a)$, deducitur

$$\overset{1}{\Gamma}(a) = \int_0^\infty e^{-x} x^{a-1} \log x \, dx$$

vel quod est artificium usitatissimum, introducendo valore ipsius log x modo invento

$$\overset{1}{\Gamma}(a) = \int_0^\infty \int_0^\infty e^{-x} x^{a-1} \left(\frac{e^{-y} - e^{-xy}}{y} \right) dx dy$$

— litteram y adhibere manifesto licet, quippe quae non sit nisi operationis quasi sustentrix — sive

$$\int_0^\infty \frac{e^{-y}}{y} dy \int_0^\infty e^{-x} x^{a-1} dx - \int_0^\infty \frac{dy}{y} \int_0^\infty e^{-x(1+y)} x^{a-1} dx$$

ideoque

$$\overset{1}{\Gamma}(a) = \Gamma(a) \int_0^\infty \left(e^{-y} - \frac{1}{(1+y)^a} \right) \frac{dy}{y}$$

id est

$$\frac{d \log \Gamma(a)}{da} = \int_0^\infty \left(e^{-y} - (1+y)^{-a} \right) \frac{dy}{y}$$

Quodsi statuerimus

$$y = \frac{1}{x} - 1$$

erit

$$\frac{d \log \Gamma(a)}{da} \int_0^1 \frac{dx}{x(1-x)} \left(e^{1-\frac{1}{x}} - x^a \right)$$

unde, substitutis deinceps $a + \frac{1}{n}$, $a + \frac{2}{n}$, . . . $a + \frac{n-1}{n}$ loco constantis (a), singularum harum aequationum additione oritur

$$\frac{d \log \psi(a)}{da} = \int_0^1 \frac{dx}{x} \left(\frac{n e^{1-\frac{1}{x}}}{1-x} - \frac{x^a}{1-x^{1/n}} \right)$$

brevitatis causa posito

$$\psi(a) = \Gamma(a)\Gamma(a+\frac{1}{n})\ \Gamma(a+\frac{2}{n}) .. \ \Gamma(a+\frac{n-1}{n})$$

Itaque quum etiam sit:

$$\frac{d \log \Gamma(na)}{da} = n \int_0^1 \frac{dx}{x(1-x)} \left(e^{1-\frac{1}{x}} - x^{na} \right)$$

vel immutato x in $x^{\frac{1}{n}}$

$$\frac{d \log \Gamma(na)}{da} = n \int_0^1 \frac{dx}{x} \left(\frac{e^{1-\frac{1}{x^{1/n}}}}{1-x^{1/n}} - \frac{x^a}{1-x^{1/n}} \right)$$

$$\frac{d \log \left\{ \frac{\psi(a)}{\Gamma(na)} \right\}}{da} = \int_0^1 \frac{dx}{x} \left(\frac{n e^{1-\frac{1}{x}}}{1-x} - \frac{e^{1-\frac{1}{x^{1/n}}}}{1-x^{1/n}} \right)$$

ideoque, cum hujus aequationis pars dextera a quantitate (a) omnino libera sit,

$$\log \left\{ \frac{\psi(a)}{\Gamma(na)} \right\} = ca + c'$$

designantibus c et c′ numeros constantes ad functionis indolem determinandos. Quod ut assequamur, hanc ingredimur viam:

Est

$$\psi(a) = \Gamma(na) . e^{ca + c'}$$

adeoque

$$\psi(a + \frac{1}{n}) = \Gamma(na + 1)\, e^{ca + \frac{c}{n} + c'}$$

cumque sit

$$\frac{\psi(a+\frac{1}{n})}{\psi(a)} = \frac{\Gamma(a+1)}{\Gamma(a)} = a, \frac{\Gamma(na+1)}{\Gamma(na)} = na$$

illarum aequationum altera per alteram dividenda impetratur

$$e^{c} = n^{-n}$$

Porro cum habeas

$$\psi(a) = e^{c'}\, \Gamma(na)\, .\, n^{-na}$$

erit

$$\psi(\frac{1}{n}) = e^{c'-1}_{\ n}$$

id est

$$e^{c'} = n\psi(\frac{1}{n}) = n\Gamma(\frac{1}{n})\ \Gamma(\frac{2}{n}) \ldots \Gamma(\frac{n-1}{n}) = n^{\frac{1}{2}} (2\pi)^{\frac{n-1}{2}}$$

Quibus constantium c et c′ vel e^{c} et $e^{c'}$ valoribus in aequatione

$$\psi(a) = \Gamma(na)\, e^{ca+c'}$$

substitutis relationem nostram (12) impetratum iri apparet. Id quoque theorema, quo nihil cogitari posse videtur elegantius, paulo post e Gaussianis principiis demonstrabimus.

Ad haec quidem illustranda ut perveniamus, ab aequatione (6) proficiscimur, ex qua enim cum sit

$$\int_{o}^{1} x^{a-1} (1-x)^{b}\, dx = \frac{\Gamma(a)\ \Gamma(b+1)}{\Gamma(a+b+1)}$$

erit statuto $\frac{x}{b}$ loco ipsius x

$$\int_{o}^{b} x^{a-1} (1-\frac{x}{b})^{b}\, dx = b^{a} \frac{\Gamma(a)\,\Gamma(b+1)}{\Gamma(a+b+1)}$$

Cujus integralis valorem, quem a duobus argumentis (a) et (b) pendentem nostraque cum functione intime cohaerentem per $\Gamma(a, b)$ designare placet, siquidem, quod in sequentibus supponemus, b est integer positivus. ita quoque possumus exprimere:

$$\Gamma(a, b) = \frac{1\ .\ 2\ .\ 3\ .\ .\ .\ b\ .\ b^{a}}{a(a+1)\ (a+2)\ .\ .\ (a+b)} = \frac{b!\ b^{a}}{[a, -1]^{b+1}}$$

Quum autem numero b in infinitum crescente potentia $(1-\frac{x}{b})^{b}$ transeat in e^{-x}, $\Gamma(a)$ videmus esse limitem functionis $\Gamma(a,b)$, ita ut fiat:

$$\Gamma(a,b) = \Gamma(a) \text{ , si } b = \infty \text{ sit.}$$

Hinc patet esse

$$\Gamma(a) = \frac{1}{a} \cdot \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot b}{(a+1)(a+2)\ldots(a+b)} \cdot b^{a} \text{ pro } b = \infty$$

vel

$$\Gamma(a+1) = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot m}{(a+1)(a+2)\ldots(a+m)} \cdot m^{a} \text{ (cond. } m = \infty) \qquad 15$$

ive si mavis

$$\Gamma(a+1) = \frac{1^{a+1}}{(a+1)} \cdot \frac{2^{a+1}}{1^{a}(a+2)} \cdot \frac{3^{a+1}}{2^{a}(+3)} \cdot \frac{4^{a+1}}{3^{a}(a+4)} \ldots \text{ in inf.}$$

Clarissimus Gaussius,[1]) seriei hypergeometricae not e discussione in illud innumerorum actorum productum incidit — quae res, quamvis mira primo adspectu esse videatur, tamen, ut infra demonstrabimus, cum natura functionis nostrae intime cohaeret — eo ipso functionem suam $\Pi(\ )$ definivit. Profecto illius producti ope omnes integralis nostri proprietates sine opera possunt demonstrari. Ut cum sit:

$$\Gamma(1+a) = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot m}{(a+1)(a+2)(a+3) \ldots (a+m)} \cdot m^{a} \text{ (cond. } m = \infty)$$

erit

$$\Gamma(1-a) = \frac{1 \cdot 2 \cdot 3 \cdot \ldots \cdot m}{(1-a)(2-a)(3-a) \ldots (m-a)} \cdot m^{-a} \text{ (cond. } m = \infty)$$

ideoque

$$\Gamma(1+a)\,\Gamma(1-a) = \frac{(1 \cdot 2 \cdot 3 \ldots m)^2}{(1^2-a^2)(2^2-a^2)(3^2-a^2) \ldots (m^2-a^2)} \text{ (cond. } m = \infty)$$

vel

$$\Gamma(1+a)\,\Gamma(1-a) = \frac{1}{(1-\frac{a^2}{1^2})(1-\frac{a^2}{2^2})(1-\frac{a^2}{3^2})(1-\frac{a^2}{m^2})} \text{ (cond. } m = \infty)$$

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[1]) Societ. reg. scient. Goetting, comment. tom II., classis mathematica: Disquisitiones generales circa seriem infinitam:

$$1 + \frac{\alpha\beta}{1\gamma}x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)}x^2 + \ldots$$

auctore C. F. Gauss.

Notissimum autem est esse

$$\sin x = x\left(1-\frac{x^2}{\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\left(1-\frac{x^2}{2^2\pi^2}\right)\ldots \text{ in inf.}$$

hincque

$$\frac{\sin a\pi}{a\pi} = \left(1-\frac{a^2}{1^2}\right)\left(1-\frac{a^2}{2^2}\right)\left(1-\frac{a^2}{3^2}\right)\ldots \text{ in inf.}$$

ita ut sit

$$\Gamma(1+a)\,\Gamma(1-a) = \frac{a\pi}{\sin a\pi}$$

Simili ratione theorema generale in aequatione (12) contentum demonstratur Sponte enim cum e relatione (14) hae n exeant aequationes:

$$\Gamma(a,b) = \frac{b!\, b^a}{a(a+1)\ldots(a+b)}$$

$$\Gamma(a-\tfrac{1}{n},b) = \frac{b!\, b^{a-\frac{1}{n}}\, n^{b+1}}{(na-1)(na+n-1)(na+2n-1)\ldots(na+bn-1)}$$

$$\Gamma(a-\tfrac{2}{n},b) = \frac{b!\, b^{a-\frac{2}{n}}\, n^{b+1}}{(na-2)(na+n-2)(na+2n-2)\ldots(na+bn-2)}$$

$$\Gamma(a+\tfrac{n-1}{u},b) = \frac{b!\, b^{a-\frac{n-1}{n}}\, n^{b+1}}{(na-\overline{n-1})(na+n-\overline{n-1})(na+2n-\overline{n-1})\ldots(na+6n-\overline{n-1})}$$

cumque sit

$$\Gamma(na, nb) = \frac{(nb)!\, b^{na}\, n^{na}}{na\,(na+1)(na+2)\ldots(na+n)(na+n+1)\ldots(na+2n)\ldots(na+bn)}$$

designando productum

$$\Gamma(a,b)\;\Gamma(a-\tfrac{1}{n},b)\;\Gamma(a-\tfrac{2}{n},b)\ldots\Gamma(a-\tfrac{n-1}{n},b)$$

per $\psi\,(a, b)$, fractio

$$\frac{\psi(a,b)}{\Gamma(na,nb)}$$

aequalis erit producto duorum factorum M et N, siquidem

$$M = \frac{(b!)^n\, b^{\frac{1-n}{2}}\, n^{(n-1)(b-1)}}{(nb)!\, n^{na}}$$

$$N = \frac{na(na+1)(na+2)\ldots(na+n)(na+n+1)\ldots(na+2n)\ldots(na+bn)}{a(a+1)(a+2)\ldots(a+b)\{(na-1)(na-2)\ldots(na-\overline{n-1})\}\{(na+n-1)(na+n-2)\ldots(na+n-\overline{n-1})\}\{\ldots\}\{\ldots\}}.$$

vel, factis, quae in promtu sunt, reductionibus

$$N = \frac{n^{b+1}}{(na-1)(na-2)\ldots(na-\overline{n-1})}$$

Illinc sequitur esse

$$(na-1)(na-2)\ldots(na-\overline{n-1})\,n^{na}\,\frac{\psi(a,b)}{\Gamma(na,nb)} = \frac{(b!)^n}{(nb)!}\,\frac{n^{nb+n}}{b^{\frac{n-1}{2}}}$$

vel

$$\frac{n^{na}\left\{a\Gamma(a,b).(a-\frac{1}{n})\,\Gamma(a-\frac{1}{n},b)\,(a-\frac{2}{n})\,\Gamma(a-\frac{2}{n},b)\ldots(a-\frac{n-1}{n})\,\Gamma(a-\frac{n-1}{n},b)\right\}}{an\,\Gamma(an,bn)} = \frac{(b!)^n\,n^{nb}}{(nb)!\,b^{\frac{n-1}{2}}}$$

cuius aequationis altera parte a numero (a) omnino libera non immutabitur altera, quicumque valor ipsi (a) tribuatur. Statuendo ergo $a=1$ adipiscimur:

$$\frac{n^n\left\{(1-\frac{1}{n})\,(1-\frac{2}{n})\ldots(1-\frac{n-1}{n})\right\}\left\{\Gamma(1,b)\,\Gamma(1-\frac{1}{n},b)\,\Gamma(1-\frac{2}{n},b)\ldots\Gamma(1-\frac{n-1}{n},b)\right\}}{n\Gamma(n,nb)}$$

sive

$$\frac{(n-1)!\left\{\Gamma(1,b)\,\Gamma(1-\frac{1}{n},b)\ldots\Gamma(1-\frac{n-1}{n},b)\right\}}{\Gamma(n,nb)}$$

quam expressionem pro $b=\infty$ transiore constat in hanc:

$$\Gamma(1-\frac{1}{n})\,\Gamma(1-\frac{2}{n})\ldots\Gamma(1-\frac{n-1}{n}) = \frac{(2\pi)^{\frac{n-1}{2}}}{\sqrt{n}}$$

Jam igitur habetur:

$$\frac{a^{na}\left\{a\,\Gamma(a)\,(a-\frac{1}{n})\Gamma(a-\frac{1}{n})\,(a-\frac{2}{n})\Gamma(a-\frac{2}{n})\ldots(a-\frac{n-1}{n})\,\Gamma(a-\frac{n-1}{n})\right\}}{an\,\Gamma(an)} = \frac{(2\pi)^{\frac{n-1}{2}}}{n^{\frac{1}{2}}}$$

vel, quia generaliter

$$(a-\frac{m}{n})\,\Gamma(a-\frac{m}{n}) = \Gamma(a-\frac{m}{n}+1) = \Gamma(a+\frac{n-m}{n})$$

est, quandoquidem $m<n$,

$$\Gamma(a)\,\Gamma(a+\tfrac{1}{n})\ \ldots\ \Gamma(a+\tfrac{n-1}{n}) = (2\pi)^{\frac{n-1}{2}}\, n^{\frac{1}{2}-na}\ \Gamma(na)$$

quae est relatio quaesita.

Antequam ad secundam disquisitionis nostrae partem transeamus, promissis stare in animo est, quis sit connexus functionem nostram inter seriemque hypergeometricam supra commemoratam, copiosius explicatum iri. Quem in finem statuimus

$$J = \int_0^1 x^{a-1}\,(1-x)^{b-a-1}\,(1-kx)^{-c}\,dx$$

Jam cum sit

$$(1-kx)^{-c} = \sum_{\varepsilon=0}^{\varepsilon=\infty} (-1)^{\varepsilon}\,\frac{[-c]^{\varepsilon}}{\varepsilon!}\,k^{\varepsilon} x^{\varepsilon} = \sum_{\varepsilon=0}^{\varepsilon=\infty} \frac{[c,-1]^{\varepsilon}}{\varepsilon!}\,k^{\varepsilon} x^{\varepsilon}$$

erit

$$J = \sum_{\varepsilon=0}^{\varepsilon=\infty} \frac{[c,-1]^{\varepsilon}}{\varepsilon!}\,k^{\varepsilon} \int_0^1 x^{a+\varepsilon-1}\,(1-x)^{b-a-1}\,dx$$

vel

$$J = \sum_{\varepsilon=0}^{\varepsilon=\infty} \frac{[c,-1]^{\varepsilon}}{\varepsilon!}\,k^{\varepsilon}\,\frac{\Gamma(a+\varepsilon)\,\Gamma(b-a)}{\Gamma(b+\varepsilon)}$$

vel positis

$$\Gamma(a+\varepsilon) = [a,-1]^{\varepsilon}\,\Gamma(a)$$

$$\Gamma(b+\varepsilon) = [b-1]^{\varepsilon}\,\Gamma(b)$$

$$J = \sum_{\varepsilon=0}^{\varepsilon=\infty} \frac{[a,-1]^{\varepsilon}\,[c,-1]^{\varepsilon}}{\varepsilon!\,[b,-1]^{\varepsilon}}\,k^{\varepsilon}\,\frac{\Gamma(a)}{\Gamma(b)}\,\Gamma(b-a)$$

Altera ex parte cum sit

$$J' = \int_0^1 x^{a-1}\,(1-x)^{b-a-1}\,dx = \frac{\Gamma(a)\,\Gamma(b-a)}{\Gamma(b)}$$

erit statuendo

$$G = \frac{J}{J'}$$

$$G = \sum_{\varepsilon=0}^{\varepsilon=\infty} \frac{[a,-1]^{\varepsilon} . [c,-1]^{\varepsilon}}{\varepsilon! \, [b,-1]^{\varepsilon}} k^{\varepsilon}$$

quo exprimitur series quaesita ab illmo. Gauss per

F (a, c, b, k)

designata. Adiutrice illa formula, quae eadem est atque haec insequens

$$\frac{\int_0^1 x^{a-1} (1-x)^{b-a-1} (1-kx)^{-c} dx}{\int_0^1 x^{a-1} (1-x)^{b-a-1} dx} = 1 + \frac{ac}{1.b} k + \frac{a(a+1)c(c+1)}{1.2.\ b(b+1)} k^2 + \frac{a(a+1)(a+2)\,c(c+1)(c+2)}{1.2.3.\ b(b+1)(b+1)} k^3 + 2.$$

adhibitis notis functionis nostrae proprietatibus multae insignes relationes evolvi possunt, ad fuuctiones ellipticas spectantes, cui rei eo libentius operam damus, quum ea, quoad sciamus, a nemine unquam tractata sit.

Statuendo einm

$a = c = \frac{1}{2}$, $b = 1$, k^2 loco k

fit

$$\frac{1}{\pi}\int_0^1 \frac{dx}{x^{1/2}(1-x)^{1/2}(1-k^2x)^{1/2}} = 1 + \frac{1^2}{2^2} k^2 + \frac{1^2.3^2 k^4}{2^2.4^2} + \ldots.$$

vel posito

$$x = \sin^2\varphi$$

$$\frac{2}{\pi}\int_0^{\frac{\pi}{2}} \frac{d\varphi}{\sqrt{(1 - k^2 \sin^2\varphi)}} = 1 + \frac{1^2}{2^2} k^2 + \frac{1^2.3^2}{2^2.4^2} k^4 + \ldots.$$

ita ut adhibita illmi Jacobi integralis elliptici primae speciei significatione sit

$$K = \frac{\pi}{2}\left\{1 + \frac{1^2}{2^2}k^2 + \frac{1^2.3^2}{2^2.4^2}k^4 + \frac{1^2.3^2.5^2}{2^2.4^2.6^2} k^6 + \ldots \text{ in inf.}\right\}$$

Simili modo statutis

$a = \frac{1}{2}$, $c = -\frac{1}{2}$, $b = 1$, k^2 loco k, $x = \sin^2\varphi$

accipitur

$$E = \frac{\pi}{2}\left\{1 - \frac{1}{2^2}k^2 - \frac{1^2.3}{2^2.4^2}k^4 - \frac{1^2.3^2.5}{2^2.4^2.6^2}k^6 - \ldots \text{ in inf.}\right\}$$

siquidem rursus *E* ellipticum secundae speciei integralr completum

$$\int_0^{\frac{\pi}{2}} d\varphi \sqrt{(1 - k^2 \sin^2\varphi)}$$

designatur [1])

Denique considerare iuvat aequationem:

$$\frac{(\Gamma(\frac{1}{4}))^3}{\Gamma(\frac{3}{4})} = k^2 \int_0^\infty \frac{dx}{x^{\frac{1}{2}}(1+x)^{\frac{1}{2}}(1+k'^2x)^{\frac{1}{2}}} \cdot \int_0^\infty \frac{dx}{x^{\frac{1}{2}}(1+x)^{\frac{1}{2}}(1+k^2x)^{\frac{1}{2}}}$$

$$+ k'^2 \int_0^\infty \frac{dx}{x^{\frac{1}{2}}(1+x)^{\frac{3}{2}}(1+k^2x)^{\frac{1}{2}}} \cdot \int_0^\infty \frac{dx}{x^{\frac{1}{2}}(1+x)^{\frac{1}{2}}(1+k^2x)^{\frac{1}{2}}}$$

quae substitutionibus longioribus illis quidem sed expeditissimis brevitatis causa non admissis obtinetur, valoribus ipsorum k et k′ inter se coniunctis aequatione

$$k^2 + k'^2 = 1$$

Illinc loco variabilis x substituto tang ψ exire videmus:

$$\frac{\pi}{2} = k^2 \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{(1-k'^2\sin^2\psi)}} \int_0^{\frac{\pi}{2}} \frac{\cos^2\psi\, d\psi}{\sqrt{(1-k'^2\sin^2\psi)}} + k'^2 \int_0^{\frac{\pi}{2}} \frac{d\psi}{\sqrt{(1-k^2\sin^2\psi)}} \int_0^{\frac{\pi}{2}} \frac{\cos^2\psi\, d\psi}{\sqrt{(1-k'^2\sin^2\psi)}}$$

qua ex relatioue, statuto

$$\cos^2\psi = 1 - \sin^2\psi$$

statim obtinetur praeclarum hoc immortalis Legendri theorema:

$$\frac{\pi}{2} = E\overset{\prime}{K} + \overset{\prime}{E}K - K\overset{\prime}{K}$$

quod ill. Gudermann in sua functionum modularium theoria compluribus aliis modis magna cum sua in ratiocinando facilitate atque elegantia demonstravit.

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[1]) Gudermann's Theorie der Modular-Functionen und Modular-Integrale, pag. 195.

II.

Evolutis, quae notatu dignissimae sunt, functionis nostrae proprietatibus sequitur, ut dicendum sit de numerica illius computatione, quae est disquisitionis nostrae pars secunda. Quo in genere est animadvertendum, secundum aequationem (2) calculum non esse instituendum nisi pro valoribus elementi intra fines o et 1 contentis, cum maiorum elementorum functio ad istos valores reduci possit. Quin adeo cum sit

$$\Gamma(a)\,\Gamma(1-a) = \frac{\pi}{\sin a\pi}$$

conspicuum est, determinatis functionis valoribus pro elementis inter fines o et $\frac{1}{2}$ sitis cuiuslibet argumenti functionem sine opera numerice inveniri.

Prima igitur culculum subducendi methodus contenta est in Gaussiano producto, supra commemorato, quod ad limitem certum convergere ut demonstretur restare videtur. Quem in finem consideremus productum (14), quod finitum esse futurum, donec integer b finitus sit, quis est quin primo adspectu videat? Illic si numerus b incremento quamvis mango augetur, ad valorem $\Gamma(a, b)$ additamentum accedere finitum facillimo sibi quisque calculo persuadebit, unde sequitur, etiam valorem ipsius $\Gamma(a+1)$ hunc:

$$\Gamma(a+1) = \frac{1^{a+1}}{(a+2)} \cdot \frac{2^{a+1}}{1^a(a+2)} \cdot \frac{3^{a+1}}{2^a(a+3)} \cdot \frac{4^{a+}}{3^a(a+4)} \cdot \ldots$$

ad certum limitem convergere: quae res quasi „a priori", probari nobis videtur, modo (a) numerum permagnum esse supponamus. Verumtamen cum producti illius convergentia haud ita sit magna; quamquam logarithmis uti calculum instituri possunt, ad aliam methodum quaerendam impellimur.

Ne haec quidem series

$$\log\Gamma(a+\tfrac{1}{2}) = a\log a - a + \frac{\log 2\pi}{2} - \frac{\overset{1}{B}}{1.2\;2a} + \frac{7\overset{2}{B}}{3.4(2a)^3} - \frac{31\overset{3}{B}}{5.6(2a)^5} + \frac{127\overset{4}{B}}{7.8(2a)^7} - + \ldots$$

litteris $\overset{1}{B}, \overset{2}{B}, \overset{3}{B}, \overset{4}{B}. \ldots$ numeros designantibus Bernouillianos, Eulero [1]) quae debetur nobis potuit satisfacere. Illi enim numeri quum initio prompte convergant illi quidem, postea autem divergant, non video, quo suo iure ill. Gauss illa serie ad limitatam praecisionem impetrandam uti licere edixerit, siquidem elementum (a) satis magnum sumeretur: id certe constat, nusquam esse de praecisionis recuperandae grado strenue disputatum.

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[1]) Euleri Institut. Calc. Diff. pag. 466.

Itaque nisi proficiscentes a forma integralis nostri hac:

$$\Gamma(a) = \int_0^1 (\log \frac{1}{1-x})^{a-1} dx$$

quadratura mechanica, quae vocatur, ab illmo. Jacobi ad summum perficientiae gradum sublata uti velimus, hoc fere modo ad finem perveniri possit.

Supra invenimus esse

$$\frac{d \log \Gamma(a)}{da} = \int_0^\infty \left\{ e^{-x} - (1+x)^{-a} \right\} \frac{dx}{x}$$

unde, facta integratione secundum variabilem (a) pro a = 1 evanescente, exit:

$$\log \Gamma(a) = \int_0^\infty \left\{ (a-1)\, e^{-x} - \frac{(1+x)^{-1} - (1+x)^{-a}}{\log (1+x)} \right\} \frac{dx}{x}$$

Cum autem sit

$$\Gamma(2) = 1$$

haec pro a = 2 odtinetur relatio:

$$0 = \int_0^\infty \left\{ e^{-x} - \frac{(1+x)^{-1} - (1+x)^{-2}}{\log (1+x)} \right\} \frac{dx}{x}$$

vel quia est

$$(1+x)^{-1} - (1+x)^{-2} = x(1+x)^{-2}$$

$$0 = \int_0^\infty \left\{ \frac{e^{-x}}{x} - \frac{(1+x)^{-2}}{\log (1+x)} \right\} dx$$

Hanc si per (1-a) multiplicatam additione cum praecedente coniunxeris, impetrabis

$$\log \Gamma(a) = \int_0^\infty \left((a-1)(1+x)^{-2} - \frac{(1+x)^{-1} - (1+x)^{-a}}{x} \right) \frac{dx}{\log (1+x)}$$

quae formula, statuto x loco ipsius log (1+a), in hanc transit simplicìorem:

$$\log \Gamma(a) = \int_0^\infty \left((a-1)\, e^{-x} - \frac{e^{-x} - e^{-ax}}{1-e^{-x}} \right) \frac{dx}{x}$$

cuius aequationis ope magna sua ingenii subtilitate illmum. Cauchy [1]) plurimas functionis $\Gamma()$ proprietates evolvisse hac data occasione observamus. Jam illam ita exhibeamus, ut sit

$$\log \Gamma(a) = \int_0^\infty (P + Qe^{-ax})\, dx$$

positis

$$P = \left((a-1) - \frac{1}{1-e^{-x}}\right)\frac{e^{-x}}{x})$$

$$Q = \frac{1}{x(1-e^{-x})}$$

valoremque ipsius Q secundum potentias ascendentes variabilis x evolvamus. Ita nanciscimur

$$Q = \frac{1}{x^2} + \frac{1}{2x} + \overset{0}{A} + \overset{1}{A}x + \overset{2}{A}x^2 + \ldots.$$

cuius expressionis partem negativis exponentibus affectam

$$(\frac{1}{x} + \frac{1}{2})\,\frac{1}{x} + q$$

statuere iuvat.

Hinc patet esse

$$\log \Gamma(a) = \int_0^\infty (P + qe^{-ax})\, dx + \int_0^\infty (Q - q)e^{-ax}\, dx$$

quorum integralium quod est posterius per $\Psi(a)$, alteram per $\Omega(a)$ obsignemus ita ut habeantur hae tres relationes:

$$\log \Gamma(a) = \Psi(a) + \Omega(a)$$

$$\Psi(a) = \int_0^\infty \left(\frac{1}{1-e^{-x}} - \frac{1}{x} - \frac{1}{2}\right)\frac{e^{-ax}}{x}\, dx$$

$$\Omega(a) = \int_0^\infty \left\{(a-1-\frac{1}{1-e^{-x}})e^{-x} + (\frac{1}{x}+\frac{1}{2})e^{-ax}\right\}\frac{dx}{x}$$

Ad inveniendum valorem functionis $\Omega(a)$ observamus esse

[1]) Cauchy, Exercises d'Analyse et de Physique mathématique, tome deuxième, pag. 378 et sqq.

$$\Omega(a)-\Omega(\tfrac{1}{2})=\int_0^\infty \Big\{(a-\tfrac{1}{2})\,e^{-x}+(\tfrac{1}{x}+\tfrac{1}{2})(e^{-ax}-e^{-\frac{1}{2}x})\Big\}\frac{dx}{x}$$

vel, cum sit

$$\log a=\int_0^\infty (e^{-x}-e^{-ax})\frac{dx}{x}$$

$$\Omega(a)-\Omega(\tfrac{1}{2})-(a-\tfrac{1}{2})\log a=\int_0^\infty \Big\{e^{-ax}(a+\tfrac{1}{x})-e^{-\frac{1}{2}x}(\tfrac{1}{2}+\tfrac{1}{x})\Big\}\frac{dx}{x}$$

Integrali autem definito

$$\int_0^\infty e^{-ax}(a+\frac{1}{x})dx=U_{(a)}$$

quod non esse potest nisi mera functio ipsius (a), secundum (a) differentiato, factisque, quae ab oculos iacent, reductionibus, invenitur esse

$$U_{(a)}=-a$$

unde haec obtinetur aequatio:

$$\Omega(a)-\Omega(\tfrac{1}{2})=(a-\tfrac{1}{2})\log a-(a-\tfrac{1}{2})=(a-\tfrac{1}{2})(\log a-1)$$

Jam cum facillimum sit deductu esse

$$\Psi(\tfrac{1}{2})=\tfrac{1}{2}(1-\log 2)$$

cumque sit

$$\log\Gamma(\tfrac{1}{2})=\tfrac{1}{2}\log\pi,$$

erit $\Omega(\tfrac{1}{2})=\tfrac{1}{2}(\log 2\pi-1)$

itaque fieri videmus

$$\Omega(a)=(a-\tfrac{1}{2})(\log a-1)+\tfrac{1}{2}(\log 2\pi-1)=(a-\tfrac{1}{2})\log a+\tfrac{1}{2}\log 2\pi-a.$$

Reliquum est, ut evolvatur valor functionis $\Psi(a)$, quae contenta est in hoc integrali

$$\Psi(a)=\int_0^\infty\Big(\frac{1}{1-e^{-x}}-\frac{1}{x}-\frac{1}{2}\Big)e^{-ax}\frac{dx}{x}$$

vel in hoc

$$\Psi(a)=\int_0^\infty\Big(1-(1-c^{-x})(\frac{1}{x}+\frac{1}{2})\Big)\frac{e^{-ax}}{1-c^{-x}}\frac{dx}{x}$$

Notissimum autem est esse

$$1-e^{-x} = \sum_{\alpha=0}^{\alpha=\infty} (-1)^{\alpha} \frac{x^{\alpha+1}}{(\alpha+1)!}$$

unde fit:

$$\frac{1-(1-e^{-x})\left(\frac{1}{x}+\frac{1}{2}\right)}{x} = \frac{1}{2}\left(\frac{1x}{3!} - \frac{2x^2}{4!} - \frac{3x^3}{5!} - \frac{4x^4}{6!} + - \dots\dots\right)$$

ita ut sit

$$\psi(a) = \frac{1}{2}\left\{\frac{1}{2.3}\int_0^\infty \frac{x\, e^{-ax}}{1-e^{-x}} dx - \frac{2}{3.4}\int_0^\infty \frac{x^2 e^{-ax}}{2!\,1-e^{-x}} dx + \frac{3}{4.5}\int_0^\infty \frac{x^3 e^{-ax}}{3!\,1-e^{-x}} - + \dots\right\}$$

Scimus autem haberi

$$\frac{e^{-ax}}{1-e^{-x}} = \sum_{\alpha=0}^{\alpha=\infty} e^{-(a+\alpha)x}$$

et esse

$$\int_0^\infty \frac{x^m}{m!} \frac{e^{-ax}}{1-e^{-x}} dx = \sum_{\alpha=0}^{\alpha=\infty} \int_0^\infty \frac{x^m}{m!} e^{-(a+\alpha)x} dx = \sum_{\alpha=0}^{\alpha=\infty} \frac{1}{(a+\alpha)^{m+1}}$$

quo fit

$$\psi(a) = \frac{1}{2}\left\{\frac{1}{2.3}\left[\frac{1}{a^2}+\frac{1}{(a+1)^2}+\frac{1}{(a+2)^2}+\dots\right] - \frac{2}{3.4}\left[\frac{1}{a^3}+\frac{1}{(a+1)^3}+\frac{1}{a+2)^3}+\dots\right]\right.$$

ideoque

$$\log \Gamma(a) = \left(a-\frac{1}{2}\right)\log a + \frac{1}{2}\log 2\pi - a + \frac{1}{2}\left(\frac{1}{2\,3}\left[\left(\frac{1}{a^2}\right)\right] - \frac{2}{3.4}\left[\left(\frac{1}{a^3}\right)\right] + \frac{3}{4.5}\left[\left(\frac{1}{a^4}\right)\right] - + \dots\right)$$

designante

$$\left(\frac{1}{a^m}\right) = \sum_{\alpha=0}^{\alpha=\infty} \frac{1}{(a+\alpha)^m}$$

cujus formulae ope pro valoribus aliquantum magnis elementi (a) log Γ(a) expeditissimo calculo computatur, quo facto ad minora elementa notis relationibus descendere licet.

Sed hanc nostram formulam, in qua infinite multae series infinitae continentur, in aliam elegantissimam formam involvi posse benigne nos docuis ill. Gudermann. Re enim ita adornata, ut statuatur:

$$\psi\,(a) = \sum_{\alpha=0}^{\alpha=\infty} \frac{1}{2}\left(\frac{1}{2.3}\,\frac{1}{(a+\alpha)^2} - \frac{2}{3.4}\cdot\frac{1}{(a+\alpha)^3} + \frac{3}{4.5}\cdot\frac{1}{(a+\alpha)^4} - \; +\right)$$

quum inveniatur esse

$$\frac{1}{2}\left(\frac{1}{2.3}\cdot\frac{1}{a^2} - \frac{2}{3.4}\cdot\frac{1}{a^3} + \frac{3}{4.5}\cdot\frac{1}{a^4} - \frac{4}{5.6}\cdot\frac{1}{a^5} + \ldots\right) = (a+\tfrac{1}{2})\log\left(1+\frac{1}{a}\right) - 1$$

erit

$$\psi(a) = \sum_{\alpha=0}^{\alpha=\infty}\left(\left(a+\alpha+\frac{1}{2}\right)\log\left(1+\frac{1}{a+a}\right) - 1\right)$$

adeoque factis, quae in promptu sunt, reductionibus

$$\log \Gamma(a+1) = \frac{1}{2}\log 2\pi - a + \left(a+\frac{1}{2}\right)\log a + \sum_{\alpha=0}^{\alpha=\infty}\left(\left(\alpha+\frac{1}{2}+a\right)\log\left(1+\frac{1}{a+\alpha}\right) - 1\right)$$

cui aequationi cum hac Gaussiana supra commemorata serie

$$\log \Gamma(a+1) = \frac{1}{2}\log 2\pi - a + \left(a+\frac{1}{2}\right)\log a + \frac{B^1}{1.2\,a} + \frac{B^2}{3.4.a^3} + \frac{B^3}{5.6.a^5} + \frac{B^5}{7.8.a^7} + - \ldots$$

in primis certe terminis magna intercedit similitudo.

Ideis confisi Gaussianis, quae in commentatione illius supra laudata nostra admiratione digna pronuntiantur, functionem nostram in fractionem continuam quandam, quae et rapide convergeret et legem sectaretur simplicem, multis modis evolvere tentavimus, sed sine prospero successu.

Satis igitur de computo numerico dixisse videmur, addito, et Legendrum et Gaussium pro sua utrumque functione tabulas valde accuratas edididisse.

IIII.

Restat ut dicendum sit de multis integralibus definitis, quae substitutionibus variis et simplicibus plerumque e functione nostra deducere possis, libenterque facimus, ut diutius hac in re versemur, quum alia integralia aliis rationibus ex aliis principiis deduci soleant. Jam ergo si in integrali

$$\Gamma(a) = \int_0^\infty e^{-x} x^{a-1} dx$$

loco variabilis x substituitur x^b, numero b positivo, erit

$$\int_0^\infty e^{-x^b} x^{ba-1} dx = \frac{1}{b}\Gamma(a)$$

unde ponendo $b = 2$ et $\frac{a}{2}$ pro (a) formula minus illa quidem generalior sed tamen multo gravior accipitur

$$(1) \qquad \int_0^\infty e^{-x^2} x^{a-1} dx = \frac{1}{2}\Gamma\left(\frac{a}{2}\right)$$

Hinc statuto $a = 1$ exit

$$(2) \qquad \int_0^\infty e^{-x^2} dx = \frac{1}{2}\sqrt{\pi}$$

quod integrale gravissimum clarissimo Laplace deberi notum est.

Facta autem in aequatione (1) suppositione, numerum (a) esse integrum, duo distinguendi casus sunt, prout (a) impar est parve. Jam esto (a) impar et $= 2n + 1$, obsignante n quemlibet integrum, non exclusa nullitate, erit

$$\int_0^\infty e^{-x^2} x^{2n} dx = \frac{1.\ 3.\ 5\ldots(2n-1)}{2^{n+1}}\sqrt{\pi}$$

quod integrale, ordinata $f(x) = e^{-x^2} x^{2n}$ pari, ita quoque exhiberi potest

$$\int_{-\infty}^{+\infty} e^{-x^2} x^{2n}\, dx = \frac{1.3.5\ldots(2n-1)}{2^n}\sqrt{\pi} \tag{3}$$

Sin (a) numerus est par et $= 2n$, erit

$$\int_{0}^{\infty} e^{-x^2} x^{2n-1}\, dx = \tfrac{1}{2}\,(n-1)! \tag{4}$$

qua in aequatione $(n-1)! = 1$ est pro $n = 1$.

Ope integralis (3) aliud valde memorabile deducitur integrale. Statuendo enim loco ipsius n omnes deinceps numeros integros inde a $n=o$ usque ad $n=\infty$ multiplicando singulas illas potentias per potentias constantis cuiusdam numeri positivi (a) aeque altas, dividendo denique singulos istos valores per ipsorum exponentium „numeros permutationum" qui dicuntur; cuiusmodi integralium alternative positivo negativoque signo afficiendorum additione impetratur:

$$\int_{-\infty}^{+\infty} e^{-x^2} \cos ax\, dx = \sum_{\alpha=o}^{\alpha=\infty} (-1)^{\alpha} \frac{a}{(2\alpha)!}\, \frac{1.3.5\ldots(2\alpha-1)}{2^{\alpha}}\sqrt{\pi}$$

$$= \sqrt{\pi} \sum_{\alpha=o}^{\alpha=\infty} (-1)^{\alpha} \frac{\left(\frac{a^2}{4}\right)^{\alpha}}{\alpha!}$$

ut sit

$$\int_{-\infty}^{+\infty} e^{-x^2} \cos ax\, dx = \sqrt{\pi}\,.\, e^{-\frac{a^2}{4}}$$

cui ab illmo. Lagrange invento integrali quod respondet hyperbolice analogum

$$\int_{-\infty}^{+\infty} e^{-x^2} \mathrm{Cos}\, ax\, dx = \sqrt{\pi}\,.\, e^{\frac{a^2}{4}}$$

et iustum est et notatu dignissimum.

Porro cum invenerimus esse

$$\int_{o}^{1} (1-x)^{b-1} x^{c-1}\, dx = \frac{\Gamma(b)\,\Gamma(c)}{\Gamma(b+c)}$$

fit statuendo $x = \sin\varphi$

$$\int_0^{\frac{\pi}{2}} (\cos\varphi)^{2b-1} (\sin\varphi)^{2c-1} d\varphi = \tfrac{1}{2} \frac{\Gamma(b)\,\Gamma(c)}{\Gamma(b+c)}$$

unde memorabilis impetratur relatio

$$\int_0^{\frac{\pi}{2}} \sin^{2a}\varphi \cos^{2b}\varphi \, d\varphi = \tfrac{1}{2} \frac{\Gamma(a+\frac{1}{2})\,\Gamma(b+\frac{1}{2})}{\Gamma(a+b+1)}$$

quacum coincidere hanc similem uberrimam

$$\int_0^{\infty} (\sin\varphi)^{2a+1} (\cos\varphi)^{2b} \frac{d\varphi}{\varphi} = \tfrac{1}{2} \frac{\Gamma(a+\frac{1}{2})\,\Gamma(b+\frac{1}{2})}{\Gamma(a+b+1)}$$

vir doctissimus O. Schlömich demonstravit.

Multo graviora haec duo integralia sunt a functione nostra, ut facile sibi quisque ultro persuadet, dependentia:

$$\int_0^{\infty} e^{-ax} \sin bx \,.\, x^{n-1} dx = A$$

$$\int_0^{\infty} e^{-ax} \cos bx \,.\, x^{n-1} dx = B$$

in quibus a, b, n positivi quilibet numeri sunto.

Differentiatione secundum quantitatem b facta habetur:

$$\frac{dA}{db} = \int_0^{\infty} e^{-ax} \cos. bx \,.\, x^{n} dx$$

$$-\frac{dB}{db} = \int_0^{\infty} e^{-ax} \sin bx \,.\, x^{n} dx$$

Integrando autem per partes eruitur generaliter

$$\int e^{-ax} \cos bx \,.\, x^{n} dx = -\frac{1}{a} \,.\, e^{-ax} \cos bx \,.\, x^{n} + \frac{1}{a} \int e^{-ax} (-b \sin bx . x^{n} + n \cos bx . x^{n-1}) \, dx$$

$$\int e^{-ax} \sin bx . x^n dx = -\frac{1}{a} e^{-ax} \sin bx . x^n + \frac{1}{a} \int e^{-ax} (b \cos bx . x^n + n \sin bx . x^{n-1}) dx$$

ideoque substitutis limitibus o et ∞

$$\frac{dA}{db} = \frac{b}{a} \cdot \frac{dB}{db} + \frac{n}{a} B$$

$$-\frac{dB}{db} = \frac{b}{a} \cdot \frac{dA}{db} + \frac{n}{a} A$$

quarum aequationum differentialium priorem multiplicando per $\frac{dA}{db}$, posteriorem per $\frac{dB}{db}$, tum vero alteram per A, alteram per B, binis subtractione inter se coniunctis fit:

$$\left(\frac{Ad}{db}\right)^2 + \left(\frac{dB}{db}\right)^2 = \frac{n}{a}\left(B \frac{dA}{db} - A \frac{dB}{db}\right)$$

$$A\frac{dA}{db} + B\frac{dB}{db} = \frac{b}{a}\left(A\frac{dB}{db} - B\frac{dA}{db}\right)$$

Statuendo autem

$$A = r \sin v$$

$$B = r \cos v$$

ubi modulus r positivus sumatur, ultro nanciscimur

$$\left(\frac{dr}{db}\right)^2 + r^2\left(\frac{dv}{db}\right)^2 = \frac{n}{a} r^2 \frac{dv}{db}$$

$$r\frac{dr}{db} = -\frac{b}{a} r^2 \frac{dv}{db}$$

e quibus aequationibus valore ipsius $\frac{dr}{db}$ eliminando impetramus

$$\frac{dv}{db} = \frac{na}{a^2 + b^2}$$

hincqne

$$v = n \text{ arc tang } \left(\frac{b}{a}\right) + c$$

cumque sit $\frac{dr}{r} = -\frac{nb}{a^2 + b^2} db$, fieri videmus

$$\log r = -\frac{n}{2} \log (a^2 + b^2) + \log c,$$

vel, si mavis

$$r = \frac{c,}{(a^2 + b^2)^{\frac{n}{2}}}$$

Ad determinandos constantium c et c, valores observamus pro $b = o$ etiam avanescere et A et v, ita ut et

$$c = 0$$

esse debeat, eaque facta suppositione cum fiat $B = r$, erit

$$\int_0^\infty e^{-ax} x^{n-1} dx = \frac{c_{,}}{a^n}$$

ideoque respectu aequationis (4) sectionis primae

$$c_{,} = \Gamma(n)$$

Jam ergo est:

$$\int_0^\infty e^{-ax} \sin b\, x . x^{n-1} dx = \frac{\Gamma(n)}{(a^2+b^2)^{\frac{n}{2}}} \sin\left(n \operatorname{arc\,tang}\left(\frac{b}{a}\right)\right) \qquad (8)$$

$$\int_0^\infty e^{-ax} \cos b\, x . x^{n-1} dx = \frac{\Gamma(n)}{(a^2+b^2)^{\frac{n}{2}}} \cos\left(n \operatorname{arc\,tang}\left(\frac{b}{a}\right)\right) \qquad (6)$$

addita conditione, ut utraque in formula sub arc. tang. $(\frac{b}{a})$ minimus subintelligatur arcus intra fines $+\frac{\pi}{2}$ et $-\frac{\pi}{2}$ situs: quae cum per se in promptu esse videatur, iam inde derivari potest, quod evanescente b et c evanescere oportuit.

Eadem theoremata profecto elegantissima hoc quoque modo possis deducere. In integrali noto

$$\int_0^\infty e^{-kx} x^{a-1} dx = \frac{\Gamma(a)}{k^a}$$

k licet esse numerum complexum formae: $k + mi$, dummodo pars realis k maneat positiva, ita ut sit

$$\int_0^\infty e^{-(k+mi)x} x^{a-1} dx = \frac{\Gamma(a)}{(k+mi)^a}$$

qua in formula potentiae $(k+mi)^a$, quae generaliter multiplex est, valorem

$$(k^2 + m^2)^{\frac{a}{2}} . e^{ia . \operatorname{arc\,tang}(\frac{m}{k})}$$

tribuendum esse potest demonstrari. Hinc fit

$$\int_0^\infty e^{-kx} . e^{-mix} . x^{a-1} dx = \frac{\Gamma(a)}{(k^2+m^2)^{\frac{a}{2}}} . e^{-ia . \operatorname{arc\,tang}(\frac{m}{k})}$$

unde, cum sit

$$e^{-xi} = \cos x - i \sin x$$

separatis realibus partibus ab imaginariis theoremata nostra exeunt.

Integralia (8) et (9), quantitati (a) tributo valore o, transeunt in haec:

$$\int_0^\infty x^{n-1} \sin bx\, dx = \frac{\Gamma(n)}{b^n} \sin\left(\frac{n\pi}{2}\right) \tag{10}$$

$$\int_0^\infty x^{n-1} \cos bx\, dx = \frac{\Gamma(n)}{b^n} \cos\left(\frac{n\pi}{2}\right) \tag{11}$$

in quibus tamen tum solum sensum inesse apparet, si $n > o$ et < 1 est.

Eadem conditione pro $b = 1$ exeunt

$$\int_0^\infty x^{n-1} \sin x\, dx = \Gamma(n) \sin\left(\frac{n\pi}{2}\right) \tag{12}$$

$$\int_0^\infty x^{n-1} \cos x\, dx = \Gamma(n) \cos\left(\frac{n\pi}{2}\right) \tag{13}$$

quae formulae pro $n = \frac{1}{2}$ transeunt in has:

$$\left.\begin{aligned} \int_0^\infty \frac{\sin x}{\sqrt{x}}\, dx &= \sqrt{\frac{\pi}{2}} \\ \int_0^\infty \frac{\cos x}{\sqrt{x}}\, dx &= \sqrt{\frac{\pi}{2}} \end{aligned}\right\} \tag{14}$$

e quibus, posito x^2 loco ipsius x, haec duo integralia gravissima exeunt:

$$\int_0^\infty \begin{matrix}\sin \\ \cos\end{matrix}\left\{x^2\right\} dx = \frac{1}{2}\sqrt{\frac{\pi}{2}} \tag{15}$$

Jam porro formulam

$$\int_0^1 (1-x)^{b-1} x^{c-1}\, dx = \frac{\Gamma(b)\,\Gamma(c)}{\Gamma(b+c)}$$

secundum c differentiando adipiscimur:

$$\int_0^1 (1-x)^{b-1} x^{c-1} \log x \, dx = \frac{\Gamma(b)\,\Gamma(c)}{\Gamma(b+c)} \left\{ \frac{\Gamma'(c)}{\Gamma(c)} - \frac{\Gamma'(b+c)}{\Gamma(b+c)} \right\}$$

unde, cum habeatur

$$\frac{\Gamma'(c)}{\Gamma(c)} = \int_0^1 \left(e^{1-\frac{1}{x}} - x^{c} \right) \frac{dx}{x(1-x)}$$

$$\frac{\Gamma'(b+c)}{\Gamma(b+c)} = \int_0^1 \left(e^{1-\frac{1}{x}} - x^{c+b} \right) \frac{dx}{x(1-x)}$$

cumque sit

$$\log x = - \log \frac{1}{x}$$

hoc prodit memorabile integrale

$$\int_0^1 (1-x)^{b-1} x^{c-1} \log \frac{1}{x} \cdot dx = \frac{\Gamma(b)\,\Gamma(c)}{\Gamma(b+c)} \int_0^1 x^{c-1} \left(\frac{1-x^{b}}{1-x} \right) dx \tag{16}$$

quod ab Eulero repertum sub hac alia forma exhiberi potest

$$\int_0^1 (1-x)^{b-1} x^{c-1} \log x \, dx = - \frac{\Gamma(b)\,\Gamma(c)}{\Gamma(b+c)} \sum_{\alpha=0}^{\alpha=b} \frac{1}{c-1+\alpha}$$

siquidem b designat integrum positivum.

Illi formulae multas possemus invenire consimiles, si functionem $\varphi(a)$ cum integrale nostro ita cohaerentem, ut sit

$$\frac{\Gamma(a+1)}{\Gamma(a+1)} = \varphi(a)$$

magis illustrare vellemus. De qua tamen cum ill. Gaussius sub finem commentationis suae tam ingenuose ac copiose disseruerit, ut notatu dignum quidquam adiungere neque sciamus neqne audeamus, huic labori supersedendum esse censuimns.

Sed eo longiores erimus in consideranda formula

$$\int_0^1 (1-x^{\lambda})^{b} x^{a-1} \, dx = \frac{\Gamma(b+1)\,\Gamma(\frac{a}{\lambda})}{\lambda\,\Gamma(b+1+\frac{a}{\lambda})}$$

quae ex integrali saepius adhibito

$$\int_0^1 (1-x)^{b-1}\, x^{c-1}\, dx = \frac{\Gamma(b)\,.\,\Gamma(c)}{\Gamma(b+c)}$$

accipitur, statutis

x^{λ} loco ipsius	x
$b+1$	b
$c+1$	c
a	$\lambda(c+1)$

Ex illa, cum numerus b omnes valores unitatem negativam superantes percurrere possit, connexus inter integralia formae

$$\int_0^1 \frac{dx}{\sqrt{(1-x^4)}},\quad \int_0^1 \frac{x^2 dx}{\sqrt{(1-x^4)}}$$

aliaque huiuscemodi, quem olim Eulerus multo cum labore invenit, ultro perspicitur. Quin in illo integrali tria integralium genera notatu dignissima ac functionis nostrae ope accuratius determinanda continentur, quae deinceps exhibituri in omni casu postulamus, ut pro λ substituatur integer positivus n.

Primum igitur statuamus $b = \frac{1}{n}$, designemusque valorem integralis

$$\int_0^1 \frac{x^{a-1}\, dx}{\sqrt[n]{(1-x^n)}}$$

ubi (a) denotet integrum positivum, per $\underset{(a)}{A}$, ita ut sit

$$\underset{(1)}{A} = \int_0^1 \frac{dx}{\sqrt[n]{(1-x^n)}},\ \underset{(2)}{A} = \int_0^1 \frac{x\, dx}{\sqrt[n]{(1-x^n)}},\ \underset{(3)}{A} = \int_0^1 \frac{x^2 dx}{\sqrt[n]{(1-x^n)}} \ \ldots\ \underset{(n)}{A} = \int_0^1 \frac{x^{n-1}\, dx}{\sqrt[n]{(1-x^n)}}$$

Ad relationes, quae inter integralia A intercedunt, constituendas illorum valores hoc modo exprimi debent:

$$\underset{(1)}{A} = \frac{\Gamma(\frac{1}{n})\ \Gamma(1-\frac{1}{n})}{n}\quad \underset{(2)}{A} = \frac{\Gamma(\frac{1}{n})\ \Gamma(1-\frac{1}{n})}{1.\ \Gamma(\frac{1}{n})},\quad \underset{(3)}{A} = \frac{\Gamma(\frac{3}{n})\ \Gamma(1-\frac{1}{n})}{2\ \Gamma(\frac{2}{n})}\ \ldots\ \underset{(n)}{A} = \frac{\Gamma(1-\frac{1}{n})}{(n-1)\ \Gamma(\frac{n-1}{n})}$$

ita ut nominatim valores ipsorum $A_{(1)}$ et $A_{(n)}$ sint

$$A_{(1)} = \frac{\pi}{n\sin\frac{\pi}{n}}, \quad A_{(n)} = \frac{1}{n-1}.$$

Per m si designamus integrum minorem quam n, erit

$$\frac{A_{(m)}}{A_{(n+1-m)}} = \frac{n-m}{m-1} \cdot \frac{\sin(m-1)\frac{\pi}{n}}{\sin m\frac{\pi}{n}}$$

quae eo tantum casu illusoria fit formula, si m = 1, pro quo habetur

$$\frac{A_{(1)}}{A_{(n)}} = \frac{(n-1)\pi}{n\sin\frac{\pi}{n}}$$

Id quoque nobis persuasimus, quodlibet integrale A, cuius index multiplo aliquo ipsius n indicem m superaret, in hoc: $A_{(m)}$ reduci posse, ope aequationis sine ulla difficultate demonstrandae:

$$A_{(\alpha n + m)} = [m, -n]^{\alpha} \, [m-1, n]^{-\alpha} \cdot A_{(m)}$$

unde posito m = 1 et tum = n, hoc ipso casu insuper α commutato in α-1, valorem exeunt

$$A_{(\alpha n + 1)} = \int_0^1 \frac{x^{\alpha n}\, dx}{\sqrt[n]{(1-x^n)}} = \frac{[\frac{1}{n}, 1]^{\alpha}}{\alpha!} \; \frac{\pi}{\sin\frac{\pi}{n}}$$

$$A_{(\alpha)} = \int_0^1 \frac{x^{\alpha n}\, dx}{\sqrt[n]{(1-x^n)}} = \frac{(\alpha-1)!}{n} \cdot [-\frac{1}{n}, 1]^{-\alpha}$$

Secunda integralium classis non minus spectabilis obtinetur, si constentem a=1, pro b autem deinceps fractiones

$$-\frac{1}{n}, \; -\frac{2}{n}, \; -\frac{3}{n}, \ldots \; -(\frac{n-1}{n})$$

substituerimus, quae est ea classis, quam olim Gaussius nonnullis verbis pertractavtt Jam igitur aequando

$$B_2 = \int_0^1 \frac{dx}{\sqrt[n]{(1-x^n)}}, \quad B_{(2)} = \int_0^1 \frac{dx}{\sqrt[n]{(1-x^n)^2}}, \quad B_{(3)} = \int_0^1 \frac{dx}{\sqrt[n]{(-x^n)^3}} \cdots B_{(n-1)} - \int_n^1 \frac{dx}{\sqrt[n]{(1-x^n)^{n-1}}}$$

hae habebuntur formulae:

$$n\,B_{(1)} = \Gamma(\tfrac{1}{n})\ \Gamma(1-\tfrac{1}{n});\quad n\,B_{(2)} = \frac{\Gamma(\frac{1}{n})\ \Gamma(1-\frac{2}{n})}{\Gamma(1-\frac{1}{n})};\ \dots\ n\,B_{(n-1)} = \frac{\Gamma(\frac{1}{n})\ \Gamma(\frac{1}{n})}{\Gamma(\frac{2}{n})}$$

unde apparet, rationem

$$\frac{B_{(m)}}{B_{(n+1-m)}} = \frac{\sin\,(m-1)\frac{\pi}{n}}{\sin\,m\,\frac{\pi}{n}}$$

esse pro omnibus valoribus intra fines 1 et (n — 1) sitis. Quae nos docet ratio, dimidia parte integralium B cognita alteram, uti supra, sine labore computandam esse quod iam inde intelligi potuit, quia connexus inter functiones

$$\Gamma(\tfrac{1}{u}),\ \Gamma(\tfrac{2}{n})\ ,\ .\ .\ .\ \Gamma(\tfrac{n-1}{u})$$

talis est, ut data functione $\Gamma(\frac{m}{n})$ et notus sit valor functionis $\Gamma(\frac{n-m}{n})$.

Integralibus autem A et B inter se relatis hanc locum obtinere videmus aequationem

$$(n-m)\int_0^1 \frac{dx}{\sqrt[n]{(1-x)^{nm}}}\int_0^1 \frac{x^{n-m}\,dx}{\sqrt[n]{(1-x)^{n}}} = m-1)\int_0^1 \frac{dx}{\sqrt[n]{(1-x)^{n-(m-1)}}}\int_0^1 \frac{x^{m-1}\,dx}{\sqrt[n]{(1-x)^{n}}}$$

cui addimus hanc notatu dignam

$$A_{(1)}\,.\,B_{(1)}\,.\,A_{(2)}\,.\,B_{(2)}\,.\ \dots\ A_{(n-1)}\cdot B_{(n-1)} = \frac{1}{(n-2)!}\left(\frac{\pi}{n\,\sin\frac{\pi}{n}}\right)$$

Jam denique tertia est classis, quae obtinetur, si pro b deinceps fractiones

$$-\frac{1}{n} - \frac{2}{n} \dots - \frac{(n-1)}{n}$$

pro (a) autem simul numeros integros inde ab unitate usque ad (n-1) substitueris. Ita enim impetrantur integralia

$$\int_0^1 \frac{dx}{\sqrt[n]{(1-x)^{n}}} = \frac{\pi}{n\ \sin\frac{\pi}{n}}\ \dots\ \int_0^1 \frac{x^{n-2}\,dx}{\sqrt[n]{(1-x)^{n-1}}} = \frac{\pi}{n\ \sin\frac{(n-1)\pi}{n}}$$

quae digna esse visa sunt, ut hoc loco commemorarentur.

VITA.

Bernardus Josephus Féaux, Monasteriensis, superioris magisterii candidatus, natus sum pridie Nonas Februarias a. MDCCCXXI patre Henrico, matre Gertrude, e gente Schmedding, quam ante sedecim menses praematura morte mihi abreptam lugeo. Fidei addictus sum catholicae. Posteaquam primis litterarum elementis in Schola Paulina imbutus sum, Gymnasium patrium, quod tum illmo. Nadermann directore florebat, praeceptoribus ordinariis deinceps Viris spectatissimis ac doctissimis, sup. ord. magistris, Phil. DD. Fuisting, Hesker, Boner, Dieckhoff, hoc ipso professore, usus frequentavi; quibus ceterisque praeceptoribus honoratissimis, optime de me meritis, gratias ago quam possum maximas. Jam cum ante hos tres annos testimonio maturitatis donatus essem, in studia matheseos inprimis et physices incubiturus Academiae patriae, illmo. Kellermann Rectore, adscriptus civibus nomen meum apud illmum Grauert, spectatissimum Ordinis Philosophorum Decanum rite professus sum. Lectionibus autem interfui per semestre hibernum illmi Gudermann: de analysi, geometria analytica, physice mathematica, illmi Grauert: de aevi recentioris culturae historia, cel. Becks: de anatomia comparata, zoologia, geognosia; exp. Schlueter: de animi immortalitate. Deinde quum a SUMMO, QUI PRAEEST REI LITTERARIAE BORUSSICAE, MAGISTRATU impetrassem, ut quod beneficium in tot litterarum studiosos iuvenes, omnibus opibus destitutos, conferretur, idem in me conferri PLACERET; stipendio insigni donatus almam Musarum sedem Rhenanam petii, ubi illmo. G Bischof

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Rectore per annum solidum in mathesi duces mihi fuerunt ill. Pluecker, ill. Argelander, cel. de Riese; in physice ill. Pluecker, in astronomia ill. Argelander, in philologia ill. Schopen. Tum vero ad Almam Universitatem Fridericam Guilelmam Berolinensem me contuli, ubi cum illmo de Raumer, Rectore, ab illmo. Trendelenburg, gratiosi Ordinis Philosophorum Decano maxime spectabili inter philosophiae studiosos receptus essem, per tria semestria in studia incumbere licuit. Docuerunt autem me: ill. Lejeune-Dirichlet theoriam numerorum et quis esset in hac theoria calculi integralis usus, ill. Ohm algebram sublimiorum, calculum variationis eiusque in doctrina de maximis et minimis usum; ill. Dirksen serierum infinitarum summationem et aequationum differentialium integrationem; cel Steiner curvarum secundi ordinis aliarumque proprietates synthetice evolvendas, cuius etiam exercitationibus geometricis interfui: cel. Minding staticen et mechanicen. In chemia autem ducem secutus sum illmum Mitscherlich, in physice celmos. Dove, A. Erman, Magnus, Poggendorff, in astromia illmum Encke, in botanice excursionibusque eo spectantibus, illmum Kunth. Jam logicen, methaphysicen, psychologiam edoctus sum ab illmo Trendelenburg, paedagogicen et didacticen a celmo. Beneke, illmo Lachmann denique Horatii epistolas atque illmo. Becker orationes Thucydideas explicantibus interfui.

Omnibus, quos nominavi, doctrina et humanitate insignibus viris, optime de ingenii mei cultura meritis, quibus addendus est cel. Joachimsthal, Phil. D. et matheseos magister Berolinensis, gratias ago maximas et, donec vivam, habebo.

THESES.

1) Kantii de spatio et tempore ideae falsae sunt.
2) Herbartii methodus philosophandi mathematica non sufficit.
3) Qui connexum integralia definita inter et numerorum theoriam mirum esse dicunt, errant.
4) Numerorum theoria nulla arithmetices pars ad ingenii facultates excolendas accommodatior est.
5) Quamquam Statice non est nisi casus particularis quidam Dynamices, tamen haec ipsa in illam reducenda.
6) Nitrogenium chemicum esse elementum verisimile non est.
7) Lapidum meteoricorum origo cosmica.

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