KEY TO ROBINSON'S NEW GEOMETRY & TRIGONOMETRY CONIC SECTIONS AND ANALYTICAL GEOMETRY.

KEY

TO

ROBINSON'S

NEW GEOMETRY AND TRIGONOMETRY,

AND

CONIC SECTIONS AND ANALYTICAL GEOMETRY.

WITH

SOME ADDITIONAL ASTRONOMICAL PROBLEMS.

DESIGNED FOR TEACHERS AND STUDENTS.

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PREFACE.

A Key to a Text-book on the Higher Mathematics, if not a new creation, is by no means a common thing. And it is a question undecided in the minds of many, whether a Key to any mathematical work is an aid, or a hindrance to the teacher.

The value of a Key depends upon the *use* to which it is applied. Class, and school-room duties so fully occupy the time of a majority of teachers, as to make a Key a great convenience, if not a necessity. And teachers of limited acquirements, as well as private students, will often find a Key of great service.

A Key should never be used to supersede investigation and labor, but to give direction to study, and make labor more effective. It should lessen the *mechanical* labor of teaching, by showing *how* to study, and *how* to teach, by giving the best forms of analysis, the best arrangement of the work, new and improved methods of operations, and often by developing methods of solution too elaborate to find a place in the Text, thus giving to those who use a Key, more extended and enlarged views of the applications of Mathematical Science.

In the preparation of the present work, great care has been taken to give full and comprehensive solutions of the examples and problems in Robinson's Plane and Solid Geometry, Plane and Spherical Trigonometry, and Analytical Geometry, and by means of cuts, more fully to explain and demonstrate the principles involved, thus making it a sort of commentary on the Text itself, and which in the hands of a good teacher may prove a valuable auxiliary in teaching.

Several pages of miscellaneous Astronomical examples have been added after Trigonometry. As soon as the New Calculus and Surveying, which are in preparation, are completed, a Key to those works will be also added to this.

JANUARY 1st., 1862.

KEY TO

ROBINSON'S

NEW GEOMETRY AND TRIGONOMETRY.

PRACTICAL PROBLEMS.

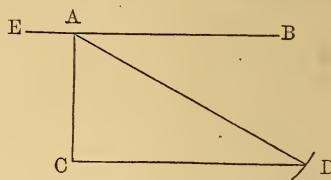
(BOOK V.—PAGE 143.)

Ex. **1.** The vertical angle being 60°, and the sum of all the angles being 180, therefore the sum of the other two angles, is 120°, and, as these angles must be equal to each other, each one must be 60°, and the triangle is *equilateral* as well as *isosceles.* Each side is therefore equal to the base 6.

Ex. **2.** As the two oblique angles of any right angled triangle equal 90°, and as one of them is 30°, the other must be 60°. Describe an equilateral triangle, each angle of which will be 60°. Divide either angle into two equal parts, by a line which will divide the opposite side into two equal parts, and the triangle into two equal triangles. The least angle, in either of these triangles is 30°, and the side opposite 30° will be half of the hypotenuse, *and this is a general truth.* Namely, *That in every right angled triangle whose least angle is* 30°, *the side opposite that angle is one-half of the hypotenuse.*

Therefore, if in this example the least side is 12, the hypotenuse is 24.

Ex. **3.** Let AB, CD, represent two parallels, and AC the perpendicular distance between them, equal 10. Take AD, in your divides, double the distance AC, and strike an arc, cutting CD in D. Join AD, and the angle ADC will be equal to 30°, because ACD is a right angled triangle, and its hypotenuse double the side AC (By Ex. 2). But $DAB=ADC$, being alternate angles, and because $DAB=30°$, its adjacent angle is $180°-30°=150°$.



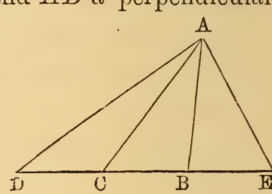
Hence the answer must be 30° and 150°.

Ex. **4.** We may use the same parallels as in (Ex. 3), to illustrate this example. Let $AC=20$, take $CD=20$, and join AD, and then will $AC=CD$, and ACD will be an isosceles triangle, and the angles ADC and CAD, equal to each other. Their sum must be 90°, therefore each one of them must be 45°. Now as $CAB=90°$, and $CAD=45°$, therefore, $DAB=45°$, and $DAE=90°+45°=135°$.

$$\text{Because } \overline{AC}^2+\overline{CD}^2=\overline{AD}^2,\ AC=20,$$

$$\overline{AD}^2=800=400\times 2. \quad \text{Whence } AD=20\sqrt{2}. \quad \textit{Ans.}$$

Ex. **5.** Let DE be one parallel, and AB a perpendicular to the other parallel. Take $BE=BC$ and BD any distance greater than BC. Assume $AD=15$, $AC=AE=10$, and $AB=8$. Now, by the well known property of the right angled triangle, we have



$$CB^2=\overline{BE}^2=\overline{10}^2-8^2=6^2. \quad \text{Whence, } CB=6=BE.$$

$$\text{Also, } \overline{DB}^2=\overline{15}^2-8^2=23\times 7=161$$

$$\left.\begin{array}{r}\text{Whence, } DB=\sqrt{161}=12.69, \text{ and } DE=18.69\\ \text{and } DC=\ \ 6.69\end{array}\right\} \textit{Ans.}$$

Ex. **6.** This problem is the same as (Ex. 5,) except in data, therefore use the same cut as before.

Make $AB=12$, $AD=20$, $AC=AE=18$.

Now, $\overline{BD}^2=(\overline{20}^2-\overline{12}^2)=32\times 8=16\times 16$. Whence, $BD=16$.

$\overline{CB}^2=\overline{BE}^2=\overline{18}^2-\overline{12}^2=30\times 6=180.\qquad CB=13.416.$

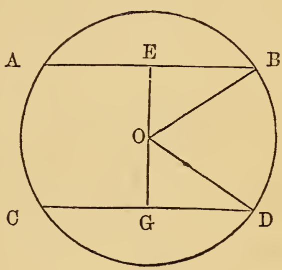
$$\left.\begin{array}{l}\text{Sum, } DE=29.416\\ \text{Diff. } DC=\ \ 2.584\end{array}\right\}\ \textit{Ans.}$$

$$\left.\begin{array}{l}\text{The area of the triangle } ADE=(29.416)6=176.496\\ \text{The area of the triangle } ADC=(\ \ 2.584)6=\ \ 15.504\end{array}\right\}\ \textit{Ans.}$$

Ex. **7.** This example produces a right angled triangle, the hypotenuse is 6, and one side 2, one half of the chord. Therefore the other side is,

$$\sqrt{6^2-2^2}=\sqrt{32}=4\sqrt{2}.\quad \textit{Ans.}$$

Ex. **8.** The two parallel chords being equal, they must be equally distant from the center, and be on opposite sides of the center.



Let AB, CD, be those paralells; and from the center draw OB, OD; and OE, OG, each perpendicular to AB, and CD; then EOG, will be one right line through the center. $EG=6$, therefore $EO=3$. $AB=8$; hence, $EB=4$; and in the right angled triangle EOB, we have $\overline{OB}^2=4^2+3^2=25$. Whence, $OB=5$. *Ans.*

Ex. **9.** This is a problem of the same kind as (Ex. 8,) and therefore we may use the same cut.

Let AB, CD, be the parallel chords. Put $AB=16$,

$CD=12$, and $EG=14$. Place $OB=OD=R$, $EO=x$, $OG=y$. Then $EG=x+y=14$, or $y=14-x$.

$$\text{And} \quad x^2+8^2=R^2$$
$$\text{Also} \quad y^2+6^2=R^2$$

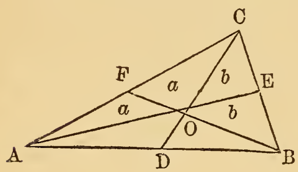
$$\text{Whence } x^2+8^2=y^2+6^2=(14-x)^2+6^2$$
$$x^2+28=\overline{14}^2-28x+x^2$$
$$\text{Or } 28x+28=\overline{14}^2$$
$$2x+2=14, \text{ or } x=6. \quad \text{Whence } y=8,$$
$$\text{And } R^2=8^2+6^2=100, \text{ or } R=10. \quad \textit{Ans.}$$

Ex. **10.** Let x represent the perpendicular required, which is the perpendicular drawn from the vertex of an isosceles triangle to the center of its base.

Then $x^2=15^2-5^2=20.10=2(10)^2$, or $x=10\sqrt{2}$. *Ans.*

Ex. **11.** Let ABC be the triangle, bisect CB in E, AC in F. And join AE and BF, and through the point of intersection O, draw CO, and produce it to D.

Because CB is bisected in E, the triangles ABE and AEC are equal. Or the triangle AEC is half the triangle ABC.



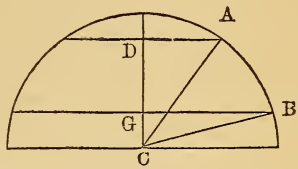
In like manner, we prove that the triangle BFC is half ABC, because AC is bisected in F. But BOC consists of two equal triangles, BOE and EOC, each of which may be represented by b, and AOC consists of the equal triangles, AOF and FOC, each of which may be represented by a. Therefore BFC consists of $2b+a$, and AEC consists of $2a+b$. Whence, $2b+a=2a+b$.

That is, $b=a$.

Now it is obvious that, BFC is equal to three triangles, each equal to a, or each equal to b.

Hence, the triangle AFB must be equal to $3a$, and AOB equal to $2a$. But the triangle ABC is equal to $6a$. That is, AOB is equal to one third of ABC.

Or, $6a$ to $2a$ as 3 to 1. *Ans.*



Ex. **12.** If the diameter of a circle is 32, its radius is 16. Hence, $CB=16$, $CA=16$, and we have two right angled triangles, CGB and CDA; $GB=10$, half of one chord, and $AD=4$, half of the other parallel chord.

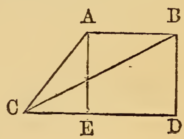
Now, these two right angled triangles give us

$$\overline{CD}^2=\overline{16}^2-4^2 \;=20\times 12=240,$$
$$\overline{CG}^2=\overline{16}^2-\overline{10}^2=26\times 6=156,$$

Whence $\quad CD=\sqrt{240}=15.49+$

And $\quad CG=\sqrt{156}=12.49+$

Diff.$=DG=3$, if the parallels be the same side of the center, but when on opposite sides, their sum 27.98 will be the distance between the parallels.



Ex. **13.** Here $CD=12$, $AB=8$, $BD=5$. Conceive CA and DB produced until they meet, thus forming a triangle of which AB is the base. Also conceive a line drawn from the vertex of this triangle perpendicular to the base, and designate it by x.

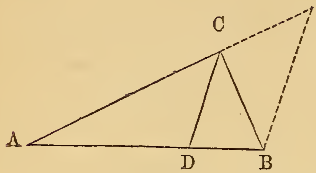
Now by proportional triangles, we shall have,

$$x : 8 :: x+5 : 12$$

And $\quad 12x=8x+40$. Whence, $x=10$.

The area of the triangle on AB as a base, is therefore

$4x=40$. The area of the trapezoid is $\frac{12+8}{2}\times 5=50$, and the area of the whole triangle on the base CD is 90. *Ans.*



Ex. 14. Let ABC be the triangle $AB=697$, $AC=813$, $CB=534$.

Now by (Th. 24, B. II.), we have

$$AD : DB : : 813 : 534$$
$$271 : 178$$

Whence, $178\ AD=271\ DB$. And $AD+DB=697$

$$178\ AD+178\ DB=697\times 178$$

That is, $271\ DB+178\ DB=697\times 178$

Or $449\ BD=697\times 178$

$$BD=\frac{697\times 178}{449}$$

Thus we find $BD=276.316$, the less part.
Whence $AD=420.684$, the greater part. } *Ans.*

Examples 15, 16, and 17, solutions in text-book.

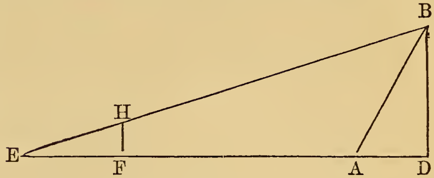
Ex. 18. By referring back to (Ex. 15), we observe, that HC must be taken from DH. But $DH=\sqrt{51}$, and $HC=\sqrt{40}$, hence if HC were taken from DH, the point C from D would then be $(\sqrt{51}-\sqrt{40})$. But $AD=16$. Whence, according to this condition,

$$\overline{AC}^2=(\sqrt{51}-\sqrt{40})^2+\overline{16}^2$$
$$=91-2\sqrt{2040}+256=347-2\sqrt{2040}.$$

Or, $\overline{AC}^2=347-2\sqrt{2040}=347-90.332=256.668.$

Whence, $AC=16.021$
Area of the triangle is $\frac{1}{2}(9\sqrt{51}+7\sqrt{40})$. } *Ans.*

Ex. **19.** Let E be the position of the observer's eye, $EF=40$ feet, $FH=24$ feet, ED the required distance which we designate by x, and $DB=$ 90 rods, which must be reduced to feet thus $90\times16\frac{1}{2}$, or 45×33.



Now by proportional triangles EFH, EDB, we have,

$$40 : 24 : : x : 45\times33$$

$$\text{Or, } 10 : 6 : : x : 45\times33 \quad \text{or,} \quad 6x=45\times330$$

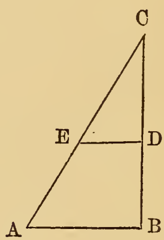
$$2x=45\times110$$

$$\text{or,} \quad x=2475 \text{ feet.} \quad \textit{Ans.}$$

Ex. **20.** Illustrated in text-book.

Ex. **21.** Let ABC be a triangle divided into two parts by the line DE, parallel to the base AB.

Let CDE be taken, as 1, and the other part $ABDE$, as 3. Then the whole triangle ABC will be represented by 4, the triangle CDE, being unity.



Now by (Th. 22, B. II.), we have,

$$\overline{CD}^2 : \overline{CB}^2 : : 1 : 4$$

$$\text{Square root,} \quad CD : CB : : 1 : 2.$$

Whence $CB=2CD$, or CD equals one half of CB, and therefore CE equals one half of CA.

Ex. **22.** Let the last triangle represent the triangle in this example also. That is, place $AB=320$, and as the angles adjacent to the base are 90° and 60°, the angle opposite at C must be 30°. The side opposite 90° must be double of the side opposite 30°, as was explained in (Ex. 2), of this key. Therefore, $AC=640$.

Again, because the angle $B=90°$.

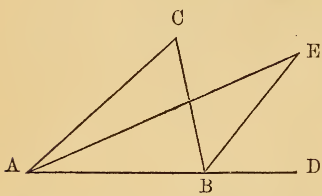
$$\overline{AB}^2+\overline{BC}^2=\overline{AC}^2$$

That is, $$\overline{320}^2+\overline{BC}^2=\overline{640}^2$$

Or, $$\overline{BC}^2=\overline{640}^2-\overline{320}^2=(320)^2(2^2-1)=\overline{320}^2\times 3.$$

Whence, $$BC=320\sqrt{3}=554.24. \quad \textit{Ans.}$$

Ex. **25.** Draw AB of any indefinite length, and from the point A draw AC, making an angle of 32° with AB, and from B draw the line BC, making with BA an angle equal to 84°. Produce AB to D. Because the sum of the three angles of any triangle is equal to two right angles, the angle C of the triangle ABC, must be

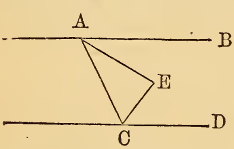


$$C=180°-(32°+84°)=64.$$

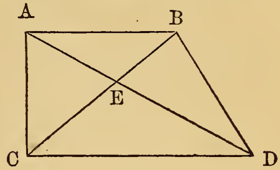
The exterior angle $CBD=180°-84°=96°$, and because EBD is one half of it, the exterior angle EBD of the triangle AEB is 48°. But the exterior angle $EBD=E+EAB=E+16°$.

Therefore, $E+16°=48°$, and $E=32°$, one half the angle C, a general result.

Ex. **26.** Let AB and CD be two parallels, and AC a line between them; because the two interior angles between two parallels are equal to two right angles, therefore, one half of each of them, added together, must make one right angle. That is, the angle EAC, and ECA together, make one right angle, therefore the third angle at E, of the triangle AEC, must be a right angle.

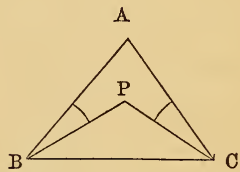


Ex. 27. Let $ABCD$ be any trapezoid, and draw its diagonals AD and CB, intersecting in E. The two triangles AEB and CED are equi-angular and similar. The opposite angles at E are equal; and because, AB and CD are parallel, the alternate angles EAB and EDC are equal, and so are the alternate angles EBA and ECD, therefore the two triangles AEB and CED, are equi-angular, and AB and CD, sides opposite the equal angles E, are homologous. Therefore, by (Th. 20, B. II.), we have

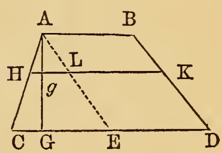


$$\overline{AB}^2 : \overline{CD}^2, \text{ as } AEB \text{ is to } CED.$$

Ex. 28. Let ABC be any triangle, the sum of its three angles is 180°. Take any point P within it, and draw PB, PC. If we designate the angle between AB and PB by a, and between AC and PC by b, then the angles at the base of the triangle PBC, will be less than the angles at the base of the triangle ABC by the angles a and b. But as the sum of the angles of every triangle make two right angles, therefore the angle BPC is greater than the angle A, by the sum of the two angles a and b.



Ex. 29. Let AB and CD be 12 and 20 respectively, and the perpendicular AG equal 8. Let HK represent the line between the parallels. Then $HL = 14\frac{1}{2} - 12 = 2\frac{1}{2}$. And we have,



$$CE : AG = HL : Ag$$

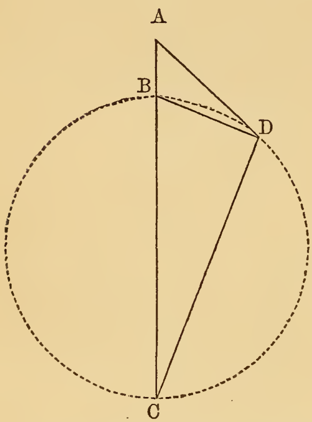
Or, $$8 : \quad 8 = 2\tfrac{1}{2} \quad : Ag.$$

Hence, $Ag=2\frac{1}{2}$, and $gG=5\frac{1}{2}$.

Area of $ABCD$ is $\frac{20+12}{2}\times 8=128$.

Area of $ABKH$ is $\frac{14.5+12}{2}\times 2\frac{1}{2}=33\frac{1}{8}$.

Area of $HKDC$ is $\frac{20+14.5}{2}\times 5\frac{1}{2}=94\frac{7}{8}$.



Ex. **30.** The triangles ACD and ADB are similar, and therefore, the sides about the angle A are proportional, giving the proportion

$$AC : AD :: AD : AB.$$

But AC expressed in feet is the product of 7956 by 5280, plus $AB=40$ feet.

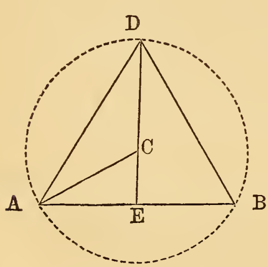
$$7956\times 5280=42007680.$$

Now the above proportion becomes,

$$42007720 : AD :: AD : 40$$

Whence, $\overline{AD}^2=1680308800$

Sq. root $AD=40992$ feet nearly, or 7 miles and 4032 feet.



Ex. **31.** As the triangle ABD in the circle is to be equilateral, it must also be equiangular. Hence, the line AC from one of the angles to the center must bisect the angle A, and each division must be 30°. $AC=2$, and the side opposite

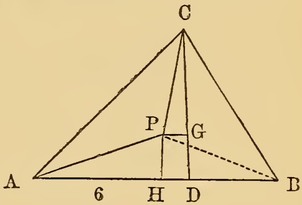
30° is half AC, therefore, $CE=1$, and $DE=3$. Place $AE=x$.

Then, the right angled triangle AEC gives $x^2+1=4$.

Whence, $x=\sqrt{3}$. But AE into DE, is the area sought.

That is, $3\sqrt{3}$, is the area of the triangle ABD.

Ex. **32.** Let $AB=12$, $AC=11$, $CB=10$. From C draw CD perpendicular to AB. Take H the middle point between A and B, and draw HP at right angles to AB, until it meets a perpendicular drawn from the middle point between A and C; then P will be the location of the well. Draw PG parallel to AB, $AH=HB=6$. PH is common to the two triangles APH and PHB, therefore, $AP=PB$, $AP=PC$ by construction. Place $PH=x$ and $HD=y$.



Then, $AD=6+y$ and $DB=6-y$.

Now $(6+y)^2+\overline{CD}^2=\overline{11}^2$, and $(6-y)^2+\overline{CD}^2=\overline{10}^2$.

By substitution, $24y=121-100=21$, or $y=\frac{7}{8}$.

This value of y substituted in the preceding equation gives

$$(5\tfrac{1}{8})^2+\overline{CD}^2=100,$$

Or, $$25+\tfrac{10}{8}+\tfrac{1}{64}+\overline{CD}^2=100,$$

Or, $$210+\tfrac{1}{8}+8\overline{CD}^2=800.$$

Whence, $8\overline{CD}^2=589.875$, and $CD=8.587$.

Secondly. In the right angled triangles AHP and CPG, we have,

$$6^2+x^2=R^2. \qquad R=AP=PC=PB.$$

And $$(8.587-x)^2+(\tfrac{7}{3})^2=R^2.$$

From these two equations, we obtain x, and afterwards R.

Ex. **33.** The required perpendicular will obviously bisect the base of the triangle, and divide the figure into two equal right angled triangles. Represent the common perpendicular by x. Then $x^2+36=400$, or, $x^2=364$, $x=19.07$.

Area sought $=(19.07)6$.

Ex. **35.** This problem is a supplement to (Ex. 34), which is explained in the text-book.

In (Ex. 34), we have the two sides of a right angled triangle (27.3) and (35.76). This problem demands the hypoteeuse to that triangle, and its division into parts in the ratio of (27.3) to (35.76). (Th. 24, B. II.)

Place $a=27.3$, and $b=35.76$, and $x+y=$ the required hypotenuse, then $x+y=\sqrt{a^2+b^2}$ (1)

But, $x : y :: a : b$. Or, $bx=ay$ (2)

Multiply (1) by b, then

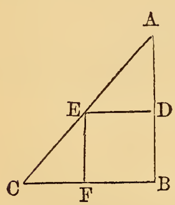
$$bx+by=b\sqrt{a^2+b^2}$$

That is, $ay+by=b\sqrt{a^2+b^2}$ $\quad y=\dfrac{b\sqrt{a^2+b^2}}{a+b}$.

And $\quad x=\dfrac{a\sqrt{a^2+b^2}}{a+b}$.

Ex. **36.** If 6 feet makes a descent of 1 foot, what distance will be required for a descent of 4 feet. Represent that distance by x.

Then, $6 : 1 :: x : 4$. Or, $x=24$. *Ans.*



Ex. **37.** Let ABC represent the triangle, and DE a side of the inscribed square.

Place $AD=x$, $DE=12=a$.

Now by proportional triangles we have,

$x : a :: x+a : BC$. Whence $BC=\dfrac{a(x+a)}{x}$

Also, $$\overline{AB}^2+\overline{BC}^2=\overline{AC}^2$$

That is, $$(x+a)^2+\frac{a^2}{x^2}(x+a)^2=\overline{35}^2$$

Or, $$x^2(x+a)^2+a^2(x+a)^2=\overline{35}^2x^2$$

That is, $$(x^2+a^2)(x+a)^2=\overline{35}^2x^2$$

Or, $$(x^2+a^2)(x^2+2ax+a^2)=(35)^2x^2$$

Whence $$(x^2+a^2)^2+2ax(x^2+a^2)=(35)^2x^2.$$

Place $y=x^2+a^2$. Then, $y^2+2ax.y=(35)^2x^2$.

$$y^2+2ax.y+a^2x^2=(a^2+(35)^2x^2$$
$$=1369x^2.$$

Square root $$y+ax=37x,$$
$$y=25x.$$

But $y=x^2+144$. Therefore, $x^2+144=25x$

$$x^2-25x=-144.$$

From this equation we find, $x=16$, or 9.

That is,

$AD=16$ or 9. But $DB=12$, whence $AB=28$ or 21. *Ans.*

To find BC we have $BC=a\dfrac{(x+a)}{x}=\dfrac{12(16+12)}{16}=21.$

Or, $\dfrac{12(9+12)}{9}=28.$

Ex. 38. Taking the triangle as in the preceding Example, we observe that the least hypotenuse in a triangle containing the same square, must require $AD=DE$, or $AB=BC$.

If $AD=DE$, AB must be equal to $2DE$, because $DB=DE$. Therefore $AB=2a$, and $BC=2a$. Each side double the side of the square.

BOOK VII.

PRACTICAL PROBLEMS.

(Page 229.)

Ex. **1.** The result sought is $\frac{1}{6}\pi D^3$. $\frac{1}{6}\pi = 0.5236$, and $D = 12$. Whence, $\frac{1}{6}\pi D^3 = (1728 \times 0.5236) = 904.78$. *Ans.*

Ex. **2.** The result required in this problem, is expressed by $\frac{1}{3}\pi H^2(3R - H)$. (See Geometry, page 229).

When applied to this problem, $H = 3$, $R = 6$, $\pi = 3.1416$. Whence,

$$\tfrac{1}{3}\pi H^2(3R - H) = (1.0472)9 \times 15 = 141.372 \text{ cubic in.} \quad \textit{Ans.}$$

Ex. **3.** To solve this problem, we place $4\pi R^2 = 68$, $2R$ being the value sought. Whence,

$$\sqrt{\frac{68}{3.1416}} = 4.652. \quad \textit{Ans.}$$

Ex. **4.** In this problem the result sought is $2\pi RH$, but in the preceding Example, the value of $2R$ was found to be 4.652 feet. $H = 2$ feet. $2\pi = 6.2832$.

Therefore, the result sought is the product of the factors,

$$(4.652)(6.2832) = 29.229 + \text{ square feet.} \quad \textit{Ans.}$$

Ex. **5.** For the solidity of any sphere, we have $\frac{1}{6}\pi D^3$, $\frac{1}{6}\pi = 0.5236$, and in this problem $D = 4.652$. Whence,

$$\tfrac{1}{6}\pi D^3 = (4.632)^3(0.5236) = 52.72. \quad \textit{Ans.}$$

For the solidity of the segment, we have the general formula, $\frac{1}{3}\pi H^2(3R-H)$, and for this example we have $H=2$ feet, and $R=2.326$. Whence, $3R-H=4.978$, and the solidity sought is the product of the three factors,

$(1.0472)4(4.978)$, which is 20.852. *Ans.*

Ex. 6. The general expression for the solidity of a segment of a sphere having two bases, is

$$\frac{1}{6}\pi H^3+\frac{\pi H}{2}(R'^2+R''^2)$$

R' and R'' being the radii of the bases, and H the perpendicular distance between them.

In this Example the radius of the sphere is 10, and $H=2$. $(R')^2$ and $(R'')^2$ must be computed.

For the greater segment, we have,

$$(R')^2=100-9, \text{ and } (R'')^2=100-25.$$

Whence,

$(0.5236)(8)+(3.1416)(166)=525.7$ cubic feet. *Ans.*

For the smaller segments, we have, $(R')^2=100-25=75$.

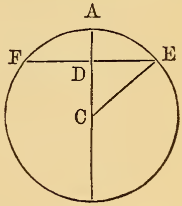
and $(R'')^2=100-49=51$.

Whence,

$4.1888+(3.1416)(126)=400.0304$ cubic feet. *Ans.*

Ex. 7. Let the circle AFB represent a central section of the sphere, FE the diameter of the segment required. Let $DE=8$, $AD=4$, and $AC=EC=R$; then in the triangle CDE,

$$DC=R-4,$$

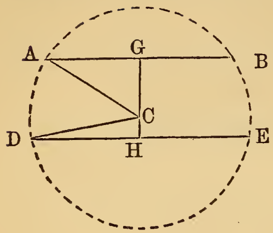


And, $(R-4)^2+8^2=R^2$. Whence, $R=10$.

Now for the solidity of the segment, we have

$$\tfrac{1}{3}\pi H^2(3R-H)=(1.0472)16(30-4)=435.6352. \quad \textit{Ans.}$$

Ex. 8. Let AB and DE be the diameter of the segment, and they are parallel, and their distance asunder being 9 inches, $AG=7$, and $DH=9$, and AC and $DC=$ each to R. Place $CH=x$, and the two right angled triangles AGC and DCH, give



$$7^2+(9-x)^2=R^2,\ 9^2+x^2=R^2.$$

Whence, $49+9^2-18x+x^2=9^2+x^2$, or $18x=49$, $x=2\frac{13}{18}$.

Now $$R^2=81+\left(\frac{49}{18}\right)^2=\frac{26244+2401}{324}=\frac{28645}{324}.$$

$$R=9.4027.\quad \textit{Ans.}$$

For the value of the segment, we have,

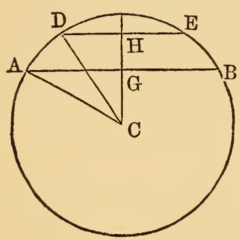
$$.5236.(729)+\frac{9.(3.1416)}{2}(81+49)=2219.5+$$

Ex. 9. In this example, the height of the segment is 6, and the solution is

$$.5236(216)+\frac{6.(3.1416)}{2}(300+144)=4297.7.$$

Ex. 10. The contents of the segments will be

$$\tfrac{1}{6}\pi(2^3)+\frac{2\pi}{2}(100+36)=431.45 \text{ cu. mi.}$$



To find the diameter,

Let $CA=x$, $CG=y$

Then, $CH=y+2$. Hence,

$$x^2=y^2+100 \qquad (1)$$

$$x^2=(y+2)^2+36 \qquad (2)$$

Whence we have

$$(y+2)^2=y^2+100$$

$$y=15.$$

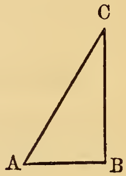
And from (1), $x=\sqrt{225+100}=18.027$, radius.

BOOK VIII.

APPLICATION OF ALGEBRA TO GEOMETRY.

PROBLEM VI.

In a right angled triangle, having given the base, and the sum of the perpendicular and hypotenuse to find these two sides.



Let ABC be the triangle, $AB=b$, and $AC+CB=S$, and $BC=x$.

Whence, $AC=S-x$.

But $\overline{AB}^2+\overline{BC}^2=\overline{AC}^2$

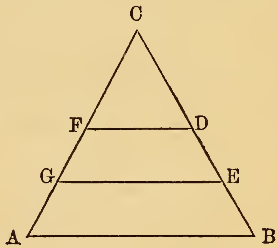
That is, $b^2+x^2=(S-x)^2=S^2-2Sx+x^2$

$$2Sx=S^2-b^2$$

$$x=\frac{S^2-b^2}{2S}=CB,\ AC=S-\frac{S^2-b^2}{2S}=\frac{S^2+b^2}{2S}.$$

PROBLEM VII.

Given the base and altitude of a triangle, to divide it into three equal parts, by lines parallel to the base.



Let ABC represent the $\triangle$. Conceive a perpendicular let drop from C to the base AB, and represent it by b. Put $2a=AB$. Then $ab=$ the area of the triangle.

Let x be the distance from C to FD; then by (Th. 22, Bk. II.),

$$x^2 : b^2 :: \tfrac{1}{3}ab : ab$$

Whence, $x : b :: 1 : \sqrt{3}$. Or, $x=\frac{b}{\sqrt{3}}$.

If x represents the distance from C to GE, then

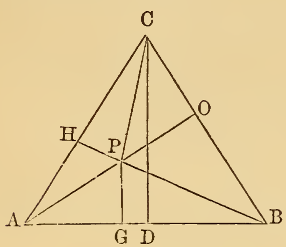
$$x^2 : b^2 : : \tfrac{2}{3}ab : ab.$$

Or, $$x : b : : \sqrt{2} : \sqrt{3}. \quad x=\frac{\sqrt{2}\cdot b}{\sqrt{3}}.$$

We perceive by this that the divisions of the perpendicular are independent of the base, and that we may divide the triangle into any required number of parts, m, n, p, etc., equal or unequal.

PROBLEM VIII.

In any equilateral triangle, given the lengths of three perpendiculars drawn from any point within to the sides, to determine the sides.



Let ABC be an equilateral triangle, and because CD is drawn perpendicular to the base, it bisects the base. Place $AD=x$. Then AB, BC, AC each equal $2x$. Take $PO=a$, $PG=b$, $PH=c$. The area of the triangle ABC equals $(CD)x$. But in the right angled triangle ADC, we have, $\overline{CD}^2+x^2=4x^2$. Whence, $CD=x\sqrt{3}$, and the area of the triangle ABC, therefore is $x^2\sqrt{3}$.

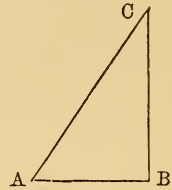
Again, the area of the triangle APB, is $\left(PG\times\frac{AB}{2}\right)$, or bx. Area $BPC=ax$ and $APC=cx$, and the sum of these three interior triangles, equals the triangle ABC.

That is, $x^2\sqrt{3}=ax+bx+cx$

Whence, $2x=\dfrac{2(a+b+c)}{\sqrt{3}}$ the length of a side.

PROBLEM IX.

In a right angled triangle, having given the base (3), *and the difference between the hypotenuse and perpendicular* (1) *to find the sides.*



Place $AB=3$, $AC-CB=1$.

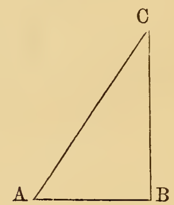
Make $CB=x$, then $AC=1+x$.

And $9+x^2=1+2x+x^2$.

Whence, $2x=8$, $x=4$. Then, $AC=5$

PROBLEM X.

In a right angled triangle, having given then hypotenuse (5), *and the difference between the base and perpendicular* (1), *to find the sides.*



Place $AC=5$, $CB-AB=1$, $AB=x$.

Then $CB=1+x$.

Now $x^2+(1+x)^2=25$.

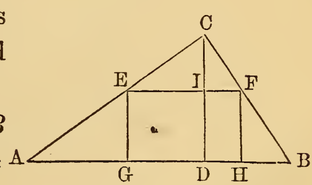
$$2x^2+2x=24.$$

Whence, $x=3$, $BC=4$. *Ans.*

PROBLEM XI.

Having given the area of a rectangle inscribed in a given triangle to determine the side of the rectangle.

When we say that a triangle is given, we mean that the base and perpendicular are given.



Let ABC be the triangle, $AB=b$, $CD=p$, $CI=x$; then $ID=p-x$.

By proportional triangles, we have,

$$CI : EF :: CD : AB$$

That is, $x : EF :: p : b.\quad EF=\frac{bx}{p}.$

By the problem $\frac{bx}{p}(p-x)=a.$ The symbol a being the given area.

Whence, $x^2-px=-\frac{ap}{b}.\quad x=\frac{1}{2}p\pm\sqrt{\frac{1}{4}p^2-\frac{ap}{b}}.$

PROBLEM XII.

In a triangle having given the ratio of the two sides, together with both the segments of the base, made by a perpendicular from the vertical angle, to determine the sides of the triangle.

Let ACB be the $\triangle$, (see last figure.) $AD=a$, $BD=b$, and $CD=x$. Then $AC=\sqrt{a^2+x^2}$, and $CB=\sqrt{b^2+x^2}$.

The ratio of AC to CB is given, and let that ratio be as 1 to r; then

$$\sqrt{a^2+x^2} : \sqrt{b^2+x^2} :: 1 : r.$$

Whence, $a^2+x^2 : b^2+x^2 :: 1 : r^2.$

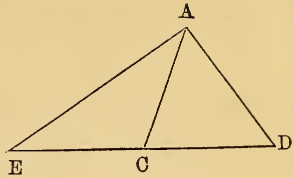
Or, $b^2+x^2=a^2r^2+r^2x^2.$

Or, $x^2=\frac{a^2r^2-b^2}{1-r^2}.$

But $AC=\sqrt{a^2+x^2}$, and as x^2 is now known, AC is known.

PROBLEM XIII.

In any triangle having given the base, the sum of the other two sides, and the length of a line drawn from the vertical angle to the middle of the base, to find the sides of the triangle.



Let ADE be the $\triangle$. Suppose C to be the middle of the base.

Put $AC=a$, DC or $CE=b$, $AE=x$, $DA+AE=c$; then $DA=c-x$.

Now by (Th. 42, B. I.), we have,

$$(DA)^2+(AE)^2=2(AC)^2+2(DC)^2$$

That is, $\quad c^2-2cx+2x^2=2a^2+2b^2.$

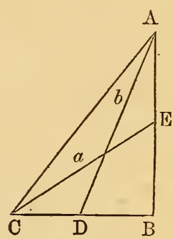
Or, $\quad 4x^2-4cx+c^2=4a^2+4b^2-c^2.$

$$2x-c=\sqrt{4a^2+4b^2-c^2}.$$

Whence x becomes known, and consequently the sides become known.

PROBLEM XIV.

To determine a right angled triangle, having given the length of two lines drawn from the acute angles to the middle of the opposite sides.



Let ABC be the triangle. Bisect AB in E, CB in D, and join $CE=a$, and $AD=b$.

Place $DB=x$, and $BE=y$.

Now in the two right angled triangles ABD, and CBE, we have,

$$4y^2+x^2=b^2 \qquad (1)$$
$$4x^2+y^2=a^2 \qquad (2)$$

Sum $\quad 5(x^2+y^2)=(a^2+b^2)$

Or, $\quad x^2+y^2=\frac{1}{5}(a^2+b^2) \qquad (3)$

But $\quad 4x^2+y^2=a^2 \qquad (4)$

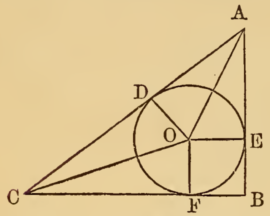
Sub. (3) from (4), and $3x^2=\frac{4}{5}a^2-\frac{1}{5}b^2$.

Whence, $\quad x=(\frac{4}{15}a^2-\frac{1}{15}b^2)^{\frac{1}{2}}$

Also, $\quad y=(\frac{4}{15}b^2-\frac{1}{15}a^2)^{\frac{1}{2}}.$

PROBLEM XV.

To determine a right angled triangle having given the perimeter and the radius of the inscribed circle.



Let ABC be the triangle, draw AO, CO to the centre of the circle, and it is obvious that the two right angled triangles AEO, ADO, are equal to each other. Also, $CFO=CDO$, and $BEOF$ is a square, each side equal to the radius of the circle.

Place $EB=BF=r$, $AE=AD=x$, and $CF=CD=y$. Designate the given perimeter by $2p$.

Then $$2x+2y+2r=2p$$

Or, $$x+y+r=p$$

Or, $$x+y=(p-r) \qquad (1)$$

Also, by the right angled triangle, we find,

$$(x+r)^2+(y+r)^2=(x+y)^2 \qquad (2)$$

Or, $$x^2+2rx+r^2+y^2+2ry+r^2=x^2+2xy+y^2.$$

Reducing and, $$2r(x+y)+2r^2=2xy.$$

Or, $$r(p-r)+r^2=xy.$$

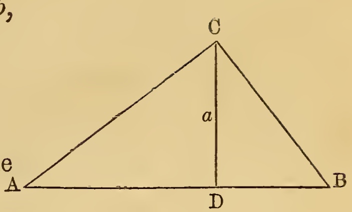
Whence, $$xy=rp \qquad (3)$$

Equations (1) and (3) will readily give the values of x and y, which solves the problem.

PROBLEM XVI.

To determine a triangle, having given the base, the perpendicular, and the ratio of the two sides.

Let ABC be the $\triangle$. $AB=b$, $CD=a$, $DB=x$. Then



$AC=\sqrt{(b-x)^2+a^2}$.

$CB=\sqrt{a^2+x^2}$.

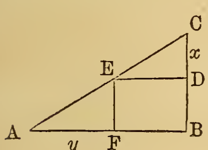
Let the given ratio of the sides be as m to n; then

$$\sqrt{(b-x)^2+a^2} : \sqrt{a^2+x^2} :: m : n.$$

This proportion will give the value of x, then AC and CB will be known.

PROBLEM XVII.

To determine a right angled triangle, having given the hypotenuse, and a side of its inscribed square.



Let ABC be the triangle, and $DE=a$, a side of the inscribed square. Place $CD=x$, and $AF=y$.

Also, let $AC=h$.

Then, $(x+a)^2+(y+a)^2=h^2$. (1)

And $x : a :: a : y$, (2)

Expanding (1), transposing, etc., gives

$$x^2+y^2+2a(x+y)=h^2-2a^2. \quad (3)$$

From (2) $2xy = 2a^2$. (4)

Sum of (3) and (4) $x^2+2xy+y^2+2a(x+y)=h^2$. (5)

Or, $(x+y)^2+2a(x+y)=h^2$. (6)

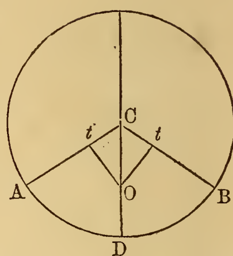
Whence, $x+y=-a\pm\sqrt{h^2+a^2}$. (7)

Equations (4) and (7) will give the value of x and y, and solve the problem.

PROBLEM XVIII.

To determine the radii of three equal circles inscribed in a given circle, touching each other, and each touching the circumference of the given circle.

Every circle consists of 360°, and therefore if three circles are to be placed in a given circle, each one of them must occupy a section of 120°, one third of 360°.



Let ACB be one of those sectors, and it is obvious that the centre of one of the interior circles must be on the radius CD of the larger circle, CD dividing the angle ACB into two equal parts.

It is also obvious that the interior circle which is placed in the sector ACB, must touch the given circle in the point D, and touch the radii CA, CB, in the points t, t, equidistant from C, and so situated that perpendiculars Ot, Ot, shall be equal to OD.

Place $OD=Ot=x$, then x represents the radius of one of the required circles, and let R represent CD, the radius of the given circle.

In the right angled triangles OtC the angle $tCO=60°$, angle $t=90°$, therefore the angle $tOC=30°$, $CO=R-x$; and in every right angled triangle, when one of the acute angles is 30°, the side opposite to that angle is one half the hypotenuse, therefore, $tC=\dfrac{R-x}{2}$

But $\qquad \overline{tO}^2+\overline{tC}^2=\overline{CO}^2.$

That is, $\qquad x^2+\dfrac{(R-x)^2}{4}=(R-x)^2.$

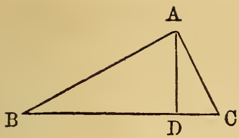
Or, $\qquad x^2=\frac{3}{4}(R^2-2Rx+x^2).$

$$x^2=3R^2-6Rx.$$

$$x=R(2\sqrt{3}-3). \quad Ans.$$

PROBLEM XIX.

In a right angled triangle, having given the perimeter, and the perpendicular let fall from the right angle on the hypotenuse, to determine the triangle; that is, its sides.



Let ABC be the $\triangle$, and represent its primeter by p. Put $AD=a$, $AB=x$, $AC=y$. Then $BC=p-x-y$.

Because BAC is a right angle,

$$x^2+y^2=p^2-2p(x+y)+x^2+2xy+y^2 \qquad (1)$$

$BC \cdot AD=2$ times the area of the triangle ABC.

Also, $AC \cdot AB=2$ times the area of the triangle ABC.

Therefore, $$a(p-x-y)=xy. \qquad (2)$$

Reducing (1), $$2p(x+y)=p^2+2xy \qquad (3)$$

Double (2), $$2ap-2a(x+y)=2xy \qquad (4)$$

By subtraction, $$(2a+2p)(x+y)-2ap=p^2 \qquad (5)$$

Whence, $$x+y=\frac{p^2+2ap}{2a+2p} \qquad (6)$$

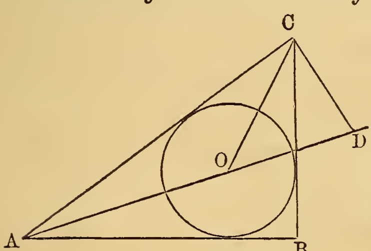
Because $BC=p-x-y$, $BC=p-\dfrac{p^2+2ap}{2a+2p}=\dfrac{p^2}{2a+2p}$

From (2) we observe that $xy=\dfrac{ap^2}{2a+2p}$ (7)

Equations (6) and (7), will readily give x and y.

PROBLEM XX.

To determine a right angled triangle, having given the hypotenuse, and the difference of two lines drawn from the two acute angles to the centre of the inscribed circle.



Let ABC be the triangle, and O the center of its inscribed circle, AO and CO being joined, the triangle AOC is formed, and AO being produced to meet a perpendicular from C, COD is an exterior angle.

AO bisects the angle at A, and CO bisects the angle ACB.

Therefore, the angle COD, is equal to half the angles A and C of the right angled triangle ABC. Consequently COD is $45°$; D being $90°$, the angle $OCD=45°$, whence $OD=CD$. Let $CO=x$, and $AO=d+x$, d being the given difference between AO, OC. Also, let $OD=y$, and $AC=h$.

Then $$2y^2=x^2 \qquad (1)$$

$$\overline{AD}^2+\overline{DC}^2=\overline{AC}^2.$$

That is, $$(x+y+d)^2+y^2=h^2 \qquad (2)$$

Expanding $x^2+y^2+d^2+2xy+2dx+2dy+y^2=h^2$.

Substituting the value of $2y^2$, as found in (1), and

$$2x^2+2xy+2d(x+y)=h^2-d^2$$

$$2x^2+\frac{2x^2}{\sqrt{2}}+2d\left(x+\frac{x}{\sqrt{2}}\right)=h^2$$

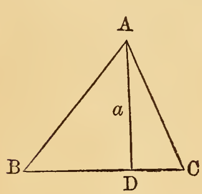
$$(2+\sqrt{2})x^2+(2+\sqrt{2})dx=h^2-d^2$$

$$x^2+dx=\frac{h^2-d^2}{2+\sqrt{2}}=m.$$

Whence, $$x=-\frac{d}{2}\pm\sqrt{m+\frac{d^2}{4}}.$$

PROBLEM XXI.

To determine a triangle, having given the base, the perpendicular and the difference of the two sides.



Let ABC be the $\triangle$. Put $BD=x$, $DC=y$, $AC=z$, $AB=z+d$, $AD=a$, $BC=b$. By the conditions,

$$x+y=b \qquad (1)$$

$$x^2+a^2=z^2+2dz+d^2 \qquad (2)$$

$$y^2+a^2=z^2 \qquad (3)$$

By subtracting, $$x^2-y^2=2dz+d^2 \qquad (4)$$

Factoring, $$(x+y)(x-y)=d(2z+d)$$

That is, $$b(x-y)=d(2z+d).$$

From this we have the proportion,

$$b : (2z+d) :: d : (x-y)$$

This proportion is the following rule given in trigonometry, viz.:

In any plane triangle, as the base is to the sum of the sides, so is the difference of the sides to the difference of the segments of the base.

We return to the solution. From (1) we have,

$$x=b-y, \text{ whence } x^2-y^2=b^2-2by.$$

From (3) $z=\sqrt{y^2+a^2}$. These values put in (4) give

$$b^2-2by=2d\sqrt{y^2+a^2}+d^2$$

$$(b^2-d^2)-2by=2d\sqrt{y^2+a^2}$$

Squaring, $(b^2-d^2)^2-4b(b^2-d^2)y+4b^2y^2=4d^2y^2+4a^2d^2$

Or, $(b^2-d^2)^2-4b(b^2-d^2)y+4(b^2-d^2)y^2=4a^2d^2$

$$(b^2-d^2)-4by+4y^2=\frac{4a^2d^2}{b^2-d^2}$$

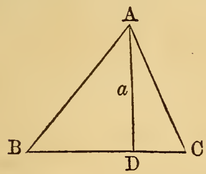
$$b^2-4by+4y^2=\frac{4a^2d^2}{b^2-d^2}+d^2=m^2$$

Whence, $b-2y=\pm m$

And, $y=b+m$, or $b-m$.

PROBLEM XXII.

To determine a triangle, having given the base, the perpendicular, and the rectangle, or product of the two sides.



Let ABC be the $\triangle$. Put $BD=x$, $DC=y$, $BC=b$, $AD=a$, and the rectangle, $(AB)(AC)=c$.

Now in the right angled triangles, ADB, ADC, we have,

$$AB=\sqrt{x^2+a^2}. \quad AC=\sqrt{y^2+a^2}.$$

Whence, $(\sqrt{x^2+a^2})(\sqrt{y^2+a^2})=c$ (1)

And, $x+y=b$ (2)

From (1), $x^2y^2+a^2(x^2+y^2)+a^4=c^2$ (3)

From (2), $x^2+y^2=b^2-2xy$ (4)

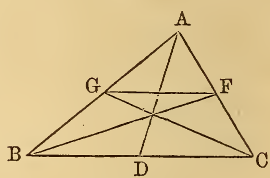
This value substituted in (3), gives

$$x^2y^2+a^2b^2-2a^2xy+a^4=c^2$$
$$x^2y^2-2a^2xy+a^4=c^2-a^2b^2$$
$$xy-a^2=\pm\sqrt{c^2-a^2b^2}$$
$$xy=a^2\pm\sqrt{c^2-a^2b^2} \quad (5)$$

Now equations (2) and (5) will give the values of x and y.

PROBLEM XXIII.

To determine a triangle, having given the lengths of the three lines drawn from the three angles to the middle of the opposite sides.



Let ABC be the $\triangle$. Place $AD=a$, $BF=b$, and $CG=c$.

Also, put $BD=x$, one half of the base, $BG=y$, one half of the side AB, and $FC=z$, one half of the side AC. Now by (Th. XLII., B. I.), we have,

$$\overline{AB}^2+\overline{AC}^2=2\overline{BD}^2+2\overline{AD}^2.$$

That is $4y^2+4z^2=2x^2+2a^2$ (1)

Similarly, $4x^2+4y^2=2z^2+2b^2$ (2)

And, $4x^2+4z^2=2y^2+2c^2$ (3)

Reduced sum, $6x^2+6y^2+6z^2=2(a^2+b^2+c^2)$

Or, $2x^2+2y^2+2z^2=\frac{2}{3}(a^2+b^2+c^2)=m$

From (1) $2y^2+2z^2=x^2+a^2$

By subtraction $2x^2=-x^2+m-a^2$

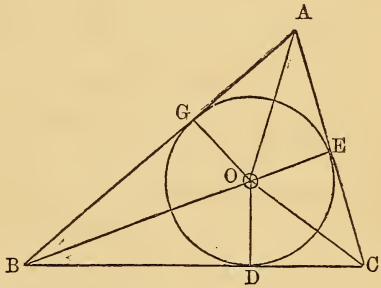
Or, $3x^2=m-a^2$. Whence, $x=\dfrac{\sqrt{m-a^2}}{\sqrt{3}}$

In like manner we find $y=\frac{\sqrt{m-c^2}}{\sqrt{3}}$.

And, $z=\frac{\sqrt{m-b^2}}{\sqrt{3}}$. Double of these values will be the sides required.

PROBLEM XXIV.

In a triangle having given the three sides, to find the radius of the inscribed circle.



Let ABC be the $\triangle$. From the center of the circle O, let fall the perpendiculars OG, OE, OD, on the sides.

These perpendiculars are all equal, and each equal to the radius required.

Let the side opposite to the angle A, be represented by a, the side opposite B, by b, and opposite C, by c. Put OE, OD, etc., equal to r.

It is obvious that the double area of the $\triangle$ BOC is expressed by ar; the double area of AOB by cr; the double area of AOC by br. Therefore, the double area of ABC is

$$(a+b+c)r.$$

From A let drop a perpendicular on BC, and call it x.

Then $ax=$ the double area of ABC. Consequently,

$$(a+b+c)r=ax \qquad (1)$$

The perpendicular from A will divide the base BC into two segments, one of which is $\sqrt{c^2-x^2}$, the other, $\sqrt{b^2-x^2}$, and the sum of these is a; therefore,

$$\sqrt{c^2-x^2}+\sqrt{b^2-x^2}=a \qquad (2)$$

$$\sqrt{c^2-x^2}=a-\sqrt{b^2-x^2}$$

$$c^2-x^2=a^2-2a\sqrt{b^2-x^2}+b^2-x^2$$

$$2a\sqrt{b^2-x^2}=a^2+b^2-c^2$$

$$\sqrt{b^2-x^2}=\frac{a^2+b^2-c^2}{2a}=m$$

Whence, $$b^2-x^2=m^2$$

Or, $$x=\sqrt{b^2-m^2}$$

This value of x put in (1), gives

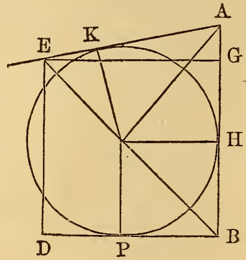
$$(a+b+c)r=a\sqrt{b^2-m^2}$$

Whence, $$r=\frac{a\sqrt{b^2-m^2}}{a+b+c},$$

the required result.

PROBLEM XXV.

To determine a right angled triangle, having given the side of the inscribed square, and the radius of the inscribed circle.



Let AB be one side of the triangle, and BG a side of the inscribed square, which we designate by a. And let $BH=r$, a radius of the inscribed circle.

Then $GH=a-r=d$.

Join AE, touching the circle at K, and BD, touching the circle at P.

Conceive the lines AE and BD produced, meeting at a point C, then ABC is the triangle sought.

Let $AH=AK=x$, and $KC=CP=y$.

Then $AC=$ the hypotenuse$=(x+y)$. (Prob. XV.)

$AB=x+r$, and $CB=y+r$.

$$(x+r)^2+(y+r)^2=(x+y)^2 \qquad (1)$$

Again $AG=x-d$, $GE=a$, and $CD=y-d$.

Now, $AG : GE :: ED : CD$

That is, $(x-d) : a :: a : y-d$.

Whence, $$xy-d(x+y)+d^2=a^2 \qquad (2)$$

Expanding and reducing (1), we obtain,

$$2r(x+y)+2r^2=2xy$$

Or, $$xy=r(x+y)+r^2 \qquad (3)$$

From (2) $$xy=d(x+y)+a^2-d^2$$

By subtraction $$0=(r-d)(x+y)+r^2+d^2-a^2$$

Whence, $$x+y=\frac{r^2+d^2-a^2}{d-r}. \qquad (5)$$

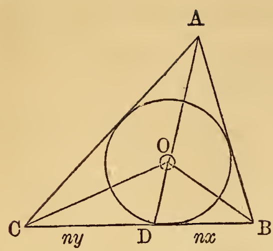
This value of $(x+y)$ placed in equation (3) and reduced, and we obtain,

$$xy=\frac{r(d^2-a^2+dr)}{d-r}. \qquad (6)$$

Equations (5) and (6) will give separate values of x and y, and thus solve the problem.

PROBLEM XXVI.

To determine a triangle and the radius of the inscribed circle, having given the lengths of three lines drawn from the three angles to the center of that circle.



Let ABC be the $\triangle$, O the center of the circle.

Put $AO=a$, $OB=b$, $OC=c$. AO bisects the angle A.

Produce AO to D. Then because the angle A is bisected,

$$CD : DB :: AC : AB.$$

Put $AB=x$, $AC=y$, and let the ratio of AB to BD be n; then $nx=BD$ and $ny=CD$.

Now as the angle C is bisected by CO, we have,

$$AC : CD :: AO : OD$$

That is, $$y : ny :: a : OD$$

Whence, $$OD=na.$$

Because AD bisects the angle A, we have, (Th. XX. Book III.)

$$xy=a^2(1+n)^2+n^2xy \qquad (1)$$

Also, $$nx^2=b^2+na^2 \qquad (2)$$

And, $$ny^2=c^2+na^2 \qquad (3)$$

From (1), $$xy=\frac{a^2(1+n)^2}{1-n^2}=\frac{a^2(1+n)}{1-n} \qquad (4)$$

The product of (2) and (3), gives

$$n^2x^2y^2=(c^2+na^2)(b^2+na^2) \qquad (5)$$

Squaring (4), and multiplying the result by n^2, also gives

$$n^2x^2y^2=\frac{a^4(1+n)^2n^2}{(1-n)^2} \qquad (6)$$

Equating (5) and (6), gives

$$(c^2+na^2)(b^2+na^2)(1-n)^2=a^4(1+n^2)n^2$$

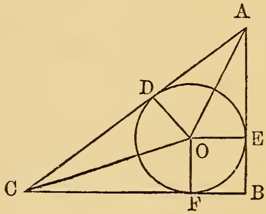
This equation contains only one unknown quantity n, but it rises to the fourth power—hence this problem is not susceptible of a solution under this notation, short of an equation of the 4th degree.

When a, b, and c, are numerically given, cases may occur in which the resulting equation may be of a low degree. When $b=c$, then $x=y$.

The three sides being determined, the radius of the inscribed circle is then found by Problem XXIV.

PROBLEM XXVII.

To determine a right angled triangle, having given the hypotenuse, and the radius of the inscribed circle.



Let ABC be the $\triangle$. Place $AE=x$, $EB=EO=r$, and $CF=y$.

Then, $AB=x+r$

$BC=y+r$

And, $AC=(x+y)=h$.

Now, $(x+r)^2+(y+r)^2=(x+y)^2$.

Expanding and reducing the above, we have

$$r(x+y)+r^2=xy$$

That is, $xy=r^2+hr$

And, $x+y=h$.

Whence, $x=\frac{1}{2}h+\frac{1}{2}\sqrt{h^2-4hr-4r}$

And, $y=\frac{1}{2}h-\frac{1}{2}\sqrt{h^2-4hr-4r^2}$.

PROBLEM XXVIII.

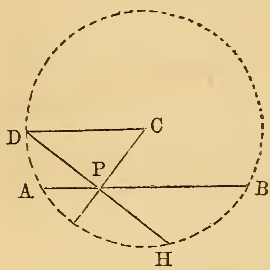
Here the problem is given in general terms—this same problem is numerically given in (Book V. Prob. 8), and is solved in this key, on page (7).

PROBLEM XXIX.

This problem involves the same relations as (Prob. 12, Book V.), which is solved in this key, on page (9).

PROBLEM XXX.

The radius of a circle being given, also the rectangle of the segments of a chord, to determine the distance from the center to the point at which the chord is divided.



Let C be the center of the circle, AB the chord, and P the given point in it. Join CP and denote that distance by x.

Through P draw the chord DH at right angles to PC. Then because CP is a line from the center perpendicular to the chord DH, it must bisect that chord. (Th. I. Book III.)

DPC is a right angled triangle, $DC=r$, then

$$DP=\sqrt{r^2-x^2}.$$

But because DH, and AB intersect at P, we have,

$$AP.PB=DP.PH=\overline{DP}^2$$

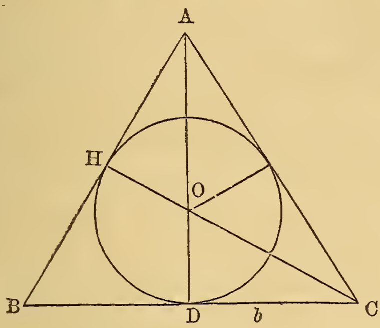
Whence, $\quad AP.PB=r^2-x^2$

Or, $$x=\sqrt{r^2-AP.PB}.$$

That is the distance from the center to the required point will be found by *subtracting the rectangle from the square of the radius of the circle, and extracting the square root of the remainder.*

PROBLEM XXXI.

If each of the two equal sides of an isosceles triangle be represented by (a), and the base by (2b), what will be the radius of the inscribed circle in terms of a *and* b.



Let ABC be the triangle, and

$AB=AC=a$, $BD=b$.

Then $AD=\sqrt{a^2-b^2}$.

Place $OD=R=HO$.

Now, by the similar triangle AHO, ABD, we have

$$AO : OH :: AB : DB$$

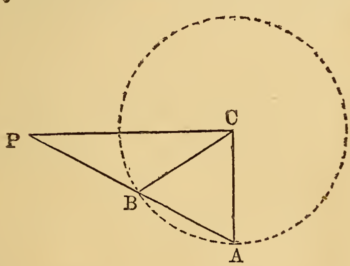
That is, $AD-R : R :: a : b$

And, $b\sqrt{a^2-b^2}-bR=aR$

Whence, $R=\dfrac{b\sqrt{a^2-b^2}}{a+b}=\dfrac{b\sqrt{a-b}}{\sqrt{a+b}}$.

PROBLEM XXXII.

From a point without a circle whose diameter is (d), a line equal to (d) is drawn to the concave terminating in the concave arc, and bisected at the first point of meeting the circumference. Required the distance of the point without from the center of the circle.



Let C be the center of the circle, and P a point without, and $AP=d$, the diameter of the circle, which is bisected by the circumference at B.

Join CA, CB, and CP. Because AP is equal to the diameter of the circle, and it is bisected in B, AB must be equal to the radius of the circle; that is, $AB=BC=CA$, and the triangle ABC equiangular.

Also $BP=BC$, therefore the angle $BCP=BPC$. But the angle $CBP=120°$, therefore the angle at P is 30°, and the angle $PCA=90°$.

Hence, $\overline{CA}^2+\overline{PC}^2=\overline{PA}^2$.

Or, $(\tfrac{1}{2}d)^2+\overline{CP}^2=d^2$.

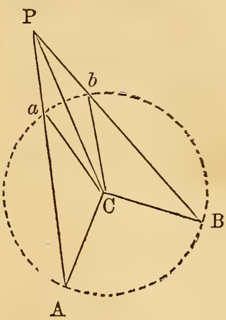
Whence, $\overline{CP}^2=\tfrac{3}{4}d^2$, or, $CP=\dfrac{d}{2}\sqrt{3}$.

the value of the line sought.

MISCELLANEOUS PROPOSITIONS.

(Page 238.)

(3.) *If from any point without a circle, two straight lines be drawn to the concave part of the circumference, making equal angles with the lines joining the same point and the center; the parts of these lines which are intercepted within the circle, are equal.*



Let P be any point without a circle, and C the center of the circle. Join PC and draw PA, PB making equal angles with PC.

Join CA, CB, Ca and Cb. We are to prove that $Aa=Bb$.

In the two triangles PCA, PCB, we have $CA=CB$, and PC common to both triangles, and the angle CPA opposite the side AC is equal to CPB opposite the equal side BC.

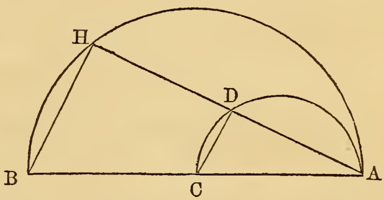
That is, the two triangles have three parts respectively equal. Therefore, the three other parts are also equal, and $PA=PB$. In like manner by the two triangles PCa, PCb, we prove $Pa=Pb$.

Whence, by subtraction,

$$PA-Pa=PB-Pb.$$

That is, $Aa=Bb$, and the theorem is demonstrated.

(4.) *If a circle be described on the radius of another circle, any straight line drawn from the point where they meet, to the outer circumference, is bisected by the interior one.*



Let AC be the radius of one circle and the diameter of another, as represented in the figure. From the point of contact A, of the two circles, draw any line, as AH; *this line is bisected in* D. Join DC and HB. Then ADC being in a semicircle, is a right angle; also, AHB is a right angle, for the same reason: therefore, DC and HB are parallel. Whence,

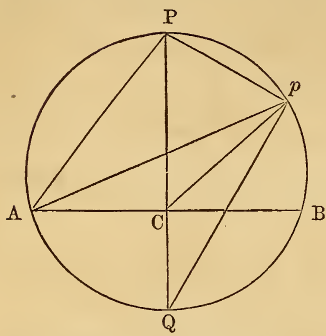
$$AD : AH : : AC : AB$$

But as AB is the double of AC, therefore AH is the double of AD, or AH is bisected in D, which was to be shown.

(5.) *From two given points on the same side of a line given in position, to draw two straight lines, which shall contain a given angle and be terminated in that line.*

Let AB be the line given in position, and P and p the given points. To make the problem definite, we take the given angle at 35°.

Join Pp, and draw PpQ a right angle. Take 35° from

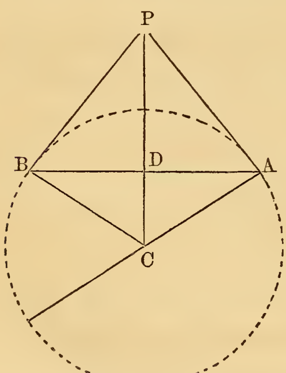


90°, and the remainder is 55°. Now, at P, make the angle pPQ equal to 55°, then the third angle of the triangle PQp will be 35°, the given angle. But it is not located in the line AB.

About the triangle PQp, describe the circle, and whence it cuts the lines AB, draw PA, pA, and the angle $PAp = PQp$, because they are angles in the same segment of a circle; and the angle is located as was required. The point B is another in the line when the angle would be the same in magnitude.

SCHOLIUM.—The given angle must be less than the angle PCp, otherwise the two lines from P, p, making the angle, would meet before they could reach the line given in position.

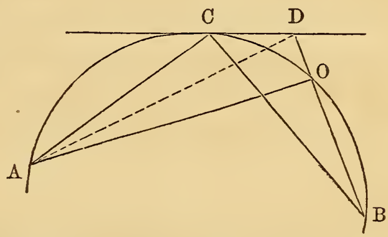
(6.) *If from any point without a circle, lines be drawn touching it, the angle contained by the tangents is double the angle contained by the line joining the points of contact, and the diameter drawn through one of them.*



Let P be a point without a circle, and PA, PB, tangents to the circle. Lines joining the center to the point of tangency, make right angles with the tangent lines. (Th. IV., Book III.). Join also PC and AB, PC is common to the two right angled triangles PAC and PBC. Hence, $PB = PA$, and the angle at P is bisected by PC.

AD is a line drawn from the right angle of the triangle *PAC* perpendicular to its hypotenuse *PC*, and divides that triangle into two similar triangles (Th. XXV., Book II., 1). Hence the angle *DAC* is equal to the angle *APC*, or *BPA*, is double of the angle *BAC*, which was to be demonstrated.

(7.) *If from any two points in the circumference of a circle, there be drawn two straight lines to a point in a tangent to that circle, they will make the greatest angle when drawn to the point of contact.*

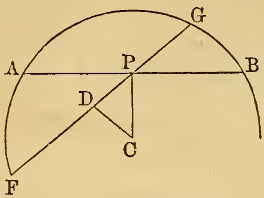


Let *A* and *B* be the two points in the circle, and *CD* a tangent line. The proposition requires us to demonstrate *that the angle ACB is greater than the angle ADB.* *ACB*=*AOB*, (Th. IX., Book III., Cor.) But *AOB* is greater than *ADB*, (Book I., Th. XII., Cor. 1). Therefore, *ACB* is also greater than *ADB*.

(8.) *From a given point within a given circle, to draw a straight line which shall make with the circumference an angle less than any angle made by any other line drawn from that point.*

NOTE.—An angle between a chord and a circumference is the same as between the chord and a tangent drawn through the same point. Thus the angle made by the chord *AB*, and the circumference at *B* is the same as between *AB* and a tangent drawn through *B*.

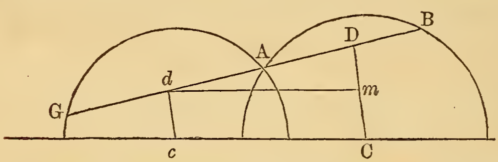
Let *C* be the center of the circle, *P* a given point within the circle not at the center. Join CP, and through P at right angles to CP draw the chord AB.



The angle made between AB and the circumference B is measured by by half the arc BGA, and this is less than any other angle made by any other line drawn through P, and the circumference.

Through P draw any other line as FPG. Now the angle which PG makes with the circumference is measured by half the arc GAF. But GAF is a greater arc than BGA, because GF is greater than AB. GF we know to be greater than AB, because CD is less than CP, CP being the hypotenuse of the right angled triangle CDP. Consequently the angle which PB makes with the circumference is less than that which PG makes with the circumference.

(9.) *If two circles cut each other, the greatest line that can be drawn through either point of intersection, is that which is parallel to the line joining their centers.*

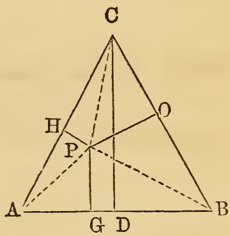


Let c and C be the centers of two circles which cut each other, and A, one of the points of intersection. Now our object is to prove that a line drawn through A, parallel to Cc, will be greater than any other line, as GB drawn through A, and not parallel to cC.

From cC draw perpendiculars cd, CD, to GB. AB is bisected in D, and AG in d, therefore, dD, is half GB. But dD is less than $dm=cC$, because dm is the hypotenuse of the right angled triangle dDm.

But if GB were drawn parallel to cC through A, then dD would be equal to cC. Consequently a line drawn through A, parallel to cC, is the greatest possible through A, which was to be shown.

(10.) *If from any point within an equilateral triangle perpendiculars be drawn to the sides, they are together equal to a perpendicular drawn from any of the angles to the opposite side.*



Let ABC be the equilateral $\triangle$, CD a perpendicular from one of the angles on the opposite side; then the area of the $\triangle$ is expressed by $\frac{1}{2}AB \times CD$. Let P be any point within the triangle, and from it let drop the three perpendiculars PG, PH, PO.

The area of the triangle APB is expressed by $\frac{1}{2}AB \times PG$. The area of the $\triangle$ CPB is expressed by $\frac{1}{2}CB \times PO$: and the area of the $\triangle$ CPA is expressed by $\frac{1}{2}CA \times PH$. By adding these three expressions together, (observing that CB and CA are each equal to AB,) we have for the area of the whole $\triangle$ ACB, $\frac{1}{2}AB(PG+PH+PO.)$

Therefore, $\quad \frac{1}{2}AB \times CD = \frac{1}{2}AB(PG+PH+PO.)$

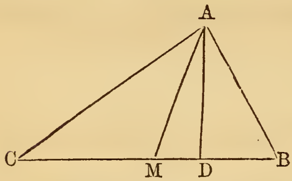
Dividing by $\frac{1}{2}AB$, gives $CD=PG+PH+PO$.

(11.) *If the points bisecting the sides of any triangle be joined, the triangle so formed will be one-fourth of the given triangle.*

If the points of bisection be joined, the triangle so formed will be similar to the given $\triangle$, (Th. XX., Book II.)

Then, the area of the given $\triangle$ will be to the area of the $\triangle$ formed by joining the bisecting points, as the square of a line is to the square of its half; that is, 2^2 to 1, or as 4 to 1. Hence the $\triangle$ formed is $\frac{1}{4}$ of the given triangle.

(12.) *The difference of the angles at the base of any triangle, is double the angle contained by a line drawn from the vertex perpendicular to the base, and another bisecting the angle at the vertex.*



Let ABC be a $\triangle$. Draw AM bisecting the vertical angle, and draw AD perpendicular to the base.

The theorem requires us to prove that the difference between the angles B and C is double of the angle MAD.

By hypothesis, the angle $CAM=MAB$.

That is, $CAM=MAD+DAB$. (1)

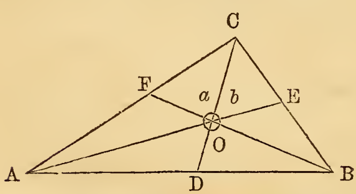
By (Th. XII., B. I., Cor. 4) $\left\{\begin{array}{ll} C+CAM+MAD=90^\circ & (2) \\ B+DAB \qquad\quad =90^\circ & (3) \end{array}\right.$

Therefore, $B+DAB=C+CAM+MAD$ (4)

Taking the value of CAM from (1), and placing it in (4), gives $B+DAB=C+MAD+DAB+MAD$.

Reducing $(B-C)=2MAD$, which verifies the text.

(13.) *If from the three angles of a triangle, lines be drawn to the middle of the opposite sides, these lines will intersect each other in the same point.*



Let ABC be a $\triangle$, bisect BC in E, AC in F.

Join AE and BF, and through their point of intersection O, draw the line CD. *Now if we prove $AD=DB$, the theorem is true.*

Triangles whose bases are in the same line, and vertices in the same point, are to one another as their bases; and when the bases are equal, the triangles are equal. For this reason the $\triangle AFO = \triangle FCO$, and the $\triangle COE = \triangle EOB$.

Put $\triangle AFO = a$; then $\triangle FCO = a$. Also, put $\triangle COE = b$, as represented in the figure.

Because CB is bisected in E, the $\triangle ACE$ is half of the whole $\triangle ABC$. Because AC is bisected in F, the $\triangle BFC$ is half the whole $\triangle ABC$.

That is, $2a + b = 2b + a$.

Whence, $a = b$, and the four triangles above the point O are equal to each other.

Let the area of the $\triangle ADO$ be represented by x, and the area of DOB by y.

Now taking CD as the base of the triangles, we have

$$2a : x :: CO : OD$$

Also, $2b =$ $2a : y :: CO : OD$

Whence, $2a : x :: 2a : y$. Or, $x = y$.

Therefore, $AD = DB$.

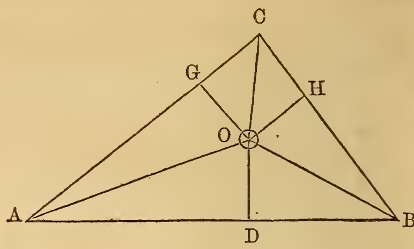
SCHOLIUM.—If the triangle, ABC be regarded as a thin lamina of matter, *its center of gravity* must be somewhere in the line AE; for the part $AEC =$ the part AEB. For a similar reason, the center of gravity must be somewhere in the line BF. Hence it must be at their intersection, O.

Again, the triangle $AEB =$ the triangle ADC. And since $x = y$, and $a = b$, we have,

$$2x + a = 2a + x, \text{ or, } a = x.$$

Hence, $\triangle AOB = \frac{1}{3} \triangle ACB$; and $OD = \frac{1}{3} CD$.

(14.) *The three straight lines which bisect the three angles of a triangle, meet in the same point.*



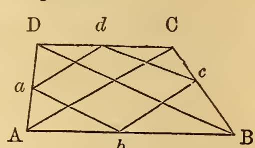
Let ABC be any triangle, bisect two of its angles A and B, the bisecting lines meeting at O.

Let fall perpendiculars from O on the three sides, OD, OG, OH.

Join CO, and now if we can demonstrate that CO bisects the angle C the proposition will be proved. The two right angled triangles AOD, AOG have a common hypotenuse, AO, and equal angles respectively, therefore, $OD=OG$. In like manner, by the two right angled triangles, DOB and OHB, we prove $OD=OH$. Whence, $GO=OH$, and the hypotenuse CO is common to each of the triangles, CGO and CHO, therefore their angles are equal, and the angle C is bisected by the line CO. Hence the bisecting lines all meet at the point O, and the proposition is proved.

Cor.—Let the student observe that OD, OG, OH, each, is equal to the radius of the inscribed circle.

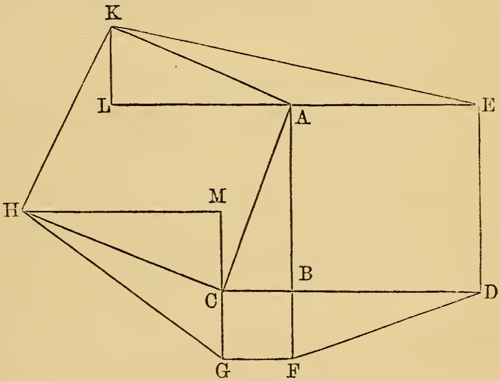
(16.) *The figure formed by joining the points of bisection of the sides of a trapezium, is a parallelogram.*



Let $ABCD$ be a trapezium. Draw the diagonals AC, BD. Bisect the sides in a, b, c, and d. Join $abcd$. We are to prove that this figure is a parallelogram.

ABD is a $\triangle$ whose sides are bisected in a and b; therefore, ab is parallel to BD, (Th. XVII., Book II.) In the same manner we can prove that dc is parallel to BD. Consequently ab and dc are parallel. It may likewise be shown that ad and bc are parallel. Hence the figure $abcd$ is a parallelogram.

(17.) *If squares be described on the three sides of a right angled triangle, and the extremities of the adjacent sides be joined, the triangles so formed are equivalent to the given triangle, and to each other.*



Let ABC be the original right angled triangle; on its sides describe the squares AD, BG, and AH.

Join KE, DF, HG, thus forming the triangles KAE, DBF, and HCG, and we are to prove each of them equal to the triangle ABC, and to each other.

We will now prove the triangle ALK equal to ABC, as follows,

The angle $\qquad LAB=90°$,

Also $\qquad KAC=90°$.

From each of these equals take away the common angle LAC, and the remaining angles KAL and CAB must be equal. Again, ALK is right angled at L, and ABC right angled at B, and the hypotenuse AK of the triangle ALK, is equal to AC the hypotenuse of ABC. Therefore, ALK is equal to ABC.

Whence, $\qquad AL=AB=AE$, and $KL=BC$.

Now, because $AL=AE$, the area of the triangle KAE is equal to the area ALK, or ABC, and this is one of the facts which was to be demonstrated.

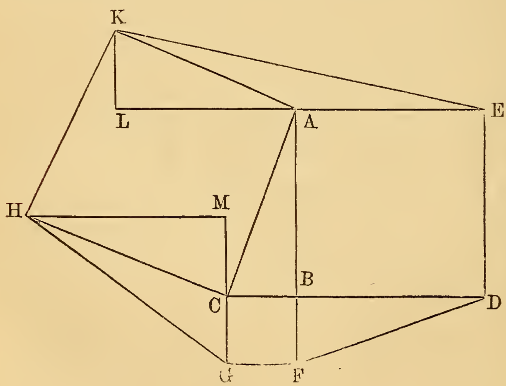
In like manner because $ACH=90°$, and $BCM=90°$, taking away the common angle MCA from each of these equals, we have the angle $HCM=ACB$; but $HC=AC$ and the triangles right angled, one at M, the other at B. Therefore, the triangle HCM is equal in all respects to ABC.

That is, $CM=CB$, but $CB=CG$, therefore $MC=CG$; consequently the area of the triangle $HCG=$ the area HCM, or ABC.

Now, because the area $KAL=ABC$
And the area $HCG=ABC$
Therefore, area $KAL=$ the area HCG.

The triangle $BDF=ABC$, because the opposite angles at B are equal, each $90°$. $AB=BD$, and $CB=BF$; and the proposition is fully demonstrated.

(18.) *If squares be described on the hypotenuse and sides of a right angled triangle, and the extremities of the sides of the former, and the adjacent sides of the others, be joined, the sum of the squares of the lines joining them, will be five times the square of the hypotenuse.*



The two triangles KLE, and HMG are right angled triangles, right angles at L, and at M. Observe also that $AE=AB=AL$.

Whence, $EL=2AB$.

Also, $KL=BC$, $BC=CG=CM$. Whence, $MG=2BC$.

But $\overline{EL}^2+\overline{KL}^2=\overline{EK}^2$

And $\overline{MG}^2+\overline{MH}^2=\overline{HG}^2$

That is, $(2AB)^2+(BC)^2=(EK)^2$ (1)

And $(2BC)^2+(AB)^2=(HG)^2$ (2)

By adding (1) and (2), observing that $(2AB)^2$ is $4\overline{AB}^2$

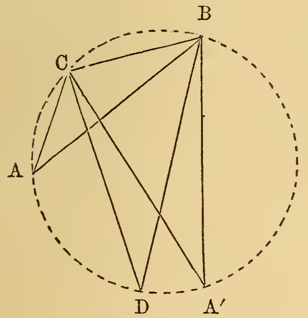
$$5\overline{AB}^2+5\overline{BC}^2=(EK)^2+(HG)^2.$$

That is, 5 times $(AC)^2=(EK)^2+(HG)^2$

which verifies the proposition.

Cor.—The area of the whole figure is double the square on AC, and 4 times the area of the triangle ABC.

(19.) *The vertical angle of an oblique angled triangle, inscribed in a circle, is greater or less than a right angle, by the angle contained between the base and the diameter drawn from the extremity of the base.*



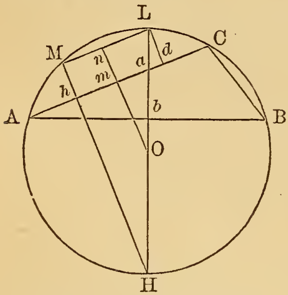
Let ACB be the vertical angle of a triangle, its base the chord AB. From the extremity of the base B, draw the diameter of the circle BD Join CD.

Because BD is a diameter the angle BCD is a right angle; therefore, the angle ACB is greater than a right angle by the angle ACD. But $ACD=ABD$, each being measured by half the arc AD.

That is, ACB is greater than a right angle by the angle ABD, which was to be demonstrated.

Or, let $A'BC$ be the triangle in a circle, and $A'CB$ its vertical angle. This is less than a right angle, by the angle $A'CD$, or $A'BD$.

(20.) *If the base of any triangle be bisected by the diameter of its circumscribing circle, and from the extremity of that diameter, a perpendicular be let fall upon the longer side, it will divide that side into segments, one of which will be equal to half the sum, and the other to half the difference of its sides.*



Let ABC be the $\triangle$, bisect its base by the diameter of the circle drawn at right angles to AB.

From the center O let fall Om at right angles to AC, it will then bisect AC. From the extremity of the diameter H, draw Hh perpendicular to AC, and consequently parallel to Om. Produce Hh to M and join ML. Complete and letter the figure as represented.

The two triangles Aab and Hha are equiangular. The angle a is common to them, and each has a right angle by construction, therefore the angle $H=$ the angle A, and the arc $ML=$ the arc CB, (Th. IX., Book III., C.); therefore $CB=ML$. The angle HML is a right angle, because it is in a semicircle, therefore, ML is parallel to AC, and ML is bisected in n.

Now $Am=\frac{1}{2}AC$. $nL=md=\frac{1}{2}ML=\frac{1}{2}BC$.

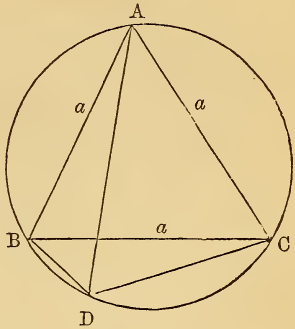
Therefore by addition, $Am+md=\frac{1}{2}(AC+CB)$.

Or, $Ad=\frac{1}{2}(AC+CB.)$

But, if Ad is the half sum of the sides, dC or Ah must be the half difference; for the half sum and half difference make the greater of any two quantities.

(21.) *A straight line drawn from the vertex of an equilateral triangle, inscribed in a circle, to any point in the opposite circumference, is equal to the sum of the two lines drawn from the extremities of the base to the same point.*

Let ABC be an equilateral triangle inscribed in a circle, and designate each side by a, as represented in the figure.



Take any point D on the arc BC, and join AD and designate it by x. Join BD and DC.

We are required to demonstrate that $x=BD+DC$.

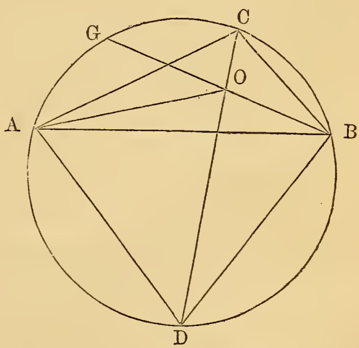
Observe that $ABDC$ is a quadrilateral inscribed in a circle, and by (Th. XXI., Book III.), we must have,

$$ax=a(BD)+a(DC).$$

Dividing each side by a, and $x=BD+DC$, which was to be demonstrated.

(22.) *The straight line bisecting any angle of a triangle inscribed in a circle, cuts the circumference in a point equidistant from the extremities of the side opposite to the bisected angle, and from the center of the circle inscribed in the triangle.*

Let ABC be a triangle in a circle, and draw CD bisecting the angle at C, and draw BG bisecting the angle at B. The intersection of these two bisecting lines at O, is the center of an imaginary circle inscribed in the triangle ABC.



The line CD bisects the angle at C by hypothesis, therefore, the arc $AD=$ the arc

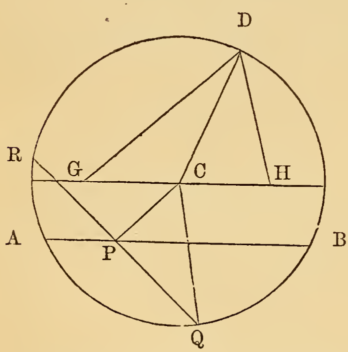
BD. Hence the chord $AD=$ the chord BD, and thus the first part of the theorem is proved.

In like manner, because BG bisects the angle B, the arc $AG=$ the arc GC.

Now in the triangle DBO, the angle DBO or DBG is measured by half the sum of the arc, $AD+AG$. And the angle DOB is measured by half of the sum of the arcs $DB+GC$. But $AD+AG=DB+GC$. Therefore, the angles DBO and DOB are equal, and the triangle is isosceles, and $DB=DO$.

In like manner we can prove that $DA=DO$, and the second point of the theorem is demonstrated.

(23.) *If from the center of a circle, a line be drawn to any point in a chord of an arc, the square of that line, together with the rectangle of the segments of the chord, will be equal to the square on the radius of the circle.*



Let C be the center of a circle, and P any point in the chord AB. Join PC, then we are required to prove that

$$\overline{PC}^2+AP.PB=\overline{CQ}^2.$$

Through P draw the chord RQ at right angles to PC, and join CQ.

Because CP is a line perpendicular from the center to a chord it bisects the chord. Hence, $RP=PQ$.

And because two chords intersect at P, we have,

$$AP.PB=RP.PQ=\overline{PQ}^2.$$

Add $\overline{CP}^2$ to each member of this last equation, and we have,

$$\overline{CP}^2+AP.PB=\overline{PQ}^2+\overline{PC}^2$$

But by the right angled triangle PCQ, we perceive that the second member is equal to $\overline{CQ}^2$. Therefore,

$$\overline{CP}^2+AP.PB=\overline{CQ}^2,$$

which was to be demonstrated.

(24.) *If two points be taken in the diameter of a circle equidistant from the center, the sum of the squares of the two lines drawn from these points to any point in the circumference, will always be the same.*

Take the same circle as in the preceding theorem, and let $GC=CH$. Take any point in the circumference as D, and join DG, DC, DH.

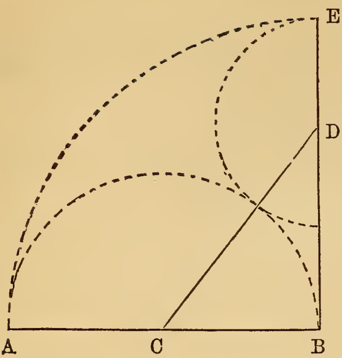
Now because the base of the triangle GDH is bisected in C, we have by (Th. XLII., Book I.),

$$\overline{GD}^2+\overline{DH}^2=2\overline{DC}^2+2\overline{GC}^2.$$

But at whatever point on the circumference D may be placed, DC and GC will always retain the same value. Therefore, the sum of the squares of the other two sides is constantly equal to the same sum.

(25.) *If on the diameter of a semicircle, two equal circles be described, and in the space enclosed by the three circumferences, a circle be inscribed, its diameter will be two thirds the diameter of either of the equal circles.*

Let AB be the diameter of one of the two equal circles and construct the figure as here represented.



We are required to show the relation between CB and ED.

Place $CB=r$ and $ED=x$.

Then $BE=2r$, and $BD=2r-x$, $CD=r+x$.

In the right angled triangle CBD, we have,

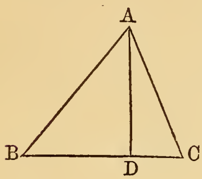
$$\overline{CB}^2+\overline{BD}^2=\overline{CD}^2.$$

That is, $$r^2+(2r-x)^2=(r+x)^2,$$

Or, $$r^2+4r^2-4rx+x^2=r^2+2rx+x^2$$

Whence, $$6rx=4r^2, \text{ and } x=\tfrac{2}{3}r.$$

(26.) *If a perpendicular be drawn from the vertical angle of any triangle to the base, the difference of the squares of the sides is equal to the difference of the squares of the segments of the base.*



Let ABC be a triangle. From A let fall AD perpendicular on BC, making two right angled triangles ADB, ADC. These triangles give

$$(AD)^2+(DB)^2=(AB)^2$$
$$(AD)^2+(DC)^2=(AC)^2$$

By subtraction $$(\overline{BD})^2-(DC)^2=(AB)^2-(AC)^2.$$

This equation demonstrates the theorem.

Cor.—By factoring the above equation, we have,

$$(BD+DC)(BD-DC)=(AB+AC)(AB-AC).$$

Observing that $BD+DC=BC$, and changing this equation into a proportion, we have,

$$BC : AB+AC=AB-AC : (BD-DC).$$

(This is Prop. 6, Plane Trigonometry, Page 256, Geometry.)

This proportion is true whatever relation exist between AB and AC. When $AB=AC$, then $BD=DC$ and the preceding proportion becomes

$$BC : 2AB :: 0 : 0.$$

This is an apparent absurdity, but we can reconcile it to truth by taking the product of the extremes and means, and we have,

$$(BC)0=(2AB)0.$$

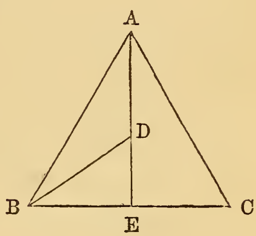
An equation obviously true,

Or, $\frac{BC}{2AB}=\frac{0}{0}$, and 0 divided by 0 is any quantity whatever,

Hence, $0 : 0 :: a :$ to any quantity whatever.

(27.) *The square described on a side of an equilateral triangle is equal to three times the square of the radius of the circumscribing circle.*

Let ABC be an equilateral triangle. Let fall the perpendicular AE on the base, that line will bisect the angle A. Draw BD bisecting the angle at B.



We will now prove $AD=BD$; then D must be the center of the circumscribing circle.

Each angle of an equilateral triangle is 60° ($\frac{1}{3}$ of 180°). Bisecting each of these angles, we have, DBA, DAB, each 30°, and therefore $AD=DB$, and hence, if D be taken as the center of circle with DB or DA as radius, that circle will circumscribe the triangle. Put $AB=2a$, then $BE=a$.

Also, place $BD=x$, then $DE=\frac{1}{2}x$. (It being the side of a right angled triangle opposite 30°). (Prob. 2 of this Key.)

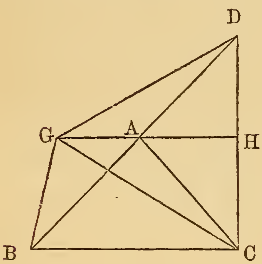
Now by the right angled triangle DBE, we have,

$$(BE)^2+(DE)^2=(BD)^2.$$

That is, $a^2+\frac{1}{4}x^2=x^2$, or, $3x^2=4a^2$.

But $(4a^2)=(AB)^2$. That is, $3x^2=(AB)^2$, which demonstrates the theorem.

(28.) *The sum of the sides of an isosceles triangle, is less than the sum of the sides of any other triangle on the same base, and between the same parallels.*



Let ABC be the isosceles triangle. $AB=AC$. Through the point A draw GAH parallel to BC.

Take G any other point on the line GH, and draw BG and GC.

We are to show that $AB+AC$ is less than $BG+GC$. Produce AB to D, making $AD=AB$, or AC.

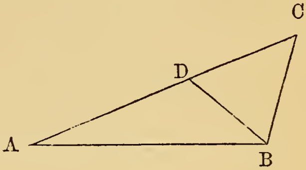
Then by reason of the parallels GH and BC, the angle DAH is equal to the angle ABC, and $HAC=ABC$.

Because $AD=AC$, the angle $ADH=$ the angle ACH.

Whence the two triangles ADH and ACH, are equal in all respects, and GH is perpendicular to DC; whence any point in the line GH is equally distant from the two points D and C.

Now the straight line $BD=BA+AC$, and because $DG=GC$, $DG+GB=GB+GC$. But $DG+GB$, the two sides of a $\triangle$ are greater than the third side DB; therefore $GB+GC$ is greater than BD, that is, greater than $BA+AC$.

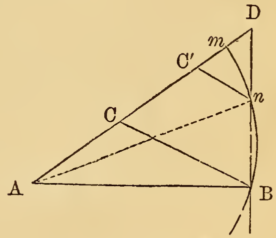
(29.) *In any triangle, given one angle, a side adjacent to the given angle, and the difference of the other two sides, to construct the triangle.*



Draw AB equal to the given side. From one extremity of the base A, draw AC indefinitely, making the angle BAC, equal to the given angle. (Prob. V., Book IV.)

Take AD equal to the given difference between the other two sides and join DB. From the point B make the angle DBC equal to the angle BDC, and the point C will be the vertex of the triangle required. Because $BC=CD$, and AD was made equal to the given difference of two sides, and AB the given side.

(30.) *In any triangle, given the base, the sum of the other two sides, and the angle opposite the base, to construct the triangle.*



Take any point A, and from it draw the line AD, equal to the sum of the sides.

At the extremity D, make the angle ADB equal to half of the given angle opposite the base. From A as a center, with the distance Am equal to the given base, describe an arc, mnB, cutting DB in n and B.

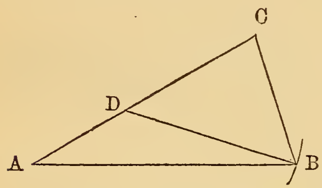
From B draw BC making the angle DBC, equal to the angle D. Also from n draw nC' parallel to BC, then ABC and AnC', are triangles corresponding to the given conditions.

For, $AB=An=Am$, the given base. The angles ACB, or $AC'n$ opposite the respective bases, AB, An, are equal to the given angle, because ACB being the exterior angle of the triangle CDB, is equal to the sum of D and DBC; but DBC was made equal to D by construction, and D was made equal to half the given angle.

The same reasoning applies to the triangle $DC'n$.

Also, $CD=CB$ by construction, therefore $AC+CB=AC+CD$, or AD the sum of the given sides.

(31.) *In any triangle, given the base, the angle opposite to the base, and the difference of the other two sides, to construct the triangle.*



Let ABC be the triangle. To discover its proper or direct construction, we must suppose the problem solved, and then analyze it.

If AB is the given base, and AD the given difference of the other sides, it is then obvious that $DC=CB$, and knowing the angle C which is given, we can determine each of the other angles of the triangle CBD.

Therefore, CDB is known, and its supplement ADB is known. Whence the angle

$$ADB=90°+\tfrac{1}{2}C.$$

Take any point A and draw AC indefinitely. From the scale of equal parts take $AD=$ to the given difference, and make the angle

$$ADB=90°+\tfrac{1}{2}C.$$

Take the given base from the scale of equal parts in the dividers, and with one foot on A as a center, strike an arc, cutting DB in B. The line AB will be the base.

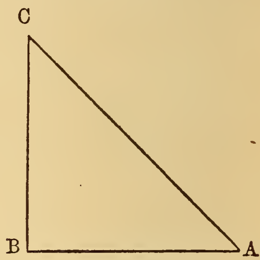
From the point B, and from the line BD make the angle $DBC=BDC$, and produce DC and BC until they meet in C, and we have the triangle ABC, as was required.

PLANE TRIGONOMETRY.

SECTION II.

(PAGE 289.)

NOTE.—In each of the following examples in this section, there are six parts, three angles A, B, and C, and the three corresponding sides a, b, c, A being opposite to a, etc. Each of the following examples may be referred to one and the same triangle. The right angle always at B, and in right angled trigonometry this part is always given, and not generally expressed. The right angle and two other parts being given, the remaining three parts can be determined.



Ex. 2. Given AC, 73.26, and the angle A, $49° 12' 20''$, to find the other parts.

From $90°$ take A, and we have $C = 40° 47' 40''$.

As sin. 90°.	10.000000		10.000000
To sin. 49° 12′ 20″	9.879129	cos.	9.815144
So is 73.26	1.864867	log.	1.864867
To BC, 55.462	1.743996		1.680011

Ans. BC, 55.46. AB, 47.87.

Ex. 3. Given AB, 469.34, and the angle A, $51° 26' 17''$, to find the other parts.

From 90° take A, and $C = 38° 33' 43''$.

	For BC.		*For AC.*
As sin. C, 38° 33′ 43″	9.794739	sin. C	9.794739
Sin. A	9.893171	sin. 90°	10.000000
AB, 469.34	2.671488	AB	2.671488
(Sin. A) (AB)	12.564659	AC	2.876749
BC, 588.74	2.769920	AC	752.92. *Ans.*

Ex. 4. Given AB, 493, and the angle C, 20° 14′, to find the other parts.

The remaining angle A is of course, 69° 46′.

	For CB.		*For AC.*
As sin. C, 20° 14′	9.538880	sin. C	9.538880
Cos. C, or sin. A	9.972338	sin. 90	10.000000
AB, 493	2.692847	AB	2.692847
	12.665185	AC, 1425.5=	3.153967
CB, 1337.53	3.126305		

Ex. 5. Given AB, 331, and the angle A, 49° 14′, to find the other parts.

The angle $C = 40° 46'$.

	For CB.		*For AC.*
As sin. C	9.814900	sin. C	9.814900
Cos. C or Sin. A	9.879311	sin. 90°	10.000000
AB	2.519828	AB	2.519828
	12.399139	AC, 506.91	2.704928
CB, 383.92	2.584239		

Ex. 6. Given AC, 45, and the angle C, 37° 22′, to find the other parts.

The angle A must be 52° 38′.

	For CB.		For AB.
As sin. 90°	10.000000	sin. 90°	10.000000
Sin. C, 37° 22′	9.783127	sin. A	9.900240
AC	1.653212	AC	1.653212
AB, 27.311	1.436339	CB, 35.764	1.553452

Ex. **7.** Given AC, 4264.3, and the angle A, 56° 29′ 13″, to find the other parts.

The angle C must be 33° 30′ 47″.

	For CB.		For AB.
As sin 90°	10.000000	sin. 90°	10.000000
Sin. A, 56° 29′ 13″	9.921041	sin. C	9.742038
AC	3.629848	AC	3.629848
CB, 3555.4	3.550889	AB, 2354.4	3.371886

Ex. **8.** Given AB, 42.2, and the angle A, 31° 12′ 49″, to find the other parts.

The angle C must be 58° 47′ 11″.

	For BC.		For AC.
As sin. C, 58° 47′ 11″	9.932088	sin. C	9.932088
Sin. A, 31° 12′ 49″	9.714522	sin. 90°	10.000000
AB	1.625312	AB	1.625312
	11.339834	AC, 49.34	1.693223
BC, 25.57	1.407746		

Ex. **9.** Given AB, 8372.1, and BC, 694.73, to find the other parts.

(By Prop. III. Plane Trig., 2d part, we have,)

$$8372.1 : 694.73 :: R : \tan. A.$$

Whence,	tan. $A=\dfrac{(694.73)R}{8372.1}$	log.	12.841816
		log.	3.922835
	tan. $A=$tan. $4^\circ\, 44'\, 37''=$		8.918981

Therefore the angle $C=85^\circ\, 15'\, 23''$

For the hypotenuse we have,

As sin. A : 694.73 : : R : AC

As before,	log. $(694.73)R=$	12.841816
	Sin. A	8.917489
	AC, 8400.9	3.924327

Ex. **10.** Given AB, 63.4, AC, 85.72, to find the other parts.

As AC : sin. 90° : : AB : sin. C

	Log. $(AB.R)$	11.802089
	Log. AC, 85.72	1.933082
	Sin. $C=47^\circ\, 41'\, 56''$	9.869007
Again,	As R	10.000000
	Is to AC	1.933082
	Sin. $A=$cos. C	9.828032
	Log. $BC=$ log. 57.69	1.761114

Ex. **11.** Given AC, 7269, and AB, 3162, to find the other parts.

As in Example 10,	Log. $(AB.R)$	13.499962
	Log. AC, 7269	3.861475
	Sin. C, $25^\circ\, 47'\, 7''$	9.638487
	A $64^\circ\, 12'\, 53''$	
As	sin. C, $25^\circ\, 47'\, 7''$	9.638487
Is to	AB, 3162	3.499972
So is	sin. $A=$cos. C	9.954450
		13.454422
	Log. $BC=$log. 6545	3.815935

Ex. **12.** Given AC, 4824, and BC, 2412, to find the other parts.

In this example the hypotenuse is double of the side BC, therefore the angle opposite BC or A, must be 30°, and the angle C, 60°.

If we call BC, 1, AC, 2, and AB, x, then $x^2+1=4$, and $x=\sqrt{3}$. Whence, $AB=2412\sqrt{3}=4178$, nearly.

Ex. **13.** In this example the hypotenuse of the right angled triangle is 94,770,000 miles, and the most acute angle only 16′ 6″, the double of the side opposite this small angle is required.

The difficulty here is to obtain the sine of 16′ 6″ to a sufficient degree of accuracy from the common table. This is explained on page 288, Text-book. Or we may operate as follows,

As	Sin. 90°		10.000000	
Is to	94,770,000	log.	7.976671	
So is	Sin. 16′ 6″=(16.1) sin. 1′		6.463726,	sin. 1′
	Log. 16.1		1.206826	
	443850		5.647223	
	2			
Sun's diameter	887700			

Ex. **14.** Solution, As 94,779,000 7.976671

	R	10.000000
	Log. 3956	3.597256
Sine of solar parallax=		5.620585=8.61″. *Ans.*

Ex. **15.** The distance from the earth to the moon, at mean distance in miles, is (60.3)(3960).

Statement, As sin. 90° : sin. 15′ 32″ : : (60.3)(3960) : M.

Observe that 15′ 32″=932″, sin. 932″=932 times sin. 1″.

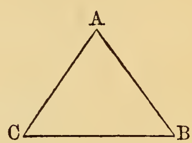
		Solution.
Log. 60.3		1.780317
Log. 3960		3.597695
Sin. 1″ (see page 288, Geom.)		4.685575
Log. 932		2.969416
Sum, less 10,	*R*, 1078.9	3.033003
	2	
Diameter of moon,	2157.8	*Ans.*

OBLIQUE ANGLED TRIGONOMETRY.

(PRACTICAL PROBLEMS.)

(Page 295.)

NOTE.—One triangle will serve for the solution of nearly all the following examples in this section. But the learner should draw his figure as near as he can according to the data given in each example. He will thus be less likely to make mistakes than he otherwise would.



Let ABC be any oblique angled triangle.

Ex. **1.** Given AB, 697, the angle A, 81° 30′ 10″, and the angle B, 40° 30′ 44″, to find the other parts.

Because the sum of the three angles make two right angles, or 180°. Therefore,

$$C=180°-(81° 30' 10''+40° 30' 44'')=57° 59' 6''.$$

As Sin. C : AB : : sin. A : BC.

Whence,	To sine A, 81° 30′ 11″		9.995207
	Add, 697		2.843233
			12.838440
	Sub. sin. C, 57° 59′ 6″		9.928350
	BC, 813	log.	2.910090

Again, Sin. C : AB : : sin. B : AC

Log. AB	2.843233
Sin. B	9.812652
	12.655885
Sin. C, as before	9.928350
AC, 534	2.727535

Ex. 2. Given AC, 720.8, the angle A, 70° 5′ 22″, and B, 59° 35′ 36″, to find the other parts.

The angle C must be 50° 19′ 2″.

As Sin. B : AC : : sin. A : $BC=\dfrac{AC.\ \sin.\ A}{\text{Sin.}\ B}$.

Also, Sin. B : AC : : sin. C : $AB=\dfrac{AC.\ \sin.\ C}{\text{Sin.}\ B}$.

Sin. A	9.973232	sin. C	9.886260
AC, 720.8	2.857815		2.857815
	12.831047		12.744075
Sin. B	9.935737	sin. B	9.935737
BC, 785.8	2.895310	AB, 643.2	2.808338

Ex. 3. Given BC, 980.1, the angle A, 7° 6′ 26″, and the angle B, 106° 2′ 23″, to find the other parts.

The angle C must be 66° 51′ 11″.

As Sin. A : BC : : sin. B : $AC=\dfrac{BC.\ \sin.\ B}{\text{Sin.}\ A}$.

Sin. A : BC : : sin. C : $AB=\dfrac{BC.\ \sin.\ C}{\text{Sin.}\ A}$.

Sin. B	9.982755	sin. C	9.963552
BC	2.991270	BC	2.991270
	12.974025		12.954822
Sin. A	9.092463	sin. A	9.092463
AC, 7613.1	3.881562	AB, 7283.8	3.862359

Ex. 4. Given AB, 896.2, BC, 328.4, and the angle C, 113° 45′ 20″ to find the other parts.

As $\qquad AB : \sin. C :: BC : \sin. A = \dfrac{BC. \sin. C}{AB}$.

Sin. C=cos. 23° 45′ 20″	9.761550
BC, 328.4	2.516403
	12.477953
AB, 896.2	2.952405
Sin. A=sin. 19° 35′ 46″	9.525548

Now having the angles A and C, we subtract their sum from 180°, and thus find the remainder B, 46° 38′ 54″. *Ans.*

With this angle we determine AC.

Ex. 5. Given AC=4627, BC, 5169, and the angle A, 70° 25′ 12″, to find the other parts.

As $\qquad BC : \sin. A :: AC : \sin. B = \dfrac{AC. \sin. A}{BC}$.

Sin. A, 70° 25′ 12″	9.974132
AC, 4627	3.665299
	13.639431
BC, 5169	3.713407
Sin. B, 57° 29′ 56″	9.926024

We now have the angles A and B, subtracting their sum from 180°, gives C=52° 4′ 52″. *Ans.*

For AB, we have,

$$\text{Sin. } A : BC :: \text{sin. } C : AB = \frac{BC. \text{ sin. } C}{\text{sin. } A}.$$

BC	3.713407
Sin. C, 52° 4′ 52″	9.897011
	13.610418
Sin. A	9.974132
AB, 4328	3.636286

Ex. **6.** Given AB, 793.8, BC, 481.6, AC, 500, to find the angles.

Note.—*Here all three sides are given, and the solution is by formulas found on page 259, Geometry.*

The angles of a triangle are denoted by A, B, C, and the sides opposite by a, b, c. A opposite a, etc.

The formulas are $\frac{1}{2}(a+b+c)=S$.

$$Cos.^2\tfrac{1}{2}A=\frac{S(S-a)}{bc};\ cos.^2\tfrac{1}{2}B=\frac{S(S-b)}{ac};\ cos.^2\tfrac{1}{2}C=\frac{S(S-c)}{ab};$$

the radius being unity.

By logarithms the formulas become

$$\text{Log. } (\cos.^2\tfrac{1}{2}A) = \text{log. } S(S-a) - \text{log. } bc. \qquad \&c.$$

$a=$ 481.6		Log. S	2.948266 } add.
$b=$ 500		Log. $(S-a)$	2.608633 }
$c=$ 793.8		Log. $S(S-a)$	5.556899 }
2)1775.4	log. b, 2.698970		} sub.
887.7$=S$	log. c, 2.899711		
a, 481.6	5.598681		5.598681 }
406.1$=(S-a)$	log. $(\cos.^2\frac{1}{2}A)$		-1.958218(2, div.
	log. $(\cos.\frac{1}{2}A)$		-1.979109

To correspond to table II., add 10.

Cos. $\frac{1}{2}A$=cos. 17° 37′ 46″	9.979109
2	
$A=$ 35° 15′ 32″.	

Having thus found one of the angles, the other may be determined by the direct proportion between the sides and the sines of the opposite angles.

Ex. 7. Given AB, 100.3, AC, 100.3, and BC, 100.3, to find the angles.

Here the sides are equal to each other, the triangle is therefore equiangular as well as equilateral. Consequently each angle is one third of 180°, or 60°.

Ex. 8. Given AB, 92.6, BC, 46.3, and AC, 71.2, to find the angles.

NOTE.—In example 6 we obtained the angle A, for radius unity. Here we will find the angle B, for radius R, whose log. is 10. The formula must be

$$\text{Cos. } \tfrac{1}{2}B = \left(\frac{R^2 S(S-b)}{ac}\right)^{\frac{1}{2}}.$$

By logarithms this becomes,

$$\text{Log. } (\text{cos. } \tfrac{1}{2}B) = \tfrac{1}{2}(\text{log. } R^2 S(S-b) - \text{log. } ac).$$

$$\begin{array}{lllll}
b = & 71.2 & & \text{Log. } R^2 = & 20 \\
a = & 46.3 & & \text{log. } S & 2.021396 \\
c = & 92.6 & & \text{log. } (S-b) & 1.529559 \\
\hline
2)\, & 210.1 & \text{log. } a, & 1.665581 & 23.550955 \\
& 105.05 = S & \text{log. } c, & 1.966611 & \\
& 71.2 & & 3.632192 & 3.632192 \\
& 33.85 = S - b. & & & 2)\,19.918763 \\
& \text{Cos. } \tfrac{1}{2}B = \text{cos. } 24^\circ\, 23'\, 45'' & & & 9.959381 \\
& & & 2 & \\
& B = & & 48^\circ\, 47'\, 30''. &
\end{array}$$

Now, $b : \sin. B :: a : \sin. A = \frac{a \sin. B}{b}$.

Log. $a=$	1.665581
Sin. $B=$	9.876402
	11.541983
Log. $b=$	1.852480
Sin. $A=\sin. 29° 17' 22''$	9.689503

Hence, $C= 101° 55' 8''$

Ex. 9. Given AB, 4963, BC, 5124, AC, 5621, to find the angles.

$a=5124$
$b=5621$
$c=4963$
15708
$S=7854$
$S-c=2891$

For the angle C

	log. R^2	20.
	log. S	3.895091
	log. $(S-c)$	3.461048
log. a, 3.709609		27.356139
log. b, 3.749814		
7.459423		7.459423
		2)19.896716
cos. $\frac{1}{2}C=\cos. 27° 23' 27.5''$		9.948358
$C= 54° 46' 55''$		

For the angle A

$c : \sin. C :: a : \sin. A$.

Sin. C	9.912202
Log. a	3.709609
	13.621811
Log. c	3.695744
Sin. $A=\sin. 57° 30' 28''$	9.926067

Ex. 10. Given AB, 728.1, BC, 614.7, AC, 583.8 to find the angles.

	$a=614.7$		log. R^2	20.000000
	$b=583.8$	log. 2.766264	log. S	2.983762
	$c=728.1$	log. 2.862191	log. $(S-a)$	2.542327
	1926.6	5.628455		25.526089
S	963.3			5.628455
	614.7			2)19.897634
$(S-a)$	348.6	cos. $\frac{1}{2}A=$cos. $27^\circ\ 16'\ 26''$		9.948817
		2		
	A	$54^\circ\ 32'\ 52''$		

Now as a : sin. A : : b : sin. $B=\dfrac{b \text{ sin. } A}{a}$.

log. b	2.766264
sin. A	9.910944
	12.677208
log. a, 614.7	2.788663
sin. $B=$sin. $50^\circ\ 40'\ 58''$	9.888545

Ex. **11.** Given AB, 96.74, BC, 83.29, and AC, 111.42, to find the angles.

	$a=$ 83.29		log. R^2	20.000000
	$b=$111.42		log. S	2.163534
	$c=$ 96.74		log. $(S-a)$	1.795428
	2)291.45	log. b, 2.046963		23.958962
S	145.725	log. c, 1.985606		
a	83.29	4.032569		4.032569
$(S-a)$	62.435			2)19.926393
		Cos. $\frac{1}{2}A=$cos. $23^\circ\ 15'\ 22.5''=$		9.963196
		2		
	$A=$	$46^\circ\ 30'\ 45''$		

As $a : \text{sin. } A :: b : \text{sin. } B$

Sin. $A=$	9.860652
b	2.046963
	11.907615
a	1.920593
Sin. B, sin. $76° 3' 46''$	9.987022

A and B being as above, C must be $57° 25' 29''$.

Ex. 12. Given AB, 363.4, BC, 148.4, and the angle B, $102° 18' 27''$, to find the other parts.

Here we have two sides and their included angle given, and we apply (Prop. VII., Plane Trig., page 257), which is

$$\left.\begin{matrix}363.4\\148.4\end{matrix}\right\}\text{sum.} : \left.\begin{matrix}363.4\\148.4\end{matrix}\right\}\text{diff.} :: \text{tan.}\left(\frac{C+A}{2}\right) : \text{tan}\left(\frac{C-A}{2}\right) \quad (1)$$

We observe that C must be greater than the angle A, because C is opposite the greater side.

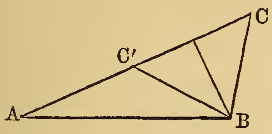
$$511.8 : 215 :: \text{tan. } 38° 50' 47'' : \text{tan. } \frac{C-A}{2}.$$

Log. 215	2.332438
Tan. $\frac{C+A}{2}$, $38° 50' 47''$	9.905986
	12.238424
Log. 511.8	2.709100
Tan. $\frac{1}{2}(C-A)$ $18° 41' 29''=$	9.529324
$\frac{1}{2}(C+A)$ $38° 50' 47''$	
Sum C $57° 32' 16''$	*Ans.*
Diff. A $20° 9' 18''$	*Ans.*

As Sin. $C : c(363.4) :: \text{sin. } B : b$.

Sin. B, $102° 18' 27''=\text{cos. } 12° 18' 27''$	9.989903
log. 363.4	2.560385
	12.550288
sin. C, $57° 32' 16''$	9.926211
log. $AC=$log. 420.8	2.624077

Ex. **13.** Given AB, 632, BC, 494, and the angle A, 20° 16′, to find the other parts, the angle C being acute.



If no mention were made requiring the angle C to be acute, the data would give the triangle ABC', for the solution as well as the triangle ABC.

In such cases it is customary to solve both triangles, and call the solution ambiguous.

As, $BC : AB :: \sin. A : \sin. C$

Or, $494 : 632 :: \sin. 20° 16' : \sin. C$

Log. sin. 20° 16′	9.539565
log. 632	2.800717
	12.340282
log. 494	2.693727
C, 26° 18′ 19″	9.646555

A and C taken from 180°, gives 133° 25′ 41″, for B. Lastly for AC, we have,

$$\text{Sin. } A : BC :: \sin. B : AC \ (b).$$

Sin. B, 133° 25′ 41″=cos. 43° 25′ 41″	9.861079
log. BC	2.693727
	12.554806
sin. A	9.539565
AC, 1035.7	3.015241

Ex. **14.** Given AB, 53.9, AC, 46.21, and the angle B, 58° 16′ to find the other parts.

As, $46.21 : \sin. B, 58° 16' :: 53.9 : \sin. C$

Sin. B, 58° 16′	9.929677
AB, 53.9	1.731589
	11.661266
AC, 46.21	1.664736
Sin. C, 82° 46′	9.996530

B and C being known, their sum, $141° 2'$ taken from $180°$, gives $38° 58'$, for the angle A. *Ans.*

As, sin. C : AB, 53.9 : sin. A, $38° 58'$: BC

Sin. $38° 58'$	9.798560
AB, 53.9	1.731589
	11.530149
Sin. C, $82° 46'$	9.996530
BC, 34.16	1.533619

Ex. **15.** Given AB, 2163, BC, 1672, and the angle C, $112° 18' 22''$, to find the other parts.

As, sin. C : AB, 2163 : : sin. A : BC, 1672.

Whence, $$\text{Sin. } A = \frac{1672 \text{ sin. } C}{2163}.$$

Sin. C=cos. $22° 18' 22''$	9.966221
log. 1672	3.223236
	13.189457
log. 2163	3.335057
Sin. A, $45° 39' 22''$	9.854400

Now, A and C being known, B must be $22° 2' 16''$.

Sin. A : 1672 : : sin. B, $22° 2' 16''$: AC.

Sin. $22° 2' 16''$	9.574283
BC, 1672	3.223236
	12.797519
Sin. A, $45° 39' 22''$	9.854400
AC, 877.2	2.943119

Ex. **16.** Given AB, 496, BC, 496, and the angle B, $38° 16'$ to find the other parts.

In this example we observe that AB=BC, therefore, the angles A and C, must be equal to each other, and the value of each must be

$$\tfrac{1}{2}(180^\circ - 38^\circ\,16') = 70^\circ\,52'.$$

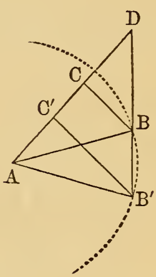
Now, Sin. $70^\circ\,52' : 496 :: \sin.\ 38^\circ\,16' : AC$.

$$AC = \frac{496.\sin.\ 38^\circ\,16'}{\sin.\ 70^\circ\,52'}.$$

Sin. $38^\circ\,16'$	9.791917
Log. 496	2.695482
	12.487399
Sin. $70^\circ\,52'$	9.975321
$AC = 325.1$	2.512078

Ex. **17.** Given AB, 428, the angle C, $49^\circ\,16'$, and $(AC + CB) = 918$, to find the other parts.

Note.—This problem is the same as 30, in Book VIII., Geometry, page 242, and its general solution is given in this Key, on page 59.



Either triangle, ABC or $AB'C'$ will correspond with the data. We will take the triangle ABC, and commence by solving the triangle ABD.

Because $CB = CD$, the angle ACB is double of the angle D.

Therefore, $D = 24^\circ\,38'$, $AD = 918$, $AB = 428$.

As $428 : \sin.\ D :: 918 : \sin.\ ABD$.

Log. 918	2.962843
Sin. $24^\circ\,38'$	9.619938
	12.582781
Log. 428	2.631444
Sin. ABB', $63^\circ\,22'\,48''$	9.951337

Or,	$ABD=$	$116° 37' 12''$
Sub.	$CBD=$	$24° 38' 00$
Diff.=	$ABC=$	$91° 59' 12''$

To ABC, add ACB, $49° 16'$, and their sum, $141° 15' 12''$, taken from $180°$ gives $A = 38° 44' 48''$.

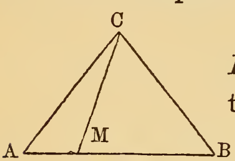
Lastly, Sin. C : AB : : sin. A : BC.

Which is, Sin. $49° 16'$: 428 : : sin. $38° 44' 48''$: BC.

Sin. $38° 44' 48''$	9.796490
Log. 428	2.631444
	12.427934
Sin. $49° 16'$	9.879529
BC, 353.5	2.548405

Now, as $(AC+BC)=918$, we have $AC=918-353.5=564.5$. In like manner, solving the triangle $AB'C'$, we shall have $AC'=353.5$, and $B'C'=564.5$.

Ex. **18.** Given a side and its opposite angle, and the difference of the other two sides, to construct the triangle and find the other parts.



Let ABC be the triangle. $AC=126$, $B=29° 46'$, and AM, the difference between AB and BC, $=43$.

From $180°$ take $29° 46'$ and divide the remainder by 2. This gives the angle BMC or BCM. BMC taken from $180°$, gives AMC.

Now in the triangle AMC, we have the two sides AC, 126, AM, 43, and the angle AMC, to find the angle A. The computation is as follows: $180° - 29° 46' = 150° 14'$; half, $= 75° 7' = BMC$. $180° - 75° 7' = 104° 53' = AMC$. Now in the $\triangle\ AMC$, we have

$$AC : AM :: \text{sin. } 104° 53' : \text{sin. } ACM$$

$126 : 43 :: \cos. 14° 53' : \sin. ACM$

Cos. 14° 53′	9.985180
Log. 43	1.633468
	11.618648
Log. 126	2.100371
Sin. ACM=sin. 19° 15′ 28″	9.518277

Now to BCM, 75° 7′ add ACM, 19° 15′ 28″, and we have ACB, 94° 22′ 28″. Subtracting 19° 15′ 28″ from 75° 7′, that is, ACM from BMC, and the difference must be

$$A = 55° 51' 32''.$$

Lastly, As sin. B : AC, 126 : : sin. A, 55° 51′ 32″ : CB.

Sin. A	9.917851
126	2.100371
	12.018222
Sin. B, 29° 46′	9.695892
BC, 210.05	2.322330

Ex. **19.** Given AB, 1269, AC, 1837, and the including angle A, 53° 16′ 20″, to find the other parts.

Solution by the same formulas as in 12.

$$\left.\begin{array}{l} b,\ 1837 \\ c,\ 1269 \end{array}\right\}\text{sum} : \left.\begin{array}{l} 1837 \\ 1269 \end{array}\right\}\text{diff.} :: \tan. \tfrac{1}{2}(B+C) : \tan. \tfrac{1}{2}(B-C).$$

$$3106 \quad : \ 568 :: \tan. 63° 21' 50'' : \tan \tfrac{1}{2}(B-C).$$

Tan. 63° 21′ 50″	10.299685
Log. 568	2.754348
	13.054033
Log. 3106	3.492201
$\frac{1}{2}(B-C)$, 20° 1′ 57″	9.561832

But $\quad \frac{1}{2}(B+C) = 63° 21' 50''$

Sum $= \quad B = 83° 23' 47''$. Diff. $C = 43° 19' 53''$.

Lastly, Sin. B : b : : sin. A : $a(BC)$

Sin. A, 53° 16′ 20″	9.903896
Log. b, 1837	3.264109
	13.168005
Sin. B, 83° 23′ 47″	9.997109
AC, 1482.16	3.170896

SECTION III.

APPLICATION OF PLANE TRIGONOMETRY.

(Page 305.)

(1.) Required the height of a wall whose angle of elevation, at the distance of 463 feet is observed to be 16° 21′.

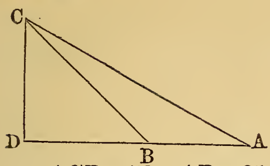
This example presents a right angled triangle, the base of which is 463, and the acute angle at the extremity of the base 16° 21′.

Let $x=$ the required height ; then

$$R : \tan. 16^\circ 21' :: 463 : x = \frac{463 \tan. 16^\circ 21'}{R}.$$

	Log. 463	2.665581
	Tan. 16° 21′	9.467413
Ans.	135.8	2.132994

(2.) We solve this example by the adjoining geometrical figure.



ABC is a triangle, CBD its exterior angle.

Whence, $CBD = ACB + A$.

That is, $31° \, 18' = ACB + 26° \, 18'$,

or, $ACB = 5°$, $AB = 214$.

$$\text{Sin. } 5° : 214 :: \text{sin. } 26° \, 18' : CB.$$

Whence, $$BC = \frac{214 . \text{sin. } 26° \, 18'}{\text{sin. } 5°}$$

Again, in the triangle CBD we have

$$\text{Sin. } 90° : CB :: \text{sin. } 31° \, 18' : CD.$$

That is, $$\text{Sin. } 90° : \frac{214 . \text{sin. } 26° \, 18'}{\text{sin. } 5°} :: \text{sin. } 31° \, 18' : CD.$$

Whence, $$CD = \frac{214 . \text{sin. } 26° \, 18' . \text{sin. } 31° \, 18'}{\text{sin. } 90° . \text{ sin. } 5°}.$$

214		2.330414
Sin. 26° 18′		9.646474
Sin. 31° 18′		9.715602
Log. Num.		21.692490
Sin. 90°	10.000000	
Sin. 5°	8.940296	18.940296
$CD = 565.2$		2.752194

(3.) Here the perpendicular of a right angled triangle is given 149.5 feet, and the vertical angle $57° \, 21'$. The opposite angle is therefore $32° \, 39'$.

As, $\text{Sin. } 32° \, 39' : 149.5 :: \text{sin. } 57° \, 21' :$ *Ans.*

Or, As $\text{Cos. } 57° \, 21' : 149.5 :: \text{sin. } 57° \, 21' :$ *Ans.*

$$Ans. = 149.5 . \frac{\text{sin. } 57° \, 21'}{\text{cos. } 57° \, 21'} = 149.5 . \text{tan. } 57° \, 21'.$$

149.5	2.174641
tan. 57° 21′	10.193307
Ans. = 233.3	2.367948

(4.) Here are two right angled triangles to be solved. The angle of depression, is equal to the angle opposite to the perpendicular.

Hence, for the distance of the mean object, we have the following proportion.

$$\text{Sin. } 48^\circ\, 10' : 138 :: \cos.\ 48^\circ\, 10' : \text{Dis.}$$

$$\text{Dis.} = 138.\frac{\cos.\ 48^\circ\, 10'}{\sin.\ 48^\circ\, 10'} = 138.\cot.\ 48^\circ\, 10'.$$

138	2.139879
cot. $48^\circ\, 10'$	9.951896
Least dis. 123.52	2.091775

The greatest dis. $=138.\cot.\ 18^\circ\, 52'$

(5.) Here is but one acute angled plane triangle, and the angle opposite to the base is found thus.

$$\begin{array}{r} 31^\circ\ 15' \\ 86^\circ\ 27' \\ \hline 117\ \ 42 \\ 180 \\ \hline 62^\circ\ 18' \end{array}$$

Let x represent the distance from one extremity of the base to the house, and y represent the distance of the other extremity. Then

$$\text{Sin. } 62^\circ\, 18' : 312 :: \sin.\ 31^\circ\, 15' : x$$

$$\text{Sin. } 62^\circ\, 18' : 312 :: \sin.\ 86^\circ\, 27' : y.$$

$$x = 312.\frac{\sin.\ 31^\circ\, 15'}{\sin.\ 62^\circ\, 18'}. \qquad y = 312.\frac{\sin.\ 86^\circ\, 27'}{\sin.\ 62^\circ\, 18'}.$$

312	2.494155	312	2.494155
Sin. $31^\circ\, 15'$	9.714978	sin. $86^\circ\, 27'$	9.999166
	12.209133		12.493321
Sin. $62^\circ\, 18'$	9.947136	sin. $62^\circ\, 18'$	9.947136
$x=182.8$	2.261997	$y=351.7$	2.546185

(6.) This is in many respects the same as (5), except in numbers.

Sin. 60° : 260 : : sin. 40° : x.

Log. 260	2.414973
Sin. 40°	9.808067
	12.223040
Sin. 60°	9.937531
192.8	2.285509

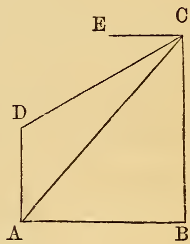
192.8 is the distance from one extremity of the base to the tree, but this line (192.8) makes an angle with the base of 80°. Let 192.8 be a hypotenuse of a right angled triangle, and the angles are 80° and 10°.

The side opposite 80° is the line or distance required.

Sin. 90° : 192.8 : : sin. 80° : perpendicular.

	192.8	2.285509
	Sin. 80°	9.993351
Ans.	190.1	2.278860

(7.) Let BC be the eminence, 268 feet, and AD the steeple. Draw CE parallel to the horizontal AB. Then $ECD=40° 3'$, $ECA=CAB=56° 18'$.



$DCA=56° 18'-40° 3'=16° 15'$, $DAC=90°-56° 18'=33° 42'$.

In the $\triangle$ ABC, we have

Sin. 56° 18′ : 268 : : sin. 90° : AC.

$$AC=\frac{268\times R}{\sin. 56° 18'}.$$

In the $\triangle$ ADC, we have the supplement to the angle ADC, equal to 16° 15′ added to 33° 42′, or 49° 57′; therefore,

As Sin. ADC : AC : : sin. DCA : AD.

That is, $\text{Sin. } 49° \ 57' : \dfrac{268 \times R}{\text{sin. } 56° \ 18'} :: \text{sin. } 16° \ 15' : AD.$

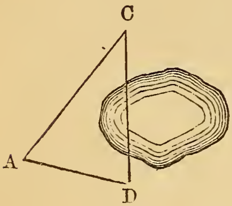
Log. $AD=$

$\log. [268.R.\text{sin. } 16° \ 15'] - \log. [\text{sin. } 49° \ 57'.\text{sin. } 56° \ 18'] =$

$[2.428135 + 10 + 9.446893] - [9.883936 + 9.920099] =$

$21.875028 - 19.804035 = 2.070993$

Whence, $AD = 117.76$ feet.



(8.) Let C and D be the two objects, and A the point at which both can be seen.

$AD=1428$, $AC=1840$, and the angle at A $\quad 36° \ 18' \ 24''$

From $\quad 180°$

Angles C and $D = 143° \ 41' \ 36''$

$\frac{1}{2}$ sum $\quad 71° \ 50' \ 48''$

Here we will apply the following theorem in trigonometry.

As the sum of two sides is to their difference, so is the tangent of half the sum of the angles at the base, to the tangent of half their difference.

Let $x=$ the half difference between D and C.

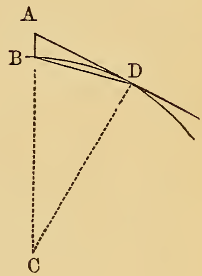
Then, $\quad 3268 : 412$

Or, $\quad 817 : 103 :: \text{tan. } 71° \ 50' \ 48'' : \text{tan. } x.$

	Log. 103	2.012837
	Tan. 71° 50′ 48″	10.484284
		12.497121
	817	2.912222
Tan. $x=$tan.	21° 1′ 55″	9.584899
	71° 50′ 48″	
Angle $D=$	92° 52′ 43″	
Angle $C=$	50° 48′ 53″	

Sin. $50^\circ\ 48'\ 53''$: 1428 : : sin. $36^\circ\ 18'\ 24''$: *CD*.

	1428	3.154728
	Sin. $36^\circ\ 18'\ 24''$	9.772400
		12.927128
	Sin. $50^\circ\ 48'\ 53''$	9.889362
Ans.	1090.85	2.037766



(9.) Let *AB* represent the mountain, and *AD* the visible distance. *AB* produced will pass through the center of the earth at *C*. From *D* draw *CD* perpendicular to *AD*. Join *BD*. *ADC* is a right angled triangle.

$CAD = 90^\circ - 2^\circ\ 13'\ 27'' = 87^\circ\ 46'\ 33''$. $ACD = 2^\circ\ 13'\ 27''$. $ADB = \frac{1}{2}ACD = 1^\circ\ 6'\ 43.5''$. $ABD = 91^\circ\ 6'\ 43.5''$.

Now in the $\triangle$ *ABD*, we have

Sin. $1^\circ\ 6'\ 43.5''$: 3 : : sin. $91^\circ\ 6'\ 43.5''$: *AD*.

Sin. $91^\circ\ 6'\ 43.5'' = $ cos. $1^\circ\ 6'\ 43.5''$	9.999919
Log. 3	0.477121
	10.477040
Sin. $1^\circ\ 6'\ 43.5''$	8.287976
Log. *AD*, 154.54	2.189064

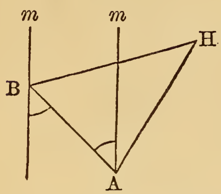
In the triangle *ADC*, we have

sin. *ACD* : *AD* : : cos. *ACD* : *CD*.

Cos. $ACD = $ cos. $2^\circ\ 13'\ 27''$	9.999674
AD	2.189064
	12.188738
Sin. $ACD = $ sin. $2^\circ\ 13'\ 27''$	8.588932
CD	3.599806
Log. 2	0.301030
Diameter, log. 7958 miles,	3.900836

(10.) Let H be the location of the headland, A the position of the ship when the first observation was taken, and B its position when the last observation was taken.

$AB=20$ miles, and mA, mB, meridians passing through the eye of the observer at the two stations.



The data gives us the angle $mAB=47° 49'$, $mBH=87° 11'$, the sum of these two angles taken from 180°, gives the angle $ABH=45°$.

The angle BAH, is obviously equal to the sum of 47° 49′, and 39° 23′, which is 87° 12′. The sum of 87° 12′ and 45° taken from 180° gives $AHB=47° 48'$.

We now have $AB=20$ miles one side, and each of the angles, A, B, and H, to find AH, and BH.

For AH Sin. 47° 48′ : 20 : : sin. 45° : AH

For BH, sin. 47° 48′ : 20 : : sin. 87° 12′ : BH.

Sin. 45°	9.849485	sin. 87° 12′	9.999481
Log. 20	1.301030		1.301030
	11.150515		11.300511
Sin. 47° 48′	9.869704		9.869704
AH, 19.09	1.280811	BH, 26.96	1.430807

(11.) By (Th. XVIII., Book III.), the length of a line drawn from the top of the tower to touch the surface of the sea, must be, $\sqrt{7960.5280.100}$, and from the same point on the sea, the line extended to the mast head of the ship must be $\sqrt{7960.5280.90}$. The problem requires the sum of these two lines.

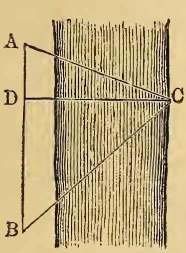
The computation by logarithms is as follows,

Log. 7960	3.900913		3.900913
Log. 5280	3.722634		3.722634
Log. 100	2.000000	log. 90	1.954243
	2)9.623547		2)9.577790
64829.6	4.811773	61502.8	4.788895
Sum in feet, 126332.4		log.	5.101514
Log.	5280		3.722634
Sum in miles,	23.92		1.378880
Add $\frac{1}{13}$ for refraction	1.84		
	25.76	*Ans.*	

(12.) As Sin. 35° : 143 feet : : cos. 35° : Dist.

Whence, $$\text{Dist.} = 143 . \frac{\cos. 35^\circ}{\sin. 35^\circ} = 143 . \cot. 35^\circ.$$

Log. 143	2.155336
Cot. 35°	10.154773
Ans. 204.22	2.310109



(13.) Let CD be the breadth of the river, or the distance sought, $AB=500$ yards, the measured base.

The angle $ABC=53°$, and BAC, 79° 12′. Whence the angle $ACB=47°\ 48'$.

Now,

Sin. 47° 48′ : 500 : : sin. 79° 12′ : BC.

Whence, $$BC = \frac{500 . \sin. 79^\circ\ 12'}{\sin. 47^\circ\ 48'}.$$

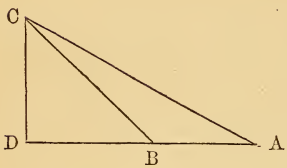
Again in the right angled triangle DBC, we have

Sin. D : BC : : sin. B : DC.

That is, Sin. 90 : BC : : sin. 53° : DC.

Whence, $$DC=\frac{500.\sin.\ 79°\ 12'.\sin.\ 53°}{\sin.\ 47°\ 48'.\sin.\ 90°}.$$

Log. 500	2.698970	
Sin. 79° 12′	9.992239	
Sin. 53°	9.902349	
	22.593558	numerator,
Log. R, sin. 47° 48′	19.869704	denominator.
Log. CD=log. 529.48	2.723854	



(14.) Let L represent the length of the inclined plane, and P its perpendicular height.

The angle CBD=46°, and CAB =31°. Whence, ACB=15°, and AB=200. The triangle ABC gives the proportion

Sin. 15° : 200 : : sin. 31° : $L=\frac{200.\sin.\ 31°}{\sin.\ 15°}$

From the triangle CBD, we obtain,

Sin. 90° : L : : sin. 46° : $P=\frac{L.\ \sin.\ 46°}{\sin.\ 90°}$.

That is, $$P \text{ or } CD=\left(\frac{200.\sin.\ 31°}{\sin.\ 15°}\right)\frac{\sin.\ 46°}{\sin.\ 90°}.$$

Log. 200	2.301030	
Log. sin. 31°	9.711839	
Log. sin. 46°	9.856934	
Log. of numerator	21.869803	
Log. of denominator	19.412996	=R.sin. 15°
Log. 286.28	2.456807	

(15.) Take the same triangle as in the preceding example.

Then, $P=\dfrac{300.\sin.\ 32^\circ.\sin.\ 58^\circ}{R.\sin.\ 26^\circ}$, Dis. $=\dfrac{300.\sin.\ 32^\circ.\cos.\ 58^\circ}{R.\sin.\ 26^\circ}$

Log. 300	2.477121		2.477121
Sin. 32°	9.724210		9.724210
Sin. 58°	9.928420	cos. 58°	9.724210
Log. num.	22.129751		21.925541
Log. denom.	19.641842		19.641842
P=307.54	2.487909	dis. 192.18	2.283699

(16.) Here we have a triangle, one side of which is 440 yards, and the adjacent angles, 83° 45′, and 85° 15′, therefore the angle opposite must be 11°.

Now, for the side of the triangle which is opposite the angle 85° 15′, we have the following proportion.

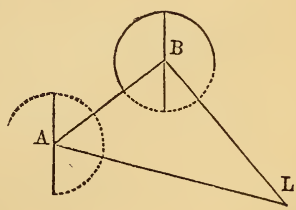
$$\text{Sin. } 11^\circ : 440 :: \text{sin. } 85^\circ 15' : x.$$

For the other side, we have

$$\text{Sin. } 11^\circ : 440 :: \text{sin. } 83^\circ 45' : y.$$

	For x.		*For y.*
Log. 440	2.643453		2.643453
Sin. 85° 15″	9.998506	sin. 83° 45′	9.997411
	12.641959		12.640864
Sin. 11°	9.280599		9.280599
2298.05	3.361360	2292.26	3.360265

(17.) Let A and B be the positions of the ship, when the observations were taken.



Then AB=12 miles in the direction north east.

Then, also AL, is the direction to the land, and BL, is another direction to the same point.

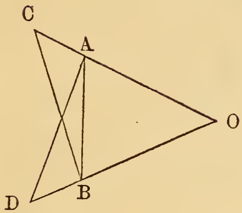
The angle LAB is 5 points of the compass, or $56° 15'$, and the angle ABL, is 9 points, or $101° 15'$.

Hence, the angle at L must be $22° 30'$.

Now, $\quad$ Sin. $22° 30' : 12 :: \sin. 56° 15' : BL$.

$$BL = \frac{12 . \sin. 56° 15'}{\sin. 22° 30'}.$$

Log. 12	1.079181
Sin. 56° 15′	9.919846
	10.999027
Sin. 22° 30′	9.582840
Ans. 26.072	1.416187



(18.) This problem requires the adjoining figure.

We must compute the angle CAB, to find BAO.

And from the triangle ABD, we must compute the angle ABD, to find ABO.

We compute the angle CAB, by the following formula (Prop. 8, Plane Trigonometry.)

$$\text{Cos. } \tfrac{1}{2}A = \sqrt{\frac{R^2 s(s-a)}{bc}}.$$

For the triangle ABC,

$$a=560,\ b=100,\ c=500.$$

Whence, $\quad s=580$, and $\quad s-a=20$.

$$\text{Cos. } \tfrac{1}{2}A = \sqrt{\frac{R^2\, 580 \times 20}{100 \times 500}} = \sqrt{\frac{R^2\, 116}{500}}$$

Log. R^2 116	22.064458
Log. 500	2.698970
	2)19.365488
Cos. $\frac{1}{2}A=$ 61° 12′ 21″ $=$	9.682744

$$\begin{array}{rl} & 2 \\ \hline A & 122^\circ\ 24'\ 42'' \\ & 180^\circ \\ \hline BAO= & 57^\circ\ 35'\ 18'' \end{array}$$

In the triangle ABD, changing D to C', then to find the angle B, we have

$$\text{Cos. } \tfrac{1}{2}B=\sqrt{\frac{R^2 s(s-b)}{ac}} \text{ a formula in which}$$

$$a=100, \quad b=550, \quad c=500.$$

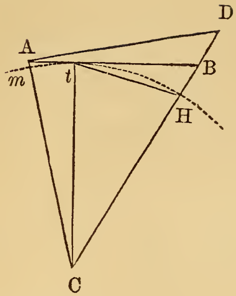
Whence, $\quad s=575$, and $s-b=25$.

$$\text{Cos. } \tfrac{1}{2}B=\sqrt{\frac{R^2\, 575\times 25}{100\times 500}}=\sqrt{\frac{R^2\, 115}{400}}.$$

$R^2 115$	22.060698
400	2.602060
	2)19.458638
Cos. $\frac{1}{2}B=$cos. 57° 34′ 31″	9.729319

$$\begin{array}{rl} & 2 \\ \hline B & 115^\circ\ \ 9'\ \ 2'' \\ & 180 \\ \hline ABO & \ 64^\circ\ 50'\ 58'' \\ BAO & \ 57^\circ\ 35'\ 18'' \\ \hline & 122^\circ\ 26'\ 16'' \\ & 180^\circ \\ \hline \text{Angle } O= & \ 57^\circ\ 33'\ 44'' \end{array}$$

Sin. 57° 33′ 44″ : 500 : : sin. 57° 35′ 18″ : $BO=500.14$.



(19.) Let ACD be 45 minutes of a degree, although in the figure before us, it is many degrees.

Let A be the position of the eye of the observer, and conceive the line AB, to touch the earth at t.

Then the angle $ACt = 4' \, 15''$.

Hence, $tCH = 40' \, 45''$.

Join tH.

Now the angle made between the chord tH and the tangent tB, is measured by half the arc, therefore, the angle $BtH = 20' \, 22\frac{1}{2}''$.

The angle DAB is $31' \, 20''$, and BD is the *visible* part of the mountain, and BH is the *invisible* part. We must first compute the tangents At, and tB.

$$R : \tan. \; 4' \, 15'' :: 3956 : tA$$

$$R : \tan. \; 40' \, 45'' :: 3956 : tB$$

Log. 3956	3.597256			3.597256
Tan. 4′ 15″	7.092115	*tan. 40′ 45″		8.073874
At 4.8907	0.689371	tB	46.8954	1.671130
		At	4.8907	
		AB	51.7861	

* The sine or tangent of an arc exceeding 20′ and less than 2° can not be computed accurately by the common method nor by the method explained on page 288, (Plane Trigonometry.)

In the present case we may employ the familiar formula.

$$\text{Sin. } 2a = 2 \text{ sin. } a . \text{cos. } a.$$

Put $$a = (40' \, 45'') \div 8 = 5' \, 5.625''.$$

Then, we have

Log. 2	.301030	
Sin. 5′ 5.625″	7.170764	(See page 288, Trig.)
Cos. 5′ 5.625″	10.000000	
	7.471794=sin. 10′ 11.25″.	

In the triangle tBH, we have

Sin. 89° 39′ 37½″ : 46.8954 : : sin. 20′ 22½″ : BH.

Log. tB		1.671130
Sin. 20′ 22½″		7.772824
		9.443954
Sin. 89° 39′ 37″		9.999992
BH, in parts of a mile	.27795	—1.443962

In the triangle ABD, we have

Sin. 88° 47′ 55″ : 51.7861 : : sin. 31′20″ : BD.

	Log. 51.7861			1.714213
(See note)	Sin. 31′ 20″			7.959727
				9.673940
	Sin. 88° 48′			9.999905
Visible part,	BD,	.47210		—1.674035
Invisible part,	BH,	.27795		
Whole mountain		0.75005	log.	—1.875090
	Log. 5280 feet in a mile,			3.722634
Height in feet, 3960				3.597724

2d Solution.

We can also solve this problem by means of the triangle ACD alone.

The angle $CAD=90° 27' 5''$. $C=45'$, therefore the angle

$$D=88° 47' 55''.$$

Sin. D : AC : : sin. CAD : CD.

Log. 2	.301030
Cos. 10′ 11.25″	9.999998
	7.772822=sin. 20′ 22.5″.
Log. 2	.301030
Cos. 20′22.5″	9.999993
	8.073845=sin. 40′45″.
Divide by cos. 40′45″	9.999971
	8.073874=tan. 40′45″.

$$\text{Sin. } 88^\circ\, 47'\, 55'' : AC :: \sin. 90^\circ\, 27'\, 5'' : CD.$$

$$CD = \frac{AC.\sin. 90^\circ\ 27'\ 5''}{\sin. 88^\circ\ 47'\ 55''}$$

$$CD - AC = \frac{AC.\sin. 90^\circ\ 27'\ 5''}{\sin. 88^\circ\ 47'\ 55''} - AC$$

$$CD - AC = \frac{AC(\sin. 90^\circ\, 27'\, 5'' - \sin. 88^\circ\, 47'\, 55'')}{\sin. 88^\circ\ 47'\, 55''} \quad (1)$$

But we have, Eq. (16), Plane Trigonometry,

$$\text{Sin. } A - \sin. B = 2 \cos. \tfrac{1}{2}(A+B)\sin. \tfrac{1}{2}(A-B).$$

Applying this to equation (1) above, and taking $AC = 3956$ miles, which is sufficiently near the truth for accuracy in the required result, the operation will be as follows:

$A=$	$90^\circ\, 27'\ \ 5''$		
$B=$	$88^\circ\, 47'\, 55''$		
	2)$179^\circ\, 15'\ \ 0$	log. 2	0.301030
	$89^\circ\, 37'\, 30''$	cos.	7.815909
$A-B=$	$1^\circ\, 39'\, 10''$		
$\frac{1}{2}(A-B)$	$49'\, 35''$	sin.	8.159047
	Log. $AC=3956$		3.597256
			$-\bar{1}.873242$
	Sin. $88^\circ\, 48'$ sub.		9.999905
Log. of $(CD-CA)$ in parts of a mile,			$-\bar{1}.873337$
Feet in a mile 5280,	log.		3.722634
Log. $(CD-CA)$ in feet,	3944.31		3.595971

We will now compute the value of Am. From (Th. 18, Book III., Geometry,) we have

$$2AC \times Am = \overline{At}^2, \text{ nearly.}$$

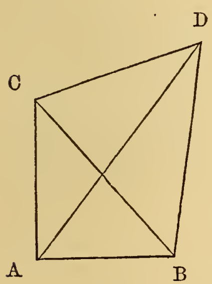
Or,
$$Am = \frac{\overline{At}^2}{2AC}.$$

And taking again, $AC=3956$, we have

		Log. $\overline{At}^2$	1.378742
Log. 2	.301030		
AC	3.597256		
	3.898286		3.898286
			-3.480456
		Log. 5280	3.722634
		Log. Am, 15.96 ft.	1.203090

Hence, $3944.31+15.96=3960.27$ feet$=HD$.

(20.) Note.—This problem is a singular and a very instructive one. We can not directly make use of the given side or line 500 yards, and we are forced to make a similar geometrical figure by assuming another base, and using the given angles.



The operation is as follows,

Take $AB=100$. Make $BAD=52° 12'$, $CAD=41° 25'$. $ABC=47° 40'$, and $CBD=48° 10'$.

These angles make the lines intersect at C and D, and CD must be computed, and if found to be 500, then 100 is the true distance between the ships. After we have computed CD, we call it a.

Then, because this figure is similar to the true figure, we have

$$a : 500 :: 100 : AB.$$

Whence, $$AB=\frac{50000}{a}.$$

The angle $ACB=38° 43'$, and $ADB=31° 58'$.

In the triangle ABC, we have

$$\text{Sin. } 38° 43' : 100 :: \sin. 93° 37' : BC \qquad (1)$$

In the triangle ABD we have

$$\text{Sin. } 31^\circ 58' : 100 :: \text{sin. } 52^\circ 12' : BD \qquad (2)$$

Log. 100 sin. 93° 37′	11.999134
Log. sin. 38° 43′	9.796206
Log. BC, 159.561	2.202928

Log. 100 sin. 52° 12′	11.897712
Log. sin. 31° 58′	9.723805
Log. BD, 149.247	2.173907

$CB+BD=308.808$. $CB-DB=10.314$.

In the triangle CBD, we have

$$308.808 : 10.314 :: \tan. 65^\circ 55' : \tan. \tfrac{1}{2}(D-C).$$

Log. 10.314	1.013427
Log. tan. 65° 55′	10.349719
	11.363146
Log. 308.808	2.489689
Tan. $\frac{1}{2}(D-C)=$ 4° 16′ 24″	8.873457

$$\tfrac{1}{2}(D+C)=65^\circ 55'$$

$$D \quad =70^\circ 11' 24''$$

$$\text{Sin. } 70^\circ 11' 24'' : 159.561 :: \text{sin. } 48^\circ 10' : CD.$$

Log. 159.561	2.202928	
Log. sin. 48° 10′	9.872208	
	12.075136	
Log. sin. 70° 11′ 24″	9.973507	
Log. CD=126.366	2.101629	sub. log. CD.
Log. 50000	4.698970	
Log. AB, 395.68	2.597341	

To obtain the true values of *CB*, *BD*, &c., we must multiply the results in this computation by 3.9568. That is, the sides of the true figure are very nearly equal to 4 times the sides of this computed figure.

SPHERICAL TRIGONOMETRY.

(Page 310.)

To solve right angled spherical triangles, the student will find it most convenient to apply *Napier's Circular points.*

Every triangle consists of *three* sides and two angles, besides the right angle, or *five parts.*

Either one of these parts taken at pleasure may be called a *middle* part. Then there will be two *adjacent parts,* and two *opposite parts.*

Then we can easily remember, that

1st. *Radius into the sine of the middle part, is equal to the product of the tangents of the adjacent parts.*

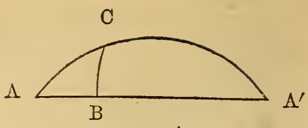
2d. *Radius into the sine of the middle part, is equal to the product of the cosines of the opposite parts.*

The parts are the two sides, *the complements of the hypotenuse,* and the *complement of the oblique angles.*

We can remember these rules by this consideration, that each rule expresses an equation.

The first member of each of these equations is *radius into some sine,* the second member is the product of *two tangents* or *two cosines.*

The student must rely on his own judgment, in selecting a part to be called the *middle part.*



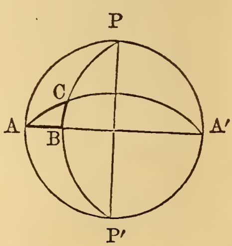
Let ABC represent any right angled spherical triangle, the right angle at B.

If the hypotenuse is greater than 90°, then the triangle will be represented by $A'BC$, and in that case one side will be greater than 90°, and the angle opposite to that side greater than 90°.

For the sake of perspicuity, we recommend the natural construction of each triangle presented for solution. This course will banish all doubts from the mind of the student as to what the results should be, &c. The solution of one triangle will in all cases be the solution of a whole hemisphere of triangles, as we are about to show by the following examples.

RIGHT ANGLED SPHERICAL TRIGONOMETRY.

(Page 356.)



In a right angled spherical triangle ABC (right angles always at B), given $AB = 118° 21' 4''$, and the angle $A = 23° 40' 12''$.

Remark.—This triangle is represented in the adjoining figure by the triangle $A'BC$, but we always operate on the triangle ABC of the figure, the one whose parts are each less than 90°.

Whence, $AB = 180° - 118° 21' 4'' = 61° 38' 56'' = c.$

To this triangle we apply equation (16), page 335, Text-book), observing that

$$\text{Sin. } A' = \text{cos. } A \text{ and tan. } b' = \text{cot. } b.$$

Whence, $\quad R.\text{cos. } A = \text{tan. } 61°\ 38'\ 56''\ \text{cot. } b.$

Or.	$\text{Cot. } b = \dfrac{R.\text{cos. } A}{\text{tan. } c}$	19.961835
		10.267932
AC	cot. $63°\ 42'\ 5''$	9.693903
	$180°$	
$A'C$	$116°\ 17'\ 55''$	

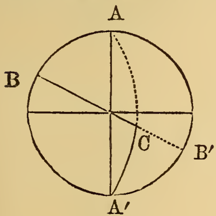
For the angle c.

As $\quad \text{Sin. } 63°\ 42'\ 5''\ :\ R\ ::\ \text{sin. } 61°\ 38'\ 56''\ :\ \text{sin. } C.$

$R.\text{sin. } c =$	19.944510
$\text{Sin. } b$	9.952549
$C = 79°\ 0'\ 34''$	9.991961

Or, $100°\ 59'\ 26''$. *Ans.*

(2.) In a spherical $\triangle$ ABC, given $AB = 53°\ 14'\ 20''$, and the angle $A = 91°\ 25'\ 53''$.



Here the proposed triangle is ABC, but we shall operate on its supplemental triangle $A'B'C$. $AB = A'B$. The angle $CA'B' = 90° - (1°\ 25'\ 53'') = 88°\ 34'\ 7''$.

Eq. (16.) $R.\ \text{cos. } 88°\ 34'\ 7'' = \text{tan. } 53°\ 14'\ 20''.\text{cot. } A'C.$

Or,	$\text{Cot. } A'C = \dfrac{R.\text{sin. } 1°\ 25'\ 53''}{\text{Tan. } 53°\ 14'\ 20''}$	18.397585
		10.126658
Cot. $A'C$	$88°\ 55'\ 51''$	8.270927
	180	
$AC =$	$91°\ 4'\ 9''$. *Ans.*	

For CB, we have $\quad R\ :\ \text{sin. } A'C\ ::\ \text{sin. } A'\ :\ \text{sin. } B'C.$

Whence, $\quad \sin. B'C = \dfrac{\sin. 88° 55' 51'', \sin. 88° 34' 7''.}{R}$

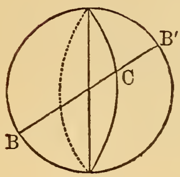
Sin. $88° 55' 51''$	9.999925
Sin. $88° 34' 7''$	9.999864
$B'C$, sin. $88° 12' 50''$	9.999789

Sup.$=BC$, $91° 47' 10''$. *Ans.*

Sin. $A'C$: sin. $90°$: : sin. AB : sin. C.

R.sin. AB	19.903707
Sin. AC=sin. $A'C$ $88° 55' 51''$	9.999925
Ans. Sin. $C=53° 15' 8''$	9.903782

(3.) In this example,



Let $AB=102° 50' 25''$, and the angle $BAC=113° 14' 37''$, but we shall operate on the supplemental triangle $AB'C$; $AB'=77° 9' 35''$, and $CAB'=66° 45' 23''$.

Eq. (16), calling A the middle part,

R.cos. $66° 45' 23''$=cot. AC.tan. $77° 9' 35''$.

$$\text{Cot. } AC = \frac{R.\text{cos. } 66°, \&c.}{\text{Tan.} AB}$$

R.cos. $66°$, &c.	19.596202
Tan. AB	10.642190
Ans. Cot. AC=cot. $84° 51' 36''$	8.954012

For $B'C$, we have

R.cos. AC=cos. AB'.cos. $B'C$.

R.cos. AC, $84° 51' 36''$	18.952258
Cos. AB' $77° 9' 35''$	9.346811
Cos. $B'C$, $66° 13' 33''$	9.605447

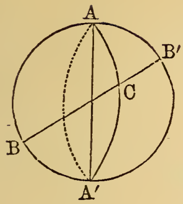
And Supplement, BC, $113° 46' 27''$. *Ans.*

For the angle C, we have

Sin. AC : R : : sin. AB' : sin. C.

Log. R.sin. AB' $77° 9' 35''$	19.989002
Sin. AC, $84° 51' 36''$	9.998250
Sin. ACB', sin. $78° 13' 4''$	9.990752

Supple. $ACB=101° 46' 56''$. *Ans.*



(4.) Let $AB'C$ in the last cut represent the triangle to be solved, $AB'=48° 24' 16''$, and $B'C$, $59° 38' 27''$.

Whence,

$R.\cos. AC = \cos. 48° 24' 16''.\cos. 59° 38' 27''.$

Cos. $48° 24' 16''$	9.822082
Cos. $59° 38' 27''$	9.703652
Cos. AC, cos. $70° 23' 42''$	9.525734. *Ans.*

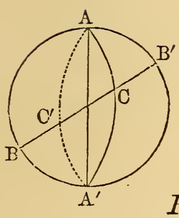
For the angles, we have the following proportions.

Sin. $70° 23' 42''$: sin. $90°$: : sin. $48° 24' 16''$: sin. C

: : sin. $59° 38' 27''$: sin. A.

Log. R.sin. $48° 24' 16''$	19.873814
Sin. $70° 23' 42''$	9.974064
Angle C, $52° 32' 56''$	9.899750

Log. R.sin. $59° 38' 27''$	19.935948
sin. $70° 23' 42''$	9.974064
Angle A, $66° 20' 40''$	9.961884. *Ans.*



(5.) In this example, AB is $151° 23' 9''$, and BC', $16° 35' 14''$.

We operate on the supplemental triangle $A'BC'$. Whence

$A'B = 28° 36' 51''$. And,

$R.\cos. A'C' = \cos. 28° 36' 51''.\ \cos. 16° 35' 14''.$

Cos. $28° 36' 51''$	9.943427	
Cos. $16° 35' 14''$	9.981540	
$A'C'$, cos. $32° 43' 9''$	9.924967	
AC', $147° 16' 51''$.		*Ans.*

For the angle C,

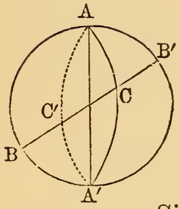
$$\text{Sin. } A'C' : \sin. 90^\circ :: \sin. A'B : \sin. A'C'B.$$

Log. $R.\sin.\ 28^\circ\ 36'\ 51''$	19.680253
Sin. $A'C'.\ 32^\circ\ 43'\ 9''$	9.732814
sin. $A'C'B,\ 62^\circ\ 22'\ 35''$	9.947439

Its supplement, or $AC'B = 117^\circ\ 37'\ 25''$

For the angle A, we have

$R.\sin.\ 16^\circ\ 35'\ 14''$	19.455568
$\sin.\ 32^\circ\ 43' 9''$	9.732813
Sin. $A = 31^\circ\ 52'\ 49''$	9.722755. *Ans.*



(6.) Here $A'B, 73^\circ\ 4'\ 31''$, $A'C', 86^\circ\ 12'\ 15''$. Required the other side and angles.

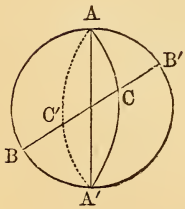
$$\text{Sin. } A'C' : \sin. 90^\circ :: \sin. A'B : \sin. BC'A'.$$

Log. $R.\sin.\ 73^\circ\ 4'\ 31''$	19.980771
$A'C'$, sin. $86^\circ\ 12'\ 15''$	9.999046
Sin. $BC'A'$, sin. $73^\circ\ 29'\ 40''$	9.981725. *Ans.*

To obtain BC', we apply one of the equations in circular parts.

$$\text{R.cos. } C' = \tan BC'.\cot. A'C'$$

Tan. $BC' = \dfrac{R.\cos.\ C'}{\cot.\ A'C'}$	19.453484 8.821819
$BC'\ 76^\circ\ 51'\ 20''$	10.631665. *Ans.*



(7.) Let $AB'C'$ represent the proposed triangle.

Then, $AB' = 47^\circ\ 26'\ 35''$, and $AC' = 118^\circ\ 32'\ 12''$.

We operate on the opposite supplemental triangle, $A'BC'$.

Now, $A'B = 47° 26' 35''$. $A'C' = 61° 27' 48''$.

Sin. $61° 27' 48''$: R. : : sin. $47° 26' 35''$: sin. C'

$$\text{Sin. } C' = \frac{R.\text{sin. } 47° 26' 35''}{\text{sin. } 61° 27' 48''} \qquad \begin{array}{r} 19.867235 \\ 9.943748 \end{array}$$

$C' = AC'B' = 56° 58' 44''$ $\quad$ 9.923487. *Ans.*

Again, we have

$R.\text{cos. } 56° 58' 44'' = \text{cot. } 61° 27' 48''.\text{tan. } BC'$.

Whence,

$$\text{Tan.} BC' = \frac{R.\text{cos. } 56° 58' 44''}{\text{cot. } 61° 27' 48''} \qquad \begin{array}{r} 19.736355 \\ 9.735427 \end{array}$$

BC', tan. $45° 3' 40''$ $\quad$ 10.000928

$180°$

$B'C'$ $\quad$ $134° 56' 20''$. *Ans.*

To find the angle A, we have

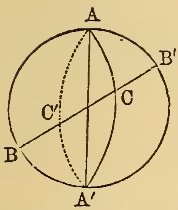
Sin. $A'C'$: sin. $90°$: : sin. BC' : sin. $BA'C'$.

Log. R.sin. BC', $45° 3' 40''$ $\quad$ 19.849948

Sin. $A'C'$, $61° 27' 48''$ $\quad$ 9.943748

Sin. $BA'C'$, $53° 40' 58''$ $\quad$ 9.906200

Supplement A, $126° 19' 2''$ $\quad$ *Ans.*



(8.) Let $AB'C'$ represent the triangle, AB', $40° 18' 23''$, and AC', $100° 3' 7''$. Required the other side and the oblique angles.

We operate as before on the supplemental triangle $A'BC'$. $AB' = A'B = 40° 18' 23''$. $A'C' = 79° 56' 53''$.

1st. For the angle C', we have

Sin. $A'C'$: sin. $90°$: : sin. $A'B$: sin. C'.

Log. R. sin. $40° 18' 23''$ $\quad$ 19.810821

$A'C'$, sin. $79° 56' 53''$ $\quad$ 9.993282

C', sin. $41° 4' 6''$ $\quad$ 9.817539. *Ans.*

To find BC', take the angle C' for a middle part, then by Napier's Circular Parts, we have

$$\text{Tan. } BC', \text{ cot. } A'C' = R.\text{cos. } 41° \ 4' \ 6''.$$

Log. R.cos. 41° 4′ 6″	19.877329
$A'C'$, cot. 79° 56′ 53″	9.248616
BC', tan. 76° 46′ 8″	10.628713

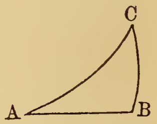
Supplement $B'C'$, 103° 13′ 52″. *Ans.*

For the angle A, we have

$$\text{Sin. } A'C' : \text{sin. } 90° :: \text{sin. } BC' : \text{sin. } A'.$$

Log. R.sin. 76° 46′ 8″	19.988316
sin. 79° 56′ 53″	9.993282
A', sin. 81° 21′ 7″	9.995034

Supplement $C'AB' = 98° \ 38' \ 53''$. *Ans.*



(9.) In the right angled spherical triangle ABC, given AC, 61° 3′ 22″, and the angle A, 49° 28′ 12″, to find the other parts.

As, Sin. 90° : sin. AC, 61° 3′ 22″ : : sin. A, 49° 28′ 12″ : sin. BC.

AC, sin. 61° 3′ 22″	9.942054
A, sin. 49° 28′ 12″	9.880852
BC, sin. 41° 41′ 32″	9.822906. *Ans.*

For AB, we have the equation,

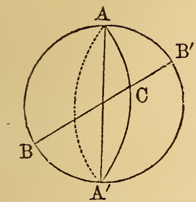
$$R.\text{cos. } AC = \text{cos. } AB.\text{cos. } BC.$$

$\text{Cos. } AB = \dfrac{R.\text{cos. } AC}{\text{cos. } BC}$	19.684803
	9.873162
Cos. 49° 36′ 6″	9.811641. *Ans.*

For the angle C, we have

Sin. AC, 61° 3′ 22″ : R. : : sin. AB, 49° 36′ 6″ : sin. C.

R.sin. AB	19.881703
sin. AC	9.942054
C, sin. 60° 29′ 20″	9.939649. *Ans.*



(10.) Here we have given one side of a right angled spherical triangle, and its opposite angle, to determine the other parts of the triangle, $AB'C$. $AB'=29° 12' 50''$, and $C=37° 26' 21''$.

In such cases the answers are said to be *ambiguous*, for the data give us no indication of which triangle is intended, $AB'C$ or $A'BC$. Because, $AB'=A'B=29° 12' 50''$, and $A'CB=ACB'=37° 26' 21''$.

To find AC, we have

Sin. AB', $29° 12' 50''$: sin. $37° 26' 21''$: : sin. AC : R.

$$\text{Sin. } AC=\frac{R.\text{sin. } 29° 12' 50''}{\text{sin. } 37° 26' 21''} \qquad \begin{array}{r} 19.688483 \\ 9.783846 \end{array}$$

AC, sin. $53° 24' 13''$ 9.904637. *Ans.*

Supplement $A'C$, $126° 35' 47''$.

To find $B'C$.

$$R.\text{cos. } AC=\text{cos. } AB' \text{ cos. } B'C.$$

$$\text{Cos. } B'C=\frac{R.\text{cos. } 53° 24' 13''}{\text{cos. } 29° 12' 50''} \qquad \begin{array}{r} 19.775372 \\ 9.940916 \end{array}$$

$B'C$, cos. $46° 55' 2''$ 9.834456. *Ans.*

Or its supplement.

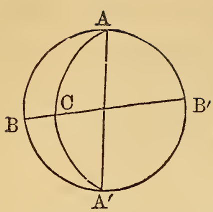
For the angle A, we have

Sin. $53° 24' 13''$: R. : : sin. $46° 55' 2''$: sin. A.

Log. R.sin. $46° 55' 2''$	19.863542
sin. $53° 24' 13''$	9.904637
Sin. A, $65° 27' 57''$	9.958905. *Ans.*

Or its supplement.

(11.) This example may require the solution of the triangle ABC, in adjoining figure, or of the triangle $A'B'C$.



But the solution of either of these supplemental triangles is effected by the triangle $A'BC$, which is a common supplemental triangle. We have

$$AB = A'B' = 100° \ 10' \ 3''.$$

$$ACB = A'CB' = 90° \ 14' \ 20''.$$

Also, $A'B = 79° \ 49' \ 57''$, and $A'CB = 89° \ 45' \ 40''$.

$$\text{Sin. } C : \text{sin. } A'B :: \text{sin. } 90° : \text{sin. } A'C.$$

Or, Sin. $89° \ 45' \ 40''$: sin. $79° \ 49' \ 57''$ :: $R.$: sin. $A'C$.

Whence,

Sin. $A'C = \dfrac{R.\text{sin. } 79° 49' \ 57''}{\text{sin. } 89° \ 45' \ 40''}$		19.993126
		9.999996
$A'C$, sin. $79° \ 50' \ 8''$		9.993130
	$180°$	
AC,	$100° \ 9' \ 52''$. *Ans.*	

By Eq. (20), Napier's Circular Parts, we have

$$R.\text{cos. } C = \text{cos. } 79° \ 49' \ 57''.\text{sin. } A.$$

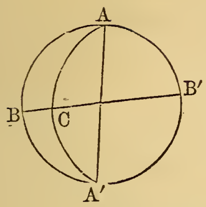
$$\text{Cos. } C = \text{cos. } 89° \ 45' \ 40'' = \text{sin. } 0° \ 14' \ 20'' = \text{sin. } 860'' = 860 \text{ sin. } 1''.$$

Log. sin. $1''$	4.685575	(See page 288 Text-book.)
Log. $R.$ 860	12.934498	
Log. $R.$ cos. C	17.620073	
Log. cos. $79° \ 49' \ 57''$	9.246810	
BAC=sin. $1° \ 21' \ 12''$	8.373263.	*Ans.*

For BC, we have

$$R. : \text{sin. } A'C :: \text{sin. } A : \text{sin. } BC.$$

Sin. $A'C$,	$79° \ 50' \ \ 8''$	9.993130
Sin. A,	$1° \ 21' \ 12''$	8.373263
BC, sin.	$1° \ 19' \ 55''$	8.366393. *Ans.*



(12.) This example may be represented by the triangle $A'BC$, or its supplemental triangle $AB'C$, in the figure.

In the triangle $A'BC$, we have

Sin. C, $61° 2' 15''$: sin. $A'B$, $54° 21' 35''$: : [R. : sin. $A'C$.

Sin. $A'C = \dfrac{R.\text{sin. } A'B}{\text{sin. } 61° 2' 15''}$		19.909925
		9.941976
$A'C$,	$68° 15' 26''$	9.967949
	$180°$	
AC,	$111° 44' 34''$. *Ans.*	

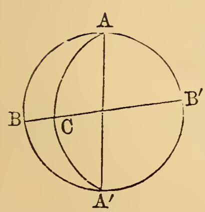
For BC, we have R.sin. BC = tan. $A'B$. cot. C.

Tan. $A'B$		10.144485
Cot. C		9.743081
Sin. BC,	$50° 31' 32''$	9.887566
	$180°$	
$B'C$ =	$129° 28' 28''$	

For the angle $BA'C$,

R.cos. $BA'C$ = tan. $A'B$.cot. $A'C$.

Tan. $A'B$		10.144485
Cot. $A'C$		9.600770
Cos. BAC = cos. $BA'C$,	$56° 12' 16''$	9.745255
	$180°$	
$B'AC$ =	$123° 47' 44''$	



(13.) This example may be made to correspond to the triangle ACB, or to the triangle $A'B'C$, because the opposite angles ACB, $A'CB'$ are equal, and $AB = A'B$.

But we shall operate on the triangle $A'BC$, which is supplemental to each of the other two.

Because $AB = 121° 26' 25''$, $A'B = 58° 33' 35''$.
Because $ACB = 111° 14' 37''$, $A'CB = 68° 45' 23''$.

For the side $A'C$, we have

$$\text{Sin. } C : \text{sin. } A'B :: R. : \text{sin. } A'C.$$

Log. R.sin. $58° 33' 35''$	19.931043
Sin. C, $68° 45' 23''$	9.969439
$A'C$, sin. $66° 15' 38''$	9.961604

Supplement, or AC, $113° 44' 22''$. *Ans.*

For the angle A, or BAC, we have

$$\text{Tan. } A'B.\text{cot. } A'C = R.\text{cos. } A' = R.\text{cos. } A.$$

Or, $\text{Cos. } A = \text{tan. } 58° 33' 35''.\text{cot. } 66° 15' 38''$.

Tan. $58° 33' 35''$	10.213698
Cot. $66° 15' 38''$	9.643246
A, cos. $43° 59' 55''$	9.856944

Or, supplement, $B'A'C$, $136° 0' 5''$. *Ans.*

For the side BC, we have

$$R. : \text{sin. } A'C :: \text{sin. } A' : \text{sin. } BC.$$

Sin. $A'C$	9.961604
Sin. A'	9.841760
BC, sin. $39° 29' 3''$	9.803364

Supplement, or $B'C$, $140° 30' 57''$ AC.

QUADRANTAL TRIANGLES.

Quadrantal spherical triangles have one side equal to 90 degrees, and all such triangles can be solved by right angled spherical trigonometry, as illustrated in the Text-book.

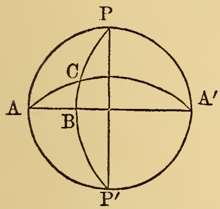
The following are not solved in the Text-book.

PRACTICAL PROBLEMS.

(Page 361.)

(1.) In a quadrantal triangle, given the quadrantal side, 90°, a side adjacent, 67° 3′, and the included angle, 49° 18′ to find the other parts.

Ans. { The remaining side is 53° 5′ 44″ ; the angle opposite the quadrantal side, 108° 32′ 29″ ; and the remaining angle, 60° 48′ 54″.



The triangle corresponding to this example is represented by APC. $AP=90°$, $PC=67°\ 3'$, and the angle at $P=49°\ 18'$.

We operate upon the triangle ABC, having $AB=P=49°\ 18'$, and $BC=90°-(67°\ 3')=22°\ 57'$.

For AC,

$$R.\cos. AC=\cos. 49°\ 18'.\ \cos. 22°\ 57'$$

Cos. 49° 18′	9.814313
Cos. 22° 57′	9.964187
AC, cos. 53° 5′ 44″	9.778500. *Ans.*

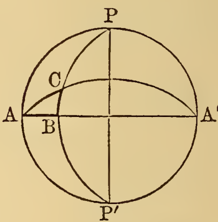
For the angles A and C, we have the proportions

$$\text{Sin. } AC : R. :: \text{sin. } AB : \text{sin. } C$$
$$:: \text{sin. } BC : \text{sin. } A.$$

For C.		*For A.*	
Log. R.sin. 49° 18′	19.879746	Log. R.sin. 22° 57′	19.590984
sin. 53° 5′ 44″	9.902894	sin. AC	9.902894
C, 71° 27′ 31″	9.976852	A, 29° 11′ 6″	9.688090
Sup. ACP, 108° 32′ 29″. *Ans.*		Com. PAC, 60° 48′ 54″ *Ans.*	

(2.) In a quadrantal triangle, given the quadrantal side, 90°, one angle adjacent, 118° 40′ 36″, and the side opposite this last-mentioned angle, 113° 2′ 28″, to find the other parts.

Ans. { The remaining side is 54° 38′ 57″ ; the angle opposite, 51° 2′ 35″ ; and the angle opposite the qaudrantal side, 72° 26′ 21″.



The triangle $A'PC$ corresponds to this example. $A'PC=A'B=118°\ 40'\ 36''$.

Whence, $AB=61°\ 19'\ 24''$, $A'C=113°\ 2'\ 28''$; therefore $AC=66°\ 57'\ 32''$. BC is the complement of PC, the side required.

For BC in the triangle ABC, we have

$$R.\cos.\ AC=\cos.\ AB.\cos.\ BC.$$

$\text{Cos. } BC=\dfrac{R.\cos.\ 66°\ 57'\ 32''}{\cos.\ 61°\ 19'\ 24''}$	19.592611
	9.681120
BC, cos. 35° 21′ 3″	9.911491

Whence, $PC=54°\ 38'\ 57''$. *Ans.*

For the angles A and C.

$$\text{Sin. } AC : R. :: \text{sin. } AB : \text{sin. } C$$

$$\text{Sin. } AC : R. :: \text{sin. } BC : \text{sin. } A.$$

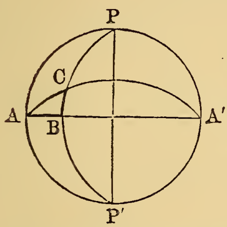
	For C.
Log. R.sin. 61° 19′ 24″	19.943168
Sin. AC, 66° 57′ 32″	9.963894
Sin. $C=72°\ 26'\ 21''$	9.979274. *Ans.*

	For A.
Log. R.sin. 35° 21′ 3″	19.762365
Sin. AC, 66° 57′ 32″	9.963894
Sin. 38° 57′ 26″	9.798471

Com. 51° 2′ 34″. *Ans.* $PA'C$.

(3.) In a quadrantal triangle given the quadrantal side 90°, and the two adjacent angles, one 69° 13′ 46″, the other 72° 12′ 4″, to find the other parts.

Ans. $\left\{\begin{array}{l}\text{One of the remaining sides is } 70^\circ\ 8'\ 39'', \text{ the other} \\ \text{is } 73^\circ\ 17'\ 29'', \text{ and the angle opposite the quad-} \\ \text{rantal side is } 96^\circ\ 13'\ 23''.\end{array}\right.$



The triangle is represented by PAC. Taking the angle $P=69^\circ\ 13'\ 46''$, then $AB=69^\circ\ 13'\ 46''$, and the angle BAC must equal $90^\circ-(72^\circ\ 12'\ 4'')$, or, $17^\circ\ 47'\ 56''=A$.

Now, by taking the angle A for a middle part, we have the equation,

$$\text{Cot. } AC.\text{tan. } AB=R.\text{cos. } A.$$

Whence,

$$\text{Cot. } AC=\frac{R.\text{cos. } A,\ 17^\circ\ 47'\ 56''}{\text{tan. } 69^\circ\ 13'\ 46''} \qquad \begin{array}{r} 19.978698 \\ 10.421044 \\ \hline \end{array}$$

$$AC,\ \text{cot. } 70^\circ\ 8'\ 39'' \qquad 9.557654. \quad \textit{Ans.}$$

For the side BC, of the triangle ABC, we have

$$R. : \text{sin. } AC :: \text{sin. } A : \text{sin. } BC.$$

Sin. AC, 70° 8′ 39″	9.973382
Sin. A, 17° 47′ 56″	9.485262
BC, sin. 16° 42′ 31″	9.458644

Complement, PC, 73° 17′ 29″. *Ans.*

For the angle C, of the triangle ABC, we nave

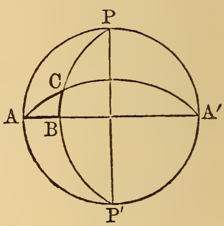
$$R.\text{cos. } AC=\text{cot. } A.\text{cot. } C.$$

R.cos. AC	19.531037
Cot. A	10.493436
Cot. C, 83° 46′ 37″	9.037601

Supplement, $PCA=96^\circ\ 13'\ 23''$. *Ans.*

(4.) In a quadrantal triangle, given the quadrantal side, 90°, one adjacent side, 86° 14′ 40″, and the angle opposite to that side, 37° 12′ 20″, to find the other parts.

Ans. { The remaining side is 4° 43′ 2″; the angle opposite 2° 51′ 23″; and the angle opposite the quadrantal side, 142° 42′ 3″.



The triangle is represented by PAC, $PC = 86° 14' 40''$, and $PAC = 37° 12' 20''$. Whence, the angle A, of the triangle $ABC = 52° 47' 40''$, and BC, 3° 45′ 20″.

For AC, we have

Sin. A : sin. BC : : R. : sin. AC.

Log. R.sin. BC, 3° 45′ 20″	18.816240
sin. A, 52° 47′ 40″	9.901170
Sin. AC, 4° 43′ 2″	8.915070. *Ans.*

For AB, we have

$$R.\cos. A = \cot. AC. \tan. AB.$$

R.cos. 52° 47′ 40″	19.781523
Cot. 4° 43′ 2″	11.083454
Tan. AB, 2° 51′ 23″	8.698069. *Ans.*

For the angle C, of the triangle ABC,

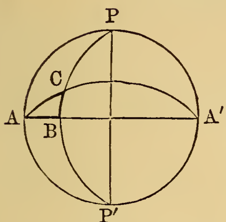
$$R.\cos. AC = \cot. A.\cot. C.$$

R.cos. 4° 43′ 2″	19.998527
cot. 52° 47′ 40″	9.880353
Cot. C, 37° 17′ 57″	10.118174

Supplement, $PCA = 142° 42' 3''$. *Ans.*

(5.) In a quadrantal triangle, given the quadrantal side, 90°, and the other two sides, one 118° 32′ 16″, the other 67° 48′ 40″, to find the other parts—the three angles.

Ans. The angles are 64° 32′ 21″, 121° 3′ 40″, and 77° 11′ 6″; the greater angle opposite the greater side, of course.



This problem requires the solution of the triangle $A'PC$, or $P'AC$, each one may by hypothesis correspond with the data. That is, $A'C$ or $P'C$=118° 32′ 16″, and PC or AC=67° 48′ 40″.

We will take $P'AC$. Then, BC= 118° 32′ 16″, less 90°.

That is, BC=28° 32′ 16″.

In the triangle ABC, we have AC and BC.

To find the angle A, or CAB, we have

Sin. AC, 67° 48′ 40″ : R. : : sin. 28° 32′ 16″ : sin. A.

Log. R.sin. 28° 32′ 16″	19.679191
sin. 67° 48′ 40″	9.966585
Sin. A, 31° 3′ 40″	9.712606

Add 90° and $P'AC$=121° 3′ 40″. *Ans.*

The side AB, is the measure of the angles P and P', and the angle C is also an angle in the triangle $P'AC$.

For AB, R. : cos. AB : : cos. BC : cos. AC.

$$\text{Cos. } AB = \frac{R.\text{cos. } AC,\ 67° 48' 40''}{\text{cos. } BC,\ 28° 32' 16''} \qquad \begin{array}{r} 19.577102 \\ 9.943743 \end{array}$$

Angle P'=AB, 64° 32′ 21″ 9.633359. *Ans.*

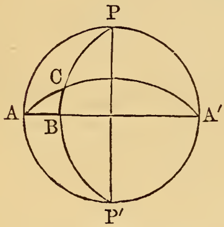
For the angle C, we have

Sin. AC : R. : : sin. AB : sin. C.

Log. R.sin. AB, 64° 32′ 21″	19.955630
Log. R.sin. AC, 67° 48′ 40″	9.966585
sin. C, 77° 11′ 6″	9.989045. *Ans.*

(6.) In a quadrantal triangle, given the quadrantal side, 90°, the angle opposite, 104° 41′ 17″, and one adjacent side, 73° 21′ 6″, to find the other parts.

Ans. $\left\{ \begin{array}{l} \text{Remaining side, } 49°\ 42'\ 16'' \text{; remaining an-} \\ \quad \text{gles, } 47°\ 32'\ 38'', \text{ and } 67°\ 56'\ 13''. \end{array} \right.$



This example refers to the triangle *APC*, because the given angle is greater than 90°. We must operate on the triangle *ABC*.

The angle $ACB = 180 - (104° 41' 17'') = 75° 18' 43''$, and *AC*, 73° 21′ 6″.

For *AB*, we have

$$R. : \sin. AC :: \sin. C : \sin. AB.$$

Whence, $\quad \text{Sin. } AB = \sin. AC . \sin. C.$

Sin. 75° 18′ 43″	9.985571
AC, sin. 73° 21′ 6″	9.981402
APC=*AB*, sin. *AB*, 67° 56′ 13″	9.966973. *Ans.*

For the side *BC*, we have

$$R. : \cos. AB :: \cos. BC : \cos. AC.$$

$\text{Cos. } BC = \dfrac{R.\cos.AC}{\cos. AB}$	19.457120
	9.574757
Cos. *BC*, 40° 17′ 44″	8.882363
Complement, *PC*=49° 42′ 16″.	*Ans.*

For the angle *A*, we have

$$\text{Sin. } AC : R. :: \sin. BC : \sin. A.$$

Log. *R*.sin. *BC*	19.810723
sin. 73° 21′ 6″	9.981402
Sin. *A*, 42° 27′ 22″	9.829321

Complement, *PAC*=47° 32′ 38″. *Ans.*

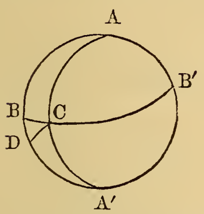
OBLIQUE ANGLED SPHERICAL TRIGONOMETRY.

PRACTICAL EXAMPLES.

(Page 367.)

NOTE.—Here, as in Plane Trigonometry, the sides are represented by a, b, c, and the angles opposite by A, B, C, that is, A opposite a, B opposite b, and C opposite c.

(1.) Given $b=118°\ 2'\ 14''$, $c=120°\ 18'\ 33''$, and the included angle $A=27°\ 22'\ 34''$, to find the other parts.



Here we have two sides, and the included angle.

This triangle is represented by ABC, but we operate on the supplemental triangle $A'BC$.

We may let fall the perpendicular CD, dividing the triangle $A'BC$ into the two right angled spherical triangles, $A'DC$, and BDC.

Or we may solve either triangle ABC or $A'BC$ directly, by applying Equations (8) and (9), (page 350, Geometry), which are

$$\text{Tan. } \tfrac{1}{2}(C+B)=\frac{\text{cot. } \frac{1}{2}A.\ \text{cos. } \frac{1}{2}(c-b)}{\text{cos. } \frac{1}{2}(c+b)} \qquad (8)$$

$$\text{Tan. } \tfrac{1}{2}(C-B)=\frac{\text{cot. } \frac{1}{2}A.\ \text{sin. } \frac{1}{2}(c-b)}{\text{sin. } \frac{1}{2}(c+b)} \qquad (9)$$

$c=120°\ 18'\ 33''$. Whence, $A'B=59°\ 41'\ 27''$.

$b=118°\ \ 2'\ 14''$

Sum $238°\ 20'\ 47''$ half sum $119°\ 10'\ 23.5''$.

Diff. $2°\ 16'\ 19''$ half diff. $1°\ \ 8'\ 9.5''$.

Cot. $\frac{1}{2}A=13° 41' 17''$	10.613406		10.613406
cos. $1° 8' 9.5''$	9.999915	sin.	8.297218
	20.613321		18.910624
Cos. $119° 10' 23.5''$	9.687932n	sin.	9.941090
Tan. $\frac{1}{2}(C+B)(83° 13' 42'')$	10.925389n		
Tan. $\frac{1}{2}(C-B)(\ 5° 19' 34'')$			8.969534

NOTE.—The cosine of an arc greater than 90° is negative. Hence the cosine 119° is minus, and we place n against the log. to show that it is negative. And since tan. $\frac{1}{2}(C+B)$ is negative also, the arc must terminate in the 2d quadrant · it is therefore the supplement of 83° 13′ 42″ ; hence

$$\begin{array}{rl} \frac{1}{2}(C+B) & 96° 46' 18'' \\ \frac{1}{2}(C-B) & 5° 19' 34'' \\ \hline \text{Sum } C, & 102° \ 5' 52'' \\ \text{Diff. } B, & 91° 26' 44'' \end{array} \Bigg\} \ Ans.$$

For the side BC, we subtract C from 180°, giving 77° 54′ 8″ for the angle $A'CB$.

Sin. 77° 54′ 8″ : sin. 59° 41′ 27″ : : sin. 27° 22′ 34″ : sin. a.

Sin. 59° 41′ 27″	9.936170
Sin. 27° 22′ 34″	9.662597
	19.598767
Sin. 77° 54′ 8″	9.990246
Sin. a, 23° 57′ 13″	9.608521. *Ans.*

(2.) Given

$$\left.\begin{array}{l} A=81° 38' 17'' \\ B=70° \ \ 9' 38'' \\ C=64° 46' 32'' \end{array}\right\} \text{to find } a, b, \text{ and } c.$$

By formula (W), (page 348, Geom.)

$$\begin{array}{rl} & 2)\underline{216° 34' 27''} \\ S & 108° 17' 13.5'' \\ & \underline{81° 38' 17''} \\ S-A & 26° 38' 56.5'' \end{array}$$

$$\sin. \tfrac{1}{2}a=\left(\frac{-\cos. S. \cos. (S-A)}{\sin. B. \sin. C}\right)^{\frac{1}{2}}.$$

NOTE.—The arc S being greater than 90°, its cosine is *minus*, and subtracting a *minus* quantity as the sign in the formula indicates, makes it plus.

Cos. S, 108°17′ 13.5″		9.496623
Cos.$(S-A)$, 26° 38′ 56.5″		9.951226
		19.447849
Sin. B, 70° 9′ 38″	9.973427	
Sin. C, 64° 46′ 32″	9.956479 sum,	19.929906
		2)-1.517943
Sin. $\frac{1}{2}a$.(Radius unity)=		-1.758971
For the radius of tables, add		10.
Tabular sin. of $\frac{1}{2}a=35°\,2'\,6.3''$		9.758971
2		
$a=70°\,4'\,13''$. *Ans.*		

Because, $\frac{\sin. b}{\sin. B}=\frac{\sin. a}{\sin. A}$.

Therefore, Sin. $b=\sin. B.\frac{\sin. a}{\sin. A}$. Sin. $c=\sin. C.\frac{\sin. a}{\sin. A}$.

Sin. a. 70° 4′ 13″	9.973179		
Sin. A, 81° 38′ 17″	9.995358		
	-1.977821		-1.977821
Sin. B	9.973427	sin. C	9.956479
b, 63° 21′ 24″	9.951248	$c=59°\,16'\,21''$	9.934300
			Ans.

(3.) Given a, 93° 27′ 34″, b, 100° 4′ 26″, and c, 96° 14′ 50″, the three sides to find the three angles.

By formula, on page 343.

$$\text{Cos. }\tfrac{1}{2}A=\left(\frac{\sin. S.\ \sin.(S-a)}{\sin. b.\ \sin. c.}\right)^{\frac{1}{2}}\text{(Radius unity.)}$$

a	$93° 27' 34''$			
b	$100° \ 4' 26''$	sin.	9.993252	
c	$96° 14' 50''$	sin.	9.997413	
2)	$289° 46' 50''$		19.990665	sum,
$S.$	$144° 53' 25''$	sin.		9.759777
	$93° 27' 34''$			
$(S-a)$	$51° 25' 51''$	sin.		9.893127
				19.652904
Sub. sum as above,				19.990665
			2)	-1.662239
Cos. $\frac{1}{2}A$,	$47° 19' 32''$			9.831120 adding 10.
	2			
A,	$94° 39' \ 4''$			

We may now obtain the other angles by Equations (8) and (9), (page 350, Geom.)

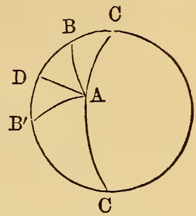
$$b = 100° \ 4' 26''$$
$$c = 96° 14' 50''$$
$$\tfrac{1}{2}(b+c) = 98° \ 9' 38''$$
$$\tfrac{1}{2}(b-c) = 1° 54' 48''$$

Cot. $\frac{1}{2}A$,	9.964707		9.964707
Cos. $\frac{1}{2}(b-c)$	9.999758	sin.	8.523587
	19.964465		18.488294
Cos. $\frac{1}{2}(b+c)$	$9.152128n$	sin.	9.995580
Tan. $\frac{1}{2}(B+C)$	$10.812337n$	tan. $\frac{1}{2}(B-C)$	8.492714

$$\tfrac{1}{2}(B+C) = 98° 45' 27''$$
$$\tfrac{1}{2}(B-C) = 1° 46' 52''$$
$$B = 100° 32' 19''$$
$$C = 96° 58' 35''$$

(4.) Given two sides, b, $84° 16'$, c, $81° 12'$, and the angle C, $80° 28'$, to find the other parts.

When this triangle is constructed, we find that the data will correspond equally well to the triangle ABC, and to $AB'C$, in the adjoining cut.



Hence the result is said to be *ambiguous*. In such cases the operator is expected to determine both results.

Observe that $AB=AB'$, hence the triangle ABB' is *isosceles*, and AD the perpendicular from A bisects BB' in D, making two right angled spherical triangles, ADC and ADB'. Their sum is the triangle ACB', and their difference ABC.

For the angles at B, or at B', we have the proportion

$$\text{Sin. } c,\ 81^\circ 12' : \text{sin. } C,\ 80^\circ 28' :: \text{sin. } b,\ 84^\circ 16' : \text{sin. } B.$$

$$\text{Sin. } B = \frac{\text{sin. } 84^\circ 16'.\text{sin. } 80^\circ 28'}{\text{sin. } 81^\circ 12'}.$$

Sin. $84^\circ 16'$	9.997822
Sin. $80^\circ 28'$	9.993960
	19.991782
Sin. $81^\circ 12'$	9.994857
Sin. B', or ABD, $83^\circ 11' 24''$	9.996925
Supplement, $96^\circ 48' 36'' = ABC$.	*Ans.*

By Equation (16), Napier's Circular Parts (page 335, Geometry), we have

$$\text{Tan. } CD.\text{cot. } b = R.\text{cos. } C = R.\text{cos. } 80^\circ 28'.$$

$$\text{Tan. } CD = \frac{R.\text{cos. } 80^\circ 28'}{\text{cot. } 84^\circ 16'} \qquad \begin{array}{r} 19.219116 \\ 9.001738 \end{array}$$

$$CD,\ \text{tan. } 58^\circ 46' 31'' \qquad 10.217378$$

In like manner, we obtain BD, from the triangle ABD.

That is, Tan. $BD = \dfrac{R.\cos.\ 83^\circ\ 11'\ 24''}{\cot.\ c.\ 81^\circ\ 12'}$ 19.074002
9.189794

BD, tan. $37^\circ\ 27'\ 2''$ 9.884208

But CD, $58^\circ\ 46'\ 31''$

Sum CB', or a, $96^\circ\ 13'\ 33''$. *Ans.*

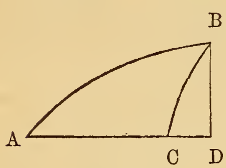
Diff. CB, or a, $21^\circ\ 19'\ 29''$. *Ans.*

For the angles A, we have

$$\frac{\text{Sin. } A}{\text{Sin. } a} = \frac{\text{sin. } C}{\text{sin. } c}, \text{ or sin. } A = \text{sin. } a . \frac{\text{sin. } C}{\text{sin. } c}$$

Sin. a. $21^\circ\ 19'\ 29''$	9.560688	sin. a. $96^\circ\ 13'\ 33''$	9.997431
Sin. C—sin. c	—1.999103		—1.999103
A, sin. $21^\circ\ 16'\ 43''$	9.559791	A, sin. $97^\circ\ 13'\ 45''$	9.996534
			Ans.

(5.) Given one side c, $64^\circ\ 26'$, and the adjacent angles A, 49°, and B, 52°, to find the other parts.



Let ABC represent the triangle, and from one extremity of the given side, let fall the perpendicular BD, making the two right angled spherical triangles, ADB and CDB.

By Equation (16), Napier's Circular Parts (Geom., p. 335), we have Tan. AD. cot. $64^\circ\ 26' = R.\cos.\ 49^\circ$.

R.cos. 49°	19.816943
Cot. $64^\circ\ 26'$	9.679795
Tan. AD, $53^\circ\ 54'$	10.137148

For the angle ABD, we have

Sin. $64^\circ\ 26'$: R. : : sin. $53^\circ\ 54'$: sin. ABD.

R.sin. $53^\circ\ 54'$	19.907406
Sin. $64^\circ\ 26'$	9.955247
Angle $ABD = 63^\circ\ 35'\ 51''$	9.952159
ABC 52	
Angle $CBD = 11^\circ\ 35'\ 51''$	

In the triangle ABD, we have

$$R. : \sin. 64^\circ 26' :: \sin. A, 49^\circ : \sin. BD.$$

Sin. 49°	9.877780
Sin. $64^\circ 26'$	9.955247
BD, sin. $42^\circ 54' 26''$	9.833027

For BC, or a of the triangle ABC, we use the equation

$$\text{Tan. } BD. \cot. BC = R{:}\cos. 11^\circ 35' 51''.$$

R.cos. $11^\circ 35' 51''$	19.991042
Tan. $42^\circ 54' 26''$	9.968246
Cot. BC, $43^\circ 29' 49''$	10.022796. *Ans.*

For AC, or b, we have

$$\text{Sin. } 49^\circ : \sin. 43^\circ 29' 49'' :: \sin. 52^\circ : \sin. AC.$$

Sin. $43^\circ 29' 49''$	9.837788
Sin. 52°	9.896532
	19.734320
Sin. 49°	9.877780
b, or AC, sin. $45^\circ 56' 46''$	9.856540. *Ans.*

For the angle C, we have

$$R.\cos. BC = \cot. CBD.\cot. BCD.$$

R.cos. $43^\circ 29' 49''$	19.860584
Cot. CBD, $11^\circ 35' 51''$	10.687769
Cot. BCD, $81^\circ 31' 56''$	9.172815

$BCA = 98^\circ 28' 4''$ *Ans.*

(6.) Result obvious.

(7.) Given two sides and an angle opposite one of them to determine the other parts.

$a = 77^\circ 25' 11''$, $c = 128^\circ 13' 47''$, and the angle $C = 131^\circ 11' 12''$.

To find the angle A, we have

$$\frac{\text{Sin. } A}{\text{Sin. } a} = \frac{\sin. C}{\sin. c} = \frac{\sin. 131^\circ 11' 12''}{\sin. 128^\circ 13' 47''} = \frac{\cos. 41^\circ 11' 12''}{\cos. 38^\circ 13' 47''}$$

Whence, Sin. A=sin. a. $\dfrac{\cos.\ 41^\circ\ 11'\ 12''}{\cos.\ 38^\circ\ 13'\ 47''}$ 9.989446 sin. a

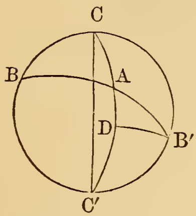
9.876546

19.865992

cos. $38^\circ\ 13'\ 47''$ 9.895166

Sin. A, $69^\circ\ 13'\ 59''$ 9.970826. *Ans.*

We have now to determine B and b, and the process will be apparent after we construct the triangle, as represented in the adjoining figure.



The data gives the triangle ABC; $AB'C$ is a supplemental triangle. From B', let fall the perpendicular $B'D$.

In the right angled triangle $AB'D$, we have the angle A, and the hypotenuse AB'. AB' being the supplement of c.

From the two triangles ADB' and $C'DB'$, we can obtain AD, and DC', and their sum taken from 180°, will give AC, or b.

The angle $DC'B$, is the supplement of C, which is

$$48^\circ\ 48'\ 48''$$

By Napier's Circular Parts, we have

$$\text{Cot. } AB'.\text{tan. } AD = R.\text{cos. } A.$$

Or, Tan. $AD = \dfrac{R.\cos.\ 69^\circ\ 13'\ 59''}{\cot.\ 51^\circ\ 46'\ 13''}$ 19.549698

9.896396

Tan. AD, $24^\circ\ 13'\ 56''$ 9.653302

Also, Tan. $C'D = \dfrac{R.\cos.\ 48^\circ\ 48'\ 48''}{\cot.\ a.,\ 77^\circ\ 25'\ 11''}$ 19.818566

9.348626

Tan. $C'D$, $71^\circ\ 16'\ 44''$ 10.469940

$AD + DC' = 95^\circ\ 30'\ 40''$

Supplement AC', $84^\circ\ 29'\ 20'' = b$. *Ans.*

Lastly, for the angle B, we have

$$\frac{\text{Sin. } B}{\text{Sin. } b}=\frac{\text{sin. } A}{\text{sin. } a}, \text{ or sin. } B=\frac{\text{sin. } b.\text{sin. } A}{\text{sin. } a}.$$

Sin. b	9.997988
Sin. A	9.970826
	19.968814
Sin. a	9.989446
Sin. B, 72° 28′ 42″	9.979368. *Ans.*

(8.) Given $a=$ 68° 34′ 13″ to find A, B, and C.

	$b=$ 59° 21′ 18″	sin. com.	065328
	$c=$112° 16′ 32″	sin. com.	033684
	2)240° 12′ 3″		
Use formula (T),	120° 6′ 1.5″	S. sin.	9.937090
(p. 343, Geom.)	68° 34′ 13″		
	51° 31′ 48.5″	$(S-a)$ sin.	9.893726
			2)19.929828
	Cos. 22° 43′ 19″		9.964914
	2		
	A, 45° 26′ 38″. *Ans.*		

Now, $\quad$ Sin. $B=$sin. $b.\dfrac{\text{sin. } A}{\text{sin. } a}$, and sin. $C=$sin. $c.\ \dfrac{\text{sin. } A}{\text{sin. } a}$.

Sin. $b=$	9.934672	sin. $c=$	9.966316
Sin. $A-$sin. $a=$	$-$1.883936		$-$1.883936
B, sin. 41° 11′ 30″	9.818608	C, 134° 53′ 55″	9.850252
			Ans.

NOTE.—We take C greater than 90°, because c was given greater than 90°. The logarithm gives 45° 6′ 5″, and its supplement is the angle required.

(9.) Same formula as applied to the preceding.

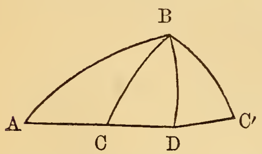
Given $a=$	$89° 21' 37''$ to find A, B, and C.		
$b=$	$97° 18' 39''$	sin. com.	003546
$c=$	$86° 53' 46''$	sin. com.	000637
	$2)273° 34' 2''$		
	$136° 47' 1''$ S.	sin.	9.835536
	$89° 21' 37''$		
	$47° 25' 24''$ $(S-a)$	sin.	9.867097
			$2)19.706816$
Cos. $\frac{1}{2}A$,	$44° 28' 40''$		9.853408
	2		
A,	$88° 57' 20''$. *Ans.*		

Now we have,

$$\text{Sin. } B=\text{sin. } b.\ \frac{\text{sin. } A}{\text{sin. } a},\ \text{and sin. } C=\text{sin. } c.\ \frac{\text{sin. } A}{\text{sin. } a}.$$

Sin. b	9.996454	sin. c	9.999362
Sin. $A-$sin. a	-1.999955		-1.999955
B, sin. $97° 21' 26''$	9.996409	$C=88° 47'' 17''$	9.999317

(10.) Given $a=31° 26' 41''$, $c=43° 22' 13''$, and the angle $A=12° 16'$, to find the other parts.



This example applies to the adjacent figure.

ABC and ABC', either one, will correspond with so much of the data as is given. Hence the result is ambiguous.

But, $$\text{Sin. } C=\text{sin. } c.\ \frac{\text{sin. } A}{\text{sin. } a}.$$

c, sin. $43° 22' 13''$	9.836774
A, sin. $12° 16'$	9.327281
	19.164055
a, sin. $31° 26' 41''$	9.717400
C', sin. $16° 14' 27''$	9.446655

Or, C, $163° 45' 33''$. *Ans.*

In the right angled spherical triangle ADB, we have

$$R. : \sin. c :: \sin. A. : \sin. BD.$$

Sum as above, omitting radius,

$$\text{Sin. } BD,\ 8° 23' 22''=9.164055.$$

In the same triangle we have,

$$R.\cos. c=\cos. AD.\cos. BD.$$

$R.\cos. c$	19.861493
Cos. BD	9.995327
Cos. $AD=42° 42' 37''$	9.866166

In the right angled triangle CBD, we have

$$R.\cos. a=\cos. CD.\cos. BD.$$

$R.\cos. a$	19.931022
Cos. BD	9.995327
Cos. CD, $30° 24' 57''$	9.935695

$AD=\ 42° 42' 37''$

Sum$=AC'=b$, $73° 7' 34''$. *Ans.*

Diff.$=AC=b$, $12° 17' 40''$. *Ans.*

To find the angles at B, we have the following proportion.

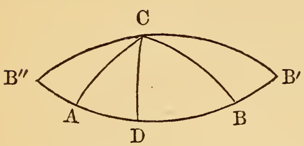
$$\text{Sin. } a : \sin. A :: \sin. 12° 17' 40'' : \sin. ABC.$$

Sin. A	9.327281
Sin. $12° 17' 40''$	9.328248
	18.655529
Sin. a	9.717400
ABC, $4° 58' 30''$	8.938129

Sin. a. : sin. A. : : sin. $73° 7' 34''$: sin. ABC'.

Sin. A	9.327281
Sin. $73° 7' 34''$	9.980888
	19.308169
Sin. a	9.717400
$C'BA$, $22° 56' 16''$	9.590769
$180°$	
ABC' $157° 3' 44''$	

(11.) In a triangle ABC, we have given, A, $56° 18' 40''$, B, $39° 10' 38''$, and AD, $32° 54' 16''$, a segment of the base made by a perpendicular let fall from the angle C, on to the side AB, to determine the triangle. The angle C being obtuse.



This is an ambiguous case. For, in the lune B'' B', let $B''=B'=39° 10' 38''$. Take a point, C, nearer to B'' than to B', and draw CB equal to CB''. Then the triangle $CB''B$ will be isosceles, and we shall have $CBB''=B''=B'$. Suppose CA, and the perpendicular CD, be drawn, making $CAB=56° 18$ $40''$, and $AD=32° 54' 16''$. Now since $B'=CBA$, the given parts belong equally to the two triangles, CAB, and CAB'. It will be observed, however, that CB and CB' are supplements of each other, because $CB=CB''$.

Tan. $32° 54' 16''$. cot. $AC=R$. cos. A. $56° 18' 40''$.

R.cos. $56° 18' 40''$	19.744045
Tan. $32° 54' 16''$	9.810931
(b) cot. AC, $49° 23' 41''$	9.933114. *Ans.*

For CD, we have

$$R. : \sin. AC :: \sin. A : \sin. CD.$$

Sin. AC	9.880363
Sin. A	9.920155
Sin. CD, $39° 10' 35''$	9.800518

Now in the right angled triangle CDB, we have

$$\text{Sin. } B : \sin. CD :: R. : \sin. CB.$$

R.sin. CD	19.800518
Sin. B	9.800525
Sin. CB, $89° 40'$	9.999993

Or, $CB' = 90° 20'$.

NOTE.—It will be observed that arcs which differ from 90° by less than 1°, can not be determined accurately to seconds, when the *sine* is used; and we can not use the cosine or tangent in this case.

For the angle ACD, we have

$$\text{Sin. } AC : R. :: \sin. AD : \sin. ACD.$$

R.sin. AD,	$32° 54' 16''$	19.734991
Sin. AC,	$49° 23' 41''$	9.880363
Sin. ACD,	$45° 41' 12''$	9.854628

To determine the angles, DCB and DCB', we have

$$R.\cos. DCB = \tan. CD.\cot. CB.$$

$$R.\cos. DCB' = \tan. CD.\cot. CB'.$$

Tan. CD = 9.911102		Tan. CD = 9.911102
Cot. CB = 7.764761		Cot. CB' = 7.764761n
Cos. DCB = 7.675863		Cos. DCB' = 7.675863n
DCB = $89° 44'$		DCB' = $90° 16'$
Add ACD = $45° 41'$		ACD = $45° 41'$
ACB = $135° 25'$	*Ans.*	ACB' = $135° 57$ *Ans.*

For the sides AB and AB', we have

$$\text{Sin. } B : \text{sin. } AC :: \text{sin. } ACB : \text{sin. } AB.$$
$$\text{Sin. } B' : \text{sin. } AC :: \text{sin. } ACB' : \text{sin. } AB'.$$

Sin. AC=	9.880363	Sin. AC =	9.880363
Sin. ACB=	9.846304	Sin. ACB'=	9.842163
	19.726667		19.722526
Sin. B=	9.800525	Sin. B'=	9.800525
Sin. AB=	9.926142	Sin. AB'=	9.922001
AB=	$122° 29'$ *Ans.*	AB'=	$123° 19'$ *Ans.*

(12.) Given the angles A, B, C, and required the sides a, b, c. (Prop. 6, Sec. I., Spherical Geometry.)

$A=80° 10' 10''$	sup.	$99° 49' 50''$		
$B=58° 48' 36''$		$121° 11' 24''$	sin. com.	.067803
$C=91° 52' 42''$		$88° 7' 18''$	sin. com.	0.000233
		2)$309° 8' 32''$		
	S,	$154° 34' 16''$	sin.	9.632852
	a,	$99° 49' 50''$		
	(S$-a$)	$54° 44' 26''$	sin.	9.911981
				2)19.612869
	Cos. $\frac{1}{2}$ A,	$50° 10' 49''$		9.806434
		2		
		$100° 11' 38''$		

Supplement $a=79° 38' 22''$. *Ans.*

NOTE.—The preceding process strictly corresponds to theory, but the result will be the same, if we take out the arc, whose sine corresponds to the given logarithm, and the double of that arc will be the side a.

Thus, the sine of $39° 49' 11''$, is 9.806434, and the double of $39° 49' 11''$, is $79° 38' 22''$, or a.

Now, $\text{Sin. } b = \text{sin. } B. \dfrac{\text{sin. } a}{\text{sin. } A}$, and $\text{sin. } c = \text{sin. } C. \dfrac{\text{sin. } a}{\text{sin. } A}$.

Sin. B,	9.932197	sin. C,	9.999767
Sin. a—sin. A.	-1.999284		-1.999284
$b=58°\ 39'\ 16''$	9.931481	$c=86°\ 12'\ 50''$	9.999051

SECTION V.

APPLICATION OF SPHERICAL TRIGONOMETRY TO THE SOLUTION OF PROBLEMS IN ASTRONOMY.

(Page 373.)

(2.) In latitude 42° 40′ N., when the sun's declination is 23° 12′ N., what time will the sun set?

Tan. 42° 40′	9.964588
Tan. 23° 12′	9.632053
Sin. 23° 16′ 7″	9.596641

This arc, 23° 16′ 7″, reduced to time at the rate of 4 minutes to one degree, gives $1^h\ 33^m\ 4^s$.

Add 6

Sun sets, $7^h\ 33^m\ 4^s$. sun rises, $4^h\ 26^m\ 56^s$.

(3.) What time will the sun set in latitude 42° 4′ north, and sun's declination 15° 21′ south?

Tan. 42° 40′	9.964588
Tan. 15° 21′	9.438554
Sin. 14° 39′ 21″	9.403142

This arc corresponds to $58^m\ 37^s$ in time, which is the interval between six o'clock and sun set, and as the observer is north, and the declination south, the sun must set before six, that is, at $5^h\ 1^m\ 23^s$, apparent time. The sun must rise same day at $6^h\ 58^m\ 37^s$, A.M.

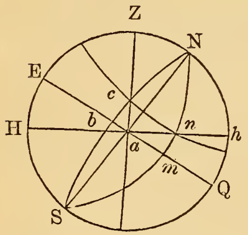
(4.) Lat. 52° 30′ north (London), and the sun's declination 18° 41′ south. Required, times of sunrise and sunset.

Tan. 52° 30′	10.115019
Tan. 18° 42′	9.529535
Sin. 26° 10′ 30″	9.644554

This arc corresponds in time to $1^h\ 44^m\ 42^s$, to which add 6^h for sunrise, and subtract it from six hours for sunset.

Whence, $\left.\begin{array}{l}\text{Rises,}\\ \text{Sets,}\end{array}\right\}\begin{array}{l}7^h\ 44^m\ 42^s\\ 4^h\ 15^m\ 18^s\end{array}\Big\}$ apparent time. *Ans.*

(5.) This problem is clearly represented by the adjoining cut. *NS* is the earth's axis, *NES* the meridian, *EQ* the equator, *cn* the parallel of declination.



The right angled spherical triangle *amn*, will give the position of the sun, and times of sunrise and sunset, and the right angled triangle, *abc*, or *ac*, will be the altitude of the sun, when east and west.

The arc *am* corresponds to the times after and before six, when the sun sets and rises, and *an* is the arc on the horizon towards the north from the east and west points, when the sun rises and sets.

In short, the solution will be as in the preceding examples, and the triangle *amn* in this cut will illustrate all of them.

Tan. D, 23° 24′	9.636226
Tan. L, 59° 56′	10.237394
Sin. 48° 22′ 32″, *am*,	9.873620

In time, $3^h\ 13^m\ 30^s$.

Adding this interval to 6 hours, gives the time of sunset, and subtracting it from 6 hours, will give the time of sunrise.

For the point *n* on the horizon, we have

Sin. *man*, or cos. lat. : sin. D. : : R. : sin. *an*.

$\text{Sin. } an = \dfrac{R.\text{sin. } D}{\text{cos. } L}$	19.598952
	9.699844
N. of E., sin. 52° 26′ 18″	9.899108

In the right angled spherical triangle *abc*, we have the angle *EaZ*, the latitude, and *bc*, the declination.

Sin. L. : sin. *bc*, 23° 24′ : : R. : sin. *ac*.

ac is the altitude of the sun when the sun is at the point *c*, east or west on the prime vertical.

$\text{Sin. Alt.} = \dfrac{R.\text{sin. } D}{\text{sin. } L}$	19.598952
	9.937238
Sin. *ac*, 27° 18′ 57″	9.661714

To find *ba*, the time before and after six o'clock, apparent time, when the sun is east and west, we have

R. : cos. *ab* : : cos. *bc* : cos. *ac*.

$\text{Cos. } ab = \dfrac{R.\text{cos. } ac}{\text{cos. } bc}$	19.948653
	9.962727
Cos. *ab*, 14° 30′ 30″	9.985926

This arc (14° 30′ 33″) reduced to time, is equivalent to 58 minutes 2 seconds. Hence the sun is east at $6^h\ 58^m\ 2^s$, A.M., and it is west at $5^h\ 1^m\ 58^s$, P.M., in latitude 59° 56′

when the sun's declination is 23° 24′ north. Problem (2,) [page 376].

By the formula on same page (Text-book), we must operate as follows,

True Alt.,	36° 12′		
Lat. north,	40° 21′	cos. com.	0.117986
North polar dis.	93° 20′	sin. com.	0.000735
	2)169° 53′		
	84° 56′ 30″	cos.	8.945320
	36° 12′		
	48° 44′ 30″	sin.	9.876070
			2)18.940111
Sin. ½P, 17° 10′ 1″			9.470055

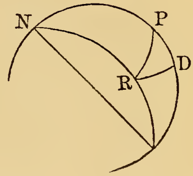
The double of this angle, or (34° 20′ 2″), changed into time, is $2^h\ 17^m\ 20^s$, the interval of time from apparent noon.

Whence, from noon,	$12^h\ 0^m\ 0^s$	
Take,	$2^h\ 17^m\ 20^s$	
	$9^h\ 42^m\ 40^s$	A.M., apparent time.

(3.) Alt.,	40° 8′		
Latitude,	21° 2′	cos. com.	0.029945
South polar dis.	108° 32′	sin. com.	0.013228
	2)169° 42′		
	84° 51′	cos.	8.953100
	40° 8′		
	44° 43′	sin.	9.847327
			2)18.853500
½P, sin. 15° 29′ 40″			9.426750
	2		
	30° 59′ 20″ = $2^h\ 3^m\ 57^s$, P.M. *Ans.*		

A Geographical Problem, (page 377, Text-book.)

Required, the number of degrees on a great circle between New Orleans and Rome, also the number of miles taking 69.16 miles to each degree.



Let P, be the north pole on the earth, N, the position of New Orleans, and R the position of Rome, which positions are

	Latitude.	*Longitude.*
New Orleans,	29° 57′ 30″ *N.*	90° *W.*
Rome,	41° 53′ 54″ *N.*	12° 28′ 40″ *E.*

The co. latitude of N. O. is $NP=60° \ 2' \ 30''$.
co. latitude of Rome is $PR=48° \ 6' \ 6''$.

And the sum of longitudes is 102° 28′ 40″, which is the angle NPR. The supplement of the angle NPR, is RPD, 77° 31′ 20″. RD, is the perpendicular let fall on NP produced.

Now, in the triangle PRD, we have

$$R. : \sin. PR, 48° \ 6' \ 6'' :: \sin. 77° \ 31' \ 20'' : \sin. RD.$$

Sin. PR	9.871767
Sin. P	9.989619
Sin. RD, 46° 36′ 53″	9.861386

$$R.\cos. PR=\cos. PD.\cos. RD.$$

$\text{Cos. } PD=\dfrac{R.\cos. PR.}{\cos. RD}$	19.824654
	9.836894
Cos. PD=cos. 13° 32′ 21″	9.987760
Add NP, 60° 2′ 30″	
Sum, or ND, 73° 34′ 51″	

In the right angled triangle NDR, we have

$$R.\cos. NR = \cos. ND.\cos. DR.$$

Cos. ND, 73° 34′ 51″	9.451268
Cos. DR, 46° 36′ 53″	9.836894
Cos. NR, 78° 48′ 15″	9.288162

That is, the distance in degrees is 78°.8041, and at the rate of 69.16 miles to each degree, gives 5450.1 English miles.

We can compute this side more concisely by using the formula

$$\text{Cos. } c = \cos. (a+b) + 2\cos.^2 \tfrac{1}{2}C.\sin. a.\sin. b.$$

Let the angle $NPR = C = 102°\ 28'\ 40''$, $a = 60°\ 2'\ 30''$, and $b = 48°\ 6'\ 6''$, $c =$ the arc NR.

Nat. Cos. $(a+b)$ = Nat. cos. 108° 8′ 36″ = −.31139.

Log. 2		0.301030
Cos. $\frac{1}{2}C$, $R=1$		−1.796627
		−1.796627
Sin. a		−1.937712
Sin. b		−1.871767
Nat. number,	0.50555	−1.703763. Sum.
Add,	−0.31139	
Nat. cos. 78° 48′ 15″	.19416.	*Ans.* same as before.

NOTE.—There are nine examples on page 379, Text-book; we shall show the solutions of only two or three of them, by a special formula, which we think the most concise, all things considered. Others may solve them by this or any one of the other methods, explained in the Geometry.

The difference of the right ascensions of the two bodies at the time designated, changed into arc, will be the included angle of a spherical triangle, and the complements of the declinations are the two sides of such a triangle; the third side is required and we designate it by x. The angle at the pole of the celestial equator, we designate by P, and a and b the sides.

Now, by the Fundamental Equation in Spherical Trigonometry, we have. (See page 342, Geom.)

$$\text{Cos. } P = \frac{\cos. x - \cos. a.\cos. b}{\sin. a.\sin. b}. \qquad (1)$$

Let D be the greater declination, and d the less.

Then D, and d, are complements of a and b, and $\cos. a = \sin. D$, and $\sin. a = \cos. D$, and the above equation becomes,

$$\text{Cos. } P = \frac{\cos. x - \sin. D.\sin. d}{\cos. D.\cos. d}.$$

Subtracting each member from unity, we obtain

$$1 - \cos. P = 1 - \frac{\cos. x - \sin. D.\sin. d}{\cos. D.\cos. d}.$$

$$1 - \cos. P = \frac{(\cos. D.\cos. d + \sin. D.\sin. d) - \cos. x.}{\cos. D.\cos. d}.$$

That is, $$2\sin.^2 \tfrac{1}{2}P = \frac{\cos.(D-d) - \cos. x}{\cos. D.\cos. d}.$$

Whence, $\text{Cos. } x = \cos.(D-d) - 2\sin.^2 \tfrac{1}{2}P.\cos. D.\cos. d \quad (F)$

This formula is essentially the same as that in the Textbook.

In that, the given sides of the triangle are used. In this, it is the complements of those sides.

EXAMPLES.

June 24, 1860. At noon, mean time.

	R. A.		Declination.
Moon,	$10^h\ 51^m\ 36.5^s$	$d=$	$3°\ 35'\ 24''$ N.
Jupiter,	$8^h\ 4^m\ 27.6^s$	$D=$	$20°\ 51'\ 36.8''$ N.
Diff.	$2^h\ 47^m\ 8.9^s$	$(D-d)=$	$17°\ 16'\ 12.8''$
Or,	$41°\ 47'\ 13.5''=P$		
	$20°\ 53'\ 36.8''=\tfrac{1}{2}P.$		

We now apply the formula (F).

Log. 2		0.301030
Log. sin. $\frac{1}{2}P$,	$20^\circ\ 53'\ 36.8''$	-1.552221
Log. sin. $\frac{1}{2}P$,	$20^\circ\ 53'\ 36.8''$	-1.552221
Log. cos. D,	$20^\circ\ 51'\ 36.8''$	-1.970557
Log. cos. d,	$3^\circ\ 35'\ 24''$	-1.999147
Nat. num.	0.237233	-1.375176
Nat. cos. $(D-d)=$	0.954915	
Diff.=Nat. cos. $x=$	$0.717682=44^\circ\ 8'\ 12''$. *Ans.*	

NOTES.—1. The three remaining problems in the same group are solved in the same manner.

2. Let the student observe that if the natural sine or cosine is required to more than 5 decimal places, the *logarithmic* sine or cosine should first be taken out; and from this, diminishing the index by 10, the number may be obtained correctly to 6 or 7 places. Conversely, if the *arc* is required from the natural sine or cosine, first find the *logarithm*, then the arc.

We shall now solve one or two in the next group, where the distances are greater than 90°, and the declinations on opposite sides of the equator. The formula employed is the same. Recollect, however, that the cosine of an arc, greater than 90°, must be taken with a *minus sign.*

Example 1st., October 6, 1860. At noon.

	R. A.		*Declination.*
Sun	$12^h\ 49^m\ 29.3^s$	$-d$	$5^\circ\ 18'\ 42.6''$ *S.*
Moon,	$5^h\ 41^m\ 20.8^s$	D	$26^\circ\ 8'\ 00$ *N.*
	$7^h\ 8^m\ 8.5^s$	$(D-d)=$	$31^\circ\ 26'\ 42.6''$
In arc,	$107^\circ\ 2'\ 7.5''=P$		
	$53^\circ\ 31'\ 3.8''=\frac{1}{2}P$.		

Log. 2		0.301030
Log. sin.	$53^\circ\ 31'\ 3.8''$	-1.905278
Log. sin.	$53^\circ\ 31'\ 3.8''$	-1.905278
Cos.	$26^\circ\ 8'$	-1.953166
Cos.	$5^\circ\ 18'\ 42.6''$	-1.998130
Nat. N.	-1.155798	-0.062882
Nat. cos. $(D-d)$	0.853139	
Diff., cos. x,	$-0.302659=107^\circ\ 37'\ 2''$.	

We will work the next example in this group by Equations (8) and (9), page 350, Text-book.

Making C the included angle, and A, the angle opposite the greater polar distance.

$\odot RA$,	$12^h\ 49^m\ 56.7^s$	Dec.	$5^\circ\ 21'\ 35.4''$ S.
☾ RA,	$5^h\ 48^m\ 30.1^s$	Dec.	$26^\circ\ 3'\ 20''$ N.
	$7^h\ 1^m\ 26.6^s$		

In degrees, $105^\circ\ 21'\ 39'' = C$

$52^\circ\ 40'\ 49.5'' = \frac{1}{2}C$.

The distance from the north pole to the center of the sun, is the sun's declination added to 90°, which we designate by a.

Therefore, $a = 95^\circ\ 21'\ 35.4''$

The moon from the same point is $b = 63^\circ\ 56'\ 40''$

Half sum, is $\frac{1}{2}(a+b) = 79^\circ\ 39'\ 7.7''$

Half diff., is $\frac{1}{2}(a-b) = 15^\circ\ 42'\ 27.7''$

Cot. $\frac{1}{2}C$,	9.882147	cot. $\frac{1}{2}C$,	9.882147
Cos. $\frac{1}{2}(a-b)$	9.983467	sin. $\frac{1}{2}(a-b)$	9.432536
	19.865614		19.314683
Cos. $\frac{1}{2}(a+b)$	9.254364	sin. $\frac{1}{2}(a+b)$	9.992878
Tan. $\frac{1}{2}(A+B)$,	10.611250	tan. $\frac{1}{2}(A-B)$	9.321805
$\frac{1}{2}(A+B)$,	$76^\circ\ 14'\ 47''$	$\frac{1}{2}(A-B)$	$11^\circ\ 50'\ 56''$
$\frac{1}{2}(A-B)$,	$11^\circ\ 50'\ 56''$		
Diff. $= B =$	$64^\circ\ 23'\ 51''$		

Lastly, for the side sought, we have the proportion

Sin. B, $64^\circ 23' 51''$: sin. b, $63^\circ 56' 40''$: : sin. $105^\circ 21' 39''$: sin. x.

Sin. b	9.953454
Sin. $C = \cos.\ 15^\circ\ 21'\ 39''$	9.984201
	19.937655
Sin. B	9.955116
Sin. $106^\circ\ 8'\ 19'' = \cos.\ 16^\circ\ 8'\ 19''$	9.982539

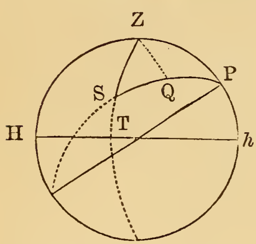
We prefer the other formula.

ASTRONOMICAL PROBLEMS.

The following Astronomical problems were included in Robinson's Geometry, as first published, but being deemed too difficult for such a work, were omitted in the New Geometry.

(1.) In latitude 40° 48′ north, the sun bore south 78° 16′ west, at 3^h 38^m P.M., apparent time. Required his altitude and declination, making no allowance for refraction.

Ans. The altitude, 36° 46′, and declination, 15° 32′ north.



Let Hh be the horizon, Z the zenith of the observer, P the north pole, and PS a meridian through the sun.

PZ is the co-latitude, 49° 12′, and PS is the co-declination or polar distance, one of the arcs sought. ZS is the co-altitude, or ST is the altitude of the sun at the time of observation.

The angle ZPS is found by reducing 3^h 38^m to degrees at the rate of 4^m to one degree; hence, ZPS=54° 30′.

Because HZS=78° 16′, PZS=101° 44′. From Z let fall the perpendicular ZQ on PS. Then in the right angled spherical △ PZQ, equation (13) gives us*

$$R.\sin. ZQ=\sin. PZ \sin. P$$

sin. PZ=sin. 49° 12′	9.879093
sin. P=sin. 54° 30′	9.910686
sin. ZQ=sin. 38° 2′ 42″	9.789779

* To apply the equations without confusion, letter each right angled spherical triangle ABC, right angled at B, then A must be written in place of P; and when operating on ZSQ, write A in place of S, and C for the angle SZQ.

To obtain the angle PZQ, we apply equation (19), which gives

$$R.\cos. PZQ = \cot. PZ.\tan. ZQ.$$

That is, $R.\cos. PZQ = \tan. 40° 48'.\tan. 38° 2' 42''.$

Tan. 40° 48′	9.936100
Tan. 38° 2′ 42″	9.893513
PZQ=cos. 47° 30′ 30″	9.829613
PZS= 101° 44′	
SZQ= 54° 13′ 30″	

To obtain ZS or its complement, we again apply (19)

$$(19) \qquad R.\cos. SZQ = \cot. ZS.\tan. ZQ.$$

That is, $R.\cos. 54° 13' 30'' = \tan. ST.\tan. 38° 2' 42''.$

R.cos. 54° 13′ 30″=	19.766761
tan. 38° 2′ 42″=	9.893513
Tan. 36° 46′, nearly	9.873248

To find PS, we take the following proportion,

$$\text{Sin. } P : \text{sin. } ZS :: \text{sin. } PZS : \text{sin. } PS.$$

That is,

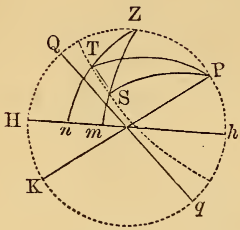
$$\text{Sin. } 54° 30' : \text{cos. } 36° 46' :: \text{sin. } 101° 44' : \text{sin. } PS.$$

Cos. 11° 44′	9.990829
Cos. 36° 46′	9.903676
	19.894505
Sin. 54° 30′	9.910686
PS, sin. 74° 28′	9.983819

Whence, the sun's distance from the equator must have been 15° 32′ north.

(2.) In north latitude, when the sun's declination was 14° 20′ north, his altitudes, at two different times on the same forenoon, were 43° 7′+, and 67° 10′+ : and the change of azimuth, in the interval, 45° 2′. Required the latitude.

Ans. 34° 21′ 14″ north.



Let PK be the earth's axis, Qq the equator, and Hh the horizon.

Also, let Z be the zenith of the observer, Sm the first altitude, Tn the second, and the angle $TZS=45° 2'$. Our first operation must be on the triangle ZTS. $ZT=22° 50'$, $ZS=46° 53'$, and we must find TS, and the angle TSZ.

From T, conceive TB let fall on ZS, making two right angled △'s; and to avoid confusion in the figure, we will keep the arc TB in mind, *and not actually draw it.*

Then the △ ZTB furnishes this proportion,

$$R : \sin. 22° 50' :: \sin. 45° 2' : \sin. TB = \sin. 15° 56' 8''.$$

To find ZB we have the following proportion,

$$R : \cos. ZB :: \cos. 15° 56' 8'' : \cos. 22° 50'.$$

Whence, we find $ZB=16° 34' 13''$. Now in the right angled spherical △ TBS, we have $TB=15° 56' 8''$, $BS=46° 53'-16° 34' 13''$, or $BS=30° 18' 47''$; and TS is found from the following proportion,

$$R : \cos. 15° 56' 8'' :: \cos. 30° 18' 47'' : \cos. TS.$$

This gives $TS=33° 53' 26''$. To find the angle TSZ, we have the proportion,

$$\text{Sin. } 33° 53' 26'' : R :: \sin. TB\ 15° 56' 8'' : \sin. TSZ.$$

Whence, the angle $TSZ=29° 29' 49''$.

The next step is to operate on the *isosceles* spherical △ PTS. We require the angle TSP.

Conceive a meridian drawn bisecting the angle at P, it will also bisect the base TS, forming two equal right angled spherical triangles.

Observe that $PS=75° 40'$ and $\frac{1}{2}$ $TS=16° 56' 43''$.

To find the angle TSP, we apply equation (19), in which $a=16° 56' 43''$, $b=75° 40'$, and the equation becomes,

$$R.\cos. TSP=\cot. 75° 40'.\tan. 16° 56' 43''.$$

Whence, $TSP=85° 32' 5''$, and $PSZ=85° 32' 5''-29° 29' 49''$ $=56° 2' 16''$.

The third step is to operate on the $\triangle$ ZSP; we *now* have its two sides ZS and SP, and the included angle.

From Z *conceive* a perpendicular arc let fall on SP, calling it ZB; then the right angled spherical triangle SZB, gives

$$R. : \sin. ZS :: \sin. ZSB : \sin. ZB.$$

That is,

$$R. : \sin. 46° 53' :: \sin. 56° 2' 16'' : \sin. ZB=\sin. 37° 15' 37''.$$

To find SB, we have the following proportion,

$$R. : \cos. SB :: \cos. ZB : \cos. ZS.$$

That is, $R. : \cos. SB :: \cos. 37° 15' 37'' : \cos. 46° 53'$.

Whence, $SB=30° 49' 18''$. Now, from PS, $75° 40'$, take SB, $30° 49' 18''$, and the difference must be BP, $44° 50' 42''$.

Lastly, to obtain PZ, and consequently ZQ the latitude, we have

$$R. : \cos. ZB :: \cos. BP : \cos. ZP=\sin. ZQ.$$

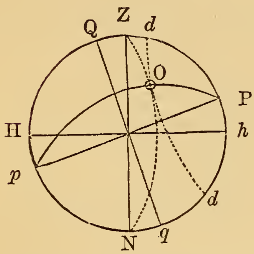
That is, $R. : \cos. 37° 15' 37'' :: \cos. 44° 50' 42'' : \sin. ZQ=$ $\sin. 34° 21' 14''$ north.

This is the result by a careful computation, and it differs $1' 14''$ from the answer given in the text-book.

This is a modification of latitude by double altitudes, but in real double altitudes the arc TS is measured from the elapsed time between the observations, and the angle TZS is not given.

(3.) In latitude 16° 4′ north, when the sun's declination is 23° 2′ north. Required the time in the afternoon, and the sun's altitude and bearing when his azimuth neither increases nor decreases.

Ans. Time, $3^h\ 9^m\ 26^s$ P.M., altitude, 45° 1′, and bearing north 73° 16′ west.



Let Pp be the earth's axis, Hh the horizon, Qq the equator, QZ and Ph, each equal to 16° 4′ north, and Qd, qd, each equal to 23° 2′; then the dotted curve dd represents the parallel of the sun's declination.

Through Z and N an infinite number of vertical circles can be drawn, one of these will touch the curve dd; let it be ZON.

At the point O where this circle touches the curve dd will be the position of the sun at the time required, and POZ will be a right angled spherical △, right angled at O. The problem requires the complement of ZO, and the time corresponding to the angle ZPO.

In the spherical △ POZ, we have

$$R.\ :\ \cos. PO\ ::\ \cos. ZO\ :\ \cos. PZ$$

That is, $R.\ :\ \sin. 23°\ 2'\ ::\ \sin.\text{ altitude}\ :\ \sin. 16°\ 4'$.

Whence, $\sin.\text{ alt.} = \dfrac{R.\sin. 16°\ 4'}{\sin. 23°\ 2'} = \sin. 45°\ 1'$ nearly. *Ans.*

To find the angle at P, we have the following proportion,

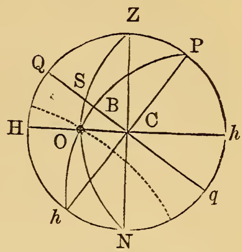
$$\text{Cos. } 16°\ 4'\ :\ R.\ ::\ \cos. 45°\ 1'\ :\ \sin. P.$$

Whence, $\sin. P = \sin. 47°\ 21'\ 40''$, and $ZPO = 47°\ 21'\ 40''$, which being changed into time, at the rate of 15° to one hour, gives $3^h\ 9^m\ 26^s$.

To find the angle PZO, we have the proportion,

$$\text{Cos. } 16°\ 4'\ :\ R.\ ::\ \cos. 23°\ 2'\ :\ \sin. PZO = \sin. 73°\ 16'.$$

(4.) The sun set south-west $\frac{1}{2}$ south, when his declination was 16° 4′ south. Required the latitude. *Ans.* 69° 1′ north.



Draw a circle as before. Let Hh be the horizon, Z the zenith, P the pole. The great circle PZH is the meridian, and ZCN at right angles to it, and of course east and west. Let BC be a portion of the equator, and BO the arc of declination. The position on the horizon where the sun set is the arc $HO=45° - 5° 37' 30'' = 39° 22' 30''$.

Consequently, the arc $OC=50° 37' 30''$.

In the right angled spherical triangle BOC, we have BC, BO given to find the angle BCO, which is the complement of the latitude, or the complement of the angle BCZ.

To find the angle BCO, we apply equation (14).

$$R.\sin. BO=\sin. OC.\sin. BCO.$$

That is, $R.\sin. 16° 4' = \sin. 50° 37' 30''.\sin. BCO.$

R.sin. 16° 4′	19.442096
Sin. 50° 37′ 30″	9.888186
Cos. 69° 1′ nearly	9.553910

Scholium.—The arc BC on the equator measures the angle BPC, corresponding to the time from 6 o'clock to sunrise or sunset. This arc is called the arc of ascensional difference in astronomy. The time of sunset is before six, if the latitude is north and the declination south, as in this example, but after six, if the latitude and declination are both north or both south.

To obtain this arc, the latitude and declination must be given; that is, BO and the angle BCO, the complement of the latitude. Here we apply (12), that is,

$$R.\sin. BC=\tan. D.\tan. L,$$

an equation in which D represents the declination, and L the latitude.

(5.) The altitude of the sun, when on the equator, was $14° 28' +$ bearing east $22° 30'$ south. Required the latitude and time. *Ans.* Latitude $56° 1'$, and time $7^h 46^m 11^s$, A.M.

Let S be the position of the sun on the equator. (See the last figure.) Draw the arc ZS, and the right angled spherical $\triangle$ ZQS is the one we have to operate upon.

Then ZS is the complement of the given altitude, and the angle QZS, is the complement of $22° 30'$. The portion of the equator between Q and S, changed into time, will be the required time from noon, and the arc QZ will be the required latitude.

First for the arc QS.

$$R : \sin. ZS :: \sin. QZS : \sin. QS.$$

That is,

$$R. : \cos. 14° 28' :: \cos. 22° 30' : \sin. QS = 63° 27' 19''.$$

But $63° 27' 19''$ at the rate of 4^m to one degree, corresponds to $4^h 13^m 49^s$ from noon—and as the altitude was marked $+$, rising, it was before noon, or at $7^h 46^m 11^s$ in the morning.

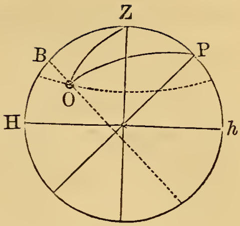
To find the arc QZ, we have the following proportion,

$$R. : \cos. 63° 27' 19'' :: \cos. QZ : \sin. 14° 28'.$$

Whence, $\cos. QZ = \cos. 56° 1'$ nearly, and $56° 1'$ is the latitude sought.

(6.) The altitude of the sun was $20° 41'$ at $2^h 20^m$ P.M. when his declination was $10° 28'$ south. Required his azimuth and the latitude.

Ans. Azimuth south $37° 5'$ west, latitude $51° 58'$ north.



This problem furnishes the spherical $\triangle PZO$, in which the side ZO is the complement of $20° 41'$ or $69° 19'$. $PO = 90° + 10° 28' = 100° 28'$, and the angle ZPO is $2^h\ 20^m$, changed into degrees at the rate of $15°$ to one hour, or $ZPO = 35°$.

Now in the triangle ZPO, we have

$$\text{Sin. } ZO : \text{sin. } ZPO :: \text{sin. } PO : \text{sin. } PZO.$$

That is,

$$\text{Cos. } 20° 41' : \text{sin. } 35° :: \text{cos. } 10° 28' : \text{sin. } PZO = \text{sin. } 37° 5'.$$

In the right angled spherical $\triangle$ BOZ, we apply equation (16).

$$(16). \qquad R.\text{cos. } 37° 5' = \text{tan. } 20° 41'.\text{tan. } BZ.$$

R.cos. $37° 5'$	19.901872
Tan. $20° 41'$	9.576958
Tan. BZ = tan. $64° 40'$	10.324914

To find PB in the right angled $\triangle$ BPO, we apply the same equation (16).

$$R.\text{cos. } 35° = \text{tan. } 10° 28' \text{ tan. } PB.$$

R.cos. $35°$	19.913365
Tan. $10° 28''$	9.266555
Cot. $12° 42'$	10.646810

But PB is obviously greater than $90°$, therefore the point B is $12° 42'$ below the equator; but from B to Z is $64° 40'$; therefore, from Z to the equator, or the latitude, is the difference between $64° 40'$ and $12° 42'$, or $51° 58'$ north. *Ans.* Lat. $51° 58'$ north.

(7.) If in August 1840, Spica was observed to set $2^h\ 26^m\ 14^s$ before Arcturus, what was the latitude of the observer, no account being taken of the height of the eye above the sea, nor of the effect of refraction? *Ans.* 36° 47′ 38″ north.

By a catalogue of the stars to be found in the author's Astronomy, or in any copy of the English Nautical Almanac, we find the positions of these stars in 1840, to have been as follows:

Spica, right ascension, $13^h\ 16^m\ 46^s$ Dec. 10° 19′ 40″ south.
Arcturus, " " $14^h\ 8^m\ 25^s$ Dec. 20° 1′ 4″ north.

Let $L=$ the latitude sought. Put $d=10°\ 19'\ 40''$, and $D=20°\ 1'\ 4''$.

The difference in right ascensions is $51^m\ 39^s$, and this would be about the time that Arcturus would set after Spica, provided the observer was near the equator or a little south of it; but as the interval observed was $2^h\ 26^m\ 14^s$, the observer must have been a considerable distance in *north latitude.* In high southern latitudes Arcturus sets before Spica.

When an observer is north of the equator, and the sun or star south of it, the sun or star will set within six hours after it comes to the meridian.

When the observer and the object are both north of the equator, the interval from the meridian to the horizon is greater than six hours.

The difference between this interval and six hours, is called the ascensional difference, and it is measured in *arc* by BC in the figure to the 4th example.

Now let $x=$ the ascensional difference of Spica corresponding to the latitude L, and $y=$ that of Arcturus corresponding to the same latitude; then by the scholium to the 4th example, calling radius unity, we shall have

$$\sin. x = \tan. L . \tan. d \qquad (1)$$
$$\sin. y = \tan. L . \tan. D \qquad (2)$$

The star Spica came to the observer's meridian at a certain time, that we may denote by M.

Then $$M+\left(6-\frac{x}{15}\right)=\text{the time Spica set.}$$

And $$M+51^m\ 39^s+\left(6+\frac{y}{15}\right)=\text{the time Arcturus set.}$$

By subtracting the time Spica set from the time Arcturus set, we shall obtain an expression equal to $2^h\ 26^m\ 14^s$.

That is, $$51^m\ 39^s+\frac{x}{15}+\frac{y}{15}=2^h\ 26^m\ 14^s.$$

Or, $$\frac{x}{15}+\frac{y}{15}=1^h\ 34^m\ 35^s \qquad (3)$$

$$x+y=15\ (1^h\ 34^m\ 35^s) \qquad (4)$$

Equation (3) expresses time. Equation (4) expresses arc. When we divide arc by 15, we obtain time, one degree being the *unit* for arc, and one hour the *unit* for time; therefore, when we multiply time by 15, we obtain arc; that is, 1^h multiplied by 15, gives 15°; hence (4) becomes

$$x+y=23^\circ\ 38'\ 45''=a$$

$$x=a-y \qquad (5)$$

That is, the *arc* x is equal to the difference of the *arcs* a and y; but to make use of these *arcs* and avail ourselves of equations (1) and (2), we must take the *sines* of the arcs, (see equation (8), plane trigonometry); then (5) becomes

$$\text{Sin. } x=\sin. a. \cos. y-\cos. a. \sin. y. \qquad (6)$$

Substituting the values of sin. x and sin. y from (1) and (2), (6) becomes

$$\text{Tan. } L. \tan. d=\sin. a. \cos. y-\cos. a. \tan. L. \tan. D \quad (7)$$

Squaring (2), $$\sin.^2 y=\tan.^2 L. \tan.^2 D.$$

Subtracting each member from unity, and observing that $(1-\sin.^2 y)$ equals $\cos.^2 y$, then

$$\text{Cos.}^2 y=1-\tan.^2 L. \tan.^2 D.$$

Or, $$\text{Cos. } y=\sqrt{1-\tan.^2 L. \tan.^2 D}.$$

This value of cos. y put in (7), gives

$$\text{Tan.}L.\text{tan.}d=\text{sin.}a.\sqrt{1-\text{tan.}^2L.\text{tan.}^2D}-\text{cos.}a.\text{tan.}L.\text{tan.}D \quad (8)$$

By transposition and division,

$$\left(\frac{\text{Tan. }d+\text{cos. }a.\text{tan. }D}{\text{sin. }a}\right).\text{tan. }L=\sqrt{1-\text{tan.}^2L.\text{ tan.}^2D}$$

Squaring,

$$\left(\frac{\text{Tan. }d+\text{cos. }a.\text{tan. }D}{\text{sin. }a}\right)^2.\text{tan.}^2L=1-\text{tan.}^2L.\text{ tan.}^2D$$

Dividing by tan.^2L, and observing that $\frac{1}{\text{tan.}^2L}=\text{cot.}^2L$ we have $\left(\frac{\text{Tan. }d+\text{cos. }a.\text{tan. }D}{\text{sin. }a}\right)^2=\text{cot.}^2L-\text{tan.}^2D$

Or, $$\text{Cot.}^2L=\text{tan.}^2D+\left(\frac{\text{tan. }d+\text{cos. }a.\text{tan. }D}{\text{sin. }a}\right)^2$$
$$=\text{tan.}^2D+\left(\frac{\text{tan. }d}{\text{sin. }a}+\frac{\text{tan. }D}{\text{tan. }a}\right)^2$$

We must now find the numerical value of the second member. Using logarithmic sines, cosines, tangents, &c., we must diminish the indices by 10, because the equation refers to radius unity.

Log. tan.D. $=-1.561485$; tan.$^2D=-1.122970=0.132730$ num

Log. tan. d	-1.260623	log. tan. D	-1.561485
sin. a	-1.603233	tan. a	-1.641318
0.454349	-1.657390	0.832083	-1.920167

$0.454349+0.832083=1.286432$ $(1.286432)^2=1.654906$

Whence, $\text{cot.}^2L=0.132730+1.654906=1.787636$

Square root, $\text{cot. }L=1.337025$

Taking the log. of this number, increasing its index by 10 will give the log. cot. in our tables.

Log. $1.337025=0.126139+10.=10.126139=\text{cot. }36° \ 47' \ 38''$.

(8.) On the 14th of November, 1829, Menkar was observed to rise $48^m\ 3^s$ before Aldebaran : what was the latitude of the observer. *Ans.* $39°\ 33'\ 53''$ north.

The position of these two stars in the heavens, November 1829, were as follows :

Menkar, right ascension, $2^h\ 53^m\ 21^s$. Dec. $3°\ 24'\ 52''$ north.
Aldebaran, " $4^h\ 26^m\ 7^s$. Dec. $16°\ 19'\ 31''$ north.

Aldebaran passes the meridian $1^h\ 32^m\ 46^s$ after Menkar. Now let M represent the time Menkar was on the meridian, then $M+1^h\ 32^m\ 46^s$ represents the time Aldebaran was on the meridian. Also, let $x=$ the arc of ascensional difference corresponding to the latitude and the star Menkar, and y that of the star Aldebaran.

Then, $M-\left(6+\frac{x}{15}\right)=$ the time Menkar rose.

And, $M+1^h\ 32^m\ 46^s-\left(6+\frac{y}{15}\right)=$ the time Aldebaran rose.

Subtracting the upper from the lower, the difference must be $48^m\ 3^s$; that is,

$$1^h+32^m\ 46^s-\frac{y}{15}+\frac{x}{15}=48^m\ 3^s$$

Whence, $\frac{x}{15}-\frac{y}{15}=-44^m\ 43^s=-0.74527.$

That is, 1^h being the unit, $44^m\ 43^s=0.74527$ of an hour, and multiplying by 15, we shall have as many degrees of arc as we have units ; therefore,

$$x-y=-(0.74527)15=-11°\ 10'\ 45''=-a$$

$$x=y-a$$

$$\sin.x=\sin.y.\cos.a-\cos.y.\sin.a \qquad (1)$$

Put $d=3°\ 24'\ 52''$, $D=16°\ 19'\ 31''$, and $L=$ the required latitude. Then by scholium to the 4th example,

$$\text{Sin}.x=\tan.d.\tan.L. \qquad \sin.y=\tan.D.\tan.L.$$

The values of sin. x and sin. y, substituted in (1), give

$$\text{Tan. } d.\text{tan. } L = \cos. a.\text{tan. } D.\text{tan. } L - \cos. y.\sin. a \qquad (2)$$

But, $\sin.^2 y = \tan.^2 D.\tan.^2 L$, and $1 - \sin.^2 y = 1 - \tan.^2 D.\tan.^2 L$.

Or, $\cos.^2 y = 1 - \tan.^2 D.\tan.^2 L$.

Or, $\cos. y = \sqrt{1 - \tan^2 D.\tan.^2 L}$.

By substituting this value of cos. y in (2) and transposing, we find

$$\text{Sin.} a \sqrt{1 - \tan.^2 D.\tan.^2 L} = (\cos. a.\tan. D. - \tan. d)\tan. L.$$

Dividing by sin. a, and observing that $\frac{\cos. a}{\sin. a} = \frac{1}{\tan. a}$, we have

$$\sqrt{1 - \tan.^2 D.\tan.^2 L} = \left(\frac{\tan. D}{\tan. a} - \frac{\tan. d}{\sin. a}\right)\tan. L.$$

Squaring and dividing by $\tan.^2 L$, and at the same time observing that $\frac{1}{\tan. L} = \cot. L$, and we shall have

$$\text{Cot.}^2 L - \tan.^2 D = \left(\frac{\tan. D}{\tan. a} - \frac{\tan. d}{\sin. a}\right)^2$$

We will now find the numerical value of the known quantities.

Log. tan. D	-1.466718	log. tan. d	-2.775712
Log. tan. a	-1.295849	log. sin. a	-1.287530
Log. 1.482072	0.170869	log. 0.307738	-1.488182

$\text{Tan.}^2 D = 0.085790 \qquad 1.482072 - 0.307738 = 1.174334$

Whence, $\cot.^2 L - 0.085790 = (1.174334)^2$.

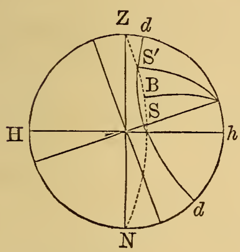
Or, $\cot.^2 L = 1.464849$.

$\cot. L = 1.210309$.

Log. cot. $L + 10 = 10.082896 = \cot.\ 39^\circ\ 33'\ 53''$. *Ans.*

(9.) In latitude $16^\circ\ 40'$ north, when the sun's declination was $23^\circ\ 18'$ north, I observed him twice, in the same forenoon, bearing north 68° 30′ east. Required the time of observation, and his altitude at each time.

Ans. Times, $6^h\ 15^m\ 40^s$, A.M., and $10^h\ 32^m\ 48^s$, A.M., altitude, 9° 59′ 33″, and 68° 29′ 43″.



Let Z be the zenith, P the north pole, and the curve dd be the parallel of the sun's declination along which it appears to revolve. Make the angle PZS' equal to 68° 30′; then the sun was at S at the time of the first observation, and at S' at the time of the second.

In the spherical △ PZS' there is given PZ, PS' and the angle PZS'; also, in the △ PZS there is given PZ, PS, and the angle PZS. Observe that PSS' is an *isosceles* △.

Describe the meridian PB bisecting the angle $S'PS$, and then we have three right angled spherical triangles, BPS, BPS', and BPZ; taking the last, we have the following proportion:

$$R. : \sin. PZ :: \sin. PZB : \sin. PB.$$

That is,

$$R. : \cos. 16° 40' :: \sin. 68° 30' : \sin. PB = \sin. 63° 2' 27''.$$

To find ZB, we take the following proportion, (see page 185, Observation 1, Robinson's Geometry),

$$R. : \cos. ZB :: \cos. BP : \cos. PZ.$$

That is, $R. : \cos. ZB :: \cos. 63° 2' 27'' : \sin. 16° 40'.$

$R.$sin. 16° 40′	19.457584
Cos. 63° 2′ 27″	9.656439
Cos. ZB, 50° 45′ 22″	9.801145

To find $S'B$, we have

$$R. : \cos. S'B : : \cos. 63^\circ\, 2'\, 27'' : \sin. 23^\circ\, 18'.$$

$R.\sin. 23^\circ\, 18'$	19.597196
$\cos. 63^\circ\, 2'\, 27''$	9.656439
$\text{Cos. } S'B,\ 29^\circ\, 15'\, 5''$	9.940757

Observe that $S'B = BS$; therefore, $ZS = 50^\circ\, 45'\, 22'' + 29^\circ\, 15'\, 5'' = 80^\circ\, 0'\, 27''$, and $ZS' = 50^\circ\, 45'\, 22'' - 29^\circ\, 15'\, 5'' = 21^\circ\, 30'\, 17''$, the complements of the altitudes. Consequently the altitude at the first observation was $9^\circ\, 59'\, 33''$, and at the second, $68^\circ\, 29'\, 43''$.

To find the time from noon at the first observation, we have the following proportion,

$$\text{Sin. } PS : \sin. PZS : : \sin. ZS : \sin. ZPS.$$

That is,

$$\text{Cos. } 23^\circ 18' : \sin. 68^\circ 30' : : \sin. 80^\circ 0' 26'' : \sin. ZPS = \sin. 86^\circ 5' 7''$$

Had the angle been 90°, the time would have been just 6^h, but the angle $3^\circ\, 54'\, 53''$ less; this corresponds to $15^m\, 40^s$, in time. Therefore, the time was $6^h\, 15^m\, 40^s$. For the time at the second observation, we have

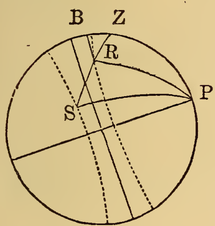
$$\text{Cos. } 23^\circ\, 18' : \sin. 68^\circ\, 30' : : \sin. 21^\circ\, 30'\, 17'' : \sin. ZPS'$$
$$[= \sin. 21^\circ\, 47'\, 57''$$

$21^\circ\, 47'\, 57'' = 1^h\, 27^m\, 12^s$ from noon, or $10^h\, 32^m\, 48^s$ apparent time in the morning.

(10.) An observer in north latitude marked the time when the stars Regulus and Spica were eclipsed by a plumb line,—that is, when they were both in the same vertical plane passing through the zenith of the observer. One hour and ten minutes afterwards, Regulus was on the observer's meridian. What was the observer's latitude?

The positions of the stars in the heavens were

Regulus, right ascension, $10^h\ 0^m\ 10^s$. Dec. $12°\ 43'$ north.
Spica, " " $13^h\ 17^m\ 2^s$. Dec. $10°\ 21'\ 20''$ south.



Let R be the position of Regulus, S the position of Spica, P the pole, and Z the zenith.

Then the side $PS=100°\ 21'\ 20''$, $PR=77°\ 17'$, and the angle $RPS=3^h\ 16^m\ 52^s$, converted into degrees; that is, $RPS=49°\ 13'$.

One hour and ten minutes reduced to arc, give $17°\ 30'$; but the stars revolve according to *siderial*, not solar time, and to reduce solar to siderial arc, we must increase it by about its $\frac{1}{365}$th part; this gives about $3'$ to add to $17°\ 30'$, making $17°\ 33'$ for the angle ZPR. Our ultimate object is to find PZ, the complement of the latitude.

In the $\triangle\ PRS$, we have the two sides PR, PS, and the included angle P, from which we must find RS and the angle SRP, and we can let a perpendicular fall from R on to the side PS and solve it by the usual way; but to show that a wide field is open for a bold operator, we will put the unknown arc $RS=x$, the side opposite $R=r$, and opposite $S=s$, and apply one of the equations in formula (S), page 191, Robinson's Geometry.

That is, $$\cos. P=\frac{\cos. x-\cos. r.\cos. s}{\sin. r.\sin. s}.$$

Whence, $$\text{Cos.} P.\sin. r.\sin. s+\cos. r.\cos. s=\cos. x.$$

We now apply this equation, recollecting that radius is unity, which will require us to diminish indices of the logarithms by 10.

Cos. P = cos. 49° 13′	-1.815046		
Sin. r = sin. 100° 21′ 20″	-1.992868	—cos.	-1.254683*
Sin. s = sin. 77° 17′	-1.989214	cos.	-1.342679
0.62680	-1.797128	.03957	-2.597362

$$\text{Cos. } x = 0.6268 - 0.03957 = .58723.$$

Whence, by the table of natural cosines, we find

$$x = 54° \ 2' \ 20''.$$

To find the angle SRP or ZRP, we have

Sin. 54° 2′ 20″ : sin. 49° 13′ : : sin. 100° 21′ 20″ : sin. ZRP.

Whence, $ZRP = 66° \ 57' \ 37''$.

Let fall the perpendicular RB on PZ produced, then the right angled spherical △ PBR gives this proportion,

R. : sin. 77° 17′ : : sin. 17° 33′ : sin. RB = sin. 17° 6′ 22″.

To find PB, we have

R. : cos. PB : : cos. 17° 6′ 22″ : cos. 77° 17′.

Whence, $PB = 76° \ 41'$.

Now, to find the angle BRP, we have

Sin. 77° 17′ : R. : : sin. 76° 41′ : sin. BRP = sin. 86° 1′.

From PRB take PRZ, and ZRB will remain ; that is,

From 86° 1′ take 66° 57′ 37″, and $ZRB = 19° \ 3' \ 23''$.

By the application of equation (12), we find that

R.sin. 17° 6′ 22″ = tan. BZ. cot. 19° 3′ 23″.

Whence, $BZ = 5° \ 48'$

And, $PZ = 76° \ 41' - 5° \ 48' = 70° \ 53'$.

The complement of 70° 53′ is 19° 7′, the latitude sought.

By this example we perceive that by the means of a meridian line, a good watch, and a plumb line, any person having a knowledge of spherical trigonometry, and having a catalogue of the stars at hand, can determine his latitude by observation.

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* Observe that r is greater than 90°, its cosine is therefore, negative in value, rendering the product, cos. r. cos. s, or .03957, negative.

KEY TO
ROBINSON'S
ANALYTICAL GEOMETRY.

CHAPTER I.

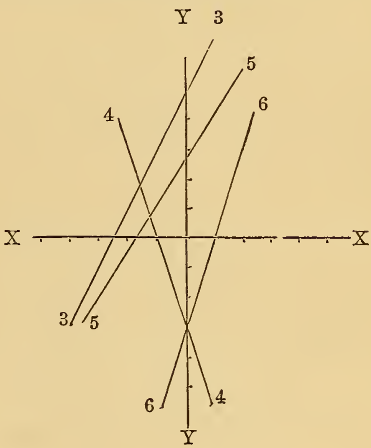
STRAIGHT LINES.

(PAGE 105.)

Ex. **3.** $y=2x+5$.

Draw the rectangular coordinate axes, XX, YY'.

Then, when $y=0$, the equation, give $x=-\frac{5}{2}=-2\frac{1}{2}$, and when $x=0$, $y=5$; hence the line represented by the equation cuts the axis of X at the distance $-2\frac{1}{2}$, and the axis of Y at the distance $+5$ from the origin or zero point.



Assuming any convenient unit, lay it off on the axis several times, in both the positive and negative directions, and draw a line cutting the axis of X at $-2\frac{1}{2}$, and the axis of Y at $+5$ from the origin. This will be (3, 3), the line required, since two points determine the position of a line.

Ex. **4.** $y=-3x-3$.

Making $y=0$ in this equation gives $x=-1$, and $x=0$ gives $y=-3$; hence, this line cuts the axis of X at -1 and the axis of Y at -3. Laying off these distances on their respective axes, and drawing through the points thus determined the line (4, 4) it is that required.

Ex. 5. $2y=3x+5$, or $y=\frac{3}{2}x+\frac{5}{2}$.

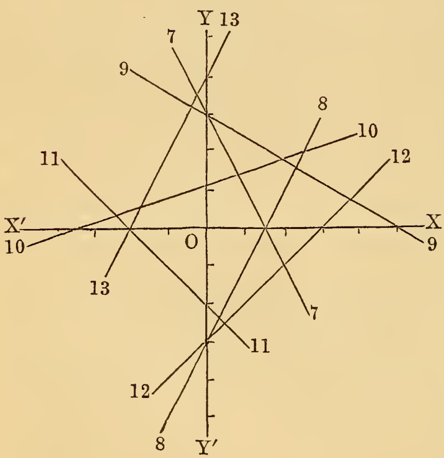
In this equation $y=0$ makes $x=-\frac{5}{3}=-1\frac{2}{3}$, and $x=0$ makes $y=\frac{5}{2}=2\frac{1}{2}$. The line (5, 5), drawn through the points in the axis, given by these values of x and y, is that represented by the equation.

Ex. 6. $y=4x-3$.

The values of x and y given by this equation, by making first, $y=0$, and then $x=0$, are $x=\frac{3}{4}$, $y=-3$. The line is (6, 6).

The lines of which equations 4 and 6 are the equations both intersect the axis of Y at the distance -3 from the origin.

Ex. 7. $y=-2x+3$.



Making in this equation $y=0$ and $x=0$, successively, we find $x=\frac{3}{2}=1\frac{1}{2}$, $y=3$, and the required line is (7, 7).

Ex. 8. $y=2x-3$.

Proceeding with this as with the preceding equation, we get $x=1\frac{1}{2}$, $y=-3$; and (8, 8) is the line responding to this equation.

In the triangle CAB since OA and OB are each equal 3,

AB, the base, is equal to 6, and because OC is perpendicular to the base and bisects it the triangle is isoceles. OC is equal to $1\frac{1}{2}$ by the construction.

Ex. **9.** $3x+5y-15=0$, or $y=-\frac{3}{5}x+3$.

The suppositions $y=0$ and $x=0$, made successively, give $x=5$, $y=3$, and the line (9, 9) drawn through the point $+5$ on the axis of X, and the point $+3$ on the axis of Y responds to the equation.

Ex. **10.** $2x-6y+7=0$, or $y=\frac{1}{3}x+\frac{7}{6}$.

From this we find the intersections of the line with the axes of X and Y respectively to be at the distances $-\frac{7}{2}$ and $+\frac{7}{6}$ from the origin, and (10, 10) is the required line.

Ex. **11.** $x+y+2=0$, or $y=-x-2$.

This line cuts both the axes at the distance -2 from the origin and makes with each an angle of 45°. It is the line (11, 11).

Ex. **12.** $-x+y+3=0$, or $y=x-3$.

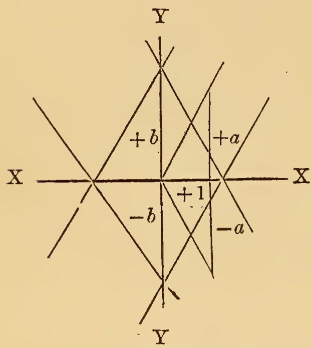
This line cuts both axes at the distance of 3 units from the origin; the axis of X at $+3$, the axis of Y at -3, it therefore makes with each an angle of 45°. It is the line (12, 12).

Ex. **13.** $2x-y+4=0$, or $y=2x+4$.

In this $y=0$ gives $x=-2$, and $x=0$ gives $y=4$; hence, the line (13, 13) which cuts the axis of X at -2, and the axis of Y at $+4$ is that which responds to the equation.

If we solve any equation of the first degree between two variables with reference to one of the variables it will take either the form $y=ax+b$, or $x=a'y+b'$. Now a denotes the tangent of the angle that the line makes with the axis of X, and a' denotes the tangent of the angle the same line makes with the axis of Y, while b is the distance from the origin to the point in which the line cuts the axis of Y, and b' the distance from the origin to the point in which the line cuts the axis of X.

To construct this line we may draw through the origin a line making with the axis of X an angle having a for its tangent, and then the line drawn parallel to this through the point on the axis of Y at the distance b from the origin will be that represented by the equation.

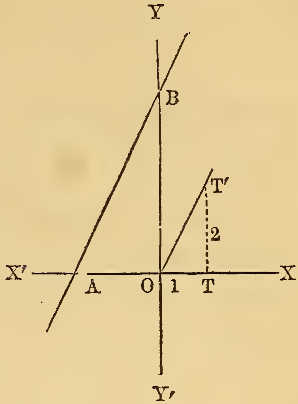


To make this construction assume any convenient unit of measure, and lay it off on the axis of X from the origin to the right, that is, in the positive direction. Through the extremity of this unit draw a parallel to the axis of Y, and mark off on this parallel from the axis of X the distance a units, above the axis of X if a is positive, below if it is negative. The line which connects the origin with the point thus determined on the parallel will make with the axis of X an angle of which a is the tangent, and it is, therefore, parallel to the required line. If, then, we lay off the distance b units on the axis of Y, above, or below the origin, according as b is positive or negative, and through the extremity of this line we draw a line parallel to that passing through the origin we shall have the required line.

Thus, to construct the line represented by the equation $2x-y+4=0$ (example 13), we solve it with reference to y and get $y=2x+4$.

We lay off the distance $OT=$ unity on the axis of X from the origin to the right and on the parallel to the axis of Y, at this distance, mark off $TT'=$ two units above the axis of X, because in this example $a=+2$.

Then make $OB=4$, draw OT' and through B draw BA parallel to OT', and the line given by the equation is constructed.



With this explanation the student will find no difficulty in applying this method of construction to all of the above examples.

(Page 108.)

To prove that equations (6) and (7) are different forms of the same equation.

By clearing equation (6) of fractions it becomes

$$yx''-y'x''-yx'+y'x'=y''x-y'x-y''x'+y'x'.$$

Canceling and transposing we find

$$yx''-yx'=y'x''+y''x-y'x-y''x'. \qquad (8)$$

Equation (7) treated in the same way gives, first,

$$yx''-y''x''-yx'+y''x'=y''x-y'x-y''x''+y'x''.$$

And finally

$$yx''-yx'=y'x''+y''x-y'x-y''x'. \qquad (9)$$

Since equations (8) and (9) are the same equations, (6) and (7) from which they were derived must represent the same line.

Equation (6) being

$$y-y'=\frac{y''-y'}{x''-x'}(x-x')$$

the suppositions $y=y'$, and $x=x'$ reduces both members to zero; hence the line passes through the first point of which the co-ordinates are y' and x'.

If in this equation we suppose $y=y''$ and $x=x''$ it becomes

$$y''-y'=\frac{y''-y'}{x''-x'}(x''-x').$$

Dividing both members of this last equation by $y''-y'$ and clearing of fractions it reduces to

$$x''-x'=x''-x' \text{ or } 0=0.$$

Therefore, the co-ordinates of both points when substituted for the variables x and y in equation (6) satisfy that equation, which is the condition that the line shall pass through these points.

(Page 109.)

Ex. 2. Placing in the equation

$$y-y'=\frac{y''-y'}{x''-x'}(x-x') \qquad (a)$$

for x', x'', and y', y'' their values, this equation becomes

$$y+1=\frac{-\frac{10}{6}+1}{4\frac{1}{2}+4}(x+4).$$

Reducing

$$y+1=-\tfrac{4}{51}x-\tfrac{16}{51}.$$

Or, $$y=-\tfrac{4}{51}x-1\tfrac{16}{51}.$$

Ex. 3. Making the substitutions in equation (a), (last example) for x', x'', and y', y'', it becomes

$$y-5=\frac{3-5}{-3-6}(x-6),$$

Or, $$y-5=\tfrac{2}{9}x-\tfrac{12}{9}.$$

Whence, $$y=\tfrac{2}{9}x+3\tfrac{2}{3}.$$

(Page 112).

Ex. 2. The equations

$$2y=5x+8$$

$$3y=-2x+6,$$

become by dividing through by the coefficients of y,

$$y = \tfrac{5}{2}x + 4$$

$$y = -\tfrac{2}{3}x + 2.$$

Making $\quad a = \tfrac{5}{2}$ and $a' = -\tfrac{2}{3}$ the formula.

$$m = \frac{a' - a}{1 + aa'}$$

Becomes

$$m = \frac{-\frac{2}{3} - \frac{5}{2}}{1 - \frac{10}{6}} = 4\tfrac{3}{4} = 4.75.$$

(Page 114.)

Ex **1.** By making $a=-2$, $a'=5$, $b=1$ and $b'=10$, in the formula

$$x_{\prime} = -\frac{b - b'}{a - a'} \text{ it becomes}$$

$$x_{\prime} = -\frac{1-10}{-2-5} = -\tfrac{9}{7}.$$

The same substitutions in

$$y_{\prime} = \frac{a'b - ab'}{a' - a} \text{ give}$$

$$y_{\prime} = \frac{5+20}{5+2} = \tfrac{25}{7} = 3\tfrac{4}{7}.$$

(Page 115.)

Ex. **1.** In the formula for the perpendicular,

$$\text{Per.} = \pm \frac{b + ax' - y'}{\sqrt{a^2+1}},$$

we must make $a=3$, $b=-10$, $x'=4$, and $y'=5$. These values being substituted, we have

$$\text{Per.} = \pm \frac{-10+12-5}{\sqrt{3^2+1}} = \pm \frac{-3}{\sqrt{10}}.$$

Multiplying the numerator and denominator of this by

$\sqrt{10}$ and passing the 3 under the radical by squaring, we have finally

$$\text{Perpendicular} = \tfrac{1}{10}\sqrt{90}.$$

Ex. **2.** Here $a=-5$, $b=-15$, $x'=4$, $y'=5$, and by placing these in the formula for the perpendicular, we find

$$\text{Per.} = \pm\frac{-15-20-5}{\sqrt{5^2+1}} = \pm\frac{-40}{\sqrt{26}}$$

$$= \tfrac{1}{26}\times 40\sqrt{26} = \frac{203+}{26} = 7.84+.$$

(Page 118.)

Ex. **3.** Dividing both of the equations

$$3y + 5x = 4$$
$$2y = 3x + 4,$$

through by y and transposing, they become

$$y = -\tfrac{5}{3}x + \tfrac{4}{3}$$
$$y = \tfrac{3}{2}x + 2.$$

Here $a=-\frac{5}{3}$, $a'=\frac{3}{2}$, $b=\frac{4}{3}$, $b'=2$. Placing these values in the formula

$$x_{\prime} = -\left(\frac{b-b'}{a-a'}\right), \text{ we get } x_{\prime} = -\frac{\frac{4}{3}-2}{-\frac{5}{3}-\frac{3}{2}} = -\frac{-\frac{2}{3}}{-\frac{10}{6}-\frac{9}{6}}$$

$$= -\frac{2\times 6}{3\times 19} = -\tfrac{4}{19}.$$

Making the same substitutions in the formula

$$y_{\prime} = \frac{a'b-ab'}{a'-a} \text{ we get } y_{\prime} = \frac{\frac{3}{2}\times\frac{4}{3}+\frac{5}{3}\times 2}{\frac{3}{2}+\frac{5}{3}} = \frac{2+\frac{10}{3}}{\frac{9}{6}+\frac{10}{6}} = \frac{\frac{16}{3}}{\frac{19}{6}} = \tfrac{32}{19}.$$

To get the natural tangent of 30° we have

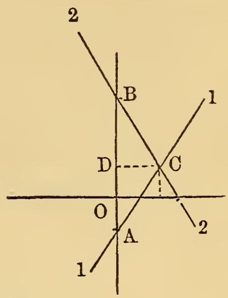
$$\text{Tan. } 30° = \frac{\sin.\ 30°}{\cos.\ 30°} = \frac{.50000}{.86603} = .5773.$$

These values of $x_{\prime}$, $y_{\prime}$, and tan. 30° placed in the equation

$$y - y_{\prime} = \tan.\ 30°(x-x_{\prime}) \text{ give}$$

$$y - \tfrac{32}{19} = 0.5773(x+\tfrac{4}{19}).$$

(Page 119.)



Ex. 4. By transposing and dividing through by the coefficients of y, the equations

$$2y-3x=-1$$
$$2y+3x=3.$$

Become

$$y=\tfrac{3}{2}x-\tfrac{1}{2}$$
$$y=-\tfrac{3}{2}x+\tfrac{3}{2}.$$

Whence $a=\frac{3}{2}$, $a'=-\frac{3}{2}$, $b=-\frac{1}{2}$, $b'=\frac{3}{2}$.

These values put in the formulæ

$$x_{\prime}=\frac{b-b'}{a'-a},\quad y_{\prime}=\frac{a'b-ab'}{a'-a},\ \text{give}\ x_{\prime}=\tfrac{2}{3},\ y_{\prime}=\tfrac{1}{2}.$$

The point C in which the lines intersect may then be constructed. Since the first line (1, 1), intersects the axis of Y at the distance $-\frac{1}{2}$, and the second line (2, 2) intersects it at the distance $\frac{3}{2}$, the distance AB is 2; and because the lines (1, 1) and (2, 2) are equally inclined to the axis of X on opposite sides, they are also equally inclined to the axis of Y on opposite sides; hence, the triangle ABC is isosceles. The side $AC=BC=\sqrt{1+(\frac{2}{3})^2}=\sqrt{\frac{13}{9}}=\frac{1}{3}\sqrt{13}=\dfrac{3.605+}{3}=1.201+.$

The perpendicular, CD, of the triangle is $\frac{2}{3}$; hence the area of the triangle is $=\frac{1}{3}\times 2=0.66+$.

(Page 119.)

Ex. 5. The equations

$$-2\tfrac{1}{5}y+3\tfrac{1}{2}x=-2\tfrac{1}{4},$$
$$2\tfrac{2}{5}y-\tfrac{2}{3}x\ =4$$

by transposing and dividing through by the coefficients of y become

$$y=\tfrac{35}{22}x+\tfrac{45}{44},$$
$$y=\tfrac{5}{18}x+\tfrac{5}{3}.$$

Whence $a=\frac{35}{22}$, $a'=\frac{5}{18}$, $b=\frac{45}{44}$, $b'=\frac{5}{3}$, substituting these in the formula for $x_{,}$ and $y_{,}$, we get

$$x_{,}=-\frac{\frac{45}{44}-\frac{5}{3}}{\frac{35}{22}-\frac{5}{18}}=-\frac{-\frac{85}{132}}{\frac{520}{396}}=\frac{3\times 17}{104}=\frac{51}{104}=0.49,$$

and $$y_{,}=\frac{\frac{5}{18}\times\frac{45}{44}-\frac{35}{22}\times\frac{5}{3}}{\frac{5}{18}-\frac{35}{22}}=\frac{\frac{25}{88}-\frac{175}{66}}{-\frac{130}{99}}=1.8.$$

The form of the equation of a line passing through two given points is

$$y-y''=\frac{y''-y_{,}}{x''-x_{,}}(x-x''),$$

and this will become the equation of the required line when we make $x_{,}=0.49$, $y_{,}=1.8$, $x''=3$, $y''=0$. In this case the expression $\frac{y''-y_{,}}{x''-x_{,}}$ reduces to $-0.7171+$.

Placing these values in the above equation we get

$$y=-0.7171x+0.7171\times 3=-0.7171x+2.1523.$$

And substituting the values of the two points in the formula.

$$D=\sqrt{(x''-x_{,})^2+(y''-y_{,})^2}.$$

We have

$$D=\sqrt{(2.51)^2+(1.8)^2}.$$

CHAPTER II.

THE CIRCLE.

(Page 139.)

Ex. **1.** In all these examples the equation to be referred to is

$$y^2\mp cy=\frac{R^2-c^2}{2}$$

Comparing the equation $x^2+11x=80$ with this, we have

$c=11, \dfrac{R^2-c^2}{2}=80$; whence $R^2=160+121=281$. $R=\sqrt{281}$ $=16.76$, with which we proceed as explained in the text.

Ex. **2.** $x^2-3x=28$.

Here $c=-3$, $\dfrac{R^2-c^2}{2}=28$; whence $R^2=56+9=65$, $R=\sqrt{65}=8.06$.

Ex. **3.** $x^2-x=2$.

In this $c=-1$, $\dfrac{R^2-c^2}{2}=2$, $R^2=5$, $R=\sqrt{5}=2.23$.

Ex. **4.** $x^2-12x=-32$.

In this $c=-12$, $\dfrac{R^2-c^2}{2}=-32$, $R^2=80$, $R=8.94$.

Ex. **5.** $x^2-12x=-36$.

This gives, $c=-12$, $\dfrac{R^2-c^2}{2}=-36$. $R^2=72$, $R=8.48$.

Ex. **6.** $x^2-12x=-38$.

Whence $c=-12$, $\dfrac{R^2-c^2}{2}=-38$, $R^2=68$, $R=8.24$.

Ex. **7.** $x^2+6x=-10$.

Whence $c=6$, $\dfrac{R^2-c^2}{2}=-10$, $R^2=16$, $R=4$.

CHAPTER IV.

THE PARABOLA.

(Page 187.)

Ex. **3.** $x^2-\frac{2}{11}x=8$.

Comparing this with the equation

$$R^2+2bR=2c-b^2$$

we see that $b=-\frac{1}{11}$, $2c-b^2=8$; whence $2c=8+\frac{1}{121}=\frac{969}{121}$,

$c=\frac{969}{242}=4+$. We construct the pole by laying off the distance 4 on the axis to the right, for the abscissa of the pole, and the distance $-\frac{1}{11}$ for the ordinate.

The values of x will be found to be $+2.9+$ and $-2.7+$.

Ex. 4. $\frac{3}{4}x^2+\frac{3}{5}x=\frac{7}{11}$, or $x^2+\frac{4}{5}x=\frac{28}{33}$.

Here $2b=\frac{4}{5}$, $b=\frac{2}{5}$, $2c-b^2=\frac{28}{33}$, $2c=\frac{28}{33}+\frac{4}{25}$, $c=.504+$. If the pole be constructed with these coordinates, the distances from the pole, above and below the axis, to the intersection of the perpendicular radius vector with the curve are $.6+$ and $1.4+$.

Ex. 5. $\frac{1}{4}y^2-\frac{1}{6}y=2$, or $y^2-\frac{2}{3}y=8$.

We find for this $2b=-\frac{2}{3}$, $b=-\frac{1}{3}$, $2c-b^2=8$, $2c=8+\frac{1}{9}$, whence $c=8.1+$. Constructing the pole with these coördinates, and drawing the perpendicular radius vector, we shall find the distances from the pole to the intersections with the curve to be $=3.17+$ and $-2.5+$.

CHAPTER VI.

INTERPRETATION OF EQUATIONS.

(Page 223.)

Ex. 1. We find the abscissas of the vertices of the diameter whose equation is $y=-x$, by placing the quantity $-2x(x-2)$ under the radical in the general value of y equal to zero; that is we make

$$-2x(x-2)=0$$

which gives the two values $x'=0$, $x''=2$. These values of x substituted in the equation of the diameter $y=-x$, give $y'=0$, $y''=-2$; hence $x'=0$, $y'=0$ are the coordinates of one vertex, and $x''=2$, $y''=-2$ are those of the other.

The abscissa of the center $x=1$, and giving x this value in the proposed equation it reduces to

$$y^2+2y+3-4=0$$

Or, $$y^2+2y=1$$

Whence, $$y=-1\pm\sqrt{2}=-1\pm1.41+,$$

That is $$y=+.41+ \text{ and } y=-2.41+.$$

(Page 236.)

Ex. **6.** In the equation

$$y^2-2xy-x^2-2y+2x+3=0,$$

we see, by comparing it with the general equation that $A=1$, $B=-2$, $C=-1$; hence $B^2-4AC>0$ and the equation represents an hyperbola.

Solving the equation with reference to y, we find

$$y=x+1\pm\sqrt{2(x^2-1)}.$$

Placing the quantity under the radical in this value of y, equal to zero we have $x=+1$, and $x=-1$ for the abscissas of the vertices of the diameter which bisects the chords parallel to the axis of Y; hence the axis of Y bisects this diameter and is the conjugate to it.

By making y equal to zero in the proposed equation it becomes

$$x^2-2x=3$$

Whence, $$x=1\pm\sqrt{3+1}=1\pm2,\ x=+3, \text{ or } -1.$$

Ex. **7.** Comparing the equation

$$y^2-2xy+2x^2-2x+4=0$$

with the general equation, we find,

$$A=1,\ B=-2,\ C=2;\ \text{therefore } B^2-4AC<0,$$

and the analytical condition for the ellipse is satisfied; but since the equation can be put under the form

$$(y-x)^2+(x-1)^2+3=0$$

we see that all real values of x and y will make the first number the sum of three positive quantities, and hence such values cannot satisfy the equation.

(Page 237.)

Ex. 8. $y^2-2xy+x^2+x=0$.

This represents the parabola, because $A=1$, $B=-2$, $C=1$ and, therefore, $B^2-4AC=0$. The curve passes through the origin, since the equation contains no absolute term and is therefore satisfied by $x=0$, $y=0$.

Solving the equation in respect to y, we find

$$y = x \pm \sqrt{-x},$$

from which we conclude that all positive values of x will give imaginary values to y, or that the curve does not extend in the direction of x positive.

If the equation be solved with reference to x we shall find

$$x = \frac{2y-1}{2} \pm \sqrt{-y^2+\frac{4y^2-4y+1}{4}} = \frac{2y-1}{2} \pm \tfrac{1}{2}\sqrt{1-4y}.$$

The radical part of this value of x shows that any positive value of y greater than $\frac{1}{4}$, will render x imaginary, but that x will be real for all negative values of y. Hence the curve extends indefinitely in the direction of x and y negative. Substituting $\frac{1}{4}$ for y, in the equation, it becomes

$$\tfrac{1}{16}-\tfrac{1}{2}x+x^2+x=0$$

Or, $\qquad x^2+\frac{1}{2}x+\frac{1}{16}=(x+\frac{1}{4})^2=0$

Whence, $\qquad x+\frac{1}{4}=0, \quad x=-\frac{1}{4}.$

A line drawn through the point whose coördinates are $y=+\frac{1}{4}$, $x=-\frac{1}{4}$, parallel to the axis of X, will limit the curve in the direction of y positive; but it will be unlimited in the direction of y negative.

Ex. 9. $y^2-2xy+x^2-2y-1=0$.

The curve represented by this equation is the parabola.

Since we have for it, $A=1$, $B=-2$, $C=1$, and, therefore,

$$B^2-4AC=0.$$

Transposing and factoring it becomes

$$y^2-2(x+1)y=-x^2+1,$$

Whence,

$$y=x+1\pm\sqrt{-x^2+1+x^2+2x+1}=x+1\pm\sqrt{2(x+1)}.$$

Also, $$x^2-2xy=-y^2+2y+1,$$

Whence, $$x=y\pm\sqrt{2y+1}.$$

The value of y shows that all negative values of x greater, numerically, than -1 will render y imaginary, and the value of x shows that all negative values of y greater, numerically, than $-\frac{1}{2}$ will render x imaginary. The curve is therefore limited in the direction of the negative axes; but it is unlimited in the opposite direction.

Ex. **10.** $y^2-4xy+4x^2=0.$

Since in this equation, $A=1$, $B=-4$, $C=4$, we have $B^2-4AC=0$ which is the condition for the parabola. But we have also the further conditions (page 219),

$$BD-2AE=0,\quad D^2-4AF=0.$$

In consequence of the first of these conditions, the parabola would reduce to two parallel straight lines, and in consequence of the second, the two straight lines reduce to one, or coincide.

The first member of the equation is a perfect square, and we therefore have

$$(y-2x)^2=0, \text{ or } y=2x.$$

Hence the line to which the parabola reduces, passes through the origin of coördinates and makes, with the axis of X, an angle, having 2 for its natural tangent. The degrees and minutes of the angle are found thus

$$\begin{array}{rr} \text{Log. } 2 = & 0.301030 \\ \text{Log. R} = & \underline{10.000000} \\ \text{Log. tangent } 63^\circ\ 26', & \underline{10.301030.} \end{array}$$

The complement of 63° 26′ is 26° 34′, which is the angle the line makes with the axis of Y.

Ex. 11. $y^2-2xy+2x^2-2y+2x=0$.

Here $A=1$, $B=-2$, $C=2$. Hence $B^2-4AC<0$, and the curve is an ellipse.

Solving the equation with reference to y, we get

$$y=x+1\pm\sqrt{-x^2+1}.$$

Placing the quantity under the radical sign equal to zero, to get the abscissas of the vertices of the diameter whose equation is $y=x+1$, we find

$$x=\pm 1, \text{ or } x=+1 \text{ and } x=-1.$$

The value of y shows that all substitutions for x, numerically greater than 1, positive or negative, will render y imaginary. Hence parallels to the axis of Y, drawn at a unit's distance from it on either side, will limit the curve.

CHAPTER VII.

ON THE INTERSECTION OF LINES AND THE GEOMETRICAL SOLUTION OF EQUATIONS.

(PAGE 243.)

Ex. 3. Given $y^3-48y=128$, to find the values of y by construction.

Making $y=nz$ and substituting in the given equation it becomes

$$n^3z^3-48nz=128,$$

Whence, $$z^3-\frac{48}{n^2}z=\frac{128}{n^3}.$$

If now, we assume $n=4$, this equation will reduce to

$$z^3-3z=2,$$

and the construction of the values of z is the same as in example 2 ; that is $z=2$, $z=-1$, $z=-1$, and since $y=nz$, and $n=4$ we have,

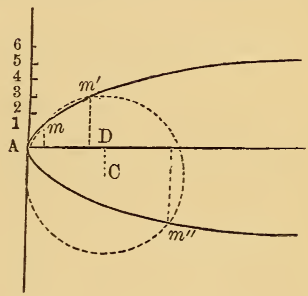
Ans. $y=+8, \ -4, \ -4.$

Ex. 4. Given $y^3-13y=-12$, to find the values of y by construction.

Comparing this equation with equation (G) page 241, we have

$$4-4a=-13, \ 8b=-12,$$

Whence, $a=4\frac{1}{4}$, $b=-1\frac{1}{2}$.



Constructing the parabola with the distance from the focus to the directix for the unit, we then lay off on its axis, in the positive direction, the distance $AD=4\frac{1}{4}$, then $DC=-1\frac{1}{2}$ will determine the center of the circle. The circumference described with C as a center and CA as a radius, cuts the parabola in the points m, m', m'', which are at the distances from the axis of the parabola $+1$, $+3$, and -4, respectively ; hence

Ans. $y=+1, \ +3, \ -4.$

The different values of x, on page 246, corresponding to the assumed values 30.0388, 25 - - - -5, &c., of y are thus found ;

Equation (A) page 245, when $y=30.0388$ becomes

$$x^3=13x+12=30.0388,$$

Or, $$x^3-13x=18.0388.$$

Whence, $4-4a=-13$, $8b=18.0388$ (see eq. (G) page 241).

Therefore, $a=4\frac{1}{4}$, $b=2.25485$.

These values of a and b will give us the center of the circumference, the intersections of which with the parabola, determine the values of x.

So when $y=25$ we shall have

$$x^3-13x=13$$

$$4-4a=-13, \quad 8b=13\,; \quad a=4\tfrac{1}{4}, \quad b=1\tfrac{5}{8},$$

and so on, the value of a being constantly equal to $4\frac{1}{4}$.

CHAPTERS VIII., and IX.

STRAIGHT LINES IN SPACE AND PLANES.

(Page 269.)

Ex. **1.** What is the distance between two points in space of which the coördinates are

$$x=1, \ y=-5, \ z=-3\,; \ x'=4, \ y'=-4, \ z'=1.$$

Substituting these values of x, y, z, x', y', z', in the formula

$$D^2=(x-x')^2+(y-y')^2+(z-z')^2$$

found on page 254 (Prop. 6), and taking the square root we get

$$D=\sqrt{(3+2)^2+(5+1)^2+(-2-6)^2}$$

$$D=\sqrt{25+36+64}=\sqrt{125}=11.180+. \qquad \textit{Ans.}$$

Ex. **2.** Of which the coördinates are

$$x=1, \ y=-5, \ z=-3\,; \ x'=4, \ y'=-4, \ z=1.$$

As in Example 1, substituting these values of the coördinates of the two points we find

$$D=\sqrt{(1-4)^2+(-5+4)^2+(-3-1)^2}$$

$$=\sqrt{9+1+16}=\sqrt{26}$$

$$=5.098+=5\tfrac{1}{10} \text{ nearly.} \qquad \textit{Ans.}$$

Ex. 3. The equations of the projections of a straight line on the coördinate planes (xz), (yz), are

$$x=2z+1, \quad y=\tfrac{1}{3}z-2,$$

required the equation of the projection on the plane (xy).

Multiplying the second equation through by 6, we have

$$6y=2z-12$$

which makes the coefficients of z the same in the two equations. Subtracting the first equation from this last, member from member, we get

$$6y-x=-13$$

Or, $$y=\tfrac{1}{6}x-2\tfrac{1}{6}. \qquad \textit{Ans.}$$

Ex. 4. The equations of the projections of a line on the coördinate planes (xy) and (yz), are

$$2y=x-5 \text{ and } 2y=z-4.$$

Required the equation of the projection on the plane (xz).

By subtracting the second of these equations from the first, member from member, we have

$$0=x-z-1, \text{ or } x=z+1. \qquad \textit{Ans.}$$

Ex. 5. Required the equations of the three projections of a straight line, which passes through two points, whose coördinates are

$$x'=2,\ y'=1,\ z'=0;\ \text{and}\ x''=-3,\ y''=0,\ z''=-1.$$

Placing these values of x', x'', y', &c., in the formulas on page 252 (Prop. 8), which are

$$x-x'=\frac{x''-x'}{z''-z'}(z-z')\ ;\ y-y'=\frac{y''-y'}{z''-z'}(z-z'),$$

the first becomes

$$x-2=\frac{-3-2}{-1}z, \text{ or } x=5z+2,$$

and the second

$$y-1=\frac{-1}{-1}z, \text{ or } y=z+1,$$

Answers.

which are the equations of the projections of the required line on the planes (xz), (yz).

We eliminate z from these equations by subtracting the first from the second after multiplying the second through by 5 : Thus

$$5y = 5z + 5$$
$$x = 5z + 2$$

Subtract and $5y - x = 5 - 2$, or $5y = x + 3$. *Ans.*

Which is the equation of the projection of the line on the plane (xy).

Ex. **6.** Required the angle included between two lines whose equations are

$$\left.\begin{array}{l} x=3z+1 \\ y=2z+6 \end{array}\right\} \text{of the 1st ; and} \left\{\begin{array}{l} x = z + 2 \\ y=-z+1 \end{array}\right\} \text{of the 2d.}$$

Referring to the formula

$$\text{Cos. } V = \frac{1+aa'+bb'}{\sqrt{1+a^2+b^2}\sqrt{1+a'^2+b'^2}}$$

found on page 257, Prop. 8, and comparing the coefficients of z in the given equations with those of the equations in that proposition, we have

$$a=3,\ a'=1\ ;\ b=2,\ b'=-1.$$

These values substituted in the formula give

$$\text{Cos. } V = \frac{1+3-2}{\sqrt{1+9+4}\sqrt{1+1+1}} = \frac{2}{\sqrt{42}} = \tfrac{1}{21}\sqrt{42}.$$

$$\sqrt{42} = 6.4808\ ;\ \text{hence } \tfrac{1}{21}\sqrt{42} = .308609.$$

That is

$$\text{Cos. } V = .308609.$$

The table gives .30846 for the cosine of $72° 2'$, and .30874 for the cosine of $72° 1'$; hence we have the proposition

$$28 : 13 :: 60'' : x'' = 28'',$$

And $V = 72° 1' 28''$. *Ans.*

Ex. 7. Find the angles made by the lines, designated in the preceding example, with the coördinate axes.

Formulas (5), (6) and (7), page 255, Prop. (7), are

$$\text{Cos. } X = \frac{a}{\pm\sqrt{1+a^2+b^2}}. \qquad (5)$$

$$\text{Cos. } Y = \frac{b}{\pm\sqrt{1+a^2+b^2}}. \qquad (6)$$

$$\text{Cos. } Z = \frac{1}{\pm\sqrt{1+a^2+b^2}}. \qquad (7)$$

In these, to find the angles the first line makes with the coördinate axes, we must make $a=3$, $b=2$.

These values placed in the above formulas give

$$\text{Cos. } X = \frac{3}{\pm\sqrt{1+9+4}} = \frac{3}{\pm\sqrt{14}}$$

$$= \frac{3\sqrt{14}}{14} = \tfrac{1}{14}\sqrt{126} = \tfrac{1}{14}(11.224+).$$

Cos. $X = .80179$. By the table, $X = 36^\circ\ 42'$. *Ans.*

$$\text{Cos. } Y = \frac{2}{\pm\sqrt{14}} = \tfrac{1}{14}\sqrt{56} = \tfrac{1}{14}(7.4833+)$$

Cos. $Y = .53452$. $y = 57^\circ\ 41'\ 20''$. *Ans.*

$$\text{Cos. } Z = \frac{1}{\pm\sqrt{14}} = \tfrac{1}{14}\sqrt{14} = \tfrac{1}{14}(3.7416+)$$

Cos. $Z = .26727$. $z = 74^\circ\ 29'\ 54''$. *Ans.*

In like manner to find the angles that the second line makes with the coördinate axes, we must make $a=a'=1$, $b=b'=-1$. Formulas (5) and (7) then give

$$\text{Cos. } X = \text{cos. } Z = \frac{1}{\sqrt{3}} = \tfrac{1}{3}\sqrt{3} = \tfrac{1}{3}(1.73205)$$

Cos. $X =$ cos. $Z = .57735$, $X=Z=54^\circ\ 44'$. *Ans.*

Since $b'=-1$, the cosine of Y must be taken with a minus sign, if the cosines of X and Z are taken with the plus sign, and the converse. But these cosines have the same numerical value; therefore Y is the supplement of $54° 44'$; that is

$$Y = 125° 16'. \qquad \textit{Ans.}$$

Ex. 8. Having given the equations of two straight lines in space, as

$$\left.\begin{array}{l} x = 3z + 1 \\ y = 2z + 6 \end{array}\right\} \text{ of the 1st; and } \left\{\begin{array}{l} x = z + 2 \\ y = -z + \beta' \end{array}\right\} \text{ of the 2d,}$$

to find the value of β', so that the lines shall actually intersect, and to find the coördinates of the point of intersection.

If the equation

$$\frac{\alpha - \alpha'}{a - a'} = \frac{\beta - \beta'}{b - b'}$$

found on page 252, Prop. 4, is satisfied by the constants in the equations of any two straight lines, such lines will intersect. Any five of these constants being given, the equation will determine the sixth. In the present example we have $\alpha=1$, $\alpha'=2$, $a=3$, $a'=1$, $b=2$, $b'=-1$, $\beta=6$, from which to find β'.

Placing these values in the above equation it becomes

$$\frac{1-2}{3-1} = \frac{6-\beta'}{2+1},$$

Whence $\qquad 2\beta' = 15; \quad \beta' = 7\tfrac{1}{2}. \qquad$ *Ans.*

The formulas for the values of x and y, the coördinates of the point of intersection given on the same page, are

$$x = \frac{a\alpha' - a'\alpha}{a - a'}, \quad y = \frac{b\beta' - b'\beta}{b - b'}.$$

Substituting in these the values of a, a', α, α', β, β', we get

$$x = \frac{6-1}{2} = 2\tfrac{1}{2}, \text{ and } y = \frac{15+6}{3} = 7.$$

The value of z may be found from any one of the given equations, by substituting for x or y, the value just found.

Take for example the 1st, $x=3z+1$; it becomes $2\frac{1}{2}=3z-1$; whence $6z=3$, $z=\frac{1}{2}$.

Hence the *Ans.* $\begin{cases} \beta'=7\frac{1}{2}, \ y=7. \\ x=2\frac{1}{2}, \ z=\frac{1}{2}. \end{cases}$

Ex. **9.** Given the equation of a plane

$$8x=3y+z-4=0,$$

to find the points in which it cuts the three axes, and the perpendicular distance from the origin to the plane.

The point in which the axis of X pierces the plane is found by making $y=0$, $z=0$, in the equation, by which we get $8x-4=0$, or $x=\frac{1}{2}$. Similarly by making $x=0$, $z=0$, we have $y=-1\frac{1}{3}$; and $x=0$, $y=0$, gives $z=4$.

Formula (7), page 276, which is

$$p=\frac{D}{\sqrt{A^2+B^2+C^2}},$$

will give the perpendicular distance from the origin to the plane by making $A=8$, $B=-3$, $C=1$, $D=-4$.

These values give us

$$p=\frac{-4}{\pm\sqrt{64+9+1}}=\frac{4}{\sqrt{74}}=\frac{4\sqrt{74}}{74}$$

$$=\frac{\sqrt{1184}}{74}=\frac{34.409}{74}=.4649+.$$

Hence the *Ans.* $\begin{cases} x_{\prime}=\frac{1}{2}, \ y_{\prime}=-1\frac{1}{3}, \ z_{\prime}=4. \\ p=.4649+. \end{cases}$

Ex. **10.** Find the equations for the intersections of the two planes,

$$3x-4y+2z-1=0,$$
$$7x-3y-z+5=0.$$

If z be eliminated between these two equations the resulting equation will be the equation of the projection of the intersection on the plane (xy). We eliminate by multiplying the second equation through by 2 and then adding it to the first, member to member; thus

$$\begin{array}{lr} & 3x-4y+2z-\ \ 1=0 \\ \text{Add} & 14x-6y-2z+10=0 \\ \hline & 17x-10y+9=0. \end{array} \qquad \text{1st } Ans.$$

We eliminate y from the given equations by multiplying the first through by 3, and the second by 4, and subtracting the first result from the second.

$$\begin{array}{lr} \text{Thus,} & 28x-12y-4z+20=0 \\ \text{Subtract,} & 9x-12y+6z-\ \ 3=0 \\ \hline & 19x-10z+23=0. \end{array} \qquad \text{2d } Ans.$$

We find the angle included between the two planes by the formula

$$\text{Cos. } V=\frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\sqrt{A'^2+B'^2+C'^2}},$$

in which we must make $A=3$, $B=-4$, $C=2$, $A'=7$, $B'=-3$, $C'=-1$. These substitutions will give

$$\text{Cos. } V=\frac{21+12-2}{\sqrt{9+16+4}\sqrt{49+9+1}}=\frac{31}{\sqrt{29}\sqrt{59}}=\frac{31}{\sqrt{1711}}$$

$$=\frac{31\sqrt{1711}}{1711}=\frac{\sqrt{1644271}}{1711}=\frac{1282.24}{1711}.$$

Hence, $\quad$ Cos. $V=.74940$; $V=41°\ 27'\ 41''$. $\qquad$ *Ans.*

Ex. **12.** The equations of a line in space are

$$x=-2z+1, \text{ and } y=3z+2.$$

Find the inclination of this line to the plane represented by the equation

$$8x-3y+z-4=0$$

The formula to be used in this example is that found on page 268, Prop. 7, which is

$$\text{Sin. } v=\frac{Aa+Bb+C}{\sqrt{1+a^2+b^2}\sqrt{C^2+B^2+A^2}}.$$

In this we must make $a=-2$, $b=3$, $A=8$, $B=-3$, $C=1$, which will give

$$\text{Sin. } v=\frac{-16-9+1}{\sqrt{1+4+9}\,\sqrt{1+64+9}}=\frac{24}{\sqrt{14}\,\sqrt{74}}=\frac{12}{\sqrt{7}\,\sqrt{37}}$$

$$=\frac{\sqrt{37296}}{259}=\frac{193.147+}{259}=.74571.$$

Whence $\qquad v=48^\circ\ 13'\ 13''.$ *Ans.*

Ex. **13.** Find the angles made by the plane whose equation is

$$8x-3y+z-4=0$$

with the coördinate planes.

The formulas to be used in this case are

$$\text{Cos. }(xy)=\frac{\pm C}{\sqrt{A^2+B^2+C^2}},\quad \cos.\ (xz)=\frac{\pm B}{\sqrt{A^2+B^2+C^2}},$$

$\cos.\ (yz)=\dfrac{\pm A}{\sqrt{A^2+B^2+C^2}}$, and are found on page 264, Prop. 3.

By comparing the given equation with the general equation of the plane, we find, $A=8$, $B=-3$, $C=1$.

These values being substituted in the first of the above formulas it becomes

$$\text{Cos. }(xy)=\frac{\pm 1}{\sqrt{64+9+1}}=\frac{\pm 1}{\sqrt{74}}=\tfrac{1}{74}\sqrt{74}$$

$$\text{Cos. }(xy)=\tfrac{1}{74}(8.6023)=.11625.$$

The Table gives $83^\circ\ 19'\ 27''$ for the angle whose natural cosine is this decimal.

Similarly the second formula gives

$$\text{Cos. }(xz)=\frac{\mp 3}{\sqrt{74}}=-\tfrac{1}{74}\sqrt{666}=-.34874,$$

and the third

$$\text{Cos. }(yz)=\frac{\pm 8}{\sqrt{74}}=\tfrac{1}{74}\sqrt{4736}=.92998,$$

which correspond respectively to the angles $110° 24' 38''$ and $21° 34' 5''$.

Hence the

$$Ans. \begin{cases} 83° 19' 27'' \text{ with the plane } (xy). \\ 110° 24' 38'' \quad `` \quad `` \quad `` \quad (xz). \\ 21° 34' 5'' \quad `` \quad `` \quad `` \quad (yz). \end{cases}$$

Ex. **15.** Find the equation of the plane which will cut the axis of Z at 3, the axis of X at 4, and the axis of Y at 5.

The equation of the required plane must have the form

$$Ax+By+Cz+D=0,$$

and such values of the coefficients A, B and C, are to be found as will cause the plane represented by the equation to cut the coördinate axes at the specified points.

Referring to the Scholium on page 260, we find the following expressions for the distances from the origin to the points in which a plane cuts the axes, viz.:

$$x=-\frac{D}{A}=OP, \quad y=-\frac{D}{B}=OQ, \quad z=-\frac{D}{C}=OR.$$

If in these we give to x, y and z, their values by the conditions of the problem, we have

$$3=-\frac{D}{C}, \quad 5=-\frac{D}{B}, \quad 4=-\frac{D}{A}.$$

Whence, $\quad C=-\frac{D}{3}, \quad A=-\frac{D}{4}, \quad B=-\frac{D}{5}.$

Substituting these values in the general equation of the plane it becomes

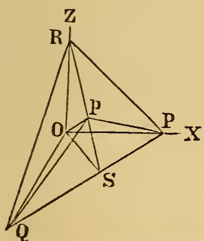
$$-\frac{D}{4}x-\frac{D}{3}y-\frac{D}{3}z+D=0,$$

Or, $$15x+12y+20z=60,$$

by multiplying through by $-\frac{60}{D}$; and by dividing this last equation through by 3 we get finally.

Ans. $5x+4y+6\frac{2}{3}z=20.$

Ex. 16. Find the equation of the plane which will cut the axis of X at 3, the axis of Z at 5, and which will pass at the perpendicular distance 2 from the origin. At what distance from the origin, will this plane cut the axis of Y?



Let PQR be the required plane cutting the axis of X at P, making $OP=3$, and the axis of Z at R making $OR=5$. Op is the perpendicular let fall from the origin upon the plane, and by condition it is equal to 2.

Conceive a plane to be passed through the axis of Z, and the perpendicular Op intersecting the required plane in the line RS, and the plane (xy) in OS. Since the plane through OR and Op is perpendicular to both the planes (xy) and PQR, it is perpendicular to their intersection, PQ. In the figure we therefore have the right angled triangles ROp, ROS, ROP, RPS, OPS and OPQ.

From the triangle ROp, we have

$$Rp = \sqrt{\overline{RO}^2 - \overline{Op}^2} = \sqrt{5^2-2^2} = \sqrt{21},$$

and from the triangles ROp, ROS,

$$Rp : OR :: OR : RS,$$

Or, $$\sqrt{21} : 5 :: 5 : RS = \frac{25}{\sqrt{21}}.$$

The triangle ROP gives

$$RP=\sqrt{\overline{RO}^2+\overline{PO}^2}=\sqrt{5^2+3^2}=\sqrt{34},$$

Whence, $$PS=\sqrt{\overline{RP}^2-\overline{RS}^2}=\sqrt{34-\tfrac{625}{21}}=\sqrt{\tfrac{89}{21}}.$$

Then from the similar right angled triangles OPS, OPQ, we have

$$PS : OP :: OP : PQ$$

Or, $\sqrt{\frac{89}{21}} : 3 :: 3 : PQ = \dfrac{9}{\sqrt{\frac{89}{21}}} = \dfrac{9\sqrt{21}}{\sqrt{89}}$.

Whence, $OQ = \sqrt{\overline{PQ}^2 - \overline{OP}^2} = \sqrt{\dfrac{81\times 21}{89} - 9} = \sqrt{\dfrac{900}{89}}$

$= \dfrac{30}{\sqrt{89}}$, which may have either sign.

The equation of the plane will be what the equation

$$Ax + By + Cz + D = 0$$

becomes, when we substitute in it the values of A, B and C, given by the equations

$$5 = -\frac{D}{C},\quad 3 = -\frac{D}{A},\quad \frac{30}{\sqrt{89}} = -\frac{D}{B}.$$

We thus get

$$-\frac{D}{3}x - \frac{\sqrt{89}D}{30}y - \frac{D}{5}z + D = 0,$$

Or, $$10x + \sqrt{89}\,y + 6z - 30 = 0,$$

which is the equation of the required plane, and it cuts the axis of Y at the distance $OQ = \dfrac{30}{\sqrt{89}}$, from the origin.

Ex. **17.** Find the equations of the intersection of the two planes whose equations are

$$3x - 2y - z - 4 = 0,$$
$$7x + 3y + z - 2 = 0.$$

By adding these equations, member to member, we eliminate z and get for our result

$$10x + y - 6 = 0$$

which is the equation of the projection of the intersection of the two planes on the plane (xy).

Multiplying the same equations through, the first by 3, and the second by 2, we have

$$9x-6y-3z-12=0,$$

Add, $$14x+6y+2z-\ 4-0$$

And our result $23x-z-16=0$, is the equation of the projection of the intersection on the plane (xz).

In like manner multiplying the first through by 7, and the second by 3, we find

$$21x-14y-7z-28=0,$$

Subtract, $$21x+\ 9y+3z-\ 6=0,$$

And our result, $23y+10z+22=0$, is the equation of the projection of the intersection on the plane (yz).

Ex. 18. Find the inclination of the planes whose equations are expressed in Example 17.

The equations of the planes are

$$3x-2y-z-4=0,$$
$$7x+3y+z-2=0.$$

The formula for the cosine of the inclination of these two planes is

$$\text{Cos. } V=\frac{AA'+BB'+CC'}{\sqrt{A^2+B^2+C^2}\sqrt{A'^2+B'^2+C'^2}},$$

found on page 267, Prop 6. In this we must make the following substitutions, viz.:

$$A=3,\ B=-2,\ C=-1;\ A'=7,\ B'=3,\ C=+1,$$

by which we get

$$\text{Cos. } V=\frac{21-6-1}{\sqrt{9+4+1}\sqrt{49+9+1}}=\frac{14}{\sqrt{14}\sqrt{59}}=\frac{\sqrt{14}}{\sqrt{59}},$$

Whence, $$\text{Cos. } V=\frac{\sqrt{14}}{\sqrt{59}}=\frac{\sqrt{14}\sqrt{59}}{59}=\frac{\sqrt{826}}{59}=.48712,$$

and by the Table this is found to correspond to the angle $60^\circ\ 50'\ 55''$, or the supplementary angle, $119^\circ\ 9'\ 5''$. *Ans.*

Ex. **19.** A plane intersects the coördinate plane (xz), at an inclination of 50°, and the coördinate plane (yz), at an inclination of 84°. At what angle will this plane intersect the plane (xy)?

We employ in this case the formula

$$\text{Cos.}^2(xy)+\cos.^2(xz)+\cos.^2(yz)=1,$$

which is found on page 264. From it we get

$$\text{Cos.}^2(xy)=1-\cos.^2(xz)-\cos.^2(yz).$$

But $\cos.(xz)=\cos.50°=.64279$, and $\cos.(yz)=\cos.84°=.10453$.

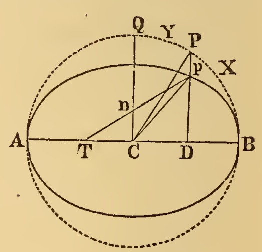
Whence, $\quad \text{Cos.}^2(xy)=1-.4241054050-.0109265209$

$$\text{Cos.}\,(xy)=\sqrt{.5758945950}=.75887,$$

which by the Table corresponds to the angle 40° 38′ 6″. *Ans.*

MISCELLANEOUS PROBLEMS.

Ex. **1.** The greatest or major axis of an ellipse is 40 feet, and a line drawn from the center, making an angle of 36° with the major axis and terminating in the ellipse, is 18 feet long; required the minor axis of this ellipse, its area and eccentricity.



Suppose AB to be the major axis, C the center, and F the focus of the ellipse, and let Cp be the line drawn from the center making an angle of 36° with AB, and terminating in the curve at p. From p let fall the perpendicular pD on the axis, and produce this perpendicular to meet the circumference described on AB, as a diameter in P, and draw CP.

Then in the right angled triangle CpD we have

$$1 : \sin. pCD :: Cp : pD = Cp.\sin. pCD.$$

$$1 : \cos. pCD :: Cp : CD = Cp.\cos. pCD.$$

In the Table we find $\sin. 36° = .58779$, $\cos. 36° = .80902$,

Hence, $pD = 18 \times .58779 = 10.58022$,

And $CD = 18 \times .80902 = 14.56236$.

From the right angled triangle CPD, we get

$$PD = \sqrt{\overline{CP}^2 - \overline{CD}^2} = \sqrt{400 - 212.062439}$$

$$= \sqrt{187.937561} = 13.709.$$

Now we have an ordinate of the ellipse drawn to its transverse axis, and the corresponding ordinate of the circle described on that axis. Calling the semi-transverse axis A, and the semi-conjugate axis B, we have,

$$A : B :: PD : pD,$$

That is $20 : B :: 13.709 : 10.58022.$

Whence, $B = \dfrac{20 \times 10.58022}{13.709} = 15.4376,$

And $2B = 30.8752.$

The area of an ellipse is measured by the product of its semi-axes multiplied by $3.14159+$.

Hence for the required ellipse, we have

$$\text{Area} = \pi AB = 3.14159 \times 20 \times 15.4376 = 969.972+.$$

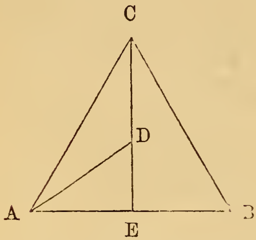
Denoting the eccentricity by E, we have

$$E = \frac{\sqrt{A^2 - B^2}}{A} = \frac{\sqrt{400 - 238.3195}}{20} = \frac{12.715}{20} = .63575.$$

Collecting our results we therefore have

$$Ans. \begin{cases} \text{Minor axis,} & 30.8752 \text{ feet.} \\ \text{Area,} & 969.972 \text{ sq. feet.} \\ \text{Eccentricity,} & .63575 \text{ feet.} \end{cases}$$

In the second miscellaneous problem it was assumed that if the side of an equilateral triangle be denoted by a, the line drawn from the center to the vertex of either angle of the triangle, will be represented by $\frac{a}{\sqrt{3}}$. To prove this, let



ABC be an equilateral triangle of which D is the center, by which we mean the center of the inscribed or circumscribed circle. Draw AD and CD, producing the latter to meet the base of the triangle in E.

Now since the angle CAE is 60° and AD bisects this angle, the angle DAE is 30°, and DE is therefore equal to one half of AD. Denote AB by a and DE by x; then $AD = 2x$.

In the right angled triangle ADE, we have

$$\overline{AE}^2 + \overline{DE}^2 = \overline{AD}^2,$$

Or, $$\frac{a^2}{4} + x^2 = 4x^2, \quad 3x^2 = \frac{a^2}{4}.$$

Hence, $$x = \frac{a}{2\sqrt{3}}, \quad 2x = AD = \frac{a}{\sqrt{3}},$$

which was to be proved.

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