Few-body quark dynamics for doubly heavy baryons and tetraquarks

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We discuss the adequate treatment of the three- and four-body dynamics for the quark model picture of double-charm baryons and tetraquarks. We stress that the variational and Born-Oppenheimer approximations give energies very close to the exact ones, while the diquark approximation might be somewhat misleading. The Hall-Post inequalities also provide very useful lower bounds that exclude the possibility of stable tetraquarks for some mass ratios and some color wave functions.

I. INTRODUCTION

There is rich literature on multiquarks, and many reviews, including [1]. The recent contributions are stimulated by the discovery of a double-charm baryon [2], which is interesting by itself and also triggers speculations about exotic double-charm mesons $QQ\bar{q}\bar{q}$. For years, the sector of flavor-exotic tetraquarks was somewhat forgotten, and even omitted from some reviews on exotic hadrons, as much attention was paid to hidden-flavor states $Q\bar{Q}q\bar{q}$. However, the flavor-exotic multiquarks have been investigated already some decades ago [3] and have motivated an abundant literature [4] that was unfortunately ignored in some recent papers.

The underlying dynamics is not exactly the same in all papers cited in [4]. Some authors consider a purely linear interaction, either pairwise or inspired by the string model, and some others include a Coulomb-like interaction and spin-dependent terms. Sometimes, the wave function contains a single color configuration, while in other papers the role of color mixing is analyzed.

In the present note, we stress that a careful treatment of the few-body problem is required before drawing any conclusion about the existence of stable states in a particular model. We, indeed, observe a dramatic spread of strategies: Some authors use the full machinery of a variational method based on correlated Gaussians or hyperspherical expansion, and others use a crude trial wave function or a cluster approximation. We shall review critically the different strategies that can be found in the literature.

Not surprisingly, the main difficulties are encountered when a multiquark state is found near its lowest dissociation threshold. The question of whether or not there is a bound state requires a lot of care. In particular, one should account for the mixing of color configurations [5,6].

We apologize for the somewhat technical character of this survey. However, we find it necessary to clarify the somewhat contradictory results in the literature. In particular, some authors who use similar ingredients obtain either stability or instability for the all-heavy configuration $QQ\bar{Q}\bar{Q}$, and in our opinion, this is because of an erroneous handling of the four-body problem in some papers.

This paper is organized as follows. In Sec. II, we briefly discuss the variational approximation, with several variants, including the hyperspherical expansion. In Sec. III, we discuss the diquark approximation, that is widely used. In Sec. IV, we discuss the Born-Oppenheimer method. In Sec. V, we comment about the approximate relation between meson, baryon, and tetraquark energy. In Sec. VI a reminder is given about the Hall-Post inequalities, and some new applications are derived for tetraquarks within potential models. The importance of color mixing is illustrated in Sec. VII. The role of the spin-dependent part of the potential is stressed in Sec. VIII. Some conclusions are proposed in Sec. IX.

II. VARIATIONAL METHODS

A. General considerations

Variational methods have been applied from the beginning of quantum mechanics, as they were already used in other fields of physics involving similar equations. A well-known example is the helium atom, for which the unperturbed wave function $\Psi_0 = \exp(-2\,r_1 - 2\,r_2)$, in an obvious notation, is already a good trial function, and can be improved, without much further computation, in the form $\Psi(\alpha) = \exp(-\alpha\,r_1 - \alpha\,r_2)$, where α is empirically adjusted, and is interpreted as the effective charge seen by each electron. See, e.g., [7].

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However, the stability of $\alpha e^- e^-$ is obvious as once the first electron is bound, there is enough attraction left to attach the second one. More delicate is the case of $H^-(pe^-e^-)$, for which the above trial function does not achieve binding, nor any factorized $f(r_1) f(r_2)$. As shown by Hylleraas, and independently by Chandrasekhar (see references in [7]), achieving binding requires either some asymmetry and restoration of symmetry, as $\exp(-\alpha r_1 - \beta r_2) + [\alpha \leftrightarrow \beta]$, or some explicit anticorrelation, such as $\exp(-\alpha r_1 - \alpha r_2 - \gamma r_{12})$, or, of course, a combination of both.

Similarly, the energy and structure of a baryon is easily calculated in any quark model, as the wave function is somewhat compact. But for a tetraquark $q_1 q_2 \bar{q}_3 \bar{q}_4$ at the edge of binding, the wave function contains antibaryonlike components with $q_1 q_2$ clustered, meson-meson components such as $q_1 \bar{q}_3 - q_2 \bar{q}_4$, and perhaps some diquark-antidiquark contributions. Thus a simplistic variational function cannot account for these three aspects.

For illustration, we shall use some toy models with increasing complexity. In the simplest version, the color wave function is frozen as $\bar{3}3$ in the $(QQ)(\bar{q}\bar{q})$ basis, and the potential is purely chromoelectric. It reads

$$H_{33} = \frac{\boldsymbol{p}_1^2 + \boldsymbol{p}_2^2}{2M} + \frac{\boldsymbol{p}_3^2 + \boldsymbol{p}_4^2}{2m} + \frac{v_{12} + v_{34}}{2}$$
$$+ \frac{v_{13} + v_{14} + v_{23} + v_{24}}{4}, \quad (1)$$

where $v_{ij} = v(r_{ij})$, with $v(r)$ being either r or $r^{0.1}$ or $\lambda r - \kappa/r$ for illustration. The analog with color $6\bar{6}$ reads

$$H_{66} = \frac{\boldsymbol{p}_1^2 + \boldsymbol{p}_2^2}{2M} + \frac{\boldsymbol{p}_3^2 + \boldsymbol{p}_4^2}{2m} - \frac{v_{12} + v_{34}}{4}$$
$$+ \frac{5}{8}(v_{13} + v_{14} + v_{23} + v_{24}). \quad (2)$$

If color mixing is accounted for, then one gets a coupled-channel problem,

$$H = \begin{pmatrix} H_{33} & H_{36} \\ H_{36} & H_{66} \end{pmatrix},$$
$$H_{36} = \frac{3(v_{14} + v_{23} - v_{13} - v_{24})}{4\sqrt{2}}. \quad (3)$$

It can be checked, that a simple one-Gaussian wave function $\exp(-a\boldsymbol{x}^2 - b\boldsymbol{y}^2 - c\boldsymbol{z}^2)$, where

$$\boldsymbol{x} = \boldsymbol{r}_2 - \boldsymbol{r}_1, \quad \boldsymbol{y} = \boldsymbol{r}_4 - \boldsymbol{r}_3, \quad \boldsymbol{z} = \frac{\boldsymbol{r}_3 + \boldsymbol{r}_4 - \boldsymbol{r}_1 - \boldsymbol{r}_2}{\sqrt{2}}, \quad (4)$$

is a set of Jacobi variables, describes somewhat well the ground state of the single-channel Hamiltonian H_{33} or H_{66}. As reviewed in [5], for $M = m$, the $6\bar{6}$ is lower than the $\bar{3}3$ one. For $M/m \gg 1$, the $\bar{3}3$ channel benefits from the QQ attraction, and becomes more favorable. However, by itself, it requires a large value of M/m to achieve stability below the $2 Q\bar{q}$ threshold. The critical value $(M/m)_c$ depends on the shape of the potential, for instance, $(M/m)_c \gtrsim 40$ for a linear interaction, $(M/m)_c \sim 15$ for a soft potential $r^{0.1}$, and $(M/m)_c \sim 7$ for an attractive Coulomb interaction.

This critical value $(M/m)_c$ is significantly lowered if one refines the wave function and introduces color mixing, i.e., uses
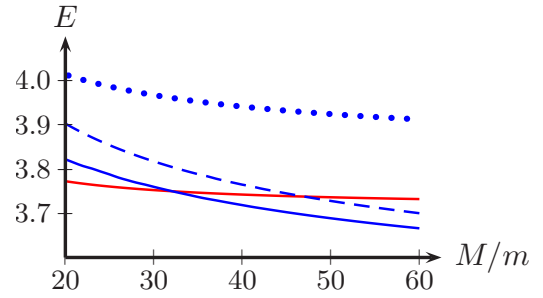

FIG. 1. Comparison of different approximations for a tetraquark bound by a linear potential with frozen color wave function, or with color mixing. Dotted blue line, pure $6\bar{6}$; dashed blue line, pure $\bar{3}3$; solid blue line, with color mixing; red line, threshold. The units are such that $m = 1$ and $v_{ij} = r_{ij}$.

H instead of H_{33} alone. Because of the different symmetry patterns of the color $\bar{3}3$ and $6\bar{6}$ states, the mixing requires an antisymmetric (under $1 \leftrightarrow 2$ or $3 \leftrightarrow 4$) wave function in one of the channels. The minimal wave function is, thus,

$$\Psi \propto \exp\left[-a_u r_{12}^2 - b_u r_{34}^2 - c_u \left(r_{13}^2 + r_{24}^2\right) - c_u' \left(r_{14}^2 + r_{23}^2\right)\right]$$
$$\pm (c_u \leftarrow c_u'), \quad (5)$$

where u stands for $\bar{3}3$ or $6\bar{6}$, and $c_u \neq c_u'$ in the antisymmetric channel ($6\bar{6}$ in practice). The effect of color mixing is illustrated in Fig. 1 (using a simple variational method, so that the actual energy might be slightly lower). It is seen that the critical value for binding is reduced to $(M/m)_c \sim 32$ by color mixing.[1]

The effect of an explicit Coulomb part in the spin-independent potential is seen in Fig. 2. The potential is chosen as $v(r) = -\kappa/r + \sigma r$ with $\kappa = 0.4$ and $\sigma = 0.2 \, \text{GeV}^2$. The light mass is taken as $m = 0.3 \, \text{GeV}$.

One remarks that the effect of color mixing is less dramatic; the explicit inclusion of a Coulomb term decreases the critical value $(M/m)_c$ significantly, here about 18 instead of about 28 (with a simple Gaussian expansion).

We shall return in Sec. VII to the problem of color mixing, with more realistic models that include a spin-spin component.

B. Correlated Gaussian expansion

A more efficient wave function is

$$\Psi = \sum_i \gamma_i \exp(-a_{11,i} \boldsymbol{x}^2 - 2 a_{12,i} \boldsymbol{x}.\boldsymbol{y} - \cdots - a_{33,i}\boldsymbol{z}^2), \quad (6)$$

which describes an overall scalar with the possibility of internal orbital excitations. The quadratic form $a_{11,i} \boldsymbol{x}^2 + 2 a_{12,i} \boldsymbol{x}.\boldsymbol{y} + \cdots$ is positive definite, and is sometimes rewritten as $\sum_{j<k} b_{jk,i} r_{jk}^2$ with all $b_{jk,i}$ positive. None of the Gaussians fulfill the requirements of permutation symmetry, but Ψ does, after optimization of the parameters.

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[1]In Fig. 1 and similar figures, the energies above the threshold are an artifact of any variational calculation based on normalizable wave functions. The proper treatment of the continuum requires dedicated techniques.
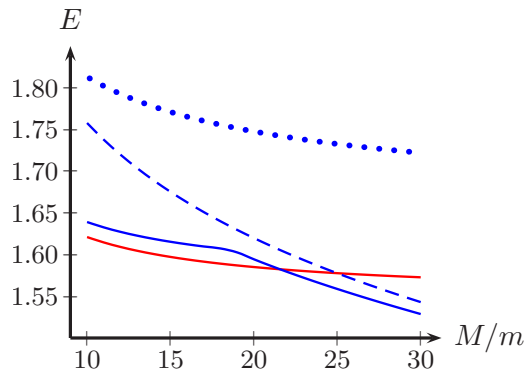
FIG. 2. Same as Fig. 1 for a Coulomb-plus-linear interaction $v(r) = -0.4/r + 0.2\,r$, where r is in GeV^{-1}. The energy E is in GeV.

A variant of (6) consists of using only diagonal Gaussians associated with the coordinates $\boldsymbol{x}$, $\boldsymbol{y}$, and $\boldsymbol{z}$, but to add diagonal terms in other sets of Jacobi coordinates, say

$$\Psi = \sum_i d_i \exp(-a_i\,\boldsymbol{x}^2 - b_i\,\boldsymbol{y}^2 - c_i\,\boldsymbol{z}^2)$$
$$+ \sum_i d_i' \exp(-a_i'\,\boldsymbol{x}^2 - b_i'\,\boldsymbol{y}^2 - c_i'\,\boldsymbol{z}^2) + \cdots, \quad (7)$$

where, for instance,

$$\boldsymbol{x}' = \boldsymbol{r}_3 - \boldsymbol{r}_1, \quad \boldsymbol{y}' = \boldsymbol{r}_4 - \boldsymbol{r}_2,$$
$$\boldsymbol{z}' \propto M\,\boldsymbol{r}_2 + m\,\boldsymbol{r}_4 - M\,\boldsymbol{r}_1 - m\,\boldsymbol{r}_3, \quad (8)$$

corresponding to different cluster decompositions [8]. In this case, the spin-isospin-color algebra is slightly more delicate.

Other variants deal with the numerical determination of the parameters. For a given set of range parameters the weights γ_i in (6) or $d_i, d_i', \ldots$ in (7) and the energy are given by a generalized eigenvalue equation. The range parameters themselves are searched for by stochastic methods [9] or as belonging to a geometric series [8]. In both cases, the method is now well functioning.

C. Hyperspherical expansion

By properly rescaling the Jacobi coordinates $\boldsymbol{x}$, $\boldsymbol{y}$, ..., the Hamiltonian describing the relative motion of the quarks can be written as

$$H = \frac{1}{\mu}\big(\boldsymbol{p}_x^2 + \boldsymbol{p}_y^2 + \cdots\big) + V(\boldsymbol{x}, \boldsymbol{y}, \ldots), \quad (9)$$

which can be read as a Schrödinger equation for a single particle of mass $\mu/2$ in a world of spatial dimension $3\,(n-1)$, where $n = 3$ for baryons, $n = 4$ for tetraquarks, etc. In general, the potential $V(\boldsymbol{x}, \boldsymbol{y}, \ldots)$ is not central, so the partial wave expansion,

$$\Psi = \sum_{[L]} R_{[L]}(r)\,\mathcal{Y}_{[L]}(\Omega), \quad (10)$$

results into an infinite set of coupled equations for the radial functions $R_{[L]}(r)$ or their reduced form $r^{5/2}\,R_{[L]}(r)$. But if one

solves with an increasing number of equations, the convergence is somewhat fast. Here, $r = (\boldsymbol{x}^2 + \boldsymbol{y}^2 + \cdots)^{1/2}$ is the hyperradius, Ω a set of $3\,n - 4$ angles, and $[L]$ denotes the "grand" angular momentum L and its associated magnetic numbers labeling the generalized spherical harmonics $\mathcal{Y}$.

The convergence is illustrated in Table IX and Fig. 2 of Ref. [10].

III. DIQUARK APPROXIMATION

The motivations for diquarks cover much more than hadron spectroscopy. See, e.g., [11] for a survey and references to pioneering articles which are sometimes ignored in the recent literature. A few decades ago, the main concern in baryon spectroscopy, was the problem of missing resonances, predicted by the quark model and not observed. Many states of the symmetric quark model disappear if baryons are constructed out of a frozen diquark and a quark. However, the missing resonances, in which of the degrees of freedom $\boldsymbol{x} = \boldsymbol{r}_2 - \boldsymbol{r}_1$ and $\boldsymbol{y} \propto \boldsymbol{r}_3 - (\boldsymbol{r}_1 + \boldsymbol{r}_2)/2$ are both excited, are not very much coupled to the typical investigation channels πN or γN which privilege states with one pair of quarks shared with the target nucleon N. In recent photoproduction experiments with improved statistics, some of the missing states have been identified, which cannot be accommodated as made of a ground-state diquark and a third quark [12]. So one of the grounds of the diquark model is somewhat weakened.

The diquark model is regularly revisited, to accommodate firmly established exotics such a the $X(3872)$, or candidates awaiting confirmation [1]. Unfortunately, some unwanted multiquarks are also predicted in this approach, though this is not always explicitly stated or even realized. The issue of multiquarks within the diquark model was raised many years ago by Fredriksson and Jandel [13][2], and is sometimes rediscovered, without any reference to the 1981 paper. The paradox is perhaps that the diquark model, that produces fewer baryon states, produces too many multiquarks!

There are many variants of the so-called diquark model. An extreme point of view is that diquarks are almost-elementary objects, with their specific interaction with quarks and between them. A whole baryon phenomenology can be built starting from well-defined assumptions about the diquark constituent masses and the potential linking a quark to a diquark. Then a diquark-diquark interaction has to be introduced as a new ingredient for the multiquark sector.

Another extreme is to view diquarks as a type of "Voodoo few-body[3]." In this empirical approach to few-body physics, to estimate the energy and wave function of, say, $(a_1 a_2 a_3)$, with masses m_i and interaction $v_{ij}(r)$, one first solves for $(a_1 a_2)$ with v_{12} alone, with energy η_{12}, then estimates the bound state of a pointlike $(a_1 a_2)$ of mass $m_1 + m_2$, or perhaps

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[2]Some technical details of that paper might be revised, but the main concern remains.

[3]Jaffe [14] reported that Bjorken used the words "Voodoo QCD" to denote several useful models of strong-interaction physics, such as vector meson dominance, and also some less convincing recipes. A correspondence with R. L. Jaffe is gratefully acknowledged.
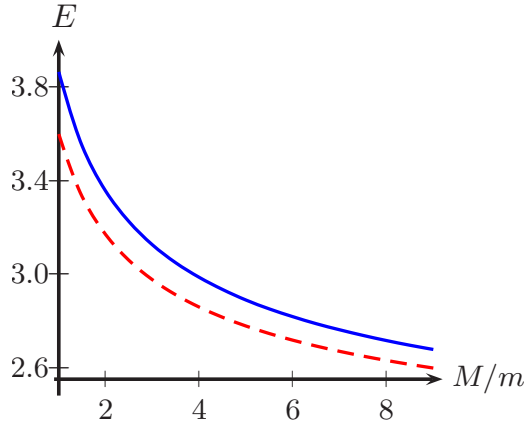
FIG. 3. Comparison of the exact energy (solid blue line) and diquark approximation (red dotted line) for a baryon (QQq) with masses M and $m = 1$, and a purely linear interaction, as a function of the mass ratio. The units are such that $m = 1$ and the potential is $\sum r_{ij}/2$.
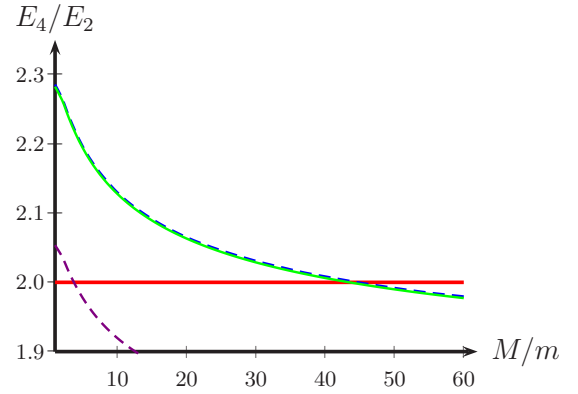

FIG. 4. Comparison of the variational upper bound (green curve) and Hall-Post lower bound (dotted blue curve), hardly distinguishable from the variational estimate at this scale, for the tetraquark Hamiltonian (1) with a linear interaction. Also shown is the naive diquark-antidiquark approximation (dashed violet curve).

$m_1 + m_2 + \epsilon_2$ in some variants, located at $\boldsymbol{R}_{12}$, interacting with a_3 through the potential $v_{13}(\boldsymbol{r}_3 - \boldsymbol{R}_{12}) + v_{23}(\boldsymbol{r}_3 - \boldsymbol{R}_{12})$, resulting in binding energy $\eta_{12,3}$, and the whole energy is given by $\eta_{12} + \eta_{12,3}$. For a four-body system, the $(a_1 a_2)$ and $(a_3 a_4)$ systems are estimated first, and then a third two-body equation is solved for $(a_1 a_2)$ interacting with $(a_3 a_4)$ via a potential $\sum' v_{ij}(\boldsymbol{R}_{12} - \boldsymbol{R}_{34})$, where Σ' denotes $i = 1,2$ and $j = 3,4$ throughout this paper, in particular in Sec. VI.

This strategy is of course fully justified for the deuterium atom considered as a pne^- system, as the internuclear motion is not significantly modified by the electron. On the other hand, this approach ruins some subtle collective binding, for instance, that of Borromean states [15]. Also, one cannot see either how H$^-(pe^-e^-)$ could become bound in this approach, or the hydrogen molecule be described as a "diproton" linked to a "dielectron"! In other cases, the method just underestimates the binding: For a $\alpha e^- e^-$ atom with a static nucleus, the first electron would get an energy -2 in natural units, and the second, only an energy -0.5, as it would endorse a full screening, while the exact energy is about -2.90. For the quark model, the effect is opposite, and, as seen below, the *ad hoc* clustering lowers significantly the energy.

A. The diquark model for double-charm baryons

In the case of double-charm baryons QQq there is obviously a QQ clustering which makes it tempting to use a two-step approach: first a (QQ) diquark and then a $(QQ)q$ quasimeson, as the diquark has the same color $\bar{3}$ as an antiquark. In Fig. 3, we compare the exact energy of QQq bound by a linear interaction $\sum r_{ij}/2$ and the diquark approximation, as a function of the heavy-to-light quark mass ratio M/m.

There is a clear overbinding. The situation does not improve too much as M/m increases: The diquark becomes more compact, but simultaneously, the total energy is more and more dominated by the heavy sector, so any systematic error in the QQ effective interaction is more visible. The problem, as already stressed in [5], is that the light quark induces

some interaction between the two heavy quarks. In the case of a harmonic confinement, $V = \sum r_{ij}^2/2$, the potential splits exactly into $V = 3(\boldsymbol{x}^2 + \boldsymbol{y}^2)/4$ if the second Jacobi variable is normalized as $\boldsymbol{y} = (2\boldsymbol{r}_3 - \boldsymbol{r}_1 - \boldsymbol{r}_2)/2$. The naive diquark approximation consists of replacing $3\boldsymbol{x}^2/4$ by $\boldsymbol{x}^2/2$, so that the contribution of the heavy quarks to the energy is reduced by a factor $(3/2)^{1/2}$. Similarly, for a linear interaction, the light quark potential, averaged over a sphere surrounding the diquark, will induce a positive contribution which is either $\propto \boldsymbol{x}^2/y$ or $\propto \boldsymbol{y}^2/x$, depending on the radius, and is omitted in the naive diquark model.

B. The diquark model for doubly heavy tetraquarks

The exercise can be repeated for the $QQ\bar{q}\bar{q}$ states. For simplicity, we consider only the case of a frozen $\bar{3}3$ color wave function, i.e., the Hamiltonian (1). Color mixing has to be introduced to have the proper threshold in the model, and it was seen in explicit calculations that the mixing with meson-meson configurations is crucial for states at the edge of stability. Nevertheless the comparison of various approximations is instructive for the toy model (1). In Fig. 4, we compare the exact solution of (1) with the approximation consisting of first computing the QQ diquark with $r_{12}/2$ alone and qq with r_{34} alone, and then $(QQ)(\bar{q}\bar{q})$ as a meson with a potential $r_{12,34}$ and constituent masses $2M$ and $2m$. The comparison is also made for a soft interaction $r^{0.1}$ in Fig. 5 and a pure Coulomb interaction in Fig. 6.

IV. BORN-OPPENHEIMER METHOD

A. General considerations

The Born-Oppenheimer method is implicit in any quark model. The quarkonium potential, for instance, is the minimal energy of the gluon field for a given separation of the quark and the antiquark. Explicit reference to Born-Oppenheimer was made, e.g., in the context of the bag model [16]. Then it was speculated that some exotic mesons are just quarkonia evolving
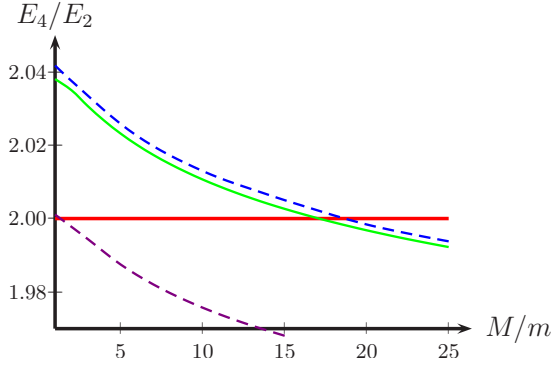
FIG. 5. Same as Fig. 4 with a potential $r_{ij}^{0.1}$ instead r_{ij}.

in a color field with gluonic or light-quark pair excitations; see, e.g., [17].

For a given interquark potential, there is also a Born-Oppenheimer approximation (BOA) for the solution of the Schrödinger equation governing double-charm baryons or double-charm tetraquarks, in analogy with the treatment of $H_2{}^+$ and H_2 in atomic physics, and it works very well, even for moderate values of the quark mass ratio M/m.

Actually, in the most naive version of BOA, the heavy quarks are frozen, and the energy of the light quark(s), supplemented by the direct QQ interaction, provides an effective potential that is independent of M. For finite M, the most significant correction comes from the recoil of the heavy quarks. This correction disappears if one applies BOA on the intrinsic Hamiltonian, free of center-of-mass motion. More precisely, in the case of baryons, let us consider

$$H_3 = \frac{\boldsymbol{p}_x^2}{M} + \frac{\boldsymbol{p}_y^2}{\mu} + V(\boldsymbol{x}, \boldsymbol{y}), \tag{11}$$

and search the solution as

$$\Psi = \varphi(\boldsymbol{x})\, \psi(\boldsymbol{x}, \boldsymbol{y}), \tag{12}$$

where $\psi(\boldsymbol{x}, \boldsymbol{y})$ is the solution of the one-body equation,

$$-\frac{\Delta_y \psi(\boldsymbol{x}, \boldsymbol{y})}{\mu} + V(\boldsymbol{x}, \boldsymbol{y})\psi(\boldsymbol{x}, \boldsymbol{y}) = \epsilon(\boldsymbol{x})\, \psi(\boldsymbol{x}, \boldsymbol{y}). \tag{13}$$

The BOA consists of neglecting in the kinetic energy operator the variations of ψ as a function of $\boldsymbol{x}$, and to deduce the first
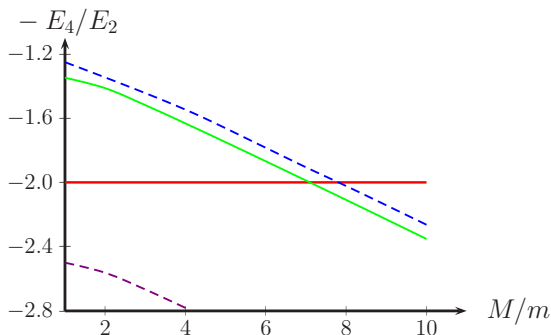

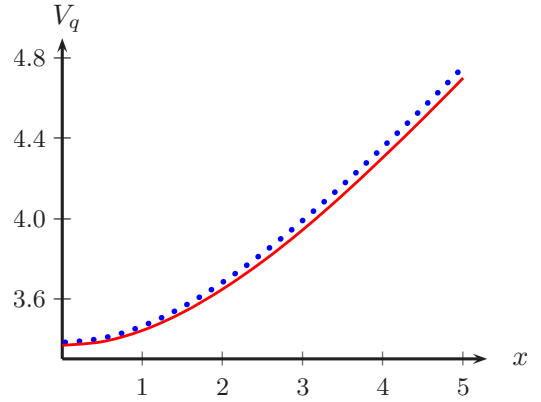FIG. 6. Same as Fig. 4 with a potential $-r_{ij}^{-1}$ instead r_{ij}. The threshold is fixed at -2.

FIG. 7. Comparison of the light quark energies for $QQ\bar{q}\bar{q}$ (solid red line) and QQq (dotted blue line) as a function of the QQ separation x. The second curve is shifted by the difference of energies $Qqq - Q\bar{q}$. The units are such $m = 1$, $M = 5$, and $v_{ij} = r_{ij}$.

levels from

$$-\frac{\Delta_x \varphi(\boldsymbol{x})}{\mu} + \epsilon(\boldsymbol{x})\, \varphi(\boldsymbol{x}) = E\, \varphi(\boldsymbol{x}). \tag{14}$$

The ground-state energy is underestimated (i.e., binding overestimated), as the last two terms of (11) are replaced by their minimum[4]. Note that if the wave function (12) is used as a trial function, one gets an upper bound for the ground state, sometimes named "variational Born-Oppenheimer."

B. Born-Oppenheimer for baryons

The validity of BOA for QQq baryons was shown in [18]. The check below is just for completeness. The light-quark energy $\epsilon(x)$ can be calculated by ordinary partial-wave expansion, which leads to coupled radial equations. One can also use a variational method, namely

$$\psi(\boldsymbol{x}, \boldsymbol{y}) = \sum_i \gamma_i [\exp(-a_i\, \boldsymbol{y}^2 - b_i\, \boldsymbol{y}.\boldsymbol{s}_i) + (\boldsymbol{s}_i \leftrightarrow -\boldsymbol{s}_i)],$$

$$\tag{15}$$

where $\boldsymbol{s}_i \parallel \boldsymbol{x}$. The matrix elements of the normalization, kinetic energy, and potential energy are given in a recent compilation [19]. The light-quark energy $V_q = \epsilon(x) - x/2$ is shown in Fig. 7, in the case of a linear potential. For $x = 0$, the result is analytic.

C. Born-Oppenheimer for tetraquarks

Here, once more, we use the toy Hamiltonian (1). It corresponds to a frozen $\bar{3}3$ color wave function. The effective potential is estimated using a trial wave function that generalizes (15) as to include two Jacobi coordinates, $\boldsymbol{y}$ and $\boldsymbol{z}$ in the light sector. For $x = 0$, the light quark energy $V_q = \epsilon(x) - x/2$ coincides with the energy of a singly heavy baryon $Q'qq$ with a flavored quark of mass $M' = 2M$. This provides a check of

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[4]These considerations can be extended to the excited states: The sum of n first levels is underestimated by BOA.

the numerics. We shall come back to this point in Sec. V. The light-quark energy is shown in Fig. 7.

V. RELATING MESONS, BARYONS, AND TETRAQUARKS

In a recent paper, Eichten and Quigg [20] use the heavy-quark symmetry to relate meson, baryon, and tetraquark energies. In a simplified version without spin effects, it reads

$$QQ\bar{q}\bar{q} = QQq + Qqq - Q\bar{q}, \tag{16}$$

where the configuration stands for the ground-state energy. For fixed m and $M \to \infty$, the identity is exact. For finite M, there is some departure. For instance, with a purely linear model, in units such that $v(r) = r$ for mesons, $\sum_{i<j} r_{ij}/2$ for baryons, and $m = 1$ and $M = 5$ in the Hamiltonian (1) with frozen $\bar{3}3$ color for tetraquarks, one gets 4.331 for the left-hand side and 4.357 for the right-hand side of (16). If one treats the tetraquark $QQ\bar{q}\bar{q}$ and the doubly heavy baryon QQq in the Born approximation, one can compare the two effective potentials as a function of the QQ separation x, the baryon one being shifted by $Qqq - Q\bar{q}$ which is independent of x. Without recoil correction, the two potentials are identical at $x = 0$. For finite M, there is slight difference, as the single q recoils against either M or $2M$, and similarly qq recoils against one or two heavy quarks.

The comparison is shown in Fig. 7. Clearly the two effective potentials are very similar, and thus give almost identical energies, up to an additive constant that corresponds to the last two terms in (16).

VI. HALL-POST INEQUALITIES

A. A brief reminder

The Hall-Post inequalities have been derived in the 1950s to relate the binding energies of light nuclei with different number of nucleons [21]. They have been re-discovered in the course of studies on the stability of matter [22], or to link meson and baryon masses in the quark model [3,23]. Before the applications to tetraquarks, we present a brief review illustrated in the three-body case, that follows the notation of [24].

The *naive* bound is deduced from the identity,

$$\frac{\boldsymbol{p}_1^2 + \boldsymbol{p}_2^2 + \boldsymbol{p}_3^2}{2m} + \sum_{i<j} V_{ij} = \left[\frac{\boldsymbol{p}_1^2 + \boldsymbol{p}_2^2}{4m} + V_{12}\right] + \cdots, \tag{17}$$

whose expectation value within the ground state of the left-hand side leads to the inequality,

$$E_3(m,V) \geqslant 3E_2(2m,V) = \frac{3}{2}E_2(m,2V), \tag{18}$$

among the ground-state energies. For instance, in a simple additive quark model with a factor 1/2, i.e., $V = \sum_{i<j} v(r_{ij})/2$, with v being the quarkonium potential, one gets $E_3(qqq) \geqslant 3E_2(q\bar{q})/2$. This implies that a baryon is heavier per quark than a meson, as seen, e.g., by comparing $\Omega^-(1672)$ and $\phi(1020)$, of quark content sss and $s\bar{s}$, respectively.

The inequality (18) never becomes an equality as it contains unbalanced center-of-mass kinetic energy. If one starts instead from the intrinsic Hamiltonians, one gets saturation in the case

of harmonic confinement. Namely,

$$\frac{\boldsymbol{p}_1^2 + \boldsymbol{p}_2^2 + \boldsymbol{p}_3^2}{2m} - \frac{(\boldsymbol{p}_1 + \boldsymbol{p}_2 + \boldsymbol{p}_3)^2}{6m} + \sum_{i<j} V_{ij}$$

$$= \left[\frac{2}{3m}\left(\frac{\boldsymbol{p}_2 - \boldsymbol{p}_1}{2}\right)^2 + V_{12}\right] + \cdots \tag{19}$$

leads to the *improved* bound,

$$E_3(m,V) \geqslant 3E_2(3m/2,V), \tag{20}$$

which is better, as the energy E_2 is a decreasing function of the mass, for given V.

For unequal masses, this "improved" bound is straightforwardly generalized as (the potential terms are omitted)

$$\sum_i \frac{\boldsymbol{p}_i^2}{2m_i} - \frac{(\sum_i \boldsymbol{p}_i)^2}{2\sum_i m_i} = \left[\frac{1}{\mu_{12}}\left(\frac{m_1\boldsymbol{p}_2 - m_2\boldsymbol{p}_1}{m_1 + m_2}\right)^2\right] + \cdots,$$

$$E_3(m_1,m_2,m_3) \geqslant \sum_{i<j} E_2(\mu_{ij}), \tag{21}$$

$$\mu_{12} = 2\frac{m_1 m_2 \sum_i m_i}{(m_1 + m_2)^2}.$$

However, this inequality is not saturated for the harmonic oscillator. It can be improved by introducing a slightly more general decomposition of the kinetic energy and optimizing some parameters. More precisely, this decomposition involves the parameters b_i, y_i and x_{ij} in the identity,

$$\sum_i \frac{\boldsymbol{p}_i^2}{2m_i} = \left(\sum_i \boldsymbol{p}_i\right) \cdot \left(\sum_i b_i \boldsymbol{p}_i\right)$$

$$+ \left[x_{12}^{-1}\left(\frac{\boldsymbol{p}_2 - y_3 \boldsymbol{p}_1}{1 + y_3}\right)^2\right] + \cdots. \tag{22}$$

For any given set $\{y_i\}$, one can determine the parameters b_i and the masses x_{ij}. If one takes the expectation value within the three-body wave function, the first term of the right-hand side disappears, and one reaches the so-called *optimized* lower bound,

$$E_3 \geqslant \max_{y_1,y_2,y_3} \sum_{i<j} E_2[x_{ij}(y_1,y_2,y_3)], \tag{23}$$

where it can be shown that the maximization automatically fulfills $y_1 y_2 y_3 = 1$.

B. Application to tetraquarks

Consider first the toy Hamiltonian (1), slightly generalized as $r_{ij} \to v_{ij} = v(r_{ij})$ for all pairs. In the case of equal masses, which can be set to $m = M$ the simple identity,

$$\sum_i \frac{\boldsymbol{p}_i^2}{2m} + \frac{v_{12} + v_{34}}{2} + \sum{}' \frac{v_{ij}}{4}$$

$$= \frac{h_{12}(m) + h_{34}(m)}{2} + \sum{}' \frac{h_{ij}(m)}{4},$$

$$h_{ij}(m) = \frac{\boldsymbol{p}_i^2 + \boldsymbol{p}_j^2}{2m} + v_{ij}, \tag{24}$$

demonstrates that for the ground-state energies,

$$E_4(m) \geqslant 2\,E_2(m), \qquad (25)$$

i.e., the tetraquark with pure chromoelectric interaction and a frozen $\bar{3}3$ color wave function, is above twice the minimum of each h_{ij}, which is the threshold energy. This is the analog of the above "naive" lower bound.

If one removes the center of mass, and starts from the decomposition,

$$\sum_i \frac{\boldsymbol{p}_i^2}{2\,m} - \frac{(\sum_i \boldsymbol{p}_i)^2}{8\,m} + \frac{v_{12} + v_{34}}{2} + {\sum}' \frac{v_{ij}}{4}$$

$$= \frac{\tilde{h}_{12}(m) + \tilde{h}_{34}(m)}{2} + {\sum}' \frac{\tilde{h}_{ij}(m/2)}{4},$$

$$\tilde{h}_{ij}(m) = \frac{1}{m}\left(\frac{\boldsymbol{p}_j - \boldsymbol{p}_i}{2}\right)^2 + v_{ij}, \qquad (26)$$

one gets the "improved" bound,

$$E_4(m) \geqslant E_2(m) + E_2(m/2), \qquad (27)$$

that is better, as $E_2(m/2) > E_2(m)$. For unequal masses, the decomposition reads

$$\frac{\boldsymbol{p}_1^2 + \boldsymbol{p}_2^2}{2\,M} + \frac{\boldsymbol{p}_3^2 + \boldsymbol{p}_4^2}{2\,m} + \frac{v_{12} + v_{34}}{2} + {\sum}' \frac{v_{ij}}{4}$$

$$= \left(\sum \boldsymbol{p}_i\right).(A(\boldsymbol{p}_1 + \boldsymbol{p}_2) + B(\boldsymbol{p}_3 + \boldsymbol{p}_4))$$

$$+ \frac{\tilde{h}_{12}(x_{12}) + \tilde{h}_{34}(x_{34})}{2} + {\sum}' \frac{\tilde{\tilde{h}}_{ij}(x,a,b)}{4},$$

$$\tilde{\tilde{h}}_{13}(x,a,b) = \frac{1}{x}\left(\frac{\boldsymbol{p}_1 - \boldsymbol{p}_3 + a\,\boldsymbol{p}_2 + b\,\boldsymbol{p}_4}{2}\right)^2 + v_{ij}, \qquad (28)$$

where the masses x_{12}, x_{34}, and x are readily calculated from the parameters A, B and a, and b. This results into

$$E_4(M,m) \geqslant \max_{A,B,a,b} [E_2(x_{12}) + E_2(x_{34}) + E_2(x)]. \qquad (29)$$

Hence a rigorous lower bound is obtained from simple algebraic manipulations and the knowledge of the two-body energy as a function of the reduced mass. For a linear interaction, (29) further simplifies into

$$E_4(M,m) \geqslant E_2(1) \max_{A,B,a,b} \left[x_{12}^{-1/3} + x_{34}^{-1/3} + x^{-1/3}\right], \qquad (30)$$

where $E_2(1) = 2.33811\ldots$ is the opposite of the first root of the Airy function. For $r^{0.1}$, the exponent $-1/3$ is replaced by $-0.1/2.1$ and $E_2(1)$ is computed numerically. The results for $E_4/E_2(1)$ as a function of M/m are shown in Figs. 4 and 5. The sum $1/M + 1/m$ is kept equal to 2 to fix the threshold energy at $2\,E_2(1)$.

VII. COLOR MIXING

The $\tilde{\lambda}_i.\tilde{\lambda}_j$ model of Eq. (1), with a pairwise potential from color-octet exchange, induces mixing between $\bar{3}3$ and $6\bar{6}$ states in the $QQ - \bar{q}\bar{q}$ basis. Perhaps the true dynamics inhibits the call for higher color representations such as sextet, octet, etc., for the subsystems of a multiquark, but for the time being, let
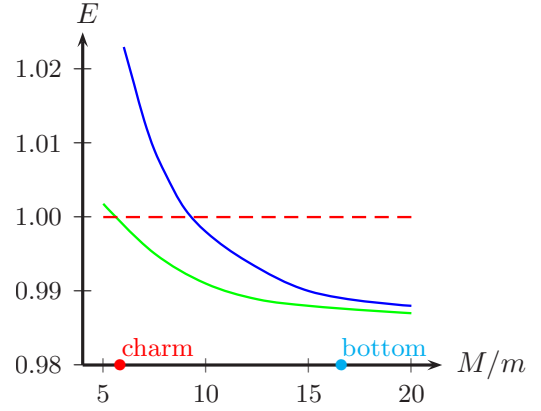

FIG. 8. Effect of color mixing on the binding of $QQ\bar{u}\bar{d}$, within the AL1 model. The tetraquark energy is calculated with only the color $\bar{3}3$ configurations (blue curve) and with the $6\bar{6}$ components (green curve).

us adopt the color-additive model. If one starts from a $\bar{3}3$ state with QQ in a spin triplet, and, for instance, $\bar{q}\bar{q} = \bar{u}\bar{d}$ with spin and isospin $S = I = 0$, then its orbital wave function is mainly made of an s wave in all coordinates. It can mix with a color $6\bar{6}$ with orbital excitations in the $\boldsymbol{x}$ and $\boldsymbol{y}$ linking QQ and $\bar{q}\bar{q}$, respectively. A minimal wave function in this sector can be chosen as

$$\Psi_6 \propto \boldsymbol{x}.\boldsymbol{y} \exp(-a\,\boldsymbol{x}^2 - b\,\boldsymbol{y}^2),$$

or

$$\Psi_6 \propto \exp\big[-a_{12}\,\boldsymbol{x}^2 - a_{34}\,\boldsymbol{y}^2 - \alpha\big(\boldsymbol{r}_{13}^2 + \boldsymbol{r}_{24}^2\big)$$
$$-\beta\big(\boldsymbol{r}_{14}^2 + \boldsymbol{r}_{23}^2\big)\big] - \{\alpha \leftrightarrow \beta\}. \qquad (31)$$

The effect of color mixing for a spin-independent interaction was shown Fig. 1 in the case of a linear potential, and in Fig. 2 for a Coulomb-plus-linear potential $V(r) = -a/r + b\,r$ with $a = 0.4$, $b = 0.2\,\mathrm{GeV}^2$, and $m = 0.3\,\mathrm{GeV}$, as a function of M/m. The gain is less pronounced for very large M/m, but for the mass ratios of interest, color mixing is crucial to achieve binding.

We now illustrate the role of color mixing for the AL1 potential (to be introduced in Sec. VIII). The energy estimated as a function of M/m without and with color mixing is shown in Fig. 8. The ground state of the $QQ\bar{u}\bar{d}$ that is a candidate for stability, with $J^P = 1^+$, has its main component with color $\bar{3}3$, and spin $\{1,0\}$ in the $QQ - \bar{u}\bar{d}$ basis. The main admixture consists of $6\bar{6}$ with spin $\{1,0\}$ and an antisymmetric orbital wave function of which (31) is a prototype, and of $6\bar{6}$ with spin $\{0,1\}$ with a symmetric orbital wave function.

The relevance of color mixing was also illustrated with realistic models in Table II of Ref. [6] and was stressed by several authors cited in [4], in particular Brink and Stancu.

VIII. SPIN-DEPENDENT CORRECTIONS

In the most advanced calculations of Ref. [4], it was acknowledged that a pure additive interaction such as (1) will not bind $cc\bar{q}\bar{q}$, on the sole basis that this tetraquark configuration
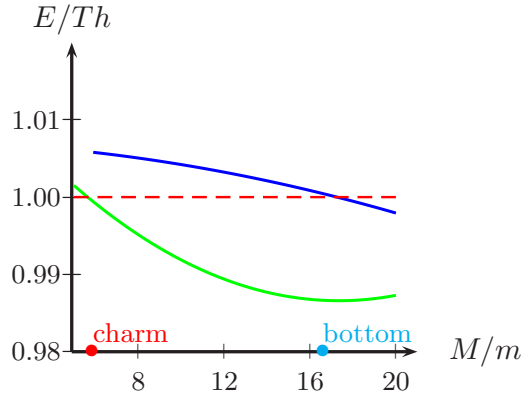
FIG. 9. Effect of the spin-spin interaction of the binding of $QQ\bar{u}\bar{d}$, within the AL1 model. The tetraquark energy is calculated with (green line) and without (blue line) the chromomagnetic term.

benefits from the strong cc chromoelectric attraction that is absent in the $Q\bar{q} + Q\bar{q}$ threshold. In the case where $qq = ud$, however, there is in addition a favorable chromomagnetic interaction in the tetraquark, while the threshold experiences only heavy-light spin-spin interaction, whose strength is suppressed by a factor m/M.

For illustration, we use the potential AL1 by Semay and Silvestre-Brac [25]. Its central part is similar to the Coulomb-plus-linear adopted in Fig. 2. Its spin-spin part is a regularized Breit-Fermi interaction, with a smearing parameter that depends on the reduced mass. More precisely,

$$V_{ij}(r) = -\frac{\kappa}{r} + \lambda r - \Lambda + \frac{2\pi\alpha}{3\,m_i\,m_j} \frac{\exp(-r^2/r_0^2)}{\pi^{3/2}\,r_0^3}\,\boldsymbol{\sigma}_i.\boldsymbol{\sigma}_j,$$

$$r_0(m_i, m_j) = A\left(\frac{2\,m_i\,m_j}{m_i + m_j}\right)^{-B}, \qquad (32)$$

$$m_q = 0.315, \quad m_c = 1.836, \quad m_b = 5.227,$$

$$\Lambda = 0.8321, \quad B = 0.2204, \quad A = 1.6553,$$

$$\kappa = 0.5069 \quad \alpha = 1.8609, \quad \lambda = 0.1653,$$

where the units are appropriate powers of GeV. The results are shown in Fig. 9 for $QQ\bar{u}\bar{d}$, as a function of the mass ratio M/m.

The system $bb\bar{u}\bar{d}$ is barely bound without the spin-spin term, though the mass ratio m_b/m_q is very large. Its acquires its binding energy of the order of 150 MeV when the spin-spin is restored.

The system $cc\bar{u}\bar{d}$ is clearly unbound when the spin-spin interaction is switched off. This is shown here for the AL1 model, but this is true for any realistic interaction, including an early model by Bhaduri et al. [26]. The case of $cc\bar{u}\bar{d}$ is actually remarkable. Here the binding requires both the color mixing of $\bar{3}3$ with $6\bar{6}$, and the spin-spin interaction. Moreover, the binding is so tiny that it cannot be obtained with a simple variational method. One needs either a fully converged expansion on a basis of correlated Gaussians, or a hyperspherical expansion up to a grand orbital momentum $K_{\max}$ of the order of 12. Semay

and Silvestre-Brac, who used their AL1 potential, missed the binding, but their method of systematic expansion on the eigenstates of an harmonic oscillator is not very efficient to account for the short-range correlations, and is abandoned in the latest quark-model calculations. Janc and Rosina were the first to obtain binding with such potentials, and their calculation was checked by Barnea et al. (see [4] for references).

IX. CONCLUSIONS

Let us summarize. The four-body problem of tetraquarks is somewhat delicate, especially for systems at the edge of stability. The analogy with atomic physics is a good guidance to indicate the most favorable configurations in the limit of dominant chromoelectric interaction. However, unlike the positronium molecule, the all-heavy configuration $QQ\bar{Q}\bar{Q}$ is not stable if one adopts a standard quark model and solves the four-body problem correctly.

The method of Gaussian expansion works somewhat well. With most current models, the matrix elements can be estimated analytically and one can study the convergence as a function of the number of terms, and the role of each spin-color configuration entering a given tetraquark state. This is also the case for the hyperspherical expansion.

The mixing of the $\bar{3}3$ and $6\bar{6}$ color configurations is important, especially for states very near the threshold. This mixing occurs by both the spin-independent and the spin-dependent parts of the potential.

Approximations are welcome, especially if they shed some light on the four-body dynamics. The diquark-antidiquark approximation is not supported by a rigorous solution of the four-body problem, but benefits of a stroke of luck, as the erroneous extra attraction introduced in the color $\bar{3}3$ channel is somewhat compensated by the neglect of the coupling to the color $6\bar{6}$ channel. The equality relating $QQ\bar{q}\bar{q}$, QQq, Qqq, and $Q\bar{q}$ works surprisingly well as long as one is restricted to color $\bar{3}3$, but does not account for the attraction provided by color mixing.

On the other hand, for asymmetric configurations ($QQ\bar{q}\bar{q}$), the Born-Oppenheimer method provides a very good approximation, and an interesting insight into the dynamics. It was probed here for a toy model with frozen color, and its extension as to include the coupling of color configurations would deserve some study.

In short, $cc\bar{u}\bar{d}$ with $J^P = 1^+$ is at the edge of binding within current quark models. For this state, all contributions to the binding should be added, in particular the mixing of states with different internal spin and color structure, and in addition, the four-body problem should be solved with extreme accuracy, for instance, by pushing the hyperspherical expansion up to a grand angular momentum $K_{\max} \geqslant 12$.

In comparison, achieving the binding of $bb\bar{u}\bar{d}$ looks easier. Still, with a typical quark model, the stability of the ground state below the threshold cannot be reached if spin effects and color mixing are both neglected. The crucial role of spin effects explains why one does not expect too many states in addition to 1^+ [10].

Needless to say that any improvement of the dynamics would be welcome. In [10], for example, this is done by

including some pion exchange in the light quark sector. A better binding is obtained for $cc\bar{u}\bar{d}$. The presence of multibody components in the interquark potential was discussed, in particular a disconnected or connected string network linking the quarks and antiquarks. This string model provides an attraction that is larger than the pairwise linear interaction $\propto \sum \tilde{\lambda}_i . \tilde{\lambda}_j \, r_{ij}$, provided there is no constraint from the Pauli principle, i.e., that the color wave function can readjust itself freely when the quarks move. This is not the case for $cc\bar{u}\bar{d}$. A good test of that model would be the stability of flavor-asymmetric configurations such as $bc\bar{u}\bar{s}$.

Note added in proof: The excess of attraction due to the pointlike approximation for diquarks was also pointed out by Kiselev *et al.* [27] in the case of doubly heavy baryons.

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