VECTOR BUNDLES AND TIGHT CLOSURE (TRIEST 2023)

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1. Lecture - Hilbert-Kunz theory and vector bundles

1.1. **Hilbert-Kunz theory.** In 1969, Kunz considered first the following function and the corresponding limit [16].

Definition 1.1. Let K denote a field of positive characteristic p, let $K \subseteq R$ be a noetherian ring and let $I \subseteq R$ be an ideal which is primary to some maximal ideal. Then the *Hilbert-Kunz function* is the function

$$\varphi_I \colon \mathbb{N} \longrightarrow \mathbb{N}, e \longmapsto \varphi_I(e) = \text{length}\,(R/I^{[p^e]})\,,$$

where $I^{[p^e]}$ is the extended ideal under the e-th iteration of the Frobenius homomorphism

$$R \longrightarrow R, f \longmapsto f^{p^e}\,.$$

Definition 1.2. Let K denote a field of positive characteristic p, let $K \subseteq R$ be a noetherian ring and let $I \subseteq R$ be an ideal which is primary to some maximal ideal of height d. Then the *Hilbert-Kunz multiplicity* of I is the limit (if it exists)

$$\lim_{e \to \infty} \frac{\text{length}\,(R/I^{[p^e]})}{p^{ed}} = \lim_{e \to \infty} \frac{\varphi_I(e)}{p^{ed}}.$$

The Hilbert-Kunz multiplicity of the maximal ideal of a local noetherian ring R is called the *Hilbert-Kunz multiplicity* of R. The existence of Hilbert-Kunz multiplicity was proven by Monsky [19].

Theorem 1.3. *Let K denote a field of positive characteristic p, let $K \subseteq R$ be a noetherian ring and let $I \subseteq R$ be an ideal which is primary to some maximal ideal. Then the Hilbert-Kunz multiplicity $e_{HK}(I)$ exists and is a positive real number.*

With the help of the Hilbert-Kunz multiplicity of a local noetherian ring one may characterize when R is regular, as the following theorem shows (which was initiated by Kunz in 1969 but finally proven by Watanabe and Yoshida in 2000 [23]).

Theorem 1.4. *Let R be a local noetherian ring of positive characteristic. Then the following hold.*

1. *The Hilbert-Kunz multiplicity of R is $e_{HK}(R) \geq 1$.*
2. *If R is unmixed, then $e_{HK}(R) = 1$ if and only if R is regular.*

1.2. **Vector bundles.** We will have a look at Hilbert-Kunz theory and tight closure (to be introduced in the next lecture) from the viewpoint of vector bundles. To motivate this concept, which exists in algebraic geometry, differential geometry, topology, mathematical physics, we go back to linear algebra. Let K be a field. We consider a system of linear homogeneous equations over K,

$$\begin{aligned} f_{11}t_1 + \cdots + f_{1n}t_n &= 0, \\ f_{21}t_1 + \cdots + f_{2n}t_n &= 0, \\ &\vdots \\ f_{m1}t_1 + \cdots + f_{mn}t_n &= 0, \end{aligned}$$

where the f_{ij} are elements in K. The solution set to this system of homogeneous equations is a vector space V over K (a linear subspace of K^n), its dimension is $n - \mathrm{rk}(A)$, where

$$A = (f_{ij})_{ij}$$

is the matrix given by these elements. Suppose now that X is a geometric object (a topological space, a manifold, a variety, a scheme, the spectrum of a ring) and that instead of elements in the field K we have functions

$$f_{ij} \colon X \longrightarrow K$$

on X (which are continuous, or differentiable, or algebraic). We form the matrix of functions $A = (f_{ij})_{ij}$, which yields for every point $P \in X$ a matrix $A(P)$ over K. Then we get from these data the space

$$V = \left\{ (P; t_1, \ldots, t_n) \mid A(P) \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = 0 \right\} \subseteq X \times K^n$$

together with the projection to X. For a fixed point $P \in X$, the fiber V_P of V over P is the solution space to the corresponding system of homogeneous linear equations given by inserting P into f_{ij}. In particular, all fibers of the map

$$V \longrightarrow X ,$$

are vector spaces (maybe of non-constant dimension). These vector space structures yield an addition[1]

$$V \times_X V \longrightarrow V, (P; s_1, \ldots, s_n; t_1, \ldots, t_n) \longmapsto (P; s_1 + t_1, \ldots, s_n + t_n) ,$$

[1] $V \times_X V$ is the fiber product of $V \to X$ with itself.

(only points in the same fiber can be added). The mapping

$$X \longrightarrow V, P \longmapsto (P; 0, \ldots, 0),$$

is called the *zero-section*. If we have just one equation with functions $f_1, \ldots, f_n$ and if $U \subseteq X$ denotes the open subset where not all f_i vanish, then we get a short exact sequence

$$0 \longrightarrow V|_U \longrightarrow U \times K^n \longrightarrow U \times K \longrightarrow 0.$$

If $D(f_i) \subseteq U$ denotes the locus where f_i does not vanish, then we get a linear isomorphism

$$V|_{D(f_i)} \longrightarrow D(f_i) \times K^{n-1}, (t_1, \ldots, t_n) \longmapsto (t_1, \ldots, t_{i-1}, t_{i+1}, , \ldots, , t_n),$$

as we can reconstruct

$$t_i = -\frac{1}{f_i}(t_1 f_1 + \cdots + t_{i-1} f_{i-1} + t_{i+1} f_{i+1} + \cdots + t_n f_n)$$

from the other variables. This local trivialization also shows that, on the intersection $D(f_i) \cap D(f_j)$, the transformation between two trivializations is linear. So locally, the object $V|_U$ is trivial, but globally it might be complicated.

We now consider the scheme version of a vector bundle and in particular of a syzygy bundle (kernel bundle), trying to keep the idea that we are dealing with objects from linear algebra, but over a varying base. Let R denote a commutative ring, let I denote an ideal and fix generators $I = (f_1, \ldots, f_n)$. This defines a short exact sequence of R-modules

$$0 \longrightarrow \mathrm{Syz}\,(f_1, \ldots, f_n) \longrightarrow R^n \stackrel{f_1, \ldots, f_n}{\longrightarrow} I \longrightarrow 0.$$

The syzygy module $\mathrm{Syz}\,(f_1, \ldots, f_n)$ is not locally free on $X = \mathrm{Spec}\,(R)$. However, on the open subset

$$U = D(I) = \bigcup_{i=1}^{n} D(f_i) \subseteq X$$

defined by the ideal, this module will be locally free, since it is free on each $D(f_i)$, using the same trivialization as above. Moreover, on U we get the short exact sequence

$$0 \longrightarrow \mathrm{Syz}\,(f_1, \ldots, f_n)|_U \longrightarrow \mathcal{O}_U^n \stackrel{f_1, \ldots, f_n}{\longrightarrow} \mathcal{O}_U \longrightarrow 0$$

of coherent sheaves. Later on, I will be an $\mathfrak{m}$-primary ideal in a local noetherian ring R and then

$$U = X \setminus \{\mathfrak{m}\}$$

will be the punctured spectrum of R. The sheaves occurring in the last sequence are locally free in the following sense.

Definition 1.5. A coherent $\mathcal{O}_X$-module $\mathcal{F}$ on a scheme X is called *locally free* of rank r, if there exists an open covering $X = \bigcup_{i \in I} U_i$ and $\mathcal{O}_{U_i}$-module-isomorphisms $\mathcal{F}|_{U_i} \cong (\mathcal{O}_{U_i})^r$ for every $i \in I$.

An equivalent concept of a locally free sheaf is the concept of a geometric vector bundle.

Definition 1.6. Let X denote a scheme. A scheme V equipped with a morphism
$$p\colon V \longrightarrow X$$
is called a *geometric vector bundle* of rank r over X if there exists an open covering $X = \bigcup_{i\in I} U_i$ and U_i-isomorphisms
$$\psi_i\colon U_i \times \mathbb{A}^r = \mathbb{A}^r_{U_i} \longrightarrow V|_{U_i} = p^{-1}(U_i)$$
such that for every open affine subset $U \subseteq U_i \cap U_j$, the transition mappings
$$\psi_j^{-1} \circ \psi_i\colon \mathbb{A}^r_{U_i}|_U \longrightarrow \mathbb{A}^r_{U_j}|_U$$
are linear automorphisms, i.e. they are induced by an automorphism of the polynomial ring $\Gamma(U, \mathcal{O}_X)[T_1, \ldots, T_r]$ given by $T_i \mapsto \sum_{j=1}^r a_{ij} T_j$.

We will work with both concepts and switch between them as needed.

1.3. **The graded case.** We will restrict now to the standard-graded case in order to work on the corresponding projective variety. Let R be a standard-graded normal domain over an algebraically closed field K. Let
$$Y \,=\, \operatorname{Proj}(R)$$
be the corresponding projective variety and let
$$I \,=\, (f_1, \ldots, f_n)$$
be an R_+-primary homogeneous ideal with generators of degrees $d_1, \ldots, d_n$. Then we get on Y the short exact sequence
$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n)(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_Y(m - d_i) \stackrel{f_1,\ldots,f_n}{\longrightarrow} \mathcal{O}_Y(m) \longrightarrow 0\,.$$
Here $\operatorname{Syz}(f_1, \ldots, f_n)(m)$ is a vector bundle, called the *syzygy bundle*, its rank is $n-1$.

Our approach to the computation of the Hilbert-Kunz multiplicity is by using the presenting sequence
$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_Y(-d_i) \stackrel{f_1,\ldots,f_n}{\longrightarrow} \mathcal{O}_Y \longrightarrow 0$$
and twists of its e-th Frobenius pull-backs, that is
$$0 \longrightarrow \operatorname{Syz}(f_1^q, \ldots, f_n^q)(m) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_Y(m - qd_i) \stackrel{f_1^q,\ldots,f_n^q}{\longrightarrow} \mathcal{O}_Y(m) \longrightarrow 0$$
(where $q = p^e$), and to relate the asymptotic behavior of
$$\operatorname{length}(R/I^{[q]}) \,=\, \dim_K (R/I^{[q]}) \,=\, \sum_{m=0}^{\infty} \dim_K (R/I^{[q]})_m$$

to the asymptotic behavior of the global sections of the Frobenius pull-backs

$$(F^{e*}(\mathrm{Syz}\,(f_1,\ldots,f_n))(m) \;=\; \mathrm{Syz}\,(f_1^q,\ldots,f_n^q)(m).$$

What we want to compute is just the cokernel of the complex of global sections of the above sequence, namely

$$\dim_K (R/I^{[q]})_m \;=\; h^0(Y,\mathcal{O}_Y(m)) - \sum_{i=1}^n h^0(Y,\mathcal{O}_Y(m-qd_i)) + h^0(Y,\mathrm{Syz}\,(f_1^q,\ldots,f_n^q)(m)).$$

The summation over m is finite (but the range depends on q), and the terms

$$h^0(Y,\mathcal{O}_Y(m)) \;=\; \dim_K \Gamma(Y,\mathcal{O}_Y(m)) \;=\; \dim_K R_m$$

are easy to control, so we have to understand the behavior of the global syzygies

$$H^0(Y,\mathrm{Syz}\,(f_1^q,\ldots,f_n^q)(m))$$

for all q and m, at least asymptotically. This is a Frobenius-Riemann-Roch problem.

By this translation of Hilbert-Kunz theory into a projective setting, we gain the following.

(1) We work with projective varieties; if we look at local rings with an isolated singularity, we even work on smooth projective varieties.
(2) We work with locally free sheaves. Taking Frobenius pull-backs is then exact. The mentioned Frobenius-Riemann-Roch problem is not specific for syzygy bundles, but should be addressed in general.
(3) We can use the advanced methods of algebraic geometry, like intersection theory, Riemann-Roch theorem, vanishing theorems, ampleness, cohomology, moduli spaces.

This is still a difficult problem in general. However, if the local normal ring has dimension two and the corresponding variety is a smooth projective curve, then our understanding is good enough to solve the main problems from Hilbert-Kunz theory. The main advantages in the curve case compared with higher-dimensional varieties are the following.

(1) The degree of a vector bundle is independent of a polarization.
(2) There are only the 0th and the first cohomology, which are directly related by Serre-duality.
(3) The Riemann-Roch theorem relates these notions.
(4) Semistability gives good criteria for having no global sections.

We introduce these concepts on a smooth projective curve C over an algebraically closed field K.

Definition 1.7. Let C denote a smooth projective curve over an algebraically closed field K. For a locally free sheaf $\mathcal{G}$ on C of rank r we define its *degree* by the degree of the determinant sheaf $\bigwedge^r \mathcal{G}$.

The determinant bundle is an invertible sheaf and corresponds therefore to a Weil divisor, say

$$D = \sum_{P \in C} n_P P,$$

its degree is defined by $\sum_{P \in C} n_P$. The degree of the curve itself is defined as the degree of $\mathcal{O}_C(1)$. The degree of bundles is additive on short exact sequences of locally free sheaves. Applying additivity to

$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n)(m) \longrightarrow \bigoplus_{i=1}^{n} \mathcal{O}_C(m - d_i) \stackrel{f_1,\ldots,f_n}{\longrightarrow} \mathcal{O}_C(m) \longrightarrow 0$$

we get

$$\deg(\operatorname{Syz}(f_1, \ldots, f_n)(m)) = ((n-1)m - \sum_{i=1}^{n} d_i) \deg(C).$$

Definition 1.8. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve C. It is called *semistable*, if $\mu(\mathcal{T}) = \frac{\deg(\mathcal{T})}{\operatorname{rk}(\mathcal{T})} \leq \frac{\deg(\mathcal{S})}{\operatorname{rk}(\mathcal{S})} = \mu(\mathcal{S})$ for all subbundles $\mathcal{T} \subseteq \mathcal{S}$.

Suppose that the base field has positive characteristic $p > 0$. Then $\mathcal{S}$ is called *strongly semistable*, if all (absolute) Frobenius pull-backs $F^{e*}(\mathcal{S})$ are semistable.

The rational number $\mu(\mathcal{S}) = \frac{\deg(\mathcal{S})}{\operatorname{rk}(\mathcal{S})}$ is called the *slope* of a vector bundle. An important property of a semistable bundle of negative degree is that it can not have any global section $\neq 0$. The semistable bundles are those for which there exists a moduli space.

Example 1.9. Let $R = K[x, y, z]/(x^3 + y^3 + z^3)$, where K is a field of positive characteristic $p \neq 3$, $I = (x^2, y^2, z^2)$, and

$$C = \operatorname{Proj}(R).$$

The equation $x^3 + y^3 + z^3 = 0$ yields the short exact sequence

$$0 \longrightarrow \mathcal{O}_C \longrightarrow \operatorname{Syz}\left(x^2, y^2, z^2\right)(3) \longrightarrow \mathcal{O}_C \longrightarrow 0\,.$$

This shows that $\operatorname{Syz}(x^2, y^2, z^2)$ is strongly semistable.

Example 1.10. Let C be the smooth Fermat quartic given by $x^4 + y^4 + z^4$, and consider on it the syzygy bundle $\operatorname{Syz}(x, y, z)$ (which is also the restricted cotangent bundle from the projective plane). This bundle is semistable. Suppose that the characteristic is 3. Then its Frobenius pull-back is $\operatorname{Syz}(x^3, y^3, z^3)$. The curve equation gives a global non-trivial section of this bundle of total degree 4. But the degree of $\operatorname{Syz}(x^3, y^3, z^3)(4)$ is negative, hence it can not be semistable anymore.

Definition 1.11. Let $\mathcal{S}$ be a vector bundle on a smooth projective curve C over an algebraically closed field K. Then the (uniquely determined) filtration

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = \mathcal{S}$$

of subbundles such that all quotient bundles $\mathcal{S}_k/\mathcal{S}_{k-1}$ are semistable with decreasing slopes $\mu_k = \mu(\mathcal{S}_k/\mathcal{S}_{k-1})$, is called the *Harder-Narasimhan filtration* of $\mathcal{S}$.

Theorem 1.12. *Let C denote a smooth projective curve over an algebraically closed field of positive characteristic p, and let $\mathcal{S}$ be a vector bundle on C. Then there exists a natural number $e \in \mathbb{N}$ such that the Harder-Narasimhan filtration of the eth Frobenius pull-back $F^{e*}(\mathcal{S})$, say*

$$0 = \mathcal{S}_0 \subset \mathcal{S}_1 \subset \ldots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = F^{e*}(\mathcal{S})$$

has the property that the quotients $\mathcal{S}_k/\mathcal{S}_{k-1}$ are strongly semistable.

This theorem is due to A. Langer [18] and holds also in higher dimension. An immediate consequence of this is that the Harder-Narasimhan filtration of all higher Frobenius pull-backs are just the pull-backs of this filtration. With these filtrations we can at least Frobenius-asymptotically control the global sections of the pull-backs and hence also the Hilbert-Kunz multiplicity. This implies the following theorem [3], [22].

Theorem 1.13. *Let R be a two-dimensional standard-graded normal domain over an algebraically closed field of positive characteristic. Let $I = (f_1, \ldots, f_n)$ be a homogeneous R_+-primary ideal with homogeneous generators of degree d_i. Let $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)$ be the syzygy bundle on $C = \operatorname{Proj}(R)$ and suppose that the Harder-Narasimhan filtration of $F^{e*}(\mathcal{S})$ is strong, and let μ_k, $k = 1, \ldots, t$, be the corresponding slopes. We set $\nu_k = \frac{-\mu_k}{\deg(C)p^e}$ and $r_k = \operatorname{rk}(\mathcal{S}_k/\mathcal{S}_{k-1})$. Then the Hilbert-Kunz multiplicity of I is*

$$e_{HK}(I) = \frac{\deg(C)}{2}\left(\sum_{k=1}^{t} r_k \nu_k^2 - \sum_{i=1}^{n} d_i^2\right).$$

In particular, it is a rational number.

Corollary 1.14. Let $R = K[x, y, z]/(H)$ be a normal homogeneous hypersurface domain of dimension two and degree δ over an algebraically closed field of positive characteristic. Then there exists a rational number ν_2, $\frac{3}{2} \leq \nu_2 \leq 2$, such that the Hilbert-Kunz multiplicity of R is

$$e_{HK}(R) = \delta\big(\nu_2^2 - 3\nu_2 + 3\big).$$

2. Lecture - Tight closure and torsors

2.1. Tight closure.

Definition 2.1. Let R be a noetherian domain of positive characteristic and let $I \subseteq R$ be an ideal. The *tight closure* of I is the ideal

$$I^* = \left\{ f \in R \mid \text{ there exists } z \neq 0 \text{ such that } zf^q \in I^{[q]} \text{ for all } q = p^e \right\}.$$

This theory was introduced by M. Hochster and C. Huneke (see [12], [13], [14], [15]). There is a direct relation between Hilbert-Kunz multiplicity and tight closure.

Theorem 2.2. *Let $(R, \mathfrak{m})$ be an analytically unramified and formally equidimensional local noetherian ring of positive characteristic, let $I \subseteq R$ be an $\mathfrak{m}$-primary ideal. Let $f \in R$. Then*

$$f \in I^* \text{ if and only if } e_{HK}((I, f)) = e_{HK}(I)\,.$$

We try to understand tight closure from the perspective of bundles and will have again a look at the syzygy bundle. Let R denote a noetherian normal domain and let $I = (f_1, \ldots, f_n)$ denote an ideal of height I at least 2 (think of a local normal domain of dimension at least 2 and an $\mathfrak{m}$-primary ideal I, or the graded version of this). Let $U = D(I) \subseteq \operatorname{Spec}(R)$ and consider again the short exact sequence

$$0 \longrightarrow \operatorname{Syz}(f_1, \ldots, f_n)|_U \longrightarrow \mathcal{O}_U^n \stackrel{f_1, \ldots, f_n}{\longrightarrow} \mathcal{O}_U \longrightarrow 0$$

of locally free sheaves on U. Another element $f \in R = \Gamma(U, \mathcal{O}_U)$ (because of the height condition) defines via the long exact sequence of cohomology the cohomology class $c = \delta(f) \in H^1(U, \operatorname{Syz}(f_1, \ldots, f_n))$. When R contains a field of positive characteristic, we try to understand tight closure in terms of this cohomology class. Quite directly, we have the eth absolute Frobenius on U. As the sheaves are locally free, we have

$$F^{e*}(\operatorname{Syz}(f_1, \ldots, f_n)) = \operatorname{Syz}(f_1^q, \ldots, f_n^q),$$

and the eth Frobenius pull-back of the cohomology class is

$$F^{e*}(c) \in H^1(D(I), F^{e*}(\operatorname{Syz}(f_1, \ldots, f_n)) \cong H^1(D(I), \operatorname{Syz}(f_1^q, \ldots, f_n^q))$$

($q = p^e$), and this is the cohomology class corresponding to f^q. By the height assumption, we have $zF^{e*}(c) = 0$ if and only if $zf^q \in (f_1^q, \ldots, f_n^q)$, and this holds for all e if and only if $f \in I^*$ by definition. This shows already that under the given conditions, tight closure does only depend on the cohomology class. In the graded case, we can also translate the tight closure question $f \in I^*$ for homogeneous data into the question whether the corresponding cohomology class $c \in H^1(Y, \operatorname{Syz}(f_1, \ldots, f_n)(m))$ on $\operatorname{Proj}(R)$ is tightly zero in the sense that $zF^{e*}(c) = 0$ holds for some homogenous $z \neq 0$ (z considered inside some ample invertible sheaf $\mathcal{O}_Y(\ell)$). This property of being tightly zero is relevant for every cohomology class in any locally free sheaf. Here, this translation is in particular helpful for inclusion results. For exclusion results we have to go another way and consider torsors.

2.2. **Torsors and forcing algebras.** We come back to the situation of a system of linear homogeneous equations over a field K with which we tried to motivate the concept of a vector bundle. However, we now consider a system of linear inhomogeneous equations,

$$\begin{aligned} f_{11}t_1 + \cdots + f_{1n}t_n &= f_1, \\ f_{21}t_1 + \cdots + f_{2n}t_n &= f_2, \\ &\vdots \\ f_{m1}t_1 + \cdots + f_{mn}t_n &= f_m. \end{aligned}$$

The solution set T of this inhomogeneous system may be empty, but nevertheless it is tightly related to the solution space of the homogeneous system. First of all, there exists an action

$$V \times T \longrightarrow T, (v,t) \longmapsto v + t\,,$$

because the sum of a solution of the homogeneous system and of a solution of the inhomogeneous system is again a solution of the inhomogeneous system. This action is a group action of the group $(V, +, 0)$ on the set T. Moreover, if we fix one solution $t_0 \in T$ (supposing that at least one solution exists), then there exists a bijection

$$V \longrightarrow T, v \longmapsto v + t_0\,.$$

This means that the group V acts simply transitive on T, and so T can be identified with the vector space V, however not in a canonical way.

Suppose now that X is a geometric object and we have functions

$$f_{ij}, f_i \colon X \longrightarrow K$$

on X (which are continuous, or differentiable, or algebraic). As before, we get for the f_{ij} a bundle with an addition and such that the fibers are vector spaces.

Then we can form the set

$$T = \left\{ (P; t_1, \ldots, t_n) \mid A(P) \begin{pmatrix} t_1 \\ \vdots \\ t_n \end{pmatrix} = \begin{pmatrix} f_1(P) \\ \vdots \\ f_n(P) \end{pmatrix} \right\} \subseteq X \times K^n$$

with the projection to X. Again, every fiber T_P of T over a point $P \in X$ is the solution set to the system of inhomogeneous linear equations which arises by inserting P into f_{ij} and f_i. The actions of the fibers V_P on T_P (coming from linear algebra) extend to an action

$$V \times_X T \longrightarrow T, (P; t_1, \ldots, t_n; s_1, \ldots, s_n) \longmapsto (P; t_1 + s_1, \ldots, t_n + s_n)\,.$$

Also, if a (continuous, differentiable, algebraic) map

$$s\colon X \longrightarrow T$$

with $s(P) \in T_P$ exists, then we can construct a (continuous, differentiable, algebraic) isomorphism between V and T. However, different from the situation in linear algebra (which corresponds to the situation where X is just one point), such a section does rarely exist.

These objects T have new and sometimes difficult global properties which we try to understand. We will work mainly in an algebraic setting and restrict to the situation where just one equation

$$f_1T_1 + \cdots + f_nT_n = f$$

is given. Then in the homogeneous case ($f = 0$) the fibers are vector spaces of dimension $n-1$ or n, and the later holds exactly for the points $P \in X$ where $f_1(P) = \ldots = f_n(P) = 0$. In the inhomogeneous case the fibers are either empty or of dimension $n-1$ or n. We give a typical example.

Example 2.3. Let X denote a plane (like $K^2, \mathbb{R}^2, \mathbb{A}^2_K$) with coordinate functions x and y. We consider an inhomogeneous linear equation of type

$$x^a t_1 + y^b t_2 = x^c y^d.$$

The fiber of the solution set T over a point $\neq (0,0)$ is one-dimensional, whereas the fiber over $(0,0)$ has dimension two (for $a,b,c,d \geq 1$). Many properties of T depend on these four exponents.

In (most of) these example, we can observe the following behavior. On an open subset, the dimension of the fibers is constant and equals $n-1$, whereas the fiber over some special points degenerates to an n-dimensional solution set (or becomes empty).

Definition 2.4. Let V denote a geometric vector bundle over a scheme X. A scheme $T \to X$ together with an action

$$\beta\colon V \times_X T \longrightarrow T$$

is called a geometric (Zariski)-*torsor* for V (or a *principal fiber bundle* or a *principal homogeneous space*) if there exists an open covering $X = \bigcup_{i \in I} U_i$ and isomorphisms

$$\varphi_i\colon T|_{U_i} \longrightarrow V|_{U_i}$$

such that the diagrams (we set $U = U_i$ and $\varphi = \varphi_i$)

$$\begin{array}{ccc} V|_U \times_U T|_U & \stackrel{\beta}{\longrightarrow} & T|_U \\ \mathrm{Id} \times \varphi \downarrow & & \downarrow \varphi \\ V|_U \times_U V|_U & \stackrel{\alpha}{\longrightarrow} & V|_U \end{array}$$

commute, where α is the addition on the vector bundle.

The torsors of vector bundles can be classified in the following way.

Proposition 2.5. *Let X denote a noetherian separated scheme and let*

$$p \colon V \longrightarrow X$$

denote a geometric vector bundle on X with sheaf of sections $\mathcal{S}$. Then there exists a correspondence between first cohomology classes $c \in H^1(X, \mathcal{S})$ and geometric V-torsors.

Remark 2.6. Let $\mathcal{S}$ denote a locally free sheaf on a scheme X. For a cohomology class $c \in H^1(X, \mathcal{S})$ one can construct a geometric object: Because of $H^1(X, \mathcal{S}) \cong \operatorname{Ext}^1(\mathcal{O}_X, \mathcal{S})$, the class defines an extension

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}' \longrightarrow \mathcal{O}_X \longrightarrow 0\,.$$

This extension is such that under the connecting homomorphism of cohomology, $1 \in \Gamma(X, \mathcal{O}_X)$ is sent to $c \in H^1(X, \mathcal{S})$. The extension yields a projective subbundle[2]

$$\mathbb{P}(\mathcal{S}^\vee) \subset \mathbb{P}(\mathcal{S}'^\vee).$$

If V is the corresponding geometric vector bundle of $\mathcal{S}$, one may think of $\mathbb{P}(\mathcal{S}^\vee)$ as $\mathbb{P}(V)$ which consists for every base point $x \in X$ of all the lines in the fiber V_x passing through the origin. The projective subbundle $\mathbb{P}(V)$ has codimension one inside $\mathbb{P}(V')$, for every point it is a projective space lying (linearly) inside a projective space of one dimension higher. The complement is then over every point an affine space. One can show that the global complement

$$T = \mathbb{P}(\mathcal{S}'^\vee) \setminus \mathbb{P}(\mathcal{S}^\vee)$$

is another model for the torsor given by the cohomology class. The advantage of this viewpoint is that we may work, in particular when X is projective, in an entirely projective setting.

Within the algebraic setting, torsors can also be realized as open subsets of spectra of forcing algebras.

Definition 2.7. Let R be a commutative ring and let $f_1, \ldots, f_n$ and f be elements in R. Then the R-algebra

$$R[T_1, \ldots, T_n]/(f_1T_1 + \cdots + f_nT_n - f)$$

is called the *forcing algebra* of these elements (or these data).

Theorem 2.8. *Let R denote a noetherian ring, let $I = (f_1, \ldots, f_n)$ denote an ideal and let $f \in R$ be another element. Let $c = \delta(f) \in H^1(D(I), \operatorname{Syz}(f_1, \ldots, f_n))$ be the corresponding cohomology class and let*

$$B = R[T_1, \ldots, T_n]/(f_1T_1 + \cdots + f_nT_n - f)$$

[2] $\mathcal{S}^\vee$ denotes the dual bundle. According to our convention, the geometric vector bundle corresponding to a locally free sheaf $\mathcal{T}$ is given by $\operatorname{Spec}\left(\oplus_{k \geq 0} S^k(\mathcal{T})\right)$ and the projective bundle is $\operatorname{Proj}\left(\oplus_{k \geq 0} S^k(\mathcal{T})\right)$, where S^k denotes the kth symmetric power.

denote the forcing algebra for these data. Then the scheme $\operatorname{Spec}(B)|_{D(I)}$ *together with the natural action of the syzygy bundle on it is isomorphic to the torsor given by c.*

Forcing algebras provide a natural framework for closure operations in general, it is however a special feature of tight closure that the induced torsor contains the relevant information.

2.3. **Tight closure and solid closure.** Forcing algebras occurred in the work of Hochster on solid closure. The following theorem of Hochster [11, Theorem 8.6] gives a characterization of tight closure in terms of forcing algebra and local cohomology.

Theorem 2.9. *Let R be a normal excellent local domain with maximal ideal* $\mathfrak{m}$ *over a field of positive characteristic. Let* $f_1, \ldots, f_n$ *generate an* $\mathfrak{m}$*-primary ideal I and let f be another element in R. Then* $f \in I^*$ *if and only if* $H^{\dim(R)}_{\mathfrak{m}}(A) \neq 0$, *where* $A = R[T_1, \ldots, T_n]/(f_1T_1 + \cdots + f_nT_n + f)$ *denotes the forcing algebra of these elements.*

If the dimension d is at least two, then

$$H^d_{\mathfrak{m}}(R) \longrightarrow H^d_{\mathfrak{m}}(B) \cong H^d_{\mathfrak{m}B}(B) \cong H^{d-1}(D(\mathfrak{m}B), \mathcal{O}_B)\,.$$

This means that we have to look at the cohomological properties of the complement of the exceptional fiber over the closed point, i.e. the torsor given by these data. If $H^{d-1}(D(\mathfrak{m}B), \mathcal{O}_B) = 0$ then this is true for all quasicoherent sheaves instead of just the structure sheaf. This property can be expressed by saying that the *cohomological dimension* of $D(\mathfrak{m}B)$ is $\leq d-2$ and thus smaller than the cohomological dimension of the punctured spectrum $D(\mathfrak{m})$, which is exactly $d-1$. So belonging to tight closure can be rephrased by saying that the formation of the corresponding torsor does not change the cohomological dimension.

If the dimension is two, then we have to look whether the first cohomology of the structure sheaf vanishes. This is true (by Serre's cohomological criterion for affineness) if and only if the open subset $D(\mathfrak{m}B)$ is an *affine scheme* (the spectrum of a ring).

The right hand side of the equivalence in Theorem 2.9 (the non-vanishing of the top-dimensional local cohomology) is independent of any characteristic assumption, and can be taken as the basis for the definition of another closure operation, called *solid closure*. So the theorem above says that in positive characteristic, tight closure and solid closure coincide. There is also a definition of tight closure for algebras over a field of characteristic 0 by reduction to positive characteristic.

2.4. **The graded two-dimensional case.** In the situation of a forcing algebra of homogeneous elements, this torsor T can also be obtained as $\operatorname{Proj}(B)$, where B is the (not necessarily positively) graded forcing algebra. In particular, it follows that the containment $f \in I^*$ is equivalent to the property that T is not an affine variety. For this properties, positivity (ampleness) properties of the syzygy bundle are crucial. We need again the concept of semistability introduced in the first lecture.

For a strongly semistable vector bundle $\mathcal{S}$ on C and a cohomology class $c \in H^1(C, \mathcal{S})$ with corresponding torsor we obtain the following affineness criterion.

Theorem 2.10. *Let C denote a smooth projective curve over an algebraically closed field K and let $\mathcal{S}$ be a strongly semistable vector bundle over C together with a cohomology class $c \in H^1(C, \mathcal{S})$. Then the torsor $T(c)$ is an affine scheme if and only if $\deg(\mathcal{S}) < 0$ and $c \neq 0$ ($F^e(c) \neq 0$ for all e in positive characteristic[3]).*

This result rests on the ampleness of $\mathcal{S}'^\vee$ occuring in the dual exact sequence $0 \to \mathcal{O}_C \to \mathcal{S}'^\vee \to \mathcal{S}^\vee \to 0$ given by c (this rests on work of Gieseker and Hartshorne(see [9], [10]). It implies for a strongly semistable syzygy bundle the following *degree formula* for tight closure.

Theorem 2.11. *Suppose that $\operatorname{Syz}(f_1, \ldots, f_n)$ is strongly semistable. Then $R_m \subseteq I^*$ for $m \geq \frac{\sum d_i}{n-1}$ and (for almost all prime numbers) $R_m \cap I^* \subseteq I$ for $m < \frac{\sum d_i}{n-1}$.*

If we take on the right hand side I^F, the *Frobenius closure* of the ideal, instead of I, then this statement is true for all characteristics. As stated, it is true in a relative setting for p large enough.

We indicate the proof of the inclusion result. The degree condition implies that $c \in \delta(f) = H^1(C, \mathcal{S})$ is such that $\mathcal{S} = \operatorname{Syz}(f_1, \ldots, f_n)(m)$ has non-negative degree. Then also all Frobenius pull-backs $F^*(\mathcal{S})$ have non-negative degree. Let $\mathcal{L} = \mathcal{O}(k)$ be a twist of the tautological line bundle on C such that its degree is larger than the degree of ω_C^{-1}, the dual of the canonical sheaf. Let $z \in H^0(Y, \mathcal{L})$ be a non-zero element. Then $zF^{e*}(c) \in H^1(C, F^{e*}(\mathcal{S}) \otimes \mathcal{L})$, and by Serre duality we have

$$H^1(C, F^{e*}(\mathcal{S}) \otimes \mathcal{L}) \cong H^0(F^{e*}(\mathcal{S}^\vee) \otimes \mathcal{L}^{-1} \otimes \omega_C)^\vee.$$

On the right hand side we have a semistable sheaf of negative degree, which can not have a non-trivial section. Hence

$$zF^{e*}(c) = 0,$$

and therefore f belongs to the tight closure.

In general, there exists an exact criterion for the affineness of the torsor $T(c)$ depending on c and the *strong Harder-Narasimhan filtration* of $\mathcal{S}$.

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[3]Here one has to check only finitely many es and there exist good estimates how far one has to go. Also, in a relative situation, this is only an extra condition for finitely many prime numbers.

Theorem 2.12. *Let C denote a smooth projective curve over an algebraically closed field K and let $\mathcal{S}$ be a vector bundle over C together with a cohomology class $c \in H^1(C, \mathcal{S})$. Let*

$$\mathcal{S}_1 \subset \mathcal{S}_2 \subset \ldots \subset \mathcal{S}_{t-1} \subset \mathcal{S}_t = F^{e*}(\mathcal{S})$$

be a strong Harder-Narasimhan filtration. We choose i such that $\mathcal{S}_i/\mathcal{S}_{i-1}$ has degree ≥ 0 and that $\mathcal{S}_{i+1}/\mathcal{S}_i$ has degree < 0. We set $\mathcal{Q} = F^{e*}(\mathcal{S})/\mathcal{S}_i$. Then the following are equivalent.

1. *The torsor $T(c)$ is not an affine scheme.*
2. *Some Frobenius power of the image of $F^{e*}(c)$ inside $H^1(X, \mathcal{Q})$ is 0.*

3. Lecture - Plus closure and trivializable bundles

3.1. Plus closure.

For an ideal $I \subseteq R$ in a domain R define its *plus closure* by

$$I^+ = \{f \in R \mid \text{there exists a finite domain extension } R \subseteq T \text{ such that } f \in IT\}.$$

Equivalent: Let R^+ be the *absolute integral closure* of R. This is the integral closure of R in an algebraic closure of the quotient field $Q(R)$ (first considered by Artin [1]). Then

$$f \in I^+ \text{ if and only if } f \in IR^+ .$$

The plus closure commutes with localization.

We also have the inclusion $I^+ \subseteq I^*$. Here the question arises:

Question: Is $I^+ = I^*$?

This question is known as the *tantalizing question* in tight closure theory.

In terms of forcing algebras and their torsors, the containment inside the plus closure means that there exists a d-dimensional closed subscheme inside the torsor which meets the exceptional fiber (the fiber over the maximal ideal) in isolated points, and this means that the so-called superheight of the extended ideal is d. In this case the local cohomological dimension of the torsor must be d as well, since it contains a closed subscheme with this cohomological dimension. So also the plus closure depends only on the torsor.

In characteristic zero, the plus closure behaves very differently compared with positive characteristic. If R is a normal domain of characteristic 0, then the trace map shows that the plus closure is trivial, $I^+ = I$ for every ideal I.

3.2. Plus closure in dimension two.

Let K be a field and let R be a normal two-dimensional standard-graded domain over K with corresponding smooth projective curve C. A homogeneous $\mathfrak{m}$-primary ideal with homogeneous ideal generators $f_1, \ldots, f_n$ and another homogeneous element f of degree m yield a cohomology class

$$c = \delta(f) = H^1(C, \mathrm{Syz}\,(f_1, \ldots, f_n)(m)).$$

Let $T(c)$ be the corresponding torsor. We have seen that the affineness of this torsor over C is equivalent to the affineness of the corresponding torsor over $D(\mathfrak{m}) \subseteq \mathrm{Spec}\,(R)$ (and to the property of not belonging to the tight closure). Now we want to understand what the property $f \in I^+$ means for c and for $T(c)$. Instead of the plus closure we will work with the *graded plus closure* $I^{+\mathrm{gr}}$, where $f \in I^{+\mathrm{gr}}$ holds if and only if there exists a finite graded extension $R \subseteq S$ such that $f \in IS$. The existence of such an S translates into the existence of a finite morphism

$$\varphi \colon C' = \mathrm{Proj}\,(S) \longrightarrow \mathrm{Proj}\,(R) = C$$

such that $\varphi^*(c) = 0$. Here we may assume that C' is also smooth. Therefore, we discuss the more general question when a cohomology class $c \in H^1(C, \mathcal{S})$, where $\mathcal{S}$ is a locally free sheaf on C, can be annihilated by a finite morphism

$$C' \longrightarrow C$$

of smooth projective curves. The advantage of this more general approach is that we may work with short exact sequences (in particular, the sequences coming from the Harder-Narasimhan filtration) in order to reduce the problem to semistable bundles which do not necessarily come from an ideal situation.

Lemma 3.1. *Let C denote a smooth projective curve over an algebraically closed field K, let $\mathcal{S}$ be a locally free sheaf on C and let $c \in H^1(C, \mathcal{S})$ be a cohomology class with corresponding torsor $T \to C$. Then the following conditions are equivalent.*

(1) *There exists a finite morphism*

$$\varphi \colon C' \longrightarrow C$$

from a smooth projective curve C' such that $\varphi^*(c) = 0$.

(2) *There exists a projective curve $Z \subseteq T$.*

Proof. If (1) holds, then the pull-back $\varphi^*(T) = T \times_C C'$ is trivial (as a torsor), as it equals the torsor given by $\varphi^*(c) = 0$. Hence $\varphi^*(T)$ is isomorphic to a vector bundle and contains in particular a copy of C'. The image Z of this copy is a projective curve inside T.

If (2) holds, then let C' be the normalization of Z. Since Z dominates C, the resulting morphism

$$\varphi \colon C' \longrightarrow C$$

is finite. Since this morphism factors through T and since T annihilates the cohomology class by which it is defined, it follows that $\varphi^*(c) = 0$. □

We want to show that the cohomological criterion for (non)-affineness of a torsor along the Harder-Narasimhan filtration of the vector bundle also holds for the existence of projective curves inside the torsor, under the condition that the projective curve is defined over a finite field. This implies that tight closure is (graded) plus closure for graded $\mathfrak{m}$-primary ideals in a two-dimensional graded domain over a finite field.

3.3. **Annihilation of cohomology classes of strongly semistable sheaves.** We deal first with the situation of a strongly semistable sheaf $\mathcal{S}$ of degree 0. The following two results are due to Lange and Stuhler [17]. We say that a locally free sheaf is *étale trivializable* if there exists a finite étale morphism $\varphi\colon C' \to C$ such that $\varphi^*(\mathcal{S}) \cong \mathcal{O}_{C'}^r$. Such bundles are directly related to linear representations of the étale fundamental group.

Lemma 3.2. *Let K denote a finite field (or the algebraic closure of a finite field) and let X be a smooth projective curve over K. Let $\mathcal{S}$ be a locally free sheaf over X. Then $\mathcal{S}$ is étale trivializable if and only if there exists some n such that $F^{n*}\mathcal{S} \cong \mathcal{S}$.*

Theorem 3.3. *Let K denote a finite field (or the algebraic closure of a finite field) and let X be a smooth projective curve over K. Let $\mathcal{S}$ be a strongly semistable locally free sheaf over X of degree 0. Then there exists a finite morphism*

$$\varphi\colon Y \longrightarrow X$$

such that $\varphi^*(\mathcal{S})$ is trivial.

Proof. We consider the family of locally free sheaves $F^{e*}(\mathcal{S})$, $e \in \mathbb{N}$. Because these are all semistable of degree 0, and defined over the same finite field, we must have (by the existence of the moduli space for vector bundles) a repetition, i.e.

$$F^{e*}(\mathcal{S}) \cong F^{e'*}(\mathcal{S})$$

for some $e' > e$. By Lemma 3.2, the bundle $F^{e*}(\mathcal{S})$ admits an étale trivialization $\varphi\colon Y \to X$. Hence the finite map $F^e \circ \varphi$ trivializes the bundle. $\square$

Theorem 3.4. *Let K denote a finite field (or the algebraic closure of a finite field) and let X be a smooth projective curve over K. Let $\mathcal{S}$ be a strongly semistable locally free sheaf over X of nonnegative degree and let $c \in H^1(X, \mathcal{S})$ denote a cohomology class. Then there exists a finite morphism*

$$\varphi\colon Y \longrightarrow X$$

such that $\varphi^*(c)$ is trivial.

Proof. If the degree of $\mathcal{S}$ is positive, then a Frobenius pull-back $F^{e*}(\mathcal{S})$ has arbitrary large degree and is still semistable. By Serre duality we get that $H^1(X, F^{e*}(\mathcal{S})) = 0$. So in this case we can annihilate the class by an iteration of the Frobenius alone.

So suppose that the degree is 0. Then there exists by Theorem 3.3 a finite morphism which trivializes the bundle. So we may assume that $\mathcal{S} \cong \mathcal{O}_X^r$. Then the cohomology class has several components $c_i \in H^1(X, \mathcal{O}_X)$ and it is enough to annihilate them separately by finite morphisms. But this is possible by the parameter theorem of K. Smith [21] (or directly using Frobenius and Artin-Schreier extensions). □

3.4. **The general case.** We look now at an arbitrary locally free sheaf $\mathcal{S}$ on C, a smooth projective curve over a finite field. We want to show that the same numerical criterion (formulated in terms of the Harder-Narasimhan filtration) for non-affineness of a torsor holds also for the finite annihilation of the corresponding cohomomology class (or the existence of a projective curve inside the torsor).

Theorem 3.5. *Let K denote a finite field (or the algebraic closure of a finite field) and let X be a smooth projective curve over K. Let $\mathcal{S}$ be a locally free sheaf over X and let $c \in H^1(X, \mathcal{S})$ denote a cohomology class. Let $\mathcal{S}_1 \subset \ldots \subset \mathcal{S}_t$ be a strong Harder-Narasimhan filtration of $F^{e*}(\mathcal{S})$. We choose i such that $\mathcal{S}_i/\mathcal{S}_{i-1}$ has degree ≥ 0 and that $\mathcal{S}_{i+1}/\mathcal{S}_i$ has degree < 0. We set $\mathcal{Q} = F^{e*}(\mathcal{S})/\mathcal{S}_i$. Then the following are equivalent.*

1. *The class c can be annihilated by a finite morphism.*
2. *Some Frobenius power of the image of $F^{e*}(c)$ inside $H^1(X, \mathcal{Q})$ is 0.*

Proof. Suppose that (1) holds. Then the torsor is not affine and hence by Theorem 2.12 also (2) holds.

So suppose that (2) is true. By applying a certain power of the Frobenius, we may assume that the image of the cohomology class in $\mathcal{Q}$ is 0. Hence the class stems from a cohomology class $c_i \in H^1(X, \mathcal{S}_i)$. We look at the short exact sequence

$$0 \longrightarrow \mathcal{S}_{i-1} \longrightarrow \mathcal{S}_i \longrightarrow \mathcal{S}_i/\mathcal{S}_{i-1} \longrightarrow 0\,,$$

where the sheaf on the right hand side has a nonnegative degree. Therefore the image of c_i in $H^1(X, \mathcal{S}_i/\mathcal{S}_{i-1})$ can be annihilated by a finite morphism due to Theorem 3.4. Hence, after applying a finite morphism, we may assume that c_i stems from a cohomology class $c_{i-1} \in H^1(X, \mathcal{S}_{i-1})$. Going on inductively we see that c can be annihilated by a finite morphism. □

Theorem 3.6. *Let C denote a smooth projective curve over the algebraic closure of a finite field K, let $\mathcal{S}$ be a locally free sheaf on C and let $c \in H^1(C, \mathcal{S})$ be a cohomology class with corresponding torsor $T \to C$. Then T is affine if and only if it does not contain any projective curve.*

Proof. Due to Theorem 2.12 and Theorem 3.5, for both properties the same numerical criterion does hold. □

These results imply the following theorem in the setting of a two-dimensional graded ring.

Theorem 3.7. *Let R be a standard-graded, two-dimensional normal domain over (the algebraic closure of) a finite field. Let I be an R_+-primary graded ideal. Then*

$$I^* = I^+ .$$

This is also true for non-primary graded ideals and also for submodules in finitely generated graded submodules. Moreover, G. Dietz [8] has shown that one can get rid also of the graded assumption (of the ideal or module, but not of the ring).

4. Lecture - Deformations and localization problem

After having understood tight closure and plus closure in the two-dimensional situation we proceed to a special three-dimensional situation, namely families of two-dimensional rings parametrized by a one-dimensional base scheme.

4.1. **Affineness under deformations.** We consider a base scheme B and a morphism

$$Z \longrightarrow B$$

together with an open subscheme $W \subseteq Z$. For every base point $b \in B$ we get the open subset

$$W_b \subseteq Z_b$$

inside the fiber Z_b. It is a natural question to ask how properties of W_b vary with b. In particular, we may ask how the cohomological dimension of W_b varies and how the affineness (the cohomological dimension of a scheme X is the maximal number i such that $H^i(X, \mathcal{F}) \neq 0$ for some quasicoherent sheaf $\mathcal{F}$. A noetherian scheme is affine if and only if its cohomological dimension is 0. Tight closure can be characterized by the cohomological dimension of torsors) may vary.

In the algebraic setting, we have a commutative K-algebra D, a commutative D-algebra S and an ideal $\mathfrak{a} \subseteq S$ (so $B = \operatorname{Spec}(D)$ $Z = \operatorname{Spec}(S)$ and $W = D(\mathfrak{a})$) which defines for every prime ideal $\mathfrak{p} \in \operatorname{Spec}(D)$ the extended ideal $\mathfrak{a}_{\mathfrak{p}}$ in $S \otimes_D \kappa(\mathfrak{p})$. Then in this situation, $D(\mathfrak{a}_{\mathfrak{p}}) \subseteq \operatorname{Spec}(S \otimes_D \kappa(\mathfrak{p}))$ is the fiber over $\mathfrak{p}$.

This question is already interesting when $B = \operatorname{Spec}(D)$ is an affine one-dimensional integral scheme, in particular in the following two situations.

(1) $B = \operatorname{Spec}(\mathbb{Z})$. Then we speak of an *arithmetic deformation* and want to know how affineness varies with the characteristic and what the relation is to characteristic zero.

(2) $B = \mathbb{A}^1_K = \operatorname{Spec}(K[t])$, where K is a field. Then we speak of a *geometric deformation* and want to know how affineness varies with the parameter t, in particular how the behavior over the special points where the residue class field is algebraic over K is related to the behavior over the generic point.

It is fairly easy to show that if the open subset in the generic fiber is affine, then also the open subsets are affine for almost all special points.

We deal with this question where W is a torsor over a family of smooth projective curves (or a torsor over a punctured two-dimensional spectrum). The arithmetic as well as the geometric variant of this question are directly related to questions in tight closure theory. Because of the above mentioned degree criteria in the strongly semistable case (see Theorem 2.11), a weird behavior of the affineness property of torsors is only possible if we have a weird behavior of strong semistability.

4.2. **Arithmetic deformations.** We start with the arithmetic situation, the following example is due to Brenner and Katzman [6].

Example 4.1. Consider $\mathbb{Z}[x,y,z]/(x^7+y^7+z^7)$ and take the ideal $I = (x^4,y^4,z^4)$ and the element $f = x^3y^3$. Consider reductions $\mathbb{Z} \to \mathbb{Z}/(p)$. Then

$$f \in I^* \text{ holds in } \mathbb{Z}/(p)[x,y,z]/(x^7+y^7+z^7) \text{ for } p \equiv 3 \mod 7$$

and

$$f \notin I^* \text{ holds in } \mathbb{Z}/(p)[x,y,z]/(x^7+y^7+z^7) \text{ for } p \equiv 2 \mod 7\,.$$

In particular, the bundle $\operatorname{Syz}(x^4,y^4,z^4)$ is semistable in the generic fiber, but not strongly semistable for any reduction $p \equiv 2 \bmod 7$. The corresponding torsor is an affine scheme for infinitely many prime reductions and not an affine scheme for infinitely many prime reductions.

In terms of affineness (or local cohomology) of quasiaffine schemes, this example has the following properties: the open subset given by the ideal

$$(x,y,z) \subseteq \mathbb{Z}/(p)[x,y,z,s_1,s_2,s_3]/\big(x^7+y^7+z^7, s_1x^4+s_2y^4+s_3z^4+x^3y^3\big)$$

has cohomological dimension 1 if $p = 3 \mod 7$ and has cohomological dimension 0 (equivalently, $D(x,y,z)$ is an affine scheme) if $p = 2 \mod 7$.

4.3. **Geometric deformations - A counterexample to the localization problem.** Let $S \subseteq R$ be a multiplicative system and I an ideal in R. Then the *localization problem* of tight closure is the question whether the identity

$$(I^*)_S = (IR_S)^*$$

holds.

Here the inclusion $\subseteq$ is always true and $\supseteq$ is the problem. The problem means explicitly:

If $f \in (IR_S)^*$, can we find an $h \in S$ such that $hf \in I^*$ holds in R?

Proposition 4.2. *Let $\mathbb{Z}/(p) \subset D$ be a one-dimensional domain, $D \subseteq R$ of finite type and I an ideal in R. Suppose that localization holds and that*

$$f \in I^* \text{ holds in } R \otimes_D Q(D) = R_{D^*} = R_{Q(D)}$$

($S = D^* = D \setminus \{0\}$ is the multiplicative system). Then $f \in I^*$ holds in $R \otimes_D \kappa(\mathfrak{p})$ for almost all $\mathfrak{p}$ in Spec D.

Proof. By localization, there exists $h \in D$, $h \neq 0$, such that $hf \in I^*$ in R. By persistence of tight closure (under a ring homomorphism), we get

$$hf \in I^* \text{ in } R_{\kappa(\mathfrak{p})} .$$

The element h does not belong to $\mathfrak{p}$ for almost all $\mathfrak{p}$, so h is a unit in $R_{\kappa(\mathfrak{p})}$ and hence

$$f \in I^* \text{ in } R_{\kappa(\mathfrak{p})}$$

for almost all $\mathfrak{p}$. □

In order to get a counterexample to the localization property we will look now at geometric deformations:

$$D = \mathbb{F}_p[t] \subset \mathbb{F}_p[t][x, y, z]/(g) = S,$$

where t has degree 0 and x, y, z have degree 1 and g is homogeneous. Then (for every homomorphism $\mathbb{F}_p[t] \to K$ to a field)

$$S \otimes_{\mathbb{F}_p[t]} K$$

is a two-dimensional standard-graded ring over K. For the residue class fields of points of $\mathbb{A}^1_{\mathbb{F}_p} = \text{Spec}\,(\mathbb{F}_p[t])$ we have basically two possibilities.

- $K = \mathbb{F}_p(t)$, the function field. This is the *generic* or *transcendental* case.
- $K = \mathbb{F}_q$, the *special* or *algebraic* or *finite* case.

How does $f \in I^*$ vary with K? To analyze the behavior of tight closure in such a family we can use what we know in the two-dimensional standard-graded situation.

In order to establish an example where tight closure does not behave uniformly under a geometric deformation, we first need a situation where strong semistability does not behave uniformly. Such an example was given, in terms of Hilbert-Kunz theory, by Paul Monsky in 1998 [20].

Example 4.3. Let

$$g = z^4 + z^2xy + z\big(x^3 + y^3\big) + \big(t + t^2\big)x^2y^2.$$

Consider

$$S = \mathbb{F}_2[t, x, y, z]/(g).$$

Then Monsky proved the following results on the *Hilbert-Kunz multiplicity* of the maximal ideal (x,y,z) in $S \otimes_{\mathbb{F}_2[t]} L$, L a field:

$$e_{HK}(S \otimes_{\mathbb{F}_2[t]} L) \;=\; \begin{cases} 3 \text{ for } L = \mathbb{F}_2(t) \\ 3 + \frac{1}{4^d} \text{ for } L = \mathbb{F}_q = \mathbb{F}_2(\alpha),\ (t \mapsto \alpha,\ d = \deg(\alpha))\,. \end{cases}$$

We consider S as an $\mathbb{F}_2[t]$-algebra, the corresponding morphism $\operatorname{Spec}(S) \to \mathbb{A}^1_{\mathbb{F}_2}$ and the corresponding smooth projective relative curve $C = \operatorname{Proj}(S) \to \mathbb{A}^1_{\mathbb{F}_2}$. The fibers are $\operatorname{Spec}\big(S_{\kappa(\mathfrak{p})}\big)$ and $C_{\kappa(\mathfrak{p})}$ respectively.

By the geometric interpretation of Hilbert-Kunz theory, the computations mentioned in Example 4.3 mean that the restricted cotangent bundle

$$\operatorname{Syz}(x,y,z) \;=\; (\Omega_{\mathbb{P}^2})|_C$$

is strongly semistable in the transcendental case, but not strongly semistable in the algebraic case. In fact, for $d = \deg(\alpha)$, $t \mapsto \alpha$, where $L = \mathbb{F}_2(\alpha)$, the d-th Frobenius pull-back destabilizes (meaning that it is not semistable anymore).

The maximal ideal (x,y,z) can not be used directly, as it is tightly closed. However, we look at the second Frobenius pull-back which is (characteristic two) just

$$I \;=\; \big(x^4, y^4, z^4\big).$$

By the degree formula, we have to look for an element of degree 6. Let's take $f = y^3z^3$. This is our example (x^3y^3 does not work). First, by strong semistability in the transcendental case, we have

$$f \in I^* \text{ in } S \otimes \mathbb{F}_2(t)$$

by the degree formula. If localization would hold, then by Proposition 4.2, f would also belong to the tight closure of I for almost all algebraic instances $\mathbb{F}_q = \mathbb{F}_2(\alpha), t \mapsto \alpha$. Contrary to that we show that for all algebraic instances, the element f belongs never to the tight closure of I.

Lemma 4.4. *Let $\mathbb{F}_q = \mathbb{F}_p(\alpha)$, $t \mapsto \alpha$, $\deg(\alpha) = d$. Set $Q = 2^{d-1}$. Then*

$$xyf^Q \;\notin\; I^{[Q]}.$$

Proof. This is an elementary but tedious computation. □

Theorem 4.5. *Tight closure does not commute with localization.*

Proof. One knows in our situation that xy is a so-called test element. Hence Lemma 4.4 shows that $f \notin I^*$ □

In terms of affineness of quasiaffine schemes (or local cohomology), this example has the following properties: the open subset given by the ideal

$$(x,y,z) \;\subseteq\; \mathbb{F}_2(t)[x,y,z,s_1,s_2,s_3]/\big(g, s_1x^4 + s_2y^4 + s_3z^4 + y^3z^3\big)$$

has cohomological dimension 1 if t is transcendental and has cohomological dimension 0 (equivalently, $D(x,y,z)$ is an affine scheme) if t is algebraic.

Corollary 4.6. Tight closure is not plus closure in graded dimension two for fields with transcendental elements.

Proof. Consider
$$R = \mathbb{F}_2(t)[x,y,z]/(g).$$
In this ring $y^3z^3 \in I^*$, but it can not belong to the plus closure. Else there would be a curve morphism
$$Y \longrightarrow C_{\mathbb{F}_2(t)}$$
which annihilates the cohomology class c and this would extend to a morphism of relative curves over $\mathbb{A}^1_{\mathbb{F}_2}$ almost everywhere. □

Corollary 4.7. There is an example of a smooth variety Z and an effective divisor $D \subset Z$ and a projective morphism
$$Z \longrightarrow \mathbb{A}^1_{\mathbb{F}_2}$$
such that $(Z \setminus D)_\eta$ is not an affine variety over the generic point η, but for every algebraic point x the fiber $(Z \setminus D)_x$ is an affine variety.

Proof. Take $C \to \mathbb{A}^1_{\mathbb{F}_2}$ to be the Monsky quartic and consider the syzygy bundle
$$\mathcal{S} = \operatorname{Syz}\left(x^4, y^4, z^4\right)(6)$$
together with the cohomology class c determined by $f = y^3z^3$. This class defines an extension
$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{S}' \longrightarrow \mathcal{O}_C \longrightarrow 0$$
and hence $\mathbb{P}(\mathcal{S}^\vee) \subset \mathbb{P}(\mathcal{S}'^\vee)$. Then $\mathbb{P}(\mathcal{S}'^\vee) \setminus \mathbb{P}(\mathcal{S}^\vee)$ is an example with the stated properties by the previous results. □

It is an open question whether such an example can exist in characteristic zero.

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