Definition 1:

$$\mathbf{x + 0} \\ = \mathbf{x}$$

Definition 2:

$$\mathbf{x + Sy} \\ = \mathbf{S(x + y)}$$

Definition 3:

$$\mathbf{x \cdot 0} \\ = \mathbf{0}$$

Definition 4:

$$\mathbf{x \cdot Sy} \\ = \mathbf{x \cdot y + x}$$

Lemma 5:

$$\mathbf{0 + x} \\ = \mathbf{x}$$

Proof by induction on x:

Base case:

$0 + 0$	
$= 0$	by Def. 1

Inductive case:

$0 + Sx$	
$= S(0 + x)$	by Def. 2
$= Sx$	by I.H.

Lemma 6:

$$\mathbf{Sx + y} \\ = \mathbf{S(x + y)}$$

Proof by induction on y:

Base case:

$Sx + 0$	
$= Sx$	by Def. 1
$= S(x + 0)$	by Def. 1

Inductive case:

$Sx + Sy$	
$= S(Sx + y)$	by Def. 2
$= ss(x + y)$	by I.H.
$= S(x + Sy)$	by Def. 2

Lemma 8:

$$\mathbf{(x + y) + z} \\ = \mathbf{x + (y + z)}$$

Proof by induction on z:

Base case:

$(x + y) + 0$	
$= x + y$	by Def. 1
$= x + (y + 0)$	by Def. 1

Inductive case:

$(x + y) + sz$	
$= S((x + y) + z)$	by Def. 2
$= S(x + (y + z))$	by I.H.
$= x + S(y + z)$	by Def. 2
$= x + (y + sz)$	by Def. 2

Lemma 10:

$$\mathbf{0 \cdot x} \\ = \mathbf{0}$$

Proof by induction on x:

Lemma 7:

$$\mathbf{x + y} \\ = \mathbf{y + x}$$

Proof by induction on y:

Base case:

$x + 0$	
$= x$	by Def. 1
$= 0 + x$	by Lem. 5

Inductive case:

$x + Sy$	
$= S(x + y)$	by Def. 2
$= S(y + x)$	by I.H.
$= Sy + x$	by Lem. 6

Lemma 9:

$$\mathbf{x \cdot (y + z)} \\ = \mathbf{x \cdot y + x \cdot z}$$

Proof by induction on z:

Base case:

$x \cdot (y + 0)$	
$= x \cdot y$	by Def. 1
$= x \cdot y + 0$	by Def. 1
$= x \cdot y + x \cdot 0$	by Def. 3

Inductive case:

$x \cdot (y + sz)$	
$= x \cdot S(y + z)$	by Def. 2
$= x \cdot (y + z) + x$	by Def. 4
$= (x \cdot y + x \cdot z) + x$	by I.H.
$= x \cdot y + (x \cdot z + x)$	by Lem. 8
$= x \cdot y + x \cdot sz$	by Def. 4

Lemma 11:

$$\mathbf{Sx \cdot y} \\ = \mathbf{x \cdot y + y}$$

Proof by induction on y:

Lemma 13:

$$\mathbf{(x \cdot y) \cdot z} \\ = \mathbf{x \cdot (y \cdot z)}$$

Proof by induction on z:

Lemma 12:

$$\mathbf{x \cdot y} \\ = \mathbf{y \cdot x}$$

Proof by induction on y:

Legend:

$S(x)$	Successor of x
Def.	Definition
Lem.	Lemma
I.H.	Induction Hypothesis

Binding Priorities:

$Sx \cdot y + z$ denotes $((S(x)) \cdot y) + z$

Used Induction Scheme:

If	$P(0)$
and	$P(x)$ always implies $P(Sx)$,
then	always $P(x)$.

Red arrow: use of lemma

Definition-uses omitted