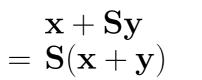
Definition 1:

 $= {f x} + {f 0}$

Definition 2:

Lemma 6:



by Def. 1

Definition 3: $\mathbf{x} \cdot \mathbf{0}$ = 0

 $(\mathbf{x} + \mathbf{y}) + \mathbf{z}$

Definition 4: $\begin{array}{l} \mathbf{x}\cdot\mathbf{S}\mathbf{y}\\ = \mathbf{x}\cdot\mathbf{y}+\mathbf{x}\end{array}$

Lemma 10: $\mathbf{0} \cdot \mathbf{x}$ = 0

Proof by induction on x:

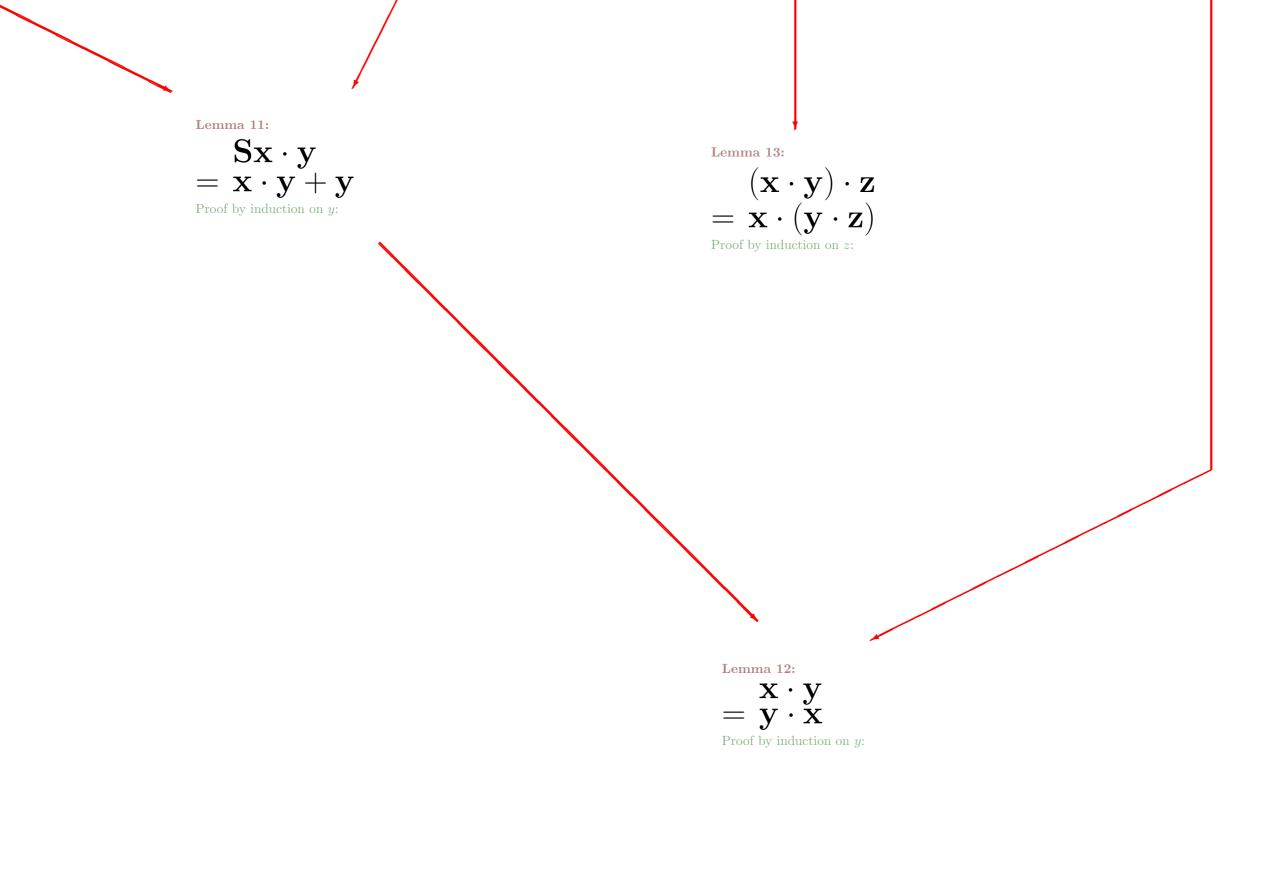
Sx + yLemma 5: $\mathbf{0} + \mathbf{x}$ $= \mathbf{S}(\mathbf{x} + \mathbf{y})$ $= \mathbf{x}$ Proof by induction on y: Proof by induction on x: Base case: Base case: Sx + 00 + 0= Sx= 0by Def. 1 = S(x+0) by Def. 1 Inductive case: Inductive case: 0 + SxSx + Sy= S(0+x) by Def. 2 = S(Sx + y) by Def. 2 = Sx by I.H. = ss(x+y) by I.H. = S(x + Sy) by Def. 2 Lemma 7: $\mathbf{x} + \mathbf{y}$ $= \mathbf{y} + \mathbf{x}$ Proof by induction on y: Base case: x + 0by Def. 1 = x= 0 + x by Lem. 5 Inductive case: x + Sy= S(x+y) by Def. 2 = S(y+x) by I.H. =Sy+x by Lem. 6

 $= \mathbf{x} + (\mathbf{y} + \mathbf{z})$ Proof by induction on z: Base case: (x+y)+0by Def. 1 = x + y= x + (y + 0)by Def. 1 Inductive case: (x+y)+sz= S((x+y)+z) by Def. 2 =S(x+(y+z)) by I.H. = x + S(y+z) by Def. 2 = x + (y + sz) by Def. 2 Lemma 9: $= \frac{\mathbf{x} \cdot (\mathbf{y} + \mathbf{z})}{\mathbf{x} \cdot \mathbf{y} + \mathbf{x} \cdot \mathbf{z}}$ Proof by induction on z: Base case: $x \cdot (y+0)$ by Def. 1 $= x \cdot y$ $= x \cdot y + 0$ by Def. 1 $= x \cdot y + x \cdot 0$ by Def. 3 Inductive case: $x \cdot (y + sz)$ $= x \cdot S(y+z)$ by Def. 2 $= x \cdot (y+z) + x$ by Def. 4

 $= (x \cdot y + x \cdot z) + x$ by I.H.

 $= x \cdot y + (x \cdot z + x)$ by Lem. 8 $= x \cdot y + x \cdot sz$ by Def. 4

Lemma 8:



Legend:

S(x) Successor of xDef. Definition Lem. Lemma I.H. Induction Hypothesis **Binding Priorities:** $Sx \cdot y + z$ denotes $((S(x)) \cdot y) + z$ Used Induction Scheme: If P(0)and P(x) always implies P(Sx), then always P(x).

Red arrow: use of lemma Definition-uses omitted