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AN OPTIMAL CONTROL FORMULATION OF THE BLASCHKE-LEBESGUE THEOREM

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TECHNICAL REPORT

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An Optimal Control Formulation of the Blaschke-Lebesgue Theorem

Abstract

The Blaschke-Lebesgue theorem states that of all plane sets of given constant width the Reuleaux triangle has least area. The area to be minimized is a functional involving the support function and the radius of curvature of the set. The support function satisfies a second order ordinary differential equation where the radius of curvature is the control parameter. The radius of curvature of a plane set of constant width is non-negative and bounded above. Thus we can formulate and analyze the Blaschke-Lebesgue theorem as an optimal control problem.

Introduction

The width of a closed convex curve in a given direction is the distance between two parallel supporting lines perpendicular to that direction. A set of constant width b has the same width in all directions. Besides circle, the best known closed convex curve of constant width b is the Reuleaux triangle of width b, i.e. a set in R^2 whose boundary consists of three congruent circular arcs of radius b. See Figure 1. The Blaschke-Lebesgue theorem states that the Reuleaux triangle has the least area of all plane convex sets of the same constant width b. The minimum area is $\frac{(\pi-\sqrt{3})b^2}{2}$. This theorem was first proved independently by Blaschke [2], and Lebesgue [17]. Besicovitch [1], Chakerian [9], and Eggleston [10;11] contain a proof of the Blaschke-Lebesgue theorem.

Optimal control theory can be applied to geometric extremum problems for plane curves as follows: The functional for which extrema are examined are geometric invariants such as area or perimeter. The system of ordinary differential equations for the control theory formulation is derived from the Frenet-Serret formulas, and the control parameter is curvature.

Klötzler [16] has used optimal control theory to study n-orbiforms. These are convex planar domains which can be rotated inside a regular n-gon under tangential contact on all sides. Plane sets of constant width can be rotated inside a square with tangential contact on all sides. Our approach to plane sets of constant width is different than that in Klötzler in the choice of the functional to be minimized.

In the following we discuss preliminary definitions related to sets of constant width and include necessary background from the theory of optimal control. We then formulate and analyze the Blaschke-Lebesgue theorem as an optimal control problem.

Preliminaries

By a convex body in R^n we mean a compact convex subset of R^n with nonempty interior. For each direction $u \in S^{n-1}$, where S^{n-1} is the unit sphere centered at the origin in R^n, we let $h(K,u)$ denote the support function of the convex body K evaluated at u. Thus,

(1) $$h(K,u) = \sup\{u \cdot x : x \in K\} ,$$

which may be interpreted as the distance from the origin to the supporting hyperplane of K having outward-pointing normal u. The width of K in direction u is given by

(2) $$W(K,u) = h(K,u) + h(K,-u) .$$

A convex body K is said to have constant width b if and only if $W(K,u) = b$, for all $u \in S^{n-1}$.

For a plane convex body K we shall use the notation $h(K,\theta) = h(K,u)$, where $u = (\cos\theta, \sin\theta)$. In this case equation (2) can be written as

(3) $$W(K,\theta) = h(K,\theta) + h(K,\theta+\pi) .$$

A result that we shall find useful is the formula of Cauchy for the Euclidean length of K, namely

$$(4) \qquad L(K) = \frac{1}{2}\int_0^{2\pi} W(K,\theta)\, d\theta \ .$$

From (4) we can obtain Barbier's theorem which states that all plane sets of constant width b have the same perimeter πb. An elementary proof of Barbier's theorem is given in Honsberger [15].

The area of K is denoted by A(K) and is given by

$$(5) \qquad A(K) = \frac{1}{2}\int_0^{2\pi} h(K,\theta)\rho(K,\theta)\, d\theta$$

where $\rho(K,\theta)$ denotes the radius of curvature of K at the point with outward normal $u = (\cos\theta, \sin\theta)$. As a consequence of the Frenet-Serret formulas we can obtain the following second order ordinary differential equation involving radius of curvature $\rho(K,\theta)$ and the support function:

$$(6) \qquad h(K,\theta) + \ddot{h}(K,\theta) = \rho(K,\theta) \ ,$$

where dot denotes differentiation with respect to θ. Detailed discussion of (4), (5) and (6) can be found in Flanders [12], or the monograph on convexity by Bonnesen-Fenchel [5].

For a plane convex body of constant width b we can use (3) to obtain

$$(7) \qquad h(K,\theta) + h(K,\theta+\pi) = b \ .$$

Substituting θ by $\theta+\pi$ in (6), we obtain

$$(8) \qquad h(K,\theta+\pi) + \ddot{h}(K,\theta+\pi) = \rho(K,\theta+\pi) \ .$$

Using (7) we have

(9) $$\ddot{h}(K,\theta) + \ddot{h}(K,\theta+\pi) = 0 .$$

Adding both sides of (6) and (8) and using (9) we conclude that for a plane convex body of constant width b the radius of curvature satisfies

(10) $$\rho(K,\theta) + \rho(K,\theta+\pi) = b .$$

Since the radius of curvature $\rho(K,\theta)$ is nonnegative for a plane convex curve, we use (10) to obtain (11) for a plane convex body of constant width b.

(11) $$0 \leq \rho(K,\theta) \leq b .$$

The idea of using optimal control is simply to minimize the area A(K) given by (5) subject to differential equation (6) and conditions (7), (10) and (11).

In R^3, the generalization of Barbier's theorem is the fact that the total mean curvature M(K) for a set of constant width b is constant. That is

(12) $$M(K) = 2\pi b .$$

We also have the following remarkable relationship for a set of constant width b:

(13) $$2V(K) = bS(K) - \frac{2\pi}{3} b^3 ,$$

where V(K) denotes the volume of K and S(K) the surface area. Chakerian [8] contains derivation of (12) and (13). The problem of minimizing the volume of a convex body of constant width in R^3 is apparently unsolved. Using (13), minimizing the volume is equivalent to minimizing the surface area. The analogues of formulas (5) and (6) for volume and the support function of a convex body in R^3 are given in Blaschke [3]. The support function H of a convex body in R^3 satisfies

(14) $$\Delta H + 2H = R_1 + R_2 ,$$

where R_1 and R_2 are principal radii of curvature and Δ is essentially the Laplacian restricted to the unit sphere. See Bonnesen-Fenchel [5].

Chakerian [9] gives the following lower bound for n-dimensional volume V(K) of a convex body K of constant width 1 in R^n:

(15) $$V(K) \geq \lambda w_n \prod_{k=3}^{n} \left(1 - \sqrt{\frac{k}{2k+2}}\right), \quad n \geq 3 ,$$

where w_n is the volume of the unit ball in R^n and $\lambda = \frac{\pi-\sqrt{3}}{2\pi}$. The volume of n-dimensional unit ball in terms of the Gamma function is given by

(16) $$w_n = \frac{\pi^{\frac{n}{2}}}{\Gamma(1 + \frac{n}{2})} .$$

Chakerian and Groemer [7] give an interesting survey article on sets of constant width. Bonnesen and Fenchel [5], Eggleston [11], and Yaglom and Boltyanskii [20] have good treatment of required background material from the theory of convex sets. These books also contain properties of sets of constant width.

Optimal Control

In the following, we sketch the theory of optimal control and give Pontryagin's maximum principle. Boltyanskii [4], Leitmann [18], and Pontryagin [19] are some interesting books on the theory of optimal control. Gelfand and Fomin [13] also contains an introduction to optimal control. Hermann [14], and Brockett [6] contain differential geometric treatments of calculus of variations and control theory.

We will consider control processes that can be described by a system of ordinary differential equations

$$(17) \qquad \frac{dx_i}{dt} = f_i(x_1,\ldots,x_n,u_1,\ldots,u_r), \quad i = 1,2,\ldots,n$$

where $x_1,\ldots,x_n$ are space coordinates which characterize the process and $u_1,u_2,\ldots,u_r$ are control parameters which determine the process.

In order to determine the process in a given time interval $[t_0,t_1]$, it is sufficient to give the control parameters as functions of time on this interval, that is

$$(18) \qquad u_j = u_j(t), \; j = 1,\ldots,r \; .$$

Assuming that the problem is well posed, for a given initial state

$$(19) \qquad x_i(t_0) = x_i^o \; ,$$

the solution of (17) is uniquely determined.

Consider the functional

$$(20) \qquad J = \int_{t_0}^{t_1} f_0(x_1,\ldots,x_n,u_1,\ldots,u_r)\; dt \; .$$

For each control (18) on $[t_o, t_1]$, the process is determined and the functional J assumes a certain value. Assuming that there is a control (18) which transfers the object from a given initial state (19) to a final state

(21) $$x_i(t_1) = x_i^1 ,$$

the object is to find

(22) $$\bar{u}_j(t), \ j = 1, \ldots, r$$

which transfers the object from (19) to (21) in such a way that the functional (20) has a minimum. In general there are restrictions on control parameters u_j. Thus we shall assume the vector u belongs to a region U in 2-dimensional Euclidean space called the control region, that is

(23) $$u \in U .$$

An admissible control is an arbitrary piecewise continuous control in the control region U.

We now state Pontryagin's maximum principle. As a good start for understanding, one may study the shortest time problem for a phase point moving from an initial point x^o to the origin in accordance with $\frac{dx_1}{dt} = x_2$, $\frac{dx_2}{dt} = u$, $|u| \leq 1$. This problem is studied in many books including Boltyanskii [4].

Maximum Principle

Let $u(t)$, $t_o \leq t \leq t_1$, be an admissible control such that the corresponding trajectory $x(t)$ which begins at the point x^o at the time t_o passes at the time t, through a point x^1. In order that u(t) and x(t) be optimal it is necessary that there exists a nonzero continuous vector function $\psi(t) = (\psi_o(t),\ldots,\psi_n(t))$ corresponding to u(t) and x(t) satisfying

$$\frac{d\psi_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad i = 0,1,2,\ldots,n, \tag{24}$$

where

$$H(\psi,x,u) = \sum_{\alpha=0}^{n} \psi_\alpha f_\alpha , \tag{25}$$

such that

(a) for every t, $t_o \leq t_1$, the function $H(\psi(t),x(t),u)$ of the variable $u \in U$ attains its maximum at the ponit $u = u(t)$:

$$H(\psi(t),x(t),u(t)) = m(\psi(t),x(t)) = \sup_{u\in U} H(\psi,x,u) . \tag{26}$$

(b) At the terminal time t_1, the relations

$$\psi_o(t_1) \leq 0, \quad m(\psi(t_1),x(t_1)) = 0 \tag{27}$$

are satisfied. It turns out that if $\psi(t)$, x(t) and u(t) satisfy systems

$$\frac{dx_i}{dt} = \frac{\partial H}{\partial x_i}, \quad i = 0,1,\ldots,n , \tag{28}$$

$$\frac{d\psi_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad i = 0,1,\ldots,n , \tag{29}$$

and condition (a), then the time functions $\psi_o(t)$ and $m(\psi(t),x(t))$ are constant. Thus (27) may be verified at any time t, $t_o \leq t \leq t_1$, and not just t_1.

Formulation

The Blaschke-Lebesgue theorem states that the Reuleaux triangle has the least area of all plane convex sets K of the same constnat width b. The minimum area is $\frac{(\pi-\sqrt{3})b^2}{2}$. In the following, without loss of generality, we assume b = 1. We also write our formulas concerning a plane convex body K without specifying K. For example (6) will be written as

$$(30) \qquad h(\theta) + \ddot{h}(\theta) = \rho(\theta) \ .$$

Thus we can rewrite formulas (5), (6), (7), (10) and (11) as follows:

$$(31) \qquad A = \frac{1}{2}\int_0^{2\pi} h(\theta)\rho(\theta)\ d\theta \ ,$$

$$(32) \qquad h(\theta) + \ddot{h}(\theta) = \rho(\theta) \ ,$$

$$(33) \qquad h(\theta) + h(\theta+\pi) = 1 \ ,$$

$$(34) \qquad \rho(\theta) + \rho(\theta+\pi) = 1 \ ,$$

$$(35) \qquad 0 \le \rho(\theta) \le 1 \ .$$

We also have the following formula for any convex body K which can be derived from Frenet-Serret formulas (Flanders [12]).

$$\dot{h}(\theta) = x(\theta) \cdot t(\theta),$$

where x denotes the position vector and t unit tangent vector at the point where the outward unit normal is given by $(\cos\theta, \sin\theta)$.

Through the endpoints of any diameter of a set K of constant width there are support lines of K perpendicular to that diameter.

Furthermore there is a diameter for which the corresponding support lines are tangent to the curve (Eggleston [11], P. 126). Let one endpoint of such a diameter be taken as origin. Using (33) and $\dot{h}(\theta)$ we obtain

(36) $$h(0) = 1, \; h(\pi) = 0 ,$$

(37) $$\dot{h}(0) = 0 , \quad \dot{h}(\pi) = 0 .$$

Substitute (33) and (34) in (31) to derive

(38) $$A = \frac{1}{2} \int_0^\pi [1 + 2h(\theta)\rho(\theta) - h(\theta) - \rho(\theta)] \, d\theta .$$

Let

(39) $$x_1(\theta) = 2h(\theta) - 1 ,$$

(40) $$x_2(\theta) = \dot{x}_1(\theta) = 2\dot{h}(\theta), \text{ and}$$

(41) $$u(\theta) = 2\rho(\theta) - 1 .$$

If we now substitute (39), (40) and (41) in (32), (35), (36), (37) and (38), we obtain the following approach to proving the Blaschke-Lebesgue theorem:

(42) $$\text{Minimize } \frac{1}{4} \int_0^\pi (1 + x_1(\theta)u(\theta)) \, d\theta ,$$

subject to

(43) $$\dot{x}_1 = x_2 ,$$

(44) $$\dot{x}_2 = u - x_1 ,$$

(45) $$x_1(0) = 1, \; x_1(\pi) = -1 ,$$

(46) $x_2(0) = 0, \; x_2(\pi) = 0$,

(47) $|u| \le 1$.

Analysis

The Blaschke selection theorem states that every infinite sequence of closed convex subsets of a bounded portion of R^n contains an infinite subsequence that converges to a closed nonempty subset (Eggleston [11], P. 64). The Blaschke selection theorem implies that the minimum area exists.

Minimizing (42) subject to (43)-(47) is equivalent to minimizing

(48) $$\int_0^{\pi} x_1(\theta)u(\theta)\, d\theta$$

under the same constraints. We proceed to use Pontryagin's maximum principle to minimize (48).

Let

(49) $$\frac{dx_o}{d\theta} = x_1(\theta)u(\theta) \; .$$

Let $\psi(\theta) = (\psi_o(\theta), \psi_1(\theta), \psi_2(\theta))$ be the auxiliary vector. Use (25) to obtain

(50) $$H = \psi_o x_1 u + \psi_1 x_2 + \psi_2(u - x_1) = (\psi_o x_1 + \psi_2)u + \psi_1 x_2 - \psi_2 x_1 \; .$$

We then use (24) to write down differential equations for ψ.

(51) $$\frac{d\psi_o}{d\theta} = -\frac{\partial H}{\partial x_o} = 0 \; ,$$

(52) $$\frac{d\psi_1}{d\theta} = -\frac{\partial H}{\partial x_1} = \psi_2 - \psi_1 u \; ,$$

(53) $$\frac{d\psi_2}{d\theta} = - \frac{\partial H}{\partial x_2} = -\psi_1 \ .$$

There exists a nonzero continuous vector $\psi(\theta)$ satisfying (51)-(53) such that

(54) $$\max_{|u|\leq 1} H(x(\theta),u,\psi(\theta)) = H(x(\theta),u(\theta),\psi(\theta)) \ .$$

Now we consider cases to analyze the maximum of H given in (50) as a linear function of u.

Case 1(a): $\psi_o \not\equiv 0$ and $\psi_o x_1 + \psi_2 \not\equiv 0$ (not identically equal to zero) lead to a contradiction.

In this case

(55) $$u = \begin{cases} 1 & \text{if} \quad \psi_o x_1 + \psi_2 > 0 \ , \\ -1 & \text{if} \quad \psi_o x_1 + \psi_2 < 0 \ . \end{cases}$$

Using (43) and (44) we obtain $x_1 = u + A \sin(\theta-\alpha)$ for some A and α. Continuity of ψ_o, x_1 and ψ_2 imply that $\psi_o x_1 + \psi_2 = \psi_o(u + A \sin(\tau-\alpha)) + \psi_2$ is continuous at a switching point τ. Continuity at τ implies

(56) $$\psi_o + \psi_o A \sin(\tau-\alpha) + \psi_2(\tau) = -\psi_o + \psi_o A \sin(\tau-\alpha) + \psi_2(\tau) \ .$$

Hence using (56) we conclude that $\psi_o = 0$ which is a contradiction.

Case 1(b): Assume $\psi_o \not\equiv 0$ and $\psi_o x_1 + \psi_2 \equiv 0$.

In this case $\psi_2 = -\psi_o x_1$. Hence $\dot{\psi}_2 = -\psi_o \dot{x}_1 = -\psi_1 = -\psi_o x_2$, and

(57) $$\dot{\psi}_1 = \psi_2 - \psi_o u = \psi_o \dot{x}_2 = \psi_o(u-x_1) \ .$$

Hence

$$-\psi_o x_1 - \psi_o u = \psi_o u - \psi_o x_1 \ ,$$

which implies u = 0. Using (41) we conclude $\rho(\theta) = \frac{1}{2}$ which corresponds to a circle of radius $\frac{1}{2}$, giving the maximum area rather than the minimum.

<u>Case 2</u>. Suppose $\psi_0 \equiv 0$.

In this case $\psi_2 \not\equiv 0$. Since $\psi_2 \equiv 0$ and (53) imply $\psi_1 = 0$. But we know that ψ is a nonzero vector. Hence using (55) we obtain

(58) $$u = \begin{cases} 1 & \text{if} \quad \psi_2 > 0 , \\ -1 & \text{if} \quad \psi_2 < 0 . \end{cases}$$

Differential equations (52) and (53) reduce to

(59) $$\frac{d\psi_1}{d\theta} = \psi_2 .$$

(60) $$\frac{d\psi_2}{d\theta} = -\psi_1 .$$

An analysis similar to Pontryagin [19, pp. 27-35] will give the switching curve in Figure 2 where

(61) $$u = \begin{cases} -1 & \text{above the cuve} \\ +1 & \text{below or on the curve} . \end{cases}$$

We can now interpret the system of differential equations (43) and (44) as equations of motion. Our objective would be to get from (1,0) to (-1,0) in such a way that we minimize (48). One can show that this is equivalent to minimizing the action of the moving object.

Optimal trajectory starts at (1,0). Since (1,0) is above the curve in Figure 2, we use u = -1 until the trajectory intersects the switching curve on the x_2 axis. We will then use u = +1 which will lead to (-1,0). If we interpret θ as time, then the total time from

(1,0) to (-1,0) is $\frac{2\pi}{3}$. We realize that $x_1 = 1$ and $x_1 = -1$ are solutions of (43)-(46) in neighborhoods of $\theta = 0$ and $\theta = \pi$ respectively. Hence in order to use the total time suggested by functional (48), the object starts at (1,0) and waits there for time $\alpha \leq \frac{\pi}{3}$. Then the object goes to $(0,-\sqrt{3})$ and then to (-1,0). During the time interval $(\frac{2\pi}{3}+\alpha,\pi)$ the object waits at (-1,0) for a period of $\frac{\pi}{3}-\alpha$. In this way the total time would be π. See Figure 3.

We use differential equations (43)-(46) and (62) below to obtain (63).

(62)
$$u = \begin{cases} 1, & 0 \leq \theta < \alpha \\ -1, & \alpha \leq \theta < \frac{\pi}{3}+\alpha \\ 1, & \frac{\pi}{3}+\alpha \leq \theta < \frac{2\pi}{3}+\alpha \\ -1, & \alpha+\frac{2\pi}{3} \leq \theta \leq \pi \ . \end{cases}$$

(63)
$$h(\theta = \frac{1 + x_1(\theta)}{2} = \begin{cases} 1, & 0 < \theta \leq \alpha \\ \cos(\theta-\alpha), & \alpha \leq \theta \leq \frac{\pi}{3}+\alpha \\ 1 + \cos(\theta+\frac{\pi}{3}-\alpha), & \frac{\pi}{3}+\alpha \leq \theta \leq \frac{2\pi}{3}+\alpha \\ 0, & \alpha+\frac{2\pi}{3} \leq \theta \leq \pi \ . \end{cases}$$

However (63) gives the support function of a Reuleaux triangle of width 1. The angle α corresponds to the fact that a rotation of a Reuleaux triangle by angle α will result in a new support function where θ is replaced by $\alpha+\frac{\pi}{3}$, $0 \leq \alpha \leq \frac{\pi}{3}$. We can now calculate (42) to obtain

(64) $$\text{Minimum area} = \frac{1}{4}\int_0^{2\pi} (1 + x_1(\theta)u(\theta))\, d\theta = \frac{\pi - \sqrt{3}}{2} .$$

Hence by comparison of case 1 and case 2 we choose the minimum, which is $\frac{\pi - \sqrt{3}}{2}$ as desired.

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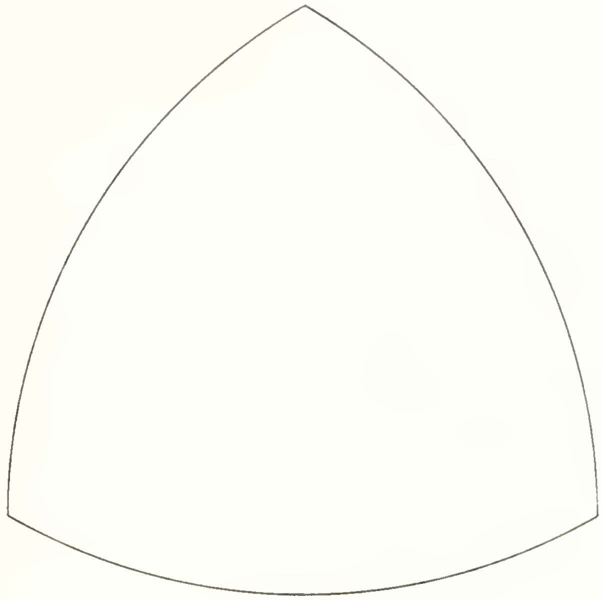

Figure 1. Reuleaux Triangle

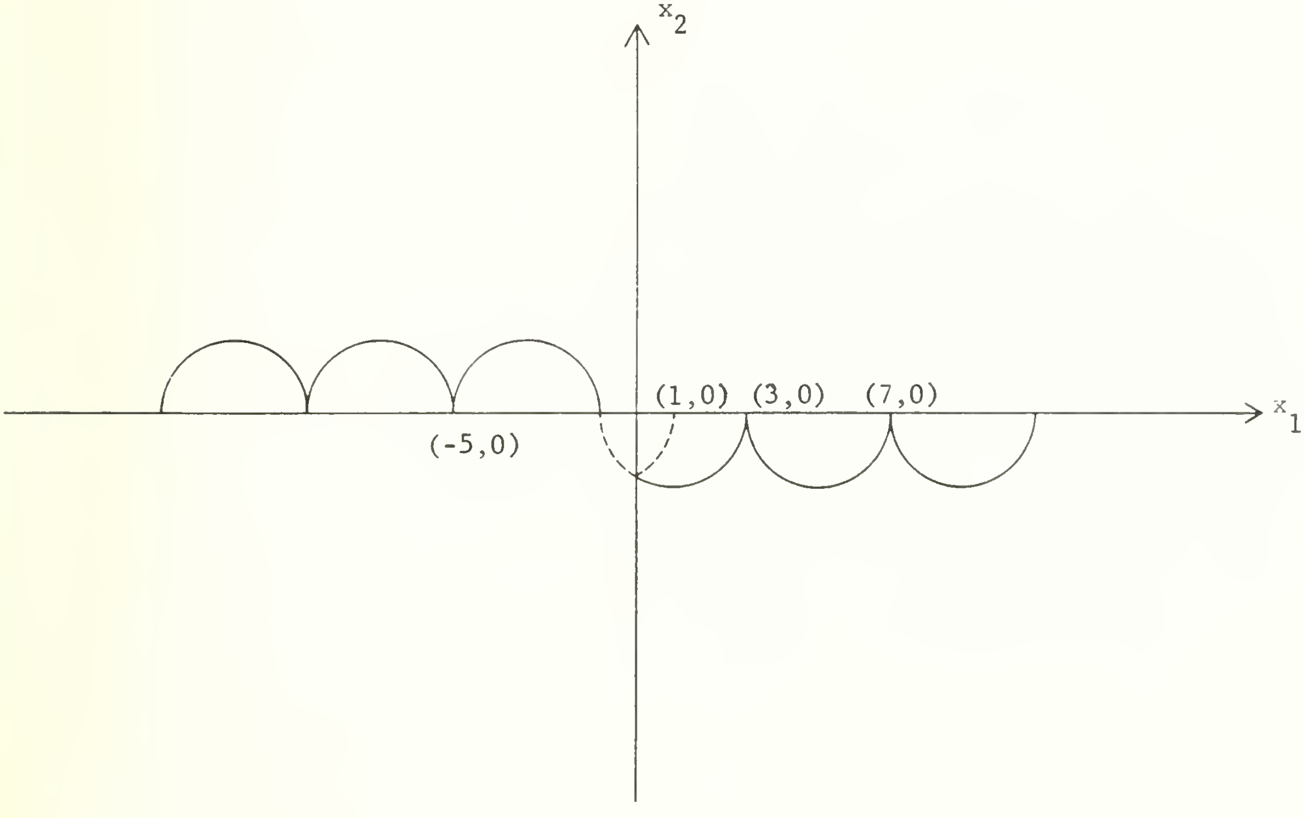

Figure 3. Dotted curve is optimal path

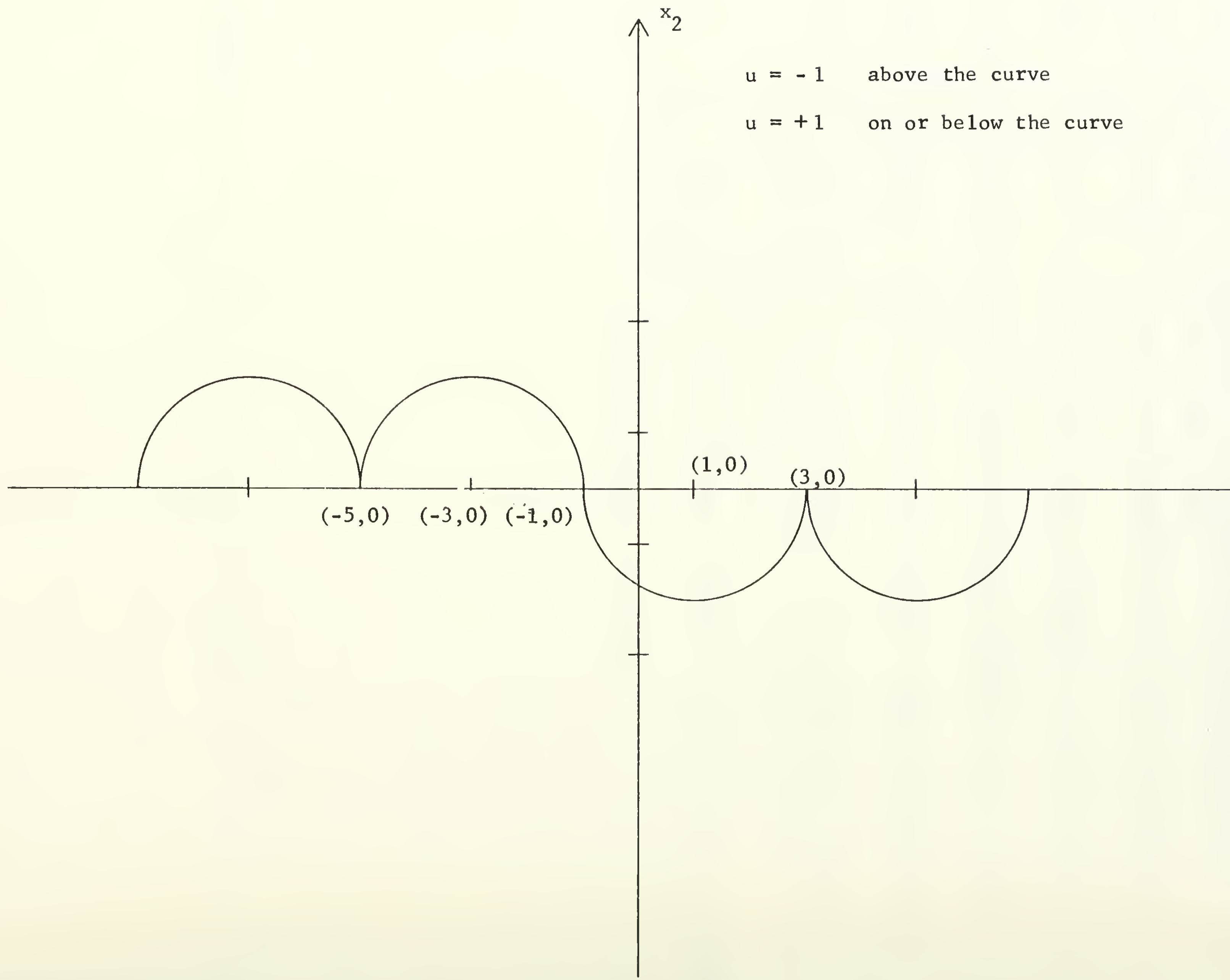

Figure 2. Switching Curve

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