ELEMENTS OF PROJECTIVE GEOMETRY

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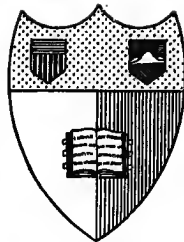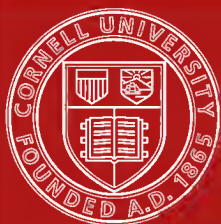
ELEMENTS OF
PROJECTIVE GEOMETRY

BY

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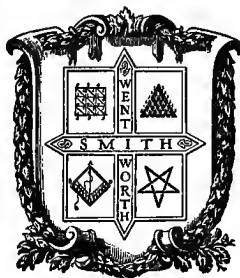
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PREFACE

This work has been prepared for the purpose of providing a thoroughly usable textbook in projective geometry. It is not intended to be an elaborate scientific treatise on the subject, unfitted to classroom use; neither has it been prepared for the purpose of setting forth any special method of treatment; it aims at presenting the leading facts of the subject clearly, succinctly, and with the hope of furnishing to college students an interesting approach to this very attractive and important branch of mathematics.

There are at least three classes of students for whom a study of the subject is unquestionably desirable; namely, those who expect to proceed to the domain of higher mathematics, those who are intending to take degrees in engineering, and those who look forward to teaching in the secondary schools. Although the value of the subject to the second of these classes has not as yet been duly recognized in America, European teachers for several decades have realized its usefulness as a theoretical basis for some of the practical work in this field. For the large number of students belonging to the third class, trigonometry, analytic geometry, and projective geometry are the three subjects essential to a fair knowledge of elementary geometry, and it is believed that the presentation given in this book is such as greatly to aid the future teacher. There is a healthy and growing feeling in America that teachers of secondary mathematics need a more thorough training in the subject matter, even at the expense of some of the theory of education which they now have. This being the case, one of the best fields for their study is projective geometry.

It is recognized that students of projective geometry have usually completed an elementary course in analytic geometry and the calculus, that they have a taste for mathematics which leads them to elect this branch of the science, and that therefore there may fittingly be some departure from the elementary methods employed in the earlier mathematical subjects. On the other hand, for some students at least, projective geometry is a transition stage to higher mathematics, and the subject should therefore be presented with due attention to the important and recognized principles which must always be followed in the preparation of a usable textbook.

It is the belief of the authors that they have followed these principles in such a way as to afford to college students a simple but sufficient introduction to this interesting and valuable branch of geometry. Especial attention has been given to the proper paging of the book, to a clear presentation of the great basal propositions, to the illustrations accompanying the text, to the number and careful grading of the exercises, and to the application of projective geometry to the more elementary field of ordinary Euclidean geometry.

CONTENTS

PART I. GENERAL THEORY

PART II. APPLICATIONS

GREEK ALPHABET

The use of letters to represent both numbers and geometric magnitudes has become so extensive in mathematics that it is convenient for certain purposes to employ the letters of the Greek alphabet. In projective geometry the Greek letters are used particularly to represent planes and angles. These letters with their names are as follows:

A	α	alpha	N	ν	nu
B	β	beta	Ξ	ξ	xi
Γ	γ	gamma	O	o	omicron
Δ	δ	delta	Π	π	pi
E	ϵ	epsilon	P	ρ	rho
Z	ζ	zeta	Σ	σ, ς	sigma
H	η	eta	T	τ	tau
Θ	θ	theta	Υ	υ	upsilon
I	ι	iota	Φ	ϕ	phi
K	κ	kappa	X	χ	chi
Λ	λ	lambda	Ψ	ψ	psi
M	μ	mu	Ω	ω	omega

ELEMENTS OF
PROJECTIVE GEOMETRY

PART I. GENERAL THEORY

CHAPTER I

INTRODUCTION

1. Orthogonal Projection. In elementary geometry the projection of a point upon a line or upon a plane is usually defined as the foot of the perpendicular from the point to the line or to the plane, and the projection of a line is defined as the line determined by the projections of all its points. This simple projection is called *orthogonal projection.*
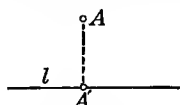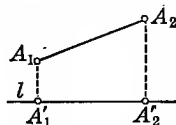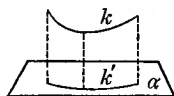
Fig. 1   Fig. 2   Fig. 3

Thus, the point A' in Fig. 1 represents the orthogonal projection of the point A upon the line l; the line $A_1'A_2'$ in Fig. 2 represents the projection of the line A_1A_2 upon the line l; and the line k' in Fig. 3 represents the projection of the curve k upon the plane α.

2. Symbols. Projective geometry, like other branches of mathematics, employs special symbols, generally using capital letters to denote points, small letters to denote lines, the first letters of the Greek alphabet, α, β, γ, δ, $\cdots$, to denote planes, and the Greek letters ϕ and θ to denote angles.

3. Parallel Projection. In a plane α which contains a line p and the points A_1, A_2, A_3, $\cdots$, A_n, if a line l is drawn making an angle ϕ with p, each of the lines through A_1, A_2, A_3, $\cdots$, A_n parallel to l makes with p the angle ϕ.
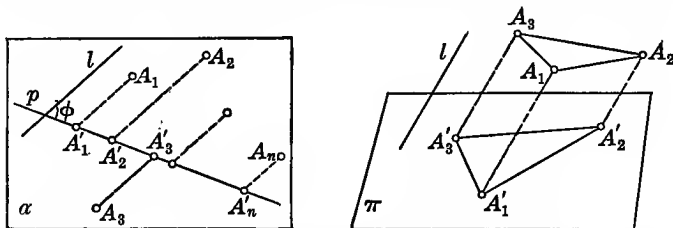
The points A_1', A_2', A_3', $\cdots$, A_n', in which these lines intersect p, are called the *projections* of A_1, A_2, A_3, $\cdots$, A_n upon p and are said to be found by *parallel projection.*

In space of three dimensions a plane figure $A_1A_2A_3$ may be projected upon a plane π by parallel projection.

4. Central Projection. If several coplanar points A_1, A_2, A_3, $\cdots$, A_n are joined to a point P of their plane, and if these lines are cut by a line p, the points of intersection
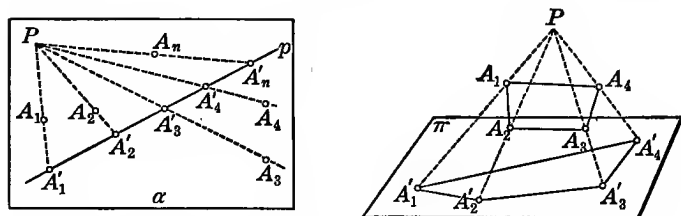
of the lines PA_1, PA_2, PA_3, $\cdots$, PA_n with the line p are called the *projections* of A_1, A_2, A_3, $\cdots$, A_n from the center P upon p and are said to be found by *central projection.*

In space of three dimensions a plane figure $A_1A_2A_3A_4$ may be projected centrally upon a plane π.

5. Projection from an Axis. Let A_1, A_2, A_3, $\cdots$, A_n be points which are not all coplanar. Their orthogonal projections upon a line p may be obtained by drawing a line from each of them perpendicular to p, or by passing a plane through each of them perpendicular to p. The latter method may be generalized by requiring merely that the planes passed through the points shall be parallel to a fixed plane which is not necessarily perpendicular to p, or by requiring that the planes passed through the points shall pass also through a fixed line p' instead of making them parallel to a fixed plane, p' being different from p. Then the points are said to be projected from an axis, and this second kind of projection is called *projection from an axis* or *axial projection*.
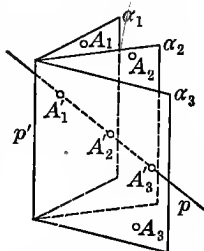
The projection of points by parallel planes is the limiting case of projection from an axis in which the axis has receded indefinitely.

6. Operations of Projection and Section. The process of finding the projection of a plane figure upon a line or plane consists of two parts. The first part is called the *operation of projection* and consists in the construction of a figure composed of lines, or of planes, or of both lines and planes, passing through the points and lines of the figure and through the center or axis of projection. These lines and planes are called the *projectors* of the points and lines of the figure, and constitute the *projector of the figure.*

The second part is called the *operation of section* and consists in cutting the projector of the figure by a line or plane called the *line of projection* or the *plane of projection.*

The center or axis of projection, and the line or plane of projection, should be so taken as not to be parts of the figure to be projected.

Exercise 1. Simple Projections

1. Draw a figure showing the orthogonal projection of a given circle upon a given plane.

The figure may be drawn freehand, and at least three cases should be considered : (1) the circle parallel to the plane ; (2) the circle oblique to the plane ; (3) the circle perpendicular to the plane. In (2) consider also the case in which the circle cuts the plane.

2. Draw a figure showing the parallel, but not necessarily orthogonal, projection of a given square upon a given plane.

3. Draw a figure showing the central projection of a given straight line or a given plane curve upon a given line.

4. Draw a figure showing the central projection of a given square upon a given plane.

Consider three cases, as in Ex. 1. Consider the case in which P is between the square and the plane as well as the case in which it is not.

5. Draw a figure in which the four vertices of a square are projected from a given axis upon a given line.

In projecting from a center P upon a plane π, describe the projectors and the projections of the following, mentioning all the noteworthy special cases under each :

6. A set of points.   **10.** Two intersecting lines.

7. A line.   **11.** Two parallel lines.

8. A triangle.   **12.** A quadrilateral.

9. A circle.   **13.** A pentagon.

14. Four points and the lines joining them in pairs.

15. A tangent to a circle at any point.

In projecting from an axis p' upon (1) *a plane π and* (2) *a line p, describe the projectors and the projections of the following, mentioning the noteworthy special cases under each :*

16. A set of points.   **18.** A line not parallel to p'.

17. A line parallel to p'.   **19.** A circle.

7. Elements at Infinity. Considering central projection only, and supposing the center P and the plane of projection π to be given, these questions now deserve attention :

1. Does every point of a plane α have a projector? Does it have a projection?

Since every point of the plane α can be joined to P by a straight line, every point of α has a projector; but since this projector may happen to be parallel to π, a point of α may have no projection.

2. Is every line which passes through P the projector of some point of α?

No; for certain of these lines may be parallel to α.

3. Does every line of α have a projector? Does it have a projection?

Consider the answers to Question 1. Draw the figure.

4. Is every plane which passes through P the projector of some line of α?

Consider the answer to Question 2. Draw the figure.

Certain exceptional cases have been suggested in connection with these questions. Their occurrence is due to the existence of parallel lines and parallel planes, and the difficulty caused by them may be removed as follows :

Every straight line is assumed to have one and only one infinitely distant point, and this point is called the *point at infinity* of the line.

Every plane is treated as having one and only one straight line situated entirely at an infinite distance, and as having all its infinitely distant points situated on that line. This line is called the *line at infinity* of the plane.

Space is treated as having one and only one plane situated entirely at an infinite distance, and as having all its infinitely distant points and lines situated on that plane. This plane is called the *plane at infinity*.

8. Illustrations. Let a line c rotate in a plane α about a point C of that plane, and when it has the positions c_1, c_2, $\cdots$, let it meet a fixed line p in the points A_1, A_2, $\cdots$. Then as long as c is not parallel to p it meets p in one point and only one. As the point of intersection becomes more and more distant, the line c becomes more and more nearly parallel to p. The limiting position of c as the point of intersection recedes infinitely is the line c' through C parallel to p. If,
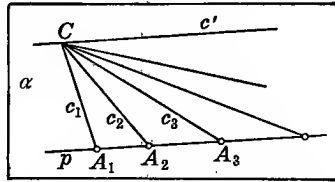
however, the rotation of c is continued ever so little beyond c', the intersection of c and p is found to be at a great distance in the other direction on p, and as the rotation proceeds farther this point of intersection comes continuously back toward A_1. Hence c' is said to meet p in a point at infinity. If there were several infinitely distant points on p, they would with C determine several lines through C parallel to p, or several of these points would be common to c' and p, or one or more of these points taken with C would fail to determine a straight line. Apparent conflict with propositions of Euclidean geometry is best avoided by the assumption that every line not situated at an infinite distance has one and only one infinitely distant point.

Now consider all points of a plane which are infinitely distant. In elementary geometry we find that the only plane locus met by every line of its plane in one and only one point is a straight line. The locus of infinitely distant points of the plane also possesses this property. Hence this latter locus is called the (straight) *line at infinity* of the plane. Similarly, the locus of the infinitely distant points in space is called the *plane at infinity*.

9. Ideas of Projector and Projection Simplified. From the considerations set forth in §§ 7 and 8 it appears that the introduction of the elements at infinity has distinct advantages arising out of the fact that, from the new point of view, statements can often be more briefly and more simply made. The greater simplicity is due to the fact that certain cases involved in the questions cease to be exceptional.

For example, in dealing with the first question in § 7 we now say that every point of a plane α has a projection upon the plane π; for if a projector happens to be parallel to π, it is regarded as meeting π in one point at infinity.

It is also clear that we may now say that all the lines through a point P, and lying in a plane determined by P and a line a, constitute the projector from the center P of all the points of a.

Similarly, we may say that all the lines and all the planes passing through P constitute respectively the projectors from the center P of all the points and all the lines of any plane not passing through P; and that all the planes through any line p form the projector from the axis p of all the points of any line not parallel to p.

Exercise 2. Projectors and Projections

1. Draw figures illustrating the four statements in the last three paragraphs above.

2. Consider the truth of the statement that two lines in a plane have one and only one common point. Illustrate the statement by a figure.

3. Do every two planes in a space of three dimensions determine a line? Explain the statement.

4. In what case do a straight line and a plane fail to determine within a finite distance exactly one point?

5. In what case do a straight line and a point fail to determine a plane?

6. Two straight lines which determine a plane determine, without exception, a point.

10. The Ten Prime Forms. As fundamental sets of elements we use the following sets, called the *ten prime forms* :

One-Dimensional Forms. 1. The totality of points of a straight line (the *base*) is called a *range of points*, a *range*, or, less frequently, a *pencil of points*.

The distinction between a line and the totality of its points may be appreciated by considering points as arranged on a line like beads on a string. Similar considerations apply to the other prime forms.

2. The totality of planes through a straight line (the *base*) is called an *axial pencil*.

This is also called a *pencil of planes* or a *sheaf of planes*.

3. The totality of straight lines in a plane and through a point of the plane is called a *flat pencil*.

In a flat pencil, either the point common to the lines or the plane containing the lines may be regarded as the base of the pencil.

The terms *range of points, axial pencil*, and *flat pencil* are used for a finite number as well as for an infinite number of elements.

Two-Dimensional Forms. 4. The totality of points in a plane (the *base*) is called a *plane of points*.

5. The totality of planes through a point (the *base*) is called a *bundle of planes*.

6. The totality of lines in a plane (the *base*) is called a *plane of lines*.

7. The totality of lines through a point (the *base*) is called a *bundle of lines*.

It is also called a *sheaf of lines*, but because the word *sheaf* is used in conflicting senses, we shall not use it.

Three-Dimensional Forms. 8. The totality of points of three-dimensional space.

9. The totality of planes of three-dimensional space.

Four-Dimensional Form. 10. The totality of lines of three-dimensional space.

Exercise 3. The Ten Prime Forms

Draw a rough sketch to illustrate each of the following:

1. Range of points.
2. Axial pencil.
3. Flat pencil.
4. Plane of points.
5. Bundle of planes.
6. Plane of lines.

Examine each of the following prime forms when the base is at infinity:

7. Flat pencil.
8. Axial pencil.
9. Bundle of lines.
10. Bundle of planes.

11. Find the central projection of a range of points; of a flat pencil; of a plane of points; of a plane of lines.

12. Find the plane section of a flat pencil; of an axial pencil; of a bundle of planes; of a bundle of lines.

13. Find the axial projection of a range of points and also of a flat pencil, the axis passing through the base.

14. Find the linear section of an axial pencil and also of a flat pencil, the line of section being in the plane.

15. Investigate the central projection of a bundle of lines; of the points of space; of the lines of space.

16. Investigate the plane section of a plane of lines; of the planes of space; of the lines of space.

17. Investigate the projection from an axis of a plane of points and also of the points of space.

18. Investigate the linear sections of a bundle of planes and also of the planes of space.

19. Apply each of the four operations to the prime forms not already considered in connection with it.

20. Examine the results of Exs. 11–19 and in each case determine whether to every element of the original figure there corresponds one element and only one element of the resulting figure, and vice versa.

11. Classification of Prime Forms.   In each of the first three classes of the ten prime forms mentioned in § 10 the prime forms of every possible pair are connected by a simple relation.

Consider first a range of points $A_1 A_2 A_3 \cdots A_n \cdots$ on a base p, and consider its projector $a_1 a_2 a_3 \cdots a_n \cdots$ from a point P exterior to p, this pro-

jector being manifestly a flat pencil. By setting up, or arranging, the infinitely many pairs of elements A_1, a_1; A_2, a_2; A_3, a_3; $\cdots$; A_n, a_n; $\cdots$, we find that for every point of the range there is a corresponding line of the flat pencil, and vice versa; and that if two points are nearly coincident, so also are the corresponding lines.
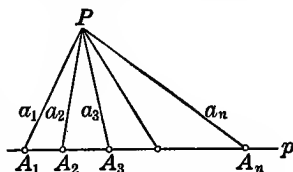
Next, make a section of an axial pencil by a plane π. From the planes α_1, α_2, α_3, $\cdots$, α_n, $\cdots$ of the axial pencil and the lines $a_1, a_2, a_3, \cdots, a_n, \cdots$ of the section of these planes by the plane π, infinitely many pairs of elements α_1, a_1; $\alpha_2, a_2; \alpha_3, a_3; \cdots; \alpha_n, a_n; \cdots$ may be set up. Evidently for every plane of the axial pencil there is a line of its section (the flat pencil), and vice versa; similarly for the range and the axial pencil.
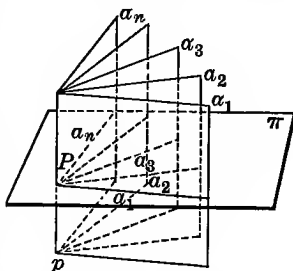
A similar conclusion may be reached regarding any two prime forms of the second class, and also regarding the two prime forms of the third class. In each case, by the setting up of the pairs, there is established a *one-to-one correspondence* between the elements of the two forms.

This is often written as a $1-1$ correspondence.

12. Perspectivity. Certain cases of one-to-one correspondence between the elements of prime forms of the same kind should also be noted. For example, in this figure if two transversals p_1, p_1' cut the lines a_1, a_2, $\cdots$, a_n, $\cdots$ of a flat pencil in the points A_1, A_1'; A_2, A_2'; $\cdots$; A_n, A_n'; $\cdots$, the ranges $A_1 A_2 \cdots$ and $A_1' A_2' \cdots$ correspond in this way.
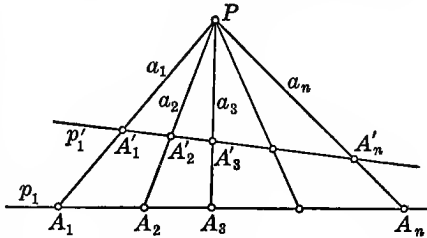
Similarly, in this figure, if two flat pencils $a_1 a_2 \cdots a_n \cdots$ and $a_1' a_2' \cdots a_n' \cdots$ are so situated that a_1, a_1'; a_2, a_2'; $\cdots$; a_n, a_n'; $\cdots$ intersect in the points A_1, A_2, $\cdots$, A_n, $\cdots$ of a range, such a correspondence exists.
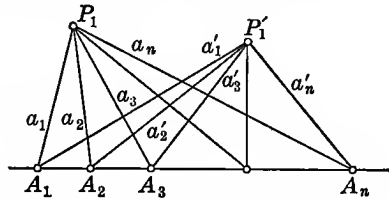
The correspondences in the cases in § 11 resulted from one operation of projection *or* one of section. In the cases just mentioned the correspondences resulted from one operation of projection *and* one of section. All these cases and other similar cases may be brought into one group by means of the following definition:

If either of two prime forms can be obtained from the other by means of one operation of projection, or one operation of section, or by means of one operation of each kind, the two forms are said to be *perspectively related*, or to be *in perspective*, or to be *perspective*.

The symbol $\overline{\overline{\wedge}}$ is often used for "is perspective with."

The perspective relation is called a *perspectivity*.

Exercise 4. Perspectivity and Projection

1. If the line a' of the plane α' is the projection, from the center P, of the line a of the plane α, then the lines a and a' intersect in a point on the line of intersection of α and α'.

2. If the angle formed by the lines a_1' and a_2' of the plane α' is the projection, from the center P, of the angle formed by the lines a_1 and a_2 of the plane α, the pairs of lines a_1, a_1'; a_2, a_2' intersect in points on the line of intersection of α and α'.

3. If two triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$ of the planes α and α' respectively are so situated that the lines A_1A_1', A_2A_2', and A_3A_3' pass through a common point P, the intersections of the pairs of sides A_1A_2, $A_1'A_2'$; A_2A_3, $A_2'A_3'$; A_3A_1, $A_3'A_1'$ are collinear.

4. If two polygons $A_1A_2 \cdots A_n$ and $A_1'A_2' \cdots A_n'$ of the planes α and α' respectively are so situated that the lines A_1A_1', A_2A_2', $\cdots$, A_nA_n' pass through a common point P, the intersections of the pairs of sides A_1A_2, $A_1'A_2'$; A_2A_3, $A_2'A_3'$; $\cdots$; A_nA_1, $A_n'A_1'$, and the intersections of the pairs of diagonals A_1A_3, $A_1'A_3'$; A_1A_4, $A_1'A_4'$; $\cdots$; A_2A_4, $A_2'A_4'$; $\cdots$; A_kA_m, $A_k'A_m'$; $\cdots$ are collinear.

It will be noticed that **Exs. 1–4** form a related set of problems, as is also the case with **Exs. 5–8.**

5. If two lines a and a' of the planes α and α' respectively intersect, either may be regarded as the projection of the other from any point exterior to both lines but in their common plane.

6. State and prove the converse of Ex. 2.

7. State and prove the converse of Ex. 3.

8. State and prove the converse of Ex. 4.

9. Given three points on a line a and a point A_1 not on the line, construct a triangle that shall have A_1 as a vertex and shall have each of its sides, produced if necessary, pass through one and only one of the three given points. How many such triangles can be constructed?

10. Investigate the problem similar to Ex. 9 in which two given points A_1 and A_2 of the plane α are to be vertices of the required triangle, and show how to construct the triangle when such a triangle exists.

11. Given the points A_1 and A_1' of two planes α and α' which intersect in a given line a, and given in the plane α a triangle constructed as required in Ex. 9, use Ex. 3 to obtain a triangle in the plane α' that shall have A_1' as a vertex and shall have sides which, produced if necessary, shall intersect the line a in the points in which this line is cut by the sides of the given triangle in α. How many constructions are possible ?

12. Investigate the cases of Ex. 11 in which a second vertex of one or of each of the triangles is also given.

13. If two triangles $A_1A_2A_3$ and $A_1'A_2'A_3'$ in the same plane α are so situated that the lines A_1A_1', A_2A_2', and A_3A_3' are concurrent, the intersections of $A_1A_2, A_1'A_2'$; $A_2A_3, A_2'A_3'$; $A_3A_1, A_3'A_1'$ are collinear.

Let A_1A_2 and $A_1'A_2'$ meet in C_3, A_2A_3 and $A_2'A_3'$ in C_1, and A_3A_1 and $A_3'A_1'$ in C_2. Take a center of projection P not in the plane α, and project the whole figure upon a plane parallel to the plane PC_3C_1, thus obtaining the line at infinity as the projection of C_3C_1. Prove that the projection of C_2 is on this line.

The development of this problem and similar problems is fully considered in Chapter V.

14. State and prove the converse of Ex. 13.

15. State and prove the proposition of plane geometry which corresponds to Ex. 4.

16. State and prove the converse of Ex. 15.

17. Given three points A_1, A_2, A_3 on a line a in a plane α, and three points A_1', A_2', A_3' on a line a' also in the plane α, find three points A_1'', A_2'', A_3'', not necessarily collinear, into which both sets of three points can be projected.

18. With the same data as in Ex. 17 find three collinear points A_1'', A_2'', A_3'' into which the first two sets of three points mentioned can be projected.

19. In Ex. 17 consider also the case in which the lines a and a' are coincident and in which the points A'_1, A'_2, A'_3 are not necessarily all distinct from the points A_1, A_2, A_3.

20. Given three points on a line a, construct a quadrilateral such that the pairs of opposite sides shall intersect in two of the given points, and such that one of its diagonals shall pass through the other point.

21. Assuming the construction asked for in Ex. 20, use Ex. 4 and Ex. 16 to obtain additional quadrilaterals fulfilling the same conditions as the first. Do the other diagonals of these quadrilaterals intersect?

22. Given two quadrilaterals so constructed as to fulfill the conditions of Ex. 20, the straight lines joining corresponding vertices of these figures are concurrent.

23. If the quadrilateral constructed in Ex. 20 moves so as continuously to fulfill the conditions stated, the other diagonal constantly passes through a fixed point.

24. Show how to find the fixed point mentioned in Ex. 23.

25. In Ex. 20, if the third of the given points bisects the segment joining the other two given points, determine the position of the fixed point mentioned in Ex. 23.

26. Given five points on a line a, construct a quadrilateral such that each of its sides and one of its diagonals, produced if necessary, shall pass through one and only one of the given points. Obtain additional quadrilaterals fulfilling the same conditions.

27. In Ex. 26 investigate the relation that the other diagonals of any two of the quadrilaterals bear to each other and to the given line a.

28. Extend the problem in Ex. 26 to the case of the pentagon.

29. Given two quadrilaterals so constructed as to fulfill the conditions of Ex. 26, the straight lines joining pairs of corresponding vertices of these figures are concurrent.

CHAPTER II

PRINCIPLE OF DUALITY

13. Principle of Duality. It is now highly desirable to consider a certain important relation between pairs of figures in space, and also between their properties. The nature of this relation, by the use of which the difficulties of the subject may be reduced by almost half, is explained by the *Principle of Duality*, or the *Principle of Reciprocity*, which may be stated as follows:

Corresponding to any figure in space which is made up of or generated by points, lines, and planes there exists a second figure which is made up of or generated by planes, lines, and points, such that to every point, every line, and every plane of the first figure there corresponds respectively a plane, a line, and a point of the second figure, and such that to every proposition which relates to points, lines, and planes of the first figure, but which does not essentially involve ideas of measurement, there corresponds a similar proposition regarding the planes, lines, and points of the second figure, and these two propositions are either both true or both false.

The two figures which are related in the manner just described, as well as the two propositions, are said to be *dual*, *reciprocal*, or *correlative*.

As a simple illustration of the principle, consider the following:

Two *points* determine a *line*.	Two *planes* determine a *line*.
Two *lines* through a *point* determine a *plane*.	Two *lines* in a *plane* determine a *point*.

14. Assumption of the Principle of Duality. The validity of the principle of duality will not be proved in this book, although it is possible so to formulate the axioms of projective geometry that they are unchanged if everywhere the words *point* and *plane* are interchanged, and thus to show this validity. Nevertheless the principle will be applied with great frequency in deriving properties of figures, and in so doing either of two courses may be adopted: On the one hand, it may be assumed that the principle is valid and is capable of a proof which is, of course, entirely independent of any results obtained by means of the principle itself; on the other hand, the principle may be used simply as the basis of a rule for formulating the dual of any proposition, the rule being justified in every case by a proof of this dual proposition.

Of these two courses the·latter is not a difficult one, for after the principle of duality has been used to derive the enunciation of the dual proposition, it may be applied to the various steps of the proof of the original proposition to obtain a new set of statements which may be examined to see if they constitute a proof of the dual. In each case it will be found that a proof is secured. The plan has the further advantage that it avoids the feeling of dissatisfaction and uncertainty attendant upon making a very general and far-reaching but apparently unjustified assumption, besides which the repeated application of the principle leads to that confidence in its validity which comes from increasing experimental evidence.

For this reason the second course has been adopted in this book; and whenever the dual of a proposition is derived by applying the principle of duality, either the proof of the dual is derived by the same means or such derivation is left to serve as an exercise for the student.

15. Derivation of Dual Propositions. If a figure or a proposition in the geometry of space is given, the first considerations which enter into the derivation of the dual figures or propositions are the facts that the point and the plane are dual elements, and that every geometric figure may be obtained by using either the point or the plane as the primary generating element. Although neither the point nor the plane has a superior claim over the other to be considered as the primary generating element, it frequently happens that the statement of a proposition is so framed as to imply that one or the other of these elements has been so used. For this reason the derivation of the dual of any proposition generally requires more than the mere interchange of the words *point* and *plane*. On the other hand, it is true, almost without exception, that propositions which have duals may be so stated that the latter may be found from the former by such interchange.

Skill in the derivation of dual figures and propositions is quickly gained, and the examples given in § 16 will assist the beginner in acquiring this skill. In general it may be said that the student will find it advantageous to consider, in every proposition he meets, the proposition which results from the method of treatment mentioned above, and then to consider whether the proof of the derived proposition can be obtained from the original proof in the same way.

In plane geometry the point and the line are dual elements, and any figure may be regarded as having been generated by either of these elements. In a general way, duals in the plane are derived by interchanging these elements.

The principle of duality for threefold space, applied to plane geometry, yields the geometry of the bundle of lines and planes, the line and the plane being dual elements.

16. Examples of Duality. The following are a few of the more simple examples of duality:

1. *Point A.*

2. *Line a.*

3. Two *points* determine a *line.*

4. Two *lines* which determine a *plane* also determine a *point.*

5. Three *points* in general determine a *plane.*

6. Several *points* which lie in a *plane.*

7. Several *lines* which lie in a *plane.*

8. A *plane triangle*; that is, three *points* and the three *lines* determined by them in pairs.

9. A *plane polygon.*

10. A *range of points.*

11. A *flat pencil.*

12. A *plane of points* and a *plane of lines.*

13. Four *points* in a *plane* and the six *lines* joining them in pairs; a complete quadrangle, or *four-point.*

14. Four *lines* in a *plane* and the six *points* determined by the various pairs of these lines; a complete quadrilateral, or *four-line.*

15. Four *points* in space and the *lines* and *planes* determined by them.

1′. *Plane α.*

2′. *Line a′.*

3′. Two *planes* determine a *line.*

4′. Two *lines* which determine a *point* also determine a *plane.*

5′. Three *planes* in general determine a *point.*

6′. Several *planes* which pass through a *point.*

7′. Several *lines* which pass through a *point.*

8′. A *trihedral angle*; that is, three *planes* and the three *lines* determined by them in pairs.

9′. A *polyhedral angle.*

10′. An *axial pencil.*

11′. A *flat pencil.*

12′. A *bundle of planes* and a *bundle of lines.*

13′. Four *planes* through a *point* and the six *lines* of intersection in pairs; a complete *four-flat.*

14′. Four *lines* through a *point* and the six *planes* determined by the various pairs of these lines; a complete *four-edge.*

15′. Four *planes* in space and the *lines* and *points* determined by them.

16. Given three *collinear points* A, B, C, find four *points* P_1, P_2, P_3, P_4 such that the *lines* P_1P_2 and P_3P_4 shall meet in A, the *lines* P_2P_3 and P_4P_1 shall meet in B, and the *line* P_1P_3 shall pass through C.

16'. Given three *coaxial planes* α, β, γ, find four *planes* π_1, π_2, π_3, π_4 such that the *lines* $\pi_1\pi_2$ and $\pi_3\pi_4$ shall lie in α, the *lines* $\pi_2\pi_3$ and $\pi_4\pi_1$ shall lie in β, and the *line* $\pi_1\pi_3$ shall lie in γ.

17. *If the line a' of the plane α' is the projection from the center P of a line a of the plane α, the lines a and a' intersect in a point of the line of intersection of α and α'.*

17'. *If the line a' through the point A' is determined by A' and the point of intersection of the plane π with the line a through the point A, the lines a and a' lie in a plane which passes through the line AA'.*

Proof. The lines a and a' lie in the plane determined by P and a, and hence they determine a point. Since their point of intersection lies in the plane α and also in the plane α', it lies in the line determined by α and α'.

Proof. The lines a and a' pass through the point determined by π and a, and hence they determine a plane. Since their plane passes through the point A and also through the point A', it passes through the line determined by A and A'.

17. Figures in a Plane and Figures in a Bundle. The Principle of Duality may also be stated for the geometry of figures in a plane, and likewise for the geometry of figures in a bundle. In the first case the *point* and the *line* are dual elements, and in the second case the *line* and the *plane*. Simple modifications of the statement of the principle in §13 yield the statements for the two cases.

Pairs of the examples of § 16 may be used to illustrate this fact. For example, the following pairs are duals in the plane : 1, 2 ; 6, 7 ; 8, 8 ; 9, 9 ; 10, 11 ; 13, 14.

The following pairs are duals in the bundle : 1', 2' ; 6', 7' ; 8', 8' ; 9', 9' ; 10', 11' ; 13', 14'.

Many other examples of duality will be found as we proceed.

Exercise 5. Principle of Duality

1. By means of the Principle of Duality obtain for three-fold space the statement and also the proof of the dual of Ex. 3, page 12.

2. Similarly, find the space dual of Ex. 4, page 12.

3. Derive and prove the space dual of Ex. 13, page 13.

4. Obtain for plane geometry the statement and proof of the dual of Ex. 13, page 13.

5. Derive the space dual of Ex. 4.

6. Verify that in the geometry of the bundle the results of Exs. 3 and 5 are dual.

7. If a proposition in plane geometry or in the geometry of the bundle has a dual, but is not self-dual in that geometry, then in the geometry of threefold space it belongs to a set of four propositions each of which is dual with two of the others.

8. If a proposition is self-dual in plane geometry, then the four propositions mentioned in Ex. 7 reduce to two.

9. Can the four propositions of Ex. 7 ever reduce to one proposition ? Discuss in full.

10. Give an example of a self-dual figure in threefold space.

11. State a simple self-dual proposition regarding the figure mentioned in the answer to Ex. 10.

12. Are all propositions regarding the self-dual figure of Ex. 10 themselves self-dual ? Discuss in full.

13. Given three planes passing through a line a of a bundle whose base is A, construct in this bundle a four-edge such that pairs of opposite edges lie in two of the given planes and one of its diagonal lines lies in the remaining plane.

Compare this example with Ex. 20, page 14. Notice that when a proposition is harder to prove than its dual the proof of the latter may be used to suggest that of the former.

CHAPTER III

METRIC RELATIONS. ANHARMONIC RATIO

18. Metric and Descriptive Properties. Properties of geometric figures are of two sorts : (1) *metric*, that is, those which relate to the measurement of geometric magnitudes ; (2) *descriptive*, that is, those which are not metric. Nearly all the propositions of ordinary elementary geometry deal with metric properties, while, speaking generally, those of projective geometry deal with descriptive properties. In fact, it is possible to exclude almost entirely from projective geometry the consideration of the metric properties of figures. On the other hand, even when the object in view is the study of the descriptive properties of figures, it frequently happens that brevity is secured by the use of metric considerations. For this reason the metric properties of figures will be used freely whenever the nature of the work is such as to make this course advisable.

If the student will consider the work which has thus far been done in this book he will see that no statement has been made that depends in any way upon measurement. The lines projected may be of any desired length, the angles may have any desired measure, and the closed figures may have any desired area. It is therefore evident that the work thus far has not been metric in any way.

In this chapter, on the other hand, we proceed to establish certain important properties which will prove to be of great service to us in subsequent work. Moreover, as we proceed, it will appear that by virtue of the propositions proved in §§ 26 and 27 the study of these properties is appropriate in connection with the descriptive properties of figures.

21

19. Relations of Line Segments. In measuring distances along a straight line attention is given to direction as well as to length. One direction along a line is selected as positive, the opposite one being negative. The direction of a line segment is called its *sense* and is indicated by the order of the end letters, AB denoting the segment of a line thought of as extending from A to B and BA the segment of a line thought of as extending from B to A. Evidently, therefore, we have

$$AB = -BA,$$

or

$$AB + BA = 0.$$

Having adopted this convention with respect to signs, many identical relations can be proved. For example, A, B, C, $\cdots$, J, K being collinear points in any order:
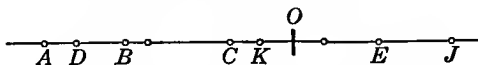
1. $AB + BC + CD + \cdots + JK + KA = 0.$
2. $AB \cdot CD + BC \cdot AD + CA \cdot BD = 0.$
3. $BC \cdot \overline{AD}^2 + CA \cdot \overline{BD}^2 + AB \cdot \overline{CD}^2 + BC \cdot CA \cdot AB = 0.$

In one method of proving these relations we employ as origin any point O on the given line. Then for any segment AB we have

$$AB = OB - OA.$$

This substitution and others of a similar nature being made in any identity of this sort, the truth of the identity becomes apparent.

The proof may be made algebraic if the measures of the segments OA, OB, $\cdots$ are denoted by the letters a, b, $\cdots$. Moreover, having an identical relation among real algebraic numbers, we may deduce a corresponding relation among line segments.

20. Relations of Angles. In measuring angles attention is given to the direction of rotation as well as to the magnitude of the angles. Rotation is called positive if it proceeds in the direction opposite to that taken by the hands of a clock; otherwise it is called negative. The direction of rotation is called the *sense* of the angle and is indicated by the order of the letters which denote its arms, the angle formed by the rotation of a line from the position of the line a to that of the line b being called the angle ab.

The ambiguity which may be felt to attach to this method of representing angles may easily be removed in the following way: Take as the standard line any line o through the intersection of the lines a and b, and let it be agreed that the angle between a and b shall be understood to mean that angle formed by the lines a and b which does not contain the line o, and that the angle oa formed by o and a shall mean the angle included between a specified one of the halves of the infinite line o which proceed from the point common to a and b and that half of a which is first reached by a positive rotation from o.
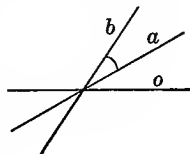
Then the algebraic identities by means of which the relations between line segments were proved, as well as all other algebraic identities, are capable of interpretations with respect to angles. In the above case $ab = ob - oa$, and by means of such identities the relations may be verified.

The dihedral angles formed by pairs of planes of an axial pencil may be treated in a similar fashion, a standard plane ω being used. In this case the angle between two planes α and β will be denoted by $\alpha\beta$ if the planes α, β, and ω have the same general positions as the lines a, b, and o in the figure shown above.

21. Anharmonic Ratio. The most useful metric element in projective geometry is called an *anharmonic ratio*. It is related to a range of four points, a flat pencil of four lines, and an axial pencil of four planes, as follows:

1. *The anharmonic ratio* $(ABCD)$ *of four collinear points* A, B, C, D, *is defined as* $\dfrac{AC}{BC} : \dfrac{AD}{BD}$.

2. *The anharmonic ratio* $(abcd)$ *of four concurrent and coplanar line segments* a, b, c, d *is defined as*

$$\frac{\sin ac}{\sin bc} : \frac{\sin ad}{\sin bd}.$$

3. *The anharmonic ratio* $(\alpha\beta\gamma\delta)$ *of four coaxial planes* α, β, γ, δ *is defined as* $\dfrac{\sin \alpha\gamma}{\sin \beta\gamma} : \dfrac{\sin \alpha\delta}{\sin \beta\delta}$.

An anharmonic ratio is also called a *cross ratio* or a *double ratio*. The anharmonic ratio $(ABCD)$ is easily remembered by writing $\dfrac{A}{B} : \dfrac{A}{B}$ and then writing C in both terms of the first fraction and D in both terms of the second fraction.

The above definition of anharmonic ratio, though not universal, has the approval of the leading authorities of the present time.

22. COROLLARY. *If A, B, C, D are collinear points, then:*

1. $(ABCD)$ *is negative when and only when the segment AB contains either C or D, but not both.*

2. $(ABCD)$ *approaches AC/BC as a limit as D recedes indefinitely in either direction.*

Exercise 6. Anharmonic Ratios

1. If $(ABCD_1) = (ABCD_2)$, D_1 and D_2 are coincident.

2. Consider Ex. 1 and § 22 for the anharmonic ratio $(abcd)$.

3. Consider Ex. 1 and § 22 for the anharmonic ratio $(\alpha\beta\gamma\delta)$.

23. Twenty-four Anharmonic Ratios. Corresponding to the order A, B, C, D of four collinear points, there has been defined the anharmonic ratio $(ABCD)$. There are, however, twenty-four possible orders for these points, that is, the 4 ! permutations of the four letters; and therefore there are twenty-four anharmonic ratios for the four points, as follows:

$$(ABCD),\ (ABDC),\ (ACBD),\ (ACDB),\ (ADBC),\ (ADCB),$$
$$(BACD),\ (BADC),\ (BCAD),\ (BCDA),\ (BDAC),\ (BDCA),$$
$$(CABD),\ (CADB),\ (CBAD),\ (CBDA),\ (CDAB),\ (CDBA),$$
$$(DABC),\ (DACB),\ (DBAC),\ (DBCA),\ (DCAB),\ (DCBA).$$

But by definition (§ 21)

$$(BACD) = \frac{BC}{AC} : \frac{BD}{AD} = \frac{AD \cdot BC}{AC \cdot BD},$$

while

$$(ABDC) = \frac{AD}{BD} : \frac{AC}{BC} = \frac{AD \cdot BC}{AC \cdot BD},$$

and so

$$(BACD) = (ABDC).$$

In like manner it may be shown that the last eighteen ratios fall into six sets of three each, all those in any set being equal to one of the first six anharmonic ratios.

Exercise 7. Anharmonic Ratios

Given the three collinear points A, B, C, proceed as follows:

1. Find the collinear point D such that $(ABCD) = 7$.

2. Find the collinear point D such that $(ABCD) = -7$.

3. Find the collinear point D such that $(ABCD) = k$.

4. Prove that $(ABCD) = (BADC) = (CDAB) = (DCBA)$.

5. Determine the several permutations of the four elements A, B, C, D which leave the value of $(ABCD)$ unchanged.

6. Which of the twenty-four ratios are equal to $(ADBC)$?

24. Relations of the First Six Ratios. The first six anharmonic ratios given in § 23 are also connected by simple relations. If $(ABCD) = x$, we have the following:

1. $(ABCD) = x.$

2. $(ABDC) = \dfrac{1}{x}.$

For $\qquad (ABCD) = \dfrac{AC}{BC} : \dfrac{AD}{BD} = \dfrac{AC \cdot BD}{AD \cdot BC} = x,$

and $\qquad (ABDC) = \dfrac{AD}{BD} : \dfrac{AC}{BC} = \dfrac{AD \cdot BC}{AC \cdot BD} = \dfrac{1}{x}.$

3. $(ACBD) = 1 - x.$

For (§ 19, 2) $AB \cdot CD + BC \cdot AD + CA \cdot BD = 0$;

whence $\qquad \dfrac{AB}{AD} \cdot \dfrac{CD}{CB} = 1 - \dfrac{CA \cdot BD}{AD \cdot CB} = 1 - \dfrac{AC}{BC} : \dfrac{AD}{BD} = 1 - x.$

But $\qquad (ACBD) = \dfrac{AB}{CB} : \dfrac{AD}{CD} = \dfrac{AB}{AD} \cdot \dfrac{CD}{CB}.$

Therefore $\quad (ACBD) = 1 - x.$

4. $(ACDB) = \dfrac{1}{1 - x}.$

For $(ACDB) = \dfrac{1}{(ACBD)}$, since we have simply interchanged the last two letters, as in 1 and 2 above. Hence the result follows from 3.

5. $(ADBC) = \dfrac{x - 1}{x}.$

For $(ACBD) = 1 - x$, by 3, where we merely interchange the second and third letters. Hence, by similar reasoning,

$$(ADBC) = 1 - (ABDC) = 1 - \dfrac{1}{x} = \dfrac{x - 1}{x}.$$

6. $(ADCB) = \dfrac{x}{x - 1}.$

For we found from 1 and 2 that the transposition of the third and fourth letters gave the reciprocal of the original anharmonic ratio, and so from 5 we have $(ADCB) = \dfrac{1}{(ADBC)} = \dfrac{x}{x - 1}.$

25. Equality of the Six Expressions. We may now determine the values of x for which any pair of the six expressions $x, \dfrac{1}{x},\ 1-x,\ \dfrac{1}{1-x},\ \dfrac{x-1}{x}$, and $\dfrac{x}{x-1}$ are equal, and therefore we may determine the values of these six expressions which correspond to the values of x so found. The results may be put in tabular form as follows:

x	$\dfrac{1}{x}$	$1-x$	$\dfrac{1}{1-x}$	$\dfrac{x-1}{x}$	$\dfrac{x}{x-1}$
1	1	0	∞	0	∞
-1	-1	2	$\frac{1}{2}$	2	$\frac{1}{2}$
$\frac{1}{2}$	2	$\frac{1}{2}$	2	-1	-1
$\dfrac{1+\sqrt{-3}}{2}$	$\dfrac{1-\sqrt{-3}}{2}$	$\dfrac{1-\sqrt{-3}}{2}$	$\dfrac{1+\sqrt{-3}}{2}$	$\dfrac{1+\sqrt{-3}}{2}$	$\dfrac{1-\sqrt{-3}}{2}$
$\dfrac{1-\sqrt{-3}}{2}$	$\dfrac{1+\sqrt{-3}}{2}$	$\dfrac{1+\sqrt{-3}}{2}$	$\dfrac{1-\sqrt{-3}}{2}$	$\dfrac{1-\sqrt{-3}}{2}$	$\dfrac{1+\sqrt{-3}}{2}$
0	∞	1	1	∞	0
2	$\frac{1}{2}$	-1	-1	$\frac{1}{2}$	2
∞	0	∞	0	1	1

It will be noticed that these values of the six distinct ratios of the 4! anharmonic ratios may be classified into three groups as follows:

1. Those in which the values of x are imaginary, the values of all the functions being also imaginary.

2. Those in which the values of x are 1, 0, or ∞, the values of the functions being also 1, 0, or ∞.

3. Those in which the values of x are -1, $\frac{1}{2}$, or 2, the values of the functions being also -1, $\frac{1}{2}$, or 2.

The first group is not concerned with the anharmonic ratio of four real collinear points, and the second does not correspond to the anharmonic ratio of four real points which are distinct. The third group is the only important one for our present purpose, and this will be considered in Chapter IV.

PG

THEOREM. PRIME FORMS RELATED BY PROJECTIONS AND SECTIONS

26. *If two prime forms of the first class are so related that either may be obtained from the other by a finite number of projections and sections, the anharmonic ratio of any four elements of one is equal to the anharmonic ratio of the corresponding four elements of the other.*

Proof. I. Let either form be obtainable from the other by means of *one* operation of projection or *one* of section.
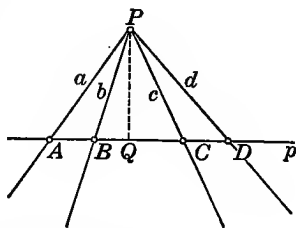
1. *Range ABCD and flat pencil abcd.*

From P, the base of the flat pencil, draw PQ perpendicular to p, the base of the range $ABCD$. Then, equating pairs of expressions for double the areas of the triangles ACP, BCP, ADP, BDP, we have

$$PA \cdot PC \sin ac = PQ \cdot AC,$$
$$PB \cdot PC \sin bc = PQ \cdot BC,$$
$$PA \cdot PD \sin ad = PQ \cdot AD,$$
$$PB \cdot PD \sin bd = PQ \cdot BD.$$

Hence $\quad \dfrac{PA}{PB} \cdot \dfrac{\sin ac}{\sin bc} = \dfrac{AC}{BC}, \quad \dfrac{PA}{PB} \cdot \dfrac{\sin ad}{\sin bd} = \dfrac{AD}{BD},$

and $\qquad\qquad \dfrac{\sin ac}{\sin bc} : \dfrac{\sin ad}{\sin bd} = \dfrac{AC}{BC} : \dfrac{AD}{BD};$

whence $\qquad\qquad (abcd) = (ABCD).$

2. *Range ABCD and axial pencil $\alpha\beta\gamma\delta$.*

From a point P in the base of the axial pencil $\alpha\beta\gamma\delta$ project the range $ABCD$, obtaining as projector the flat pencil $abcd$.

Then $(ABCD) = (abcd)$. But in Case 3 it will be shown that $(abcd) = (\alpha\beta\gamma\delta)$. Hence $(ABCD) = (\alpha\beta\gamma\delta)$.
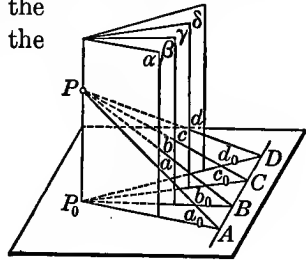
3. *Flat pencil abcd and axial pencil $\alpha\beta\gamma\delta$.*

Through a point P_0 in the base of the axial pencil $\alpha\beta\gamma\delta$ pass a plane perpendicular to this base, cutting the planes α, β, γ, δ in the lines a_0, b_0, c_0, d_0 and the lines a, b, c, d in the points A, B, C, D. From the definition of the angles between the planes it follows that

$$(\alpha\beta\gamma\delta) = (a_0 b_0 c_0 d_0).$$

But $(a_0 b_0 c_0 d_0) = (ABCD)$

$$= (abcd).$$

Hence $(\alpha\beta\gamma\delta) = (abcd)$.

II. Let either form be obtainable from the other by means of several operations of projection or of section, or of both.

Let two prime forms f_1 and f_{n+1} be obtainable, either from the other, by means of n operations, and let the prime forms which are successively produced beginning with f_1 be $f_2, f_3, \cdots, f_n, f_{n+1}$. Also let e_k, e_k', e_k'', e_k''' and e_{k+1}, e_{k+1}', e_{k+1}'', e_{k+1}''' be corresponding sets of four elements of two consecutive prime forms f_k, f_{k+1}.

Then, by Part I, $(e_k e_k' e_k'' e_k''') = (e_{k+1} e_{k+1}' e_{k+1}'' e_{k+1}''')$.

This relation is true for all values of k from 1 to n.

Hence $(e_1 e_1' e_1'' e_1''') = (e_{n+1} e_{n+1}' e_{n+1}'' e_{n+1}''')$,

and the truth of the theorem is established.

27. COROLLARY. *If two prime forms of the first class are perspective, the anharmonic ratio of any four elements of one form is equal to that of the four corresponding elements of the other form.*

Exercise 8. Relations of the Ratios

1. Show how the table on page 27 is obtained, verifying each result.

2. Any two letters in the anharmonic ratio $(ABCD)$ may be interchanged without affecting the value of the ratio, provided the other two letters are also interchanged.

In the six expressions $x, \dfrac{1}{x}, 1-x, \dfrac{1}{1-x}, \dfrac{x-1}{x}, \dfrac{x}{x-1}$ *make successively the following substitutions for x and note the recurrence of the original forms of the expressions:*

3. x'.   **5.** $1-x'$.   **7.** $\dfrac{x'-1}{x'}$.

4. $\dfrac{1}{x'}$.   **6.** $\dfrac{1}{1-x'}$.   **8.** $\dfrac{x'}{x'-1}$.

9. If the point O and three nonconcurrent lines a, b, c are in one plane, draw a line through O which shall cut a, b, c in points A, B, C such that $(OABC) = k$, any given number.

10. Solve the dual of Ex. 9 in plane geometry.

11. Solve the space dual of Ex. 9.

12. If $A_1, A_2, B_1, B_2, C_1, C_2, D_1, D_2, X, Y$ are collinear points, and if $(A_1A_2XY)=(B_1B_2XY)=(C_1C_2XY)=(D_1D_2XY)=-1$, it follows that $(A_1B_1C_1D_1)=(A_2B_2C_2D_2)$.

13. If $A_1, A_2, \cdots, A_n, X, Y$ are $n+2$ collinear points, then $(A_1A_2XY)(A_2A_3XY)\cdots(A_{n-1}A_nXY)(A_nA_1XY)=1$.

14. If A_1, A_2, A_3, X, Y are five coplanar points, and if $A_1(A_2A_3XY)$ denotes the anharmonic ratio of the four lines from A_1 to A_2, A_3, X, Y, it follows that the product of the anharmonic ratios $A_1(A_2A_3XY), A_2(A_3A_1XY), A_3(A_1A_2XY)$ is 1.

15. Generalize the result found in Ex. 14.

16. If A_1, A_2, A_3, A_4, X, Y are concyclic points, it follows that the anharmonic ratios $X(A_1A_2A_3A_4)$ and $Y(A_1A_2A_3A_4)$ are equal.

CHAPTER IV

HARMONIC FORMS

28. Harmonic Range. When four collinear points A, B, C, D are so situated that $(ABCD) = -1$, the four points are said to constitute a *harmonic range*.

In the same way we may define a *harmonic flat pencil* and a *harmonic axial pencil*.

Any one of these three forms is spoken of as a *harmonic form*, and it follows (§ 26) that every form derived from a harmonic form by a finite number of projections and sections is a harmonic form. If three elements of a harmonic form are given, it is evident that the fourth element, called the *fourth harmonic* to the three, is uniquely determined.

Moreover, since the anharmonic ratio is here negative, from the above definition the elements of the first pair, say A and B, separate those of the second pair, say C and D.

That is, since $(ABCD) = -1, \dfrac{AC \cdot BD}{AD \cdot BC} = -1$, and $\dfrac{AC}{BC} = -\dfrac{AD}{BD}$, so that the two ratios have opposite signs. Therefore, either C or D divides AB internally and the other divides it externally.

The elements of either of these pairs, A and B, or C and D, are said to be *conjugates* or *harmonic conjugates* with respect to the other pair. They are also said to be *harmonically separated* by the elements of the other pair.

Since from the definition (§ 21) the anharmonic ratios $(ABCD)$ and $(CDAB)$ are equal, it follows that the relation between the pairs of elements A, B and C, D is symmetric with respect to them.

Exercise 9. Harmonic Ranges

1. Given three collinear points A, B, C, with C bisecting AB, determine the fourth harmonic D.

2. Consider Ex. 1 when C is the point at infinity.

3. Consider Ex. 1 when A, B, C are any collinear points.

4. Given three concurrent lines a, b, c, with c bisecting the angle ab, determine the fourth harmonic d.

Compare Ex. 4 with Ex. 1.

5. Consider Ex. 4 when a, b, c are any concurrent lines.

Compare Ex. 5 with Ex. 3.

6. If $(ABCD) = -1$, the four points A, B, C, D may, by a finite number of projections and sections, be projected into the positions A, B, D, C.

7. If $ABCD$ is a harmonic range, the line segments AC, AB, and AD are connected by the proportion $AC : AD = AC - AB : AB - AD$.

8. If $AB_1C_1D_1$ and $AB_2C_2D_2$ are harmonic ranges on different bases, the lines B_1B_2, C_1C_2, D_1D_2 are concurrent, and the lines B_1B_2, C_1D_2, C_2D_1 are also concurrent.

9. If $A_1B_1C_1D_1$ and $A_2B_2C_2D_2$ are harmonic ranges, and if A_1A_2, B_1B_2, C_1C_2 are concurrent at O, then D_1D_2 also passes through O.

10. If A, B, C, D, O, P are points on a circle and are so placed that the pencil $O(ABCD)$ is harmonic, the pencil $P(ABCD)$ is also harmonic.

11. If $ABCD$ is a harmonic range, and if O is the midpoint of CD, then $\overline{OC}^2 = OA \cdot OB$.

12. Use the result of Ex. 11 to find a pair of points which shall be harmonic conjugates with respect to two given pairs of collinear points A_1, B_1; A_2, B_2.

13. Given four coplanar lines, draw when possible a line which shall cut them in a harmonic range.

29. Complete Quadrangle. The figure formed by four points in a plane, no three of which are collinear (as P, Q, R, S in the figure below), and the six lines determined by them is called a *complete four-point* or *complete quadrangle*.

Any two of the six lines of a complete quadrangle which do not intersect in one of the original four points are called *opposite sides*. The intersections of opposite sides are called *diagonal points*, and they are the vertices of the *diagonal triangle* of the complete quadrangle.

THEOREM. HARMONIC PROPERTY OF A QUADRANGLE

30. *If four collinear points A, B, C, D are so situated that two opposite sides of a complete quadrangle pass through A, two opposite sides pass through B, and the two remaining sides pass through C and D respectively, then $(ABCD) = -1$.*
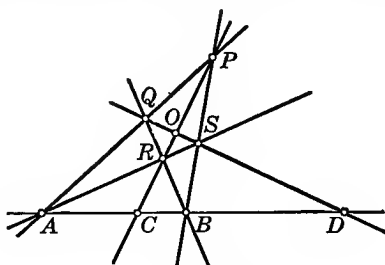
Let P, Q, R, S be the vertices of the complete quadrangle, and let PQ, RS pass through A; PS, QR through B; PR through C; and QS through D.

Then $(ABCD) = (SQOD) = (BACD) = \dfrac{1}{(ABCD)}.$

Hence $(ABCD)^2 = 1,$

and $(ABCD) = -1.$

$(ABCD)$ cannot be equal to $+1$, since no two points are coincident, as would then be the case.

31. Complete Quadrilateral. The figure formed by four lines in a plane, no three of which are concurrent (as *p*, *q*, *r*, *s* below), and the six points determined by them is called a *complete four-side* or *complete quadrilateral*.

Any two of the six points of a complete quadrilateral which do not both lie on one of the original four lines are called *opposite vertices*. The lines determined by pairs of opposite vertices are called *diagonal lines*, and they determine the *diagonal triangle*.

The student should compare this figure with that of the complete quadrangle in § 30, and should notice also the duality suggested by § § 29 and 31, the dual elements being the point and line.
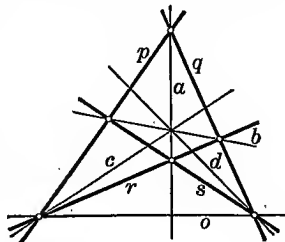
THEOREM. HARMONIC PROPERTY OF A QUADRILATERAL

32. *If four concurrent lines a, b, c, d are so situated that two opposite vertices of a complete quadrilateral are on a, two opposite vertices on b, and the two remaining vertices on c and d respectively, then* $(abcd) = -1$.

Let *p*, *q*, *r*, *s* in the figure above be the sides of the complete quadrilateral, and let *p* and *q*, and also *r* and *s*, intersect on *a*; *p* and *s*, and also *q* and *r*, intersect on *b*; *p* and *r* intersect on *c*; and *s* and *q* intersect on *d*.

Then $$(abcd) = (sqod) = (bacd) = \frac{1}{(abcd)}.$$

Hence $$(abcd)^2 = 1,$$

and $$(abcd) = -1.$$

Why cannot $(abcd) = +1$? Students should compare this proof, step by step, with that of § 30.

Exercise 10. Quadrangles and Quadrilaterals

1. State and prove the converse of § 30.

2. State and prove the converse of § 32.

3. Two vertices of the diagonal triangle of a complete quadrangle are harmonically separated by the points in which the line determined by them is cut by the remaining pair of opposite sides of the quadrangle.

4. By interchanging certain elements, it is possible to derive § 32 from § 30 and Ex. 2 above from Ex. 1. Derive a proposition in this way from Ex. 3 and investigate its truth.

5. From § 30 derive a theorem respecting the complete four-flat and prove it.

In the geometry of space the figure dual to the complete quadrangle is called the *complete four-flat*, and, similarly, the *complete four-edge* in Ex. 6 is dual to the complete quadrilateral.

6. As in Ex. 5, from § 32 derive a theorem respecting the complete four-edge and prove it.

7. The six points, other than the diagonal points, in which the diagonal lines meet the sides of a complete quadrangle lie in sets of three on each of four lines.

8. From the result in Ex. 7 prove the existence of a complete quadrilateral which has the same diagonal triangle as any complete quadrangle.

9. Prove the plane duals of Exs. 7 and 8.

10. In this figure QS is parallel to AB. Show that PC is a median and is divided harmonically.
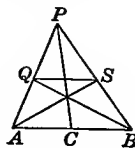
Consider D, the intersection of QS and AB, to have moved to infinity.

11. In the complete quadrangle shown in § 30 show that $AQ \cdot PS \cdot BC = - AC \cdot BS \cdot PQ.$

12. As in Ex. 11, show that $AQ \cdot PS \cdot BD = AD \cdot BS \cdot PQ.$

13. Using § 30, prove Ex. 12, page 30.

33. Descriptive Definitions of Harmonic Forms. The harmonic forms might originally have been defined in a purely descriptive fashion based upon the facts just developed. Thus, a harmonic range might have been defined as a set of four collinear points so situated that through each of the first two points there pass two opposite sides of a complete quadrangle, and through each of the other two points there passes one of the remaining sides of the quadrangle. Similar definitions might have been given for the harmonic flat pencil and the harmonic axial pencil. These are the definitions which are usually adopted when it is desired to avoid as far as may be possible the use of considerations based upon measurement.

Exercise 11. Harmonic Forms

1. Given three collinear points A, B, C, construct the fourth harmonic D from the descriptive definition of § 33.

In the constructions on this page use only an ungraduated ruler.

2. Given three concurrent lines a, b, c, construct the fourth harmonic d from the descriptive definition of § 33.

3. Given a line segment AB and an indefinite line parallel to AB, bisect AB.

4. Given a line segment AB, its midpoint C, and any point O not in the line of AB, through O draw a line parallel to AB.

5. Given two intersecting lines and the bisector of one of the angles formed by them, construct the bisector of the supplementary angle formed by the lines.

6. Given a line segment AB divided at C in the ratio $m : n$, construct a point D that divides the segment AB externally in the same ratio.

7. Dualize for space the descriptive definitions of a harmonic range and a harmonic flat pencil.

CHAPTER V

FIGURES IN PLANE HOMOLOGY

34. Homologic Plane Figures. Further interesting applications of the anharmonic ratio and illustrations of its significance occur in homologic plane figures.

Given two figures in a plane, if to every point of one figure there corresponds a point of the other, if to every line of one there corresponds a line of the other, if the lines joining corresponding points of the two figures are concurrent, and if the intersections of corresponding lines are collinear, the two figures are said to be *homologic*, or *in (plane) homology*.
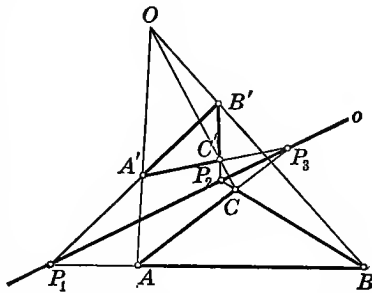
The point in which all lines joining corresponding points are concurrent is called the *center of homology*; the line which contains all intersections of corresponding lines is called the *axis of homology*.

In the above illustration the two given figures are the triangles *ABC*, *A'B'C'*. The corresponding points indicated are the three pairs of vertices, but any number of other pairs of points may be chosen. The corresponding lines are *AB* and *A'B'*, *BC* and *B'C'*, *CA* and *C'A'*. The center of homology is *O*, and the axis of homology is *o*.

THEOREM. FIGURES IN HOMOLOGY

35. *If two figures are in plane homology, it follows that:*

1. *All sets of four collinear points consisting of the center of homology, a point on the axis of homology, and two corresponding points of the figure have a common anharmonic ratio.*

2. *All sets of four concurrent lines consisting of a line through the center of homology, the axis of homology, and two corresponding lines of the figure have a common anharmonic ratio.*

3. *These two common anharmonic ratios are equal.*
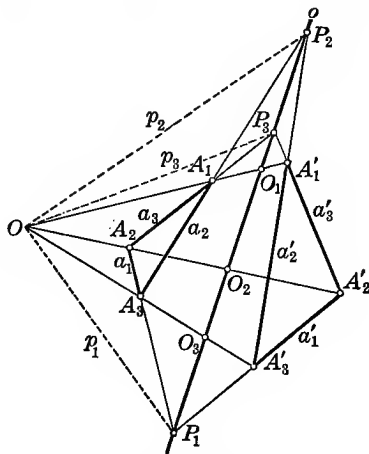
Proof. Let A_1, A_2, A_3 and A_1', A_2', A_3' be corresponding sets of three points of two homologic figures; and let OA_1A_1', OA_2A_2', OA_3A_3' intersect o in O_1, O_2, O_3 respectively; also let A_1A_2, $A_1'A_2'$ meet in P_3; A_2A_3, $A_2'A_3'$ meet in P_1; and A_3A_1, $A_3'A_1'$ meet in P_2. Then it is evident that

$$(OO_1A_1A_1') = (OO_2A_2A_2') = (OO_3A_3A_3')$$
$$= (p_3oa_3a_3') = (p_1oa_1a_1') = (p_2oa_2a_2').$$

36. Constant of Homology. The common value of the two anharmonic ratios found in the theorem of § 35 is called the *constant of homology* for the two figures.

Given a center O and an axis of homology o, we can construct a figure homologic with any given figure in such a way that to any point A of the given figure there shall correspond any selected point A' on the line OA, or that to any line a of the given figure there shall correspond any selected line a' concurrent with o and a. For the point A' selected there is a value of the constant of homology. Conversely, for any assigned value of the constant of homology, the center and axis being given, one and only one point A' corresponds to any given point A of a given figure. The fact is that *when the center, axis, and constant of homology are given, one and only one figure homologic with a given figure can be constructed.*

Examples of such constructions are given on page 40.

There are several notable special cases of the homologic relation. One such case is that of *harmonic homology* in which the constant of homology is -1.

Another case is that in which the axis of homology is the line at infinity. Then all pairs of corresponding lines are parallel, and the figures are *similar*. In this case the constant of homology $(OO_1A_1A_1')$ becomes OA_1/OA_1', which may be shown to be the *ratio of similitude*. If in addition the constant is -1, the center O is a center of symmetry for the figure composed of the two homologic figures.

A third case is that in which the center of homology is a point at infinity. Then the constant is O_1A_1'/O_1A_1. If also the homology is harmonic, the axis of homology is an axis of symmetry for the figure composed of the two homologic figures.

Exercise 12. Figures in Homology

1. Given a center of homology O, an axis of homology o, and any triangle $A_1A_2A_3$, construct, homologic with $A_1A_2A_3$, a triangle that has a vertex A_1' at a given position on OA_1.

As to the possibility of this construction, see Ex. 13, page 13.

Given a triangle $A_1A_2A_3$, the center of homology O, the axis of homology o, and the following constants of homology, construct the figures homologic with $A_1A_2A_3$:

2. 3.   **3.** -3.   **4.** -1.   **5.** 1.

6. In each of Exs. 2–5, if $A_1'A_2'A_3'$ is the result of the construction, use the same center, axis, and constant of homology to construct the figure homologic with $A_1'A_2'A_3'$.

7. If there are three coplanar figures f_1, f_2, f_3, and if for a given center and a given axis of homology two of them, f_1 and f_2, are homologic with the constant c_3, and if f_2 and f_3 are homologic with the constant c_1, then f_1 and f_3 are homologic with the constant of homology c_2 equal to $c_3 \cdot c_1$.

8. Consider carefully Exs. 2–7 for the case in which O is a point at infinity and the case in which o is the line at infinity.

9. For a given center, axis, and constant of homology construct the line corresponding to the line at infinity.

This line is called the *vanishing line*.

10. In Exs. 2–5 determine the vanishing line.

11. Under what conditions is the vanishing line at infinity?

Given a circle, the axis of homology o, and the center of homology O, construct the figure homologic with the circle under each of the following conditions:

12. The vanishing line does not meet the circle.

13. The vanishing line is a secant of the circle.

14. The vanishing line is a tangent to the circle.

CHAPTER VI

PROJECTIVITIES OF PRIME FORMS

37. Projective One-Dimensional Prime Forms. Whenever there exists between the elements of two one-dimensional prime forms a one-to-one correspondence such that, by means of a finite number of operations of projection and section, it is possible to pass simultaneously from all the elements of one prime form to the corresponding elements of the other, the two prime forms are said to be *projectively related* or to be *projective.*

The correspondence existing between two projective one-dimensional prime forms is called a *projectivity.*

The symbol $\overline{\wedge}$ is frequently used for "is projective with."

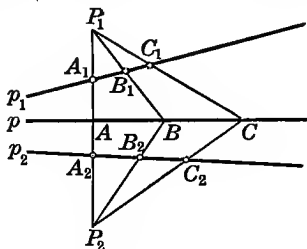### THEOREM. PROJECTIVE PRIME FORMS

38. *Prime forms which are projective with the same prime form are projective with each other.*

Proof. If n_1 operations yield a form f_2 from a form f_1, and if n_2 operations yield f_3 from f_2, then $n_1 + n_2$ operations yield f_3 from f_1.

It is not the purpose of this book to discuss the projectivities of prime forms other than one-dimensional ones, but it may be stated that between the prime forms of higher dimensions there exist relations which have the same general character as those just defined, and that these relations are also called projectivities. A complete study of projective geometry would include the consideration of these higher projectivities and of many important geometric propositions relating to them.

41

THEOREM. PROJECTIVITY OF TRIADS

39. *Between two one-dimensional prime forms, each of which consists of three elements in a specified order, there exists a projectivity.*

Proof. If either of the prime forms is not a range, it is possible by operations of projection and section to obtain from it a range which is projective with it. Hence it is necessary to prove the proposition only for the case in which both prime forms are ranges of three points.

This theorem is what was formerly called a lemma, a proposition inserted merely for the purpose of leading up to a fundamental theorem; in this case, the one given in § 40. The proof involves the consideration of the three cases below.

1. *The ranges may be coplanar and upon different bases.*

Let the ranges be $A_1B_1C_1$ on the base p_1 and $A_2B_2C_2$ on the base p_2, and let both ranges be in the same plane.

Draw the line through A_1 and A_2, and on it take any points P_1 and P_2, not coincident with A_1 and A_2 respectively.

Draw P_1B_1, P_1C_1, P_2B_2, P_2C_2; and let P_1B_1, P_2B_2 intersect at B, and let P_1C_1, P_2C_2 intersect at C.
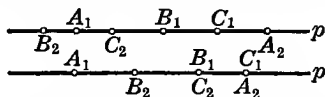
Through B and C draw the line p, cutting A_1A_2 at A.

Then   range $A_1B_1C_1 \overline{\wedge}$ range $ABC \overline{\wedge}$ range $A_2B_2C_2$.

Hence   range $A_1B_1C_1 \overline{\wedge}$ range $A_2B_2C_2$.   § 38

2. *The ranges may be coplanar and upon the same base.*

Let the ranges $A_1B_1C_1$ and $A_2B_2C_2$ be on the same base p.

It is here assumed that the points A_2, B_2, C_2 may be the same except for order as the points A_1, B_1, C_1, or may be partly or wholly distinct from these points.

In any case from a center P, exterior to the base p, project $A_2B_2C_2$ upon a new base p'. Then apply Case 1 to show that the range so obtained on the base p' is projective with the range $A_1B_1C_1$. It then follows that the range $A_2B_2C_2$ is projective with the range $A_1B_1C_1$.

3. *The ranges may not be coplanar.*
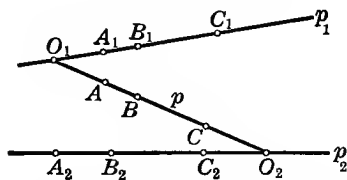
Let the bases p_1 and p_2 of the ranges $A_1B_1C_1$ and $A_2B_2C_2$ not be in any one plane.

Join any point O_1 of the base p_1 to any point O_2 of the base p_2 by the line p. Select three points A, B, C on p.

Then, by Case 1, it follows that

$$\text{range } A_2B_2C_2 \mathbin{\overline{\wedge}} \text{range } ABC,$$

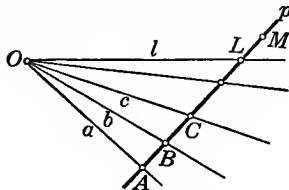and $$\text{range } A_1B_1C_1 \mathbin{\overline{\wedge}} \text{range } ABC.$$

Therefore $$\text{range } A_2B_2C_2 \mathbin{\overline{\wedge}} \text{range } A_1B_1C_1.$$

From these results the existence of a projectivity follows.
PG

40. *Between two one-dimensional prime forms there exists one and only one projectivity in which three elements of one form, in a specified order, correspond to three elements of the other form, also in a specified order.*

Proof. Let us first consider three special cases.

1. *Suppose that the two prime forms are a flat pencil and the range obtained by cutting the pencil by a line.*

Let the lines of the flat pencil be a, b, c, $\cdots$, l, $\cdots$ and let these lines be cut by a line p in the points A, B, C, $\cdots$, L, $\cdots$. Then these prime forms are perspective in such a way that a, A; b, B; c, C; $\cdots$; l, L; $\cdots$ are pairs of corresponding elements.

Assume that, if possible, a second projectivity exists in which the first three of these pairs of elements correspond, but in which l corresponds to M and not to L.

Then, from the perspectivity,

$$(abcl) = (ABCL);$$

and, from the second projectivity,

$$(abcl) = (ABCM).$$

Then $$(ABCL) = (ABCM),$$

which is impossible unless $L \equiv M$.

Hence the second projectivity cannot exist, and the only projectivity existing between the forms is the perspectivity.

2. *Suppose that the two prime forms are an axial pencil and the range obtained by cutting the pencil by a line.*
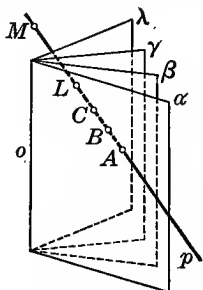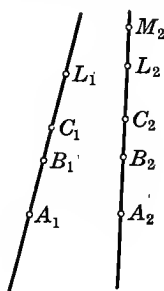

FIG. 1   FIG. 2

Let the planes of the axial pencil be α, β, γ, $\cdots$, λ, $\cdots$ (Fig. 1), and let p cut the planes in A, B, C, $\cdots$, L, $\cdots$.

A proof similar to that on page 44 should be given by the student.

3. *Suppose that the two prime forms are ranges.*

Let A_1, B_1, C_1 (Fig. 2) be three points of the first range, and let them correspond respectively to the points A_2, B_2, C_2 of the second. A projectivity exists between these sets of three points, and this projectivity can be extended to include the whole of both ranges. Let L_1 correspond to L_2.

If possible let there be a second projectivity connecting the ranges, in which L_1 corresponds to M_2, a point other than L_2. Then, from the two projectivities,

$$(A_1B_1C_1L_1) = (A_2B_2C_2L_2)$$

and $$(A_1B_1C_1L_1) = (A_2B_2C_2M_2).$$

Therefore $$(A_2B_2C_2L_2) = (A_2B_2C_2M_2),$$

which is impossible unless $L_2 \equiv M_2$.

Hence in this case two projectivities cannot exist.

4. *Consider now the general case.*

Let the two forms be f, f', and let the three pairs of corresponding elements be 1, 1'; 2, 2'; 3, 3'.
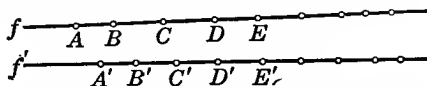

If f is a range, let the elements 1, 2, 3, 4, 5, $\cdots$ be A, B, C, D, E, $\cdots$; but if f is not a range, let the elements 1, 2, 3, 4, 5, $\cdots$ be cut by a line in the points A, B, C, D, E, $\cdots$. Similarly, if f' is a range, let the elements 1', 2', 3', 4', 5', $\cdots$ be A', B', C', D', E', $\cdots$; but if f' is not a range, let the elements 1', 2', 3', 4', 5', $\cdots$ be cut by a line in the points A', B', C', D', E' $\cdots$.

Between the ranges $ABCD \cdots$ and $A'B'C'D' \cdots$ there is just one projectivity in which A, A'; B, B'; C, C' are corresponding elements. Let D, D' be corresponding elements.

Then   form $1234 \cdots \overline{\wedge}$ range $ABCD \cdots$

$\overline{\wedge}$ range $A'B'C'D' \cdots \overline{\wedge}$ form $1'2'3'4', \cdots$.

Hence   form $1234 \cdots \overline{\wedge}$ form $1'2'3'4' \cdots$.

Suppose that, if possible, between f and f' there is a second projectivity in which 1, 1'; 2, 2'; 3, 3' are pairs of corresponding elements and 4, 5' are corresponding elements.

Then   range $ABCD \cdots \overline{\wedge}$ form $1234 \cdots$,

and   form $1'2'3'5' \cdots \overline{\wedge}$ range $A'B'C'E' \cdots$;

whence   range $ABCD \cdots \overline{\wedge}$ range $A'B'C'E' \cdots$.

Accordingly there would be a second projectivity between the ranges $ABC \cdots$ and $A'B'C' \cdots$, in which A, A'; B, B'; C, C' are pairs of corresponding points. This, however, is impossible. Hence the theorem is true in all cases.

41. COROLLARY. *There is one projectivity and only one projectivity between one-dimensional forms on the same base which makes three distinct elements of a one-dimensional form correspond each to itself. This projectivity makes every element of the form correspond to itself.*

Exercise 13. Projectivities of Triads

1. If A, B_1, B_2, C_1, C_2 are five distinct points on a line, find a set of projections and sections, minimum in number, which connects the triads AB_1C_1 and AB_2C_2.

2. Examine Ex. 1 for the case in which B_1 coincides with B_2.

3. If A, B, C are three points on a line, find the set of projections and sections, minimum in number, which connects these points with themselves in any selected order.

Consider Ex. 3 for each of the six possible orders of the points.

4. If A_1, A_2, B_1, B_2, C_1, C_2 are six distinct collinear points, find a set of projections and sections, minimum in number, which connects the triads $A_1B_1C_1$ and $A_2B_2C_2$.

5. If A, B_1, B_2, C_1, C_2 are five points, no four of which are coplanar, find a set of projections and sections, minimum in number, which connects the triads AB_1C_1 and AB_2C_2.

This is a special case of a more general problem.

6. In how many ways can a projectivity between the triads specified in Ex. 5 be established?

7. If A_1, A_2, B_1, B_2, C_1, C_2 are six points in space, no four of which are coplanar and no three of which are collinear, the triads $A_1B_1C_1$ and $A_2B_2C_2$ are projective. Specify a set of projections and sections which constitutes such a projectivity.

8. Investigate the possibility of establishing a projectivity between the triads specified in Ex. 5 such that any fourth point D_1 in the plane AB_1C_1 shall correspond to any fourth point D_2 in the plane AB_2C_2.

THEOREM. METRIC CONDITION FOR A PROJECTIVITY

42. *If between the elements of two one-dimensional prime forms there exists a one-to-one correspondence such that the anharmonic ratio of every set of four elements of one prime form is equal to that of the set of four corresponding elements of the other prime form, the correspondence is a projectivity.*
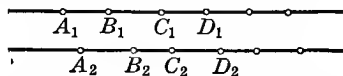
Proof. It is sufficient to consider the case of two ranges because if either of the given prime forms is not a range, it is possible by operations of projection and section to obtain from it a range which is projective with it.

From § 39 it follows that a projectivity exists between three points A_1, B_1, C_1 of the first range and the three corresponding points A_2, B_2, C_2 of the second range. Then if a fourth pair of corresponding points D_1, D_2 on the ranges are found, it follows by hypothesis that

$$(A_1B_1C_1D_1) = (A_2B_2C_2D_2).$$

Also let range $A_1B_1C_1D_1 \mathbin{\overline{\wedge}}$ range $A_2B_2C_2D_2'$.

Then $(A_2B_2C_2D_2') = (A_1B_1C_1D_1),$ § 26

and so $(A_2B_2C_2D_2') = (A_2B_2C_2D_2).$

Accordingly D_2' coincides with D_2; and therefore

range $A_2B_2C_2D_2 \mathbin{\overline{\wedge}}$ range $A_1B_1C_1D_1.$

Thus, any fourth point D_1 of the first range has the same corresponding point D_2 in the given one-to-one correspondence as it has in the projectivity between the ranges which is determined by the triads of points $A_1B_1C_1$ and $A_2B_2C_2$. Hence the correspondence is this projectivity.

If it is given that two one-dimensional prime forms are projective, it should be noted that it is necessary only that three elements of one of them and the corresponding elements of the other be specified in order that by the construction of § 39 the whole projectivity may be established ; and this construction furnishes a standard method of establishing a projectivity.

Associated with the general results of §§ 39–42, several which are special in their nature and application are considered in §§ 43–46.

Exercise 14. Projectivity of Prime Forms

1. Given three points A_1, B_1, C_1 of a range and the corresponding points A_2, B_2, C'_2 of a second range projective with the first, the bases being different, construct the point D_2 of the second range corresponding to a fourth point D_1 of the first.

2. Consider Ex. 1 when the point D_1 is at infinity.

3. Consider Ex. 1 when the ranges are coplanar and D_1 is their intersection.

4. Obtain simplified constructions for Exs. 1 and .2 when A_1 and A_2 coincide at the intersection of the ranges.

5. Consider Ex. 1 when the bases are coincident.

6. Consider Ex. 5 when A_1, B_2; A_2, B_1; C_2, D_1 are all pairs of coincident points.

7. Prove the theorem suggested by the result of Ex. 6.

8. If two ranges are projective, to every harmonic range in one of them there corresponds a harmonic range in the other.

9. Assuming that if four points A, B, C, D are properly divided into pairs a common pair of harmonic conjugates with respect to these pairs exists, prove the converse of Ex. 8.

10. Investigate the question of dividing a set of four collinear points into pairs so that a common pair of harmonic conjugates may exist.

THEOREM. PROJECTIVE RANGES IN PERSPECTIVE

43. *Two projective ranges whose bases are not coplanar
are perspective.*
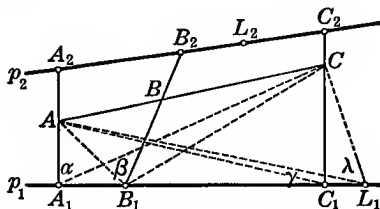
Proof. Let $A_1 B_1 C_1 \cdots L_1 \cdots$ on the base p_1 and $A_2 B_2 C_2$
$\cdots L_2 \cdots$ on the base p_2 be projective ranges in which
A_1, A_2; B_1, B_2; C_1, C_2; $\cdots$ are corresponding elements,
their bases not being coplanar.

On the line $A_1 A_2$ take a point A, and let C be the point
in which the line $C_1 C_2$ intersects the plane determined by
the point A and the line $B_1 B_2$. Let the line AC intersect
the line $B_1 B_2$ in the point B.

Denote by α, β, γ, $\cdots$, λ, $\cdots$ the planes determined by the
line ABC and the points A_1, B_1, C_1, $\cdots$, L_1, $\cdots$ respectively.

The axial pencil $\alpha\beta\gamma \cdots \lambda \cdots$ is perspective with the
range $A_1 B_1 C_1 \cdots L_1 \cdots$ and cuts the line p_2 in a range pro-
jective with the range $A_1 B_1 C_1 \cdots L_1 \cdots$.

Moreover, in the projectivity thus established the points
A_2, B_2, C_2 correspond to the points A_1, B_1, C_1 respectively.
This projectivity is therefore the same as the one that was
originally assumed to exist between the ranges (§ 40).

Hence from the range $A_1 B_1 C_1 \cdots$ on the base p_1 it is
possible to pass to the range $A_2 B_2 C_2 \cdots$ on the base p_2
by one operation of projection from the axis ABC and
one operation of section by the line p_2. Hence the pro-
jectivity existing between the ranges is a perspectivity.

THEOREM. CONDITION FOR PERSPECTIVE RANGES

44. *Two projective ranges on different bases in a plane
π are perspective if and only if the point common to them is
self-corresponding in the projectivity.*
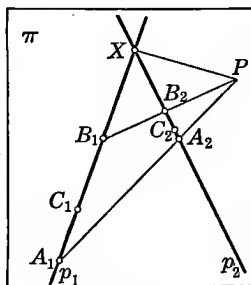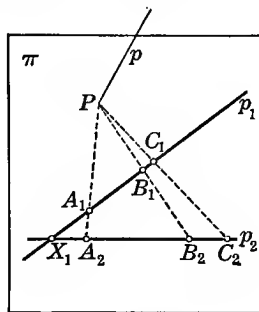
Fig. 1   Fig. 2

Proof. First, let the ranges $A_1B_1C_1 \cdots$ (Fig. 1) on the
base p_1 and $A_2B_2C_2 \cdots$ on the base p_2 be projective in such
a way that X, the intersection of the two ranges, is self-
corresponding in the projectivity.

Let the lines A_1A_2 and B_1B_2 intersect at P, and draw PX.

The projectivity is completely determined by the corre-
spondence of A_1, B_1, X with A_2, B_2, X, but this correspond-
ence is established by the projection of A_1, B_1, X from the
center P and the section of the resulting flat pencil by
the line p_2. Hence the ranges are perspective.

Next, let the ranges be perspective (Fig. 2).

Let the intersection of p_1 and p_2, regarded as a point of
the first range, be denoted by X_1. Since its correspond-
ing point would be found by cutting the projector of X_1
from a center P in the plane π, or from an axis p not in
this plane, as the case may be, by the line p_2, it is the
intersection X_1. That is, X_1 is self-corresponding.

45. *Two projective (but not perspective) ranges on different bases in a plane are both perspective with the same range on any line that does not pass through the intersection of the bases of the given ranges.*
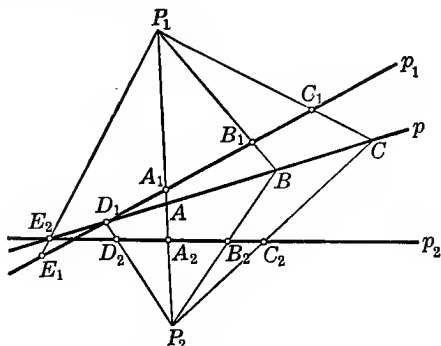
Proof. The construction used in Case 1 of § 39 establishes the existence of a range which is perspective with each of the two given coplanar ranges. It is now necessary to show that the points P_1 and P_2 can be so selected that the line ABC shall be any line which does not pass through the intersection of p_1 and p_2.

Let p be any line not concurrent with p_1 and p_2. Let the intersection of p_1 and p be D_1, and that of p_2 and p be E_2. Suppose that the point D_2 corresponding to D_1, and the point E_1 corresponding to E_2, have been found by the method of Case 1 of § 39, or by any other suitable method. The problem now is to choose points P_1 and P_2 such that the line ABC shall coincide with the line p or D_1E_2.

Draw D_2D_1, and let this line meet A_2A_1 in P_2. Draw E_1E_2, and let this line meet A_1A_2 in P_1. Join P_1 to B_1, C_1, $\cdots$, and let these lines meet D_1E_2 in B, C, $\cdots$.

Then   range $ABCD_1E_2 \cdots \overline{\underline{\wedge}}$ range $A_1B_1C_1D_1E_1 \cdots$.

Hence range $ABCD_1E_2 \cdots \overline{\wedge}$ range $A_2B_2C_2D_2E_2 \cdots$.

But the triad AD_1E_2 is perspective with the triad $A_2D_2E_2$, and these triads determine completely the nature of the projectivity between the range $ABCD_1E_2 \cdots$ and the range $A_2B_2C_2D_2E_2 \cdots$.

Therefore the two ranges $A_1B_1C_1D_1E_1 \cdots$ on p_1 and $A_2B_2C_2D_2E_2 \cdots$ on p_2 are perspective with a range on the given line p.

Exercise 15. Projective and Perspective Forms

1. State and prove the dual of § 43.

State and prove the duals of each of the following :

2. § 44, in the plane.   **5.** § 45, in the plane.

3. § 44, in space.   **6.** § 45, in space.

4. Ex. 3, in the bundle.   **7.** Ex. 6, in the bundle.

Solve the duals of each of the following :

8. Ex. 1, page 49, in space.

9. Ex. 3, page 49, in the plane.

10. Ex. 3, page 49, in space.

11. Ex. 1, page 49, in the plane, considering the case in which the ranges are coplanar.

12. If three ranges in a plane have concurrent bases and if two of the ranges are perspective with the third, these two ranges are perspective with each other.

13. Three fixed lines p_1, p_2, p_3 radiate from A_1, one of three fixed collinear points A_1, A_2, A_3. A point P_1 moves on p_1, and the lines A_2P_1, A_3P_1 cut p_2, p_3 respectively in P_2, P_3. Show that P_2P_3 has a fixed point.

46. Special Case. In the construction described in § 39 the only limitation upon the position of the points P_1 and P_2 was that P_1 should not be at A_1 and that P_2 should not be at A_2. The necessity for this limitation is due to the complete failure of the construction which would result from the coincidence of the lines P_1A_1, P_1B_1, P_1C_1. We shall now consider a special case which is noteworthy because it puts in evidence a line that has important relations to all pairs of corresponding points of the ranges and furnishes a basis for several simple constructions.
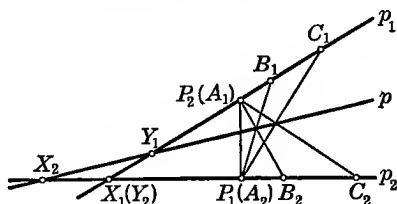
Let P_1 be taken at A_2, and P_2 at A_1. Then the line p is determined by the intersection of $P_1B_1(A_2B_1)$ and P_2B_2 (A_1B_2) and the intersection of $P_1C_1(A_2C_1)$ and $P_2C_2(A_1C_2)$. It contains likewise the intersections of A_2D_1, A_1D_2; A_2E_1, A_1E_2; and so on. It can now be shown that this line p can be located independently of the placing of P_1 and P_2 on the particular line A_1A_2.

If the ranges $A_1B_1C_1 \cdots$ and $A_2B_2C_2 \cdots$ are projective but not perspective, the point of intersection of p_1 and p_2 will not be self-corresponding. Regarding this point as belonging to the range $A_1B_1C_1 \cdots$, call it X_1. In the other range there will correspond to X_1 a point X_2. Regarding the intersection as a point of range $A_2B_2C_2 \cdots$, call it Y_2. To it will correspond a point Y_1 of the first range. Then A_1X_2 and A_2X_1 intersect on the line p. But their intersection is X_2.

Similarly, Y_1, the intersection of A_1Y_2 and A_2Y_1, is on the line p. Accordingly the line p is the line determined by the points of the first and second ranges which correspond to the point of intersection of the ranges regarded as a point of the second and first ranges respectively. It follows that the line p is determined by the projectivity between the ranges and bears the same relation to any two corresponding points of the ranges that it does to A_1, A_2. Accordingly not only do A_1B_2, A_2B_1; A_1C_2, A_2C_1; A_1D_2, A_2D_1; $\cdots$ intersect on p, but so do B_1C_2, B_2C_1; B_1D_2, B_2D_1; $\cdots$; C_1D_2, C_2D_1; C_1E_2, C_2E_1; $\cdots$; and so on.

If the ranges $A_1B_1C_1 \cdots$ and $A_2B_2C_2 \cdots$ are perspective, their common point X is self-corresponding. Let p be the line joining X to the intersection of A_1B_2 and A_2B_1.

We then see that the flat pencils $A_1(A_2B_2C_2 \cdots X \cdots)$ and $A_2(A_1B_1C_1 \cdots X \cdots)$ are projective and have a common line A_1A_2. Hence these pencils are perspective, and the axis of perspective is p. Hence, as in the other case, A_1B_2, A_2B_1; A_1C_2, A_2C_1; A_1D_2, A_2D_1; $\cdots$; B_1C_2, B_2C_1; $\cdots$; C_1D_2, C_2D_1 all intersect on p.

Exercise 16. Perspective Forms

1. If a simple reëntrant hexagon has its first, third, and fifth vertices on one straight line of a plane π and has its second, fourth, and sixth vertices on a second straight line of that ·plane, the intersections of the first and fourth, the second and fifth, and the third and sixth sides are collinear.
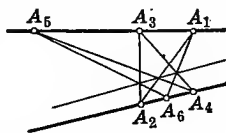
2. State and prove the proposition dual in a plane to Ex. 1.

3. State and prove the proposition dual in space to Ex. 1.

4. State and prove the proposition dual in the bundle to Ex. 3

PROBLEM. LINE TO AN INACCESSIBLE POINT

47. *To draw a line which shall pass through the inaccessible intersection of two given straight lines and also through a given point O of their plane.*
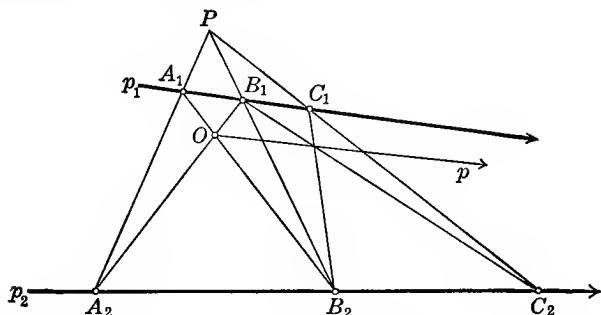
Solution. To draw a line which shall satisfy the first condition, let the given lines be p_1 and p_2. Select a point P in the plane of the lines p_1 and p_2, but not on either line, and through P pass three lines cutting p_1 in A_1, B_1, C_1 and p_2 in A_2, B_2, C_2.

Since the ranges $A_1B_1C_1$ and $A_2B_2C_2$ are perspective, the pairs of lines A_1B_2, A_2B_1; A_1C_2, A_2C_1; B_1C_2, B_2C_1 intersect on a line p which passes through the intersection of the lines p_1 and p_2 (§ 46).

Now to satisfy the second condition we must so choose the point P that the line p will pass through O.

Through O draw two lines and let them intersect p_1 in A_1 and B_1 and intersect p_2 in A_2 and B_2. Take P to be the intersection of A_1A_2 and B_1B_2. Through P draw another line cutting p_1 in C_1 and p_2 in C_2. Then the line p which is determined by the intersections of two of the three pairs of lines A_1B_2, A_2B_1; A_1C_2, A_2C_1; B_1C_2, B_2C_1 satisfies the two given conditions.

PROBLEM. LINE PARALLEL TO A GIVEN LINE

48. *Given two parallel lines and a point in their plane,* *to draw through the point a line parallel to the given lines.*

The solution is left for the student, who should write out the proof after the method used in § 47.

The case in § 49 is somewhat different.

PROBLEM. LINE PARALLEL TO A GIVEN LINE

49. *Given in a plane a point O, a line p_1 not passing through O, and a parallelogram none of whose sides is known to be parallel to p_1, to draw through O a line parallel to p_1.*
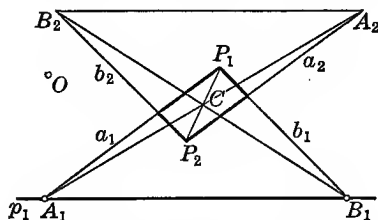
Solution. This problem may be reduced to the preceding one as follows:

Let the adjacent sides a_1 and b_1 of the parallelogram meet p_1 in A_1 and B_1 respectively. Let C be any point on the diagonal P_1P_2, and let the lines CA_1 and CB_1 meet a_2 and b_2 in A_2 and B_2 respectively.

Then the triangles $P_1A_1B_1$ and $P_2A_2B_2$ are homologic, and since the intersections of P_1A_1, P_2A_2 and of P_1B_1, P_2B_2 are at infinity, the axis of homology is at infinity.

Hence A_2B_2 is parallel to p_1.

We now have two parallel lines A_2B_2 and p_1, and by means of § 48 we can draw through the given point O a line parallel to the given line p_1.

THEOREM. SIMILAR RANGES

50. *If in two projective ranges the points at infinity are corresponding points, the ratios of all pairs of corresponding segments are the same, and conversely.*

Proof. Let the points at infinity of two ranges be J_1 and J_2, and suppose that the points J_1, A_1, B_1, C_1, $\cdots$ of the first range correspond respectively to the points J_2, A_2, B_2, C_2, $\cdots$ of the second range.

Then $\qquad\qquad (A_2B_2C_2J_2) = (A_1B_1C_1J_1),$

or $\qquad\qquad A_2C_2 : B_2C_2 = A_1C_1 : B_1C_1.$ $\qquad$ § 22

Hence $\qquad\quad A_2C_2 : A_1C_1 = B_2C_2 : B_1C_1.$

Similarly, it can be shown that

$$A_2B_2 : A_1B_1 = A_2C_2 : A_1C_1 = A_2D_2 : A_1D_1 = \cdots$$
$$= B_2C_2 : B_1C_1 = B_2D_2 : B_1D_1 = \cdots = r,$$

where r is independent of the pair of corresponding segments involved.

Conversely, suppose that all the pairs of corresponding segments have the same ratio.

Let A_1, A_2; B_1, B_2; C_1, C_2; J_1, K_2 be pairs of corresponding points, J_1 being the point at infinity of the first range.

Then $\qquad\qquad (A_1B_1C_1J_1) = (A_2B_2C_2K_2),$

or $\qquad\qquad \dfrac{A_1C_1}{B_1C_1} = \dfrac{A_2C_2}{B_2C_2} : \dfrac{A_2K_2}{B_2K_2}.$

But $\qquad\quad A_1C_1 : A_2C_2 = B_1C_1 : B_2C_2.$

Therefore $\qquad A_2K_2 : B_2K_2 = 1,$

and hence K_2 must be at infinity and should be called J_2.

Accordingly J_1 and J_2, the points at infinity of the two ranges, are corresponding points.

51. Similar Ranges. Two projective ranges whose points at infinity are corresponding points are said to be *similar*.

An example of similar ranges is furnished by sections of a flat pencil by parallel lines, or by sections of a flat pencil of parallel rays by any two lines.

If similar ranges are on the same base, the point at infinity is a self-corresponding point. If the ratio of corresponding segments (§ 50) is $1 : 1$, there is no other such point, but in any other case a second point exists.

52. Congruent Ranges. Similar ranges in which the ratio of corresponding segments is unity are said to be *congruent*.

If two similar ranges have parallel bases, their common point at infinity is self-corresponding and the ranges are perspective.

53. Similar and Congruent Pencils. The terms *similar* and *congruent* are applied to certain special cases of flat pencils and axial pencils, the mere mention of this fact being sufficient for the present treatment.

Two projective flat pencils whose bases are at infinity are said to be *similar* if linear sections of these pencils are similar ranges.

In this and in similar cases the student should draw the figure.

Two projective flat pencils whose bases are in the finite part of the plane are said to be *equal* or *congruent* if every pair of lines of one pencil contains an angle equal to the angle contained by the corresponding pair of lines of the other pencil.

Similar and congruent axial pencils are defined in the same way.

Flat pencils and axial pencils are said to be *proper* or *improper* according as their bases are or are not in the finite part of space.

PG

Exercise 17. Projectivities

1. There exist infinitely many sets of projections and sections which connect a prime form with itself.

2. There exist infinitely many sets of projections and sections which connect two projective prime forms.

3. A plane quadrangle is projective with a properly chosen parallelogram.

4. A plane quadrangle is projective with a properly chosen square.

5. A plane quadrangle is projective with every square.

6. Every two plane quadrangles are projective. Specify a set of operations that connects the quadrangles.

7. A triangle and any point in its plane, but not in its perimeter, are projective with any equilateral triangle and its center.

8. Solve the duals of Ex. 6.   .

9. Two ranges on the same base are projective if they are so related that every pair of corresponding points is a harmonic conjugate with respect to a given pair of points on the base.

Apply Ex. 12 on page 30.

10. If J_1 and K_2 are the points at infinity of two projective ranges $A_1 B_1 C_1 \cdots$ and $A_2 B_2 C_2 \cdots$, J_2 and K_1 being the points corresponding to them, then $A_1 K_1 \cdot A_2 J_2 = B_1 K_1 \cdot B_2 J_2 = \cdots$.

11. If the ranges in Ex. 10 are on the same base, and if O_1 is the midpoint of $J_2 K_1$ and if O_2 is the point corresponding to O_1, then $O_1 A_1 \cdot O_1 A_2 - O_1 A_1 \cdot O_1 J_2 - O_1 K_1 \cdot O_1 A_2 + O_1 K_1 \cdot O_1 O_2 = 0$.

12. If A_1 and A_2 in Ex. 11 coincide with A, it follows that $\overline{O_1 A}^2 + O_1 K_1 \cdot O_1 O_2 = 0$.

13. If P_4 is a fixed point on the side $P_1 P_2$ of a given triangle $P_1 P_2 P_3$, and if a moving line cuts $P_1 P_2$ in P_4, $P_2 P_3$ in Q, and $P_3 P_1$ in R, $P_1 Q$ and $P_2 R$ meeting in P, then $P_3 P$ cuts $P_1 P_2$ in a fixed point.

14. Generalize Ex. 12, page 53.

15. Given $A_1B_1C_1 \cdots$, $A_2B_2C_2 \cdots$, and $A_3B_3C_3 \cdots$, three projective but not perspective ranges whose bases p_1, p_2, p_3 are coplanar and concurrent, prove that, if any triad of corresponding points is collinear, there exists a range $ABC \cdots$ which is perspective with each of the three given ranges.

16. If P_1, P_2, P_3 in Ex. 15 are the centers from which the three ranges on p_1, p_2, p_3 respectively are projected into the range $ABC \cdots$, each of the sides of the triangle $P_1P_2P_3$ cuts a pair of the given ranges in points which correspond in one of the projectivities.

17. If in Ex. 16 P moves along the base p of the range $ABC \cdots$, the lines P_1P, P_2P and P_3P trace on any fourth line of the plane three projective ranges which have one point that is self-corresponding for all three projectivities.

18. If X_1, X_2; Y_1, Y_2; A_1, A_2 are pairs of corresponding points of two coplanar projective ranges, and if (§ 46) X_1 and Y_2 coincide at the intersection of the bases, determine the position of B_2 which corresponds to any fourth point B_1 of the first range, basing the solution on § 46.

19. Solve the plane dual of Ex. 18.

20. Solve the space dual of Ex. 18.

In solving the remainder of the problems in this exercise, use only the ungraduated ruler. These exercises are chiefly due to Steiner, one of the founders of the science of projective geometry.

21. Given a segment AB extended to twice its length, divide it into any number of equal parts.

22. In an indefinite straight line given a segment AB divided at C in the ratio of two given whole numbers, draw through any given point a line parallel to the given line.

23. Given two parallel lines p and p', and given on p a segment AB, from any given point on p lay off a segment of p equal to any given multiple of AB.

24. Divide AB in Ex. 23 in the ratio of two given integers.

25. Given a parallelogram, divide any segment in its plane in the ratio of two given integers.

26. In a plane, given three parallel lines which cut a fourth line in the ratio of two given integers, draw through a given point a line parallel to a given line.

27. If two parallel line segments which have to each other the ratio of two given integers are given, draw a line parallel to a given third line.

28. Given two parallel lines and a segment divided in the ratio of two given integers, draw through a given point a line parallel to a given fourth line.

29. Given two nonparallel segments, each divided in the ratio of two given integers, draw a line parallel to a given line.

30. Given a square, a point, and a line, all in one plane, draw through the point a line perpendicular to the given line.

31. Given a square and a right angle, both in the same plane, bisect the angle.

32. Given a square and an angle, both in the same plane, construct any multiple of the angle.

In each of the following problems, in addition to the data mentioned, one circle fully drawn and its center are assumed to be given.

33. Through a given point draw a line parallel to a given line.

34. Through a given point draw a line perpendicular to a given line.

35. Through a given point draw a line which shall make with a given line an angle equal to a given angle.

**36.** From a given point draw a line parallel and equal to a given line.

37. Determine the intersections, if any, of a given line and a circle of given center and radius.

38. Determine the intersections, if any, of two circles of given centers and given radii.

CHAPTER VII

SUPERPOSED PROJECTIVE FORMS

54. Superposed Projective Forms. Hitherto only incidental reference has been made to the existence of projective forms on the same base, but the general results obtained apply to them except when the contrary is indicated. It is proposed now to consider some special properties of one-dimensional projective prime forms on the same base. Such forms are called *superposed forms.*
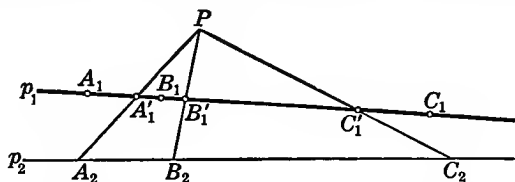
In the first place, the existence of superposed projective forms is established if, when two projective ranges are on different bases, as $A_1B_1C_1 \cdots$ on the base p_1 and $A_2B_2C_2 \cdots$ on the base p_2, the second range is projected on the base p_1 from a center P so taken as not to be on the line A_1A_2. The result is a range $A_1'B_1'C_1' \cdots$ on the base p_1 which is projective but not identical with $A_1B_1C_1 \cdots$, since A_1 and A_1' are not coincident.

In the second place, if each of two superposed projective forms is operated upon by section or projection, the resulting forms are superposed and projective.

THEOREM. EXISTENCE OF SUPERPOSED PROJECTIVE FORMS

55. *There exist pairs of projective prime forms which have common bases and which have not all their corresponding elements coincident.*

The proof of this theorem is evident from § 54.

THEOREM.   INVARIANCE UNDER PROJECTION

56. *If from two superposed projective prime forms two other superposed prime forms are obtained by projection or by section, the latter forms are projective.*

The proof of this theorem is evident from § 54.

57. Self-corresponding, or Double, Elements. It has been shown (§ 39) that three points A_1, B_1, C_1 of a line are projective with any three points of the same line, even though some or all of the two sets of three are the same; and it follows from § 40 that the correspondence of the three pairs of points establishes for all points of the line a projectivity in which some of the points may coincide with their corresponding points. It is evident that similar considerations apply to flat pencils and to axial pencils.

Elements of superposed projective prime forms that coincide with those to which they correspond are called *self-corresponding elements*, or *double elements*; and the determination of the number of such elements is a problem of importance. Since from superposed flat pencils or axial pencils we may by section obtain superposed ranges whose self-corresponding points are situated on the self-corresponding elements of the pencils, the discussion of this question is substantially the same for all one-dimensional prime forms, and hence will be limited to the case of superposed ranges.

THEOREM. SELF-CORRESPONDING ELEMENTS

58. *The number of self-corresponding elements of two distinct superposed projective one-dimensional prime forms is two, one, or none, and all three of these cases occur.*

Proof. The proof consists of the following four parts:

1. *There may be two self-corresponding elements.*

If A_1, B_1, C_1, C_2 are four points on a line p, the triads $A_1B_1C_1$ and $A_1B_1C_2$ determine a projectivity between distinct ranges on p with A_1 and B_1, but not with either C_1 or C_2, as self-corresponding points.

2. *There may be just one self-corresponding element.*
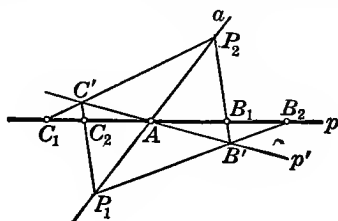
On a line p take four points A, B_1, B_2, C_1, and through A pass two lines a and p'. On a take any point P_1. Let the line P_1B_2 cut p' in B', the line $B'B_1$ cut a in P_2, the line P_2C_1 cut p' in C', and the line P_1C' cut p in C_2.

The triads AB_1C_1 and AB_2C_2 determine a projectivity between distinct ranges on p in which the point A is self-corresponding. Moreover, each of these ranges on p is perspective with the range $AB'C' \cdots$ on p' and hence, if U_1 and U_2 are corresponding points of the range on p_1, it follows that P_1U_2 and P_2U_1 must intersect on p'. This would not happen if U_1 and U_2 were coincident, except at A. Therefore A is the only self-corresponding point.

3. *There may be no self-corresponding element.*

Taking a range $A_1 B_1 C_1 \cdots$ on a base p, let its projector from a point P, not in the base, be the pencil $a_1 b_1 c_1 \cdots$. Through P draw the lines a_2, b_2, c_2, $\cdots$, making any fixed angle, say $30°$, with a_1, b_1, c_1, $\cdots$ respectively, and let these lines meet the base p in A_2, B_2, C_2, $\cdots$.

Then   range $A_1 B_1 C_1 \cdots \overline{\wedge}$ pencil $a_1 b_1 c_1 \cdots$

$\overline{\wedge}$ pencil $a_2 b_2 c_2 \cdots$

$\overline{\wedge}$ range $A_2 B_2 C_2 \cdots$.

Since the ranges $A_1 B_1 C_1 \cdots$ and $A_2 B_2 C_2 \cdots$ have no corresponding element, this part of the theorem is proved.

4. *There cannot be three self-corresponding elements.*

It follows from § 41 that if three elements are self-corresponding, all elements are self-corresponding, and the forms are coincident. Hence if the forms are distinct, there cannot be three self-corresponding elements.

In selecting the triads of elements which determine the projectivity between superposed forms we may include self-corresponding elements. Thus, for ranges the projectivity is determined if the two self-corresponding points X, Y and a pair of corresponding points A_1, A_2 are given, for then the triads of corresponding points are XYA_1 and XYA_2. Also, if one self-corresponding point X and two pairs of corresponding points A_1, A_2; B_1, B_2 are given, the determination is complete. Here the triads are XA_1B_1 and XA_2B_2. In each of these cases simple constructions serve to determine additional pairs of corresponding points.

In this connection the student may review the solutions of Exs. 1 and 2, page 47, which furnish constructions for these cases.

59. Classes of Projectivities. The projectivity between superposed forms is said to be *hyperbolic* when there are two self-corresponding elements, *parabolic* when there is one, and *elliptic* when there is none.

THEOREM. ANHARMONIC RATIO

60. *If two superposed projective ranges have two self-corresponding points, the anharmonic ratio of these two points with any pair of corresponding points is independent of the choice of the latter.*

Proof. Let X, Y be self-corresponding points of two superposed projective ranges, and let A_1, A_2; B_1, B_2 be any two pairs of corresponding points.

Then $$(XYA_1B_1) = (XYA_2B_2).$$

Therefore $$\frac{XA_1}{YA_1} : \frac{XB_1}{YB_1} = \frac{XA_2}{YA_2} : \frac{XB_2}{YB_2},$$

and hence $$\frac{XA_1}{YA_1} : \frac{XA_2}{YA_2} = \frac{XB_1}{YB_1} : \frac{XB_2}{YB_2}.$$

Therefore $$(XYA_1A_2) = (XYB_1B_2).$$

Exercise 18. Superposed Ranges

Given a range $XYA_1B_1 \cdots$, find two points A_2, B_2 of a range on the same base, projective with the given range, such that X, Y shall be self-corresponding points and the ratio (XYA_1A_2) shall be as follows:

1. 4. **2.** -4. **3.** -1. **4.** 1.

5. Consider Exs. 1 and 3 when Y is the point at infinity.

In Exs. 1–5 observe the situation of pairs of corresponding points with respect to the self-corresponding points.

6. Construct two superposed ranges in which the point at infinity shall be the only self-corresponding point.

7. Given a self-corresponding point and two pairs of corresponding points, construct the other self-corresponding point of two superposed projective ranges.

THEOREM. CONGRUENCE OF PROJECTIVE RANGES

61. *Two superposed projective ranges that have the point at infinity as their only self-corresponding point are congruent.*

Proof. Let $A_1B_1C_1 \cdots$ and $A_2B_2C_2 \cdots$ be two projective ranges on the same base p, the only self-corresponding point of the ranges being the point J at infinity. Let X_1 and X_2 be corresponding points not at infinity.

Then $(A_1B_1X_1J) = (A_2B_2X_2J)$;

whence $\dfrac{A_1X_1}{B_1X_1} = \dfrac{A_2X_2}{B_2X_2}$.   § 22

Therefore $\dfrac{A_1B_1}{B_1X_1} = \dfrac{A_1X_1}{B_1X_1} - 1 = \dfrac{A_2X_2}{B_2X_2} - 1 = \dfrac{A_2B_2}{B_2X_2}$,

or $\dfrac{A_1B_1}{A_2B_2} = \dfrac{B_1X_1}{B_2X_2}$.

Suppose now that, if possible, X_1 is taken so that

$$B_1X_1 : B_2X_1 = A_1B_1 : A_2B_2.$$

Then $\dfrac{B_1X_1}{B_2X_1} = \dfrac{B_1X_1}{B_2X_2}$,

and hence X_1 and X_2 must coincide.

But this yields a self-corresponding point distinct from J, which is contrary to the hypothesis.

Hence the choice of X_1 in the finite part of the line, as assumed above, must be impossible. But it is possible unless $A_1B_1 : A_2B_2 = 1$.

Hence $\dfrac{A_1B_1}{A_2B_2} = 1$, or $A_2B_2 = A_1B_1$.

It follows then that any two corresponding segments are equal and that the ranges are congruent.

An interesting alternative proof is given in § 62.

62. Alternative Proof. Let p' be a line parallel to p, and hence intersecting p at J. Let the range $A_1B_1 \cdots J$ be projected from any center P_2 upon the base p', the resulting range being $A_1'B_1' \cdots J$.

Because the latter range is perspective with $A_1B_1 \cdots J$, the point J at infinity must be self-corresponding (§ 44). Therefore the range $A_2B_2 \cdots J$ is projective with the range

$A_1'B_1' \cdots J$ in such a way that the point J at infinity is self-corresponding, and hence these ranges are perspective.

The center of perspective of these ranges must be on $A_1'A_2$. Let it be P_1. Then B_2 must be the intersection of $B_1'P_1$ and p. Draw P_1P_2.

It will be proved that P_1P_2 is parallel to p and p'.

If P_1P_2 is not parallel to p and p', let it meet p in X_1 and p' in X_1'. Then

$$\text{range } A_1B_1 \cdots X_1 \cdots J \underset{\wedge}{=} \text{range } A_1'B_1' \cdots X_1' \cdots J$$
$$\underset{\wedge}{=} \text{range } A_2B_2 \cdots X_1 \cdots J.$$

Hence the given ranges have two self-corresponding points X_1 and J, which is contrary to the hypothesis, and so P_1P_2 is parallel to p.

From similar triangles it follows that

$$\frac{A_1B_1}{A_1'B_1'} = \frac{P_2A_1}{P_2A_1'} = \frac{P_1A_2}{P_1A_1'} = \frac{A_2B_2}{A_1'B_1'},$$

and hence that $\qquad A_1B_1 = A_2B_2.$

Similarly, any two corresponding segments are equal. Hence the ranges are congruent.

Compare the above figure with the one in § 58.
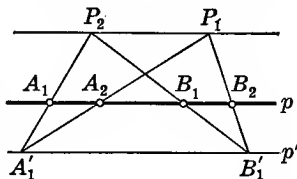
THEOREM. ANGLE SUBTENDED BY CORRESPONDING POINTS

63. *Given two superposed projective ranges having no self-corresponding point, it is possible to find a point at which all pairs of corresponding points in the range subtend equal angles.*

Proof. Let $A_1 B_1 C_1 \cdots$ and $A_2 B_2 C_2 \cdots$ be projective ranges on a base p, and let J_1 and K_2 be the point at infinity of p. Let K_1 and J_2 be the points of the two ranges which correspond to the point at infinity.

Bisect $K_1 J_2$ in O_1, and let O_2 of the second range correspond to O_1 of the first range.

Then, J_1 and K_2 being the point at infinity and A_1, B_1 being any points of the first range, we have

$$(A_1 B_1 J_1 K_1) = \frac{A_1 J_1}{B_1 J_1} : \frac{A_1 K_1}{B_1 K_1} = 1 : \frac{A_1 K_1}{B_1 K_1} = \frac{B_1 K_1}{A_1 K_1},$$

and $\qquad\qquad (A_2 B_2 J_2 K_2) = \dfrac{A_2 J_2}{B_2 J_2};$ $\qquad$ § 22

and since $(A_1 B_1 J_1 K_1) = (A_2 B_2 J_2 K_2)$, we have

$$A_1 K_1 \cdot A_2 J_2 = B_1 K_1 \cdot B_2 J_2 = \cdots = O_1 K_1 \cdot O_2 J_2.$$

Hence

$$(O_1 K_1 - O_1 A_1)(O_1 J_2 - O_1 A_2) - O_1 K_1 \cdot (O_1 J_2 - O_1 O_2) = 0,$$

and $\qquad O_1 K_1 \cdot O_1 J_2 - O_1 A_1 \cdot O_1 J_2 - O_1 K_1 \cdot O_1 A_2$

$$+ O_1 A_1 \cdot O_1 A_2 - O_1 K_1 \cdot O_1 J_2 + O_1 K_1 \cdot O_1 O_2 = 0.$$

If now A_1 and A_2 should coincide at A, a self-corresponding point, then, since $O_1 J_2 = - O_1 K_1$, O_1 being the midpoint of $J_2 K_1$, it would follow that

$$\overline{O_1 A}^2 = - O_1 K_1 \cdot O_1 O_2 = O_1 J_2 \cdot O_1 O_2.$$

Then $O_1 J_2$ and $O_1 O_2$ would agree in sign; that is, O_1 would not lie between J_2 and O_2.

On the other hand, if O_1 does not lie between O_2 and J_2, a point A, related to O_1, O_2, J_2 as above, may be found and will be self-corresponding. Hence, when there is no self-corresponding point, O_1 must be within the segment O_2J_2.

Let K_1, J_2, O_1, O_2 be indicated on the base. Erect at O_1 the perpendicular O_1P meeting at P the semicircle whose diameter is O_2J_2.

Through the point P draw a line parallel to p.
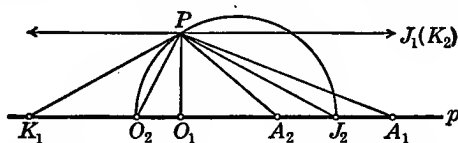
Then angle $O_2PJ_2 = 90°$, and angles K_2PK_1 (acute), J_2PJ_1, and O_2PO_1 are equal. But the triads $K_1J_1O_1$ and $K_2J_2O_2$ determine the projectivity of the ranges.

Let A_1, A_2 be any pair of corresponding points.

The projectors from P of the ranges $K_2O_2J_2A_2$ and $K_1O_1J_1A_1$ are pencils which by the rotation of the second through the common value θ of the three angles K_2PK_1, J_2PJ_1, O_2PO_1 could be made to have three common lines while the projectivity would not be destroyed.

Then all the corresponding lines would be made to coincide; and by the rotation of PA_2 through the angle θ it would be brought into the position PA_1.

Hence angle $A_2PA_1 = \theta$, and the theorem follows.

In Case 3 of § 58 the existence of superposed projective ranges having no self-corresponding point was established by means of an example. In this example the two ranges could have been generated simultaneously by the intersection of their base with the arms of an angle of constant size rotating about its vertex. It has just been established that every pair of such superposed projective ranges can be generated in this way.

THEOREM. INVOLUTION OF ELEMENTS

64. *If the projectivity between two superposed projective one-dimensional forms is such that when any one element is taken as belonging to the first form that element has the same corresponding element as it has when it is taken as belonging to the second form, then every element has this property.*

Proof. We shall consider the theorem for ranges only, the proof being similar for other forms.

Between two triads $A_1 B_1 C_1$ and $A_2 B_2 C_2$ that lie on a base p there exists a projectivity which determines two superposed projective ranges on this base. In connection with this projectivity every point of the line p may be given two names. Thus, a point might be L_1 and R_2. Moreover, the original triads $A_1 B_1 C_1$ and $A_2 B_2 C_2$ may have some points in common.

Suppose now that B_1 is taken to be the same as A_2, and B_2 the same as A_1. Let a point D_1 be taken to be the same as C_2. The theorem is then proved if we can show that D_2 coincides with C_1.

From the conditions of the case

$$(A_1 A_2 C_1 C_2) = (A_1 B_1 C_1 D_1) = (A_2 B_2 C_2 D_2) = (A_2 A_1 C_2 D_2);$$
whence $\qquad A_1 C_1 : A_2 C_1 = A_1 D_2 : A_2 D_2,$
and D_2 coincides with C_1.

65. Involution. When two superposed prime forms are connected by a projectivity such as that described in the above theorem, they are said to form an *involution*.

Elements of an involution which correspond to each other are said to be *conjugate*.

The projectivity is also called an *involution*.

66. COROLLARY. *Every projection and every section of an involution of elements yields an involution.*

THEOREM. ANHARMONIC RATIOS IN HYPERBOLIC INVOLUTION

67. *In a hyperbolic involution the anharmonic ratio of the two self-corresponding elements and any pair of corresponding elements is −1.*

Proof. If X, Y are the self-corresponding points and A_1, A_2 are corresponding points of a point involution, then

$$(XYA_1A_2) = (XYA_2A_1) = \frac{1}{(XYA_1A_2)}.$$

Hence $(XYA_1A_2)^2 = 1$, and $(XYA_1A_2) = -1$.

(XYA_1A_2) cannot be equal to 1, for X, Y, A_1, A_2 are distinct points.

The classification of projectivities into hyperbolic, parabolic, and elliptic applies to the involutions, and it is easy to prove that involutions of all these classes exist. Hence §§ 58–61 apply in the case of involutions. In particular, the value of the anharmonic ratio mentioned in § 60 has been determined in the above theorem.

The converse of this theorem is easily proved; and hence, when any two elements of a range or pencil are given, it becomes easy to determine any desired number of pairs of corresponding elements of an involution of which the two given elements shall be self-corresponding. A similar remark may be made regarding the more general case of § 60.

68. COROLLARY 1. *In a hyperbolic point involution the point at infinity, if not self-corresponding, corresponds to a point midway between the self-corresponding points.*

69. COROLLARY 2. *In a hyperbolic point involution, if the point at infinity is a self-corresponding point, the other self-corresponding point bisects the line joining every pair of corresponding points.*

70. COROLLARY 3. *In a hyperbolic involution of lines (or planes), the line (or plane) which bisects the angle between the self-corresponding lines (or planes) has for its corresponding line (or plane) the one which is perpendicular to it.*

71. Involution Determined. An involution is determined whenever enough is known to establish two determining triads of the projectivity. Consequently the following data are sufficient for this purpose:

1. Two pairs of corresponding elements.

2. One self-corresponding element and a pair of corresponding elements.

3. Two self-corresponding elements.

Considering the argument for ranges only, the other cases admitting of similar treatment, we see that the following triads determine projectivities:

In Case 1 the triads $A_1A_2B_1$, $A_2A_1B_2$, where A_1, A_2; B_1, B_2 are corresponding pairs of points.

In Case 2 the triads XA_1A_2, XA_2A_1, where X is a self-corresponding point and A_1, A_2 are a pair of corresponding points.

In Case 3 the triads XYA_1, XYA_2, where X, Y are self-corresponding points and A_1, A_2 are a pair of harmonic conjugates with respect to them.

72. Center of Involution. Let the pairs of points J, O; A_1, A_2; B_1, B_2 (J being the point at infinity) be corresponding.

Then
$$(A_1B_1\,O\,J) = (A_2B_2J\,O);$$

whence
$$\frac{A_1O}{B_1O} = \frac{B_2O}{A_2O},$$

and
$$OA_1 \cdot OA_2 = OB_1 \cdot OB_2 = \cdots.$$

Then in a point involution the point corresponding to the point at infinity is such that the product of its distances from any pair of corresponding points is independent of the choice of the pair. This point is called the *center of the involution.*

73. Two Cases of Point Involution. A further examination of the relations just found suggests two possible cases:

1. The product $OA_1 \cdot OA_2$ may be negative.
2. The product $OA_1 \cdot OA_2$ may be positive.

In Case 1 no self-corresponding point X can exist; for if it did we should have $\overline{OX}^2$ negative, which is impossible for real values of OX. The involution is therefore elliptic.

Also, from the relation $OA_1 \cdot OA_2 = OB_1 \cdot OB_2$, if each product is negative, it follows that the point O separates every pair of corresponding points. Moreover, if OA_1 is longer than OB_1, it is evident that OA_2 is shorter than OB_2, and so any two pairs of corresponding points, as A_1, A_2; B_1, B_2, mutually separate each other.

In Case 2 by similar reasoning we establish that the involution is hyperbolic with the self-corresponding points equidistant from O, that any two corresponding points lie in the same direction from O, and that no two pairs of corresponding points mutually separate each other.

These metric properties are sometimes used to define elliptic and hyperbolic point involutions.

Exercise 19. Point Involutions

1. Choosing two pairs of corresponding points that will determine an elliptic involution, find by construction a third pair of corresponding points and also find the center.

2. Solve Ex. 1 for a hyperbolic involution.

3. Find a pair of corresponding points of an involution of which X (a given point) and J (the point at infinity) are the self-corresponding points.

4. Given one self-corresponding point of an involution and a pair of corresponding points, find by construction the other self-corresponding point.

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THEOREM. LINE INVOLUTION

74. *Every involution of lines has one pair of correspond-
ing lines at right angles; if it has two pairs at right angles,
all its pairs are at right angles.*
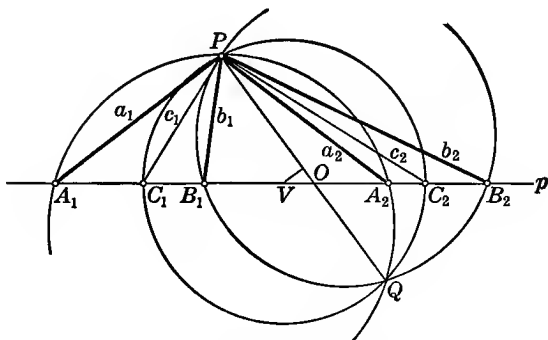
Proof. Let a_1, a_2; b_1, b_2 be pairs of corresponding lines
of an elliptic involution on a base P. If a_1, a_2 or b_1, b_2
are at right angles, the first part needs no proof.

If neither of these pairs of lines is at right angles, cut
the four lines by any line p in the points A_1, A_2, B_1, B_2.

Describe the circles PA_1A_2 and PB_1B_2 meeting again
in Q. Let PQ cut p in O, and let the perpendicular bisector
of PQ cut p in V.

Describe the circle whose center is V and whose radius
is VP, or VQ, and let it cut p in C_1 and C_2.

Then $OC_1 \cdot OC_2 = OP \cdot OQ$
$$= OA_1 \cdot OA_2 = OB_1 \cdot OB_2.$$

Hence the lines c_1, or PC_1, and c_2, or PC_2, belong to the
original involution. Furthermore, they are at right angles,
because C_1PC_2 is a semicircle.

For the case of a hyperbolic involution see § 70.

Again, if two pairs of corresponding lines are at right angles, these pairs separate each other, and the involution is elliptic.

Suppose now that the lines a_1, a_2 and also the lines b_1, b_2 are at right angles. Then the segments PA_1A_2 and PB_1B_2 are semicircles, and their common chord PQ is at right angles to p at O. Now if c_1, c_2, any other pair of corresponding lines of the involution, cut p at C_1, C_2, then

$$OC_1 \cdot OC_2 = OP \cdot OQ,$$

and the segment C_1PC_2 is a semicircle.

Accordingly, c_1, c_2, any two corresponding lines of the involution, are at right angles.

Exercise 20. Review

1. On a given base p, if each of two ranges is projective with a third range in such a way that X and Y are self-corresponding points for both projectivities, then these two ranges are projective with each other, and X and Y are self-corresponding points in the projectivity connecting them.

2. In Ex. 1, if the anharmonic ratios (§ 60) associated with the two projectivities are r_1 and r_2, find the anharmonic ratio associated with the third projectivity.

3. Interpret the results of Exs. 1 and 2 when the first two projectivities are involutions.

4. A projectivity between superposed forms is determined if two self-corresponding elements and the anharmonic ratio of these elements and two corresponding elements are given.

5. Given two self-corresponding points X and Y of two superposed projective ranges, find a third range on the same base between which and each of the other two ranges there exists a projectivity for which X and Y are self-corresponding points. Consider the number of solutions.

6. A point involution is completely determined by its center and one self-corresponding point.

7. A point involution is completely determined by its center and a pair of corresponding points.

8. The circles which pass through two given fixed points determine, on any line which cuts them, corresponding points of an involution.

9. If A, B, C, D are fixed coplanar but noncollinear points and a point O moves in their plane in such a way that the anharmonic ratio of the pencil $O(ABCD)$ is a given constant k, the lines OC and OD meet the line AB in corresponding points of two superposed projective ranges.

10. Consider Ex. 9 for the case in which $k = -1$.

11. In Ex. 9 the path of O intersects the line AB in A and B and in no other points, and hence it passes through all four of the points A, B, C, D.

12. Every line through one of the points A, B, C, D in Ex. 9 cuts the path of O in one and only one additional point.

13. In Ex. 12 find the other point in which the path of O is cut by any given line through A.

14. In Ex. 9, assuming that as a point moving along a curve approaches a fixed point the secant through the two points approaches the tangent to the curve at the fixed point, construct the tangent to the path of O at the point A.

15. If A_1, A_2, A_3 are noncollinear points, a_1, a_2, a_3 nonconcurrent lines, P_1 a point on a_1, $P_1 A_1$ and a_2 intersect in P_2, $P_2 A_2$ and a_3 in P_3, and $P_3 A_3$ and a_1 in P_1', then, as P_1 and P_1' move along a_1, they trace two superposed projective ranges.

16. In Ex. 15 under what circumstances (if any) do the lines $P_1 A_1$, $P_2 A_2$, and $P_3 A_3$ form a triangle with its vertices on the three given lines ?

17. What modifications of the data in Ex. 15 render it certain that in all positions the lines $P_1 A_1$, $P_2 A_2$, and $P_3 A_3$ form a triangle with its vertices on the same three given lines ?

PART II. APPLICATIONS

CHAPTER VIII

PROJECTIVELY GENERATED FIGURES ·

75. Statement of the General Problem. Application of the properties of prime forms will now be made to the study of a problem which is connected with a somewhat wide range of topics in geometry. This problem, to the discussion of which the remainder of the book is devoted, may be stated in these words:

To determine the character of all geometric configurations whose generating elements are determined by corresponding elements of two projective one-dimensional prime forms.

The problem divides naturally into a number of cases according as the two projective one-dimensional prime forms are any one of the following ·pairs:

1. Two ranges, considered in §§ 84 and 85.
2. Two flat pencils, considered in §§ 86–89.
3. Two axial pencils, considered in §§ 90 and 95.
4. A range and a flat pencil, considered in § 96.
5. A range and an axial pencil, considered in § 97.
6. A flat pencil and an axial pencil, considered in § 98.

76. Locus of a Point. If a point moves in space subject to a given law, the figure consisting of all the points with which the moving point may coincide, and of no others, is called the *locus of the point.*

77. Envelope of a Plane. If a plane moves in space subject to a given law, the figure which is tangent to all the planes with which the moving plane may coincide, and to no others, is called the *envelope of the plane.*

78. Envelope of a Line. If a line moves in a plane subject to a given law, the figure which is tangent to all the lines with which the moving line may coincide, and to no others, is called the *(plane) envelope of the line.*

79. Generation of a Figure by a Line. If a line moves in space subject to a given law, the figure consisting of all the lines with which the moving line may coincide, and of no others, is said to be *generated by the moving line.*

80. Order of a Figure. The greatest number of points of a figure that lie on a line which is not entirely in the figure is called the *order of the figure.*

Thus a circle may be met by a line in two, one, or no points. Consequently the order of the circle is two. Similarly, the order of a straight line is one. The order of a plane is also one, while that of a sphere is two.

81. Class of a Figure in Space. The greatest number of tangent planes of a figure in space which pass through a line that does not have all the planes through it tangent to the figure is called the *class* of the figure.

Thus a sphere may have tangent to it two, one, or no planes which pass through a straight line, and hence the sphere is of class two.

82. Class of a Figure in a Plane. The greatest number of tangent lines which can be drawn to a plane figure from any point in its plane is called the *class* of the figure.

Thus, of the lines in a plane which pass through a given point two, one, or none may be tangent to a given circle, and hence the circle is of class two.

83. Dual Elements. From the point of view of the Principle of Duality the following are corresponding elements :

1. In geometry of three dimensions, a *point on a figure* and a *plane tangent to a figure.*

It can also be shown that a line on a figure is self-dual.

2. In geometry of the plane, a *point on a figure* and a *line tangent* to a figure.

3. In geometry of the bundle, a *line on a figure* and a *plane tangent* to a figure.

Exercise 21. Preliminary Definitions

1. What is the envelope of a system of tangents to a given circle ? What is the dual of a circumscribed polygon ?

2. What is the order and what is the class of the projector of a circle from a point not in its plane ?

The student may consult the chapters on higher plane curves in texts on elementary analytic geometry, such as the one in this series, and determine the orders and the classes of the curves considered there.

3. In Ex. 2 what is the order and what is the class of any plane section of the figure ?

4. Find the surface generated by a line so moving as to be constantly parallel to and at a given distance from a given line, and state the order and the class of this surface.

5. In Ex. 4 consider the various plane sections of the surface, stating the order and the class of each.

6. Find the locus in space of a point which so moves as to be constantly at a given distance from the nearest point of a given line segment, stating the order and the class of the locus.

7. Find the order and the class of the plane sections of the figure obtained in Ex. 6.

8. Find the envelope of a plane which so moves as to be constantly at a given distance from the nearest point of a given line segment, stating the order and the class of the locus.

84. *The envelope of the line which so moves as always to contain corresponding points of two coplanar projective ranges is of the second class, unless the ranges are superposed and without a self-corresponding point, in which case the envelope is the common base. If the ranges are perspective, the envelope consists of two points, one of which is common to the ranges. If the ranges are not perspective and not superposed, the envelope is tangent to the base of each range at that point of it which corresponds to the point common to the ranges when this common point is regarded as belonging to the other range.*

Proof. The two projective ranges referred to in § 75 may or may not have one common element, and in this theorem we consider two ranges having such an element. In this case the ranges must evidently be coplanar.

The element determined by two corresponding points of the ranges is a straight line, and hence in this case the problem is that of determining the envelope of a line which so moves as always to contain two corresponding points of two coplanar projective ranges.

These ranges may be (1) superposed; (2) not superposed but perspective; (3) neither superposed nor perspective.

1. *Let the ranges be superposed.*

Then all pairs of corresponding but not self-corresponding points determine the common base of the ranges; and the lines through any self-corresponding point are infinitely many. The only figure that has all these lines and no others as tangents consists of the one or two self-corresponding points or, if there is no self-corresponding point, consists of the common base of the ranges.

2. *Let the ranges be not superposed but perspective.*

Then all lines determined by distinct corresponding points pass through the center of perspective, and every line through the point common to the ranges joins that point to its corresponding point, that is, to itself. Hence the envelope consists of two points; namely, the center of perspective and the point common to the ranges.

3. *Let the ranges be neither superposed nor perspective.*

To the common point X_1, or Y_2, there correspond in the ranges two points X_2 and Y_1. Since the base p_1 of the first range joins Y_2 to Y_1, the envelope is tangent to p_1; similarly, it is tangent to p_2.

Consider now V_1 and V_2, two corresponding points nearly coincident with X_1 and X_2 respectively. The line v joining them is nearly coincident with p_2. If, now, V_1 approaches coincidence with X_1, v and V_2 approach p_2 and X_2. But if a moving tangent to a curve approaches a fixed tangent as a limiting line, the intersection of these two approaches the point of contact of the fixed tangent as a limiting point. Hence in this case X_2 is the point of contact of p_2 with the envelope. Similarly, Y_1 is the point of contact of p_1.
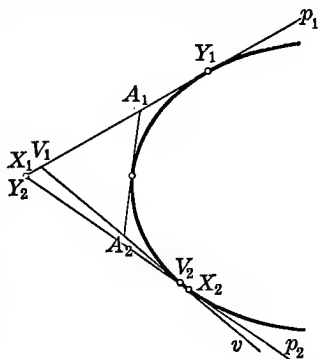
Finally, the class of the envelope is two; for two tangents to the envelope, namely, p_1 and A_1A_2, pass through a point A_1. If through any point O there should pass three tangents to the envelope, O would be a center of perspective for the ranges; but the ranges are not perspective.

THEOREM. RANGES WITH NO COMMON ELEMENT

85. *The lines which join corresponding points of two pro-jective ranges that have no common point are the intersections of corresponding planes of two projective axial pencils which have no common plane.*
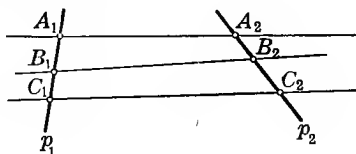
Proof. If two projective ranges have no common element, their bases cannot meet, and therefore the ranges cannot be coplanar.

Let $A_1B_1C_1 \cdots$ and $A_2B_2C_2 \cdots$ be projective ranges on the bases p_1 and p_2 which are not coplanar.

Let $\alpha_1, \beta_1, \gamma_1, \cdots$ be the planes determined by the line p_1 and the points $A_2, B_2, C_2, \cdots$ respectively, and let $\alpha_2, \beta_2, \gamma_2, \cdots$ be the planes determined by the line p_2 and the points $A_1, B_1, C_1, \cdots$ respectively. Then we have

$$\text{axial pencil } \alpha_1\beta_1\gamma_1 \cdots \overline{\wedge} \text{ range } A_2B_2C_2 \cdots$$
$$\overline{\wedge} \text{ range } A_1B_1C_1 \cdots$$
$$\overline{\wedge} \text{ axial pencil } \alpha_2\beta_2\gamma_2 \cdots.$$

Also the line A_1A_2 is the intersection of the planes α_1 and α_2. Moreover, if the axial pencils had a common plane, the ranges along their axes would both be in this plane. But this is contrary to hypothesis.

Hence the proof is complete.

As a consequence of this theorem the discussion of the figure which is generated by these two projective ranges may be deferred until we consider the figure which is generated by projective axial pencils that have no common plane (§ 95).

THEOREM. COPLANAR FLAT PENCILS

86. *The locus of the point which so moves as always to be common to two corresponding lines of two coplanar projective flat pencils is of the second order, unless the pencils are superposed and without a self-corresponding line, in which case the locus is the common vertex. If the pencils are perspective, the locus consists of two straight lines, one of which is common to the pencils. If the pencils are not superposed, the locus contains the base of each pencil and at each of these points is tangent to the line of the corresponding pencil which corresponds to the common line of the pencils when this common line is regarded as belonging to the other pencil.*

Proof. Two flat pencils may or may not have a common base. In the former case the common base may be a plane containing both pencils or a point which is the vertex of both pencils. We shall now deal with the first of these subcases, the second being discussed in § 87.

Essentially, then, the theorem involves the problem of finding the locus of the intersection of two coplanar projective flat pencils. It is evident that these pencils may be (1) superposed; (2) not superposed but perspective; (3) neither superposed nor perspective.

1. *Let the pencils be superposed.*

In this case the flat pencils have in common not only their planes but also their vertices. The points common to the corresponding lines include in any case the common vertex and also include all points of any self-corresponding lines of the pencil. Hence in this case the locus is the one or two self-corresponding lines of the pencils or, in case there is no self-corresponding line, the common point of all the lines of the pencil.

2. *Let the pencils be not superposed but perspective.*

In this case the pencils have a self-corresponding line, all the points of which are in the locus. In addition the intersections of pairs of corresponding lines are in the locus. But these intersections are on a straight line, and accordingly the locus is a pair of straight lines.

3. *Let the pencils be neither superposed nor perspective.*
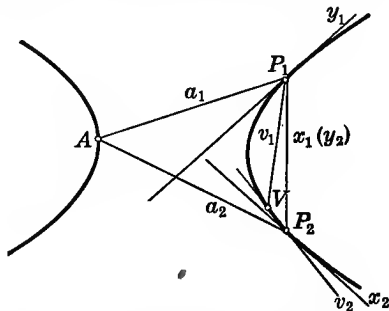
Since the common line is not self-corresponding, let it be called x_1 and y_2. In the two pencils it has corresponding to it the lines y_1 and x_2.

The base P_1 of the first pencil, being the intersection of the lines y_1 and y_2, is on the locus. Similarly, the base P_2 of the second pencil is on the locus.

Consider now a line v_1 that is nearly coincident with x_1, and the corresponding line v_2 that is nearly coincident with x_2. The point V of the locus determined by v_1 and v_2 is nearly coincident with P_2. If, now, the point V approaches P_2 along the locus, the line v_2, or VP_2, approaches x_2. But the limiting position of a secant VP_2 as V approaches P_2 is the tangent to the locus at P_2. Hence the locus is tangent to the line x_2 at P_2. Similarly, it follows that the line y_1 is tangent to the locus at P_1.

Finally, the order of the locus is two; for any line of the first pencil, as a_1, meets the locus at P_1 and at A, the intersection of a_1 and its corresponding line a_2. Moreover, if any line o cuts the locus in three points, the triads of lines of the pencils which would intersect in these points would be perspective, and so would the complete pencils. But the pencils are not perspective. The theorem is, therefore, completely proved.

Exercise 22. Ranges and Pencils

1. Two fixed lines AO_1B and CO_2D are each perpendicular to O_1O_2, and a moving line cuts them in P_1 and P_2 so that the ratio $O_1P_1 : O_2P_2$ is a constant. Find the envelope of the moving line. Consider the case in which O_1P_1 and O_2P_2 are equal.

Compare the ranges traced by P_1 and P_2.

2. Two fixed lines intersect at right angles at O, and a moving line cuts them at equal distances from O. Find the envelope of the moving line.

3. Examine Ex. 1, substituting the condition that O_1P_1 and O_2P_2 maintain a constant difference.

4. Examine Ex. 2, substituting the condition that the distances from O maintain a constant difference.

5. Examine Ex. 2, substituting the condition that one distance exceeds a given multiple of the other by a fixed amount.

6. Two lines revolve at the same angular velocity in opposite senses about the fixed points O_1 and O_2 respectively. Initially they make angles of 90° and 45° respectively with the line O_1O_2. Find the locus of their intersection.

7. Consider Ex. 6, substituting the condition that initially the lines coincide.

8. Examine Exs. 6 and 7 on the assumption that the lines revolve in the same sense.

THEOREM. FLAT PENCILS WITH A COMMON VERTEX

87. *The envelope of the plane, which so moves as always to contain corresponding lines of two projective flat pencils that are not coplanar but have a common vertex, is of the second class. If the flat pencils are perspective, the envelope consists of two straight lines, one of which is common to the pencils. If the pencils are not perspective, the envelope is a surface tangent to the plane of each flat pencil along the line which corresponds to the common line of the pencil when this common line is regarded as belonging to the other flat pencil. All the planes and the surface generated pass through the common vertex of the pencil.*
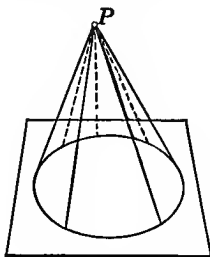
Proof. Since each pair of corresponding lines of the given flat pencils determines both a point and a plane, we are concerned with the problem of finding the aggregate of elements, either points or planes, determined by corresponding lines of two projective flat pencils which have a common vertex. The first of these cases is trivial.

It may here be assumed that the flat pencils are not coplanar, as the case of coplanar pencils has just been discussed.

From the point of view of loci the points determined by corresponding lines either will be the common vertex alone or, if the common line happens to be self-corresponding, will be this common line itself.

On the other hand, any two corresponding lines determine a plane which passes through the common vertex. If we cut the whole configuration by a plane that does not pass through the common vertex, we obtain two projective coplanar ranges and the lines joining corresponding points. This section of the envelope is then one of the figures described in § 84. Consequently the envelope sought is the projector, from the common base of the two flat pencils, of one of these figures. If the flat pencils are perspective, this projector consists of two straight lines through the common vertex. If the flat pencils are not perspective, the projector is a conic surface and is tangent to the plane of each flat pencil. This is evident from the fact that the plane of either pencil is determined by the common line of the pencils and x, that one of its own lines which corresponds to the common line. The plane of this pencil has the line x in common with the envelope.

Through the common line of the pencils there pass the two planes of the pencils, and these are tangent to the envelope. If through any line there should pass more than two tangent planes, the given pencils would be perspective.

88. Flat Pencils having no Common Base. The discussion of the case of projective flat pencils that have a common line but are not coplanar and do not have a common vertex can be given quickly. No intersection of corresponding lines can occur except on the common line. The common line may be self-corresponding, in which case any point on it is common to corresponding lines, and any plane through it contains corresponding lines. If the common line is not self-corresponding, there is in each pencil a line corresponding to it. With these lines it determines two points, the vertices of the pencils, and two planes, the planes of the pencils.

89. Projective Flat Pencils having no Common Element.
Figures generated by means of projective flat pencils having no common line are so simple as not to demand special study. They will be noticed in passing, but no formal theorem regarding them need be stated.

Since pencils which lie in the same plane or in different planes that intersect in a line of the pencils have a common element, it follows that the pencils under consideration lie in planes whose intersection does not pass through the vertex of either pencil. Upon this line the pencils determine ranges which may be either identical or distinct.

If the ranges are identical, their common base is the locus of the intersections of corresponding lines of the pencils. Likewise each pair of corresponding lines determines a plane through the line joining the bases of the flat pencils; and the envelope of these planes is this line. Hence the figure determined is either the line determined by the vertices of the pencils or the line determined by the planes of the pencils, according to the point of view.

If the superposed ranges mentioned above are not identical, the only pairs of corresponding lines of the flat pencils which determine elements are those which meet in the self-corresponding points of the superposed ranges. These determine *two*, *one*, or *no* points on the line common to the planes of the pencils, or *two*, *one*, or *no* planes through the line joining the vertices. Hence, from the point of view of loci, the figure generated by means of the projective flat pencils consists of two, one, or no points; and from the point of view of envelopes, the figure consists of two, one, or no planes. Though neither of the configurations obtained has any special interest for us, it is clear that they conform in a general way to the type of figures which we obtain in the other cases.

THEOREM. AXIAL PENCILS WITH A COMMON PLANE

90. *The surface generated by the line which so moves as always to be contained in corresponding planes of two axial pencils that have a common plane is of the second order unless the axial pencils are superposed and are without self-corresponding planes, in which case the surface degenerates into the common axis of the pencils. If the axial pencils are perspective, the surface consists of two planes, one of which is common to the pencils. If the pencils are neither perspective nor superposed, the surface contains the axis of each pencil and along each of these axes is tangent to the plane which corresponds to the common plane of the pencil when this common plane is regarded as belonging to the other pencil. The generating line continually passes through the intersections of the axes of the pencils.*

Proof. Two projective axial pencils may have or may not have a common element. In this theorem we consider only the former case. Evidently the axes of the two pencils are coplanar. Then the pencils may be (1) superposed; (2) not superposed but perspective; (3) neither superposed nor perspective.

1. *Let the axial pencils be superposed.*

Then all pairs of corresponding planes intersect in the axis, but any line in a self-corresponding plane may be regarded as common to two coincident corresponding planes. The surface generated by the lines common to corresponding planes consists, therefore, of *one* or *two* self-corresponding planes if there are such planes or, if there are no such planes, this surface degenerates into a line, the common axis of the pencil. This last case is of minor importance only.

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2. *Let the axial pencils be not superposed but perspective.*

The axes of the pencils intersect in a point O which lies in every plane of both pencils. Then the line determined by any pair of corresponding planes passes through this point. Moreover, if a plane is passed so as not to contain this point, it cuts the axial pencils in coplanar perspective flat pencils, and the locus of the intersections of corresponding lines of these pencils is two straight lines, one of which is the line through the bases of the flat pencils (§ 86). Consequently the surface generated by the intersections of corresponding planes of the axial pencils is the projector of the two straight lines from the point O; that is, the surface is two planes, one of which is the common plane of the axial pencil.

3. *Let the axial pencils be neither superposed nor perspective.*
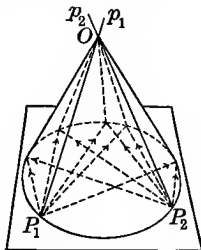
As in the previous case, the axes p_1 and p_2 intersect in a point O through which passes every line determined by corresponding planes of the axial pencils. A plane which does not pass through O cuts the axial pencils in coplanar projective but not perspective flat pencils whose bases are P_1 and P_2, the locus of whose intersections is described by § 86. The surface generated by the lines common to corresponding planes of the axial pencils is the projector from the point O of the locus just mentioned.

91. Regulus. Any three straight lines, no two of which are coplanar, are met by infinitely many straight lines which, taken as an aggregate, are said to form a *regulus*.
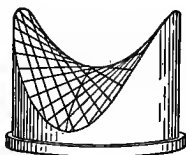
The three given lines are called the *directrices* of the regulus, and the lines which meet the directrices are called the *generators* of the regulus.
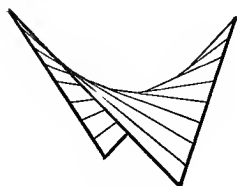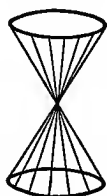
REGULUS   QUADRIC SURFACE
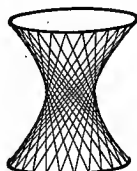
SKEW QUADRIC RULED SURFACES

92. Quadric Surface. The aggregate of the points of the lines of a regulus constitute a surface called a *quadric surface*, and the generators of the regulus are also called the *generators* of this surface.

There are quadric surfaces which are not constituted in this way.

93. Ruled Surface. A surface generated by the movement of a straight line is called a *ruled surface*.

94. Skew Ruled Surface. A ruled surface in which no two consecutive generators intersect is called a *skew ruled surface*.

For a full discussion of skew ruled surfaces see § 187.

THEOREM. AXIAL PENCILS WITH NO COMMON ELEMENT

95. *The lines of intersection of two projective axial pencils which have no common element generate a skew ruled surface of the second order in which lie the bases of the pencil. Every section of this surface by a plane through a generating line is a pair of straight lines. All other plane sections are curves of the second order.*

Proof. The proof may be divided into five parts:

1. *The ruled surface is skew.*

Each generator intersects each axis. If two generators intersect, they determine a plane which contains two points and therefore all points of each axis. This plane must then be a common element of the pencils, which is contrary to hypothesis. Hence no two generators intersect, and so the surface is skew.

2. *The bases of the two axial pencils lie in the surface.*

Through any point A of the base of either axial pencil there pass all the planes of that pencil and one plane α of the other. Hence A is on the line determined by the plane α and its corresponding plane. Hence the surface contains every point of the base of either pencil.

3. *Every section of the surface by a plane which does not pass through a generating line is a curve of the second order.*

Any plane π which does not contain a generating line cuts the axial pencils in two projective flat pencils. If these flat pencils were perspective, their common line would be self-corresponding and the plane π would cut two planes of the axial pencils in their common line, that is, in a generating line; and this is contrary to hypothesis. Then (§ 86) the section is a curve of the second order.

4. *Every section of the surface by a plane through a gen-erating line is a pair of straight lines, one of which is the generating line and the other of which meets every one of the generating lines mentioned above.*

Let the surface be generated by the projective axial pencils $\alpha_1\beta_1\gamma_1 \cdots$ and $\alpha_2\beta_2\gamma_2 \cdots$.

Any plane that is an element of one of these axial pencils intersects the surface in two straight lines, of which one line is the base of its pencil, and the other line is the line of intersection of this plane with the plane that corresponds to it in the other pencil.

Now let a plane π that does not belong to either axial pencil be passed through a generating line a which is deter-mined by the planes α_1 and α_2 of the axial pencils.

This plane cuts the axial pencils in projective flat pencils $ab_1c_1 \cdots$ and $ab_2c_2 \cdots$. It cuts the generating lines and hence the surface generated in the locus determined by these flat pencils. The latter have a self-corresponding line a, and accordingly they are perspective.

Consequently the locus determined by these flat pencils consists of two lines; namely, a and the line in which lie the points of intersection of the lines b_1, b_2; c_1, c_2; $\cdots$.

The line containing the points of intersection of the pairs of lines b_1, b_2; c_1, c_2; $\cdots$ intersects every one of the generating lines determined by the axial pencils. For if we consider the line determined by the corresponding planes β_1, β_2, we find that it passes through the intersec-tion of the lines b_1, b_2. Similarly, it may be shown that it meets the line determined by any other pair of corre-sponding planes except α_1, α_2. The line and a, the inter-section of α_1, α_2, are in the same plane π, and hence the statement is established.

5. *The surface itself is of the second order.*

Let any line p which is not a generating line intersect the surface in a point P. Pass a plane π through the line p. The section of the surface is a curve of the second order, and consequently the line p cannot meet it in more than two points. Furthermore, it cannot meet the surface in points not in this curve. Hence p cannot meet the surface in more than two points. That many lines actually meet the surface in two points follows from Case 4. Hence the surface is of the second order.

96. A Range and a Flat Pencil. If a range and a flat pencil are projective, they may or may not be coplanar.

If the range and pencil are coplanar, each point of the range taken with the corresponding line of the flat pencil determines the plane containing both the range and the flat pencil; and this plane is the figure sought.

If the range and pencil are not coplanar, let the range be $A_1 B_1 C_1 \cdots$ on the base p_1, and the flat pencil $a_1 b_1 c_1 \cdots$ on the base P_1. The plane determined by A_1 and a_1 is the same as the plane determined by $P_1 A_1$ and a_1. Hence this case yields the same result as that of two noncoplanar projective flat pencils which have a common base P_1 (§ 87).

97. A Range and an Axial Pencil. The case of a range and an axial pencil may also be considered briefly.

A point and a plane have not been considered as determining a third element. If, however, the point is in the plane, they may be said to determine either the point or the plane; and since all, two, one, or none of the points of a range might lie on the corresponding planes of an axial pencil projective with the range, the configuration sought in the problem might be regarded as consisting of points or planes as indicated. The case is not important.

98. A Flat Pencil and an Axial Pencil. The case of a flat pencil and an axial pencil demands but little attention. There are two subcases, a somewhat trivial one in which the base of the flat pencil is on the base of the axial pencil, and another subcase which has been dealt with from a different point of view.

A little consideration shows that the first subcase may be regarded as yielding all, two, one, or none of the elements of the flat pencil, for these elements contain all points common to pairs of corresponding elements of the flat pencil and the axial pencil.

In the second subcase the lines of the flat pencil and the corresponding planes of the axial pencil determine a locus of points. But the plane of the flat pencil cuts the axial pencil in a second flat pencil projective with the first; and the locus in question is also determined by the intersections of corresponding lines of this second flat pencil and the given flat pencil. Hence the locus is the same as that described in § 86.

99. Summary of Results. From the discussion in this chapter there have come to notice the following figures, exclusive of certain trivial ones:

1. Certain plane curves of the second class.
2. Certain plane curves of the second order.
3. Certain conical surfaces of the second class.
4. Certain conical surfaces of the second order.
5. Certain ruled surfaces of the second order in space.

The study of these curves and surfaces will be undertaken with a view to exhibiting the symmetry among them and to establishing their more striking properties, as well as with a view to showing the power of methods which are based upon the principles that have been set forth.

Exercise 23. Review

1. If two points of a circle are each joined to four other points of the circle, the anharmonic ratios of the two pencils so formed are equal.

2. If a point moving on a circle is constantly joined to two fixed points on the circle, the flat pencils generated by the two joining lines are projective.

3. If a variable tangent to a circle meets two fixed tangents, the ranges traced by the intersections are projective.

4. A line so moves as to cut the sides BC, CD of a square $ABCD$ in points X, Y such that the angle XAY is constant. Find the nature of the envelope of the moving line.

5. A line so moves as always to be at a constant distance from a fixed point. Find the nature of the envelope of the line.

6. A wire fence consists of a number of horizontal strands of wire at equal intervals, crossed by a number of vertical strands also at equal intervals between each pair of posts. One of two posts is pushed into an oblique position. What sort of surface passes through all the wires between the two posts ?

7. Two lighthouses are in a north-and-south line. The lamps revolve at the same uniform rate in the same angular sense, and each lamp throws two shafts of light in opposite directions. If the lamps are so adjusted that when one light shines north and south the other shines northeast and southwest, find the nature of the locus of the spot illuminated simultaneously by both lights. Has this locus any infinitely distant points ?

8. Consider Ex. 7 when the lamps are so adjusted that periodically the shafts of light coincide.

9. Consider Exs. 7 and 8 when the two lamps revolve in opposite senses.

10. In Ex. 7, if one light rotates twice as quickly as the other, do the rays generate projective flat pencils ?

11. A number of lamps, each of which throws two shafts of light in opposite directions and rotates at a uniform rate, are to be placed so that their rays shall all continually converge upon an object which moves along a circle. If all the lamps have the same angular rate, specify a possible arrangement.

12. In Ex. 11 specify an arrangement and adjustment in which all the lamps do not have the same angular rate.

13. Each of the circles $c_1, c_2, \cdots, c_n$ is tangent to the circle next preceding and to the one next following but to no others of the set, and $P_1, P_2, \cdots, P_n$ are variable points on the respective circles such that each line $P_k P_{k+1}$ contains the point of contact of the circles c_k, c_{k+1}. If O_1 is any point on c_1, and O_n is any point on c_n, find the nature of the locus of the intersection of $O_1 P_1$ and $O_n P_n$, the figure being plane.

14. Consider Ex. 13 with the omission of the restriction that no circle shall be tangent to any other except the one next preceding and the one next following.

15. Each of the circles $c_1, c_2, \cdots, c_n$ is tangent to the circle next preceding and to the one next following but to no others of the set, and $t_1, t_2, \cdots, t_n$ are variable tangents to the respective circles such that the point $t_k t_{k+1}$ is on the common tangent of c_k, c_{k+1}. If o is any fixed tangent to c_1, and o_n is any fixed tangent to c_n, find the nature of the envelope of the line joining the intersection of o_1, t_1 and that of o_n, t_n.

16. Each of two circles in one plane is divided into ten equal arcs. For each circle the tangent at one point of division and the secants through that point and the other points of division are drawn. If these two sets of lines are produced indefinitely, their 100 points of intersection lie in sets of ten upon 10 curves of the second order.

17. At the same moment two trains leave a junction on straight diverging lines and travel at uniform rates. If two passengers, one on the rear platform of each train, watch each other, find the envelope of their line of sight.

18. Solve Ex. 17 with the modification that the trains leave nearly but not quite at the same time.

19. Each of the right circular cones $C_1, C_2, \cdots, C_n$ is tangent along a straight line to the one following it. The variable lines $p_1, p_2, \cdots, p_n$ on the respective cones are such that each plane $p_k p_{k+1}$ contains the line of contact of the cones C_k, C_{k+1}. If o_1, o_n are any fixed lines lying on C_1, C_n respectively, find the surface generated by the intersection of the planes $o_1 p_1, o_n p_n$.

20. As a train is running along a straight track at a uniform rate, an automobile moves, also at a uniform rate, down a hill along a straight road which passes beneath the railroad. Find the figure generated by the line joining two fixed points, one on the train and the other on the automobile.

21. Initially a plane cuts two fixed intersecting planes perpendicularly in the lines o_1 and o_2. As it moves it cuts these planes in the lines p_1, p_2, which cut o_1 and o_2 at their intersection and always make equal angles with them. Find its envelope.

Examine the flat pencils traced by p_1 and p_2.

22. If $ABCD$ is a regular tetrahedron, and a line so moves as always to intercept on AB and CD equal distances from A and C, find the surface generated by the line.

23. Consider Ex. 22 if the line continually divides AB and CD proportionally.

24. A sloping telephone wire and an electric-light pole cast upon the side of a house shadows which intersect. If the wind causes the source of light to swing in a straight line, find the path traced by the intersection of the shadows.

25. Consider Ex. 24 if the source of light swings in a circle which intersects the wire and the pole.

26. Two fixed lines o_1 and o_2 intersect and pierce a plane ω at the points O_1 and O_2. Two planes π_1 and π_2 revolve about o_1 and o_2 respectively so that their intersections with ω describe equal angles in the same time. Find the nature of the envelope of the intersections of π_1 and π_2.

CHAPTER IX

FIGURES OF THE SECOND ORDER

100. Purpose of the Discussion. The results obtained in the preceding chapter may be given a more general as well as a more compact and symmetric form. It will be noted that these results relate to three types of figures, namely, figures in a plane; figures in a bundle, or conical figures; and figures in space. These three types of figures will be considered separately.

101. Plane Figures. It has already been shown that the figure obtained as the envelope of the line joining corresponding points of coplanar projective ranges is a curve of the second class, and the one obtained as the locus of the intersections of corresponding lines of projective flat pencils is a curve of the second order. Whether all curves of the second class and all curves of the second order may be generated in this way, whether the ranges and flat pencils which give rise to the curves in question have special situations relative to those curves, and what relation, if any, exists between curves of the second class and those of the second order, are questions whose answers will exhibit clearly the importance and generality of the results obtained. These questions will now be discussed, but for the sake of brevity the treatment will be limited, the parts of the argument which are omitted being indicated. The student should not lose sight of the omission, and he should later seek to complete the argument which answers the questions.

102. Generalization of Results. By a line of reasoning which will not be given here the following can be proved:

Every curve of the second class is the· envelope of the lines joining corresponding points of two coplanar projective ranges whose bases are tangent to the curve.

The application of the Principle of Duality for the plane to this result immediately leads to the further result:

Every curve of the second order is the locus of the inter-sections of corresponding lines of two coplanar projective flat pencils whose bases are on the curve.

Accordingly it is clear that the developments in the preceding chapter relate to all curves of the second class and to all of the second order.

Consider the second question suggested in § 101. We have already seen that the bases of the projective ranges by means of which the curves of the second class were obtained are tangent to these curves at certain of their points. It will be shown (§ 106) that any two tangents to a curve of the second class may be taken as the bases of projective ranges such that the given curve is the envelope of the lines joining corresponding points of these ranges. Correspondingly, it may be shown that any two points on a curve of the second order may be taken as bases of projective flat pencils such that the given curve is the locus of the intersections of corresponding lines of these pencils; and because to most students the idea of the locus of points is more familiar than that of an envelope, the latter proposition will be established first.

The proofs of these statements regarding curves of the second order depend upon an auxiliary proposition which will now be stated and proved. This theorem will later (§ 120) be generalized.

THEOREM. AUXILIARY PROPOSITION

103. *If P_2, P_3, P_4, P_6 are four points on a curve of the second order which is the locus of the intersections of corresponding lines of the two coplanar projective flat pencils whose bases are P_1 and P_5, then the pairs of lines P_1P_2, P_4P_5; P_2P_3, P_5P_6; P_3P_4, P_6P_1 determine collinear points.*
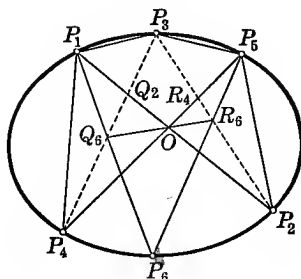
Proof. By hypothesis the points on the curve in question determine projective flat pencils whose bases are P_1 and P_5. These flat pencils may be denoted by $P_1(P_2P_3P_4P_6)$ and $P_5(P_2P_3P_4P_6)$.

Cut these pencils by the lines P_3P_4 and P_2P_3 respectively, and let the resulting ranges be $Q_2P_3P_4Q_6$ and $P_2P_3R_4R_6$.

These ranges are not only projective but also have a self-corresponding element P_3. Hence they are perspective, and the lines P_2Q_2 (or P_1P_2), P_4R_4 (or P_4P_5), and Q_6R_6 pass through O, the center of perspective.

Therefore the points Q_6, R_6 are collinear with the intersection of P_1P_2 and P_4P_5. But Q_6 is the intersection of P_3P_4 and P_6P_1, and R_6 is the intersection of P_2P_3 and P_5P_6.

Hence the proposition is proved.

By means of the above theorem the desired proposition, known as Steiner's theorem, may now be proved.

THEOREM. STEINER'S THEOREM

104. *Every curve of the second order is the locus of the intersections of coplanar projective flat pencils whose bases are any two points of the curve.*
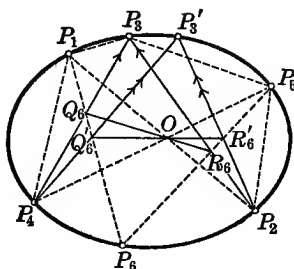
Proof. Let P_1, P_5 be the bases of the projective flat pencils which generate the curve, and let P_2, P_4, P_6 be any three fixed points on the curve.

Let P_3 be any point moving along the curve and occupying successive positions, as P_3, P_3', $\cdots$.

It will be shown that the flat pencils generated by the moving lines $P_2 P_3$ and $P_4 P_3$ are projective.

For each position of P_3 the pairs of lines $P_1 P_2$, $P_4 P_5$; $P_2 P_3$, $P_5 P_6$; $P_3 P_4$, $P_6 P_1$ determine as collinear the fixed point O and the points Q_6, R_6 (§ 103).

As P_3 moves, Q_6 and R_6 move along the fixed lines $P_1 P_6$ and $P_5 P_6$ and trace ranges on them.

Then flat pencil $P_2 (P_3 P_3' \cdots) \overline{\overline{\wedge}}$ range $R_6 R_6' \cdots$ (on $P_5 P_6$)

$\overline{\overline{\wedge}}$ range $Q_6 Q_6' \cdots$ (on $P_1 P_6$)

$\overline{\overline{\wedge}}$ flat pencil $P_4 (P_3 P_3' \cdots)$.

Accordingly the curve is the locus of the intersections of corresponding lines of the projective flat pencils whose bases are P_2 and P_4, any two points of the curve.

THEOREM. AUXILIARY PROPOSITION

105. *If t_2, t_3, t_4, t_6 are four tangents to a curve of the second class which is the envelope of the lines joining corresponding points of two coplanar projective ranges whose bases are t_1 and t_5, then the pairs of points t_1t_2, t_4t_5; t_2t_3, t_5t_6; t_3t_4, t_6t_1 determine concurrent lines.*

This theorem is the dual of the theorem of § 103 and leads to the dual of Steiner's theorem. It will later (§ 121) be generalized into a highly important proposition. The proof is left for the student.

THEOREM. DUAL OF STEINER'S THEOREM

106. *Every curve of the second class is the envelope of the lines joining corresponding points of the coplanar projective ranges whose bases are any two tangents to the curve.*

It is particularly important that by means of the Principle of Duality, or otherwise, the student should follow out in detail the proof of this proposition, as well as the proof of the proposition which immediately precedes it (§ 105). There is not much difficulty in obtaining the steps of the proof as duals of the corresponding steps of the proof in § 104, but the figure and the verification of the various steps of the argument in connection with this figure require close attention on the part of the student.

107. Relations between Curves of the Second Order and Curves of the Second Class. The two sorts of plane curves which have been obtained can now be shown to be identical. In other words, it can be shown that every curve of the second order is of the second class, and conversely. Only one of these proofs will be given, since the other can be derived from it by means of the Principle of Duality. In this proof use will be made of a limiting case of the proposition in §103, in which two of the four arbitrarily chosen points on the curve have moved up to coincidence with the bases of the pencils, and this will first be established.

THEOREM. INSCRIBED QUADRANGLE

108. *If P_3, P_6 are two points on a curve of the second order which is generated by two coplanar projective flat pencils whose bases are P_1 and P_5, the tangents at P_1 and P_5, the tangents at P_3 and P_6, and the pairs of opposite sides P_1P_3, P_5P_6; P_3P_5, P_6P_1 of the inscribed quadrangle $P_1P_3P_5P_6$ intersect in collinear points.*
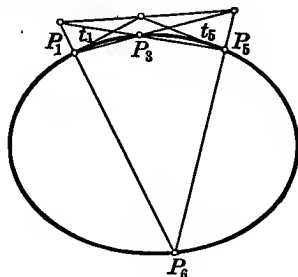
Proof. In the figure of § 103 let P_2 and P_4 move along the curve so as to approach P_1 and P_5 as limiting points.

Then P_1P_2 and P_4P_5 approach the tangents to the curve at P_1 and P_5 respectively, and O approaches the intersection of the tangents at P_1 and P_5 as a limiting point.

The hexagon $P_1P_2P_3P_4P_5P_6$ approaches the quadrangle $P_1P_3P_5P_6$, together with the tangents at P_1 and P_5.

During the motion of the points and lines the intersections of the pairs of lines P_1P_2, P_4P_5; P_2P_3, P_5P_6; P_3P_4, P_6P_1 remain collinear, and the limiting positions which they approach are collinear.

Since P_1, P_5 are not special points on the curve (§ 104), it follows that the tangents at P_3, P_6 meet on the same line.

Then the theorem of § 103 takes the form of this theorem.

This proposition will be used as auxiliary to the proof of the identity of curves of the second order with those of the second class.

Exercise 24. Steiner's Theorem and Related Theorems

1. In the theorem of § 103, if P_1, P_2, P_4, P_6 are fixed points, and if P_3 and P_6 so move that the intersection R_6 of P_2P_3, P_5P_6 moves on a fixed line through the intersection of P_1P_2, P_4P_6, then the intersection Q_6 of P_3P_4, P_6P_1 moves on the same line, and Q_6, R_6 trace on this line two superposed projective ranges.

2. In Ex. 1 find the self-corresponding points of the superposed projective ranges.

3. Prove the proposition which is related to the theorem of § 108 as Ex. 1 is related to the theorem of § 103.

4. Solve the problem which is related to Ex. 3 as Ex. 2 is related to Ex. 1.

5. If t_3, t_6 are two tangents to a curve of the second class which is the envelope of the lines joining corresponding points of two coplanar projective ranges whose bases are t_1, t_5, the points of contact of t_1, t_5, the points of contact of t_3, t_6, and the pairs of opposite vertices t_1t_3, t_5t_6; t_3t_5, t_6t_1 of the circumscribed quadrilateral $t_1t_3t_5t_6$ determine concurrent lines.

6. A variable hexagon $P_1P_2P_3P_4P_5P_6$ inscribed in a curve of the second order so moves that P_1P_2, P_4P_5; P_2P_3, P_5P_6 always intersect in fixed points O and R respectively. Find the locus of the intersection of P_3P_4, P_6P_1.

7. Prove the duals of Exs. 1, 3, and 6 for the plane.

8. Solve the dual of Ex. 2 for the plane.

9. By an argument independent of that given in this chapter prove Steiner's theorem for the special case of the circle.

10. By an argument independent of that referred to in § 106 prove the theorem stated there for the special case of the circle.

11. P_1, P_2, P_3, P_4, P_5 are five fixed coplanar points no three of which are in a straight line; find the locus of a point P which so moves that the intersections of the pairs of lines P_1P_2, P_4P_5; P_2P_3, P_5P; P_3P_4, PP_1 are constantly collinear.

PG

THEOREM. IDENTITY OF CURVES

109. *Every plane curve of the second order is of the second class, and conversely.*
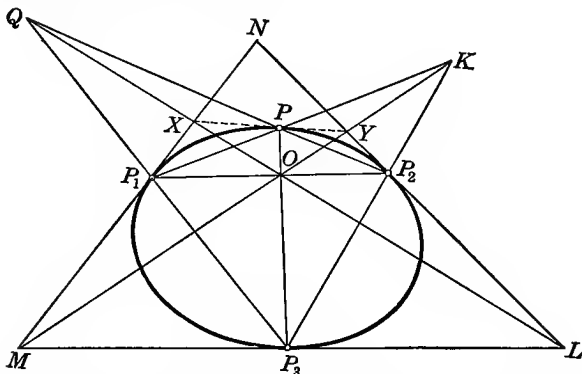
Proof. When a curve of the second order is given, the pencils of lines drawn from two of its points to all of its points are projective. From any three pairs of corresponding lines all additional pairs can be obtained by the method of § 39. In particular, since the tangent at either of the two points corresponds to the line joining the two points, it can be drawn by the same method; and therefore, when the whole curve is given, the tangents at as many points as may be desired can be drawn.

Let a curve of the second order be given, and select on it any three points P_1, P_2, P_3. Draw P_1P_2, P_2P_3, P_1P_3 and the tangents MP_1N, NP_2L, and LP_3M. The projectivity is determined by the triads P_1M, P_1P_2, P_1P_3 and P_2P_1, P_2L, P_2P_3.

Consider the curve as the locus of a moving point P. It will be shown that as P moves along the curve the tangent at P so moves as to meet the tangents at P_1 and P_2 in corresponding points X and Y of two projective ranges.

Let O, K be the intersections of the sides P_1P_2, P_3P and P_1P, P_2P_3 of the quadrangle $PP_1P_2P_3$ inscribed in the given curve. For this inscribed quadrangle, M is the intersection of tangents at opposite vertices and O, K are intersections of pairs of opposite sides, and hence it follows from § 108 that the points M, O, K are collinear. By means of a like reasoning the points O, K, Y are proven collinear. Hence the points M, O, Y are collinear.

Similarly, the points L, O, X may be proved collinear.

Since P_1, P_2, P_3 are fixed points, the tangents at these points are also fixed. As the point P moves, so do the lines P_1P, P_2P, P_3P, XY, P_3O, LO, MO, and the points O, X, Y. The lines LO, MO generate perspective flat pencils with bases at L and M, and the points X and Y trace on the lines MP_1N and LP_2N ranges perspective with these pencils and hence projective with each other. Therefore, as P moves, the tangent at P so moves as always to meet the tangents at P_1 and P_2 in corresponding points of two projective ranges.

The given curve is the envelope of the tangent at P and, by § 104, is of the second class.

The converse of this proposition being also its dual for the plane, its proof is also the dual of the above argument. The theorem is therefore established.

110. Conic. A plane curve which is of the second order and second class can be shown to be a curve ordinarily called a *conic section* or a *conic*.

The proof will not be given, but, independently of its other uses, the word *conic* will be employed to designate any curve of the second order. The use of properties of conics not deduced from the definition here given will be avoided.

A conic section, or conic, is a curve of the second order and second class.

THEOREM. ORDER AND CLASS OF SURFACES (IN THE BUNDLE)

111. *Every surface (in the bundle) that is of the second order is of the second class, and conversely.*

Proof. A comparison of the problems whose discussion led to §§ 84 and 90 shows that these theorems deal with figures and state results that, for threefold space, are dual. Similarly, the theorems of §§ 86 and 87 are dual. It was indicated, but not proved, that § 84 is true for all plane curves of the second class and that § 87 is true for all plane curves of the second order. Correspondingly, §§ 90 and 87 can be shown to apply to all surfaces (in the bundle) that are of the second class and all surfaces (in the bundle) that are of the second order. Moreover, the identity of the curves of the second order with curves of the second class having been established in § 109, the identity of these surfaces of the second order with those of the second class may be established by reasoning dual to that of § 109. This involves the derivation of an auxiliary theorem dual for space to that of § 103 and one that is dual to the limiting case of the latter as worked out in § 108. The student will find that the principal difficulty is connected with the drawing of appropriate figures for these cases.

In this way the classes of figures numbered 3 and 4 in § 99 are shown to be identical.

112. Quadric. Since a surface (in the bundle) of the second order and second class may be thought of as generated by the motion of a line that always passes through a fixed point, it is said to be a conic surface or a cone; and on account of its order and class it is called a *quadric conic surface*.

A quadric conic surface or quadric cone is a surface (in the bundle) of the second order and second class.

THEOREM. SKEW RULED SURFACES

113. *Every ruled surface (not in the plane or in the bundle) of the second order is of the second class, and conversely.*

Proof. In §§ 85 and 95 certain ruled surfaces of the second order that exist in threefold space but not in the plane or in the bundle were found to be generated by means of both projective ranges and projective axial pencils. It can further be shown that all ruled surfaces of the second order can be so generated. Moreover, a complete discussion of these surfaces from the point of view of both methods of generation would have led to results similar to those obtained for the conics and for the quadric cones. Among other things it would have appeared that the planes which pass through a point P not on the surface, and are tangent to the surface, would generate a quadric cone, and that of these planes not more than two, but in some cases two, would pass through a line that contains P. Hence these surfaces are of the second class.

The converse may also be proved.

The discussion indicates that the class of figures numbered 5 in § 99 is self-dual in threefold space.

114. Summary. We have now shown that the configurations whose generating elements are determined by corresponding elements of two projective one-dimensional prime forms are as follows:

1. *All plane curves of the second order and second class (conics).*

2. *All conic surfaces of the second order and second class (quadric cones).*

3. *All ruled surfaces of the second order and second class not in the plane or in the bundle.*

Exercise 25. Review

1. State and prove the dual for space of § 103.

2. State and prove the dual for the bundle of Ex. 1.

3. Compare the dual for space of the theorem in § 105 with the result of Ex. 2.

4. Give the dual for the plane of the statement and proof of Ex. 11, page 107.

5. Give the dual for space of the statement and proof of the theorem in § 108.

6. Compare the space dual of the result of Ex. 5, page 107, with the dual for the bundle of the result of Ex. 5, above.

7. Derive the dual for space of Ex. 1, page 107.

8. Establish three projectivities between flat pencils which shall lead to the generation of conics having respectively two, one, and no points of intersection with any given straight line.

In this connection consider § 58.

9. Consider Ex. 8 for the case in which the given straight line is the line at infinity of a given plane.

The student will observe that the solution of this problem establishes the existence of conics with two, one, and no points at infinity.

10. Establish three projectivities between ranges which shall lead to the development of conics having respectively two, one, and no tangents whose points of contact are on any straight line.

11. Consider Ex. 10 for the case in which the given straight line is the line at infinity of a given plane.

12. Prove the dual for space of Ex. 6 on page 107.

13. Solve the dual for the plane of Ex. 8.

14. Consider Ex. 13 for the case in which the given point is at infinity in a given direction.

15. Solve the dual for space of Ex. 8.

16. Five concurrent lines, no three of which are in any one plane, all lie on one conic surface of the second order.

17. Prove the proposition regarding five parallel lines which corresponds to Ex. 16.

18. Establish three projectivities between axial pencils which shall lead to the generation of conic surfaces having their vertices at infinity and having as right sections curves with two, one, and no points at infinity respectively.

19. Establish between two given ranges which are not in the same plane a projectivity such that if the surface generated is cut by any given plane in the finite part of space, the section shall be two straight lines.

20. In a bundle π_1, π_2, π_3, π_4, π_5 are five fixed planes, no three of which are coaxial. Find the envelope of a plane π which moves so that the planes determined by the intersections of π_1, π_2 and π_4, π_5, of π_2, π_3 and π_5, π, and of π_3, π_4 and π, π_1 are constantly coaxial.

21. Consider Ex. 19 for the case in which the given plane is at infinity.

22. Given a plane and two projective axial pencils which have no common element, establish between other pencils a projectivity which shall lead to the generation of a surface that shall be the projector from a given center of the intersection of the given plane and the surface generated by the given axial pencils.

23. Derive the dual of Ex. 22 for space.

24. Given in a plane three nonconcurrent bases p_1, p_2, p_3 passing through the points A_1, A_2, A_3 respectively, specify three projectivities which connect ranges on the bases p_1, p_2; p_2, p_3; p_3, p_1 respectively, and which are such that the three conics that they determine shall coincide and also be tangent to the three lines p_1, p_2, p_3 at A_1, A_2, A_3 respectively.

25. Examine Ex. 24 for the case in which the three points A_1, A_2, A_3 are collinear.

26. In Ex. 24 select such lines p_1, p_2, p_3 and such points A_1, A_2, A_3 that the conic generated shall be a circle.

27. Given in a bundle three non-coaxial planes π_1, π_2, π_3 passing through the lines a_1, a_2, a_3 respectively, specify three projectivities which connect flat pencils in the planes π_1, π_2; π_2, π_3; π_3, π_1 respectively, and which are such that the three conic surfaces they determine shall coincide and shall be tangent to the planes π_1, π_2, π_3 along the lines a_1, a_2, a_3.

28. Solve the dual of Ex. 27 for the bundle.

29. Consider Ex. 19 for the case in which the section by the plane at infinity is to be two straight lines.

30. Establish such a projectivity between two given axial pencils, not in the same bundle, that if the surface generated is cut by any given plane in the finite part of space, the section shall be circular.

31. Find two axial pencils, not in the same bundle, between which such a projectivity may be established that the corresponding surface generated shall be cut by a given plane in a given circle of that plane.

32. Given two projective axial pencils, not in a bundle, pass a plane which shall cut the surface generated by them in two straight lines.

33. Given three bases p_1, p_2, p_3 in space, no two of which intersect, specify three projectivities between ranges on the bases p_1, p_2; p_2, p_3; p_3, p_1 respectively, such that the three skew ruled quadric surfaces determined by them shall coincide.

34. Solve the dual for space of Ex. 33.

35. Develop completely the proof of the theorem corresponding to the theorem of § 109 for the case of figures of the second order in the bundle.

36. For the case of figures of the second order in threefold space describe accurately the figure for the theorem corresponding to that of § 109, and outline the proof.

CHAPTER X

CONICS

115. Determination of Conics by Certain Conditions. Some of the more important properties of the curves and surfaces to which attention has been drawn in the preceding chapters will now be deduced. The conics will be dealt with much more fully than the other figures because of their more frequent application and also because, after their properties have been set forth, the corresponding properties of the quadric cone may be obtained by means of the Principle of Duality.

Chapters X–XII are devoted to the conics.

In Chapter XIII there is given a discussion of quadric cones, this being confined to a few topics in addition to those suggested by the developments obtained for the conics. Notwithstanding this limitation, students should give due attention to these properties of quadric cones.

In Chapter XIV will be found a brief introduction to the study of the properties of skew quadric ruled surfaces. A thorough study of these figures may well be deferred until the student has an opportunity to approach the subject from the point of view of analytic geometry also, when a comparison of the analytic and synthetic treatments will heighten the interest.

The first body of facts to be established relates to sets of data which completely determine conics. It constitutes the important theorem stated in § 116, which consists of six simple propositions.

THEOREM. A CONIC DETERMINED

116. *In a plane there is one and only one conic which has one of the following properties :*

1. *It passes through five given points, no four of which are collinear.*

2. *It passes through four given points, no three of which are collinear, and at any one of these points is tangent to a given line which passes through this point but not through any other of the given points.*

3. *It passes through three given points, not collinear, and at each of two of these points is tangent to a given line which passes through this point but not through any other of the three given points.*

4. *It passes through two given points, and at each of these is tangent to a given line which passes through that point but not through the other given point, and in addition is tangent to a third given line which is not concurrent with the other two given lines.*

5. *It passes through a given point and is tangent at that point to a given line through the point, and is tangent to each of three other given lines so situated that of four given lines no three are concurrent.*

6. *It is tangent to five given lines, no four of which are concurrent.*

Without doubt the student will generally use in his work an abridgment of this statement. The longer statement given above may be regarded as an interpretation of the shorter one in § 117, making clear the meaning of the determination of a figure by means of certain data. The student will find that very frequently in geometry this abridged form of statement is used in the sense expressed more fully by the other one. Occasional expansions of shorter statements into the corresponding longer ones are well worth the attention of the student.

THEOREM. ALTERNATIVE STATEMENT OF § 116

117. *A conic is determined by any one of the following sets of elements that are associated with it:*

1. *Five of its points.*

2. *Four of its points and the tangent at one of these points.*

3. *Three of its points and the tangents at two of these points.*

4. *Three of its tangents and the points of contact on two of these tangents.*

5. *Four of its tangents and the point of contact on one of these tangents.*

6. *Five of its tangents.*

Proof. We shall deal with the cases in the above order.

1. *A conic is determined by five of its points.*
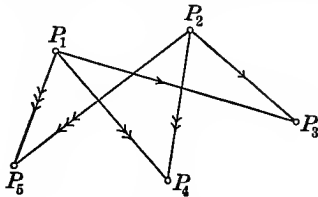
Let P_1, P_2, P_3, P_4, P_5 be five points of a plane, no four of them being collinear. Join P_1 and P_2 to P_3, P_4, P_5.

The triads of lines P_1P_3, P_1P_4, P_1P_5 and P_2P_3, P_2P_4, P_2P_5 determine a projectivity between the flat pencils whose bases are P_1, P_2, and hence they determine a conic through the five points. Any conic through these points could be generated from the projectivity determined by the same triads (§ 104) and would be the same as the one mentioned.

If three of the five points are situated on a line l, the other two points should be taken as the bases of the pencils. In this case the conic consists of the line l and the line through the bases.

2. *A conic is determined by four of its points and the tangent to it at one of these points.*
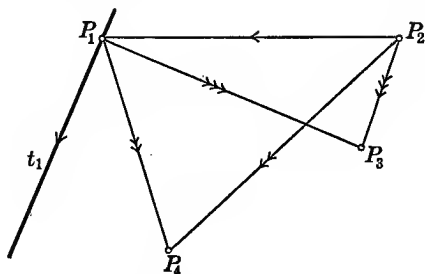
Let P_1, P_2, P_3, P_4 be four points of a plane, no three of them being collinear, and in the plane let t_1 be any line which passes through P_1 but which does not pass through any other of the four given points.

Draw from P_1 the lines P_1P_3, P_1P_4, and draw from P_2 the lines P_2P_1, P_2P_3, P_2P_4.

Consider P_1 and P_2 as bases of flat pencils. The lines t_1, P_2P_1; P_1P_3, P_2P_3; P_1P_4, P_2P_4 being taken as corresponding, one and only one projectivity is thereby established between the flat pencils. This projectivity determines one conic passing through the points P_1, P_2, P_3, P_4 and having the line t_1 as its tangent at P_1.

No other conic can fulfill these conditions, since in that case the conic would also be generated from the projectivity just mentioned, and hence this conic would coincide with the first one.

Therefore the second statement is proved.

If three of the four points other than the one at which the tangent is given are on a line l, the conic consists of the given tangent and the line l. If P_1 and two only of the other points are collinear, no conic is determined.

3. *A conic is determined by three of its points and the tangents to it at two of these points.*
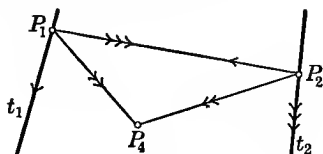
Let P_1, P_2, P_4 be three noncollinear points of a plane, and in the plane let t_1 and t_2 be lines which pass through P_1 and P_2 respectively but through no other of the three given points. Draw the lines P_1P_2, P_1P_4, and P_2P_4.

The triads t_1, P_1P_2, P_1P_4 and P_2P_1, t_2, P_2P_4 determine a projectivity between the pencils whose bases are P_1 and P_2, and this projectivity determines a conic passing through P_1, P_2, P_4 and tangent at P_1 and P_2 to t_1 and t_2 respectively.

As in the other cases, it may be shown that there is only one such conic. Hence the third statement is proved.

If the three points are on a line l and if one of the given tangents is l, the conic consists of l and the other tangent. If neither of the tangents coincides with l or both tangents coincide with l, the conic may be thought of as the line l taken twice.

4. *A conic is determined by three of its tangents and the points of contact of two of these tangents.*

Since Nos. 3 and 4 are dual in the plane, the proof of No. 4 follows at once.

5. *A conic is determined by four of its tangents and the point of contact of one of these tangents.*

Since Nos. 2 and 5 are dual in the plane, the proof of No. 5 follows at once.

6. *A conic is determined by five of its tangents.*

Since Nos. 1 and 6 are dual in the plane, the proof of No. 6 follows at once.

118. Construction of Conics. The statements of § 117 establish the existence of conics that fulfill certain conditions, and suggest but do not solve the problem of constructing the conics under these various conditions. The solution of this problem can be based upon the notion of projectivity involved in § 117, but it can also be based upon two very celebrated theorems which will be considered on page 121. After these theorems have been proved, the problems of the construction of conics will be treated from both points of view.

Before considering these two theorems, however, it will be found necessary to make some extension of the common notion of a hexagon with which the student is familiar from elementary geometry.

119. Hexagon. If any six coplanar points are taken in a given order, the figure formed by the lines through all pairs of successive points, as well as through the first and last points, is called a *hexagon.*
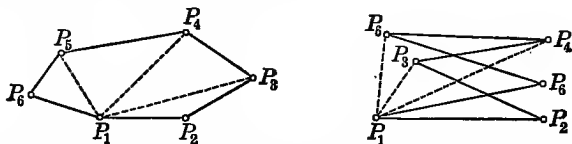
As in the ordinary case of the hexagon, the first and fourth, the second and fifth, and the third and sixth sides are called *opposite sides.*

Thus, in the above figures the pairs of opposite sides are P_1P_2, P_4P_5; P_2P_3, P_5P_6; P_3P_4, P_6P_1.

In each of the above figures the diagonals from P_1 are P_1P_3, P_1P_4, and P_1P_5.

A similar generalization applies to each of the other polygons. It thus appears that opposite sides of a quadrilateral may intersect and that a diagonal may lie wholly outside a polygon.

THEOREM. PASCAL'S THEOREM

120. *If a hexagon is inscribed in a conic, the three intersections of the three pairs of opposite sides are collinear.*
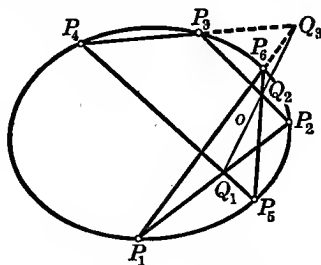
Proof. This is the theorem of § 103 with the restriction upon P_1, P_5 removed by Steiner's theorem (§ 104).

Unless a cross hexagon is taken, the figure is usually very large. The proposition is due to Blaise Pascal (1623–1662).

THEOREM. BRIANCHON'S THEOREM

121. *If a hexagon is circumscribed about a conic, the three lines joining the three pairs of opposite vertices are concurrent.*
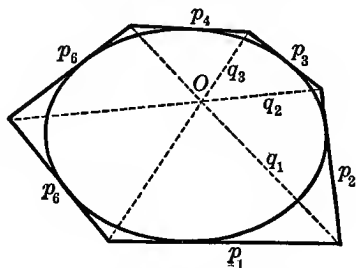
Proof. This is simply a generalization of § 105.

The student should write out the proof of this theorem.
The proposition is due to Charles Julien Brianchon (1785–1864).

122. Pascal Line. The line containing the points of intersection of the three pairs of opposite sides of a hexagon in a conic is called the *Pascal line* of the hexagon.

123. Brianchon Point. The point of concurrence of the three lines joining the opposite vertices of a hexagon about a conic is called the *Brianchon point* of the hexagon.

124. Converses of the Theorems of Pascal and Brianchon. The converses of the theorems of Pascal and Brianchon can be established as in the exercise below, and each of them may then be given a different interpretation. Thus, if six coplanar points are chosen and joined to form a hexagon, a conic passes through any five of them. Does it pass through the sixth point? It does if and only if the three points of intersection of the pairs of opposite sides are collinear. Hence Pascal's theorem and its converse imply the necessary and sufficient conditions for the passing of a conic through six given coplanar points. Brianchon's theorem can be interpreted in a corresponding fashion.

Exercise 26. Theorems of Pascal and Brianchon

1. State and prove the converse of Pascal's theorem.

2. If two pairs of opposite sides of a hexagon inscribed in a conic are parallel, the other two opposite sides are parallel.

3. A hexagon is to be inscribed in a conic in such a way that a given line shall be its Pascal line. Determine the maximum number of sides of the hexagon that may be given, and solve the problem.

4. Solve Ex. 3 for the case when the given line is at infinity.

5. State and prove the converse of Brianchon's theorem.

6. Circumscribe a hexagon about a given conic in such a way that a given point shall be its Brianchon point, as many of the vertices of the hexagon as possible being given in advance.

125. Limiting Cases of the Theorems of Pascal and Brianchon. There are several limiting cases of the theorems of Pascal and Brianchon which have useful applications and which require mention at this point. They arise out of approach to coincidence of vertices of an inscribed hexagon of a conic and also of sides of a circumscribed hexagon.

Let $P_1P_2P_3P_4P_5P_6$ be a hexagon inscribed in a conic. If P_2 approaches P_1 along the conic, the line P_1P_2 approaches the tangent t_1 at the point P_1, and the hexagon approaches the figure composed of the pentagon $P_1P_3P_4P_5P_6$ and the tangent t_1 to the conic at P_1. The pairs of opposite sides are t_1, P_4P_5; P_1P_6, P_5P_6; P_3P_4, P_6P_1. These pairs determine collinear points.

Similarly, P_2 may approach P_1 and either P_4 approach P_3 or P_5 approach P_4, yielding an inscribed quadrilateral and tangents to the conic at two of the vertices of the quadrilateral. A third case is that in which P_2 approaches P_1, P_4 approaches P_3, and P_6 approaches P_5.

In each case the propriety of extending Pascal's theorem, and others, to limiting cases in which two distinct elements are allowed to become coincident is left for the student's consideration.

In the case of a circumscribed hexagon, if one side approaches coincidence with a second, their point of intersection approaches a limiting position at the point of contact of the second side. There arise out of the approach of sides to coincidence a number of limiting cases of Brianchon's theorem which can be worked out and which will be needed from time to time.

Other limiting cases of these propositions are those in which the points or lines of the figures are not all in the finite part of the plane. For example, one or two vertices of the Pascal hexagon and one side of the Brianchon hexagon may be at infinity. Coincident elements and infinitely distant elements may be present in one hexagon.

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PROBLEM. CONIC THROUGH FIVE POINTS

126. *Given five points in a plane, no four of them being collinear, construct the conic which is determined by them.*
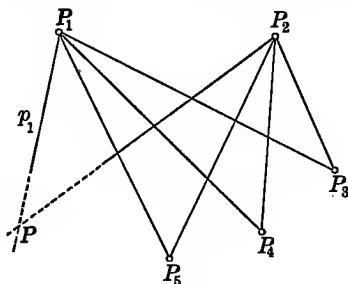
Solution. This problem admits of two simple solutions.

1. *Method based on a projectivity.*

Let the given points be P_1, P_2, P_3, P_4, P_5. Then in any chosen direction from any point, as P_1, there can be found another point of the conic which is not collinear with two of the others. Let the chosen direction be along the line p_1, and let the point to be found be called P. Draw P_1P_3, P_1P_4, P_1P_5, P_2P_3, P_2P_4, P_2P_5. The triads of lines P_1P_3, P_1P_4, P_1P_5 and P_2P_3, P_2P_4, P_2P_5 determine the projectivity between two flat pencils which generate the required conic.

The point P is the intersection of p_1 and its corresponding line of the pencil whose base is P_2; and it may be found by the method used in § 39, Case 1. Draw this line and produce it to meet p_1, thus determining P.

By varying the position of the line p_1 any number of additional points of the conic may be found.

Evidently it is not feasible to obtain all the points of the conic by this method, nor is the method convenient in practice. In this respect it is similar to the method of plotting in analytic geometry.

2. *Method based on Pascal's theorem.*

Let the given points be P_1, P_2, P_3, P_4, P_5. As before, another point P can be found on a chosen line p_1 that passes through any one of the points, as P_1.
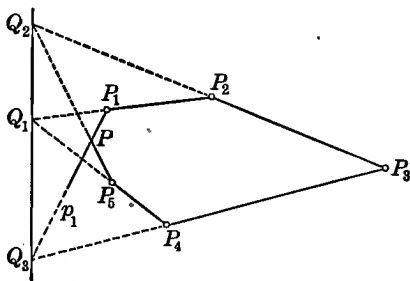
The given points P_1, P_2, P_3, P_4, P_5 and the point P, which is to be found, are the vertices of a hexagon inscribed in the conic determined by the five given points.

Then P_1P_2, P_4P_5; P_2P_3, P_5P; P_3P_4, PP_1 intersect on the Pascal line of this hexagon. Of these six lines, P_5P is not given and PP_1 is the given line p_1. The Pascal line is determined by the intersection of P_1P_2, P_4P_5 and that of P_3P_4, p_1.

Draw the Pascal line and let it meet P_2P_3 in Q_2. Draw Q_2P_5. This line Q_2P_5 must coincide with the line P_5P and must intersect the line p_1 in the required point P.

Since every line through P_1 determines a point on the conic, it is possible to locate any number of points.

This method, like the first one, bears a certain resemblance to the method of plotting in analytic geometry. From the point of view of convenience it is decidedly superior to the first method.

The student will observe that, since the conic is of the second order, the line p_1 cuts it in one and only one point other than P_1, and also that in either solution of the problem the use of the ruler alone is sufficient.

127. *Given four points in a plane, no three of them collinear, and a line passing through one and only one of these points, construct the conic which passes through the given points and at one of them is tangent to the given line.*
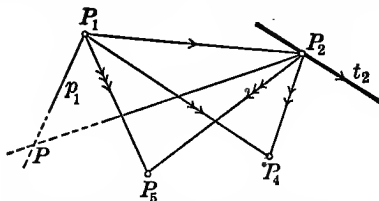
Solution. As was the case in § 126, there are two simple methods of construction. In each method any number of additional points of the conic may be found by determining where the conic would be cut by lines which pass through one of the points.

1. *Method based on a projectivity.*

Let the given points be P_1, P_2, P_4, P_5, and let the given line be t_2 passing through the point P_2. Draw any line p_1 through the point P_1. Join P_1 to each of the points P_2, P_4, P_5, and join P_2 to each of the points P_4, P_5.

The triads $P_1 P_2$, $P_1 P_4$, $P_1 P_5$ and t_2, $P_2 P_4$, $P_2 P_5$ determine a projectivity between the flat pencils whose bases are P_1, P_2, and the required conic is the locus of the intersections of corresponding lines of the projective pencils.

The line through P_2 which corresponds to p_1 of the pencil whose base is P_1 can be determined by the method used in § 39, Case 1, and P, the intersection of this line with p_1, is the point required.

By varying the position of p_1 any number of points of the conic may be found.

2. *Method based on Pascal's theorem.*

Let P be the point on p_1 that is to be found. It is determined if the direction of P_5P can be determined.
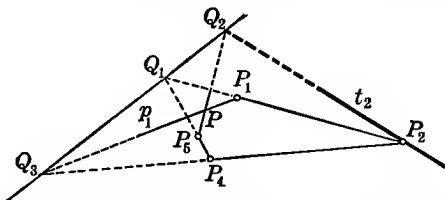
The pentagon $P_1P_2P_4P_5P$ is inscribed in the required conic, and the line t_2 is tangent to the conic at P_2.

The intersections of P_1P_2, P_4P_5; t_2, P_5P; and P_2P_4, PP_1 (or p_1) are on the Pascal line.

Produce P_1P_2 and P_4P_5 to meet at Q_1, and produce P_2P_4 and p_1 to meet in Q_3. Draw the Pascal line. Let t_2 meet this line in Q_2; join P_5 and Q_2.

Then the lines P_5Q_2 and P_5P are coincident and the intersection of P_5Q_2 and p_1 is the required point P.

PROBLEM. THREE POINTS AND TWO TANGENTS

128. *Given in a plane three noncollinear points and two lines, each of which passes through one and only one of the given points, construct the conic which passes through the three given points and at each of two of them is tangent to the given line through that point.*

The solution is left for the student. It should be effected by two methods, as in the two preceding theorems. As in the other problems the second method is to be preferred for practical reasons.

An appreciation of the superior convenience of the second method is best secured by making the actual construction necessary for finding by the first method the line through P_2 which corresponds to p_1 of the first pencil.

PROBLEM. THREE TANGENTS AND TWO POINTS

129. *Given three nonconcurrent lines in a plane, and on each of two of these lines a point which is not on any other of the three, construct the conic which is tangent to each of the given lines and has each of the given points as the point of contact of the given line on which it lies.*

Of what problem is this the dual? The solution is left for the student.

PROBLEM. FOUR TANGENTS AND ONE POINT

130. *Given four lines in a plane, no three of them concurrent, and a point on one but not on two of them, construct the conic which is tangent to each of these lines and has the given point as the point of contact of the given line on which it lies.*

Of what problem is this the dual? The solution is left for the student.

PROBLEM. FIVE TANGENTS

131. *Given five lines in a plane, no four of them concurrent, construct the conic which is tangent to each.*

Of what problem is this the dual? The solution is left for the student.

PROBLEM. CONSTRUCTING A TANGENT

132. *Given five or more points of a conic, construct the tangent to the conic at any one of these points.*

The solution is left for the student.

PROBLEM. FINDING A POINT OF CONTACT

133. *Given five or more tangents to a conic, determine by construction the point of contact of any one of these tangents.*

The solution is left for the student.

Exercise 27. Problems of Construction

1. If two projective flat pencils generate a circle, they are congruent.

2. Using the result in Ex. 1, find any number of additional points of a circle when three of its points are given.

3. Find any number of points of a circle when two of its points and the tangent at one of them are given.

4. Solve the problem in § 126 when one of the five points is at infinity in a given direction.

5. Solve the problem in § 126 when two of the points are at infinity in given directions.

6. Solve the problem in § 127 when the given line is at infinity.

7. Solve the problem in § 127 when one of the four points is at infinity in a given direction.

8. Solve the problem in § 128 when one of the points is at infinity in a given direction and the tangent at that point is given to be the line at infinity.

9. Solve the problem in § 129 when the two given points are at infinity.

10. Solve the problem in § 131 when one of the five given lines is the line at infinity.

11. Solve the problem in § 132 when the point at which the tangent is to be constructed is at infinity in a given direction.

12. Solve the problem in § 133 when the given tangent whose point of contact is to be found is the line at infinity.

13. If a parallelogram is inscribed in a conic, the tangents to the conic at the vertices form a parallelogram circumscribed about the conic.

14. If P_1, P_2, P_3, P_4, P_5 are fixed points and P moves on the conic determined by them, find the envelope of the Pascal line of the hexagon $P_1P_2P_3P_4P_5P$.

THEOREM. INVOLUTION ON COMPLETE QUADRANGLE

134. *If a straight line cuts all the sides of a complete quadrangle but does not pass through any vertex, it cuts the three pairs of opposite sides of the quadrangle in conjugate points of an involution.*
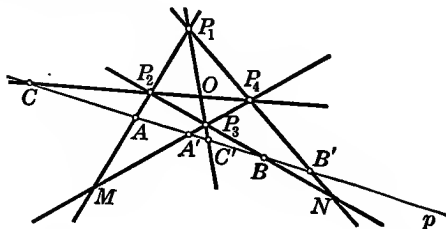
Proof. Let a straight line p cut the pairs of opposite sides of the complete quadrangle whose vertices are P_1, P_2, P_3, P_4 in A, A'; B, B'; C, C'; and let M, N, O be the diagonal points of the quadrangle.

Then   range $ABA'C \overline{\wedge}$ flat pencil $P_2(MP_3A'P_4)$

$\overline{\wedge}$ range $MP_3A'P_4$

$\overline{\wedge}$ flat pencil $P_1(MP_3A'P_4)$

$\overline{\wedge}$ range $AC'A'B'$.

But   range $ABA'C \overline{\wedge}$ range $A'CAB$.   § 23

Hence   range $AC'A'B' \overline{\wedge}$ range $A'CAB$.

Accordingly, A, A'; B, B'; C, C' are conjugate points of an involution on p.   § 65

This theorem is auxiliary to, and is in fact a special case of, an interesting and important theorem which was first established by the French geometer Girard Desargues (1593–1662).

Desargues's theorem offers another line of approach to some of the preceding constructions and to other similar problems. In particular, on page 133, it is applied to the solution of § 126.

THEOREM. DESARGUES'S THEOREM

135. *If a complete quadrangle is inscribed in a conic, and if a straight line cuts the conic in two points distinct from each other and from the vertices of the quadrangle, these two points form a conjugate pair of the involution of points on the line, which is determined by the intersections of the line with the pairs of opposite sides of the quadrangle.*
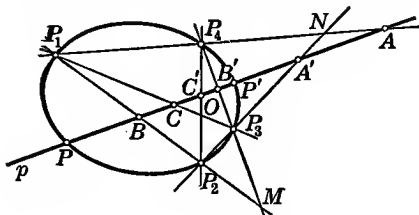
Proof. Let the complete quadrangle $P_1P_2P_3P_4$ be inscribed in a conic, and let a line p which does not pass through any of the four vertices cut the conic in P, P' and the pairs of opposite sides in A, A'; B, B'; C, C'.

Then $\qquad$ range $PBP'A \overline{\wedge}$ flat pencil $P_1(PP_2P'P_4)$

$\qquad\qquad \overline{\wedge}$ flat pencil $P_3(PP_2P'P_4)\overline{\wedge}$ range $PA'P'B'$.

But $\qquad$ range $PA'P'B' \overline{\wedge}$ range $P'B'PA'$. $\qquad$ § 23

Therefore $\qquad$ range $PBP'A \overline{\wedge}$ range $P'B'PA'$.

Hence P, P'; A, A'; B, B' are conjugate points of an involution on p determined by the pairs A, A'; B, B'.

The involution formed by the intersections of p with the pairs of opposite sides of the quadrangle $P_1P_2P_3P_4$ is also determined by the pairs of points A, A'; B, B'.

Accordingly, P, P' are conjugate points of the involution of points determined by the intersections of the line p with the pairs of opposite sides of the quadrangle $P_1P_2P_3P_4$.

136. Restatement of Desargues's Theorem. It should be noted that many conics pass through the four points P_1, P_2, P_3, P_4 and that to each of such conics Desargues's theorem applies. Moreover, it should be remembered that the pairs of lines P_1P_2, P_3P_4; P_1P_4, P_2P_3 are degenerate conics through the four points. Hence Desargues's theorem is capable of restatement as follows:

The infinitely many conics, including pairs of lines, which pass through four given coplanar points, no three of which are collinear, determine on any line which intersects them (but does not pass through any one of the points) infinitely many pairs of points of an involution.

137. COROLLARY. *If the involution determined by the conics is hyperbolic, two of the conics which pass through the four points touch the straight line; if it is elliptic, no conic through the four points is tangent to the straight line.*

Exercise 28. Application of Desargues's Theorem

1. What sort of involution is determined upon a side of the diagonal triangle of the quadrangle mentioned in the theorem of § 135 ?

2. Test the validity of the proof of Desargues's theorem when it is applied to a line through one of the given points, say P_2.

Given four points in a plane, no three of which are collinear, show how to draw a straight line subject to each of the following conditions:

3. There shall be two conics passing through the four points and tangent to the line.

4. There shall be one conic passing through the four points and tangent to the line.

5. There shall be no conic such as described in Ex. 4.

PROBLEM. CONIC THROUGH FIVE POINTS

138. *Given five points, no four of which are collinear, construct the conic which is determined by them.*
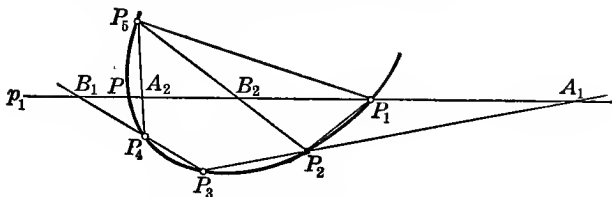
Solution. Let us consider two cases.

1. *No three of the five given points are collinear.*

Let the five points be P_1, P_2, P_3, P_4, P_5. Draw P_1P_2, P_2P_3, P_3P_4, P_4P_5, P_5P_1, P_5P_2, and any transversal p_1 through P_1.

It is now required to find the point P in which this line again cuts the conic determined by the five given points.

The points A_1, A_2, B_1, B_2, in which p_1 is cut by the lines P_2P_3, P_4P_5, P_3P_4, P_5P_2, determine an involution in which P_1 and the required point P are corresponding points.

Hence the point P can be determined from the projectivity between the ranges in the involution. For instance, the three known points A_1, B_1, A_2 and the required point P are projective with the four known points A_2, B_2, A_1, P_1. Various special devices for finding P based upon the method of § 39, Cases 1 and 2, can be found.

By varying the position of the line p_1 any number of points on the conic can be found.

2. *Three of the five given points are collinear.*

In this case the required conic is a pair of straight lines, one the line through the three points and the other the line through the other two points.

139. *Given two superposed projective one-dimensional prime forms, construct the position of the self-corresponding elements.*
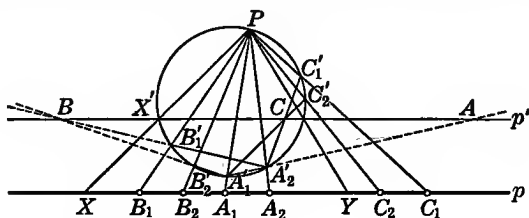
Solution. If the superposed prime forms are not ranges, it is possible by operations of projection and of section to obtain from them two superposed projective ranges. Hence it is necessary to solve the problem only for the case in which the prime forms are ranges.

Let $A_1 B_1 C_1$ and $A_2 B_2 C_2$ be two triads of corresponding points of superposed projective ranges on a base p.

Describe any circle coplanar with the line p. Join any point P of the circle to each of the six given points, and let these lines cut the circle again in A_1', B_1', C_1 and A_2', B_2', C_2'. Join A_1' to A_2', B_2', C_2', and A_2' to B_1', C_1'.

Then   flat pencil $A_2' (A_1' B_1' C_1' \cdots)$

$\overline{\wedge}$ flat pencil $P(A_1' B_1' C_1' \cdots)$ § 53

$\overline{\wedge}$ range $A_1 B_1 C_1 \cdots$

$\overline{\wedge}$ range $A_2 B_2 C_2 \cdots$

$\overline{\wedge}$ flat pencil $P (A_2' B_2' C_2' \cdots)$

$\overline{\wedge}$ flat pencil $A_1' (A_2' B_2' C_2' \cdots)$.

But the flat pencils $A_2' (A_1' B_1' C_1' \cdots)$, $A_1' (A_2' B_2' C_2' \cdots)$ have a self-corresponding element, and hence are perspective.

Let X' be a point, if there be any, in which p', the axis of perspectivity, cuts the circle, and let PX' meet p at X.

In the four flat pencils previously mentioned the corresponding lines are $A_1'X'$, PX', $A_2'X'$, PX', and hence in the superposed pencils whose bases are at P, PX' is a self-corresponding line. Therefore X is a self-corresponding point of the ranges on p.

Conversely, it is true that, corresponding to each self-corresponding point of the ranges on the line p, there is an intersection of the line p' with the circle.

Hence, to find the self-corresponding points on p we join P to the intersections of p' and the circle, and produce these lines to intersect p. There may be no, one, or two intersections with p, and each of these intersections is a self-corresponding point.

140. Constructions of the Second Order. All constructions made before § 139 were effected wholly by the use of straight lines, and at every stage the results were uniquely determinate; that is, all the problems had one and only one solution. If the solutions of the problems analogous to these constructions are effected by the methods of analytic geometry, it is found that only equations of the first degree are used. For this reason these and similar problems are said to be of the first order.

Beginning with § 141, attention will be given to problems whose solutions by the method of analytic geometry would involve the use of at least one equation of the second degree, as in § 139. Correspondingly, each construction will require the use (at least once) of a curve of the second order, and for simplicity the circle will be taken.

The problem in § 139 furnishes a basis for others, and hence has been deferred from its most natural place, which was in connection with the treatment of superposed projective forms in Chapter VII.

### PROBLEM. INTERSECTIONS OF A LINE AND A CONIC

141. *Given any of the sets of elements mentioned in § 117 as determining a conic, construct the intersections (if there are any) of the conic with a given straight line p in its plane.*

Solution. If the set of elements is not five points, by means of §§ 127–133 find five points P_1, P_2, P_3, P_4, P_5 on the conic. Join any two of the points, as P_1 and P_2, to P_3, P_4, P_5, and let these lines meet the line p in P_3', P_4', P_5' and P_3'', P_4'', P_5''.

These triads determine superposed projective ranges on p. By the method of § 139 find the self-corresponding points (if there are any) of these ranges.

Since these self-corresponding points are common to corresponding lines of the projective flat pencils whose bases are P_1 and P_2, they are on the conic. Moreover, they are the only points of p which are on the conic. There may, therefore, be two, one, or no intersections.

PROBLEM. TANGENTS FROM A POINT

142. *Given any of the sets of elements mentioned in § 117 as determining a conic, construct the tangents (if there are any) to the conic from a given point P in its plane.*

Solution. If the set of given elements is not five tangents to the conic, by means of §§ 127–133 find five tangents t_1, t_2, t_3, t_4, t_5 to the conic. The tangents t_3, t_4, t_5 cut t_1, t_2 in triads of points which, being joined to P, determine a projectivity between flat pencils whose base is P.

Find the self-corresponding lines of these pencils. Any such line passes through P and also joins corresponding points of the projective ranges on t_1, t_2 which serve to generate the conic. Hence the line is tangent to the conic. The number of these lines is two, one, or none.

PROBLEM. FOUR POINTS AND A TANGENT

143. *Construct a conic which shall pass through four given points, no three of which are collinear, and shall be tangent to a given line that does not contain any of the points.*
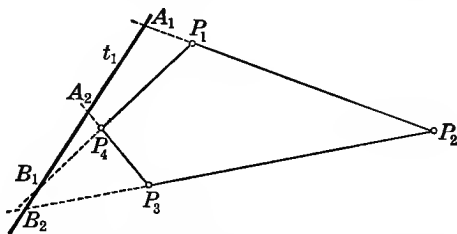
Solution. Let the given points be P_1, P_2, P_3, P_4, and let the given line be t_1. Let t_1 cut the lines P_1P_2, P_3P_4 in A_1, A_2· and cut the lines P_1P_4, P_2P_3 in B_1, B_2.

Find the self-corresponding points of the involution on t_1 which is determined by these pairs of points. Through the four points and any self-corresponding point P_5 construct a conic (§ 138). This conic is tangent to t_1 at P_5. For if it cuts t_1 in a second point P_6, then the point P_5 is not self-corresponding, and this is contrary to fact.

Hence, for every self-corresponding point of the involution on t_1 one conic can be constructed.

There may be no, one, or two self-corresponding points (§ 139). Hence no, one, or two conics may be constructed.

THEOREM. FOUR POINTS AND A TANGENT

144. *The number of conics which pass through four given points, no three of which are collinear, and are tangent to a given straight line which does not pass through any of the points, is none, one, or two.*

The proof is left for the student.

145. *Construct a conic which shall be tangent to each of four given straight lines, no three of which are concurrent, and which shall pass through a given point exterior to the lines.*

This problem is the dual of § 143 and may be solved as such. The solution is left for the student. Likewise, a theorem dual to § 144 results from the proof of the construction.

146. Special Case of Desargues's Theorem. To complete a set of constructions which include §§ 126, 129–133, 143, and 145, two others are necessary, and these are given in §§ 148 and 149. In order to solve these two problems special cases of Desargues's theorem (and its dual) may be used.

Instead of the four distinct points of the conic considered in Desargues's theorem, let the first and second points move up to coincidence, and also let the third and fourth points move up to coincidence. Then the line joining the first and second points and that joining the third and fourth points become tangents to the conic. Also the lines joining the first and third points, the second and fourth points, the first and fourth points, and the second and third points move into coincidence upon the chord of contact of the two tangents mentioned.

147. *Two straight lines and the conics which are tangent to them at two given points intersect a given line that does not pass through either of these points in pairs of points of an involution, one of the self-corresponding points of which is the intersection of the given line with the chord of contact.*

The proof is left for the student.

PROBLEM. THREE POINTS AND TWO TANGENTS

148. *Construct a conic which shall pass through each of three given noncollinear points and be tangent to each of two given lines that do not pass through any of the points.*
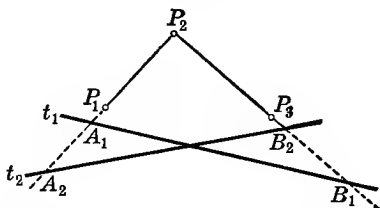
Solution. Let the points be P_1, P_2, P_3, and let the lines be t_1, t_2. Let t_1, t_2 cut the line P_1P_2 in A_1, A_2 and the line P_2P_3 in B_1, B_2. We shall first find the points of contact of t_1, t_2 with the conic.

Find the self-corresponding points (if there are any) of the involutions determined by P_1, P_2 and A_1, A_2 and by P_2, P_3 and B_1, B_2 respectively. Let the line through M_1, one of the first of these, and M_2, any one of the second, cut t_2 in P_4 and t_1 in P_5. Pass a conic through P_2, and tangent to t_1 and t_2 at P_5 and P_4 respectively (§ 128).

Since the point corresponding to P_2 in the involution on A_1A_2 is completely determined by M_1 and the pair of points A_1, A_2, it follows that this conic must pass through P_1 (§ 147), for the involution determined on P_1P_2 by conics tangent at P_4 and P_5 to t_2 and t_1 is also determined by a self-corresponding point and the pair of points A_1, A_2. Similarly, this conic can be shown to pass through P_3.

Hence, for every possible pair of points P_4, P_5 one conic may be constructed.

There may be no, one, two, or four pairs of points, as P_4, P_5 (§ 144), and for each a conic may be constructed.

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THEOREM. THREE POINTS AND TWO TANGENTS

149. *The number of conics which pass through three given noncollinear points and which are tangent to two given lines that do not contain any of the points is none, one, two, or four, as the case may be.*

The student should write out the proof, which is essentially that of § 148.

PROBLEM. THREE TANGENTS AND TWO POINTS

150. *Construct a conic which shall be tangent to three given nonconcurrent lines and shall pass through two given points which are exterior to the lines.*

The student should write out the solution, which is simply the dual of that of § 148.

Exercise 29. Review

1. If the sides of an angle of constant size rotating about a fixed vertex intersect respectively two fixed lines, the line joining these intersections envelops a conic.

2. Two vertices of a variable triangle move along two fixed lines, and the three sides respectively pass through three fixed collinear points. Find the locus of the third vertex.

3. Consider Ex. 2 for the case in which the three fixed points are not collinear.

4. If two triangles are in plane homology, the intersections of the sides of one triangle with the noncorresponding sides of the other lie on a conic.

5. State Pascal's theorem for the case in which the first and second, the third and fourth, and the fifth and sixth vertices have become coincident.

6. The complete quadrilateral formed by four tangents to a conic, and the complete quadrangle formed by their four points of contact, have the same diagonal triangle.

7. If a variable quadrangle $P_1P_2P_3P_4$ inscribed in a conic has as fixed points P_1, P_2, and the intersection of P_1P_2, P_3P_4, the other vertices of its diagonal triangle move along the same fixed straight line.

8. If P_3, P_5 are fixed points on a given conic, and if P is a moving point, as P moves along the conic the Pascal line of the hexagon, consisting of the triangle PP_3P_5 and the tangents to the conic at the points P_1, P_3, P_5, envelops a conic.

9. If P_1, P_3, P_4 are fixed vertices of a complete quadrilateral whose fourth vertex P moves along a given conic through P_1, P_3, P_4, all the vertices of the diagonal triangle trace straight lines and all the sides pass through fixed points.

10. If the points P_2 and P_3 trace superposed projective ranges on the base AB of a fixed triangle ABC, if P_1 is a fixed point not on any side of the triangle, if P_1P_2 meets AC in P_4, and if P_1P_3 meets BC in P_5, find the locus of P, the intersection of AP_5 and BP_4.

11. In Ex. 10 find the envelope of P_4P_5.

12. State Desargues's theorem for the case in which a pair of the four given coplanar points become coincident.

13. State Desargues's theorem for the case in which two pairs of the given coplanar points become coincident.

14. Three sides, AB, AD, CD respectively, of a variable quadrangle inscribed in a given conic pass through three given points of a line. Find the envelope of BC.

15. Extend Ex. 14 to the case of a simple inscribed polygon having $2\,n$ sides.

16. From the data of § 127 construct, by means of § 147, tangents at additional points of the conic.

17. Prove the dual of § 149, namely, that the number of conics which can be constructed under the conditions of § 150 is none, one, two, or four.

18. Solve the dual of § 139 for the plane.

19. If the lines p_1 and p_2 are drawn through the vertices P_1 and P_2 respectively of a given quadrangle, the conics which pass through the vertices of the quadrangle determine perspective ranges on p_1 and p_2.

20. If the lines p_1 and p_2 are drawn through the vertex P_1 of a given quadrangle, the system of conics which pass through the vertices of the quadrangle determine projective ranges on p_1 and p_2.

21. State and prove the dual of Ex. 19 for the plane.

22. Construct a conic which shall pass through two given points P_1 and P_2, shall be tangent to a given line t_3 at the point P_3, and shall be tangent to a second given line t_4.

Apply Ex. 12 for the line t_4. Find the self-corresponding points of the involution.

23. Construct a conic which shall be tangent to a given line t_1 at the point P_1, to t_2 at P_2, and to t_3.

24. Consider the problem of § 141 for the case in which the given line is the line at infinity.

25. Consider the problem of § 143 for the case in which the given line is the line at infinity.

26. Consider the problem of § 148 for the case in which one of the given lines is the line at infinity.

27. Solve the dual of Ex. 22 for the plane.

28. Construct a triangle which shall be inscribed in a given triangle and have its sides pass through three given points.

Observe that if a triangle has two vertices, as required, but not the third, the sides through the latter cut a side of the given triangle in corresponding points of superposed projective ranges.

29. Construct a triangle which shall be inscribed in a given conic and have its sides pass through three given points.

30. If a conic can be described through the six vertices of two given triangles, another conic can be described which shall be tangent to the six sides of the two given triangles.

CHAPTER XI

CONICS AND THE ELEMENTS AT INFINITY

151. Classification of Conics. In the discussion of conics in the preceding chapter no classification was made, nor was any account taken of the fact that on certain occasions the term *straight line* may mean " straight line at infinity " and the term *point* may mean " point at infinity." These considerations can be associated very advantageously.
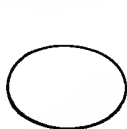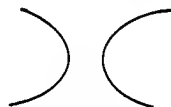
ELLIPSE   PARABOLA   HYPERBOLA

In projective geometry, conics are classified by means of their relations to the line at infinity. This line, like any other, may intersect a conic in no, one, or two points, and hence conics are divided into three classes as follows:

1. *Ellipses*, or conics that do not intersect the line at infinity.

2. *Parabolas*, or conics that intersect the line at infinity in one point (or are tangent to the line at infinity).

3. *Hyperbolas*, or conics that intersect the line at infinity in two distinct points.

While these conics are familiar to the student from his work in analytic geometry, the study of conics will now be considered from a different point of view.

152. Elements at Infinity. In the interpretation of the results already obtained, in so far as the ellipse is concerned, it will be seen that the expressions *point on the curve* and *tangent to the curve* always mean a point and a line in the finite part of the plane.

On the other hand, in connection with the parabola, one and only one tangent, and one and only one point of the curve (the point of contact of that tangent), may be taken to be at infinity.

In the case of the hyperbola there are two points on the curve which are at infinity, but the line at infinity is not a tangent. At each of the infinitely distant points of the curve there is, however, a tangent which has no infinitely distant point except its point of contact.

It follows that in the cases of the parabola and hyperbola the interpretations of the theorems of Pascal and Brianchon and of similar theorems obtained by the methods of projective geometry vary according as all or only part of the elements are assumed to be in the finite part of the plane.

In the light of the procedure indicated, the results which have been obtained are capable of restatements which vary for the three types of conics, but which have a great interest, because they bring these results into clearer relation to those obtained by the methods of analytic geometry.

153. Asymptote. A line, not the line at infinity, which is tangent to a conic at an infinitely distant point is called an *asymptote*.

In this figure *a* is an asymptote.

Every hyperbola has, then, two asymptotes, and the other conics have none, though sometimes the parabola is said to have the line at infinity as an asymptote. This latter form of statement is convenient when geometry is treated algebraically, but it will not be adopted in this text.
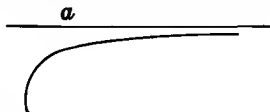
154. Special Interpretations. As indicated in § 152, each of the results that have been derived for the conics should be examined for interpretations based upon the relations of the elements at infinity to the three types of conics. The great variety of results that can be obtained prevents a systematic and detailed reexamination in this place of all the theorems and constructions that have been derived. A few of these will be obtained, but for the most part their derivation must be left to the student, a work which will prove both interesting and profitable.

Of the elements (points and lines) which determine a conic not more than two points and not more than one line may be at infinity, except in the limiting case of coincident points or coincident tangents. The existence of one infinitely distant point on a conic determines that the curve is not an ellipse, and the existence of two such points determines that the curve is a hyperbola. Similarly, the tangency of the line at infinity to the curve determines it to be a parabola.

On the other hand, when all the given determining elements are in the finite part of the plane, the conic may prove to be of any one of the three types. The determination of the character of the conic of which certain elements are given is a particularly interesting case. It is the problem of § 141 as modified in Ex. 24, page 142.

In view of what is said above, we shall now restate the important theorem of § 116.

In each case the student should draw the figure and satisfy himself that the statement is correct and that it is a special case of one of the corresponding statements in §§ 116 and 117. He should also supplement the results here set forth by the others which can be obtained if a thorough examination of the theorem is made for its various interpretations.

THEOREM. A CONIC DETERMINED

155. 1. *There is one and only one conic (parabola or hyperbola) which passes through four points in the finite part of a plane and one infinitely distant point in a specified direction.*

This follows from the theorem stated in § 116, No. 1.

2. *There is one and only one hyperbola which passes through three points in the finite part of a plane and has given directions for its asymptotes.*

This follows from the theorem stated in § 116, No. 1.

3. *There is one and only one parabola which passes through three noncollinear points in the finite part of a plane and has its infinitely distant point in a given direction.*

This follows from the theorem stated in § 116, No. 2.

4. *There is one and only one hyperbola which passes through any point in the finite part of the plane and has two given straight lines as asymptotes.*

This follows from the theorem stated in § 116, No. 3.

5. *There is one and only one hyperbola which has two given lines as asymptotes and is tangent to a third line which is not parallel to either of the others.*

This follows from the theorem stated in § 116, No. 4.

6. *There is one and only one parabola which is tangent to each of three nonconcurrent lines lying in the finite part of the plane and has its infinitely distant point in a given direction.*

This follows from the theorem stated in § 116, No. 5.

7. *There is one and only one parabola which is tangent to any four lines of a plane, no three of which are concurrent and no two of which are parallel.*

This follows from the theorem stated in § 116, No. 6.

THEOREM. SPECIAL INTERPRETATION OF PASCAL'S THEOREM

156. *The chords from a point on a hyperbola to each of two other points on the hyperbola intersect the lines through these two points parallel to one of the asymptotes, the intersections being collinear with the intersection of the tangents at the two points.*

The proof of this theorem is included in the proof given in § 157.

THÉOREM. FURTHER INTERPRETATION OF PASCAL'S THEOREM

157. *The chords from a point on a parabola to each of two other points on the parabola intersect the lines from these two points to the infinitely distant point of the curve, the intersections being collinear with the intersection of the tangents at the two points.*

Proof. These two theorems are closely related, being obtained by applying to such conics the case of Pascal's theorem in which two pairs of vertices coincide. If a conic is known to have one point at infinity, it may be either a hyperbola or a parabola.

Consider a hexagon inscribed in a conic in such a way that the first and second vertices coincide, the fourth and fifth vertices coincide, and the sixth vertex is at infinity. We may also assume that the curve is a hyperbola or that it is a parabola. If it is a hyperbola the sides of the hexagon which intersect at the infinitely distant point are parallel to the same asymptote. In either case one pair of opposite sides is a pair of tangents.

The two theorems considered above are merely statements of Pascal's theorem for the two cases described, the terms used being appropriate in connection with these two kinds of conics.

THEOREM. SPECIAL INTERPRETATION OF BRIANCHON'S THEOREM

158. *Given five tangents to a parabola, the line parallel to the first tangent and concurrent with the third and fourth tangents cuts the line parallel to the fifth tangent and concurrent with the second and third tangents on the line joining the intersection of the first and second tangents to that joining the fourth and fifth.*

Proof. In Brianchon's theorem (§ 121), simply let one of the tangents be the line at infinity, and the proof follows at once.

Pascal's theorem and Brianchon's theorem have a large number of special interpretations. Of these we have space for only the three given in §§ 156–158. They have been selected not because of their intrinsic importance but because they indicate the method of procedure.

159. Special Constructions. On account of the elements at infinity the problems which were considered in Chapter X may also be given special statements for certain cases. Thus a point at infinity may be specified by its direction; and since a hyperbola is determined by means of any three of its points in the finite part of the plane and its two points at infinity, it is determined by the three points mentioned and the directions of the two points at infinity. These latter directions are also the directions of the asymptotes.

In actual constructions certain special situations arise. Thus, drawing a line to a given infinitely distant point is the same as drawing a line parallel to a given line. To effect this with the ungraduated ruler it is necessary to have additional data, as in the exercises on pages 99 and 100. Problems involving considerations of this sort will be considered in §§ 160–162.

PROBLEM. CONSTRUCTION OF THE HYPERBOLA

160. *Given in a plane three noncollinear points and two pairs of parallel lines, each pair having a direction different from those of the lines joining the three points and also different from that of the other pair, construct a hyperbola through the three given points and having asymptotes parallel to the two given pairs of lines.*

The student should write out the solution, making appropriate modifications of the methods employed in §§ 126 and 138.

PROBLEM. DETERMINATION OF A CONIC

161. *Given a set of elements sufficient for the determination of a conic, determine the nature of the conic and the directions of its infinitely distant points (if there are any).*

Among the constructions of the second order the construction in § 141 deserves attention in this connection, and this problem is one of its special forms. If elements sufficient for the determination of the conic are given, the finding of the intersections of the conic with the line at infinity includes determining whether the conic is an ellipse, a parabola, or a hyperbola.

As in the original case, if five points of the conic are not given they may be found by construction. Let them be P_1, P_2, P_3, P_4, P_5. The triads of lines P_1P_3, P_1P_4, P_1P_5 and P_2P_3, P_2P_4, P_2P_5 determine the projectivity by means of which any number of additional points of the conic may be found.

A difficulty now arises in following the original construction, because the line p is at infinity. The triads of points of the superposed projective ranges on this line that are determined by the triads of the flat pencil are now not available from the point of view of construction by the ruler. If, however, the possibility of drawing lines parallel to all given lines is assumed, the resulting difficulty disappears. For the purpose of drawing the necessary parallels, the compasses must be used more freely than in the construction in § 141. With this difference the construction follows as before, and the student should write out the solution in full.

PROBLEM. CONSTRUCTION OF THE PARABOLA

162. *Given four points in a plane, no three of them colinear, construct the parabola which passes through these points.*

By § 144 the number of such parabolas is none, one, or two.

The solution of this problem, which is based on that of § 143, is left for the student.

Exercise 30. Elements at Infinity

As suggested on pages 144–148, investigate with respect to the elements at infinity the following cases already considered:

1. § 120.	**5.** § 128.	**9.** Page 142, Ex. 19.
2. § 121.	**6.** § 149.	**10.** Page 142, Ex. 22.
3. § 126.	**7.** Page 141, Ex. 10.	**11.** Page 142, Ex. 28.
4. § 127.	**8.** § 147.	**12.** Page 142, Ex. 29.

In Exs. 1–12 practice in special interpretation, not the finding of important results, is the object.

13. Using the compasses only once and the ruler, find five points in the finite part of the plane which shall determine an ellipse other than a circle.

14. Consider Ex. 13 for the case of a parabola.

15. Consider Ex. 13 for the case of a hyperbola.

16. In the finite part of the plane find four points through which no parabola passes.

17. In the finite part of the plane find four points through which two parabolas pass.

18. Find four points through which both a circle and a parabola pass.

19. On a straight line p passing through a given point P_1, find a point P such that through P_1, P and two other given points P_2, P_3 there pass (1) two parabolas; (2) one parabola; (3) no parabola. In each case indicate all possible positions of P.

CHAPTER XII

POLES AND POLARS OF CONICS

163. Polar of a Point. In § 165 it will be proved that if three lines, concurrent at a point O, cut a conic in A_1, A_2; B_1, B_2; C_1, C_2, the harmonic conjugates A, B, C of O with respect to A_1, A_2; B_1, B_2; C_1, C_2 are collinear. Since OC_1C_2 may be any line through O, it follows that all harmonic conjugates of O with respect to the pairs of points in which lines through O cut the conic are on the line determined by A and B, two of these conjugates.
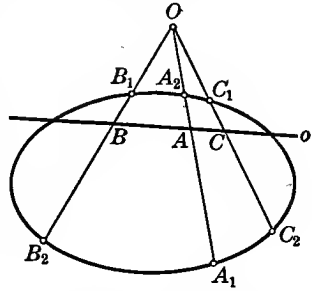
The line thus determined by two harmonic conjugates is called the *polar* of the point O with respect to the conic.

164. Pole of a Line. Suppose that there are given a conic and a line o. If from a point of the line o two tangents are drawn to the conic, then k, the harmonic conjugate of o with respect to the two tangents, may be constructed. This line will be spoken of as a *harmonic conjugate of o with respect to the conic*, or simply as a *harmonic conjugate of o*. The point of intersection of two harmonic conjugates of o will be called the *pole of o*.

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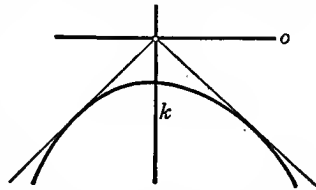
THEOREM. TRIANGLES IN HOMOLOGY

165. *If through a point O three lines are drawn cutting
a conic in the pairs of points A_1, A_2; B_1, B_2; C_1, C_2, the
triangles $A_1B_1C_1$ and $A_2B_2C_2$ are in harmonic homology.*
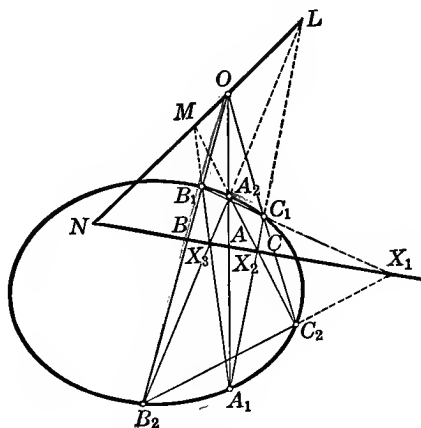
Proof. The triangles $A_1B_1C_1$ and $A_2B_2C_2$ are in homology,
as is indicated in the note under **Ex. 13**, page 13.

The axis of homology passes through X_1, X_2, X_3, the
intersections of B_1C_1, B_2C_2; C_1A_1, C_2A_2; A_1B_1, A_2B_2.

The Pascal line of the hexagon $A_1C_1C_2A_2B_2B_1$ contains
the points L, O, M, which are the intersections of the oppo-
site sides A_1C_1, A_2B_2; C_1C_2, B_2B_1; C_2A_2, A_1B_1. Moreover,
the Pascal line cuts the line X_1X_2 in a point N.

Also   range $LMON \overset{\wedge}{=}$ range A_2A_1OA

$\overset{\wedge}{=}$ range OAA_2A_1.   § 23

But $(LMON) = -1$.   § 30

Therefore $(OAA_1A_2) = 1 \div (OAA_2A_1) = -1$.   § 24

Hence the constant of homology is -1.

Exercise 31. Poles and Polars

Prove the theorem of § 165 for the following triangles:

1. $A_1B_1C_2$, $A_2B_2C_1$. **2.** $A_1B_2C_1$, $A_2B_1C_2$. **3.** $A_1B_2C_2$, $A_2B_1C_1$.

4. In the figure of § 165 A_1B_2, A_2B_1; A_1C_2, A_2C_1; B_1C_2, B_2C_1 also intersect on the line $X_1X_2X_3$.

5. Draw a large and accurate figure consisting of the figure of § 165 and the additional lines which would be introduced in proving Exs. 1, 2, and 3.

It is suggested that the sets of lines introduced on account of Exs. 1–3 be given distinctive colors.

6. State and prove the dual of § 165 for the plane.

7. Construct carefully the figure which is the dual of that required in Ex. 5.

8. If two triangles are homologic and the constant of homology is -1, the six vertices are on a conic.

9. If two triangles are homologic and the constant of homology is -1, the six sides are tangent to a conic.

10. By means of § 165 find a figure harmonically homologic with any polygon inscribed in a conic.

11. Inscribe in a conic a polygon which shall be harmonically self-homologic.

12. Use § 165 to obtain a line that bisects a given set of parallel chords of a conic.

13. Given a circle or a carefully drawn ellipse, parabola, or hyperbola, show experimentally that the polar of a given point O is the same line whatever pair of three given lines through O is used in constructing it.

14. Construct the polar o of a given point O of a conic, and then find the intersection of the polars of two points on o.

15. Construct the polar of a vertex of the diagonal triangle of a complete quadrangle inscribed in a conic.

166. Properties of a Polar. The polar of a point with respect to a conic has the following important properties:

1. *The polar of a point O with respect to a conic contains all harmonic conjugates of O with respect to the conic.*

In the preceding discussion the polar, though determined by A and B only, contains C, no matter what is the direction of OC_1C_2. Accordingly, any two of the harmonic conjugates of O determine a line through all of them.

In each of these cases the student should draw the figure and be certain that the suggested proof is clearly followed.

2. *The polar of a point O with respect to a conic contains the other intersections of opposite sides of any inscribed complete quadrangle of which O is a diagonal point.*

In the discussion of § 165, $A_1B_1A_2B_2$ is any inscribed quadrangle of which O is a diagonal point; and the other intersections of pairs of opposite sides of this quadrangle are situated on the axis of homology, which coincides with the polar of O.

3. *The polar of a point O with respect to a conic contains the intersections of pairs of tangents to the conic at the points in which any line through O cuts the conic.*

Let the line OB_1B_2 (§ 165) approach coincidence with the line OA_1A_2. Then B_1A_2 approaches the tangent at A_2, and B_2A_1 approaches the tangent at A_1; and at all stages these lines intersect on the polar of O.

4. *The polar of a point O with respect to a conic contains the points of contact of the tangents (if there are any) from O to the conic.*

If the line OA_1A_2 rotates about O and approaches the position of tangency to the conic, the points A_1, A, A_2 approach coincidence at the point of contact.

167. Properties of a Pole. Applying the Principle of Duality to the statements of § 166, we have the following :

1. *The pole of a line o with respect to a conic is on all harmonic conjugates of o with respect to the conic.*

The student should draw the figure in each of these cases and should write out the duals of the proofs suggested in § 166.

2. *The pole of a line o with respect to a conic is on the other lines joining opposite vertices of any circumscribed complete quadrilateral of which o is a diagonal line.*

3. *The pole of a line o with respect to a conic is on the chord of contact (produced if necessary) of the tangents from any point of o to the conic.*

4. *The pole of a line o with respect to a conic is the intersection of the tangents to the conic at the points (if there are any) in which the line o cuts the conic.*

A comparison of the properties of pole and polar as stated in §§ 166 and 167 leads to various interesting conclusions. A few of these are stated in §§ 168–171.

Exercise 32. Construction of Poles and Polars

1. Give a construction based upon § 166, 2, for the polar of a given point with respect to a given conic.

2. Construct the tangents to a conic from a given point O.

3. Find the pole of a given line with respect to a conic.

4. At a given point on a conic draw a tangent to the conic.

5. For any conic construct the polar o of a given point O, and then find the pole of the line o.

The figure should be drawn very carefully.

6. With respect to a given conic find the polar o_1 of a given point O_1, the polar o_2 of a given point O_2 on o_1, and the polar o_3 of the intersection of o_1 and o_2.

PG

Theorem. Relation of Pole and Polar

168. *If a point O is the pole of the line o, the line o is the polar of the point O.*
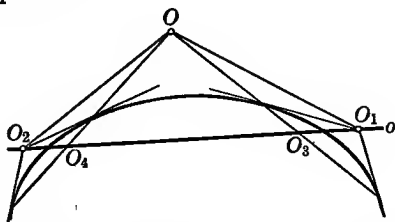
Proof. Let O be the pole of the line o, and let O_1, O_2 be points on o. Let the chords of contact of the tangents to the curve from O_1 and O_2 cut o in O_3 and O_4. These chords pass through O (§ 167, 3). Draw O_1O and O_2O.

Since the lines O_1O, O_2O are conjugates of the line o (§ 167, 1), the points O_3, O_4 are conjugates of O, and therefore o is the polar of O (§ 164).

169. Inside and Outside of a Conic. If a point O moves up to a position on a conic, its polar o becomes a tangent; but if O is not on the conic, either no tangent or two tangents pass through it. According as O is on two tangents or on no tangent, it is said to be *outside* or *inside* the curve.

If O is outside the curve, its polar cuts the conic in the two points of contact of the tangents from O to the conic.

Suppose O is inside the curve; then, since the polar meets all tangents to the conic, infinitely many of its points are outside the curve. Moreover, the polar does not meet the curve; for if it did, tangents could be drawn from O to the intersections. The proof of the theorem that every point of the polar, that is, every harmonic conjugate of O, is outside the conic is, however, somewhat complicated and will be omitted from this book, the fact being assumed.

THEOREM. POINTS ON A POLAR

170. *If the point O_1 is on the polar of the point O_2, then O_2 is on the polar of the point O_1.*
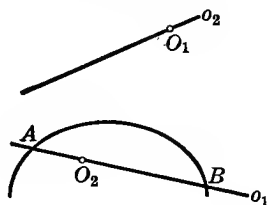
Proof. Of the points O_1 and O_2, at least one must be outside the conic. For if O_1 and O_2 are both inside the conic, then (§ 169) every point of o_2, including O_1, is outside the conic, which is contrary to the hypothesis made.

Let O_2 be outside the conic. Then § 166, 3 and 4, yields the desired conclusion.

Let O_2 be inside the conic. Then O_1, being on o_2, is outside the conic. Let o_1 cut the conic in the points A, B. It will cut the line $O_1 O_2$ in O_2; for otherwise the tangents of which $O_2 A$ is the chord of contact would not meet on o_2, as they must by § 166, 3. Hence O_2 is on o_1.

171. COROLLARY. *If a point O_1 traces out a range whose base is o_2, its polar o_1 traces out a flat pencil whose base is O_2, the pole of o_2.*

The relation between the range traced by the point O_1 and the flat pencil described by o_1 is stated in the theorem of § 177.

172. Conjugate Points and Conjugate Lines. Two points so situated that each is on the polar of the other are said to be *conjugate*, and two lines so situated that each contains the pole of the other are said to be *conjugate*.

Accordingly, harmonic conjugates are special cases of conjugates.

173. Self-Polar, or Self-Conjugate, Triangle. If any point O_1, not on a conic, is taken, its polar o_1 may be found. On this line let any point O_2, not on the conic, be taken, and let its polar o_2 be found. Then o_2 passes through O_1. Let the intersection of o_1 and o_2 be O_3.

The polar o_3 of O_3 is the line O_1O_2, and O_3 is conjugate to both O_1 and O_2. The triangle $O_1O_2O_3$ is such that each side is the polar of the op-
posite vertex. Every such triangle is said to be *self-polar*, or *self-conjugate*, with respect to the conic. Evidently there is an infinite number of triangles which are self-polar with respect to a given conic.
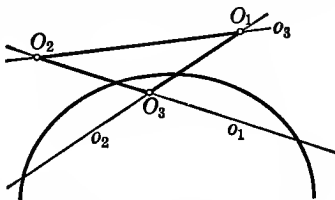
No self-polar triangle has two vertices inside the conic; for if one vertex is inside the curve, its polar in which the other two vertices lie is entirely outside the curve.

If we should attempt to construct a self-polar triangle all of whose vertices are outside the conic, we might choose a point O_1 outside the conic and draw its polar o_1. Let A, B be the two points in which this line would cut the conic. Then the other vertices O_2, O_3 would be on o_1 and would be separated by A, B. If we should take O_2 outside the conic, it would remain to determine whether O_3 would be inside or outside the conic; that is, whether tangents could be drawn from O_3 to the conic. The considerations adduced thus far would not enable us to give a sufficiently brief but complete discussion of this question, but an application of principles of continuity, which have not been developed in this book, would enable us to go farther and to establish the proposition that every self-polar triangle has one and only one vertex within the conic.

THEOREM. DIAGONAL TRIANGLE

174. *The diagonal triangle of a complete quadrangle inscribed in a conic is self-polar ; and, conversely, a self-polar triangle is the diagonal triangle of an inscribed complete quadrangle.*
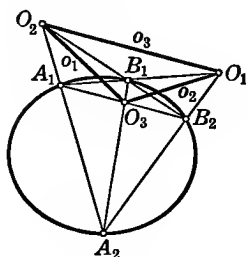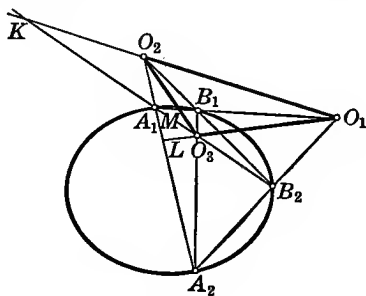
FIG. 1   FIG. 2

Proof. Let $O_1O_2O_3$ (Fig. 1) be the diagonal triangle of a complete quadrangle $A_1A_2B_2B_1$ inscribed in a given conic. From § 166, 2, it follows that $O_1O_2O_3$ is a self-polar triangle.

Conversely, let $O_1O_2O_3$ (Fig. 2) be a self-polar triangle with respect to a given conic.

From A_1, any point on the conic, draw A_1O_2, A_1O_3, and let them meet the conic again in A_2, B_2. Draw O_2B_2, O_1B_2, O_1A_1, O_3A_2, and let O_1A_1, O_3A_2 meet in B_1. Also let O_3A_1, O_1O_2 meet in K, let O_2A_1, O_3O_1 meet in L, and let O_1A_1, O_2O_3 meet in M.

Then $(A_1B_2KO_3) = -1 = (A_1A_2O_2L)$.

Hence range $A_1B_2KO_3 \overline{\wedge}$ range $A_1A_2O_2L$,

and the points A_2, B_2, O_1 are collinear.

Again, since range $A_1B_1MO_1 \overline{\wedge}$ harmonic range $A_1A_2O_2L$ and M is on the polar of O_1, then B_1 is on the conic.

Hence the quadrangle $A_1A_2B_2B_1$ is inscribed.

THEOREM. RANGES AND THEIR CONJUGATES

175. *On a line the range of points and the range of their conjugates with respect to a conic constitute an involution.*
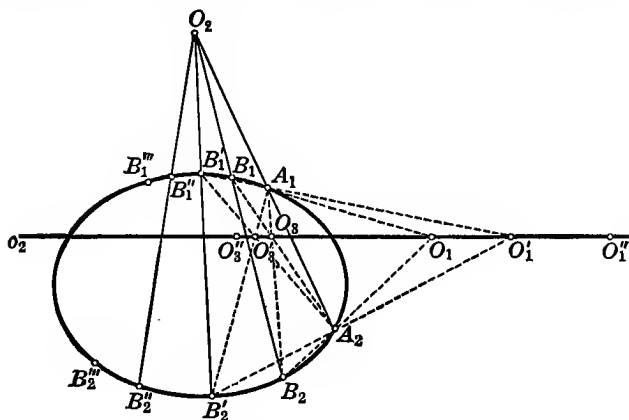
Proof. Let O_2 be the pole of a line o_2 with respect to a given conic. Through O_2 draw lines cutting the conic in A_1, A_2; and also in B_1, B_2; B_1', B_2'; B_1'', B_2''; $\cdots$. Then $O_2 O_1 O_3$ is the diagonal triangle of $A_1 A_2 B_2 B_1$; $O_2 O_1' O_3'$ of $A_1 A_2 B_2' B_1'$; $\cdots$. The pairs of points O_1, O_3; O_1', O_3'; $\cdots$ are conjugate, and, associating with $A_1 A_2$ all lines through O_2, we obtain all pairs of conjugates on o_2.

But   range $O_1 O_1' O_1'' \cdots O_3 O_3' O_3'' \cdots$

$\overline{\overline{\wedge}}$ pencil $A_1 (O_1 O_1' O_1'' \cdots O_3 O_3' O_3'' \cdots)$

$\overline{\wedge}$ pencil $A_2 (O_1 O_1' O_1'' \cdots O_3 O_3' O_3'' \cdots)$

$\overline{\wedge}$ pencil $A_2 (B_2 B_2' B_2'' \cdots B_1 B_1' B_1'' \cdots)$

$\overline{\wedge}$ pencil $A_1 (B_2 B_2' B_2'' \cdots B_1 B_1' B_1'' \cdots)$

$\overline{\wedge}$ range $O_3 O_3' O_3'' \cdots O_1 O_1' O_1'' \cdots$.

THEOREM. PENCILS AND THEIR CONJUGATES

176. *The pencil of lines through a point and the pencil of their conjugates with respect to a conic constitute an involution.*

The proof of this dual of § 175 is left for the student.

The properties of poles and polars furnish one basis for the establishment of the validity of the Principle of Duality for figures in a plane, and Poncelet practically used them in this way.

THEOREM. POINT DESCRIBING A RANGE

177. *If a point describes a range, the polar of the point with respect to a conic describes a flat pencil which is projective with that range.*
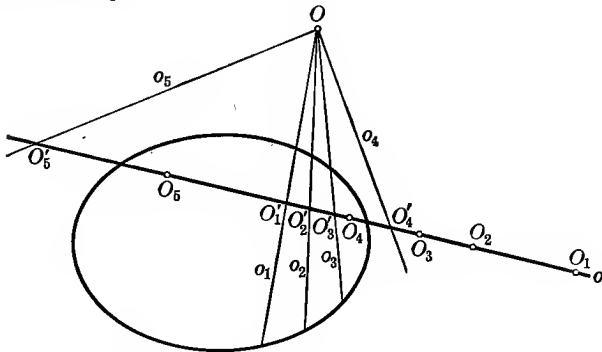
Proof. Let O be the pole of the range, and let the point take the positions $O_1,\ O_2,\ O_3,\ O_4,\ \cdots$. Then its polar always passes through o. The polars of $O_1,\ O_2,\ O_3,\ O_4 \cdots$ intersect o, the base of the range, in $O_1',\ O_2',\ O_3',\ O_4',\ \cdots$, the conjugates of $O_1,\ O_2,\ O_3,\ O_4,\ \cdots$ respectively.

Then   range $O_1 O_2 O_3 O_4 \cdots \overline{\wedge}$ range $O_1' O_2' O_3' O_4' \cdots$ § 175

$\overline{\overline{\wedge}}$ pencil $o_1 o_2 o_3 o_4 \cdots$.

Hence   range $O_1 O_2 O_3 O_4 \cdots \overline{\wedge}$ pencil $o_1 o_2 o_3 o_4 \cdots$.

178. Duality in Plane Figures. It is now possible to indicate a line of argument by which the Principle of Duality may be established for plane geometry. In the plane of a given figure, composed of, or generated by, points and lines, take any conic and construct the polar of every point and the pole of every line. Then a new figure is obtained in which there is a point for every line and a line for every point of the first figure, and in which to the intersection of any two lines of the first figure there corresponds the line determined by the poles of these two lines in the second. To any locus of points in the first figure there corresponds an envelope of lines in the second. Hence it is evident that a *duality in figures* exists.

179. Duality in Properties of Figures. Likewise, if any nonmetric proposition is true for some or all of the points and lines of the first figure in § 178, it follows that this figure cannot be constructed by choosing arbitrarily in the plane the sets of points and lines which constitute it, but that, certain points and lines being selected, the choice of the remaining ones is restricted. For the proposition, by its assertion of a relation, or of relations, existing among the points and the lines of a given figure, is a denial of the possibility of choosing all of them arbitrarily. Hence the second figure is not merely a set of lines and points, each chosen arbitrarily in the plane. In fact, there exists a certain limitation upon the choice of the lines and points of the second figure, and the statement of this limitation constitutes the proposition correlative to the one regarding the first figure. Hence there is a *duality in properties of figures.*

180. Polar Reciprocal or Polar Dual. A figure obtained from a given figure by the method explained in § 178 is called the *polar reciprocal* or *polar dual* of the given figure.

181. Center and Diameters of a Conic. As in some preceding cases, useful metric relations are obtained by consideration of the elements at infinity. Thus, the line at infinity has a pole, and from the property of the harmonic range this pole is seen to bisect every chord of a conic which passes through it. Because of this symmetry of the curve with respect to the pole of the line at infinity, this point is called the *center* of the conic. If the conic is a parabola, the center is also the point of contact of the parabola with the line at infinity; and since this point is at infinity and the notion of symmetry loses its usual force, the parabola is generally said not to have a center. In the case of the other conics the center is not at infinity, the center of the ellipse being inside the curve and that of the hyperbola being outside. In the case of the latter it is the intersection of two tangents to the curve whose points of contact are at infinity. These tangents are, of course, the asymptotes.

Again, every point on the line at infinity has a polar which passes through the center. The polar of a point at infinity is called a *diameter* of a conic. In the case of a parabola, since all the points at infinity are on the line at infinity, the diameters intersect in a point at infinity and hence are parallel. Also, any point on the line at infinity being chosen, all lines through that point are parallel; and if any one of these lines meets the conic, the harmonic conjugate of the chosen point at infinity that is situated on that line bisects the segment of it which is intercepted by the curve. Hence, every diameter bisects each of the set of parallel chords of the conic which passes through its pole. Likewise, the tangents at the points in which a diameter cuts the conic pass through the pole of that diameter; that is, they are parallel to the chords bisected by the diameter.

182. Conjugate Diameters and Principal Axes. If two diameters are conjugate lines with respect to a conic, each diameter passes through the pole of the other, which is the point at infinity, and so each is parallel to the chords bisected by the other. Such diameters are called *conjugate diameters*. According to § 176 they constitute an involution, and since, in general, only one pair of corresponding lines of an involution is 'at right angles, in general only one pair of conjugate diameters is at right angles. These two diameters are called the *principal diameters* and form the *principal axes*. Moreover, it follows from § 74 that if there are more pairs of conjugate diameters which are at right angles, all pairs have this property. In this case it can be shown that the conic is a circle.

When the involution of diameters is hyperbolic, the self-corresponding elements are the asymptotes; and these separate harmonically every pair of conjugate diameters.

Two conjugate diameters and the line at infinity constitute a self-polar triangle, and of such a triangle just two sides cut the conic. Hence both of two conjugate diameters of an ellipse meet the curve, but only one of any two conjugate diameters cuts the hyperbola.

Exercise 33. Review

1. Find the locus of the harmonic conjugates of a given point with respect to a given pair of straight lines.

2. Find the point which is the harmonic conjugate of a given point with respect to each of two given pairs of straight lines.

3. Find the point which has an infinite number of harmonic conjugates with respect to each of two given pairs of straight lines, and find the locus of these conjugates.

4. Find three points each of which is conjugate to the other two with respect to a pair of opposite sides of a given complete quadrangle.

5. Each of the three points found in Ex. 4 is conjugate to the other two with respect to any conic which passes through the four vertices of the quadrangle.

6. Given any five points, no four of which are collinear, construct, with the ruler only, the polar of a given point with respect to that conic.

7. Solve the dual of Ex. 6.

8. Construct a self-polar triangle for a given conic, using the ruler only.

9. Construct the common self-polar triangle for all conics which pass through four given points.

10. If all parts of a figure which consists of a conic and a self-polar triangle are erased except the triangle and two points of the curve, reconstruct the figure.

11. Through three given points construct a conic which has a given point and a given line as pole and polar.

12. Solve the dual of Ex. 11.

13. Through two given points construct a conic which is tangent to a given line and has a given point and a given line as pole and polar.

14. Through four given points construct a conic which has a given pair of points as conjugates.

15. Through four given points and through some pair (not specified) of points of a given involution on a straight line construct a conic.

16. Given four points P_1, P_2, P_3, P_4 and two fixed lines p_1, p_2 passing through P_1 and P_2 respectively, find the envelope of the line joining the other intersections of these two lines with a variable conic through the four points.

17. Given the two points common to two conics and three other points of each, find all the intersections of the conics.

18. Given five points of a conic, find any diameter and the diameter conjugate thereto.

19. Given five points on a conic, construct the center, the axis, and the asymptotes of the conic.

The student should consult § 70 and § 74.

20. If a conic has more than one pair of conjugate diameters which are at right angles, the conic is a circle.

21. Given six points on a conic and the tangents at these points, the Pascal line of the inscribed hexagon is the polar of the Brianchon point of the circumscribed hexagon.

22. If the pole of the line P_1P_2 with respect to one conic which passes through the points P_1, P_2, P_3, P_4 coincides with the pole of P_3P_4 with respect to a second conic through these points, the pole of P_1P_2 with respect to the second conic coincides with the pole of P_3P_4 with respect to the first.

23. Construct a conic which has a given triangle as a self-polar triangle and a given point and a given line as pole and polar.

24. Construct a conic which has a given point as center and a given self-polar triangle.

25. Construct a conic which has a given pair of lines as conjugate diameters and a given point and a given line as pole and polar.

26. Construct a conic that has each side of a given pentagon as polar of the vertex opposite to it.

27. In the construction of Ex. 26 find the polar reciprocal of the conic determined by the vertices of the pentagon.

28. If a moving point traces a given conic, find the envelope of the polar of the point with respect to another given conic.

29. The lines joining the vertices of a triangle to the corresponding vertices of the triangle polar to it are concurrent.

CHAPTER XIII

QUADRIC CONES

183. Properties of Quadric Cones. A large number of the properties of quadric cones can be derived as duals of those of the conic. Thus, it is evident that a quadric cone is determined by its relation to certain sets of five planes and lines. Moreover, the theorems of Pascal, Brianchon, and Desargues, and their limiting cases, have duals which relate to the tangent planes and generating lines of the quadric cones.

In the problems of construction of the first order the possibility of drawing lines through pairs of points and of finding points as the intersections of lines was assumed. For the corresponding problems of this chapter there should be assumed the possibility of drawing lines common to pairs of planes and of constructing planes determined by pairs of lines.

In the problems of the second order the constructibility of at least one conic, ordinarily a circle, was assumed. At this point the corresponding assumption is that of the constructibility of at least one quadric cone. On the basis of these assumptions the problems analogous to those of Chapter X can be adequately treated.

Similarly, the theory of polar planes and lines of quadric cones follows from that of poles and polars of conics, and can be made to furnish an evidence of the truth of the Principle of Duality for the bundle. The development of the subject along these lines is left as an exercise.

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THEOREM. SECTIONS OF QUADRIC CONES

184. *A quadric cone the vertex of which is not infinitely distant may be so cut by a plane as to yield an ellipse, a parabola, or a hyperbola.*
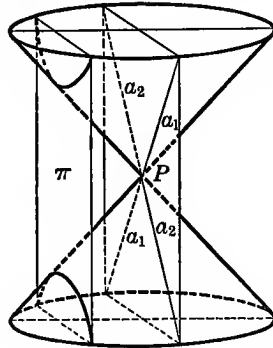
FIG. 1

Proof. Every quadric cone can be generated by means of the lines of intersection of corresponding planes of projective axial pencils that belong in the same bundle, and every plane section of a quadric cone is a conic.

Suppose now that a quadric cone (Fig. 1) is generated from the intersections a_1, a_2, a_3, $\cdots$ of pairs of corresponding planes α_1, α_1'; α_2, α_2'; α_3, α_3'; $\cdots$ of two axial pencils whose bases are the lines p and p'; and let the base of the bundle be P, a point not at infinity.

First, to secure a plane section which is a hyperbola, let π be a plane not through P but parallel to a_1 and a_2.

This plane cuts the cone in a conic, and since the plane cuts a_1 and a_2 at infinity, the conic has two distinct points at infinity. The conic is not a pair of straight lines, since its projector from P is not a pair of planes.

Hence the conic is a hyperbola.

Next, to obtain a section which is a parabola, let a_1' be the plane tangent to the cone (Fig. 2) along the generator a_1.

This plane contains the whole of the line a_1 but meets any other generator, as a_2, in only one point, namely, P.

Cut the cone by a plane a_1 parallel to a_1'. This plane cuts any generator other than a_1, as a_2, at a finite distance from P, and it cuts a_1 at infinity.

Hence the section of the cone has one and only one point at infinity, and the conic is a parabola.
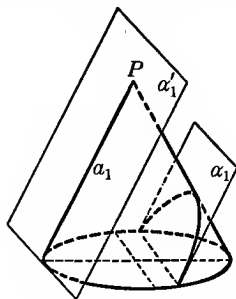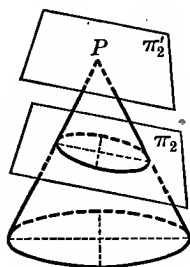
Fig. 2   Fig. 3

Finally, to obtain a section which is an ellipse, let π_2' be a plane (Fig. 3), through P but not coincident with any plane of either axial pencil, and let π_2 be a plane not through P but parallel to π_2' at a finite distance from it.

The intersection of any pair of corresponding planes, as a and a', since it meets π_2' at P, cannot be parallel to π_2.

Hence the intersection of π_2 and the cone has no point at infinity, and the conic is an ellipse.

If the vertex of the cone is at infinity, the generating lines are all parallel. It will be seen (§ 186) that the surface must contain no, one, or two infinitely distant generators. In these cases the sections made by planes not parallel to the generators will be all ellipses, all parabolas, or all hyperbolas respectively. Hence the theorem fails.

185. Axes of a Quadric Cone. Let o be a line through the vertex of a quadric cone, and let ω be the polar plane of o with respect to the cone. In ω there is one line, and there may be many lines, perpendicular to o. Through o and a line o' in ω perpendicular to o pass a plane π, and let l and l' be the lines in which the plane π cuts the cone. Then the lines o and o' are conjugates with respect to l and l', and, being perpendicular to each other, they bisect the angles formed by l and l'. Hence, through any line o there is one plane which cuts the cone in lines that form an angle of which a is the bisector.

If and only if the line o is perpendicular to its polar plane ω, all planes through o cut the cone in lines that make an angle of which o is the bisector. In this case the line o is an *axis of symmetry* with respect to the cone.

Manifestly every axis of symmetry is perpendicular to its polar plane. Also, a line o' parallel to o, an axis of symmetry, cuts a cone in two points, and the segment joining these points is bisected by the polar plane, since the axis of symmetry, the line joining the vertex to the intersection of the plane ω with o', and the lines l and l' are harmonic. Hence the polar plane of an axis of symmetry is a *plane of symmetry*.

It can be shown that every quadric cone has one axis o_1 of symmetry and a plane ω_1 of symmetry which is perpendicular to it. The plane ω_1 cuts the cone in a conic that has two principal axes which we may call o_2 and o_3. Of the lines o_1, o_2, o_3 each pair is conjugate to the third line and determines the polar plane of this line. Moreover, each of these polar planes is perpendicular to its polar line, and hence to the other two polar planes. There are, therefore, three axes of symmetry, perpendicular each to each, and three planes of symmetry, perpendicular each to each.

186. Cylinders. Hitherto it has been assumed that the axes of the generating axial pencils intersect in the finite part of space. If, however, the axes of the generating axial pencils are parallel, then all the generating lines are parallel to them, and the vertex of the surface is at infinity. In this case the surfaces generated are called *cylinders*.
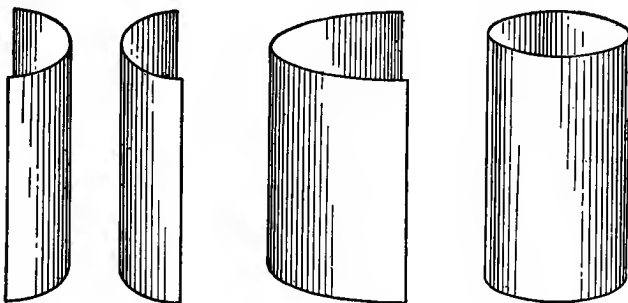
HYPERBOLIC CYLINDER   PARABOLIC CYLINDER   ELLIPTIC CYLINDER

Cylinders are classified with reference to their relation to the plane at infinity. The plane at infinity may cut the cylinder in two, one, or no straight lines. In these cases a section perpendicular to the generating lines of the cylinder is a hyperbola, a parabola, or an ellipse respectively; and the cylinder is said to be *hyperbolic*, *parabolic*, or *elliptic*, as the case may be.

In the case of a cylinder the plane at infinity and certain of its lines are in the bundle to which the cylinder belongs. The plane at infinity has a polar line which is an axis of symmetry and is called the *axis of the cylinder*. It can be shown that certain planes through this axis and all planes perpendicular to it are planes of symmetry. In the case of the parabolic cylinder the polar line of the plane at infinity is at infinity and lies in the cylinder.

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Exercise 34. Quadric Cones

1. Prove the theorem regarding quadric cones which corresponds to Steiner's theorem regarding conics.

2. Every conic surface of the second class is of the second order. Prove also the converse.

3. The hexahedral angle whose faces are determined by the six pairs of alternate edges of another hexahedral angle which is inscribed in a quadric cone has its faces tangent to a quadric cone.

4. Inscribe in a quadric cone a trihedral angle whose three edges shall be in three given planes.

5. If a variable simple four-flat so moves as always to be circumscribed about a given quadric cone, while three of its edges move each in one of three fixed coaxial planes, then the fourth edge moves on a fourth fixed plane coaxial with the three given cones.

6. State the properties of polar lines and planes of quadric cones corresponding to those of poles and polars of conics which are given in §§ 166 and 167.

7. Find the points of intersection of a given straight line with a quadric cone of which five determining elements are also given.

8. In a bundle $\alpha_1, \beta_1, \gamma_1$ and $\alpha_2, \beta_2, \gamma_2$ are two sets of fixed coaxial planes. Two planes π_1 and π_2 so move that the lines determined by π_1 and α_1, π_2 and α_2; π_1 and β_1, π_2 and β_2; π_1 and γ_1, π_2 and γ_2 lie in three coaxial planes. Find the surface generated by the line common to π_1 and π_2.

9. In a bundle the edges of a trihedral angle, whose planes are α, β, γ and which is self-polar with respect to a given quadric cone, determine with any line o the planes α', β', γ'. If the polar plane of o is ω, the pairs of lines determined by ω with α and α', by ω with β and β', and by ω with γ and γ' form a pencil in involution.

CHAPTER XIV

SKEW RULED SURFACES

187. Skew Ruled Surfaces. The third set of figures which were projectively generated was found to consist of the ruled surfaces of the second order that are not conic. Before discussing these we shall consider the classification of ruled surfaces in general.

One classification of ruled surfaces is based upon the law governing the motion of the generating line. At any instant the motion of this line may be a revolution about one of its own points or it may be a displacement by virtue of which the line immediately ceases to intersect its present position. In the former case it is sometimes said that every pair of consecutive generators intersect, and in the latter case it is said that no two consecutive generators intersect. Surfaces generated in the first way are called *developable surfaces*, and those generated in the second way are called *skew surfaces*. Cones and cylinders are examples of developable surfaces, but they are of a special type, inasmuch as each of their generators intersects every other one. Likewise, skew ruled surfaces of the second order and second class are special in character, for no generator intersects any other of the set, even though they be not consecutive. Generators usually cut other generators of the set if the latter are not consecutive.

In this chapter only a few specially interesting facts regarding the surfaces of the second order and second class will be established.

188. *Every skew ruled surface generated by the intersections of corresponding planes of two projective axial pencils has also a second set of generating lines whose relation to the surface is similar to that of the first set. Each member of either set of generators intersects no others of its own set, but intersects every one of the other set. Through every point of the surface there pass two generators, one of each set.*
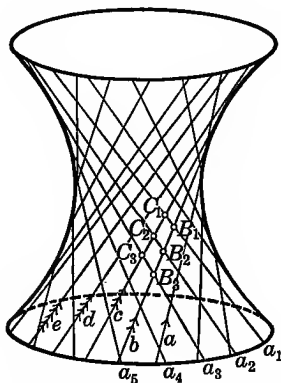
Proof. Every plane through any generating line a of the surface cuts the surface along the line a and also along a second line a_1, and nowhere else.

Moreover, the line a_1 cuts each of the generating lines that have been noted. The infinitely many planes through a cut the surface in infinitely many lines $a_1, a_2, a_3, \cdots$, each of which cuts every one of the generators; and every point of the surface lies on one and only one of the new lines.

No two of these lines intersect; for if they did, all the generators would lie in the plane determined by them.

Consider now the two sets of points $B_1, B_2, B_3, \cdots$ and $C_1, C_2, C_3, \cdots$ in which the lines $a_1, a_2, a_3, \cdots$ intersect two other generators b and c.

These sets of points are the intersections of the generators b and c with the planes of the axial pencil whose base is a and whose planes pass through the lines $a_1, a_2, a_3, \cdots$; and consequently they constitute perspective (but not coplanar) ranges.

Then the axial pencil whose base is b and whose planes pass through $C_1, C_2, C_3, \cdots$ and the axial pencil whose base is c and whose planes pass through $B_1, B_2, B_3, \cdots$, being perspective respectively with the ranges $C_1 C_2 C_3 \cdots$ and $B_1 B_2 B_3 \cdots$, are projective with each other.

Corresponding planes of these axial pencils intersect in the lines $a_1, a_2, a_3, \cdots$, which are therefore generating lines of a skew quadric ruled surface.

The latter skew ruled surface must coincide with the original one, since the lines $a_1, a_2, a_3, \cdots$ contain all the points of the original ruled surface, and no others. Hence these lines must constitute a second set of generators for that ruled surface.

189. COROLLARY. *The lines of either set of generators determine projective ranges on any two lines of the other set.*

190. Conjugate Reguli. Two reguli which are related as are the two in § 188 are called *conjugate reguli.*

The theorem of § 188 may be restated as a corollary to this definition as shown below.

191. COROLLARY. *Every skew ruled surface of the second order carries two conjugate reguli. Each line of either regulus intersects no lines of its own regulus, but intersects each line of the other regulus. Through every point of the surface there pass two lines, one from each regulus.*

192. *Given three straight lines no two of which are coplanar, there exists one and only one skew quadric ruled surface of which each of these lines is a generator.*
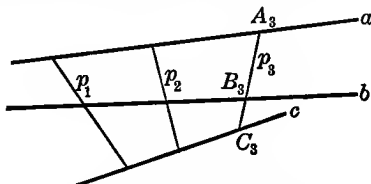
Proof. Let a, b, c be three straight lines no two of which are coplanar.

Through each point of a one line and only one can be drawn to meet all three of the lines.

Let p_1, p_2, and p_3 be three lines which meet a, b, c.

Then p_1 together with the three lines a, b, c, and p_2 together with the same lines, determine triads of planes of axial pencils whose bases are p_1 and p_2. These triads determine a projectivity between the pencils, and this projectivity determines one skew quadric ruled surface of which a, b, c are generators.

Any two corresponding planes of the projective axial pencils intersect in a line that meets p_1 and p_2. This line also meets p_3; for if p_3 meets a, b, c in A_3, B_3, C_3 respectively, the two axial pencils above mentioned are each perspective with the same range on p_3, the perspectivities being determined by the correspondence of A_3, B_3, C_3 to the planes p_1a, p_1b, p_1c in the one case, and the planes p_2a, p_2b, p_2c in the other. It follows that corresponding planes cut p_3 in the same point, and hence their line of intersection cuts p_3.

Accordingly, the surface is uniquely determined by the three generators a, b, c.
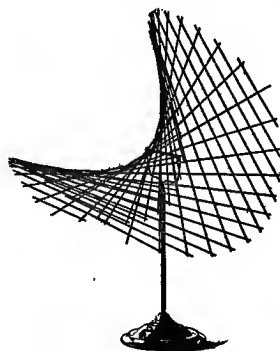
Hyperbolic Paraboloid, the conjugate reguli being formed by rods. Certain sections are parabolas, other sections are hyperbolas. The curvature of the surface is not secured by the pressure of one set of rods upon the other
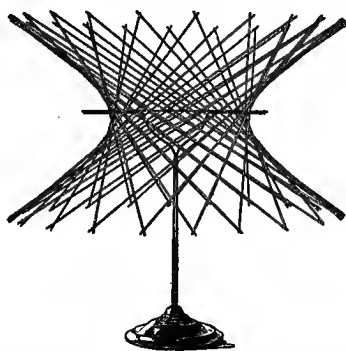
Hyperboloid of One Sheet, the conjugate reguli being formed by straight rods. The surface in the neighborhood of its center of symmetry is shown. Horizontal sections are ellipses, vertical sections are hyperbolas

193. Skew Quadric Ruled Surfaces Classified. These surfaces are classified according to the nature of their sections by the plane at infinity. As has been shown, any plane section of one of these surfaces is a conic and degenerates into two straight lines if the cutting plane contains a generator. The surfaces are, therefore, of two sorts:

1. *Hyperbolic paraboloids*, or those whose intersections with the plane at infinity are pairs of generators.

2. *Hyperboloids of one sheet*, or those whose intersections with the plane at infinity are nondegenerate conics.

It may be noted that if a hyperbolic paraboloid is regarded as generated by the joining lines of corresponding points of two projective ranges, the points at infinity of the ranges are found to be corresponding points. Hence in this case (and in this case only) the ranges are similar. Accordingly, if corresponding points of two similar (but not coplanar) ranges are connected by threads, a good model of a hyperbolic paraboloid may be constructed.

To exhibit both sets of generating lines it is better to use a quadrilateral $ABCD$ hinged at two opposite vertices, as B and D, so that the triangles ABD, CBD can be adjusted to lie in different planes. Congruent ranges can be taken on AB and CD and also on BC and DA. Corresponding points can be joined by strings, and in this manner an excellent model can be constructed with very little trouble.
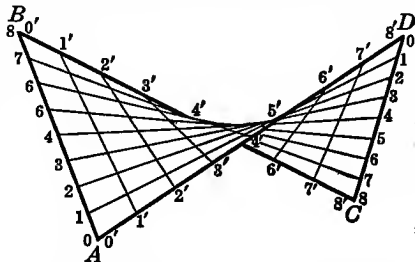
Directions for constructing a string model of the hyperboloid of one sheet are not so easily given. The existence of such a surface is evident, since it is generated by the lines joining corresponding points of projective ranges which are not coplanar and not similar.
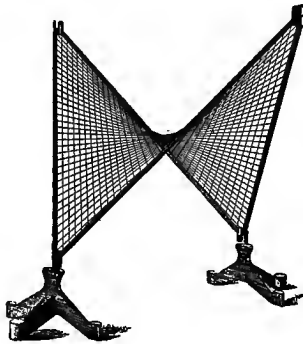
Hyperbolic Paraboloid, the conjugate reguli being formed by straight rods. The surface near the vertex, or saddle point, is shown. Certain sections through the vertex are parabolas, others are hyperbolas

Hyperbolic Paraboloid, the conjugate reguli being formed by strings. If every string in either set is cut, the strings in the other set retain their positions

Exercise 35. Skew Ruled Surfaces

1. A regulus is determined by two nonintersecting lines and three noncollinear points, no two of which are coplanar with any of the lines.

2. If a regulus contains a line at infinity, the conjugate regulus also contains a line at infinity.

3. Determine three pairs of quadric ruled surfaces which have in common two given noncoplanar lines and also respectively no, one, and two generators of the other set.

4. Determine a quadric ruled surface which contains two given noncoplanar lines and a given point exterior to them. How many such surfaces are there ? Find any additional lines which are common to such surfaces.

5. If four generators of a regulus cut one generator of the conjugate regulus in a harmonic range, they cut every generator of the conjugate regulus in a harmonic range.

Four generators of a regulus which have the property mentioned in Ex. 5 are called *harmonic generators.*

6. Given any three lines in space, no two of which are coplanar, find a fourth line which, with the three given lines, constitutes a set of harmonic generators of a regulus.

7. If a line so moves as constantly to intersect each of two noncoplanar lines and also to remain parallel to a given plane, the line generates a hyperbolic paraboloid.

8. If a range and a flat pencil which do not lie in the same plane or in parallel planes are projective, and if from each point of the range a line is drawn parallel to the corresponding line of the flat pencil, these parallel lines all lie on a hyperbolic paraboloid.

9. The locus of the harmonic conjugates of any point with respect to a ruled surface is a plane.

10. The lines (or planes) of any bundle which are tangent to a quadric ruled surface generate a quadric cone.

HISTORY OF PROJECTIVE GEOMETRY

The history of geometry may be divided roughly into four periods: (1) The synthetic geometry of the Greeks, including not merely the geometry of Euclid but the work on conics by Apollonius and the less formal contributions of many other writers; (2) the birth of analytic geometry, in which the synthetic geometry of Desargues, Kepler, Roberval, and other writers of the seventeenth century merged into the coordinate geometry of Descartes and Fermat; (3) the application of the calculus to geometry, — a period extending from about 1650 to 1800, and including the names of Cavalieri, Newton, Leibniz, the Bernoullis, L'Hôpital, Clairaut, Euler, Lagrange, and D'Alembert, each one, especially after Cavalieri, being primarily an analyst rather than a geometer; (4) the renaissance of pure geometry, beginning with the nineteenth century and characterized by the descriptive geometry of Monge, the projective geometry of Poncelet, the modern synthetic geometry of Steiner and Von Staudt, the modern analytic geometry of Plücker, the non-Euclidean hypotheses of Lobachevsky, Bolyai, and Riemann, and the laying of the logical foundations of geometry, — a period of remarkable richness in the development of all phases of the science.

It is in this fourth period that projective geometry has had its development, even if its origin is more remote. The origin of any branch of science can always be traced far back in human history, and this fact is patent in the case of this phase of geometry. The idea of the projection

of a line upon a plane is very old. It is involved in the treatment of the intersection of certain surfaces, due to Archytas, in the fifth century B.C., and appears in various later works by Greek writers. Similarly, the invariant property of the anharmonic ratio was essentially recognized both by Menelaus in the first century A.D. and by Pappus in the third century. The notion of infinity was also familiar to several Greek geometers, so that various concepts that enter into the study of projective geometry were common property long before the science was really founded.

One of the first important steps to be taken in modern times, in the development of this form of geometry, was due to Desargues, a French architect. In a work on conic sections, published in 1639, Desargues set forth the foundation of the theory of four harmonic points, not as done today, but based on the fact that the product of the distances of two conjugate points from the center is constant. He also treated of the theory of poles and polars, although not using these terms. In 1640 Pascal, then only a youth of sixteen, published a brief essay on conics setting forth the well-known theorem that bears his name.

The descriptive geometry of Monge is a kind of projective geometry, although it is not what we ordinarily mean by this term. He was a French geometer of the period of the Revolution, and had been in possession of his theory for over thirty years before the publication of his "Géométrie Descriptive" (1795). It is true that certain of the features of this work can be traced back to Desargues, Taylor, Lambert, and Frézier, but it was Monge who worked out the theory as a science. Inspired by the general activity of the period, but following rather in the steps of Desargues and Pascal, Carnot treated chiefly of the metric relations of figures. In particular he investigated

these relations as connected with the theory of transversals, — a theory whose fundamental property of a four-rayed pencil goes back to Pappus, and which, though revived by Desargues, was set forth for the first time in its general form by Carnot in his "Géométrie de Position" (1803), and supplemented in his "Théorie des Transversales" (1806). In these works Carnot introduced negative magnitudes, the general quadrilateral, the general quadrangle, and numerous other similar features of value to the elementary geometry of today.

Projective geometry had its origin somewhat later than the period of Monge and Carnot. Newton had discovered that all curves of the third order can be derived by central projection from five fundamental types. But in spite of this the theory attracted so little attention for over a century that its origin is generally ascribed to Poncelet. A prisoner in the Russian campaign, confined at Saratoff on the Volga (1812–1814), "privé," as he says, "de toute espèce de livres et de secours, surtout distrait par les malheurs de ma patrie et les miens propres," Poncelet still had the vigor of spirit and the leisure to conceive the great work, "Traité des Propriétés Projectives des Figures," which he published in 1822. In this work was first made prominent the power of central projection in demonstration and the power of the principle of continuity in research. His leading idea was the study of projective properties, and as a foundation principle he introduced the anharmonic ratio, — a concept, however, which dates back to Menelaus and Pappus, and which Desargues had also used. Möbius, following Poncelet, made much use of the anharmonic ratio in his "Barycentrische Calcul" (1827), but he gave it the name *Doppelschnitt-Verhältniss (ratio bisectionalis)*, a term now in common use under Steiner's abbreviated

form *Doppelverhältniss*. The name *anharmonic ratio* or *anharmonic function* (*rapport anharmonique*, or *fonction anharmonique*) is due to Chasles, and *cross-ratio* was suggested by Clifford. The anharmonic point-and-line properties of conics have been elaborated by Brianchon, Chasles, Steiner, Dupin, Hachette, Gergonne, and Von Staudt. To Poncelet is due the theory of *figures homologiques*, the perspective axis and perspective center (called by Chasles the axis and center of homology), an extension of Carnot's theory of transversals, and the *cordes idéales* of conics, which Plücker applied to curves of all orders. Poncelet also discovered what Salmon has called "the circular points at infinity," thus completing and establishing the first great principle of modern geometry, — the principle of continuity. Brianchon (1806), through his application of Desargues's theory of polars, completed the foundation which Monge had begun for Poncelet's theory of reciprocal polars (1829).

Steiner (1832) gave the first complete discussion of the projective relations between rows, pencils, etc., and laid the foundation for the subsequent development of pure geometry. He practically closed the theory of conic sections, of the corresponding figures in three-dimensional space, and of surfaces of the second order. With him opens the period of special study of curves and surfaces of higher order. His treatment of duality and his application of the theory of projective pencils to the generation of conics are masterpieces.

Cremona began his publications in 1862. His elementary work on projective geometry (1875) is familiar to English readers in Leudesdorf's translation. The recent contributions have naturally been of an advanced character, seeking to lay more strictly logical foundations for the science, and in this line the American work by Professors Veblen and Young is noteworthy.

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