THE ELECTROMAGNETIC VECTORS.

By H. Bateman.

§ 1. An electromagnetic field in the ether is usually specified by the values at each point and at each instant of two vectors E and H, but a more general specification is obtained by using the two vectors

$$\begin{aligned} F &= E + \frac{1}{c}(v \times H), \\ Q &= H - \frac{1}{c}(v \times E), \end{aligned} \qquad (1)$$

where v is an arbitrary velocity representing at each point and at each instant the velocity of an imaginary recording instrument and c is the velocity of light.

These vectors are of fundamental importance in electromagnetic theory for F is usually supposed to represent the force which the field would exert on a minute unit electric charge moving with velocity v and Q may be supposed to represent the force which the field would exert on a minute unit magnetic charge if such a thing could exist and move with velocity v.

On account of the importance of these vectors F and Q it will be worth while to get a clear conception of the way in which they vary when the field remains constant and v varies.

Let lines OE, OH, be drawn to represent the instantaneous values of E and H at any point in magnitude and direction and let circles of radii H and E and center O be drawn in planes at right angles to OH and OE respectively. If v is less than c the vector F is represented in magnitude and direction by the line FE where F is some point within the first circle while Q is represented by QH where Q is some point within the second circle. When v is greater than c the same construction may be used but the points F and Q may now lie outside their respective circles.

It is clear from this construction that F is a minimum when F is parallel to H and that Q is a minimum when Q is parallel to E.

When E and H are perpendicular and E is greater than H the point E lies in the plane of the first circle and outside the circle, consequently when v is less than c the direction of F lies within a certain angle bounded by two straight lines in the directions of s_1 and $-s_2$ where s_1 and s_2 are the two real unit vectors s which satisfy the relations[1]

$$E + (s \times H) = ks, \qquad E \cdot s = k. \tag{2}$$

The vector Q can under the same conditions take any direction in a plane perpendicular to E.

In the case of the field of a moving electric pole one of the two vectors s is in the direction of the radius from the effective position of the pole, that is the point from which a disturbance travelling with velocity c must start in order to reach the point of observation O at time t.

When the electric pole moves with uniform velocity along a straight line the second vector s is in the direction of the radius to that position of the pole which can be reached by a disturbance starting from O at time t.

In the general case when E and H are not necessarily perpendicular, the directions of F and Q for $v < c$ are confined to certain quadric cones. If E is greater than H the angle of Q's cone is greater than that of F's. These two cones have the same focal lines which are in the direction of the two real unit vectors[2] s which satisfy the equations (2).

Relative to one of these vectors s the field vectors E and H may be resolved into longitudinal components $E \cdot s$ and $H \cdot s$ parallel to s and transverse components represented by the vectors $E - (E \cdot s)s$ and $H - (H \cdot s)s$ perpendicular to s. These latter components are perpendicular to one another and equal in magnitude as may be seen from the relations

$$(E \cdot H) = (E \cdot s)(H \cdot s) \tag{3}$$

$$E^2 - (E \cdot s)^2 = H^2 - (H \cdot s)^2 \tag{4}$$

which are easily derived from (2).

We may also divide up the energy in the field as follows:

Longitudinal electric energy $\frac{1}{2}(E \cdot s)^2$,
Longitudinal magnetic energy $\frac{1}{2}(H \cdot s)^2$,
Transverse electric energy $\frac{1}{2}E^2 - \frac{1}{2}(E \cdot s)^2$,
Transverse magnetic energy $\frac{1}{2}H^2 - \frac{1}{2}(H \cdot s)^2$.

It is clear from equation (4) that the transverse electric energy is equal to the transverse magnetic energy and that the Lagrangian function $\frac{1}{2}(E^2 - H^2)$ may be supposed to arise entirely from the longitudinal part

[1] They also satisfy the relations embodied in the vector equation

$$H - (s \times E) = s(s \cdot H).$$

[2] These vectors have been introduced for different purposes on previous occasions. Proc. London Math. Soc., Ser. 2, Vol. 8, 1910, p. 469; Vol. 10, 1911, pp. 7, 96; Mess. of Math., Vol. 14, 1915, p. 112.

of the field. It should be noticed that the longitudinal part of the field is *conjugate*[1] to the transverse part; in other words the invariants (for transformations of the theory of relativity) $E^2 - H^2$ and $(E \cdot H)$ for the total field are the sums of the corresponding invariants for the longitudinal and transverse parts. It should also be noticed that the energy in the total field is the sum of the energies of the longitudinal and transverse parts, consequently all the principal characteristics of the two component fields are additive.

The transverse field evidently has the characteristics of a pure radiant field or self conjugate field and in the case of an electric pole which at one instant has no velocity but a finite acceleration the transverse field relative to the vector s, which is in the direction of the radius vector from the pole, is the so-called wave of acceleration while the longitudinal field is the electrostatic field. When the electric pole has a finite velocity this description fails for the longitudinal field is the electric field represented by the radial component of the electric vector, while part of the transverse field depends on the velocity.

It is interesting to notice that the resolution of the energy into longitudinal and transverse energies is the same as far as the magnitudes are concerned whichever vector of type s is used. It should also be noticed that the total longitudinal energy

$$\tfrac{1}{2}(E \cdot s)^2 + \tfrac{1}{2}(H \cdot s)^2 = \tfrac{1}{2}[(E^2 - H^2)^2 + 4(E \cdot H)^2]^{1/2} \tag{5}$$

is an invariant and is proportional in fact to Cunningham's principal stress. The longitudinal field disappears entirely when

$$E^2 - H^2 = 0 \qquad \text{and} \qquad (E \cdot H) = 0, \tag{6}$$

that is, in the case of a pure radiant field or self-conjugate field. The two vectors of type s then coincide in direction with Poynting's vector.

It should be noticed that $v = cs$ is a special velocity for which both F and Q are parallel to v. The different possible directions of a velocity v such that F is parallel to v are of some interest and may be found as follows:

When F is parallel to v the vector $v \times E$ is perpendicular to F and is therefore perpendicular to $v \times H$. This means that the direction of v lies on a quadric cone which is the locus of a line OL for which the planes EOL and HOL are perpendicular. This cone has the lines OE, OH in one of its principal planes and has its circular sections perpendicular to them. It is clear that this cone is also the locus of directions for which Q can be parallel to v. If v and w have the same direction and v is a

[1] For the definition of conjugate fields see Bull. Amer. Math. Soc., Vol. 21, March, 1915, p. 299.

velocity for which F is parallel to v while w is a velocity for which Q is parallel to w, v and w are connected by the relation $vw = c^2$, consequently they cannot both be less than c and they cannot be equal unless they are both equal to c.

As OL moves round the cone the velocity v takes all values between O and ∞ twice over and so for any velocity there are generally two directions for which either F or Q is in the direction of v. Two generators of this cone are, of course, in the direction of the two real unit vectors s which satisfy equations (2). The directions of v for which F is perpendicular to v evidently lie in a plane perpendicular to E, similarly the directions of v for which Q is perpendicular to v lie in a plane perpendicular to H.

The directions of v for which F and Q are in the same direction are of some interest as these velocities v are possible velocities of the ether in the theory of E. Cunningham.[1] The directions are confined to two planes, one of which contains the two real unit vectors of type s and Poynting's vector while the other contains the two imaginary vectors of type s and Poynting's vector. These planes are in fact the two real planes through the real and imaginary focal lines of the two cones[2] already mentioned which limit the directions of F and Q respectively for $v < c$. As we have remarked elsewhere,[3] the extremity of a line OV representing the vector v lies on one of two straight lines which are polar lines with regard to a sphere center O and radius c, these lines intersect at right angles the line through O in the direction of Poynting's vector.

The lines FE representing the corresponding vectors F are such that F lies on one of two straight lines parallel to Poynting's vector. These lines may be obtained by drawing through E two planes parallel to the two planes previously mentioned and finding where these planes meet the plane through O perpendicular to H. There is a similar construction for the corresponding vectors Q.

When Q is perpendicular to F we have an equation

$$(c^2 - v^2)(E\cdot H) = -(v\cdot E)(v\cdot H),$$

for v which indicates that the extremity of a line OV representing this velocity lies on a quadric surface having the cone swept out by OL as asymptotic cone. The region for which v is less than c is bounded by the two planes through O perpendicular to E and H and consequently by the two diametral circular sections of the quadric. If we are given the direction of F the corresponding value of v for which Q is perpendicular

[1] For this remark see S. B. McLaren, Phil. Mag., 26, 1913, p. 636.

[2] These cones are supposed to have their vertices at O.

[3] Phil. Mag., 34, 1917, p. 405; Mass. of Math., 14, 1915, p. 110.

to F is uniquely determined except in the case when OV is in the plane containing OH and the line through O in the direction of Poynting's vector. The correspondence between the points F and Q when F is perpendicular to Q is a one to one quadratic transformation. The point at infinity in the direction of Poynting's vector and the points in which the plane OEH meets the circle are the singular points in each plane. This completes the description of the geometrical properties of an electromagnetic field which is supposed to be explored by a crowd of imaginary observers moving in an arbitrary manner.

We have been acting on the assumption that F and Q are the quantities that are observed directly and that E and H are secondary quantities which may be derived from them. The quantities E and H are usually regarded as the primary quantities on account of the simplicity of the field equations which they satisfy but it should be noticed that the electromagnetic laws can also be expressed directly in terms of F and Q. This is a known result but it may be worth while to recall it at this point.

§ 2. In the year 1908 two very important papers on electromagnetic theory were published. One of these was Minkowski's paper on the electrodynamical equations for moving bodies,[1] a paper which soon influenced mathematical thought very considerably and received world wide attention. The other paper was by Mr. Richard Hargreaves, of Southport, England, and was entitled "Integral forms and their connection with physical equations." This paper which is perhaps the more important of the two, contains two new presentations of the principles of electromagnetism in terms of space-time integrals. This at once places the time coördinate on the same level as the other coördinates and suggests the idea of space-time vectors just as in Minkowski's work. The chief importance of Mr. Hargreaves' work lies, however, in the fact that it throws light at once upon the nature of the solutions of the electromagnetic equations and that the principles are presented in a form which is independent of the choice of the space and time coördinates. The last circumstance enables one to obtain the transformations of the theory of relativity in a simple and natural manner and makes it easy to obtain the invariants by a simple application of the methods of the absolute calculus of Ricci and Levi Civita.[3] The first two theorems which are usually written in the form

[1] H. Minkowski, Gött. Nachr., 1908.

[2] R. Hargreaves, Cambr. Phil. Trans., Vol. 21, 1908, p. 107. Some interesting developments and applications of Hargreaves' theorems have been made in an enthusiastic way by M. de Donder in Belgium, Bull. de l'Acad. roy. de Belgique (Classe des Sciences), 1909, p. 66; 1911, p. 3; 1912, p. 3.

[3] H. Bateman, Proc. London Math. Soc., Vol. 8, 1910, p. 223.

$$\iint [E_x d(y, z) + E_y d(z, x) + E_z d(x, y) - cH_x d(x, t) \\ - cH_y d(y, t) - cH_z d(z, t)] \\ = \iiint [\rho d(x, y, z) - \rho w_x d(y, z, t) - \rho w_y d(z, x, t) - \rho w_z d(x, y, t)]$$

$$\iint [H_x d(y, z) + H_y d(z, x) + H_z d(x, y) + cE_x d(x, t) \\ + cE_y d(y, t) + cE_z d(z, t)] = 0$$

indicate the invariance of the electromagnetic equations under conditions in which the integrands in the integrals are invariants. Now the first integral certainly vanishes when the moving surface of integration is made up of moving lines of electric force[1] and this indicates the invariance of moving lines of electric force under the transformations of the theory of relativity. This theorem is true for all transformations of coördinates when the ideas of general relativity are adopted and the vectors in the second of the two equations are not assumed to be the same as in the first.

A moving line of electric force thus assumes the form of a definite physical entity when we adopt the view that invariants are symbols for physical entities which are independent of the measuring apparatus or method of observation.

We have already shown that Mr. Hargreaves' theorems can be presented in a form in which ordinary surface and volume integrals are used[2] and as this gives the desired expression of the electromagnetic laws in terms of the vectors F and Q the results will be quoted here in vector notation.

Let us assume that throughout a certain region of space a time is associated with each point in space by means of a relation of type $t = f(x, y, z)$, where f is a uniform function. This time may be supposed to be the time at which an observation is made at the point in question. We shall suppose, moreover, that the velocity of the observer or observing instrument is determined by the equation

$$v = c^2 \Delta f, \tag{7}$$

then with the notation of § 1, we have the two theorems

$$\int F_n d\sigma = \int \rho \left[1 - \frac{1}{c^2}(v \cdot w) \right] d\tau, \tag{8}$$

$$\int Q_n d\sigma = 0, \tag{9}$$

where $d\sigma$ denotes an element of surface, $d\tau$ an element of volume, the suffix n denotes that the normal component of the vector is taken, ρ

[1] Phil. Mag., Vol. 34, 1914, p. 405.

[2] See last reference but one.

denotes the volume density of electricity and ρw the convection current. In these theorems the surface integrals are supposed to be taken over a closed surface and the volume integral over the volume enclosed by this surface. In the ether of course the volume integral disappears and we have

$$\int F_n d\sigma = \int Q_n d\sigma = 0.$$

The equation (8) may be regarded as a generalization of Gauss' theorem and may be assumed to hold whether the behavior of the quantities involved permits an application of the ordinary form of Green's theorem or not.[1] Calling $F_n d\sigma$ the flux across the element of surface $d\sigma$ the theorem may be interpreted to mean that the flux across a closed surface is equal to the charge inside. This, of course, is in a generalized sense, for it must be remembered that quantities are measured at different points of space at different times. Since the flux across a surface made up of moving lines of electric force is zero we may conclude that the flux across a cross section of a tube made up of moving lines of electric force is constant along the tube provided there is no electricity within the tube between the two sections under consideration at the times specified by the law of observation $t = f(x, y, z)$. We must generally assume that observations are made throughout a region of space by ∞^3 observers. This is necessary for instance if the observations are to be simultaneous but in some cases it is convenient to assume that sets of observations are made by ∞^2 observers travelling along specified paths. In particular, if the observers are supposed to travel along straight lines with the velocity of light the velocity v may represent the velocity of the observer and the time $t = f(x, y, z)$ the time at which he reaches the point (x, y, z).

Let us take the case in which v is at each point in the direction of one of the unit vectors s in an electromagnetic field and that sets of observations are made by imaginary observers who travel along straight lines with the velocity of light. The vectors F and Q then represent the longitudinal part of the field and the electromagnetic laws give us the properties of this part of the field alone. Thus if the field is that of a moving electric pole and the observers are supposed to start from a particular position of this pole and travel away from it with the velocity of light our theorems enable us to study the properties of the field which is left when the transverse wave is subtracted from the total field. In other words our observers do not record the effects of the transverse wave because they travel with it at the speed of light.

[1] In this way we may avoid the difficulty that occurs on p. 21 of my book "Electrical and Optical Wave Motion." The above theorems are more general than the electromagnetic equations.

Mr. Hargreaves' second set of theorems indicate the way in which space time integrals change when a small variation is made of the region of integration; they give for instance the rate of change of the flux across a closed surface when the surface is in motion.[1] Let us suppose that the change of times and region of integration are made by moving the observer at each point (x, y, z) to some consecutive point $(x + \delta x, y + \delta y, z + \delta z)$ at a velocity u which is a function of x, y, z, t and let the new time of observation be the time at which each observer arrives at his new position.

We shall quote Hargreaves' theorem for the case of a volume integral which in our notation is written in the form

$$\int \rho \left[1 - \frac{1}{c^2}(v \cdot w) \right] d\tau.$$

ρ and w are supposed here to satisfy the equation of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div} \rho w = 0,$$

so that the integral may be regarded as representing the total amount of electricity within a closed moving surface when the observations are made at times specified by the law $t = f(x, y, z)$, and w represents the velocity of the electricity, ρ its density. The rate of change of this volume integral is now represented by an integral of type

$$\int F_n' d\sigma$$

over the boundary of the closed surface where

$$F' = E' + \frac{1}{c}(v \times H')$$

and E' and H' are defined by the equations

$$E' = \rho(u - w), \qquad H' = \frac{1}{c}\rho[w \times u].$$

This result is of interest because it indicates that if we start with the idea of a fluid electricity moving according to the law of continuity we can, by a process analogous to differentiation, derive from it a field of two vectors E' and H' which are expressible directly in terms of the quantities ρ and ρw which specify the flow of electricity in the original field. The quantities ρ' and $\rho' w'$ which are connected with E' and H' by the equations

$$\operatorname{div} E' = \rho', \qquad c \operatorname{rot} H' = \frac{\partial E'}{\partial t} + \rho' w',$$

[1] This provides us with the generalized forms of two theorems given by H. A. Lorentz, Encyklopädie der Math. Wiss., Bd. V., § 13, 1903, p. 119 and J. Larmor, Proc. Int. Congr. of Math., Cambridge, 1912, Vol. 1.

are different from ρ and ρw because the total amount of electricity within a closed surface in the field E', H' represents the rate of change of the amount of electricity within this closed surface in the original field. Strictly we ought not to use the same word electricity in the two cases because the quantities ρ and ρ' are not of the same dimensions. To avoid confusion we may call ρ' density of electricity and ρ density of *proto-electricity*. It must be clearly understood that quantity of electricity represents the rate at which a quantity of proto-electricity appears to change in magnitude when successive observations are made by a crowd of imaginary observers moving in such a way that the velocity of the observer at (x, y, z, t) is $u(x, y, z, t)$. This vector u may of course be the same as the vector v but it is not necessary to assume that the velocity of an observer is the same as that of his instrument of observation.

It should be noticed that in our derived field E', H' we have

$$E'\cdot H' = 0 \qquad \text{and} \qquad E' > H'$$

where the inequality holds provided at least one of the veloctiies u, w is less than c. To prove this inequality let us draw lines OU, OW to represent the velocities u and w in magnitude and direction. The quantity E'^2 is then represented by the square of the distance UW while H'^2 is represented by the square of the distance UW multiplied by the square of the perpendicular from O on UW and divided by the square of c. Clearly then E' is greater than H' if the perpendicular is less than c and this is certainly true if one of the quantities u, w is less than c. If both u and w are greater than c it does not necessarily follow that E' is not greater than H' for the line UW may still cut a sphere of radius c in real points and so be at a distance from O less than c.

The idea that quantity of electricity represents a rate of change of a quantity of proto-electricity may perhaps account for the existence of two types of electricity, positive and negative, because even if a quantity of proto-electricity is always represented by a positive number, its rate of change may be either positive or negative.

This suggests that the plan of deriving electromagnetic fields from proto fields of moving proto-electricity by a process analogous to differentiation may be very useful. It should be remarked at the outset that the type of differentiation considered above is only a particular case of a more general type of differentiation in which two proto fields which differ slightly from one another are subtracted and a limiting process carried out after all the small quantities obtained are divided by the same small quantity.

Of course in a proto field we can consider two vector functions E and H which are connected with ρ and ρw by the electromagnetic laws and it seems likely that fields which satisfy the electromagnetic equations and do not possess the characteristics of the types of electromagnetic fields with which we are familiar may be really proto fields. The fields whose singularities consist of moving curves are a case in point for when two such fields are superposed so that the singular curves overlap it is possible to obtain a cancelling of singularities with the result that in the total field there are only point singularities. In other words line charges in the proto field may give rise to isolated electric charges in the derived field. In the original form of Sir Joseph Thomson's theory of moving Faraday tubes the analysis indicates that a Faraday tube always consists of the same particles of electricity.[1] This result may seem strange to most scientists but the explanation is that a field of Thomson's type is probably a proto field and that really a Faraday tube always consists of the same particles of *proto-electricity* while when an electromagnetic field is derived from this proto field by a process analogous to differentiation the electric charges appear only at the ends of the tube. The foregoing remarks indicate that the interpretation of the electric and magnetic vectors in terms of proto-electricity may be simpler than any interpretation in terms of the electricity of their own field and this we shall now endeavor to show. It may be remarked that the dynamical laws of motion for proto-electricity are unknown, it is possible that they are of the first order instead of the second and that the accelerations in the Newtonian laws arise in the process of differentiation by which an electromagnetic field is derived from a proto field.

§ 3. To begin with it will be convenient to assume that a proto-electric charge can be either positive or negative. The negative charges can be eliminated eventually, by adding positive constants to all the charges, but this may not be necessary.

Let us suppose that a positive charge e and a negative charge $-e$ separate at a point O and travel along straight lines with the velocity of light and let the points A and B respectively represent their positions at some time t. If T denotes the time which has elapsed from the moment of separation up to this instant, we have

$$OA = OB = cT.$$

Now consider two imaginary observers, one of whom travels with the point A while the other travels with the point B. Let the unit vectors s_1 and s_2 specify the directions of motion of A and B. If we imagine

[1] See H. Bateman, Phil. Mag., Vol. 34, 1917, p. 405.

a field of two vectors E and H to exist at the two moving points A and B and nowhere else, A may be supposed to record the vectors

$$F_1 = E + (s_1 \times H),$$
$$Q_1 = H - (s_1 \times E),$$

while B records the vectors

$$F_2 = E + (s_2 \times H),$$
$$Q_2 = H - (s_2 \times E).$$

Now a natural vector for A to record is $ecTs_1$ and a natural vector for B to record is $-ecTs_2$; we shall therefore consider the consequences of assuming that

$$\begin{aligned} E + (s_1 \times H) &= -ecTs_1, & H - (s_1 \times E) &= 0, \\ E + (s_2 \times H) &= ecTs_2, & H - (s_2 \times E) &= 0. \end{aligned} \tag{10}$$

These equations are quite consistent with one another and they indicate that s_1 and s_2 are the two vectors of type s defined by equations (2) for the two points where the field exists. Solving these equations we get

$$E = ecT\frac{s_2 - s_1}{1 - s_1 \cdot s_2}, \qquad H = ecT\frac{s_1 \times s_2}{1 - s_1 \cdot s_2}. \tag{11}$$

The electric vector is thus in the direction of the line AB but is not simply proportional to the length of this line for there is an additional factor depending on the angle between the vectors s_1 and s_2. The magnetic vector H is perpendicular to both these vectors and to E.

It should be noticed that

$$E^2 - H^2 = e^2c^2T^2, \qquad E \cdot s_1 = -ecT, \qquad E \cdot s_2 = ecT,$$

$$E^2 = e^2c^2T^2\frac{2}{1 - s_1 \cdot s_2}, \qquad H^2 = e^2c^2T^2\frac{1 + s_1 \cdot s_2}{1 - s_1 \cdot s_2},$$

$$E \times H = e^2c^2T^2\frac{s_1 + s_2}{1 - s_1 \cdot s_2}.$$

If the flow of energy is supposed to be indicated by Poynting's vector a puzzling result is obtained for since the field exists only at the two moving points A and B it is difficult to understand how energy can get away from these points, which it would do if the flow were in the direction of the vector $s_1 + s_2$. The probable explanation of this paradox is that the flow is modified owing to the presence of the charges and the work which is done on them by the field. We are interested in the behavior of the total energy and so the concealed energy which is equivalent to the work done must be taken into account and its influence on the total flow must also be considered.

We shall consequently take the total energy at A to be a multiple of[1]

$$\tfrac{1}{2}(E^2 + H^2) + \tfrac{1}{2}(E\cdot s_1)^2 = E^2$$

and the flow of energy to be represented by the same multiple of

$$c(E \times H) + cE(E\cdot s_1) = cE^2 s_1.$$

Similarly the total energy at B will be assumed to be a multiple of

$$\tfrac{1}{2}(E^2 + H^2) + \tfrac{1}{2}(E\cdot s_2)^2 = E^2$$

and the flow of energy the same multiple of

$$c(E \times H) + cE(E\cdot s_2) = cE^2 s_2.$$

This makes the flow of energy take place in the directions of motion of A and B which seems right. The lack of symmetry with respect to E and H arises on account of the fact that $(H\cdot s_1)$ and $(H\cdot s_2)$ are both zero.

Let us now use our expressions for E and H to build up an electromagnetic field in the ether. Instead of supposing that the separation of charges takes place at only one point at one instant we shall suppose that a separation is continually taking place in the neighborhood of a moving point S whose coördinates at time τ are $\xi(\tau)$, $\eta(\tau)$, $\zeta(\tau)$. We shall suppose that in the process of separation which takes place during the interval of time from τ to $\tau + d\tau$ concentrated charges of magnitudes $f'(\tau)d\tau$ and $-f'(\tau)d\tau$ are fired out in the directions specified by the unit vectors s_1 and s_2 respectively and that diffuse charges which will just balance the concentrated charges are shot out in all directions in such a way that each compensating diffuse charge is distributed uniformly throughout the shell bounded by the two spheres of radii $c(t - \tau)$ and $c(t - \tau - d\tau)$ whose centers are the positions of S at times τ and $\tau + d\tau$ respectively. The density of the charge compensating $f'(\tau)d\tau$ is easily seen to be

$$\frac{f'(\tau)}{4\pi r\nu},$$

where

$$\nu = \xi'(\tau)[x - \xi(\tau)] + \eta'(\tau)[y - \eta(\tau)] + \zeta'(\tau)[z - \zeta(\tau)] - c^2(t - \tau),$$
$$r = c(t - \tau).$$

On account of its displacement this charge provides us with the field

$$E_1 = \frac{f'(\tau)}{4\pi\nu}\,\frac{s - s_1}{1 - s\cdot s_1}, \qquad H_1 = \frac{f'(\tau)}{4\pi\nu}\,\frac{s_1 \times s}{1 - s\cdot s_1},$$

where rs denotes the vector whose components are $x - \xi(\tau)$, $y - \eta(\tau)$, $z - \zeta(\tau)$, respectively. It should be noticed that the *density* of the

[1] The term $\frac{1}{2}(E\cdot s_1)^2$ represents the potential energy gained on account of work done against the force.

diffuse charge is used here in place of e and so E_1, H_1 are in a sense densities of the corresponding quantities when e is used. The density of the diffuse charge compensating $-f'(\tau)d\tau$ is likewise

$$\frac{-f'(\tau)}{4\pi r\nu}$$

and this provides us with the field

$$E_2 = -\frac{f'(\tau)}{4\pi\nu}\frac{s - s_2}{1 - s\cdot s_2}, \qquad H_2 = -\frac{f'(\tau)}{4\pi\nu}\frac{s_2 \times s}{1 - s\cdot s_2}.$$

Superposing the two fields the densities of the two diffuse distributions of charge cancel out but we are left with an electromagnetic field specified by the vectors

$$\begin{aligned} E &= \frac{f'(\tau)}{4\pi\nu}\left[\frac{s - s_1}{1 - s\cdot s_1} - \frac{s - s_2}{1 - s\cdot s_2}\right], \\ H &= \frac{f'(\tau)}{4\pi\nu}\left[\frac{s_1 \times s}{1 - s\cdot s_1} - \frac{s_2 \times s}{1 - s\cdot s_2}\right]. \end{aligned} \tag{12}$$

In these expressions it must be remembered that τ is defined in terms of x, y, z and t by means of the equation

$$[x - \xi(\tau)]^2 + [y - \eta(\tau)]^2 + [z - \zeta(\tau)]^2 = c^2(t - \tau)^2, \qquad \tau \leq t.$$

If we introduce the auxiliary quantities

$$\alpha = \frac{f(\tau)}{4\pi}, \qquad \beta = \log\frac{1 - s\cdot s_1}{1 - s\cdot s_2},$$

it is easy to see that

$$E = \frac{1}{c}\left(\nabla\alpha\frac{\partial\beta}{\partial t} - \nabla\beta\frac{\partial\alpha}{\partial t}\right), \qquad H = \nabla\alpha \times \nabla\beta, \tag{13}$$

and that

$$\text{rot } H = \frac{1}{c}\frac{\partial E}{\partial t}, \qquad \text{div } E = 0,$$

$$\text{rot } E = -\frac{1}{c}\frac{\partial H}{\partial t}, \qquad \text{div } H = 0.$$

This means that our field vectors satisfy Maxwell's equations and that these equations are a consequence of our assumption that proto-electric charges travel along straight lines with the velocity of light and our additional hypothesis for the specification of the electric and magnetic vectors.[1] This hypothesis is that the field vectors at the two moving points A, B are the same at any instant, that the longitudinal components

[1] Another derivation of Maxwell's equations from elementary assumptions is given by Leigh Page, Am. Jour. Sci., 38, 1914, p. 169. He uses the theory of relativity and the assumption that each electric point charge is a center of uniformly diverging tubes of strain.

of the electric vector are $-ecT$ and $+ecT$ respectively and that the longitudinal components of the magnetic vector are zero. By longitudinal components we mean components parallel to s_1 and s_2. Another way of looking at the matter is to regard the coefficients of ecT in formulæ (11) as direction ratios of the moving line AB. Since each vector has three components there are six direction ratios.

The field specified by the formulæ (12) is a simple generalization of the one described by Heaviside[1] in 1901. If the function $f'(\tau)$ varies continuously the concentrated charges form two moving curves along which the above expressions for E and H are infinite but it is not certain that these expressions are valid for these moving curves. Since the field contains line charges it should, perhaps, be regarded as a proto field from which an ordinary electromagnetic field may be derived by a process analogous to differentiation. An appropriate process is described fully in a paper which will appear shortly in the Proceedings of the London Mathematical Society. In the simple case when the line charges are stationary except for motion along the line the process is simply a differentiation with regard to one coördinate z, for we may write

$$\alpha = t - \frac{r}{c}, \qquad \beta = \log\frac{r-z}{r+z}, \qquad r^2 = x^2 + y^2 + z^2.$$

The resulting field is the electrostatic field of an electric pole at the origin.

It should be noticed that a field of type (12) from which the field of a moving electric pole may be obtained by a process analogous to differentiation is a pure radiant field in which the vectors E and H satisfy the conditions

$$(E\cdot H) = 0, \qquad E^2 - H^2 = 0.$$

As this field is of an elementary nature the character of the energy ought to be determined and this is a matter which we shall now discuss.

§ 4. I think most authorities agree that the flow of energy in an ordinary electromagnetic field may be represented by means of Poynting's vector[2] $S = cE \times H$ and that the volume density of electromagnetic energy is $W = \frac{1}{2}(E^2 + H^2)$.

If no electricity is present so that no work is being done at the point under consideration the question arises as to whether the transfer of energy can be represented as a motion of all the energy in a single direction.

The fact that the equation of continuity

$$\frac{\partial W}{\partial t} + \operatorname{div} S = 0$$

[1] O. Heaviside, The Electrician, Nov. 29, 1901. Electromagnetic Theory, Vol. III., p. 122.

[2] For a recent discussion of the matter see a paper by G. H. Livens, Phil. Mag., 34, 1917, p. 385.

is satisfied suggests that it is sometimes possible to adopt this view, and as the velocity of motion

$$\frac{c}{W}\{E^2H^2 - (E \cdot H)^2\}^{1/2}$$

is never greater than c, the velocity of light, this way of looking at the matter seems at first quite reasonable. It is of some interest, however, to compare this velocity with Cunningham's velocity of the ether,[1] which we shall denote by the symbol u.

Let us consider the hypothesis that the energy in a field can be regarded as energy of motion of a single entity or a group of entities having practically the same motion when the velocity of flow is a possible velocity of the ether.

Taking first the simple case in which $E \cdot H = 0$, the velocity u is governed by one of the equations $H = u \times E$, $cE + u \times H = 0$ but the component of u perpendicular to S is indeterminate. If now u is in the direction of S, u is equal to either $c \cdot H/E$ or $c \cdot E/H$ and it is clear that u is equal to the velocity of flow of the energy only in the case when $E^2 = H^2$.

A similar result is obtained when we do not make the initial assumption $E \cdot H = 0$; for in this case the velocity u is governed by the equation[2]

$$ch = u \times e,$$

where

$$h = \lambda H + \mu E, \qquad e = \lambda E - \mu H,$$

and λ and μ are chosen so that $e \cdot h = 0$. Since $e \times h = (\lambda^2 + \mu^2)(E \times H)$ it appears that when u is in the direction of S, $u \cdot e = 0$ and we have

$$c^2h^2 = u^2e^2.$$

Now

$$e^2 + h^2 = (\lambda^2 + \mu^2)(E^2 + H^2)$$

and

$$e^2h^2 = (\lambda^2 + \mu^2)^2\{E^2H^2 - (E \cdot H)^2\},$$

hence u is equal to the velocity with which the energy flows only when $e^2 = h^2$ and then it follows that $E^2 = H^2$ and $E \cdot H = 0$.

Thus it is only in a simple radiant field for which the above equations are satisfied that the energy may be supposed to flow with the same speed as the ether. I have adopted the view elsewhere[3] that the energy in such a field is entirely kinetic energy or energy of motion but this view is rather unorthodox and may perhaps be questioned. The point at issue depends on the definition of the different kinds of energy and also

[1] Proc. Roy. Soc. London, 83, 1909, p. 110. The Principle of Relativity, Ch. XV.

[2] See H. Bateman, Mess. of Math., 14, 1915, p. 111.

[3] Proc. National Academy of Sciences, May, 1918.

on the question whether the usual identification of magnetic energy with kinetic energy is valid for all types of field. Perhaps it is advisable to distinguish between kinetic energy and energy of motion, as is done for instance by Page,[1] and to use the term kinetic energy only when energy can be expressed in the form $T = \frac{1}{2}mu^2$ where u is a velocity and m the transverse mass. According to this view the kinetic energy in the electromagnetic field of a moving electron can be represented by $\frac{1}{2}H^2$ per unit volume, this being the usual expression adopted by Larmor and others. This view seems to be strengthened by the form of the Lagrangian function in the principle of least action but it may not be right to adopt the expression $\frac{1}{2}H^2$ for the kinetic energy per unit volume in the ideal case of a magnetic particle for we may expect from symmetry that $\frac{1}{2}E^2$ is the correct expression. Since, moreover, a simple radiant field cannot generally be derived by superposition from the fields of electric poles moving with velocities less than c we cannot conclude from the above that the kinetic energy per unit volume is in this case represented by $\frac{1}{2}H^2$, although there may be some other way of arriving at this result. At any rate it does not seem right to assume that $\frac{1}{2}H^2$ per unit volume completely represents the *apparent energy of motion*[2] in the field, for it seems reasonable to adopt the view that there is no apparent energy of motion when the field is static and there is no flow of energy. Assuming that a field in the ether is static when there is no flow of energy a static field may be characterized by either $E = 0$ or $H = 0$, the general requirement being that E should be parallel to H. When a field is static it is possible for Cunningham's velocity u to be zero and conversely if u can be zero the field is static.

According to the above view there is no apparent energy of motion either in an electrostatic field or a static magnetic field but in both cases there is concealed energy of motion if we adopt the ideas of Thomson and Hertz. This concealed energy of motion may perhaps be brought into evidence by building up the static fields from simple radiant fields of the type for which $E^2 = H^2$ and $E \cdot H = 0$, for in each field of this type there is certainly energy of motion.

In what follows we shall adopt the hypothesis that the energy in a simple radiant field is entirely energy of motion. In justification of this we may, perhaps, reason as follows, but the argument is not very conclusive.

In the case when $cH = u \times E$ and u is parallel to S the kinetic energy

[1] Am. Jour. of Science, 40, Aug., 1915.

[2] This term is used because according to the views of Sir Joseph Thomson—The Applications of Dynamics to Physics and Chemistry, 1888, p. 15—all energy is ultimately kinetic energy or energy of motion.

$\frac{1}{2}H^2$ and the momentum $(1/c^2)S$ can be accounted for by assuming that a particle with transverse mass $m = E^2/c^2$ moves with velocity $u = c \cdot H/E$ in the direction of S. Now in the case of a moving electron the energy which it has acquired in virtue of its motion may be expressed in terms of the velocity u and transverse mass m by means of the formula[1]

$$mc^2[1 - \sqrt{1 - \beta^2}][1 + \tfrac{1}{4}\sqrt{1 - \beta^2}], \tag{14}$$

where $\beta = u/c$. If this formula is supposed to be applicable in the case of a particle whose transverse mass remains finite when $u = c$ the expression for the energy of motion becomes mc^2 and this is equal to E^2 in the case of the electromagnetic field. The view that the energy is entirely energy of motion when $E^2 = H^2$ and $E \cdot H = 0$ is thus in accordance with the above expression. Of course if the transverse mass is finite when $u = c$ the particle must be supposed to have no energy when it is at rest.

So far no general expression for the energy of motion in terms of the field vectors has been obtained which will satisfy the condition that the total energy of motion in a moving electron's field is equal to the expression (14).

In a general electromagnetic field it is probably not permissible to regard the momentum and kinetic energy in an electromagnetic field as arising from the motion of a mass in the direction of Poynting's vector. To elucidate matters a little let us consider the result of superposing two fields $(-E, -H)$ and $(E + dE, H + dH)$ which differ very slightly and which are both simple radiant fields. Let the unit vectors s and $s + ds$ indicate the direction of Poynting's vector S in the two fields, then on account of the assumed property of the fields we have the relations

$$E + s \times H = 0, \qquad dE + s \times dH + ds \times H = 0,$$
$$s \cdot E = 0, \qquad s \cdot H = 0.$$

Since $s \cdot ds = 0$ we may write $ds = AE + BH$ and so

$$ds \times H = AE \times H = \lambda s;$$

this means that the vector s satisfies equations of type

$$dE + s \times dH = ks, \qquad s \cdot dE = k,$$

and so is one of the unit vectors (of the type considered in § 1) for the resultant field (dE, dH).

The energy $\frac{1}{2}(dE)^2 + \frac{1}{2}(dH)^2$ in the resultant field is generally very much less than the sum of the energies

$$\tfrac{1}{2}E^2 + \tfrac{1}{2}H^2, \qquad \tfrac{1}{2}(E + dE)^2 + \tfrac{1}{2}(H + dH)^2$$

[1] See for instance, L. Page, Amer. Jour. Sci., Vol. XL., Aug., 1915, p. 119. It should be mentioned that in the derivation of the formula there is an assumption with regard to the size of the electron when at rest as compared with its size when in motion.

in the component fields, so that there is a loss of energy of amount $E(E + dE) + H(H + dH)$ due to interference. This energy may perhaps be regarded as concealed energy of motion. The question now arises how two fields in which energy flows practically in the direction of the unit vector s can give a resultant field in which energy flows in a direction different from s. The answer is that the flow at right angles to s in the resultant field arises from the motion of a very large amount of energy with a very small component velocity perpendicular to s.

This may be the explanation of the perplexing fact that in the field of a moving electron the direction of Poynting's vector is quite different from that of the radius from the effective position of the electron. Indeed the field of a moving electron can be built up from simple radiant fields in each of which the flow of energy to a point is actually or very nearly along the radius just mentioned and it is just because the total amount of energy in these radiant fields is vastly greater than the energy usually attributed to the electron's field that the slight deviations in the paths from the mean radius can give rise to an appreciable transverse flow.

In our opinion then there is a colossal amount of concealed energy of motion in the field of an electron or positive nucleus of an atom. Whether this energy will ever become available or not we are quite unable to say. Of course it must be remembered that our theory of the structure of an electric field is based on the idea that the electric charge of an electron is continually being renewed[1] by electric separation (*i. e.*, the breaking up of minute doublets) and that the charge remains constant because a steady state has been reached. At present there is no way of deciding between this theory and the usual theory that an electron always consists of the same particles of electricity, but in support of the new theory it may be claimed that it gives a simple geometrical reason for the shape of the lines of electric force of a moving electric pole[2] and provides a possible explanation of gravitation as an effect due to an extremely slight fluctuation of the charges on electrons and positive nuclei in what may be slight deviations from the steady state of renewal of these charges.

The chief reason for pursuing a theory of this kind is the hope that it may throw some light on the nature of force and the real meaning of the dynamical equations of motion. It is very probable that the equations of motion are fundamentally of a geometrical nature[3] implying

[1] An idea somewhat similar to this is adopted by the late S. B. McLaren in his theory of gravitation, Phil. Mag., Vol. 26, 1913, p. 636.

[2] Mess. of Math., Vol. 47, 1918, p. 161.

[3] An attempt to express the fundamental laws of physical phenomena by geometrical considerations has been made recently by H. A. Lorentz, Amst. Proc., 19, pp. 1341–1369; 20,

the existence of certain incidences and correspondences with perhaps a minimum principle thrown in. While the general problem still baffles us some useful information may perhaps be gained by considering cases in which we can actually find an equivalent of the energy lost when work is done.

We shall commence by considering an electromagnetic field in which the usual equations

$$c \text{ rot } H = \frac{\partial E}{\partial t} + \rho v, \qquad \text{div } E = \rho,$$
$$c \text{ rot } E = -\frac{\partial H}{\partial t} + \sigma w, \qquad \text{div } H = -\sigma, \tag{15}$$

are satisfied and as usual $\sigma = 0$, $\sigma w = 0$. We shall assume, however, that ρ and ρv have the forms

$$\rho = -\frac{1}{c}\frac{\partial \psi}{\partial t}, \qquad \rho v = c\nabla\psi. \tag{16}$$

The usual form of the energy equation is, moreover, obtained from the relation

$$\rho(v \cdot E) + \frac{\partial}{\partial t}[\tfrac{1}{2}(E^2 + H^2)] + \text{div } [c(E \times H)] = 0, \tag{17}$$

where the first term represents the rate at which work is being done by the field on the electric charges present.

Now

$$c\text{div } (\psi E) = -\psi\frac{\partial \psi}{\partial t} + \rho(v \cdot E),$$

hence the energy equation may be written in the form

$$\frac{\partial}{\partial t}[\tfrac{1}{2}(E^2 + H^2 + \psi^2)] + \text{div } [c(E \times H) + c\psi E] = 0,$$

and the career of the energy which has been transformed into work is not lost sight of if we assume that the total amount of energy per unit volume is $\frac{1}{2}(E^2 + H^2 + \psi^2)$ and that the total flow of energy is specified by the vector

$$c(E \times H) + c\psi E.$$

The lack of symmetry with respect to E and H can be avoided if we write

$$\sigma = -\frac{1}{c}\frac{\partial \chi}{\partial t}, \qquad \sigma w = c\nabla\chi,$$

pp. 2–34, 1917. Further interesting geometrical developments may enter in the study of the growth of simple correspondences between different parts of the field figure and in estimating the closeness of fit of an imperfect correspondence.

and add the term $-(\sigma w\cdot H)$ to the left-hand side of (17) for then

$$c\,\mathrm{div}\,(\chi H) = \chi\frac{\partial\chi}{\partial t} + \sigma(w\cdot H),$$

and the energy equation may be written in the form

$$\frac{\partial}{\partial t}[\tfrac{1}{2}(E^2 + H^2 + \psi^2 + \chi^2)] + \mathrm{div}\,[c(E\times H) + c\psi E - c\chi H] = 0.$$

Similarly the usual momentum equations of type

$$\frac{\partial X_x}{\partial x} + \frac{\partial X_y}{\partial y} + \frac{\partial X_z}{\partial z} + \frac{1}{c}\frac{\partial X_t}{\partial t}$$
$$= \rho\left[E_x + \frac{1}{c}(v_yH_z - v_zH_y)\right] - \sigma\left[H_x - \frac{1}{c}(w_yE_z - w_zE_y)\right],$$

in which

$$X_x = E_x^2 + H_x^2 - \tfrac{1}{2}(E^2 + H^2),\qquad X_z = E_xE_z + H_xH_z,$$
$$X_y = E_xE_y + H_xH_y,\qquad X_t = E_zH_y - E_yH_z$$

may be written in the form

$$\frac{\partial X_x'}{\partial x} + \frac{\partial X_y'}{\partial y} + \frac{\partial X_z'}{\partial z} + \frac{1}{c}\frac{\partial X_t'}{\partial t} = 0,$$

where

$$X_x' = X_x + \tfrac{1}{2}(\psi^2 + \chi^2),\qquad X_z' = X_z + \psi H_y + \chi E_y,$$
$$X_y' = X_y - \psi H_z - \chi E_z,\qquad X_t' = X_t + \psi E_x - \chi H_x.$$

If we write

$$M = H + iE,\qquad -i\phi = \psi + i\chi,$$

the fundamental equations may be written in the form

$$c\,\mathrm{rot}\,M = -i\left(\frac{\partial M}{\partial t} + c\nabla\phi\right),$$
$$\mathrm{div}\,M + \frac{1}{c}\frac{\partial\phi}{\partial t} = 0, \tag{18}$$

and an interesting class of solutions is obtained by adding the additional equation[1] $M^2 = \phi^2$.

In this case we may write

$$M = \phi\frac{s_1 - s_2 + i(s_1\times s_2)}{1 - (s_1\cdot s_2)},$$

where s_1 and s_2 are unit vectors which are easily seen to satisfy equations (2) of § 1. Assuming that these are real and writing l for the x-com-

[1] Cf. H. Bateman, Mess. of Math., 47, 1918, p. 161.

ponent of s_2, X for the vector with components X_x, X_y, X_z, and K for the quantity

$$2\,\frac{\psi^2+\chi^2}{1-(s_1\cdot s_2)}$$

we have

$$\begin{aligned}\tfrac{1}{2}(E^2+H^2+\psi^2+\chi^2) &= K,\\ \psi E-\chi H+E\times H &= Ks_1,\\ \psi E-\chi H-E\times H &= -Ks_2,\\ X &= -Kls_1,\\ X_t &= -Kl.\end{aligned}$$

This means that the career of the energy transformed into work is not lost sight of if we suppose that an amount of energy K flows with the velocity of light in the direction of the unit vector s_1 and that momentum $-Ks_2$ flows with the velocity of light in the same direction. This momentum may perhaps be supposed to arise from the motion of particles in directions differing very slightly from the direction s_1, a condition which suggests the existence of two or more superposed fields which differ very slightly in properties. If a field of the present type is supposed to arise from the breaking up of minute electric and magnetic doublets and a rectilinear motion of their constituents with the velocity of light the dynamical equations of motion may simply imply that certain groups of particles travel along without losing any of their energy or momentum. In the general case, however, it is more probable that the dynamical laws tell us what happens when old groups of particles are broken up and new ones formed. It should be mentioned that a more general type of field has been found in which the career of the energy transformed into work at an ordinary point of space is not lost sight of. In this case we take

$$\rho v = c\nabla\alpha + c\lambda s_1, \qquad \rho = -\frac{1}{c}\frac{\partial\alpha}{\partial t}+\lambda,$$

$$\sigma w = c\nabla\beta + c\mu s_1, \qquad \sigma = -\frac{1}{c}\frac{\partial\beta}{\partial t}+\mu,$$

where s_1 is one of the unit vectors connected with the field by equations (2) of § 1. If now α and β are defined by the equations

$$\alpha = (s_1\cdot E), \qquad \beta = -(s_1\cdot H),$$

we have as before with α and β in place of ψ and χ

$$(\rho v\cdot E) = c\,\operatorname{div}(\alpha E) + \alpha\frac{\partial\alpha}{\partial t},$$

$$(\sigma w\cdot H) = c\,\operatorname{div}(\beta H) - \beta\frac{\partial\beta}{\partial t},$$

whatever λ and μ may be, but these must be chosen, of course, so that the equations of continuity

$$\frac{\partial \rho}{\partial t} + \operatorname{div}(\rho v) = 0, \qquad \frac{\partial \sigma}{\partial t} + \operatorname{div}(\sigma w) = 0$$

are satisfied. The energy in the field is now

$$T = \tfrac{1}{2}(E^2 + H^2 + \alpha^2 + \beta^2) = \tfrac{1}{2}(E^2 + H^2) + \tfrac{1}{2}[(E^2 - H^2)^2 + 4(E \cdot H)^2]^{1/2},$$

and the flow of energy

$$S = c[(E \times H) + E(s_1 \cdot E) + H(s_1 \cdot H)] = cTs_1.$$

On the other hand we have

$$X_t = [E_x(s_1 \cdot E) + H_x(s_1 \cdot H) - (E \times H)_x] = - T(s_2)_x,$$

where s_2 is the second unit vector connected with the field. We also have $X = - Ts_1(s_2)_x$, hence a total amount of energy T flows with the velocity of light in the direction of s_1 and a total momentum $-(1/c)Ts_2$ flows with the velocity of light in the same direction. It is probable that this type of field also arises from the breaking up of electric and magnetic doublets and the rectilinear motion of their constituents with the velocity of light. The above argument probably breaks down at points where the breaking up of the doublets occurs.

Returning to the previous type of field we note that equations (18) may be satisfied by writing

$$M = \frac{1}{c}\frac{\partial \Gamma}{\partial t} + i \operatorname{rot} \Gamma + \nabla \Phi,$$

$$\phi = - \operatorname{div} \Gamma - \frac{1}{c}\frac{\partial \Phi}{\partial t},$$

where Φ and the components of Γ are solutions of the wave equation. If $\chi = 0$ so that there is no real magnetism we have simply a field in which the volume density of electricity is proportional to the rate of change of a function ψ and the convection current is also derivable from ψ by differentiation. Let us call ψ the *electric storage*. It is important now to notice that *there may be a finite constant value of* ψ *and no electromagnetic field*, for if in the above equations we write

$$\Gamma = - \nabla \theta, \qquad \Phi = \frac{1}{c}\frac{\partial \theta}{\partial t},$$

we have clearly $M = 0$ and

$$\phi = \nabla^2 \theta - \frac{1}{c^2}\frac{\partial^2 \theta}{\partial t^2}.$$

It is only necessary then to choose θ to be a solution of the equation

$$\nabla^2\theta - \frac{1}{c^2}\frac{\partial^2\theta}{\partial t^2} = \text{constant},$$

in order to ensure that Φ and the components of Γ may be solutions of the wave-equation. The quantity ϕ is then constant and so therefore is ψ while there is no electromagnetic field.

This result indicates that in any field of type (18) we can regard the electric storage ψ as a positive quantity.

Electric storage is in some respects analogous to quantity of proto-electricity but the function ψ is an absolute invariant under the transformations of the theory of relativity. It seems more reasonable to regard quantity of proto-electricity as the rate of change of electric storage; the existence of both positive and negative charges of proto-electricity then seems quite natural.

Summary.

1. An electromagnetic field is studied geometrically in relation to a moving observer and various vectors are located with the aid of the two cones which at each point limit the directions of the forces acting on electric and magnetic charges moving with velocities less than that of light.

2. The electromagnetic laws are expressed directly in terms of the forces on unit electric and magnetic charges in motion and some deductions relating to lines of force are made from Hargreaves' theorems for space time integrals. One of Hargreaves' theorems suggests that quantity of electricity may represent a rate of change of another entity—quantity of proto-electricity, and electromagnetic fields are regarded as derivable from proto-electromagnetic fields containing line charges by a method analogous to differentiation.

3. Field vectors satisfying Maxwell's equations are constructed from the assumption that an aggregate of particles travel along straight lines with the velocity, and from a further hypothesis regarding the nature of the vectors.

4. A critical discussion is given on the nature of the energy in an electromagnetic field and the nature of its flow. A theory is developed which indicates that there is a colossal amount of concealed energy in the field of a moving electron.

Some examples are given which indicate what becomes of the energy which is apparently lost or transformed when work is done by an electromagnetic field on the electric charges within it.

Throop College, Pasadena.