X. *On the action of the second surfaces of transparent plates upon light.*

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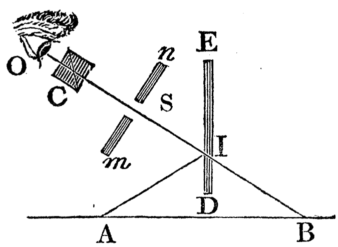
IN a paper on the Polarization of Light by Reflexion, published in the Philosophical Transactions for 1815, I showed that the Law of the Tangents was rigorously true for the second surfaces of transparent bodies, provided that the sine of the angle of incidence was less than the reciprocal of the index of refraction. The action of the second surfaces of plates at angles of incidence different from the maximum polarizing angle, was studied by M. ARAGO, who conducted his experiments in the following manner.

"With respect to this phænomenon," says M. ARAGO, "a remarkable result of experiment may here be noticed; that is, that *in every possible inclination* $A = A'$ *.

"Let us suppose that a plate of glass E D (Fig. 1.) is placed in the position that the figure represents before a medium A B of a uniform tint; for instance, a sheet of fine white paper. The eye placed at O, will receive simultaneously the ray I O reflected at I, and the ray B I O transmitted at the same point. Place at *m n* an opaque diaphragm blackened, and perforated by a small hole at S. Lastly, let the eye be furnished with a doubly refracting crystal C, which affords two images of the aperture.

Fig. 1.



"If now, by means of a little black screen placed between B and I, we stop the ray B I which would have been transmitted, the crystal properly placed will give an ordinary image $= A + \frac{1}{2} B$, and an extraordinary image

* A is the light polarized by reflexion, and A′ that polarized by refraction.

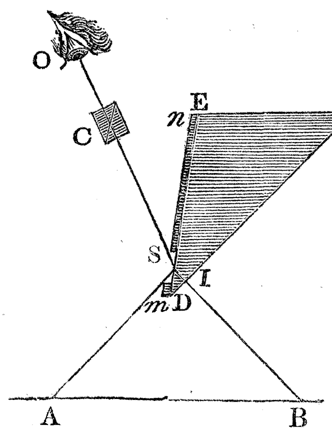
$= \frac{1}{2}$ B. But if the screen were placed between A and I, and the ray A I were intercepted, we should still have two images of the hole, and their intensities would be $\frac{1}{2}$ B′ and A′ + $\frac{1}{2}$ B′ respectively. Consequently, without any screen, if the whole of the reflected light A I O, and the transmitted B I O are allowed to arrive at the eye, we shall have for the ordinary image A + $\frac{1}{2}$ B + $\frac{1}{2}$ B′, and for the extraordinary image $\frac{1}{2}$ B + A′ + $\frac{1}{2}$ B′.

"Now it appears from actually making the experiment, that the two images are perfectly equal, *whatever may be the angle formed by the ray* A I *with the plate of glass* which can only be because A is always equal to A′. Consequently

"The quantity of polarized light contained in the pencil transmitted by a transparent plate is exactly equal, to the quantity of light polarized at right angles, which is found in the pencil reflected by the same plate."

We have no doubt that M. Arago obtained these results, particularly near the polarizing angle, at which limit they are rigorously true; but at all other angles of incidence they are wholly incorrect. When we consider, indeed, the nature of the experiment which has been lauded for its elegance and ingenuity, we shall see reason to pronounce its results as nothing more than coarse estimates, in which the apparent equality of the two images is the effect either of imperfect observation or of some unrecognized compensation.

Fig. 2.



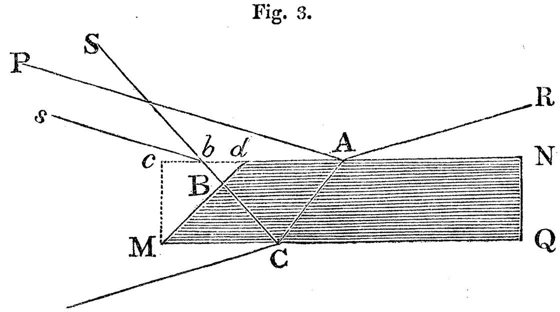
If we make the experiment in the manner shown in Fig. 2. with a colourless and well annealed prism of glass E F D, in place of a plate of glass; and make the ray B I enter the surface F D perpendicularly at I, we get rid of all sources of error, and we obtain, what is really wanted, the result for a single surface. In this case the experiment is not disturbed by the light reflected from the inner surfaces of the prism, which is all thrown off from the pencil which enters the eye.

In M. Arago's form of the experiment, part of the ray B I (Fig. 1.) undergoes reflexions within the plate, and there comes along with it to the eye, at O, a portion of light polarized in the plane of reflexion: in like manner the part of the pencil AI that enters the plate, undergoes partial reflexions, and the part reflected from the first surface carries along with it another portion of light polarized in the plane of reflexion, so that four portions of light polarized in the plane of reflexion reach the eye, while only two portions reach it polarized at right angles to the plane of reflexion, viz. those which are polarized by the refraction of each of the surfaces of the plate. Now the part of the pencil A I which suffers a first reflexion from each of the surfaces of the plate, is, as we shall presently show, defective in polarized light compared with that which has experienced two refractions, so that it requires the above additional quantities to produce a compensation with the transmitted pencil BO. If this is not the true cause of the apparent compensation, that is, if M. Arago took means to exclude the reflected pencils which seem to have produced the compensation, we must then ascribe the equality of the two images to inaccuracy of observation.

But even if we admit that M. Arago's experimental results are correct with regard to plates, it necessarily follows that they cannot be true with regard to surfaces; for it is obvious from the slightest consideration of the subject, that the phænomena of the one can never be interchangeable with those of the other.

In order to demonstrate these views by an analysis of the changes which the intromitted light experiences from the two refractions and the intermediate reflexion of a transparent plate, I took a plate of glass of the shape M N (Fig. 3.) having an oblique face M *d* cut upon one of its ends. A ray of light R A, polarized + 45° and − 45°, was made to fall upon it at A, at an angle of incidence of nearly 83°, so that the inclination of the planes of polarization of the reflected ray A P was about 36½°. Now the ray A C after reflexion in the direction C S, without any refraction at B, where it emerges perpendicularly to M *d*, would also have had the inclination of its planes of polarization equal to 36½° if there had been no intermediate refraction at A; but this refraction alone being

Fig. 3.



capable of producing an inclination of 53° or a rotation of $53° - 45° = 8°$, and this rotation being in an opposite direction from that produced by the second reflexion at C, the inclination of the planes of polarization for the ray C S is nearly $44\frac{1}{2}°$, the reflexion at C having brought back the ray A C almost exactly into the state of natural light.

Without changing either the light or the angle, I cemented a prism M*cd* on the face M *d*, so that *c d* was parallel to *d* N, and I found that the second refraction at *b*, equal to that at A, changed the inclination of the planes of polarization to 53°; that is, the two refractive actions at A and *b* had overcome the action of reflexion at C, and the pencil *bs* actually contained light polarized perpendicular to the plane of reflexion.

In order to put this result to another test, I took a plate M *c* N Q (Fig. 3.) of the same glass, which separated the pencil *b s* reflected at the second surface, from the parallel pencil A P reflected from the first surface, and I found that at an angle of 83°, the value of the inclination I or φ for the ray was about $37\frac{1}{2}°$, while the value of I for the ray *b s* was nearly 55°, an effect almost equal to the refractive action of a plate at 83° of incidence.

When the pencil R A is incident on the first surface at the polarizing angle or 56° 45′, the rotation produced by refraction at A is about 2°, or the inclination $I = 45° + 2° = 47°$; but the maximum action of the polarizing force at C is sufficient to make $I = 0°$ whether x is 45° or 47°. Hence CB is completely polarized in the plane of reflexion, and the refractive action at *b* is incapable of changing the plane of polarization when $I = 0°$: the reason is therefore obvious why the two rotations at A and *b*, of 2° each, produce no effect at the maximum polarizing angle.

If we now call

φ = Inclination to the plane of reflexion produced by the 1st refraction at A,

φ' = Inclination produced by the reflexion at C,

φ'' = Inclination produced by the 2nd refraction at *b*,

We shall have

$$\text{Cot}\,\varphi = \cos\,(i - i')\ ;\ \text{or}\ \tan\,\varphi = \frac{1}{\cos\,(i - i')}$$

$$\text{Tan}\ \varphi' = \tan x \left(\frac{\cos(i+i')}{\cos(i-i')}\right. = \frac{\cos(i+i')}{(\cos(i-i'))^3}$$

$$\text{Cot}\ \varphi'' = \cot x\ (\cos(i-i')) = \frac{(\cos(i-i'))^3}{\cos(+i')}$$

These formulæ are suited to common light where $x = 45°$, but when x varies they become

$$\text{Cot}\ \varphi = \cot x\ (\cos(i-i'))$$

$$\text{Tan}\ \varphi' = \tan x \left(\frac{\cos(i+i')}{(\cos(i-i'))^2}\right)$$

$$\text{Cot}\ \varphi'' = (\cot x \left(\frac{(\cos(i-i'))^3}{\cos(i+i')}\right).$$

Resuming the formula for common light, viz. $\cot \varphi'' = \frac{(\cos(i-i'))^3}{\cos(i+i')}$, it is obvious that when $(\cos(i-i'))^3 = \cos(i+i')$, $\cot \varphi'' = 1$, and $\varphi'' = 45°$; that is, the light is restored to common light.

In glass where $m = 1.525$ this effect takes place at $78° 7'$; a little below $78°$ in diamond; and a little above $80°$ in water.

At an angle below this, φ becomes less than $45°$, and the pencil contains light polarized in the plane of reflexion; while at all greater angles φ is above $45°$, and the pencil contains light polarized perpendicular to the plane of reflexion. Hence we obtain the following curious law.

"A pencil of light reflected from the second surfaces of transparent plates, and reaching the eye after two refractions and an intermediate reflexion, contains at all angles of incidence from $0°$ to the maximum polarizing angle, a portion of light polarized in the plane of reflexion. Above the polarizing angle the part of the pencil polarized in the plane of reflexion diminishes till $\cos(i+i') = (\cos(i-i'))^3$, when it disappears, and the whole pencil has the character of common light. Above this last angle the pencil contains a quantity of light polarized perpendicularly to the plane of reflexion, which increases to a maximum and then diminishes to zero at $90°$."

Let us now examine the state of the pencil C S' that has suffered only one refraction and one reflexion. Resuming the formula $\tan \varphi' = \frac{\cos(i+i')}{(\cos(i-i'))^2}$, it is evident that when $(\cos(i-i'))^2 = \cos(i+i')$, $\varphi' = 45°$, and consequently the light is restored to common light. This takes place in glass at an angle of 82° 44′. At all angles beneath this the pencil contains light polarized in the plane of reflexion; but at all angles above it, the pencil contains light polarized perpendicular to the plane of reflexion, the quantity increasing from 82° 44′ to its maximum, and returning to its minimum at 90°.

By comparing these deductions with the formula and table for reflected light given in my paper On the Laws of the Polarization of Light by Refraction, the following approximate law will be observed. When

$\text{Cos}\,(i - i') = \cos\,(i + i')$ All the incident light is reflected.

$(\text{Cos}\,(i - i'))^2 = \cos\,(i + i')$ Half the incident light is reflected.

$(\text{Cos}\,(i - i'))^3 = \cos\,(i + i')$ A third of the incident light is reflected.

$(\text{Cos}\,(i - i'))_n = \cos\,(i + i')$ An nth part of the incident light nearly is reflected.

This law deviates from the truth by a regular progression as n increases, and always gives the value of the reflected light in defect. Thus

Angles of Incidence.	Values of n.	Differences.
82° 44′	2	0
78 34	3	12
75 38	4	21
68 56	8	38
66 4	11	43
61 22	20	50

Let us now apply the results of the preceding analysis to M. Arago's experiment shown in Fig. 1. Suppose the angle of incidence to be 78° 7′, and let the light polarized by reflexion at A (Fig. 3.) be $= m$, and that polarized by one refraction also $= m$. Then since the pencil bs is common light, the polarized light in the whole reflected pencil AP, bs is $= m$, whereas the light polarized by the two refractions is $= 2m$; so that M. Arago's experiment makes two quantities appear equal when the one is double that of the other. If the angle exceeds 78° 7′, the oppositely polarized light in the pencil bs will neutralize a portion of the polarized light in the pencil AP, and the ratio of the oppositely polarized rays which seem to be compensated in the experiment, may be that of $3m$ or even $4m$ to 1.

Having thus determined the changes which light undergoes by reflexion from plates, it is easy to obtain formulæ for computing the exact quantities of polarized light at any angle of incidence, either in the pencil CBS or *bs*.

The primitive ray RA being common light, AC will not be in that state, but will have its planes of polarization turned round a quantity x by the refraction at A; so that $\cot x = \cos(i - i')$. Hence we must adopt for the measure of the light reflected at C the formula of FRESNEL for polarized light whose plane of incidence forms an angle x with the plane of reflexion. The intensity of AC being known from the formula for common light, we shall call it unity, then the intensity I of the two pencils polarized $-x$ and $+x$ to the plane of reflexion will be

$$\mathrm{I} = \frac{\sin^2(i - i')}{\sin^2(i + i')} \cos^2 x + \frac{\tan^2(i - i')}{\tan^2(i + i')} \sin^2 x \quad \text{and}$$

$$\mathrm{Q} = \mathrm{I}\left(1 - 2\,\frac{\left(\frac{\cos(i + i')}{(\cos(i - i'))^2}\right)^2}{+1\left(\frac{\cos(i + i')}{(\cos(i - i'))^2}\right)^2}\right)$$

In like manner if we call the intensity of CB $= 1$, we shall have

$$\mathrm{Tan}\, x = \frac{\cos(i + i')}{(\cos(i - i'))^2}$$

and the intensity I of the transmitted pencil *bs*

$$\mathrm{I} = 1 - \frac{\sin^2(i - i')}{\sin^2(i + i')} \cos^2 x + \frac{\tan^2(i - i')}{\tan^2(i + i')} \sin^2 x \quad \text{and}$$

$$\mathrm{Q} = \left(\mathrm{I}\ 1 - 2\,\frac{\left(\frac{(\cos(i - i'))^3}{\cos(i + i')}\right)^2}{1 + \left(\frac{(\cos(i - i'))^3}{\cos(i + i')}\right)^2}\right)$$

I shall now conclude this paper with the following Table computed from the formulæ in pages 148, 149, and showing the state of the planes of polarization of the three rays AC, CS, and *bs*.

Angle of Incidence on the First Surface.	Angle of Refraction at First Surface, and Angle of Incidence on Second Surface.	Inclination of Plane of Polarization of A C Fig. 3.	Inclination of Plane of Polarization of C S Fig. 3.	Inclination of Plane of Polarization of *bs* Fig. 3.
° ′	° ′	° ′	° ′	° ′
0 0	0 0	45 0	45 0	45 0
32 0	20 33	45 34	32 20	32 51
40 0	25 10	45 58	24 12	24 56
45 0	27 55	46 17	17 49	18 38
56 30	33 30	47 22	0 0	0 0
67 0	37 34	48 57	18 20	20 50
70 0	38 30	49 33	23 34	27 6
75 0	39 46	50 45	32 22	37 48
78 37	40 29	51 49	38 10	44 59
79 0	40 33	51 56	38 49	45 46
80 0	40 42	52 16	40 27	47 46
83 0	41 5	53 21	44 39	53 40
86 30	41 23	54 47	50 58	60 13
90 0	41 58	56 29	56 29	66 19

Allerly, December 31*st*, 1829.