A DESIGN PROCEDURE FOR LINEAR, PASSIVE RC-FILTER NETWORKS

by

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T A B L E   O F   C O N T E N T S

ABSTRACT

Filter networks using resistance and capacitance elements only are frequently desirable in installations where the inherent superiority of an LC-filter is nullified by the factors of coil weight and mutual interaction. It is the purpose of this thesis to present a practical procedure for the synthesis of satisfactory RC-filters.

The magnitude of any ideal transfer admittance or impedance is first plotted as a function of ω, the real frequency. By a change of variable, $\omega = \tan \phi/2$, the infinite range of ω is reduced to a finite range of ϕ. Using a modified form of Fourier Analysis, the ideal transfer function is approximated, in an oscillatory manner, by a finite trigonometric polynomial. A reverse variable change transforms this trigonometric polynomial to a rational function of ω^2 expressed as the quotient of two polynomials of equal degree. This function, when squared, approximates the squared magnitude of the ideal transfer function, and is converted, for synthesis purposes, to a function of λ, the complex frequency.

The transfer function in the variable λ is separated into realizable factors, and each factor, or stage, is synthesized, by a modification of Guillemin's basic method, as a group of unbalanced-ladder networks connected in parallel. The number of elements in each network, the number of component networks in each stage, and the number of stages depend

on the degree of the original approximating trigonometric polynomial. Amplifier and cathode-follower sections are used to eliminate the constant attenuation common to RC-networks and to isolate individual stages. The product of the transfer functions of these stages then gives the prescribed overall transfer admittance or impedance.

Limitations on this synthesis method, and a procedure for obtaining practical element values are discussed. Then, as an illustrative example of the entire procedure, a simple, low-pass RC-filter is designed, built, and tested to conform closely to the predicted frequency characteristic. Other, more complicated filters are calculated to indicate the various design considerations and the variations that can be obtained in attenuation and in width and flatness of the pass-band.

CHAPTER I

INTRODUCTION

Low cost, compactness, and immunity against stray pick-up are practical advantages that have led to the increasing importance of RC-networks for filtering and coupling, despite the acknowledged general superiority of performance of LC-circuits. Moreover, in the frequency range below 100 cycles/sec. the size, weight, and low Q of inductances make it almost mandatory to resort to RC-components.

The problem with RC-networks is to obtain sharp frequency discrimination. Thiessen,[1] Fritzenger,[2] and Toshniwal[3] resort, as have nearly all investigators in this field, to feed-back to obtain sharp cut-off features. This has the disadvantage that changing tube characteristics may change the performance of the network. Linvill[4] obtains a more stable frequency characteristic by using tubes as equivalent to -1:1 transformers. It is the purpose of this paper to present a complete practical method of synthesis of linear, passive RC-networks with satisfactory frequency characteristics independent of tube coefficients.

This method of design follows the four basic steps in procedure outlined by Linvill[4] in his excellent, concise history of synthesis theory. The first step is a statement of the limitations on a driving-point or transfer function imposed by the requirements of physical realizability. Secondly, a rational function is found which satisfies these requirements and approximates the desired function of frequency within allowable limits. The third step is the realization of a network having this driving-point or transfer function. Finally, more practical equivalents to this network are developed if required.

These steps are specifically applied in synthesizing low-pass RC-filters from the transfer function. High-pass and band-pass filters follow directly by complementary procedure. The method is also applicable to equalizing networks or to any desired function of frequency.

It may be emphasized at this point that one is designing directly for the insertion loss. As in conventional filter design, this method requires considerable computational effort, but has the advantage of giving exact results.

The first step may be stated briefly. The zeros of the transfer function of any R-C network may have multiple order and may lie anywhere in the complex plane; the poles must be simple and are restricted to the negative real axis. The other three steps will be the subject of the succeeding three chapters.

CHAPTER II

A SOLUTION TO THE APPROXIMATION PROBLEM

The Method of Approximation and Specification of the Rational Transfer Function in Factored Form

Guillemin has suggested that an ideal magnitude of transfer function may be approximated by a periodic function of (ϕ). Then by means of a change of variable ($\omega = \tan \phi/2$), one may obtain a function of (ω^2), where (ω) is the real frequency, in the form of a quotient of two polynomials of equal degree.

In approximating the ideal transfer function, which has been specified by the purpose of the network to be realized, it is desirable to do so with a Tschebyscheff approximation which assures equal minima of attenuation in the attenuation band and equal maxima of attenuation in the pass band. Weirstrass[5] proved in 1885 that a continuous function f(x) of period 2π could be represented by a finite trigonometric series g(x) such that $|f(x) - g(x)| < \epsilon$ for all values of (x) in the interval. It appears, however, that when approximating a discontinuous function to the same tolerance with trigonometric polynomials of equal degree, the polynomial which exhibits Tschebyscheff behavior assures the maximum range over which the trigonometric representation remains within the desired tolerance. The degree of the function g(x) and the selected tolerance together are interacting restraints which specify the range of (x) over which one may approximate to the ideal characteristic.

The primary object of this work is the design of a low pass filter. The transfer function selected is the ideal admittance characteristic, $F^1(\phi)$.

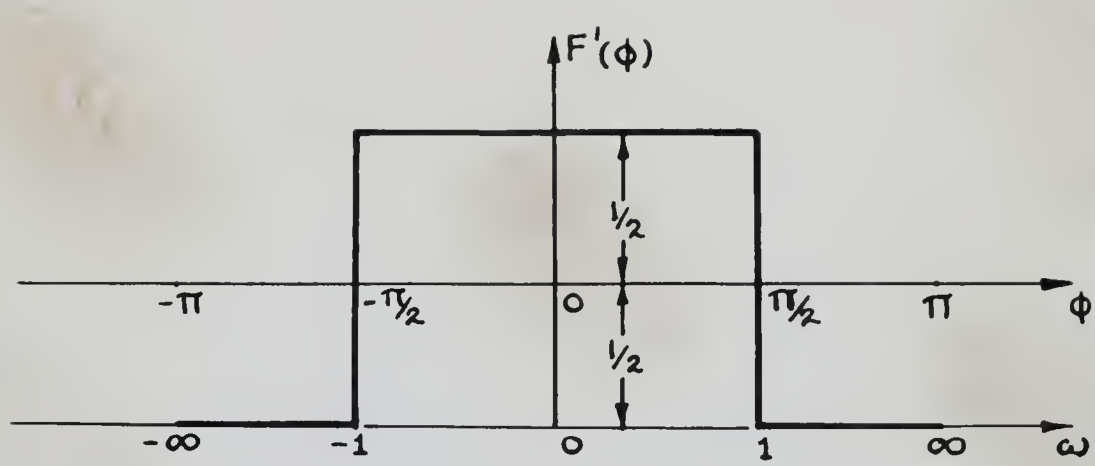


FIG II-1

This square wave must be approximated with uniform tolerance by the polynomial $f^1(\phi)$. It is expedient to use modified Fourier coefficients which represent the approximation to the square wave. Since this approximation contains only odd harmonics, one obtains:

$$f^1(\phi) = a_1^1 \cos \phi + a_3^1 \cos 3\phi + - - - + a_n^1 \cos n\phi \quad \text{(II-1)}$$

where $\underline{n}$ is always odd. By means of the relation

$$\cos n\phi = 2 \cos (n-1)\phi \cos \phi - \cos (n-2)\phi \quad \text{(II-2)}$$

$a_n \cos n\phi$ yields a term in $\left(\frac{1 - \omega^2}{1 + \omega^2}\right)^n$ and the transform

$f^1(\phi) \quad f^1(\omega^2)$ gives $f^1(\omega^2) = \dfrac{P^1(\omega^2)}{(1 + \omega^2)^n}$. It therefore

becomes necessary to modify the denominator of $f^1(\omega^2)$ in order that there be no multiple poles, because of the constraint placed on the poles by the requirements of physical realizability of R-C network transfer functions, namely, the poles must all be simple and lie on the negative real axis of the complex frequency (λ) plane. (Later a method of accepting multiple poles will be explained). This modification can be accomplished by the separation of the poles of $f^1(\omega^2)$ such that the product of pairs of poles equals unity as shown below:

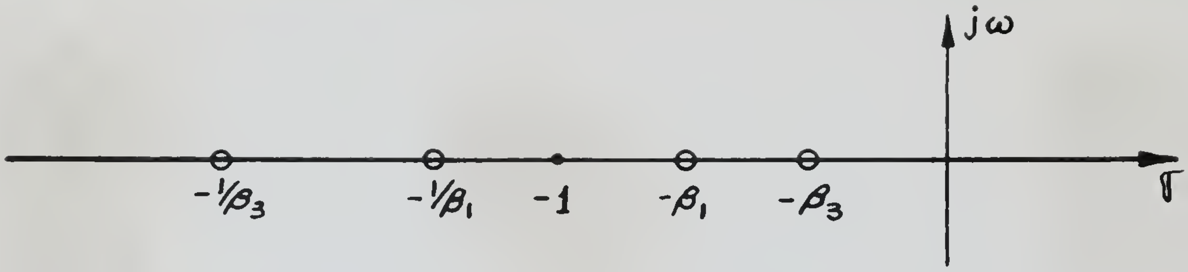


FIG II -2

Accordingly $(1 + \omega^2)^{n-1}$ is modified to $B_{n-1}(\omega^2)$ where

$$B_{n-1}(\omega^2) = (\omega^2 + \beta_1^2)(\omega^2 + \frac{1}{\beta_1^2})(\omega^2 + \beta_3^2)(\omega^2 + \frac{1}{\beta_3^2}) \cdots$$

$$\cdots (\omega^2 + \beta_{n-2}^2)(\omega^2 + \frac{1}{\beta_{n-2}^2}) \qquad \text{(II-3)}$$

This change has the effect of multiplying $f^1(\omega^2)$ by

$\frac{(1+\omega^2)^{n-1}}{\beta_{n-1}(\omega^2)} = \frac{1}{f_2(\omega^2)}$ which has the value unity at $\omega = 0$ and

$\omega = \infty$ and is smallest at $\omega = 1$. It is then necessary to precorrect $f^1(\omega^2)$ in order to avoid the effect of $f_2(\omega^2)$. To make the correction one transforms $f_2(\omega^2)$ into $f_2(\emptyset)$. It can be shown by algebraic manipulation and the transform $(\omega + \frac{1}{\omega}) = \frac{2}{\sin \emptyset}$ that

$$\frac{(1+\omega^2)^2}{(\omega^2+\beta_u^2)(\omega^2+\frac{1}{\beta_u^2})} \longrightarrow \frac{1}{1+\frac{m_u^2}{2}(1-\cos 2\emptyset)} \text{ where } \frac{m_u^2}{2} = \frac{1}{8}\left(\beta_u^2 + \frac{1}{\beta_u^2} - 2\right)$$

Therefore:

$$f_2(\emptyset) = \left[1 + \frac{m_1^2}{2}(1-\cos 2\emptyset)\right] \cdots \left[1 + \frac{m_{n-2}^2}{2}(1-\cos 2\emptyset)\right] \qquad \text{(II-4)}$$

The approximation $f^1(\emptyset) \cong F^1(\emptyset)$ is now modified to

$$f(\emptyset) = \frac{f_1(\emptyset)}{f_2(\emptyset)} \cong F^1(\emptyset) \qquad \text{(II-5)}$$

where $f(\emptyset) \rightarrow f(\omega^2)$.

The number $(\frac{n-1}{2})$ of odd harmonics used to approximate $F^1(\emptyset)$ fixes the number (n) of poles of $f(\omega^2)$ and the number (n-1) of poles of $f_2(\omega^2)$. The values β_u^2 are selected so that the networks resulting from the synthesis procedure have desirable element values. With these values of β_u^2, $f_2(\emptyset)$

is formed. To obtain $f_1(\phi)$, the product $F^1(\phi) \times f_2(\phi)$ is plotted as indicated.

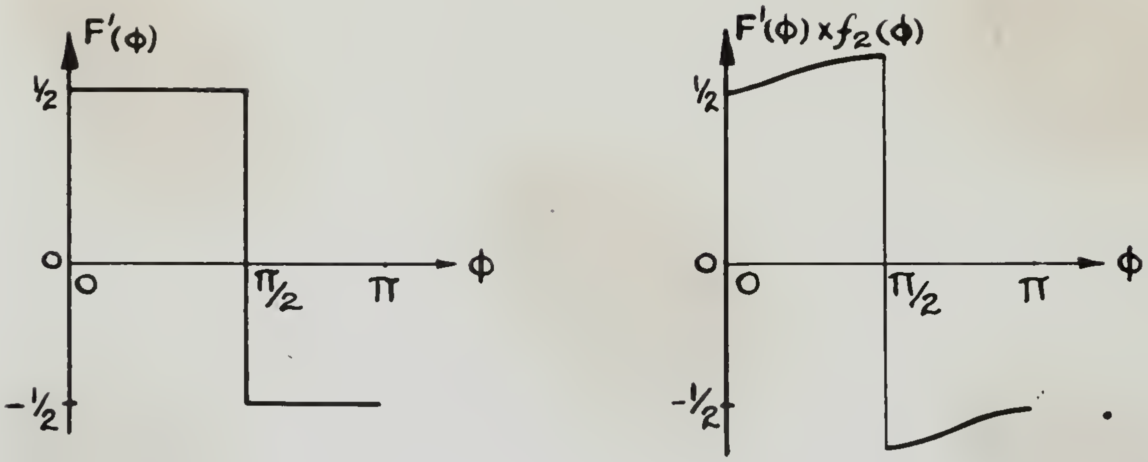


FIG II-3

In order that the extrema of $f_1(\phi)/f_2(\phi)$, in the range $0 < \phi < \pi$, shall deviate from $F^1(\phi)$ by $\pm\epsilon$, it is necessary that $f_1(\phi)$ approximate $F^1(\phi) \times f_2(\phi)$ in the same range in such a manner that points of inflection have the values $f_2(\phi)\left[1/2 \pm\epsilon\right]$. Two curves, A and B (Fig. II-8), are plotted having the values $f_2(\phi)\left[1/2 \pm\epsilon\right]$ respectively, and by a cut and try process $f_1(\phi)$ is obtained such that it approximates $F^1(\phi) \times f_2(\phi)$ within the limits of these two curves. This gives

$$f_1(\phi) = a_1 \cos \phi + a_3 \cos 3\phi + \dots + a_n \cos n\phi$$

The choise of the separation of the poles of $f^1(\omega^2)$ such that the product of pairs of poles is unity is now apparent.

It can be shown that only if this is true is $f_2(\phi)$ geometrically even about $\phi = \pi/2$ and only under this condition will $f_1(\phi)$ be odd about $\phi = \pi/2$, a necessary condition to permit corresponding Tschebyscheff behavior in both the attenuation and pass bands.

A constant $a_0 = 1/2$ is added to $f(\phi)$ to obtain the following desired result:

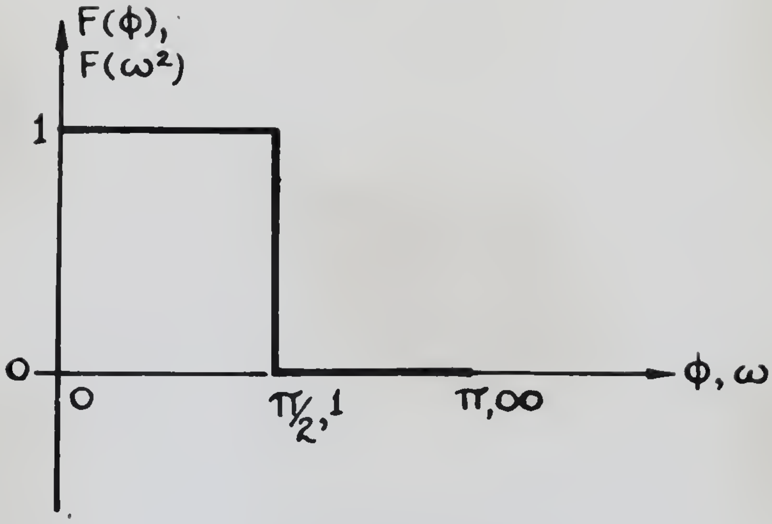


FIG II-4

Two procedures can now be followed in obtaining the rational transfer function in factored form.

<u>Procedure I</u>

$$\text{Form } f(\phi) = \frac{f_1(\phi)}{f_2(\phi)}$$

$$\text{Form } F(\phi) = 1/2 + f(\phi) = \frac{1/2\ f_2(\phi) + f_1(\phi)}{f_2(\phi)}$$

By means of relation II-2 obtain

$$F(\emptyset) = \frac{A_0 + A_1 \cos \emptyset + A_2 \cos^2 \emptyset + A_3 \cos^3 \emptyset + \dots + A_n \cos^n \emptyset}{f_2(\emptyset)} =$$

$$= \frac{f_1^A(\emptyset)}{f_2(\emptyset)}$$

Now factor $f_1^A(\emptyset)$ into its roots

$$\cos \emptyset_1 = C_1; \ \cos \emptyset_2 = C_2; \ \dots; \ \cos \emptyset_n = C_n$$

By means of the relation $\omega_u^2 = \frac{1 - \cos \emptyset_u}{1 + \cos \emptyset_u}$ obtain $\omega_1^2; \ \omega_2^2; \ \dots \ \omega_n^2$

Now:

$$f_1^A(\emptyset) \rightarrow \frac{P(\omega^2)}{(1 + \omega^2)^n}$$

$$f_2(\emptyset) \rightarrow \prod_{u=1}^{u=n-2} \left[\frac{(\omega^2 + \beta_u^2)(\omega^2 + \frac{1}{\beta_u^2})}{(1 + \omega^2)^2} \right]$$

Then:

$$\frac{f_1^A(\emptyset)}{f_2(\emptyset)} \rightarrow F(\omega^2) = \frac{(\omega^2 - \omega_1^2)(\omega^2 - \omega_2^2) \dots (\omega^2 - \omega_n^2)}{(\omega^2 + \beta_1^2)(\omega^2 + \frac{1}{\beta_1^2}) \dots (\omega^2 + 1) \dots (\omega^2 + \beta_{n-2}^2)(\omega^2 + \frac{1}{\beta_{n-2}^2})}$$

<u>Procedure II</u>

Form $f(\emptyset) = \frac{f_1(\emptyset)}{f_2(\emptyset)} = \frac{B_1 \cos \emptyset + B_3 \cos^3 \emptyset + \dots + B_n \cos^n \emptyset}{f_2(\emptyset)} = \frac{f_1^B(\emptyset)}{f_2(\emptyset)}$

By means of the relation $\cos^u \phi = \dfrac{(1-\omega^2)^u}{(1+\omega^2)^u}$

$$\frac{f_1^B(\phi)}{f_2(\phi)} \rightarrow f(\omega^2) = \frac{D_{2n}\omega^{2n} + D_{2n-2}\omega^{2n-2} + \dots + D_0}{(\omega^2+\beta_1^2)(\omega^2+\frac{1}{\beta_1^2})\dots(\omega^2+1)\dots(\omega^2+\frac{1}{\beta_{n-2}^2})(\omega^2+\beta_{n-2}^2)}$$

Then form and factor into its roots $F(\omega^2) = 1/2 + f(\omega^2)$.

An examination of both procedures reveals that less computational labor is involved if Procedure I is followed. In either case, great accuracy is required. This is particularly true in Procedure I because of the transform $\omega = \tan \phi/2$.

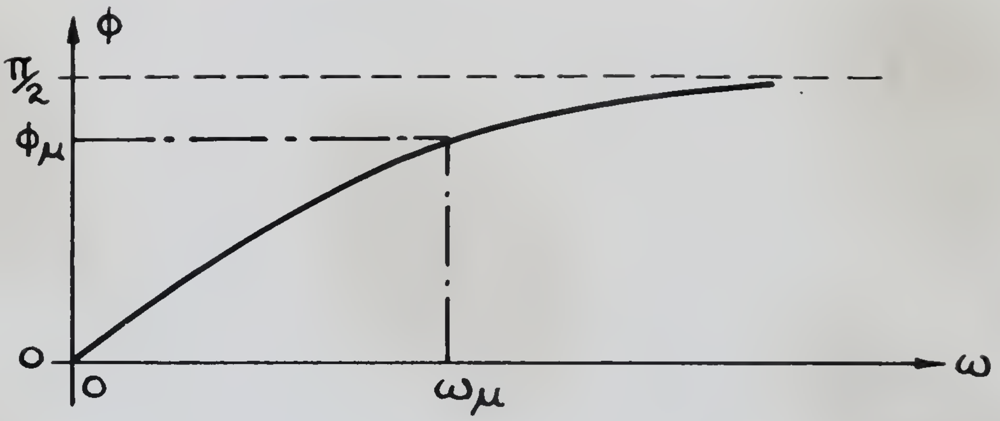


FIG II -5

It can be seen that a small error in determining the value of ϕ_u may give a large error in the value of ω_u.

A modification of Procedure I is possible so that one always factors a polynomial in ω^2 rather than in $\cos\phi$. In the polynomial $f_1^A(\phi)$ make the substitution $\cos^u(\phi) = \dfrac{(1-\omega^2)^u}{(1+\omega^2)^u}$, express $f_1^A(\phi)$ as $\dfrac{P(\omega^2)}{(1+\omega^2)^n}$ and factor $P(\omega^2)$.

Having obtained the rational transfer function $F(\omega^2)$ in factored form, it is now squared and one makes the identification

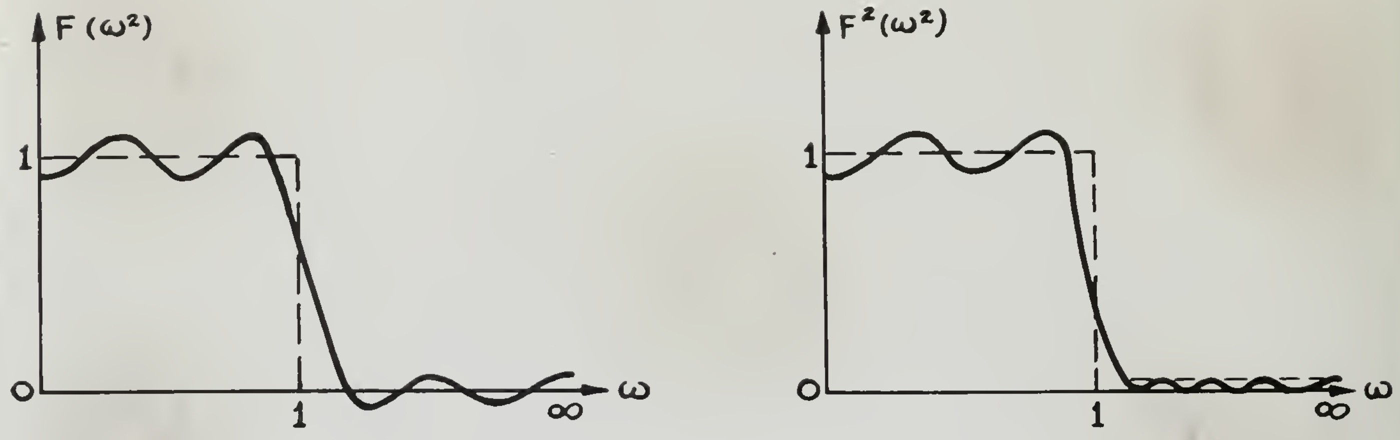


FIG II-6

$|Y_{12}(\lambda)|^2 = F^2(\omega^2)$ [Since $|Y_{12}|^2$ is always positive, one sees the necessity for squaring $F(\omega^2)$.]

$$\text{Now } Y_{12}(\lambda) = \frac{p(\lambda)}{q(\lambda)} = \frac{p_o + p_1\lambda + p_2\lambda^2 + \dots + p_n\lambda^n}{q_o + q_1\lambda + q_2\lambda^2 + \dots + q_n\lambda^n} = \frac{m_1(\lambda) + n_1(\lambda)}{m_2(\lambda) + n_2(\lambda)}$$

where m_1 and m_2 are the even parts of p and q, and n_1 and n_2 are the odd parts. Then $p(-\lambda) = m_1 - n_1$ and $q(-\lambda) = m_2 - n_2$. Therefore, for pure imaginary values of (λ), one has

$$|Y_{12}(\lambda)|^2_{\lambda=j\omega} = \left(\frac{m_1^2 - n_1^2}{m_2^2 - n_2^2}\right)_{\lambda=j\omega} = F^2(\omega^2).$$

The problem of finding $p_o \dots p_n$ and $q_o \dots q_n$ from $F^2(\omega^2)$ may be done easily in the manner of Gewertz[6] as follows.

Substitute $-\lambda^2 = \omega^2$ in $F^2(\omega^2)$. Since $F^2(\omega^2)$ is already in factored form, this gives the λ^2-roots of $(m_1^2 - n_1^2)$ and $(m_2^2 - n_2^2)$; the λ-roots are therefore obtained by inspection. To form $Y_{12}(\lambda)$, allot the left-half-plane λ-roots of $(m_1^2 - n_1^2)$ and $(m_2^2 - n_2^2)$ to $p(\lambda)$ and $q(\lambda)$, respectively.

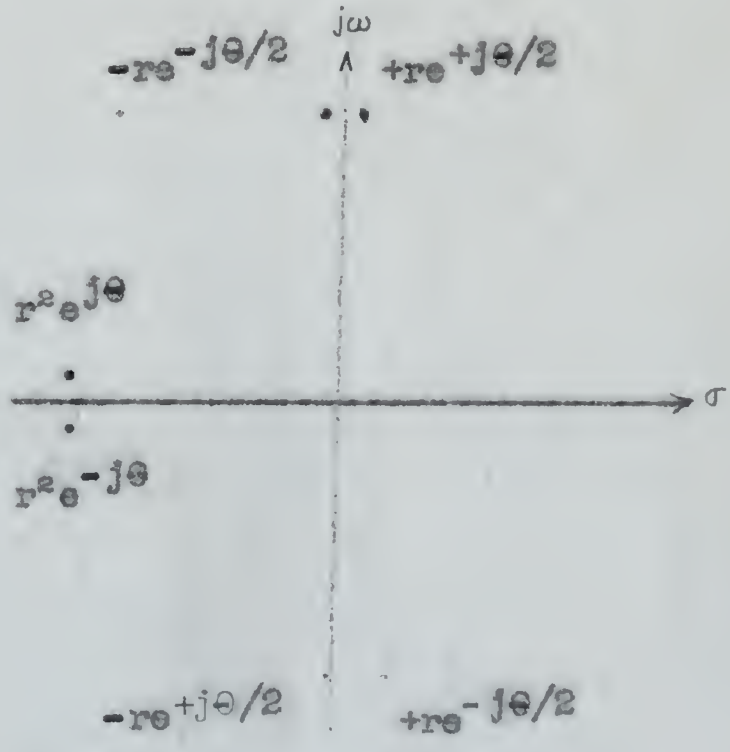
As an illustration of this method, an example consisting of the three possible types of λ^2-roots follows:

$$\text{Given; } (m_1^2 - n_1^2) = \underset{\text{I}}{(\omega^4 + C\omega^2 + D)^2}\,\underset{\text{II}}{(\omega^2 + B^2)^2}\,\underset{\text{III}}{(\omega^2 - A^2)^2}$$

I $\qquad (\lambda^4 - C\lambda^2 + D)(\lambda^4 - C\lambda^2 + D) = 0$

$$\lambda^2 = r^2 e^{\pm j\theta}; \quad r^2 e^{\pm j\theta}$$

$$\lambda = \pm\, r e^{\pm j\theta/2}; \ \pm\, r e^{\pm j\theta/2}$$

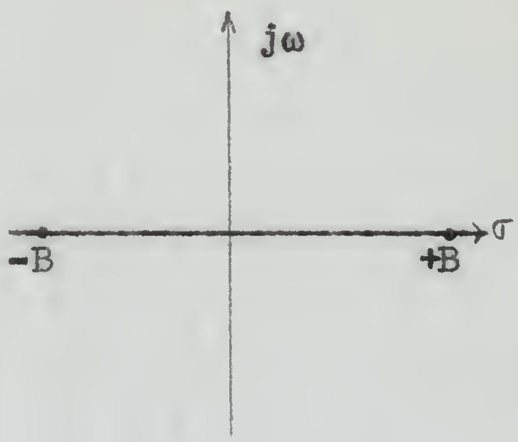


Hence $(\omega^4 + C\omega^2 + D)^2 \longrightarrow \left[(\lambda + re^{j\theta/2})(\lambda + re^{-j\theta/2})\right]^2 =$

$$= \left[\lambda^2 + (e^{j\theta/2} + e^{-j\theta/2})r\lambda + r^2\right]^2$$

II $\quad (-\lambda^2 + B^2)(-\lambda^2 + B^2) = 0$

$$\lambda = \pm B; \ \pm B$$

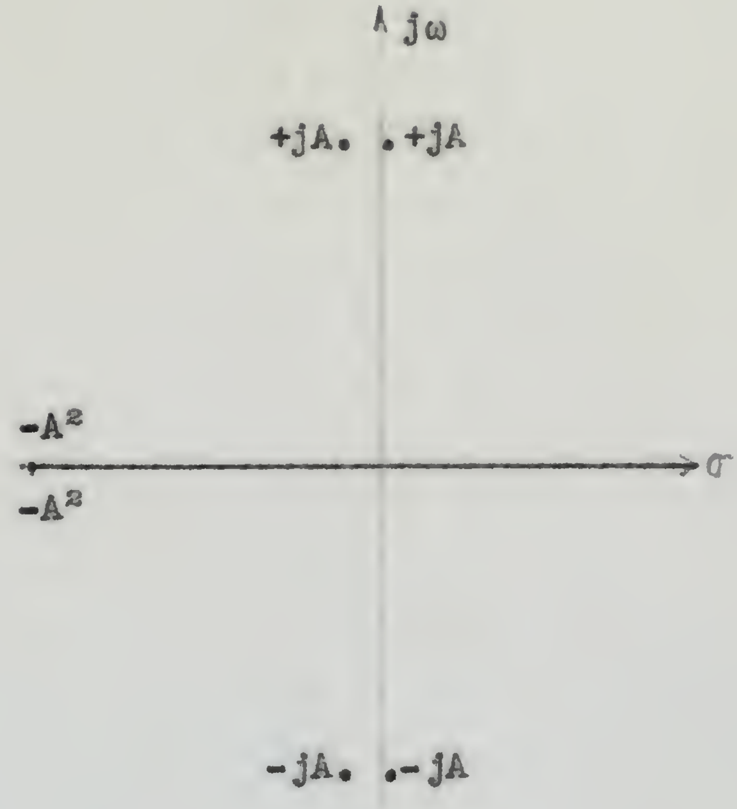


Hence $(\omega^2 + B^2)^2 \longrightarrow (\lambda + B)^2$

III $\quad (-\lambda^2 - A^2)(-\lambda^2 - A^2) = 0$

$$\lambda^2 = -A^2; \quad -A^2$$

$$\lambda = \pm jA; \ \pm jA.$$



Hence $(\omega^2 - A^2)^2 \longrightarrow (\lambda + jA)(\lambda - jA) = \lambda^2 + A^2$, and it is seen that III is a limiting case of I where $r^2e^{j\theta}$ coincides with $r^2e^{-j\theta}$. This also shows that negative real λ^2-roots must be of even multiplicity, which corresponds to positive real ω^2-roots being of even multiplicity. From the above one can now write: $m_1 + n_1 = (\omega^2 + E\lambda + F)^2(\lambda + B)^2(\lambda^2 + A^2)$

$$\text{where } E = \sqrt{C + 2\sqrt{D}}$$
$$F = \sqrt{D}$$

An alternative method naturally presents itself. If a constant (ϵ) is added to $F(\omega^2)$, $F'(\omega^2)$ is obtained as shown below.

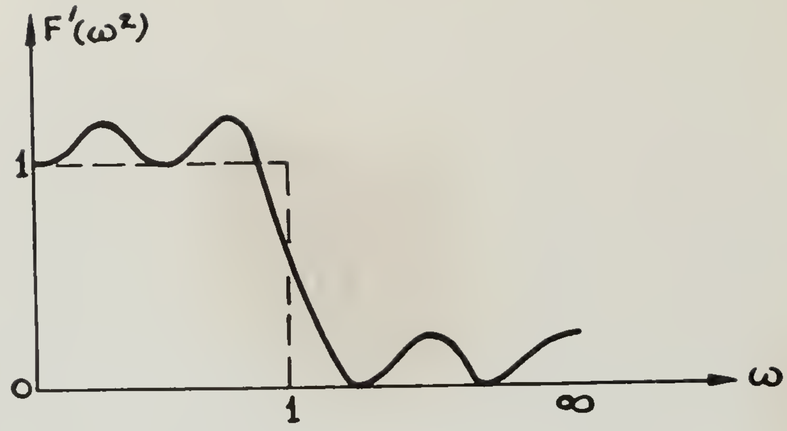


FIG II-7

The identification

$$\left| Y_{12}(\lambda) \right|^2_{\lambda=j\omega} = F'(\omega^2)$$

can then be made, since $F'(\omega^2)$ is always positive. One then proceeds to find $Y_{12}(\lambda)$ as before. It must be realized that in the first case $\left| Y_{12}(\lambda) \right|_{\lambda=j\omega} = \left| F(\omega^2) \right|$ while in the alternative case $\left| Y_{12}(\lambda) \right|_{\lambda=j\omega} = \sqrt{F'(\omega^2)}$.

Example:

The ideal transfer admittance is to be approximated with a trigonometric polynomial of third degree and to a tolerance of 0.07.

Select $\beta_1 = 5/8$, then;

$$f_2(\omega^2) = \frac{(\omega^2 + \beta_1^2)(\omega^2 + 1/\beta_1^2)}{(\omega^2 + 1)^2} = \frac{(\omega^2 + 0.3906)(\omega^2 + 2.560)}{(\omega^2 + 1)^2}$$

$$f_2(\emptyset) = 1.11883 - 0.11883 \cos 2\emptyset$$

ϕ	$f_2(\phi)$	$F^1(\phi)$ x $f_2(\phi)$	ϵ x $f_2(\phi)$	A $f_2(\phi)$ x $[1/2 + \epsilon]$	B $f_2(\phi)$ x $[1/2 - \epsilon]$
0°	1.00000	0.500	0.070	0.570	0.430
20°	1.02781	0.514	0.072	0.586	0.442
40°	1.09815	0.549	0.077	0.626	0.472
50°	1.13951	0.570	0.080	0.650	0.490
70°	1.20985	0.605	0.085	0.690	0.520
90°	1.23766	0.619	0.087	0.706	0.532

$F^1(\phi)$ x $f_2(\phi)$ and the curves A and B are plotted in Fig. II-8. By trial $f_1(\phi) = 0.6637 \cos \phi - 0.2337 \cos 3\phi$ is obtained which approximates $F^1(\phi)$ x $f_2(\phi)$ as shown in this same figure.

$$F(\phi) = \frac{1}{2} + \frac{f_1(\phi)}{f_2(\phi)} = \frac{\left[\frac{1}{2}\ 1.11883 - 0.11883 \cos 2\phi\right] + 0.6637 \cos \phi - 0.2337 \cos 3\phi}{f_2(\phi)}$$

$$F(\phi) = \frac{\frac{1}{2}\left[1.11883 - 0.11883\ (2\cos^2\phi - 1)\right] + 0.6637 \cos\phi - 0.2337(4\ \cos^3\phi - 3\ \cos\phi)}{f_2(\phi)}$$

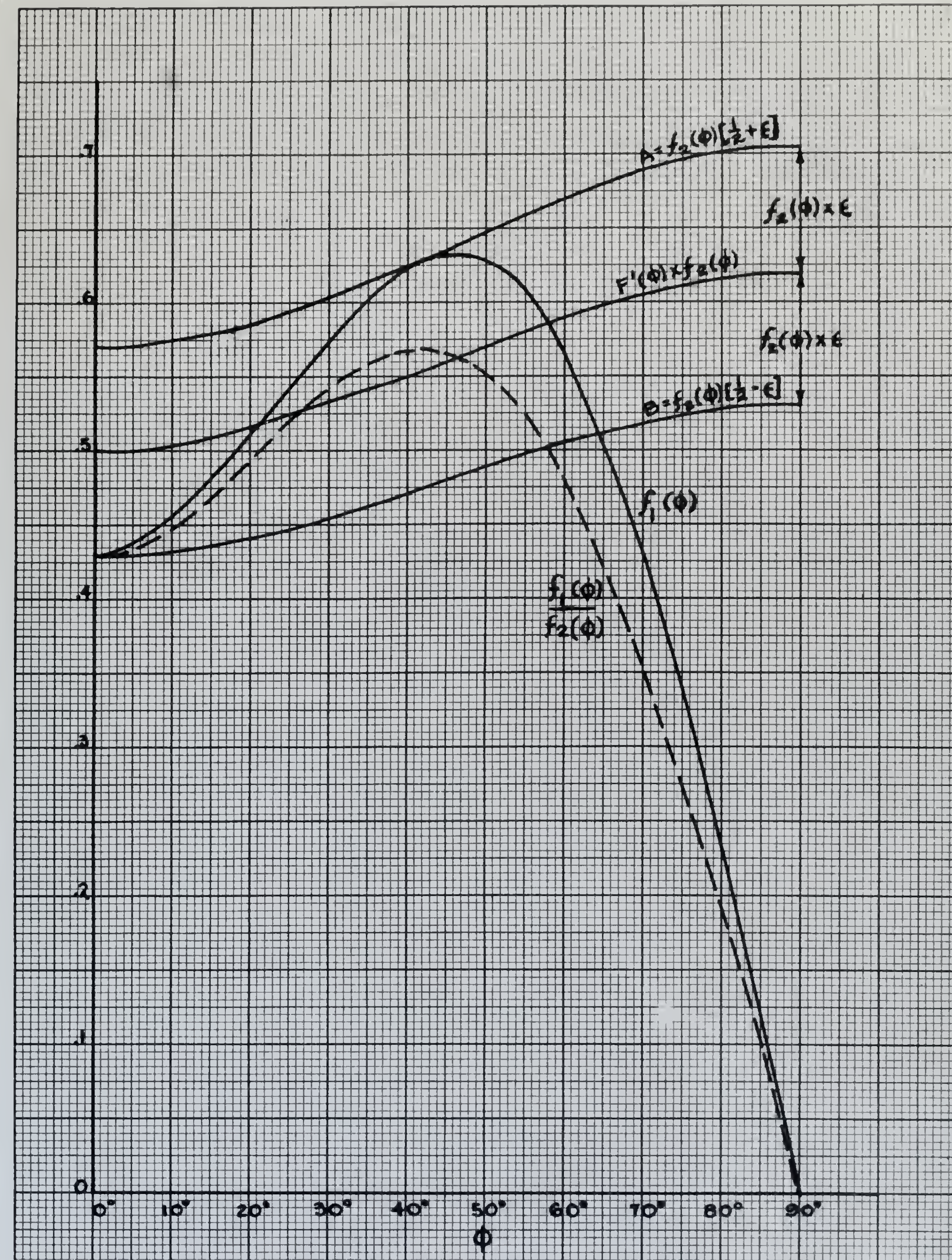
$$F(\phi) = \frac{-0.9348\cos^3\phi - 0.11883\cos^2\phi + 1.3648\cos\phi + 0.61883}{f_2(\phi)} = \frac{f_1^A(\phi)}{f_2(\phi)}$$

The roots of $f_1^A(\phi)$ are;

$\cos \phi_1 = -0.93153$; $\cos \phi_2 = -0.5318$; $\cos \phi_3 = 1.33624$

$$F(\omega^2) = \frac{(\omega^2 - 28.21)(\omega^2 - 3.272)(\omega^2 + 0.14392)}{(\omega^2 + 0.3906)(\omega^2 + 1)(\omega^2 + 2.56)}$$

$$Y_{12}(\lambda) = \frac{(\lambda^2 + 28.21)(\lambda^2 + 3.272)(\lambda + 0.379)^2}{(\lambda + 0.625)^2(\lambda + 1)^2(\lambda + 1.6)^2}$$



EXAMPLE OF APPROXIMATION PROCEDURE

FIG II-8

CHAPTER III

THE NETWORK REALIZATION

The preceding chapter has given the rational transfer admittance function, $Y_{12}(\lambda)$, expressed as a quotient of two finite polynomials of equal degree. As the third step in the general synthesis procedure, it now becomes necessary to synthesize a network or combination of networks which will realize this $Y_{12}(\lambda)$ function. Two steps are necessary:

(A) To divide the $Y_{12}(\lambda)$ function into realizable factors.

(B) To develop a procedure for synthesizing RC-networks to realize the individual factors.

A. The Division of $Y_{12}(\lambda)$ into Factors.

From Chapter II we may write the general form of $\left|Y_{12}\right|^2$ as:

(1) When $\frac{n+1}{2}$ is an even integer: (n is degree of trigonometric approximating polynomial)

$$\left|Y_{12}\right|^2 = \left[\frac{(\omega^2 + A^2)(\omega^2 - B_1)\ldots(\omega^2 - B_{\frac{n+1}{2}})(\omega^4 + C_1\omega^2 + D_1)\ldots}{(\omega^2 + \beta_1^2)(\omega^2 + \frac{1}{\beta_1^2})(\omega^2 + \beta_3^2)\ldots(\omega^2 + 1)\ldots}\right.$$

$$\left.\frac{\ldots(\omega^4 + C_{\frac{n-3}{4}}\,\omega^2 + D_{\frac{n-3}{4}})}{\ldots(\omega^2 + \beta_{n-2}^2)(\omega^2 + \frac{1}{\beta_{n-2}^2})}\right]^2 \qquad \text{(III-1}$$

(2) When $\frac{n+1}{2}$ is an odd integer:

$$\left|Y_{12}\right|^2 = \left[\frac{(\omega^2-\beta_1)\ldots(\omega^2-B_{\frac{n+1}{2}})(\omega^4+C_1\omega^2+D_1)\ldots(\omega^4+C_{\frac{n-1}{4}}\omega^2+D_{\frac{n-1}{4}})}{(\omega^2+\beta_1^2)(\omega^2+\frac{1}{\beta_1^2})\ldots(\omega^2+1)\ldots(\omega^2+\beta_{n-2}^2)(\omega^2+\frac{1}{\beta_{n-2}^2})}\right]^2$$

(III-2)

Now, according to the method outlined in Chapter II, one obtains:

(1')

$$Y_{12}(\lambda) = \left[\frac{(\lambda+A)^2(\lambda^2+B_1)\ldots(\lambda^2+B_{\frac{n+1}{2}})(\lambda^2+E_1\lambda+F_1)^2\ldots}{(\lambda+\beta_1)^2(\lambda+\frac{1}{\beta_1})^2\ldots(\lambda+1)^2\ldots}\right.$$

$$\left.\frac{\ldots(\lambda^2+E_{\frac{n-3}{4}}\lambda+F_{\frac{n-3}{4}})^2}{\ldots(\lambda+\beta_{n-2})^2(\lambda+\frac{1}{\beta_{n-2}})^2}\right] \qquad \text{(III-3)}$$

Where $F_1 = \sqrt{D_1}$
$E_1 = \sqrt{C_1 + 2\sqrt{D_1}}$

(2')

$$Y_{12}(\lambda) = \left[\frac{(\lambda^2+B_1)\ldots(\lambda^2+B_{\frac{n+1}{2}})(\lambda^2+E_1\lambda+F_1)^2\ldots(\lambda^2+E_{\frac{n-1}{4}}\lambda+F_{\frac{n-1}{4}})^2}{(\lambda+\beta_1)^2(\lambda+\frac{1}{\beta_1})^2\ldots(\lambda+1)^2\ldots(\lambda+\beta_{n-2})^2(\lambda+\frac{1}{\beta_{n-2}})^2}\right]$$

(III-4)

From these general formulae for $Y_{12}(\lambda)$ it is obvious that one must divide $Y_{12}(\lambda)$ into a product of at least two transfer functions, since for physical realizability of an RC-network, the poles of the transfer admittance function must be simple.

For case (1'), a symmetrical division gives:

$$Y_{12}(\lambda) = Y_{12}^{(S_1)}(\lambda) \times Y_{12}^{(S_2)}(\lambda)$$

$$Y_{12}^{(S_1)}(\lambda) = \frac{p^{(S_1)}(\lambda)}{(\lambda+\beta_1)(\lambda+\frac{1}{\beta_1})\ldots(\lambda+1)\ldots(\lambda+\beta_{n-2})(\lambda+\frac{1}{\beta_{n-2}})} \qquad \text{(III-5)}$$

$$Y_{12}^{(S_2)}(\lambda) = \frac{p^{(S_2)}(\lambda)}{(\lambda+\beta_1)(\lambda+\frac{1}{\beta_1})\ldots(\lambda+1)\ldots(\lambda+\beta_{n-2})(\lambda+\frac{1}{\beta_{n-2}})}$$

For case (2'), a symmetrical division is not possible. However, one can have:

$$Y_{12}(\lambda) = Y_{12}^{(S_1)}(\lambda) \times Y_{12}^{(S_2)}(\lambda) \times \ldots \times Y_{12}^{(S_k)}(\lambda) \qquad \text{(III-6)}$$

As before, it is necessary that $Y_{12}^{(S_u)}(\lambda)$ have only simple poles. Furthermore, as a result both of the approximation procedure used and of the synthesis procedure to be developed in the next section, the numerator and denominator polynomials of $Y_{12}^{(S_u)}(\lambda)$ should be of the same degree.

The statement made in Chapter II that multiple poles could be accepted in the overall transfer function now becomes apparent. $Y_{12}^{(S_u)}(\lambda)$ must have only simple poles; $Y_{12}(\lambda)$, however, may have poles of any even multiplicity up to k.

The group of networks obtained which realize $Y_{12}^{(S_u)}(\lambda)$ is now termed a stage of the filter. By a cascade of stages separated by an amplifier (necessary because of the constant attenuation in the stage) and a cathode-follower to present an impedanceless source to the succeeding stage, the overall function, $Y_{12}(\lambda)$, is realized.

B. The Basic Synthesis Procedure

The basic procedure for synthesizing RC-networks from a transfer admittance function, $Y_{12}^{(S_u)}(\lambda)$, expressed as a rational quotient of finite polynomials of the same degree, having no multiple poles, is contained in Guillemin's paper, "Procedure for Synthesizing RC-Networks."[7] Excerpts, in a summarized form, from this paper follow.

Given:

$$Y_{12}^{(S_u)}(\lambda) = \frac{p(\lambda)}{q(\lambda)} = \frac{a_0 + a_1\lambda + a_2\lambda^2 + \dots + a_m\lambda^m}{(\lambda+\beta_1)(\lambda+\beta_2)\dots(\lambda+\beta_m)}\,;\; 0<\beta_1<\beta_2<\dots<\beta_m.$$

Form the polynomial:

$$q_1(\lambda) = A(\lambda+\alpha_1)(\lambda+\alpha_2)\dots(\lambda+\alpha_m);\quad (\alpha_u > 0)$$

in which

$$\beta_1<\alpha_1<\beta_2<\alpha_2<\dots<\alpha_{m-1}<\beta_m<\alpha_m.$$

Choose a nonzero value for A such that

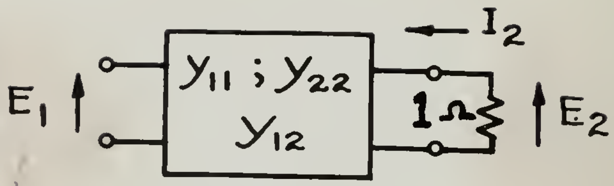
$$A\,\alpha_1\,\alpha_2\dots\alpha_m < \beta_1\,\beta_2\dots\beta_m\,.$$

Next form $q_2(\lambda) = q(\lambda) - q_1(\lambda)$. Then write

$$Y_{12}^{(S_u)} = \frac{p/q_1}{1 + \frac{q_2}{q_1}} = \frac{y_{12}}{1 + y_{22}},$$

and make the identifications

$$y_{12} = \frac{p}{q_1}, \quad y_{22} = \frac{q_2}{q_1}.$$



$$Y_{12}^{(S\mu)} = \frac{I_2}{E_1} = -\frac{E_2}{E_1}$$

FIG III-1

Write:

$$\frac{1}{y_{22}} = \frac{d_m\lambda^m + d_{m-1}\lambda^{m-1} + \dots + d_1\lambda + d_0}{h_m\lambda^m + h_{m-1}\lambda^{m-1} + \dots + h_1\lambda + h_0} = \frac{q_1}{q_2}$$

$$= r_1 + \frac{d'_{m-1}\lambda^{m-1} + d'_{m-2}\lambda^{m-2} + \dots + d'_1\lambda + d'_0}{h_m\lambda^m + h_{m-1}\lambda^{m-1} + \dots + h_1\lambda + h_0} = r_1 + \frac{q'_1}{q_2} = r_1 + \frac{1}{\frac{q_2}{q'_1}}.$$

By repeating this process, the following continued fraction expansion is obtained:

$$\frac{1}{y_{22}} = r_1 + \cfrac{1}{c_1\lambda + \cfrac{1}{r_2 + \cfrac{1}{c_2\lambda + \cdots + \cfrac{1}{r_m + \cfrac{1}{c_m\lambda + \cfrac{1}{r_{m+1}}}}}}}.$$

To this form for y_{22} there corresponds a ladder network as shown below:

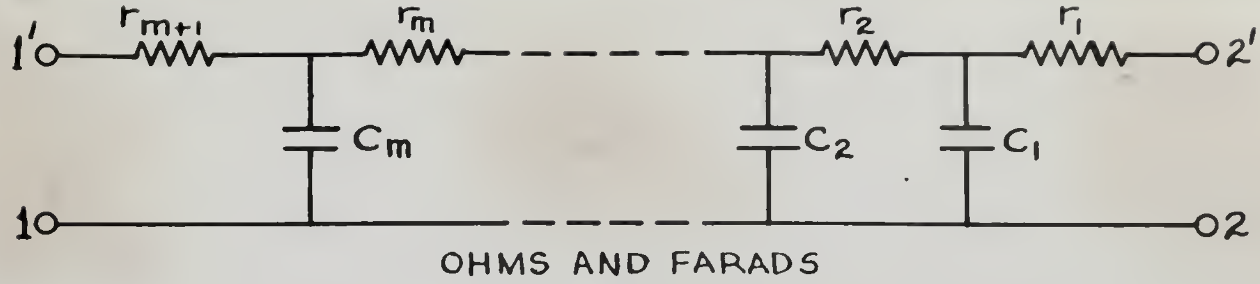


FIG III - 2

This network has the short-circuit driving-point admittance y_{22} and the short circuit transfer admittance $y_{12}^{(o)} = A_o/q_1$ in which A_o is a positive real constant. By considering the admittance of the network as λ approaches zero, it is obvious that $A_o = \dfrac{d_o}{r_1 + r_2 + r_3 \ldots + r_m + r_{m+1}}$

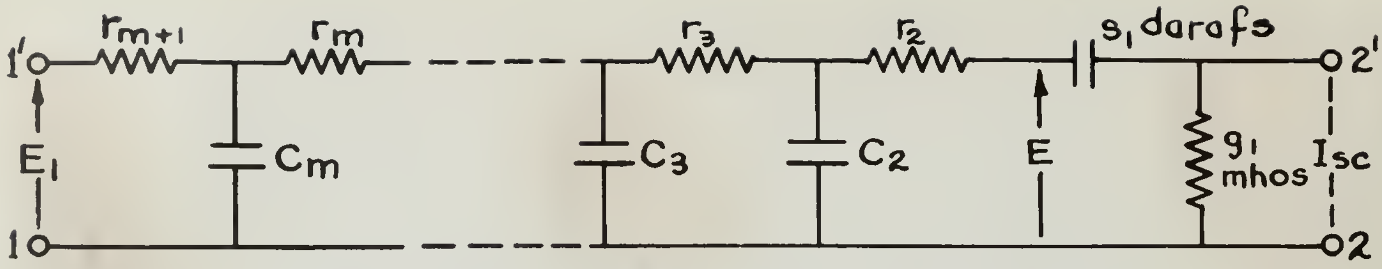
Now consider again the function y_{22} in the form:

$$y_{22} = \frac{q_2(\lambda)}{q_1(\lambda)} = \frac{h_o + h_1\lambda + \ldots h_m\lambda^m}{d_o + d_1\lambda + \ldots d_m\lambda^m} = g_1 + \cfrac{1}{\cfrac{q_1}{q_2'}} = g_1 + \cfrac{1}{s_1\lambda^{-1} + \cfrac{q_1'}{q_2'}}$$

At this point one reverts to the procedure for the first network, finally obtaining:

$$y_{22} = g_1 + \cfrac{1}{s_1\lambda^{-1} + r_2 + \cfrac{1}{c_2\lambda + \cfrac{\cdot}{\cdot \; \cdot + \cfrac{1}{r_m + \cfrac{1}{c_m\lambda + \cfrac{1}{r_{m+1}}}}}}},$$

to which there corresponds the network:



OHMS AND FARADS EXCEPT AS NOTED

FIG III-3

The value of r's and c's are, of course, different from those obtained for the previous network. The present network has the short-circuit driving-point admittance y_{22} and the short-circuit transfer admittance $y_{12}^{(1)} = A_1\lambda/q_1$ in which A_1 is a positive real constant. A_1 again is determined by letting λ approach zero. If E_1 is applied, the impedances of the shunt capacitance branches are so large compared with the series resistances that E_1 approaches E. Then the short-circuit current

$$I_{sc} = \frac{E\lambda}{s_1} = \frac{E_1\lambda}{s_1} \text{ and}$$

$$y_{12} = \frac{I_{sc}}{E_1} = \frac{\lambda}{s_1} = \frac{A_1\lambda}{d_o} \text{ as } \lambda \text{ approaches 0.}$$

A_1 is therefore d_o G_1.

The next expansion of y_{22} follows the pattern of the previous one for <u>two</u> cycles and then finishes as in the first case. Continuing in this way, one finally obtains a network having the functions y_{22} and $y_{12}^{(m)} = \frac{A_m\lambda^m}{q_1}$. A_m is obtained by the same reasoning as used in evaluating A_o and A_1. This network has the form:

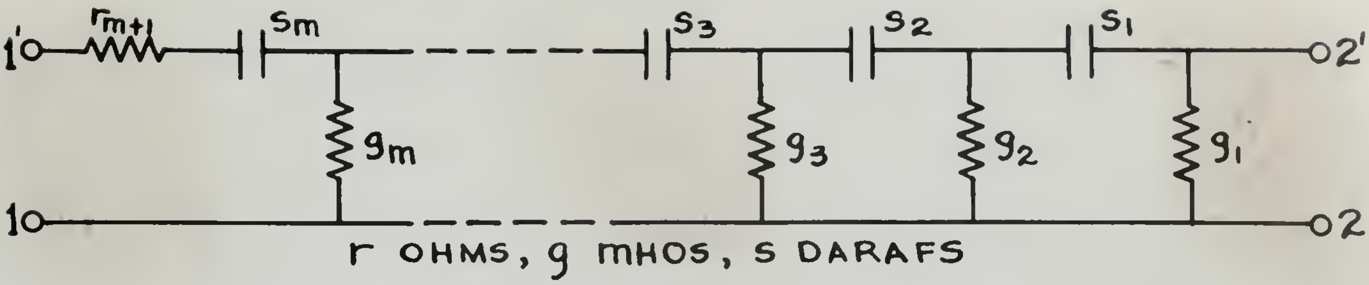


FIG III-4

Now it should be recalled that the y_{12}-function of the desired network is

$$y_{12} = \frac{p(\lambda)}{q_1(\lambda)} = \frac{a_o + a_1\lambda + a_2\lambda^2 + \ldots + a_m\lambda^m}{q_1(\lambda)} ,$$

or

$$y_{12} = \frac{a_o y_{12}^{(o)}}{A_o} + \frac{a_1 y_{12}^{(1)}}{A_1} + \ldots + \frac{a_m y_{12}^{(m)}}{A_m} .$$

If the admittance levels of the networks found for $y_{12}^{(o)}$, $y_{12}^{(1)}$, etc., are multiplied respectively by the factors a_o/A_o, a_1/A_1, etc., then it is clear that the subsequent parallel connection of these networks yields a resultant one having the desired y_{12}-function. However, the y_{22}-function of this resulting network will evidently be the desired y_{22} multiplied by the constant.

$$G = \frac{a_o}{A_o} + \frac{a_1}{A_1} + \ldots + \frac{a_m}{A_m} .$$

To avoid this result one should multiply the admittance levels of the networks found for $y_{12}^{(o)}$, $y_{12}^{(1)}$, ... etc., by the factors a_o/A_oG, a_1/A_1G, ... etc. respectively. The parallel connection of the resultant component networks then yields a network having the short-circuit driving-point admittance y_{22} and the short-circuit transfer admittancy y_{12}/G. The transfer admittance for a one-ohm load becomes $Y_{12}^{(S_u)}/G$, which is the desired function except for a constant multiplier.

It may be emphasized at this point that the fundamental networks are obtained solely from the denominator polynomial of $Y_{12}^{(S_u)}(\lambda)$; hence, with these networks any numerator polynomial can be synthesized.

C. The Modified Synthesis Procedure.

Guillemin has suggested a modification of the above method, to halve the number of networks required to realize a given transfer admittance function, by realizing two terms of the numerator polynomial of $Y_{12}^{(S_u)}(\lambda)$ in each network. This modified method is developed below.

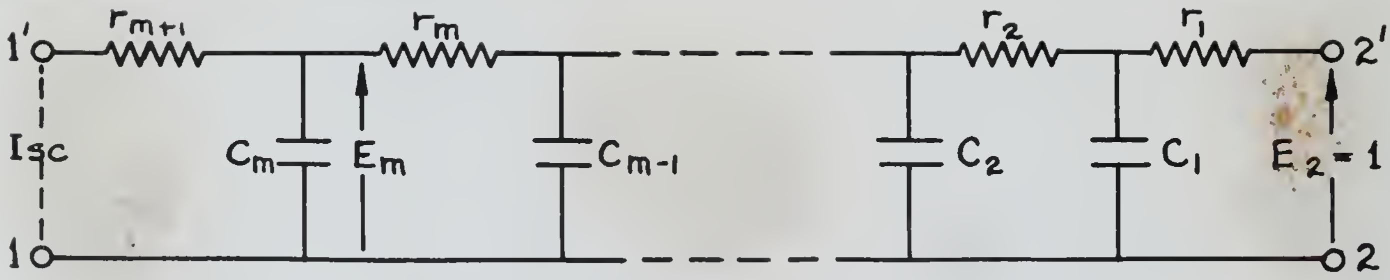


FIG III-5

Figure III-5 shows the first network developed, by the basic method. This network has the short-circuit driving-point admittance y_{22} and the short-circuit transfer admittance $y_{12}^{(o)} = A_o/q_1$. If a unit voltage at E_2 is applied:

$$y_{21}^{(o)} = y_{12}^{(o)} = A_o/q_1 = \frac{I_{sc}}{E_2} = I_{sc} = \frac{E_m}{r_{m+1}}$$

The elements at the short-circuited end of this network may be modified as shown in Fig. III-6 below without changing the short-circuit driving point admittance y_{22}, provided we keep:

$$g_a + g_b = g_{m+1}$$

$$C_a + C_b = C_m$$

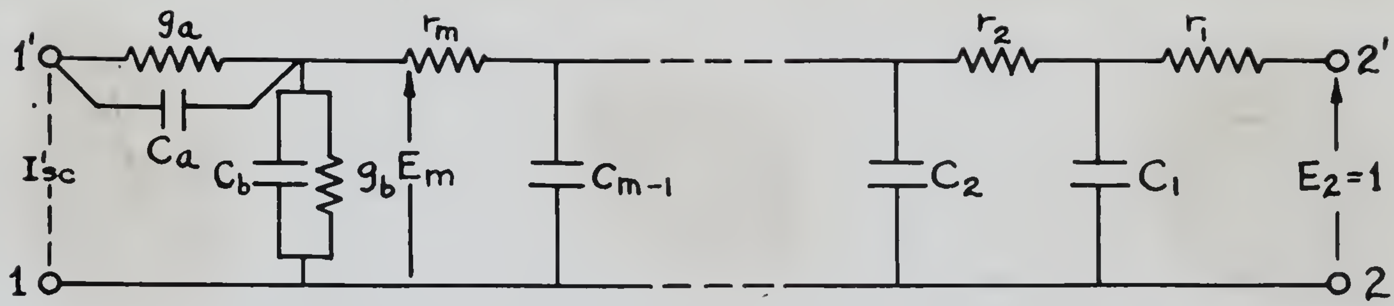


FIG III-6

The short-circuit transfer admittance, $y_{12}^{(o,1)}$, is now:

$$y_{21}^{(o,1)} = y_{12}^{(o,1)} = \frac{I'_{sc}}{E_2} = I'_{sc} = E_m(g_a + C_a\lambda)$$

Then

$$y_{12}^{(o,1)} = y_{12}^{(o)} \times \frac{(g_a + C_a\lambda)}{g_{m+1}} = A_o \left[\frac{g_a}{g_{m+1}}\right] \frac{(1 + \frac{C_a}{g_a}\lambda)}{q_1}$$

Now if $\frac{a_1}{a_o} = \frac{C_a}{g_a}$

then $$y_{12}(0,1) = \frac{\frac{A_o\, g_a}{a_o g_{m+1}}(a_o + a_1\lambda)}{q_1} = \frac{(\frac{1}{G_o})(a_o + a_1\lambda)}{q_1}$$

Similarly, $$y_{12}^{(2,3)} = \frac{(\frac{1}{G_2})(a_2\lambda^2 + a_3\lambda^3)}{q_1}, \text{ etc.}$$

As in the basic method, it is necessary, in order to maintain the correct y_{22}, to accept a constant multiplier in the short-circuit transfer admittance, i.e., $\frac{y_{12}}{G}$ where $G = G_o + G_2 + G_4 + \ldots$ $\ldots G_{m-1}$. Consequently, one multiplies the admittance levels of the modified networks found for $y_{12}^{(0,1)}$, $y_{12}^{(2,3)}$, ... etc., by the factors $\frac{G_o}{G}$, $\frac{G_2}{G}$ ... etc., respectively, and connects in parallel the resultant component networks to realize $\frac{Y_{12}^{(S_u)}(\lambda)}{G}$.

Actually, it is not necessary to separate both C_m and g_{m+1} into series and shunt branches. The three equations that must be satisfied are:

$$g_a + g_b = g_{m+1}$$

$$C_a + C_b = C_m$$

$$\frac{a_1}{a_o} = \frac{C_a}{g_a}$$

Now consider the ratio C_m/g_{m+1}. If $\frac{C_m}{g_{m+1}} = \frac{a_1}{a_o}$, select $C_a = C_m$, $C_b = 0$, $g_a = g_{m+1}$, $g_b = 0$.

If $\frac{C_m}{g_{m+1}} < \frac{a_1}{a_o}$ (the usual case), select $C_a = C_m$, $C_b = 0$,

$g_a = C_m \frac{a_o}{a_1}$, $g_b = g_{m+1} - g_a$. Finally, if $\frac{C_m}{g_{m+1}} \quad \frac{a_1}{a_o}$, select $g_a = g_{m+1}$, $g_b = 0$, $C_a = g_{m+1} \frac{a_1}{a_o}$, $C_b = C_m - C_a$.

Consequently, not more than one additional element, either C or R, is required for any component network.

It may be noted here that for the general <u>m-pole stage</u>, the maximum number of elements required, including the terminal resistance, is:

<u>m odd</u> $N_{e_{max}} = m^2 + 2m + 2$

of which not more than $\frac{m^2 + 3m + 4}{2}$ are resistances

and not more than $\frac{m^2 + 2m + 1}{2}$ are capacitances

<u>m even</u> $N_{e_{max}} = m^2 + 2m + 3$

of which not more than $\frac{m^2 + 4m + 4}{2}$ are resistances

and not more than $\frac{m^2 + 3m}{2}$ are capacitances

CHAPTER IV

PRACTICAL NETWORK DESIGN

Since the individual networks are already in the desirable[8] form of unbalanced-ladder structures, all that is necessary in the fourth step of the synthesis procedure is to insure that all final element values are physically realizable. To retain generality in the choice of cut-off frequency and impedance level, the overall spread of element values within each normalized stage must be controlled. Specifically, the physical limitation imposed is that in a single normalized stage the largest element value must not be more than approximately 10^5 times greater than the smallest element value. This overall spread is a function of various arbitrarily selected constants throughout the design procedure. Empirical criteria must therefore be developed to serve as guides in the design procedure so that the resultant networks can actually be built.

The factors affecting the spread of element values are listed in the order in which they occur during the design procedure as follows:

(1) The initial decision as to the attenuation and pass-band-width requirements, which fixes the number of poles of the overall transfer admittance function, $Y_{12}(\lambda)$.

(2) The selection of the number of stages to be used in obtaining $Y_{12}(\lambda)$.

Since $Y_{12}(\lambda) = Y_{12}^{(S_1)}(\lambda) \times Y_{12}^{(S_2)}(\lambda) \times \ldots$
$\ldots Y_{12}^{(S_u)}(\lambda) \ldots Y_{12}^{(S_k)}(\lambda)$, attention can now be confined to $Y_{12}^{(S_u)}(\lambda)$.

(3) The selection of the poles of $Y_{12}^{(S_u)}(\lambda)$.

(4) The selection of the poles of the short-circuit transfer admittance, $y_{12}^{(S_u)}(\lambda)$.

(5) The selection of the constant multiplier $A^{(S_u)}$ in forming $q_1^{(S_u)}(\lambda)$, which fixes the zeros of the short-circuit driving point admittance, $y_{22}^{(S_u)}(\lambda)$.

Stringent performance specifications require that a trigonometric polynomial of high degree be used in the approximation procedure. The degree (n) of this polynomial fixes the number (2n) of poles in $Y_{12}(\lambda)$ and hence determines the complexity of the resulting filter. For a minimum guaranteed attenuation of 20 to 30 decibets, a 6-pole filter will have a pass-band width of 65-60 per cent of the cut-off frequency, a 14-pole filter 81 to 76 per cent, and a 38-pole filter 95 to 94 per cent.

With n determined, an inspection of equations III-3 and III-4 shows the analytic form of $Y_{12}(\lambda)$ and indicates possible subdivisions into stages. The resultant spread of element values and the excessive gain required indicate a maximum practical limit of seven poles per stage. To reduce the number of amplifier sections, it is desirable to use a minimum number of stages. However, the use of more than this minimum number requires fewer total elements with a smaller overall spread. The process of synthesis is also greatly simplified because less complex networks are required.

To obtain a small spread of element values, the poles of $Y_{12}^{(S_u)}$ should be well separated. It is this factor which limits the maximum number of poles per stage to seven. For ease of approximation and to increase slightly the pass-band width, the poles should be close to unity. These opposing considerations both indicate the use of a minimum number of poles per stage. Consequently, the decisions on the number of stages and the number and location of poles in each stage must be made simultaneously, and are compromise selections.

An empirical procedure only can be given for locating the poles of $y_{12}^{(S_u)}(\lambda)$. Experience indicates that $\alpha_i^{(S_u)}$ should be slightly less than $\beta^{(S_u)}{}_{i+1}$ and that $\alpha_m^{(S_u)}$ should be of the order of four times $\beta_m^{(S_u)}$.

The selection of $A^{(S_u)}$ is the most important factor in determining the spread of element values. Since $A\alpha_1\alpha_2 \ldots \alpha_m < \beta_1\beta_2 \ldots \beta_m$, the allowable range of $A^{(S_u)}$ is limited. It is always possible within this range, however, to choose a value of A such that the elements in each component network oscillate about the value of R_1 and C_1, providing a minimum spread. The first selection of $A^{(S_u)}$ should be

$$A^{(S_u)} = \frac{1}{1 + (\alpha_1\alpha_2 \ldots \alpha_m)} \qquad \text{(IV-1)}$$

The polynomials $q_1^{(S_u)}(\lambda)$ and $q_2^{(S_u)}(\lambda)$ are now formed and the fundamental networks synthesized. If the resultant spread of element values is too great, the problem must be begun again, relocating the poles of $Y_{12}^{(S_u)}(\lambda)$ with greater separation. If the spread is satisfactory, the admittance levels of the fundamental networks are next corrected to synthesize $Y_{12}^{(S_u)}(\lambda)$. This process increases the spread of element values; consequently it may be necessary to adjust A slightly to reduce the spread or to lower the amplifier gain required.

CHAPTER V

DESIGN AND CONSTRUCTION OF A SIMPLE, LOW-PASS RC-FILTER

To serve as an illustrative example of the entire synthesis procedure, the steps in designing and constructing a simple RC-filter are detailed in the following sections. The filter was tested to conform closely to the predicted frequency characteristics.

Design.

It was desired that the filter: (1) be as simple in design and construction as possible; (2) possess Tschebyscheff behavior, with at least 23 decibels minimum guaranteed attenuation in the attenuation-band; and (3) have the maximum pass-band compatible with the required attenuation and the limitations of practical construction. The first specification is met by selecting a six-pole filter. Tschebyscheff behavior with the required attenuation is obtained by the given approximation procedure, with ϵ equal to 0.07. The pass-band width is largely fixed by these previous selections, but is slightly decreased from the maximum theoretically possible by the necessity of locating β_1 and $1/\beta_1$ (Fig. V-1) sufficiently far from unity that practical element values may be obtained.

The graphical procedure to obtain the approximating function is given as an example in Chapter II, and yields:

$$F^2(\omega^2) = \left[\frac{(\omega^2 - 28.21)(\omega^2 - 3.272)(\omega^2 + 0.1439)}{(\omega^2 + 0.3906)(\omega^2 + 1)(\omega^2 + 2.56)} \right]^2$$

from which:

$$Y_{12}(\lambda) = \frac{(\lambda^2 + 28.21)(\lambda^2 + 3.272)(\lambda + 0.379)^2}{(\lambda + 0.625)^2(\lambda + 1)^2(\lambda + 1.6)^2}$$

Separating into stages:

$$Y_{12}(\lambda) = Y_{12}^{(S_1)}(\lambda) \times Y_{12}^{(S_2)}(\lambda)$$

$$= \frac{(\lambda^2 + 3.272)(\lambda + 0.379)}{(\lambda + 0.625)(\lambda + 1)(\lambda + 1.6)} \times \frac{(\lambda^2 + 28.21)(\lambda + 0.379)}{(\lambda + 0.625)(\lambda + 1)(\lambda + 1.6)}$$

This symmetrical separation of $Y_{12}(\lambda)$ allows the same fundamental networks to be used for both stages.

To synthesize $Y_{12}^{(S_1)}(\lambda)$, the poles (α_1, α_2, and α_3) of the short-circuit transfer admittance, Y_{12}, are selected as shown in Fig. V-1

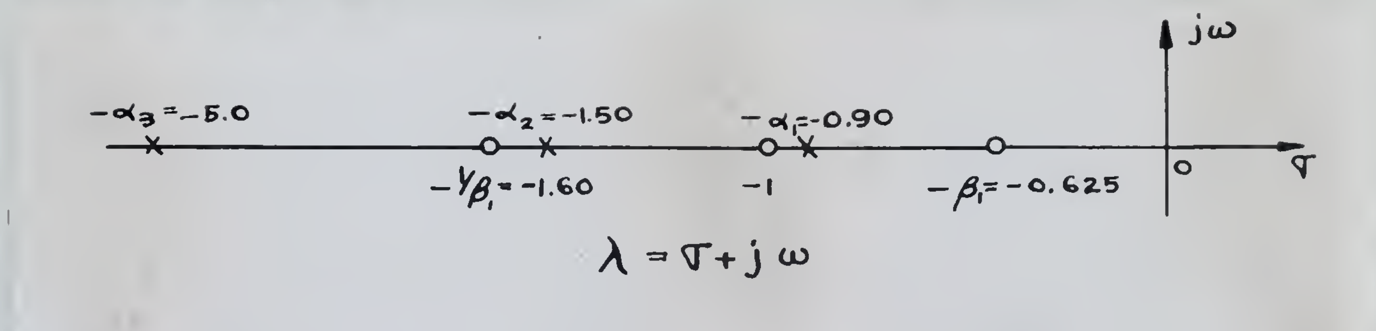


FIG V-1

Form $q_{(\lambda)} = (\lambda + \beta_1)(\lambda + 1)(\lambda + \frac{1}{\beta_1}) = \lambda^3 + 3.225\,\lambda^2 + 3.225\,\lambda + 1.000$

Form $q_1(\lambda) = A\,(\lambda + \alpha_1)(\lambda + \alpha_2)(\lambda + \alpha_3) = A\,(\lambda^3 + 7.40\lambda^2 + 13.35\,\lambda + 6.75)$

As the first trial:

$$A = \frac{1}{1 + \alpha_1 \alpha_2 \alpha_3} = 0.129$$

The resulting networks have a very small spread of element values. However, by accepting a larger spread, the gain required is reduced by selecting A as 0.05.

Then:

$$q_1(\lambda) = 0.05\ \lambda^3 + 0.370\ \lambda^2 + 0.6675\ \lambda + 0.3375$$

$$q_2(\lambda) = q_{(\lambda)} - q_{1(\lambda)} = 0.95\ \lambda^3 + 2.855\ \lambda^2 + 2.5575\ \lambda + 0.6625$$

Identifying y_{22} as $q_2(\lambda)/q_1(\lambda)$, one obtains by continued-fraction expansion the fundamental networks shown in Fig. V-2.

To obtain the admittance-level multiplying factors, one writes:

$$y_{12}^{(S_1)}(\lambda) = \frac{p(\lambda)}{q(\lambda)}$$

$$p(\lambda) = (\lambda^2 + 3.272)(\lambda + 0.379) = \lambda^3 + 0.379\ \lambda^2 + 3.272 + 1.240$$
$$= a_3\lambda^3 + a_2\lambda^2 + a_1\lambda + a_0$$

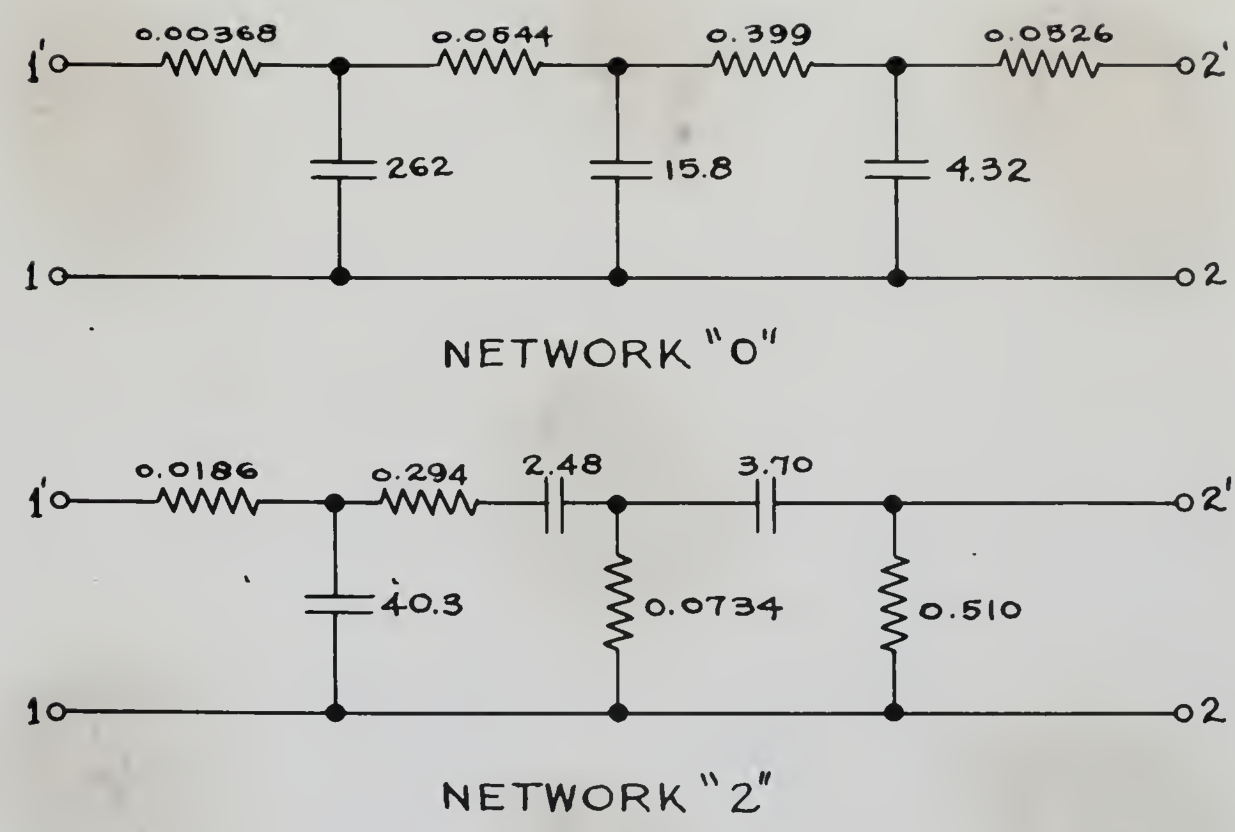


FIG V-2

For Network "0"

$C_3 = 262.6; \quad g_4 = 271.3; \quad \frac{C_3}{g_4} < \frac{a_1}{a_0}$

$C_a = 262.6; \quad C_b = 0$

$g_a = C_3 \frac{a_0}{a_1} = 262.6 \times 0.379 = 99.5$

$g_b = g_4 - g_a = 171.8$

VALUES IN OHMS AND FARADS

FIG V-3

$$A_o = \frac{d_o}{r_1 + r_2 + r_3 + r_4} = 0.6625$$

$$G_o = \frac{a_o \, g_4}{A_o \, g_a} = 5.10$$

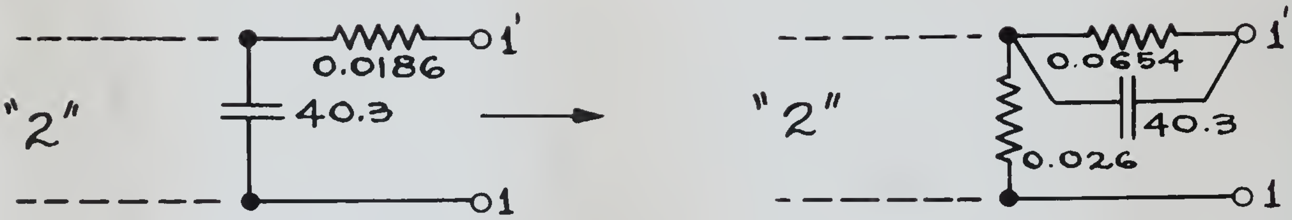
<u>For Network "2"</u>

$C_3 = 40.3; \quad g_4 = 53.7; \quad C_3/g_4 < a_3/a_2$

$C_a = 40.3; \quad C_b = 0$

$$g_a = C_3\frac{a_2}{3a_3} = 40.3 \times 0.379 = 15.3$$

$$g_b = g_4 - g_a = 38.4$$



VALUES IN OHMS AND FARADS

FIG V-4

$$A_2 = d_o C_1 C_2 R_2 = 0.2266$$

$$G_2 = \frac{a_2 \ g_4}{A_2 \ g_a} = 5.87$$

Then

$$G = G_o + G_2 = 10.97$$

$$G_o/G = 0.465; \quad G_2/G = 0.5355$$

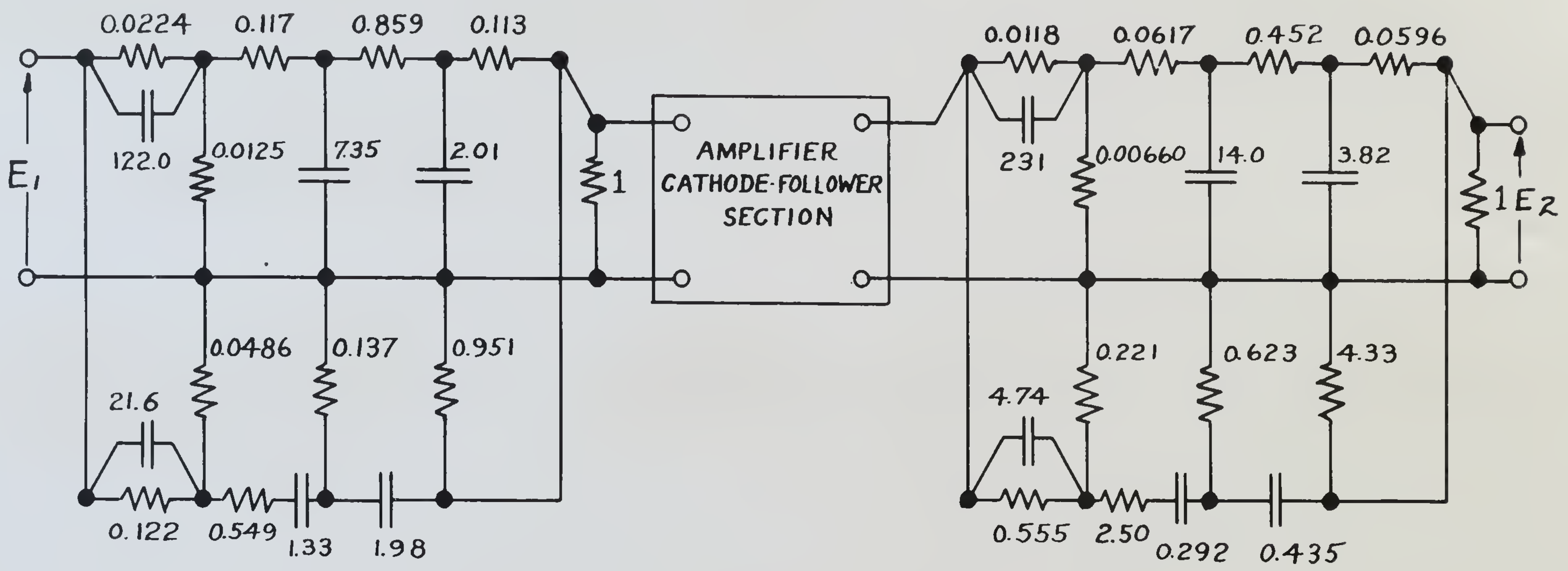
The admittance level of network "0" is now multiplied by 0.465; that of network "2" by 0.5355. The two networks connected in parallel and terminated in a one-ohm resistance have the transfer admittance $Y_{12}^{(S_1)}(\lambda)/10.97$.

To synthesize $Y_{12}^{(S_2)}(\lambda)$ the same fundamental networks can be used. The admittance-level multiplying factors are obtained as before, giving $G_o = 44.0$, $G_2 = 5.87$ and $G = 49.87$. The second-stage networks in parallel have the transfer admittance $Y_{12}^{(S_2)}(\lambda)/49.87$.

The gain required for the amplifier is computed as

$$\left(\frac{G}{a_o}\right)^{(S_1)} \times \left(\frac{G}{a_o}\right)^{(S_2)} = 41.2$$

The completed design, still on a normalized basis, is illustrated in Fig. V-5.



NORMALIZED SIX POLE, LOWPASS RC-FILTER
VALUES IN OHMS AND FARADS

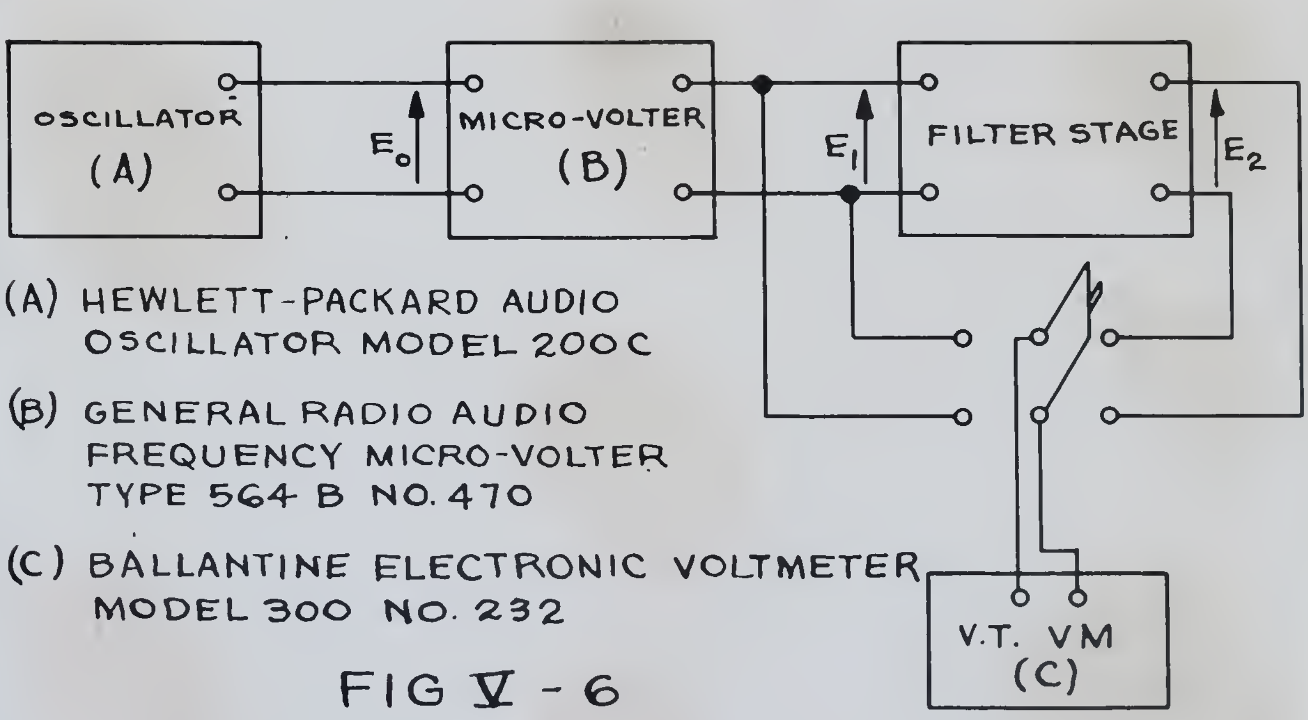
FIG. V - 5

Construction and Testing.

It was decided to test the filter at a cut-off frequency of 1592 cycles/sec ($\omega = 10^4$) and at an impedance level of 10^4 ohms. Therefore all elements of the normalized filter were modified to meet these requirements by multiplying resistances by 10^4 and dividing capacitances by 10^8.

Standard elements were combined to obtain the calculated values within one or two per cent, the measurements being made with a General Radio Impedance Bridge.

The two stages were tested separately, using the comparison method illustrated in Fig. V-6.



(A) HEWLETT-PACKARD AUDIO OSCILLATOR MODEL 200C

(B) GENERAL RADIO AUDIO FREQUENCY MICRO-VOLTER TYPE 564 B NO. 470

(C) BALLANTINE ELECTRONIC VOLTMETER MODEL 300 NO. 232

FIG V - 6

Over the entire frequency range, the input voltage, E_0, to the microvolter is held constant. At any frequency for which a measurement is desired, the vacuum-tube voltmeter is used to read the output voltage, E_2, of the filter stage. The D.P.D.T. switch is then thrown to the input side of the filter, and attenuation is introduced in the microvolter until its output voltage, E_1, is reduced to E_2. The attenuation-meter of the microvolter is graduated to read directly the ratio E_2/E_1.

The resultant overall frequency characteristic is plotted in Fig. V-7.

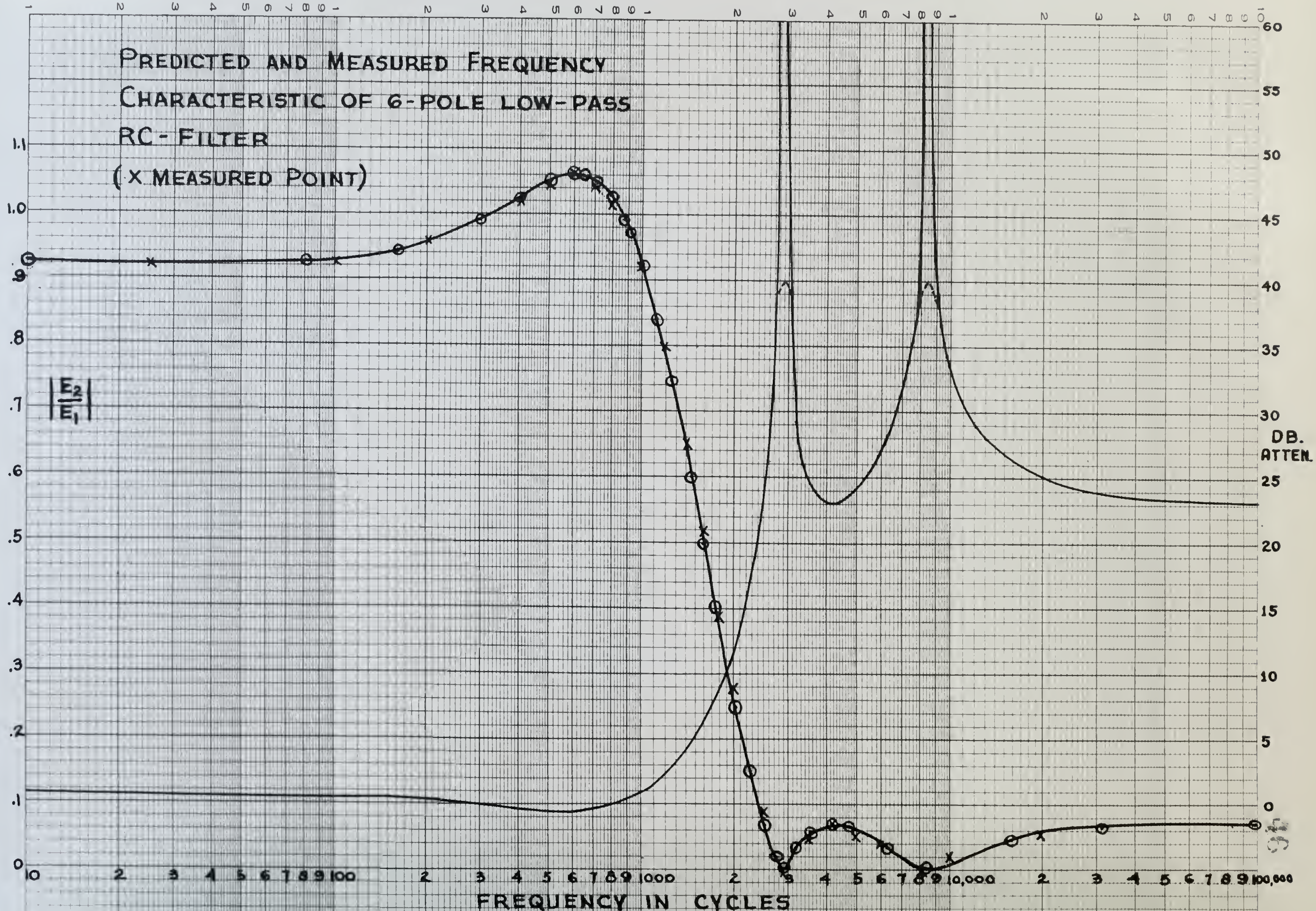
Results.

The extent to which the filter meets the original specifications may be summarized as follows:

(1) Simplicity in design and construction is only partially attained. In addition to the amplifier, cathode-follower section, a total of 4 networks, using 34 elements, is required. This complexity of structure is probably the greatest defect of the filter, but appears to be inherent in any RC design. The overall spread of element values is controlled so that no difficulty is encountered in obtaining any individual element. An examination of the structure shows that the critical elements are located at the output end of each stage. Tests indicate that the filter is insensitive to small variations even in these critical elements, and that an accuracy of $\pm$ 5 per cent in element values suffices.

(2) The frequency characteristic of the filter exhibits Tschebyscheff behavior with a guaranteed minimum attenuation of 23 decibels.

(3) A pass-band of 63 per cent of the cut-off frequency is obtained.



PREDICTED AND MEASURED FREQUENCY
CHARACTERISTIC OF 6-POLE LOW-PASS
RC-FILTER
(X MEASURED POINT)
$\left|\frac{E_2}{E_1}\right|$
DB. ATTEN.
FREQUENCY IN CYCLES
10
100
1000
10,000
100,000

FIG V-7

CHAPTER VI

DESIGN OF OTHER LOW-PASS RC-FILTERS

A. The 10-Pole Filter

Symmetrical subdivision of $Y_{12}(\lambda)$ for the 10-pole filter is not possible if the approximating function is $F^2(\omega^2)$. However, the alternate approximating procedure described in Chapter II, where one obtains the function $F'(\omega^2)$, can be used here to overcome this difficulty.

The β's and α's are chosen as indicated in Fig. VI-1.

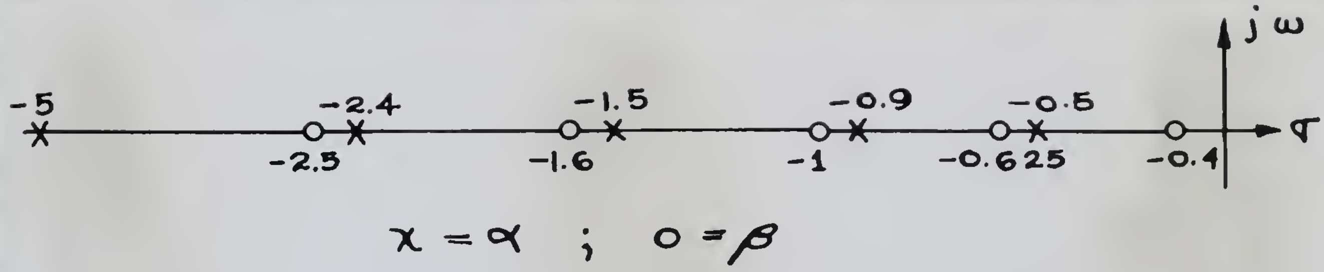


FIG VI - 1

and the constant A is selected as 0.10. The resulting fundamental networks then obtained are illustrated in Fig. VI-2.

The tolerance (ϵ) is selected as 0.03, and following the procedure of Chapter II one obtains

$$f_2(\phi) = (1.11833 - 0.11833 \cos 2\phi)(1.55125 - 0.55125 \cos 2\phi)$$

and

$$f_1(\phi) = 0.906 \cos \phi - 0.536 \cos 3\phi + 0.160 \cos 5\phi.$$

The latter function approximates $F'(\phi) \times f_2(\phi)$ as shown in Fig. VI-3.

Then

$$\frac{f_1^{A'}(\phi)}{f_2(\phi)} = \frac{0.53f_2(\phi) + f_1(\phi)}{f_2(\phi)}$$

and

$$f_1^{A'}(\phi) = 2.56 \cos^5\phi + 0.138871 \cos^4\phi - 5.344 \cos^3\phi - 0.938026 \cos^2\phi + + 3.314 \cos\phi + 1.379155$$

$$= (\cos\phi + 0.641)^2(2.56 \cos^2\phi - 5.703 \cos\phi + 3.3366)(\cos\phi + 1)$$

$$(\cos\phi + 0.641)^2(a_2\cos^2\phi + a_1 \cos\phi + a_0)(\cos\phi + 1)$$

The quadratic factor is in the desired form, except for a constant multiplier (this constant is $\sum_{u=0}^{u=2}(-1)^u a_u$), and is converted directly to the quadratic form in (ω^2). The factor $(\cos\phi + 1)$ corresponds to an infinite ω^2-root and hence reduces the numerator of $F'(\omega^2)$ to the fourth degree.

$$F'(\omega^2) = \frac{3.005\ (\omega^2 - 4.571)^2(\omega^4 + 0.135\ \omega^2 + 0.0169)}{(\omega^2 + 0.3906)(\omega^2 + 2.56)(\omega^2 + 1)(\omega^2 + 0.16)(\omega^2 + 6.25)} \qquad \text{(VI-1)}$$

$$Y_{12}(\lambda) = \frac{(\lambda^2 + 4.571)(\lambda^2 + 0.6285\lambda + 0.13)}{(\lambda + 0.625)(\lambda + 1.6)(\lambda + 1)(\lambda + 0.4)(\lambda + 2.5)}$$

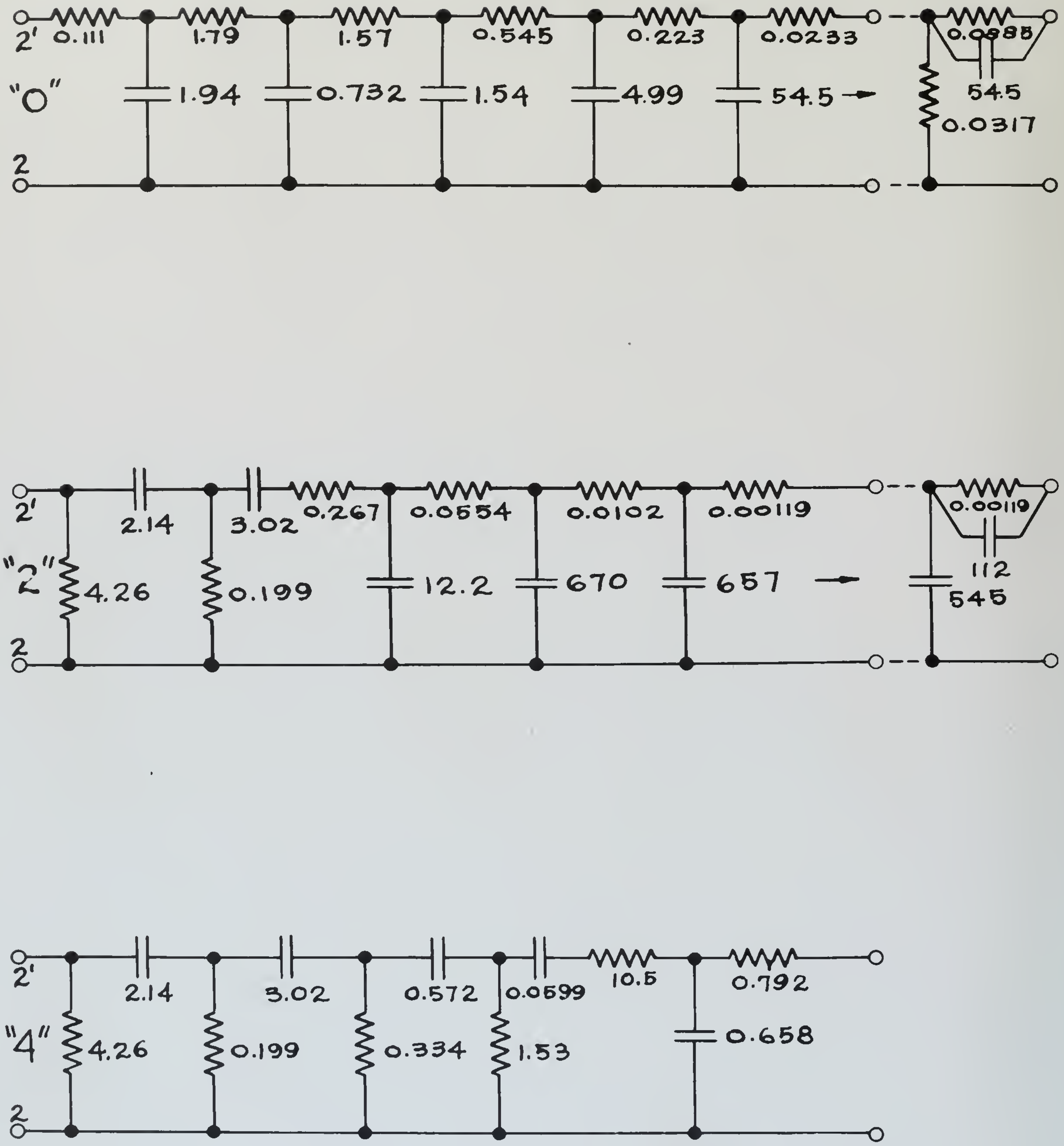
$$= \frac{\lambda^4 + 0.6285\lambda^3 + 4.701\lambda^2 + 2.873\lambda + 0.5942}{(\lambda + 0.625)(\lambda + 1.6)(\lambda + 1)(\lambda + 0.4)(\lambda + 2.5)}$$

The fundamental networks are then modified to realize this transfer function, and the resulting normalized filter is shown in Fig. VI-4. It is necessary to cascade two of these filters to obtain the frequency characteristic of Fig. VI-5, giving a voltage ratio equal to $F'(\omega^2)$. The final ten-pole filter has a guaranteed minimum attenuation of 24.4 db., a pass-band width of 68% of the cut-off frequency, and requires a gain of 955 per stage.

The relative merits of this alternative procedure may now be summarized:

(1) Greater accuracy is required in the graphical approximation work.

(2) A completely symmetrical subdivision into stages is possible, greatly reducing the computational work necessary, and giving more flexibility in design.



VALUES IN OHMS AND FARADS
FUNDAMENTAL NETWORKS

FIG VI - 2

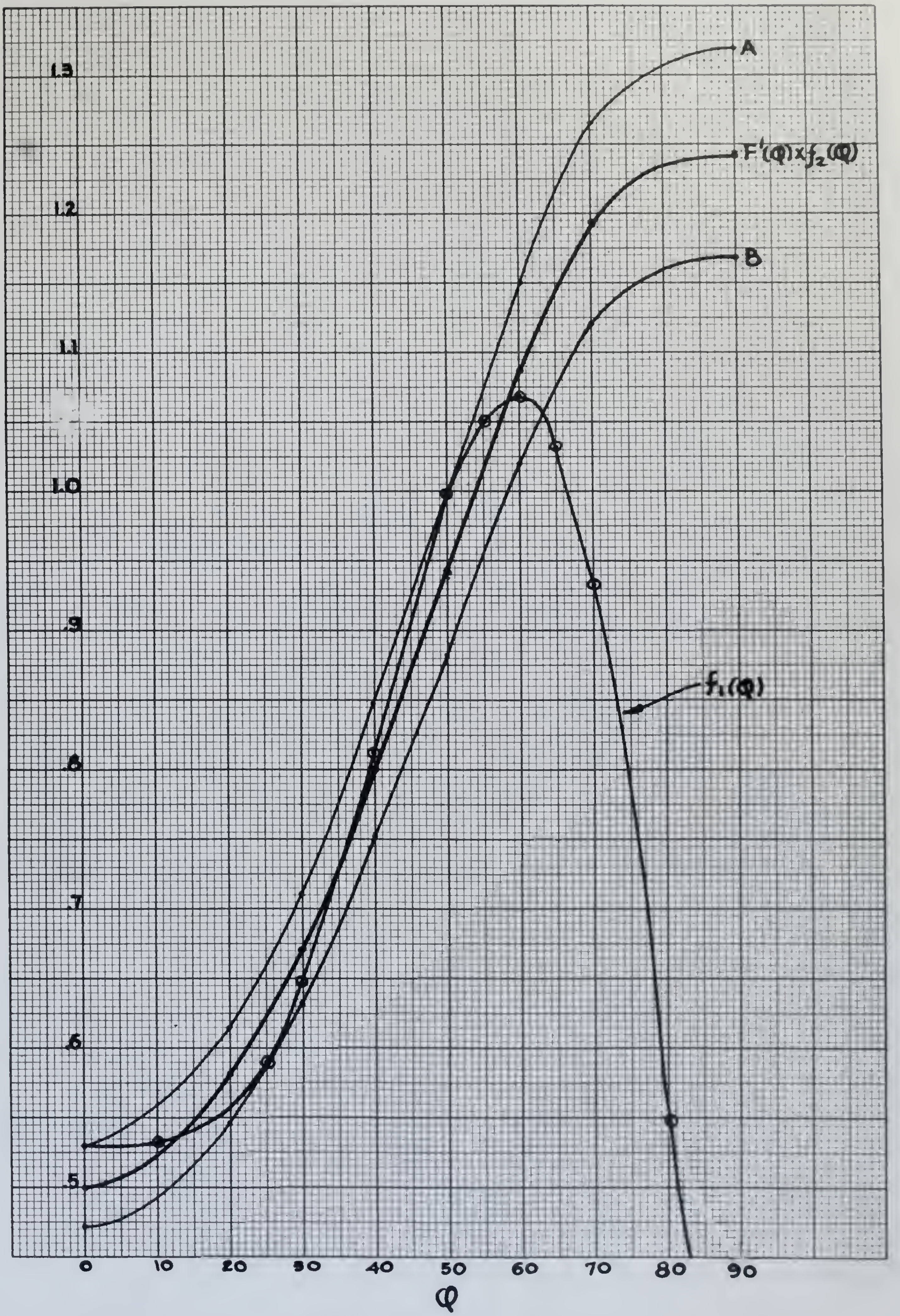
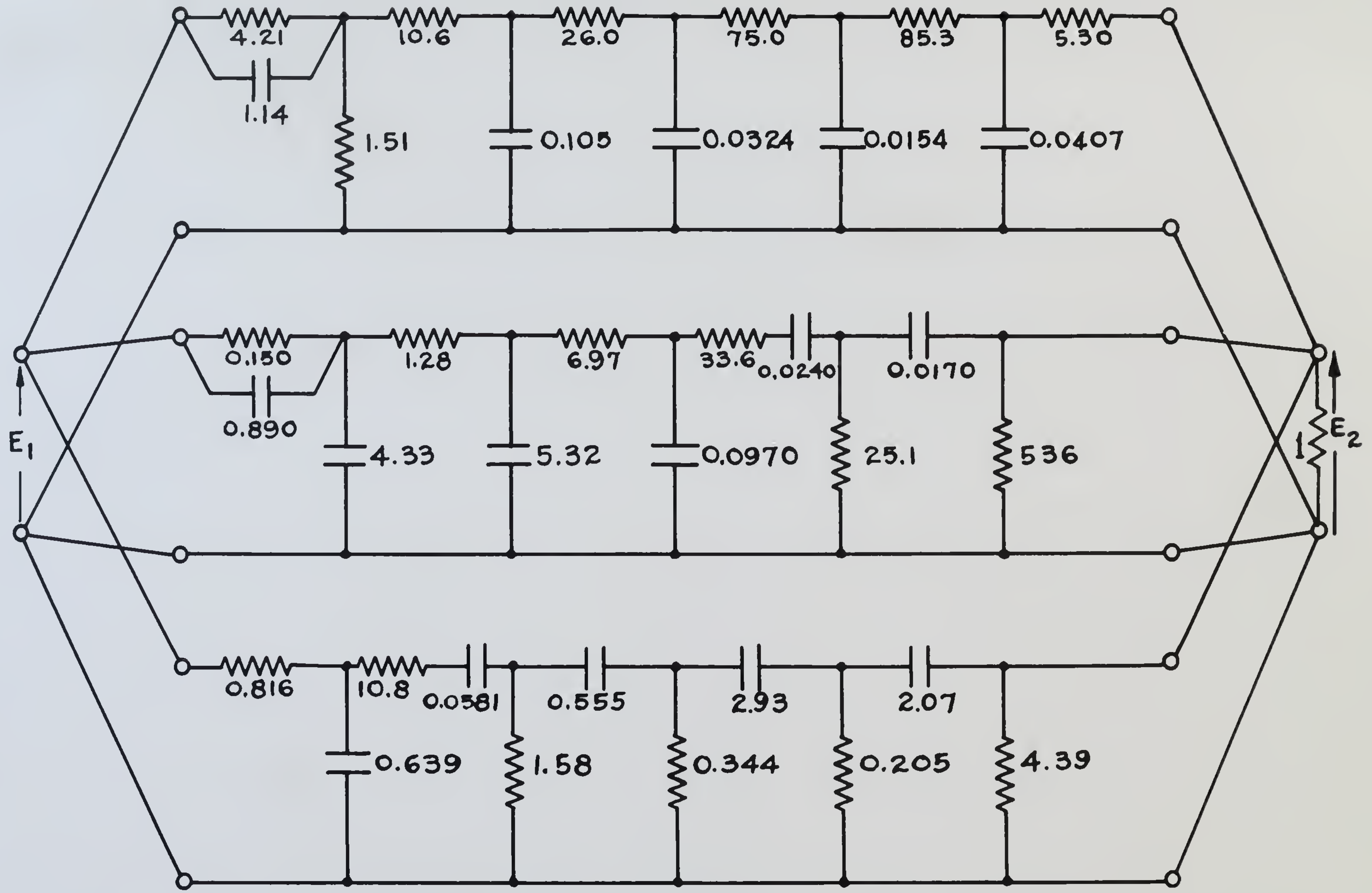
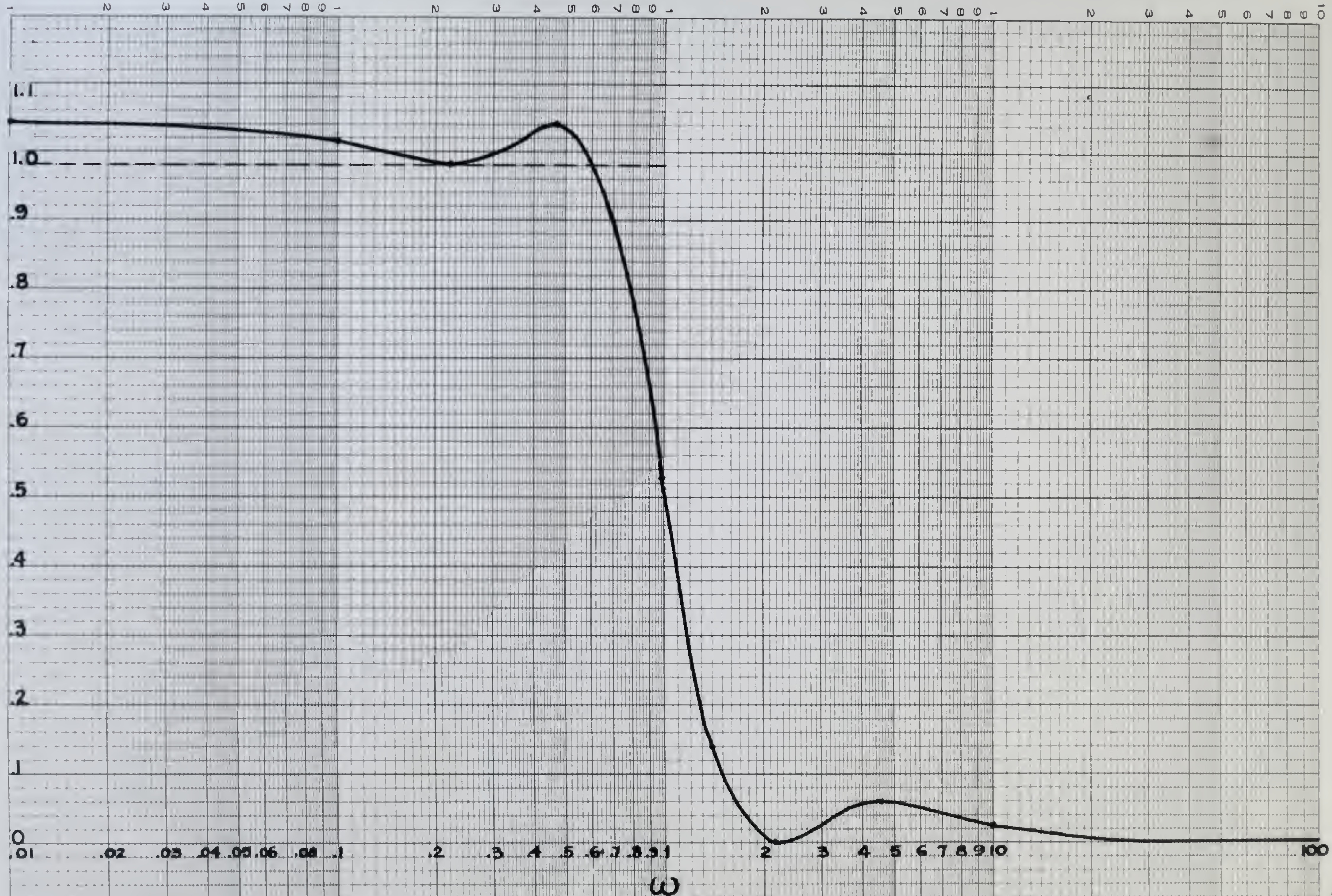


FIG VI - 3



VALUES IN OHMS AND FARADS
NORMALIZED FILTER

FIG VI-4



NORMALIZED FREQUENCY CHARACTERISTIC FOR THE 10 POLE FILTER

FIG VI-5

B. The 14-Pole Filter

It is possible to design the 14-pole filter with two 7-pole stages, using either approximation procedure. However, to illustrate other methods of design, it was decided to use four stages.

The β's and α's are chosen as indicated in Fig. VI-6.

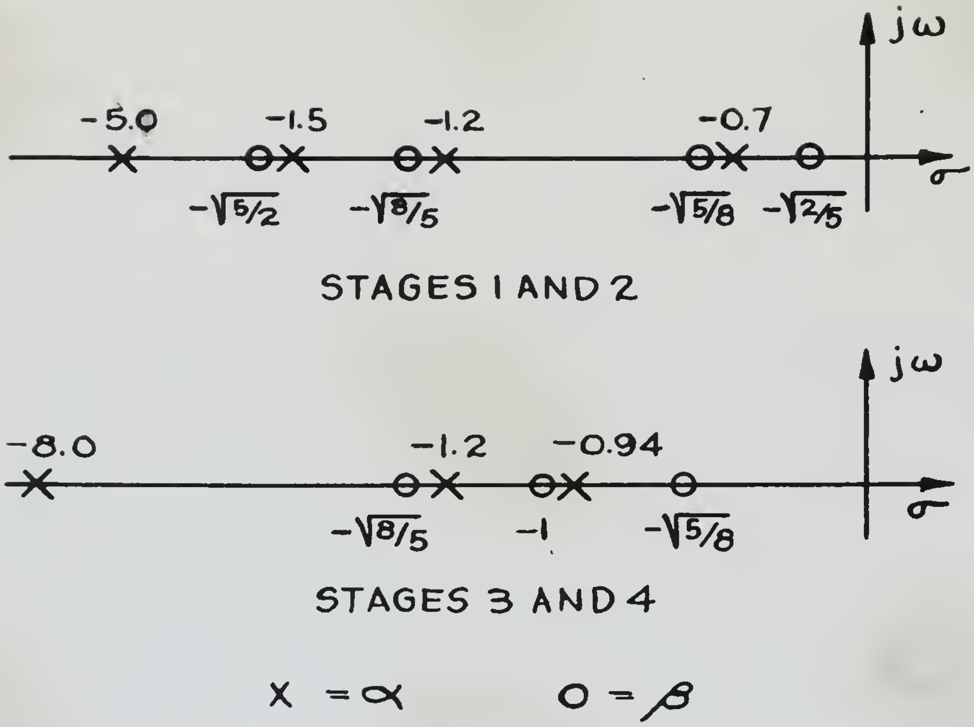


FIG VI-6

$A^{(S_{1,2})}$ is selected as 1/7 and $A^{(S_{3,4})}$ as 0.09. The resulting fundamental networks for the 3-pole and 4-pole stages are illustrated in Fig. VI-7.

The tolerance (ε) is selected as 0.05. Then

$$f_2(\phi) = f_2^{(S_{1,2})}(\phi) \times f_2^{(S_{3,4})}(\phi)$$

$$= \left[(1.028125-0.028125 \cos 2\phi)(1.1125-0.1125 \cos 2\phi)\right] \times \times \left[1.028125-0.028125 \cos 2\phi\right]$$

Graphically one obtains

$$f_1(\phi) = 0.7033 \cos\phi - 0.294 \cos 3\phi + 0.1387 \cos 5\phi - 0.098 \cos 7\phi$$

This function approximates $F'(\phi) \times f_2(\phi)$ as shown in Fig. VI-8.

Then

$$F(\phi) = \frac{0.5f_2(\phi) + f_1(\phi)}{f_2(\phi)}$$

$$= \frac{(-6.272 \cos^7\phi - 0.000356 \cos^6\phi + 13.1952 \cos^5\phi + 0.01531 \cos^4\phi - 9.438 \cos^3\phi - 0.19829 \cos^2\phi + 2.9648 \cos \phi + 0.68334)}{f_2(\phi)}$$

$$F(\omega^2) = \frac{(0.05\omega^{14} - 5.52942\omega^{12} + 77.6224\omega^{10} - 303.67776\omega^8 + 353.5902\omega^6 - 49.4169\omega^4 + 13.87938\omega^2 + 0.95)}{(\omega^2+0.4)(\omega^2+0.625)^2(\omega^2+1)(\omega^2+1.6)^2(\omega^2+2.5)}$$

$$\left|Y_{12}\right|^2 = \left[\frac{.05(\omega^2-3.52)(\omega^2-10.2)(\omega^2-1.86)(\omega^2-94.9)(\omega^2+0.0593)(\omega^4-0.167\omega^2+0.0556)}{(\omega^2+0.4)(\omega^2+0.625)^2(\omega^2+1)(\omega^2+1.6)^2(\omega^2+2.5)}\right]^2$$

$$Y_{12}(\lambda) = Y_{12}^{(S_1)}(\lambda) \times Y_{12}^{(S_2)}(\lambda) \times Y_{12}^{(S_3)}(\lambda) \times Y_{12}^{(S_4)}(\lambda)$$

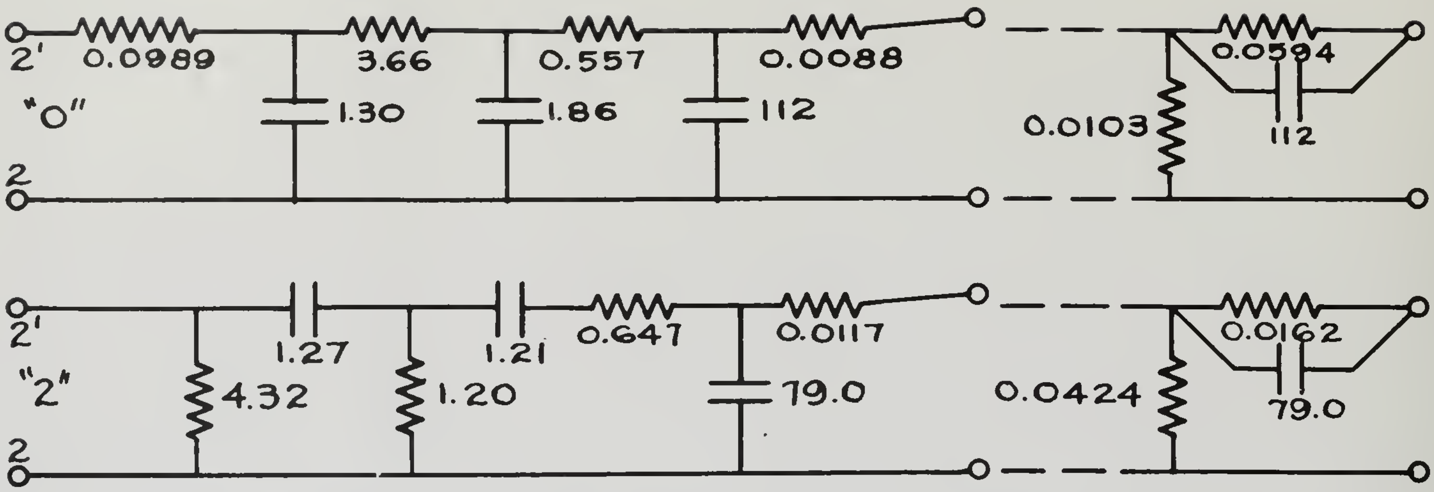
$$Y_{12}^{(S_1)}(\lambda) = \frac{\lambda^4 + 13.7\lambda^2 + 35.9}{(\lambda + \sqrt{2/5})(\lambda + \sqrt{5/8})(\lambda + \sqrt{8/5})(\lambda + \sqrt{5/2})}$$

$$Y_{12}^{(S_2)}(\lambda) = \frac{\lambda^4 + 96.7\lambda^2 + 176}{(\lambda + \sqrt{2/5})(\lambda + \sqrt{5/8})(\lambda + \sqrt{8/5})(\lambda + \sqrt{5/2})}$$

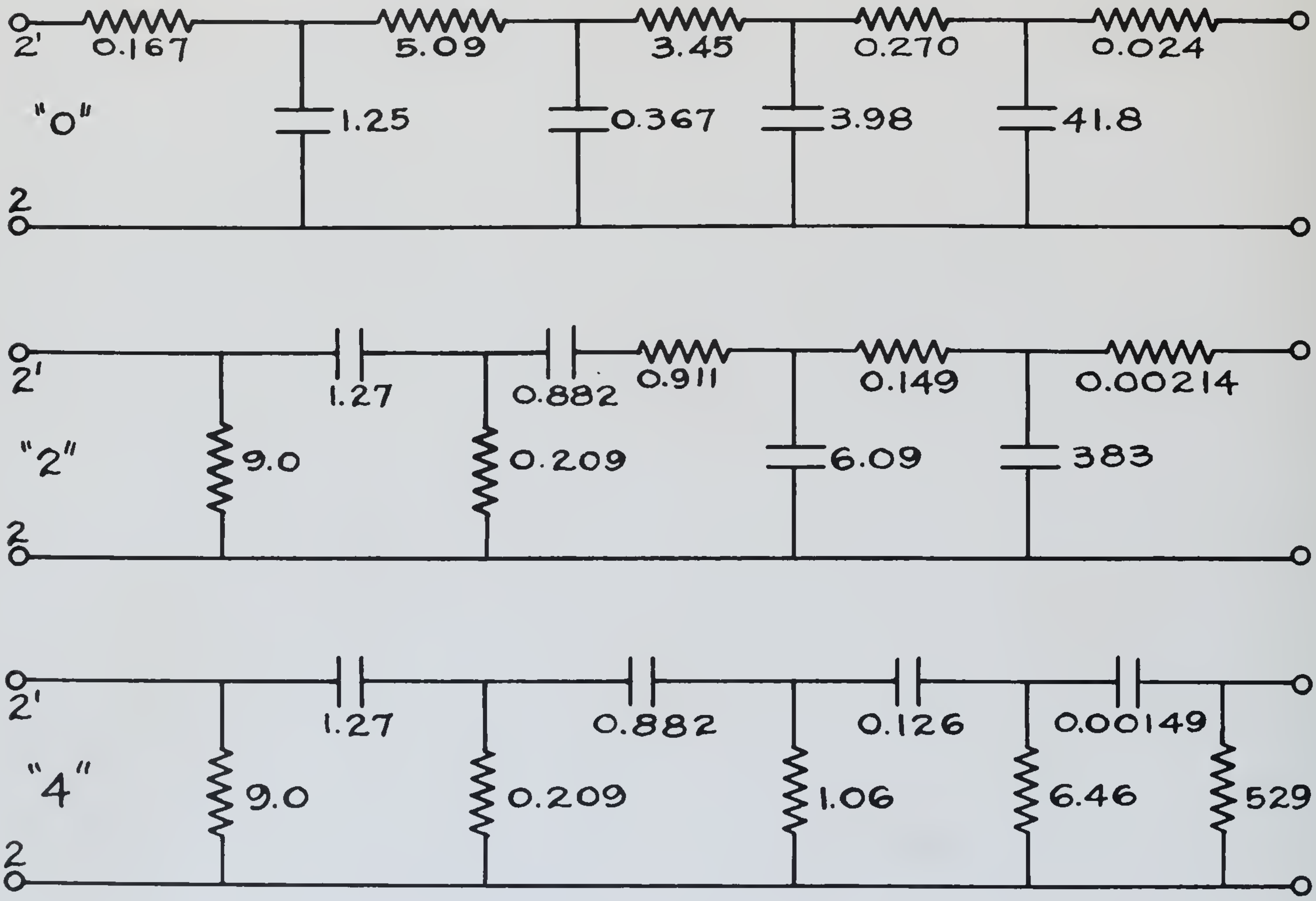
$$Y_{12}^{(S_3)}(\lambda) = Y_{12}^{(S_4)}(\lambda) = \frac{\lambda^3 + 0.784\lambda^2 + 0.364\lambda + 0.0548}{(\lambda + \sqrt{5/8})(\lambda + 1)(\lambda + \sqrt{8/5})}$$

The fundamental networks are now modified to realize these transfer functions, and the resulting normalized filter is shown in Fig. VI-9. The frequency characteristic is illustrated in Fig. VI-10. The completed 14-pole filter has a guaranteed minimum attenuation of 26 db. and a pass-band width of 77 per cent of the cut-off frequency.

In this example, the poles of $Y_{12}(\lambda)$, i.e., the β's, were chosen very close to unity to keep the magnitude of $f_2(\phi)$ near unity, because it was felt that this was desirable from the graphical approximation standpoint. Consequently the normalized networks have a large overall spread of element values. Investigations show that this spread can be reduced by at least a factor of ten by slightly varying the α's and A's. Moreover, the use of alternate network forms in the 4-pole stages will further reduce the overall spread. The impedance-level of any stage can also be adjusted to provide for physical realizability of elements. Subsequently it was found that, except for a very slight decrease in the pass-band width, there was no objection to a wider separation of the β's, which would give more desirable element values. Thus, although the networks obtained are practicable and may be improved, a complete redesign of the filter, by selecting a wider separation of the β's, might be desirable.



FUNDAMENTAL NETWORKS FOR 3-POLE STAGES



FUNDAMENTAL NETWORKS FOR 4-POLE STAGES
VALUES IN OHMS AND FARADS

FIG VI-7

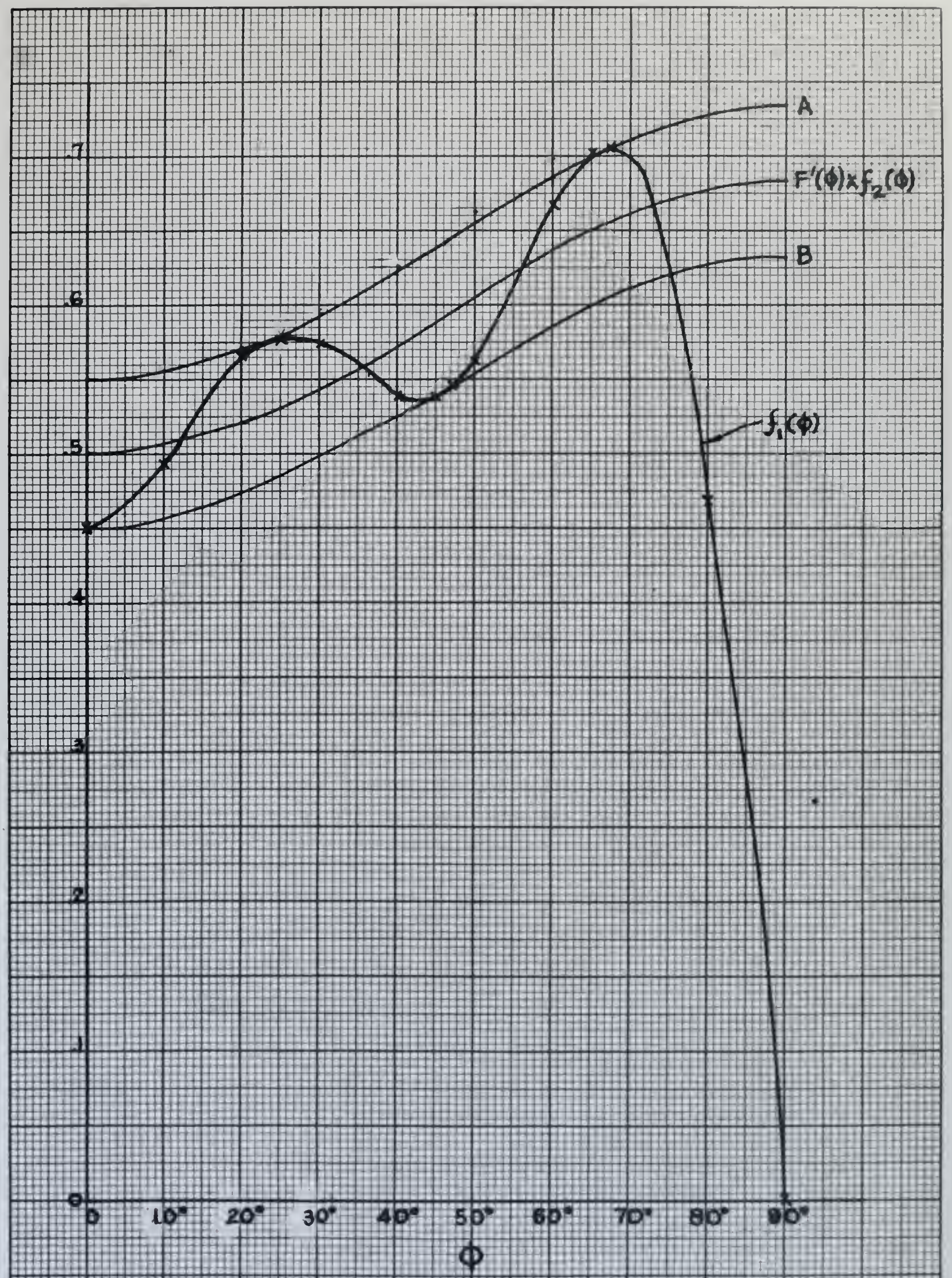
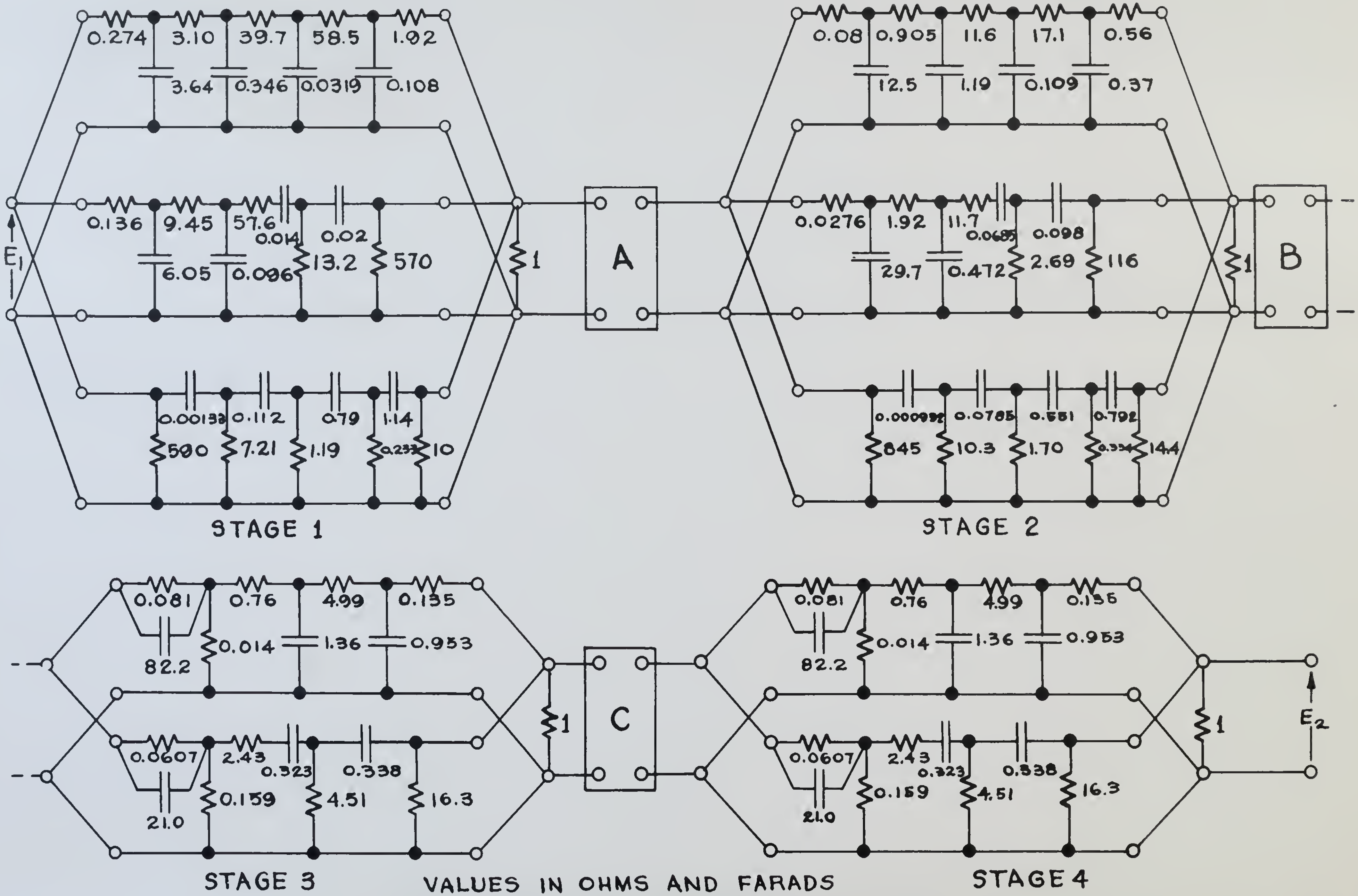
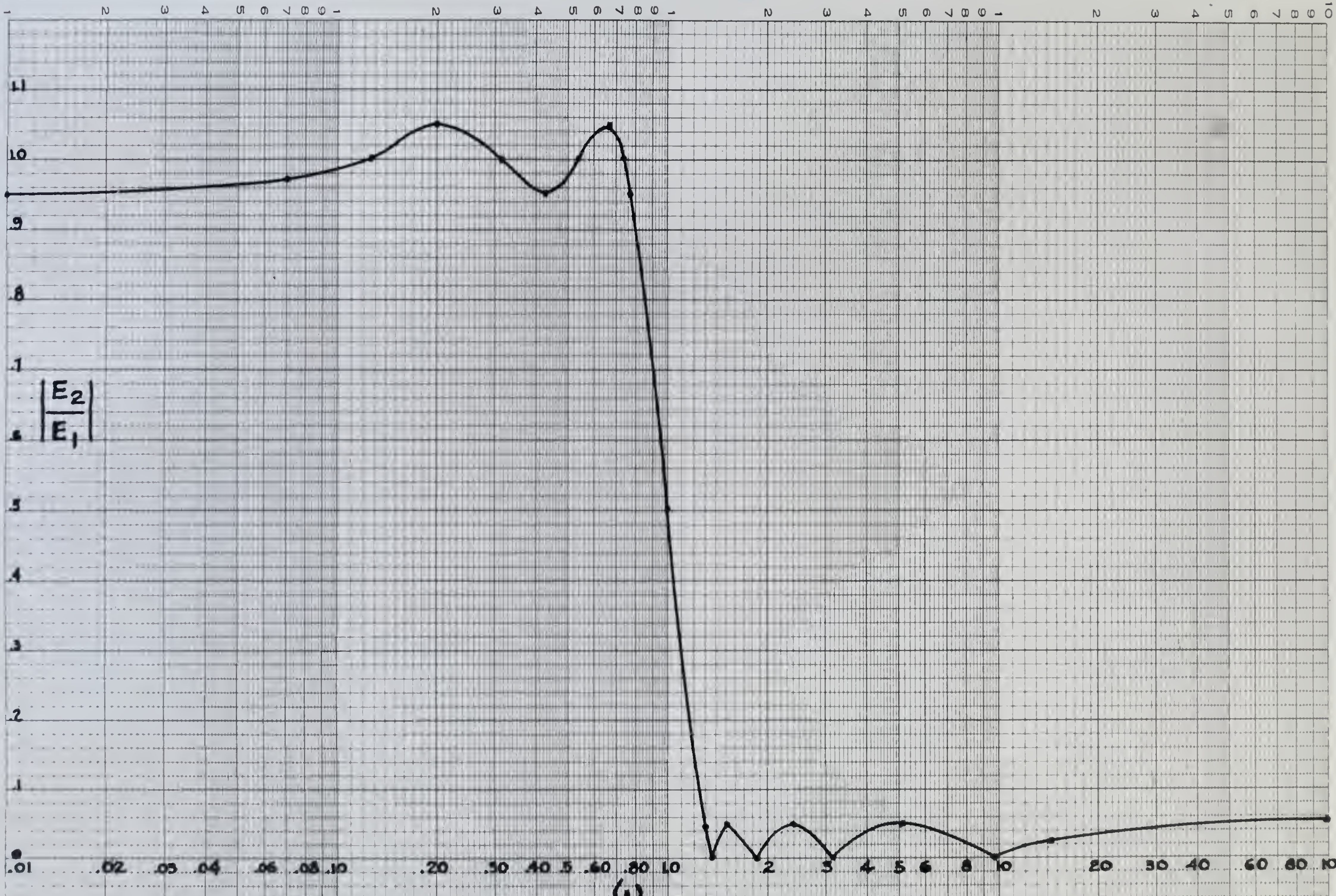


FIG VI-8



VALUES IN OHMS AND FARADS
A,B,C, - AMPLIFIER, CATHODE - FOLLOWER
NORMALIZED 14 POLE FILTER

FIG VI - 9



NORMALIZED FREQUENCY CHARACTERISTIC FOR THE 14 POLE FILTER

FIG VI-10

CHAPTER VII

CONCLUSION

A practical design procedure for linear, passive RC-networks with prescribed frequency characteristics is developed in this paper. The low-pass filters specifically presented compare favorably in performance with RC-filters based on feedback principles. Their purpose is to replace conventional L-C wave filters in low-frequency applications or in cases where other practical considerations rule out the use of inductances.

The most promising field for future work in connection with this procedure appears to lie in improving the solution to the approximation problem. A combination of the graphical and analytical solutions with an experimental attack seems feasible. An approximate graphical solution may be used with the analytical form of the transfer function (Equations III-3, III-4) to obtain the general location of the poles and zeros of this function; the exact locations may then be determined by the use of an electrolytic tank as described by Linvill[4]. This combination of the graphical and experimental solutions would greatly reduce the time and labor required by either method alone. Since such a tank is useful in the solution of conjugate potential function problems as well as in network analysis and synthesis, the development of a precise, large-scale electrolytic tank is recommended as a profitable project.

APPENDIX A

AN ANALYTICAL SOLUTION TO THE APPROXIMATION PROBLEM

In Chapter II a graphical method was presented of obtaining a rational function of (ω^2) to approximate with Tschebyscheff behavior an ideal transfer function. It is possible to obtain similar results by an algebraic solution in the (ω) domain alone. Basically, this solution consists of writing the analytic form of $F'(\omega^2)$, (Fig. II-7), and then applying sufficient known or assumed values to solve for all unknown constants. When $F'(\omega^2)$ is of the third or fifth degree, this method requires less time than the graphical solution, and gives exact results. For $F'(\omega^2)$ of higher degree, approximations become necessary, and the procedure is still laborious; hence the graphical method is, in general, preferable.

The third-degree approximation function may be written as:

$$F'(\omega^2) = \frac{2\epsilon\left(\omega^2 + \frac{1}{2\epsilon K^2}\right)(\omega^2 - K)^2}{(\omega^2 + \beta_1^2)(\omega^2 + 1)(\omega^2 + 1/\beta_1^2)} \qquad \text{(A-1)}$$

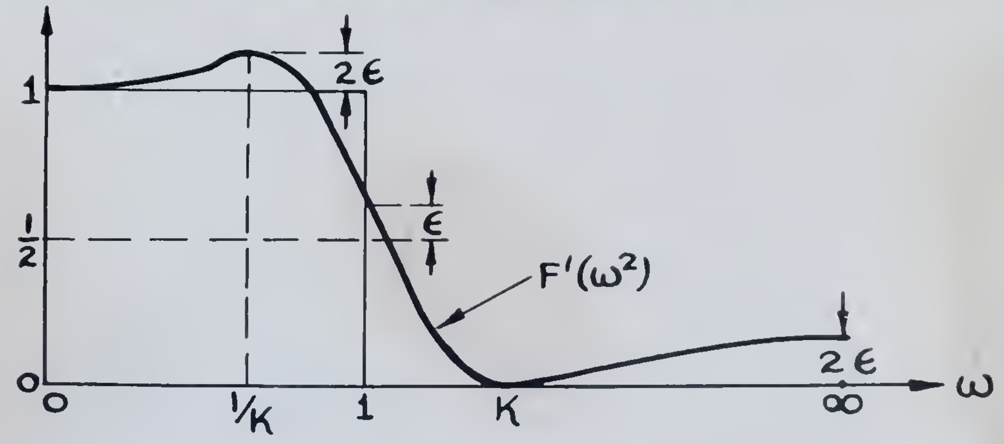


FIG A-1

At $\omega^2 = 1$, the value of $F'(\omega^2)$ is $\frac{1}{2} + \epsilon$. Applying this condition:

$$F'(1) = \frac{2\epsilon(1 + \frac{1}{2\epsilon K^2})(1 - K)^2}{(1 + \beta_1^2)(2)(1 + 1/\beta_1^2)} = \frac{1 + 2\epsilon}{2}$$

Clearing:

$$K^4 - 2K^3 - \frac{(1 + \beta_1^2 + 1/\beta_1^2)(1 + 2\epsilon)}{2\epsilon} K^2 - \frac{K}{\epsilon} + \frac{1}{2\epsilon} = 0 \quad \text{(A-2)}$$

Factor for the largest positive real root.

Example

Select $\epsilon = 0.05$; $\beta_1 = 0.707$

$K^4 - 2K^3 - 38.5K^2 - 20K + 10 = 0.$

$K = 7.48$

$$F'(\omega^2) = \frac{0.1(\omega^2 + 0.1788)(\omega^2 - 7.48)^2}{(\omega^2 + 0.5)(\omega^2 + 1.0)(\omega^2 + 2.0)}$$

$$Y_{12}(\lambda) = \frac{(\lambda + 0.423)(\lambda^2 + 7.48)}{(\lambda + 0.707)(\lambda + 1.0)(\lambda + 1.414)}$$

Two identical stages are necessary to realize the curve of Fig. A-1 as the voltage ratio E_2/E_1.

If the transfer function $F(\omega^2)$ of Fig. II-6 is desired, it may be obtained from the relation

$$F(\omega^2) = F'(\omega^2) - \epsilon \quad \text{(A-3)}$$

The fifth-degree approximating function may be written as:

$$F'(\omega^2) = \frac{N(\omega^2 - K)^2(\omega^4 + B\omega^2 + C)}{(\omega^2 + \beta_1^2)(\omega^2 + \frac{1}{\beta_1^2})(\omega^2 + 1)(\omega^2 + \beta_3^2)(\omega^2 + \frac{1}{\beta_3^2})} \quad \text{(A-4)}$$

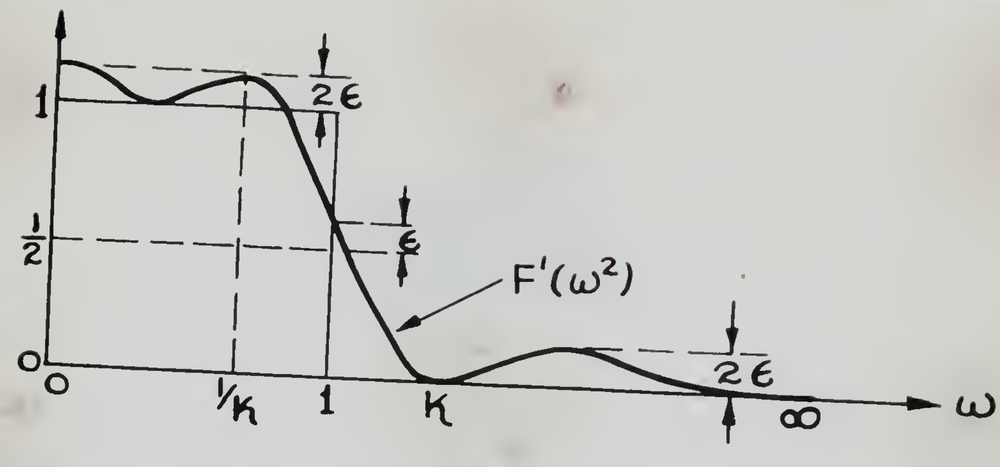


FIG A-2

The three known conditions applied are:

$$F'(0) = N\,K^2\,C = 1 + 2\epsilon$$

$$F'(1) = \frac{N\,(1 - K)^2\,(1 + B + C)}{(1 + \beta_1^2)(1 + \frac{1}{B_1^2})(2)(1 + \beta_3^2)(1 + \frac{1}{\beta_3^2})} = \frac{1 + 2\epsilon}{2}$$

$$F'(\tfrac{1}{K}) = \frac{N\,(\frac{1}{K} - K)^2(\frac{1}{K^2} + \frac{B}{K} + C)}{(\frac{1}{K} + \beta_1^2)(\frac{1}{K} + \frac{1}{\beta_1^2})(\frac{1}{K} + 1)(\frac{1}{K} + \beta_3^2)(\frac{1}{K} + \frac{1}{\beta_3^2})} = 1 + 2\epsilon$$

Substituting $K^2C = (1 + 2\epsilon)/N$, and clearing:

$$1 + B = C\left[\frac{K^2(1+\beta_1^2)(1+\frac{1}{\beta_1^2})(1+\beta_3^2)(1+\frac{1}{\beta_3^2})}{(1-K)^2} - 1\right] \qquad \text{(A-5)}$$

$$1 + KB = C\left[\frac{K^4(\frac{1}{K}+\beta_1^2)(\frac{1}{K}+\frac{1}{\beta_1^2})(\frac{1}{K}+1)(\frac{1}{K}+\beta_3^2)(\frac{1}{K}+\frac{1}{\beta_3^2})}{(\frac{1}{K}-K)^2} - K^2\right] \quad . \qquad \text{(A-6)}$$

An arbitrary selection of the frequency $\omega^2_{max} = \frac{1}{K}$ is now made, and equations A-5 and A-6 are solved simultaneously for B and C. The constant N is not required for synthesis. The value of ε may be obtained by evaluating $F'(\omega^2)$ at ω^2_{min}. Since the value of $F'(\omega^2)$ changes slowly in the vicinity of ω^2_{min}, it is satisfactory to estimate ω^2_{min}, using the relation $\omega^2_{min} = \tan\left(\frac{1}{2}\tan^{-1}\omega^2_{max}\right)$. (A-7)

Example

Select $\omega^2_{max} = 0.271 = \frac{1}{K}$

Select $\beta_1 = 5/8$; $\beta_3 = 2/5$

$1 + B = 78.5\ C$ (A-5)

$1 + 3.69\ B = 92.6\ C$ (A-6)

$C = 0.01364$, $B = 0.0682$

$\omega^2_{min} = 0.06$ (A-7)

$\frac{F'(0.06)}{F'(0)} = \frac{0.1618}{0.1860} = \frac{1}{1+2\varepsilon}$ $\qquad$ $\varepsilon = 0.075$, $N = 6.18$

$$F'(\omega^2) = \frac{6.18(\omega^2-3.69)^2(\omega^4+0.0682\,\omega^2+0.01364)}{(\omega^2+0.3906)(\omega^2+2.56)(\omega^2+1.0)(\omega^2+0.16)(\omega^2+6.25)}$$

$$Y_{12}(\lambda) = \frac{6.18\ (\lambda^2 + 3.69)(\lambda^2 + 0.55\ \lambda + 0.117)}{(\lambda + 0.625)(\lambda + 1.6)(\lambda + 1.0)(\lambda + 0.4)(\lambda + 2.5)}$$

If $\epsilon = 0.075$ is unsatisfactory, another value of ω^2_{max} may be chosen, or the poles β_1 and β_3 may be reselected. Thus, selecting $\omega^2_{max} = 0.220$, $\beta_1 = 5/8$, $\beta_3 = 2/5$, one obtains:

$$F'(\omega^2) = \frac{3.052(\omega^2 - 4.545)^2(\omega^4 + 0.1331\omega^2 + 0.0168)}{(\omega^2 + 0.3906)(\omega^2 + 2.56)(\omega^2 + 1.0)(\omega^2 + 0.16)(\omega^2 + 6.25)}$$

which is practically identical with the function obtained graphically, Eq. VI-1, and gives $\epsilon = 0.03$.

APPENDIX B

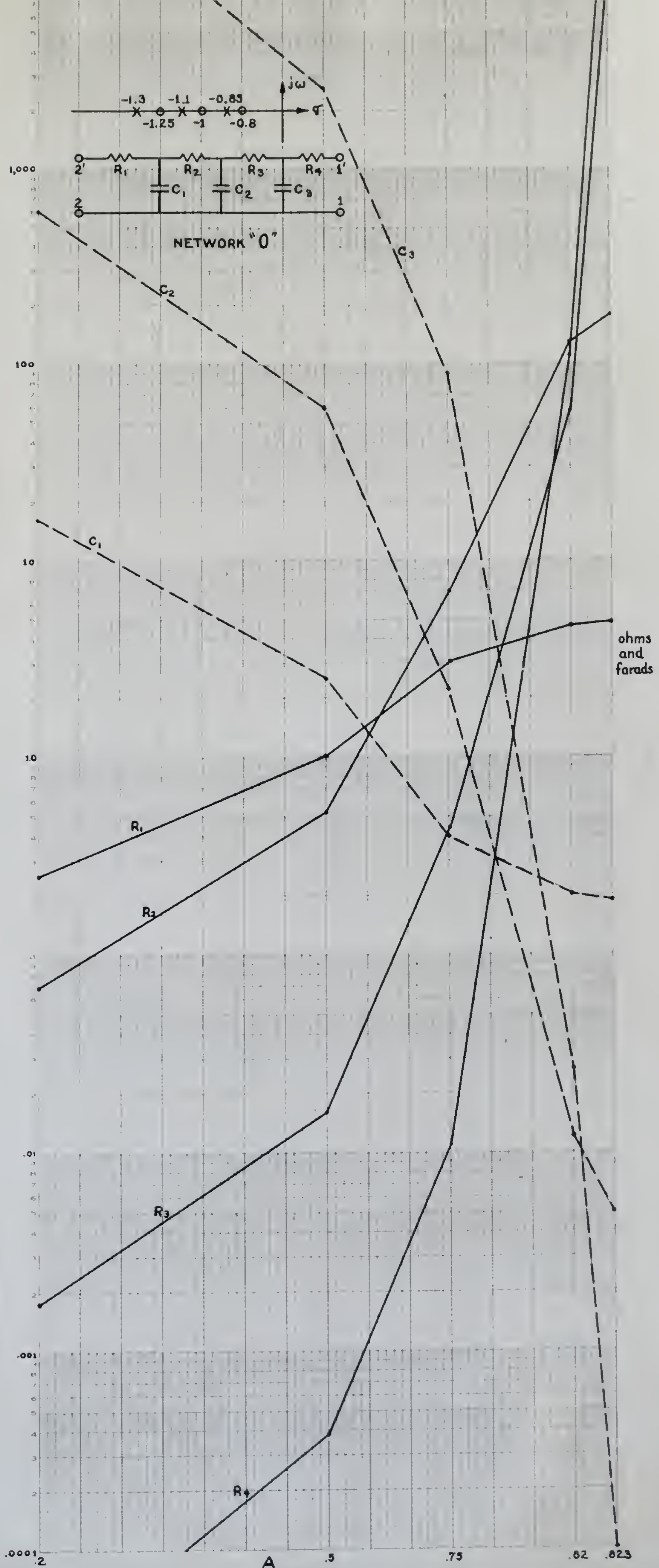
VARIATION IN ELEMENT VALUES AS A RESULT OF CERTAIN ARBITRARY CONSTANTS

As an illustration of the selection of certain arbitrary constants (β's, α's, and A) in the synthesis procedure and their influence on the element values in resulting networks, two cases are demonstrated.

In Case I (Fig. B-1), the β's are chosen near unity and the α's near the β's. Resulting element values in Network "O" as a function of A are then given. For the conditions of Case I, a small spread of element values is obtained only with an extremely critical value of A.

In Case II (Fig. B-2), the β's are more widely separated and the α's moved well to the left of the β's. The resulting element values for Networks "O" and "2" as a function of A are then given, illustrating that for these conditions a small overall spread of element values is obtained over a wide range of values of A. For the conditions of Case II, the choice of $A = \frac{1}{1 + (\alpha_1 \alpha_2 \ldots \alpha_m)}$ (IV-1) places one near the optimum value of A.

When the β's and α's for the m-pole networks are selected as illustrated in Case II, the use of the general formula IV-1 results in networks having approximately the minimum spread of element values obtainable under the established conditions.



jω
σ
-1.3
-1.1
-0.85
-1.25
-1
-0.8
2'
R₁
R₂
R₃
R₄
1'
C₁
C₂
C₃
2
1
NETWORK "0"
C₃
C₂
C₁
R₁
R₂
R₃
R₄
ohms and farads
1,000
100
10
1.0
.01
.001
.0001
.2
.5
.75
.82
.823
A

FIG. B-1

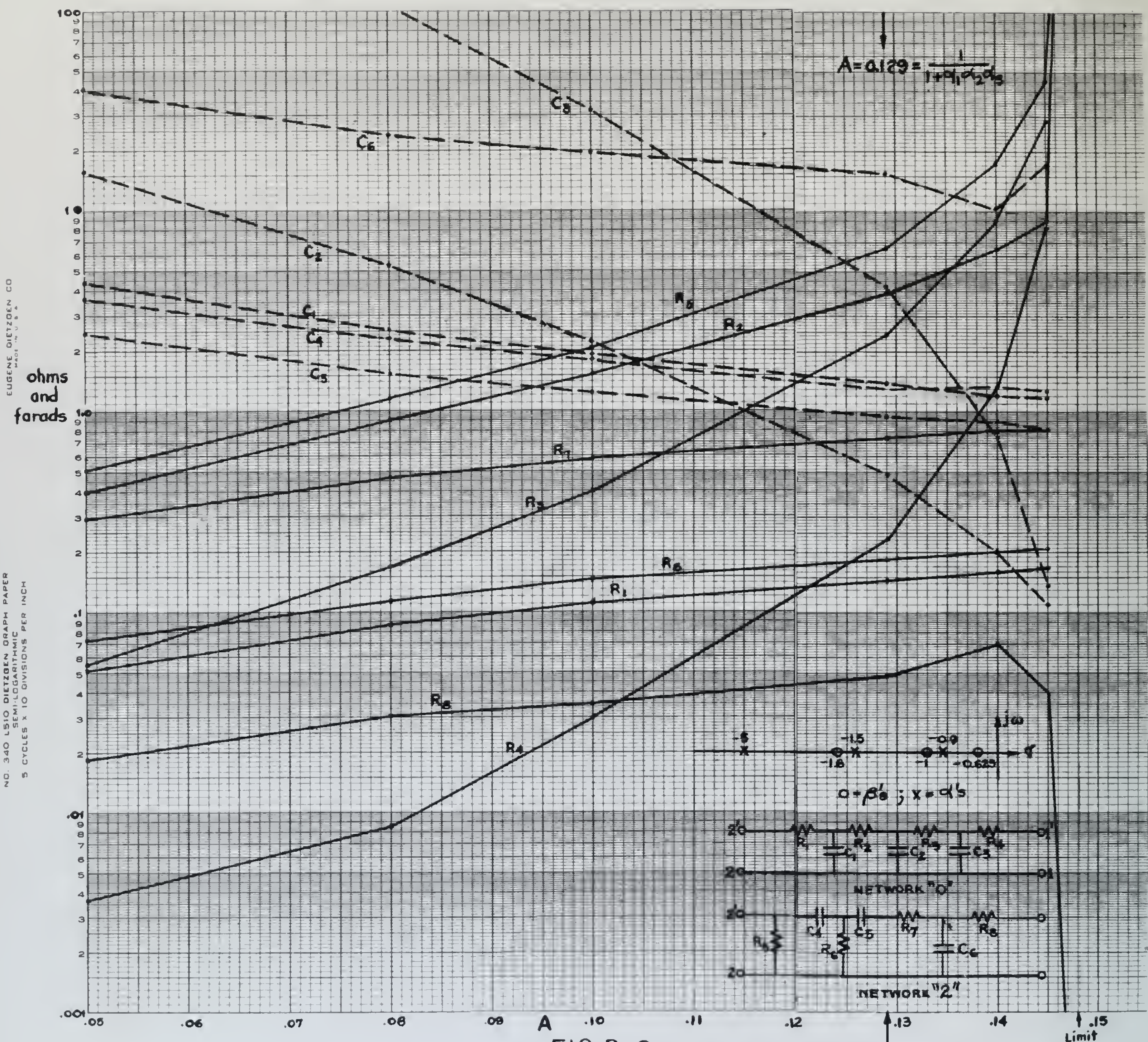

A = 0.129 = $\frac{1}{1+\alpha_1\alpha_2\alpha_3}$
ohms and farads
A
.05
.06
.07
.08
.09
.10
.11
.12
.13
.14
.15
Limit
C_1
C_2
C_3
C_4
C_5
C_6
R_1
R_2
R_3
R_4
R_5
R_6
R_7
R_8
○ = β's ; x = α's
NETWORK "0"
NETWORK "2"
EUGENE DIETZGEN CO.
NO. 340 L510 DIETZGEN GRAPH PAPER
SEMI-LOGARITHMIC
5 CYCLES X 10 DIVISIONS PER INCH

FIG B-2

BIBLIOGRAPHY

1. Thiessen, G.J. ..... "R-C Filter Circuits", Journal of the Acoustical Society of America, 16, pp 275-9, April, 1945.

2. Fritzinger, G.H. ....."Frequency Discrimination by Inverse Feed-back", Proc. I.R.E., vol. 26, pp 207-25, Feb., 1938.

3. Toshniwal, B.D. ..... "The Realization of a Band-Pass Amplifier by Utilizing Feedback Principles", S. M. Thesis, M.I.T., Sept., 1939.

4. Linvill, J.G. ..... "An Experimental Approach to the Approximation Problem for Driving-Point and Transfer Functions", S. M. Thesis, M.I.T., 1945.

5. Scarborough, J.B. ..... "Numerical Mathematical Analysis", John Hopkins Press, 1930.

6. Gewertz, Chas. M:Son ....."Network Synthesis", Williams and Wilkins Co., 1933.

7. Guillemin, E.A. ..... "Synthesis of RC-Networks", unpublished memorandum, 1945.

8. Guillemin, E.A. ..... "Communication Networks", Vol. 2, John Wiley and Sons, 1935.

9. Darlington, S. ..... "Synthesis of Reactance 4-Poles Which Produce Prescribed Insertion Loss Characteristics," Journal of Mathematics and Physics, Vol. 18, pp 257-353, 1939.

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