In this report, we give a counterexample to show that mutual Galois embeddings do not imply order isomorphy, even for complete lattices and linear orders.

We take the definitions from [BKS00, Sect.1.1, p.8].

Definition 1. *(Partially, totally ordered space)* The pair $\langle A, \leqslant_A \rangle$ is called a partially ordered space if A is a set and $\leqslant_A$ is a partial order on A, that is, a reflexive, antisymmetric, and transitive relation on A.

If $\leqslant_A$ is also connex, we call it a total order on A. □

Definition 2. *(Monotone, isotone function)* If $\langle A, \leqslant_A \rangle$ and $\langle B, \leqslant_B \rangle$ are partially ordered spaces, a function $f : A \to B$ is called monotone if for each $x_1, x_2 \in A$ with $x_1 \leqslant_A x_2$, we have $f(x_1) \leqslant_B f(x_2)$. Such an f is called isotone if moreover $f(x_1) \leqslant_B f(x_2)$ implies $x_1 \leqslant_A x_2$ for each $x_1, x_2 \in A$. □

Definition 3. *(Galois connection, embedding)* If $\langle A, \leqslant_A \rangle$ and $\langle B, \leqslant_B \rangle$ are partially ordered spaces, a Galois connection between them is a pair of monotone functions $\varepsilon : A \to B$ and $\pi : B \to A$ such that

1. for each $x \in A$, we have $\pi(\varepsilon(x)) \geqslant_A x$, and dually
2. for each $y \in B$, we have $\varepsilon(\pi(y)) \leqslant_B y$.

If moreover 1. can be sharpened to

1'. for each $x \in A$, we even have $\pi(\varepsilon(x)) = x$,

then the pair $\langle \varepsilon, \pi \rangle$ is called a Galois embedding. □

Definition 4. *(Counterexample construction)* Consider closed intervals of real numbers, and let $A = [0, 1]$ and $B = [0, 1] \cup [2, 3]$, both with the usual order.

Let $\varepsilon_1 : A \to B$ be defined by $\varepsilon_1(x) = x$, and $\pi : B \to A$ by

$$\pi_1(y) = \begin{cases} y & \text{if} \quad 0 \leqslant y \leqslant 1 \\ 1 & \text{if} \quad 1 < y \end{cases} .$$

Let $\varepsilon_2 : B \to A$ be defined by $\varepsilon_2(y) = {}^{y}\!/_{3}$, and $\pi_2 : A \to B$ by

$$\pi_2(x) = \begin{cases} 3x & \text{if} \quad 0 \leqslant x \leqslant {}^{1}\!/_{3} \\ 1 & \text{if} \quad {}^{1}\!/_{3} < x < {}^{2}\!/_{3} \\ 3x & \text{if} \quad {}^{2}\!/_{3} \leqslant x \leqslant 1 \end{cases} .$$

□

The definitions of A, B, ε_1, π_1, ε_2, π_2 are illustrated in Fig. 1.

Obviously, $\langle A, \leqslant_A \rangle$ and $\langle B, \leqslant_B \rangle$ are partially ordered spaces. Moveover, their orders are even linear, and make both A and B a complete lattice.

The functions ε_1 and ε_2 are trivially monotone. The monotonicity of π_1 and π_2 can be seen from their graphs in Fig. 1.

Lemma 5. $\langle \varepsilon_1, \pi_1 \rangle$ is a Galois embedding between A and B.

Proof. We make case distinctions according to the function definitions.
Let $x \in [0, 1]$, then

$$\begin{array}{lll} & \pi_1(\varepsilon_1(x)) & \\ = & \pi_1(x) & \\ = & x & \text{since } 0 \leqslant x \leqslant 1 \end{array}$$

Let $y \in [0, 1]$, then

$$\begin{array}{lll} & \varepsilon_1(\pi_1(y)) & \\ = & \varepsilon_1(y) & \text{since } 0 \leqslant y \leqslant 1 \\ = & y & \\ \leqslant & y & \end{array}$$

Let $y \in [2, 3]$, then

$$\begin{array}{lll} & \varepsilon_1(\pi_1(y)) & \\ = & \varepsilon_1(1) & \text{since } 1 < y \\ = & 1 & \\ \leqslant & y & \end{array}$$

Lemma 6. $\langle \varepsilon_2, \pi_2 \rangle$ is a Galois embedding between B and A.

Proof. We make case distinctions according to the function definitions.
Let $y \in [0, 1]$, then

$$\begin{array}{lll} & \pi_2(\varepsilon_2(y)) & \\ = & \pi_2(y/3) & \\ = & 3y/3 & \text{since } 0 \leqslant y/3 \leqslant 1/3 \\ = & y & \end{array}$$

Let $y \in [2, 3]$, then

$$\begin{array}{lll} & \pi_2(\varepsilon_2(y)) & \\ = & \pi_2(y/3) & \\ = & 3y/3 & \text{since } 2/3 \leqslant y/3 \leqslant 1 \\ = & y & \end{array}$$

Let $x \in [0, 1/3]$, then

$$\begin{array}{cll} & \varepsilon_2(\pi_2(x)) & \\ = & \varepsilon_2(3x) & \text{since } 0 \leqslant x \leqslant 1/3 \\ = & 3x/3 & \\ \leqslant & x & \end{array}$$

Let $x \in (1/3, 2/3)$, then

$$\begin{array}{cll} & \varepsilon_2(\pi_2(x)) & \\ = & \varepsilon_2(1) & \text{since } 1/3 < x < 2/3 \\ = & 1/3 & \\ \leqslant & x & \end{array}$$

Let $x \in [2/3, 1]$, then

$$\begin{array}{cll} & \varepsilon_2(\pi_2(x)) & \\ = & \varepsilon_2(3x) & \text{since } 2/3 \leqslant x \leqslant 1 \\ = & 3x/3 & \\ \leqslant & x & \end{array}$$

$\square$

However, the partially ordered spaces $\langle A, \leqslant \rangle$ and $\langle B, \leqslant \rangle$ cannot be isomorphic, since $\leqslant$ is a dense order when restricted to A, but not when restricted to B:

If $\psi : [0, 1] \to [0, 1] \cup [2, 3]$ was an isotone bijection, then

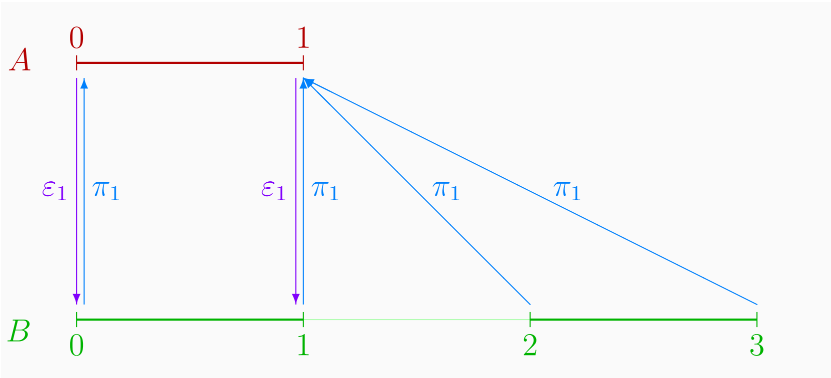
$$\begin{array}{lcccccl} & 1 & & < & & 2 & \\ \Rightarrow & \psi^{-1}(1) & & < & & \psi^{-1}(2) & \\ \Rightarrow & \psi^{-1}(1) & < & x & < & \psi^{-1}(2) & \text{for some } x\text{, since } (\leqslant) \text{ is dense on } [0,1] \\ \Rightarrow & \multicolumn{5}{l}{1 = \psi(\psi^{-1}(1)) < \psi(x) < \psi(\psi^{-1}(2)) = 2} & , \end{array}$$

where the last condition contradicts the range of ψ.

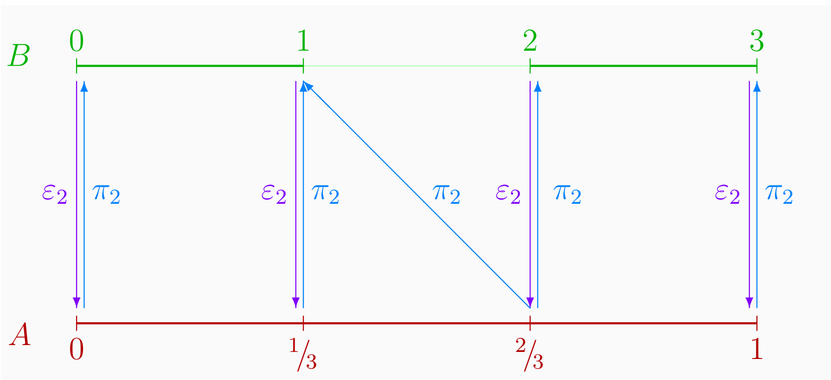
References

[BKS00] Jochen Burghardt, Florian Kammüller, and Jeff W. Sanders. Isomorphism of Galois embeddings. GMD Report 122, GMD, Dec 2000.

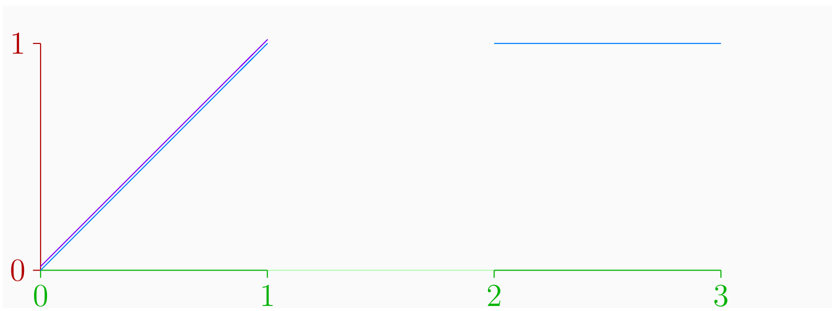
Jochen Burghardt, 2 Jun 2022



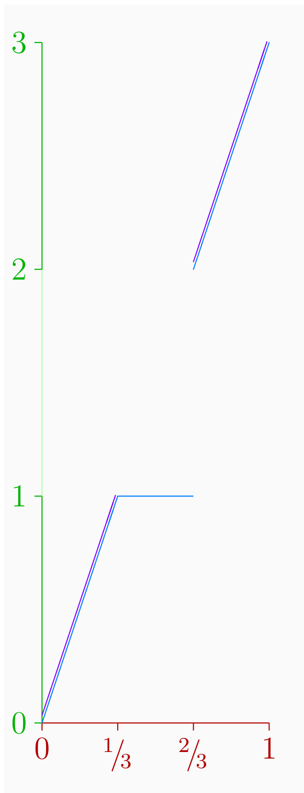
First Galois embedding



Second Galois embedding (A magnified)



Graph of ε_1, π_1



Graph of ε_2, π_2

Figure 1: Galois embeddings in the counterexample