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**Metallic Means : Beyond the Golden Ratio,  
New Mathematics and Geometry of all Metallic Ratios based upon Right Triangles,  
The Formation of the “Triads” of Metallic Means,  
And their Classical Correspondence with Pythagorean Triples and  $p \equiv 1 \pmod{4}$  Primes,  
Also the Correlation between Metallic Numbers and the Digits 3 6 9  
Triangles – Triads – Triples - & 3 6 9**

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## Abstract

This paper brings together the newly discovered generalised geometry of all Metallic Means and the recently published mathematical formulae those provide the precise correlations between different Metallic Ratios. The paper also puts forward the concept of the “Triads of Metallic Means”. This work also introduces the close correspondence between Metallic Ratios and the Pythagorean Triples as well as Pythagorean Primes. Moreover, this work illustrates the intriguing relationship between Metallic Numbers and the Digits 3 6 9.

**Keywords:** Fibonacci sequence, Pi, Phi, Pythagoras Theorem, Divine Proportion, Silver Ratio, Golden Mean, Right Triangle, Metallic Numbers, Pell Numbers, Lucas Numbers, Golden Proportion, Metallic Ratio, Metallic Triples, 3 6 9, Pythagorean Triples, Pythagorean Primes, Golden Ratio, Metallic Mean

## Introduction

The prime objective of this work is to synergize the following couple of newly discovered aspects of Metallic Means:

- 1) The Generalised Geometric Construction of all Metallic Ratios: cited by Wikipedia in its page on “Metallic Mean”[1]. This generalised geometric substantiation of all Metallic Means was published in January 2021 [5]
- 2) The Mathematical Formula that provides the precise correlation between different Metallic Means. This explicit formula has been recently published in the month of May 2021 [6]

The *synergism* between above two features of Metallic Means unveils an intriguing pattern of Metallic Ratios, which asserts that the mathematical implications of these Means have not been fully appreciated so far.



Is Golden Ratio to be perceived only in handful of Botanical formations ?..... and the Silver Ratio just to be observed in the man-made A4, A5, A6 paper sizes.....?

Nope! The whole family of Metallic Ratios is certainly far more important than that. And, the abovementioned Geometry and Mathematics synergically enables us to recognize the full worth of these Metallic Means, as follows.

As a brief introduction, each Metallic Mean  $\delta_n$  is the root of the simple Quadratic Equation  $X^2 - nX - 1 = 0$ , where  $n$  is any positive natural number.

Thus, the fractional expression of the  $n^{\text{th}}$  Metallic Ratio is  $\delta_n = \frac{n + \sqrt{n^2 + 4}}{2}$

Moreover, each Metallic Ratio can be expressed as the continued fraction:

$$\delta_n = n + \frac{1}{n + \frac{1}{n + \frac{1}{n + \dots}}} ; \text{ And hence, } \delta_n = n + \frac{1}{\delta_n} \quad \dots\text{References: [2] [3] [4]}$$

### GEOMETRIC CONSTRUCTION OF ALL METALLIC MEANS :

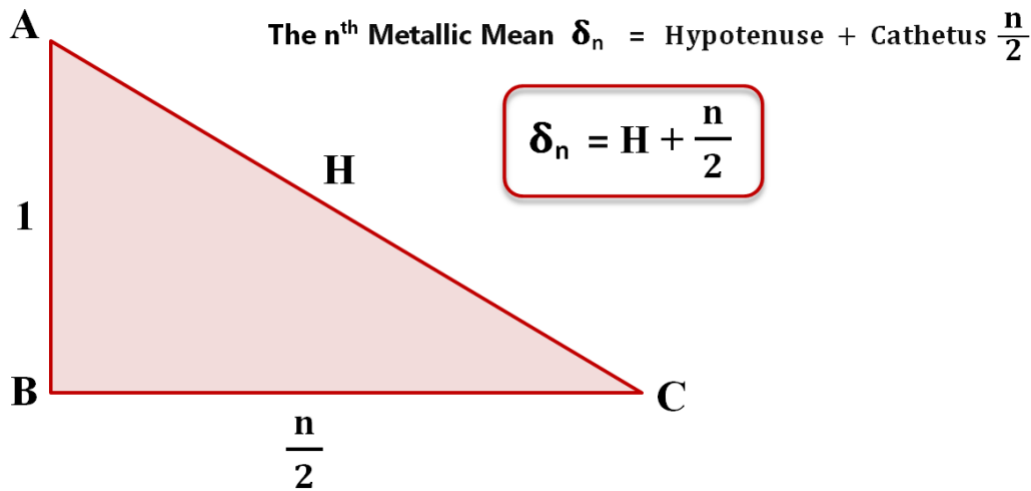
Golden Ratio in regular Pentagon, and Silver Ratio in regular Octagon, but contrary to natural expectations, in *Polygons, Diagonals, and the Bronze Mean*, Antonia Redondo Buitrago clearly demonstrated that the Bronze Mean does not appear as a side to diagonal relationship in a polygon [7] [8]. And for long time, there remained the question of whether or not the higher Metallic Means appear as a ratio between line segments? Now the question has been answered positively. [1] [5]

Each Metallic Ratio can be constructed geometrically with a special Right Angled Triangle. Any  $n^{\text{th}}$  Metallic Mean can be represented by the Right Triangle having its catheti  $1$  and  $\frac{n}{2}$ . Hence, the right triangle with one of its catheti =  $1$  may substantiate any Metallic Mean, having its second cathetus =  $\frac{n}{2}$ , where  $n = 1$  for Golden Ratio,  $n = 2$  for Silver Ratio,  $n = 3$  for Bronze Ratio, and so on. Such Right Triangle provides the precise value of  $n^{\text{th}}$  Metallic Mean by the generalised formula:

$$\text{The } n^{\text{th}} \text{ Metallic Mean } (\delta_n) = \text{Hypotenuse} + \text{Cathetus } \frac{n}{2}$$

Such Right Triangle not only provides for the accurate geometric construction and precise fractional expression of any  $n^{\text{th}}$  Metallic Mean ( $\delta_n$ ), but its every geometric feature is the prototypical form of that Metallic Mean [5] [9] [10]. The characteristic geometry of such Right Triangle having its catheti  $1$  and  $\frac{n}{2}$ , is resplendent with the corresponding  $n^{\text{th}}$  Metallic Mean ( $\delta_n$ ) embedded in its every aspect. For example, the remarkable expression of Golden Ratio in every geometric feature of  $1:2:\sqrt{5}$  triangle, including all its angles and side lengths, its 'Incenter-Excenters Orthocentric system', its Gergonne and Nagel triangles, and also the Nobbs points and the Gergonne line, various triangle centers as well as the Incircle of  $1:2:\sqrt{5}$  triangle, make this triangle the quintessential form of the Golden Ratio ( $\phi$ ) and also of the fourth Metallic Mean ( $\phi^3$ ). [5]





**Figure 1:** Generalised Right Triangle to represent any  $n^{\text{th}}$  Metallic Mean  $\delta_n$

**MATHEMATICAL CORRELATIONS AMONG DIFFERENT METALLIC RATIOS :**

If  $k, m$  and  $n$  are three positive integers such that  $\frac{mn + 4}{m - n} = k$

then, it is observed that

$\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$  where  $\delta_k, \delta_m$  and  $\delta_n$  are the  $k^{\text{th}}, m^{\text{th}}$  and  $n^{\text{th}}$  Metallic Means respectively. This explicit formula, among the several other formulae those give the precise mathematical relations between different Metallic Means, has been recently published in the work mentioned in Reference: [6]

**METALLIC MEANS’ “TRIADS” :**

The abovementioned explicit formula gives the “Triads” of Metallic Means, as  $[\delta_n, \delta_m, \delta_k]$ ,

where  $\frac{mn + 4}{m - n} = k$  and  $\frac{kn + 4}{k - n} = m$

hence,  $\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$  and also  $\frac{\delta_k \times \delta_n + 1}{\delta_k - \delta_n} = \delta_m$

For example, if  $n=6$ , the three integers 6, 11 and 14 satisfy the prerequisite  $\frac{mn + 4}{m - n} = k$  ; Hence, the three Meralllic means  $\delta_6, \delta_{11}$  and  $\delta_{14}$  form a **Triad**, such that  $\frac{\delta_{11} \times \delta_6 + 1}{\delta_{11} - \delta_6} = \delta_{14}$  and also  $\frac{\delta_{14} \times \delta_6 + 1}{\delta_{14} - \delta_6} = \delta_{11}$

Just like  $n=6$  exemplified above, every integer forms such multiple triads, as shown below for first ten integers.

**Table 1: "Triads" of Metallic Means formed by the First Ten Metallic Means:**

<b>n</b>	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>	<b>1</b>
<b>m</b>	<b>2</b>	<b>6</b>						
<b>k</b>	<b>6</b>	<b>2</b>						

<b>n</b>	<b>2</b>	<b>2</b>	<b>2</b>	<b>2</b>	<b>2</b>	<b>2</b>	<b>2</b>	<b>2</b>
<b>m</b>	<b>3</b>	<b>4</b>	<b>6</b>	<b>10</b>				
<b>k</b>	<b>10</b>	<b>6</b>	<b>4</b>	<b>3</b>				

<b>n</b>	<b>3</b>	<b>3</b>	<b>3</b>	<b>3</b>	<b>3</b>	<b>3</b>	<b>3</b>	<b>3</b>
<b>m</b>	<b>4</b>	<b>16</b>						
<b>k</b>	<b>16</b>	<b>4</b>						

<b>n</b>	<b>4</b>	<b>4</b>	<b>4</b>	<b>4</b>	<b>4</b>	<b>4</b>	<b>4</b>	<b>4</b>
<b>m</b>	<b>5</b>	<b>6</b>	<b>8</b>	<b>9</b>	<b>14</b>	<b>24</b>		
<b>k</b>	<b>24</b>	<b>14</b>	<b>9</b>	<b>8</b>	<b>6</b>	<b>5</b>		

<b>n</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>	<b>5</b>
<b>m</b>	<b>6</b>	<b>34</b>						
<b>k</b>	<b>34</b>	<b>6</b>						

<b>n</b>	<b>6</b>	<b>6</b>	<b>6</b>	<b>6</b>	<b>6</b>	<b>6</b>	<b>6</b>	<b>6</b>
<b>m</b>	<b>7</b>	<b>8</b>	<b>10</b>	<b>11</b>	<b>14</b>	<b>16</b>	<b>26</b>	<b>46</b>
<b>k</b>	<b>46</b>	<b>26</b>	<b>16</b>	<b>14</b>	<b>11</b>	<b>10</b>	<b>8</b>	<b>7</b>

: Shaded Triads have been exemplified above.



<b>n</b>	<b>7</b>	<b>7</b>	<b>7</b>	<b>7</b>	<b>7</b>	<b>7</b>	<b>7</b>	<b>7</b>
<b>m</b>	<b>8</b>	<b>60</b>						
<b>k</b>	<b>60</b>	<b>8</b>						

<b>n</b>	<b>8</b>	<b>8</b>	<b>8</b>	<b>8</b>	<b>8</b>	<b>8</b>	<b>8</b>	<b>8</b>
<b>m</b>	<b>9</b>	<b>10</b>	<b>12</b>	<b>25</b>	<b>42</b>	<b>76</b>		
<b>k</b>	<b>76</b>	<b>42</b>	<b>25</b>	<b>12</b>	<b>10</b>	<b>9</b>		

<b>n</b>	<b>9</b>	<b>9</b>	<b>9</b>	<b>9</b>	<b>9</b>	<b>9</b>	<b>9</b>	<b>9</b>
<b>m</b>	<b>10</b>	<b>14</b>	<b>26</b>	<b>94</b>				
<b>k</b>	<b>94</b>	<b>26</b>	<b>14</b>	<b>10</b>				

<b>n</b>	<b>10</b>	<b>10</b>	<b>10</b>	<b>10</b>	<b>10</b>	<b>10</b>	<b>10</b>	<b>10</b>
<b>m</b>	<b>11</b>	<b>12</b>	<b>14</b>	<b>18</b>	<b>23</b>	<b>36</b>	<b>62</b>	<b>114</b>
<b>k</b>	<b>114</b>	<b>62</b>	<b>36</b>	<b>23</b>	<b>18</b>	<b>14</b>	<b>12</b>	<b>11</b>

It may be noticed from above Table that every  $n^{\text{th}}$  Metallic Mean can give precise values of various Metallic Means by the formula  $\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$ , maximum upto  $(n^2 + n + 4)^{\text{th}}$  Metallic Mean:

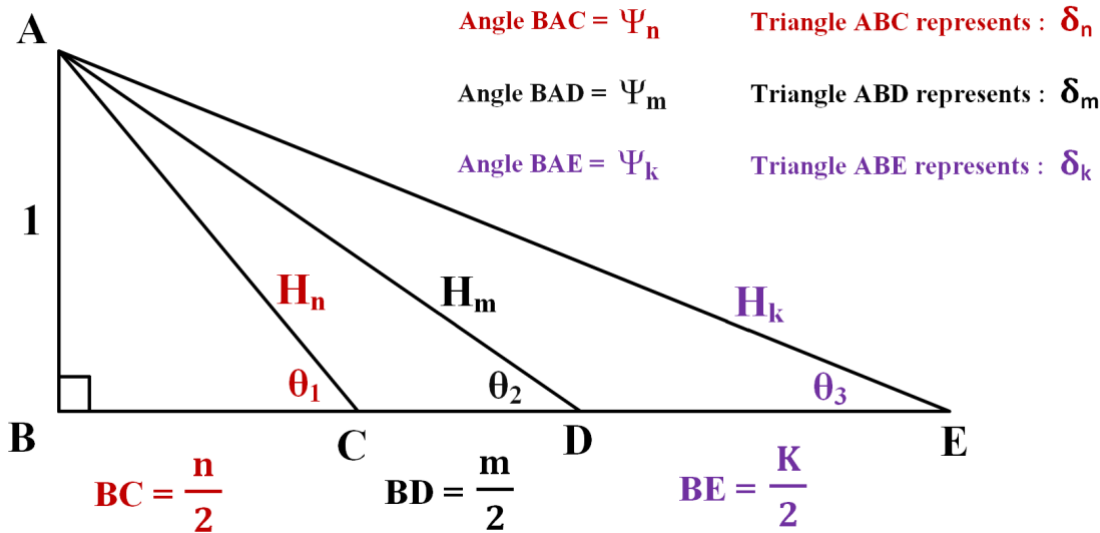
$$m_{\text{max}} = k_{\text{max}} = (n^2 + n + 4)$$

Also noticeably, the Even Integers ( Even  $n_s$  ) form comparatively more Triads than the Odd  $n_s$  . Several such patterns about these Triads will be discussed in detail, in the subsequent parts of this paper.

More importantly, such Triads can be represented geometrically, as shown below.



For instance, the **Triad** [  $\delta_n, \delta_m, \delta_k$  ] is illustrated geometrically in following **Figure 2**.



**Figure 2:** Three Right Triangles representing the “Triad of Metallic Means”

Remarkably, in above Figure 2: if the three Metallic Means  $\delta_n, \delta_m$  and  $\delta_k$  constitute a **Triad**, then

$$\theta_1 = \theta_2 + \theta_3$$

In other words,

$$\arctan \frac{2}{n} = \arctan \frac{2}{m} + \arctan \frac{2}{k}$$

Solving it gives the couple of correlations:  $\frac{mn + 4}{m - n} = k$  and  $\frac{kn + 4}{k - n} = m$

And hence satisfy the prerequisite for  $\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$  and also  $\frac{\delta_k \times \delta_n + 1}{\delta_k - \delta_n} = \delta_m$

Moreover, entire geometry of such Triad-Triangles is resplendent with the precise correlation among the three Metallic Means  $\delta_n, \delta_m$  and  $\delta_k$ . Following intriguing relations are observed in above Figure 2. Consider the larger acute angles of the three triangles;

$$\Psi_m + \Psi_k = 2 \arctan \delta_n$$

$$\Psi_m - \Psi_n = 2 \arctan \frac{1}{\delta_k}$$

$$\Psi_k - \Psi_n = 2 \arctan \frac{1}{\delta_m}$$

Also, it is observed that,  $\Psi_k - \Psi_m = 2 \arctan \frac{1}{\delta_x}$  where  $x = \frac{km + 4}{k - m}$  and the mean  $\delta_x$  lies somewhere between  $\delta_k$  and  $\delta_m$ . However, unlike  $n, m$  and  $k$ , this  $X$  may or may not be an integer. In certain cases, the  $X$  is an integer, for example, in case of  $n=4$  and the Triad  $[\delta_4, \delta_6, \delta_{14}]$ , the value of  $X$  is 11, and hence interestingly enough, it forms new Triad of Metallic Means  $[\delta_6, \delta_{11}, \delta_{14}]$  with  $n=6$  i.e. a new  $[m, x, k]$  Triad is generated from  $[n, m, k]$ .

Moreover, the Hypotenuses of the Triad-Triangles:  $H_n, H_m$  and  $H_k$  in Figure 2 exhibit following relations.

$$\frac{H_k}{H_m} = \frac{2}{m - n} \times H_n$$

However, most interesting aspect of these **Triads** of Metallic Means is that they exhibit a classical correspondence with **Primitive Pythagorean Triples**.

### Triads of Metallic Means and the Primitive Pythagorean Triples :

Each of the abovementioned Triad of Metallic Mean  $[\delta_n, \delta_m, \delta_k]$  is associated with a particular primitive Pythagorean triple, depending upon the value of  $n$ : the smallest of three integers  $[k, m, n]$ . For example, all Triads with  $n=6$  are associated with 3-4-5 Pythagorean Triple. For instance, consider the Triad  $[\delta_6, \delta_{11}, \delta_{14}]$ . If this Triad is constructed geometrically, as in above **Figure 2**, then it is observed that:

$$(\theta_1 + \theta_2 + \theta_3) = \text{The Smaller Acute Angle of 3-4-5 Pythagorean Triangle i.e. } \arctan \frac{3}{4}$$

$$(\Psi_n + \Psi_m + \Psi_k) - 180^\circ = \text{The Larger Acute Angle of 3-4-5 Pythagorean Triangle i.e. } \arctan \frac{4}{3}$$

Likewise, all Triads with  $n=3$  or  $n=10$  are related to 5-12-13 Pythagorean Triple, and so on.

$$\text{Consider the fractional expression of the } n^{\text{th}} \text{ Metallic Mean : } \delta_n = \frac{n + \sqrt{n^2 + 4}}{2}$$

The radical  $(n^2 + 4)$  in this Fractional expression of  $\delta_n$  determines the Pythagorean Triple associated with such Triad of Metallic Means:  $[\delta_n, \delta_m, \delta_k]$  with  $n$  being the smallest of the three integers  $k, m$  and  $n$ . Any such Triad of Metallic Means  $[\delta_n, \delta_m, \delta_k]$  is associated with such Primitive Pythagorean Triple whose Hypotenuse is Factor of the Radical  $(n^2 + 4)$ , with following observed sub-rules :

**If  $n$  is Odd :** Hypotenuse of associated Pythagorean Triple =  $(n^2 + 4)$

**If  $n$  is Even and multiple of Four,  $n=4x$  :** Hypotenuse of associated Pythagorean Triple =  $nx + 1$

**If  $n$  is Even but not Multiple of Four :** Smaller Cathetus of associated Pythagorean Triple =  $\frac{n}{2}$



Following **Table 2** shows associated Pythagorean Triples for first few values of  $\mathbf{n}$ :

**Table 2:** Primitive Pythagorean Triples associated with “Triads” of Metallic Means: depending upon  $\mathbf{n}$

$\mathbf{n}$	Associated Pythagorean Triple
1	3-4-5
2	None
3	5-12-13
4	3-4-5
5	20-21-29
6	3-4-5
7	28-45-53
8	8-15-17
9	36-77-85
10	5-12-13
11	44-117-125
12	12-35-37
13	52-165-173
14	7-24-25

The classical correspondence between these Triads of Metallic Means and the corresponding Primitive Pythagorean Triple is manifested as follows.

Consider again the **Figure 2** representing the **Triad** of Metallic Means [  $\delta_n, \delta_m, \delta_k$  ].

$(\theta_1 + \theta_2 + \theta_3) =$  The **Smaller** Acute Angle of associated Pythagorean Triple.

$(\Psi_n + \Psi_m + \Psi_k) - 180^\circ =$  The **Larger** Acute Angle of associated Pythagorean Triple.

**Special cases :**

- For  $\mathbf{n=1}$  : the right Triangle that represents Golden Ratio has its Cathetus 1 longer than its Second Cathetus  $\frac{n}{2}$ . Hence, for the Triad formed with  $n=1$  [  $\delta_1, \delta_2, \delta_6$  ] :  
 $(\theta_1 + \theta_2 + \theta_3) - 90^\circ =$  The Smaller Acute Angle of associated Pythagorean Triple 3-4-5.  
 And,  $(\Psi_n + \Psi_m + \Psi_k) - 90^\circ =$  The Larger Acute Angle of associated Pythagorean Triple.
- Similarly, in case of  $\mathbf{n=2}$  : the Triangle representing Silver Ratio is an Isosceles Right Triangle. Hence, for Triad formed with  $n=2$ , like [  $\delta_2, \delta_3, \delta_{10}$  ] or [  $\delta_2, \delta_4, \delta_6$  ]:  
 $(\theta_1 + \theta_2 + \theta_3) = 90^\circ$ ,  
 and  $(\Psi_n + \Psi_m + \Psi_k) = 180^\circ$





And hence there is no associated Pythagorean Triple for  $n=2$ .

3) For  **$n=3$  or  $4$**  :

$(\theta_1 + \theta_2 + \theta_3) =$  The **Larger** Acute Angle of associated Pythagorean Triple 3-4-5.

And,  $(\Psi_n + \Psi_m + \Psi_k) - 180^\circ =$  The **Smaller** Acute Angle of associated Pythagorean Triple.

Note the Right Triangles those represent the First and Fourth Metallic Means:  $\delta_1$  ( that is Golden Ratio  $\phi$ ) and  $\delta_4$  (which equals  $\phi^3$ ), are similar triangles.

In all the Triads formed with  $n=5$  onwards, ( $n = 5, 6, 7, \dots$  and so on) :

**$(\theta_1 + \theta_2 + \theta_3) =$**  The **Smaller** Acute Angle of associated Pythagorean Triple.

Similarly,  **$(\Psi_n + \Psi_m + \Psi_k) - 180^\circ =$**  The **Larger** Acute Angle of associated Pythagorean Triple.

Moreover, an intriguing relation is also observed between the Hypotenuses of the Three Triangles in **Figure 2** ( viz.  $H_n, H_m$  and  $H_k$ ) and the Hypotenuse of associated Pythagorean Triple **H**.

If  $n$  is Odd : 
$$\frac{H_k}{H_m} \times (m-n) = \sqrt{H}$$

If  $n$  is Even and multiple of Four : 
$$\frac{H_k}{H_m} \times (m-n) = 2\sqrt{H}$$

If  $n$  is Even but not Multiple of Four : 
$$\frac{H_k}{H_m} \times (m-n) = 2\sqrt{2}\sqrt{H}$$

Thus, consider the following couple of facts. **First:** the generalised geometric substantiation of all Metallic Means based upon Right Triangles, as elaborated in the work mentioned in References [1] and [5]. And **second:** the close correspondence between the abovementioned Triads of Metallic Ratios and the Primitive Pythagorean Triples, as described here. These couple of facts clearly highlight the underlying proposition that the Metallic Means are more closely associated with; and more holistically represented by the **"Right Angled Triangles"**, rather than Pentagon, Octagon or any other  $(n^2+4)$ -gon.

### Metallic Means and Pythagorean Primes : The Prime Families of Metallic Means

From the close correspondence between Metallic Means and Pythagorean Triples described so far, it becomes obvious that various Metallic Means are also closely associated with different Pythagorean Primes.

Consider the radical  **$(n^2+4)$**  in the Fractional expression of the  $n^{\text{th}}$  Metallic Mean ( $\delta_n$ ). By Fermat's Theorem on Sums of Two Squares, this radical is an integer multiple of a prime of the form  $p \equiv 1 \pmod{4}$ . The **Greatest Prime Factor** of this radical  $(n^2 + 4)$  is a Pythagorean Prime, as shown below in **Table 3**.



<b>n</b>	<b><math>n^2+4</math></b>	<b>Greatest Prime Factor of <math>(n^2+4)</math> : A Pythagorean Prime</b>
1	5	5
2	8	2
3	13	13
4	20	5
5	29	29
6	40	5
7	53	53
8	68	17
9	85	17
10	104	13
11	125	5
12	148	37
13	173	173
14	200	5
15	229	229
16	260	13

**Table 3:** The Greatest Prime Factors of the radical  $(n^2 + 4)$

It is noticeable from above table that various values of  $n$  have common Greatest Prime Factor of  $(n^2 + 4)$ .

For example, for 3<sup>th</sup>, 10<sup>th</sup> and 16<sup>th</sup> Metallic Means, the common Greatest Prime Factor of the radical  $(n^2 + 4)$  is **13**.

Also, for 1<sup>st</sup>, 4<sup>th</sup>, 6<sup>th</sup>, 11<sup>th</sup> Metallic Means, the common Greatest Prime Factor of the radical  $(n^2 + 4)$  is **5**.

Hence, the different Metallic Means can be classified into various groups corresponding to the Greatest Prime Factors (GPF) of the radical  $(n^2+4)$ . This GPF is necessarily a Pythagorean Prime  $(4x + 1)$ , as shown below in **Table 4**.

**Table 4 : Prime Families of Metallic Means**

<b>Greatest Prime Factor (GPF) of <math>[n^2+4]</math></b>	<b><math>n_s</math> of the associated Metallic Means (<math>\delta_n</math>)</b>
<b>5</b>	<b>1, 4, 6, 11</b>
<b>13</b>	<b>3, 10, 16, 29</b>
<b>17</b>	<b>8, 9, 26, 43</b>
<b>29</b>	<b>5, 24, 34</b>
<b>37</b>	<b>12, 25, 49</b>
<b>41</b>	<b>18, 23, 59</b>

Noticeably, the Hypotenuse of associated Pythagorean Triple is a factor of  $(n^2+4)$ , and the associated Pythagorean Primes, as shown in Table 4 are the Greatest Prime Factors of  $(n^2+4)$ . Hence note: the 8<sup>th</sup> and the 9<sup>th</sup> Metallic Means both have Pythagorean Prime 17 as the GPF of their  $(n^2+4)$ , however they have different associated Primitive Pythagorean Triples, as shown in Table 2.

Moreover, beside the Greatest Prime Factors indicated in above Table, multiple Pythagorean Primes constitute the factors of various  $(n^2+4)$  radicals.

$$(59)^2 + 4 = (41 \times 17 \times 5)$$

$$(49)^2 + 4 = (37 \times 13 \times 5), \text{ and so on.}$$

Noticeably, the Greatest Prime Factor for  $n=43$  is 109

$$(43)^2 + 4 = 109 \times 17$$

However, why it has been included in the Prime 17 Family in above Table, that will become obvious with following couple of illustrations.



Remarkably, the Metallic Means belonging to same Prime Family exhibit very distinctive relations among themselves, as illustrated below.

For instance, consider the Prime Family of Metallic Means associated with the Pythagorean Prime **13** :  $\delta_3$ ,  $\delta_{10}$  and  $\delta_{16}$

Metallic Means associated with the Pythagorean Prime **13** :  $\delta_3$ ,  $\delta_{10}$  and  $\delta_{16}$

**Prime 13**

$$\begin{matrix} 10+3 & = & 13 & = & 16-3 \\ \boxed{\delta_{10}} & & \boxed{\delta_3} & & \boxed{\delta_{16}} \end{matrix}$$

Hence, the Pythagorean Prime of this Family **13** =  $\frac{3^2 + 4}{1}$  : Note the Digit **1** in Denominator.

Here, the **3<sup>rd</sup>** and **(3 + 1)<sup>th</sup>** Metallic Means give the precise value of **16<sup>th</sup>** Mean :  $\frac{\delta_4 \times \delta_3 + 1}{\delta_4 - \delta_3} = \delta_{16}$

Similarly, the **3<sup>rd</sup>** and **(3 - 1)<sup>th</sup>** Metallic Means give the precise value of **10<sup>th</sup>** Mean :  $\frac{\delta_3 \times \delta_2 + 1}{\delta_3 - \delta_2} = \delta_{10}$

Moreover,  $\frac{13 - 1}{2} = 6$  :this 6 forms a new Triad with members 10 and 16 of the family: [ $\delta_6$ ,  $\delta_{10}$ ,  $\delta_{16}$ ]:

$$\frac{\delta_{16} \times \delta_6 + 1}{\delta_{16} - \delta_6} = \delta_{10} \quad \text{and} \quad \frac{\delta_{10} \times \delta_6 + 1}{\delta_{10} - \delta_6} = \delta_{16}$$

Further, beyond 3, 10 and 16, more members would be added in the family of Pythagorean Prime 17, next member in the family would be 29 which equals the sum of all smaller members in the family;

$3 + 10 + 16 = 29$  and, the Prime 13 =  $29 - 16$

And,

$2 \times 16 + 3 + 1 = 36$  :this 36 forms a new Triad with members 16 and 29 of the family [ $\delta_{16}$ ,  $\delta_{29}$ ,  $\delta_{36}$ ]:

$$\frac{\delta_{36} \times \delta_{16} + 1}{\delta_{36} - \delta_{16}} = \delta_{29} \quad \text{and} \quad \frac{\delta_{29} \times \delta_{16} + 1}{\delta_{29} - \delta_{16}} = \delta_{36}$$

Likewise, consider another example for illustration and comparison. the Metallic Means associated with the Pythagorean Prime **17** :  $\delta_8$ ,  $\delta_9$  and  $\delta_{26}$

Metallic Means associated with the Pythagorean Prime **17** :  $\delta_8$ ,  $\delta_9$  and  $\delta_{26}$

**Prime 17**

$$\delta_8 + \delta_9 = 17 = 26 - \delta_9$$

$\delta_8$

$\delta_9$

$\delta_{26}$

And, the Prime 17 =  $\frac{9^2 + 4}{5}$  : Note the Digit **5** in Denominator.

Here, the **9<sup>th</sup>** and **(9 + 5)<sup>th</sup>** Metallic Means give the precise value of **26<sup>th</sup>** Mean :  $\frac{\delta_9 \times \delta_{14} + 1}{\delta_{14} - \delta_9} = \delta_{26}$

Similarly, the **9<sup>th</sup>** and **(9 - 5)<sup>th</sup>** Metallic Means give the precise value of **8<sup>th</sup>** Mean :  $\frac{\delta_9 \times \delta_4 + 1}{\delta_9 - \delta_4} = \delta_8$

Moreover,  $\frac{17 - 5}{2} = 6$  : this 6 forms a new Triad with members 8 and 26 of the family: [ $\delta_6, \delta_8, \delta_{26}$ ]

$$\frac{\delta_{26} \times \delta_6 + 1}{\delta_{26} - \delta_6} = \delta_8 \quad \text{and} \quad \frac{\delta_8 \times \delta_6 + 1}{\delta_8 - \delta_6} = \delta_{26}$$

Further, beyond 8, 9 and 26, more members would be added in the family of Pythagorean Prime 17, like next member in the family would be 43 which equals the sum of all smaller members in the family;

$$8 + 9 + 26 = 43 \quad \text{and, the Prime 17} = 43 - 26$$

And,

$$2 \times 26 + 9 + 5 = 66 \text{ :this 66 forms a new Triad with members 26 and 43 of the family: } [\delta_{26}, \delta_{43}, \delta_{66}]$$

$$\frac{\delta_{66} \times \delta_{26} + 1}{\delta_{66} - \delta_{26}} = \delta_{43} \quad \text{and} \quad \frac{\delta_{43} \times \delta_{26} + 1}{\delta_{43} - \delta_{26}} = \delta_{66}$$

Such several distinctive correlations are observed among the Metallic Means belonging to the same Pythagorean Prime Families, and these correlations are bound to generate more such intriguing mathematical formulae, which may provide the precise relations between different Metallic Ratios.

### Special Pattern regarding Digits 3, 6 and 9 in the Realm of Metallic Means :

The proponents of Vortex Based Mathematics will continue to make irrational claims, and the opponents will continue to debunk them on grounds of the Base-10 Number System. Let the both camps do their jobs with missionary zeal.

Author’s objective is just to appreciate the beauty of numbers and the special attributes of the digits 3, 6 and 9, especially their unique patterns in the realm of Metallic Means.

For instance, consider the Integer Sequences and the corresponding Lucas Sequences associated with various Metallic Means [11] [12]. Remarkably, the Digital Roots of every Fourth Terms of Fibonacci, Lucas, Pell and Pell-Lucas sequences are **3, 6 or 9**.  $F_4, F_8, F_{12}, \dots, L_2, L_6, L_{10}, \dots, P_4, P_8, P_{12}, \dots$  and  $PL_2, PL_6, PL_{10}, \dots$  all have their Digital Roots 3, 6 or 9, and it holds true for the Integer Sequences as well as corresponding Lucas Sequences associated with any  $n^{th}$  Metallic Mean  $\delta_n$ ; provided  $n$  is not multiple of 3.

If  $n$  is multiple of 3 i.e. digital root of  $n$  itself is 3, 6 or 9, like Bronze Ratio  $\delta_3$  or the Aluminium Ratio  $\delta_6$  or the Ninth Mean  $\delta_9$  or the Twelfth Mean  $\delta_{12}$  and so on : here, not just every fourth, but the Alternate terms of associated Integer Sequences and corresponding Lucas Sequences have the digital roots 3, 6, or 9.

Now, lets incorporate here the idiosyncratic formula  $\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$

For instance, consider the **Table 1** provided on the Fourth and Fifth page of this paper.

It is noticeable in the Table 1 : if  $n$  is not multiple of 3, the alternate values of  $m$  and  $k$  have their digital roots 3, 6, or 9.

And, if  $n$  is multiple of 3 : None of the associated  $m_s$  and  $k_s$  have their digital roots 3, 6, or 9.

More remarkably, the number of Triples ( or the numbers of  $m_s$  and  $k_s$  ) increase noticeably for  $n = 6$  and 9

For Even  $n_s$  : the number of Triples exhibit noticeable rise at  $n = 6, 16, 26, \dots$  and so on.

For Odd  $n_s$  : the number of Triples exhibit noticeable rise at  $n = 9, 19, 29, \dots$  and so on.

For illustration of more such intriguing patterns embedded in the metallic Means, consider following couple of examples, based upon the formula  $\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k$

Consider the Triads of Metallic Means formed with  $n = 6$ , as shown below.

<b>n</b>	<b>6</b>	<b>6</b>	<b>6</b>	<b>6</b>	<b>6</b>	<b>6</b>	<b>6</b>	<b>6</b>
<b>m</b>	<b>7</b>	<b>8</b>	<b>10</b>	<b>11</b>	<b>14</b>	<b>16</b>	<b>26</b>	<b>46</b>
<b>m-n</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>5</b>	<b>8</b>	<b>10</b>	<b>20</b>	<b>40</b>



Note the bottom row in above table which contains the values of **(m-n)**.

The numbers in this (m-n) row exhibit 1 – 2 – 4 – 5 – 8 – 10×( 1 – 2 – 4 ) pattern, and remarkably the digits 3, 6 and 9 are conspicuous by their absence from this row !

And what’s about digit 7 ? Consider another example with n = 34, as shown below.

<b>n</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>	<b>34</b>
<b>m</b>	<b>35</b>	<b>36</b>	<b>38</b>	<b>39</b>	<b>42</b>	<b>44</b>	<b>54</b>	<b>63</b>	<b>74</b>	<b>92</b>	<b>150</b>	<b>179</b>	<b>266</b>	<b>324</b>	<b>614</b>	<b>1194</b>
<b>m-n</b>	<b>1</b>	<b>2</b>	<b>4</b>	<b>5</b>	<b>8</b>	<b>10</b>	<b>20</b>	<b>29</b>	<b>40</b>	<b>58</b>	<b>116</b>	<b>145</b>	<b>232</b>	<b>290</b>	<b>580</b>	<b>1160</b>

As any n<sup>th</sup> Metallic Mean  $\delta_n$  can give the precise values of other Metallic Means  $\delta_m$  and  $\delta_k$  by the formula:

$$\frac{\delta_m \times \delta_n + 1}{\delta_m - \delta_n} = \delta_k; \text{ with } k_{\max} = m_{\max} = n^2 + n + 4;$$

Hence,  $(m-n)_{\max} = (n^2 + 4)$  which is the all important Radical in the fractional expression of n<sup>th</sup> Metallic Mean  $\delta_n$

In above table with n = 34,  $(m-n)_{\max} = 1160$

Consider the Prime Factorization of this  $(m-n)_{\max}$  :  $1160 = 1 \times 2 \times 2 \times 2 \times 5 \times 29$

Note the bottom row (m-n) in above table, the numbers in the row exhibit the characteristic pattern based upon these factors 1, 2, 5 and 29. Numbers in the row are the 1 – 2 – 4 – 5 – 8 – 10×( 1 – 2 – 4 ) multiples of the prime factors 1, 2, 5 and 29.

Noticeably, the digit 7 is present not directly as (m-n), but it’s present only as the Digital Roots of certain (m-n) values; for instance the red shaded number 232 in above example. Such presence of 7 as Digital Root of (m-n) values is observed with n = 8, 10, 11, 14, 16, 22, 26, 29, 34, 36, 39, and so on. Remarkably, with n = 26, 36, 39, etc. the digital roots of multiple (m-n) values are found to be 7.

However, the digits 3, 6 and 9 are invariably missing from this pattern, they are neither present directly, nor as the digital roots of any (m-n) or (k-n) values.

For Even n<sub>s</sub> : the typical 1 – 2 – 4 – 5 – 8 – 10×( 1 – 2 – 4 ) pattern is observed.

For Odd n<sub>s</sub> : pattern based upon product of the Prime Factors of ( n<sup>2</sup> + 4 ) is observed.

In either case, the digits 3, 6 and 9 are conspicuous by their absence from these (m-n) or (k-n) values.

Moreover, if n is multiple of 3 : the digital roots of [(m-n) + (k-n)] are invariably 4 – 5 – 4 – 5

And, if n is not multiple of 3 : the digital roots of [(m-n) + (k-n)] are invariably 3 – 6 – 9 , as shown below.



For example, consider  $n = 30$  ( digital root of  $n$  is 3 )

<b>n</b>	30	30	30	30	30	30	30	30
<b>m</b>	31	32	34	38	143	256	482	934
<b>k</b>	934	482	256	143	38	34	32	31
<b>m-n</b>	1	2	4	8	113	226	452	904
<b>k-n</b>	904	452	226	113	8	4	2	1
<b>Digital Root of (m-n) + (k-n)</b>	<b>5</b>	<b>4</b>	<b>5</b>	<b>4</b>	<b>4</b>	<b>5</b>	<b>4</b>	<b>5</b>

Likewise, consider  $n = 29$  ( digital root of  $n$  is other than 3, 6 or 9 )

<b>n</b>	29	29	29	29	29	29
<b>m</b>	30	34	42	94	198	874
<b>k</b>	874	198	94	42	34	30
<b>m-n</b>	1	5	13	65	169	845
<b>k-n</b>	845	169	65	13	5	1
<b>Digital Root of (m-n) + (k-n)</b>	<b>9</b>	<b>3</b>	<b>6</b>	<b>6</b>	<b>3</b>	<b>9</b>

**Conclusion:**

This paper apposed and synergised the newly discovered couple of aspects of the Metallic Ratios, viz. the generalised geometry of Metallic Means premised upon Right Triangles and the mathematical correlation between different Metallic Ratios. And these two novel aspects not only gave rise to the formations of the distinctive Triads of Metallic Means, but they also unveiled the close associations between such Triads and the primitive Pythagorean Triples, as well as Pythagorean Primes. Moreover, this work illustrated the intriguing patterns of Digits 3 6 9 in realm of Metallic Numbers. Triangles – Triads – Triples - & 3 6 9.....!

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Author is grateful to Wikipedia for citing his work on generalised geometric construction of all Metallic Means, in the Wiki page "Metallic Mean". That's the greatness and magnificence of **Wikipedia**, the free encyclopedia, which cares most about the real substance in any manuscript, rather than the so called academic degree of author. Citation by Wikipedia has really inspired and motivated the author to further discover the various aspects of Metallic Means. Thanks Wikipedia!

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