FREQUENCY RESPONSE ANALYSIS OF NONLINEAR DYNAMIC SYSTEMS

by

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UNITED STATES
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THESIS

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ABSTRACT

A graphical technique to predict the frequency response of nonlinear transmission functions is presented. The technique is applied to nonlinear transmission functions with accurate results. The technique utilizes the magnitude ratio curve as a function of the system parameters developed in the algebraic methods.

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1. INTRODUCTION

An often used criterion applied in the design of control systems is that of the frequency response of the system. It is the objective of this paper to investigate a technique to predict the frequency response of nonlinear transmission functions. The nonlinearities involved will be changes in the system parameters and will be functions of the output signal magnitude only.

2. PARAMETER PLANE METHOD

Use of the parameter plane methods [1] for frequency response of linear time-invariant systems was first derived by Dr. G. J. Thaler and Dr. A. G. Thompson [2]. These methods were later utilized to obtain parameter plane plots by J. R. Rommel, R. H. Cradit, and G. Glavis [3] , [4] .

To provide a basic understanding of the use of the parameter plane methods in the frequency response of dynamical systems, the derivation pertaining to frequency response presented in [2] will be presented below.

Given a system transfer function of the form

$$T(S) = \frac{N(S)}{D(S)} \tag{2.1}$$

Where N(S) and D(S) are of the form $A_o + A_1S + A_2S^2 + \ldots + A_{n-1}S^{n-1} + A_nS^n$, and in general the coefficients A_o to A_n are nonlinear functions of two system parameters α and β. A_N can be expressed as $A_N = B_N + C_N\beta + D_N\alpha + E_N\alpha\beta$ (2.2)

and any or all of the coefficients B_N, C_N, D_N, and E_N may be zero.

The squared magnitude function (m^2) can be obtained by

$$m^2 = |T(S)|^2 = T(S)\,T(-S) \tag{2.3}$$

Separating the polynomials N(S) and D(S) into their even and odd parts

$$T(S) = \frac{N_e(S) + N_o(S)}{D_e(S) + D_o(S)} \tag{2.4}$$

and

$$T(-S) = \frac{N_e(S) - N_o(S)}{D_e(S) - D_o(S)} \tag{2.5}$$

Then by taking the product of (2.4) and (2.5) and letting $S = j\omega$ the squared magnitude function can be obtained.

The resulting product is

$$T(S)\,T(-S) = \frac{N_e^2(S) - N_o^2(S)}{D_e^2(S) - D_o^2(S)} \tag{2.6}$$

Examination of (2.6) shows that both the numerator and denominator functions are even polynomials in S and letting $S = j\omega$ both are even functions in ω.

Then the squared magnitude function is

$$m^2 = \frac{\sum_{k=0}^{k=n} A_k\,\omega^{2k}}{\sum_{k=0}^{k=n} B_k\,\omega^{2k}} \tag{2.7}$$

where A_k and B_k are coefficients of the form of equation (2.2).

As done in [3] and [4], by holding one parameter (either α or β) at a constant value, magnitude curves as functions of the variable parameter can be plotted. Using the computer program developed in [4] an example of the magnitude curves for the second order system, shown in figure (2.1), are presented in figure (2.2). Here the magnitude curves represent the absolute ratio value of X_1 to X_o, where X_1 and X_o denote the instantaneous displacement of the point of support and the mass from their equilibrium condition. It is easily seen that the transfer function for this second order system is,

$$T(S) = \frac{K}{MS^2 + BS + K} \qquad (2.8)$$

For the constant omega curves shown in figure (2.2), β = K and α = B, also for illustrative purposes, β = M = 1.0.

The next section will utilize the magnitude ratio curves, developed from the parameter plane theory, to establish a technique in predicting the frequency response of dynamic systems.

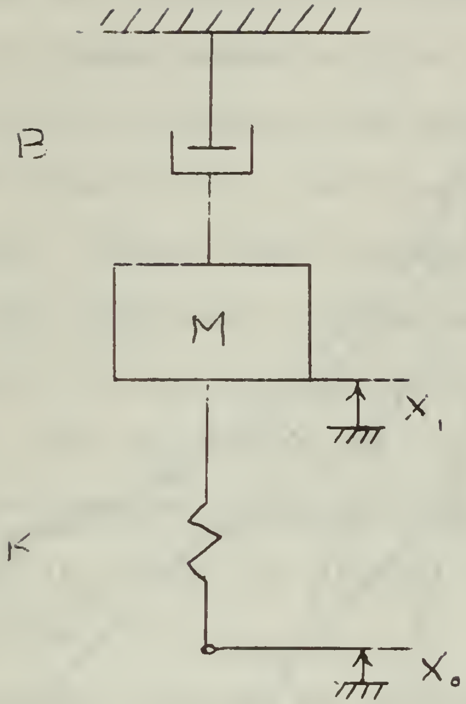

Figure 2.1 Spring-Mass-Damper, Second Order Dynamic System

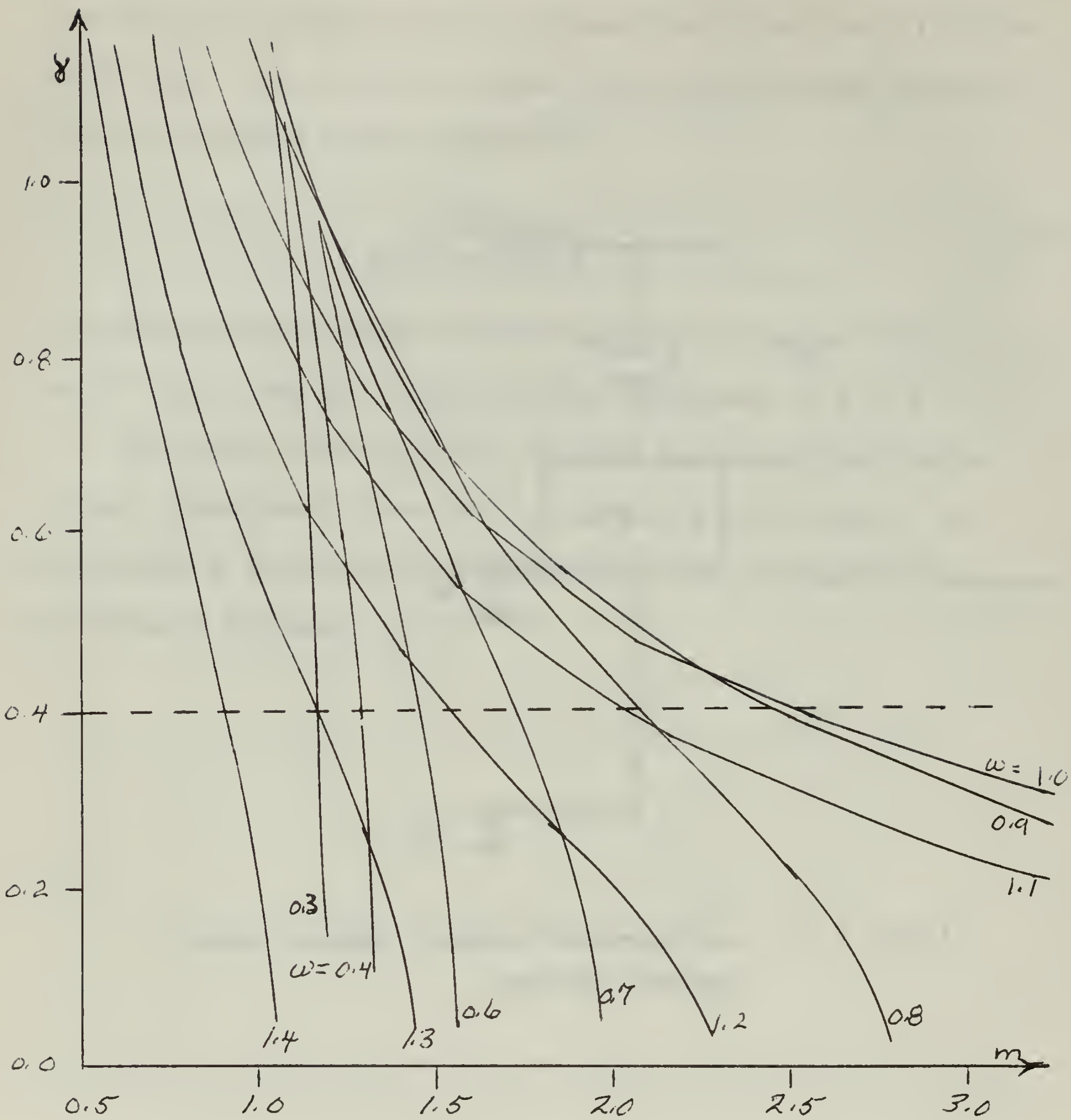

Figure 2.2 Magnitude Curves for

$$m = \left| \frac{\beta}{S^2+\alpha S+\beta} \right|$$

α varying and β = 1.0

ALPHA (α) vs Magnitude (m)

3. TECHNIQUE IN OBTAINING FREQUENCY RESPONSE OF A NONLINEAR DYNAMIC SYSTEM

Before introducing the nonlinear aspect into the dynamic system, the technique for predicting the frequency response of a linear system will be reviewed. Using the curves, which have resulted from the parameter plane methods, in figure (2.2), the frequency response curve for the linear, second order system of figure (2.1) was obtained. This frequency response was obtained by fixing $\alpha = 0.4$ and at the various intersections with the constant omega curves, the value of m was read off. As a basic check to this frequency response, the second order systems was simulated in an IBM 360 digital computer, utilizing the DSL/360 simulation program [5]. This method of obtaining the frequency response may be applied to a more complex system, and with excellent results as shown in figure (3.1). But this is quite an involved method, requiring a digital computer, compared with say, the Bode diagram method. The advantage here is the feature of varying the system parameter and obtaining a family of frequency response curves for ranges of the system parameters.

To investigate the possibility of predicting the frequency response of nonlinear dynamic systems, the nonlinear feature was introduced by letting one of the system parameters be a function of the output of the system. For this investigation, the system parameter will be considered a function of the output amplitude only.

The identical magnitude curves for the second order system are reproduced in figure (3.2) for the purpose of illustrating the frequency response technique. In a manner similar to that used with the linear case, the curve or variance of the system parameter(in this case α) with magnitude is drawn on the magnitude curves. For example, as shown in figure (3.2), α is described by curve

A or $\alpha = 0.6,$ for $m \leq 1.0$ (3.1)

and $\alpha = 0.4 + 0.2m$ for $m > 1.0$ (3.2)

With the function describing α superimposed on the magnitude curves, it should be possible to predict the frequency response of the system as α varies with the output amplitude. This method was investigated by entering the curves at a specific omega and at the intersection of that omega curve and the describing curve for α the magnitude was read off. For example, at $\omega = 1.2$ in figure (3.2), $m = 1.125$. This technique is continued until a frequency response curve or family of curves is obtained for the system transfer function.

Initial investigation was performed using the above described technique for the second order dynamic system shown in figure (2.1). With the magnitude curves drawn for the transfer function of equation (2.8), various functions of magnitude for α were superimposed on the curves. For each magnitude function the frequency response was obtained for the system with α set equal to that magnitude function. For a check on this method, the system with α equal to a function of X_1, (the output signal) was simulated in the digital computer using the DSL/360 program [5]. The predicted and simulated value did not check out with any degree of accuracy. Further investigation uncovered two inaccuracies in the method. First it was determined by the fact that the magnitude ratio was X_1 to X_o and was not the direct displacement across the system parameter being described by a function of X_1. Later investigation proved that the transmission function used, must be defined such that the signal across the nonlinear parameter is the system output. The complex theory or reasoning behind this inaccuracy is not completely known at this time and should be the subject of further investigation as described in section 5 of this paper.

To overcome this first inaccuracy a new transmission function was derived for the second order system. This new transmission function was based upon the fact that the system parameter to be varied was α (the damping coefficient for the second order system). Since the damping force is physically proportional to the velocity across the damper rather than the displacement across the damper, the velocity $\dot{X}_1$, was selected as the output signal and the new system transmission function becomes

$$T(S) = \frac{\dot{X}_1}{X_o} = \frac{\beta S}{S^2 + \alpha S + \beta} \qquad (3.3)$$

For the purpose of drawing the magnitude curves, β is set equal to 1.0 in equation (3.3) and the curves are plotted for the system parameter α varying, figure (3.3). In obtaining the correct frequency response utilizing the technique presented here the following procedure is used. First having selected the describing function curve for the system parameter in an intelligent manner, superimpose this describing function curve upon the magnitude ratio curves as done here in figure (3.3) for curve A. Then entering the magnitude ratio curves along the describing function curve for the varying system parameter, at the intersection of this curve with that of a desired frequency curve, for example point B in figure (3.3), the corresponding magnitude ratio is read off by dropping a vertical to a point immediately below the point of intersection to the abscissa. For point B, figure (3.3), the magnitude ratio is 0.75 and the corresponding frequency is 0.6 rads. Successive points, such as C, D, and E are read off the curves in figure (3.3) and plotted as the frequency response of the system, as shown in figure (3.4) where points B, C, D, and E are corresponding points selected from figure (3.3). Also shown in figure (3.4) for comparison purposes is the simulated results of the frequency response of the system with α varying and utilizing the correct transmission function. The results of the two methods, predicted technique and simulated technique, compare very favorably. Curve B, figure (3.3) describes an additional describing function curve superimposed upon the magnitude ratio curves. Following the described technique above using curve B, the new frequency response of the system was found as shown in figure (3.5). The above two examples demonstrate that it is possible to predict the frequency response of the nonlinear system if the correct transmission function is applied.

The second inaccuracy uncovered in the initial investigation was that no attention was given to the dependency of the technique on the amplitude of the input signal. This inaccuracy was investigated by varying the input signal amplitude. It was determined that the technique was independent of the input signal amplitude since the magnitude ratio of the input to output signal is involved.

Table (3.1) demonstrates the independence of the method to the input signal amplitude. This table is based upon data obtained in the simulation run for equation (3.3) of the second order system.

Table (3.1)

Frequency	Input Signal Amplitude	Magnitude ratio
1.4	1.0	1.79428
1.4	0.5	1.79392
1.4	2.0	1.79428
0.8	1.0	0.82243
0.8	0.5	0.82242
0.8	2.0	0.82243
0.5	1.0	0.41562
0.5	0.5	0.41562
0.5	2.0	0.41562

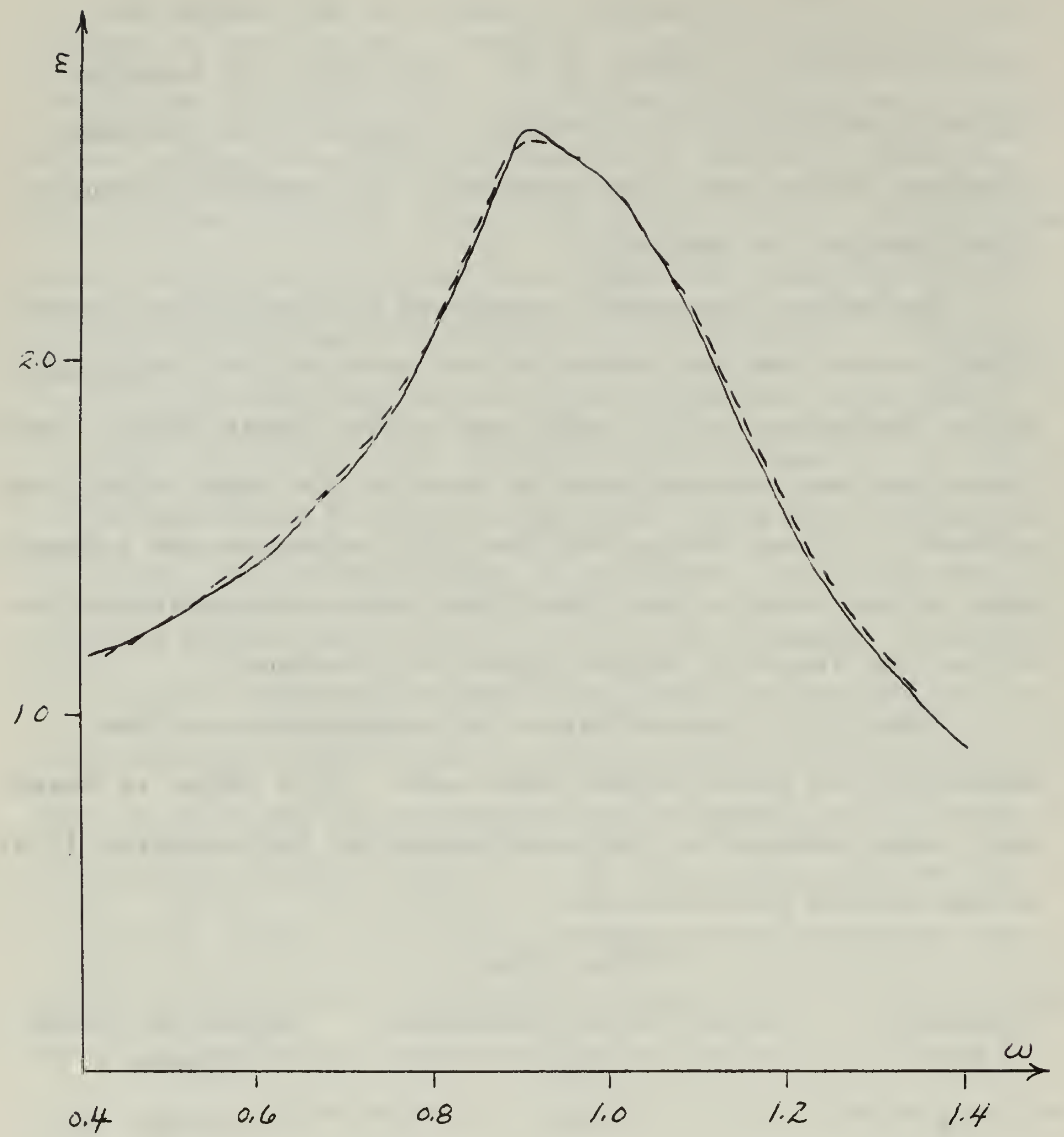

Figure 3.1 Frequency Response for $m = \left| \dfrac{\beta}{S^2+\alpha S+\beta} \right|$,

$\alpha = 0.4$, $\beta = 1.0$

Magnitude (m) vs Frequency (ω)

Note on graph put "— Predicted Response"

-- Simulated Response

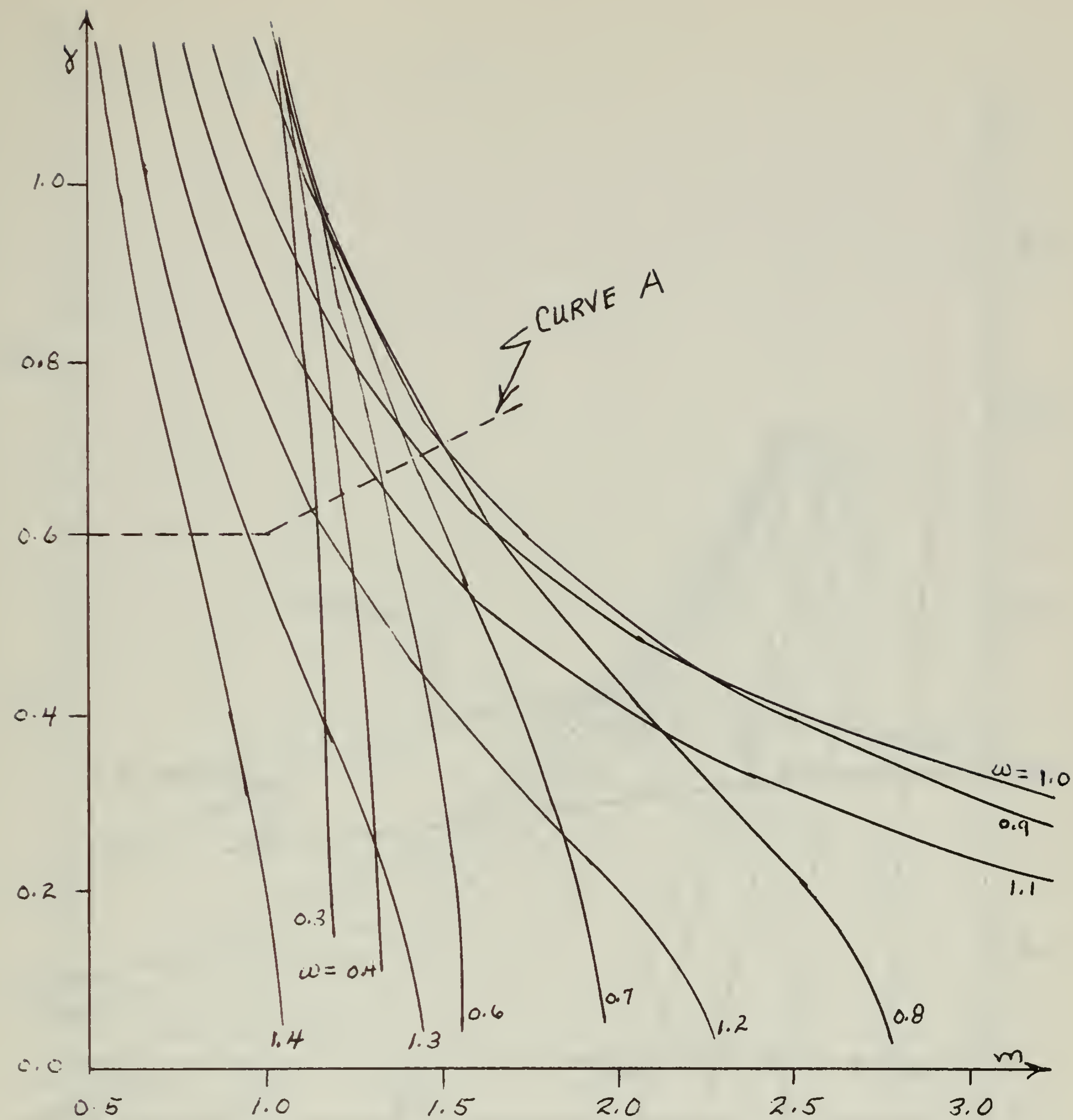

Figure 3.2 Magnitude Curves for

$$m = \left| \frac{\beta}{S^2+\alpha S+\beta} \right| ,$$

α varying and β = 1.0

ALPHA (α) vs Magnitude (m)

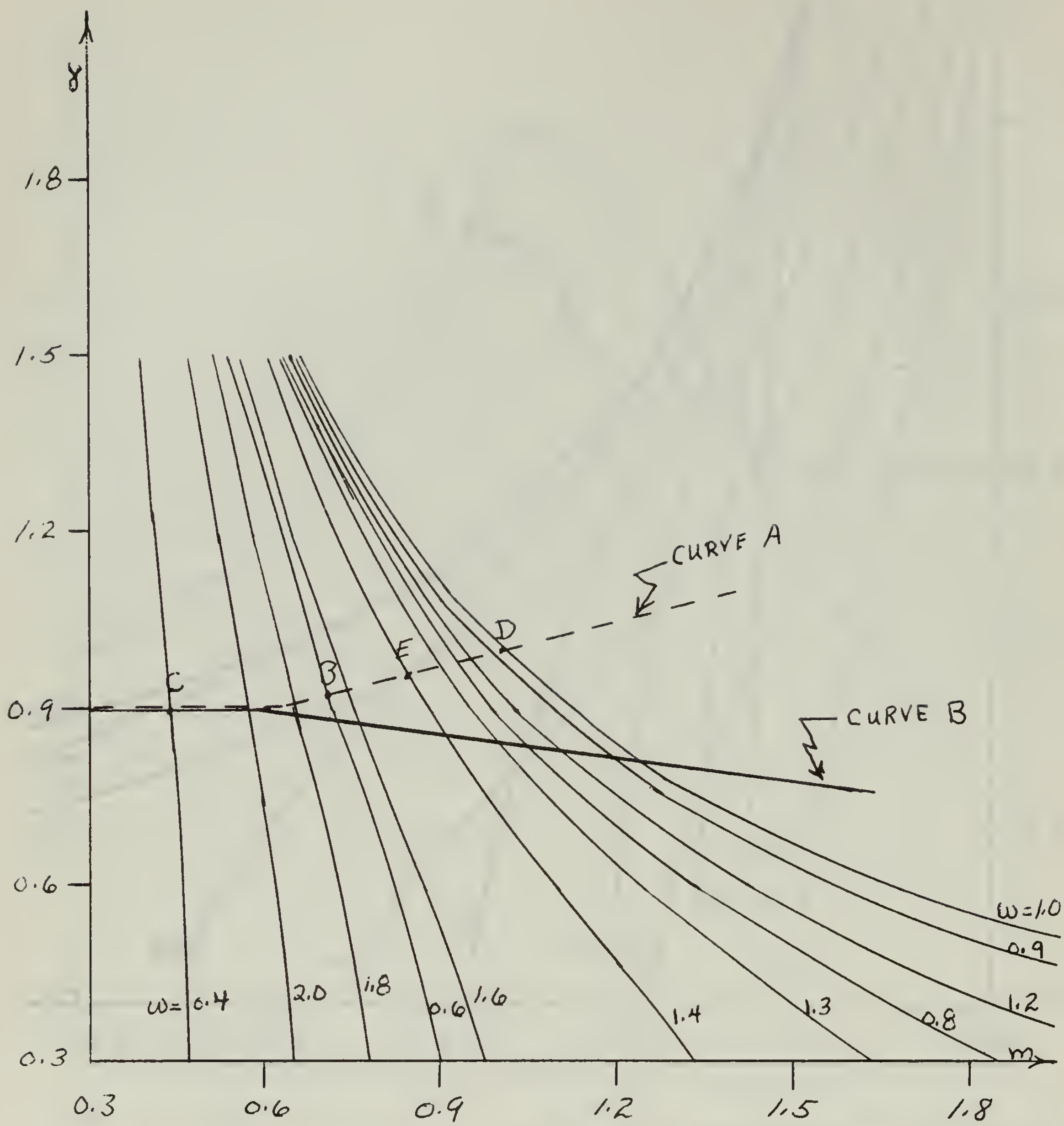

Figure 3.3 Magnitude Curves for

$$m = \left| \frac{\beta S}{S^2+\alpha S+\beta} \right|$$

α varying and β = 1.0

ALPHA (α) vs Magnitude (m)

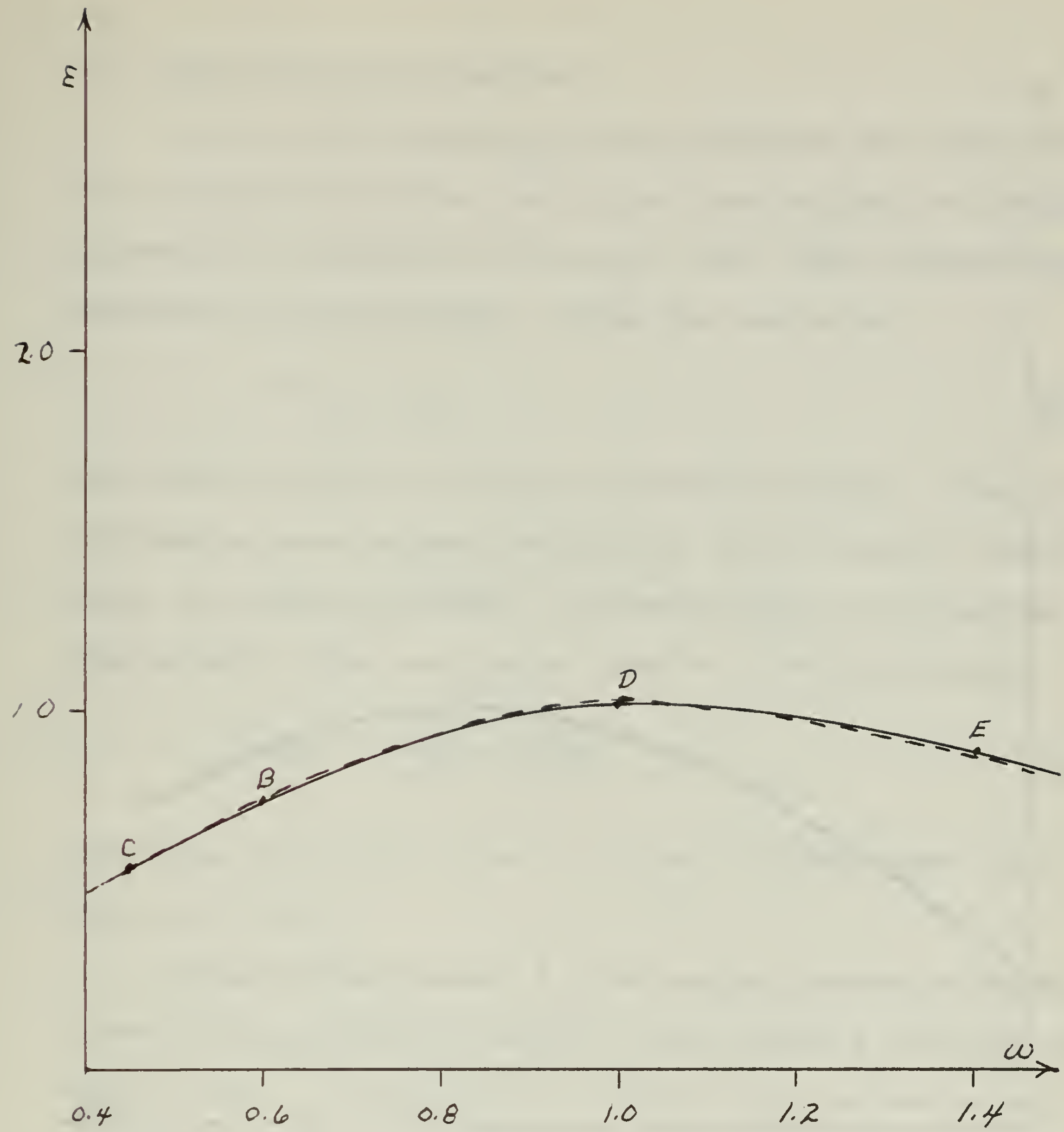

Figure 3.4 Frequency Response for $m = \left| \dfrac{\beta S}{S^2+\alpha S+\beta} \right|$,

α = curve A Figure 3.3, β = 1.0

Magnitude (m) vs Frequency (ω)

Note: ___ Predicted Response
-- Simulated Response

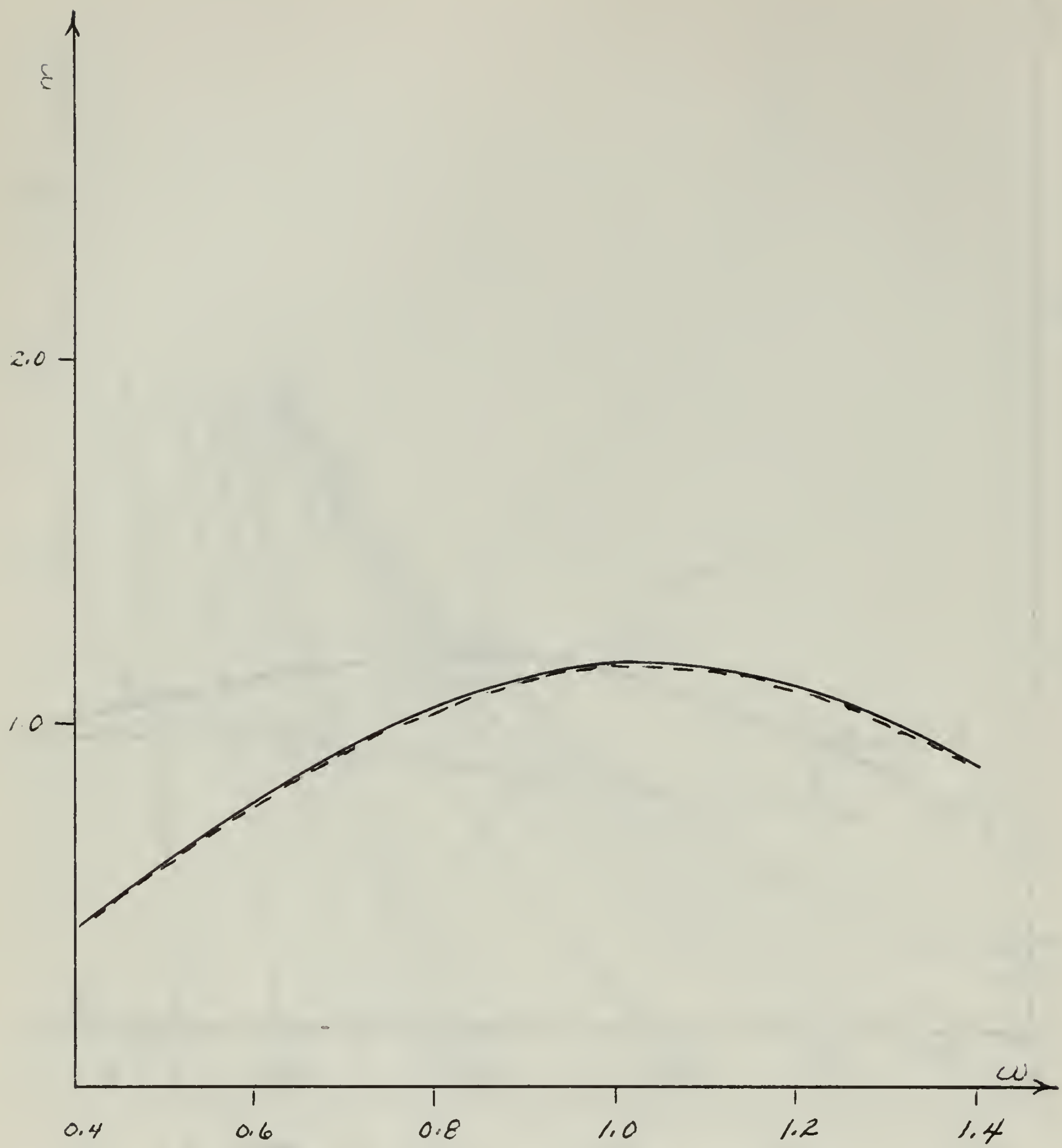

Figure 3.5 Frequency Response for $m = \frac{\beta S}{S^2+\alpha S+\beta}$

α = curve B Figure 3.3, β = 1.0

Magnitude (m) vs Frequency (ω)

Note: ___ Predicted Response
-- Simulated Response

4. APPLICATION OF TECHNIQUE

Further investigation of the technique was applied to the second order system, but this time letting the system parameter K (the spring constant) vary. This necessitates defining the displacement across the spring as

$$Y = X_1 - X_o \tag{4.1}$$

and resolving for the correct transfer function. Using Y, the displacement across the spring, as the output signal, since the spring constant is proportional to the spring displacement, the new system transfer function becomes

$$T(S) = \frac{-(S^2 + \alpha S)}{S^2 + \alpha S + \beta} \tag{4.2}$$

Where now the output signal is Y and the magnitude ratio is that of Y to X_o.

The magnitude versus β (the varying system parameter) curves are presented in figure (4.2), where α has been set equal to one. Figures (4.3) and (4.4) for curves A and B of figure (4.2) show the results of the frequency response techniques for β being set equal to two different functions (curves A and B of figure (4.2) of the output signal Y). Figures (4.3) and (4.4) also show the simulated frequency response curve for comparison of the technique.

The frequency response of the fourth order system of figure (4.1) was investigated for the case of the spring constant K, varying as a function of the output signal (the physical displacement of the spring in this case).

The results of both the predicted and simulated frequency response techniques are shown in figure (4.6) and (4.7) for the parameter describing function curves A and B of figure (4.5).

This case of the fourth order system covers the product coefficient case of the parameter plane methods and further demonstrates the applicability of the frequency response techniques. For the fourth order system, the system transmission function is

$$T(S) = \frac{-(S^4 + 2S^3 + 2\alpha S^2)}{S^4 + 2S^3 + (2\alpha + \beta)S^2 + \beta S + \alpha\beta} \tag{4.3}$$

where α is set equal to 1.0 and β is allowed to vary as a function of the output displacement Y. Where Y is defined as

$$Y = X_1 - X_o \tag{4.4}$$

For this fourth order system the following physical parameters were related to α and β as follows:

$$\alpha = K_2$$

$$\beta = K_1$$

and for illustration purposes $M_1 = M_2 = B_2 = 1.0$.

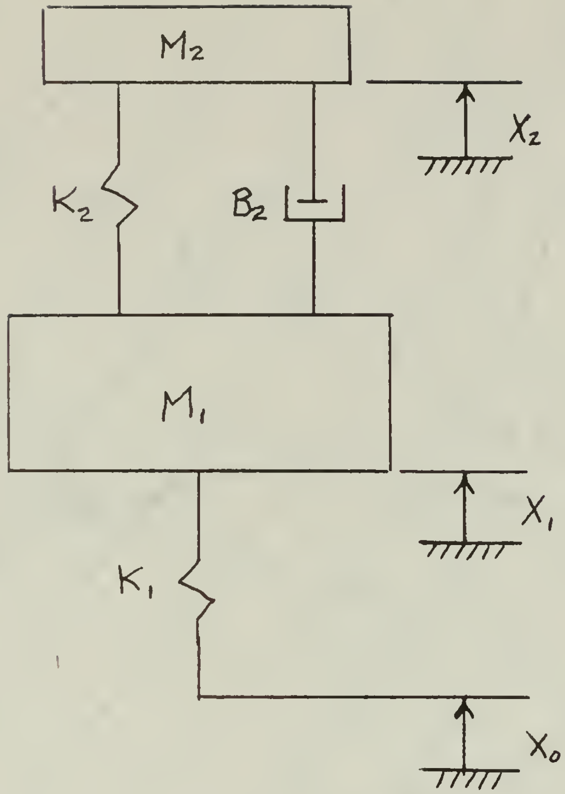

Figure 4.1 Spring-Mass-Damper Fourth Order Dynamic System

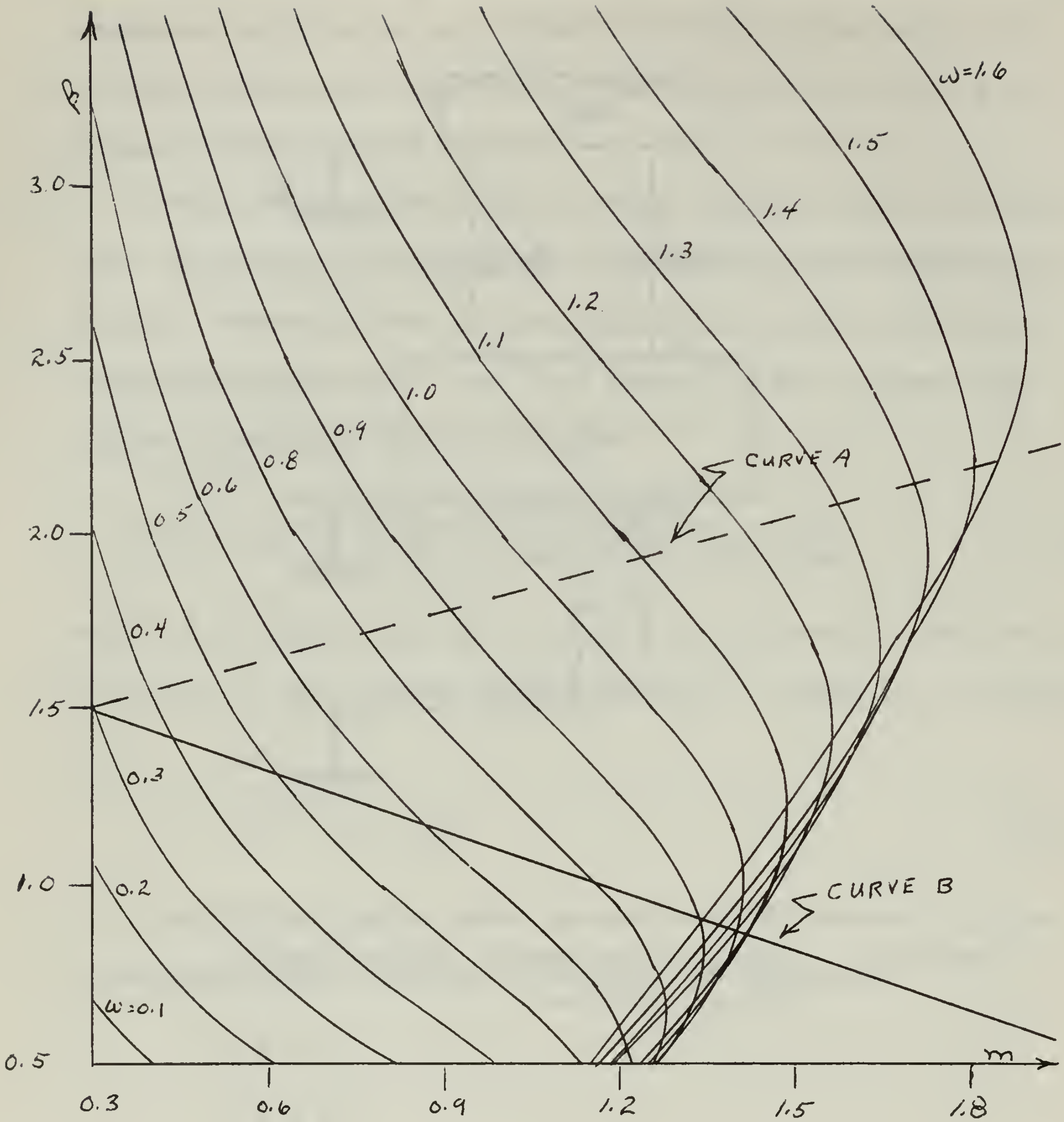

Figure 4.2 Magnitude Curves for

$$m = \left| \frac{S^2+\alpha S}{S^2+\alpha S+\beta} \right|$$

β varying and $\alpha = 1.0$

BETA (β) vs Magnitude (m)

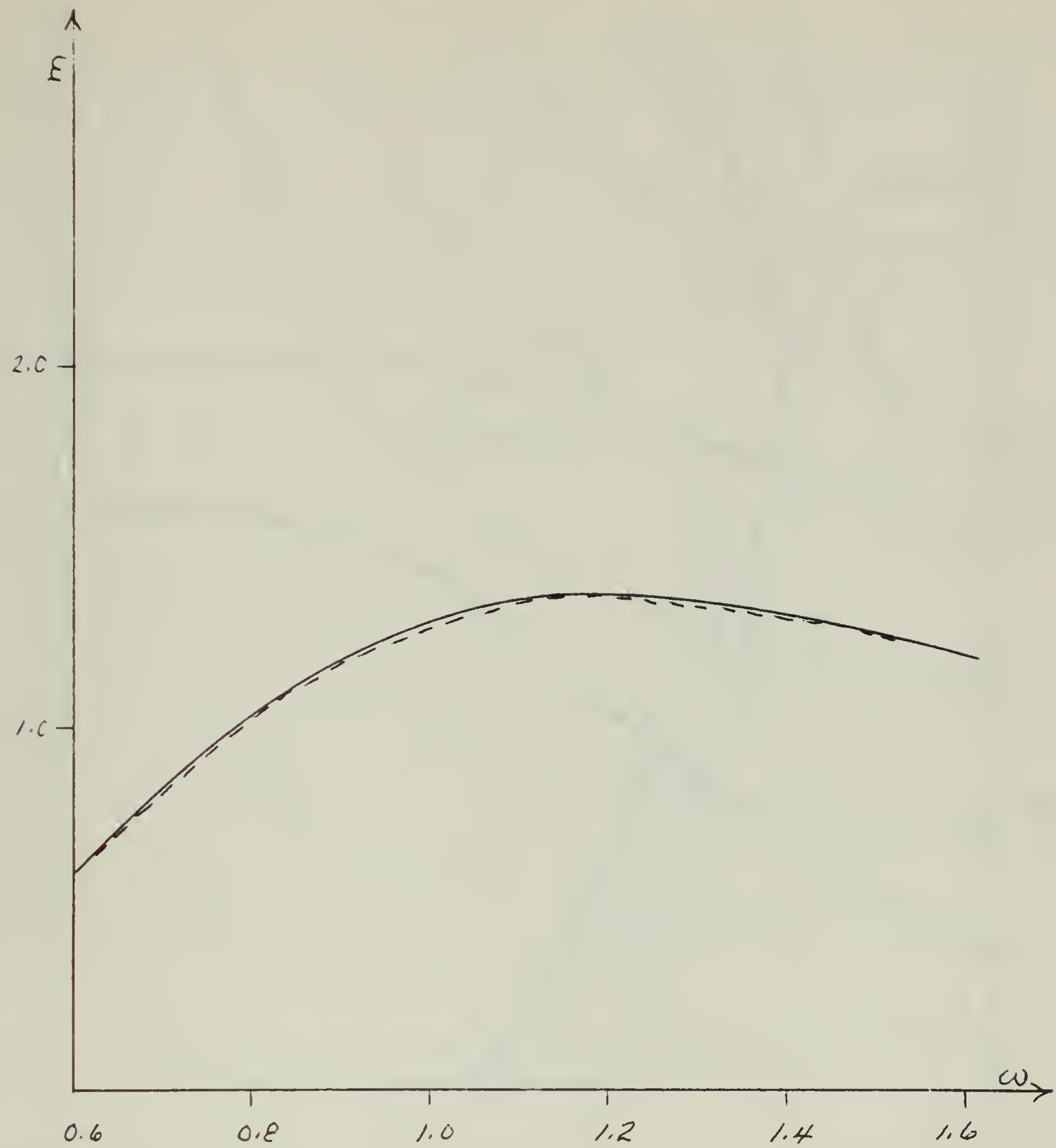

Figure 4.3 Frequency Response for $m = \left| \frac{S^2+\alpha S}{S^2+\alpha S+\beta} \right|$

β = curve B Figure 4.2, α = 1.0

Magnitude (m) vs Frequency (ω)

Note: __ Predicted Response
-- Simulated Response

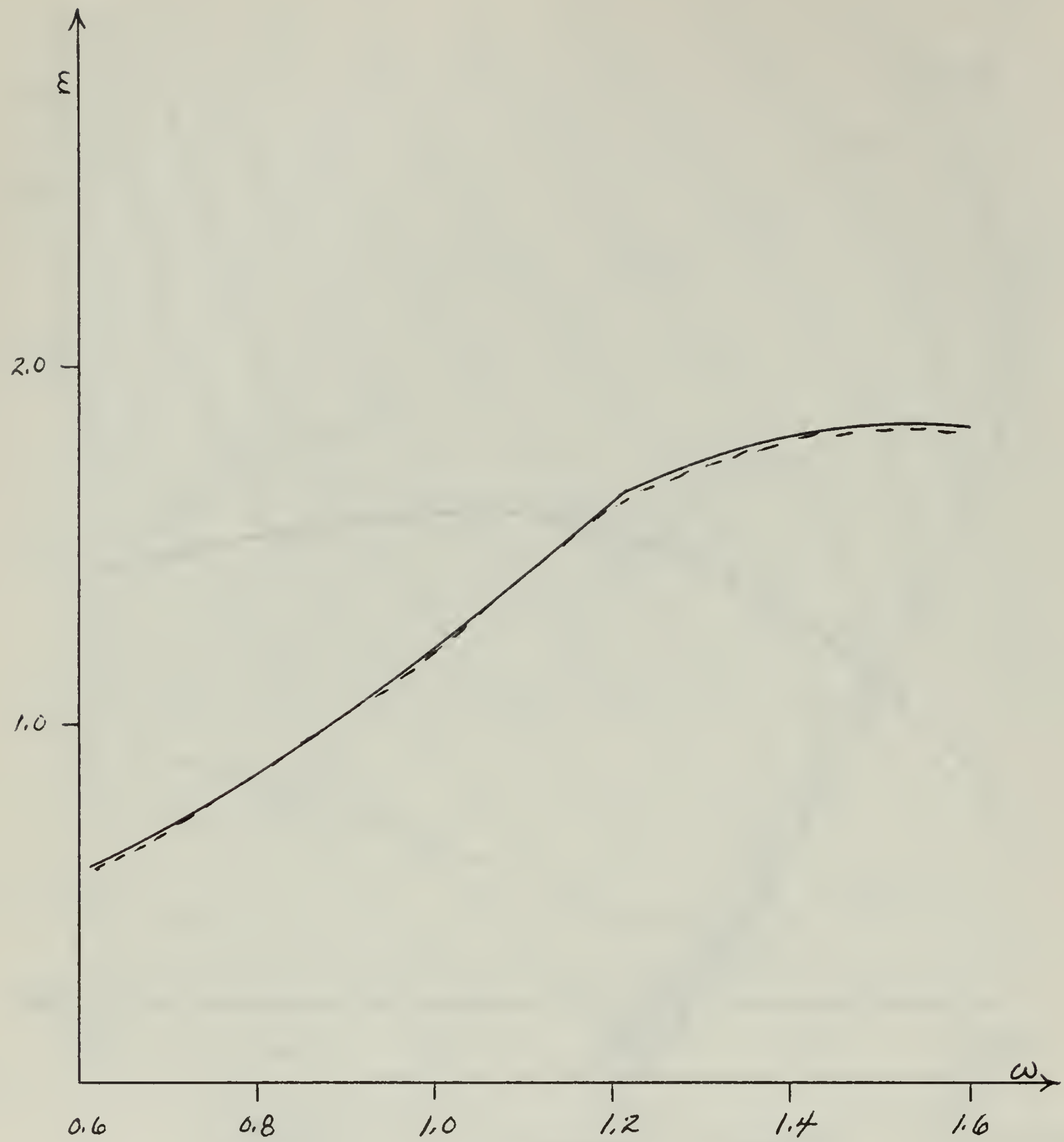

Figure 4.4 Figure Response for $m = \left| \dfrac{S^2+\alpha S}{S^2+\alpha S+\beta} \right|$

β = curve A Figure 4.2, α = 1.0

Magnitude (m) vs. Frequency (ω)

—— Predicted Response
-- Simulated Response

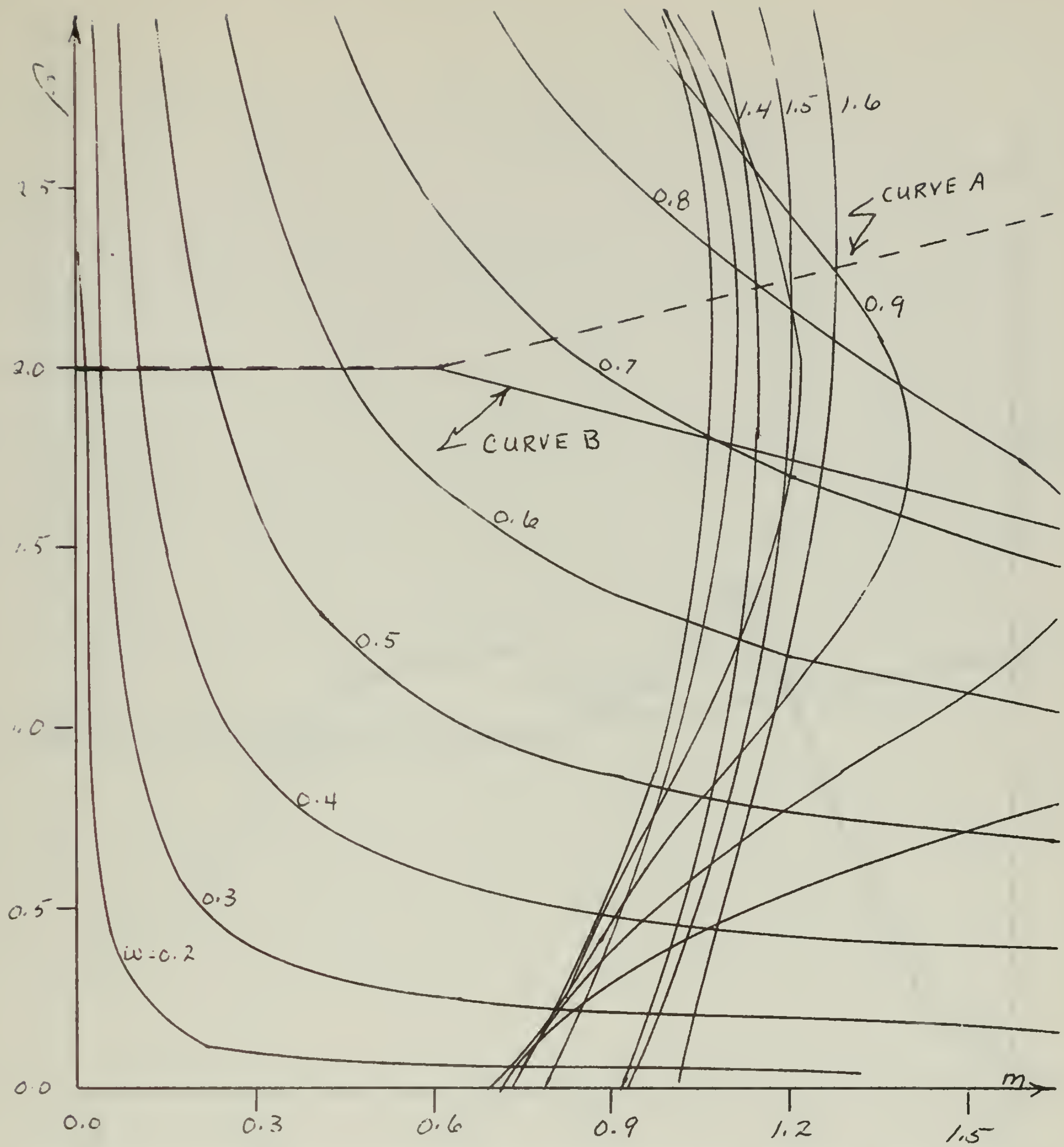

Figure 4.5 MAGNITUDE CURVES for

$$m = \left| \frac{S^4+2S^3+2\alpha S^2}{S^4+2S^3+(2\alpha+\beta)S^2+\beta S+\alpha\beta} \right|$$

β varying and $\alpha = 1.0$

BETA (β) vs Magnitude (m)

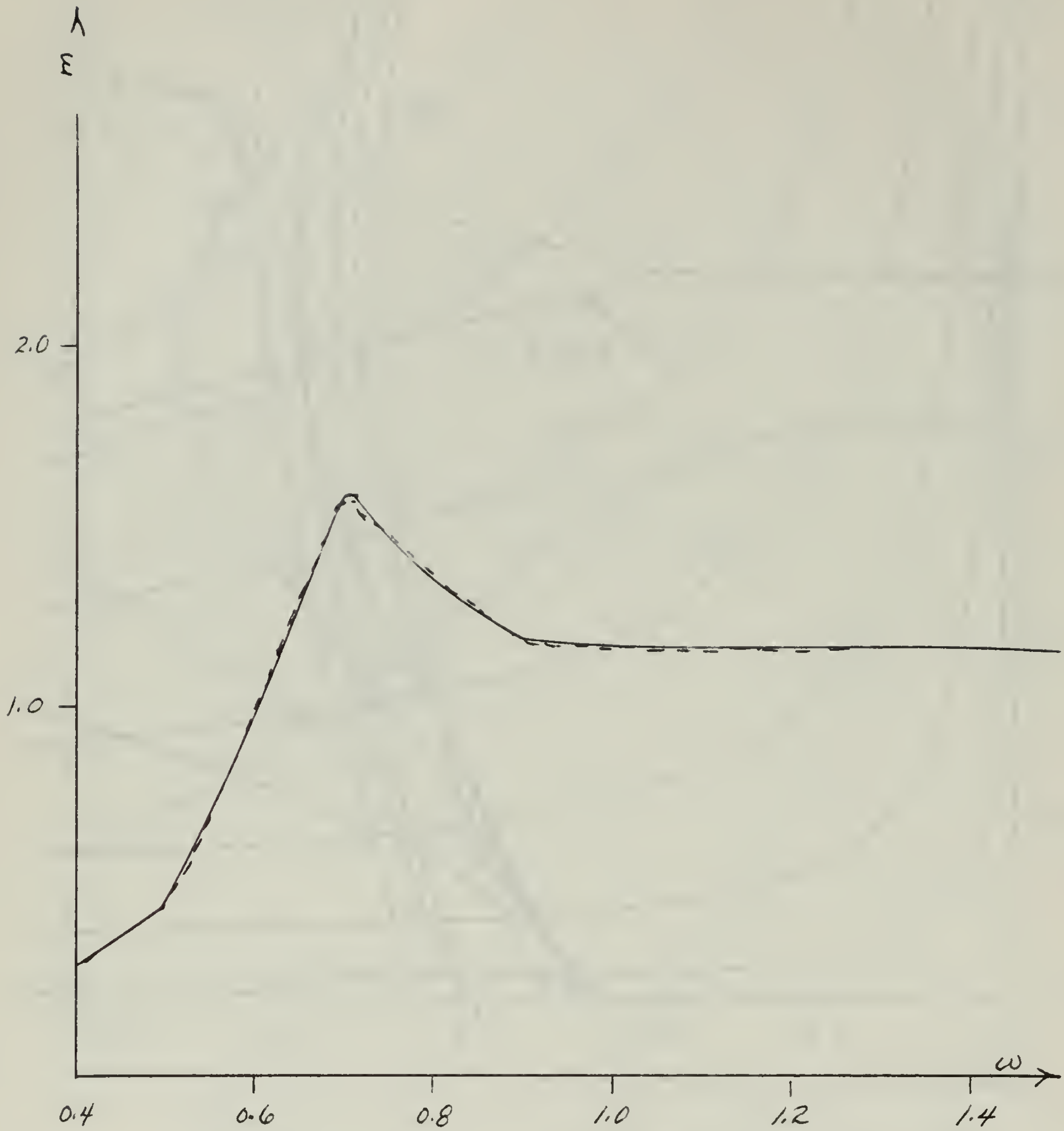

Figure 4.6 Frequency Response for $m = \left| \dfrac{S^4+2S^3+2\alpha S^2}{S^4+2S^3+(2\alpha+\beta)S^2+\beta S+\alpha\beta} \right|$

β = curve B Figure 4.5, α = 1.0

Magnitude (m) vs Frequency (ω)

Note — Predicted Response
-- Simulated Response

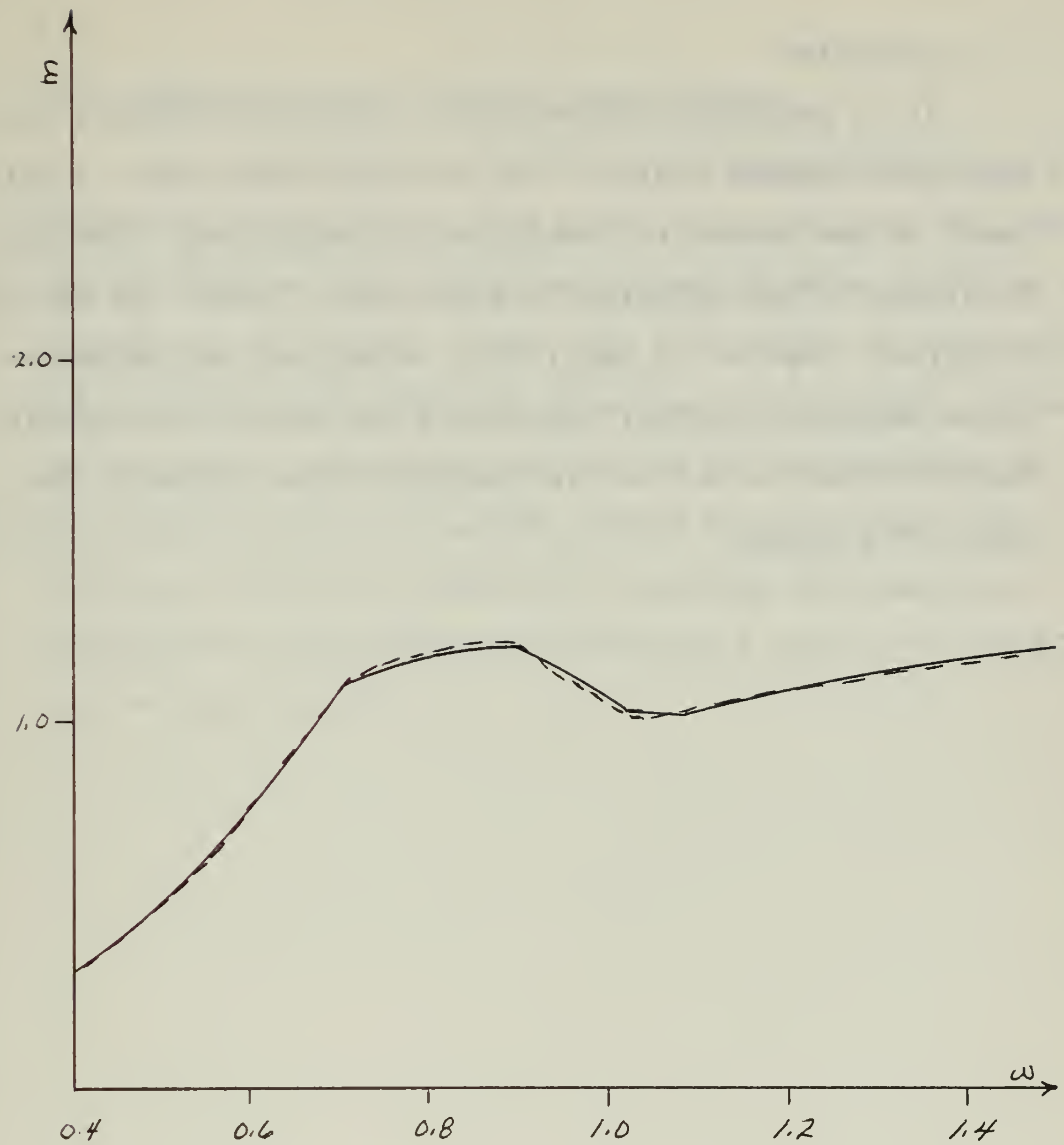

Figure 4.7 Frequency Response for

$$m = \left| \frac{S^4+2S^3+2\alpha S^2}{S^4+2S^3+(2\alpha+\beta)S^2+\beta S+\alpha\beta} \right|$$

β = curve A Figure 4.5, α = 1.0

Magnitude (m) vs Requency (ω)

Note — Predicted Response
-- Simulated Response

5. CONCLUSION

It is possible to predict the frequency response of a nonlinear dynamic system. The technique described in this paper is one method for making such predictions. The limitations to the technique are two fold. First the use of a digital computer is required in obtaining the parameter plane magnitude curves, and second the proper transmission function must be defined in applying these curves to the specific problem.

6. SUGGESTIONS FOR FURTHER INVESTIGATION

It is believed that additional investigation into the technique presented in this paper for predicting the frequency response of a nonlinear dynamic system should be continued on two fronts. First the limitations in the choice of a transmission function should be investigated. Secondly (but more important) application of the technique to meaningful physical problems should be undertaken. The technique should be applied to a problem with physical significance and developed further as a tool to be used by the design engineer.

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13. ABSTRACT

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