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EUCLID

100 Sqr of Roofe
and floors in the Same
and Stable

656 yards of Brick work
200 yards of garden wall

600 yds of Out: wall

144 [illegible]

MR. *John Warner*, Accomptant Surveyour and Teacher of *Mathematicks*, Lives at the Corner of *Hemlock-Court* in *Ca-ry-street*, by Lincolns Inn, *London*.

THE

ENGLISH EUCLIDE,

BEING

The Firſt SIX ELEMENTS

OF

GEOMETRY.

THE

ENGLISH EUCLIDE,

BEING

The First SIX ELEMENTS

OF

GEOMETRY,

Translated out of the GREEK,

WITH

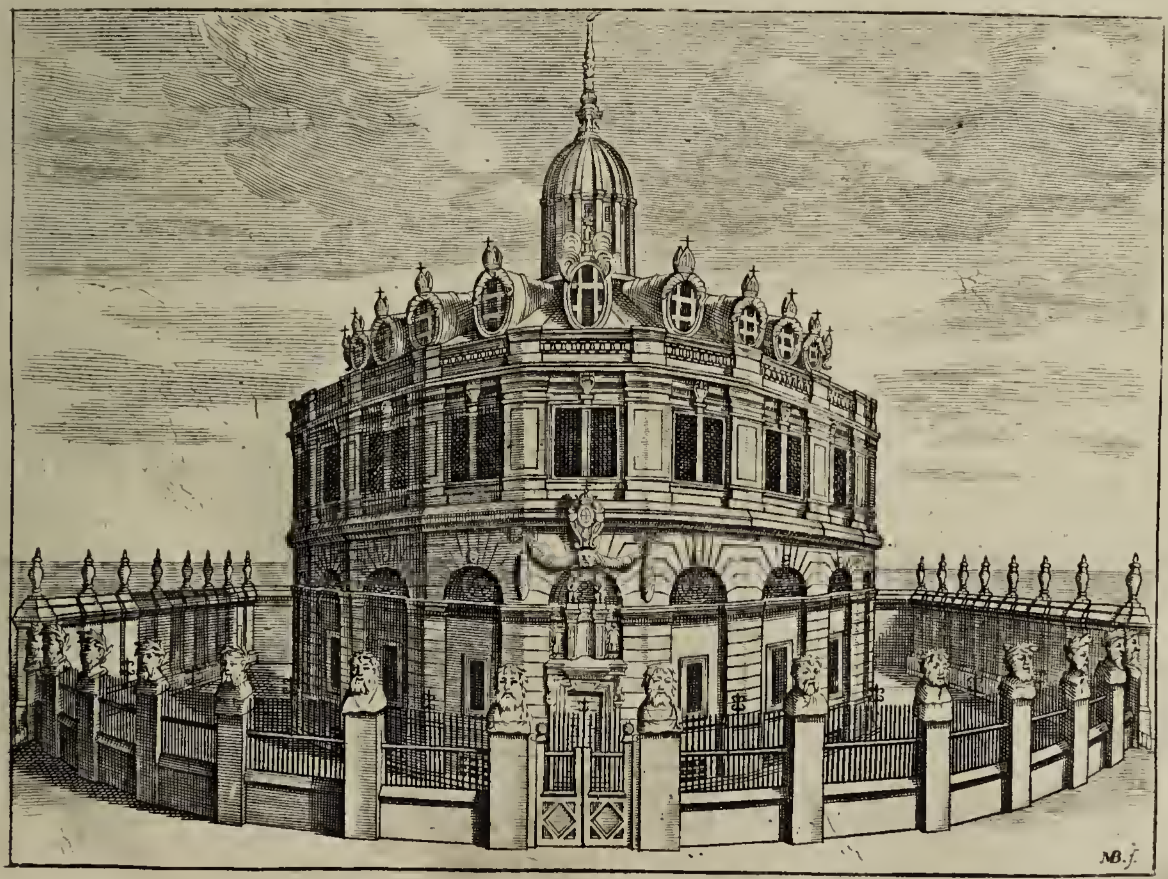
Annotations and useful Supplements,

By EDMUND SCARBURGH M. A.

Chaplain to his Grace the Duke of BUCKINGHAMSHIRE Ld Privy Seal;

Prebendary of SARUM;

And Rector of UPWEY in the County of DORSETT.



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TO HIS ROYAL HIGHNESS

GEORGE

HEREDITARY PRINCE OF

DENMARK, NORWAY, *and of the* GOTHS *and* VANDALS;

DUKE OF

Schleswick, Holstein, Stormar, Dickmarsh and *Cumberland*;

EARL OF

Oldenburgh, Delmanhorst, and *Kendal*;

Baron of *Wokingham*;

Generalissimo of All Her Majesties Forces;

Lord High Admiral of *ENGLAND*,

AND

Knight of the most Noble Order of the GARTER.

May it please Your ROYAL HIGHNESS,

THIS Piece was once intended for the Service of Her Majesty, and Your Royal Highness in the Education of A PRINCE, which was then the Hope, and Joy, and Glory of the *British* Nation; on whose Character I would enlarge, but that I fear to touch upon ſo Soft and Tender a Part, leaſt whilſt I endeavour to adorn His Memory, I renew Your Grief, and revive again the ſorrowful Image of That Inexpressible Loss.

After which, the greateſt ſatisfaction I have taken in Compoſing, and Finiſhing this Work, hath been from my hopes of having a proper Opportunity of expreſſing to the World, the high Eſteem, Reſpect, and Veneration I have for *Your Royal Highneſs.*

Your glorious Deeds of War, and mighty Atchievments in the Field of Battle; a Brother Reſcued, and a Kingdom Sav'd, are Glories proper for another Pen. But Your wonderful Zeal and Courage in preſerving theſe Kingdoms, Your entire Love to our Countrey, and unparallel'd Affection for our moſt Excellent, and Glorious Queen, are Vertues, and Merits, which none, that Love our Crown or Countrey, can conceal; and I am proud to have an Opportunity to joyn in the general Applauſe and Celebration of Them.

By theſe Vertuous, and engaging Arts, Great Prince! You have won the Common and Univerſal Love and Eſteem of the whole Nation; which, however Divided in other Matters, are yet Entirely *One*, and firmly *United* in their juſt *Eſteem*, and *Honour* for Your Princely Vertues, and Merits to their Countrey.

It ſpeaks a mighty, and powerful Charm, Great Sir, to Unite ſuch Divided Hearts; and nothing but Your incomparable Vertue, and Goodneſs, and thoſe infinite Obligations You are ſtill laying upon our

 Countrey,

Countrey, could ever have eſtabliſhed ſuch an Univerſal Conſent and Agreement in the Hearts of a *People*, ſo little acquainted with the pleaſures of *Union*, either in *Intereſt*, or *Affection*.

Long may You live, GREAT PRINCE! the common Object of all our Praiſes, and of our Prayers to Almighty God for the Preſervation of ſo valuable a Life. How dear it is to us hath already publickly appeared in the Solemn Prayers and Supplications of Our Church, and is daily acknowledged in ſecret by the private Interceſſions of Many, who earneſtly beg of God to continue ſo Bright and Illuſtrious an Example of Vertue and Goodneſs amongſt us; and preſerve to an Age, that hath ſo few of them, the Incomparable Pattern of THE BEST OF HUSBANDS, THE BEST OF MASTERS, and THE BEST OF FRIENDS.

A Character of Your *Royal Highneſs* which all Men do profeſſedly agree in, and of which our Family in particular have had the Cleareſt and the Nobleſt Demonſtrations, having long had the Honour and Happineſs to attend Your *Royal Highneſs*, and be Eye-witneſſes of thoſe Vertues, which others Admire and Celebrate at a diſtance.

How unable I am, GREAT SIR! to do Juſtice to Your Illuſtrious Character is ſufficiently ſeen in the poor Attempt I have already made: but I humbly beg leave to aſſure Your *Royal Highneſs* that no Man hath a truer, or greater Zeal for Your *Glory* and *Honour*; no Man hath a juſter Reſpect, and Veneration for *Your*

Perſon, nor does any Man ſend up more Ardent Supplications to Heaven for the Encreaſe of Your *Health*, and long Continuance of THAT IMPORTANT, and BELOVED LIFE, on which the Joy, and Glory of THE BEST OF QUEENS, The Happineſs of Her Reign, and the Proſperity of Her People do ſo much Depend. In theſe things no Man exceedeth him, who is, GREAT PRINCE,

Your Royal Highneſs's

Moſt Obedient,

Moſt Faithful,

And moſt humble Servant,

EDMUND SCARBURGH.

THE
PREFACE.

LONG Prefaces how pleasing soever to the Writer's own humour, who is very apt to be favourable to his own Productions, yet are seldom agreeable to his Readers: nor can all his Courtesy, Insinuation, and affected study to please and invite, make any tolerable amends for the tedious and ungrateful Fatigue.

If the Piece be well perform'd, and answers the design and intention of it, the Reader is uneasy to be detain'd from its perusal; if it be not, all the plausible Pretences and Excuses in the World will never recommend it to the Approbation of a Man of Taste and Judgement.

It were very easy to run out, and Harangue the Reader in commendation of this Excellent Study, so highly celebrated by the Ancients, and so much in the Esteem and Fashion of the Age we live in: But this is a design that the best Wits of every Age have perform'd to admiration, and have left no colours for any new Pretender to adorn it with; therefore waving any attempt of that Nature, and all the customary Modes of Formal and Ceremonious Apologyes, *I shall apply my self to inform the Reader, that shall please to peruse it,* What *he may exspect, and* What *he will find to his satisfaction in this following* Work.

First, *A plain, but I hope a just and exact* Translation *from the* Original *into Our* Mother Tongue, *without neglecting too much the Turn and Idiom of the Language it was at first written in.*

Secondly, *He will find* Such Illustrations *for the benefit of Younger Students, and* Such Annotations *annext to the most difficult Places, as may serve to clear the Author's Sense, and explain it to the Capacity of the meanest Reader, that is never so little conversant in these Studies.*

Thirdly, *and lastly, He will see* Our Author, The Great *and* Noble Elementator *himself,* Vindicated *from the many captious* *and unreaſonable Objections brought againſt Him by ſome Severe, and over Critical Commentators.*

If I have perform'd THIS, *I have my deſire in Compleating and Publiſhing a* WORK, *which was deſign'd, and begun many years ago at the Command of my much* HONOURED FATHER, *and* THAT ILLUSTRIOUS PRELATE, *the late Lord* BISHOP OF SARUM. A Prelate *whoſe* Piety, Charity, Hoſpitality, Friendſhip, *and* Wonderful Attainments *as well in the Politer Arts, as Deeper Searches in Divinity, made* HIM ONE *of the Glories of his time, who hath deriv'd a laſting Honour on* THAT SEE, *and left a moſt incomparable Example of* Charity *and* Munificence, *to all his Reverend Succeſſors in* That Church. A PRELATE *whoſe Memory will be for ever* BLESSED, *not only in his own Dioceſs, but wherever Piety, and Learning, and ſweetneſs of Converſation have any Name, and whoſe Authority, I dare promiſe my ſelf, will not a little recommend this Undertaking to the World. For tho' his Lordſhip did not live to ſee it finiſhed, yet He, in his perfect Health, highly approv'd the Deſign, and laid his earneſt Commands upon me to Compleat it.*

MY FATHER, *whoſe name (I preſume) is not the leaſt in the Regiſter of Men Learned, and Famous in theſe Studies, liv'd to have the peruſal, and Correction of the greateſt part of this Work, which may in ſome meaſure recommend it to the* Judicious Reader, *and vindicate the Piece from the imputation of being* a Common, *and* Uſeleſs Performance. *He had the honour in his Life time to have the Acquaintance, and Converſation of the moſt Celebrated Maſters of theſe Sciences, and had made ſo Large, and early an Advance, and Progreſs in theſe Studies, as to deſerve that kind and honourable Character from the Learned and Judicious* Mr Oughthred *in his Preface to his* Clavis Math. "*A Man, ſays He, of a* "*pleaſant and obligeing Temper and Converſation, of a piercing* "*Wit, and penetrating Judgement; ſo admirably verſed in Ma-* "*thematical Studies, and of ſo happy and ſtrong a Memory withal,* "*that he was able upon any occaſion to Repeat, and Apply every* "*Propoſition in* Euclide, Archimedes, *and ſeveral other Ancient* "*Maſters in theſe Studies.*" *A Character, which not only ſpeaks the high Eſteem that excellent* Author *had of Him; but ſhews*

HIM *likewise to be no improper Person, upon whose* Authority, *and* Direction, *a performance of this nature might be undertaken.*

But neither did He live to see it finish'd; For my many concerns, and unavoidable Avocations, kept me many years from pursuing This Work: *till at last I had a very fair Prospect of making it serviceable to the Ever to be Lamented* DUKE *of* GLOCESTER, Whose Death *put an Other, and almost Final stop to this Work; till being more at leisure, and continually stirr'd up by the remembrance of those Worthy and Excellent* Persons, *that had recommended the Compleating of it, I resolv'd to go thro', and Publish it to the World, having at the same time the Encouragement, and Recommendation of the* The GREAT D^r^ WALLIS, *and his Learned Friend the* WORTHY D^r^ GREGORY; *Men whose very names are of Virtue to keep the Work from blushing, and not only shelter it from Censure, but Recommend it to the Approbation of Men of Judgement.*

For the First *was undoubtedly* THE GREATEST MASTER *of* THIS SCIENCE, *that hath appear'd in any of these later Ages; The honour of Our Countrey, and Admiration of Others, whose Character can never be more fully, or lively expressed than in that just and excellent description of Him by the Learned and Judicious* M^r^ Oughthred; "*A Person (says He) adorn'd with all ingenu-* "*ous, and excellent Arts, and Sciences; Pious and Industrious, of* "*a deep and diffusive Learning, and an accurate Judgement in all* "*Mathematical Studies, and Happy and Successful to Admiration* "*in Decyphering the most difficult and intricate Writings.*" *Which was indeed his Peculiar Honour, and the greatest Argument of a most subtle and searching* Wit *and* Judgement.

As for the Latter, The Learned Professor of Astronomy, *as* He *wants no Commendation to the Present, so will He not fail to leave a* Noble *and* Lasting Character *to future Ages, and Live for ever in his many learned Discoveries, and incomparable Performances in* ASTRONOMY.

An

An Index of the Authors mention'd in the Annotations.

The Reader is desir'd to Correct these few Escapes of the Printer, according to the following ERRATA.

Page 5. *line* 7. and elsewhere, for Extream, or Extreams *read* Extreme, or Extremes. *p.* 11. *l.* 10. *r.* Superficies. *p.* 45. *l.* 27. *r.* ABC. *p.* 48. *l.* 30. for Proposition *r.* Proportion. *p.* 53. *l.* 11. for DH *r.* DG. *p.* 63. *l.* 38. *r.* δεικτικῶς. *p.* 66. *l.* 21. *r.* to DA. *p.* 68. *l.* 42. for it *r.* is. *p.* 81. *l.* 32. to strait *add* line. *p.* 87. *l.* 1. to point *add* A. *p.* 88. *l.* 22. for in *r.* is. *p.* 96. *l.* 13. for known *r.* know. *ib.* *l.* 39. *r.* to a Rect. *p.* 108. *l.* 12. *r.* Parall. *p.* 110. *l.* 5. after subtending *dele* ,. *p.* 181. *l.* 29. for diving *r.* dividing. *p.* 184. in hends *dele* s. *p.* 186. *l.* 3. *r.* lyable. *p.* 204. *l.* 18. *r.* of D. *p.* 214. *l.* 12. for 3 *r.* $\frac{9}{3}$. *p.* 219. *l.* 21. *dele* and. *p.* 224. *l.* 2. to hath *add* to.

THE

THE FIRST ELEMENT OF GEOMETRY.

DEFINITIONS, OR Expoſitions of Geometrical Names, and Terms of Art.

DEFINITION I.

A *Point is That of which there is no part.*

ANNOTATIONS.

EUCLIDE begins the Elements of Geometry in a true Elementary method of Doctrine from the moſt ſimple Things, and Notions in Geometry to the more and more Compound.

And becauſe Magnitude the ſubject of Geometry hath not a *Minimum* to begin with, as Number hath an *Unite*, therefore nothing elſe could be more ſimple in theſe Mathematical Speculations of Magnitudes, than firſt to put a Point as void of Magnitude. Which notwithſtanding is a proper beginning to enter into the preſent matter, as the firſt ſtep towards Geometry: and It is in ſome manner like to a Cypher in Arithmetic, which may be call'd an Arithmetical Point.

The definition of a Point is plainly negative, and no otherwiſe informs us what a Point is, than by telling us what 'tis not. Yet in many Things that are in nature moſt ſimple Theſe kinds of Negative Definitions are ſufficiently inſtructive; tho' not to the Eſſence of the Thing defined, yet very well to the Uſe that is to be made thereof.

So here a Point is defined by a negation of Parts. Which Definition in reſpect of Magnitude, That was next to be conſidered as diviſible into parts, is Inſtructive, or Preparatory to the right underſtanding of the Doctrine of Magnitudes; and lays down what conception of a Point is hereafter uſed, or uſeful in Geometry, namely, *To have no parts.* Which is ſufficient for the preſent to a Geometrician. Let the Philoſophers diſpute further, as they pleaſe, about the reality of a Point, or the nature of an Atome.

But now in the firſt place, we are to underſtand, that A Point made with a Pen or any other Inſtrument is but an imperfect, and groſs Notation of a Mathematical Point here defin'd by *Euclide.* Which is abſtracted from Matter, and Quantity, and only to be conceiv'd in the Mind. Yet thoſe kinds of material, and viſible Points being put as the leaſt object of our Sight, may be allowed for a Note of a Mathematical point, which is the leaſt object of our Underſtanding in theſe Geometrical Contemplations.

And to go on with the name. PUNCTUM, A POINT is among the Greeks vulgarly Νυγμὴ & Στιγμή. *A Prick, Note*, or *Mark*. By *Plato, Ariſtotle* and other Philoſophers, Στιγμὴ is taken in a ſtrict Geometrical ſenſe for *an Indiviſible Mark*, or a *Notation of an Indiviſible Thing*. Which is the ſame with a *Mathematical Point*. But *Euclide* names it moſt properly Σημεῖον, *A Signe*. That is, ſuch a Signe, as denotes in ſpace *An Impartible Here, or There*.

Now ſpace is an Infinite, and Unmoveable Diffuſion every way: ſuch as can afford a Locality to any one Thing without the Reſiſtency, or Ceſſion of any other. Πάντων ὑποδοχὴ, the Receptacle of all whatſoever is, or can be. Thus *Virgil* makes old *Silenus* ſing the Beginnings of Things.

Namque canebat uti Magnum per Inane coacta
Semina terrarumque, animæque, mariſque fuiſſent.

Virgils Magnum Inane, that Immenſe, and Empty ſpace both the Philoſophers, and Mathematicians, put as the *Primum Conceptibile*, The Firſt Conceiveable in the Being of Things. Call it ſpace Phyſical, or Mathematical. For we diſpute not here with *Democritus*, and *Epicurus*, whether Space be a Thing in Nature diſtinct from Body poſited in Space. Only the Mathematician gives a Being to It, as in general abſtracted from any kind of Body, that may fill, or poſſeſs a Space.

In this Univerſal Space *A Point, or Signe is a certain Poſition without any Quantity*. It is *An Indiviſible Ubi* to be put at pleaſure any where. *An Ubi Ubilubet*. And becauſe whereſoever a Point be put, the ſame is conceived to be ἀμερὲς, καὶ ἀδιάστατον, void of Parts, and Interval; therefore it is the moſt Simple, the Firſt, and Leaſt Thing imaginable in Space. Μιμεῖται τὴν ἀκροτάτην τῶν ὄντων φύσιν, ſays *Proclus* moſt acutely. A Point repreſents the Utmoſt nature of Things. That is, a Point is of a nature ſo Subtile, that it has the very Extremity of Being, or the Next to Nothing. So *Lucretius*,

——— *Punctum ſine partibus exſtat,*
Et minimâ conſtat naturâ. ———

But yet a Point hath ſuch a Being in Nature, how little ſoever it may ſeem, that from very many Inſtances it evidently ſhews it ſelf to Be.

In the ordinary uſe of Burning Glaſſes there is vulgarly taken notice of a certain Burning point. And ſuch a point It is, as proves it ſelf really to be. In Loadſtones it is commonly known that there are Polar Points, called North and South. In the deſcent of Bodies towards the Earth, and in all Parts of the Earth, while they every way preſſing together do fall into an Orb, or Globe, there muſt ariſe a reſpect to a certain Point; Inſomuch that upon this very Point the Maſs of the Earth,

Moles Telluris
Ponderibus librata ſuis immobilis hæret.

And in the like manner, what is more manifeſt to be, than the middle point in the Balance? Than the point of *Æquilibrium*, or Equal-poiſe in every Body? And it is no part of the Body. For that every part of a Body is a Body: and whatſoever is Body, has a point of *Æquilibrium*. So then in every Body ſuch a Point there is. Which Point being no part of that Body, muſt be a meer Mathematical Point.

A Point therefore has a Being, tho' Indiviſible. Yet it is not the only Thing to be conceived indiviſible in Nature, But in *Geometry* It is the only indiviſible Thing.

And in reſpect of this its Indiviſibility, for Illuſtration ſake, a Point in Geometry is compared to an *Unite* in number, and to an *Inſtant* in Time. Both Theſe being alike conceived under the ſame notion of Indiviſibility with a Point, tho' in other reſpects all the Three be much different from one another. But becauſe the Indiviſible natures of Unity, and Inſtant are more obvious to common Apprehenſions; Theſe do well enough illuſtrate the Indiviſible Being of a Geometrical Point.

By the *Pythagoreans*, who bring all things into the Myſtery of Numbers, A Point is ſaid to be Μονὰς θέσιν ἔχουσα. A Monade, or *Unity having poſition*. Indeed to *have Poſition*, or *Situation* is the only *poſitive conception* to be made of a Point: Whoſe Exiſtence is in its Locality: As *Proclus* ſays, οἷον ἐν τόπῳ γέγονε. *Exiſtit tanquam in loco*, Here,

Here, or, There. But now Unity properly taken, as the principle of Number, has nothing of Position. As *Proclus* goes on, ἄθετος ἡ Μονὰς, καὶ πάντος ἔξω τόπου. And upon this consideration Unity is στιγμῆς ἁπλουστέρα, more Simple than an Indivisible Point, for that a Point exceeds the Indivisible simplicity of Unity by the additament of Position. And therefore in this metaphorical Definition, by Unity is only to be understood Indivisibility, or Vacancy of parts. So that, *A Point is to be conceived such a kind of Unity, such a Monade, or Indivisible Being, as in Space to possess an impartible Place, or Position.*

And to proceed in a familiar comparison between a Point in Geometry, and an Instant in Time, there is such an Agreement in their Indivisibility, that even Life it self, our very Being is but a Point. For only τὸ νῦν, This Point of Time, The Indivisible Instant, The present Moment is. And what was, and what shall be, is not.

Thus much of a Point. Πρός τε τὴν ὕπαρξιν αὐτῆς, καὶ τὴν ἀνυπαρξίαν. Whether of Something, or of Nothing.

The Geometricians do commonly note a Point by some one Letter of the Alphabet. As the Point A, or the Point B.

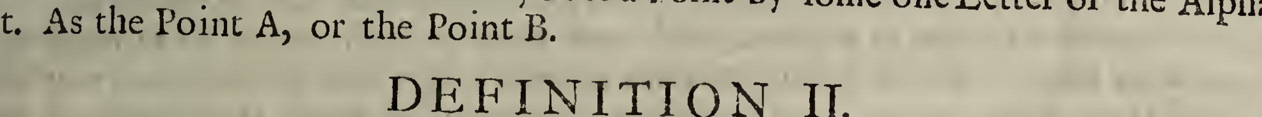
DEFINITION II.

A Line is Length without Breadth.

ANNOTATIONS.

A Point was the first thing towards Elementary Geometry posited in Space, and That without any Quantity. The next and first most simple Thing with Quantity is a Line, That is, *A meer and simple Extense; or solely Length.* In which Definition there is a positive notion of Length, restrained by a negation of the next immediate composition made of Length and Breadth. And herein for the present It confines our Imagination to one single consideration of the mensuration of Magnitudes, and That according to *Length alone.* whereby we understand what conception of a *Line* is ever and only to be made use of in Geometry, LENGTH ALONE.

We may help our imagination in the conception of a Mathematical Line after this manner. Put a Point (I mean here a Thing, whose quantity is not considered) and let it be supposed to move from one place to another. Then shall the same leave an imaginary Track only Long, which is called a Line. As if the Point A be imagined to move from A to B, It shall trace forth the Line, named A B.



Not that hereby a Line is *A Flux of a Point*, as some define It: (for motion creates not magnitude; tho' the nature of the *several Dimensions* of magnitude, as abstracted from Matter, may be well conceived by motion) but only by this Instance we do give *some Idea of Length*, or of that kind of magnitude, which is called a *Line*, not considering the subject wherein It is.

And therefore, to explain further this present matter, we are not to conceive that a Line described from the motion of a Point does consist of Points. For the motion, or flux of a Point adds not Point to Point in its progression, and thereby constitutes a Line, as the links of a Chain make up a Chain: but only does *by the continuity of its motion* represent to our imagination a *Continued Line*, or Length without Breadth.

The Conception of Length arising from the motion, and the negation of Breadth from the nature of the moving Point, which is conceived such, as to be void of Length, and Breadth, or any thing of Magnitude; and therefore in its imaginary flux cannot form any representation of Breadth; but only of a pure Mathematical Length.

Again, forasmuch as a Thing cannot be made of Nothing; and therefore ten thousand Nothings cannot make One Thing: so a magnitude cannot be made of that Thing, which in it self is no magnitude; tho' never so many of Them were put together. If therefore a Point be ſuppoſed to be put to a Point, there cannot be any thing elſe conceived, then that They both muſt wholely enter into one another, and ſo ſtill be put as one Point. For otherwiſe if they join to one another, they muſt join by ſome common Extreme: but a Point having no part, can have no Extreme, no Beginning, Middle, or End. Wherefore upon the ſuppoſition of Two, and ſo by conſequence of more points put together, there can ſtill nothing more reſult or be imagined, than a Point. Therefore whenſoever we ſuppoſe two diſtinct points, ſome kind of interval muſt be alſo ſuppoſed to be between them. So that the flux of a point proceeds not from point to point; but immediately paſſeth into Length. And therefore from the common Inſtance of a fluent point we are not to conceive, that a Line does conſiſt of points; or that a point is any part of a Line. But as every motion begins and ends where there is no motion, ſo every Length begins and ends where there is no Length.

But tho' there be many irrefragable Demonſtrations that a Line cannot be conſtituted of Points; yet *Theodoſius* hath as clearly demonſtrated, that a Sphere toucheth a plain ſuperficies only in a point, and how then It doth not in its motion on a Plain trace forth a line of points, *Clavius* acknowledges that he hath not met with a ſatisfactory Explication. And therefore *Fromondus* his Book *De Compoſitione Continui* is by him not amiſs entitled *Labyrinthus*. Thus the Reaſon of Man muſt ſubmit to the Incomprehenſible Secrets, which by an Inſcrutable, Infinite Wiſdom, are planted in the Nature and Frame of Things.

OF the Application of Number to Magnitude, and the Uſe Thereof in the Menſuration of Magnitudes.

But now in Diſcrete Quantity the matter is far otherwiſe. Where Μονὰς, *Monas*, an UNITE, or *Monade*, tho' It be as indiviſible and partleſs as a point; yet It is a conſtituent part of every number: and therefore every *Number is a Multitude conſiſting of certain Units*, or Monades. Wherefore it ought to be obſerved, that in comparing Magnitudes with Numbers a point in Magnitudes does not anſwer to an Unite in Numbers; but rather to a Cypher. Which is no more a part of Number, than a point is a part of Magnitude. Whereas an Unite is not only a conſtituent part, but alſo the *Leaſt Part* of every number, and therefore It is the natural meaſure of all numbers. *A Meaſure being the Minimum omnium in eodem genere.* But becauſe in Magnitudes there is not a *Minimum*; therefore Magnitude has no *Natural* meaſure. Yet here we ſupply a Natural meaſure ſufficiently for our uſe in conſtituting, by common conſent, ſome certain known magnitude for a meaſure. Τὸ ἑστηκὸς μέτρον, *A Stated Meaſure* the ancient Geometricians call It. And upon ſuch a Stated Meaſure there follows a juſt agreement between Numbers and Magnitudes. For in the menſuration of any magnitude, what part ſoever of a magnitude is taken by conſent for a Meaſure, the ſame truly anſwers to an Unite in Numbers. For it is in this caſe a *ſuppoſed Minimum* and put like an Unite for the Leaſt Part to be conſidered in That magnitude; by which ſuppoſed Leaſt Part *we do agree* to eſtimate the quantity of the Whole. As in the menſuration of any Length, let a *Yard*, or a *Foot*, or an *Inch*, &c. be agreed upon as a known magnitude for a Certain and a Standing Meaſure. And for Inſtance, let an *Inch* be put for the meaſure of Length. The *Inch* is now a ſuppoſed *Minimum*, and becomes to be of the like nature and uſe, that an Unite is in Numbers. For as a number has its value from the multitude of Unites which are numerated to be collected in It: ſo a Line or Length ſhall have its value from the multitude of Inches, which are numerated to be contained in the ſame. As if a Line contain twelve Inches, or eight Inches, or four Inches, The ſame is valued according to 12, 8, or 4. The Line in a reſpect had to an Inch, as the number in a reſpect had to an Unite. So after this manner *The Menſuration and Eſtimation of all magnitudes is made by the numeration of the Stated meaſure, as a number is eſtimated by the numeration of its Units.*

And This is the firſt and real ground of correſpondence between Numbers, and their mutual application to one another. Which ought to be perfectly underſtood, and

and remembred for the common uſe that is hereafter to be made thereof throughout all Geometry.

Therefore ſpeaking upon this matter in general, Let the Stated Meaſure of any kind of Magnitude be called THE MEASURING UNITE, or THE GEOMETRICAL UNITE: For very uſeful it is, and moſt requiſite, that It ſhould have ſome ſetled Name.

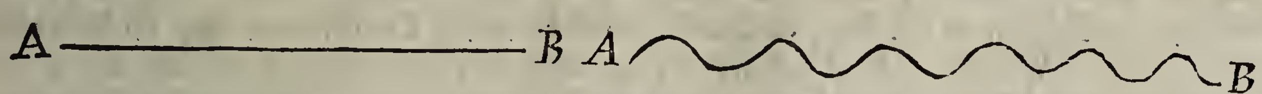
DEFINITION III.

THe Extreams, or Ends of a Line are Points.

ANNOTATIONS.

A Line was defined to be Length without Breadth, that is, *Length alone*: and ſo taken abſolutely without any reſpect to Termination; which is a Secondary notion, and a Supervenient Mode, or Qualification of Magnitude.

In the firſt place therefore *Euclide* defined a line without limitation. But afterwards if a line be any ways determined, as *Here*, or *There*, or *Any where*, He tells us now that in the termination of a line the Extreams, or Ends thereof are underſtood to be points; ſuch as he had before defined to be void of parts. For in common ſenſe Length cannot be terminated by any thing having length, becauſe it will again be demanded, What limits This Length, and ſo forth infinitely. Therefore a line muſt be conceived to be limited by ſomething void of Length, and by conſequence of Magnitude, which Thing is called A Point. As for example, of a limited line A B the Extreams, or Ends are the imaginary points named A, and B, that is as much as to ſay, that the line A B begins *Here* at A, and ends *There* at B.



So that *a Finite line is a continued Length determined by two points, Here and There.*

DEFINITION IV.

*A Strait Line is That * which lies evenly to the Points within It ſelf.*

* Ἧς ἐξ ἴσου τοῖς ἐφ' ἑαυτῆς σημείοις κεῖται.

ANNOTATIONS.

As for Inſtance. In a line let there be taken points at pleaſure how many ſoever. A, B, C, D, E, F, &c.

If the line be ſuch as *Euclide* ſets forth by the name of a Strait line, then we

are to conceive, that the whole line κεῖται ἐξ ἴσου, or ὁμαλῶς. Jacet equabiliter, *Lyes evenly*, that is, juſt in one and the ſame poſition to all thoſe points: ſo that from any point to any point there is no where made in the tract of the line any change of Poſition on one ſide or other, upward or downward, or any various way. But that every imaginable part of the line does bear a like Site and Reſpect to all points imaginable in the ſame line.

Whereas in a crooked line Every aſſigned part has a different reſpect of Situation to all aſſignable points in the crooked line. *Here* one way, *There* another; in an infinite variety of poſitions of the One to the Other, of the Parts of the line to the Points, and of the Points to the Parts of the line. But in That, which *Euclide* calls a Strait line, there is to be conceived One and the ſame Equability throughout in the poſition of the whole, and of all its parts to every point in the ſame line.

The Uſe of This Definition.

So that *firſt*, if any ſtrait line be ſuppos'd to lye upon another ſtrait line, It ſhall neceſſarily follow upon this uniform conſtitution of all ſtrait lines, that their intermedial parts muſt be congruous, that is, coincident, and every where agreeing exactly with one another, ſo that They be as one ſtrait line.

And *ſecondly*, by a natural conſequence from this Definition of a ſtrait line *Archimedes* before his Books *De Sphærâ & Cylindro* does aſſume as a Common Notion, That

A Strait line is the Leaſt of all Lines having the ſame Extreams.

And theſe are the two ſpecial uſes made of *Euclid*'s Definition in ſeveral Geometrical Demonſtrations. As for the *Firſt*, in Prop. 4th. and 8th. El. I. in Prop. 24. El. III. &c. Of the *ſecond*, in thoſe admirable Books of *Archimedes* concerning the *Sphere* and *Cylinder*.

But This Propoſition of *Archimedes* ſome Commentators cite for Another Definition of a Strait Line, wherein they are much miſtaken. That Great Geometrician ſo well underſtood the ELEMENTATOR in this his moſt accurate Definition, that he went not about to mend It, and to give a new Elementary Definition, which was much below his thoughts, and preſent Matter. But from thence, as his Subject requir'd, did aſſume this Propoſition, as a natural and immediate Conſectary, and puts It for a granted Maxim, not a Definition. Hereupon therefore it will not be unuſeful to conſider farther of Mathematical Definitions.

Of Definitions Mathematical and Philoſophical.

Definitions may be taken two ſeveral ways. Firſt, there may be an Idea, Image or Conception in our Mind of a Thing, which we cannot expreſs to others, but in many words. Now to this Thing expreſſed at large we would give in brief a certain Appellation or a ſettled Name; ſo that whenſoever this Name is mentioned, we intend thereby that the Thing be in ſuch manner conceived, as It was at firſt expreſſed, or as we ſay, defined. Mathematical Definitions are to be taken in this ſenſe. Which indeed admits of no diſpute; for that it is free for every man to give what Name he pleaſes to his own Conceptions. He is only afterwards bound to uſe the Name always in the ſame ſignification, which at firſt he gave unto It.

Again there are many Things of which Men have a common Idea, and alſo a Name commonly received to ſignify every ſuch Thing. But yet upon ſearching more curiouſly into the nature of Things we do often require a more perfect Explanation of the Eſſence and Intrinſecal conſtitution of thoſe Things. This kind of Explanation is call'd a Philoſophical Definition. Which ought to be an Analyſis of the Thing into the Eſſential principles of which It is compounded. As when we demand what is Water? What is Fire? Of theſe Things there is among men a common Idea, I mean, of *The Whole* or *Totum Phyſicum*, and a like uſe; Alſo in every Language a certain Name; Notwithſtanding which knowledge of Ours, we do by theſe queſtions require ſomething more to be inſtructed in, concerning their original conſtitution. That is, we would reſolve the Whole into its conſtituent parts, and lay open the ſecret Compoſition of their Natures. But whether the Intellect of Man can pierce ſo deep into the Intrinſic State of Things, as to give an Eſſential Definition of any Thing; is to me ſo much unknown, that altho' *Ariſtotle* rightly teaches what That Demonſtration, which he calls Διότι ought to be, yet I believe the moſt ſubtile wits in the world never fully diſcern'd the Eſſential frame of any Thing, or the Natural progreſs of Cauſes and Effects, whereby to be enabled to give juſt Definitions of Things, and accordingly Demonſtrations Διότι in the Courſe of Nature. Our Definitions being only the Analyſis of our own imperfect Conceptions of Things, rather than of the Things themſelves.

For to Know the Divine Mechaniſm of this material World belongs only to THAT Eternal, Almighty Self-Being, τὸ ὄν, τὸ ἕν, καὶ ἑνὸς νȣ̃. The *very* Being, The One, and of *one only* Thought, Which (The ſame with Himſelf) together Knows and makes all Beings.

What Intuition into the Eſſence of Things, Spiritual Creatures may have is alltogether unconceiveable by Man. Certain it is, that Humane Underſtanding proceeds

proceeds upon Conceptions inadequate to Nature, taking Things only by Parts, and from their outward Appearances. Λογισμὸν οἶδα πραγμάτων διαίρεσιν. Is the pure Iambic of *S. Greg. Nazianzen*, *Reaſoning I know to be the Diviſion of Things*. Our Ratiocination indeed is made after that manner, ſtep by ſtep, to lead us into ſome kind of knowledge. And in obſerving what appears the Firſt, moſt Simple, and immediate Emanation of a Thing, we do from Thence determine the nature of the Thing in It ſelf, and diſtinguiſh It from all Others. And hereupon we form in words a Propoſition, commonly called an Eſſential Definition. It ſerves indeed as well for our uſe in Reaſoning; if we can from thence as *A Cauſe* gradually deduce all other Affections and Properties obſervable in the ſame Thing.

Thus there are theſe two kinds of Definitions. The firſt is by Logicians called *Definitio Nominis*. Which is of moſt uſe among the Mathematicians, who giving Names to their Conceptions, do in their Definitions put *A Name* for the *Subject* of the Propoſition, and *The Thing* which is to be underſtood by that Name is made the *Predicate*. And in This kind of Definition there is only imply'd that *ſuch a Thing* is ſo *Named*.

The ſecond is called *Definitio Rei*, uſed generally by Philoſophers, in which the *Subject* of the Propoſition is the *Totum Phyſicum*, or a Thing conceived in groſs under a natural compoſition, and ſignified by a certain Name: And the *Predicate* is, or ought to be the Eſſential conſtituent Parts of that Phyſical Compound.

Thus a Mathematical Definition conſiſts of *the Name and the Thing*, and a Philoſophical Definition of *the Thing named and the Eſſential parts* of the ſame Thing.

And now to our preſent matter. Firſt, as the Mathematicians underſtand their Definitions, This of a ſtrait line is with all the Reſt to be received alike by Geometricians without exception. For that with Them, as we have ſaid, there is only put or ſuppoſed ſuch or ſuch a Thing, and a Name is given to It. As here A Length is put, which is to be conceived to lye evenly to all its Points; and ſuch a Length is called *a Strait Line*. Now firſt againſt this Notion of Evenneſs in Length or an Equable interjacency of every part in a certain Length, there can be urged nothing as impoſſible or incomprehenſible; and therefore It is at preſent to paſs as a Legitimate Suppoſition. And next in calling this kind of Length a Strait Line, The Name is free and arbitrary as all Names are.

But again, if as Philoſophers we take the Subject of the Propoſition for a *Thing*, and here intend to define a Strait Line; as a Thing commonly known in It ſelf, and alſo by That Name; yet to clear our Conception therein, we would Analyſe, or Reſolve a Strait Line into the *Eſſential Grounds of Rectitude*, I ſay, even in this Philoſophical acceptation *Euclid's* Definition will appear moſt accurate upon this very reaſon, that whatſoever Notion put by any One Philoſopher, or Mathematician for the Definition of a ſtrait line, or whatſoever Properties and Affections are attributed to a ſtrait line, They do All evidently ariſe from *Euclid's Definition*, as from the *Nature of Rectitude*, and the Eſſential Conſtitution of a ſtrait line. For *Euclid's Notion of a ſtrait line does conſiſt in the Equability of its poſition to all imaginable points in the ſame line.* And there ariſes from This conception ſuch a Community, or rather Identity of Conſtitution in all ſtrait lines, that they being conſidered abſolutly in themſelves, *As Strait*, do differ from one another only in Situation, and variety of Place. So that changing in our imagination the place of one ſtrait line into the place of another (which is called Ἐφάρμοσις, *Epharmoſis*, or an Adaptation of one line upon an other) there follows,

Firſt, that *All ſtrait lines are Congruous to one another.*

And This is, as we have noted before, the Primary uſe of *Euclid's* Definition: Or rather, This is not ſo much to be accounted a Confectary, as rather the ſame notion with *Euclid's* Definition, tho' in different words. For to conceive ſtrait lines to be ſuch as to have every where an equable interjacency of all their parts to all their points; Or ſtrait lines to be ſuch as to have every where a mutual Congruability of Themſelves, and all their parts to one another is in effect the ſame Thing.

Secondly, That *A ſtrait line is the leaſt of all lines between the ſame points.*

 Which

Which notion informs not what a ſtrait line is in It ſelf; but only what It is in compariſon to lines not ſtrait. *Archimedes* makes uſe of This as a natural Conſectary from *Euclid's* Definition. Διότι, ſays *Proclus*, ἐξ ἴσου κεῖται τοῖς ἐφ' ἑαυτῆς σημείοις; διὰ τοῦτο ἐλαχίστη ἐστὶ τῶν τὰ αὐτὰ πέρατα ἐχουσῶν. *Becauſe a ſtrait line lyes evenly to all its points, for that very reaſon It is the leaſt of all lines having the ſame Ends. For if any other were leſs, The firſt did not lye evenly between its Ends.* εἰ γὰρ εἴη τις ἐλάττων οὐκ ἐξ ἴσου κείσεται τοῖς πέρασιν ἑαυτῆς.

Thirdly, From *Euclid's* Definition ſays *Proclus* 'tis manifeſt, that

Only The ſtrait line does poſſeſs a Space equal to That, which is between Its points. Μόνην τὴν εὐθεῖαν ἴσον κατέχειν διάστημα τῷ μεταξὺ τῶν ἐπ' αὐτῆς σημείων. And *Proclus* gives this reaſon; *For how much One point is diſtant from the Other, ſo much is the magnitude of the ſtrait line terminated by the ſame points.* Οσον γὰρ ἀπέχει θάτερον, τῶν σημείων θατέρου τοσοῦτον τὸ μέγεθος τῆς εὐθείας τῆς ὑπ' αὐτῶν περατουμένης. This is, ſays *Proclus*, ἐξ ἴσου κεῖσθαι. *That is, A line to lye Evenly between two points, is the ſame Notion as a line to be equal to the Intermedial ſpace between two points.* Therefore

Fourthly, *A ſtrait line is the natural meaſure of Diſtance between Point and Point, or Here, and There.*

Fifthly, *A ſtrait line is Ordinate between its Extreams*, ἐπ' ἄκρων ἐστὶ τεταγμένη, ſays *Proclus.*

Or if we take it as S^r^ *Henry Savile* corrects it, τεταμένη. Then thus it is. *A ſtrait line is ſtretched to the utmoſt between its Extreams.* And therefore

Sixthly, *A ſtrait line is ſuch, whoſe Extreams cannot in our Imagination be moveable further from each other, preſerving the Quantity of the ſame line.*

Whereas the Extreams of any crooked line may without change of its quantity be further and further diduced, till the crooked line be ſtretched to a ſtrait line.

Seventhly, *A ſtrait line is The only ſingular between the ſame Extreams.*

That is, there can be but one ſtrait line between the ſame points, whereas of crooked lines there may be infinite. Laſtly from *Euclid's* Definition there is obſerv'd an other Property of a ſtrait line relating to Viſion: *To wit,*

Eighthly, *A ſtrait line is That, All whoſe Intermedial Parts do obviate the Extreams.* Ης τὰ μέσα πάντα τοῖς ἄκροις ἐπιπροσθεῖ. As *Proclus* ſpeaks from *Plato.* And the meaning is, that if One Extream be ſuppoſed a Lucid point, and the other Extream an Eye, all the interjacent Parts of a ſtrait line ſhall obviate, or ſtand in the way, and obſtruct the Radiation of the Lucid point unto the other Extream; ſo that It cannot be viſible to the Eye in that place. As for inſtance, we find the Eclipſe of the Sun to be made by the direct interpoſition of the Moon between the Sun and our Sight, All Three then lying in a ſtrait line.

Theſe Notions, or Conceptions, and whatſoever other Attributes are by any One given to a ſtrait line, They are only Conſectaries from *Euclid's* Definition. Which for this very reaſon ſhows It ſelf to be the Primary Conception of Rectitude in lines, for that It comprehends all ſtrait lines in general, whether taken finite, or infinite. Whereas Thoſe Other here now mentioned are Secondary, relating only to a finite ſtrait line as determined between two Extreams. And moſt or all of Them are uſeleſs in Geometry.

Thus have we fully ſet forth the ſeveral notions of Mathematicians and Philoſophers concerning a ſtrait line, becauſe Many of Them have thought It worth their While to buſy Themſelves therein, and eſpecially two Great Men of our Age, M^r^ *Hobs* and the incomparably Learned D^r^ *Wallis.* And beſides we have the rather inſiſted upon this Argument, for that It gave us a juſt occaſion to Expound the Nature, and Difference of Definitions Mathematical and Philoſophical. So that in the right underſtanding of Them Both, our younger Students might be provided againſt the Cavils made upon ſome of *Euclid's* Definitions. Laſtly, It is obſervable, that Things, the more common They are and ſeem moſt known, are the moſt troubleſome to be defined. For that there is in every Man One and the ſame anticipated Idea of Familiar Things, whereby they are better known within Us, than Words can make them known unto Us. The Things themſelves ſtamping a clearer Image of Themſelves into our Imagination, than any

words

words can imprint. *Adeo difficile eſt* (to ſpeak with S^r *Henry Savile*) *rem maxime perſpicuam perſpicue definire.*

A ſtrait line altho' it be no where exiſting by it ſelf, yet there is nothing of a more common conception, and in more frequent practice. For the Diſtance of Places from one another, Land, and Sea, Heavens, and Earth, the height of Buildings, Mountains, Clouds, Comets, Planets, &c. is only conſidered and meaſured *Directly by Length*: which is nothing elſe, but what *Euclide* means by a Mathematical ſtrait line. And thus in very many caſes of Humane Affairs this notion is neceſſary, and applyed to a real uſe.

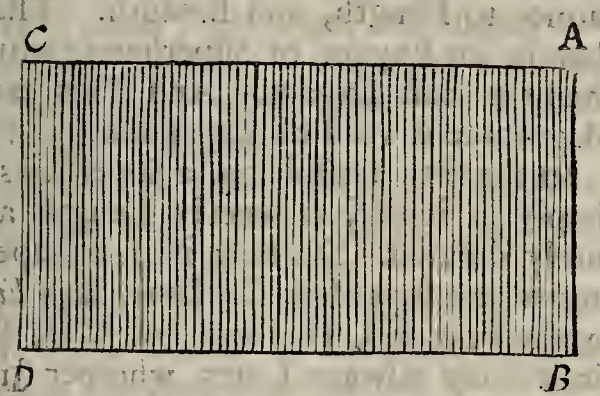
A Thread ſtretched by a Plummet, The morning Rays, and Beams of the Sun (tho' in themſelves refracted) do in ſome manner repreſent That, which we call Rectitude in lines.

DEFINITION V.

A Superficies is That, which has length and breadth only.

ANNOTATIONS.

The Mathematical conception of a Superficies, or Surface, may be thus explained by the motion of a Line. For if a line be ſuppoſed to move tranſverſly, it ſhall trace forth a certain Length, and Breadth called a Superficies; *Long* by reaſon of the Length of the moving line; and *Broad* by reaſon of the *Motion* of the ſame line *ſideways.*



As if the line AB, be imagined to move from AB to CD, it ſhall trace forth the Superficies ABCD.

But now from this illuſtration of the nature of a Mathematical Superficies, by the tranſvers motion of a line, we are not to conceive, That a Superficies conſiſts of an Aggregation of lines, for the like reaſon as were before given, That a line was not made of an Aggregation of Points. But if any Ones curioſity leads him towards an inquiſition into this ſubtile Argument, let him read That Book under *Ariſtotle's* name *De lineis Inſecabilibus*, with his Commentator *Pachemerius.* Alſo *Sextus Empiricus adverſus Mathematicos*, a Greek Sceptic Philoſopher, and of late Writers *Libertus Fromundus De Compoſitioni Continui*, which Treatiſe he juſtly entitles *Labyrinthus.*

Thus *Euclide* proceeds from a Point to a Line, and from a Line to a Superficies. And as a Line was ſaid to be an *Extenſe One way only*, and Therefore can have but One Dimenſion, or One way of Menſuration according to Length; ſo a Superficies is an *Extenſe Two Ways*, which Ways are diſtinguiſhed by the names of Length and Breadth, and therefore a Superficies has Two Dimenſions, or Two Ways of Menſuration according to Length and Breadth. And this in one word may be called AN EXPANS; as a line was called Simply an EXTENSE. Moreover a Body, or Solid, is an Extenſe three ways, (that is every way) and therefore It has three Dimenſions, or three ways of Menſuration by the names of Length, Breadth, and Depth, or Thickneſs. And that there are no more, than three Dimenſions (as in this matter *Ariſtotle* ſays, *τὰ τρία πάντα, Three are All*) *Galileus* does demonſtrate in the beginning of his firſt Dialogue of the Syſtem of the World. A Book, that deſerves to live for ever with the World.

Yet in this matter the Geometrician does acknowledge with the Philoſopher, that there is in Nature nothing exiſting of real magnitude but Body. Only there are ſeperate conſiderations of the *Menſuration* of Body, which we make to our ſelves for our own uſe, and do, as we have ſaid, ſignifie by the names of *Length, Breadth*, and *Depth.* Of which three varieties of Dimenſions every Man has naturally a clear, and diſtinct Idea.

Now the most obvious, and sensible representation of a Mathematical Superficies is a Shadow ; And the common Extream of a Superficies partly shaded, and partly enlightned, represents a Mathematical Line. Moreover the general Object of our Sight is only a very Mathematical Superficies Illuminated. For we see not into the Body, or substance of Things. And to this Consideration aptly answers the Etymologie of the Greek name Ἐπιφάνεια, *Apparentia*, the Apparence and Surface of a thing.

But farther, the use of this Mathematical Superficies is made most manifest in the Estimation, and Mensuration of Lands, wherein the Surface only is considered, and valued, and nothing of the Depth. Which shews, That the Mathematical conception of a Superficies, is also a real Notion, and of general use, in like manner as That is of a Mathematical Line.

DEFINITION VI.

THe *Extreams of a Superficies are Lines.*

ANNOTATIONS.

As a Line cannot be limited by a Line ; so upon the same reasons a Superficies cannot be limited by a Superficies. A point puts a stop to Length ; but It cannot to Length, and Breadth. That therefore must limit Length and Breadth, that is, an Expans, or Superficies, which is something more then a Point, yet less than the least Expans. And This can be nothing else but a Line, which only Magnitude is void of Expansion.

In the fifth Definition, a Superficies was first considered in its own nature, and simply in it self, as meerly Length and Breadth, but undetermined, and as infinitely diffused. Yet now if in a Superficies thus at large, there be a limitation any where supposed, the same, says *Euclide*, is conceived to be a Line. As if *Here*, by the line AB, If *There*, by CD, If *Elsewhere* by EF, GH, &c. These limitations being allways Lines, whether strait or crooked, conjoin'd, or not conjoin'd, one line, or many ; for these conditions come not as yet to be considered.

Only we are to conceive *A Superficies to be a Continuity of Length and Breadth, or an Expans determinable by Lines. As a Line is a Continuity of Length determinable by Points.*

Lastly, we are to observe, That This Definition of the Extreams of a Superficies and the Third before of the Extreams of a Line are not properly to be accounted Definitions ; alltho' They be commonly numbred among Them. For in neither of them is any new Geometrical Term defined : but they are only necessary consequences, and common notions, Resulting from the Definitions of a Line and a Superficies. For Length or a Magnitude of one Dimension must be terminated by something void of Length or any Dimension : which is a Point. And Length and Breadth or a Magnitude of two Dimensions, must be terminated by something wanting one of those Two. Which is a Line, or Length without Breadth. As *Proclus* well says, Πᾶν τὸ μέγεθος ὑπὸ τοῦ ἀμεγέθους περατοῦται. Whatsoever is partible, the same is terminated by That, which is impartible, to wit, as It is used for a term or limit, so It has no magnitude, but is Impartible, tho' in another respect the same may be a partible magnitude. As a line in respect of a Superficies, and a Superficies in respect of a Solid. And briefly, Whatsoever terminates an Other the same is less compound than the terminated magnitude by one dimension. So *Proclus* in general states this matter. Τὸ περατοῦν τοῦ περατουμένου μιᾷ λείπεται διαστάσει. *Terminans à Terminato superatur uno intervallo.*

DEFI-

DEFINITION VII.

*A Plain Superficies, or A Plane is That * which lies Evenly to the ſtrait lines within It ſelf.*

* Ἥτις ἐξίσου ταῖς ἐφ' ἑαυτῆς εὐθείαις κεῖται.

ANNOTATIONS.

After the definition of a Line, *Euclide* defines a Strait Line, and here accordingly, after the definition of a Superficies, he defines a Plain Superficies, which anſwers to a Superficies in general, as a ſtrait Line to a Line in general.

And as a ſtrait line was defined to lye evenly to the points which are in the ſame line; ſo a Plain Superficies is defined to lye evenly to the ſtrait lines which are in the Superfices. For that a Plain Superficies has a like equable reſpect to all poſſible ſtrait lines in the ſame Plain, as a ſtrait line has to all poſſible points in the ſame Line.

Whereas curved ſurfaces, as a *Conical*, and a *Cylindrical* ſuperficies, have in every part a different ſituation in a reſpect to the ſtrait lines which may be ſeated in them: and therefore they do not lye ἐξίσου (as *Euclide* expreſſes) *in an even poſition* to their ſtrait lines, as all plain ſuperficies do. And we are therefore to conceive from *Euclid's* Definition, that a Plain Superficies is ſuch as lyes every where ſo juſt, and even, that if we imagine ſtrait lines to be every way ſeated in a plain ſuperficies the ſtrait lines ſhall wholely, and in every part touch the ſuperficies, ſo as to lye juſt in it with a mutual agreement to one another. As *Sextus*, the Sceptic cites p. 101. lib. 3. *adverſus Geometras*, Ἐπίπεδον τυγχάνειν, οὗ ἡ καθαρμοζομένη εὐθεῖα πᾶσι τοῖς μέρεσι ἅπτεται. *Planum id eſſe per quod circumacta linea recta omni ex parte eidem congruit.* And upon this natural conception of a plain Superficies, Mechanics uſe to apply the edge of a ſtrait Ruler to a ſuperficies, thereby to examine, whether that ſtrait line does in all its parts agree with, and every where touch the ſuperficies; and accordingly they judge of the exactneſs of the Plane.

If we conceive a ſtrait line to move tranſverſly It traces forth *a Superficies.* And if It move in ſuch manner tranſverſly as that every point thereof deſcribes a ſtrait line; It truly repreſents to our imagination an *Exact. Plane.*

Now the PLANE, which *Euclide* has here defined, is the ἕδρα Γεωμετρική, That *Geometrical Seat*, and noble *Table*, wherein the whole matter of the Firſt Six Elements, and all the admirable ſpeculations of Plain Geometry are placed.

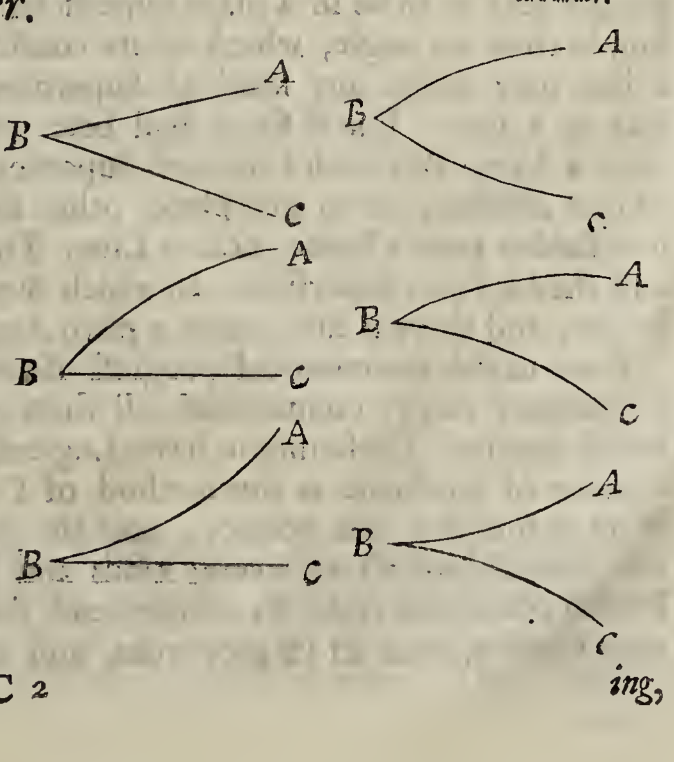
Of Plain Angles.

DEFINITION VIII.

*A Plain Angle is in a plain Superficies an Inclination of two Lines to one another, * meeting together, and not directly ſituated One to the Other.*

* Ἁπτομένων ἀλλήλων.

ANNOTATIONS.



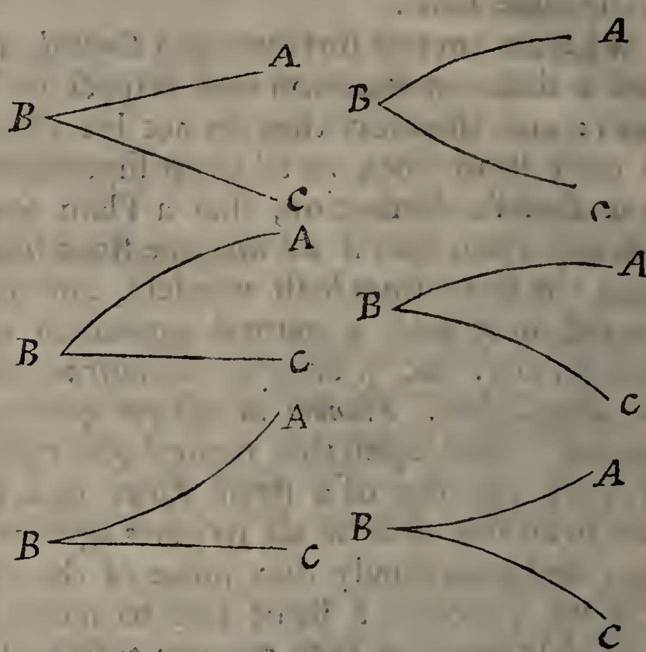
Let there be two lines not directly ſituated one to the other; as AB, CB, Touching one another, or meeting together in the point B. Of theſe *Two Concurring lines* AB, CB, Their *Inclination to one another* is called an Angle.

But an Angle is not to be ſaid to be *The Concours of two Inclining lines.* As the late and much famed French Logician, tranſpoſing *Euclid's* words, makes Him to ſay, that an Angle is *la Rencontre de deux lignes Inclinées.* Part. 4. Chap. 4. *De L' Art de penſer, Of the Art of Think-*

 ing,

ing, as the Author entitles his Logic. Whereas if he had rendred *Euclid*, he ſhould have ſaid, that an Angle is *la Inclination de deux lignes Rencontrées*. For *Euclid's Inclination* of two Concurring lines is diviſible, and menſurable; but *la Rencontre*, *the Concurs* of two inclining lines, being in a point is indiviſible: as the French Logician rightly obſerves; but hereupon he unjuſtly taxes *Euclide* with a fault of his own making, by changing *Euclid's* words, and ſenſe. And it ſeems ſtrange to me That the Author of a moſt Excellent Logical Inſtitution ſhould not advert the difference in Expreſſion and Signification, between the Inclination of two Concurring Lines, and the Concurs of two Inclining Lines: for that Inclination manifeſtly denotes a diviſible Thing; and Concurs an Indiviſible Point.

Now lines are ſaid to *Incline* to one another, when in Each line Part after Part comes nearer and nearer to the Other line, that is, when The lines do continuedly approach toward one another.



An Angle is commonly ſignified by three letters, as the Angle A B C. Where the middle letter ever denotes the point of Concurs of the two lines containing the Angle. Yet in the Notation of an Angle, we are to obſerve that The *Greeks* ſay *not* γονία ἡ αβγ. The Angle A B C, but more properly ἡ ὑπὸ αβγ, that is, ἡ ὑπὸ αβ, βγ περιεχομένη γωνία. The Angle contained by the lines A B, B C. But the repetition of β the *Greeks* omitt for brevity ſake. And for the more brevity it is tranſlated the Angle A B C, inſtead of the angle contained by A B C, or A B, B C.

In the definition of an angle *Euclide* puts the *Simple* word κλίσις, *Cliſis*, with which the Latins were not acquainted; never ſaying *Clinatio*; but always uſing ſome compound word, as *Acclinatio*, *Inclinatio*, *Declinatio*, &c. *Euclids* word κλίσις, is generally tranſlated *Inclinatio*. But I ſhould rather have choſen *Acclinatio*, as beſt anſwering to *Euclid's* word πρὸς ἀλλήλας τῶν γραμμῶν κλίσις. For πρὸς ἀλλήλας κλίσις, is more properly to be rendred, *ad ſe invicem Acclinatio*, than *Inclinatio*. *ſeſeque acclinat ad illam* is Latine; But *inclinat* is Barbariſm. Yet in this matter *Inclinatio* is the word in uſe, and to the power of cuſtom in Speech we muſt ſubmit.

Euclide having laid down a Plain Superficies as the Platform for his Elementary Geometry, a *Raſa Tabula* aptly diſpoſed to receive any impreſſion, does begin with a Plain Angle, It being the *moſt Simple* of all other delineations, whoſe only and proper ſeat is to be in a plain Superficies. For tho' a Line be a delineation more ſimple than an angle, which to its conſtituton requires Two lines, and alſo that a line may be in any kind of Superficies; yet a Superficies is not the neceſſary ſeat of a line. For if ſo, it had been immethodical to have defined a Superficies after a Line. But both Line and Superficies were taken ſingly and without reſpect to one another, or to any place, other than That of *Univerſal Space*. In which, firſt *Euclide* puts a Point. next a Line. Then a ſtrait Line. afterward a Superficies. and then a Plain Superficies. In which Smooth Mathematical field the Elementator begins, and therein firſt places a plain Angle.

Now in this Geometrical progreſs *Euclide* has not (as ſome Commentators impertinently have) enumerated all ſorts of Lines, of Superficies, of Angles, into which Each of Theſe might have Logically been divided, and ſubdivided. That manner of Doctrine is the method of Philoſophers, and very proper to Them. Who define firſt the Science, and the Subject they Treat of; and again, divide the Subject into all its ſeveral kinds and ſpecies, as They ought to do; for that Philoſophers undertake to comprehend the Whole Nature of their Subject in every part thereof, with all Its properties, and affections. Whereas the Mathematicians,

paſſing

paſſing over thoſe Logical Diviſions, do ſelect only what is neceſſary to their preſent Purpoſe. As *Euclide* in this place ſpecifies only thoſe Things, which are moſt eaſy in themſelves, and beſt ſerviceable for an entrance into Geometry. So the Mathematician orders the ſame ſubject in one way, and the Philoſopher in another: yet both equally right as to their ſeveral Ends. But *Ramus*, not juſtly conſidering the difference, that ever was, and ought to be had, between the Mathematicians and Philoſophers in handling the ſame ſubject, endeavours to join both methods together. He therefore firſt undertakes to define *Geometry* as the *Science* he treats of, and next *Magnitude* as the *Subject*. Then divides Magnitudes into Commenſurable and Incommenſurable, Rational and Irrational, and ſo forth in a Logical Form. Which Notions being very ſubtile, and above the capacities of young Scholars, are in no manner agreeable to an Introduction into Geometry, and the proper way of Teaching and Learning the ſame. Indeed *Ramus*, makes a very ingenious excuſe for himſelf, when he ſays, *Magis Logicam in Mathematico themate exercui*, *quam Mathematicam in ſuo pulvere ſerioque uſu tractavi.* To which I muſt return, *Non igitur mirum eſt*, *quod tam infeliciter tractavit Geometriam.* For in his Logical ordering of the Elements by new Definitions, by impertinent Diviſions, and Subdiviſions, and in the general courſe of his Propoſitions, He has ſo diſordered *The Elements*, that you ſcarce meet with a juſt Mathematical demonſtration: But only a perſwading face or ſome ſemblance of Truth, not a demonſtrative Conviction. And of all our late Transformers of *Euclide*, He is the moſt Ungeometrical in Demonſtration, how Exact ſoever in his Logical, or rather Verbal method, and diſpoſal of his Propoſitions, He may pretend to be. And therefore to go on with *Euclide.*

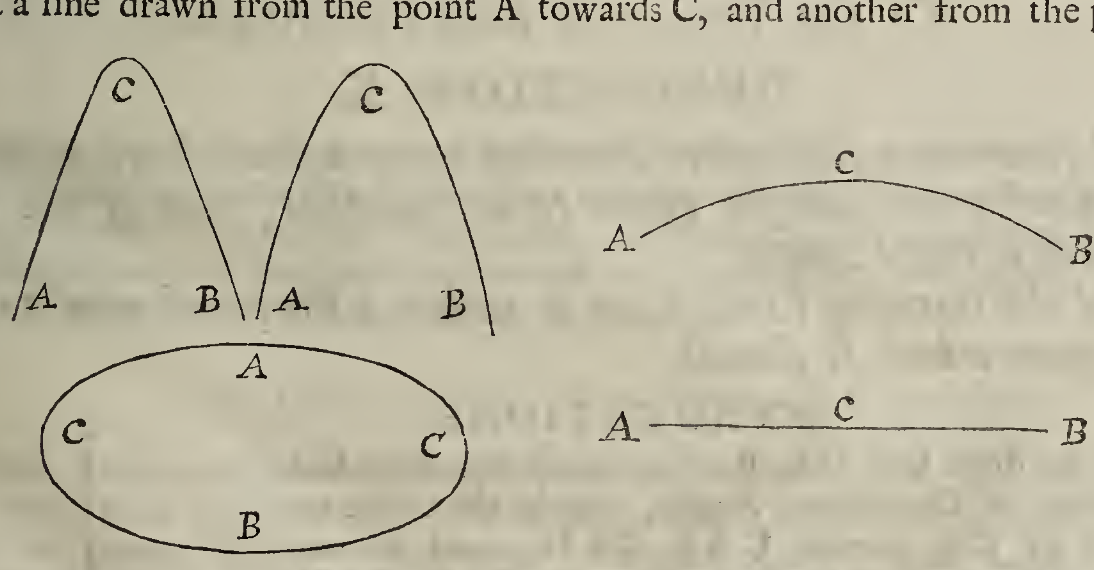
Of the Nature and Conſtitution of a Plain Angle.

For the Conſtitution of a plain angle. Firſt, there muſt be a κλίσις, as *Euclide* calls it, an *Inclination*, *Vergency*, *Leaning* or *Tendency*, of Two lines one to the other. By which words is meant, that the Two lines do one way continually approach nearer and nearer to one another.

Secondly, upon this approaching there is to follow a concurſe, or meeting together of thoſe Two lines.

Thirdly, this concurſe is to be in ſuch a manner, as that, upon meeting, The Two lines lye not ἐπ' εὐθείας, that is, ἐπ' εὐθείας ἰδοῦ, in a current or continued way towards one another, ſo as to join together in *one and the ſame line*; but that, after the point of concurſe if each line be produced in its *Proper courſe*, They ſhall ſtill be Two lines, and depart again from one another. Otherwiſe, it may ſo be as in theſe following Examples.

That a line drawn from the point A towards C, and another from the point B



towards C may in this tract, have an Inclination, Vergency, and Tendency to one another, yet in their meeting at the point C make not an angle, but that the line A C is ſo joined to the line B C that both together make one continued line A C B, and ſo no angle at all, becauſe that an angle does require Two diſtinct

 lines.

lines. If the lines thus meeting be Both ſtrait, then they make one ſtrait line. And This is εὐθεῖα κλίσις, *a ſtrait Inclination* of two ſtrait lines to one another, which upon meeting become One ſtrait line. If both lines be crooked, then may they often make *ſome One continued crooked line*; as in theſe Inſtances, and the Like; whereof there may be infinite varieties. Therefore, in the Inclination, and Approaching of two lines to one another, *Euclide* made This ſpecial Caution, for the conſtitution of an angle, That, at their meeting, They be not ſituated ἐπ' εὐθείας, that is in ſuch a courſe, as to be coincident, and continued one with the other if they be both farther produced. For this phraſe ἐπ' εὐθείας κεῖσθαι, to lye ſtrait-wiſe, is not only applicable to ſtrait, but alſo to crooked lines, where, upon meeting, their curvature remains unbroken, ἄνδυ κλάσεως as *Proclus* ſays, *without fraction*, that is, In ſuch a ſort, as that if the Two Concurring Lines be both ſtill continued onwards in their proper courſe, they do not divert, or deviate again from each other, but do mutually paſs into one another, and become one ſingle line *continued in ſua ſpecie*. The Two lines meeting in this manner are ſaid to lye ἐπ' εὐθείας, ſtrait-wiſe to one another, and the Curvature is not by Geometricians conceived, to come under the notion of an angle.

Ἐπ' εὐθείας, ſtrait-wiſe, was a brief, and vulgar manner of ſpeech, for ἐπ' εὐθείας ὁδȣ (as is before noted) that is, *in a ſtrait way*; as *recta proficiſci, To go ſtrait forward*, is for *recta via proficiſci*. And in the ſame ſenſe that the Greeks uſed ἐπ' εὐθείας, we now uſe to ſay, *ſtrait a long, ſtrait forward*, tho' there be ſome flexures and windings in the way.

DEFINITION IX.

AND when the Lines containing the Angle are ſtrait, the Angle is called a ſtrait-lin'd Angle.

ANNOTATIONS.

That is, when the Lines, which have to one another this mutual Inclination, and Concurſe, are ſtrait, then the Inclination of Thoſe Concurring ſtrait Lines is called a ſtrait-lin'd Angle.

Euclide did juſt before define a plain angle in general. Yet here He next takes notice only of *Plain ſtrait-lin'd angles*, paſſing over, after the manner of Mathematicians, the Logical diviſion of Angles into Plain and Solid. And again omits the enumeration of the ſeveral ſorts of all Plain Angles: for that the conſideration of every One of Them was beſides his preſent Matter, which he had confined to the ſimple ſpeculation of Plain ſtrait-lin'd Angles.

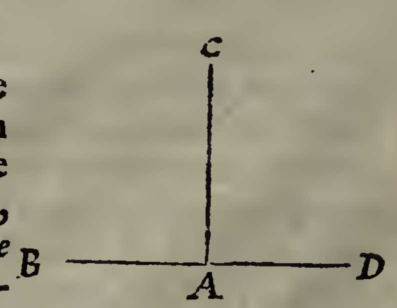
Of the Variety of ſtrait-lin'd Angles.

DEFINITION X.

ALſo when a ſtrait line ſtanding upon a ſtrait line, makes the Conſequent Angles equal to one another, each of the equal Angles is a Right Angle.

And the ſtanding ſtrait Line *is called a Perpendicular to that Line upon which It ſtands.*

ANNOTATIONS.



As if the ſtrait line C A, ſtanding upon the ſtrait line B D makes the Conſequent Angles, that is, the angles on each ſide of C A, namely C A B, C A D, equal to one another, Then each of thoſe equal angles C A B, C A D, is ſaid to be a Right Angle. And the *ſtanding ſtrait Line* C A is called a Perpendicular to the ſtrait line B D upon which It ſtands.

Theſe

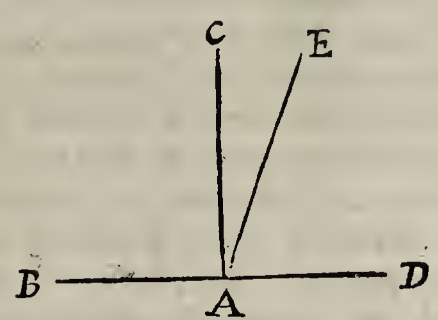
These angles CAB, CAD, which side by side are conjoyn'd to one another, are in general (whether Equal or Unequal) called by the Greeks αἱ ἐφεξῆς γωνίαι, and the Latines *Anguli Deinceps*, Which I think not unaptly rendred, the *Consequent Angles*; for that they do in order mutually follow each other.

And further to consider this matter Philosophically, The Equality of the Consequent angles is the natural and immediate cause of Their Rectitude. That is, because the strait line CA standing upon the strait line BD makes its Inclination both ways, *Here* towards B, *There* towards D equally, and just alike to the line BD, therefore the angles CAB, CAD, are both necessarily to be conceived Right, and the standing line CA to be manifestly Upright to the line BD. Whereupon, It is by the Greeks properly called Κάθετος, a *Cathetus*, That is, a line justly seated, or fitted both ways equally in respect of the line upon which It stands: To which therefore It is said to be a Normal Line.

We may here observe, That the English Tongue is as happy as the Greek, by rendring εὐθεῖα γραμμὴ *a strait line*, and ὀρθὴ γωνία *a right angle*; whereas the Latines have only one word for Both, *Recta linea*, and *Rectus angulus*, which commonly our English Translators follow, saying *a Right line*, as well as *a Right angle*. But I Wish that our Mathematical Writers would hereafter use this distinction, according to the exactness of the Greeks; seeing, that our Language affords two such proper words, as STRAIT, and RIGHT, answering to εὐθὺς and ὀρθός; And always say *a Strait Line*, and *a Right Angle*, like to εὐθεῖα γραμμὴ, & ὀρθὴ γωνία.

DEFINITION XI.

AN Obtuse Angle is That which is greater than a Right Angle. As EAB *is greater than* CAB.



DEFINITION XII.

AN Acute Angle is That which is less than a Right Angle. As EAD *is less than* CAD.

ANNOTATIONS.

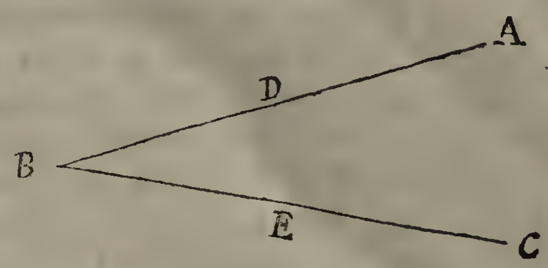
Of *Euclid's* three sorts of strait-lin'd angles the *Right angle* is made the Rule and Standard to know the Others by. For that the *Right* is always unchangeable, and the same. Whereas the *Obtuse*, and *Acute* are capable of Increasing, and Decreasing infinitely; and only determined by THIS, That all Obtuse are greater, and all Acute angles are less than a Right angle.

Of the Quantity of a Strait-lin'd Angle.

Now whereas *Euclide* defines an Obtuse angle to be greater, and an Acute angle to be less than a Right angle, the Commentators have much labour'd *Wherein to state the Quantity of a strait-lin'd Angle*. For the Quantity of an angle is estimated neither by the Length of the lines which contain the angle, nor by the Superficies which lyes between those lines, nor by the Point of Concurse of the two lines. For the angle ABC is not greater than the angle DBE, but is still the same in quantity; altho' the Line BA be greater than BD, and BC than BE,

 and

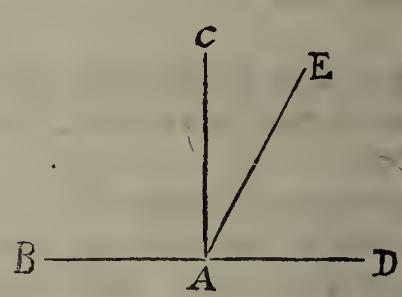
and the Superficies between A B, B C, be larger than the Superficies between D B, B E. So that in any plain ſtrait-lin'd angle the Containing Lines with the Interjacent Superficies may both together be infinitely increaſed forward, yet the Angle remaineth the very ſame.



Again, for the Point of Concurs, That cannot come into the conſideration of the quantity of an angle, ſeeing that a Point is in it ſelf no quantity.

There remains then nothing more to be conſidered in an Angle beſides the Inclination of the concurring lines. And thus much in the firſt place muſt be acknowledg'd in common Reaſon, that *only That can be Eſſential to the Quantity of an Angle, which being altered, does alſo alter the Quantity of an Angle.* Now neither the Length of the containing lines, nor the Interjacent ſpace being altered, does alter the angle, as we have already noted. But if the Inclination of the containing lines towards one another be any way altered, the angle is likewiſe altered in its quantity. As if B C be conceived to move inwardly towards B A, or outwardly from It, the angle A B C will become greater or leſs. Therefore It neceſſarily follows, that *The Quantity of a ſtrait-lin'd angle muſt conſiſt in the Quantity of the Inclination of the two ſtrait lines which contain the angle.* This Quantity of an angle is well ſaid to be, Διάςημα ὑπὸ τὴν κλίσιν. *A Diſtance of Inclination.* As *Sextus* the Sceptic cites It, Lib. 3. *adverſus Geometras*, p. 102.

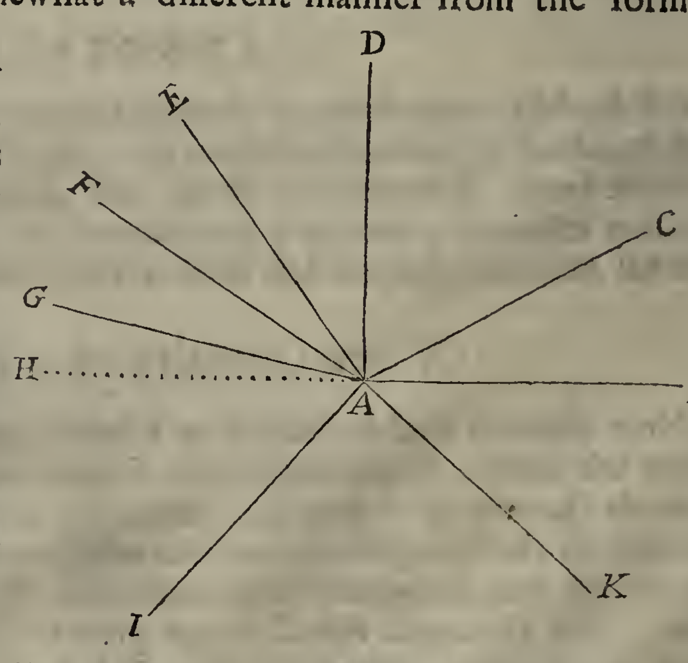
Now therefore when *Euclide* ſays that an Obtuſe angle is greater, and an Acute angle is leſs than a Right angle, It is to be underſtood, that the *Amplitude*, or *Diſtance of Inclination*, is in an Obtuſe angle greater than That, which is in a Right angle, and in an Acute angle It is leſs.



As the Inclination of the lines B A, E A to one another is greater than the Inclination of B A, C A to one another. And the Inclination of the lines E A, D A to one another is leſs than the Inclination of C A, D A to one another. And thus *from the Dilatation, or Coarctation of the Inclination of the containing lines, One angle is ſaid to be greater, or leſs than another.* But now, How *This Diſtance of Inclination, as an Angular Quantity* may be meaſured, and eſtimated, ſhall be ſhown hereafter in its due place.

A Second Conſideration of a ſtrait-lin'd Angle.

But to illuſtrate farther, & in ſomewhat a different manner from the former, the nature, and quantity of a plain ſtrait-lin'd angle, Let us ſuppoſe any point, as A in a plain Superficies, and from it to be drawn Two ſtrait lines at any wideneſs, or Aperture, as the lines A B, A C. Here the angle B A C may be ſaid to be the Aperture or Divarication of the lines AB, AC, which contain the angle B A C, ſo drawing A D, A E, A F, A G, the angle B A D is the Aperture of the lines A B, A D, which contain the angle B A D. In like manner the ſeveral other angles are the Apertures, or Divarications, of their containing lines; Which, may continually be enlarged, and opened more, and more increaſing the angles till the line A H comes to lye directly with



A B,

AB making one ſtrait line HB. So that this εὐθεῖα κλίσις, or *ſtrait Inclination* of the two lines HA, BA, to one another, cannot conſtitute an angle in their meeting at the point A. But furthermore, if there be drawn the line AI, there is again made the angle BAI, which is the Aperture of the lines AB, AI; and the like is in BAK; and ſo continually we may approach towards AB making the Apertures (that is, the Angles) leſs and leſs, till the approaching line be coincident, and one and the ſame with AB.

A Second Definition of a ſtrait-lin'd Angle.

Wherefore *a ſtrait-lin'd Angle* may be ſaid to be *The Aperture of two ſtrait lines drawn from a point, and not lying directly to one another.*

And how the Quantity of an angle may be eſtimated by the *Aperture* of the containing lines, as well, as by their *Inclination* to one another ſhall be ſhewn hereafter.

A Third Conſideration of a ſtrait-lin'd Angle.

Moreover, becauſe theſe ſeveral Angles, or Apertures do together make up *the whole ſpace about the point A*, from hence may ariſe a Third Conception of a ſtrait-lin'd angle, and its Quantity. For ſeeing that all angles that can be poſſibly ſtated from the point A, do together compleat the whole ſpace *about the ſame point*; therefore every particular angle contains ſome certain portion of the SPACE ABOUT the point A.

A Third Definition of a ſtrait-lin'd Angle.

Wherefore we may thus again define a plain ſtrait-lin'd angle. A plain ſtrait-lin'd angle is *A Portion of a plain ſpace about a Point, contained by two ſtrait lines, drawn from the ſame Point, and not directly ſituated to one another.*

This AMBIT, or CIRCUMCIRCA, does as juſtly anſwer to the nature of an angle, as either *Inclination*, or *Aperture*. For as in theſe Two conceptions, ſo likewiſe in this Third the Quantity of an angle is not concern'd either in the Length of the containing lines, or in the ſpace Interjacent between thoſe lines: but only in the *ſpace Circumjacent about the angular point*, Which *Circumcirca* being changed, as either amplyfied, or coarcted, the angle accordingly becomes Greater, or Leſs, as It does upon the Notion of Inclination, or Aperture. And how by a portion of the *Circumambient ſpace about a point*, the Quantity of an Angle may be alſo eſtimated, ſhall likewiſe be explained hereafter.

Thus have we Three ways whereby to ſet forth the Nature, and Quantity of a ſtrait-lin'd Angle. Either from the conſideration *of the ſpace about a Point*, or *of the Aperture of two ſtrait lines drawn from a Point*, or rather with *Euclide*, of the Inclination of two ſtrait lines meeting in a Point. Which is the moſt proper, and the only Notion of an angle uſeful in all Geometrical demonſtrations.

But now after all theſe conſiderations, if it be at laſt queſtioned in what *Predicament*, that is, in what Rank, or Claſſis of Things, whether in Quantity, Quality, or Relation an Angle is to be placed, *Proclus* well anſwers to this Logical, and Philoſophical Queſtion, that an Angle is not ſolely in any One of Theſe: but upon the ſeveral conceptions, which jointly compleat the Entire Notion of an Angle, It does belong to all the Three. For the Geometrician ſpeculates not the nature of Magnitude in it ſelf, as Magnitude, like to the Philoſophers: but only conſiders ſome certain kinds of Magnitudes with their demonſtrable Proprieties, and Qualities, and with the mutual Relations that Magnitude may have to Magnitude. And ſo there is a Compound Idea of all magnitudes Mathematically conſidered. An angle has a Quantity whereby one angle is greater, or leſs than another. It has alſo a Quality by its Form. And in the Inclination of the lines themſelves there is a mutual reſpect to one another.

Of a Term and a Figure.

DEFINITION XIII.

A *Term is the Extream of a Thing.*

DEFINITION XIV.

A *Figure is That, which is comprehended by One, or Many Terms.*

ANNOTATIONS.

Def. 13. In the termination, or limitation of Magnitudes, *Euclide* uſes two ſeveral words, πέρας an *End, Limit*, or *Extream*; and ὅρος a *Term*, or *Bound*. And here he ſays that ὅρος, a *Term* is πέρας an *Extream*. Πέρας he applys to a Line, and alſo to a Superficies, when both Line and Superficies were at firſt laid down as underſtood: but ὅρος is appropriated to a Magnitude every way determined and bounded.

For when *Euclide* defines *a Line to be Length only*, he puts there a line as an infinite, or indeterminate Length. But then again when a line is conſidered as to be determined, he ſays, that the *Ends*, or πέρατα of a Line are Points.

So likewiſe in the fifth Definition, *Euclide* ſets forth a Superficies under the ſimple conception of *Length, and Breadth*, that is *an Expans*, or *magnitude of Two Dimenſions* in general, and Indeterminate. But he afterward adds, That the *Extreams*, or πέρατα of a Superficies are Lines. He does not here mean by the word πέρατα *a Comprehenſion*, or an entire Encloſure of a Superficies by one, or many Lines; for that had been to make a *Figure* before he had defined It: But only ſo, as that of a Superficies indefinite, an Extream, Limit, or πέρας, as he calls it, in what part ſoever it be placed, *Here*, or *There*, is a Line. A Line being as a Barr in an indefinite Superficies, like as a Point is a ſtop in an indefinite Line.

But ὅρος relates to τὸ ὁριζόμενον, to That which is ſuppoſed *every way* encloſed. As *Proclus* ſays, that ὅρος is περιοχὴ τοῦ χωρίου, *The Comprehenſion of a ſpace*. For it is σχήματος ὅρος the *Bound*, or *Term of a Figure*; not ὅρος ἐπιφανείας, *the Bound of a Superficies*; but πέρας ἐπιφανείας, *the Limit of a Superficies*, This, *or* That way.

And accordingly, the bounds of Lands, or Territories are properly called ὅροι, but not πέρατα. So that where the ſubject is τὸ περιεχόμενον, or τὸ πάντοθεν ὁριζόμενον, a Magnitude *comprehended*, or *every way terminated*, as a Geometrical Figure is, the word ὅρος is properly uſed; and not πέρας. So that ὅρος may be ſaid to be πέρας, tho' every πέρας ought not to be called ὅρος. And thus much for the explication of πέρας and ὅρος to make the beſt of this matter.

For altho' it appears by *Proclus*, that, in his time, This was among the reſt of the Definitions; yet I cannot allow it to be *Euclids*. For firſt, it is manifeſt, that the Definition it ſelf is needleſs, becauſe, if it were wholly omitted, yet nothing that follows, is made thereby either imperfect, or obſcure. And beſides, it is no more a proper Geometrical word of Art, to require a Definition, than it was the common word, in that time, not only among the Land-meters, or Surveyors, but among all the people who had any Lands to diſtinguiſh, and appropriate to themſelves from thoſe of their Neighbours. He might have as well defined, πέρας to be τελευτή an *Ending*, or ἄκρον an *Extream*, as ὅρος to be πέρας. Which, as I have ſaid, was no more, at that time than what every *Greek* underſtood. For both ὅρος & πέρας were words of common uſe, and equally of known ſignification, and are in ſuch a ſenſe here taken. It ſeems moſt likely, that πέρας being before uſed in the Definitions of a Line and Superficies, and now ὅρος in the Definition of a Figure, ſome One noted in the margin, That ὅρος was alſo πέρας. And this afterwards came into the Text for a Definition; as the like has happen'd to many other Authors. For indeed, it is rather the by-note of a Lexicographer, than the Definition of a Geometrician. And yet This too ſignifies very little, unleſs ſomething more were added to πέρας, for the interpretation of ὅρος. As that ὅρος

is

is πέρας συγκλεῖον *an enclosing Extream*; which indeed may pass for a Glossary Exposition of ὅρος, but ought not to be accounted a Geometrical Definition. This also was the Judgement of that most Learned and Renown'd Gentleman S^r^ *Henry Savile*, the Munificent Founder of the two Famous Mathematical Professors Places, in the University of *Oxford*, upon this subject in his most excellent Lectures, introductory to the Elements, *That This had not the apparence of a Definition*; *Non habet Definitionis faciem.* Lect. 5. p. 101.

Def. 14. Figure is by *Aristotle* accounted the fourth Species of Quality; and rightly, being there taken abstractedly for the *Qualification of Magnitude by Termination, or outward Form.* But the Geometrician considers the *Quantum Figuratum*, the *Magnitude Figured.* So that, in every Figure, there is jointly taken τὸ περιεχόμενον, & ἡ περιοχή, *the Inclosed*, and *the Inclosure*, or τὸ ὁριζόμενον, & ὁ ὅρος, *Terminatum*, and *Terminus*, That is, *the Magnitude Bounded*, and *the Magnitude Bounding.* Both which together do constitute a Geometrical Figure.

In general, a Figure is χωρίον πανταχόθεν ὁριζόμενον, *A Space every way terminated.* So that first, *A Space comes to be a Figure by Termination*: And next, *Every Figure has its specification from the manner of its Termination.*

The first distinction here is, of Figures comprehended by One Term only, or by more than One. Which Division being most Simple, *Euclide* has, with great artifice, couched within his Definition, when he says, That *A Figure is a Space comprehended by one, or many Terms.* Whereas the Definition were perfect, if he had only said, That *A Figure is a Bounded Space.* But moreover, he very subtily draws in the primary difference of plain Figures from the number of their Terms, which are lines, *One, or more than One*, in order to the following Definitions of Figures, under One term, or Two, or Three, or Four, &c.

Thus from a Line *Euclide* passes to an Angle, and from an Angle to a Figure, for that an angle is in its nature between Them both. An Angle being something more than a Line, as having two concurring lines: yet is something wanting of a Figure, as not having a Compleat termination. And in that respect an *Angle is but a Semifigurate.* And so is justly placed between a Line and a Figure.

Of a Circle, Circumference, Center, and Raies, Diameter and Semicircle.

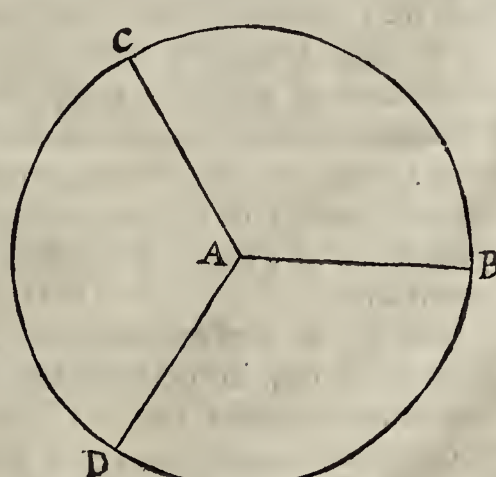
DEFINITION XV.

A Circle is a plain Figure comprehended by One Line, which is called a Circumference, unto which all strait lines, falling from One Point *of* Those *lying within the figure are equal to one another.*

DEFINITION XVI.

And This Point is called the Center of the Circle.

ANNOTATIONS.



By a plain figure, *Euclide* means a figure situated in a plain superficies, such as he had before defined. Suppose then *in a* Plane a Figure comprehended by one line B C D B, which is called the *Circumference.* And from some *One Point* seated within this Figure, as suppose from the point A, Let the strait lines how many soever, as A B, A C, A D, &c. falling on the Circumference at the points B, C, D, be all equal to one another, then such a Figure, says *Euclide*, is a Circle. And the Point A is called the *Center* of It.

Here also it is observable, that the lines A B, A C, A D, &c. may, in one word of *Cicero's*, very pro-

 perly,

perly, be named RADII, RAIES, which by *Euclide* are always called αἱ ἐκ τοῦ κέντρου, *The lines from the Center*, That is to ſay, *from the Center to the Circumference.*

And what *Cicero* has ſaid out of *Plato's Timæus*, in deſcribing the Spherical figure of the World, does almoſt word for word agree with *Euclid's* definition of a Circle, *Cujus omnis extremitas paribus à medio radiis attingitur. Cic. Fragment. de Univerſo.*

The great VIETA ſays, *Radius elegans eſt verbum Ciceronis*: Which *Ovid* alſo uſes in the ſame ſenſe, as elegantly in his Deſcription of the Sun's Chariot,

Aureus axis erat, temo aureus, aurea ſummæ
Curvatura rotæ, Radiorum *argenteus ordo.*

The Circle, Circumference, and Area.

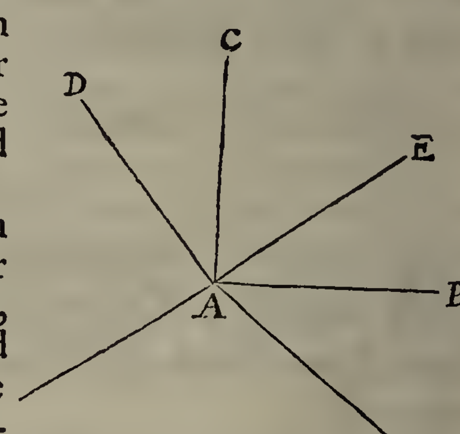
Moreover, in the conſideration of this Figure, we are rightly to diſtinguiſh between the *Circle*, the *Circumference*, and the *Area*, or the comprehended Space. The Circle is the Whole together, Area, and Circumference.

The Circumference has many properties peculiar to it ſelf, and very diſtinct from thoſe of the Circle, as hereafter will appear. And This we may firſt obſerve, that altho' the Circumference be conceived without any breadth, yet by reaſon of its curvature, It is *Concave*, and *Convex*, accordingly as it relates to What lies within, and to What is without the Circle. And ſome peculiar affections of the Concave and Convex Circumference, *Euclide* ſets forth in the 8th Propoſition of his Third Element.

Of the Circumference of a Circle applyed to the Menſuration of ſtrait-lin'd Angles.

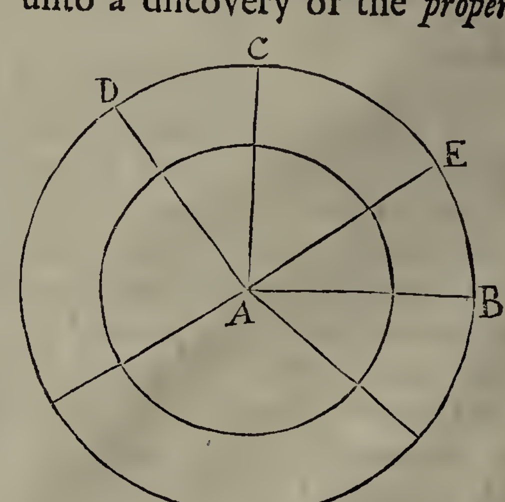
Laſtly, it will not be immethodical in this place to take a further conſideration of the properties and uſe of the Circumference of a Circle, in relation to the ſtrait-lin'd Angles before defin'd, and the menſuration of their Quantities.

Firſt then, in an infinite plain ſpace (the ſeat of plain Geometry) if we ſuppoſe a Point, as A, from whence every way do flow ſtrait lines infinitely, or indeterminately, there ſhall be contained by them the Three ſorts of ſtrait-lin'd angles, Right, Obtuſe, and Acute. As for Inſtance BAC, BAD, BAE, &c.



Here therefore for to eſtimate by ſome certain Meaſure the *Quantity of an Angle*, We are to conſider how, and by what means This infinite plain Space, wholly poſſeſſed by plain ſtrait-lin'd angles ſeated about a given Point, may be brought under ſome Comprehenſion and Boundary capable of menſuration: and therewithal the ſtrait-lin'd angles, which fill this Space, may be likewiſe meaſured.

Now the Definition of a Circle eaſily leads us unto a diſcovery of the *proper Meaſure of an Angular Quantity*: and that It may moſt fitly be found in a Circular Figure.



For *firſt*, from any Point as a Center, the Circumference of a Circle does comprehend entirely the undetermined circumambient Space about that point: and by This comprehenſion renders the Ambit capable of menſuration in that the Circumference of a circle is diviſible, and ſo certainly menſurable.

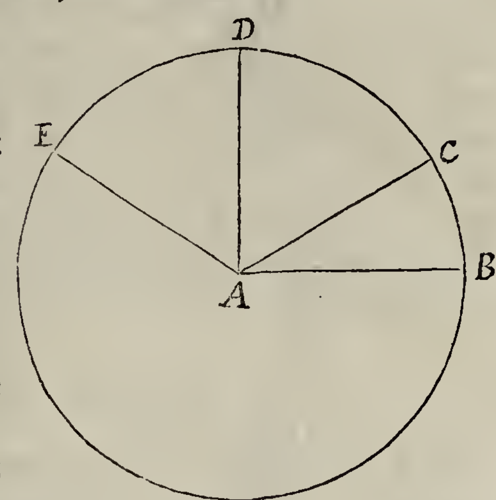
Secondly, at the center of a Circle like as at a Point in an infinite plain Space, may be conſtituted ſtrait-lin'd angles of any ſort, and quantity whatſoever, which are all contained by the Raies of the Circle. As in the foregoing Diagram, Let now the infinite plain Space with all the

the Angles about the point A be determined by the Circumference of any Circle, whose center is the said point. And therefore forasmuch as the circumference of a Circle may from any point bound an infinite plain Space uniformly round about, and therewithal does encompass all strait-lin'd angles constituted at that point, which taken together compleat the Same Space: and as the angles are greater, or less, so the parts of the whole circumference, which are intercepted by the Strait lines containing those angles, become also greater, or less; It will appear in common Sense, That the Circumference of a Circle may justly be put as the Proper Measure of the quantity of these angles. Thus therefore the Measure of a strait-lin'd angle is defined.

Definition of the Measure of strait-lin'd Angles.

The Measure of a Strait-lin'd angle is an Arch of the circumference of a Circle, described from the angular point as a Center, and intercepted by the strait lines which contain the Angle.

As the Arch BC is the measure of the angle BAC, and the Arch CD of the angle CAD, so BD of BAD, BE of BAE: Every Arch measuring the quantity of its correspondent angle from the center A, at which point all strait-lin'd angles, Right, Obtuse, and Acute, of what quantity soever, may be placed. But this mensuration of an angle is not to be taken in the strictest sense. For whereas it must be acknowledg'd, that *Mensura, & Mensuratum sunt in eodem genere. The Measure, and the Measured magnitude are of the same kind*: yet we cannot say, that an Arch and an Angle are truly of the same kind.



But because 'tis evident, and demonstrated, that Angles and Arches do mutually increase, and decrease alike, so that if the Angle BAD be double, or triple of the Angle BAC, then the Arch BD is double, or triple of the Arch BC. And on the contrary, if BD be double, or triple of BC, then BAD is double, or triple of BAC.

Therefore an Arch of the Circumference of a Circle serves aptly as the Measure of an Angle in all common Uses, and Mathematical Speculations, or Geometrical Practices appertaining to Astronomy, The Mensuration of Distances, Surveying of Lands, Navigation, Fortification, Gunnery, &c.

Of the Division of the Circumference into certain Parts, and the Use Thereof.

Now that these Arches, and by consequence the angles at the center, which are measured by these Arches, might be brought under a certain value, and estimation, therefore the most ancient Geometricians at first divided the Circumference of a Circle into 60 parts. Afterwards It was found more convenient to divide the same into six times 60, that is into 360 parts, commonly called *Degrees*; by which an Arch of the circumference, with its correspondent Angle might be estimated.

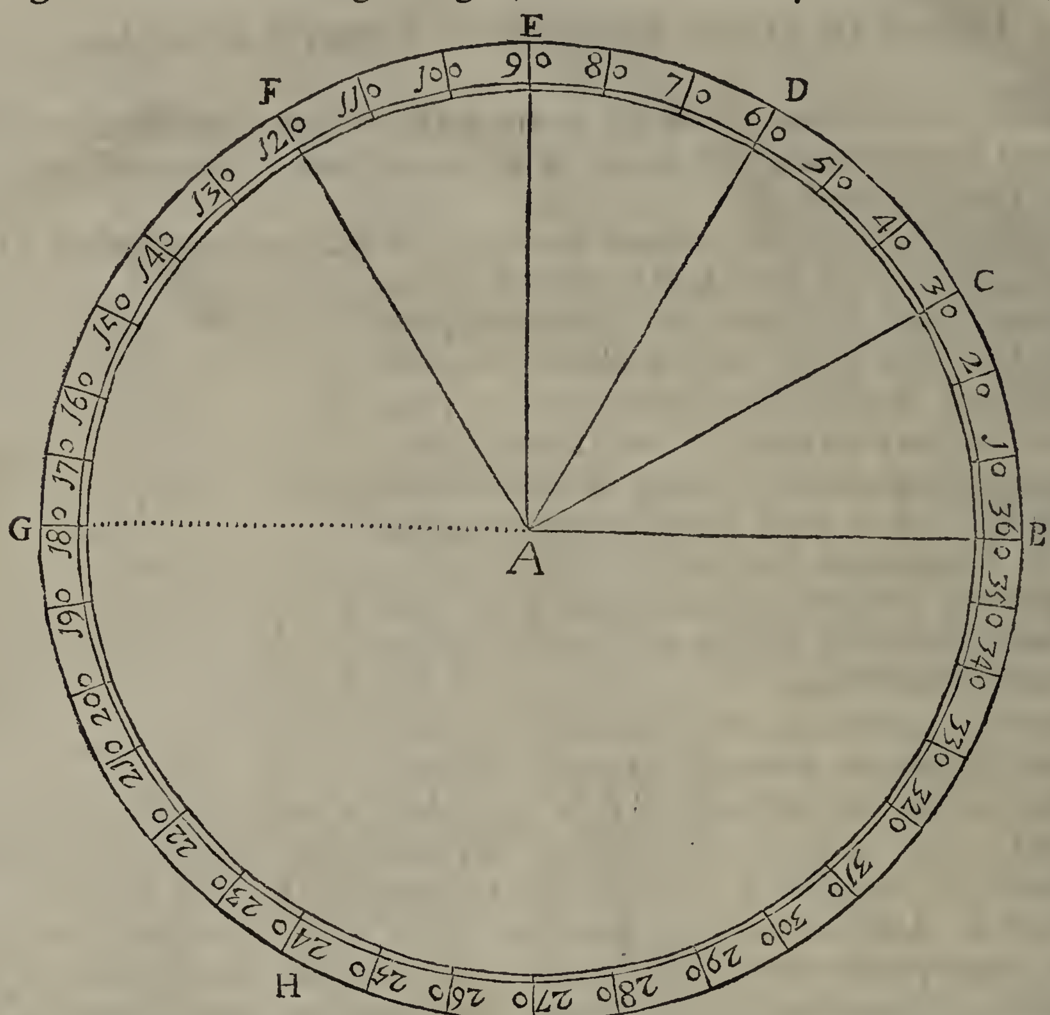
So that, for example, we say, That the Angle BAC is 30 Degrees, if the Arch BC be 30 such parts, as of which the whole circumference is divided into 360. Again, if the Arch BD be 60 Degrees, then the Angle BAD is accounted 60 Degrees. In like manner, the Angle BAE is 90 Degrees, if the Arch BE be a Quadrant, or the fourth part of the whole circumference; for 90 is the fourth part of 360. *And This particularly is the measure of the quantity of a Right Angle.*

Again, by the same reason, the Obtuse angle BAF is said to be 120 Degrees, if the Arch BF be 120 of the 360 parts of the Circumference, &c.

 Thus

Thus the Circumference of a Circle is uſed as a correſpondent meaſure to *Euclid's* three ſorts of ſtrait-lin'd angles, Right, Obtuſe, and Acute.

But to take this matter into further conſideration. Of the Obtuſe angle B A F the meaſure is the arch B E F. Whoſe Complement to make up the whole circumference is the arch F G H B. Which arch encompaſſes the remaining Space about the point A, and anſwers the contrary way to the adverſe, or external Face of the Obtuſe angle B A F; and is indeed the meaſure of the outward angle F A B. For the lines B A, F A, according to *Euclid's* definition, make not only the Obtuſe angle B A F, which is greater than one Right angle, and meaſured by the arch B E F; but alſo make the contrary way the adverſe, or external angle F A B, which is greater than two Right angles, and meaſured by the arch F G H B.



So the lines B A, E A make the Right angle B A E, meaſured by the Quadrantal arch B E, and alſo make the external angle E A B, which is equal to three Right angles, and meaſured by the arch E G H B. Again, the lines B A, C A make the Acute angle B A C, meaſured by the arch B C; and alſo make the external angle C A B, which is greater than three Right angles, and meaſured by the arch C E G H B. But of theſe kind of angles, being uſeleſs in Geometry, *Euclide* thought not fit to take any notice.

Paſſing therefore This over, and to return to the partition of the Circumference into 360 Degrees, we muſt further know, that for the more exact menſuration of an angle, every Degree is again divided into 60 *Prime Minutes*; every ſuch Minute into 60 *Seconds*; every Second into 60 *Thirds*, and ſo forth into ſmaller and ſmaller particles. For in meaſuring of Things, the more minute the *Rata Menſura, the ſtated Meaſure* is made, the more accurate will be the menſuration of the quantity of the thing meaſured. Therefore in Eſſaies of fine Gold, in the valuation of Pearl and precious Stones, we come to the hundreth part of a Grain, or ſometimes nearer. So in Aſtronomical calculations of the motions of the Planets, we often make a Sexagenary ſubdiviſion of the circumference of a Circle into Fourths and Fifths, &c, for the more exact ſtating the Computation of their Courſes, and Periods.

And here obſerve, That the firſt diviſion of 360 Degrees into Minutes, cuts the circumference of a Circle into ———————————————— 21600 parts.

The

The next of Seconds into ——————————————— 1296000 parts.
The following of Thirds into ——————————————— 77760000 parts.
And so forth continually we may proceed in subdivisions by 60, which will make an exactness more than sufficient for any common use. The Circumference of a Circle might have been divided into any other Number of parts: As our worthy Countryman M^r^ *Henry Briggs* has laid down the Form of a Decimal division, and subdivision. But 360 was thought by the ancient Mathematicians to be the most convenient number, because It admits of many several divisions precisely. As

360.	180.	120.	90.	72.	60.	45.	40.	36.	30.	24.	20.
1.	2.	3.	4.	5.	6.	8.	9.	10.	12.	15.	18.

In like manner the number 60 has these divisions.

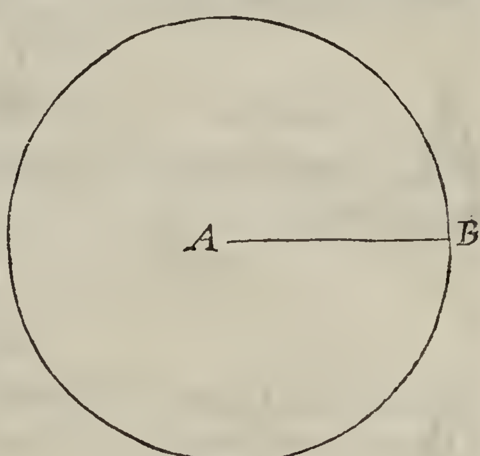
60.	30.	20.	15.	12.	10.
1.	2.	3.	4.	5.	6.

And the choice of This number 60 before all Others, was made upon this account, that no number can be divided into 3, 4, and 5, but 60, and the Multiples of Sixty. Which makes Computation more free from Fractions.

Altho' these animadversions may seem enough, or too much in this place; yet before we pass from the Subject now in hand, it will be requisite to add something more in the behalf of *Euclide*; upon this account, That some of the late Commentators, or rather Transformers of the Elements do cavil at this most accurate Definition of a Circle, for that it does not readily appear, say They, Whether there may possibly be a Figure of such conditions as *Euclide* here lays down. And therefore they would rather define a Circle, from its *Genesis*, or Structure, whereby both the existence, and nature of the Figure may be declared together, after this manner.

Another Definition of a Circle.

A Circle is a Figure described in a Plane by the revolution of a finite strait line (as AB) upon one of its extream points (A) being fixt, till the line (AB) return to the place from whence it began to move.



The fixt point (A) is called the Center of the Circle. And *The Curve Line* designed by the other extream point (B) of the *moving Line* (AB) is called the Circumference.

Now say They, This Definition makes it manifest, that there is *A Figure having a Point within, from whence all strait lines drawn to the extremity thereof are equal to one another*; because by the construction of a Circle laid down in this Definition, *The same strait line* circumvolved about the fixt point must necessarily be every where equal to it Self, and therefore all the Raies from that point shall be equal to one another.

This indeed is true, and most evident. But yet it will appear upon better consideration to be very inartificially put in this place instead of *Euclid's* Definition. For first, to except against *Euclid's* Definition, because it may be doubted, whether there may be a figure so qualified as he sets forth, shews, that They do not rightly apprehend the nature of Mathematical definitions. For to add something more to what has been already said concerning Mathematical Definitions, we are to understand that *In every one of these Definitions there is laid down the Notion of some Thing appertaining to Geometry, under a certain Name, and Term of Art.* But whether such a Thing may really be, is not of the present consideration. Only *Euclide* intends, that by such a Name, whensoever he uses it, we are to imagine such a Thing; Let It be, or not be. For indeed *Geometrical Definitions are only Suppositions of Things under a certain Name.* And therefore They are for the preſent to be allowed, unleſs That, which is ſuppoſed, and laid down, has in the words themſelves a manifeſt contradiction, or in the conception of the Thing an apparent impoſſibility. This is all that the Mathematicians mean, or require in their Definitions: And Theſe Innovators therefore ought in the firſt place to have obſerved, that before *Euclide* makes Uſe of any Thing by him defined, he firſt makes evident the exiſtence of the Same.

As before he makes any uſe of a Right angle, or of a Perpendicular Line (againſt which they might have had the like exception) he demonſtrates in Prop. 11, and 12. El. I. their Being, and how to effect Them. So likewiſe concerning a Circle, and all other Figures here defined, nay, even a Strait line It ſelf, *Euclide* hath ſo provided, that firſt an Aſſent ſhall be given to their Being, before any conſideration is had of their Uſes, and Properties.

Again, to examine farther this Definition. It ſhews indeed the natural Geneſis, but not the nature of a Circle, from which the Properties and Affections of Circles can only be demonſtrated. For this Geneſis is no ways applicable to all thoſe demonſtrations. As in the firſt place it will appear, if when a Circle is named we put this Definition inſtead of the *Definitum* a Circle (for the Definition and *Definitum* are always convertible) and then let be examined what Properties of a circle, or what Propoſition in the Third Element (where the Affections of circles are Specially handled) can be demonſtrated *from the Revolution of a ſtrait line on a fixed point*, Which is the Primary Notion that This Definition impoſes on Us. Whereas *Euclid's Equal lines from the Center* are in all thoſe Elementary Propoſitions ſerviceable toward their demonſtrations. I much wonder therefore at *Borellus*, an excellent Geometrician, to allow in his *Euclides Reſtitutus* of This Definition, and change *Euclid's* into an Axiom. Which he was forced to do for the conſtant uſe thereof in all demonſtrations, and Wholly to lay aſide his own new framed, and unapplicable Definition. This can be only ſaid for it, that from the Geneſis of a circle it doth manifeſtly follow that all ſtrait lines from the Center are equal to one another. But now to make This a Secondary notion derived from the Geneſis of a Circle, and to put the Geneſis for the Primary ſtanding Definition, which can never be uſed as a Definition, is very abſurd, and an intolerable Overſight in a Geometrician.

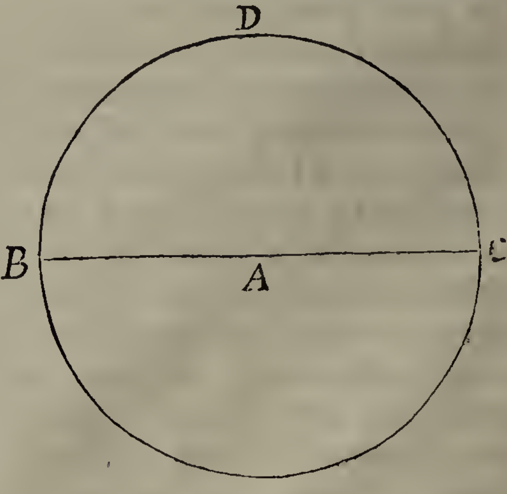
DEFINITION XVII.

A Diameter of a Circle is a ſtrait Line drawn through the Center, and terminated both ways by the Circumference of the Circle.

Which alſo cuts the Circle into halves.

ANNOTATIONS.

As the ſtrait line B C drawn through the Center A, and terminated both ways by the circumference at the points B, C, is called a Diameter of the Circle.



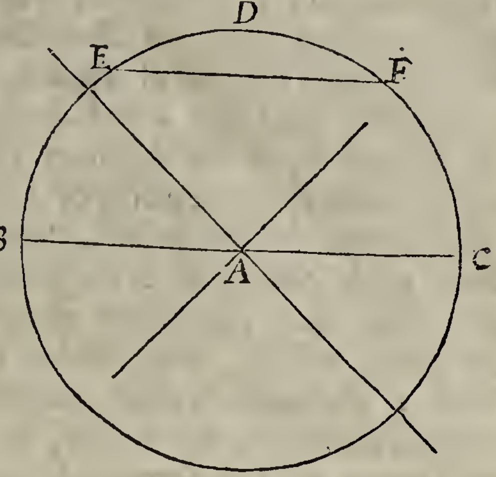
In this Definition three conditions are laid down to determine That Line, which *Euclide* calls a Diameter of a Circle.

1. That a ſtrait line be drawn through the Center.
2. That it be both ways terminated by the circumference.
3. That it cuts the Circle into halves.

The two former conditions, Each by Themſelves, are too general, and inſufficient to determine the Diameter.

For as in the Figure it appears there may be finite ſtrait lines paſſing through the Center, and yet not terminated by the circumference, but all variouſly ending within

within and without the Circle; So again, there may be infinite ſtrait lines terminated by the circumference which yet paſs not through the Center, and are called Chords or commonly SUBTENSES, in reſpect to the Arches, which they Subtend: as the ſtrait line EF is called the *Subtenſe* of the Arch EDF.



Not any one therefore of theſe two conditions can by it ſelf determine a Diameter; But both together, namely the paſſing through the Center, and the termination in the circumference do fully ſpecificate this Line, and make up a compleat Definition of a Diameter of a Circle.

Note that here is added particularly, *of a Circle*, ſaying the Diameter of a Circle. Becauſe there are many other figures hereafter to be conſidered, which have alſo their *peculiar Diameters*, different in ſeveral reſpects, from This of a Circle.

Now for the third condition, that the Diameter cuts the circle into halves. This truly is altogether needleſs, except there might be ſome other line having the two former conditions, which notwithſtanding did not cut the circle into halves; And therefore it was neceſſary to add this third condition, for the juſt determination of the Diameter of a Circle.

There might as well have been added, That the Diameter is the greateſt line in a Circle. For this notion does immediately concern the Diameter it ſelf, whereas the other only declares how it affects the Circle.

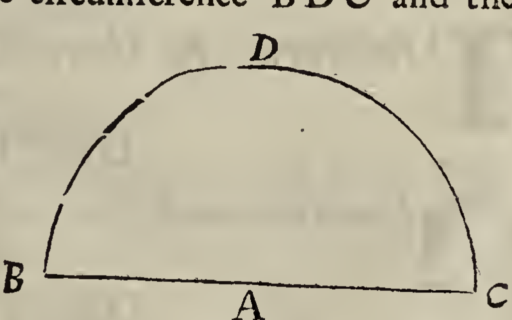
The truth is, They are both demonſtrable Propoſitions and in this place alike impertinent. But as This laſt is demonſtrated in the 15th Propoſition of the Third Element; ſo *Thales* has demonſtrated the Other here ſubjoyn'd, as ſhall be ſhewn in due place. Moreover, this addition of Biſection of the Circle anticipates the following definition of a Semicircle, contrary to the exact method of the ancient Mathematicians. It is therefore certainly none of *Euclids*, but ſome marginal note, how old ſoever it may be, that happen'd to be Tranſcribed into the Text.

DEFINITION XVIII.

A Semicircle is a Figure comprehended by a Diameter, and That part of the Circumference of the Circle, which is intercepted by the ſame Diameter.

ANNOTATIONS.

As in the figure BDCB, comprehended by the circumference BDC and the Diameter BC, is named a Semicircle. This is The ſecond of Figures, made indeed by diſſimilar and incompatible lines, yet ſuch as are the moſt ſimple in their kind, a ſtrait line, and the circumference of a Circle.



Euclide having in the definition of a Circle ſtated a Center, does next by the poſition of that point define a Diameter: and then from the Diameter a Semicircle.

The Diameter is the ſecond ſtrait line to be conſider'd in a Circle. For the Raies are the primary ſtrait lines, and eſſential in the notion of a Circle, wherein the Diameter is not at all concern'd. And tho' it happens, that a Radius be half of a Diameter; yet It ariſes not from the Diameter, as the Semicircle does from the Circle: but the Radius is put abſolute in it ſelf, without any reſpect to the Diameter, or dependence on It: And both Radius and Diameter do immediately relate to the Circle; each diſtinctly, and upon very different conceptions, without any relation to one another. So that tho' from *κύκλος a Circle*, *Euclide* uſes the word *ἡμικύκλιον a Semicircle*, yet from *διάμετρος a Diameter*, he never ſays *ἡμιδιαμέτριον*, or *ἡμιδιάμετρος, a Semidiameter.* Which ſome modern Writers not well conſidering have, inſtead of *Radius*, or in *Euclid's* phraſe, *The line from the Center*, prepoſterouſly uſed the word *Semidiameter*, as if It did ariſe from the biſection of the Diameter, as the primary Line; Whereas the Radius is before the Diameter in the natural conception of a circular figure.

After the definition of a Semicircle, *Proclus* inſerts for a definition, That *The Center of the Semicircle is the ſame with the Center of the Circle.* Or rather he ſhould have ſaid, that *The Center of the Semicircumference* is the ſame with the Center of the Circle it ſelf. For indeed the Center *of the Circle* may be ſaid to be alſo *the Center of the whole Circumference, or Semicircumference*, or any part thereof, when It is conſidered meerly by it ſelf, as a Line. But it cannot be ſo properly ſaid to be the Center of the Semicircle, or Semicircular Figure. However, This is not to be receiv'd as a definition of *Euclids*, but an Annotation, either of *Proclus* himſelf; Or elſe it might happen in his particular Copy of *Euclide* to be transferr'd from a marginal Note into the Text. For it is found only in his Commentaries, and not in any other Greek Manuſcript, nor extant in the *Baſil* Greek Edition of *Euclide*.

But moreover, that *Euclide* is ſometimes thus corrupted, by transferring Marginal Notes into the Text, will manifeſtly appear in this very place, where in the *Baſil* Greek Edition (Which as we have ſaid has not *Proclus* his definition of the Center of a Semicircle) is put the 6th. Definition of the Third Element, concerning the Segment of a Circle in general, notwithſtanding that the ſame definition is alſo found in its proper place among the definitions of the third Book. And we may conjecture, that *Euclide* having here defined a Semicircle, which is one kind of Segment of a Circle, ſome One noted what he had elſewhere ſaid of other Segments of a Circle, which by the inadvertency of a Tranſcriber was afterwards order'd among the definitions of this Firſt Book. For every Book of *Euclide* has its peculiar ſubject different from the reſt, and is therefore accounted a diſtinct Element, having Definitions proper to the matter It treats of, which are laid down at the entrance into every Book.

Of ſtrait-lin'd Figures.

DEFINITION XIX.

STrait-lin'd Figures *are Thoſe, which are comprehended by ſtrait lines.*

DEFINITION XX.

T*Rilateral, by three ſtrait lines.*

DEFINITION XXI.

Q*Uadrilateral, by four ſtrait lines.*

DEFINITION XXII.

M*Ultilateral are comprehended by more ſtrait lines than four.*

ANNOTATIONS.

A plain Superficies is made figurate by certain Bounds, or Terms, which incloſe the ſame. And therefore *Euclide* hath placed the definitions of plain Figures in an order anſwering to the ſimplicity of their Terms; firſt defining a Circle, being

ing a Figure under One Term; next a Semicircle, a Figure under Two Terms; then, in order, Figures of Three Terms, Four Terms, &c.

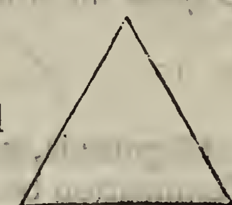
We are here alſo to obſerve the property of ſeveral words uſed in this place, which by degrees are properly changed from one to the other, altho' the ſame thing be ſignified. The ſimple words originally are γραμμαὶ, Lines, and πλευραὶ, Sides, between which EUCLIDE makes this diſtinction. For ſpeaking in general of ſtrait-lin'd Figures, he ſays, εὐθύγραμμα σχήματα. *Strait lin'd Figures.* But next when he does divide, and ſpecificate *The ſtrait-lin'd Figures*, he ſays not τρίγραμμα & τετράγραμμα, Trigramma & Tetragramma, that is, *Trilineal* and *Quadrilineal*, or *Three lin'd* and *Four lin'd Figures*; but τρίπλευρα & τετράπλευρα, Tripleura & Tetrapleura, that is, *Trilateral* and *Quadrilateral*, or *Three ſided*, and *Four ſided Figures*; changing the general word *Lines* into the particular name *Sides*. And again in the *Specification of Trilateral Figures*, he gives to Them anew the name TRIANGLE in the manner following.

Of Trilateral Figures.

DEFINITION XXIII.

AN Equilateral Triangle is a Figure which hath three equal Sides.

ANNOTATIONS.

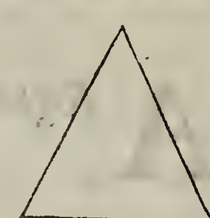


That is, A ſtrait-lin'd Trilateral Figure of *three equal ſides*, I call an *Equilateral Triangle.*

DEFINITION XXIV.

AN Equicrural Triangle is That which hath only two Sides equal.

ANNOTATIONS.

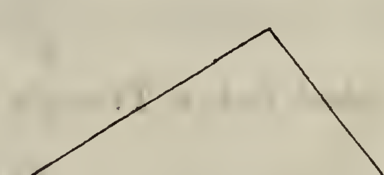


That is, A ſtrait-lin'd Trilateral Figure of only *two equal ſides*, I call an *Equicrural Triangle.*

DEFINITION XXV.

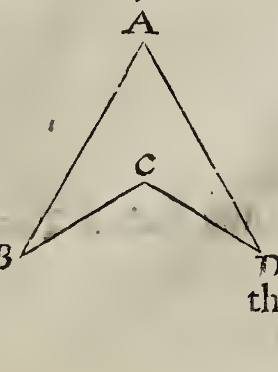
A Scalene Triangle is That which hath the three ſides unequal.

ANNOTATIONS.



That is, A ſtrait-lin'd Trilateral Figure of *three unequal ſides*, I call a *Scalene Triangle.*

After the diviſion of ſtrait-lin'd Figures according to the number of their ſides into Trilateral, Quadrilateral, and Multilateral, *Euclide* begins with the Trilateral, being the firſt of all ſtrait-lin'd Figures. And the Trilateral Figures he divides into ſeveral Species from all the poſſible changes that can be made, of their three ſides; Which is into three kind of Triangles. For now theſe *particular Trilateral Figures* he calls *Triangles.* And by a Triangle is to be conceived a Figure of three ſides, tho' the word implyes three Angles; And only that Figure is named a Triangle, which is Trilateral. For there may be a Figure, which has only three angles yet is not Trilateral, but comprehended by four lines, or more. As the Figure A B C D comprehended by the four lines A B, B C, C D, D A, has notwithſtanding only three angles at A, B, and D. For the angle B C D is not commonly taken to belong to the Figure; but to have its reſpect



the

the contrary way wholly without the figure: And this is call'd ἀκιδοειδὲς, The Arrow-headed Figure. There may be after this manner Triangles of five ſides, and ſix ſides. In general ſuch kind of figures are called κοιλογώνια, Hollow-angled Figures. But concerning this double face of an angle we have already taken notice, that it is uſeleſs in Geometry.

Now in every Triangle we are to obſerve, and diſtinguiſh the names of *Sides*, *Leggs*, and *Baſe*, or Fulciment upon which the Other two ſides are ſuppoſed to ſtand. In a Triangle of *three equal ſides*, if *any one* of the ſides be put for a *Baſe*, the *Other two* ſhall make *equal Leggs*. This therefore has three ſeveral changes of two equal Leggs: And ſo takes not a Name from *the equality of two Leggs*, which may be three ways variable, but from *the equality of all the three ſides*, and is called an Equilateral Triangle.

In a Triangle of *two only Equal Sides*, the Third Side is called the Baſe, and the *Other Two* the *Equal Leggs*. And in this caſe the Baſe, and Equal Leggs are determined, being only to be made one ſingle way: Therefore a Triangle only of two equal Sides is ſpecially call'd ἰσοσκελὲς, *Iſoſceles*, or *Equicrural*.

But in a Triangle of *three unequal Sides*, let Any ſide be put for a Baſe, the Other two ſhall make unequal Leggs every way: And therefore It is called Σκαληνὸν Τρίγωνον, *Scalenum Triangulum*, A *Lame*, or *Haulting* triangle, πανταχόθεν χωλεύει, *Undequaque claudicat*, ſays *Proclus*. So a rugged, and uneven way (as it is cited in *Eraſmus* his Adagies out of *Plutarch*) is called Σκαληνὴ ὁδὸς, *A Scambling way*.

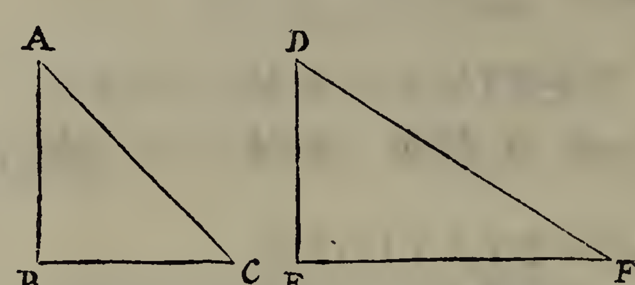
Moreover of Trilateral Figures.

Magnitude, as before hath been noted, is in It ſelf indeterminate, and only by Termination becomes Figurate, therefore *Euclide* has firſt diſtinguiſhed Trilateral figures from the condition of their Terms in reſpect of their Equality, or Inequality to one another.

And next, for the Quality of their Angles he lays down here another diſtinction, and denomination. As follows.

DEFINITION XXVI.

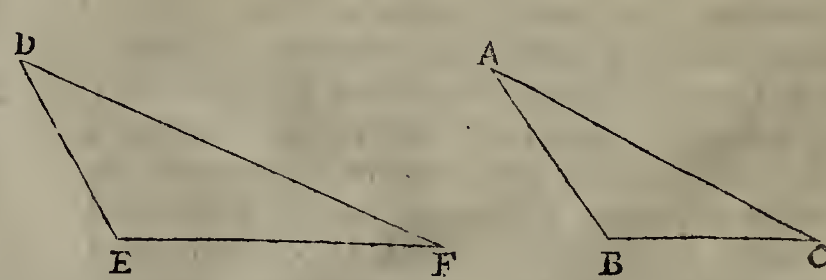
A *Right angled Triangle hath a Right angle.*



And ſuch a Triangle may be Equicrural, as ABC, or Scalene, as DEF.

DEFINITION XXVII.

A*N Obtuſe angled Triangle hath an Obtuſe angle.*



And ſuch a Triangle may be Equicrural as ABC, or Scalene as DEF.

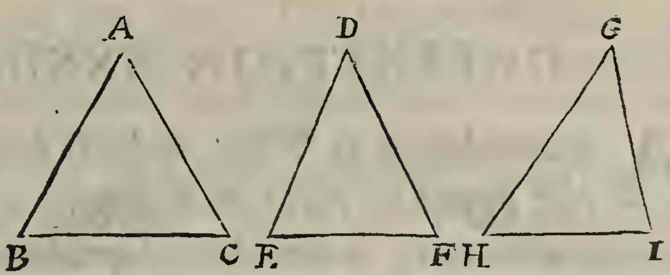
DEFI-

DEFINITION XXVIII.

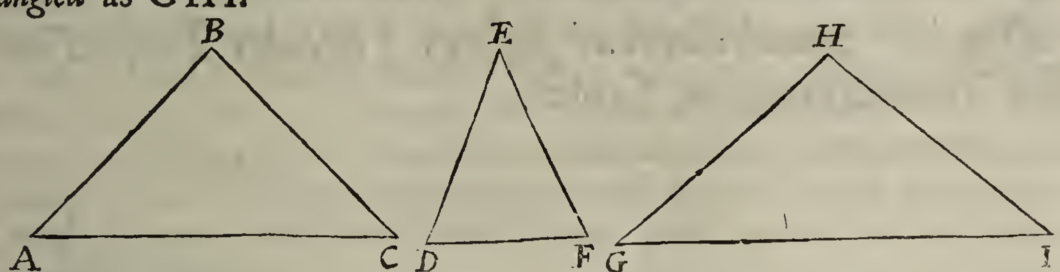
AN *Acute angled Triangle hath three Acute angles.*

ANNOTATIONS.

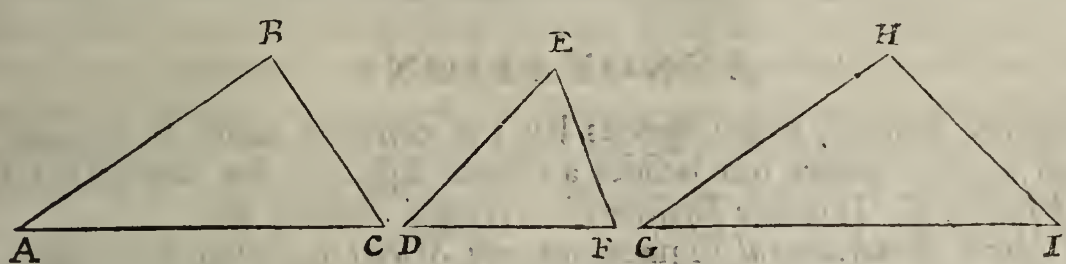
And ſuch a Triangle may be Equilateral as A B C, or Equicrural as DEF, or Scalene as G H I.



From theſe diviſions It appears, that there are ſeven ſorts of Strait-lin'd Triangles. For the EQUILATERAL Triangle is *Singular*, and only *Acute angled.* The EQUICRURAL may be *Right angled* as A B C, or *Acute angled* as D E F, or *Obtuſe angled* as G H I.



And ſo alſo may the SCALENE Triangle be *Right angled* as A B C, or *Acute angled* as D E F, or *Obtuſe angled* as G H I.

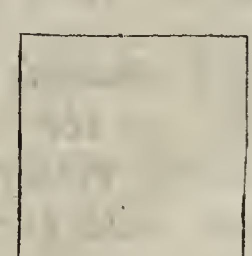


To a Triangle which has *Three equal angles*, or *Two equal angles*, or *All Three unequal angles*, *Euclide* has given no name, as he hath to a Triangle, which has *Three equal ſides*, or *Two equal ſides*, or *All Three ſides unequal*, becauſe a Triangle of three equal angles is ever Equilateral, and a Triangle of two equal angles is ever Equicrual, and a Triangle of three unequal angles is ever Scalene, as will be ſhewn hereafter. And therefore if Triangles had received a diſtinction and name from the number of their Equal, or Unequal angles, as they have from the number of their Equal and Unequal ſides, there had been given two Names to the ſame thing.

Of Quadrilateral Figures.

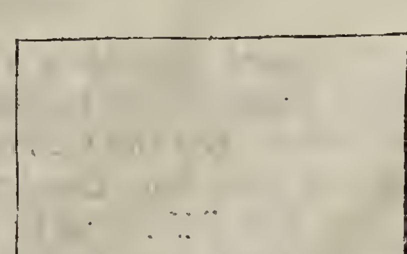
DEFINITION XXIX.

A *Square is That, which is both Equilateral, and Rectangular.*



DEFINITION XXX.

AN *Oblong is That, which is Rectangular, but not Equilateral.*

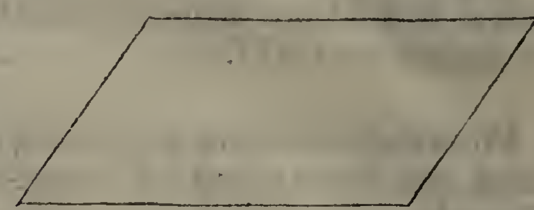


DEFINITION XXXI.

A Rhombus is That, which is Equilateral, but not Rectangular.

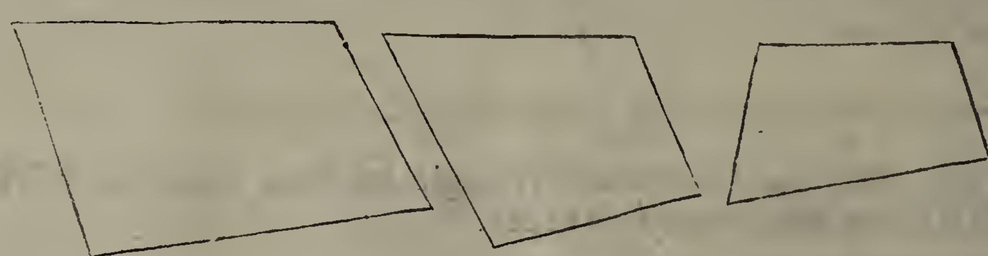
DEFINITION XXXII.

A Rhomboeid is That, which having the oppoſite ſides and angles equal to one another, is neither Equilateral, nor Rectangular.



DEFINITION XXXIII.

Let all other Quadrilateral figures [beſides Theſe Four] be called Trapeziums, *or* Tablets.



ANNOTATIONS.

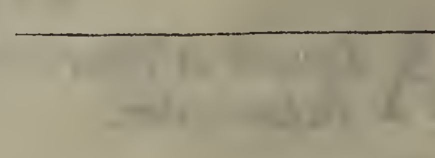
Of *Trilateral* Figures, every Species has the common name of Τρίγωνον, a *Triangle.* And as the Species has beſides a proper Epithete for diſtinction ſake, as Triangle *Equilateral*, Triangle *Equicrural*, Triangle *Scalene*, &c.

But of theſe *Quadrilateral* Figures *One only Species* is called Τετράγωνον, a *Quadrangle*, and That, from its ſingular equality of all its Sides; and the Rectitude of all its Angles. The other Quadrilateral Figures, tho' they be all Quadrangular; yet have they not the general name of Quadrangle with an Epithete annexed as the Triangles have. But each Species has an other diſtinct Appellation, as a Quadrilateral *Oblong*, *Rhombus* and *Rhomboeid.* Which three with that Figure called κατ' ἐξοχὴν, Τετράγωνον, a *Quadrangle*, and commonly tranſlated a *Quadrate*, or *Square*, make the four Regular Quadrilateral Figures. All the other Quadrilateral Figures have one Name in general *Trapezium*, *A Tablet.*

Parallels.

DEFINITION XXXIV.

Parallels are ſtrait lines, which being in the ſame Plane, and produced infinitely either way, do neither way meet One with the Other.



ANNOTATIONS.

The word *Parallels* is a meer Geometrical term of Art, which according to a literal expoſition of Παράλληλοι, is, *Lineæ ad ſe invicem poſitæ. Lines placed againſt one another.* But the Geometrical Notion and Thing to be conceived by this word requires theſe four Conditions.

1. Parallels are to be ſtrait lines.
2. They muſt lye in the ſame Plane.

For if they be in different Planes, as One in a Plane above, The Other in a Plane

Plane beneath, then the ſtrait lines may be infinitely drawn forth both ways, and never meet, yet are They not ſuch as *Euclide* calls Parallels.

3. They are producible infinitely both ways.
4. After this imaginary production both ways infinitely, (that is, indefinitely, or indeterminately further and further at pleaſure) They are never to meet together.

Let therefore the Plane be one and the ſame, The production free, and both ways infinitely, εἰς ἄπειρον, *In infinitum*, without any reſtriction or qualification in the manner of the Production, and neither way let there be any Concurrence; Then the Strait Lines having theſe Conditions are called PARALLELS.

In Paralleliſm therefore the Subject is Strait Lines in the ſame Plane, and the Attribute of theſe Strait Lines is Nonconcurrency. So that upon any mention of Parallels the Geometrical Notion to be conceived under that name of Parellels is Nonconcurring Strait Lines in one and the ſame Plane.

This Definition, at preſent, only ſuppoſes ſuch Lines to be in nature, but before theſe kind of Nonconcurring ſtrait lines are brought into any uſe, *Euclide* demonſtrates in the 27th. Propoſition of this Firſt Element, that there are ſuch kinds of ſtrait lines (called by him Parallels) which both ways infinitely produced ſhall never meet, and in the 31ſt. Prop. he ſhews how to draw them.

But now in this matter of Paralleliſm there are two miſtakes made by moſt Interpreters. One in reſpect of the name and tranſlation of the word παράλληλοι. The Other is in the Notion of the Thing, which is to be conceived under that Name and Term of Art.

For firſt concerning the Name, many Tranſlators, *Latines*, and Others, take *Parallels* to be of the ſame ſignification with *Equidiſtant lines*, ſaying, *Parallels*, or *Equidiſtant lines*, are &c. going then forward with *Euclid's* Definition; As if the Subject of the Definition, or the Name Parallels ſignified Equidiſtant lines, and that γραμμαὶ παράλληλοι, and γραμμαὶ ἴσον ἀφιστάμεναι, or ἰσοδιαστηθεῖσαι were indifferently to be taken for the ſame; and that here *Euclide* had defined Equidiſtant lines by Nonconcurrency. Whereas *Euclide* neither uſes the word Equidiſtant, nor by the word Parallels underſtands Equidiſtant lines: But only lays down the Conditions of ſome certain ſtrait lines: Which Lines in a *Signal Term of Art* he calls *Parallels.*

And becauſe the Latines have not any word, that anſwers to It, therefore the Greek name ought to be retained, ſaying, *Lineæ Parallelæ*, for γραμμαὶ παράλληλοι; and not *Lineæ Æquidiſtantes*. For, as we have ſaid, by the words γραμμαὶ παράλληλοι, is only ſignified γραμμὴ παρὰ γραμμήν, *linea adverſus lineam*, *Line againſt Line*, or Line to Line. Therefore commonly *Archimedes*, *Apollonius*, *Pappus*, ſometimes *Euclide* alſo, ſay, ἐστὶν ἡ ΑΒ παρὰ τὴν ΓΔ, when they mean, That AB is parallel to CD; which phraſe of ἡ ΑΒ παρὰ τὴν ΓΔ, cannot in any propriety of Speech among the Greeks, bear the interpretation of Equidiſtant lines: Neither was it ſo underſtood by Thoſe ancient Geometricians.

Again, the ſecond miſtake is of Thoſe, who rightly take the word Parallels for γραμμὴ παρὰ γραμμήν, LINE TO LINE. and meerly for a Term of Art, not as a common word ſignifying Equidiſtant lines, but, changing *Euclid's* Notion, do define *the Term Parallels* by the conception of *Equidiſtancy*, in the place of *Euclid's Nonconcurrency.*

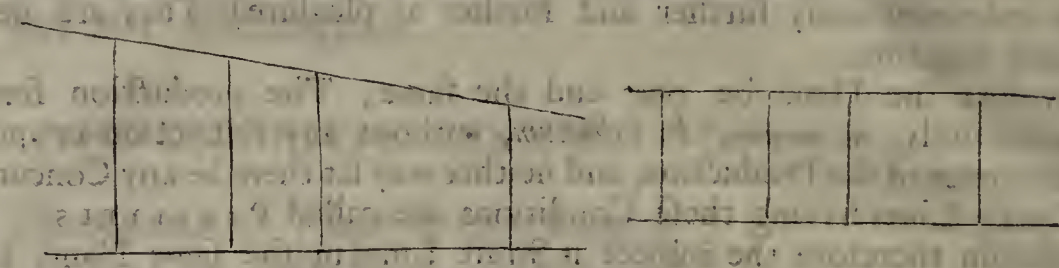
So *Poſidonius*, as we find in the Commentaries of *Proclus* on this matter, thus defines Parallels.

Another Definition of Parallels.

Parallels are ſtrait lines in one Plane, neither Inclining nor Reclining, But having all the Perpendiculars equal, which are drawn from the points of either of the lines unto the other line.

But all ſtrait lines, which make the Perpendiculars leſs, ſhall at length meet together. For the Perpendiculars only can determine the Altitude of ſpaces, and the diſtances of Lines; wherefore the Perpendiculars being equal, the Diſtances of the

the Lines are equal. But the Perpendiculars being made greater or leſs, the Diſtance is made greater or leſs, and they ſhall meet together that way, where the Perpendiculars are leſs.



Thus much *Poſidonius*, who defines Parallels to be Equidiſtant ſtrait lines from the equality of all their Perpendiculars.

But now we are next to conſider, and demand of *Poſidonius* and his Followers, to what uſe the Notion of Equidiſtancy ſerves in theſe Elements. We find nothing advanced by *Poſidonius* or Others in the Doctrine of Paralleliſm upon this new Definition: And certainly if from Equidiſtancy any Thing had been made better, and more firm than from Nonconcurrency, ſuch a kind of improvement could not have altogether periſhed.

But yet thus much we do acknowledge, that *Euclid's Nonconcurring ſtrait lines are Equidiſtant*; and moreover, that Equidiſtancy is the Phyſical Cauſe of their Nonconcurrence. But yet the Equidiſtance of Parallels is no where apply'd to any Propoſition throughout all theſe Elements. The only Notion of Parallels uſed or uſeful in Geometry, is *An unlimited production of ſtrait lines both ways without concurrence.* And therefore this affection of Nonconcurrency, *Euclide* the great Maſter of his Art, lays down to be only repreſented to our Imagination upon the naming of Parallels. But whether theſe ſtrait lines called Parallels be equidiſtant, or not, or what is the diſtance of Parallels, he thought not fit to conſider, becauſe thoſe Conſiderations ſerved to no further uſe in any of his Geometrical demonſtrations. If it be ſaid that ſtrait lines every where equidiſtant ſhall never meet, 'tis true and obvious. But this is to define a Geometrical Term in one ſenſe, and to uſe it in another: To define Parallels by Their *Equidiſtance*, but ever after to apply them in their Noncurrence. A groſs and intolerable abſurdity in the Mathematics, or in the Definitions of any Science.

We are moreover to obſerve, that altho' *Equidiſtancing ſtrait lines* be the proper cauſe of their Nonconcurrence, yet *Equidiſtancy* is not in general the adequate, or only cauſe of Nonconcurrence; ſo that whatſoever lines infinitely produced either way ſhall never meet, the ſame are to be always equidiſtant. For on the contrary, it is certain, that in one and the ſame Plane there may be two lines produced infinitely, which Lines ſhall never meet together, tho' they be not Equidiſtant, but do continually approach nearer and nearer to one another. As the *Conchoïdal* line of *Nicomedes* deſcrib'd by *Pappus* in *Prop.* 22. *lib.* IV. *Math. Collect.* And alſo by *Eutocius* in his Commentary upon *Prop.* 1. *lib.* II. *Archimed. de Sphæra & Cylindro*, which curved Line draws nearer continually *to a certain ſtrait line*, with which notwithſtanding it ſhall never meet.

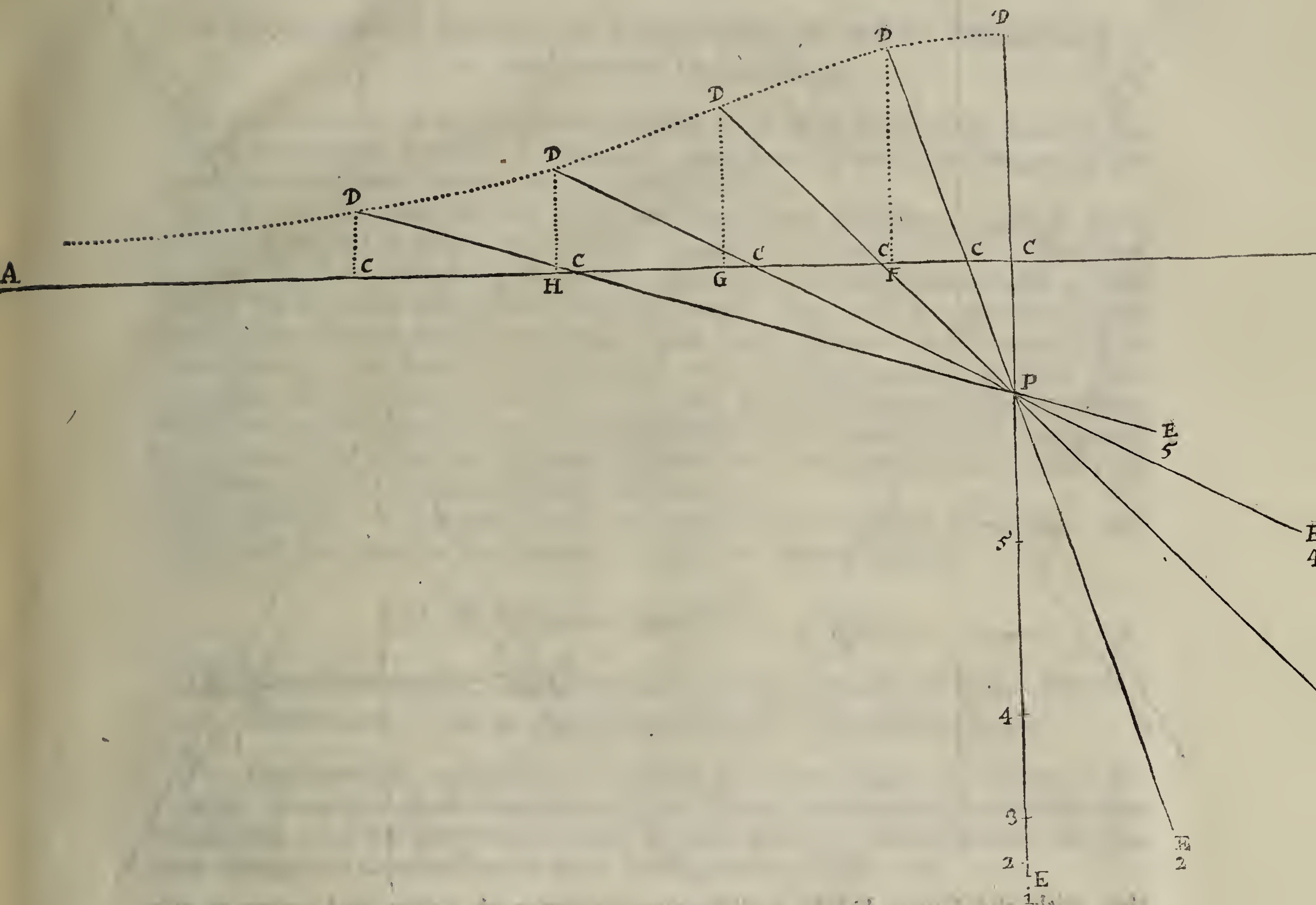
And becauſe this Propoſition ſeems very ſtrange, and yet may be eaſily made evident to any common underſtanding, without a ſtrict Mathematical demonſtration; therefore to ſatisfy the Curious (Others may paſs it over) we ſhall here only explain it, referring the legitimate proof therefore to *Pappus* and Others.

In a plain Superficies let a fixt point be put as P: And let there be a ſtrait line without the point P, both ways infinite, as AB. Then from the point P ſuppoſe a ſtrait line PCD to make right angles with the line AB at C; and that the ſame line DCP be directly continued infinitely from P towards E.

Now imagine the line DCPE to move along upon the line AB either way, towards A, or towards B, in ſuch manner that the point C may lye always in the line AB, and thereby keep the line DC, in every place, of the ſame length.

And

And moreover let the ſame line DCPE be conceived to paſs along through the point P. Now upon this ſuppoſition we are to note, firſt, That in the line DCPE *The point* C is determined to the line AB: and ſecondly, that *The Whole line* DCPE is determined to the point P.



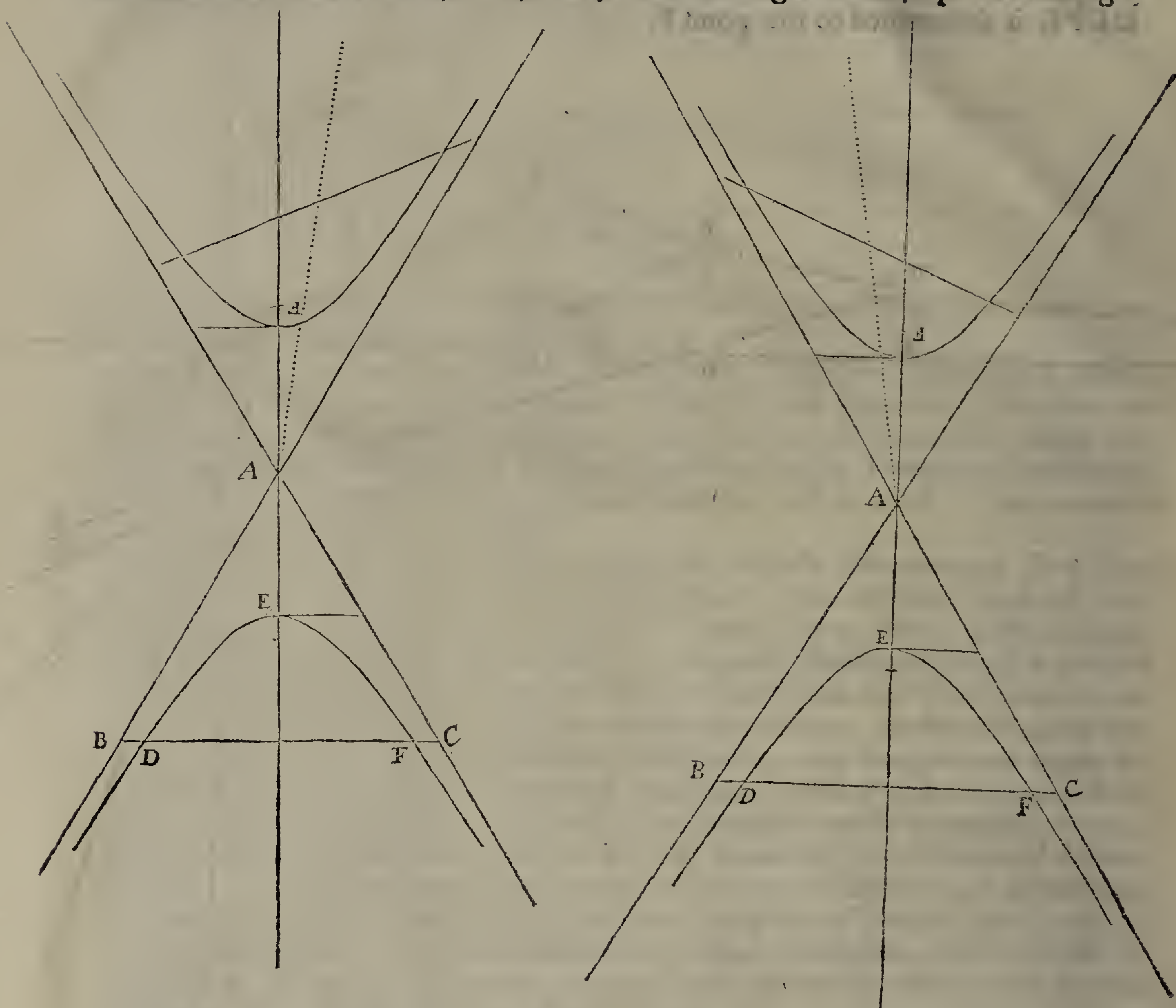
For in the motion of the line DCPE, the part DC ſhall always lye beyond AB. Firſt, for that the point C is moved ſtill forward in the line AB, ſo as to keep CD every where at the ſame length: And ſecondly, becauſe the ſtrait line DCPE can never come to be coincident with the ſtrait line AB; for that the point P, through which the ſtrait line DCPE always paſſes, is fixed at a certain diſtance from AB.

But now It is manifeſt, that in this motion of DCPE, the point D inclines continually nearer and nearer to AB, making the Perpendiculars DF, DG, DH, &c. ſhorter and ſhorter.

Wherefore the point D deſcribing the *Conchoïdal* curve line DDD &c. ſhall never bring that line to meet with AB, tho' it draws continually nearer and nearer to AB.

The Point P may be called the Pole. AB the Normal Line. DCPE the Arrow. The Point D the Arrow-head, which deſcribes the Conchoïdal line. The Point C the Button which holds the line CD at the ſame length. And under theſe names this matter may be fitly explained, and diſcourſed of: ſo as to be eaſily underſtood and acknowledged for a certain truth: Which indeed is founded upon the ſubtilty of Magnitude, being in its nature infinitely diviſible. Whereof among many Others this Conchoïdal line is a demonſtrative Argument.

Such alſo are the *Aſymptotes* of an Hyperbola; As for example, let the curve line DEF be an Hyperbola. There is a certain point, as A, from which may be drawn certain ſtrait lines, AB, AC, which being infinitely produced toge-



ther with the Curve DEF, ſhall continually approach nearer and nearer to the ſame; yet ſhall They never meet one with the other: As *Apollonius* demonſtrates in *Prop.* 1. *Lib.* II. *Of the Conic Elements.* And then in *Prop.* 4. he ſhows moreover how *within any two ſtrait lines making an angle an Hyperbola may be ſtated, to which the ſame lines ſhall be Aſymptotes.* Theſe ſtrait lines are called *Aſymptotes*, that is, *lines Noncoincident* from this property, that altho' they come nearer and nearer infinitely to the Hyperbolical curve line, yet ſhall they never meet with it. A moſt true and wonderful Myſtery in Geometry.

And thus much for the Explication of the Definition of Parallels, and of all the Other Definitions of the Firſt Element of Geometry.

Of Mathematical Propoſitions

Demonſtrable and Indemonſtrable.

In the Mathematical Sciences are uſed two kind of Propoſitions called *Problems*, and *Theorems*.

A Problem is a Practical Propoſition, in which Something is propoſed to be done.

As, To find the Center of a Given Circle.

This Propoſition is called a Problem: And it is the firſt Propoſition of the Third Element.

Element. Where, according to the Queſtion propoſed, Firſt the Center of any circle is by a Geometrical Practice certainly found, and after that, the Problem is demonſtratively proved to be Done, or Effected, namely, that the Center of the Given Circle is found.

A Theorem is a ſpeculative Propoſition, in which Something is pronounced to be True.

As, the Diameter of a Circle is the greateſt of all ſtrait lines in the ſame Circle.

This Propoſition is called a Theorem: And It is the 15th. Propoſition of the Third Element; and there demonſtrated to be True.

For in the firſt place we are to be informed, that whatſoever in theſe Geometrical Elements is propoſed, whether in form of a Problem, or a Theorem, the ſame is either undeniably demonſtrated; Or it is at the Firſt aſſented to from its own ſelf-evidence without any further proof. For in all humane Reaſonings every argumentation muſt be grounded upon ſome Thing, which is in it ſelf Indemonſtrable, that is to ſay, is incapable to be made more manifeſt to us from any other thing, than it is evident of it ſelf to every common underſtanding. For if there were not a power of ſelf-evidence in ſome things, which force upon Us an immediate Aſſent, no rational diſcourſe, nor any demonſtration could ever be framed, or have an unqueſtionable Beginning.

And therefore in this place there are premiſed ſuch general Principles, upon which, and the like, the Mathematician builds his Demonſtrations.

Of Mathematical Principles.

Of Principles in the Mathematics, ſome are Problems, ſome Theorems: like as the Demonſtrable Propoſitions are.

The Problematical, or Practical Principles are called Αἰτήματα, Petitions, or Poſtulates. Becauſe in theſe Propoſitions ſome Things are required, or poſtulated to be Effected, or done, without any proof of their Being, or Conſtruction; for that their Being, and Conſtruction is moſt ſimple, and manifeſt. *As*

To draw a ſtrait Line from point to point.

The Theoretical, or Speculative Principles are called Κοιναὶ ἔννοιαι, Common Notions as being obvious Conceptions, and generally received. For when Men from the particular Experience of their Senſes, have naturally an agreement in their perceptions, and uſe of Things, they are then by one, and the ſame common, and innate Reaſon, alike enabled to deduce from thoſe ſenſations the ſame Univerſal Propoſitions; which therefore, whenſoever propoſed to Others, and the words underſtood, are preſently without heſitation Aſſented to. As that

The Whole is greater than its Part.

Such kind of Propoſitions the Latine Philoſophers call Maxims, the Greek Ἀξιώματα, *Axioms*, or *Dignities*, for that They are ſentences of Worth, ſo Signal, and ſo Dignified, as to carry their own Authority, and Credit along with themſelves, whereby to force an Univerſal Aſſent.

Now *Euclide* thus begins the Principles of Geometry.

The

The Practical Principles Poftulates, or Petitions.

POSTULATE I.

LEt it be granted, *From any point unto any point to draw a Strait Line.*

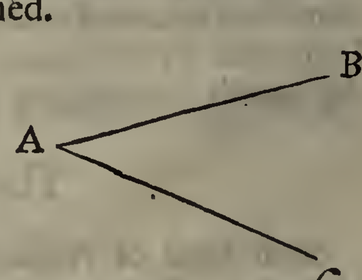
ANNOTATIONS.

Euclide here firft tacitely prefumes the putting of a point any where: or jointly with the drawing of a ftrait line he does alfo poftulate the putting of a point, or points at pleafure. And moreover becaufe a point put to a point is ftill but a point; therefore if two points be diftinctly put, as in this Poftulate, then it is a point Here, and a point There; fo that fome kind of Length, or Space muft be conceived to lye between Here and There, when *Euclide* fays, From a point to a point.

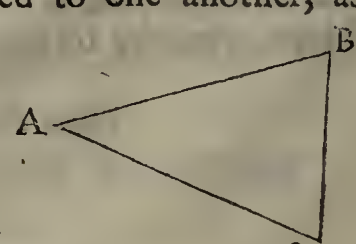
Now in the 4th Definition *Euclide* tells us, what kind of Length he means by the name of a ftrait line: And in this Poftulate he requires that fuch a Length, as he calls a ftrait line, may be put, and join any two points together. This indeed is *Æquum Poftulatum*, a very reafonable Requeft, and as juftly to be granted, as a point Here, and a point There. For between Here, and There, tho' we may make infinite deviations, and By paths; yet we naturally conceive but One only Singular, and direct Way, and that can be nothing elfe, but what *Euclide* calls a ftrait line.

Moreover we are to know, that *Euclide* only means a Mental Duction, or Pofition of that ftrait line between any two points; not a draught of the hand from point to point by the help of a Ruler: Which does but imperfectly imitate that Geometrical exactnefs, which we conceive in a Mathematical ftrait line; For a ftrait line actually never was, nor ever can be drawn. But in Practical Geometry fufficient it is *Pro Accurato ponere quam proxime Accuratum.* And not only this poftulated Problem; but all Geometrical Problems, whether poftulated as Principles, or are from Principles to be demonftrated, are likewife all fuppofed to be effected in our Imagination only, without the help of our outward Senfes, or of a Manual operation, or any material Inftrument. Yet the Ruler, and Compaffes have always been allowed to a Geometrician; not becaufe Geometry needs them, but only to affift our Underftanding by the Mechanical conftruction of a fenfible Figure, whereby we may go the eafier through an Intellectual demonftration.

And further from this Poftulate, it is efpecially to be obferved, that the Exiftence of ftrait-lin'd Angles, and alfo of ftrait-lin'd Figures, as well as of ftrait lines themfelves, do naturally follow, and is here tacitely prefumed.



For if from any point a ftrait line may be drawn to any point, as from the point A to the point B: It is likewife as evident, that again from the fame point A an Other ftrait line AC may be drawn to an Other point, as C: and fo make an Angle, as BAC. And therefore it had been frivolous to have poftulated the making of an Angle in general.



Again, if there be put three points not directly fituated to one another, as A, B, C, then by this Poftulate, They may be joined by three ftrait lines, and fo there is conftituted a Figure of three fides, called a Triangle. Wherefore *Euclide* neither Poftulates, nor Demonftrates the conftruction of a Triangle in general. In like manner if there be put four points, or more at pleafure; and they be all joined by ftrait lines drawn from point to point, there will in common fenfe arife Quadrilateral, and Multilateral ftrait-lin'd Figures. So that their exiftence is evident

dent from this Poſtulate without any further demonſtration of their ſtructure.

But for the ſeveral ſpecies of ſtrait-lin'd Figures defined by *Euclide*, as an Equilateral Triangle, or Square, &c. Alſo a Right Angle, Parallel line, &c. *Euclide* never makes uſe of any of them, till he hath firſt manifeſted their Being, and in particular their Conſtruction. So accurate is the *Elementator* in his Method, as neither to be ſuperfluous nor deficient in any matter.

POSTULATE II.

T*O Continue a finite Strait Line directly onward.*

ANNOTATIONS.

After that a ſtrait line is allowed to be drawn from any point to any point; It is as much, or more evident, that a Finite ſtrait line, that is, a ſtrait line determined by two extream points, may be from thoſe points conceived to be either way farther produced and continued at pleaſure in the ſame direct courſe: And therefore the continuation of a ſtrait line ought in common reaſon to be granted as well as a ſtrait line, which is the only Thing here required.

Conſentaneous to the Second Poſtulate, This likewiſe might be added.

To put two ſtrait Lines directly One to the Other.

That is, to conceive two ſtrait Lines ſo ſituated to each other, that they may both together make one ſtrait Line. This is frequently made uſe of in the Elements. But becauſe in this caſe there are only two Given ſtrait Lines imagined to be placed in a certain poſition towards one another; and not any other Magnitude *de novo* created, as in the foregoing Poſtulate there is to a Given ſtrait Line a New One in a direct continuation to be joined; Therefore *Euclide* does not poſtulate This as an Other, and diſtinct practical Principle: but upon occaſion aſſumes the liberty of Poſition in Lines Given; Sometimes of one ſtrait Line to an other directly; Sometimes of One ſtrait Line upon an other, as it may beſt ſuit to the demonſtration of thoſe Propoſitions, which require ſuch an *Apparatus* of Situation toward their demonſtrations.

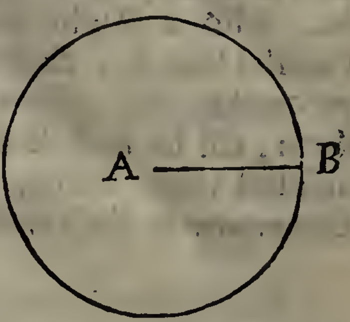
POSTULATE III.

F*Rom any Center, and to any Diſtance to deſcribe a Circle.*

ANNOTATIONS.

In the 5th. Definition *Euclide* means, that by a Circle ſhould be conceived a Figure bounded on one ſingle Term, having within it a Point, from whence all ſtrait Lines drawn unto that Term are equal to one another, that is, having within a middle Point, which he names a Center, every way equally diſtant from that Term, which he names a Circumference. Now this Definition imports no more than that whenſoever he mentions a Circle, we are to conceive ſuch a kind of Figure. But here now *Euclide* farther poſtulates the conſtruction of a Circle (according to this Definition) to be granted him, as a Figure of an eaſie conſtruction, and very manifeſt of it ſelf to be in nature. For in order to make plain the formation of a Circle, he firſt puts a point, which in relation to a Circle he had before, and ſo does here call a Center. Secondly from that point he ſuppoſes any Diſtance to be taken (Diſtance in this place is underſtood in the common acceptation of the word: and therefore it is not by *Euclide* defined among the

Geometrical Terms of Art) Now the natural conception of Diſtance is the ſhorteſt Tract between things diſtant from one another: So that the diſtance from any point to any point, is in common ſenſe taken by a ſtrait, and not by any crooked Line. And it is already granted in the firſt Poſtulate, that from any point to any (how near, or how far ſoever off) a ſtrait line may be drawn. Put then the Center A, and from the Point A, let be drawn the ſtrait line A B to ſet forth any diſtance. Now the Center and diſtance being thus laid down, the deſcription of a Circle which is the thing required to be granted, does apparently ariſe. For if we conceive the line AB to move on the point A, as fixed and immoveable, till the ſame line return to the place, from whence it began to move; then 'tis evident, that this imaginary motion of the line A B hath deſcribed a Circle, whoſe Center is A: and that the point B hath delineated the bound, or circumference of the ſame Circle. For becauſe the line AB in its revolution muſt be coincident, and the very ſame with all the ſtrait lines that can be imagined to be drawn from the point A to the circumference; therefore they muſt all be equal to AB, and to one another, according to the definition of a Circle.



Thus therefore the ſtructure of a Circle needs not any artifice, or ratiocination to prove its Being: but it is in it ſelf ſo ſimple, and obvious, That it may be as juſtly poſtulated, as the Duction of a ſtrait line from point to point. For indeed they are alike evident, there being in nature only two ſimple Motions, the Strait, and the Circular, and thereby are created the moſt ſimple of all Lines, and Figures, a ſtrait Line, and a Circle.

Moreover, in relation to this intellectual conſtruction of a Circle, we may obſerve that the Radius of a Circle, is by *Ariſtotle* in his Mechanics always named ἡ γράφουσα τὸν κύκλον, *The Line deſcribing the Circle.* Whereas *Euclid's* phraſe is ἡ ἐκ τοῦ κέντρου, *The Line from the Center. Ariſtotle's* Appellation reſpects the *Geneſis*, and *Euclid's* the Definition of a Circle. And again, from this περιφορὰ, or Circumlation of the Radius, in the creation of a Circle, the curve Line deſcribed, and bounding the Circle is call'd περιφέρεια, a *Periphery*, a name very proper to the nature of the Thing. For any curve Line, which by a regulated Motion returns into it ſelf, is ſignificantly called a Periphery, or Circumference. But it is named by *Archimedes* in his *Cyclometries*, ἡ περίμετρος, as meaſuring the Circle in its Ambit round about: like as ἡ διάμετρος, *the Diameter* is ſo called, for that it meaſures the Circle throughout at its utmoſt wideneſs. *Archimedes* therefore calls the Bound of a Circle the Perimeter, in order to the menſuration of the Area of a Circle, which was his preſent buſineſs; And *Euclide* the Periphery from the manner of its Generation.

Now anſwerable to this ſpeculative Formation of a Circle, is the Mechanical deſcription thereof made by help of the Compaſſes, and firſt invented by *Perdix* the Nephew of *Dædalus*, as *Ovid* reports. Which Inſtrument with its uſe he has moſt accurately expreſſed, *Metamorph. Lib.* VIII.

> Perdix *ex uno duo ferrea brachia nodo*
> *Junxit, ut æquali ſpatio diſtantibus Ipſis,*
> *Altera pars ſtaret, pars altera duceret Orbem.*

Theſe three eaſy Problems, firſt to draw a ſtrait Line, Then to continue the ſame at pleaſure, And laſtly to deſcribe a Circle, are the only practical Principles laid down by *Euclide*, to effect all his Geometrical Conſtructions, and all thoſe excellent and ſubtil Problems, which are demonſtrated in theſe Elements.

Laſtly 'tis ſpecially to be remarked, that *Euclide* moſt judiciouſly choſe rather to make the Geneſis of a Circle to be a Poſtulate, than with ſome of our modern Geometricians, an uſeleſs, unapplicable Definition: the abſurdity whereof we have ſhewn before.

A ſtrait Line therefore, and a Circle are the only Inſtruments of plain Geometry, and the only two Things, which *Euclide* Poſtulates to be granted him. But *Solid* Geometry requires moreover to make uſe of *Conic Sections*, the *Parabola*,

Hyperbola,

Hyperbola, and *Ellipsis*, for the Effection of Problems of an higher, or more compound nature, than what can be perform'd by a strait Line, and a Circle: which difference in this matter ought to be well considered and observed.

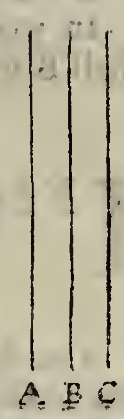
For a foul error it is in a Geometrician (whereof some of our Moderns have been too guilty) to undertake the Solution of such kind of Problems (as the Duplication of a Cube, and the like) by strait Lines, and Circles, whereas they may be readily effected by the *Conic Sections*, which are as truly Mathematical, as a Circle, and have a Genesis, as purely Geometrical: Both arising from simple Motions: the Circle being created by a mental revolution of a strait Line upon a fixed point; and the *Conic Sections* from a mental Motion of a Plane cutting a Conical Superficies. As is shewn in the Conic Elements of *Apollonius*.

The Speculative Principles, Common Notions, or Axioms.

AXIOM I.

Things equal to the same, are equal to one another.

ANNOTATIONS.



As if A be equal to B, and C be equal to B; Then shall A and C be equal to one another.

Now to express this briefly in Characters, commonly named *Symbols*, or *Species* after the manner of Analysts, with which to be timely acquainted is very useful to a Geometrician;

Let the Sign of Equality be this $=$

Then shall the Proposition be thus signified.

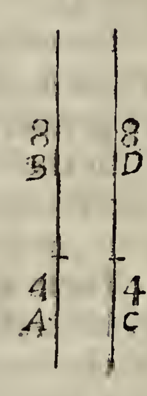
If $A=B$ and $C=B$. Then shall $A=C$.

In words thus. If A be equal to B and C to B, then shall A be equal to C.

AXIOM II.

If Equals be added to Equals the Wholes are equal.

ANNOTATIONS.



Let A be equal to C, and B to D, then shall A and B added together be equal to C and D added together.

In Symbols thus it is.

Let the Sign of Addition be this $+$

Then shall the Proposition be thus signified.

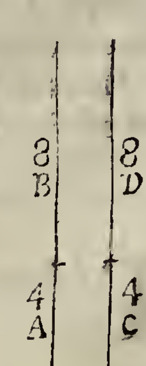
If $A=C$ and $B=D$, then $A+B=C+D$.

In words thus. If A be equal to C, and B to D, then A more B (that is more *by* B) shall be equal to C more D (that is more *by* D.)

AXIOM III.

If Equals be taken from Equals, the Remainders are equal.

ANNOTATIONS.



Let A, B, be equal to C, D, and A be equal to C: then A taken from A, B, and C taken from C, D, shall leave the remainders B, D, equal to one another.

To express this in Symbols,

Let the Sign of Subtraction be this $-$

Then shall the Proposition be thus signified.

If $A+B=C+D$, and $A=C$, then $A+B-A=C+D-C$; That is, $B=D$.

In words thus. If A more B be equal to C more D. And A be equal to C, then A more B leſs A (or leſs *by* A) ſhall be equal to C more D, leſs C (or leſs *by* C) That is, B ſhall be equal to D.

AXIOM IV.

IF Equals be added to Unequals, the Wholes are unequal.

ANNOTATIONS.

Let B, D be unequal to one another, B the greater D the leſſer; and let A be equal to C. Then ſhall A added to B be unequal to C added to D. So that A, B, together is greater then C, D, together, the inequality being ſtill the ſame as before it was between B and D.

To expreſs this in Symbols,

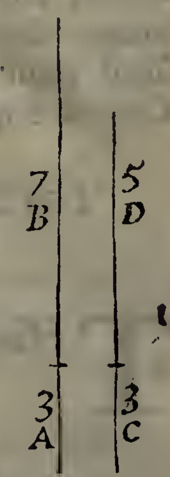
Let the Sign of Greater be this $>$.

Let the Sign of Leſſer be this $<$.

Then ſhall the Propoſition be thus ſignified.

If $B > D$, and $A = C$, then $B + A > D + C$.

In words thus. If B be greater than D, and A be equal to C, then ſhall B more A be greater than D more C.



AXIOM V.

IF Equals be taken from Unequals, the Remainders are unequal.

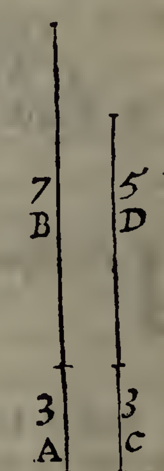
ANNOTATIONS.

Let A, B; C, D, be unequal to one another, A, B, the Greater, C, D, the Leſſer, and let A be equal to C. Then A taken from A, B, and C taken from C, D, ſhall leave the Remainders B, D, unequal to one another; B the Greater, D the Leſſer, the inequality remaining the ſame that it was at firſt.

In Symbols thus it is.

If $A, B > C, D$, and $A =, C$, then $A, B - A > C, D - C$. That is, $B > D$.

In words thus. If A, B, be Greater than C, D, and A be equal to C. Then A, B, Leſs A ſhall be greater than C, D, leſs C. That is, B ſhall be greater than D.



In theſe four laſt Axioms 'tis naturally evident, that *Equality* and *Inequality* are not changeable by Addition, or Subtraction of Equals. So that after ſuch Additions, or Subtractions, the things are as at firſt Equal, or Unequal.

Theſe are the general Maxims of this nature, which were thought fit by *Euclide* to be laid down in Form. Altho' there be uſed hereafter ſome other Propoſitions of the very ſame kind: Principles indeed as evident and as neceſſary as the foregoing. But becauſe they are not Primary Notions, but only manifeſt Conſectaries from theſe here now mentioned, or elſe that they are not of ſo general an uſe; *Euclide* at the preſent paſſes thoſe over, and only aſſumes them upon occaſion, as the matter in hand requires. Which order is leſs troubleſome to Beginners, and therefore ought rather to be followed, than that of *Clavius*, and ſome Others after him, who have gathered theſe kind of Principles out of ſeveral places in *Euclide*, and do uſually pack them all together; without a juſt conſideration had of Principles *Primative*, or *Derivative*, *more*, or *leſs General.*

AXIOM VI.

THings which are Double of the ſame are equal to one another.

AXIOM

AXIOM VII.

Things which are Halves of the Same, are equal to one another.

ANNOTATIONS.

In theſe two Axioms *Euclide* inſtances only in the *Duple*, and *in the Half*. For in laying down theſe common notions, he judged it ſufficient to put the Principal notion in General, leaving the Conſectaries, which do naturally follow to every ones common underſtanding. Alſo ſuperfluous it is, and beſides too trivial, to intermix with the general Principles every obvious Conſequence. For in the naming Double and Half, who does not preſently conceive the ſame evident truth in Triples, Quadruples, Quintuples, &c. And ſo in a Third, a Fourth, a Fifth, or any like part of the ſame thing, or *of Equal Things*: Which laſt alſo *of Equals*, *Euclide* could have as eaſily added, as his Commentators. But he would here intimate that what is ſaid of one and the ſame Thing, is alike to be underſtood of Equal Things; for that *Identity and Equality, are to be indifferently taken, and uſed in Geometrical demonſtrations.*

The Axioms hitherto laid down are more general, and common to ſeveral Sciences: But theſe which follow are purely Geometrical.

AXIOM VIII.

Magnitudes Congruous one with the other are equal.

ANNOTATIONS.

That is, If two Magnitudes be imagined to be applyed One upon the Other, and after this mental application it be demonſtrated, that neither does any ways exceed the other; but that the Intermedial parts of the One do agree with the Intermedial parts of the Other, and the Extreams with the Extreams, then theſe Magnitudes are ſaid to be Congruous. And Geometricians do juſtly aſſume for a rational Principle, that Magnitudes being ſo far proved Congruous, are then to be concluded equal to one another. Thus *Euclide* is to be underſtood in the Speculative and true Geometrical uſe of this Axiom concerning Congruous Magnitudes.

But moreover, there goes along with this Speculation a very natural, and common Mechanical uſe of the ſame Axiom. For in the practice of Artiſans They Mechanically fitting one Magnitude to an other do judge by their Eye, or Hand how one agrees with the other, and accordingly do determine their equality. As in the menſuration of Magnitudes by a Foot, a Cubit, a Perch, &c.

The like Mechanical Congruency is made uſe of in the meaſure of Liquids, of Grain, and the like, by Pints, Gallons, Pecks, Buſhels, &c. The equality of theſe kind of things being judged by congruous Veſſels, or Places, which do contain them. So that this Mechanical Application of Magnitude to Magnitude is a natural, and univerſal practice: And in common ſenſe Congruency is a ſtanding Rule of Equality.

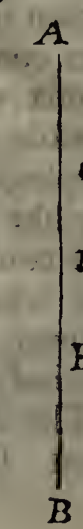
But the Geometrical Congruency in this place underſtood, ariſes only from an Intellectual application of one Magnitude to an other: And then after ſuch an application there is made by argumentation a demonſtrative proof of their Congruency, both in their Extreams, and Intermedial parts, without any judgement taken from our outward Senſes. So that their Congruency being thus rationally demonſtrated, we do then from this natural and common notion of Congruency, conclude their Equality. As we ſhall find in Prop. 4th. and 8th. of the firſt Element: In Prop. 24th. of the third Element &c.

AXIOM IX.

THe Whole is greater than its Part.

ANNOTATIONS.

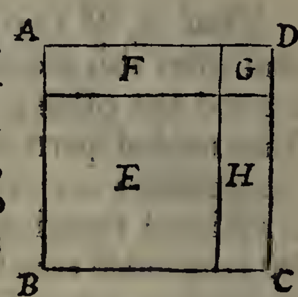
Every magnitude is in it ſelf *Quid Unum & Continuum*, only one entire Thing: Neither to be called Great, or Little, a Whole, or a Part. But becauſe magnitude is infinitely diviſible into ſmaller magnitudes, therefore every magnitude, tho' in it ſelf but one, may yet by imagination be ſuppoſed to conſiſt of certain leſs magnitudes contained within the ſame. *Theſe* are called the *Parts*, and *That* the *Whole*. Theſe parts are not really ſeparate from the whole; but are all united by common terms, which are conceived the End of one part, and the Beginning of the other. As let AB be put for any finite magnitude, whoſe Extreams are A, B. Now this magnitude AB is in It ſelf but one: yet It may be diſtinguiſhed into parts, as, AC, CD, DE, EB, whoſe intermedial limits are C, D, E; here C being the End of AC, and the Beginning of CD, ſo D of CD, DE, and E of DE, EB. Thus the parts are to be taken, and underſtood in continued magnitudes. And the ſame truth is alike manifeſt in any *Diſcrete*, *Collective*, or *Aggregate Totum*. As in numbers in a Peck of Corn, and the like, 'tis naturally evident that the Whole is greater than its Part.



Moreover, it is commonly added as an Axiom by the Commentators, That

The Whole is equal to all its Parts, or all the Parts are equal to the Whole.

Yet *Euclide* having no occaſion to uſe theſe Propoſitions in theſe direct Terms; but only upon ſome particular argumentation to infer *ex Diagrammate*, from the Diagram it ſelf, that ſuch and ſuch Parts all together are the Whole; As E, F, G, H, are the whole Square ABCD, It ſeemed not proper to place This ſignally for a diſtinct Maxim among the reſt of his General, and Common Notions.



AXIOM X.

ALL Right angles are equal to one another.

ANNOTATIONS.

Truth, and *Rectitude* have the ſame property. For as one Truth cannot be more true than an other; ſo one ſtrait Line cannot be more ſtrait than an other: nor one Right angle more Right than an other. So *Martianus Capella lib.* VI. *de Nuptiis Philologiæ & Mercurii*, ſays, *Angulorum natura triplex eſt, Nam aut Juſtus eſt, aut Anguſtus, aut Latus.* The Acute, and Obtuſe are here called *Anguſtus*, & *Latus*; *Quorum uterque ſemper eſt mobilis*, ſays he, always changeable in their increaſe, or decreaſe; there being no Obtuſe angle, but that there may be a more, or leſs Obtuſe, nor any Acute angle, but that there may be a more, or leſs Acute: Only the Right angle *Juſtus eſt & ſemper Idem.*

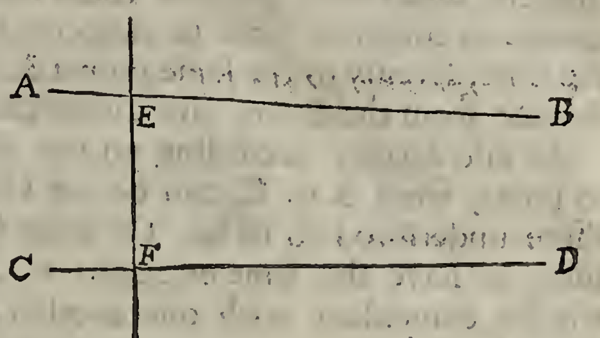
AXIOM XI.

IF upon two ſtrait lines a ſtrait line falling, does make the internal angles on the ſame ſide leſs than two right angles; Thoſe ſtrait lines being infinitely produced, ſhall meet on that ſide where the angles are leſs than two Right angles.

AN-

ANNOTATIONS.

Upon the lines A B, C D, let the ſtrait line E F fall, making the internal angles B E F, D F E, leſs than two Right angles: If then A B, C D, be the ſame way produced indeterminately, that is, onward, and onward in an undetermined free courſe, it is here put as a manifeſt notion, that the ſtrait lines A B, C D, ſhall at length meet together towards the parts B, D.



Tho' this Propoſition be a moſt certain truth, yet it hath been generally excepted againſt for want of the juſt *Evidence* of a Principle, tho' not of the *Certainty* of the Thing. But yet let us conſider that here are put two ſtrait lines A B, C D, under ſuch conditions, which do clearly ſhew that They have a *Tendency*, and *Inclination* towards one another: And therefore it may be juſtly aſſumed for a common notion, That two ſtrait lines inclining each to the other, and being that way infinitely produced, ſhall at length meet together. This is the ſubſtance of *Euclid's* 11th. Axiom, tho' expreſſed in other words more ſuitable to the Form of his demonſtrations.

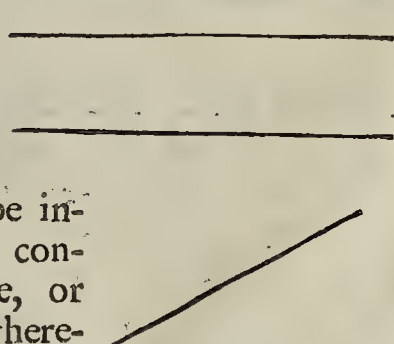
But here are made many Objections. In the firſt place, it has been already ſhewn, that there may be two Inclining lines infinitely produced, and continually approaching nearer and nearer to one another, which notwithſtanding ſhall never meet together. A thing very ſtrange, and at the firſt view hardly credible: yet afterwards certainly found to be true. Wherefore ſeeing that Inclination, and perpetual Approximation force not a Concurrence, it may be doubted, whether the ſame may not alſo happen in *ſtrait lines inclining, and continually approaching towards one another*: Inſomuch at leaſt that the Concurrence of two Inclining ſtrait lines cannot well be admitted for an evident Principle. Beſides this, there are ſeveral other Objections, of which we ſhall have occaſion to ſpeak hereafter. Only at preſent, I ſay that they who have endeavoured to mend this matter, have with much trouble, and diſturbance of *Euclid's* excellent method taken great pains to little better purpoſe.

AXIOM XII.

T*Wo ſtrait lines do not comprehend a Space.*

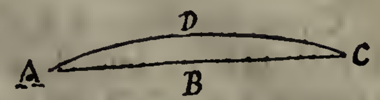
ANNOTATIONS.

If the two lines be Parallels, there is an open ſpace both ways between them, which is neither way boundable by thoſe ſtrait lines, for that parallels are Nonconcurring ſtrait lines. Again, if the two ſtrait lines any where meet, there is made a ſtrait-lin'd angle: But then the angular ſpace is not thereby comprehended, being one way infinitely open, nor is it imaginable, that the ſame ſpace can again be incloſed by a progreſs of Thoſe two ſtrait lines, which contain the angle; unleſs there be conceived ſome Flexure, or Vergency of the ſame ſtrait lines towards one another, whereby they being produced at pleaſure may again meet together. But this is to deſtroy the natural conception of Rectitude in the ſtrait lines themſelves. Therefore in common ſenſe there muſt naturally intervene a third line for the incloſing of a ſpace.



Again, for as much as every ſtrait line does in all its parts lye Evenly to all its points [Def. 4.] therefore two diſtinct ſtrait lines cannot have the ſame Extream points.

points. As let the ſtrait line A B C have its extream points A, C: then the line A D C, having the ſame extreams (and ſo both together bounding a ſpace) ſhall not be likewiſe a ſtrait line, for that the intermedial parts of thoſe two lines A B C, A D C, cannot in common ſenſe be conceived to lye alike in both the lines, Evenly to the ſame points A, C, but that the parts of the One of them muſt deviate from the Even, and direct Courſe, which lyes between the points A, C.



As alſo farther according to the *firſt Poſtulate*, a ſtrait line drawn from point to point, from A to C, can be but One, and the ſame ſingular ſtrait line; and is there underſtood ſo to be. Or if we ſuppoſe two ſtrait lines applyed to one another, to have the ſame extreams A, C, theſe ſtrait lines, *quatenus* ſtrait, muſt be wholly coincident with one another, ſo that they cannot intercept any imaginable ſpace.

Here *Euclide* ends the Principles.

In the Definitions (which are by Commentators commonly accounted among the Principles) was laid down the ſubject Matter, or the particular Things to be treated of in the firſt Part of theſe Geometrical Elements.

After the Definitions next follow the Poſtulates, and Axioms, that is, Principles *Practical*, and *Speculative*. Theſe are rightly called *Principles*, as being the foundation upon which this Science builds its demonſtrations, and are in the firſt place made uſe of in the doctrine of thoſe Magnitudes, and Figures expounded in the foregoing Definitions. For to ſpeak properly Theſe Definitions are not Principles of common Reaſoning premiſed for demonſtration ſake; but in truth they are a ſmall part of the ample Subject of Geometry.

As in the Definitions of an Angle, of a Circle, of Trilateral, and Quadrilateral Figures, of Parallel Lines, is explained what kind of Things are to be underſtood by thoſe ſeveral Names. And theſe Things are here laid open for an Entrance into Geometry, as being Matters moſt ſimple, and eaſily to be taught, and apprehended. But now in purſuit of a perfect underſtanding of Them, the Poſtulates, and Axioms ſerve as Natural, and general Principles of Reaſoning, whereby we are enabled to demonſtrate the manner of their Conſtruction, their Properties, and Affections. The contemplation whereof is the buſineſs of this firſt Element of Geometry.

The Poſtulates, and Axioms, or common Notions, are clearly intelligible to every ones capacity, altho' ſome Annotations made with Examples, and Inſtances may be to Beginners uſeful for Eaſe, and Illuſtration ſake. But for the Propoſitions themſelves, there is nothing in them further, or otherwiſe to be expounded, than what the literal ſenſe, and common meaning of the words import. As *Tertullian* ſays upon a like occaſion, *Definitiones, ac ſententiæ, quarum aperta eſt natura, non aliter Sapiunt quam Sonant.*

THE

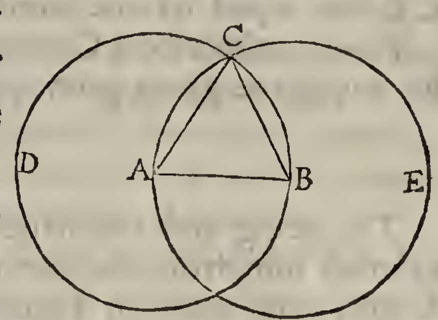
THE FIRST ELEMENT.

PROPOSITION I.

On a given finite ſtrait Line to conſtitute an Equilateral Triangle.

Expoſition. Let the given ſtrait Line be AB.

Recognition. It is required on the Line AB to conſtitute an Equilateral Triangle.



Conſtruction. The Center A, and the diſtance AB, let be deſcribed the Circle BCD. [by Poſtul. 3.] And again the Center B, and the diſtance BA, let be deſcribed the Circle ACE. [by Poſtul. 3.] Then from the point C where the Circles cut one another, to the points A, B, let be drawn the ſtrait Lines CA, CB, [by Poſt. 1.]

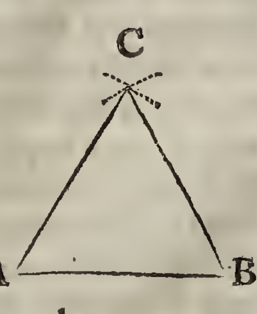
Determination. I ſay that the Triangle ABC is Equilateral.

Demonſtration. Foraſmuch as the point A is the Center of the Circle BCD, therefore the Line AC is equal to the Line AB. [Def. 15.] Again, becauſe the point B is the Center of the Circle ACE, therefore the Line BC is equal to the Line AB. [Def. 15.] But it has been proved that the line CA is equal to the line AB; therefore each of the lines CA, CB, is equal to AB. But things equal to one and the ſame thing are alſo equal to one another, [Ax. 1.] and therefore CA is equal to CB: wherefore the three lines CA, AB, BC, are equal to one another.

Concluſion. Therefore the Triangle AB is Equilateral, and is conſtituted on the given finite ſtrait line AB. Which was to be done.

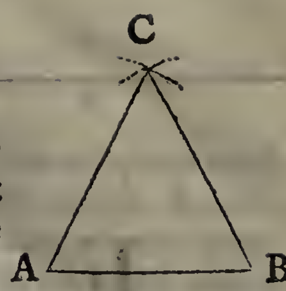
The Practice.

To make an Equilateral Triangle.



Open the Compaſſes to the length of the Given line AB, and fixing one foot on the point A, deſcribe on either ſide of the line AB an Arch. Again, fixing a foot on the point B, deſcribe on the ſame ſide an other Arch cutting the former; And from the point of Interſection C, draw CA, CB. Thus is made the Equilateral Triangle ABC.

By the like practice may be formed an Equicrural Triangle.



To make an Equicrural Triangle.

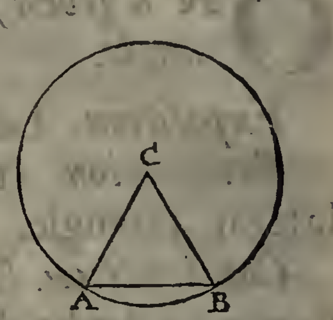
Open the Compaſſes to any diſtance beyond half of the line A B, and deſcribe Arches as before. Then from C the point of Interſection draw C A, C B, making an Equicrural Triangle A C B.

A practical Corollary.

From hence it is manifeſt, how to effect this following Problem.

A Problem.

By any two given Points to deſcribe a Circle, whoſe circumference ſhall paſs by the given Points.



Let the given points be A, B. And [by Poſtul. 1.] drawing the line A B, let thereon be conſtituted an Equilateral Triangle A B C, [by Prop. 1.] and [by the 3d. Poſtulate] from the point C as a Center, and at the diſtance either of C A, or C B, a Circle being deſcribed, the circumference ſhall paſs by the given points A, B, for that the lines C A, C B, are equal to one another.

How to deſcribe a Circle, whoſe circumference ſhall paſs by any three given points, not lying in a ſtrait line, will be ſhewn hereafter.

ANNOTATIONS.

The being and the ſtructure of a Circle, *Euclide* has before poſtulated to be granted him, from the natural ſimplicity, and evidence of its generation. The next ſimple, and uniform Figure, is an Equilateral ſtrait-lin'd Triangle; with which therefore the *Elementator* begins. Whoſe Geneſis, becauſe there is uſed ſome Artifice, and Compoſition in the work, beyond that of a Circle, requires a demonſtration to prove the Triangle conſtructed to be Equilateral. And this *Euclide* evidently deduces from a Circle, that primary Figure already poſtulated to be allowed without any demonſtration. For Mathematicians proceed by Propoſition after Propoſition: Firſt from certain Propoſitions as natural Principles, unto others as they may, with moſt facility and evidence, be deduced one from the other: The foregoing Propoſitions ſerving to demonſtrate the following.

The method in all Propoſitions, Problems, as well as Theorems, is much the ſame, and conſiſts of certain diſtinct parts. As for example, in this firſt Problem the parts are thus to be diſtinguiſhed.

I. *The Propoſition.* Which propoſes in general Terms, a Thing given, and a Thing required. In Theorems the *Quæſitum* is a thing required to be demonſtrated as an undeniable truth. In Problems the *Quæſitum* is firſt required to made, or conſtructed; and then the conſtruction of the ſame is to be undeniably demonſtrated. As in this Problem, *On a given ſtrait line to conſtitute an Equilateral Triangle.* The ſubject given is any finite ſtrait line in general: the thing required is an Equilateral Triangle to be conſtituted upon that given line.

II. *The Expoſition* of the thing given. This *Expoſitio Dati* is an inſtance in ſpecial of what in the Propoſition was given in general. As in the Propoſition was given a *finite ſtrait line* in general. Then next in the Expoſition, is laid down in particular a finite ſtrait line, as the finite ſtrait line A B. The uſe of the Expoſition is to facilitate the whole matter of the Propoſition and Demonſtration, by ſetting it forth in a ſenſible Diagram for the readier information of the Intellect; which may apply the ſame as univerſally as it was propounded; for that the line A B may denote any finite ſtrait line whatever.

III. *The Recognition* of the thing required. After the Expoſition laid down in a ſingle inſtance, there follows in all, and *only* in Problems, that part which we call *Recognitio Quæſiti*, a Recognition of the thing required: The phraſe is, Δεῖ δή, *Oportet.*

Oportet. In which *Oportet* we are reminded, what ought next to be done in particular upon the expoſed inſtance. As upon this expoſed *Datum* the line A B, it is now required in particular to conſtitute an Equilateral Triangle; ſo that in this Recognition the ſides of the Equilateral Triangle are confined to the length of the expoſed line A B, and the like in all Problems there is included in this *Oportet*, a Confinement of the *Quæſitum* to the quantity of the *Expoſed Datum.*

This, which, as a diſtinct part, we call Recognition, is by *Clavius* annexed to the Expoſition, and by *Daſypodius* called the firſt Determination. Yet it cannot properly belong to either; for that the thing required is not as yet in being. But in reference to the foregoing *Datum* now expoſed, here is next a ſpecial Deſignation of what is to be, as it was before in general required in the Propoſition: And for a diſtinction from that part, called properly the Determination, we have named it *Recognitio Quæſiti.*

IV. *The Conſtruction.* This is a Geometrical Operation made out of the Poſtulates by an intellectual drawing of ſtrait lines, and deſcribing Circles. In Problems the conſtruction does both effect the thing required: and alſo ſerves to demonſtrate the ſame to have been rightly effected. As in the Conſtruction of this Problem. The Equilateral Triangle A B C, is firſt tacitly conſtructed by deſcribing the Circles B C D, A C E, and drawing the lines C A, C B. Then from the manner of its Conſtruction the Triangle is next pronounced, and determined to be Equilateral.

V. *The Determination* of the thing required. This Part is only a Declaration that the thing required is now in a ſpecial Diagram exhibited. The phraſe is λέγω, I ſay, or Pronounce. As here in *Specie* the Triangle A B C (having been juſt before tacitly conſtructed upon the expoſed *Datum* A B) is now determined, and pronunced to be Equilateral: Which the following Demonſtration makes Apparent.

VI. *The Demonſtration* of the Propoſition. This is the glorious part of a Mathematical Propoſition, wherein is made an undeniable and indubitable proof of the thing propoſed, and exhibited in a particular Diagram. As here is demonſtrated, that the Triangle A B C is Equilateral

The form of Argumentation is for brevity ſake made in *Enthymems*; which may be reduced into perfect *Syllogiſms*: as *Cunradus Daſypodius* has ſet forth the firſt ſix Elements in a compleat Syllogiſtical form.

VII. *The Concluſion.* This is firſt particular in reference to the preſent Diagram, on which the Demonſtration proceeded: as, *Therefore the Triangle* A B C *is Equilateral, and conſtituted on the given line* A B. So that this particular Concluſion conſiſts of the Determination, and Expoſition.

And becauſe after the ſame manner there may on any other finite ſtrait line be conſtituted an Equilateral Triangle, therefore from the Logical Rule of Induction, there is laſtly a general Concluſion deduced, that *on any given finite ſtrait line may be conſtituted an Equilateral Triangle,* which is only the Propoſition repeated as being now demonſtrated. Therefore in a Problem there is ſubjoined to the Concluſion, ὅπερ ἔδει ποιῆσαι. *Which was the thing propounded to be done.* And in all Theorems, ὅπερ ἔδει δεῖξαι. *Which was the thing propounded to be demonſtrated.*

This is the regular courſe uſed by Geometricians in their Forms of Doctrine. So that in Problems there may be ſeven parts according to the forenamed Order. The Propoſition, Expoſition, Recognition, Conſtruction, Determination, Demonſtration, Concluſion. But in Theorems there can only be ſix, and in this Order. The Propoſition, Expoſition, Determination, Conſtruction, Demonſtration, Concluſion. The different nature of Problems, and Theorems requiring ſuch a difference in the number, and order of their parts, and ſpecially *in the diſpoſal of the Determination*; Which part in Problems ever follows the Conſtruction: but in Theorems follows immediately the *Expoſition.*

It is commonly, but improperly ſaid, that every perfect Propoſition has all theſe parts. Whereas all Propoſitions, which are rightly demonſtrated, are alike perfect; For that one Demonſtration cannot be more a Demonſtration than an other: But thus it is, every Propoſition requires not all theſe parts. For ſome Theorems need no Conſtruction, by adding ſtrait lines, or Circles to the Original Scheme of the Propoſition, becauſe the Scheme it ſelf is often ſufficient for the Demonſtration of the Propoſition. But where it is not, there is then ſuperadded a Conſtruction, which is only as an *Apparatus*, or Preparation made on purpoſe to help out the Demonſtration of the Theorem, and belongs not to the Propoſition either as any part of the *Datum*, or *Quæſitum*. Now in theſe Caſes, that Theorem is rather to be accounted more perfect than any ways defective, which is not forced to ſeek out a Conſtruction, for the ſetting forth of its Demonſtration.

Again, in Problems the Determination is oftentimes omitted, as being in it ſelf not abſolutely neceſſary, tho' for the more perſpicuity it be convenient. So in theſe three firſt Problems the Determination is not in the Text of *Euclide*, yet in 9, 10, 11, and 12th. Propoſitions, which are Problems, the Determination is expreſly ſet forth. Therefore accordingly we have inſerted in this, and the two following Problems the Determination, to clear the matter for the eaſe of Beginners, that after the Conſtruction, and before the Demonſtration, there might be ſet forth, and determined, what by the Conſtruction has been effected, and is next to be demonſtrated.

Moreover, ſome Problems have only a *Quæſitum*, and not a *Datum*, upon which in particular to work. And ſo there is neither an Expoſition, nor a Recognition to follow, but only a Conſtruction, Demonſtration and Concluſion. As the tenth Propoſition of the fourth Element, which is a Problem requiring *To conſtitute an Equicrural Triangle, having each of the Angles at the Baſe double to the remaining Angle.* Here now is required to be conſtructed ſuch a kind of Triangle without any thing given. Yet notwithſtanding there is no imperfection, or any defect in this Propoſition; but that it is a ſubtil and admirable Problem. Therefore we are to underſtand, that every Propoſition of the Elements is according to its nature compleat in it ſelf, whether it has, or needs not to have all the forementioned Parts.

Of the DATUM in Geometrical Propoſitions.

In Geometrical Propoſitions, a thing may be given four ſeveral ways: In Poſition, In Specie or Form, In Magnitude, In Propoſition.

A point having no Magnitude is only given in Poſition, that is to ſay, HERE. But every kind of Magnitude may be given all the four ways, jointly, or ſeverally, in all, or in ſome.

A thing is ſaid to be given in Poſition, when it is reſtrained to a certain Situation. As the Data in all Problems are: where whatſoever is given ought not, ſo much as in our imagination, to be removed from its given Situation; but according to the reſtriction of the Poſition given the Problem is to be performed.

Now on the contrary, in Theorems the Diagram of the Propoſition is not tyed to a certain Poſition, but left as indifferent. For tho' Poſition be a ſpecial condition in the Structure of a Problem, whereby it is to be regulated: yet in a Theorem it appertains not at all to the Truth, or Falſity of the Theorem: and therefore the Poſition of the Diagrams is alterable at diſcretion. Inſomuch that in ſome Theorems, where two things are given, there a certain Poſition of the one to the other, is ſometimes a means to help out the Demonſtration: And therefore an arbitrary poſition of the Data in Theorems, is allowed to the Demonſtrator for a kind of Conſtruction. As *Euclide* ſometimes applys Figure to Figure, ſometimes conceives two ſtrait lines ſo directly ſituated to one another, as to become one ſtrait line; with ſuch like choice and change of poſition, as may beſt ſerve to the demonſtration of the preſent Theorem.

In this Problem the line AB is not only *given in Poſition*, as the *Datum* in every Problem is, but alſo 'tis given in Specie, being propoſed a ſtrait line, to ſpecificate it from a crooked.

And moreover, 'tis given in Magnitude as a finite ſtrait line, which alſo is underſtood to be ſo given under a certain Termination, that it may be compared as Equal, Greater, or Leſſer, than an other: and in like manner any other may be compared

pared to it. For 'tis not meant to be given in ſuch, or ſuch a ſingular Quantity, as of one, or two, or three Inches, or Feet, in relation to any ſtated meaſure; but here That is taken to be given in Magnitude, which is propoſed under ſome certain limitations, ſo that it may be ſaid to be equal, or unequal to an other Magnitude. Yet after ſuch an Indefinite manner the limited Magnitude is put, that it may be conceived to repreſent any Magnitude of the ſame kind, in any quantity whatſoever. As in this Propoſition when we ſay, Let the given line be A B, there is meant by A B any length, by what meaſure ſoever eſtimated. Whether by Inch, Foot, &c. or not at all eſtimated by any diſtinct Quantity. So that in this Caſe by an Example, or Inſtance expoſed in particular, the Univerſality of the Propoſition, and its Demonſtration are not deſtroyed, but ſtill remain in their full Latitude, and general Extent.

The Methods of Compoſition and Reſolution, as they are uſed by Geometricians.

There are two ways of Reaſoning, whereby Man comes to the knowledge of things; And both the ways are eſtabliſhed upon the ſame Foundation, that is, upon the Dictates of Nature, or common Notions among Mankind. The difference here only is, that in one way we begin our Diſcourſes from thoſe Natural Dictates, and in the other we end with them; treading the ſame path forward and backward. Both theſe ways we thus explain.

Ratiocination proceeding from Natural, that is, Self-evident Principles of Truth unto other Truths, made known to us from them, and ſo going onward by the help of theſe diſcovered Truths, to infer and make manifeſt Truths more remote, and as yet unknown, is called the Method of *Compoſition*, or *Syntheſis*. For that in this way of Reaſoning, we do from Notions moſt plain and ſimple, gather up by degrees Notions more intricate, and compounded: framing out of ſimple Materials firm, and ſtately Edifices.

Ratiocination proceeding from the Suppoſition of things uncertain, whether True or Falſe, Poſſible or Impoſſible; and which by demonſtrative Conſequences deduced from that Suppoſition, does neceſſarily come to a manifeſt Truth, or Falſity, Poſſibility, or Impoſſibility, is called the Method of *Reſolution*, or *Analyſis*. For that a doubtful Suppoſition is hereby reſolved into a certainty of Truth, or Untruth, of Being, or not Being.

If the Iſſue of our Argumentation terminates in an acknowledged Truth, then are we aſſured of the Being, and Verity of the Suppoſition upon which we argued.

And we may again take a beginning from the ſame Truth, wherein the Reſolution reſted; and from thence as a Principle, proceed in the Method of Compoſition, making a return in the very ſame ſteps, which in the Reſolution of the Suppoſition were traced out before: till at length we arrive at that thing, which was at firſt Suppoſed, demonſtrating in the common Compoſitive Method of Geometricians the Truth of Theorems, and the Geometrical effection of Problems. Thus Compoſition and Reſolution, or Syntheſis and Analyſis, anſwer one another; the Analyſis ending where the Syntheſis begins, and the Syntheſis ending where the Analyſis begins. Like to an Aſcent and Deſcent made in the ſame path ſtep by ſtep.

Reſolution, or Analyſis, is properly the Method of Invention, and the ready way of diſcovering the Truth, or Falſity of a Propoſition, in any Art or Science.

Compoſition, or Syntheſis, is the Method of Doctrine, or the way of Teaching, and therefore in this Method from allowed Principles, all Arts and Sciences are uſually delivered. Like as laſting Buildings are raiſed upon ſure Foundations; Whereof the Doctrine of theſe Elements is a moſt perfect pattern.

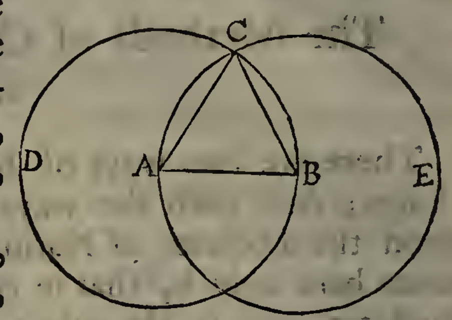
Therefore of this firſt Problem, we ſhall take a review, and nicely obſerve, by what Gradations of Compoſition it is to be effected and demonſtrated.

And for an Entrance into this Inquiry, we are firſt to conſider what from the

 preceding

preceding Principles can be made uſe of, upon the given line A B, to effect this Problem, and conſtruct an Equilateral Triangle.

Firſt, the given line A B being finite, there are given the ends thereof. So that three Things are given, two Points, and an interjacent ſtrait line. Next therefore we are to ſearch out, what can ariſe to our purpoſe, from theſe three Data. And recollecting the former Definitions, Poſtulates, and Axioms, (for in the Mathematics memory and reaſoning are to go together) there muſt among the reſt occur to our remembrance the third Poſtulate; *From any point unto any diſtance to deſcribe a Circle.* And here in theſe preſent *Data* aptly appear a given Point A, or B, and a given Diſtance, namely the ſtrait line A B given; therefore of the line A B taking one of the Extreams, as the Point A for a Center to a Circle, and the given line A B for a given Diſtance, we poſtulate a Circle to be deſcribed.



Again, taking the point B, the other extream of the given line A B for a Center, and the given line for the ſame diſtance we poſtulate another Circle to be deſcribed. 'Tis evident, that *This Circle muſt cut the former, for that the line* A B *lyes wholly within both the Figures, and is a common Radius to both Circles.*

Now by the mutual Interſections of theſe Circles, there occurs a point common to both Circumferences, and let it be ſigned the point C.

Here then are three known points, A, B, C, and the line A B. What can, from theſe four things known, be reaſonably deduced for the making an Equilateral Triangle, is next to be thought on.

The firſt Poſtulate cannot but come readily into our mind, which allows the drawing of a ſtrait line from point to point. So that we are prompted to draw from the Found point C to the given points A, B, the two ſtrait lines C A, C B.

Here at laſt is made a ſtrait-lin'd Figure of three ſides called a Triangle, which we are next to conſider of what kind, or condition it may be: and whether anſwerable to the ſolution of the Problem.

The two Circles juſt now before deſcribed by the ſame Radius A B, and B A, are obvious to our conſideration, and the Idea or Notion of a Circle delivered in the 15th. Definition, that the lines from the Center to the Circumference are all equal to one another. This Idea does readily lead us to infer, that the line C A drawn from the point C in the circumference of the Circle B C D to the point A the Center, is equal to the given line A B the deſcribing Radius, or the primary line from the Center of the ſame Circle.

In like manner, and by the ſame means we are inſtructed to argue, that the line C B drawn from the point C in the circumference of the Circle A C E to the point B the Center, is equal to the given line B A, the deſcribing Radius of the Circle A C E.

Wherefore finding that both C A and C B, are each equal to A B, we do naturally ſuggeſt to our ſelves the firſt Axiom, that things equal to the ſame are equal to one another, ſo that C A and C B, are equal to one another. And therefore all the three lines C A, A B, B C, are equal to one another, making an Equilateral Triangle, according to the 23d. Definition.

Thus in the Method of Compoſition from the third and firſt Poſtulates, from the fifteenth Definition, and firſt Axiom, we have fully ſet forth the Conſtruction and Demonſtration of an Equilateral Triangle on a given finite ſtrait line. And have together ſhown, upon what eaſy rational Grounds and Natural Suggeſtions this Problem may in this way alone be invented. So the like may be done in many other following Problems; for that the Invention of their Conſtructions and Demonſtrations, is not to be far fetched, depending only upon ſome few foregoing Propoſitions, which may at once be brought into memory, and fitly applyed to preſent Uſe.

But Problems more abſtruſe and intricate, tho' they may by a well exerciſed Geometrician be performed wholly in this Compoſitive Method, yet it is not the *readieſt*

readiest way to discover how a perplexed Problem may be extricated and effected.

In these Cases, instead of making our Gradations by *Composition* from the Principles, and other Propositions arising from them; Geometricians contrarywise order the matter after the Method of *Resolution*, and in the first place *do suppose the very Thing to be already done, which is propounded to be done.* FACTUM PUTANT QUOD FACIENDUM EST. And then they examine what can by just consequence be deduced from the *same Supposition*; demonstratively inferring one thing after another until they fall upon something, which evidently shews how to effect the Problem, or that it is impossible to be effected.

If by legitimate Argumentations we are brought to an impossibility; It is thereupon concluded in common reason, that the Problem is impossible to be effected; and that the *Thing supposed* is inconsistent with Nature. But if we meet with no such Obstacle, then are we assured the Problem is feasible, and that from the Supposition of the thing already effected, We may by necessary Inferences clear the way, and come to a certainty how to effect the same.

And therefore to give a glympse of Light into this *Admirable Method of Resolution*, take here an Example thereof in the Invention of this first Problem. Altho' by reason of its Simplicity it is readily found out in the former Method of *Composition*, as we have already explained.

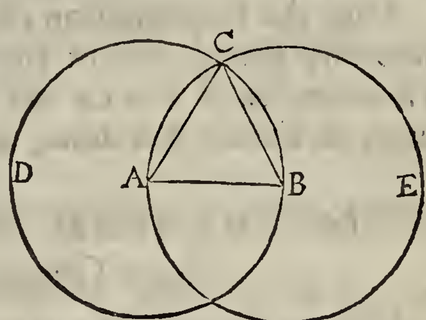
PROBLEM I.

On a given finite strait line to constitute an Equilateral Triangle.

Investigated by the Method of Resolution.

In all Problems we are first to consider wherein the STRESS of the Problem does consist. Therefore well pondering the nature of this Question, it will appear, that the main matter is to find a Point without the given line, from whence two strait lines being drawn to the ends of the given line, shall each be equal to the same.

Now to begin, *Puta factum.* Suppose this done: and the point found let it be C, and the Triangle ABC, be Equilateral, having the sides CA, CB, each equal to AB the given line.



Having thus supposed the Thing in Question: Now for the Solution of it, as of every Problem, there is to be used a dexterous Sagacity of Thought in searching out something *latent* in the Question, which is in some sort known unto Us, and from whence we may by degrees arguing from one thing to another, make in the end a perfect discovery of That, which is wholly unknown. For in humane reasoning we can attain to the Cognition of Things unknown, and under inquiry, only so far, as they partake of, and secretly contain within themselves the nature of other things already known, from which we must argue; or else one thing could never be deduced from an other.

And therefore this Præexistency of the knowledge of something in the very things unknown, and sought for, is the foundation of all our Ratiocinations, and in this Case thus leads Us on.

Because the line AB is not barely supposed, as are the lines CA, CB, but is actually given in Position, and Magnitude: therefore upon this Datum, with a respect likewise had to the Tenor of the Supposition it self, the Force and Perspicacity of our Mind is to be exercised, in bringing forth something relating both to what is actually Given, and to what is only Supposed, that may open a way toward the Invention of the Thing Required.

Now of all the several Subject Matters of Geometry, and of the Figures before defined, there is not as yet any of them in being besides a strait Line and a Circle: both which are postulated to have a being. Therefore from one, or both of these two we are to begin the Work.

From a given strait line nothing else can arise, but either the *Continuation* thereof by the *Second Postulate*; or by its *Circumlation* the Generation of a Circle according to the

the *Third Poſtulate.* It is at firſt view manifeſt, that the Continuation of a ſtrait line cannot ſerve to the framing of a Triangle. There is therefore nothing elſe exiſting but a Circle to help towards this matter.

Now we having had before an Idea of a Circle delivered to us, and ſuperadding the conception of a Circle to the given ſtrait line: Let us for an Eſſay try what may ariſe from them both, that is, From a ſtrait line given and a Circle ſuppoſed.

Suppoſe then by Poſtulate the third from the Center A, at the diſtance of A B given, a Circle to be deſcribed. It muſt now occur to our thoughts from the Notion of a Circle in Def. 15. that the circumference thereof ſhall paſs by the point C, the end of the ſuppoſed line A C, for that A C was ſuppoſed equal to the given Line A B. Let therefore be deſcribed the Circle B C D paſſing by the point C in the line A C.

Again, the Center B, and diſtance B A being likewiſe given, if we ſuppoſe a Circle to be deſcribed, the circumference thereof ſhall paſs by the point C, the end of the ſuppoſed line B C, for that B C was ſuppoſed equal to B A. Let therefore be deſcribed the Circle A C E paſſing by the point C in the line B C.

And becauſe the ſame point C is common to the ſuppoſed lines A C, B C, and is moreover in the circumference of the Circle B C D, and alſo in the circumference of the Circle A C E; and that the Circles B C D, A C E, have nothing common but their Interſection, therefore the point C is in the Interſection of the Circles B C D, A C E, now deſcribed.

Wherefore the point C is found: And *thereby* the Equilateral Triangle is found.

For the three Points A, B, C, being now known, and in Poſition given; the three ſides of the Equilateral Triangle are alſo given *by the firſt Poſtulate*, From any point to any point to draw a ſtrait line. Let therefore from the point C, thus found, be drawn the ſtrait lines C A, C B (lines only before ſuppoſed) which lines by Ax. 1. are equal to one another, becauſe each of them is equal to A B, by Def. 15. And ſo all the three ſtrait lines are equal to one another, C A, A B, B C.

Thus the Conſtruction of an Equilateral Triangle on a given finite ſtrait line, is naturally found out in the Method of Reſolution. From whence does ariſe a Theorem, as a Rule of Practice, how demonſtratively to conſtruct the ſame, as *Euclide* has done, in the Method of Compoſition, after this manner.

The Theorem deduced from this Reſolution, and teaching how Geometrically to effect the Problem.

If from the ends of a given line be deſcribed two Circles, at the diſtance of the given line, and from the point of their Interſection be drawn two ſtrait lines to the ends of the given line, there ſhall be conſtituted an Equilateral Triangle on that given line.

In the progreſs of this Problem, tho' it be very eaſy, and its Invention obvious, ſo that it might have been carryed on in ſhort by continued Inferences without any Comment, or interwoven Obſervations: Yet here as we paſs from one thing to an another by ſeveral Gradations, we have thought fit to intermix ſome Advertiſements on purpoſe to direct, and ſet forth, after what manner, and with what circumſpection Problems more abſtruſe and difficult, ought to be managed in the Method of Reſolution: Encompaſſing in our thoughts all things poſſible to be comprehended within the nature of the Queſtion, by which Sagacious ſearch moſt wonderful and occult Truths may out of that Obſcurity, wherein they lye involved, be brought to Light, and as it were hammered out.

Ut Silicis venis abſtruſum excudimus ignem.

PROPOSITION II.

At a Given Point to put a ſtrait line equal to a ſtrait Line Given.

Let the given Point be A, and the given ſtrait line be B C.

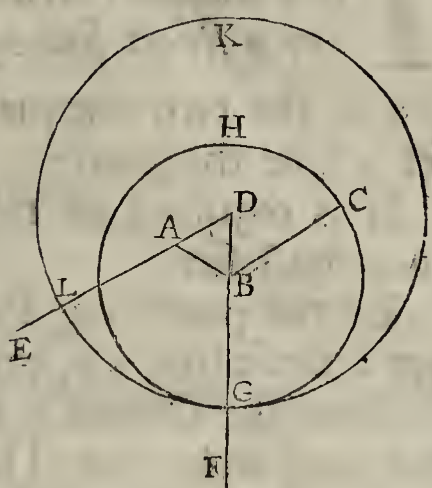
It

It is required at the point A, to put a ſtrait line equal to the ſtrait line BC.

From the point A, to the point B, let be drawn the ſtrait line AB. [Poſt. 1.]

And on the line AB let be conſtituted the Equilateral Triangle DAB. [by Prop. 1.]

And to DA, DB, let be continued directly the ſtrait lines AE, BF, [by Poſt. 2.] Then the Center B, and diſtance BC let be deſcribed the Circle CGH, [Poſt. 3.] And again the Center D, and diſtance DH let be deſcribed the Circle GLK. [Poſt. 3.]



I ſay that at the given point A, is put a ſtrait line AL equal to the given ſtrait line BC.

Foraſmuch as the point B is the Center of the Circle CGH, therefore BC is equal to BG, [Def. 15.] Again, becauſe the point D is the Center of the Circle GLK, therefore DL is equal to DG. [Def. 15.] Of which DA is equal to DB, [by *Conſtr.*] wherefore the remainder AL is equal to the remainder BG. [Ax. 3.] But it has been proved that BC is equal to BG. Therefore each of the lines AL, BC, is equal to BG.

But things equal to one and the ſame thing, are equal to one another. [Ax. 1.] And therefore AL is equal to BC.

Wherefore at the given point A is put a ſtrait line AL, equal to the given ſtrait line BC. Which was to be done.

ANNOTATIONS.

In this Problem there are three Caſes, according to the various ſituation of the given point A, in reſpect to the given line B C.

For the point A is either without the line B C, as in the Figure uſed in the demonſtration.

Or it is within the given line, or elſe at one end of it. In which laſt Caſe a Circle only deſcribed from the ſame end as the Center, and to the diſtance of the given line, effects the Problem without the conſtruction of an Equilateral Triangle.

The Practice.

The practice is obvious. For opening the Compaſſes to the length of the given line B C, and then placing one foot on the given point A, ſet forth the ſame length with the other foot to L, and draw A L. This is the ſenſible and Mechanical Operation.

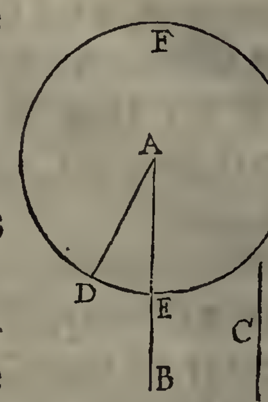
But we are again to be reminded upon this occaſion, that in the pure Geometrical Solution of Problems, no uſe is to be made of Ruler and Compaſſes, or of any outward Senſe. And moreover, that whatſoever things are given in a Problem ought to remain in the poſition given. And according to that ſtated poſition of the *Data*, and the Tenor of the Problem, every thing is to be tranſacted in the Mind, as if we neither uſed our Hands, nor Eyes. And therefore by ſome of the Ancient Geometricians Problems are alſo called Theorems, for that their Operations are only Speculative and Intellectual, in a ſubject wholly abſtracted from Matter. Yet we are to know, that they are the ſure and demonſtrative Grounds of Material and Manual Practices, in the Menſuration of all kinds of Magnitude in Architecture, Fortification, Navigation; In all ſorts of Mechaniſm, infinitely uſeful to Mankind, both for Neceſſity and Curioſity.

PROPOSITION III.

TWo unequal ſtrait lines being Given, to take from the Greater a ſtrait line equal to the Leſser.

Let the two unequal ſtrait lines given be AB, and C; of which let AB be the greater.

It is required to take from AB the greater, a ſtrait line equal to C the leſſer.



By the preceding Problem, let at the point A be put a ſtrait line AD, equal to the ſtrait line C.

Then the Center A, and diſtance AD, let be deſcribed the Circle DEF. [by Poſt. 3.]

I ſay, that from the ſtrait line AB the Greater, is taken a ſtrait line AE equal to C the Leſſer.

Foraſmuch as the point A is the Center of the Circle DEF, therefore the line AE is equal to the line AD. But the line C is equal to the line AD: ſo that alſo AE is equal to C. [Ax. I.]

Therefore two unequal ſtrait lines being given, AB and C, there is taken from AB the greater, a line equal to C the leſſer. Which was to be done.

The Practice.

The practice of this is as before: Opening the Compaſſes to the length of the given line C, and ſetting the ſame off from the point A to E.

Theſe three Propoſitions are only miniſterial Problems, and therefore here premiſed for their general uſe through all Geometry.

PROPOSITION IV.

IF two Triangles have two ſides equal to two ſides, each to each, and have an angle equal to an angle, namely that, which is contained by the equal lines.

Then ſhall they have the baſe equal to the baſe, and Triangle ſhall be equal to Triangle, and the remaining angles ſhall be equal to the remaining angles, each to each, under which are ſubtended equal ſides.

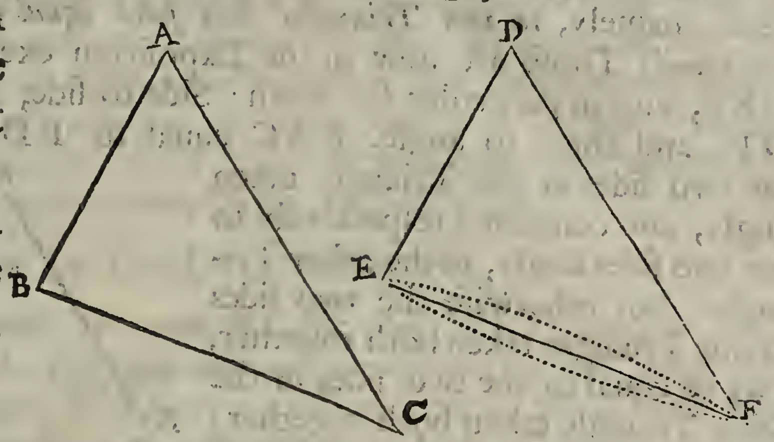
Expoſition. Let the two Triangles be ABC, DEF, having the two ſides AB, AC, equal to the two ſides DE, DF, each to each, namely AB to DE, and AC to DF, and the angle BAC equal to the angle EDF.

Determination. I ſay, that the baſe BC, is equal to the baſe EF. And the Triangle ABC ſhall be equal to the Triangle DEF, and the remaining angles ſhall be equal to the remaining angles, each to each, under which are ſubtended the equal ſides, namely the angle

ABC

ABC to the angle DEF, and the angle ACB to the angle DFE.

Construction. For the Triangle ABC, being applyed to the Triangle DEF: and the point A put on the point D; and the strait line AB on the strait line DE;



Demonstration. Then shall the point B agree with the point E; for that AB is equal to DE. [by *Supposition.*]

Now AB agreeing with DE, AC shall also agree with DF, for that [by *Supposition*] the angle BAC, is equal to the angle EDF.

So that also the point C shall agree with the point F; for that the line AC is likewise equal to DF, [by *Supposition*]

But now also the point B had agreed with the point E; so that the base BC shall agree with the base EF.

For the point B agreeing with the point E, and the point C with the point F, if the base BC shall not agree with the base EF; Then two strait lines shall comprehend a space. Which is impossible, [by Ax. 12.]

The particular Conclusion. Wherefore the base BC shall agree with the base EF; and therefore shall be equal to it, [by Ax. 8.]

So that also the whole Triangle ABC, shall agree with the whole Triangle DEF, and therefore shall be equal to it, [by Ax. 8.]

And the remaining angles shall agree with the remaining angles, and therefore shall be equal to them, namely the angle ABC to the angle DEF, and the angle ACB to the angle DFE.

The general Conclusion. If therefore two Triangles have two sides equal to two sides, each to each, and have an angle equal to an angle; namely, That which is contained by the equal lines: Then shall they have the base equal to the base, and Triangle shall be equal to Triangle, and the remaining angles shall be equal to the remaining angles, each to each, under which are subtended equal sides. Which was to be Demonstrated.

Annotation on the Proposition.

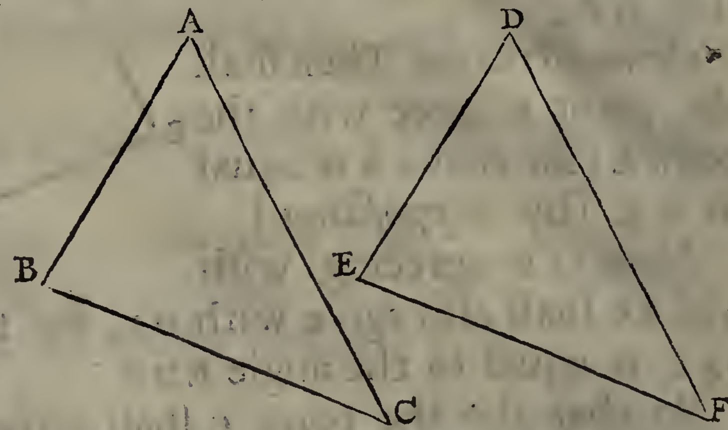
This Proposition is the first Theorem, and Foundation of all Geometry, which therefore with the whole manner of its demonstration, we shall specially endeavour to explain.

In a Triangle there are *Seven Things* to be considered.

The three Sides, the three Angles, and the Area, or Space comprehended by the sides. Of these Seven here are three Things in one Triangle, namely, *two Sides, and the angle contained,* which are supposed equal to three the like Things in the other Triangle. And from the supposed equality of these Three, is demonstrated the equality of all the other Four, that is, of the third side to the third side, of Area to Area, and of the two other angles to the two other angles, each to its correspondent angle.

Annotation on the Expoſition.

What Things were in the firſt part of the Propoſition *Given*, or *Suppoſed* in general (namely, *in two Triangles two ſides equal to two ſides, and the contained angles equal*) Thoſe are now in the Expoſition exemplyfied in the Triangles ABC, DEF, and in particular ſet forth: Side to ſide, AB equal to DE, AC equal to DF, and angle to angle, BAC equal to EDF (Each to each &c.) That is, the two ſides in one Triangle taken ſingly, and compared reſpectively to the two ſides ſingly, in the other Triangle. For otherwiſe the two ſides in one Triangle taken both together, may be equal to the two ſides in the other Triangle taken both together; that is, the ſumm of the two ſides in one Triangle may be equal to the ſumm of the two ſides in the other Triangle, and the angles contained be equal to one another: Yet from thence the other parts in thoſe Triangles cannot be proved equal to one another. As in one of the Triangles, if one ſide be 2, and the other be 5, which together are 7: And in the other Triangle if one ſide be 3, and the other be 4, making alſo together 7: And let them contain equal angles, yet the Triangles ſhall not in the other parts be equal to one another. Therefore it is here ſpecially ſaid ***Each to Each***, ***ſingula latera ſingulis lateribus***, ἑκατέραν ἑκατέρᾳ.



Annotation on the Determination.

As the *Expoſition* did ſet forth particularly, in the Triangles ABC, DEF, what was in the firſt part of the Propoſition ſuppoſed, and given in general Terms: ſo what was in the ſecond part of the Propoſition laid down in general, as to be demonſtrated from what was ſuppoſed, and given in general, That is now in the *Determination* ſpecified in the particular parts of the ſame Triangles ABC, DEF: and pronounced to be true. Saying, The baſe BC is equal to the baſe EF, and the Triangle ABC &c. Now the truth thereof is to be made good by the following Demonſtration.

Annotation on the Conſtruction.

After the Expoſition and Determination, there often follows in Theorems a *Conſtruction*, wherein are added to the ſimple Figure of the Propoſition ſome ſtrait lines, or Circles, or both, to make way for the Demonſtration. This Propoſition requires no ſuch kind of Conſtruction, having all lines requiſite to the demonſtration, laid down at firſt in the Expoſition and Determination. But here *Euclide* uſes another ſort of Conſtruction, or *Apparatus* towards his Demonſtration, by a mental Application of one Figure to an Other: which is by the Greeks called ἐφάρμοσις, *Epharmoſis*; an Adaptation, Appoſition, or Application of one magnitude to another: And in the manner following is thus made uſe of.

For the Triangle ABC *being applyed to the Triangle* DEF; *and the point* A *put on the point* D.] It is not meant, that this Application be made by the uſe of Hands and Eyes: but by an imaginary poſition, firſt of the point A on the point D; and then of the line AB on the line DE, as followeth.

And the ſtrait line AB *on the ſtrait line* DE.] In the 4th. Def. *Euclide* tells us, that by a ſtrait line we are to conceive ſuch a line as lyes evenly to all its points. So that upon this conception of a ſtrait line, if we imagine one ſtrait line to be placed upon another ſtrait line, we muſt alſo conceive, that no part of the one does any where ſwerve, or any ways deviate from the other; And as agreeable to this conception of a ſtrait line, we may obſerve the exactneſs of *Euclid's* expreſſion,

ſion, how that the *lines* AB, DE, which before in the Expoſition of the Propoſition were called *Sides*, in relation to a Triangle, are now in reſpect of their Application to one another, called *ſtrait Lines*, *not Sides*, becauſe from the notion of a *ſtrait line* it follows, that if ſtrait lines be apply'd to one another, They muſt be imagined to be entirely coincident with one another. Thus far proceeds the Conſtruction by way of Application, or *Epharmoſis*. That is, *firſt the imaginary poſition of the point* A *on the point* D. *And ſecondly of the ſtrait line* AB, *on the ſtrait line* DE. From whence by a juſt Ratiocination *Euclide* demonſtrates the equality of the two Triangles, in the Whole, and in every Part. As followeth.

Annotation on the Demonſtration.

Then ſhall the point B *agree with the point* E, *for that* AB *is equal to* DE.]

Here is firſt to be obſerved *the difference between the agreeing of two ſtrait lines, and the agreeing of their extream points.* The Former is in reſpect of being ſtrait lines, the Other in reſpect of their equality to one another. For all ſtrait lines are conceived to have their parts Congruous: And all equal ſtrait lines to have alſo their extreams Congruous. And therefore *Euclide* upon the Application of the ſtrait lines AB, DE, to one another, having preſumed from the notion of Rectitude the mutual agreement of their intermedial parts; He next urges the agreement of their extream points from their ſuppoſed equality. For the extream point A of the line AB, being conceived to lye on the extream point D of the line DE, and the line AB on the line DE; if of the line AB the other extream point B, does not agree with the other extream point E of the line DE: then one of theſe points falls ſhort of the other: ſo that one of the lines ſhall be a part of the other line. And becauſe the lines AB, DE, are ſuppoſed equal, therefore the part ſhall be equal to the whole. Which is impoſſible by the 9th. Axiom. Therefore the point B ſhall agree with the point E.

Now AB *agreeing with* DE, *alſo* AC *ſhall agree* with DF; *for that the angle* BAC *is equal to the angle* EDF.]

For otherwiſe, if AC agrees not with DF; then ſhall AC fall either within, or without DF: ſo that one of the angles BAC, EDF, ſhall be a part of the other. And becauſe the angles BAC, EDF, are ſuppoſed equal; therefore the part ſhall be equal to the whole. Which is impoſſible. Therefore the line AC ſhall agree with the line DF.

So that alſo the point C *ſhall agree with the point* F.]

For the ſame reaſon as before, that the point B did agree with the point E, AC being ſuppoſed equal to DF, as AB was to DE. Here again is to be obſerved, how *Euclide* diſtinguiſhes between the agreement of the lines AC, DF, and the agreement of the extream points C and F. For from the equality of the angles BAC, EDF, he proves the coincidency of the lines AC, DF: And from the equality of the lines AC, DF, he proves the coincidency of the extream points C, F.

For the point B *agreeing with the point* E, *and the point* C *with the point* F, *if the baſe* BC *ſhall not agree with the baſe* EF.]

Then muſt BC fall either within, or without EF, ſo that the ſtrait lines BC, EF, ſhall comprehend a ſpace. Which by Ax. 10. is impoſſible. Therefore BC ſhall agree with EF.

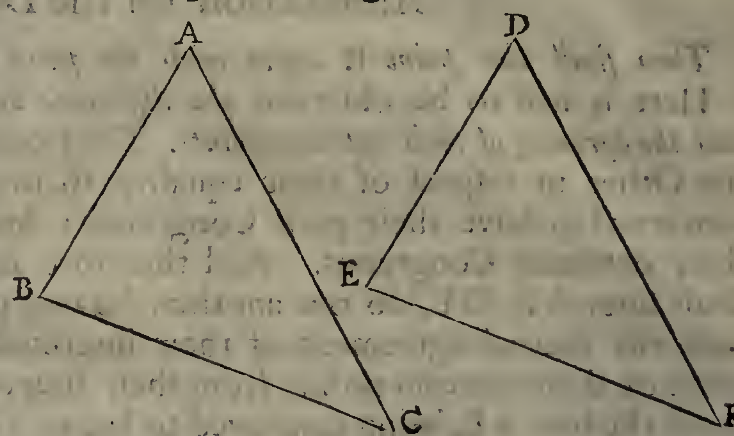
The ſubtil changes made in the courſe of this demonſtration are remarkable. Firſt, from the Definition of a ſtrait is implyed the coincidency of the intermedial parts of two ſtrait lines, which are applyed to one another, namely of AB and DE. Secondly, from the ſuppoſed equality of two ſtrait lines, is proved the agreement of their extream points. Otherwiſe the part would be equal to the whole. Thirdly, from the ſuppoſed equality of two angles, is proved the coincidency of their containing lines. Otherwiſe the part again would be equal to the whole. And laſtly, *from the agreement of the two extream points* B *with* E, C *with* F, *of two ſtrait lines* BC, EF, is concluded *the agreement of the lines Themſelves*, namely of BC and EF. Otherwiſe two ſtrait lines would comprehend a Space.

Annotation on the Conclufion.

We are here to obferve a double Conclufion. The firft particular, The other general. The *particular Conclufion* is the *foregoing Determination* repeated word for word, but fo as being now demonftrated; and therefore concluded true in thofe particular Triangles ABC, DEF. The *general Conclufion* is the *Propofition it felf* repeated, and concluded with, as juftly following upon that particular Conclufion already demonftrated.

For according to the Mathematical method, as before in Problems, fo here in Theorems, is in the firft place laid down the Propofition in general terms.

Then next, of this Propofition there is made an *Expofition*, and a *Determination* in a particular Inftance, or Example. An Expofition of what is given: and a Determination or Specification of what is to be proved. As here in the two Triangles ABC, DEF, is fet forth particularly by name, *every Thing given*, and *every Thing propofed to be demonftrated*. As we have diftinguifhed them in the feveral *Paragraphs*, of *Expofition*, and *Determination*.



Now upon this Inftance in the Triangles ABC, DEF, the demonftration proceeded: and therefore *Euclide* firft concludes particularly, as to that Inftance in the faid Triangles ABC, DEF. And thereupon implying the like reafon in the like matter, He concludes by the Logical Rule of Induction, with the Propofition it felf in general, as at firft it was laid down. For what is now particularly demonftrated in the Triangles ABC, DEF, the fame may be proved after the fame manner in any other two Triangles, which have the fame conditions. And therefore the Propofition is in general concluded to be true.

This is the regular form of a Geometrical demonftration in all Theorems; which we have here explained at large, once for all.

Advertifement.

Thus with more Artifice than is commonly taken notice of, *Euclide* manages this Demonftration. And indeed the nearer any Propofition comes to a Principle, or an evident Truth, as this Propofition doth, the more difficult it is to be demonftrated; becaufe the *Mediums* for the proof of fuch Propofitions, which are fo near to Principles, can be but few, and thofe *Mediums* not much more manifeft than the Propofition it felf. Therefore in thefe kind of Propofitions the manner and management of the Demonftration is more exquifite, and requires an extraordinary nicety on the Mafters part clearly to demonftrate, and a greater attention in the Scholar, for the right underftanding of the fubtilty of fuch a demonftration.

Wherefore, fome for want of due confideration have unjuftly cavill'd at the demonftration of this 4th.Propofition, as being in a manner Mechanical. Whereas they have not rightly confidered, what in the Application of magnitude to magnitude is Geometrical, or purely Mathematical, and what is Mechanical. To apply any Meafure, as a Foot, Cubit, &c. to any other magnitude, as Carpenters, and fuch like Artificers do, or in general to adapt one magnitude to another, and then from their *vifible and fenfible Congruency* or *Incongruency*, to conclude their equality or inequality, This indeed is plainly Mechanical. But in the demonftration of this Propofition there is nothing of this kind, no ufe made either of Hands or Eyes. Only there is of two Triangles compared together an imaginary pofition of a point on a point, and a ftrait line on a ftrait line: and from this mental Application the demonftration takes its beginning, and by clear Ratiocination from one neceffary Inference to another, proves the entire Congruency of the two Triangles, equally convincing

Cæcos,

Cæcos, and *Oculatos*: for that here the Congruency is not enforced by any evidence of Senſe; but only by an Intellectual demonſtration. And the *Epharmoſis*, or Application of one Triangle to the other, is alſo intellectual, and neceſſarily to be preſuppoſed, in order to the proof of their *Exact* Congruency. Now if ſuch an *Epharmoſis* could juſtly be excepted againſt, then the 8th. Axiom of the equality of Congruous magnitudes, which even *Ramus* himſelf allows to be *maxime Geometricum*, were altogether uſeleſs in Geometry. For that there muſt in ſome manner an Application be conceived of ſome part of one magnitude, to ſome part of another magnitude, in the way of a Conſtruction or *Apparatus* to the demonſtration, before we can proceed to demonſtrate the Congruency of all its parts, and from *Thence* to conclude the equality of the whole to the whole, by the 8th. Axiom.

Briefly then to determine in this Caſe.

Congruency is a natural Rule of equality; and the certainty of Congruency, if made from Senſe, is Mechanical; if by a juſt Reaſoning proved, it is truly Mathematical. Which diſtinction, if ſome of our Commentators had well obſerv'd, they would not in this matter have made ſuch frivolous Objections againſt *Euclide*, nor committed ſuch Paralogiſms in their vain Attempts to amend his demonſtrations of this kind.

PROPOSITION V.

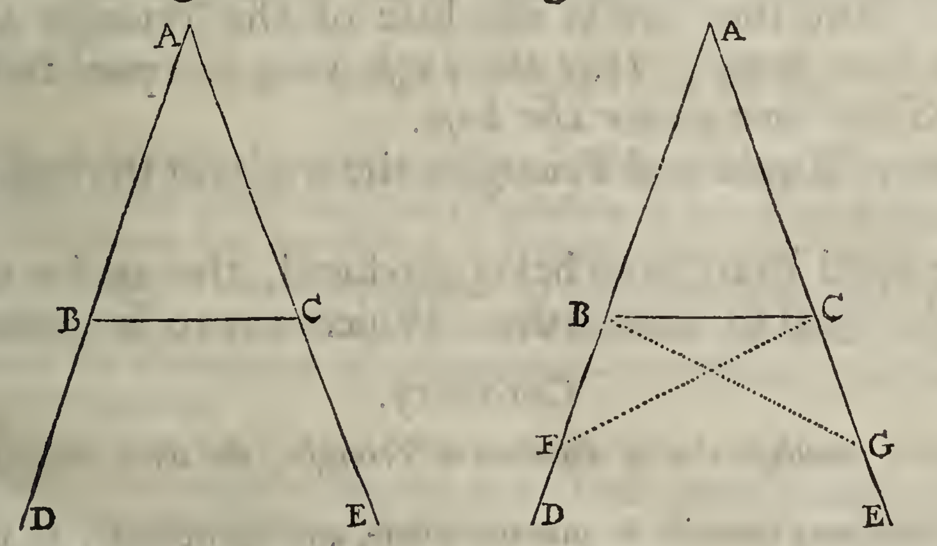
OF Equicrural Triangles, the angles at the baſe are equal to one another.

And the equal ſtrait lines being produced, the angles under the baſe ſhall be equal to one another.

Let there be an Equicrural Triangle ABC, having the ſide AB equal to the ſide AC; and to the ſtrait lines AB, AC, let be continued directly the ſtrait lines BD, CE, [by Poſt. 2.]

I ſay, that the angle ABC is equal to the angle ACB.

And alſo the angle CBD, to the angle BCE.



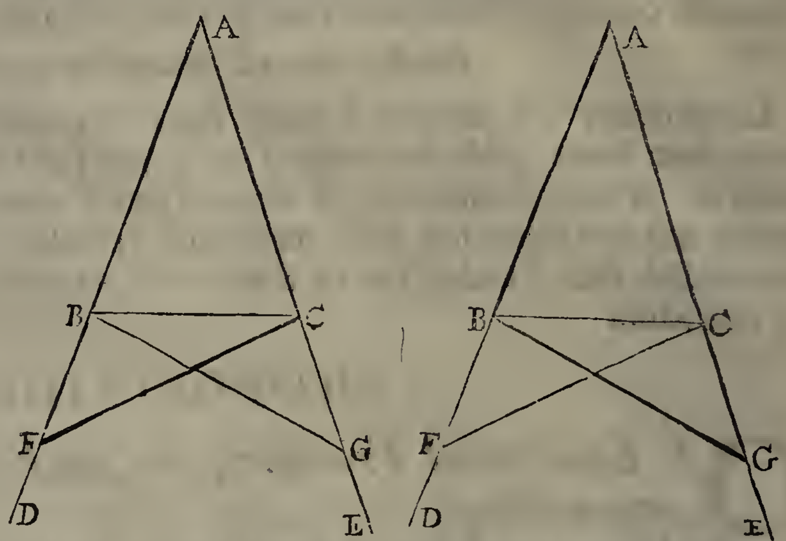
For in the line BD, let any point be taken, as F. Then from the greater AE, let be taken AG equal to AF the leſs, [by Prop. 3.] and let be joyned the ſtrait lines FC, GB, [by Poſt. 1.]

Now foraſmuch as AF is equal to AG [by *Conſtruction*] and AB to AC; [by *Suppoſition*]. Therefore there are two lines FA, AC, equal to the two lines GA, AB, each to each, and they contain a common angle FAG. Therefore [by Prop. 4.] the baſe FC is equal to the baſe GB; and the Triangle AFC ſhall be equal to the Triangle

AGB,

AGB, and the remaining angles ſhall be equal to the remaining angles, each to each, under which are ſubtended equal ſides, namely the angle ACF to the angle ABG, and the angle AFC to the angle AGB.

And now becauſe the whole line AF is [by *Conſtruction*] equal to the whole line AG, of which AB is equal to AC, [by *Suppoſition*] Therefore the remainder BF is equal to the remainder CG, [Ax. 3.] But alſo it has been proved, that FC is equal to GB.



There are therefore the two lines BF, FC, equal to the two lines CG, GB, each to each, and the angle BFC equal to the angle CGB, and BC is their common baſe; therefore the triangle BFC, ſhall be equal to the triangle CGB; and the remaining angles ſhall be equal to the remaining angles, each to each, under which are ſubtended equal ſides. Wherefore the angle FBC is equal to the angle GCB. And the angle BCF to the angle CBG.

Now whereas the whole angle ABG has been proved equal to to the whole angle ACF, of which the angle CBG is equal to the angle BCF.

Therefore the remaining angle ABC is equal to the remaining angle ACB. And they are at the baſe of the Triangle ABC. *But alſo it has been proved, that the angle* FBC, *is equal to the angle* GCB. *And they are under the baſe.*

Therefore of Equicrural Triangles, the angles at the baſe are equal to one another.

And the equal ſtrait lines being produced, the angles under the baſe ſhall be equal to one another. Which was to be demonſtrated.

Corollary.

From hence 'tis manifeſt, that of Equilateral Triangles, the three angles are equal to one another.

For every ſide may in order be put for a baſe, and accordingly, as in an Equicrural Triangle, the three angles may be proved equal to one another.

This Propoſition ſeems very difficult to young Geometricians, by reaſon of the croſs interfering of the Triangles, which are compared to one another. We have therefore endeavoured to clear the matter by making ſeparate Figures, in which the ſeveral Triangles compared together, may be more eaſily diſtinguiſhed.

PROPOSITION VI.

IF two Angles of a Triangle be equal to one another, then ſhall the ſides ſubtended under the equal Angles be equal to one another.

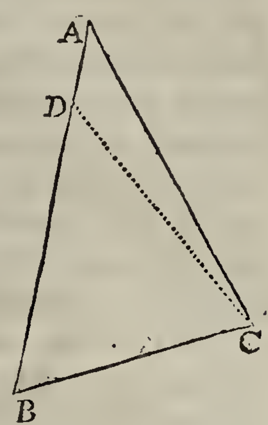
Let

Let the Triangle be ABC, having the angle ABC equal to the angle ACB.

I ſay, that the ſide AC is equal to the ſide AB.

For if AC be unequal to AB, one of them is the greater. Let the greater be AB.

And from AB the greater, let be taken DB equal to AC the leſs, [by Prop. 3.] and let be joyned DC.



Now foraſmuch as DB is equal AC, and BC is common; therefore there are the two lines DB, BC, equal to the two lines AC, CB, each to each, and the angle DBC is equal to the angle ACB, [by Suppoſition] Therefore [by Prop. 4.] the baſe DC is equal to the baſe AB; and the Triangle ABC ſhall be equal to the Triangle DCB, the greater to the leſs. Which is abſurd.

Wherefore AB is not unequal to AC. Equal therefore it is.

If therefore two Angles of a Triangle be equal to one another, then ſhall the ſides ſubtended under the equal angles be equal to one another. Which was to be Demonſtrated.

Corollary.

From hence 'tis manifeſt, that if the three Angles of a Triangle be equal to one another, the Triangle ſhall be Equilateral.

ANNOTATIONS.

Of the Converſion of Geometrical Propoſitions.

This Sixth Propoſition is the Converſe of the Fifth.

One Propoſition is ſaid to be the Converſe of another, when the Poſition of one is the Concluſion of the other; and the Concluſion of this is the Poſition of that. As the fifth Propoſition *puts two ſides of a Triangle equal to one another, and thence concludes, that their oppoſite Angles are equal.* So this ſixth Propoſition *puts two Angles of a Triangle equal to one another, and thence concludes that their oppoſite ſides are equal.* Thus in the Converſion of Geometrical Propoſitions, the Poſitions and Concluſions are reciprocal.

The fifth Propoſition might have been here entirely converted, I mean in that latter part alſo of the Angles under the Baſe: from whoſe equality the Triangle may likewiſe be demonſtrated to be Equicrural, as *Proclus* has ſhewn. But this Converſion being uſeleſs, *Euclide* does omit. Or rather, if the Form of this demonſtration be ſtrictly examined, it will appear that the ſecond part of the fifth Propoſition, was not *Euclid's*. But that becauſe *Euclide*, to prove the equality of the Angles at the Baſe, doth firſt prove the equality of the Angles under the Baſe; This might give an occaſion to ſome one afterward, for the ſubjoining of this property of an Equicrural Triangle to *Euclid's* Original Propoſition. And this conjecture is very probable, for that the equality of the Angles under the *Baſe*, is not made uſe of in any of the following Propoſitions. Now certainly *Euclide* never laid down an Elementary Propoſition uſeleſs in any part thereof: nor ever put that for one part of the Propoſition, which is only uſed as a Medium to prove the other.

Of the two kinds of *Analyſis* uſed in Geometrical Propoſitions.

The demonſtration of this ſixth Propoſition is much different from thoſe before. All which were made from true and known Principles, to prove other Truths unknown in the common Synthetical Method, but contrarywiſe this Theorem is wholly demonſtrated Analytically.

We have before diſcourſed of the two Methods of Compoſition and Reſolution; and ſhewn the uſe of the Analytical Method in the Invention and Solution of *Euclid's* firſt Problem: Now here again, we have occaſion to take more particular notice of a twofold conſideration of Analyſis, not in reſpect of its Nature and Method in the courſe of Reaſoning, which remains the ſame in both; but of the different iſſue or event, that may ariſe from any Analyſis by a juſt Ratiocination. For if the Analyſis of a Suppoſition ends in an acknowledg'd Truth, then the thing ſuppoſed was true. But if the Analyſis of a Suppoſition ends in a certain Untruth, then the thing ſuppoſed was falſe. Therefore from theſe different endings, True or Falſe, there are given two denominations to *Analyſis*. One is ſaid to be *Conſtructive*, the other *Deſtructive*.

The Conſtructive Analyſis is ſo called, for that ending in a manifeſt Truth, we may again upon this true foundation in the Synthetical Method, conſtruct a demonſtration of *that Thing*, from whoſe bare *Suppoſition* the Analyſis took its beginning. And in theſe Caſes alone the Methods of Reſolution and Compoſition anſwer to one another, the Conſtructive Analyſis having always a correſpondent Syntheſis: So that what Truths we find out by the help of the Analytical Method, the ſame when ſo found we teach to others in the Sythetical Method. This *Marinus Gethaldus* has ſhewn in his admirable Books *De Reſolutione & Compoſitione Mathematica*. And Examples thereof we may find among the *Scholia* in the 13th. Element of *Euclide*.

The Deſtructive Analyſis is ſo called, for that ending in an evident Falſity, it deſtroys the Suppoſition, from whence the Analyſis began to argue.

Now unto the Deſtructive Analyſis there cannot be made any return in the Synthetical Method, becauſe upon Falſity, wherein this Analyſis ends, that is, upon Nothing, or no Foundation, cannot be raiſed any Structure.

Syntheſis therefore is but of one ſimple conſideration and denomination, for that it can only argue from Principles of Truth laid down. As we muſt build upon ſome foundation, as well Speculatively, as Mechanically.

But Analyſis, which may argue demonſtratively from Suppoſitions either True or Falſe, and ſo accordingly muſt reſt at Principles of Truth or Falſity, is therefore from its different ending diſtinguiſhed, and (as before ſaid) denominated Conſtructive and Deſtructive. *Quia* SUPPOSITIONEM *ponit Analyſis, aut deſtruit.* Not as CLAVIUS ſays, *Quia* PRINCIPIA *ponit, aut deſtruit.* For from what Suppoſition can the Principles be either confirmed or deſtroyed: But contrarily, theſe Principles muſt either confirm, or deſtroy the Suppoſition.

The Conſtructive Analyſis is ſimply called Analyſis. The Deſtructive Analyſis is commonly called ἀπαγωγὴ εἰς ἀδύνατον, *an Abduction or Reduction to Impoſſibility.* Which way of Argumentation Geometricians thus make uſe of.

If a Propoſition, altho' it be true, yet cannot be readily proved ſo to be: then They uſually put, or *Suppoſe* its *Contradictory*, and diſproving *This*, there neceſſarily follows the acknowledgement of the Propoſition firſt laid down. For that of two Contradictory Propoſitions, if one be proved falſe, the other muſt be true, their being no Medium between Being and not Being.

So in this ſixth Propoſition are put *two Angles of a Triangle, equal to one another*: and upon this Poſition it is pronounced, that *Their ſubtending ſides are alſo equal.* But now the equality of their ſubtending ſides is not here directly proved. Only 'tis urged, that if the ſides be not equal, then they muſt be unequal.

And now *ſuppoſing* the ſides unequal, then *Euclide* demonſtratively proceeds thereupon, and by gradual Conſequences reſolves *this Suppoſition* into a manifeſt impoſſibility, proving, That *in a Triangle, if the ſides ſubtending the equal Angles be unequal, then*

then the whole ſhall be equal to its part: which is a manifeſt untruth; for that *The whole is greater than its part*, is a Principle of manifeſt Truth.

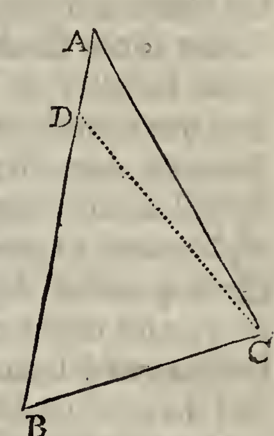
Therefore the *Suppoſition of the inequality of the two ſides which ſubtend the two equal Angles of a Triangle*, being neceſſarily reſolved into ſuch an abſurdity, was not true. For *Ex vero nil niſi verum*, From Truth no Untruth can follow.

Wherefore according to the ſixth Propoſition, Two Angles of a Triangle being equal, the ſides ſubtended under the equal Angles, are therefore Logically concluded equal, for that it is demonſtrated, that they cannot poſſibly be unequal. Equal therefore they are.

But in this Deſtructive Analyſis, very often the Deduction to an impoſſibility is *Immediate*, and the abſurdity forthwith ſeen, and urged againſt the falſe poſition without any further Argumentation. As before we may obſerve in the Cloſe of the 4th. Prop. When we ſay, *If the baſe* B C *does not agree with the baſe* D F, *then two ſtrait lines ſhall comprehend a ſpace*; which Inference is immediate and contradicts the 10th. Axiom. Whereupon the abſurdity is preſently diſcovered; and therein that Suppoſition is immediately overthrown as untrue. And this is ſtrictly to be called Deduction to Impoſſibility.

But now again, if the Propoſition it ſelf be not in the beginning of the demonſtration, undertaken to be proved oſtenſively, in ſome parts thereof, as the fourth Propoſition is: but that in its ſtead, an Hypotheſis contradictory to the Propoſition, is put at the very firſt (as it is in this and the next Propoſition, with many others) then the conſequent abſurdity does not immediately appear: but after intermediate Inferences which are made upon that Hypotheſis *Oſtenſively*, ſtep by ſtep, we do fall at length into an Impoſſibility, and thereupon deſtroy the ſame Hypotheſis, from whence we began to argue. And this is an *exact* Analyſis: which to explain more ſignally, let us review the Analyſis of this ſixth Propoſition, and particularly obſerve the ſeveral gradations of Argumentation, made upon the Suppoſition.

For now of the Triangle A B C if the ſides A B, A C, be not allowed to be equal, notwithſtanding that their oppoſite Angles at C and B were ſuppoſed to be equal: Then ſuppoſing them to be unequal, *Euclide* (by the 3d. Propoſition) takes from the greater A B a line B D equal to the leſs A C. Next (by the firſt Poſtulate) he draws a ſtrait line D C; and ſo there is made the Triangle D C B. After this Conſtruction he proves (by the 4th. Propoſition) that the Triangles A B C, D C B, are equal to one another. Hitherto the demonſtration goes from the Suppoſition fairly on δεικτικῶς, *Oſtenſively*, proving the equality of the Triangles A B C, D C B.



But here now after this Oſtenſive Proof, the next and immediate Reſult is, that one of theſe Triangles is but a part of the other; the Triangle D C B a part of the Triangle A B C, and yet upon the Suppoſition is juſtly demonſtrated equal to the ſame. But this Equality and Inequality of the ſame things being inconſiſtent with Nature, muſt overthrow and null that Hypotheſis, which by a poſitive demonſtration, is reſolved into this Impoſſibility; And therefore upon ſuch an Evidence certain it is, that of the Triangle A B C, the ſides A B, A C, cannot be unequal, and therefore equal.

But becauſe there are ſome who allow not this Apagogical method of Argumentation to be ſo ſatisfactory, as the uſual Synthetical manner of demonſtration; therefore to clear this matter, we ſhall for the vindication of Geometricians, examine further the nature and force of all Demonſtrations Mathematical and Philoſophical: As we have done before concerning Definitions Mathematical and Philoſophical.

Of Demonſtrations Mathematical and Philoſophical, as to Their ſeveral ends.

Every Man can reaſon no otherwiſe, than from the Ideas of outward things, according as he receives them, and makes reflections on them. When the Ideas are the ſame in ſeveral perſons, then they cannot but reaſonably agree in their judgements: when they are different, as being differently received, then Men muſt diſagree; tho' both may argue well, according to their own Ideas and Sentiments of thoſe things. Beſides, for want of due Obſervations, both may happen to be in the wrong, and that the very Truth is much otherwiſe. For we can only affirm, or deny, as things are apprehended by us: whether the things do in themſelves agree, or not agree, one with the other.

Now many things there are which ſo clearly and diſtinctly preſent themſelves to our common underſtanding, that they beget in us a ready and certain judgement of their Truth, or Falſity: and do ſo much urge our Natural reaſon upon their Self-evidence, that without any doubting we affirm, or deny. When with our Affirmation, or Negation, there is accordingly a real agreement, or diſagreement between the things themſelves, then is this Perception of ours true Knowledge, and an immediate conformity of the Mind of Man to the apparent Truth of things. Theſe are then taken by us for Principles of knowledge, and ſo made uſe of in our common diſcourſings.

But again, numberleſs other things there be, whoſe Truths are leſs apparent, and with ſearch and labour of Mind to be diſcovered; for which, we muſt, as in a Labyrinh, uſe a clew of Silk to find them out. This is our innate Power of reaſoning, or Natural Logic, whereby every Man from the ſimple Apprehenſion of things, and thoſe clear and independent Sentiments of Humane Underſtanding, is enabled to diſcourſe, to hunt out and judge ſo exactly, as to acquire in many matters an aſſurance of Truth, and to reſt therein. For tho' we are not born with Irrefragable Propoſitions in our Brains, as with Eyes in our Head; yet we come into the World with the advantage of ſuch a Mind, that from thoſe Eyes, and our other Senſes, by the Natural Sagacity and Power of inferring one thing from an other, we can and do advance our Thoughts in the knowledge of things, far above all whatſoever our Senſations can reach unto.

But for a farther improvement of this our common Reaſon, *Ariſtotle* has, with great ſubtilty of Thought fram'd the Inſtrument of Inſtruments; and reduced the the looſe and familiar Diſcourſings of Men into an infallible Art, both for the *Form* of Argumentation by Syllogiſms, as a Touch-ſtone to diſcover how truely or fallaciouſly they are made; and alſo for the matter thereof: having gathered all ſorts of Arguments under certain Heads, and digeſted them into their proper Claſſes, called *Topics*. In which he ſhews what kind of Arguments are only *Probable*, and beget in us that which is called OPINION, the miſchievous Mother of Diſpute and Brangling. What again are *Demonſtrative*, and infallibly Convincing, and do give us a Certainty of Knowledge. Of which there are two Degrees.

For this Knowledge, or Cognition is either abſolutely perfect, and fully ſatiſfying our Intellect; or elſe in ſome degree is leſs ſatisfactory: I mean not in the certainty of the Truth; but only in the manner of our Knowledge of this Truth. For beſides *the certain Knowledge of a thing to be*; we naturally deſire to know moreover, *Why it is ſo*, from the immediate conſtituent cauſe of ſuch a Being.

To demonſtrate any thing ἐπιστημονικῶς, Scientifically, from its immediate Eſſential Cauſe, is the utmoſt perfection of humane Knowledge, if we could arrive unto it in a thorough and Methodical Contemplation of any matter.

This Scientifical demonſtration called by *Ariſtotle* Διότι, that is *From the very immediate Cauſe by which a thing is made to be what it is*, the Philoſophers pretend unto: but with how little ſucceſs and ſatisfaction to our Underſtanding, their performances are advanced, does evidently appear from the ſo many different Sects of Jarring Philoſophers.

Whereas the Geometrican leaves nothing diſputable, or uncertain; tho' his demonſtration is very ſeldome Διότι, at which he aims not with Philoſophers, but thinks it ſufficient ſo far to ſatisfy our underſtanding, as undeniably to demonſtrate τὸ ὅτι, the *Quod ſit*, *The thing to be*; tho' it appears not in the demonſtration of that Being, *Why it is*, or *How it comes to be.* If he had undertook this buſineſs,

the

the Geometrician would have become as doubtful and miſerable a Diſputer, as the Philoſopher.

But yet we may obſerve, that the Geometrician does ſometimes give a demonſtration Διότι: as in the firſt Propoſition. For there *Euclide* proves the Triangle to be Equilateral, becauſe the ſides thereof conſiſt of the Rays of the ſame Circles: which is the immediate cauſe of their equality. Neither indeed could this firſt demonſtration be otherwiſe than Scientifical, or Διότι; for that an Equilateral Triangle being the Figure next in nature to a Circle, in reſpect of its ſimplicity and uniformity, ſo that there is no other medium poſſible to intervene between them; therefore it could have no other production, nor any other proof thereof, than from a Circle. The Demonſtrations likewiſe of the two following Propoſitions are of the ſame kind from the ſame cauſe.

But generally the Geometrician takes τὸ ὅτι, evidence and certainty of being to be abundantly ſufficient, both for our pleaſure and contentment in theſe Speculations, wherein we have an indubitable knowledge of ſome Truths in the ſecret and admirable properties of Magnitudes and Numbers. Altho' the famous *French* Logician, blames much the Geometricians, for being thus defective in their demonſtrations: which he would have to be all Scientifical and Διότι, and that the Elements ought to have been ſo methodized. And this he deſpairs not of; but that at ſome time it may be accompliſhed. This is indeed the fullneſs of Science, which only can make perfect the ſtate of humane Nature: But ſuch a knowledge it is, that ſeems beyond the reach of any finite Being. A chain of things (to ſpeak with *Homer*) faſtned to the foot of *Jove*; the frame of whoſe Links is only known to Him the Maker. I wiſh therefore that this *French* Writer had ſhewed himſelf in this point, as great a Geometrician as he is a Logician in others; and given ſome Specimen of that Perfection, which he requires in Geometry. Of which I ſhall have but little hopes, till he can ſhew me how an Acorn comes to be an Oak, and not an Elm.

But for this matter the ancient Geometricians, well exerciſed in demonſtrable Speculations, fully knew their own ſtrength, and contented Themſelves with the certainty of Truth, rather than to venture at the Cauſes of that Truth, which wou'd be ever lyable to diſpute.

Euclide therefore never attempted to order theſe Elementary Propoſitions, in a natural dependence on one another, as the cauſes of each others Truth. Yet are they ſo diſpoſed, as to be ſure Guides to lead us along infallibly from one Truth unto an other, and in that order ſerve to prove indubitably a Thing to be, from the Evidence and Force of ſome thing before acknowledged, tho' not as a Cauſe of its Being; yet as a Cauſe of our neceſſary Aſſent to ſuch a Being.

Demonſtration τȣ Διότι, can be but one in any matter, for that of the ſame thing there is but one only Eſſential and Immediate Cauſe.

Demonſtration τȣ ὅτι, may be very various, ſome ſhort and clear, others more puzzel'd and wandring about; according to the Sagacity of thoſe, who endeavour to find out the readieſt and apteſt Mediums to prove the matter propoſed. So that many times of the ſame Geometrical Propoſition, there may be ſeveral demonſtrations: And even a total change (as many have made to no purpoſe) of *Euclid's* Order in theſe Elements.

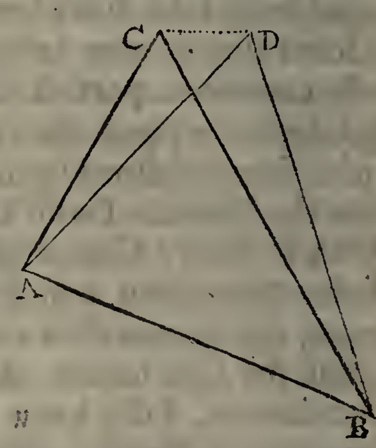
To conclude therefore concerning the Forms of Geometrical demonſtrations. Foraſmuch as τȣ ὅτι is what generally the Geometrician uſes, both in the Synthetical and Analytical Method, and that in either way the proof of a Thing to be, is made no more Scientifically or Διότι, than the proof of a Thing not to be; we are not much to prefer the one before the other; but may as the matter requires, for facility and brevity, make uſe of either; The courſe of Argumentation in both being equally Oſtenſive, and the Mediums of the ſame Nature and Force, that is, Propoſitions already demonſtrated, or allowed Principles. 'Tis only walking through them forward or backward, ſometimes from known Truths to prove an undiſcovered Truth in the Synthetical Method: And ſometimes by the ſame known Truths to diſprove an undiſcovered Untruth in an Analytical Deduction to Impoſſibility. A form of Argumentation moſt proper to overthrow a falſe Poſition: and thereby to eſtabliſh the contradictory Propoſition for a certain Truth. This

This then is the glory of the Geometrician, to demonſtrate upon clear and unqueſtionable grounds, either Synthetically or Analytically. And beſides, it is his wiſdom not to adventure with the Philoſopher, at a natural and neceſſary Series of things, from the immediate Cauſes to their immediate Effects, in which attempt all Philoſophers have hitherto failed; but by irreſiſtible reaſoning exactly to perform what he undertakes, that is, to have ſuch a Maſtery over our Intellect as to convince. Inſomuch that every Man ſhall in reaſon ſubmit, and as readily yield his Aſſent, as that he knows he thinks: And before all other Sciences, this is the power and preference of Geometry.

PROPOSITION VII.

O*N the ſame ſtrait line, to the ſame two ſtrait lines cannot be conſtituted* two other *ſtrait lines, equal each to each, at another and another point, both points ſeated the ſame way, and the* other two *lines having the ſame ends with the two firſt lines.*

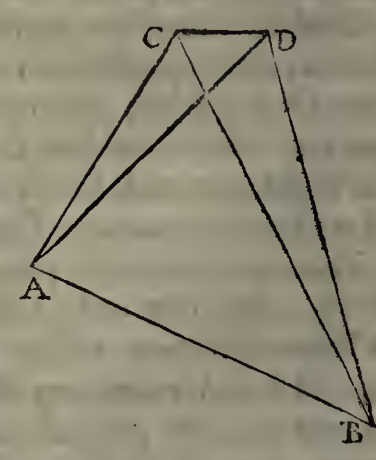
For if it be poſſible, on the ſtrait line AB, to the two ſtrait lines AC, CB, let two other ſtrait lines AD, DB, equal each to each, be conſtituted at another and another point, as at C and D, the points C, D, ſeated † the ſame way: And the lines AD, DB, having the ſame ends A, B, with the two firſt lines AC, CB. So that CA be equal DA: both having the ſame end A; alſo CB be equal to DB: both having the ſame end B.



And now let be joyned CD.

Firſt then in the Triangle ACD.

Foraſmuch as AC is equal to AD [by Suppoſition] therefore the angle ACD is equal to the angle ADC [Prop. 5.]



But the angle ACD *is greater then the angle* BCD, *a part of the ſame angle* ACD. Wherefore the angle ADC (*equal to* ACD) is alſo greater than the angle BCD: * And therefore the angle BDC, *being likewiſe greater than its part the angle* ADC, is MUCH GREATER than the ſame angle BCD. *But again in the Triangle* BCD.

Becauſe BC is equal to BD [by Suppoſition] therefore the angle BDC, is equal to the angle BCD. [Prop. 5.]

* But the angle BDC has been now proved MUCH GREATER than the angle BCD. And it is impoſſible *to be equal and greater than the ſame.*

Therefore on the ſame ſtrait line to the ſame two ſtrait lines, cannot be conſtituted two other ſtrait lines, equal each to each, at another and another point, both points ſeated the ſame way, and the other two lines having the ſame ends with the two firſt lines. Which was to be demonſtrated.

† That is, on the ſame ſide of the line AB. For if the point C lyes

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on one ſide of AB, and the point D on the contrary, then two equal lines may be conſtituted at thoſe points, and have the ſame ends with the two firſt lines. As may be eaſily conceiv'd, if we imagine the Triangle ADB, to be turned over on the other ſide of the line AB.

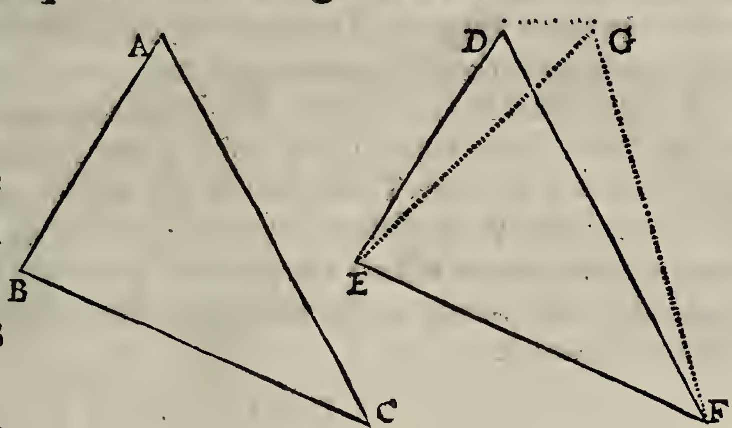
PROPOSITION VIII.

If two Triangles have two ſides equal to two ſides, each to each. And have alſo the baſe equal to the baſe; then ſhall they have the angle equal to the angle, contained by the equal lines.

Let the two Triangles be ABC, DEF, having the two ſides AB, AC, equal to the two ſides DE, DF, each to each, that is, AB to DE, and AC to DF. And let them alſo have the baſe BC equal to the baſe EF.

I ſay, that the angle BAC is equal to the angle EDF.

For the Triangle ABC being apply'd to the Triangle DEF, and the point B put on the point E, and the ſtrait line BC, on the ſtrait line EF; then the point C ſhall agree with the point F; for that BC is equal to EF.



Now BC agreeing with EF; BA, AC, ſhall alſo agree with ED, DF. For if the baſe BC ſhall agree with the baſe EF, and the ſides BA, AC, do not agree with the ſides ED, DF, but change their ſituation, as EG, GF.

Then on the ſame ſtrait line to the ſame two ſtrait lines, ſhall be conſtituted the ſame way, two other ſtrait lines, equal each to each, at another and another point, having the ſame ends. But they cannot be ſo conſtituted. [by Prop. 7.]

Wherefore the baſe BC agreeing with the baſe EF, the ſides BA, AC, ſhall not diſagree with the ſides ED, DF; therefore they ſhall agree.

So that alſo the angle BAC ſhall agree with the angle EDF, and therefore ſhall be equal to it. [Ax. 8.]

If therefore two Triangles have two ſides equal to two ſides, each to each, and have alſo the baſe equal to the baſe, then ſhall they have the angle equal to the angle, contained by the equal lines. Which was to be demonſtrated.

Advertiſement.

Obſerve here and elſewhere, a conſtant propriety of Speech among the *Greek* Geometricians, that the ſame ſtrait lines when they relate to a Figure, are ever called Sides, πλευραὶ, but when referr'd to an angle contained by thoſe ſides, then they retain, or reſume the appellation of ſtrait lines, εὐθειῶν γραμμῶν, not πλευρῶν.

ANNOTATIONS.

This eighth Propſition is in effect the Converſe of the fourth, in which two ſides of a Triangle with the contained angle, were ſuppoſed equal to the like parts of an other

other Triangle; and from thence was proved the equality of their bafes, &c. Now here two fides of a Triangle with the bafe, are fuppofed equal to the like parts of an other Triangle, and from thence is proved the equality of the angles contained by the equal fides. And to compleat the Converfe, *Euclide* might have gone on, and faid, alfo the whole Triangle is equal to the whole Triangle, &c. as in the 4th. Propofition. But this evidently following of it felf, needed not to be repeated.

The demonftration of this Propofition, like as that of the 4th. depends upon the eighth Axiom of Congruous Magnitudes. And here alfo in this demonftration, as before in the fourth, may be clearly obferved the difference between a Geometrical and a Mechanical Congruency; the Mechanical being manifefted only by Senfe, and the Geometrical only by the force of Reafon. As in this 8th. Propofition the Triangles A B C, D E F, are proved to be Congruous, not from an evidence of Senfe, but from the 7th. Propofition; which is a Theorem moft rationally demonftrated: and indeed inferted among thefe Elementary Propofitions chiefly for that purpofe, there being hereafter no further ufe made thereof.

In the fifth and fixth Propofitions, the parts of a fingle Triangle are compared to one another: Firft, in the fifth from two equal fides is proved an equality of their oppofite angles: Then in the fixth, from two equal angles, is demonftrated an equality of their fubtending fides.

In the fourth and eighth Propofitions, are compared two Triangles to one another: and from three equal parts given in each Triangle, is demonftrated an equality between the two Triangles in the whole, and in every remaining part.

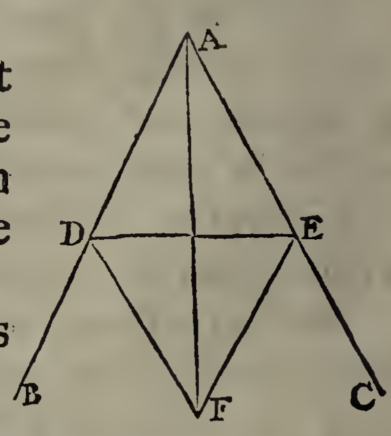
Thefe four Propofitions are the Fundamental Theorems of the Elements: And the Ground upon which they ftand is the Axiom of Congruency; which Mathematical Congruency ought therefore to be rightly underftood, according as we have before declared.

PROPOSITION IX.

TO cut a given ftrait-lin'd angle into halves.

Let the given ftrait-lin'd angle be B A C.

It is required to cut the fame into halves.



Let there be taken in the line A B, any point as D. And from the line A C, let the line A E be taken off equal to A D, and draw D E. Then on D E let be conftituted an Equilateral Triangle D E F: and draw A F.

I fay, that the angle B A C is cut into halves by the ftrait line A F.

For becaufe A D is equal A E, and A F common, therefore there are the two lines D A, A F, equal to the two lines E A, A F, each to each; and the bafe D F, is equal to the bafe E F, therefore the angle D A F, is equal to the angle E A F. [Prop. 8.]

Wherefore the given ftrait-lin'd angle B A C, it cut into halves by the ftrait line A F. Which was to be done.

The Practice.

From the point A at any diftance whatever in the lines A B, A C, take A D equal to A E; then from the points D and E, with the fame opening of the Compaffes, let be defcribed two Arches interfecting each other, fuppofe at the point F, having drawn the ftrait line A F, the angle B A C is divided into two equal parts.

PROPO-

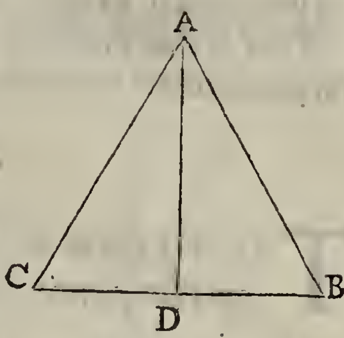
PROPOSITION X.

TO cut a given finite ſtrait line into halves.

Let the finite ſtrait line given be CB.

It is required to cut CB into halves.

Let there be conſtituted upon CB an Equilateral Triangle CAB, and let the angle CAB be cut into halves, by the line AD.



I ſay, that the ſtrait line CB is cut into halves in the point D.

For becauſe CA is equal to AB, and AD common, therefore there are the two lines CA, AD, equal to the two lines BA, AD, each to each, and the angle CAD, is equal to the angle BAD: Therefore the baſe CD, is equal to to the baſe DB.

Wherefore the given finite ſtrait line CB, is cut into halves in the point D. Which was to be done.

The Practice.

Opening the Compaſſes to any diſtance greater than half the line CB, from the points C and B, with the ſame opening of the Compaſſes let be deſcribed two Arches interſecting one another on each ſide the line CB, a ſtrait line drawn betwixt the points of interſection will divide the line into two equal parts.

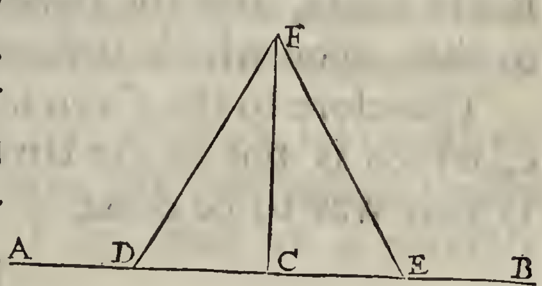
PROPOSITION XI.

TO a given ſtrait line, from a point given in the ſame to draw a ſtrait line at Right angles.

Let the given ſtrait line be AB, and the point given in the ſame be C.

It is required from the point C unto AB to draw a ſtrait line at Right angles.

Let there be taken in the line AC any point as D, and to CD let there be put an equal line CE: Then on DE let be conſtituted an Equilateral Triangle FDE. And draw the line CF.



I ſay, that to the given ſtrait line AB from the point C given in the ſame, is drawn at Right angles the ſtrait line CF.

For becauſe CD is equal to CE, and CF common; therefore there are the two lines CD, CF, equal to the two lines CE, CF, each to each, and the baſe DF is equal to the baſe EF; wherefore the angle DCF is equal to the angle ECF, and theſe are conſequent angles. But when a ſtrait line ſtanding upon a ſtrait line, makes the conſequent angles equal to one another, then each of thoſe equal angles is a Right angle. Wherefore each of the angles DCF, ECF, is a Right angle.

Therefore to the given ſtrait line AB, from the point C given in the ſame, is drawn at Right angles the ſtrait line CF. Which was to be done.

The Practice.

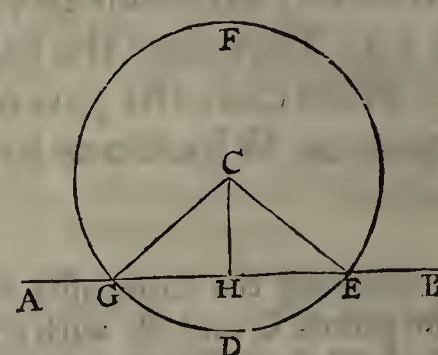
On each side of the given point C, in the line AB, take CD and CE equal to one another, then from the points D and E, with any opening of the Compasses greater than CD or CE, let be described two Arches interfecting each other, suppose at the point F, the strait line FC drawn betwixt the points F and C, shall be Perpendicular to the line AB.

This is the practice with Ruler and Compasses, but the readiest way of drawing Perpendiculars both in this and the next Proposition, is by a Square.

PROPOSITION XII.

TO a given infinite strait line from a given point which is not in the same line, to draw a Perpendicular Strait line.

Let the given infinite strait line be AB, and let the given point which is not in the same line be C



It is required to AB, from the point C to draw a perpendicular strait line.

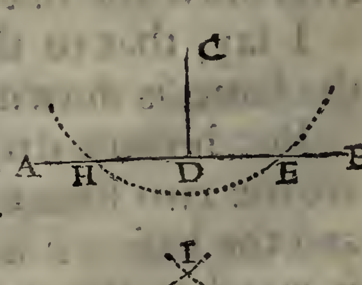
Let there be taken on the other side of the line AB, any point as D. Now the Center C, and the distance CD, let be described a Circle EFG. Then let the line EG be cut into halves in the point H, and let be joyned CG, CH, CE.

I say, that to AB, from the point C is drawn a perpendicular CH.

For because GH is equal to HE and HC common; therefore there are the two lines GH, HC, equal to the two lines EH, HC, each to each; and the base CG is equal to the base CE: wherefore the angle CHG, is equal to the angle CHE, and these are consequent angles. But when a strait line standing upon a strait line makes the consequent angles equal to one another: then each of those equal angles is a Right angle, and the standing strait line is called a perpendicular to that, upon which it stands.

Therefore to the given infinite strait line AB from the given point C, which is not in the same line, is drawn the perpendicular CH. Which was to be done.

The Practice.



From the point C at any distance greater than the nearest CD, let be described an Arch, cutting the line AB at the points H and E, then from the points H and E, with the same opening of the Compasses, let be described two Arches interfecting each other at the point I, the Ruler being laid to the points I and C draw CD, it shall be perpendicular to AB from the point C.

PROPOSITION XIII.

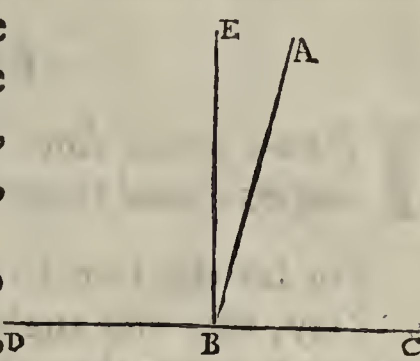
IF a strait line standing any ways upon a Strait line makes angles, it shall make either two Right angles, or angles equal to two Right.

For let a strait line AB standing upon a strait line CD, make angles,

gles, as CBA, ABD. I ſay, that the angles CBA, ABD, are either two Right angles, or equal to two Right.

For if the angle CBA, be equal to the angle ABD, then are they two Right angles, [by the 10th. Def.]

But if not, let from the point B, be drawn BE at right angles to CD. Wherefore CBE, EBD, are two Right angles.



Now whereas the angle CBE, is equal to the two angles CBA, ABE, let the angle EBD be added in common. Wherefore the angles CBE, EBD, are equal to the three angles CBA, ABE, EBD.

Again, whereas the angle DBA, is equal to the two angles DBE, EBA, let the angle ABC, be added in common. Wherefore the angles DBA, ABC, are equal to the three angles DBE, EBA, ABC.

But the angles CBE, EBD, were proved equal to the ſame three angles; And things equal to one and the ſame, are equal to one another. Therefore alſo the angles CBE, EBD, are equal to the angles DBA, ABC.

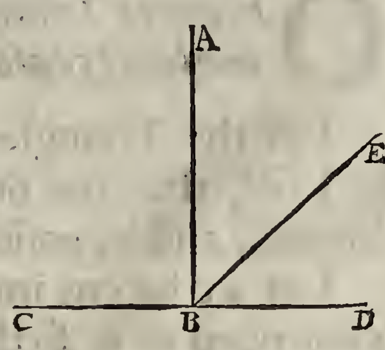
But the angles CBE, EBD, are two Right angles, wherefore alſo the angles DBA, ABC, are equal to two Right angles. Therefore if a ſtrait line ſtanding any ways upon a ſtrait line makes angles, it ſhall make either two Right angles, or angles equal to two Right. Which was to be demonſtrated.

PROPOSITION XIV.

If to a ſtrait line, and to a point in the ſame, two ſtrait lines not lying the ſame way, do make the conſequent angles equal to two Right angles, Thoſe ſtrait lines ſhall be directly placed to one another.

For to the ſtrait line AB, and to a point in the ſame B, let two ſtrait lines BC, BD, not lying the ſame way, make the conſequent angles ABC, ABD, equal to two Right angles. I ſay, that BD is directly placed to BC.

For if BD be not directly placed to BC; let BE be directly placed to BC.



Now foraſmuch as the ſtrait line AB, ſtands upon the ſtrait line CBE: therefore the angles ABC, ABE, are equal to two Right angles. [Prop. 13.]

But alſo the angles ABC, ABD, are equal to two Right angles [by Suppoſition] wherefore the angles CBA, ABE, are equal to the angles CBA, ABD.

Let the common angle CBA be taken away. Therefore the remaining angle ABE, is equal to the remaining angle ABD, the leſs to the greater: which is impoſſible. Therefore BE is not directly placed to BC.

In like manner may we prove, that there is not any other line besides BD. Therefore BC is directly placed to BD.

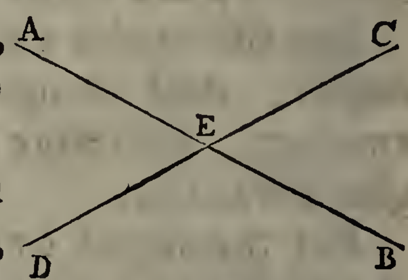
If therefore to a strait line, and to a point in the same, two strait lines not lying the same way, do make the consequent angles equal to two Right angles, Those strait lines shall be directly placed one to the other. Which was to be demonstrated.

PROPOSITION XV.

IF two strait lines cut each other, they shall make the Vertical angles equal to one another.

For let the two strait lines AB, CD, cut each other in the point E.

I say, that the angle AEC, is equal to the angle DEB: and the angle CEB, is equal to the angle AED.



Forasmuch as the strait line AE, stands upon the strait line CD, making the angles CEA, AED, therefore the angles CEA, AED, are equal to two Right angles.

Again, because the strait line DE stands upon the strait line AB, making the angles AED, DEB, therefore the angles AED, DEB, are equal to two Right angles.

But the angles CEA, AED, were proved equal to two Right angles. Wherefore the angles CEA, AED, are equal to the angles AED, DEB: let the common angle AED be taken away; then the remaining angle CEA, is equal to the remaining angle BED. In like manner it may be proved, that the angles CEB, DEA, are equal. If therefore two strait lines cut each other, they shall make the Vertical angles equal to one another. Which was to be demonstrated.

Corollary.

From hence 'tis manifest, that if strait lines, how many soever, cut one another in the same point, they shall make the angles at the section equal to four Right angles.

PROPOSITION XVI.

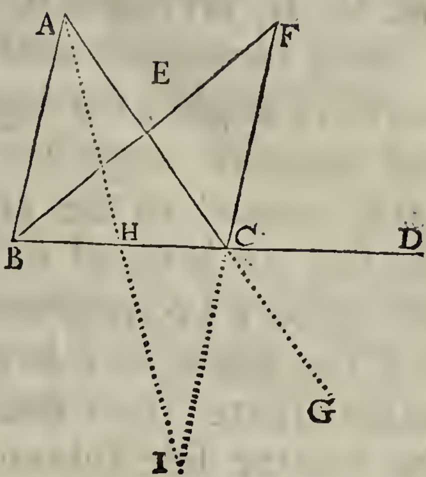
OF every Triangle, any one side being produced, the outward angle is greater than either of the inward, and opposite angles.

Let the Triangle be ABC, and let the side BC be produced to D.

I say, that the outward angle ACD, is greater than either of the inward, and opposite angles CBA, BAC.

Let AC be cut into halves in the point E; and drawing BE let it be continued to F, so that EF be put equal to BE: And let be joyn'd FC. Forasmuch as AE is equal to EC, and BE, to EF: therefore there are the two lines AE, EB, equal to the two lines CE, EF, each to each, and the angle AEB, is equal to the angle FEC (for they are Vertical angles)

angles) Therefore the baſe AB, is equal to the baſe FC, and the Triangle ABE is equal to the Triangle FEC; and the remaining angles are equal to the remaining angles, each to each, under which are ſubtended equal ſides, [Prop. 4.] therefore the angle BAE, is equal to the angle ECF. But the angle ECD, is greater than the angle ECF. Therefore the angle ECD is greater than the angle BAE: That is, the outward angle ECD, or ACD, is greater than the inward and oppoſite angle BAC.



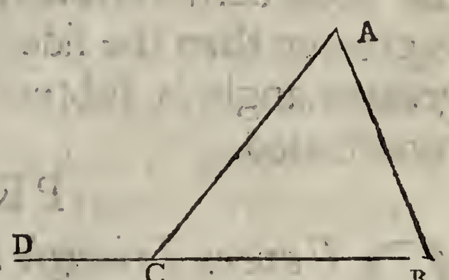
In like manner, the ſide AC being produced to G, and the ſide BC being cut into halves in the point H: and drawing AH, let it be continued to I, ſo that HI be put equal to AH: and let be joyned IC; It will be demonſtrated as before, that the outward angle BCG, is greater than the inward and oppoſite CBA. But the angle BCG, is equal to the Vertical angle ACD; therefore the angle ACD, is alſo greater than the angle CBA: And it has been proved greater than the angle BAC. Therefore of every Triangle any one ſide being produced, the outward angle is greater than either of the inward and oppoſite angles. Which was to be demonſtrated.

PROPOSITION XVII.

Of every Triangle two angles taken together every way, are leſs than two Right angles.

Let the Triangle be ABC. I ſay, that any two angles of the Triangle ABC, are leſs than two Right angles.

For let BC be produced to D: And becauſe of the Triangle ABC the outward angle ACD, is greater than the inward and oppoſite angle ABC; let be added in common the angle ACB: therefore the angles ACD, ACB, are greater than the angles ABC, BCA. But the angles ACD, ACB, are equal to two Right angles; therefore the angles ABC, BCA, are leſs than two Right angles. In like manner we may demonſtrate, that the angles BAC, ACB, are leſs than two Right angles: And alſo that the angles CAB, ABC, are leſs than two Right angles. Therefore of every Triangle two angles taken together every way, are leſs than two Right angles. Which was to be demonſtrated.



PROPOSITION XVIII.

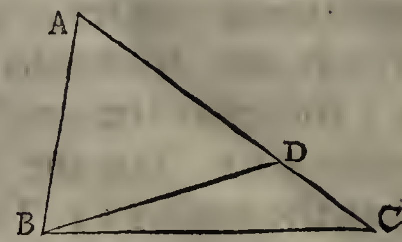
Of every Triangle the greater ſide ſubtends the greater angle.

Let the Triangle be ABC, having the ſide AC greater than the ſide

AB. I ſay, that alſo the angle ABC, is greater than the angle BCA.

For becauſe AC is greater than AB, let AD be put equal to AB; and let be joyned BD.

Now foraſmuch as of the Triangle BDC, the outward angle ADB is greater than the inward and oppoſite angle DCB; and that the angle ADB, is equal to the angle ABD, (becauſe the the ſide AB is equal to the ſide AD) Therefore the angle ABD, is greater than the angle ACB.



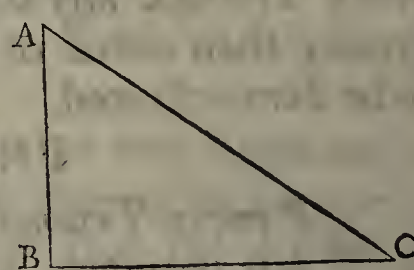
But the angle ABC is greater than the angle ABD, wherefore it is much greater than the angle ACB. Therefore of every Triangle the greater ſide ſubtends the greater angle. Which was to be demonſtrated.

PROPOSITION XIX.

OF every Triangle under the greater angle, is ſubtended the greater ſide.

Let the Triangle be ABC, having the angle ABC, greater than the angle BCA. I ſay, that the ſide AC is greater than the ſide AB. For if not, then AC is either equal to AB, or leſs than it.

But AC is not equal to AB; for that then the angle ABC ſhould be equal to the angle ACB. But it is not equal [by Suppoſition] therefore AC is not equal to AB. Neither yet is AC leſs than AB, for then alſo the angle ABC ſhould be leſs than the angle ACB. But it is not leſs [by Suppoſition.] therefore AC is not leſs than AB. And it has been demonſtrated, that it is not equal: therefore the ſide AC is greater than the ſide AB. Wherefore of every Triangle under the greater angle is ſubtended the greater ſide. Which was to be demonſtrated.

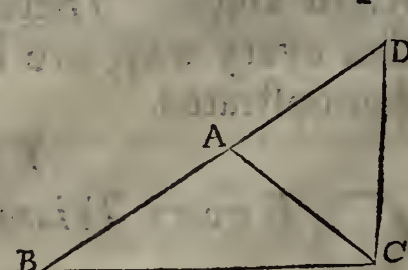


PROPOSITION XX.

OF every Triangle two ſides taken together any way, are greater than the remaining ſide.

Let the Triangle be ABC. I ſay, that two ſides of the Triangle ABC taken together any way, are greater than the remaining ſide. Namely BA, AC, than BC: and AB, BC, than CA: and BC, CA, than AB. For let BA be produced to the point D, and to CA let AD be put equal, then let be joyned DC.

Now foraſmuch as DA is equal to AC, therefore the angle ADC is equal to the angle ACD. But the angle BCD is greater than the angle ACD: therefore the angle BCD is greater than the angle ADC. And becauſe DCB is a Triangle having the angle BCD greater than the angle BDC; and that under the



the greater angle is ſubtended the greater ſide: therefore DB is greater than BC. But DB is equal to BA, AC: therefore BA, AC, are greater than BC.

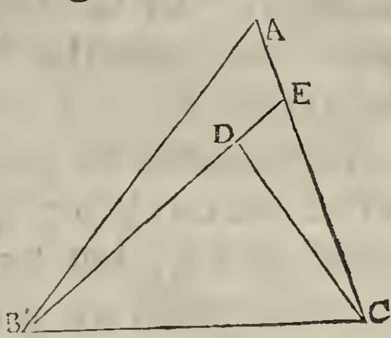
After the ſame manner ſhall we demonſtrate, that AB, BC, are greater than CA; And BC, CA, than AB. Therefore of every Triangle two ſides taken together every way, are greater than the remaining ſide. Which was to be demonſtrated.

PROPOSITION XXI.

If from the Ends of any one ſide of a Triangle be conſtituted two ſtrait lines within the Triangle: The conſtituted lines ſhall be leſs than the two remaining ſides of the Triangle: But ſhall contain a greater angle.

For on one of the ſides BC, of the Triangle ABC, from the ends B, C, let be conſtituted within, two ſtrait lines BD, DC.

I ſay, that BD, DC, are leſs than BA, AC, the two remaining ſides of the Triangle ABC: but do contain an angle BDC greater than the angle BAC.



For let BD be produced to E.

Now foraſmuch as of every Triangle two ſides are greater than the remaining ſide; therefore of the triangle BAE the two ſides BA, AE, are greater than BE. Let EC be added in common, therefore BA, AC, are greater than BE, EC.

Again, becauſe of the Triangle CED, the two ſides CE, ED, are greater than CD: let DB be added in common, therefore CE, EB, are greater than CD, DB. But BA, AC, have been proved greater than BE, EC: therefore BA, AC, are much greater than BD, DC.

Again, foraſmuch as of every Triangle the outward angle is greater than the inward and oppoſite: therefore of the Triangle CED the outward angle BDC, is greater than DEC. And by the ſame reaſon, of the Triangle BAE, the outward angle BEC, is greater than BAE. But BDC has been prov'd greater than DEC, that is, BEC; therefore the angle BDC is much greater than the angle BAC.

If therefore upon any one ſide of a Triangle, be from the ends thereof conſtituted two ſtrait lines within the Triangle, the conſtituted lines are leſs than the two remaining ſides of the Triangle: but ſhall contain a greater angle. Which was to be demonſtrated.

ANNOTATIONS.

This Propoſition is of much uſe in *Optics* concerning Viſual Rays, and Angles; in that the ſame Object ſhall appear greater, or leſſer, as upon various diſtances, the angles received in the Eye are greater, or leſſer. And therefore in general 'tis to be noted that every Thing appears to us leſs, than it really is in magnitude.

The like uſe of this Propoſition is made in Perſpective, Picture, and Architecture: where Images, Statues, Columns, &c. are proportioned according to their heights

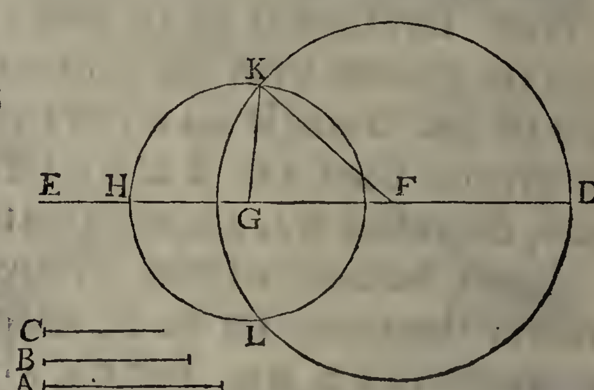
heights and diſtances. *Phidias* his Statue of *Minerva* was very famous in this point, which ſeemed near hand ſo monſtrous to the vulgar: but ſeated in that part of the Temple where it was deſign'd to be placed, it appear'd moſt beautiful. We have the like example of Pictures in the Banqueting Room at *White-Hall.* Thus many Geometrical Propoſitions, which ſeem trivial, have excellent Uſes.

PROPOSITION XXII.

Of three ſtrait lines, which are equal to three given ſtrait lines, to conſtitute a Triangle.

But any two taken every way, ought to be greater than the remaining line.

Let three given ſtrait lines be A, B, C, of which let any two taken every way, be greater than the remaining line: namely A, B, than C. A, C, than B, and B, C, than A. It is required to conſtitute a Triangle of ſtrait lines equal to A, B, C. Let be put a ſtrait line DE, terminated at D, but interminate towards E; and let DF be put equal to A, and FG equal to B, and GH to C. Now the Center F, and diſtance FD, let be deſcribed the circle DKL: And again, the Center G, and diſtance GH, let be deſcribed the circle HLK: and let be joyned KF, KG. I ſay, that the Triangle KFG, is conſtituted of three ſtrait lines equal to A, B, C.

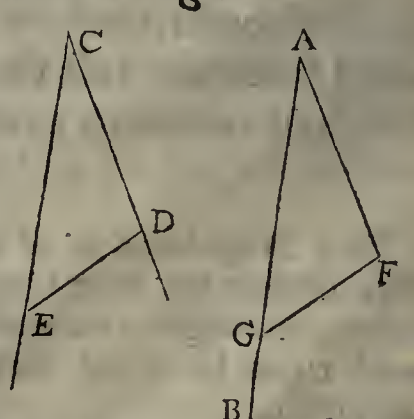


For becauſe the point F is the Center of the circle DKL, therefore FD is equal to FK; but FD is equal to A, wherefore alſo FK is equal to A.

Again, becauſe the point G is the Center of the circle HLK, therefore GH is equal to GK: but GH is equal to C: wherefore alſo GK is equal to C; and FG alſo is equal to B. Wherefore the three ſtrait lines KF, FG, GK, are equal to the three ſtrait lines A, B, C. Therefore of the three ſtrait lines KF, FG, GK, which are equal to the three given ſtrait lines A, B, C, is conſtituted the Triangle KFG. Which was to be done.

PROPOSITION XXIII.

To a given ſtrait line, and to a point in the ſame, to conſtitute a ſtrait-lin'd angle equal to a given ſtrait-lin'd angle.



Let the given ſtrait line be AB, and in the ſame let the point be A; and let the given ſtrait lin'd angle be DCE. It is required to the given ſtrait line AB, and in it to the point A, to conſtitute a ſtrait-lin'd angle equal to the given ſtrait-lin'd angle DCE.

Let there be taken in each line CD, CE, any points; as D, E; and let be drawn the ſtrait line DE. Now of three ſtrait lines which are equal to the three ſtrait lines

CD,

CD, DE, CE; let be conſtituted a Triangle AFG, ſo that CD be equal to AF; and CE to AG; as alſo DE to FG.

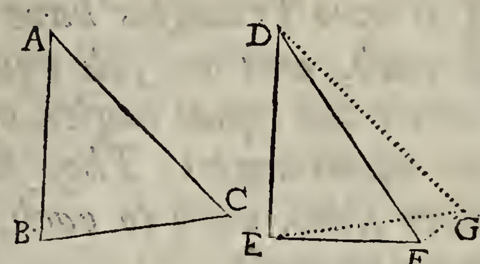
Then foraſmuch as the two ſtrait lines DC, CE, are equal to the two ſtrait lines FA, AG, each to each, and the baſe DE is equal to the baſe FG: therefore the angle DCE is equal to the angle FAG. [Prop. 8.] Therefore to a given ſtrait line AB, and to a point in the ſame A, is conſtituted a ſtrait-lin'd angle FAG, equal to a given ſtrait-lin'd angle DCE. Which was to be done.

PROPOSITION XXIV.

If two Triangles have two ſides equal to two ſides, each to each: And have the angle greater than the angle, which is contained by the equal lines: They ſhall alſo have the baſe greater than the baſe.

Let the two Triangles be ABC, DEF, having two ſides AB, AC, equal to two ſides DE, DF, each to each, that is, AB to DE; AC to DF: And let the angle BAC be greater than the angle EDF: I ſay, that the baſe BC is greater than the baſe EF.

Foraſmuch as the angle BAC is greater than the angle EDF, let to the ſtrait line DE, and to a point in the ſame D, be conſtituted the angle EDG, equal to the angle BAC. And to either of the lines AC, DF, let DG be put equal, and let be joyned GE, GF.



Now becauſe AB is equal to DE, and AC to DG; therefore there are the two lines BA, AC, equal to the two lines ED, DG, each to each; and the angle BAC, is equal to the angle EDG, therefore the baſe BC, is equal to the baſe EG. Again, becauſe DG is equal to DF, therefore the angle DFG is equal to the angle DGF, [Prop. 5.] But DGF is greater than its part EGF, therefore alſo DFG is greater than EGF. But EFG is greater than its part DFG, therefore EFG is much greater than EGF.

And becauſe there is the Triangle EGF, having the angle EFG greater than the angle EGF, and that under the greater angle is ſubtended the greater ſide: therefore the ſide EG is greater than EF. But EG is equal to BC: wherefore BC is greater than EF.

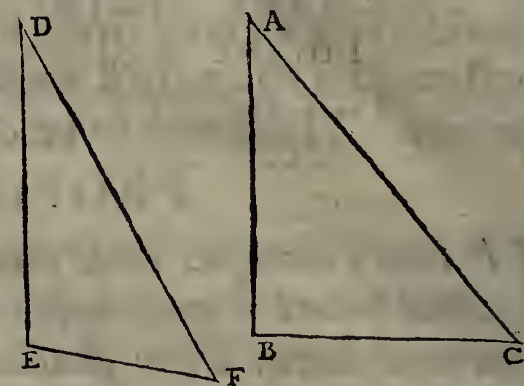
If therefore two Triangles have two ſides equal to two ſides, each to each: and have the angle greater than the angle, which is contained by the equal lines: They ſhall alſo have the baſe greater than the baſe. Which was to be demonſtrated.

PROPOSITION XXV.

If two Triangles have two ſides equal to two ſides, each to each, and have the baſe greater than the baſe: They ſhall alſo have the angle greater than the angle, which is contained by the equal lines.

Let the two Triangles be ABC, DEF, having two ſides AB, AC, equal to two ſides DE, EF, each to each, that is, AB to DE; AC to DF: And let the baſe BC be greater than the baſe EF: I ſay, that the angle BAC is greater than the angle EDF.

For if not, then BAC is either equal to EDF, or leſs. But the angle BAC is not equal to the angle EDF: For then the baſe BC ſhould be equal to the baſe EF. [Prop. 4.] But it is not ſo [by Suppoſition] therefore the angle BAC is not equal to the angle EDF.



But neither is it leſs. For then the baſe BC ſhould be leſs than the baſe EF [Prop. 24.] But it is not ſo [by Suppoſition] therefore the angle BAC is not leſs than the angle EDF; and it has been proved, that it is not equal. Therefore the angle BAC is greater than the angle EDF.

If therefore two Triangles have two ſides equal to two ſides, each to each, and have the baſe greater than the baſe: They ſhall alſo have the angle greater than the angle, which is contained by the equal lines. Which was to be demonſtrated.

PROPOSITION XXVI.

If two Triangles have two angles equal to two angles, each to each, and one ſide equal to one ſide, either THAT *which is between the equal angles, or* THAT, *which is ſubtended under one of the equal angles: They ſhall alſo have the other ſides equal to the other ſides, each to each, and the remaining angle equal to the remaining angle.*

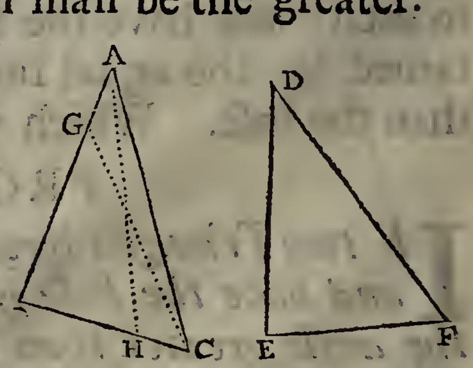
Let the two Triangles be ABC, DEF, having two angles ABC, BCA, equal to two angles DEF, EFD, each to each, that is, ABC to DEF, and BCA to EFD.

And let them have one ſide equal to one ſide.

Firſt, the ſide between the equal angles: that is, BC equal to EF. I ſay, That they ſhall have the other ſides equal to the other ſides, each to each, namely AB to DE; and AC to DF: and the remaining angle BAC, equal to the remaining angle EDF.

For if AB be unequal to DE, one of them ſhall be the greater.

Let AB be the greater, and let BG be put equal to ED, and let be joyned GC. Now foraſmuch as BG is equal to ED, and BC to EF, therefore there are the two lines BG, BC, equal to the two lines ED, EF, each to each, and the angle GBC is equal to the angle DEF; therefore the baſe GC, is equal to the baſe DF. [Prop. 4.] And the Triangle GBC ſhall be equal to the Triangle DEF, and the remaining angles ſhall be equal to the remaining angles,



angles, each to each, under which are ſubtended equal ſides. Therefore the angle BCG, is equal to the angle EFD. But EFD was ſuppoſed equal to BCA; wherefore BCG is equal to BCA: The leſs to the greater; which is impoſſible. Therefore AB is not unequal to DE, equal therefore it is.

But alſo BC is equal to EF; therefore there are the two lines AB, BC, equal to the two lines DE, EF, each to each, and the angle ABC is equal to the angle DEF, wherefore the baſe AC is equal to the baſe DF: and the remaining angle BAC is equal to the remaining angle EDF.

Now again, let the ſides ſubtended under the equal angles be equal, as AB to DE; I ſay, that alſo the other ſides ſhall be equal to the other ſides, that is, AC to DF, and BC to EF: And alſo the remaining angle BAC, ſhall be equal to the remaining angle EDF. For if BC be unequal to EF, one of them is the greater.

Let BC (if poſſible) be the greater, and let BH be put equal to EF, and let be joyned AH.

Now foraſmuch as BH is equal to EF, and AB to DE: therefore there are the two lines AB, BH, equal to the two lines DE, EF, each to each, and they contain equal angles: wherefore the baſe AH is equal to the baſe DF, and the Triangle ABH: is equal to the Triangle DEF. And the remaining angles ſhall be equal to the remaining angles, each to each, under which are ſubtended equal ſides; therefore the angle BHA is equal to EFD. But EFD is equal to BCA, therefore alſo BHA is equal to BCA: that is, of the Triangle AHC, the outward angle BHA, is equal to the inward and oppoſite BCA: which is impoſſible. Therefore BC is not unequal to EF, equal therefore it is. But alſo AB is equal to DE: therefore there are the two lines AB, BC, equal to the two lines DE, EF, each to each, and they contain equal angles: wherefore the baſe AC, is equal to the baſe DF, and the Triangle ABC, is equal to the Triangle DEF, and the remaining angle BAC, is equal to the remaining angle EDF.

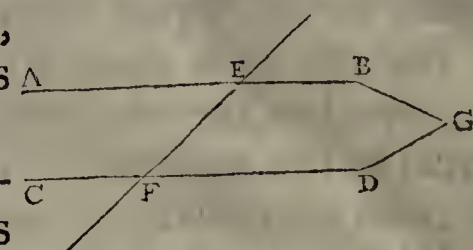
If therefore two Triangles have two angles, equal to two angles, each to each, and one ſide equal to one ſide, either that which is between the equal angles, or that which is ſubtended under one of the equal angles: they ſhall &c. Which was to be demonſtrated.

PROPOSITION XXVII.

If a ſtrait line falling on two ſtrait lines, makes the alternate angles equal to one another, the ſtrait lines ſhall be Parallels one to the other.

For let the ſtrait line EF, falling on the two ſtrait lines AB, CD, make the Alternate angles AEF, EFD, equal to one another: I ſay, that AB is Parallel to CD.

For if not, then AB, CD, being produced ſhall meet either on the parts of B, D; or on the parts of A, C. Let them be produced and meet on the parts of B, D, in the point G. Therefore of the Triangle GEF, the outward angle AEF is greater than the inward and oppofite angle EFG. [Prop. 16.] But it is alfo equal [by Suppofition] which is impoffible; therefore AB, CD, being produced fhall not meet on the parts of B, D.



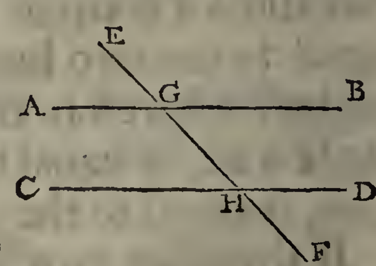
After the fame manner fhall be demonftrated, that they meet not on the parts of A, C. But meeting in neither part, they are Parallels, [Def. 35.] therefore AB is Parallel to CD.

Wherefore if a ftrait line falling on two ftrait lines, makes the Alternate angles equal to one another, the ftrait lines fhall be Parallels, one to the other. Which was to be demonftrated.

PROPOSITION XXVIII.

If a ftrait line falling on two ftrait lines, makes the outward angle equal to the inward and oppofite on the fame parts: Or the inward angles on the fame parts equal to two Right angles; the ftrait lines fhall be Parallels one to the other.

For let the ftrait line EF falling on the two ftrait lines AB, CD, make the outward angle EGB, equal to the inward and oppofite, and on the fame parts, namely to the angle GHD: or the inward angles on the fame parts, namely BGH, GHD, equal to two Right angles. I fay, that AB is Parallel to CD.



For becaufe EGB is equal to GHD [by Suppofition] and EGB is equal to the Vertical angle AGH; therefore alfo AGH is equal to GHD, and they are Alternate, therefore AB is Parallel to CD.

Again, becaufe BGH, GHD, are equal to two Right angles [by Suppofition.] and alfo AGH, BGH, are equal to two Right angles; [Prop. 13.] therefore AGH, BGH, are equal to BGH, GHD. Let BGH common be taken away, then the remaining angle AGH is equal to the remaining angle GHD. And they are Alternate, therefore AB is Parallel to CD.

If therefore a ftrait line falling on two ftrait lines, makes the outward angle equal to the inward and oppofite on the fame parts: or the inward angles on the fame parts equal to two Right angles: the ftrait lines fhall be Parallels one to the other. Which was to be demonftrated.

PROPO-

PROPOSITION XXIX.

ON Parallel Lines a ſtrait line falling doth make the Alternate angles equal to one another.

And the Outward angle equal to the Inward, and Oppoſite on the ſame parts.

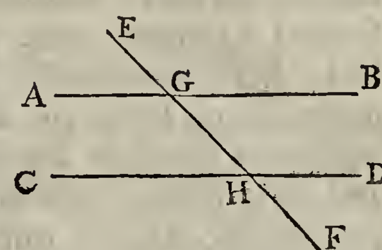
And the Inward angles on the ſame parts equal to two Right.

For on the Parallels AB, CD, let the ſtrait line EF fall, I ſay that it makes the Alternate angles AGH, GHD equal.

And the Outward angle EGB equal to GHD the Inward and Oppoſite angle on the ſame parts.

And the Inward angles on the ſame parts BGH, GHD, equal to two Right angles.

For if the angle AGH be unequal to GHD; one of them is the greater. Let the greater be AGH.



Now foraſmuch as AGH is greater than GHD, let be added in common BGH. Therefore AGH, BGH are greater than BGH, GHD [Ax. 4.] But AGH, BGH are equal to two Right angles [Prop. 13.]; therefore BGH, GHD are leſs than two Right angles. But ſtrait lines infinitely produced from Angles leſs than two Right, do meet together, [Ax. 11.] therefore AB, CD, infinitely produced ſhall meet together. But meet they do not; for that they are ſuppoſed Parallels; therefore AGH is not unequal to GHD: equal therefore it is; *And* AGH, GHD, *are Alternate angles.*

But again, AGH is equal to *the Vertical angle* EGB, [Prop. 15.] therefore alſo EGB is equal to GHD. *The Outward angle equal to the Inward and Oppoſite.*

Let now be added in common BGH: therefore EGB, BGH are equal to BGH, GHD. But EGB, BGH are equal to Two Right angles [Prop. 13.] therefore alſo BGH, GHD, *The Inward angles,* are equal to two Right.

Wherefore a ſtrait falling on Parallel lines does make the Alternate angles equal to one another: And the Outward angle equal to the Inward, and Oppoſite on the ſame parts: And the Inward angles on the ſame parts equal to two Right. Which was to be demonſtrated.

ANNOTATIONS.

This Propoſition is the Converſe of the two preceding; and in the three laſt Propoſitions is compriſed the Fundamental Doctrine of *Parallelijm*; wherein three Specificative and Convertible Properties of Parallels are laid down.

Firſt, From the Equality of the Alternate angles the Lines are proved Parallels, in Prop. 27.

Secondly, From the Equality of the Outward angle to the Inward and Oppoſite: And then next,

Thirdly, From the Equality of the two Inward angles to Two Right, the Lines are alſo proved Parallels, in Prop. 28.

So that in these two Propositions is demonstrated, that all strait lines having any one of these three properties are PARALLELS, that is, NON-CONCURRING STRAIT LINES.

Now the following 29th. Proposition is the Converse of the two preceding, and demonstrates, that all Parallels have all these three properties.

But in this 29th. Proposition, the first and main part of the Demonstration dedends wholly upon the 11th. Axiom, which tho' it be certainly true; yet for that it is lyable to dubitation, and some Objections may be made against it; this Demonstration hath not clearly passed without some reprehension.

For besides what hath been said before in the Annotations upon this 11th. Axiom, *Euclide* himself in the 17th. Proposition, doth in effect demonstrate, That *two strait lines meeting together, being cut by a strait line, are drawn from angles less than two Right.* And say they, it might be as reasonably required of *Euclide*, to have demonstrated the Converse, That *two strait lines drawn from angles less then two Right, shall meet together*, which is the 11th. Axiom, and assumed for a Principle without any Demonstration.

Again, in the 28th. Proposition it is demonstrated, That *if the two inward angles be equal to two Right, then the lines are Parallels.* But also it seems as requisite and reasonable to have demonstrated, That *if the two inward angles be less than two Right, then the lines are not Parallels*; but at length shall meet together: This Supposition having no more Natural evidence then the other. There have been in all Ages several Attempts made to remove this stumbling block: But too tedious they are to be here examined. You may peruse what *Proclus* has ventured at in his Commentaries on Prop. 29. and what *Clavius* has laboriously perform'd. What a strange notion of Parallels *Borellus* has fram'd in his *Euclides Restitutus*, at Prop. 14. Lib. I. and what others have endeavoured herein. There are likewise two Translations of *Euclide* into Arabic, one of *Nasaradinus* printed at *Rome*. The other of ——— never Printed, a Copy whereof is in the *Oxford* Library. In both of them much Labour is taken to clear this Matter.

After these great Geometricians, we shall with pardon adventure upon this Matter; and in lieu of *Euclid's* 11th. Axiom bring into the Elements the consideration of the DISTANCE OF PARALLELS, and their EQUIDISTANCES toward one another. For altho' in our Annotations upon the Definition of Parallels, we have shewn, that the name *Parallels* ought not in *Euclid's* Sense to be Translated *Equidistant lines*; or by that name should be conceived *Equidistant strait lines*, but only *Nonconcurring strait lines*: yet we do not so wholly exclude the Notion of Equidistancy in the doctrine of Parallelism, but that there may be a just use made thereof; tho' Equidistancy be not taken into the Definition of Parallels.

First then it is observable, that vulgarly Parallels are conceived to be Equidistant strait lines; altho' the Geometrician puts only the notion of Nonconcurrency into the Definition, without any regard had to the Equidistancy of Parallels; and this is done upon very good reason. For a *Nonconcurrency* in some strait lines is a Notion generally useful throughout all Geometry: therefore *Euclide* among the rest of his Definitions proper to his first Element, has laid down this Notion of Nonconcurrency under the name of Parallels. So that Parallels and Nonconcurring strait lines may be substituted indifferently for one another in any demonstration, as the *Definitum* and *Definition* ought to be. But Parallels and Equidistant strait lines cannot be so indifferently taken and used; notwithstanding the vulgar conception of them.' Yet some particular use may be made in Geometry of the Equidistancy of Parallels, as we shall shew; if according to the vulgar conception it be admitted among the other common Notions, that Parallels are equidistant strait lines: And so this to be received for a Maxim from *Euclid's* Definition of Parallels, as he has from the Definition of a Right angle put for an Axiom, that all Right angles are equal to one another.

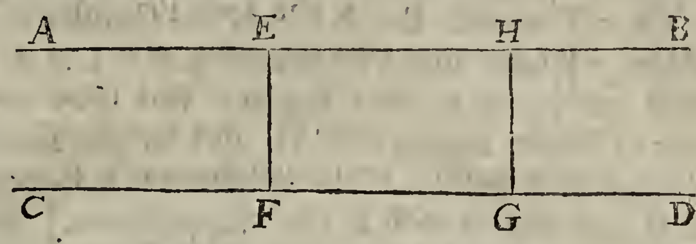
To proceed then in this matter, we shall as aforesaid, add to *Euclide* only a Definition of the distance of Parallels, and instead of his 11th. Axiom assume their Equidistancy as a common Notion.

DEFI-

DEFINITION XXXV.

The diſtance of Parallels is a ſtrait line, drawn from any point in either Parallel, perpendicular to the other.

As of the Parallels AB, CD, the diſtance is the ſtrait line EF, drawn from the point E in the line AB, perpendicular to the line CD.



And again, the diſtance of the ſame Parallels is the line GH drawn from the point G in the line CD, perpendicular to the line AB, and ſo forth infinitely.

This Notion, or Definition of diſtance is agreeable to the 4th. Definition of the third Element, and to the 4th. Definition of the ſixth Element.

AXIOM XII.

Parallels are every where equally diſtant from one another.

That is, the Perpendiculars drawn from any point in either of the Parallels to the other, are equal to one another. As in the Parallels AB, CD, the line EF perpendicular to CD is equal to GH perpendicular to AB. So every where from any points in *the One*, the perpendiculars to *the Other*, are mutually equal to one another.

We have formerly ſhewn how *Poſidonius* has defined Parallels from the equality of their perpendiculars; yet we find not what advantage was further made of that Definition, toward the amendment of *Euclid's* demonſtration, or for any other uſe he makes thereof in Geometry. But according to *Euclid's* Definition, the Notion of two ſtrait lines in the ſame plane produced both ways infinitely, which ſhall never meet, is as proper and common a ſubject of Geometry, as Angles and Figures are, and of as general an extent.

Yet furthermore we acknowledge, that the Equidiſtance of theſe ſtrait lines is a Notion concomitant with that of Nonconcurrency, and that they mutually put one another, as a cauſe puts the effect, and an effect puts the cauſe. So that in Parallels Artificers do in Architecture, and other the like matters, reſpect their Equidiſtancy, as beſt ſuiting with their buſineſs: whereas the Geometrician makes uſe only of their Nonconcurrency. And our great Geometrician the Famous Savilian Profeſſor of Geometry in *Oxford* Dr *Wallis* ſays, *Parallelifmus & Æquidiſtantia vel idem ſunt, vel certe ſe mutuo comitantur.*

Seeing therefore that theſe Notions are naturally, and in common Senſe immediately conjoyn'd, we do retain *Euclid's* Definition of Parallels, and have *aſſumed for a Geometrical Axiom their Equidiſtancy.*

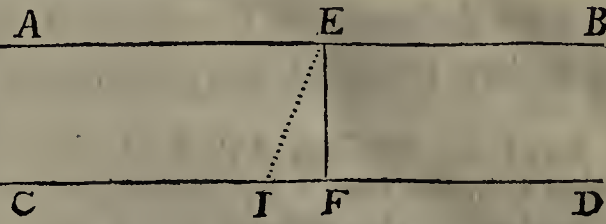
If this may be ſo allowed, or at leaſt admitted, as a more clear and obvious Notion than the 11th. Ax. of *Euclide*, then ſhall we briefly demonſtrate that troubleſome part of the 29th. Propoſition, concerning the equality of the Alternate angles in Parallels, without any uſe of the 11th. Axiom.

The Demonſtration of the equality of the Alternate Angles in Parallels.

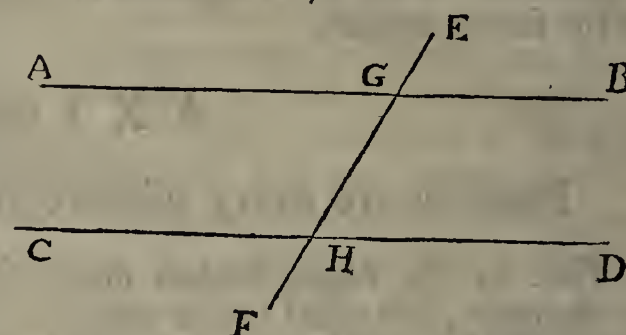
On the Parallels AB, CD, let firſt the ſtrait line EF fall on AB at Right angles.

I ſay, that EF likewiſe falls on CD at Right angles; and therefore makes the Alternate angles equal, and the outward angle equal to the inward and oppoſite, and the two inward angles equal to two Right. For if EF falls not at Right angles

gles on C D, let E I fall at Right angles on C D, [by Prop. 12.] therefore E I is the diſtance of the Parallels A B, C D, [Def. 35.] Likewiſe F E falling on A B at Right angles, [by Suppoſition] is the diſtance alſo of the ſame Parallels. Wherefore F E, E I, are equal [by Ax. 12. that Parallels are Equidiſtant ſtrait lines.] And becauſe of the Triangle E F I, the ſides E F, E I, are equal, therefore the angles at the baſe E F I, E I F are equal. But E I F is by Conſtruction a Right angle, wherefore E F I is a Right angle, ſo that the angles at the baſe are equal to two Right. But they are leſs [by Prop. 17.] therefore E I is not at Right angles to C D, and by the ſame reaſon no other can be drawn from the point E beſides E F. Wherefore E F is at Right angles to C D, and alſo it is at Right angles to A B [by Suppoſition.] therefore all the angles at E and F are Right, and equal to one another.



Again, on the Parallels A B, C D, let the ſtrait line E F fall otherwiſe at adventure. I ſay, that it makes the Alternate angles A G H, G H D, equal to one another.

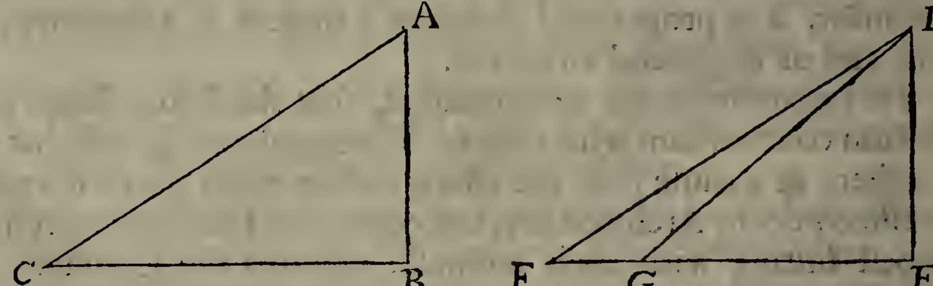


Now for the demonſtration thereof we ſhall premiſe this Lemma.

A Lemma is a Propoſition taken in by the by, to make way for the proof of ſome Principal Propoſition.

L E M M A.

If two Right angl'd Triangles A B C, D E F, have the ſide A C, ſubtending the right angle B, equal to the ſide D F, ſubtending the right angle E: and a ſide A B about the right angle B, equal to the ſide D E, about the right angle E, then ſhall they have the remaining ſide B C, equal to the remaining ſide E F. For if B C be not equal to E F, then one of them is the greater. Let E F be the greater, and from the greater E F take the line E G equal to B C the leſs; and let be drawn G D.



Foraſmuch then as E G is made equal to B C, and E D is equal to B A, [by Suppoſition] and they contain right angles at E and B, therefore the baſe D G ſhall be equal to baſe A C [Prop. 4.] But A C is equal to D F [by Suppoſition] therefore D G is equal to D F: ſo that in the Equicrural Triangle D G F, the angles at the baſe D F G, D G F, are equal to one another. But D G F is greater then the right angle D E G, (*the outward greater then the inward and oppoſite*, by Prop. 16.) therefore the angles D G F, D F G, are greater than two Right: which is impoſſible [by Prop. 17.] therefore the line B C is not unequal to E F, equal therefore they are to one another. Which was to be demonſtrated.

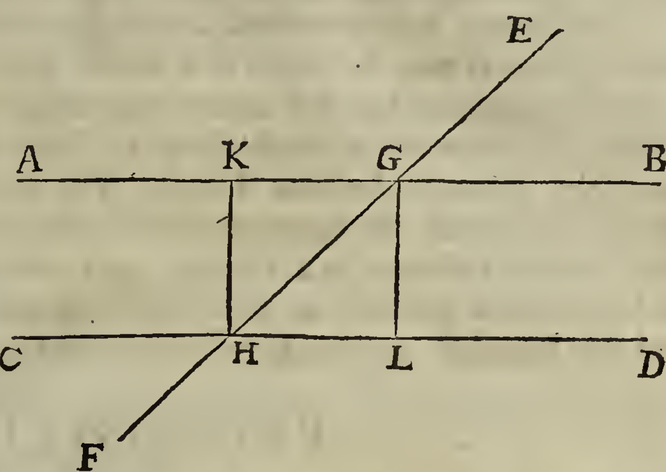
After the demonſtration of this Lemma, we thus further proceed to prove the equality of the Alternate angles in any Oblique Section.

On the Parallels A B, C D, let the ſtrait line E F fall at adventure. I ſay, that it makes the Alternate angles A G H, G H D, equal to one another: and alſo the Alternate angles C H G, H G B, equal to one another.

For from the point H to the line A B, draw a perpendicular H K [by Prop. 12.] Again, from the point G to the line C D draw a perpendicular G L.

Now foraſmuch as in the right angl'd Triangles H K G, G L H, the line H K is equal to the line G L, for that each is the diſtance of the ſame Parallels, [Def. 35. and

35. and Ax. 12.] and H G, subtending the right angles at K and L, common, therefore the remaining side K G, is equal to the remaining side L H (by the precedent Lemma). Wherefore there are the two lines H K, K G, equal to the two lines G L, L H, each to each, and they contain equal angles, namely Right; therefore [by Prop. 4.] the angles K G H, G H L, are equal; that is, in the Parallels A B, C D, the Alternate angles A G H, GHD, are equal to one another.



Again, because the angles A G H, H G B, are equal to two Right [Prop. 13.]; and likewise C H G, G H D, are equal to two Right, therefore A G H, H G B, are equal to C H G, G H D. Taking therefore away the equal Alternate angles A G H, G H D, the remaining Alternate angles C H G, H G B, are equal to one another.

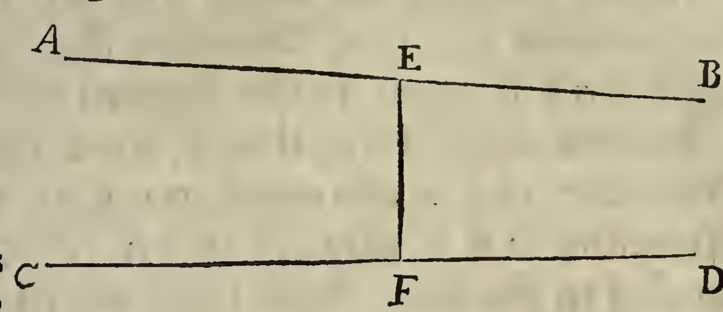
Wherefore on Parallel lines a strait line falling, doth make the Alternate angles equal to one another, &c. Which was to be demonstrated. *Now what follows in Euclide* is without exception.

Having thus demonstrated this 29th. Proposition without the help of the 11th. Axiom; we shall next demonstrate that 11th. Axiom.

A Demonstration of the Eleventh Axiom of *Euclide*.

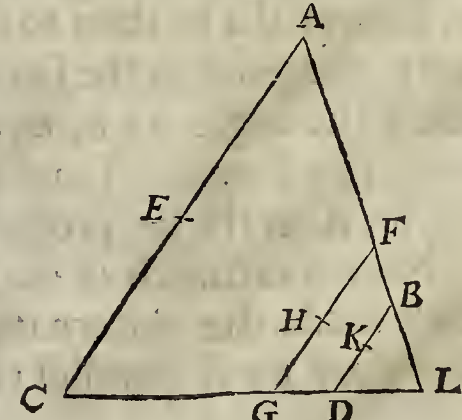
If on two strait lines A B, C D, a strait line E F falling, doth make the inward angles toward the same parts E F D, F E B, less than two Right: I say, that the lines A B, C D, being infinitely produced toward the parts of B, D, where the angles are less then two Right, shall meet together.

Forasmuch as the angles A E F, F E B, are equal to two Right [Prop. 13.]; and C F E, E F D, are equal to two Right, therefore these four are equal to four Right angles. But EFD, F E B, are less than two Right [by Supposition]; therefore the angles E F C, F E A, are greater then two Right. Wherefore the lines B A, D C, being infinitely produced toward the parts of A, C, shall that way never meet [by Prop. 17.]. If now they meet not toward the parts of B, D, then the lines A B, C D, are Parallels [Def. 34.]; and Parallels have the two inward angles toward the same parts equal to two Right [by Prop. 29.]. But the angles E F D, F E B, are supposed less; and to be less and equal to the same is impossible; Therefore the lines A B, C D, being infinitely produced toward the parts of B, D, shall meet together. Which was to be demonstrated.



Notwithstanding this, it is demonstrable that two strait lines drawn from angles less than two Right, may in some manner be for ever prolonged; yet shall they never meet together.

For let the strait lines A B, C D, be cut by A C making the inward angles B A C, D C A, less than two Right. Now let A C be cut into halves, or otherwise in E: and equal to E A let be put A F, and to E C, C G; than draw F G. Again, let F G be cut in H, and equal to H F, let be put F B, and to H G, G D; then draw B D. I say, that the lines A B, C D, may for ever be *Thus* prolong'd, yet never shall they meet together. For if possible, let them meet in the point L; therefore B D being cut in K, the line K B shall be



equal

equal to BL, and KD to DL. Wherefore of the Triangle BDL the ſides DL, LB, ſhall be equal to the third ſide BD, which is impoſſible by the 20^{th}. Propoſition. Therefore the lines AB, CD, drawn from angles leſs than two Right, may for ever be prolonged, and never meet together. Which was to be demonſtrated.

From hence it is manifeſt, that Magnitude is infinitely diviſible: and that an infinite progreſs may be made in a finite Space.

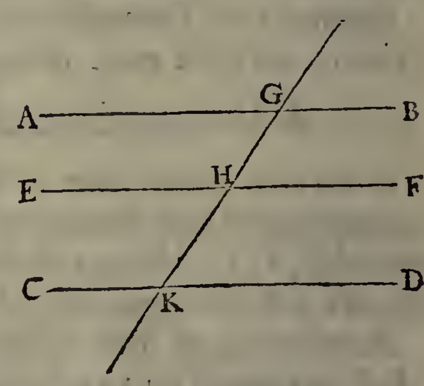
And moreover for the better underſtanding of *Euclide* in this matter, we are to diſtinguiſh between a production of lines *εἰς ἄπειρον, in infinitum,* Infinitely, and *ἀπειράκις, Infinities,* Infinite Times. The former is an unlimited, free courſe of prolongation, ſuch as Geometricians always underſtand by *εἰς ἄπειρον*. The other here in this Inſtance is a limited and reſtrained prolongation, made ſtep by ſtep, and in ſuch a manner as that the ſteps are ſhorter, and ſhorter made continually, and the lines are approaching nearer and nearer; yet ſo as never to meet together.

PROPOSITION XXX.

STrait lines Parallel to the ſame ſtrait line, are alſo Parallel to one another.

Let each of the lines AB, CD, be parallel to EF: I ſay, that AB is parallel to CD.

For let a ſtrait line GK fall upon them.



Now foraſmuch as the ſtrait line GK falls on the parallels AB, EF; therefore the angle AGH is equal to the Alternate angle GHF, [Prop. 29.]

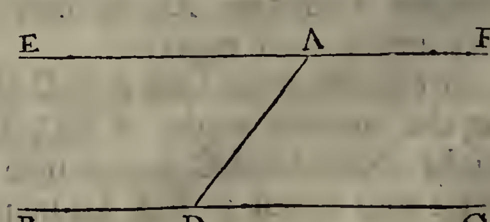
Again, becauſe the ſtrait line GK falls on the parallels EF, CD; therefore the outward angle GHF is equal to the inward and oppoſite GKD, [Prop. 29.]

But the angle AGH, that is, AGK has been proved equal to GHF. Therefore AGK is alſo equal to GKD: and they are Alternate angles; wherefore AB is parallel to CD [Prop. 27.]. Therefore ſtrait lines parallel to the ſame ſtrait line, are parallel to one another. Which was to be demonſtrated.

PROPOSITION XXXI.

BY a given point to draw a ſtrait line parallel to a ſtrait line given.

Let the given point be A, and the given ſtrait line be BC. It is required by the point A, to draw a ſtrait line parallel to BC. In the line BC, let be taken any point as D, and let be joyn'd AD: then to the ſtrait line DA, and to the point in the ſame A, let be conſtituted the angle DAE, equal to the angle ADC, [by Prop. 23.] and to the ſtrait line EA, let directly be produced the line AF.



Now foraſmuch as on the lines BC, EF, the ſtrait line AD falling, hath made the Alternate angles EAD, ADC, equal to one another; therefore EF is parallel to BC, [Prop. 27.]; wherefore by the given point

point is drawn the ſtrait line EAF, parallel to the given ſtrait line BC. Which was to be done.

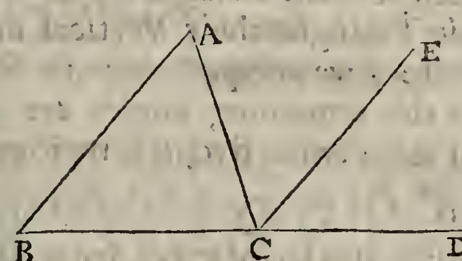
PROPOSITION XXXII.

Of every Triangle one of the ſides being produced, the outward angle is equal to the inward and oppoſite.

And the three inward angles of a Triangle are equal to two Right.

Let the Triangle be ABC, and one of the ſides BC, be produced to D.

I ſay, that the outward angle ACD is equal to the two inward and oppoſite CAB, ABC. And of that Triangle the three inward angles ABC, BCA, CAB, are equal to two Right.



For by the point C, let CE be drawn parallel to AB. [Prop. 31.] Now foraſmuch as AB is parallel to CE, and on them falls AC; the Alternate angles BAC, ACE, are equal to one another. [Prop. 29.] Again, becauſe AB is parallel to CE, and on them falls the ſtrait line BD; the outward angle ECD, is equal to the inward and oppoſite ABC. But it hath been prov'd that ACE is equal to BAC; therefore the whole outward angle ACD is equal to the two inward and oppoſite BAC, ABC.

Let the angle ACB be added in common, therefore the angles ACD, ACB, are equal to the three angles ABC, BAC, ACB: But the angles ACD, ACB, are equal to two Right; [Prop. 13.] therefore ABC, BAC, ACB, are alſo equal to two Right.

Therefore of every Triangle one of the ſides being produced, the outward angle is equal to the two inward and oppoſite.

And the three inward angles of a Triangle, are equal to two Right. Which was to be demonſtrated.

Corollaries.

1. Of an Equilateral Triangle all the three angles are given.

For each angle is a third part of two Right angles, that is, 60 Degrees of 180; or two third parts of one Right angle, that is 60 of 90 Degrees.

2. Of an Equicrural Triangle if one angle be given, the other two are alſo given.

For the angles at the baſe are equal, and the third angle compleats, or makes up two Right angles, that is, 180 Degrees, or twice 90.

3. Of a Scalene Triangle, if two angles be given, the third is alſo given; and if one angle be given, the ſumm of the other two is alſo given.

For theſe angles added to the given angle, compleat two Right angles. As if the given angle be 60, the ſumm of the other two is 120, which together make 180, or two Right angles.

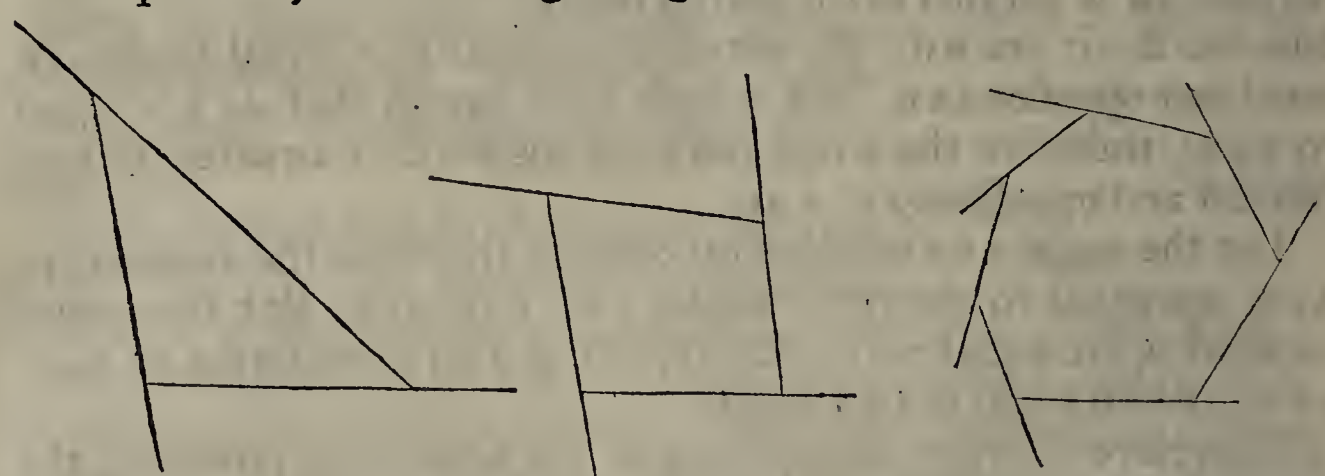
4. Of a Scalene Right-angl'd Triangle, if one of the acute angles be given, the other is alſo given.

For each of the Acute angles is the Complement of the other to a Right angle. As if one be 60 Degrees, the other is 30, which together Compleat 90 Degrees, or a Right angle.

5. Of every Multilateral Figure the inward Angles are equal to twice ſo many Right Angles, leſs by four, as is the number of their Sides.

For from any point within the Multilateral Figure, let ſtrait lines be drawn to every angle, then ſhall there be made ſo many Triangles as is the number of the Sides. As in a Figure of five Sides, there ſhall be five Triangles, which contain twice five, or ten Right angles. And of theſe Triangles their Vertical angles about the point within, are always equal only to four Right angles: wherefore the remaining angles are equal to ſix Right angles, that is, to twice five, leſs by four. And the like in all other Multilateral Figures.

6. Of every Multilateral Figure, the outward angles are altogether equal only to four Right angles.

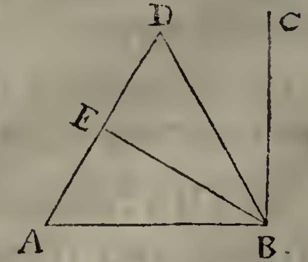


For each inward angle with it's outward, are together equal to two Right angles; and all the inward angles are equal to twice ſo many Right angles, leſs by four as in the number of their Sides: Therefore all the outward angles are equal only to four Right angles. The ſame is likewiſe manifeſt in all Quadrilateral and Trilateral Figures.

ANNOTATIONS.

A Problem.

To divide a Right angle into three equal angles.



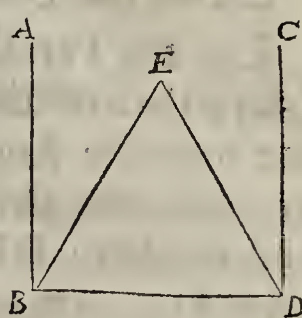
From hence 'tis manifeſt, how to triſect a Right angle. For let A B C be a Right angle, and on A B let be conſtituted an Equilateral Triangle A B D. Now becauſe the angle A B D is two third parts of the Right angle A B C, [*by the firſt Corollary*]; therefore the angle D B C is one third of the ſame Right angle. Again, let the angle A B D be biſected by the line B E, then ſhall each angle A B E, E B D, be a third part of the Right angle. Wherefore the Right angle A B C is divided into three equal angles A B E, E B D, D B C.

Archimedes lays the foundation of his menſuration of a Circle upon the Triſection of a Right angle, and the diviſion of an Equilateral Triangle into two Right-angl'd Triangles; in each of which one Acute angle is known to be the double of the other, one to be 60, the other 30 Degrees: and the Side ſubtending the Right angle to be alſo double of the Side ſubtending the leaſt angle, that is, A B to be double of A E. Upon which grounds he demonſtraſtratively proceeds to his Immortal Glory.

If

If every ſtrait lin'd angle, Obtuſe and Acute, could likewiſe be Geometrically Triſected, it would alſo be of excellent uſe. But this lyes in the ſame obſcurity with the Quadrature of a Circle, and the Duplication of a Cube; and the pretenders to the Solutions of theſe Problems have all hitherto ſhamefully miſcarry'd in their vain attempts, and overweening opinion of themſelves.

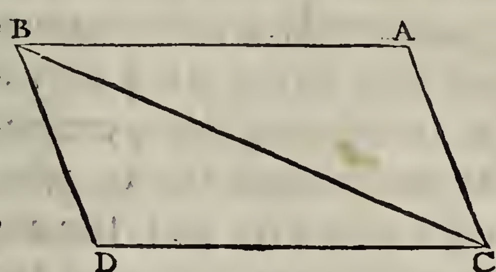
Laſtly, to look into the Phyſical reaſon, why the three angles of a Triangle are equal to two Right, it may thus plainly appear. For let the lines A B, C D, be at Right angles to B D. If they be ſuppoſed to incline toward each other till they meet in the point E; then what is by this inclination diminiſhed from the Right angles A B D, C D B, the ſame is again reſtored in the angle B E D; ſo that the three angles E B D, B D E, B E D, are equal to the two Right angles A B D, C D B.



PROPOSITION XXXIII.

*Strait lines, which * the ſame way joyn equal and parallel lines, they alſo are equal and parallel.*

Let the equal and parallel lines be AB, CD, and the ſtrait lines, which the ſame way joyn them be AC, BD. I ſay, that AC, BD, are alſo equal and parallel: For let be drawn BC. Now foraſmuch as AB is parallel to CD, and on them falls BC, the Alternate angles ABC, BCD, are equal to one another; and becauſe AB is equal to CD, and BC common: therefore the two lines AB, BC, are equal to the two lines BC, CD, and the angle ABC is equal to the angle BCD, therefore the baſe AC is equal to the baſe BD, and the Triangle ABC is equal to the Triangle BCD, and the remaining angles ſhall be equal to the remaining angles, under which are ſubtended equal ſides: therefore the angle ACB is equal to the angle CBD. And becauſe on the two ſtrait lines AC, BD, the ſtrait line BC falling, hath made the Alternate angles ACB, CBD, equal; therefore AC is parallel to BD, [Prop. 27.]; and it hath been proved to be alſo equal to the ſame. Therefore ſtrait lines, which the ſame way joyn equal and parallel lines, they alſo are equal and parallel. Which was to be demonſtrated.



ANNOTATIONS.

* *Which the ſame way,*] That is, from the point A to the point C, and from the point B to the point D: not croſs-ways from A to D, and from B to C.

Becauſe the two ſtrait lines, which joyn equal, and parallel lines are here prov'd to be equal, and parallel to one another, therefore the comprehended ſuperficies now found to be bounded by parallel lines, is called a *Parallelogram ſpace*: as follows in the next Propoſition. Therefore it is not properly ſaid to be a Parallelogram Figure, but a Parallelogram Space, as incloſed by parallel lines, which Space, or Area, is the thing conſidered in all *Euclid's* Propoſitions concerning Parallelograms. And a ſtrange overſight it was in *Clavius* (otherwiſe a moſt faithful Expoſitor) to give a particular definition of a Parallelogram, as a diſtinct Figure, after *Euclide* had defined all the kinds of Quadrilateral Figures. *Quandoque bonus dormitat Homerus.*

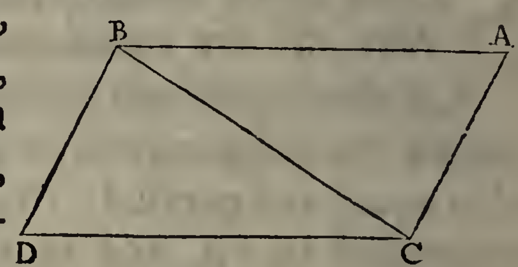
This Theorem plainly diſcovers the natural Origin and Geneſis of Parallelogram ſpaces, from two equal and parallel lines conjoyn'd by two other ſtrait lines. A notion very remarkable.

PROPOSITION XXXIV.

OF Parallelogram Spaces the oppoſite ſides, and alſo the oppoſite angles, are equal to one another.

And the Diameter cuts the ſame into halves.

Let the Parallelogram be ACDB, and the Diameter thereof BC. I ſay, that of the Parallelogram ACDB, the oppoſite Sides, and alſo the oppoſite Angles, are equal to one another. And the Diameter BC, cuts the ſame into halves. Foraſmuch as AB is parallel to CD, and on them falls the ſtrait line BC, therefore the Alternate angles ABC, BCD, are equal to one another. Again, becauſe the line AC is parallel to the line BD, and on them falls the ſtrait line BC, therefore the Alternate angles ACB, CBD, are equal to one another.



There are then the two Triangles ABC, CBD, having the two angles ABC, BCA, equal to the two angles BCD, CBD, each to each; and one ſide equal to one ſide, that is, the ſide adjacent to the equal angles, namely BC common to both. Therefore [by Prop. 26.] they ſhall have the remaining ſides equal to the remaining ſides, each to each, and the remaining angle equal to the remaining angle: wherefore the ſide AB is equal to the ſide CD, and AC to BD: and the angle BAC to the angle BDC. And becauſe the angle ABC is equal to the angle BCD, and the angle CBD to the angle ACB; therefore the whole angle ABD, is equal to the whole angle ACD: and it is proved, that the angle BAC, is equal to the angle BDC. Therefore of Parallelogram Spaces the oppoſite Sides and alſo the oppoſite Angles, are equal to one another.

I ſay alſo that the Diameter cuts the ſame into halves.

Foraſmuch as AB is equal to CD, and BC common, therefore there are two lines AB, BC, equal to the two lines BC, CD, each to each; and the angle ABC, is equal to the angle BCD, wherefore alſo the Baſe AC is equal to the Baſe BD, and therefore the Triangle ABC is equal to the Triangle BCD: wherefore the Diameter BC cuts the Parallelogram ACDB into halves. Which was to be demonſtrated.

ANNOTATIONS.

The name of Parallelogram Spaces, we have noted to be literally formed (as in common ſpeech) from the termination of Planes made by parallel lines; and this name extends only to the Square, Oblong, Rhombus, and Rhomboeid: wherefore after the Definitions of theſe four Quadrilateral Figures, *Euclide* defines not a Parallelogram; for that he had then inartificially defin'd anew, what was before defined. But now upon this common affection here demonſtrated, he does comprehend under that one name the Square, Oblong, Rhombus, and Rhomboeid: ſo that what Properties at any time are demonſtrated upon Parallelograms in general, that is Parallelogram ſpaces, do alike belong to all theſe four Figures.

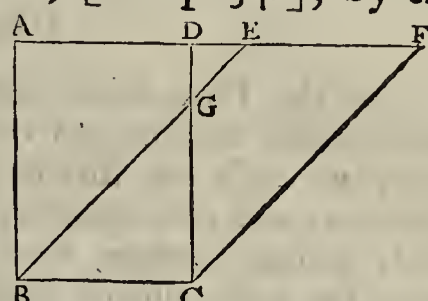
Euclide proceeds after the ſame manner in Solids, at Prop. 24. and 25. El. XI. where having laid down in diſtinct words a Solid comprehended by parallel Planes, he he next after this in one compound word, calls the ſame Solid a *Parallelepipedon*; without premiſing any Definition.

The Commentators therefore were not in this point well adviſed, who give a formal Definition of a Parallelogram, as if it were a Figure of an other kind, than what had been already defined by *Euclide*.

PROPOSITION XXXV.

Parallelograms on the ſame Baſe, and in the ſame Parallels, are equal to one another.

Let the Parallelograms be ABCD, EBCF, on the ſame Baſe BC, and in the ſame Parallels AF, BC. I ſay, that the Parallelogram ABCD, is equal to the Parallelogram EBCF. Foraſmuch as ABCD is a Parallelogram; therefore AD is equal to BC, [Prop. 34.]; by the ſame reaſon alſo EF is equal to BC: ſo that AD is equal to EF, and DE is common; therefore the whole AE is equal to the whole DF; but alſo AB is equal to DC. Wherefore the two lines EA, AB, are equal to the two lines FD, DC, each to each, and the angle FDC is equal to the angle EAB, the outward to the inward; therefore the Baſe EB is equal to the Baſe FC, and the Triangle EAB, is equal to the Triangle FDC. Let DGE common to both be taken away: then ſhall the Trapezium ABGD be equal to the Trapezium EGCF. Let the Triangle GBC be added in common: therefore the whole Parallelogram ABCD, is equal to the whole Parallelogram EBCF: wherefore Parallelograms on the ſame Baſe, and in the ſame Parallels, are equal to one another. Which was to be domonſtrated.



ANNOTATIONS.

Of Geometrical Places.

When in Theorems, or Problems, the ſame thing may be alike in ſeveral places *Indeterminately*, then is this call'd the *Geometrical Place* of that Theorem, or Problem, and theſe kind of Propoſitions are call'd Local Theorems, and Local Problems. As in this 35. Prop. it evidently appears, that to the Parallelogram ABCD, there may be infinite other equal Parallelograms, on the ſame baſe AB, in the ſame parallel lines: ſo that of one of the parallels the whole line, as AF, infinitely produced, is the common Place of this Equality in Parallelograms ſeated on the ſame baſe: The like alſo is on equal baſes. And moreover in Triangles on the ſame, or equal baſes; as it is demonſtrated in the 36, 37, 38, and 41. following Propoſitions; This is ſaid to be *Locus planus ad lineam rectam.* Likewiſe there are Geometrical Plane Places of the ſame nature, found in the Circumference of a Circle. As if it be required to draw from the ends of a ſtrait line two ſtrait lines, which ſhall contain a Right angle; 'tis evident by Prop. 21. and 31. El. III. that in a Semicircle every one of the angles is a Right angle, ſo that the Circumference of a Semicircle, is the Geometrical Place of a Right angle. This is ſaid to be *Locus planus ad Circumferentiam circuli*, and the Problem called a plane Problem, or a Problem *in loco plano.*

Beſides theſe plane Places in ſtrait lines, and the Circumference of a Circle, there are alſo *Loci ſolidi*, Geometrical ſolid Places, which admit of ſuch ſolid Problems. Theſe are found in the *Conic Sections*, namely, the *Parabola*, *Hyperbola*, and *Ellipſis*. They

They are called solid Places, and solid Problems, notwithstanding that these Figures lye in a plain superficies, because they have their Origin in a solid Figure, as the CONE: and are made by the cutting of a Conical Superficies with a Plane: as the Conic Elements of *Apollonius* shew, how these Figures are seated and created in a Conic Body. *Prop.* 11, 12, 13. Lib. I.

There are also Lineary Problems differing much from these Solid, and Plain Problems, tho' they be described in a simple plain superficies; but not by a simple motion, as is the strait line, and Circle. They are therefore in a special manner called *Lineary*, because their solutions are effected by certain lines arising from compounded, and involved motions. Such is the Helix or Spiral line of *Archimedes*, the Conchoid of *Nicomedes*, the *Lineæ Tetragonizantes*, or *Quadratrices*, with divers others described by the Ancients, and Moderns. See *Pappus* after *Prop.* 4. Lib. III. and *Prop.* 30. Lib. IV.

Geodæsia, or the Mensuration of Plain Figures.

Elementary Annotations.

Upon this Proposition, and some of the next following, is grounded the Doctrine of the Mensuration of all Plain Figures, as to their superficial Content, or *Area*; which is one sort of practical Geometry deduced from these Speculative Elements; and of a necessary use in many human Affairs. This Doctrine is commonly named *Geodæsia*, from the Partition and Distribution of Lands; it being one of the most valuable Matters handled in this part of Geometry: And with us particularly called the *Art of Surveying*. But the use of the word *Geodæsia*, like as the word *Geometria*, is enlarged beyond its original signification, and extended to the general Doctrine of the Mensuration of all sorts of Figures in a plain superficies. And to this use fully answers the name *Epipedometria*, or *Planometria*, an easier word, tho' Critically not so proper, as being compounded of Latin and Greek.

Now in all kind of Mensuration, whatsoever is taken for a measure whereby to estimate and value any proposed quantity, the same must be certain and determined. In *Discrete* quantity it is an *Unite*, which naturally measures all Numbers. In *Continued* quantity, as Magnitude, it must be a *supposed Unite* to measure Magnitudes. I say *supposed*; for that Magnitude being a quantity infinitely divisible, has no indivisible unite in it self, whereby to measure Magnitudes, as Number has an indivisible unite to measure Numbers. But instead thereof we make to our selves by mutual agreement some certain measures, as an Inch, or Foot, in every kind of Magnitude, which as a Geometrical unite may answer to an unite in Numbers, so that in Magnitudes the Geometrical measure is only a *supposed Unite taken by Consent*.

As *some one* strait line is put to measure Lengths: And let this measure be called the *Lineal Unite*.

Some one plain Figure to measure plain Figures: And let this be called the *Superficial Unite*.

Some one Solid Figure to measure Solids: And let this be called the *solid Unite*.

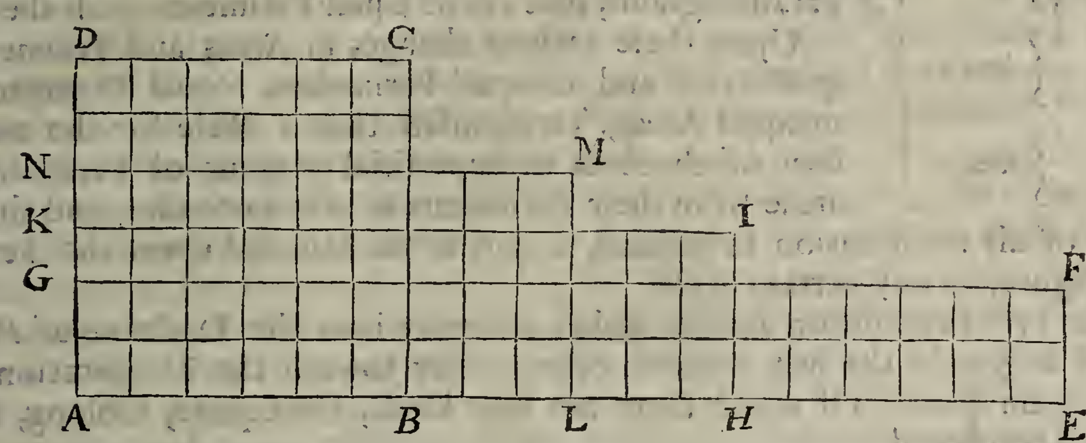
The value then, or estimate of any Magnitude is made from the multitude or number of the Geometrical measuring Unites, which that Magnitude shall contain: Be they Lineal, Superficial, or Solid Unites, according to the species of the Magnitude, as it is either a length, a superficies, or a solid, which is proposed by some certain measure to be estimated.

In the Mensuration of Lengths, there is no other trouble than to agree upon what *known Length* the Lineal Unite, or measuring Line shall be. Whether Inch, Foot, Yard, Pearch, or any other Civil and Political measure, according to the custom of the Place.

But in the Mensuration of Planes, which is according to Length and Breadth, it is not only a known superficial quantity to be agreed upon, but also *what Figure of a known superficial quantity*, is most proper to be the common measure of all plain Figures.

This matter requires some Artifice, in regard of divers mistakes that may arise in in the Management of it. For no certain rule of Menſuration can be made from the circumambient bounds, or Perimeter of a plain ſtrait-lin'd Figure (as vulgarly may be imagined) becauſe ſuch plain Figures may be of equal quantity in their Areas, yet of very unequal Perimeters: and contrarily of equal Perimeters, yet of very unequal Areas.

As for Example, let the Figure ABCD be right angl'd, and have the ſide AB 6 inches in length, and the ſide BC as much in breadth: and accordingly let the Figure be divided by parallel lines. So now it is eaſily demonſtrated from the Diagram, that the four ſides of this Figure (which are its Perimeter) ſhall be 24 inches; and the whole Area ſhall contain 36 ſquare inches: As is alſo found by multiplying 6 into 6; that is, by drawing the length AB into the breadth BC; which is the general Rule of all ſuperficial Menſurations; for that every figurate ſuperficies is to be meaſured by the two dimenſions of length and breadth.



Again, let the Oblong AEFG, have the ſide AE 18 inches in length, and the ſide EF 2 inches in breadth: ſo the Perimeter ſhall be 40 inches; yet the Area is but 36 ſquare inches, and equal to the Area of the ſquare ABCD, whoſe Perimeter is but 24 inches.

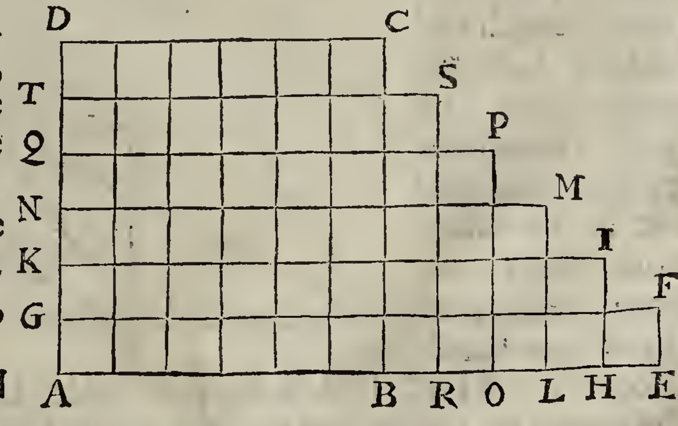
Likewiſe let the Oblong AHIK have the ſide AH 12 inches, and HI 3 inches, the Perimeter then ſhall be 30 inches, and the Area ſtill 36 ſquare inches.

Again, let the Oblong ALMN have the ſide AL 9 inches, and the ſide LM 4 inches, then the Perimeter ſhall be 26 inches; and the Area as before 36 ſquare inches.

Areas equal all 36.			Perimeters all unequal.
	AB 6	Perimeter 24.	
	BC 6		
	AE 18	Perimeter 40.	
	EF 2		
	AH 12	Perimeter 30.	
	HI 3		
	AL 9	Perimeter 26.	
	LM 4		

In theſe Figures we have the Areas equal, and the Perimeters unequal: But the Perimeter of the ſquare is the the leaſt; and in Oblongs of equal Areas with the ſquare, where they differ moſt from a ſquare Figure, that is, where the difference between the length and breadth is the moſt, there the Perimeter is the greateſt; and as the difference becomes leſs and leſs, ſo the Perimeter is leſs and leſs, till in the ſquare it is the leaſt of all.

Again on the contrary, let the ſquare ABCD be as before; and let AEFG the Oblong, have the ſide AE 11 inches in length, and the ſide EF, *One* inch in breadth, then the Perimeter is 24 inches, and equal to the Perimeter of the ſquare ABCD; yet the Area is only 11 ſquare inches, whereas that of the ſquare is 36.



Likewiſe let the Oblong AHIK have the ſide AH 10 inches, and the ſide HI 2 inches; the Perimeter is again 24 inches, but the Area 20 ſquare inches.

So farther, let the Oblong ALMN have the ſide AL 9 inches, and the ſide LM 3 inches: wherefore the Perimeter is alſo 24 inches; but the Area is 27 ſquare inches.

Again, let the Oblong AOPQ have the ſide AO 8 inches, and the ſide OP 4 inches: the Perimeter here is 24 inches; but the Area 32 ſquare inches.

Laſtly, let the Oblong ARST have the ſide AR 7 inches, and the ſide RS 5 inches: the Perimeter is ſtill 24 inches; but the Area is 35 ſquare inches.

Perimeters equal, all 24.	Sides		Areas all unequal.
	AB	6	Area 36
	BC	6	
	AE	11	Area 11
	EF	1	
	AH	10	Area 20
	HI	2	
	AL	9	Area 27
	LM	3	
	AO	8	Area 32
	OP	4	
	AR	7	Area 35
	RS	5	

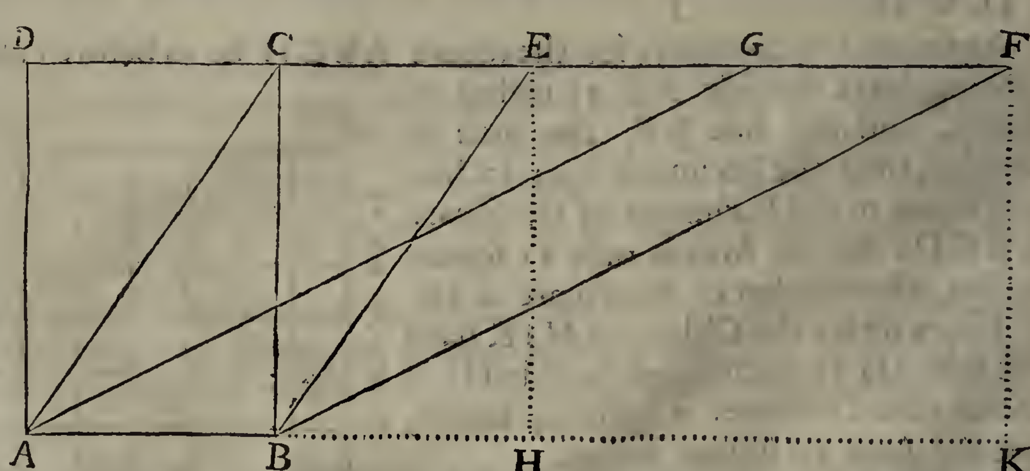
In theſe Figures we have the Perimeters equal, and the Areas unequal: but the Area of the ſquare is the greateſt. And obſerve that the nearer any Rectangle comes to a ſquare Figure, that is, where the difference between the length and breadth is the leſs, there the Rectangles of equal Perimeters are the more Capacious: ſo that where the difference is nothing at all, that is, where the Figure is a ſquare, there the Area is the greateſt in reſpect of all the Parallelograms, that are of equal Perimeters with the ſquare.

Upon theſe various changes in Areas and Perimeters, equal Areas and unequal Perimeters, equal Perimeters and unequal Areas, 'tis manifeſt that a Rule for the menſuration of the Area or Superficial content of Parallelograms, made from their Perimeters is very uncertain, and therefore the way of all menſuration in general, is not to be founded upon the Perimeter of the Figure, as any certain Rule.

In this 35th. Propoſition *Euclide* makes an entry into the Doctrine of *Planometry*, and begins in the firſt place to open a way toward the Menſuration of all Parallelogram ſpaces: Of which there are four kinds, the Square, Oblong, Rhombus, and Rhomboeid.

Firſt then, whereas it is here demonſtrated, that all Parallelogram ſpaces on the ſame baſe, and in the ſame parallels, are in their Space, or Area equal to one another; altho' it be evident that their Perimeters may be infinitely unequal, one Perimeter ſtill greater then another, as their angles are more and more oblique one than another, and accordingly two of their ſides are equally prolonged more and more infinitely; therefore no certain meaſure of the *equal Areas* of theſe Parallelograms can be taken from their *unequal Perimeters*.

To clear this Matter from its firſt ground, we are to recollect that a ſuperficies is a Magnitude of two dimenſions taken tranſverſly to one another, in length and breadth: and therefore every ſuperficial Figure is to be eſtimated by its *proper* length and breadth. It remains then to find out the proper length and breadth of theſe various Parallelograms, wherein they may all agree for their juſt eſtimation, in regard that they are in Area all equal to one another: and therefore ſome one kind of menſuration according to their *proper* length and breadth ought to be ſought, which ſhall be to every Parallelogram the ſame in quantity, and alſo common to them all: howſoever elſe they be differing from one another in their Perimeters, and the Obliquity of their Angles.

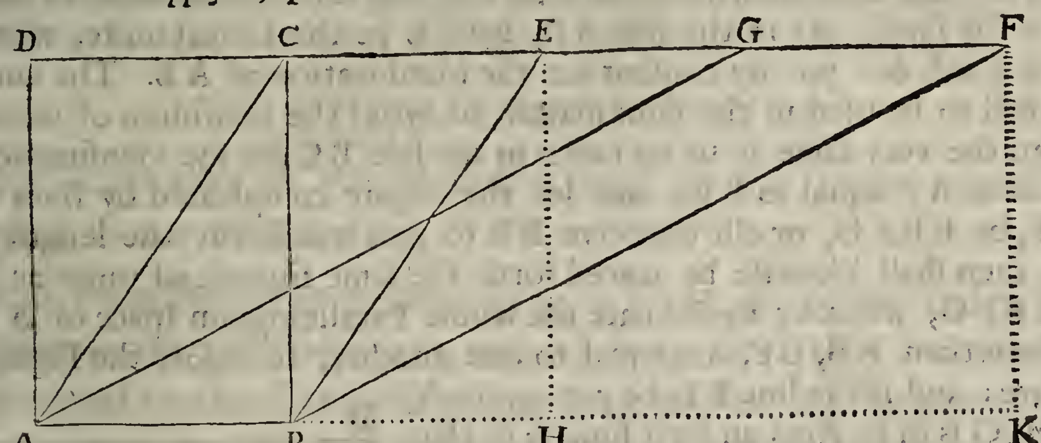


Upon enquiry it will be found manifeſt, that the proper length and breadth for the menſuration of theſe Parallelogram ſpaces, ought not to be taken from their oblique ſides. As of the oblique Parallelogram ABEC, if AB be put for its length (note that among Geometricians the names of length and breadth are indifferently apply'd to the longer, or to the ſhorter line) Let I ſay, the ſide AB be the length of the Figure ABEC, then the ſide

side BE ought not to be esteemed its proper measuring breadth. For besides that this falls into the former erroneous way of measuring by the Perimeter, the absurdity is also farther made thus manifest. If of the Parallelogram space ABEC, the side BE be its proper breadth, then likewise in the Parallelogram ABFG, having the same length AB, the side BE should as well be accounted the proper breadth thereof. Now because these Parallelogram spaces are equal to one another, and have the same length AB, therefore their breadths BE, BF, should also be equal to one another: but BE, BF, are easily demonstrated to be unequal; therefore their proper breadths are not to be esteemed by the sides BE, BF. In general therefore the oblique position of length to breadth, is a way altogether uncertain and undeterminable, as being infinitely variable, and so unfit for any Rule or common Practice in these superficial Mensurations.

Forasmuch then that neither from the Perimeter, nor from length and breadth taken obliquely towards one another, can be formed a Rule for the Mensuration of Parallelogram spaces; it necessarily follows that length and breadth are only to be taken at Right angles each to other, a way one and the same unalterable, commonly known, and easily practiced. And hereupon 'tis manifest, that the Right angl'd Parallelogram on the same base, and in the same parallels, is the *Standard* unto which all the other oblique angl'd Parallelograms are to be referred for their Mensurations. As the Areas of ABEC, ABFG, &c. are all to be known from the Area of the Rectangle ABCD; for that any two of its sides, which contain an angle, as AB, BC, or AD, DC, being at Right angles to one another, are the very proper length and breadth of this Parallelogram; one whereof being drawn into the other, brings forth the Area, which in this 35th. Prop. is demonstrated to be equal to all possible oblique angl'd Parallelograms on the same base, and in the same parallels.

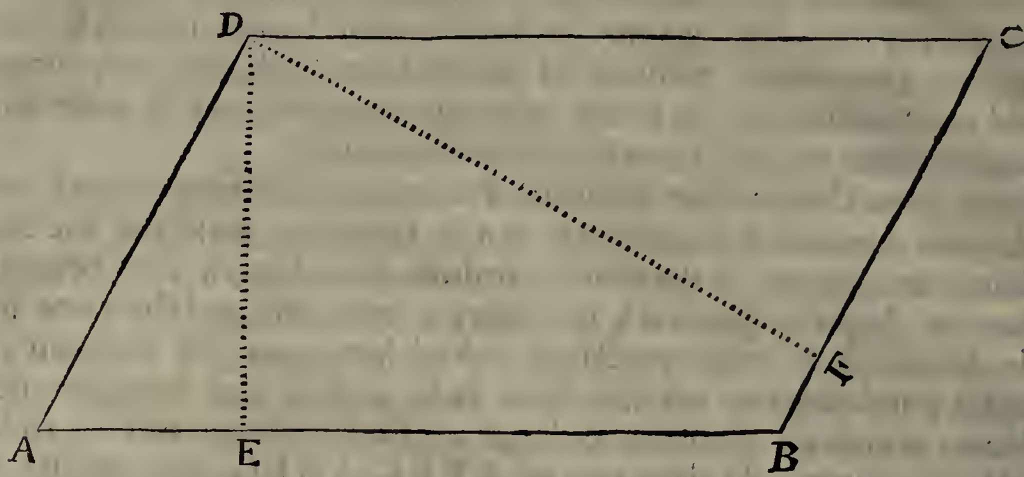
Therefore for the Mensuration of an oblique angl'd Parallelogram, it must be reduced to its Equivalent Rectangle: And this is done *by drawing from any one side a perpendicular to the opposite*, produced if need be.



As in the oblique Parallelogram ABEC, from the side CE, let be drawn EH perpendicular to the opposite side AB the base produced. Here then EH is equal to CB, or DA, the opposite sides of the Rectangle ABCD, for that they are parallel by Prop. 28. and therefore equal by Prop. 34. So that the perpendicular EH is the proper breadth of ABEC. And as the Rectangle ABCD is measured according to its proper length and breadth, by the base AB drawn into the perpendicular BC, so is the Rhomboeid ABEC measured by the same base AB, drawn into the perpendicular HE equal to BC, or AD: Likewise ABFG is measured by AB into KF. Thus the perpendicular is the only true, and common breadth of all Parallelograms on the same base, and in the same parallels. Therefore the Rule for the Mensuration of oblique angl'd Parallelograms is this.

In oblique angl'd Parallelograms, the base, and a perpendicular to the base, from the opposite side, drawn into one another, give the Area of the Parallelogram.

By the base is meant any one side of the Parallelogram taken two ways; either by by letting fall a perpendicular from the longer ſide upon its oppoſite as a baſe, or from the ſhorter ſide upon its oppoſite as a baſe. For in the Rhomboeid A B C D the perpendicular may be D E upon the baſe A B, or the perpendicular D F upon the baſe B C. From theſe two different cadencies of the perpendicular upon the baſe, the Rectangles are changed both in baſe and perpendicular: yet each Rectangle is equal to the ſame Rhomboeid by this 35th. Prop. And the Rhomboeid is indifferently eſtimated by either Rectangle.

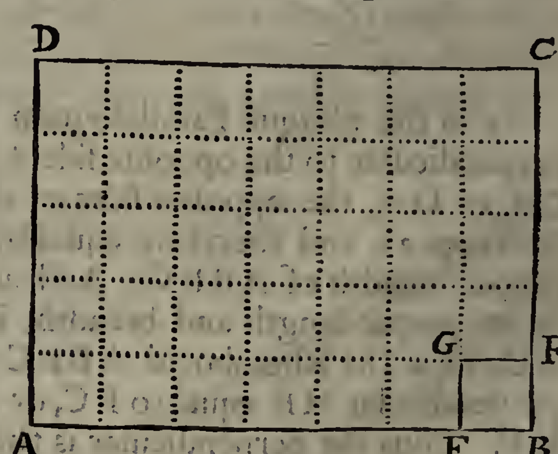


The perpendicular with its baſe; as D E with A B, or D F with B C, are called the *Latera recta*, or the upright ſides of the oblique Parallelogram, becauſe they make a right angled Parallelogram equal to the oblique.

Now for the Menſuration of a Rectangle, we are firſt to know the quantity of its length and breadth, that is, the diſtinct quantity of two ſides containing any of its angles. As for Inſtance of the Rectangle A B C D, we are to known the diſtinct quantity of the lines A B and B C.

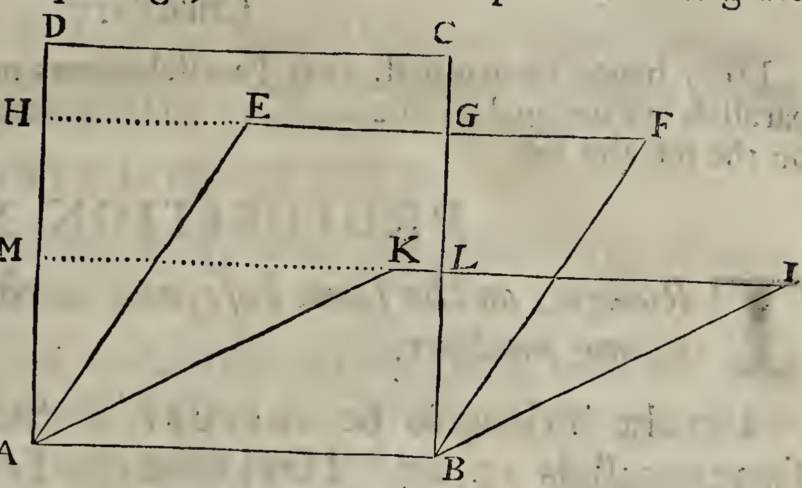
To find therefore the quantity of any propoſed length, we muſt have recourſe to ſome certain and known meaſure of lengths, for a *Lineal unite* to make an account by the ſame. As in the ſide A B, let B E be the Lineal unite, whether Inch, Foot, or Yard, &c. put by conſent for the Menſuration of A B. The ſame meaſure then is ſtill to be uſed in the ſame matter, to avoid the confuſion of meaſures; and therefore the very ſame is to be taken in the ſide B C for the Menſuration thereof; and let it be B F equal to B E; and let the Figure compleated by lines parallel to E B, B F, be E B F G, or elſe conceive E B to paſs tranſverſly the length B F, equal to B E, then ſhall likewiſe be traced forth the ſame ſuperficial unite or meaſuring Plane E B F G, whereby to eſtimate the whole Parallelogram ſpace of A B C D.

Whereas then E B, B F, are equal to one another, therefore the Figure E B F G is a ſquare: and if the line E B be put an inch, then EBFG is in its Area an inch ſquare; ſo that of the rectangle A B C D, if the ſide A B contain E B ſeven times, that is, ſeven Lineal units, as 7 inches, and according to the ſame meaſure the ſide B C 5 inches, then the whole ſpace ſhall contain 35 ſuperficial units, or ſquare inches, each of them equal to E B F G. And this at once is found by Multiplying 7 into 5; that is, by drawing the length A B into the breadth B C: The general ground of all ſuperficial Menſurations.



Thus have we ſhewn how an oblique Parallelogram ſpace, is firſt to be reduced to Rectangle for its Menſuration; and then how all Rectangular Spaces, Squares, and Oblongs, muſt be meaſured by ſome certain Square ſpace, whoſe ſide is a known meaſure of lengths.

Laſtly, therefore to confirm this matter, let there be put ſome known meaſure of lengths, which we call the LINEAL UNITE, and let it be the line A B, ſuppoſe a foot: and to A B let be put A D equal and at *Right angles*. Again, let A E be be put equal to AB, and at *an oblique angle*, then let be compleated the Figures ABCD, ABFE. Therefore ABCD is a ſquare, and ABFE a Rhombus, having every ſide equal to AB the *Lineal unite*, or ſuppoſed foot. Yet the Rhombus ABFE, is not equal in Area to the Square ABCD, but only to a part thereof, namely, to the Oblong ABGH by this 35th. Prop. And in this caſe the varieties are endleſs; as in the Rhombus ABIK, which is ſtill a leſs part of ABCD, and but equal to the Oblong ABLM, and ſo forth infinitely; therefore there is no certainty but in the ſquare Figure.



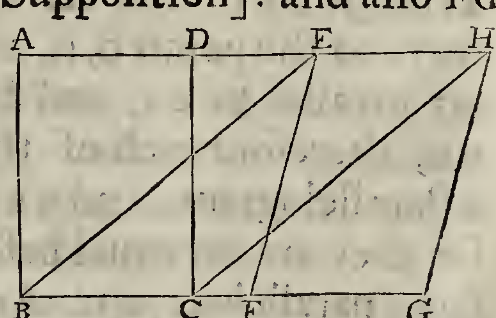
And as the *Square* is uſed in the Menſuration of *Planes*, ſo upon the like reaſons the *Cube* is uſed in the Menſuration of *Solids*. And in general it is to be obſerved, that whatever meaſure is at firſt put for lengths, the ſame paſſeth for length, breadth, and depth, and forms the meaſuring Square or Cube; be the Lineal unite either Inch, Foot, Yard, Perch, &c.

Thus for the Menſuration of all kinds of Magnitudes, as they are of *one*, or *two*, or *three* Dimenſions, there is in common practice conſtituted ſome certain meaſure conformable to each Dimenſion. And (as we muſt begin with the moſt ſimple Dimenſion) for lengths there is firſt made an agreement upon ſome *Lineal unite*: next, to continue in a certainty of meaſure, from the ſame Lineal unite is to ariſe the *Square unite* for Planes; and from the Square unite the *Cubic unite* for Solids. As to Inſtance in particular, a Lineal Inch, or Foot, &c. from this a Square Inch, or Foot, &c. then laſtly, a Cubic Inch, or Foot, &c. to be the Meaſuring unite according to the Dimenſions of the Magnitude, which is to be eſtimated by ſuch or ſuch a meaſure, Inch, or Foot, &c. ſuitable to its proper Dimenſion.

PROPOSITION XXXVI.

Parallelograms on equal baſes, and in the ſame parallels are equal to one another.

Let the Parallelograms be ABCD, EFGH, on equal baſes BC, FG, and in the ſame parallels AH, BG. I ſay, that the Parallelogram ABCD is equal to the Parallelogram EFGH. For let be joyn'd BE, CH. Now foraſmuch as BC is equal to FG [by Suppoſition]: and alſo FG is equal to EH [by Prop. 34.]; therefore BC is equal to EH; but alſo they are parallels by Suppoſition, and BE, CH, joyn the ſame. Now lines which the ſame way joyn equals and parallels, are alſo equal and parallel [Prop. 33.]. Therefore EB, CH, are equal and parallel: therefore EBCH is a Parallelogram, and is equal to ABCD; for it hath the ſame baſe BC, and is in the ſame parallels BG, AH. By the ſame reaſon, EFGH is equal to the ſame EBCH; ſo that alſo the Parallelogram ABCD, is equal to the Parallelogram EFGH.



Therefore Parallelograms on equal baſes, and in the ſame parallels, are equal to one another. Which was to be demonſtrated.

Corollary.

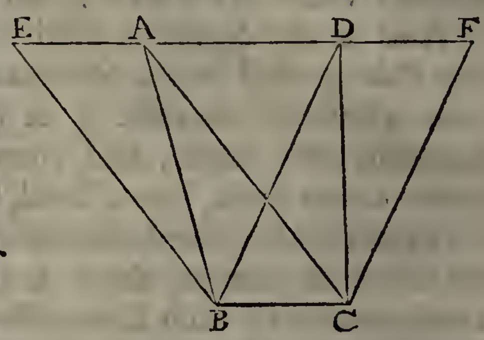
Corollary.

From hence 'tis manifeſt, that Parallelograms on unequal baſes, and in the ſame parallels are unequal to one another: on the greater baſe the greater Parallelogram, on the leſs the leſs.

PROPOSITION XXXVII.

Triangles on the ſame baſe, and in the ſame parallels are equal to one another.

Let the Triangles be ABC, DBC, on the ſame baſe BC, and in the ſame parallels AD, BC. I ſay, that the Triangle ABC, is equal to the Triangle DBC. Let AD be produced both ways to the points E, F, and by B let be drawn BE parallel to CA, and by C, CF parallel to BD: therefore each of theſe, EBCA, DBCF, is a Parallelogram, and EBCA is equal to DBCF: for they are on the ſame baſe BC, and in the ſame parallels BC, EF. And the Triangle ABC is half of the Parallelogram EBCA; for the Diameter AB cuts the ſame into halves. And the Triangle DBC is half of the Parallelogram DBCF; for the Diameter DC, cuts the ſame into halves: but the halves of equals are equal to one another: wherefore the Triangle ABC is equal to the Triangle DBC.

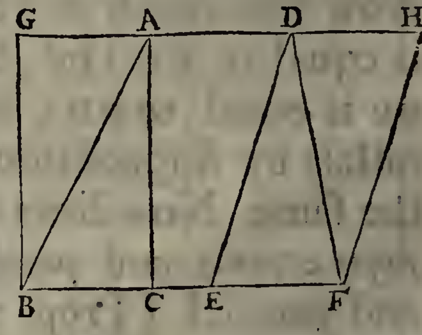


Therefore Triangles on the ſame baſe, and in the ſame parallels, are equal to one another. Which was to be demonſtrated.

PROPOSITION XXXVIII.

Triangles on equal baſes, and in the ſame parallels are equal to one another.

Let the Triangles be ABC, DEF, on equal baſes BC, EF, and in the ſame parallels BF, AD. I ſay, that the Triangle ABC is equal to the Triangle DEF. For let AD be produced both ways to the points G, H, and by B let be drawn BG parallel to CA, and by F, FH parallel to DE, therefore each of theſe GBCA, DEFH, is a Parallelogram. And GBCA is equal to DEFH; for they are on equal baſes BC, EF, and in the ſame parallels BF, GH. And the Triangle ABC is the half of the Parallelogram GBCA; for the Diameter AB cuts the ſame into halves. And the Triangle DEF is the half of the Parallelogram DEFH; for the Diameter DF cut the ſame into halves; but the halves of equals are equal to one another: wherefore the Triangle ABC is equal to the Triangle DEF.



Therefore Triangles on equal baſes, and in the ſame parallels, are equal to one another. Which was to be demonſtrated.

Corollary.

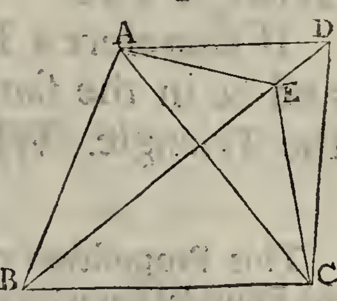
Corollary.

From hence 'tis manifest, that Triangles on unequal bases, and in the same parallels are unequal to one another: on the greater base the greater Triangle, on the less the less.

PROPOSITION XXXIX.

Equal Triangles on the same base, and the same way seated, are in the same parallels.

Let the equal Triangles be ABC, DBC, on the same base BC, and the same way seated. I say, that they are in the same parallels. For let be joyn'd AD; I say, that AD is parallel to BC. For if not, by the point A let be drawn AE parallel to BC, and let be joyn'd EC: therefore the Triangle ABC is equal to the Triangle EBC; for they are on the same base BC, and in the same parallels BC, AE. But ABC is equal to DBC: therefore also DBC is equal to EBC, the greater to the less: which is impossible. Therefore AE is not parallel to BC. In like manner may we prove that no other is besides AD: wherefore AD is parallel to BC.

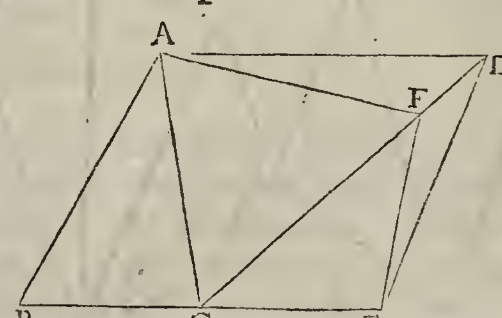


Therefore equal Triangles on the same base, and the same way seated, are in the same parallels. Which was to be demonstrated.

PROPOSITION XL.

Equal Triangles on equal bases, and the same way seated, are in the same parallels.

Let the Triangles be ABC, DCE, on equal bases BC, CE, and the same way seated. I say, that they are in the same parallels. For let be joyn'd AD: I say, that AD is parallel to BE. For if not, by the point A let be drawn AF parallel to BE, and let be joyn'd FE: therefore the Triangle ABC is equal to the Triangle FCE; for they are on equal bases BC, CE, and in the same parallels BE, AF. But the Triangle ABC is equal to the Triangle DCE, therefore also the Triangle DCE, is equal to the Triangle FCE, the greater to the less: which is impossible. Therefore AF is not parallel to BE. In like manner we may prove that no other is besides AD: wherefore AD is parallel to BE.



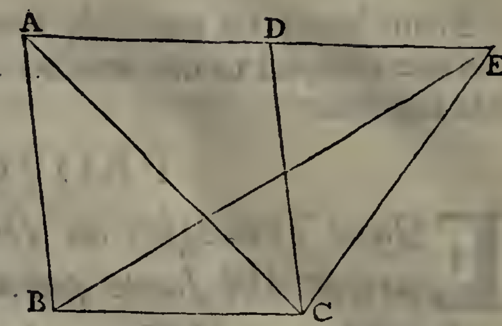
Therefore equal Triangles on equal bases, and the same way seated, are in the same parallels. Which was to be demonstrated.

PROPOSITION XLI.

If a Parallelogram shall have the same base with a Triangle, and be in the same parallels, the Parallelogram shall be double of the Triangle.

For let the Parallelogram ABCD have the same base with the Triangle

Triangle EBC, and be in the ſame parallels BC, AE. I ſay, that the Parallelogram ABCD, is double of the Triangle EBC. For let be joyn'd AC. Now the Triangle ABC is equal to the Triangle EBC; for they are on the ſame baſe BC, and in the ſame parallels BC, AE. But the Parallelogram ABCD, is double of the Triangle ABC; for the Diameter AC cuts the ſame into halves. So that the Parallelogram ABCD, is alſo double of the Triangle EBC.



If therefore a Parallelogram have the ſame baſe with a Triangle, and be in the ſame parallels, the Parallelogram ſhall be double of the Triangle. Which was to be demonſtrated.

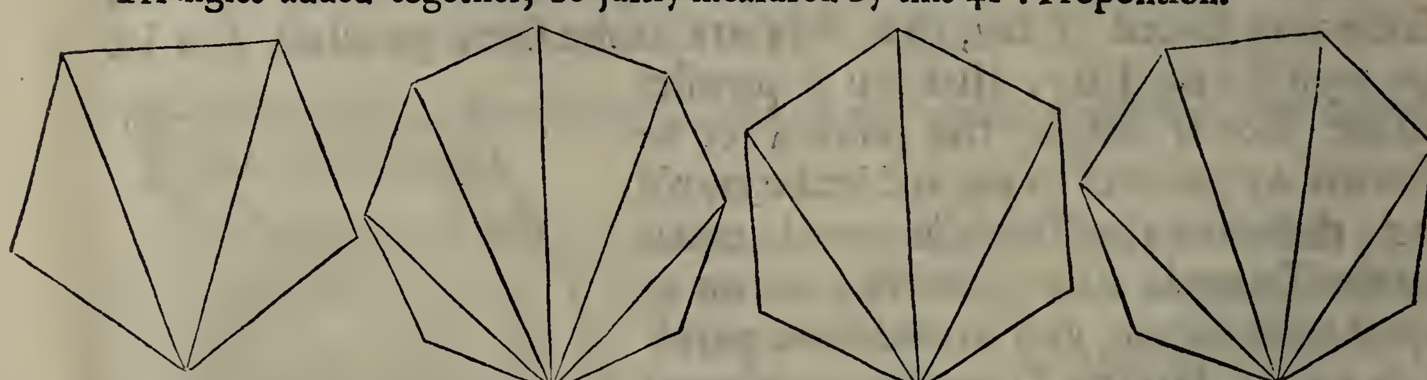
ANNOTATIONS.

This Propoſition compleats the Doctrine for Menſuration of plain Surfaces: the Foundation whereof we have fully laid down at Prop. 35th. And whereas it was there ſhewn, that the Rule, by which all Parallelograms on equal baſes, and in the ſame parallels are to be meaſured, was to multiply the baſe into the perpendicular: Now here 'tis further demonſtrated, that the Parallelogram on the ſame baſe with the Triangle, and in the ſame parallels is the double of the Triangle: wherefore half of the Parallelogram is equal to the Triangle. And therefore

To find the Area of a Triangle;

Let a perpendicular from any angle of a Triangle to the baſe, be multiply'd into half the baſe, it ſhall give the Area of the Triangle.

And foraſmuch as a Triangle is the moſt ſimple of all rectilineal Figures, therefore all rectilineal Spaces may be reſolved into Triangles, and from the particular Triangles added together, be juſtly meaſured by this 41ſt. Propoſition.



Now to reduce any Multilateral Figure into the feweſt Triangles, note, that every Multilateral Figure may be divided into ſo many Triangles, leſs by two, as is the number of its Sides. As a Pentagon into 5. Triangles, leſs by 2, that is, into three Triangles. An Hexagon into four. An Heptagon into five. An Octogon into ſix, &c. As theſe Figures make apparent.

Corollaries.

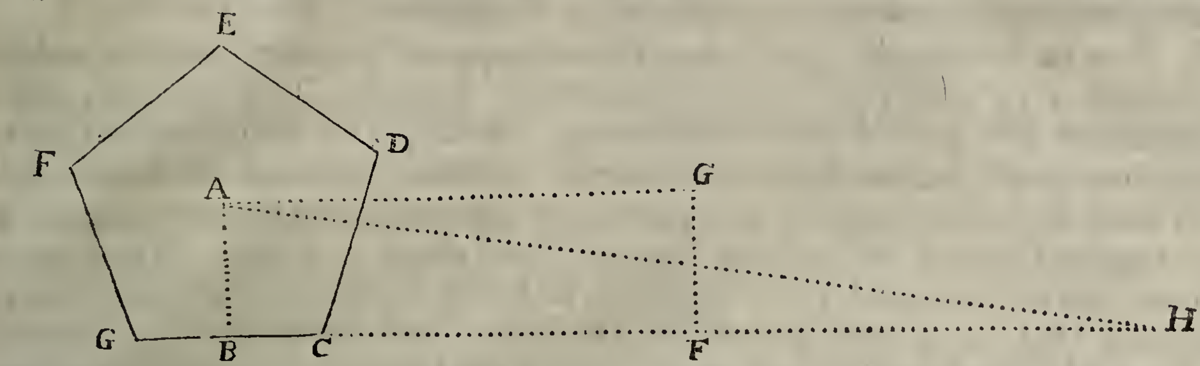
1. For the Menſuration of any Multilateral Figure.

Now to find the Area of any Multilateral Figure, let the Figure be divided after the moſt convenient manner into the feweſt Triangles, and each Triangle be meaſured by its baſe and perpendicular, according to the foregoing Rule in multiplying the perpendicular into half of the baſe; then ſhall theſe Triangles added together give the Area of the Multilateral Figure. For further inſtructions in theſe kind of Matters, recourſe is to be had to the Writers of Practical Geometry. Here we have only touched ſlightly on the Uſes of theſe Elementary Propoſitions.

2. For

2. For the Menſuration of a Regular Multilateral Figure.

From hence 'tis manifeſt, that in a Regular Multilateral Figure a perpendicular from the Center to any of the Sides multiply'd into half the Perimeter gives the Area.

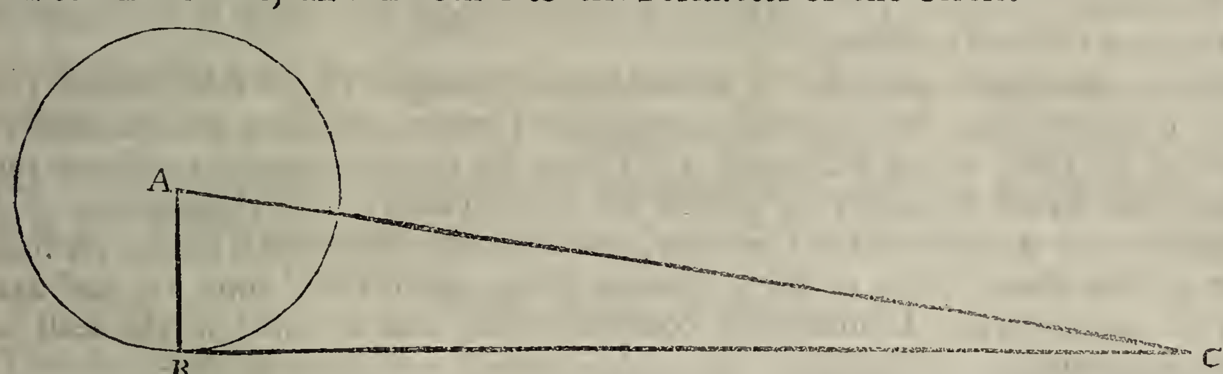


As AB multiply'd into BCDE, half the Perimeter and equal to BF, is equal to the Area of the Figure, that is, to the Parallelogram ABFG. For that of each Triangle the perpendicular AB multiply'd into half the baſe gives the Area: All which Triangles together are equal to the whole Multilateral Figure.

Therefore alſo the Right angl'd Triangle ABH, having the Side BH equal to the whole Perimeter BCDEFG, is equal to the ſame Multilateral Figure.

3. For the Menſuration of a Circle.

And as there is found this equality of Areas between a Regular Polygon and ſuch a Right angl'd Triangle; ſo *Archimedes* hath demonſtrated the ſame between a Circle and a Right angl'd Triangle, one of whoſe Sides about the Right angle is equal to the Radius, and the other to the Perimeter of the Circle.



Now the Phyſical reaſon of this agreement between a Rectilineal Figure and a Circle ſeems to be, for that a *Circle* is, as it were, *a Regular Polygon conſiſting of infinite equal ſides.* So that a Triangle, one of whoſe ſides about the Right angle, is equal to the Radius, and the other equal to the Perimeter, is in Area equal to the Circle, like as it is in Regular Polygons. As of the Triangle ABC, if the ſide BC be ſuppoſed equal to the Perimeter of the Circle, then ſhall the Triangle ABC be equal to the Circle, as *Archimedes* hath demonſtrated. But how Geometrically to exhibite a ſtrait line equal to the Perimeter of a Circle, and to demonſtrate the ſame (as in this Inſtance BC) to be a line equal to the Perimeter, *Hic labor, hoc opus.*

Archimedes therefore makes a further attempt toward this Matter, and in his wonderful Book of *Spiral Lines*, demonſtrates in Prop. 18. that if to the term of a Spiral line deſcribed by the firſt Revolution of the *Genetrix*, a Tangent be drawn: and from the Original, or Central point of the ſame be likewiſe drawn a ſtrait line at Right angles to the *Genetrix*, and produced till it meets with the Tangent, then ſhall this ſtrait line be equal to the Perimeter of the Circle, whoſe Radius is the line that deſcribes the *Helix.*

But how to draw the Tangent is a work left unfiniſhed. And till a Tangent to an Helix be Geometrically demonſtrated, a ſtrait line equal to the Perimeter of a Circle remains unknown.

Archimedes having in theſe methods proceeded Geometrically toward the inveſtigation of a ſtrait line equal to the circumference of a Circle, without a plenary ſatisfaction, endeavours next to come to a nearneſs of equality, ſo far as it might be eaſily practicable, and ſufficient for common uſe.

He begins with the Triſection of a Right angle, or a Quadrant of the circumference of a Circle (commonly ſignified by the number of 90 Degrees) ſo that each Segment is a third part of the Quadrant, or 30 Degrees of 90, and therefore a 12th. part of the whole circumference, that is, of 360 Degrees.

Firſt, Now he biſects this angle, and by conſequence the Arch, which maketh each Segment a 24th. part of the circumference. *Secondly*, This biſected, makes each Segment a 48th. part of the circumference. *Thirdly*, This again biſected, makes each Segment a 96th. part of the circumference: In which Segment *Archimedes* reſts.

Thus from the Biſections of a 12th. part of the circumference thrice repeated, he takes a regular Polygon of 96 ſides circumſcribed about a Circle. Then he demonſtrates the Perimeter of this Polygon to be to the Diameter of the Circle as 22. to 7. almoſt, that is, to be triple of the Diameter, and moreover the overplus above the triple to be almoſt $\frac{1}{7}$ part, or $\frac{10}{70}$ parts of the Diameter: which is the ſame thing; whether the Diameter de divided into 7. or 70. equal parts. And becauſe the circumference of the contained Circle is leſs than the Perimeter of the circumſcribed Polygon, therefore the overplus of the circumference above the triple, is much leſs than $\frac{1}{7}$ or $\frac{10}{70}$ of the Diameter.

For of two unequal magnitudes the leſſer hath a leſs proportion to a third magnitude, than the greater hath to the ſame. As a Groat hath a leſſer proportion to a Penny, than a Shilling hath to a Penny.

Again, he takes a regular Polygon of 96 ſides inſcribed in a Circle, then he demonſtrates the Perimeter of this Polygon to be alſo triple of the Diameter of the Circle, and the overplus above the triple to be greater than $\frac{10}{71}$ parts of the Diameter.

And becauſe the circumference of the Circle is greater than the Perimeter of the contained Polygon, therefore the overplus above the triple is much greater than $\frac{10}{71}$ parts of the Diameter.

So that *Archimedes* uſes, firſt a circumſcribed Polygon of 96 ſides, whoſe Perimeter is greater than the circumference of the Circle; and then an inſcribed Polygon of 96 ſides, whoſe Perimeter is leſs than the circumference: and from their proportions to the Diameter he proves the ſame a *Fortiori*, that the quantity of the Circumference is triple of the Diameter, and ſomewhat leſs than $\frac{10}{70}$ parts, yet ſomewhat greater than $\frac{10}{71}$ parts of the Diameter, being firſt divided into 70, and again into 71. equal parts. Limits eaſily comprehended, and expoſed in the leaſt and feweſt numbers.

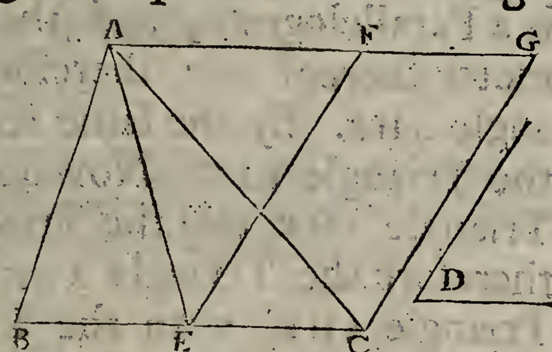
The Moderns indeed have brought this within cloſer bounds; but in great numbers more nice than neceſſary. The numbers of *Adrianus Metius* are for uſefulneſs next to thoſe of *Archimedes*. He ſtates the circumference unto the Diameter, as 355 to 113, that is triple, and moreover 16 parts almoſt of the Diameter divided into 113 equal parts, and thus to be noted $\frac{355}{113}$, or 3 $\frac{16}{113}$ almoſt. Whereas $\frac{1}{7}$ multiply'd into 16, makes only $\frac{16}{112}$: which being greater than $\frac{16}{113}$ is not ſo near to the juſt and preciſe Truth, as theſe numbers of *Adrianus Metius*. But *Archimedes*, who in his Book entitled *Pſammites*, or *Arenarius*, does by moſt artificial Calculations, beginning from a round ſmall Poppy-ſeed give us this vaſt number, which reduced into our Decimal form of Notation, is 10000000,00000000,00000000,00000000,00000000, 00000000,00000000,00000000. Which he demonſtrates to exceed the number of the Sand of the Seas, if the whole World within the Spherical Concave of the Stars, (which he takes to be as large as the ancient Syſtem of *Ariſtarchus* was, and after two thouſand years revived moſt ingeniouſly by *Copernicus*) conſiſted only of ſuch a Maſs of Sand; He, I ſay, could have come to any nearer and nearer terms at pleaſure, if he had thought it neceſſary or convenient. But when after all attempts and labour whatſoever, he knew the matter muſt end in a bare Approximation, like a great and prudent Maſter of his Art, reſts within the readieſt and moſt uſeful limits. And all the endeavours of our late Geometricians reach no further, than proceeding in Fractions of greater and greater numbers to bring the overplus above the triple to be leſſer and leſſer than $\frac{1}{7}$ or $\frac{10}{70}$, yet ſtill to be greater than

than $\frac{10}{71}$. Which *Archimedes* hath demonstrated to be the standing limit on the other side, unto which endless approaches may be nearer and nearer made to very little purpose.

PROPOSITION XLII.

Unto a given Triangle to constitute an equal Parallelogram in an angle equal to a given strait-lin'd angle.

Let the given Triangle be ABC, and the given strait-lin'd angle D. It is required to constitute a Parallelogram equal to the Triangle ABC, in an angle equal to the strait-lin'd angle D. Let BC be cut into halves in E, and let be joyn'd AE, then to the strait line EC, and to a point in the same E, let be constituted the angle CEF, equal to the angle D [by Prop. 23.]. And by A, let be drawn AG parallel to EC, and by C, CG parallel to EF. Therefore FECG is a Parallelogram.



Now forasmuch as BE is equal to EC, therefore the Triangle ABE is equal to the Triangle AEC. For they are on equal bases BE, EC, and in the same parallels BC, AG. Therefore the Triangle ABC is double of the Triangle AEC. But also the Parallelogram FECG is double of the Triangle AEC; for it hath the same base and is in the same parallels. Therefore the Parallelogram FECG is equal to the Triangle ABC, and hath the angle CEF equal to the given angle D.

Wherefore to the given Triangle ABC, there is constituted an equal Parallelogram FECG in the angle CEF, which is equal to the angle D. Which was to be done.

ANNOTATIONS.

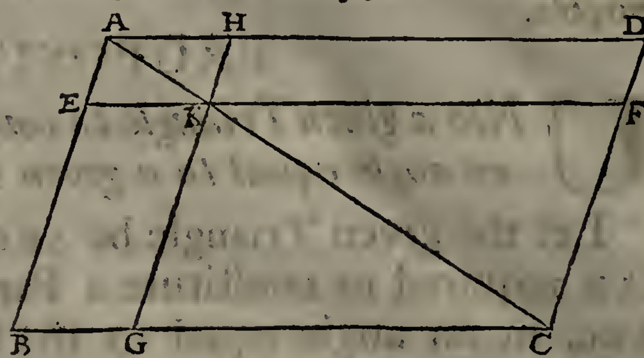
This Problem concerns the Transformation of Figures one into another; and begins with transmuting a Triangle (the most simple of strait lin'd Figures) into an equal Parallelogram. And by consequence there is imply'd the like transmutation of all Rectilineal spaces into equal Parallelograms; for that every Multilateral Figure may for this end be divided into Triangles. As *Euclide* hath done in the following 45^{th}. Proposition.

And forasmuch as all Parallelogram spaces are by the Diameter divided into two Triangles, therefore their four angles are equal to four Right [by Prop. 32.]. And because the opposite angles are equal, [Prop. 34.] therefore if one angle of a Parallelogram be given, all the four are given, and determined. For if the given angle be Right, the other three are also right angles. If the given angle be Obtuse, the opposite is also Obtuse, and equal to it, and the other two Acute angles are likewise equal to one another, and together with the two Obtuse do make, or compleat four Right angles. And the like again on the contrary, if the given angle be Acute: so that in one angle given, a Parallelogram is ever to be understood as determined in all its four angles.

PROPOSITION XLIII.

Of every Parallelogram space the complements of the Parallelograms about the Diameter, are equal to one another.

Let the Parallelogram space be ABCD, and the Diameter of the ſame be AC, and about AC let the Parallelograms be EH, FG, and what are called the complements be BK, KD. I ſay, that the complement BK is equal to the complement KD. Foraſmuch as ABCD is a Parallelogram, and the Diameter thereof AC: therefore the Triangle ABC is equal to the Triangle ADC. Again, becauſe EKHA is a Parallelogram, and the Diameter thereof AK: therefore the Triangle AEK is equal to the Triangle AHK. By the ſame reaſon alſo the Triangle KGC is equal to the Triangle KFC. Now becauſe the Triangle AEK is equal to the Triangle AHK, and the Triangle KGC, is equal to the Triangle KFC, therefore the Triangle AEK with the Triangle KGC, is equal to the Triangle AHK, with the Triangle KFC. But the whole Triangle ABC is equal to the whole Triangle ADC: wherefore the remaining complement BK, is equal to the remaining complement KD.



Therefore of every Parallelogram ſpace the complements of the Parallelograms about the Diameter, are equal to one another. Which was to be demonſtrated.

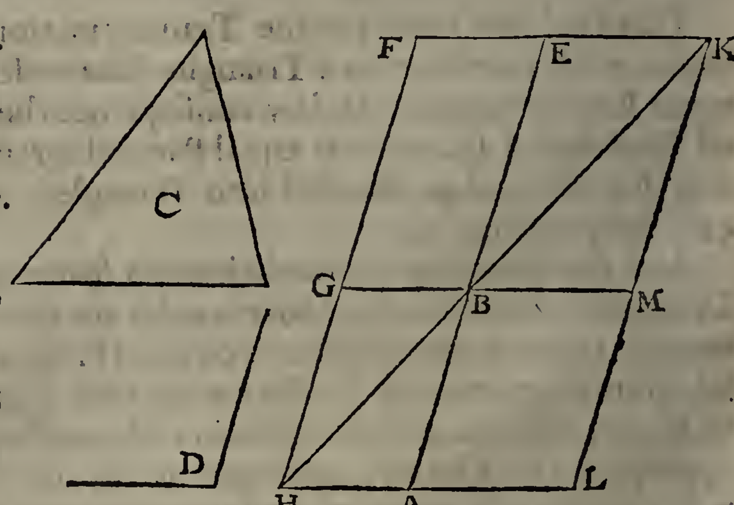
PROPOSITION XLIV.

Unto a given ſtrait line to apply a Parallelogram equal to a given Triangle, in a given ſtrait lin'd angle.

Let the given ſtrait line be AB, and the given Triangle C, and the given ſtrait lin'd angle D. It is required unto the given ſtrait line AB, to apply a Parallelogram equal to the given Triangle C, in an angle equal to D.

Let be conſtituted the Parallelogram BEFG equal to the Triangle C in the angle EBG, which is equal to D. [by Prop. 42.]

And let BE be put directly to AB, and FG be produced to H. Then by A to either of the lines BG, EF, let AH be drawn parallel, and let be joyn'd HB.



Now foraſmuch as the ſtrait line HF falls on the parallels AH, EF, therefore the angles AHF, HFE, are equal to two Right [Prop. 29.]; wherefore BHG, GFE, are leſs than two Right. But lines infinitely produced from angles leſs than two Right ſhall meet: therefore HB, FE, being produced ſhall meet. Let them be produced, and meet in K: then by the point K to either of the lines EA, FH, let KL be drawn parallel, and let HA, GB, be produced to the points L, M.

There-

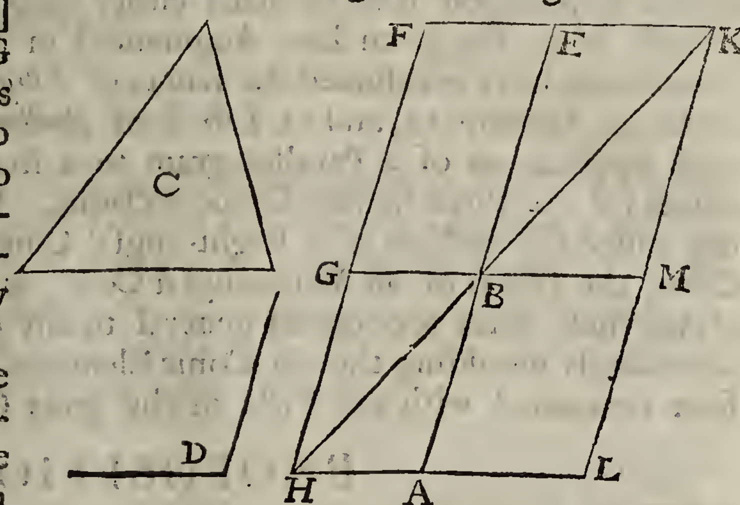
Therefore HLKF is a Parallelogram, and HK the Diameter thereof, and the Parallelograms *about* HK are AG, ME, and the Parallelograms called *complements* are LB, BE: therefore LB is equal to BF [Prop. 43.]. But BF is equal to the Triangle C; therefore also LB is equal to C. And because the angle GBE, is equal to ABM, and also GBE is equal to the angle D; therefore ABM is equal to the angle D.

Therefore unto the given strait line AB, is apply'd the Pallelogram LB, equal to the given Triangle C, in the angle ABM, which is equal to the given angle D. Which was to be done.

ANNOTATIONS.

It is required unto the given line AB *to apply a Parallelogram equal to the given Triangle* C.] That is, to constitute a Parallelogram, one of whose Sides shall be the given line AB, and the Parallelogram be also equal to the given Triangle C.

And let BE *be put directly* to AB] That is, let the Parallelogram BEFG be so constructed, that one of the sides containing the angle EBG, equal to the given angle D, be put directly to AB the given line, unto which a Parallelogram equal to the given Triangle C is required to be apply'd. Now this is to be thus effected.



Produce AB to E, and to the line EB, and to the point B let be constituted the angle EBG, equal to the given angle D [by Prop. 23.]. Then let the Parallelogram BEFG, be constituted equal to the given Triangle C [by Prop 42.]; And let FG the opposite side to EB, be produced indefinitely toward H. And so proceeding onward according to *Euclid's* construction in compleating the Diagram, and applying the required Parallelogram LB to the given line AB, as it is at the first posited in any Situation whatsoever given.

For note, that the Application of that Parallelogram which is required to be equal to the given Triangle, ought to be made *ad Datam Rectam*, that is, to the very line AB in its position, and not to an other line, which shall be put equal to it; as *Clavius* hath in his Exposition of this Proposition without just cause deviated from *Euclide*. But afterward in his Scholion he rightly corrects himself, *ex sententia Euclidis*, and there follows the general Law of Problems, that *The thing is always to be effected according to the Position given.* As in the Use of this Problem it will every where be found necessary, and even in the next following Proposition.

It is farther to be observed, that in the Construction of this Problem there are made four Parallelograms Equiangl'd to one another, and to the whole. *viz.* Two about the Diameter of the whole, and their two Complements. One of which, the Parallelogram FB is first [by Prop. 42.] constituted equal to the given Triangle C: by which means the other Complement BL, equal to the former (and therefore equal to the same given Triangle) is apply'd to the given line AB, according to the full Tenor of this eminent Problem.

Advertisement.

The 42[d]. Proposition hath shewed how to constitute a Parallelogram equal to a given Triangle, in an angle equal to a given angle.

Now in this Problem there is moreover required to apply such a Parallelogram also to a given strait line, as well as in a given angle. That is, the given line is to be one side of the apply'd Parallelogram, and an angle of that Parallelogram is to be equal to the given angle. And always remind, what hath been before noted, that in Parallelograms, if one angle be given, all the four are given, becauſe the oppoſite angles are equal, and the two inward are equal to two Right, by Prop. 34. and 29. So that in this Problem the Parallelogram is three ways reſtrained. I. In the given line the Parallelogram is confined to one certain ſide. II. It is determined in Area, or Magnitude, in that it is to be equal to a given Triangle. III. In the one given angle all the four angles are determined; wherefore of this Parallelogram nothing is left undetermined, but the other ſide whereby to effect this Problem.

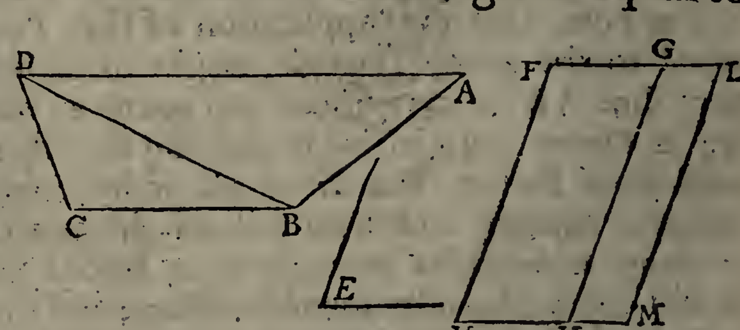
From this *exact* Application, or *Paraboliſm* of a Parallelogram to a given ſtrait line *preciſely*, is one of the Conic Sections named *Parabola*. As *Apollonius* ſhews in Prop. 11. Lib. I. of his Conic Elements. The ſtrait line to which the Application is made in the Conic Elements, is by *Mydorgius* properly called the PARAMETER.

There is likewiſe in Prop. 28, and 29. El. VI. of *Euclide*, an Application of a given Parallelogram unto a given ſtrait line, and in a given angle: but beſides theſe Reſtraints, there are added more ſtrict conditions of Defect, and Exceſs, that is, the Application is to be made either unto a part of the given line in a certain Defect, or to the given line Augmented in a certain Length, or Exceſs; which Conditions have occaſioned the names of *Ellipſis*, and *Hyperbola*, to the other Conic Sections, As Prop. 12, and 13. Lib. I. of *Apollonius* ſet forth. So that *Euclid's* three fold Application of a Parallelogram to a ſtrait line, has afforded to *Apollonius* names for the three famous Conic Sections. Whereas more anciently, the *Parabola* was called the Section of a Right-angl'd Cone, the *Hyperbola* of an Obtuſe-angl'd Cone, the *Ellipſis* of an Acute-angl'd Cone, as we find in *Archimedes*. But for applying theſe three Sections in general to any one Cone of whatſoever angle, and accordingly moulding the old Conic Elements, *Apollonius* was in his time, and ever ſince renowned with the Title of the *great Geometrician*.

PROPOSITION XLV.

TO a given Rectilineal ſpace to conſtitute an equal Parallelogram in an angle equal to a given ſtrait-lin'd angle.

Let the given Rectilineal ſpace be ABCD, and the given ſtrait-lin'd angle E. It is required to conſtitute a Parallelogram equal to the Rectilineal ſpace ABCD, in an angle equal to E. For let be joyn'd DB; and let be conſtituted [by Prop. 42.] the Parallelogram FH equal to the Triangle ADB in the angle HKF, which is equal to E.



Then unto the ſtrait line GH, let be apply'd the Parallelogram GM, equal to the Triangle DBC in the angle GHM, which is equal to E, [by Prop. 44.]

Now foraſmuch as the angle E is equal to each of the angles FKH, GHM; therefore FKH is equal to GHM. Let KHG be added in common, therefore the angles FKH, KHG, are equal to the angles KHG, GHM. But the angles FKH, KHG, are equal to two Right [Prop. 29.]; therefore alſo KHG, GHM, are equal to two Right. Now to the ſtrait line GH, and to a point in the ſame H, the two ſtrait lines KH, HM, not lying the ſame way, make the conſequent angles equal to two Right; therefore KH is direct to HM [Prop. 14.]. And becauſe the

ſtrait

ſtrait line HG falls on the parallels KM, FG, therefore the Alternate angles MHG, HGF, are equal to one another [Prop. 29.]. Let HGL be added in common, therefore MHG, HGL, are equal to HGF, HGL. But the angles MHG, HGL, are equal to two Right: therefore FG is direct to GL [Prop. 14.]. And becauſe KF is equal, and parallel to HG, and likewiſe HG to ML; therefore alſo KF is equal, and parallel to ML [Prop. 30.]. And the ſtrait lines KM, FL, joyn the ſame, therefore KM, FL, are alſo equal, and parallel [Prop. 33.]; wherefore KFLM is a Parallelogram. And becauſe the Triangle ABD, is equal to the Parallelogram HF, and BDC to GM, wherefore the whole Rectilineal ſpace ABCD, is equal to the whole Parallelogram KFLM. Therefore the Parallelogram KFLM is conſtituted equal to the Rectilineal ſpace ABCD in the angle FKM, which is equal to the given angle E. Which was to be done.

ANNOTATIONS.

And let be conſtituted the Parallelogram FH *equal to the Triangle* ADB.] To effect this Problem, the given Rectilineal ſpace is to be divided into Triangles: and firſt to one of theſe Triangles there is to be conſtituted an equal Parallelogram. As here the Parallelogram FH is conſtituted equal to the Triangle ADB, by Prop. 42. But is not required as *Clavius* propoſes to be apply'd to any certain ſtrait line, tho' the other Parallelograms are; and after this manner following.

Then unto the ſtrait line GH, *let be apply'd the Parallelogram* GM, *equal to the Triangle* DBC, *in the angle* GHM, *which is equal to* E.] Altho' the firſt Parallelogram FH, equal to the Triangle ADB, was not confined to any given ſtrait line; yet the next Parallelogram GM equal to the Triangle DBC, is of neceſſity to be apply'd to the ſtrait line GH, as it lyes in a given poſition; that by this means there might be conſtituted from ſuch particular Parallelograms one entire Parallelogram KFLM, equal to the given Rectilineal ſpace ABCD. And ſo forward, if the Rectilineal ſpace required a diviſion into more Triangles, Parallelogram is after Parallelogram to be apply'd to ſuch a certain ſtrait line, that makes ſucceſſively one common Side; and all of them are equal to one another, and the Parallelograms are together equal to the whole Parallelogram ſpace, which was required to be conſtituted equal to a given Rectilineal ſpace.

Advertiſement.

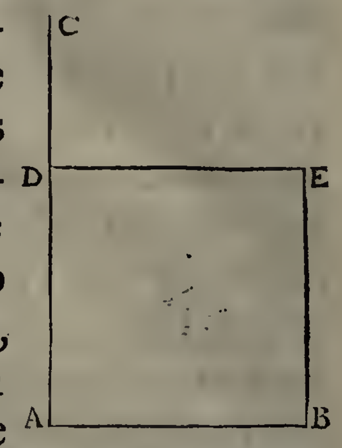
The Application of Rectilineal ſpaces to a given ſtrait line, if alſo the given angle be a Right angle, does truely anſwer to the operation of Diviſion in Arithmetic. For the given plain ſpace is as the Dividend in Numbers, the given ſtrait line as the Diviſor, and the other Side of the Parallelogram emergent from this Application is as the Quotient. And again, as the Diviſor multiply'd into the Quotient makes up the Dividend, ſo the given ſide drawn into the emergent ſide, gives the Area of the Parallelogram.

PROPOSITION XLVI.

ON a given ſtrait line to deſcribe a ſquare.

Let the given ſtrait line be AB. It is required on the ſtrait line AB to deſcribe a ſquare. To the ſtrait line AB from a given point in the

the ſame A; let be drawn at Right angles the line AC, and to AB let AD be put equal: then by the point D, let DE be drawn parallel to AB, and by the point B let BE be drawn parallel to AD; therefore ADEB is a Parallelogram: wherefore AB is equal to DE, and AD to BE [Prop. 33.]. But alſo BA is equal to AD; therefore the four lines BA, AD, DE, EB, are equal to one another: wherefore the Parallelogram ADEB is Equilateral: I ſay, it is alſo Right angl'd. For becauſe on the parallels AB, DE, falls the ſtrait line AD, therefore the angles BAD, ADE, are equal to two Right [Prop. 29.]. But BAD is a Right angle, alſo ADE is a Right angle, and of Prallelogram ſpaces the oppoſite ſides and angles are equal to one another [Prop. 34.]; wherefore each of the oppoſite angles ABE, BED, is a Right angle, therefore ADEB is Right angl'd. But alſo it hath been prov'd Equilateral: therefore it is a Square; And it is deſcrib'd on the ſtrait line AB. Which was to be done.



ANNOTATIONS.

Euclide, before he makes uſe of a ſquare, does demonſtrate the Being and Conſtruction of ſuch a Figure. And therefore hath here premiſed this problem in order toward the demonſtration of the next following Theorem, which is the firſt wherein ſquares are concern'd.

And according to his 29th. Definition he now ſhews how to deſcribe a Figure of four Equal Sides, and four Right Angles on any given ſtrait line.

Thus having demonſtrated this Figure, he does hereafter upon occaſion juſtly aſſume from the nature of a Square, that *Squares deſcribed on equal ſtrait lines are equal to one another*. As likewiſe that *Equal Squares* are deſcribed on equal ſtrait lines. Altho' *Commandinus* and *Clavius* after *Proclus*, have thought fit to demonſtrate theſe moſt natural Conceptions: which were before as evident of themſelves, and immediately conjoyn'd in common ſenſe with the definition of a Square.

PROPOSITION XLVII.

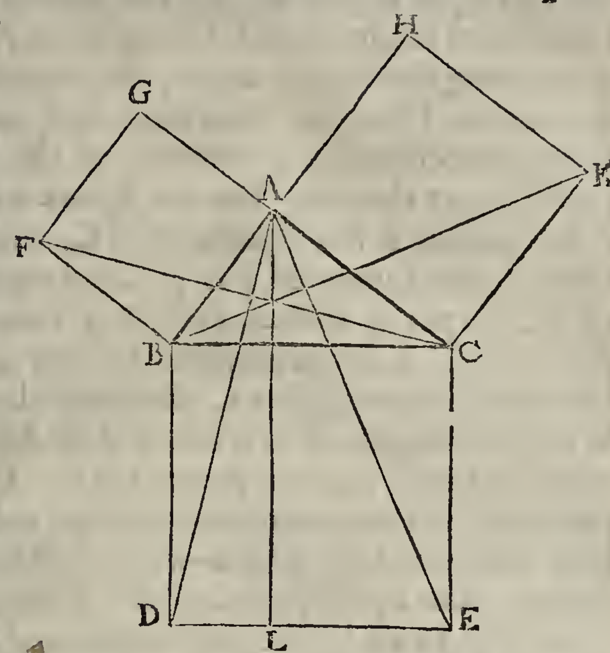
IN a Right angled Triangle, the Square of the ſide ſubtending the Right angle, is equal to the Squares of the ſides containing the Right angle.

Let the Right angl'd Triangle be ABC, having the Right angle BAC. I ſay, that the ſquare of BC, is equal to the ſquares of BA, AC. For on BC, let be deſcrib'd the ſquare BDEC, and on AB, AC, the ſquares GB, HC, and by A let AL be drawn parallel to either of the lines BD, CE, and let be joyn'd AD, FC.

Now foraſmuch as each of the angles BAC, BAG, is a right angle, and to the ſtrait line BA, and to a point in the ſame A, the two ſtrait lines AC, AG, not lying the ſame way, make the conſequent angles equal to two Right, therefore CA is direct to AG: by the ſame reaſon alſo AB is direct to AH. And becauſe the angle DBC is equal to the angle FBA, for each is a right angle, let the angle

ABC

ABC be added in common, therefore the whole angle DBA is equal to the whole angle FBC. And becauſe the two lines DB, BA, are equal to the two lines CB, BF, each to each, and the angle DBA is equal to the angle FBC, therefore the baſe AD is equal to the baſe FC, and the Triangle ABD is equal to the Triangle FBC.



Now the Parallelogram BL is double of the Triangle ABD, for they have the ſame baſe BD, and are in the ſame parallels BD, AL [Prop. 41.]. Alſo the ſquare GB is double of the Triangle FBC, for they have the ſame baſe FB, and are in the ſame parallels FB, GC. Now the doubles of equals are equal to one another; Therefore the Parallelogram BL is equal to the ſquare GB.

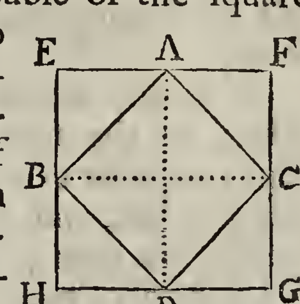
In like manner AE, BK, being joyn'd, may be proved that the Parallelogram CL, is equal to the ſquare HC; therefore the whole ſquare BDEC is equal to the two ſquares GB, HC, and the ſquare BDEC, is deſcribed on BC, and GB, HC on BA, AC; wherefore the ſquare of the ſide BC, is equal to the ſquares of the ſides BA, AC.

Therefore in Right angl'd Triangles, the ſquare of the ſide ſubtending the Right-angle, is equal to the ſquares of the ſides containing the Right angle. Which was to be demonſtrated.

ANNOTATIONS.

This Propoſition among Geometricians moſt famous, is ſaid to have been found out by *Pythagoras*, and the Invention publickly celebrated with a Sacrifice to the Muſes. Yet the hint from whence the diſcovery of this Truth might firſt ariſe, ſeems to be very obvious.

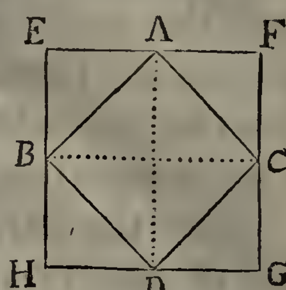
For in this Figure the ſquare EFGH, is apparently double of the ſquare ABDC; but EFGH is deſcribed on EF, which is equal to BC, the ſide ſubtending the Right angle BAC of the Equicrural Triangle ABC; and the ſquare ABDC is deſcribed on either of the ſides AB, AC, containing the Right angle BAC, of the ſame Equicrural Triangle ABC. It is therefore hereupon very reaſonable to conceive, that the ſame property might likewiſe belong to Scalene Right-angl'd Triangles, and give the occaſion of a farther enquiry into this matter.



Thus Geometricians often happen to diſcover a Truth, before they have framed a legitimate demonſtration of it: and find out their Propoſitions one way (which they uſually conceal) but prove them in an other. We have an Example of this kind in the Remains of *Archimedes*, who ſhews, how firſt he found the Quadrature of a Parabola Mechanically, as he calls it, and afterwards gives a Geometrical demonſtration.

Now *Euclid's* demonſtration reaches in general all Right-angled Triangles, Equicrural, and Scalene: and is very eaſy, natural and immediate, being framed from the two Fundamental Propoſitions 35^{th} and 41^{st}: by which all plain Surfaces are meaſured, and proved to be equal, or unequal to one another.

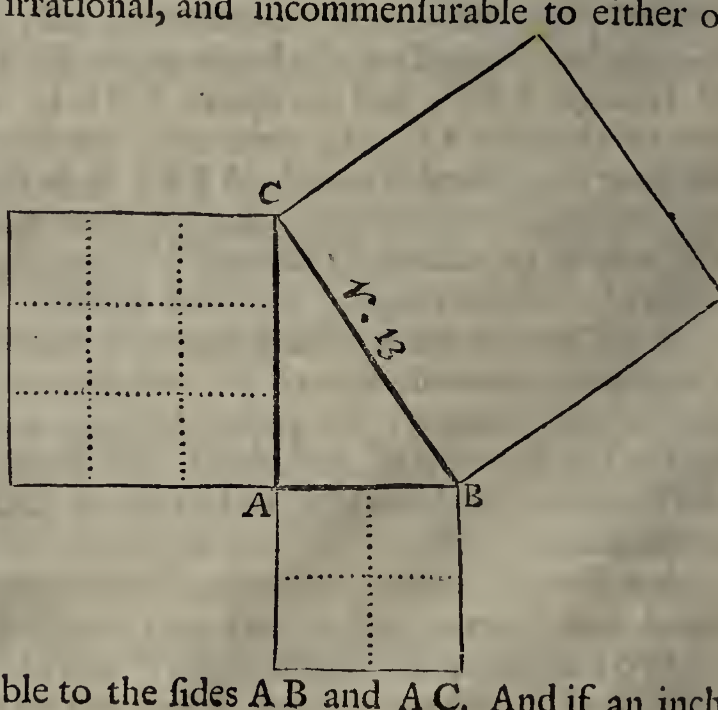
But forasmuch as no square number added to it self, can make a square number, nor any square number be the double of an other square number: therefore in an Equicrural Right-angl'd Triangle, as A B C, if the length of the sides A B, or A C, containing the Right angle, be expressed by the number of any known measure, as 2 inches, Feet, &c. then the length of the side B C subtending, the Right angle is not expressible by a number of the same measure, or by any possible part thereof. For the square of 2 is 4, which added to it self makes 8 the square of B C, whose length is less than 3, because the square of 3 is 9. And on the contrary, if the length of B C be put 4 Inches, Feet, &c. then the length of either side A B or A C is not expressible by any part of the same measure. For the square of 4 is 16, therefore the square of A B or A C, is 8, and the length of the side A B or A C, is less than 3, and expressible by no part or parts of B C. For tho' B C be divided into 4000000 &c. of parts of the same measure; yet the case is still as before, and not one of those parts shall measure A B, which will ever be less than 3000000 &c. of these parts, and greater than 2999999 &c. of the same.



So that in an Equicrural Right-angl'd Triangle, if the side subtending the Right angle be *Rational*, that is, *be expressed in quantity by a certain Number, and Measure*, then each of the other sides is *Irrational* to that Measure, and cannot be expressed in quantity by any number of parts belonging to the same.

And because every square is divisible by the Diameter into two equal and Equicrural Right-angl'd Triangles, therefore if the side of a square be Rational, the Diameter is Irrational: and if the Diameter be put Rational, the side is Irrational. And the 10th. Element of *Euclide* ends with a demonstration, that the side of a Square, and the Diameter, are to one another incommensurable, that is, whatsoever length any certain times repeated shall make up exactly, and measure either the side, or the Diameter, the same tho' never so small, cannot precisely measure the other, but repeated shall fall under, or over it. As let any length exactly measure A B from the point A to the point B; the same shall not precisely measure B C, from B to C; but repeated will either come short of the point C, or pass beyond it. And this is meant by incommensurable Magnitudes. For tho' every finite Magnitude is in it self mensurable, yet all finite Magnitudes are not capable of the same measure so as to have their quantities signified by any one and the same, as appears from this 47th. Proposition.

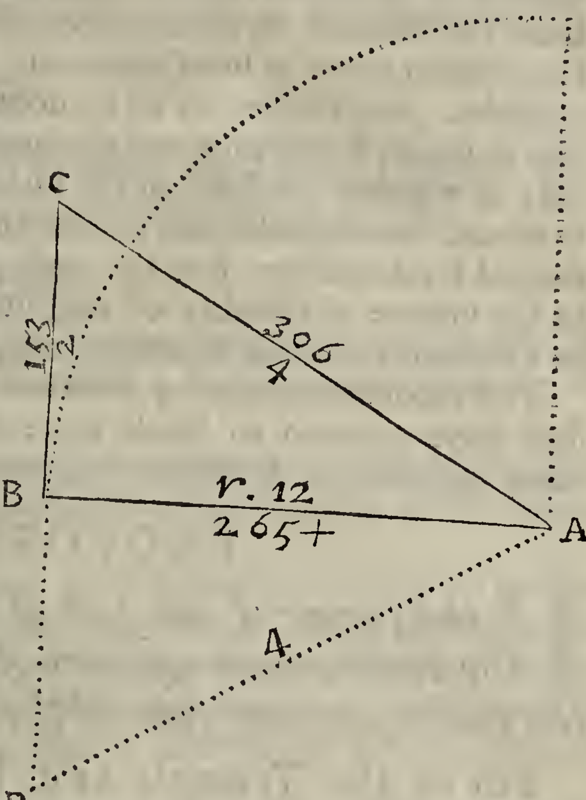
There are infinite other strait lines of the like nature: some of which are the whole subject of that most subtile 10th. Element. As also here, not only in all Equicrural but in all Scalene Right-angl'd Triangles, where the squares of the sides containing the Right angle added together, make not a square number, there the side subtending the Right angle is irrational, and incommensurable to either of them.



As in the Right-angl'd Triangle A B C, let the sides containing the Right angle be of a certain length, as A B 2 inches, and A C 3 inches; their squares are then 4, and 9, which added together, make the square 13. wherefore the side B C subtending the Right angle is greater than 3, whose square is but 9, and less than 4, whose square is 16, so that the quantity of B C, is only expressible by a surd number between 3 and 4, and is no part, nor any possible fraction, that can be made out of 3, or 4; but is called the Square Root of 13; and therefore irrational it is, and incommensurable to the sides A B and A C. And if an inch were

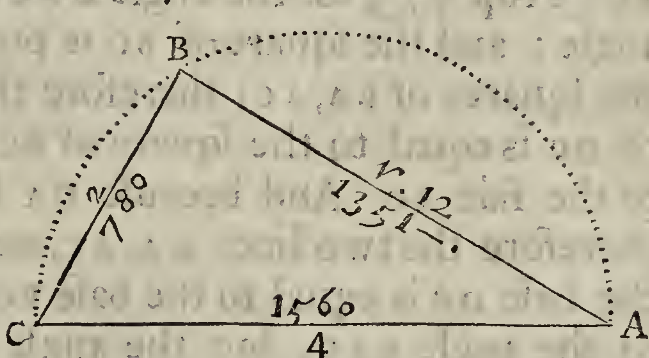
were divided into an hundred thouſand millions of parts, the caſe would be ſtill the ſame, and not one of thoſe particles ſhould exactly meaſure, and make up the line B C.

Again, let a Right-angl'd Triangle A B C, be the half of an Equilateral Triangle A C D, each of whoſe ſides let be put 4, ſo that A C is 4, and B C the half of A C is 2. Now the ſquare of A C is 16, the ſquare of B C is 4, and therefore the ſquare of A B is 12, or 16 leſs 4, whoſe ſide A B is a ſurd between 3 and 4, and called the Square Root of 12. Yet theſe Squares are commenſurable, and the ſquare of A C is quadruple of the ſquare B C, and the ſquare of A B is triple of the ſame; tho' the ſide A B be irrational and incommenſurable to the other ſides A C, B C. This is the *Archimedéan* Right-angl'd Triangle expoſed in ſmall numbers, ſuch as plainly give the mutual proportions of the ſides and ſquares, on which his Cyclometries are founded; where the ſurd ſide A B he makes the Radius of his Circle. The numbers of *Archimedes* are theſe, A C is put 306, therefore B C the half of A C is 153, and the Radius A B is a ſurd, ſomewhat greater than 265. For the ſquare of A C 306 is 93636: and the ſquare of B C 153, is 23409; which ſubſtracted from the ſquare 93636, there remains 70227, which is the ſquare of A B: ſo that A B is ſomewhat greater than 265, whoſe exact ſquare is 70225, and only leſs than 70227, the ſquare of A B by two units: and therefore A B is a ſurd, ſomewhat inſenſibly greater than 265, the Root of the ſquare number 70225.



Now from the ſides, and ſquares of a Right-angl'd Triangle, which is half of an Equilateral, *Archimedes* demonſtrates by repeated Biſections of angles, and arches, and juſt Calculations thereupon, that a Polygon of 96 ſides circumſcribed about a Circle, and by conſequence that the Circumference of the Circle, is triple, and moreover ſomewhat leſs than $\frac{1}{7}$ or $\frac{10}{70}$ of the Diameter.

And further, upon this occaſion to declare fully the uſe *Archimedes* makes of this 47th. Prop. in the menſuration of a Circle, let there be again ſuch an other Right-angl'd Triangle A B C, being the half of an Equilateral. And let A C ſubtending the Right angle A B C, be the Diameter of a Circle, and ſuppoſed to conſiſt of 1560 equal parts: therefore B C the half of A C is 780. And the ſide A B is a ſurd, ſomewhat leſs than 1351.



For the ſquare of A C 1560 is 2433600: and the ſquare of B C 780 is 608400: which ſubſtracted from the ſquare 2433600, there remains 1825200, which is the ſquare of A B: ſo that A B is ſomewhat leſs than 1351, whoſe exact ſquare is 1825201, and only exceeds 1825200 the ſquare of A B by a ſingle unite; and therefore A B is a ſurd, ſomewhat inſenſibly leſs than 1351 the Root of the ſquare number 1825201.

Now from the Sides, and Squares of this Right-angl'd Triangle, *Archimedes* demonſtrates, that a Polygon of 96 ſides inſcribed in a Circle, and by conſequence that the Circumference of the Circle, is triple, and moreover ſomewhat greater than $\frac{10}{71}$ parts of the Diameter.

Thus far upon this 47th. Prop. we have explained the Grounds of *Archimedes* his Cyclometries taken from an Equilateral Triangle, where the angles are always known, and the Sides may be put of any quantity at pleaſure, and out of which is

 made

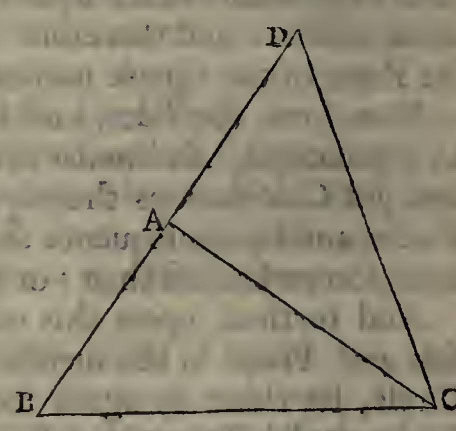
made ſuch a Right-angl'd Triangle of known angles and ſides, that whoever is but moderately exerciſed in the uſe of numbers, may now with eaſe go forward through the reſt of *Archimedes* his demonſtrations, which would otherwiſe be ſomewhat obſcure and difficult to younger Students. For whoſe ſake alſo we have from this Propoſition given in brief ſome eaſy, and uſeful Notions of incommenſurable magnitudes. And farther 'tis to be noted, that between incommenſurable magnitudes it is at the firſt arbitrary and changeable at pleaſure, which of them be put Rational; as whether the Side, or the Diameter of a Square. But theſe are of themſelves in nature incommenſurable to one another, which is the apparent cauſe of their mutual Irrationality. But the cauſe of incommenſurability is more ſubtil, and lyes in the infinite diviſibility of magnitude, which hath not in nature *a Minimum* to be a common meaſure of magnitudes, as an Unite is of Numbers.

This incommenſurability between magnitudes of the ſame kind (whereof we ſhall have occaſion to ſpeak more at large in the Fifth Element) is one of the many inexplicable Myſteries in Geometry. *Habet enim Geometria miracula ſua.*

PROPOSITION XLVIII.

If the ſquare of one ſide of a Triangle be equal to the ſquares of the two remaining ſides of the Triangle, the angle contained by the two remaining ſides of the Triangle, is a Right angle.

For of the Triangle ABC, let the ſquare of BC be equal to the ſquares of BA, AC. I ſay, that the angle BAC is a Right angle. For from the point A to the line AC, let be drawn at Right angles the line AD; and to BA let AD be put equal, and DC be joyn'd. Now forasmuch as DA is equal to AB, therefore alſo the ſquare of DA is equal to the ſquare of AB. Let the ſquare of AC be added in common; therefore the ſquares of DA, AC, are equal to the ſquares of BA, AC. But the ſquare of DC is equal to the ſquares of DA AC [Prop. 47.]; for the angle DAC is a Right angle; and the ſquare of BC is put equal to the ſquares of BA, AC; therefore the ſquare of DC is equal to the ſquare of BC. So that alſo the ſide DC is equal to the ſide BC. And becauſe DA is equal to AB, and AC common; therefore the two lines DA, AC, are equal to the two lines BA, AC, and the baſe DC is equal to the baſe BC; therefore the angle DAC is equal to the angle BAC; but the angle DAC is a Right angle, therefore alſo BAC is a Right angle.



If therefore the ſquare of one ſide of a Triangle be equal to the ſquares of the two remaining ſides of the Triangle, the angle contained by the two remaining ſides of the Triangle is a Right angle. Which was to be demonſtrated.

ANNOTATIONS.

This Propoſition is the Converſe of the precedent. And foraſmuch as here is demonſtrated, that *of a Triangle if the ſquare of one ſide be equal to the ſquares of the other two, then ſhall that be a Right angled Triangle*: wherefore to form a Right angled Triangle of Rational, and Commenſurable ſides, we are to find three ſtrait lines,

lines, whose quantities may be expressed in numbers of one, and the same measure common to all the three sides; so that also of those numbers the square of one shall be equal to the squares of the other two, that is, to find two square numbers, which added together shall make a square number according to this Problem.

A Problem.

From two given numbers to derive three numbers, where the square of one shall be equal to the squares of the other two.

The Solution.

Of the two given numbers the SUM of the squares, the DIFFERENCE of the squares, and TWICE THE RECTANGLE make a Right angl'd Triangle of Rational, and Commensurable sides.

In *Species* the Rule is thus set forth. Let the given numbers be A, and E: then shall $Aq + Eq$. $Aq - Eq$. 2 A into E make a Right angl'd Triangle of Rational, and Commensurable sides.

As to begin with 1 and 2. The square of 1 is 1. The square of 2 is 4. The sum of the squares, 4 + 1 is 5. The difference of the squares 4 — 1 is 3. The Rectangle is 1 multiply'd into 2, which makes 2: and 2 twice taken is 4. So that the sides of this Right angl'd Triangle are 3, 4, 5. Where the square of 5 is 25, and equal to the two squares 16, and 9, whose sides are 4, and 3. This is the first Triangle of this sort.

Again, the second is this. Let be given the numbers 2, and 3. Now the square of 2 is 4. The square of 3 is 9. The sum of the squares 9 + 4 is 13. The difference of the squares 9 — 4 is 5. The Rectangle of 2 multiply'd into 3 makes 6, which twice taken is 12. So that the sides of this Right angl'd Triangle are 5, 12, 13. Where the square of 13 is 169, and equal to the two squares 144, and 25, whose sides are 12, and 5.

The third of these kind of Triangles, made from 3 and 4, is 7, 24, 25. The fourth from 4 and 5, is 9, 40, 41. The fifth from 5 and 6 is 11, 60, 61. and so forth infinitely. Where note, that in such kind of Triangles the two greater sides only differ by an unite; and the sum of these is always equal to the square of the least side.

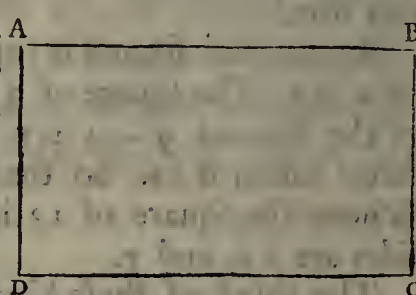
THE SECOND ELEMENT.

DEFINITIONS.

DEFINITION I.

Every Right angl'd Parallelogram is ſaid to be contained by any two of the ſtrait lines, which contain a Right angle.

Altho' a Right angl'd Parallelogram be comprehended, or encompaſſed by four ſtrait lines, yet becauſe by Prop. 34. El. I. the oppoſite ſides of Parallelograms are equal to one another, therefore any of the two ſides, which contain a Right angle, are ſaid to contain the whole Parallelogram.



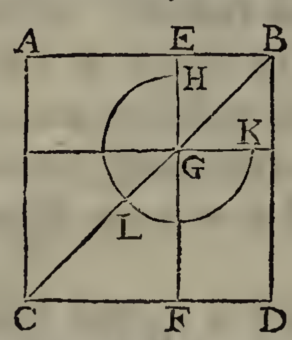
For A B being equal to D C, and A D to B C, therefore A B, and B C, or B C and C D, or C D, and D A, or D A, and A B, are indifferently ſaid to contain the whole Right angl'd Parallelogram A B C D.

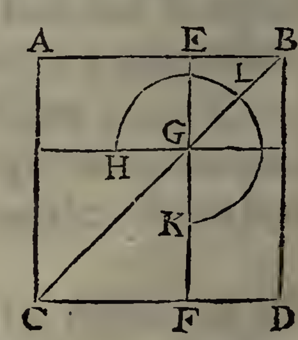
Obſerve alſo, that a Parallelogram ſpace is for bevity ſake, often noted by two oppoſite letters, as the Parallelogram A C, or D B; by either of which is ſignify'd the Parallelogram A B C D.

DEFINITION II.

In every Parallelogram ſpace, any one of the Parallelograms about the Diameter, together with the two Complements, ſhall be called a Gnomon.

What are Parallelogram ſpaces about the Diameter, and what are Complements, is before declared in Prop. 43. El. I. Now here, the two Complements with either of the Diagonal ſpaces, taken together, are for brevity ſake, in one word, as a term of Art, called a GNOMON; as in the Parallelogram ſpace A D, the two Complements A G, G D, with any one of the Parallelograms about the Diameter, either C G, or B G, are called the GNOMON H L K.





Becauſe this Second Element treats of the powers of ſtrait lines, and of the Section of ſtrait lines into ſeveral Segments, which are in power variouſly compared to one another, it will be requiſite to add to *Euclide* theſe two following Definitions.

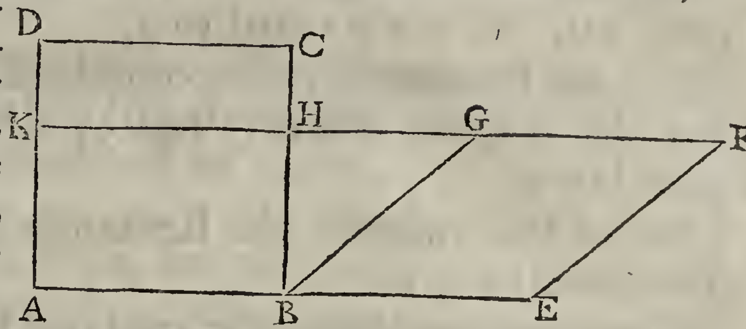
DEFINITION III.

The power of a ſtrait line is the ſquare of the ſame line.

As on the line A B, let there be deſcribed the ſquare A B C D, then the ſquare A B C D is ſaid to be the power of the line A B.

Geome-

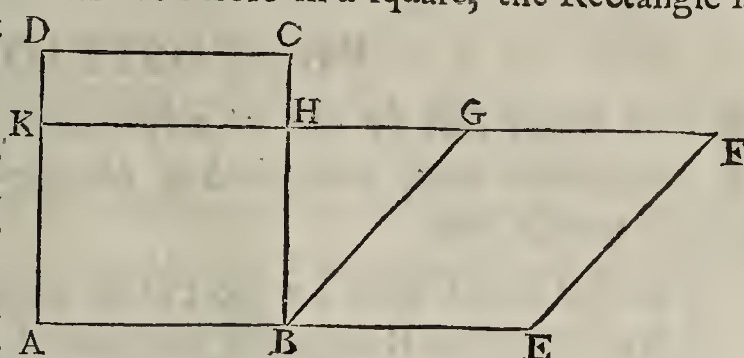
Geometricians have with good reaſon uſed the term *Power*, in this ſenſe, becauſe the ſquare is the greateſt Parallelogram ſpace that can be deſcribed on one and the ſame line. For produce the line AB, and put BE equal to AB; and on BE deſcribe the Rhombus BEFG, 'tis manifeſt, that the Rhombus BEFG is leſs than the ſquare ABCD. For producing the line FG, to K, the Rhombus BEFG is equal to the Rectangle ABHK [by Prop. 36. El. I.] which is but a part of the ſquare ABCD.



DEFINITION IV.

The power of two ſtrait lines is the Rectangle contained by thoſe lines.

As the Rectangle ABCD, contained by the lines AB, BC, is ſaid to be the power of the lines AB, BC. Becauſe here as before in a ſquare, the Rectangle is the greateſt Parallelogram ſpace that can be comprehended by any two and the ſame lines. For produce the line AB, and put BE equal to AB, and from the point E draw EF equal to BC, and compleat the Parallelogram Figure; then by the ſame Prop. 36. El. I. the Rhomboides BEFG is equal to the Rectangle ABHK, which is but a part of the Rectangle ABCD: yet the Rectangle ABCD, and the Rhomboides BEFG, are comprehended by the ſame, that is, by equal lines, for that BE is equal to AB, and EF to BC. Wherefore only the Rectangle ABCD is ſaid to be the power of the lines AB, BC.

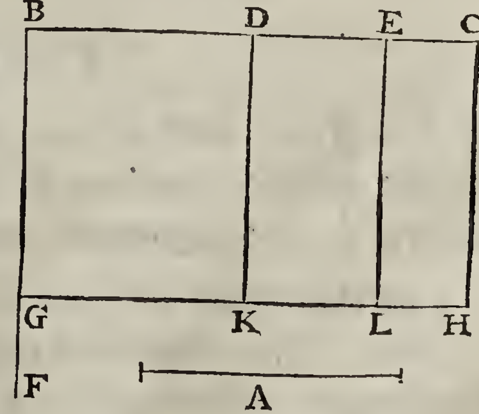


PROPOSITION I.

If there be two ſtrait lines, and one of them be cut into how many Segments ſoever, the Rectangle contained by the two ſtrait lines, is equal to the Rectangles contained by the undivided line, and the ſeveral Segments of the other line.

Let there be two ſtrait lines A, BC, and let BC be cut at adventure in the points D, E. I ſay, that the Rectangle contained by the lines A, BC, is equal to the Rectangles contained by A, BD, and by A, DE, and alſo by A, EC.

For from the point B, let there be drawn BF, at right angles to BC; and let BG be put equal to A [Prop. 3. I.]: Then by the point G let GH be drawn parallel to BC. Again, by the points D, E, C, let there be drawn DK, EL, CH, parallels to BG.



Now the Rectangle BH is equal to the Rectangles BK, DL, EH; and BH is contained by A, BC, for it is contained by GB, BC, but GB is equal to A.

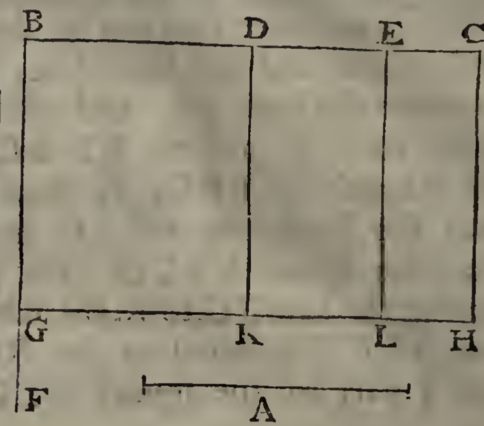
Again,

Again, the Rectangle BK is contained by A, BD, for it is contained by GB, BD, but GB is equal to A.

Alſo the Rectangle DL is contained by A, DE, for DK, that is BG, [Prop. 34. El. I.] is equal to A.

And in like manner the Rectangle EH is contained by A, EC.

Wherefore the Rectangle contained by A, BC, is equal to the Rectangles contained by A, BD, and by A, DE, and alſo by A, EC.



Therefore if there be two ſtrait lines, and one of them be cut into how many Segments ſoever, the Rectangle contained by the two ſtrait lines, is equal to the Rectangles contained by the undivided line, and the ſeveral Segments of the other line. Which was to be domonſtrated.

PROPOSITION II.

IF a ſtrait line be cut at adventure, the Rectangles contained by the whole line, and each of the Segments are equal to the ſquare of the whole line.

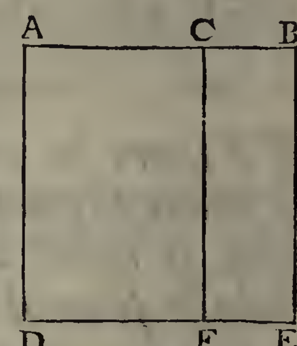
Let the ſtrait line AB be cut at adventure in the point C.

I ſay, that the Rectangle contained by AB, AC, together with the Rectangle contained by AB, BC, is equal to the ſquare of AB.

For on the line AB, let there be deſcribed the ſquare ADEB; and by C, draw CF parallel to either AD, or BE.

Now AE is equal to AF, CE; but AE is the ſquare of AB, and AF is the Rectangle contained by AB, AC; for it is contained by DA, AC; but DA is equal to AB.

Alſo CE is contained by AB, BC, for BE is equal to AB; wherefore the Rectangle contained by AB, AC, together with the Rectangle contained by AB, BC, is equal to the ſquare of AB.



Therefore if a ſtrait line be cut at adventure, the Rectangles contained by the whole line, and each of the Segments are equal to the ſquare of the whole line. Which was to be demonſtrated.

PROPOSITION III.

IF a ſtrait line be cut at adventure, the Rectangle contained by the whole, and one of the Segments is equal to the Rectangle contained by the Segments, and the ſquare of the foreſaid Segment.

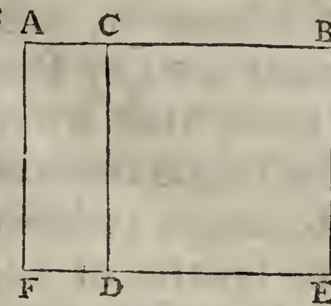
Let the ſtrait line AB be cut at adventure in the point C.

I ſay, that the Rectangle contained by AB, BC, is equal to the Rectangle contained by AC, CB, together with the ſquare of CB.

For on the line CB let there be deſcribed the ſquare CDEB, and produce

produce ED to F: Then by the point A, draw [by Prop. 31. El. I.] AF, parallel to either CD, or BE.

Now the Rectangle AE is equal to the Rectangle AD, and the square CE; and AE is the Rectangle contained by AB, BC; for it is contained by AB, BE, but BE is equal to BC.



Again, AD is contained by AC, CB, for DC is equal to CB; also DB is the square of CB; wherefore the Rectangle contained by AB, BC, is equal to the Rectangle contained by AC, CB, together with the square of CB.

Therefore if a strait line be cut at adventure, the Rectangle contained by the whole and one of the Segments, is equal to the Rectangle contained by the Segments, and the square of the foresaid Segment. Which was to be demonstrated.

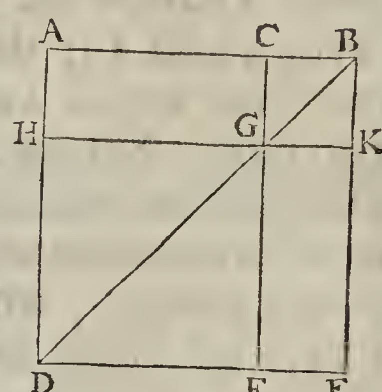
PROPOSITION IV.

If a strait line be cut at adventure, the square of the whole is equal to the squares of the Segments, and to the Rectangle twice contained by the Segments.

Let the strait line AB be cut at adventure in the point C.

I say, that the square of AB is equal to the squares of AC, CB, and to the Rectangle twice contained by AC, CB.

For, on the line AB let there be described the square ADEB, and draw the line BD. Then by the point C, let there be drawn parallel to either AD, or BE, the line CGF: and by the point G, draw HK, parallel to AB, or DE.



Now because CF is parallel to AD, and there falls on them the line BD, therefore the outward angle BGC is equal to the inward and opposite BDA; but the angle BDA is equal to the angle DBA, because the side AB is equal to the side AD [by Prop. 5. El. I.]; wherefore also the angle CGB is equal to the angle CBG, and therefore the side CB, is equal to the side CG [by Prop. 6. El. I.]; but CB is equal to GK, and CG to BK [by Prop. 34. El. I.]; therefore also GK is equal to KB, and the Parallelogram CGKB is Equilateral.

I say also, that it is Rectangular. For because CG is parallel to BK, and there falls on them the line CB, therefore the inward angles KBC, GCB, are equal to two right angles [Prop. 29. El. I.]. But the angle KBC is a right angle [by Construction]; wherefore the angle GCB is a right angle, and also the opposite angles CGK, GKB, are right angles [Prop. 34. El. I.]; wherefore the Parallelogram CGKB is Rectangular. And it hath been proved to be Equilateral; therefore CGKB is a square, and it is described on the line CB.

By the same reason also the Parallelogram HF is a square, and it is defcribed on the line HG, that is on AC; therefore HF, CK, are the fquares of AC, CB, the Segments of the line AB.

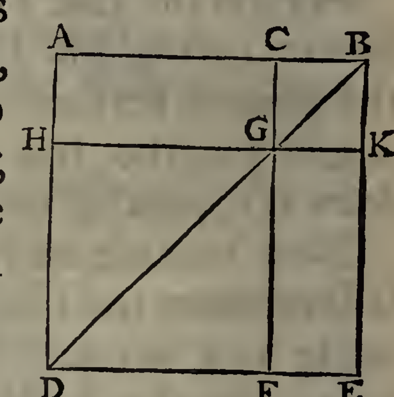
Moreover, becaufe the Complement AG is equal to the Complement GE [by Prop. 43. El. I.], and AG is contained by the lines AC, CB, for that GC is equal to CB; therefore alfo GE is equal to the Rectangle contained by AC, CB; wherefore AG, GE, are equal to the Rectangle twice contained by the lines AC, CB: But alfo HF, CK, are the fquares of AC, CB.

Therefore the four Parallelograms HF, CK, AG, GE, are equal to the fquares of AC, CB, and to the Rectangle twice contained by AC, CB. But HF, CK, AG, GE, are the whole ADEB, which is the fquare of AB; wherefore the fquare of AB is equal to the fquares of AC, CB, and to the Rectangle twice contained by AC, CB.

Therefore, if a ftrait line be cut at adventure, the fquare of the whole is equal to the fquares of the Segments, and to the Rectangle twice contained by the Segments. Which was to be demonftrated.

Otherwife.

I fay, that the fquare of AB is equal to the fquares of AC, CB, and to the Rectangle twice contained by AC, CB. For in the fame Figure, becaufe AB is equal to AD, therefore the angle ABD, is equal to the angle ADB [Prop. 5. El. I.]. And forafmuch as the three angles of every Triangle are equal to two right angles [Prop. 32. El. I.]; therefore of the Triangle ABD, the three angles ABD, ADB, BAD, are equal to two right. But the angle BAD is a right angle; therefore the remaining angles ABD, ADB, are equal to one right angle, and they are alfo equal to one another; wherefore each of the angles ABD, ADB, is the half of a right angle. But the angle BCG is a right angle [Prop. 29. El. I.]; for it is equal to the inward and oppofite angle at A; wherefore the remaining angle CGB is the half of a right angle; and therefore the angle CGB is equal to the angle CBG, and the fide CB is equal to the fide CG [Prop. 6. El. I.]; but CB is equal to GK, and CG to BK [Prop. 34. El. I.]; wherefore the Parallelogram CK is Equilateral.



And becaufe the angles BCG, CBK, are right angles, therefore CK is alfo a fquare. And it is defcribed on the line CB.

Likewife by the fame reafon, the Parallelogram HF is a fquare, and equal to the fquare of AC; wherefore HF, CK, are fquares, and equal to the fquares of AC, CB.

Moreover, becaufe the Rectangle AG is equal to the Rectangle GE [Prop. 43. El. I.], and AG is contained by the lines AC, CB, (for that CG is equal to CB) therefore alfo GE is equal to the Rectangle contained by AC, CB; wherefore AG, GE, are equal to the Rectangle twice

twice contained by AC, CB; but also HF, CK, are equal to the squares of AC, CB.

Therefore HF, CK, AG, GE, are equal to the squares of AC, CB, and to the Rectangle twice contained by AC, CB. But HF, CK, AG, GE, are the whole AE, which is the square of AB; therefore the square of AB is equal to the squares of AC, CB, and to the Rectangle twice contained by AC, CB. Which was to be demonstrated.

Corollaries.

1. *From hence 'tis manifest, that in Squares, the Parallelograms, which are about the Diameter, are also Squares.*

2. And if the side of a Square be cut into halves, then the Complements are also Squares. And the Square of the whole line is quadruple to the Square of the half.

On the fourth Proposition.

This is a most remarkable Proposition, and of excellent and various uses: The Analysis or Resolution of a Square, which, by Arithmeticians, is called the Extraction of the Square-root, wholly depends upon it. But because all things are resolved into those parts whereof they are at first constituted, it will be requisite, to begin with the Genesis or Construction of every compound Square.

The better to explain this matter, we shall apply to the Scheme of this fourth Proposition, by the addition of E, making it $A+E$: Then square it, that is, draw $A+E$ into $A+E$, thus,

$$\begin{array}{l} A+E \\ A+E \\ \hline Aq+AE \\ \quad +AE+Eq \\ \hline Aq+2AE+Eq \end{array}$$

This Genesis of a Square exactly answers to *Euclid's* Proposition, and back again, by Analysis, shews how the most simple division of a Square must necessarily fall into these kinds of parts. But indeed *Euclide* begins with the square it self, and then resolves it, by dividing it into any two parts, because in magnitude, there is no Square singly the First, as Unity is in Numbers, which answers to all Powers. And therefore dividing a line, on which a Square is described, into any two parts, he shews of what parts the most simple that may be a Square can consist. And accordingly the practice of extracting the Square root answers to it. As for example.

Every Square is encreased by the addition of twice the root for the Complements, and 1 for the Diagonal Square, thus $2\surd+1$. If there be put Aq, it is encreased $2A+1$: So that $Aq+2A+1$, is the next Square.

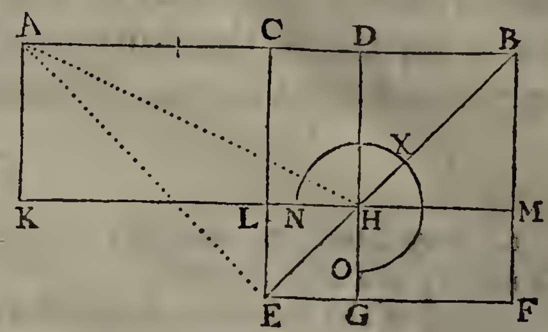
PROPOSITION V.

If a strait line be cut into two equal Segments, and two unequal, the Rectangle contained by the unequal Segments of the whole, together with the Square of the line between the Sections, is equal to the Square of half the line.

Let the strait line AB be cut into two equal Segments, at the point C, and into two unequal, at the point D.

I say, that the Rectangle contained by AD, DB, together with the square of CD, is equal to the square of CB.

For on the line CB, let there be defcribed the fquare CEFB, and draw the line BE. Then by the point D, let there be drawn parallel to either CE, or BF, the line DHG, and by the point H, draw KM, parallel to CB, or EF. And again, by the point A, draw AK, parallel to CL, or BM.



Now forafmuch as the Complement CH is equal to the Complement HF [Prop. 43. El. I.], let DM be added in common to both; therefore the whole CM is equal to the whole DF. But CM is equal to AL, for that AC is [by Suppofition] equal to CB: wherefore alfo AL is equal to DF. Add in common CH, therefore the whole AH is equal to DF, DL.

But AH is the Rectangle contained by AD, DB, for that DH is equal to DB [by Coroll. 1. Prop. 4. El. II.], and DF, DL, is the Gnomon NXO; therefore the Gnomon NXO is equal to the Rectangle contained by AD, DB.

Again, add in common LG, which is equal to the fquare of CD; therefore the Gnomon NXO, and LG, are equal to the Rectangle contained by AD, DB, and to the fquare of CD.

But the Gnomon NXO, and LG, are the whole fquare CEFB, which is defcribed on the line CB; therefore the Rectangle contained by AD, DB, together with the fquare of CD is equal to the fquare of CB.

If therefore a ftrait line be cut into two equal Segments, and two unequal, the Rectangle contained by the unequal Segments of the whole, together with the fquare of the line between the Sections, is equal to the fquare of half the line. Which was to be demonftrated.

PROPOSITION VI.

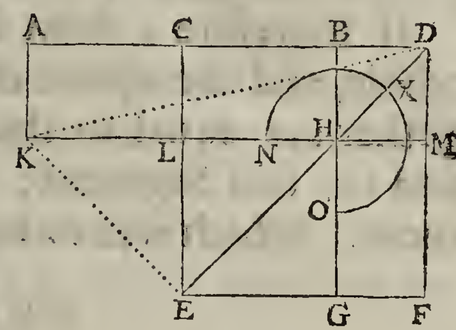
If a ftrait line be cut into two equal Segments, and to it be added another ftrait line directly, the Rectangle contained by the whole with the adjunct (as one line) and by the adjunct, together with the fquare of the half, is equal to the fquare defcribed on the half and the adjunct, as one line.

Let the ftrait line AB be cut into two equal Segments at the point C, and to AB, let BD be added directly. I fay, that the Rectangle contained by AD, DB, together with the fquare of CB is equal to the fquare of CD.

For on the line CD, let there be defcribed the fquare CEFD, and draw the line DE. Then by the point B, let there be drawn parallel to either CE or DF, the line BHG. And by the point H, draw KLM, parallel to AD, or EF: And alfo by the point A draw AK, parallel to CL, or DM.

Now

Now forasmuch as AC is equal to CB [by Supposition], therefore the Rectangle AL is equal to the Rectangle CH [by Prop. 36. El. I.]. But CH is equal to HF [Prop. 43. El. I.]; wherefore AL is equal to HF. Let CM be added in common; therefore the whole AM, is equal to the Gnomon NXO.



But AM is the Rectangle contained by AD, DB, for that DM is equal to DB [by Corol. 1. Prop. 4. El. II.]; wherefore the Gnomon NXO is equal to the Rectangle contained by AD, DB.

Again, add in common LG, which is equal to the square of CB; therefore the Rectangle contained by AD, DB, together with the square of CB, is equal to the Gnomon NXO, and to the square LG.

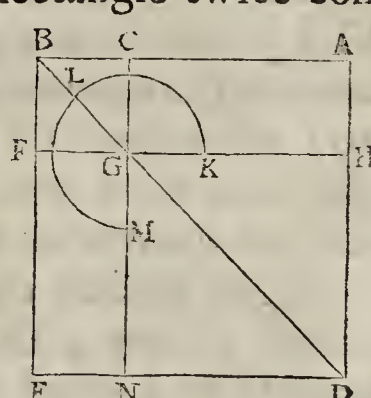
But the Gnomon NXO, and LG are the whole square CEFD, which is described on the line CD; therefore the Rectangle contained by AD, DB, together with the square of CB, is equal to the square of CD.

If therefore a strait line be cut into two equal Segments, and to it be added another strait line directly, the Rectangle contained by the whole with the adjunct (as one line) and by the adjunct, together with the square of the half, is equal to the square described on the half and the adjunct, as one line. Which was to be demonstrated.

PROPOSITION VII.

If a strait line be cut at adventure, the square of the whole, and the square of one of the Segments, both these squares together are equal to the Rectangle twice contained by the whole, and the said Segment, and also to the square of the remaining Segment.

Let the strait line AB be cut at adventure in the point C. I say, that the squares of AB, BC, are equal to the Rectangle twice contained by AB, BC, and to the square of AC.



For on the line AB, let there be described the square ADEB, and let the Figure be constructed.

Now forasmuch as AG is equal to GE [Prop. 43. El. I.], add in common CF: therefore the whole AF is equal to the whole CE: wherefore AF, CE, are double to AF. But AF, CE, are the Gnomon KLM, and the square CF; therefore the Gnomon KLM, and the square CF, are double to AF.

But the Rectangle twice contained by AB, BC, is double to AF, for BF is equal to BC; therefore the Gnomon KLM, and the square CF are equal to the Rectangle twice contained by AB, BC.

Again, let there be added in common HN, which is the square of AC; therefore the Gnomon KLM, and the squares CF, HN, are equal to the Rectangle twice contained by AB, BC, and to the ſquare of AC.

But the Gnomon KLM, and the ſquares CF, HN, are the whole ADEB, and CF, which are the ſquares of AB, BC; therefore the ſquares of AB, BC, are equal to the Rectangle twice contained by AB, BC, together with the ſquare of AC.

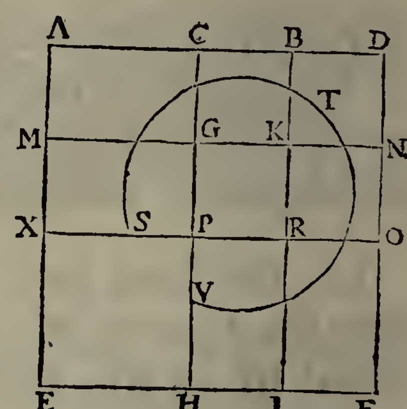
If therefore a ſtrait line be cut at adventure, the ſquare of the whole, and the ſquare of one of the Segments, both theſe ſquares together are equal to the Rectangle twice contained by the whole, and the ſaid Segment; and alſo to the ſquare of the remaining Segment. Which was to be demonſtrated.

PROPOSITION VIII.

If a ſtrait line be cut at adventure, the Rectangle four times contained by the whole, and one of the Segments, together with the ſquare of the other Segment, is equal to the ſquare deſcribed on the whole, and the ſaid Segment, as one line.

Let the ſtrait line AB be cut at adventure in the point C. I ſay, that the Rectangle four times contained by AB, BC, together with the ſquare of AC, is equal to the ſquare deſcribed on AB, BC, as one line.

For let the line AB be produced directly to D, and let BD be put equal to CB; then on AD let there be deſcribed the ſquare AEFD, and let the double Figure be conſtructed.



Now foraſmuch as CB is equal to BD, but alſo CB is equal to GK, and BD to KN [Prop. 34. El. I.], therefore GK is equal to KN. By the ſame reaſon alſo PR is equal to RO.

And becauſe CB is equal to BD, and GK to KN, therefore the Rectangle CK is equal to the Rectangle KD; and GR to RN [Prop. 36. El. I.]. But CK is equal to RN (becauſe they are the Complements of the Parallelogram CO) wherefore alſo KD is equal to GR: therefore the four Rectangles CK, KD, GR, RN, are equal to one another; and theſe four therefore are quadruple to CK.

Again, becauſe CB is equal to BD, but alſo BD is equal to BK [by Coroll. 1. Prop. 4. El. II.] that is, to CG, [by Prop. 36. El. I.]; and alſo becauſe CB is equal to GK, that is, to GP [by Corol. 1. Prop. 4. El. II.]; therefore CG is equal to GP.

And becauſe CG is equal to GP, and PR to RO, therefore the Rectangle AG is equal to the Rectangle MP, and PL to RF [by Prop. 36. El. I.].

But MP is equal to PL (becauſe they are the Complements of the Parallelogram ML); wherefore alſo AG is equal to RF.

There-

Therefore the four Rectangles AG, MP, PL, RF, are equal to one another; and theſe four therefore are quadruple to AG.

But the four Rectangles CK, KD, GR, RN, have been proved quadruple to CK.

Wherefore the eight Rectangles which contain the Gnomon STV, are quadruple to the Rectangle AK.

And becauſe AK is contained by AB, BC, for EK is equal to BD, that is to BC, therefore the Rectangle four times contained by AB, BC, is quadruple to AK.

But the Gnomon STV hath been proved quadruple to AK; therefore the Rectangle four times contained by AB, BC, is equal to the Gnomon STV.

Let there be added in common XH, which is equal to the ſquare of AC; wherefore the Rectangle four times contained by AB, BC, together with the ſquare of AC, is equal to the Gnomon STV, and XH.

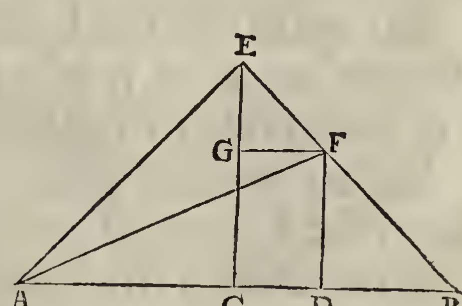
But the Gnomon STV, and XH are the whole ſquare AEFD, which is deſcribed on the line AD; therefore the Rectangle four times contained by the lines AB, BC, together with the ſquare of AC, is equal to the ſquare of AD, that is, to the ſquare deſcribed on AB, BC, as one line.

If therefore a ſtrait line be cut at adventure, the Rectangle four times contained by the whole, and one of the Segments, together with the ſquare of the other Segment, is equal to the ſquare deſcribed on the whole and the ſaid Segment, as one line. Which was to be demonſtrated.

PROPOSITION IX.

If a ſtrait line be cut into two equal Segments, and two unequal, the ſquares of the unequal Segments of the whole line are double to the ſquare of the half, and to the ſquare of the line between the Sections.

Let the ſtrait line AB be cut into two equal Segments at the point C, and into two unequal, at the point D. I ſay, that the ſquares of AD, DB, are double to the ſquares of AC, CD.



For from the point C to the line AB, let there be drawn at right angles CE, and let it be put equal to either AC, or CB; then draw AE, EB. And by the point D, let there be drawn, parallel to CE, the line DF, and alſo by the point F, draw FG parallel to AB. Then let there be drawn AF.

Now foraſmuch as AC is equal to CE, therefore the angle EAC is equal to the angle AEC. And becauſe the angle at C is a right angle, therefore the remaining angles AEC, EAC, are equal to one right angle [Prop. 32. El. I.], and they are equal to one another: wherefore

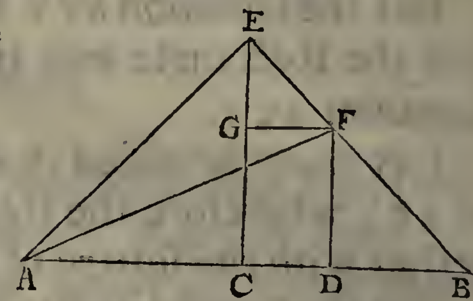
fore each of the angles AEC, EAC, is half of a right angle.

By the fame reafon alfo each of the angles BEC, EBC, is half of a right angle.

Therefore the whole angle AEB is a right angle.

And forafmuch as GEF is half of a right angle, but EGF is a right angle [Prop. 29. El. I.], (becaufe it is equal to the inward and oppofite ECB) therefore the remaining angle EFG is alfo half of a right angle.

Wherefore the angle FEG is equal to EFG, and therefore the fide EG, is equal to the fide FG. [Prop. 6. El. I.].



Again, forafmuch as the angle at B is half of a right angle, and that FDB is a right angle (becaufe it is equal to the inward and oppofite ECB) therefore the remaining angle BFD, is alfo half of a right angle.

Wherefore the angle at B is equal to the angle DFB, fo that alfo the fide DF is equal to the fide DB [Prop. 6. El. I.].

Now becaufe AC is equal to CE, and fo the fquare of AC is equal to the fquare of CE, therefore the fquares of AC, CE, are double to the fquare of AC. But the fquare of AE is [Prop. 47. El. I.] equal to the fquares of AC, CE [for ACE is a right angle by Conftruction]; wherefore the fquare of AE is double to the fquare of AC.

Again, forafmuch as EG is equal to GF; and fo the fquare of EG is equal to the fquare of GF; therefore the fquares of EG, GF, are double to the fquare of GF. But the fquare of EF is equal to the fquares of EG, GF; therefore the fquare of EF is double to the fquare of GF. But the line GF is equal to CD [Prop. 34. El. I.]; *wherefore the fquare of* EF *is double to the fquare of* CD.

But *the fquare of* AE *is double to the fquare of* AC.

Therefore the fquares of AE, EF, are double to the fquares of AC, CD.

But the fquare of AF is equal to the fquares of AE, EF, [for AEF is a right angle]; therefore the fquare of AF is double to the fquares of AC, CD.

But again, to the fquare of AF are equal the fquares of AD, DF, (for the angle at D is a right angle) therefore the fquares of AD, DF, are double to the fquares of AC, CD.

But DF is equal to DB, wherefore the fquares of AD, DB, are double to the fquares of AC, CD.

If therefore a ftrait line be cut into two equal Segments, and two unequal, the fquares of the unequal Segments of the whole line, are double to the fquare of the half, and to the fquare of the line between the Sections. Which was to be demonftrated.

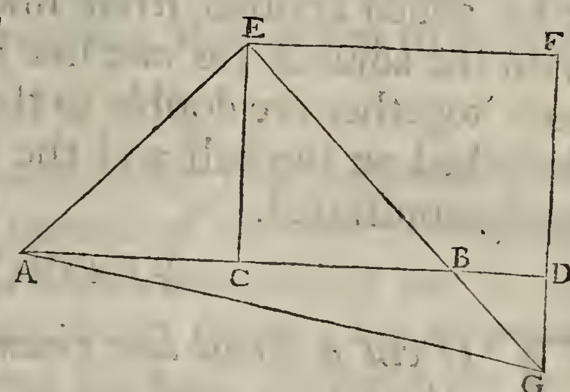
PROPO-

PROPOSITION X.

If a strait line be cut into two equal Segments, and to it be added another strait line directly, the square of the whole, with the adjunct (as one line) and the square of the adjunct, these both together are double to the square of the half, and to the square described on the half, and the adjunct, as one line.

Let the strait line AB be cut into two equal Segments, at the point C, and to AB let BD be added directly. I say, that the squares of AD, DB, are double to the squares of AC, CD.

For from the point C to the line AB, let there be drawn at right angles CE, and let it be put equal to either AC, or CB; then draw AE, EB. And by the point E let there be drawn, parallel to CD, the line EF, and also by the point D, draw DF parallel to CE. Now because there falls on the parallels EC, FD, the strait line EF; therefore the inward angles CEF, EFD, are equal to two right angles; wherefore the angles FEB, EFD, are less than two right angles: but lines infinitely produced from angles less than two right do meet together; wherefore EB, FD, produced towards B, D shall meet. Let them be produced, and meet at the point G. Then draw AG.



Now forasmuch as AC is equal to CE, therefore the angle AEC is equal to the angle EAC. And the angle at C is a right angle; wherefore each of the angles EAC, AEC, is half of a right angle.

By the same reason also each of the angles BEC, EBC, is half of a right angle.

Therefore the whole angle AEB is a right angle.

And forasmuch as EBC is half of a right angle, therefore DBG is also half of a right angle [Prop. 15. I.]; but BDG is a right angle, because it is equal to the alternate angle DCE [Prop. 29. I.]; therefore the remaining angle DGB is half of a right angle; wherefore DGB is equal to DBG. So that also the side BD is equal to the side DG [Prop. 6. I.].

Again, forasmuch as EGF is half of a right angle, and that the angle at F is a right angle, because it is equal to the opposite angle at C [Prop. 34. El. I.]; therefore the remaining angle FEG is half of a right angle. Wherefore EGF is equal to GEF; so that also the side GF is equal to the side EF.

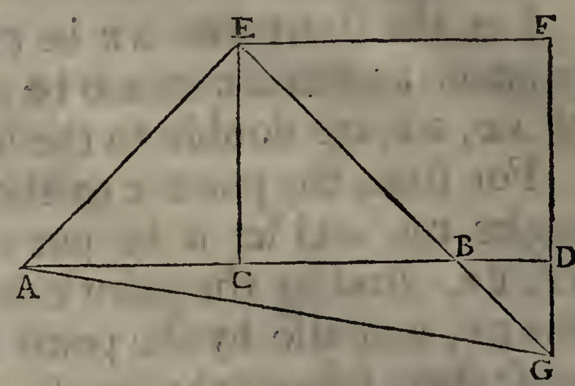
Now because EC is equal to CA; and so the square of EC, to the square of CA; therefore the squares of EC, CA, are double to the square of CA. But the square of EA is equal to the squares of EC, CA [Prop. 47. I.].

Therefore the square of EA is double to the square of AC.

Again,

Again, forasmuch as GF is equal to EF, and so the square of GF to the square of EF; therefore the squares of GF, EF, are double to the square of EF. But the square of EG is equal to the squares of GF, EF.

Therefore the square of EG is double to the square of EF. But EF is equal to CD; therefore the square of EG is double to the square of CD.



But the square of EA has been proved double to the square of AC.

Therefore the squares of EA, EG, are double to the squares of AC, CD.

But the square of AG is equal to the squares of EA, EG; wherefore the square of AG is double to the squares of AC, CD.

But to the square of AG are equal the squares of AD, DG; therefore the squares of AD, DG, are double to the squares of AC, CD.

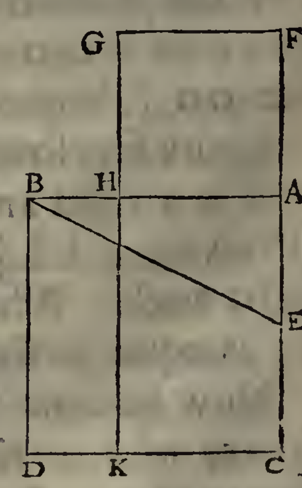
But DG is equal to DB; wherefore the squares of AD, DB, are double to the squares of AC, CD.

If therefore a strait line be cut into two equal Segments, and to it be added another strait line directly, the square of the whole, with the adjunct (as one line) and the square of the adjunct, these both together are double to the square of the half, and to the square described on the half and the adjunct, as one line. Which was to be demonstrated.

PROPOSITION XI.

To cut a strait line given, so that the Rectangle contained by the whole, and one of the Segments may be equal to the square of the other Segment.

Let the strait line given be AB. It is required so to cut AB, that the Rectangle contained by the whole AB, and one of the Segments may be equal to the square of the other Segment.



On AB let there be described the square ABDC: and let AC be divided into two equal Segments at the point E, then draw BE: and let CA be produced to F, and EF be put equal to EB. Then on AF, let there be described the square AFGH; and let GH be produced to K.

I say, that AB is cut in the point H, so that the Rectangle contained by the whole AB, and by the Segment BH, is equal to the square of AH, the other Segment.

For whereas the strait line AC is cut into two equal Segments at the point E; and there is added to it AF; therefore the Rectangle

con-

contained by CF, FA, together with the ſquare of AE is equal to the ſquare of EF [Prop. 6. El. II.].

But EF is equal to EB; therefore the Rectangle contained by CF, FA, together with the ſquare of EA, is equal to the ſquare of EB.

But to the ſquare of EB are equal the ſquares of BA, AE; for the angle at A is a right angle.

Therefore the Rectangle contained by CF, FA, together with the ſquare of AE, is equal to the ſquares of BA, AE.

Let the common ſquare of AE be taken away. Therefore the remaining Rectangle contained by CF, FA, is equal to the ſquare of AB; and the Rectangle contained by CF, FA, is the Rectangle FK (for AF is equal to FG) and alſo the ſquare of AB is the ſquare AD; therefore FK is equal to AD.

Let the common Rectangle AK be taken away. Therefore the remainder FH is equal to the remainder HD.

But HD is the Rectangle contained by AB, BH (for AB is equal to BD by Conſtruction), and FH is the ſquare of AH.

Therefore the Rectangle contained by AB, BH, is equal to he ſquare of AH.

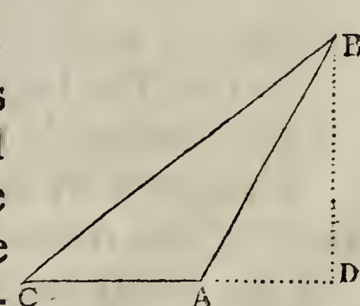
Wherefore the ſtrait line given AB, is cut in the point H, ſo that the Rectangle contained by the whole AB, and by the Segment BH, is equal to the ſquare of the other Segment AH. Which was to be done.

PROPOSITION XII.

IN Obtuſe angl'd Triangles, the ſquare of the ſide ſubtending the Obtuſe angle, is greater than the ſquares of the ſides containing the Obtuſe angle, by the Rectangle twice contained by one of the ſides about the Obtuſe angle, on which being produced a perpendicular falls; and by the line intercepted without, between the perpendicular and the Obtuſe angle.

Let ABC be an Obtuſe angl'd Triangle, having the Obtuſe angle BAC; and from the point B on the ſide CA produced, let there be drawn the perpendicular BD.

I ſay, that the ſquare of BC is greater than the ſquares of BA, AC, by the Rectangle twice contained by CA, AD.



For becauſe CD is cut at adventure in the point A, therefore the ſquare of CD is equal to the ſquares of CA, AD, and to the Rectangle twice contained by CA, AD [Prop. 4. El. II.]. Add in common the ſquare of DB; therefore the ſquares of CD, DB, are equal to the ſquares of CA, AD, DB, and to the Rectangle twice contained by CA, AD.

But to the ſquares of CD, DB, is equal the ſquare of CB (for the angle at D is a right angle), and to the ſquares of AD, DB, is equal the ſquare of AB.

Therefore the ſquare of CB is equal to the ſquares of CA, AB, and to the Rectangle twice contained by CA, AD.

So that the ſquare of CB is greater than the ſquares of CA, AB, by the Rectangle twice contained by CA, AD.

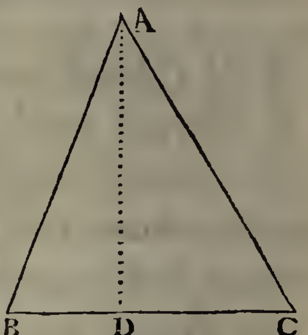
Therefore in Obtuſe angl'd Triangles, the ſquare of the ſide ſubtending the Obtuſe angle, is greater than the ſquares of the ſides containing the Obtuſe angle, by the Rectangle twice contained by one of the ſides about the Obtuſe angle, on which being produced a perpendicular falls; and by the line intercepted without, between the perpendicular and the Obtuſe angle. Which was to be demonſtrated.

PROPOSITION XIII.

IN Acute angl'd Triangles the ſquare of the ſide ſubtending the Acute angle, is leſs than the ſquares of the ſides containing the Acute angle, by the Rectangle twice contained by one of the ſides about the Acute angle, on which a perpendicular falls; and by the line intercepted within, between the perpendicular and the Acute angle.

Let ABC be an Acute angl'd Triangle, having an Acute angle at B; and from the point A, on the ſide BC, let there be drawn the perpendicular AD.

I ſay, that the ſquare of AC is leſs than the ſquares of CB, BA, by the Rectangle twice contained by CB, BD.



For becauſe CB is cut at adventure in the point D, therefore the ſquares of CB, BD, are equal to the Rectangle twice contained by CB, BD, and to the ſquare of DC [Prop. 7. El. II.]. Add in common the ſquare of AD.

Therefore the ſquares of CB, BD, DA, are equal to the Rectangle twice contained by CB, BD, and to the ſquares of DA, DC.

But to the ſquares of BD, DA, is equal the ſquare of AB (for the angle at D is a right angle), and to the ſquares of DA, DC, is equal the ſquare of AC.

Therefore the ſquares of CB, BA, are equal to the ſquare of AC, and to the Rectangle twice contained by CB, BD.

So that the ſingle ſquare of AC is leſs than the ſquares of CB, BA, by the Rectangle twice contained by CB, BD.

Therefore in Acute angl'd Triangles the ſquare of the ſide ſubtending the Acute angle, is leſs than the ſquares of the ſides containing the Acute angle, by the Rectangle twice contained by one of the ſides about the Acute angle, on which a perpendicular falls; and by the line intercepted within, between the perpendicular and the Acute angle. Which was to be demonſtrated.

PROPO-

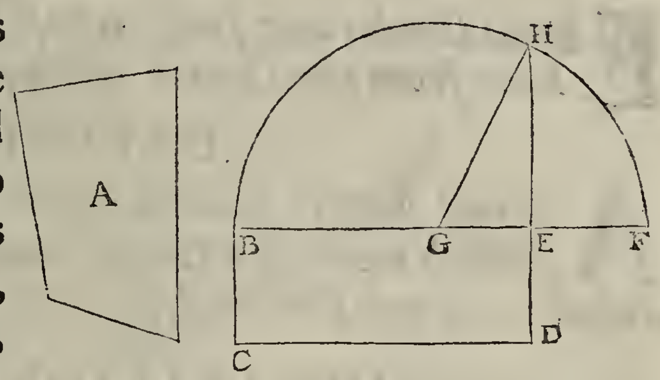
PROPOSITION XIV.

TO conſtitute a ſquare equal to a right-lin'd Figure given.

Let the right-lin'd Figure be A. It is required to conſtitute a ſquare equal to A.

By Prop. 45. El. I. let the Rectangled Parallelogram BD be conſtituted equal to the right-lin'd Figure A.

If therefore the line BE be equal to the line ED, then what was required is now done, for there is conſtituted the ſquare BD, equal to the right-lin'd Figure A.



But if not, then one of the lines BE, ED, is the greater. Let BE be the greater; and let it be produced to F; and let EF be put equal to ED. Then let BF be cut into halves at the point G; and to the center G, and the diſtance one of the lines GB, GF, let there be deſcribed the Semicircle BHF; and let DE be produced to H. Then let GH be joyn'd.

Now foraſmuch as BF is cut into two equal Segments at G, and into two unequal at E,; therefore the Rectangle contained by BE, EF, together with the ſquare of GE, is equal to the ſquare of GF [Prop. 5. El. II.]. But GF is equal to GH, therefore the Rectangle contained by BE, EF, together with the ſquare of GE, is equal to the ſquare of GH. But to the ſquare of GH, are equal the ſquares of GE, EH, therefore the Rectangle contained by BE, EF, together with the ſquare of GE, is equal to the ſquares of GE, EH. Let the common ſquare of GE be taken away. Therefore the remaining Rectangle contained by BE, EF, is equal to the ſquare of EH. But the Rectangle contained by BE, EF, is the Parallelogram BD, becauſe EF is equal to ED.

Therefore the Parallelogram BD is equal to the ſquare of EH. But the Parallelogram BD is equal to the right-lin'd Figure A; wherefore the right-lin'd Figure A, is equal to the ſquare of EH.

Therefore there is conſtituted a ſquare deſcribed on EH equal to A, the right-lin'd Figure given. Which was to be done.

THE THIRD ELEMENT.

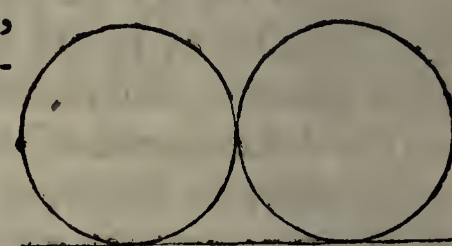
DEFINITIONS.

DEFINITION I.

Equal Circles are ſuch, whoſe Diameters are equal; or, whoſe lines from the Center are equal.

DEFINITION II.

A Strait line is ſaid to touch a Circle, which meeting a Circle, and being produced, cuts not the Circle.



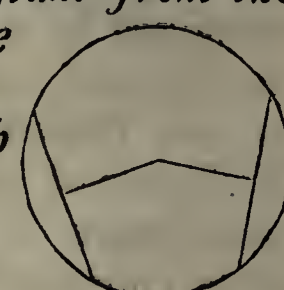
DEFINITION III.

Circles are ſaid to touch one another, which meeting together, do not cut one another.

DEFINITION IV.

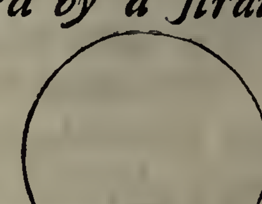
In a Circle ſtrait lines are ſaid to be equally diſtant from the Center, when the perpendiculars drawn from the Center to the ſame lines are equal.

And that line is ſaid to be more diſtant; on which the greater perpendicular falls.



DEFINITION V.

A Segment of a Circle is a Figure comprehended by a ſtrait line, and the circumference of the Circle.
The ſtrait line is called the baſe of the Segment.

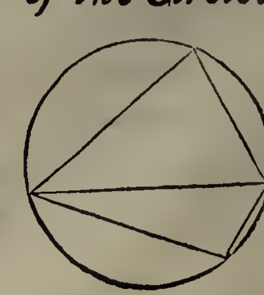


DEFINITION VI.

An angle of a Segment is that which is contained by a ſtrait line, and the circumference of the Circle.

DEFINITION VII.

An angle in a Segment is when in the circumference of a Segment ſhall be taken a point, and from that point to the ends of the line, which is the baſe of the Segment, ſhall be drawn ſtrait lines: it is the angle contained by thoſe ſame lines.

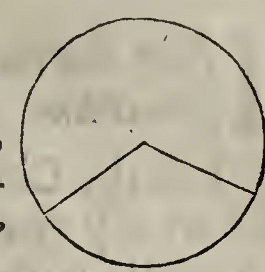


DEFI-

DEFINITION VIII.

AND when the ſtrait lines containining the angle do aſſume a circumference, the angle is ſaid to inſiſt upon that circumference.

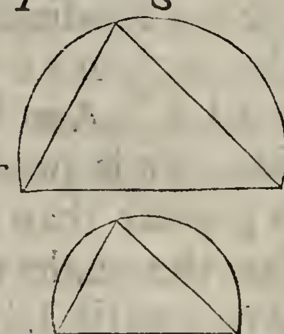
DEFINITION IX.



A Sector of a Circle is a Figure, which, when an angle ſhall be ſtated at the center of the Circle; is comprehended by the ſtrait lines containing that angle, and by the circumference aſſumed under the ſame lines.

DEFINITION X.

LIke Segments of Circles are ſuch, which receive equal angles. Or in which the angles are equal to one another.



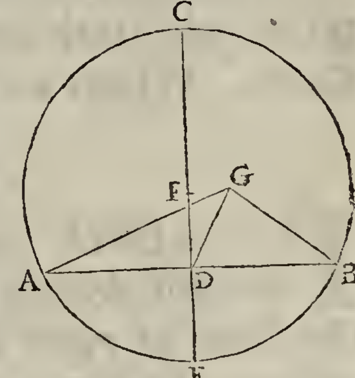
That is, if the Segments of two Circles are ſuppoſed like, or ſo proved, then we are to allow the angles in thoſe Segments to be equal. Or contrarily, if the angles be ſuppoſed, or proved equal, then we are to allow the Segments to be like to one another.

PROPOSITION I.

TO find the center of a given Circle.

Let the given Circle be ABC. It is required of the Circle ABC, to find the center.

Let in the Circle be drawn at adventure any ſtrait line as AB; and let it be cut into halves in the point D; and from D draw DC at right angles to AB, and produce it to E. Then let CE be cut into halves in F. I ſay, the point F is the center of the Circle ABC. For if not: let it, if poſſible, be the point G; and let GA, GD, GB, be joyn'd. Now foraſmuch as AD is equal to DB and DG is common; therefore there are the two lines AD, DG, equal to the two lines DB, DG, each to each, and the baſe GA is equal to the baſe GB, for they are from the center G; wherefore the angle ADG is equal to the angle GDB [Prop. 8. El. I.]. But when a ſtrait line ſtanding upon a ſtrait line, makes the angles on each ſide equal to one another, each of the equal angles is a right angle; therefore GDB is a right angle. But alſo FDB is a right angle [by Conſtruction]; therefore FDB is equal to GDB, the greater to the leſs, which is impoſſible. Wherefore G is not the center of the Circle ABC. And in like manner may we prove that no other point can be beſides F.



Therefore the point F is the center of the Circle ABC. Which was to be done.

Corol-

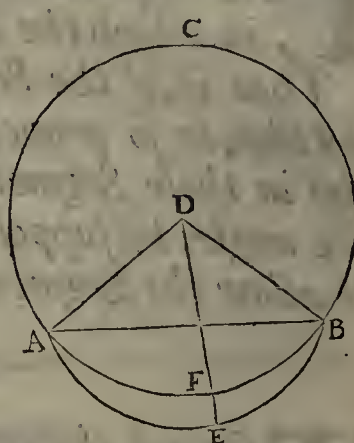
Corollary.

From hence it is manifest, that if in a Circle a ſtrait line cuts a ſtrait line into halves, and at right angles; the center of the Circle is in the cutting line.

PROPOSITION II.

IF in the circumference of a Circle any two points be taken, the ſtrait line joyning the ſame points ſhall fall within the Circle.

Let the Circle be ABC, and in the circumference thereof let be taken any two points A, B. I ſay, that the ſtrait line drawn from A to B falls within the Circle. For if not: let it, if poſſible, fall without, as AEB, and let the center of the Circle ABC be taken, and let it be D. Then let AD, DB, be joyn'd, and let DF be produced to E.



Now foraſmuch as DA is equal to DB, therefore the angle DAE is equal to the angle DBE [Prop. 5. El. I.]. And becauſe of the Triangle DAE, one ſide AEB is produced; therefore the angle DEB is greater than the angle DAE [Prop. 16. El. I.]. But the angle DAE is equal to the angle DBE [Prop. 5. El. I.]; therefore DEB is greater than DBE. But under the greater angle is ſubtended the greater ſide [Prop. 19. El. I.]; therefore BD is greater than DE. But DB is equal to DF, wherefore DF is greater than DE, the leſs than the greater, which is impoſſible; therefore the ſtrait line drawn from A to B, ſhall not fall without the Circle. In like manner we may ſhew that it ſhall not fall upon the circumference: therefore it ſhall fall within.

If therefore in the circumference of a Circle any two points be taken, the ſtrait line joyning the ſame points ſhall fall within the Circle. Which was to be demonſtrated.

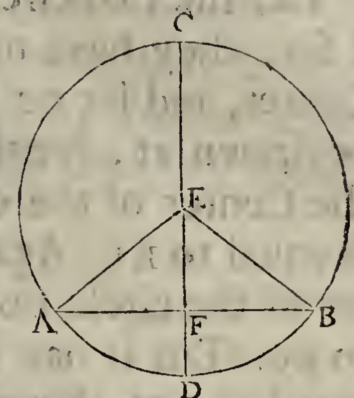
PROPOSITION III.

IF in a Circle a ſtrait line drawn through the center cuts a ſtrait line not drawn through the center into halves; it ſhall alſo cut the ſame at right angles. And if it cuts it at right angles, it ſhall alſo cut the ſame into halves.

Let the Circle be ABC, and in the ſame let the ſtrait line CD, drawn through the center, cut the ſtrait line AB not drawn through the center, into halves in the point F. I ſay, that it alſo cuts the ſame at right angles. Let the center of the Circle ABC be taken, and let it be E, and let EA, EB, be joyn'd. Now becauſe AF is equal to FB [by Suppoſition], and FE is common; therefore two to two are equal, and the baſe EA is equal to the baſe EB; wherefore the angle AFE is equal to the angle BFE [Prop. 8. El. I.]. But when a ſtrait line ſtanding upon a ſtrait line, makes the angles on each ſide equal

equal to one another, each of the equal angles is a right angle; wherefore each of the angles AFE, BFE, is a right angle. Therefore CD drawn through the Center cutting AB not drawn through the Center into halves, does alſo cut it at right angles.

But again, let the ſtrait line CD cut the ſtrait line AB at right angles. I ſay, that it alſo cuts the ſame into halves: which is, that AF is equal to FB. For the ſame conſtruction being made, becauſe in the Triangle EAB, EA, a line from the Center is equal to EB; therefore the angle EAF is equal to the angle EBF [Prop. 5. El. I.]; but the right angle AFE is equal to the right angle BFE: therefore there are the Triangles EFA, EFB, having two angles equal to two angles, and one ſide equal to one ſide, namely EF common to both, which ſubtends one of the equal angles: Therefore they ſhall have the remaining ſides equal to the remaining ſides [Prop. 26. El. I.]; wherefore AF is equal to BF.

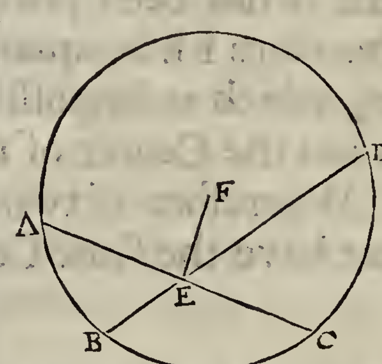


If therefore in a circle a ſtrait line drawn through the Center cuts a ſtrait line not drawn through the Center into halves; it ſhall alſo cut the ſame at right angles. And if it cuts it at right angles, it ſhall alſo cut it into halves. Which was to be demonſtrated.

PROPOSITION IV.

If in a circle two ſtrait lines not drawn through the center, cut one another, they ſhall not cut one another into halves.

Let the circle be ABCD, and in the ſame let the ſtrait lines AC, BD, not drawn through the Center, cut one another in the point E. I ſay, they do not cut one another into halves; ſo that AE be equal to EC, and BE to ED. Let the Center of the circle ABCD be taken, and be it F, and let EF be joyn'd. Now foraſmuch as the ſtrait line FE drawn through the Center, cuts the ſtrait line AC not drawn through the Center into halves, it ſhall alſo cut the ſame at right angles [Prop. 3. El. III.]; therefore FEA is a right angle. Again, becauſe FE cuts the ſtrait line BD not drawn through the Center into halves; it ſhall alſo cut the ſame at right angles; therefore FEB is a right angle. But it has been proved, that FEA is a right angle; therefore FEA is equal to FEB; the leſs to the greater, which is impoſſible; wherefore AC, BD, do not cut one another into halves.

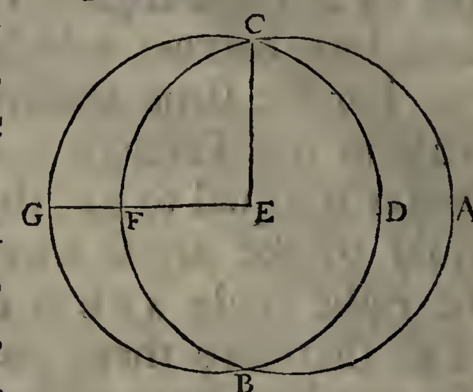


If therefore in a circle two ſtrait lines not drawn through the Center, cut one another, they ſhall not cut one another into halves. Which was to be demonſtrated.

PROPOSITION V.

If two circles cut one another, they ſhall not have the ſame Center.

Let the two circles ABC, CDG, cut one another in the points B, C. I ſay, they have not the ſame Center. For, if poſſible, let E be the Center, and let EC be joyn'd; alſo let EFG be drawn at adventure. Now becauſe E is the Center of the circle ABC, therefore EC is equal to EF. Again, becauſe E is the Center of the circle CDG, therefore EC is equal to EG. But it has been proved, that EC is equal to EF; therefore EF is equal to EG; the leſs to the greater, which is impoſſible. Therefore the point E is not the Center of the circles ABC, CDG.

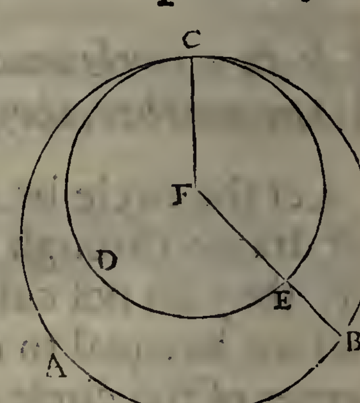


Wherefore if two circles cut one another, they ſhall not have the ſame Center. Which was to be demonſtrated.

PROPOSITION VI.

If two circles touch one another within, they ſhall not have the ſame Center.

Let the two circles ABC, CDE, touch one another within in the point C. I ſay, they have not the ſame Center. For if poſſible, let the Center be F, and let FC be joyned: alſo let FEB be drawn at adventure. Now foraſmuch as F is the Center of the circle ABC, therefore FC is equal to FB. Again, becauſe F is the Center of the circle CDE, therefore FC is equal to FE. But it has been proved that FC is equal to FB; therefore FE is equal to FB; the leſs to the greater, which is impoſſible. Therefore the point F is not the Center of the circles ABC, CDE.



Wherefore if two circles touch one another within, they ſhall not have the ſame Center. Which was to be demonſtrated.

PROPOSITION VII.

If in the Diameter of a circle be taken any point, which is not the Center of the circle; and from that point do fall upon the circle any ſtrait lines: the greateſt ſhall be that, in which the Center is; and the remaining part ſhall be the leaſt. Of the others the nearer to the line through the Center, is always greater than the more remote.

And two only equal lines can from the ſame point fall upon the circle, on each ſide of the leaſt line.

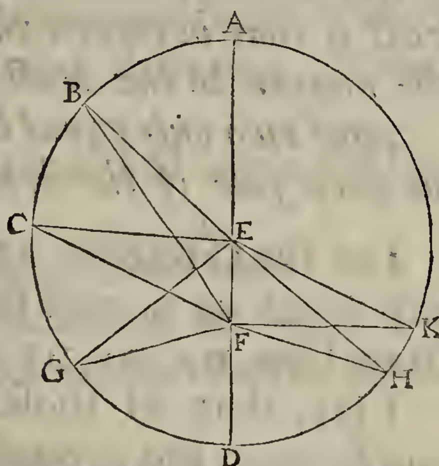
Let the circle be ABCD, and the Diameter thereof be AD, and in

AD,

AD, let be taken any point as F, which is not the center of the Circle. But let the center of the Circle be E; and from F let the ſtrait lines FB, FC, FG, fall upon the circle ABCD.

I ſay, that the greateſt is FA, and the leaſt FD: Of the others FB is greater than FC, and FC than FG. For let BE, CE, GE be joyn'd.

Now becauſe two ſides of every Triangle are greater than the remaining ſide [Prop. 20. El. I.]; therefore BE, EF, are greater than FB; but AE is equal to BE, therefore BE, EF, are equal to AF; therefore AF is greater than BF. Again, becauſe BE is equal to CE, and FE common, therefore there are the two lines BE, EF, equal to the two lines CE, EF. But the angle BEF is greater than the angle CEF, therefore the baſe BF, is greater than the baſe CF [Prop. 24. El. I.]. And by the ſame reaſon CF is greater than FG. Again, becauſe GF, FE, are greater than EG, and EG is equal to ED; therefore GF, FE, are greater than ED. Let EF common be taken away, therefore the remaining line GF, is greater than the remaining line FD; Wherefore FA is the greateſt line, and FD the leaſt: And FB is greater than FC, and FC than FG.



I ſay alſo, that from the point F, two only equal lines can fall upon the Circle ABCD, on each ſide of the leaſt line FD.

For to the line EF, and to the point E in the ſame, let be conſtituted the angle FEH, equal to the angle GEF [Prop. 23. El. I.], and let EH be drawn. Foraſmuch then as GE is equal to EH, and EF common; therefore there are the two lines GE, EF, equal to the two lines HE, EF, and the angle GEF is equal to the angle HEF: therefore the baſe FG is equal to the baſe FH [Prop. 4. El. I.].

I ſay now, that from the point F there cannot fall upon the Circle any other line equal to FG.

For if poſſible, let FK ſo fall, and becauſe FK is equal to FG, FH is equal to FG; therefore FK is equal to FH: The nearer to the line through the Center, equal to the more remote: which is impoſſible.

Or alſo thus. Let EK be joyn'd, and becauſe GE is equal to EK, FE common, and the baſe GF equal to the baſe FK; therefore the angle GEF is equal to the angle KEF. But the angle GEF is equal to the angle HEF; therefore the angle HEF is equal to the angle KEF: The leſs to the greater, which is impoſſible. Wherefore from the point F any other line cannot fall upon the Circle equal to FG; therefore one only.

If therefore in the Diameter of a Circle be taken any point, &c. Which was to be demonſtrated.

PROPOSITION VIII.

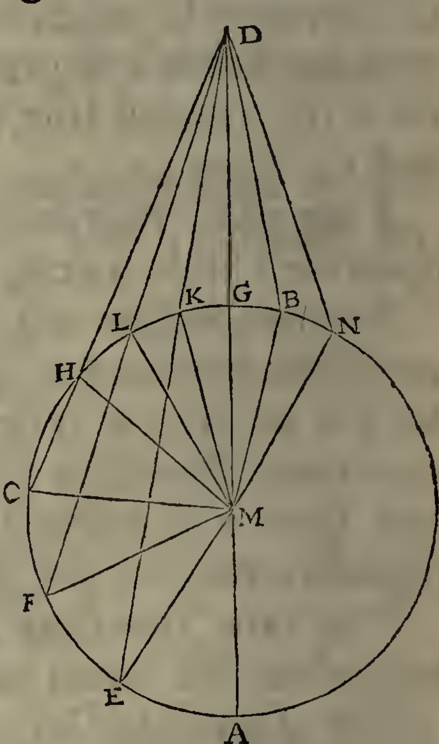
IF without a Circle be taken any point, and from that point be drawn to the Circle any ſtrait lines, of which one is through the Center, and the reſt at adventure. Of thoſe that fall upon the Concave circumference, the greateſt is that through the Center. Of the others the nearer to the line through the Center, ſhall be always greater than the more remote.

But of thoſe lines which fall upon the Convex circumference, the leaſt is that between the point and the Diameter. Of the others the nearer to the leaſt line, is always leſs than the more remote.

And two only equal lines can from that point fall upon the Circle, on each ſide of the leaſt line.

Let the Circle be ABC, and without the Circle ABC let be taken any point as D, and from the ſame let be drawn to the Circle the ſtrait lines DA, DE, DF, DC, and let DA be through the Center.

I ſay, that of thoſe which fall upon AEFC, the Concave circumference, the greateſt is DA, the line through the Center. And always the nearer to the line through the Center ſhall be greater than the more remote; that is DE, than DF, and DF, than DC.



But of thoſe lines which fall upon HLKG, the convex circumference the leaſt is DG, between the point D and the Diameter AG. And the nearer to DG the leaſt line, is always leſs than the more remote; that is, DK than DL, and DL than DH.

Let the Center of the Circle ABC be taken, and be it M, and let be joyned ME, MF, MC, MK, ML, MH. Now foraſmuch as AM is equal to ME, let MD common be added; therefore AD is equal to EM, MD. But EM, MD, are greater than ED [Prop. 20. El. I.]; therefore AD is alſo greater than ED. Again, becauſe ME is equal to MF, and MD common, therefore EM, MD, are equal to FM, MD, and the angle EMD, is greater than the angle FMD; therefore the baſe ED is greater than the baſe FD [Prop. 24. El. I.]. And in like manner may we prove, that FD is greater than CD. Therefore DA is the greateſt, and DE is greater than DF, and DF than DC.

And again, becauſe MK, KD, are greater than MD, and MG is equal to MK; therefore the remaining line KD, is greater than the remaining line GD, ſo that GD is leſs than KD, therefore GD is the leaſt. And becauſe of the Triangle MLD, upon one of the ſides MD are conſtituted within the ſame Triangle two ſtrait lines MK, KD; therefore MK, KD, are leſs than ML, LD [Prop. 21. El. I.]; of which MK is is equal to ML; therefore the remaining line DK, is leſs than the remaining line DL. In like manner we may ſhow that DL is leſs than DH; therefore DG is the leaſt: and DK is leſs than DL, and DL than DH.

I ſay alſo that two only equal lines can from the point D fall upon the Circle on each ſide of the leaſt line DG.

For to the line MD, and in it to the point M, let be conſtituted the angle DMB, equal to the angle KMD, and let DB be joyned. Now becauſe MK is equal to MB, and MD common; therefore there are the two lines KM, MD, equal to the two lines BM, MD, each to each, and the angle KMD is equal to the angle BMD; therefore the baſe DK is equal to the baſe DB [Prop. 4. El. I.].

I ſay now, that from the point D there cannot fall upon the Circle any other line equal to DK.

For if poſſible let a line ſo fall, and be it DN. Now foraſmuch as DK is equal to DN, and DK is equal to DB; therefore DB is equal to DN: The nearer to DG the leaſt, equal to the more remote, which has been proved to be impoſſible.

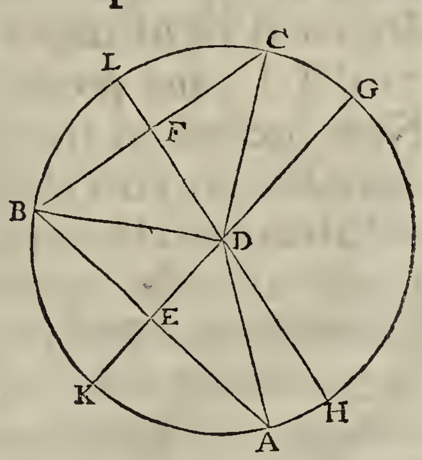
Or otherwiſe, let MN be joyned: and becauſe KM is equal to MN, and MD common, and the baſe DK is equal to the baſe DN, therefore the angle KMD is equal to the angle DMN [Prop. 8. El. I.]; but the angle KMD is equal to the angle BMD; therefore the angle BMD is equal to the angle NMD: the leſs to the greater, which is impoſſible. Wherefore not more than two equal ſtrait lines can from the point D fall upon the Circle ABC, on each ſide of the leaſt line DG.

If therefore without a Circle be taken any point, &c. Which was to be demonſtrated.

PROPOSITION IX.

If within a Circle be taken any point, and from that point do fall upon the Circle more than two equal ſtrait lines, that point taken is the Center of the Circle.

Let the Circle be ABC, and within the ſame the point be D: and from D let upon the Circle ABC fall more than two equal ſtrait lines, as DA, DB, DC. I ſay, that the point D is the Center of the Circle ABC. For let AB, BC, be joyned, and be cut into halves at the points E, F. And DE, DF, being joyned, let them be produced to the points G, K; H, L. Now foraſmuch as AE is equal to EB, and ED common; therefore there are the two lines AE, ED, equal to the two lines BE, ED,



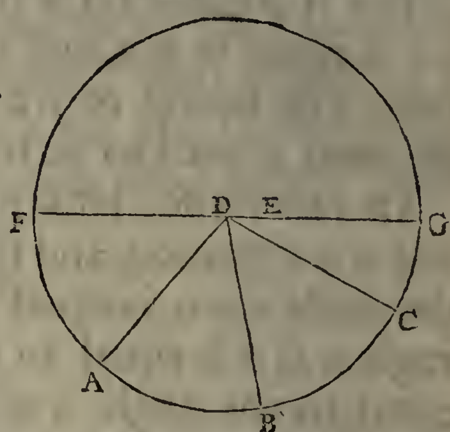
and the baſe DA is equal to the baſe DB (by Suppoſition); therefore the angle AED is equal to the angle BED: wherefore each of the angles AED, BED, is a right angle; therefore GK cutting AB into halves, does

does also cut it at right angles. And because if in a Circle a strait line cuts a strait line into halves, and at right angles, the center of the Circle is in the cutting line; therefore in the line GK is the center of the Circle ABC [by Coroll. Prop. I. El. III.]. And by the same reason also the center of the Circle ABC, is in the line HL. But the lines GK, HL, have no other common point than D; therefore the point D is the center of the Circle ABC.

If therefore in a Circle be taken any point, &c. Which was to be demonstrated.

Otherwise.

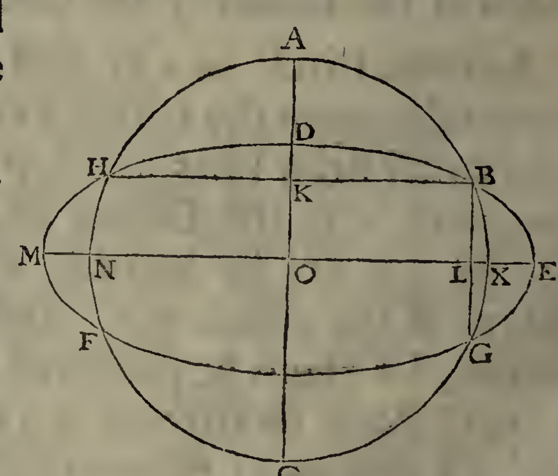
For within the Circle ABC let be taken any point as D: and from D let upon the Circle ABC, fall more than two equal strait lines, as DA, DB, DC. I say, that D the point taken is the center of the Circle ABC. For if not; let it, if possible, be E, and DE being joyned, let it be produced to the points F, G; then the line FG is the Diameter of the Circle ABC. Now forasmuch as in FG the Diameter of the Circle ABC, is taken a point D, which is not the center of the Circle, the greatest shall be DG, and DC greater than DB, and DB than DA [Prop. 7. El. III.]. But they are also equal, which is impossible: Therefore E is not the center of the Circle ABC. In like manner may we prove, that no other point can be the center besides D: therefore D is the center of the Circle ABC.



PROPOSITION X.

A Circle does not cut a Circle in more points than two.

For if it be possible let the Circle ABC, cut the Circle DEF in more points than two, as, B, G, F, H, and BG, BH, being joyned, let them be cut into halves in the points K, L, and from K, L, to the lines BG, BH, let be drawn at right angles KC, LM, [Prop. II. El. I.] and produce them to A, E. Now because in the Circle ABC, the strait line AC cuts the strait line BH into halves, and also at right angles; therefore in AC is the center of the circle ABC [Corol. Prop. I. El. III.]. Again, because in the same Circle ABC, the strait line NX cuts the strait line BG into halves, and also at right angles; therefore in NX is the center of the Circle ABC. But it has been proved to be also in AC, and in no other point doe the strait lines AC, NX, agree than in O: therefore the point O is the center of the Circle ABC.

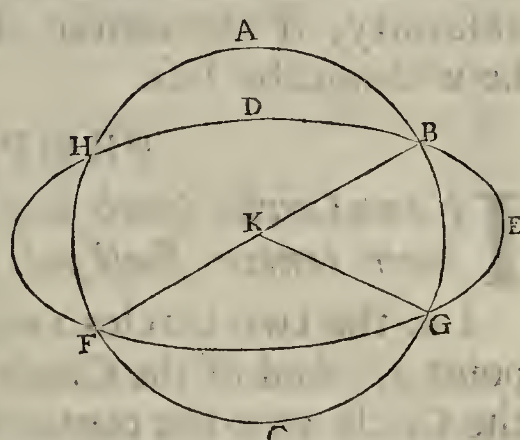


ABC. In like manner may we prove, that O is the center of the Circle DEF. Therefore of two Circles cutting one another, there is the ſame center O, which is impoſſible [Prop. 5. El. III.].

Therefore a Circle does not cut a Circle in more points than two. Which was to be demonſtrated.

Otherwiſe.

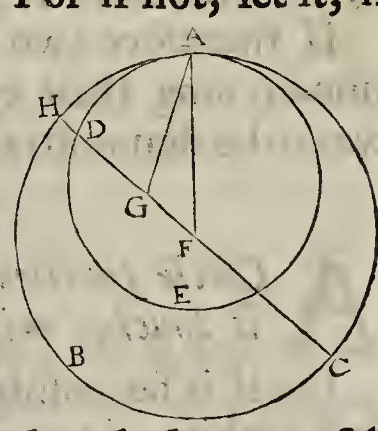
Let again the Circle ABC cut the Circle DEF in more points than two, as, B, G, F, H, and of the Circle ABC let the center be taken, as K, and let KB, KG, KF, be joyned. Now becauſe within the Circle DEF, is taken the point K, and from K do fall upon the Circle DEF, more than two equal ſtrait lines, namely, KB, KG, KF; therefore the point K is the center of the Circle DEF [Prop. 9. El. III.]; but K is alſo the center of the Circle ABC: wherefore of two Circles cutting one another, there is the ſame center K; which is impoſſible. Therefore a Circle does not cut a Circle in more points than two. Which was to be demonſtrated.



PROPOSITION XI.

IF two Circles touch one another within, and the centers of the ſame be taken, the ſtrait line joyning their centers being produced, ſhall fall upon the Contact of the Circles.

Let the two Circles ABC, ADE, touch one another within in the point A. And of the Circle ABC, let be taken the center F: alſo of the Circle ADE, the center G. I ſay, that the ſtrait line drawn from G to F, being produced falls upon the point A. For if not, let it, if poſſible, fall as FGDH, and let AF, AG be joyned. Now foraſmuch as AG, GF, are greater than FA, that is, than FH (for FA is equal to FH, both being from the center) let FG common be taken away; therefore the remaining line AG is greater than the remaining line GH. But AG is equal to GD, therefore GD is greater than GH; the leſs than the greater, which is impoſſible. Wherefore the ſtrait line drawn from F to G being produced, does not fall beſide the point of Contact A, therefore it muſt fall upon it.

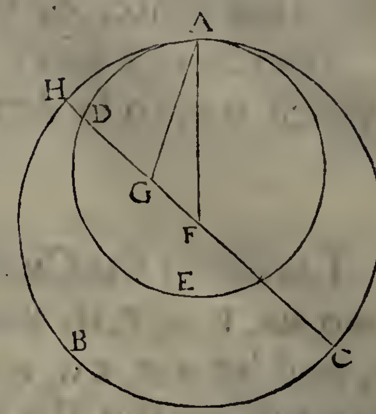


If therefore two Circles touch one another within, and the centers of the ſame be taken, the ſtrait line joyning their centers being produced does fall upon the Contact of the Circles. Which was to be demonſtrated.

Otherwiſe.

Otherwiſe.

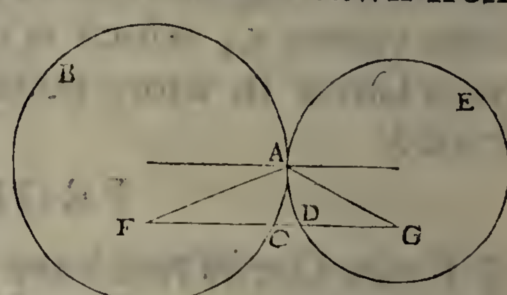
But let it fall as GFC, and produce directly CFG to the point H; and let AG, AF, be joyned. Now foraſmuch as AG, GF, are greater than AF, and AF is equal to FC, that is, to FH, let FG common be taken away; therefore the remaining line AG, is greater than the remaining line GH, that is, GD greater than GH; the leſs than the greater, which is impoſſible. In like manner may we prove the ſame abſurdity, if the center of the greater Circle be without the leſs.



PROPOSITION XII.

If two Circles touch one another without, the ſtrait line joyning their centers ſhall paſs through the Contact.

Let the two Circles ABC, ADE, touch one another without in the point A. And of the Circle ABC, let the center F be taken: alſo of the Circle ADE the center G. I ſay, that the ſtrait line drawn from F to G, ſhall paſs through the Contact in A. For if not, let it, if poſſible, paſs as FCDG, and let AF, AG, be joyned. Now foraſmuch as the point F is the center of the Circle ABC; therefore FA is equal to FC. Again, becauſe the point G is the center of the Circle ADE, therefore GA is equal to GD. But it has been proved, that FA is equal to FC, therefore FA, AG, are equal to FC, DG. So that the whole FG is greater than FA, AG; but it is leſs [Prop. 20. El. I.], which is impoſſible. Therefore the ſtrait line drawn from F to G, cannot paſs otherways but through the Contact in A: wherefore it paſſes through the Contact.



If therefore two Circles touch one another without, the ſtrait line joyning their centers ſhall paſs through the Contact. Which was to be demonſtrated.

PROPOSITION XIII.

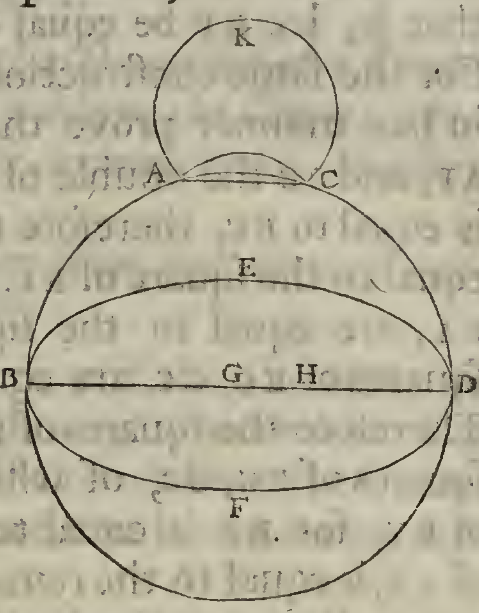
A Circle touches not a Circle in more points than one, whether it touches within or without.

For if it be poſſible, let the Circle ABDC, touch the Circle EBFD, firſt within, in more points than one, as in B, D. And of the Circle ABDC, let the center G be taken; alſo of the Circle EBFD the center H; wherefore the ſtrait line drawn from G to H, ſhall fall upon the points B, D [Prop. 11. El. III.]. Let it fall as BGHD. Now foraſmuch as the point G is the center of the Circle ABDC, therefore BG is equal to GD; wherefore BG is greater than HD, and BH much greater than HD. Again, becauſe the point H is the center of the

Circle

Circle EBFD, therefore BH is equal to HD. But it has been proved to be much greater, which is impossible; therefore a Circle touches not a Circle within, in more points than one.

I say also that neither without. For if possible, let the Circle ACK touch the Circle ABDC without, in more points than one, as in A, C: and let AC be joyned. Now forasmuch as in the circumference of each of the Circles ABDC, ACK, are taken any two points A, C; the strait line AC joyning the same points shall fall within each of the Circles [by Pr. 2. El. III.]; but AC falling within the Circle ABDC, must fall without the Circle ACK; which is absurd, (viz. to fall within and without the same Circle ACK:) therefore a Circle touches not a Circle without, in more points than one. And it has been proved that it touches not within.

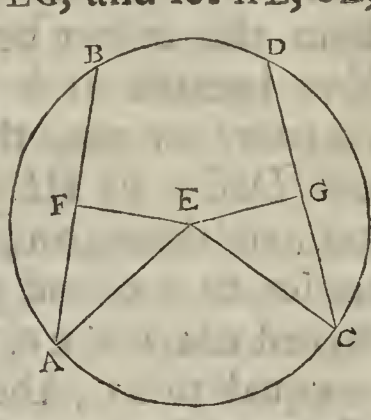


Therefore a Circle touches not a Circle in more points than one, whether it touches within, or without. Which was to be demonstrated.

PROPOSITION XIV.

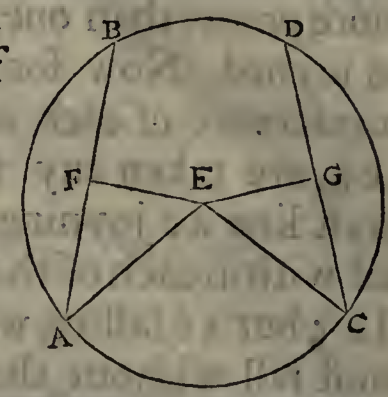
In a Circle equal strait lines are equally distant from the center. And lines equally distant from the center, are equal to one another.

Let a Circle be ABDC, and in the same let the equal strait lines be AB, CD. I say, that they are equally distant from the center. For of the Circle ABCD let the center be taken, and be it E; then from E let be drawn to AB, CD, the perpendiculars EF, EG, and let AE, CE, be joyned. Now forasmuch as the strait line EF drawn through the center, cuts the strait line AB not drawn through the center, at right angles, it also cuts the same into halves [Prop. 3. El. III.]; wherefore AF is equal to FB; therefore AB is the double of AF. And by the same reason also CD is the double of CG. But AB is equal to CD [by Supposition], therefore AF is equal to CG [Ax. 7.]. And because AE is equal to EC, therefore the square of AE is also equal to the square of EC. But the squares of AF, FE, are equal to the square of AE [Prop. 47. El. I.]: for the angle at F is a right angle. But also the squares of EG, GC, are equal to the square EC; for the angle at G is a right angle. Therefore the squares of AF, FE, are equal to the squares of CG, GE, of which the square of AF is equal to the square of CG; for the line AF is equal to the line CG: Therefore the remaining square of FE, is equal to the remaining square of EG; therefore EF is equal to EG. But in a Circle strait



lines are said to be equally distant from the center, when the perpendiculars drawn from the center to the same lines are equal [Def. 4. El. III.]; therefore AB, CD, are equally distant from the center.

But now let the lines AB, CD, be equally distant from the center, that is, let EF be equal to EG. I say, that AB is also equal to CD. For the same construction being made, we shall in like manner prove that AB is the double of AF, and CD the double of CG. And because AE is equal to EC, therefore the square of AE is also equal to the square of EC. But the squares of EF, FA, are equal to the square of AE, and the squares of EG, GC, are equal to the square of EC. Therefore the squares of EF, FA, are equal to the squares of EG, GC, of which the square of EG is equal to the square of EF, for EF is equal to EG; and therefore the remaining square of AF, is equal to the remaining square of CG: wherefore AF is equal to CG. But AB is the double of AF, and CD the double of CG; wherefore AB is equal to CD [Ax. 6.].

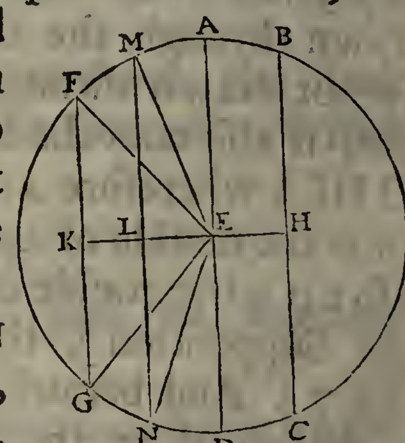


Therefore in a Circle equal strait lines are equally distant from the center. And lines equally distant from the center, are equal to one another. Which was to be demonstrated.

PROPOSITION XV.

IN a Circle the greatest line is the Diameter. And of the others, always the nearer to the center is greater than the more remote.

Let the Circle be ABCD, and the Diameter thereof be AD, and the center E. Now to the center E let BC be nearer, and FG more remote. I say, that AD is the greatest, and BC greater than FG. Let from the center be drawn to BC, FG, the perpendiculars EH, EK. Now because BC is nearer to the center, and FG more remote, therefore EK is greater than EH [Def. 5. El. III.]. Let EL be put equal to EH, and through L, let LM be drawn at right angles to EK, and produced to N: then let be joyned EM, EN, EF, EG. Now forasmuch as EH is equal to EL, therefore BC is equal to MN [Prop. 14.]. Again, because AE is equal to EM, and DE to EN, therefore AD is equal to ME, EN; but ME, EN, are greater than MN [Prop. 20. El. I.]; therefore AD is greater than MN, but MN is equal to BC, therefore AD is greater than BC. And because the two lines ME, EN, are equal to the two lines FE, EG, and the angle MEN is greater than the angle FEG, therefore the base MN is greater than the base FG [Prop. 24. El. I.]. But MN has been proved equal to BC, wherefore BC is greater than FG. The greatest therefore is AD the Diameter, and BC is greater than FG.



There-

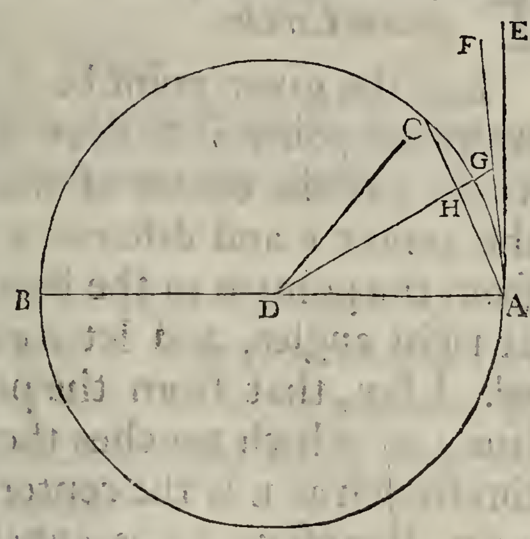
Therefore in a Circle the greateſt line is the Diameter. And of the others, always the nearer to the center is greater than the more remote. Which was to be demonſtrated.

PROPOSITION XVI.

A Strait line drawn at right angles to the Diameter of a Circle, from an extremity thereof ſhall fall without the Circle: And in the place between the ſtrait line and the circumference, an other ſtrait line ſhall not fall.

And the angle of the Semicircle is greater than any Acute ſtrait-lin'd angle; and the remaining angle is leſs.

Let the Circle be ABC about the center D, and the Diameter be AB. I ſay, that the ſtrait line drawn at right angles to AB from the point A the extremity of the Diameter, ſhall fall without the Circle. For if not, let it, if poſſible, fall within as AC, and let DC be joyned. Now foraſmuch as DA is equal to DC, therefore the angle DAC is equal to the angle ACD [Prop. 5. El. I.]. But DAC is a right angle [by Suppoſition], therefore ACD is alſo a right angle, wherefore DAC, ACD, are equal to two right angles, which is impoſſible [Prop. 17. El. I.]. Therefore a ſtrait line drawn from the point A at right angles to AB, ſhall not fall within the Circle. In like manner we may ſhew that it ſhall not fall upon the circumference; therefore it muſt fall without, as AE.



I ſay, that in the place between the ſtrait line AE, and the circumference CHA, another ſtrait line ſhall not fall. For if poſſible, let it fall as FA, and from the point D to the line FA, let be drawn the perpendicular DG. Now becauſe AGD is a right angle, and DAG is leſs than a right angle; therefore AD is greater than DG [Prop. 19. El. I.]. But DA is equal to DH, therefore DH is greater than DG, the leſs than the greater, which is impoſſible. Therefore in the place between the ſtrait line and the circumference, another ſtrait line ſhall not fall.

I ſay moreover, that the angle of the Semicircle, which is contained by the ſtrait line BA, and the circumference CHA, is greater than any Acute ſtrait-lin'd angle; and the remaining angle contained by the circumference CHA, and the ſtrait line AE, is leſs than any Acute ſtrait-lin'd angle. For if there be any Acute ſtrait-lin'd angle greater than the angle contained by the ſtrait line BA, and the circumference CHA, and any other leſs than the angle contained by the circumference CHA; and the ſtrait line AE, then in the place between the circumference CHA and the ſtrait line AE, ſhall fall a ſtrait line, which ſhall make one ſtrait-lin'd angle greater than the

 angle

angle contained by the ſtrait line BA, and the circumference CHA, and another ſtrait-lin'd angle leſs than the angle contained by the circumference CHA and the ſtrait line AE. But ſuch a line cannot fall; therefore there ſhall not be any Acute ſtrait-lin'd angle greater than the angle contained by the ſtrait line BA, and the circumference CHA, nor any leſs than the angle contained by the circumference CHA and the ſtrait line AE. Which was to be demonſtrated.

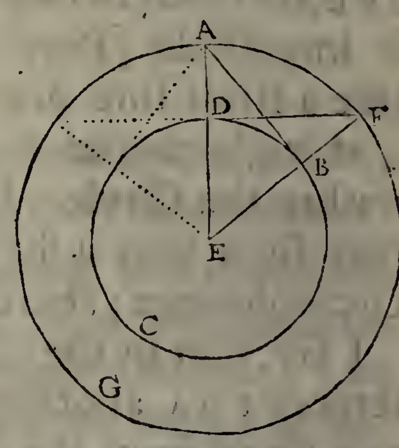
Corollary.

From hence it is manifeſt, that a ſtrait line drawn at right angles to the Diameter of a Circle, from the extremity thereof doth touch the Circle [*Def.* 2.]. *And that a ſtrait line touches a Circle in one point only: Becauſe a ſtrait line concurring with a Circle in two points, has been proved to fall within the ſame* [*Prop.* 2.]

PROPOSITION XVII.

F*Rom a given point to draw a ſtrait line, which ſhall touch a given Circle.*

Let the given point be A, and the given Circle BCD. It is required from the point A, to draw a ſtrait line which ſhall touch the Circle BCD. Let the center of the Circle be taken as E, and draw AE; then the center E and diſtance EA, let the Circle AFG be deſcribed; and from the point D to the line EA, let DF be drawn at right angles, and let EBF, and AB be joyned. I ſay, that from the point A, is drawn the line AB, which touches the Circle BCD. Now foraſmuch as E is the center of the Circles BCD, AFG, therefore EA is equal to EF, and ED to EB. There are then the two lines AE, EB, equal to the two lines FE, ED, and they contain a common angle at E; wherefore the baſe DF is equal to the baſe AB, and the Triangle DEF equal to the Triangle BEA, and the remaining angles to the remaining angles; therefore the angle EBA is equal to the angle EDF: but EDF is a right angle, therefore EBA is alſo right; and EB is from the center. But what is drawn at right angles to the Diameter of a Circle from an extremity thereof does touch the Circle: Wherefore AB does touch the Circle.

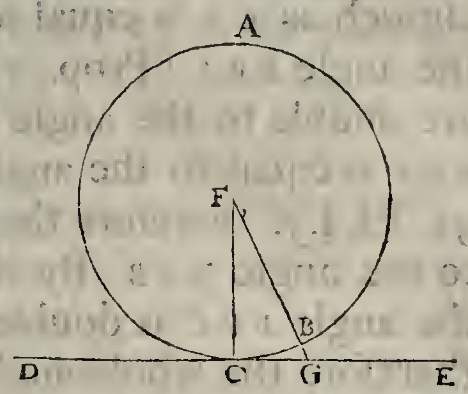


Therefore from the given point A is drawn a ſtrait line AB, touching the given Circle BCD. Which was to be done.

PROPOSITION XVIII.

I*F a ſtrait line touches a Circle, and from the center be drawn a ſtrait line to the Contact, that line ſhall be perpendicular to the Tangent.*

Let the ſtrait line DE touch the Circle ABC in the point C; and of the Circle ABC, let the center F be taken; then from F to C let be be drawn FC. I ſay, that FC is perpendicular to DE. For if not; let from the point F, be drawn FG perpendicular to DE. Now becauſe the angle FGC is a right angle, therefore GCF is an acute angle [Prop. 32. El. I.]. But under the greater angle is ſubtended the greater ſide [Prop. 19. El. I.]; therefore FC is greater than FG; but FC is equal to FB, therefore FB is greater than FG; the leſs than the greater, which is impoſſible. Therefore FG is not perpendicular to DE.



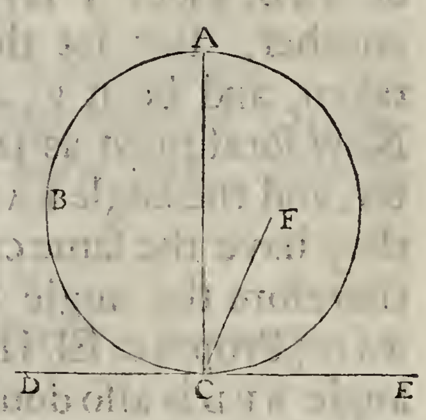
In like manner may we prove, that no other line can be beſides FC; therefore FC is perpendicular to DE.

If therefore a ſtrait line touches a Circle, and from the center be drawn a ſtrait line to the Contact, That is perpendicular to the Tangent. Which was to be demonſtrated.

PROPOSITION XIX.

IF a ſtrait line touches a Circle, and from the Contact be drawn a ſtrait line at right angles to the Tangent, the Center of the Circle ſhall be in the ſame line.

Let the ſtrait line DE touch the Circle ABC, in the point C, and from C let CA be drawn at right angles to DE. I ſay, that the center of the Circle is in the line CA. For if not, let it, if poſſible, be F, and let CF be joyned. Now foraſmuch as the ſtrait line DE touches the Circle ABC, and from the center is drawn to the Contact the line FC: wherefore FC is perpendicular to DE [Prop. 18. El. III.]; and therefore FCE is a right angle; but ACE is alſo a right angle [by Suppoſition]; wherefore FCE is equal to ACE; the leſs to the greater, which is impoſſible. Therefore F is not the center of the Circle ABC. And in like manner may we prove, that no other can be beſides a point in AC.



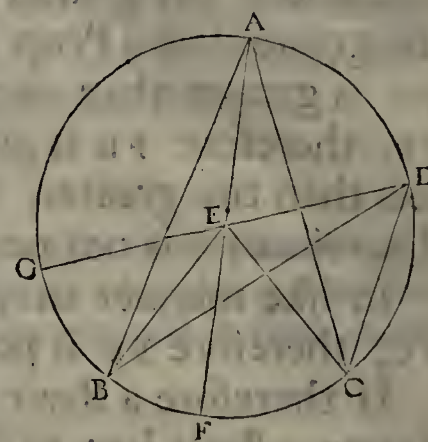
If therefore a ſtrait line touches a Circle, and from the Contact be drawn a ſtrait line at right angles to the Tangent, the center of the Circle ſhall be in the ſame line. Which was to be demonſtrated.

PROPOSITION XX.

IN a Circle, the angle at the center is double to the angle at the circumference, when the angles have the ſame circumference for their baſe.

Let the Circle be ABC, the angle at the center BEC, and at the circumference BAC; and let them have the ſame circumference

BC for their baſe. I ſay, that the angle BEC is double to the angle BAC. For drawing the line AE, let it be produced to F. Now foraſmuch as EA is equal to EB, therefore the angle EAB, is equal to the angle EBA [Prop. 5. El. I.]: wherefore the angles EAB, EBA, are double to the angle EAB. But the angle BEF is equal to the angles EAB, EBA [Prop. 32. El. I.]: therefore the angle BEF, is double to the angle EAB. By the ſame reaſon alſo the angle FEC is double to the angle EAC; therefore the whole angle BEC, is double to the whole angle BAC. Again, let the angle at the circumference be declined beyond the center E, and let it be BDC: then drawing DE, let it be produced to G. In like manner may we prove, that the angle GEC is double to the angle GDC; of which angles GEB is double to GDB: wherefore the remaining angle BEC, is double to the remaining angle BDC.

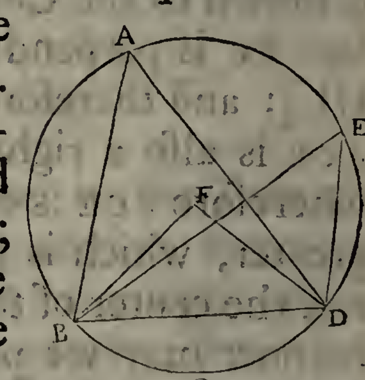


Therefore in a Circle, the angle at the center is double to the angle at the circumference, when the angles have the ſame circumference for their baſe. Which was to be demonſtrated.

PROPOSITION XXI.

IN a Circle angles in the ſame Segment are equal to one another.

Let the Circle be ABCD, and in the Segment BAED, let the angles be BAD, BED. I ſay, that the angles BAD, BED, are equal to one another. For let the center of the Circle be taken, and be it F; and let BF, FD, be joyned. Now foraſmuch as the angle BFD is at the center, and the angle BAD at the circumference, and they have the ſame circumference for a baſe BCD; therefore the angle BFD is double to the angle BAD [Prop. 20. El. III.]. By the ſame reaſon the angle BFD is alſo double to the angle BED; therefore the angle BAD is equal to the angle BED [Ax. 7.]



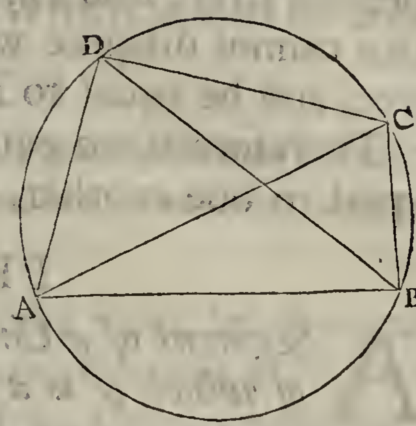
Therefore in a Circle angles in the ſame Segment are equal to one another. Which was to be demonſtrated.

PROPOSITION XXII.

OF four ſided Figures in Circles, the oppoſite angles are equal to two right.

Let the Circle be ABCD, and in the ſame the four ſided Figure be ABCD. I ſay, that the oppoſite angles are equal to two right angles. Let AC, BD, be joyned. Now foraſmuch as, of every Triangle the three angles are equal to two Right [Prop. 32. El. I.]; therefore of the

the Triangle ABC, the three angles CAB, ABC, BCA, are equal to two Right. But the angle CAB, is equal to the angle BDC, for they are in the same Segment BADC. And the angle ACB is equal to the angle ADB, for they are in the same Segment ADCB. Therefore the whole angle ADC, is equal to the angles BAC, ACB; let the common angle ABC be added; therefore the angles ABC, BAC, ACB, are equal to the angles ABC, ADC. But the angles ABC, BAC, ACB, are equal to two Right: therefore ABC, ADC, are also equal to two Right. In like manner may we prove that BAD, DCB, are also equal to two Right angles.

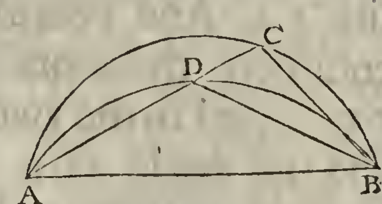


Therefore of four sided Figures in Circles, the opposite angles are equal to two Right angles. Which was to be demonstrated.

PROPOSITION XXIII.

Upon the same strait line two Segments of Circles like, and unequal, cannot be constituted the same way.

For if possible, upon the same strait line AB let two like, and unequal Segments of Circles ACB, ADB, be constituted the same way: then let ADC be drawn, and CB, BD, be joyned. Now forasmuch as the Segment ACB is like to the Segment ADB; and like Segments of Circles are such, which receive equal angles [Def. 11. El. III.]; therefore the angle ACB, is equal to the angle ADB; the outward to the inward, which is impossible.

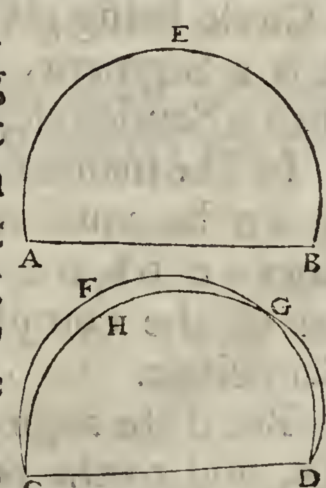


Therefore upon the same strait line two Segments of Circles like, and unequal, cannot be constituted the same way. Which was to be demonstrated.

PROPOSITION XXIV.

Upon equal strait lines like Segments of Circles, are equal to one another.

For upon equal strait lines AB, CD, let like Segments of Circles be AEB, CFD. I say, that the Segment AEB, is equal to the Segment CFD. For the Segment AEB being apply'd to the Segment CFD, and the point A put upon the point C, and the line AB upon CD; then shall the point B agree with the point D; for that AB is equal to CD. Now the strait line AB agreeing with the strait line CD, the Segment AEB shall also agree with the Segment CFD. For if AB shall agree with CD, and the Segment AEB shall not agree with the Segment CFD, then shall it differ from CFD, as CHGD. But a Circle cuts not a Circle in more points than two, yet here



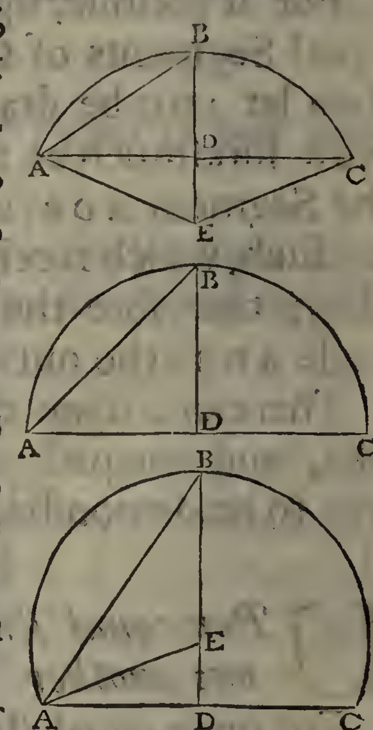
here the Circle CHGD, cuts the Circle CFD, in more points than two, namely in C, G, D, which is impossible [Prop. 10. El. III.]. Wherefore the strait line AB, agreeing with the strait line CD, the Segment AEB cannot disagree with the Segment CFD: wherefore it shall agree, and be equal to it.

Therefore upon equal strait lines like Segments of Circles, are equal to one another. Which was to be demonstrated.

PROPOSITION XXV.

A Segment of a Circle being given to describe unto it the Circle, of which it is a Segment.

Let the given Segment of a Circle be ABC. It is required to describe unto it the Circle of which ABC is a Segment. Let AC be cut into halves in D, and from the point D let DB be drawn at right angles to AC, and let AB be joyned. Now then the angle ABD is either greater than the angle BAD, or equal to it, or less. First, let it be greater, and to the strait line BA, and to the point A in the same, let the angle BAE be constituted equal to the angle ABD [Prop. 23. El. I.], and let BD be produced to E, and EC be joyned. Now forasmuch as the angle ABE, is equal to the angle BAE; therefore the strait line EB is equal to the strait line EA. And because AD is equal to DC, and DE common, therefore there are the two lines AD, DE, equal to the two lines CD, DE, each to each, and the angle ADE is equal to the angle CDE, for each of them is a right angle; therefore the base EA is equal to the base EC. But it has been proved, that EA is equal to EB, therefore EB is also equal to EC, wherefore the three lines EA, EB, EC, are equal to one another; therefore the center E, and distance any one of the lines EA, EB, EC, a Circle being described shall pass through the other points, and the Circle shall be described to the given Segment. Therefore a Segment of a Circle being given, there is described unto it the Circle of which it is a Segment. And it is manifest that the Segment ABC is less than a Semicircle, for that E the center of the same falls without.



In like manner if the angle ABD, be equal to the angle BAD, and so AD be equal to either of the lines BD, DC; therefore the three lines AD, DB, DC, are equal to one another, and D shall be the center of the compleated Circle; and the Segment ABC shall be a Semicircle.

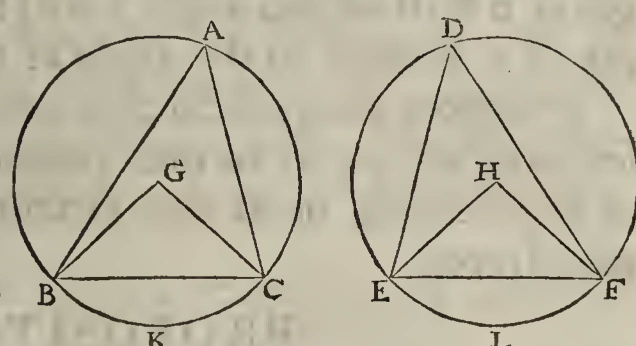
But if the angle ABD, be less than the angle BAD, and to the line BA, and to the point A in the same an angle be constituted equal to ABD; the center shall fall within the Segment ABC, and in the line

line BD; and the Segment shall be greater than a Semicircle. Therefore a Segment of a Circle being given there is described unto it, the Circle, of which it is a Segment. Which was to be done.

PROPOSITION XXVI.

IN equal Circles, equal angles insist upon equal circumferences, whether the insisting angles be at the centers, or at the circumferences.

Let the equal Circles be ABC, DEF, and in the same let the equal angles at the centers be BGC, EHF, and the circumferences BAC, EDF. I say, that the circumference BKC, is equal to the circumference ELF. For let BC, EF, be joyned. Now forasmuch as the Circles ABC, DEF, are equal; therefore the lines from the centers are equal: wherefore there are the two lines BG, GC, equal to the two lines EH, HF, and the angle at G is equal to the angle at H: therefore the base BC is equal to the base EF [Prop. 4. El. I.]. And



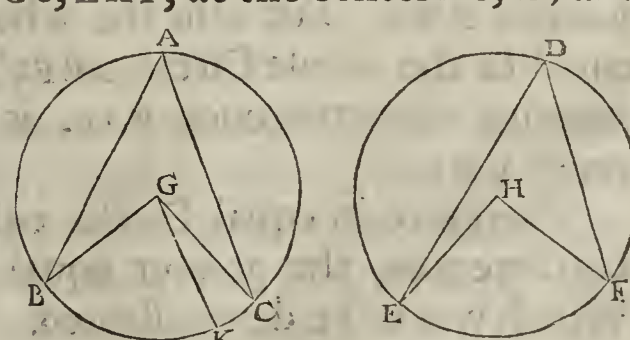
because the angle at A is equal to the angle at D, therefore the Segment BAC is like to the Segment EDF: and they are upon equal strait lines BC, EF. But upon equal strait lines like Segments of Circles are equal to one another [Prop. 24. El. III.]; therefore the Segment BAC, is equal to the Segment EDF. But also the whole Circle ABC, is equal to the whole Circle DEF: wherefore the remaining Segment BKC, is equal to the remaining Segment ELF, and therefore the circumference EKC, is equal to the circumference ELF.

Wherefore in equal Circles, equal angles insist upon equal circumferences, whether the insisting angles be at the centers, or at the circumferences. Which was to be demonstrated.

PROPOSITION XXVII.

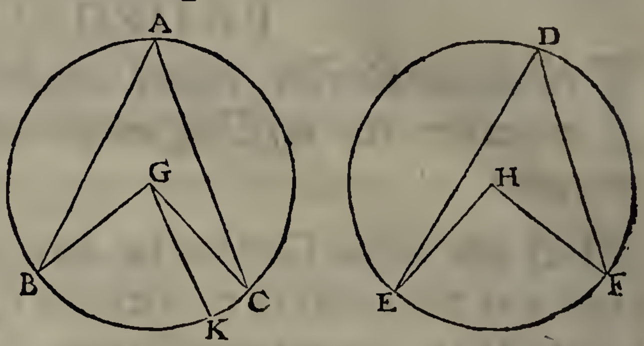
IN equal Circles, angles insisting upon equal circumferences are equal to one another; whether the insisting angles be at the centers, or at the circumferences.

For in the equal Circles ABC, DEF, and upon the equal circumferences BC, EF; let insist the angles BGC, EHF, at the centers G, H; and at the circumferences the angles BAC, EDF. I say, that the angle BGC is equal to the angle EHF, and the angle BAC, to the angle EDF. For if the angle BGC be equal to the angle EHF, it is manifest that also the angle BAC, is equal to the



angle EDF [Prop. 20. El. III.]. But if not, one of them is the greater.

Let the greater be BGC; and to the line BG, and in the ſame to the point G, let be conſtituted the angle BGK equal to the angle EHF [Prop. 23. El.I.]. But equal angles inſiſt upon equal circumferences, when they are at the center [Prop. 26.El.III.]; therefore the circumference BK is equal to the circumference EF. But EF is equal to BC [by Suppoſition]; therefore alſo BK is equal to BC; the leſs to the greater, which is impoſſible; wherefore the angle BGC, is not unequal to the angle EHF; therefore it is equal. Now the angle at A is the half of the angle BGC, and the angle at D half of the angle EHF [Prop. 20.El.III.]; wherefore the angle at A is equal to the angle at D [Ax. 7.].

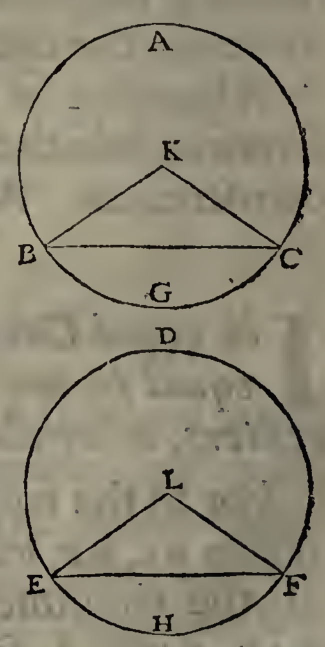


Therefore in equal Circles, angles which inſiſt upon equal circumferences are equal to one another, whether the inſiſting angles be at the centers, or at the circumferences. Which was to be demonſtrated.

PROPOSITION XXVIII.

IN equal Circles equal ſtrait lines take off equal circumferences, the greater equal to the greater, the leſs to the leſs.

Let the equal Circles be ABC, DEF; and in the ſame let the equal ſtrait lines be BC, EF, taking off the greater circumferences BAC, EDF, and the leſs BGC, EHF. I ſay, that BAC the greater circumference, is equal to EDF the greater circumference, and BGD the leſs circumference, is equal to EHF the leſs. For let the centers of the Circles be taken K, L; and let be joyned KB, KC; LE, LF. Now foraſmuch as the Circles are equal, therefore the lines from the centers are equal. There are then the two lines BK, KC, equal to the two lines EL, LF, and the baſe BC equal to the baſe EF; therefore the angle BKC is equal to the angle ELF [Prop. 8. El. I.]. Now equal angles inſiſt upon equal circumferences when they are at the centers [Prop. 26. El. III.]: wherefore the circumference BGC, is equal to the circumference EHF. But alſo the whole Circle ABC, is equal to the whole Circle DEF; wherefore the remaining circumference BAC, is equal to the remaining circumference EDF.



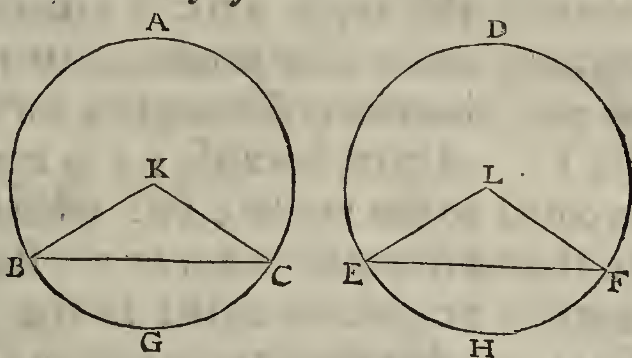
Therefore in equal Circles equal ſtrait lines take off equal circumferences, the greater equal to the greater, the leſs to the leſs. Which was to be demonſtrated.

PROPO-

PROPOSITION XXIX.

IN equal Circles, under equal circumferences are ſubtended equal ſtrait lines.

Let the equal Circles be ABC, DEF, and in the ſame let be taken equal circumferences BGC, EHF: and let be joyned the ſtrait lines BC, EF. I ſay, that BC is equal to EF. Let the centers of the Circles be taken K, L, and let be joyned KB, KC; LE, LF. Now foraſmuch as the circumference BGC, is equal to the circumference EHF; therefore the angle BKC, is equal to the angle ELF [Prop. 27. El. III.]. And becauſe the Circles ABC, DEF, are equal, therefore the lines from the centers are equal. There are then the two lines BK, KC, equal to the two lines EL, LF, and they contain equal angles, wherefore the baſe BC, is equal to the baſe EF [Prop. 4. El. I.].

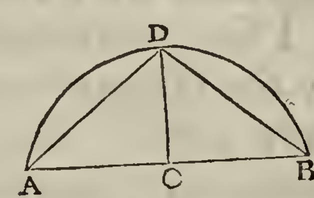


Therefore in equal Circles, under equal circumferences are ſubtended equal ſtrait lines. Which was to be demonſtrated.

PROPOSITION XXX.

TO cut a given circumference into halves.

Let the given circumference be ADB; it is required to cut the circumference ADB into halves. Let AB be joyned, and cut into halves in the point C, and from the point C to the line AB, let be drawn at right angles CD, and let be joyned AD, DB. Now foraſmuch as AC is equal to CB, and CD common: therefore there are the two lines AC, CD, equal to the two lines BC, CD, and the angle ACD equal to the angle BCD, for each of them is a right angle; therefore the baſe AD is equal to the baſe DB [Prop. 4. El. I.]. Now equal ſtrait lines take off equal circumferences, the greater to the greater, the leſs to the leſs [Prop. 28. El. III.]; and each of the circumferences AD, DB, is leſs than a Semicircle; wherefore the circumference AD is equal to the circumference DB.



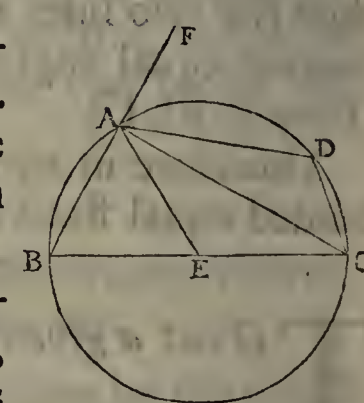
Therefore the given circumference is cut into halves. Which was to be demonſtrated.

PROPOSITION XXXI.

IN a Circle the angle in the Semicircle is a right angle. But the angle in the greater Segment is leſs than a right: And the angle in the leſs Segment is greater than a right angle. And moreover the angle of the greater Segment is greater than a right angle: and the angle of the leſs Segment is leſs than a right angle.

Let the Circle be ABCD, and the Diameter thereof be BC, and the

the center E. Then taking any point in the circumference as A, let be joyned BA, AC, AD, DC. I ſay, that the angle in the Semicircle BAC is a right angle: And the angle in the Segment ABC greater than the Semicircle, namely, the angle ABC is leſs than a right angle: And the angle in the Segment ADC leſs than the Semicircle, namely, the angle ADC is greater than a right angle. Let AE be joyned, and BA be produced to F. Now foraſmuch as BE is equal to EA, therefore the angle EAB is equal to the angle EBA [Prop. 5. El. I.]. Again, becauſe EA is equal to EC, therefore the angle ACE is equal to the angle CAE: wherefore the whole angle BAC is equal to the two angles ABC, ACB. But the angle FAC without the Triangle ABC, is alſo equal to the two angles ABC, ACB [Prop. 32. El. I.]; wherefore the angle BAC is equal to the angle FAC, each therefore of them is a right angle [Def. 10. El. I.]. Therefore in the Semicircle BAC the angle CAB is a right angle.



And becauſe of the Triangle ABC the two angles ABC, BAC, are leſs than two right [Prop. 17. El. I.], and BAC is a right angle; therefore the angle ABC is leſs than a right angle: and it is in the Segment ABC greater than the Semicircle.

And becauſe in a Circle the Figure ABCD is quadrilateral, and of quadrilateral Figures in Circles, the oppoſite angles are equal to two right angles [Prop. 22. El. III.]. Therefore the angles ABC, ADC, are equal to two right angles, and ABC is leſs than a right angle; therefore the remaining angle ADC, is greater than a right angle: and it is in the Segment ADC leſs than the Semicircle.

I ſay, moreover, that the angle of the greater Segment contained by the circumference ABC, and the ſtrait line AC is greater than a right angle: And the angle of the leſs Segment contained by the circumference ADC and the ſtrait line AC, is leſs than a right angle. This is of it ſelf very manifeſt. For becauſe the angle contain'd by the ſtrait lines CA, AB, is a right angle, therefore the angle contain'd by the ſtrait line CA, and the circumference ABC, is greater than a right angle. Again, becauſe the angle contain'd by the ſtrait lines CA, AF, is a right angle, therefore the angle contain'd by the ſtrait line CA, and the circumference ADC, is leſs than a right angle. In a Circle therefore the angle in the Semicircle is a right angle: but the angle in the greater Segment is leſs than a right angle: and the angle in the leſs Segment is greater than a right angle. And moreover the angle of the greater Segment is greater than a right angle; and the angle of the leſs Segment, is leſs than a right angle.

Otherwiſe.

That the angle BAC is a right angle. Becauſe the angle AEC is double of the angle BAE, for it is equal to the two inward and oppoſite

posite angles [Prop. 32. El. I.]: and also the angle AEB is double of the angle EAC [Prop. 32. El. I.]; therefore the angles AEB, AEC are double of the angle BAC. But the angles AEB, AEC, are equal to two right angles [Prop. 13. El. I.]; therefore the angle BAC is a right angle. Which was to be demonstrated.

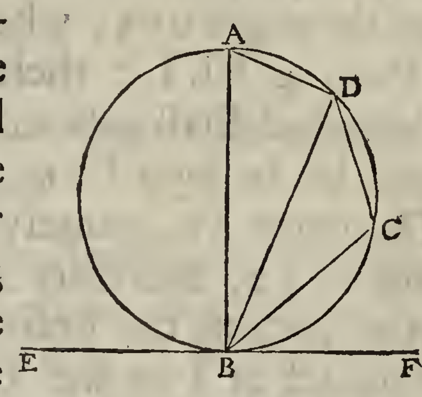
Corollary.

From hence it is manifest, that if of a Triangle one angle be equal to two it is a right angle: For that its consequent angle is equal to the same; and when consequent angles are equal, they are right angles.

PROPOSITION XXXII.

IF a strait line touches a Circle, and from the Contact to the Circle be drawn a strait line cutting the Circle, the angles which it makes with the Tangent line, shall be equal to the angles in the alternate Segments of the Circle.

Let the strait line EF touch the Circle ABCD in the point B: and from B to the Circle ABCD let be drawn any strait line as BD cutting the Circle. I say, that the angles which the line BD makes with the Tangent line EF, shall be equal to the angles in the alternate Segments of the Circle, that is, the angle FBD is equal to the angle constituted in the Segment DAB: and the angle EBD is equal to the angle in the Segment DCB. From the point B to the line EF, let be drawn at right angles the line BA; and in the circumference BD, let be taken any point, as C, and let be joyned AD, DC, CB. Now forasmuch as the strait line EF touches the circle ABCD, in the point B; and from the Contact at B is drawn the strait line BA at right angles to the Tangent, the center of the Circle ABCD is in BA [Prop. 19. El. III.]; therefore BA is the Diameter of the Circle ABCD; and the angle ADB in the Semicircle is a right angle [Prop. 31. El. III.]; therefore the remaining angles BAD, ABD, are equal to one right angle. But the angle ABF is a right angle; wherefore the angle ABF is equal to the angles BAD, ABD. Let the common angle ABD be taken away: therefore the remaining angle DBF is equal to the angle BAD in the alternate Segment of the Circle. And because in a Circle the Figure ABCD is quadrilateral, therefore the opposite angles are equal to two right angles [Prop. 22. El. III.]: wherefore the angles DBF, DBE, are equal to the angles BAD, BCD, of which BAD has been prov'd equal to DBF; therefore the remaining angle DBE, is equal to the angle DCB in the alternate Segment of the Circle.



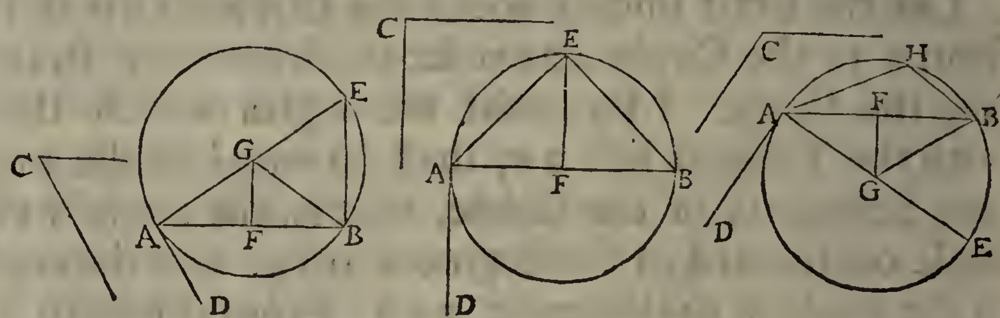
If therefore a strait line touches a Circle, and from the Contact to the Circle be drawn a strait line cutting the Circle, the angles which it makes with the Tangent line, shall be equal to the angles in the alternate Segments of the Circle. Which was to be demonstrated.

PROPOSITION XXXIII.

UPon a given ſtrait line to deſcribe a Segment of a Circle, which may receive an angle equal to a given ſtrait-lin'd angle.

Let the given ſtrait line be AB, and the given ſtrait-lin'd angle be at C. It is required upon the ſtrait line AB to deſcribe a Segment of a Circle, which may receive an angle equal to the angle at C. Now the angle at C, is either an Acute, or a Right, or an Obtuſe angle.

Firſt, let it be Acute, as in the firſt Figure. And to the ſtrait line AB, and to the point A, let the angle BAD be conſtituted equal to the angle C [by Prop. 23. El. I.]; therefore the angle BAD is an Acute angle. Now from the point A to AD, let AE be drawn at right angles; and let AB be cut into halves in the point F [by Prop. 11. El. I.]: then from the point F to AB let FG be drawn at right angles, and let GB be joyned. Now foraſmuch as AF is equal to FB, and FG common; therefore there are the two lines AF, FG, equal to the two lines BF, FG, and the angle AFG, is equal to the angle GFB; wherefore the baſe AG, is equal to the baſe BG [Prop. 4. El. I.]; therefore the center G, and diſtance GA, a Circle deſcribed ſhall paſs alſo by B. Let it be deſcribed, and be it ABE, and let be joyn'd EB. Now foraſmuch as from the extremity of the Diameter AE, namely from the point A to AE, is drawn at right angles AD, therefore AD does touch the Circle [Prop. 16. El. III.]. And becauſe the ſtrait line AD touches the Circle ABE, and from the Contact at A to the Circle ABE, is drawn the line AB; therefore the angle DAB, is equal to the angle AEB in the alternate Segment. But the angle DAB, is equal to the angle at C [by Conſtruction]; wherefore the angle at C is equal to the angle AEB. Therefore upon the given ſtrait line AB is deſcribed a Segment of a Circle AEB, receiving an angle AEB, equal to the given angle at C.



But now let the angle at C be a right angle. And again, let it be required upon the ſtrait line AB, to deſcribe a Segment of a Circle, which may receive an angle equal to the right angle at C.

Let again the angle BAD be conſtituted equal to the right angle at C, as in the ſecond Figure. And let AB be cut into halves in the point F; then from the center F, and to the diſtance of either FA, or FB, let the Circle AEB be deſcrib'd; therefore the ſtrait line AD touches the Circle AEB, for that the angle at A is a right angle; and the angle BAD is equal to the angle in the Segment AEB: for being in a Semicircle, it is alſo a right angle. But the angle BAD is equal

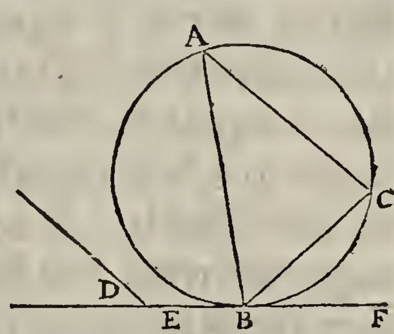
equal to the angle at C; wherefore the angle AEB in the Segment, is equal to the angle at C. Therefore again upon the ſtrait line AB, is deſcribed a Segment of a Circle AEB, receiving an angle equal to the angle at C.

But again, let the angle at C be an Obtuſe angle, and to the ſtrait line AB, and to the point A, let the angle BAD be conſtituted equal to C, as in the third Figure. Then to AD let AE be drawn at right angles; and again, let AB be cut into halves in the point F; and from F let FG be drawn at right angles to AB, and let GB be joyned. Becauſe again, AF is equal to FB, and FG common; therefore there are the two lines AF, FG, equal to the two lines BF, FG, and the angle AFG is equal to the angle BFG; wherefore the baſe AG, is equal to the baſe BG: therefore the center G, and diſtance GA, a Circle deſcribed ſhall paſs alſo by B. Let it paſs as ABE. Now becauſe to the Diameter AE, and from the extremity thereof is drawn at right angles AD, therefore AD does touch the Circle ABE, and from the Contact at A is drawn AB; therefore the angle BAD is equal to the angle conſtituted in the alternate Segment of the Circle AHB. But the angle BAD is equal to the angle at C; wherefore the angle in the Segment AHB, is equal to the angle at C. Therefore upon the given ſtrait line AB, is deſcribed a Segment of a Circle AHB, receiving an angle equal to the angle at C. Which was to be done.

PROPOSITION XXXIV.

From a given Circle to take off a Segment, which may receive an angle equal to a given ſtrait-lin'd angle.

Let the given Circle be ABC, and the given ſtrait-lin'd angle be at D. It is required from the Circle ABC, to take off a Segment, which may receive an angle equal to the angle at D. Let be drawn EF touching the Circle ABC in the point B [Prop. 17. El. III.], and to the ſtrait line EF, and to the point in it B, let the angle FBC be conſtituted equal to the angle at D [Prop. 31. El. I.]. Now foraſmuch as the ſtrait line EF touches the Circle ABC, in the point B; and from the Contact at B is drawn the line BC: therefore the angle FBC, is equal to the angle conſtituted in the alternate Segment BAC. But the angle FBC is equal to the angle at D; wherefore the angle in the Segment BAC, is equal to the angle at D.

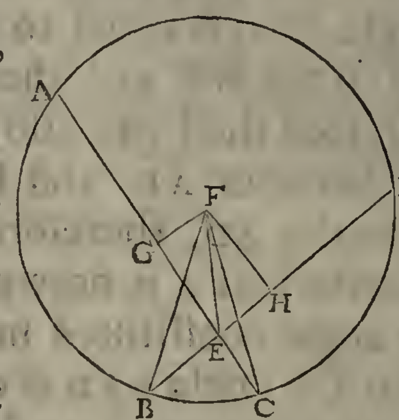


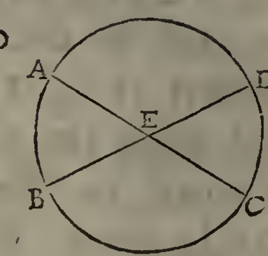
Therefore from the given Circle ABC is taken off the Segment BAC, receiving an angle equal to the given ſtrait-lin'd angle at D. Which was to be done.

PROPOSITION XXXV.

IF in a Circle two ſtrait lines cut one another, the Rectangle contained by the Segments of the one, is equal to the Rectangle contained by the Segments of the other.

For in the Circle ABCD, let the two ſtrait lines AC, BD, cut one another in the point E. I ſay, that the Rectangle contained by AE, EC, is equal to the Rectangle contained by DE, EB. If AC, BD, paſs through the center, ſo that E be the center of the Circle ABCD, then it is manifeſt that the lines AE, EC; DE, EB, being equal, the Rectangle alſo contained by AE, EC, is equal to the Rectangle contained by DE, EB.





But now let the ſtrait lines AC, DB, not paſs through the center. And let the center of the Circle ABCD be taken, and be it F, and from F to the ſtrait lines AC, DB, let perpendiculars be drawn FG, FH: and let be joyned FB, FC, FE. Now foraſmuch as the ſtrait line GF drawn through the center cuts the ſtrait line AC, not drawn through the center, at right angles, it ſhall alſo cut the ſame into halves [Prop. 3. El. III.]; therefore AG is equal to GC. And becauſe the ſtrait line AC is cut into equal parts in G, and into unequal parts in E: therefore the Rectangle contained by AE, EC, together with the ſquare of EG is equal to the ſquare of GC [Prop. 5. El. II.], let be added in common the ſquare of GF: therefore the Rectangle contained by AE, EC, together with the ſquares of EG, GF, is equal to the ſquares of CG, GF. But the ſquare of FE, is equal to the ſquares of EG, GF [Prop. 47. El. I.]; and the ſquare of FC is equal to the ſquares of CG, GF; therefore the Rectangle contained by AE, EC, together with the ſquare of EF, is equal to the ſquare of FC. But FC is equal to FB; therefore the Rectangle under AE, EC, together with the ſquare of FE, is equal to the ſquare of FB. By the ſame reaſon the Rectangle under DE, EB, together with the ſquare of FE, is equal to the ſquare of FB. But it has been proved, that the Rectangle under AE, EC, together with the ſquare of FE, is equal to the ſquare of FB; therefore the Rectangle under AE, EC, together with the ſquare of FE, is equal to the Rectangle under DE, EB, together with the ſquare of FE. Let the ſquare of FE common, be taken away; therefore the remaining Rectangle contained by AE, EC, is equal to the Rectangle contained by DE, EB.

If therefore in a Circle two ſtrait lines cut one another, the Rectangle contained by the Segments of the one, is equal to the Rectangle contained by the Segments of the other. Which was to be demonſtrated.

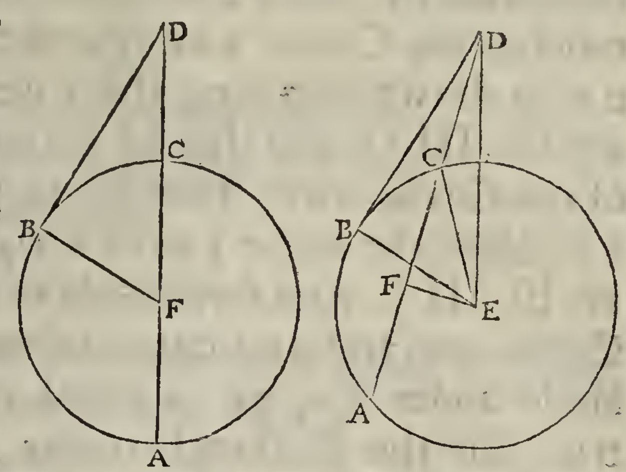
PRO-

PROPOSITION XXXVI.

IF without a Circle be taken any point, and from the ſame do fall on the Circle two ſtrait lines, of which one does cut the Circle, the other does touch it: The Rectangle contained by the whole Secant, and the outward Segment between the point and convex circumference, ſhall be equal to the ſquare of the Tangent.

Let without the Circle ABC be taken any point as D, and from D let the two ſtrait lines DCA, DB, fall on the Circle ABC: and let DCA cut the Circle ABC, and DB touch it. I ſay, that the Rectangle contained by AD, DC, is equal to the ſquare of DB. Now DCA either paſſes through the center, or not.

Firſt, let it paſs through the center, and let F be the center of the Circle ABC. Then let be joyned FB; therefore the angle FBD is a right angle [Prop. 18. El. III.]. Now foraſmuch as the ſtrait line AC is cut into halves in F, and to it is added CD; therefore the Rectangle contained by AD, DC, together with the ſquare of FC is equal to the ſquare of FD [Prop. 6. El. II.]. But FC is equal to FB; therefore the Rectangle of AD, DC, together with the ſquare of FB, is equal to the ſquare of FD. But the ſquare of FD is equal to the ſquares of FB, BD, for the angle FBD, is a right angle: therefore the Rectangle contained by AD, DC, together with the ſquare of FB, is equal to the ſquares of FB, BD. Let the ſquare of FB common be taken away; therefore the remaining Rectangle under AD, DC, is equal to the ſquare of the Tangent DB.



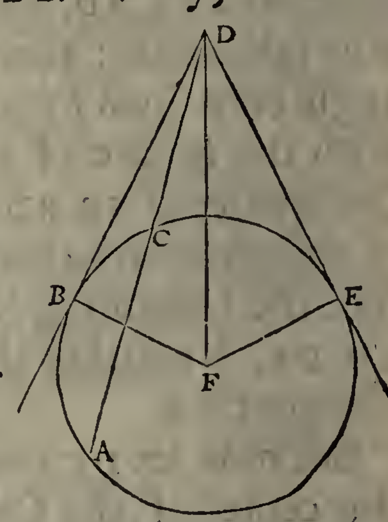
But now let DCA not paſs through the center of the Circle ABC: and let the center E be taken, and from E to AC let be drawn a perpendicular EF; and let be joyned EB, EC, ED. Now the angle EBD is a right angle [Prop. 18. El. III.]. And foraſmuch as the ſtrait line EF drawn through the center cuts the ſtrait line AC, not drawn through the center at right angles; it ſhall alſo cut the ſame into halves [Prop. 3. El. III.]; therefore AF is equal to FC. And becauſe the ſtrait line AC is cut into halves in F, and to it is added CD, therefore the Rectangle contained by AD, DC, together with the ſquare of FC, is equal to the ſquare of FD [Prop. 6. El. II.]. Let be added in common the ſquare of EF; therefore the Rectangle under AD, DC, together with the ſquares of CF, FE, is equal to the ſquares of DF, FE. But the ſquare of DE is equal to the ſquares of DF, FE, for EFD is a right angle; and the ſquare of CE is equal to the ſquares of CF, FE: Therefore the Rectangle contain'd by AD, DC, together with the ſquare of CE, is equal to the ſquare of ED. But CE is equal to EB; therefore the Rectangle under AD, DC, together with the ſquare of EB, is equal to the ſquare of ED. But the ſquares of EB, BD, are equal to the ſquare of ED, for the angle EBD is a right angle: therefore the Rectangle contained by AD, DC, together with the ſquare of EB, is equal to the ſquares of EB, BD. Let the ſquare of EB common be taken away; therefore the Rectangle contained by AD, DC, is equal to the ſquare of DB.

If therefore without a Circle be taken any point, &c. Which was to be demonſtrated.

PROPOSITION XXXVII.

IF without a Circle be taken any point, and from the ſame do fall upon the Circle two ſtrait lines; of which one does cut the Circle; the other does fall upon it: and the Rectangle contained by the whole Secant and the outward Segment between the point and the convex circumference, be equal to the ſquare of the incident line; the incident line ſhall touch the Circle.

Let without the Circle ABC be taken any point as D; and from D let the two ſtrait lines DCA, DB, fall upon the Circle ABC, and let DCA cut the Circle, and DB fall upon it: Alſo let the Rectangle contained by AD, DC be equal to the ſquare of DB. I ſay, that DB touches the Circle ABC. For let the ſtrait line DE be drawn touching the Circle ABC [Prop. 17. El. III.]: and let be taken F the center of the Circle ABC; then let be joyned FE, FB, FD. Now the angle FED is a right angle [Prop. 18. El. III.]: And foraſmuch as DE touches the Circle ABC, and DCA cuts it; therefore the Rectangle under AD, DC, is equal to the ſquare of DE. But the Rectangle under AD, DC, is put equal to the ſquare of DB; wherefore the ſquare of DE, is equal to the ſquare of DB, and therefore DE is equal to DB. But FE is equal to FB; there are then the two lines DE, EF, equal to the two lines DB, BF, and the baſe FD is common; therefore the angle DEF is equal to the angle DBF [Prop. 8. El. I.]. But DEF is a right angle, therefore DBF is alſo a right angle. Now BF being produced is the Diameter, but a ſtrait line drawn at right angles to the Diameter, from the extremity thereof touches the Circle ABC. In like manner the ſame ſhall be demonſtrated, if the center were in AC it ſelf.



If therefore without a Circle be taken any point, &c. Which was to be demonſtrated.

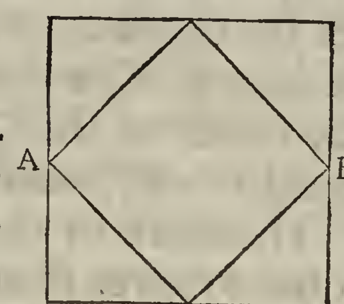
THE FOURTH ELEMENT.

DEFINITIONS.

DEFINITION I.

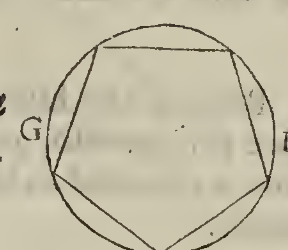
A Strait-lin'd Figure is ſaid to be inſcribed in a ſtrait-lin'd Figure, when every angle of the inſcribed Figure touches every ſide of the Figure, in which it is inſcribed.

DEFINITION II.



Likewiſe a Strait-lin'd Figure is ſaid to be circumſcribed about a Strait-lin'd Figure, when every ſide of the circumſcribed Figure touches every angle of the Figure, about which it is circumſcribed.

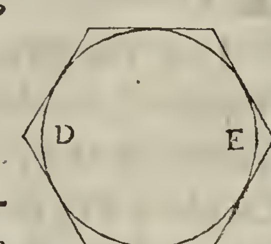
DEFINITION III.



A Strait-lin'd Figure is ſaid to be inſcribed in a Circle, when every angle of the inſcribed Figure touches the circumference of the Circle.

DEFINITION IV.

A Strait-lin'd Figure is ſaid to be circumſcribed about a Circle, when every ſide of the circumſcribed Figure touches the circumference of the Circle.



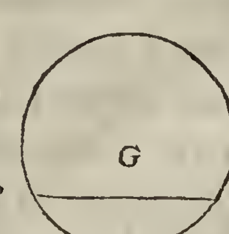
DEFINITION V.

Likewiſe a Circle is ſaid to be inſcribed in a ſtrait-lin'd Figure, when the circumference of the Circle touches every ſide of the Figure, in which it is inſcribed.

DEFINITION VI.

A Circle is ſaid to be circumſcribed about a ſtrait-lin'd Figure, when the circumference of the Circle touches every angle of the Figure, about which it is circumſcribed.

DEFINITION VII.



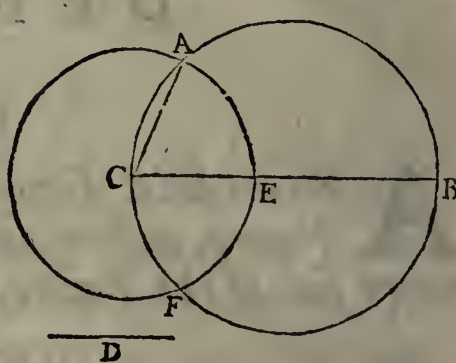
A Strait line is ſaid to be adapted in a Circle, when the extremes of the line are in the circumference of the Circle.

PROPOSITION I.

IN a given Circle to adapt a ſtrait line equal to a given ſtrait line, which is not greater than the Diameter of the Circle.

Let the given Circle be ABC, and the given ſtrait line, which is not greater than the Diameter of the Circle, be D. It is required in the Circle ABC, to adapt a ſtrait line equal to the ſtrait line D.

Let there be drawn BC, the Diameter of the Circle ABC. Now if BC be equal to D, then that is done which was propoſed. For in the Circle ABC is adapted the line BC, equal to the given line D.



But if not, then BC is greater than D [by Suppoſition]; and let there be put CE equal to D. Then to the center C, and diſtance CE, let the Circle EAF de deſcribed, and let CA be drawn. Now foraſmuch as the point C is the center of the Circle EAF, therefore CA is equal to CE, but D is equal to CE; wherefore alſo D is equal to CA.

Therefore in the given Circle ABC, is adapted a ſtrait line CA, equal to given ſtrait line D, which is not greater than the Diameter of the Circle. Which was to be done.

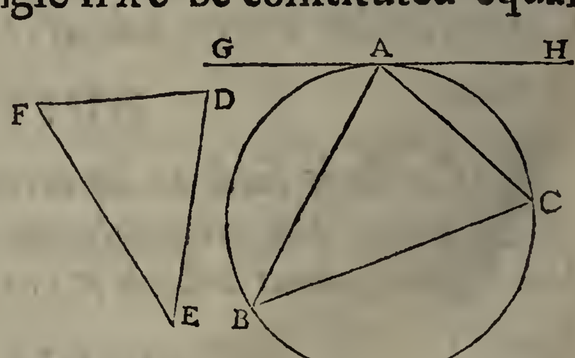
ANNOTATIONS.

Becauſe the Diameter is the greateſt line in a Circle [Prop. 15. El. III.]; therefore this proviſo, or limitation, is here made, that the given line, to which an equal line is required to be adapted in the Circle, ought not to be greater than the Diameter.

PROPOSITION II.

IN a given Circle to inſcribe a Triangle equiangled to a given Triangle.

Let the given Circle be ABC, and the given Triangle DEF. It is required in the Circle ABC, to inſcribe a Triangle equiangled to the Triangle DEF. Let there be drawn a ſtrait line GAH, touching the Circle ABC, in the point A [by Prop. 17. El. III.]. Then to the line AH, and to the point in it A, let the angle HAC be conſtituted equal to the angle DEF [by Prop. 23. El. I.]. Again, to the line GA, and to the point in it A, let the angle GAB be conſtituted equal to the angle DFE; and draw BC. Now foraſmuch as a ſtrait line HAG, touches the Circle ABC, and from the Contact is drawn AC; therefore the angle HAC, is equal to the angle ABC, in the alternate Segment of the Circle [Prop. 32. El. III.]. But the angle HAC is equal to the angle DEF [by Conſtruction]; therefore the angle ABC, is equal to the angle DEF. By the ſame reaſon alſo



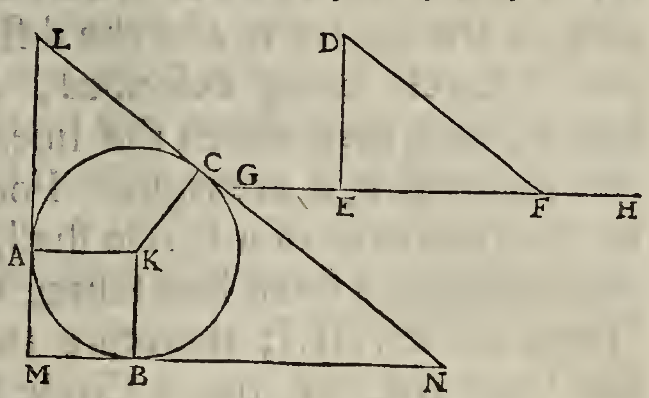
alſo the angle ACB, is equal to the angle DFE; therefore the remaining angle BAC is equal to the remaining angle EDF: wherefore the Triangle ABC, is equiangled to the Triangle DEF, and is inſcribed in the Circle ABC.

Therefore in a given Circle is inſcribed a Triangle equiangled to a given Triangle. Which was to be done.

PROPOSITION III.

ABout a given Circle to circumſcribe a Triangle equiangled to a given Triangle.

Let the given Circle be ABC, and the given Triangle DEF. It is required about the Circle ABC, to circumſcribe a Triangle equiangled to the Triangle DEF. Let EF be produced both ways to the points G, H; and of the Circle ABC let the center K be taken [by Prop. 1. El. III.]; and let a ſtrait line KB be drawn at pleaſure. Now to the line KB, and to the point in it K, let there be conſtituted the angle BKA, equal to the angle DEG [by Prop. 23. El. I.], and alſo the angle BKC, equal to the angle DFH. Then by the points A, B, C, let there be drawn the ſtrait lines LAM, MBN, NCL, touching the Circle ABC [by Prop. 17. El. III.]. Now foraſmuch as LM, MN, NL, touch the Circle ABC, in the points A, B, C; and from K, the center, to the points A, B, C, are drawn KA, KB, KC; therefore the angles at the points A, B, C, are right angles [Prop. 18. El. III.]. And becauſe the four angles of the quadrilateral Figure AMBK, are equal to four right angles, for that it is divided into two Triangles (by ſuppoſing a ſtrait line drawn from K to M, making two Triangles KAM, KBM, each of which have their three angles equal to two right, Prop. 32. El. I.), of which the angles KAM, KBM, are right angles; therefore the remaining angles AKB, AMB, are equal to two right angles. But the angles DEG, DEF, are equal to two right angles [Prop. 13. El. I.]; therefore the angles AKB, AMB, are equal to the angles DEG, DEF, of which AKB is equal to DEG: wherefore the remaining angle AMB, is equal to the remaining angle DEF. In like manner may be demonſtrated that the angle LNM, is equal to the angle DFE; therefore alſo the remaining angle MLN, is equal to the remaining angle EDF: wherefore the Triangle LMN, is equiangled to the Triangle DEF; and it is circumſcribed about the Circle ABC.

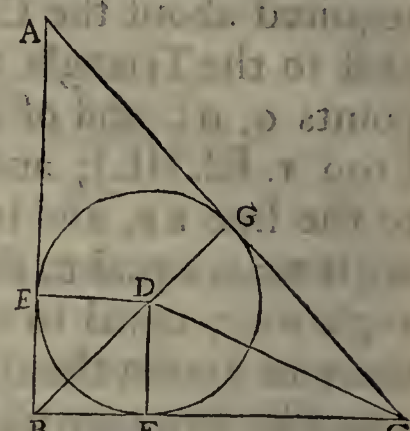


Therefore about a given Circle is circumſcribed a Triangle equiangled to a given Triangle. Which was to be done.

PROPOSITION IV.

IN a given Triangle to inſcribe a Circle.

Let the given Triangle be ABC. It is required in the Triangle ABC, to inſcribe a Circle. Let the angles ABC, BCA, be cut into halves by the ſtrait lines BD, CD [by Prop. 9. El. I.], and let them meet together in the point D; and from the point D let there be drawn to the lines AB, BC, CA, the perpendiculars DE, DF, DG [by Prop. 12. El. I.]. Now foraſmuch as the angle ABD is equal to the angle CBD (for that the angle ABC is cut into halves) and the right angle BED, is equal to the right angle BFD: there are then two Triangles, EBD, FBD, having two angles equal to two angles, and one ſide equal to one ſide, namely BD common to both, and ſubtended under one of the equal angles; therefore they ſhall have the remaining ſides equal to the remaining ſides [Prop. 26. El. I.]; wherefore DE ſhall be equal to DF. By the ſame reaſon DG is alſo equal to DF; therefore the three lines DE, DF, DG, are equal to one another; wherefore to the center D, and the diſtance any one of the lines DE, DF, DG, a Circle being deſcribed, ſhall paſs through the remaining points, and ſhall touch the lines AB, BC, CA, becauſe the angles at the points E, F, G, are right. For if the Circle ſhall cut them, then to the Diameter of a Circle ſhall, from the extremity, be drawn, at right angles, a ſtrait line falling within the Circle, which is abſurd [Prop. 16. El. III.]; therefore to the center D, and diſtance one of the lines DE, DF, DG, a Circle being deſcribed, ſhall not cut the lines AB, BC, CA: wherefore it ſhall touch them, and there ſhall be a Circle inſcribed in the Triangle ABC.

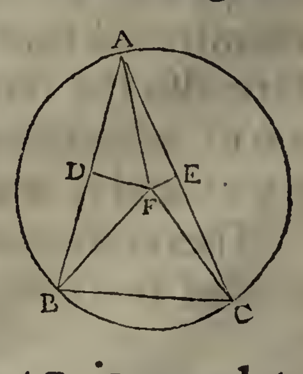


Therefore in the given Triangle ABC, is inſcribed the Circle EFG. Which was to be done.

PROPOSITION V.

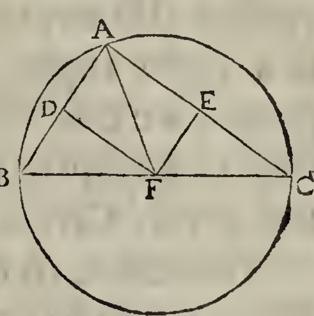
ABout a given Triangle to circumſcribe a Circle.

Let the given Triangle be ABC. It is required about the given Triangle ABC, to circumſcribe a Circle. Let AB, AC, be cut into halves in the points D, E [by Prop. 10. El. I.], and from the points D, E, let there be drawn at right angles, to AB, AC, the lines DF, EF [by Prop. 11. El. I.]. Now theſe lines ſhall meet either within the Triangle ABC, or in the line BC, or without it. Firſt, let them meet within, at the point F, and let FB, FC, FA, be joyned. Now foraſmuch as AD is equal to



DB,

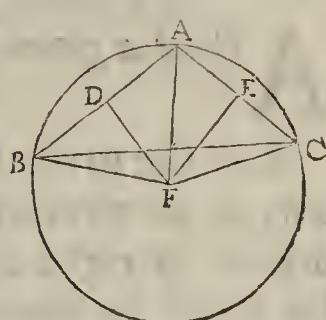
DB, and DF common, and at right angles; therefore the baſe AF, is equal to the baſe FB [Prop. 4. El. I.]. In like manner we ſhall demonſtrate, that FC is alſo equal to FA, ſo that alſo BF is equal to FC; therefore theſe three FA, FB, FC, are equal to one another: wherefore to the center F, and diſtance, any one of the lines FA, FB, FC, a Circle being deſcribed, ſhall paſs alſo through the remaining points; and there ſhall be circumſcribed a Circle about the Triangle ABC: And let it be deſcribed, as ABC.



But again, let DF, EF, meet in the line BC, at the point F, as it is in this Figure, and let AF be joyned.

In like manner we ſhall demonſtrate, that the point F is the center of a Circle circumſcribed about the Triangle ABC.

Laſtly, let DF, EF, meet without the Triangle ABC, at the point F, as in this laſt figure; and let FA, FB, FC, be joyn'd.



Now foraſmuch as AD is equal to DB, and DF common, and at right angles, therefore the baſe AF is equal to the baſe FB. In like manner we ſhall demonſtrate, that FC is equal to FA; ſo that alſo BF is equal to FC: therefore again to the center F, and diſtance any one of the lines FA, FB, FC, a Circle being deſcribed, ſhall paſs alſo through the remaining points; and ſhall be circumſcribed about the Triangle ABC: And let it be deſcribed as ABC.

Therefore about a given Triangle a Circle is circumſcribed. Which was to be done.

Corollary.

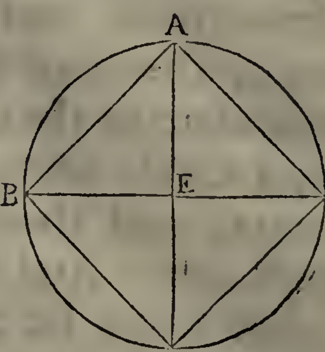
And it is manifeſt, that when the center of the Circle falls within the Triangle, then the angle BAC, *being in a Segment greater than the Semicircle, is leſs than a right angle [Prop.* 31. *El.* III.]. *But when it falls in the line* BC, *being in the Semicircle, then* BAC *ſhall be a right angle. And when the center falls without* BC, *then the angle* BAC, *being in a Segment leſs than the Semicircle, is greater than a right angle. So that when the given angle is leſs then a right angle, then the lines* DF, EF, *ſhall fall within the Triangle. But when it is a right angle, they ſhall fall in* BC: *And when greater than a right angle, they ſhall fall without* BC.

PROPOSITION VI.

IN a given Circle to inſcribe a ſquare.

Let the given Circle be ABCD. It is required in the Circle ABCD, to inſcribe a ſquare. Of the Circle ABCD, let the Diameters AC, BD, be drawn at right angles to one another; and let be joyned AB, BC, CD, DA. Now foraſmuch as BE is equal to DE, for the center is E; and EA is common, and at right angles; therefore the baſe AB is equal

equal to the baſe AD. And by the ſame reaſon either of the lines BC, CD, is equal to either of the lines AB, AD; therefore the quadrilateral Figure ABCD, is equilateral. I ſay, that it is alſo Rectangular. For becauſe the line BD is the Diameter of the Circle ABCD, therefore BAD is a Semicircle: wherefore the angle BAD is a right angle [Prop. 31. El. III.]. By the ſame reaſon alſo every one of the angles ABC, BCD, CDA, is a right angle: therefore the quadrilateral Figure ABCD is rectangular. But it has been proved to be equilateral; therefore it is a ſquare, and it is inſcribed in the given Circle ABCD.

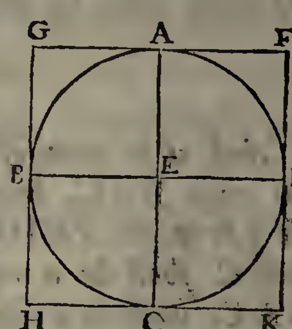


Therefore in the given Circle ABCD, is inſcribed the ſquare ABCD. Which was to be done.

PROPOSITION VII.

About a given Circle to circumſcribe a ſquare.

Let the given Circle be ABCD. It is required about the Circle ABCD, to circumſcribe a ſquare. Let two Diameters AC, BD, be drawn at right angles to one another; and by the points A, B, C, D, let there be drawn the lines FG, GH, HK, KF, touching the Circle ABCD [Prop. 17. El. III.]. Now foraſmuch as FG touches the Circle ABCD, and from E the center, to the Contact at A, is joyned EA; therefore the angles at A are right angles [Prop. 18. El. III.]; and by the ſame reaſon, the angles at the points B, C, D, are alſo right angles. Now becauſe AEB is a right angle, and that EBG is alſo a right angle; therefore GH is parallel to AC [Prop. 28. El. I.], and by the ſame reaſon AC is alſo parallel to FK. In like manner we ſhall demonſtrate, that either of the lines GF, HK, is parallel to the line BED: wherefore GK, GC, AK, FB, BK, are Parallelograms; and therefore GF is equal to HK; as alſo GH to FK [Prop. 34. El. I.].



"Note, thus far is only proved, that the oppoſite ſides, namely, "GF is equal to HK, as alſo GH to FK. Next is to be proved, that all "four are equal to one another.

Now becauſe AC is equal to BD; but AC is equal to each of the lines GH, FK; and BD is equal to each of the lines GF, HK; wherefore alſo each of the lines GH, FK, is equal to each of the lines GF, HK; therefore the quadrilateral Figure FGHK is equilateral.

I ſay, that it is alſo rectangular. For becauſe GBEA is a Parallelogram, and AEB is a right angle; therefore AGB is alſo a right angle. In like manner we ſhall demonſtrate, that the angles at the points H, K, F, are right angles; therefore the quadrilateral Figure

FGHK

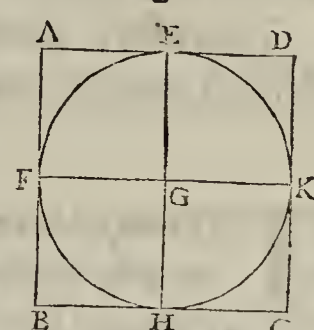
FGHK is rectangular. But it has been proved to be equilateral; therefore it is a ſquare, and it is circumſcribed about the Circle ABCD.

Therefore about a given Circle a ſquare is circumſcribed. Which was to be done.

PROPOSITION VIII.

IN a given ſquare to inſcribe a Circle.

Let the given ſquare be ABCD. It is required in the ſquare ABCD to inſcribe a Circle. Let both of the lines AB, AD, be cut into halves in the points E, F, and by E let there be drawn EH parallel to either of the lines AB, DC [by Prop. 31. El. I.]; and by F, let there be drawn FK, parallel to either of the lines AD, BC; therefore every one of the ſpaces AK, KB, AH, HD, AG, GC, BG, GD, is a Parallelogram; and therefore their oppoſite ſides are equal [Prop. 34. El. I.].



Now foraſmuch as AD is equal to AB, and of AD the half is AE, and of AB the half is AF; therefore AE is equal to AF; and alſo the oppoſite ſides are equal; therefore FG is equal to GE. In like manner we ſhall demonſtrate, that either of the lines GH, GK, is equal to either of the lines FG, GE; therefore the four lines GE, GF, GH, GK, are equal to one another. Wherefore to the center G, and diſtance any one of the lines GE, GF, GH, GK, a Circle being deſcribed, ſhall alſo paſs by the remaining points, and ſhall touch the ſtrait lines AB, BC, CD, DA, for that the angles at E, F, H, K, are right angles. For if the Circle cut the lines AB, BC, CD, DA, then to the Diameter of a Circle a ſtrait line being drawn at right angles from the extremity thereof, ſhall fall within the Circle; which is abſurd [Prop. 16. El. III.]; therefore to the center G, and diſtance any one of the lines GE, GF, GH, GK, a Circle being deſcribed does not cut the lines AB, BC, CD, DA; wherefore it ſhall touch them, and ſhall be inſcribed in the ſquare ABCD.

Therefore in a given ſquare a Circle has been inſcribed. Which was to be done.

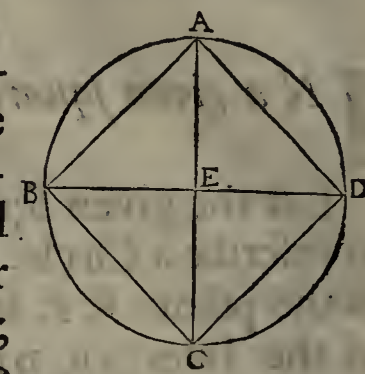
PROPOSITION IX.

ABout a given ſquare to circumſcribe a Circle.

Let the given ſquare be ABCD. It is required about the ſquare ABCD, to circumſcribe a Circle. For the lines AC, BD, being drawn, let them cut one another in F. Now foraſmuch as DA is equal to AB, and AC is common; therefore there are the two lines DA, AC, equal to the two lines BA, AC, and the baſe DC is equal to the baſe CB; therefore the angle DAC, is equal to the angle BAC; wherefore

the

the angle DAB is cut into halves by AC. In like manner we ſhall demonſtrate, that every one of the angles ABC, BCD, CDA, is cut into halves by the lines AC, DB. And becauſe the angle DAB is equal to the angle ABC; and of the angle DAB, the half is the angle EAB, and of the angle ABC, the half is the angle EBA; therefore the angle EAB is equal to the angle EBA. So that the ſide EA is equal to the ſide EB [Prop. 6. El. I.]. In like manner we ſhall demonſtrate, that either of the lines EC, ED, is equal to either of the lines EA, EB: wherefore the four lines EA, EB, EC, ED, are equal to one another; therefore to the center E, and diſtance any one of the lines EA, EB, EC, ED, a Circle being deſcribed, ſhall paſs alſo through the remaining points, and ſhall be circumſcribed about the ſquare ABCD. Let it be deſcribed as ABCD.



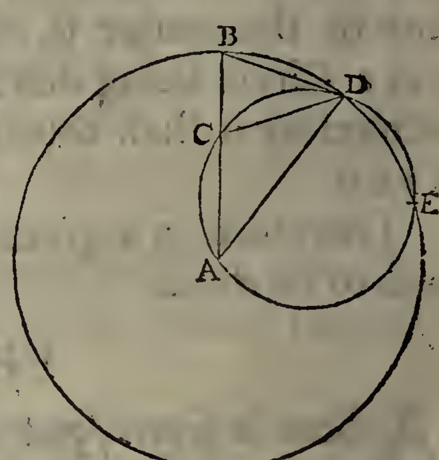
Therefore about a given ſquare a Circle has been circumſcribed. Which was to be done.

PROPOSITION X.

TO conſtitute an equicrural Triangle having each of the angles at the baſe, double to the remaining angle.

Let there be put a ſtrait line AB, and let it be cut in the point C, ſo that the Rectangle contained by AB, BC, be equal to the ſquare of CA [by Prop. 11. El. II.]. Then to the center A, and diſtance AB, let the Circle BDE be deſcribed; and in the Circle BDE, let be adapted [by Prop. 1. El. IV.], the ſtrait line BD equal to AC, which is not greater than the Diameter of the Circle BDE; and let DA, DC, be joyned. Alſo about the Triangle ACD let be circumſcribed the Circle ACD.

I ſay, that of the equicrural Triangle BAD, each of the angles ABD, ADB, is double to the angle BAD.



Foraſmuch as the rectangle AB, BC, is equal to the ſquare of AC; and AC is equal to BD; therefore the rectangle AB, BC, is equal to the ſquare of BD. And whereas there has been taken a point B, without the Circle ACD; and from the point B, on the Circle ACD, have fallen the two ſtrait lines BCA, BD, whereof one does cut, and the other does fall upon it; and the rectangle AB, BC, is equal to the ſquare of BD; therefore the line BD ſhall touch the Circle ACD [Prop. 37. El. III.]. Now foraſmuch as BD does touch, and from the Contact at D, is drawn DC; therefore the angle BDC is equal to the angle in the alternate Segment of the Circle, that is, to DAC [Prop. 32. El. III.]. Now becauſe the angle BDC is equal to DAC,

let

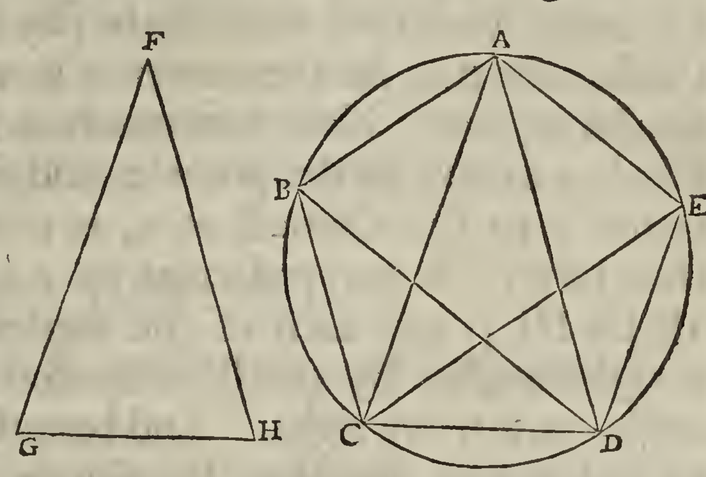
let there be added CDA, common to both; therefore the whole BDA is equal to the two angles CDA, DAC. But to CDA, DAC, is equal the outward angle BCD, therefore BDA is equal to BCD: but BDA is equal CBD, for that the ſide AD is equal to AB; that is DBA is equal to BCD; therefore the three angles BDA, DBA, BCD, are equal to one another. And becauſe the angle DBC is equal to the angle BCD; therefore the ſide BD is equal to the ſide DC. But BD is put equal to AC; wherefore alſo AC is equal to CD; ſo that the angle CDA is equal to the angle DAC; therefore the angles CDA, DAC, are double to the angle DAC: but BCD is equal to CDA, DAC, and therefore BCD is double to DAC. But BCD is equal to each of the angles BDA, DBA; wherefore each of the angles BDA, DBA, is double to DAB.

Therefore there is conſtituted an equicrural Triangle ADB, having each of the angles at the baſe double to the remaining angle. Which was to be done.

PROPOSITION XI.

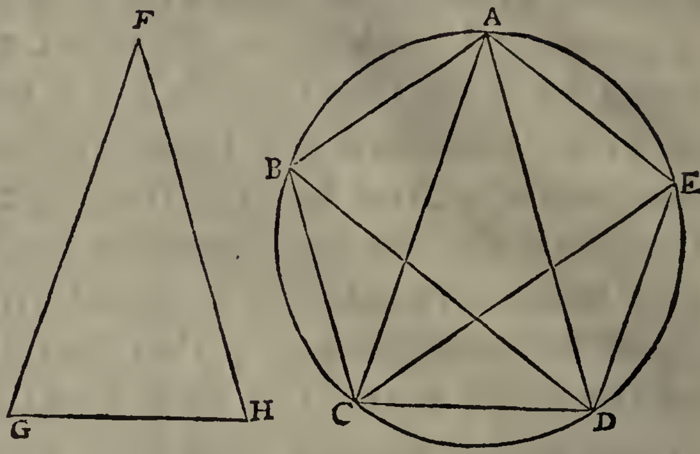
IN a given Circle to inſcribe an equilateral and equiangled Pentagon.

Let the given Circle be ABCDE. It is required in the Circle ABCDE, to inſcribe an equilateral and equiangled Pentagon. Let there be put an equicrural Triangle FGH, having each of the angles at G, H, double to the angle at F; and let there be inſcribed in the Circle ABCDE, the Triangle ACD, equiangled to the Triangle FGH; ſo that to the angle at F may be equal the angle CAD: And to either of the angles at G, H, be equal either of the angles ACD, CDA, and therefore either of the angles ACD, CDA, is double to the angle CAD. Now let each of the angles ACD, CDA, be cut into halves by the ſtrait lines CE, DB [by Prop. 9. El. I.],



and let there be drawn AB, BC, CD, DE, EA. Foraſmuch then as each of the angles ACD, CDA, is double to CAD, and they have been cut into halves by the lines CE, DB; therefore theſe five angles DAC, ACE, ECD, CDB, BDA, are equal to one another. But equal angles inſiſt on equal circumferences [Prop. 26. El. III.]; therefore the five circumferences AB, BC, CD, DE, EA, are equal to one another. But under equal circumferences are ſubtended equal ſtrait lines [Prop. 29. El. III.]; wherefore the five ſtrait lines AB, BC, CD, DE, EA, are equal to one another; therefore the Pentagon ABCDE is equilateral. I ſay, that it is alſo equiangular. For becauſe the circumference AB, is equal to the circumference DE, let BCD be added in common:

therefore the whole circumference ABCD is equal to the whole circumference EDCB. And the angle AED insists on the circumference ABCD; as also the angle BAE insists on the circumference EDCB; therefore the angle BAE is equal to the angle AED [Prop. 27. El. III.]. By the same reason also every one of the angles ABC, BCD, CDE, is equal to each of the angles BAE, AED: wherefore the Pentagon ABCDE is equiangular. But it has been proved to be equilateral.

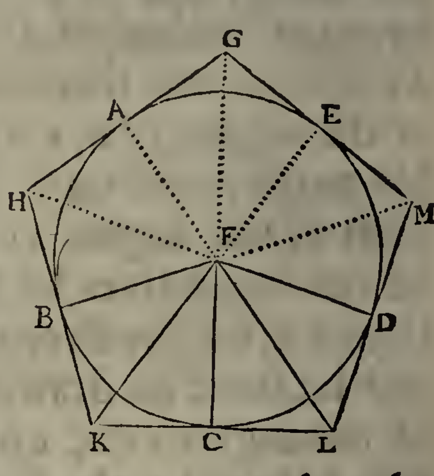


Therefore in a given Circle an equilateral and equiangled Pentagon has been inscribed. Which was to be done.

PROPOSITION XII.

About a given Circle to circumscribe an equilateral and equiangled Pentagon.

Let the given Circle be ABCDE. It is required about the Circle ABCDE, to circumscribe an equilateral and equiangled Pentagon. Let the points of the angles of the inscribed Pentagon be conceived to be ABCDE; so that the circumferences AB, BC, CD, DE, EA, are equal. And by the points A, B, C, D, E, let there be drawn GH, HK, KL, LM, MG, touching the Circle [by Prop. 17. El. III.]: and of the Circle ABCDE, let the center F be taken, then let FB, FK, FC, FL, FD, be joyned. Now forasmuch as the strait line KL toucheth the Circle ABCDE, in the point C, and from the center F to the Contact at C, is drawn FC; therefore FC is perpendicular to KL [Prop. 18. El. III.]; and each of the angles at C is a right angle: By the same reason also the angles at B, D are right. And because FCK is a right angle, therefore the square of FK is equal to the squares of FC, CK. By the same reason also, the square of FK is equal to the squares of FB, BK; therefore the squares of FC, CK, are equal to the squares of FB, BK, of which the square of FC is equal to the square of FB; therefore the remaining square of CK is equal to the remaining square of BK: wherefore BK is equal to CK. And because FB is equal to FC, and FK common, therefore there are the two lines BF, FK, equal to the two lines CF, FK, and the base BK is equal to the base CK; wherefore the angle BFK is equal to the angle KFC [Prop. 8. El. I.]. And also the angle BKF is equal to the angle FKC; therefore the angle BFC is double to the angle KFC, and the angle



BKC

BKC is double to the angle FKC. By the ſame reaſon alſo, the angle CFD is double to the angle CFL, and the angle CLD is double to the angle CLF. And becauſe the circumference BC is equal to the circumference CD; therefore the angle BFC is equal to the angle CFD [Prop. 27. El. III.]. But the angle BFC is double to the angle KFC, and the angle DFC is double to the angle LFC; therefore the angle KFC is equal to the angle CFL. Now there are two Triangles FKC, FLC, having two angles equal to two angles, each to each, and one ſide equal to one ſide, namely FC, common to both; therefore ſhall they have the remaining ſides equal to the remaining ſides, and the remaining angle equal to the remaining angle [Prop. 26. El. I.]; therefore the line KC is equal to the line CL, and the angle FKC to the angle FLC. Now foraſmuch as KC is equal to CL, therefore KL is double to KC. By the ſame reaſon HK ſhall be proved double to BK: and now becauſe BK has been proved equal to KC, and that KL is double to KC, as alſo HK to BK; therefore HK is equal to KL.

In like manner every one of the lines GH, GM, ML, ſhall be proved equal to each of the lines HK, KL; therefore the Pentagon GHKLM, is equilateral.

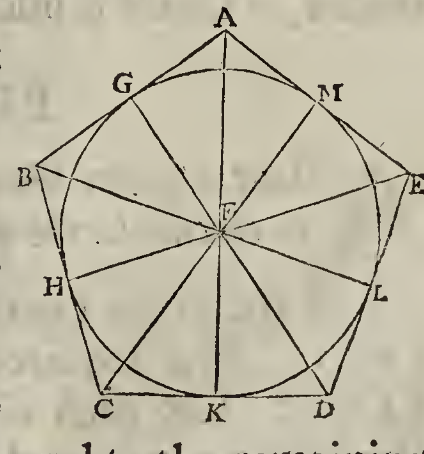
I ſay, that it is alſo equiangled. Foraſmuch as the angle FKC is equal to the angle FLC, and that the angle HKL, has been proved double to the angle FKC, as alſo KLM, double to FLC: therefore the angle HKL is equal to the angle KLM.

In like manner, every one of the angles KHG, HGM, GML, ſhall be proved equal to each of the angles HKL, KLM; wherefore the five angles GHK, HKL, KLM, LMG, MGH, are equal to one another; therefore the Pentagon GHKLM, is equiangled. But it has been proved equilateral; and it is circumſcribed about the Circle ABCDE. Which was to be done.

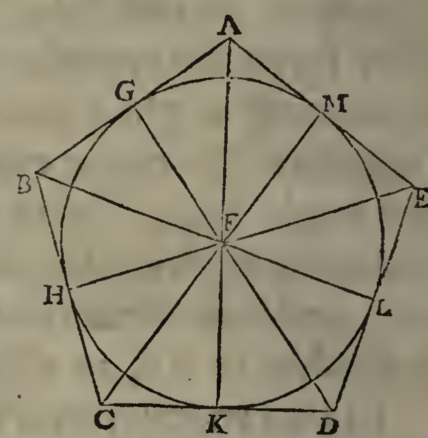
PROPOSITION XIII.

In a given Pentagon, which is equilateral and equiangled, to inſcribe a Circle.

Let the given Pentagon, which is equilateral and equiangled, be ABCDE It is required in the Pentagon ABCDE, to inſcribe a Circle. Let each of the angles BCD, CDE, be cut into halves by the lines CF, DF; and from the point F, wherein the lines CF, DF, do meet, let there be drawn FB, FA, FE. Now foraſmuch as BC is equal to CD, and CF common; therefore there are two lines BC, CF, equal to two lines DC, CF, and the angle BCF is equal to the angle DCF; therefore the baſe BF is equal to the baſe DF, and the Triangle BFC is equal to the Triangle DFC, and the remaining angles are equal to the remaining



angles, under which are ſubtended equal ſides [Prop. 4. El. I.]; therefore the angle CBF is equal to the angle CDF. And becauſe the angle CDE is double to the angle CDF; but the angle CDE is equal to the angle ABC, and CDF, to CBF; wherefore the angle CBA is double to the angle CBF; and therefore the angle ABF is equal to the angle FBC: therefore the angle ABC is cut into halves by the line BC. In like manner ſhall be demonſtrated that each of the angles BAE, AED, are cut into halves by the lines FA, FE.



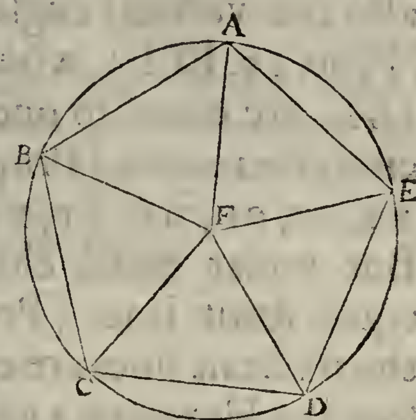
Now from the point F to the lines AB, BC, CD, DE, EA, let be drawn the perpendiculars FG, FH, FK, FL, FM. Now becauſe the angle HCF is equal to the angle KCF, and the right angle FHC, is equal to the right angle FKC; therefore there are two Triangles FHC, FKC, having two angles equal to two angles, and one ſide equal to one ſide, namely FC common to both, and ſubtended under equal angles; wherefore they ſhall have the remaining ſides equal to the remaining ſides [Prop. 26. El. I.]; therefore the perpendicular FH is equal to the perpendicular FK. In like manner ſhall be demonſtrated, that alſo every one of the lines FL, FM, FG, is equal to either of the lines FH, FK; therefore the five lines FG, FH, FK, FL, FM, are equal to one another. Wherefore to the center F, and diſtance any one of the lines FG, FH, FK, FL, FM, a Circle being deſcribed, ſhall paſs alſo through the remaining points, and ſhall touch the lines AB, BC, CD, DE, EA, becauſe that the angles at the points G, H, K, L, M, are right angles. For if the Circle ſhall not touch, but cut them, then it ſhall happen, that to the Diameter of a Circle, a ſtrait line being drawn at right angles from the extremity thereof, does fall within the Circle; which has been proved abſurd [Prop. 16. El. III.]; therefore to the center F, and diſtance any one of the lines FG, FH, FK, FL, FM, a Circle being deſcribed, ſhall not cut the lines AB, BC, CD, DE, EA; wherefore it ſhall touch them. Let it be deſcribed, as GHKLM.

Therefore in a given Pentagon, which is equilateral, and equiangled, a Circle is inſcribed. Which was to be done.

PROPOSITION XIV.

About a given Pentagon, which is equilateral, and equiangled, to circumſcribe a Circle.

Let the given Pentagon, which is equilateral, and equiangled, be ABCDE. It is required about the Pentagon ABCDE, to circumſcribe a Circle. Let each of the angles BCD, CDE, be cut into halves by the lines CF, DF; and from the point F, wherein the lines CF, DF, do do meet, let there to the points B, A, E, be joyned the lines FB, FA, FE. As it was in the foregoing Propoſition ſo may it here be demonſtrated, that every one of the angles CBA, BAE, AED, is cut into halves by the lines FB, FA, FE.



Now foraſmuch as the angle BCD is equal to the angle CDE, and of the angle BCD, the half is FCD, and of CDE, the half is CDF; therefore the angle FCD is equal to the angle FDC; ſo that the ſide FC is equal to the ſide FD.

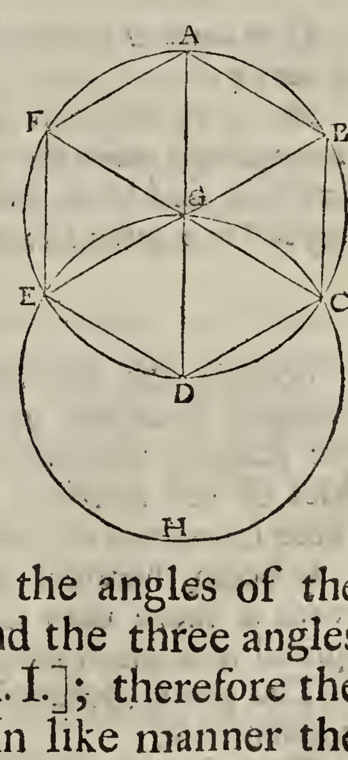
In like manner ſhall be demonſtrated, that every one of the ſides FB, FA, FE, is equal to each of the ſides FC, FD. Wherefore the five lines FA, FB, FC, FD, FE, are equal to one another; therefore to the center F, and diſtance, any one of the lines FA, FB, FC, FD, FE, a Circle being deſcribed, ſhall paſs alſo through the remaining points, and ſhall be circumſcribed about the Pentagon ABCDE, which is equilateral and equiangled. Let it be circumſcribed, and be the Circle ABCDE.

Therefore about a given Pentagon, which is equilateral and equiangled, a Circle is circumſcribed. Which was to be done.

PROPOSITION XV.

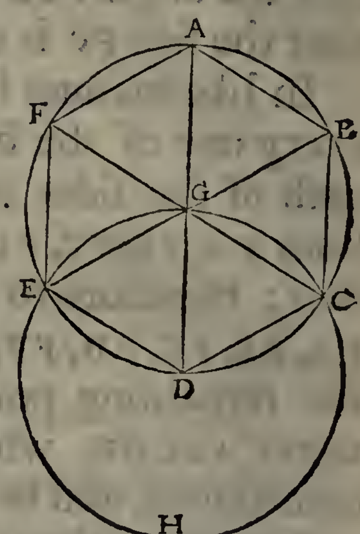
IN a given Circle to inſcribe an equilateral, and equiangled Hexagon.

Let the given Circle be ABCDEF. It is required in the Circle ABCDEF, to inſcribe an equilateral, and equiangled Hexagon. Of the Circle ABCDEF, let the Diameter AD be drawn, and the center G be taken. Now to the center D, and diſtance DG, let the Circle EGCH be deſcribed, and EG, CG being joyned, let them be produced to the points B, F, and let AB, BC, CD, DE, EF, FA, be joyned. I ſay, that the Hexagon ABCDE, is equilateral and equiangled. Now foraſmuch as the point G is the center of the Circle ABCDEF; therefore GE is equal to GD. Again, becauſe D is the center of the Circle EGCH, therefore DE is equal to DG. But GE has been proved equal to GD; wherefore GE is equal to ED; therefore the Triangle EGD is equilateral, and the three angles EGD, GDE, DEG, are equal to one another. Now becauſe in equicrural Triangles, the angles of the baſe are equal to one another [Prop. 5. El. I.], and the three angles of a Triangle are equal to two right [Prop. 32. El. I.]; therefore the angle EGD is a third part of two right angles. In like manner the angle DGC ſhall be proved a third of two right angles; and becauſe



the

the line CG, ſtanding upon the line EB, makes the collateral angles EGC, CGB, equal to two right angles [Prop. 13. El. I.]; therefore the remaining angle CGB is alſo a third of two right angles: wherefore the angles EGD, DGC, CGB, are equal to one another: And alſo the vertical angles BGA, AGF, FGE, are equal to EGD, DGC, CGB [Prop. 15. El.I.]; wherefore the ſix angles EGD, DGC, CGB, BGA, AGF, FGE, are equal to one another: But equal angles inſiſt upon equal circumferences [Prop. 20. El. III.]; therefore the ſix circumferences AB, BC, CD, DE, EF, FA, are equal to one another. But under equal circumferences are ſubtended equal ſtrait lines [Prop. 29. El. III.]; therefore the ſix ſtrait lines are equal to one another: wherefore the Hexagon ABCDEF is equilateral.



I ſay, that it is alſo equiangled. For becauſe the circumference AF is equal to the circumference ED, let there be added the circumference ABCD common; therefore the whole circumference FABCD, is equal to the whole circumference EDCBA. But the angle FED inſiſts upon the circumference FABCD, and the angle AFE upon the circumference EDCBA; therefore the angle AFE is equal to the angle FED [Prop. 27. El. III.]. In like manner ſhall be demonſtrated, that the remaining angles of the Hexagon ABCDEF, are every one equal to either of the angles AFE, FED; therefore the Hexagon ABCDEF, is equiangled. But it has been proved alſo equilateral; and it is inſcribed in the Circle ABCDEF.

Therefore in a given Circle, an equilateral, and equiangled Hexagon is inſcribed. Which was to be done.

Corollary.

From hence it is manifeſt, that the ſide of an inſcribed Hexagon, is equal to the Radius of the Circle.

And if by the points A, B, C, D, E, F, *we draw Tangents to the Circle, there ſhall be circumſcribed about the Circle an equilateral and equiangled Hexagon, according to what hath been ſaid of the Pentagon. And moreover, by the like as hath been ſaid of the Pentagon, we ſhall in a given Hexagon inſcribe a Circle, and alſo circumſcribe.*

Corollary 2. added.

Becauſe the circumference of a Circle is greater than the Perimeter of any Polygon inſcribed in it; and every ſide of an inſcribed Hexagon is equal to the Radius; therefore the circumference of the Circle, being greater than the ſix ſides of the inſcribed Hexagon, is alſo greater than the ſix Radii, that is, than three Diameters of the Circle.

It being therefore manifeſt from this Propſition of *Euclide*, that the circumference is more than triple of the Diameter, Geometricians have in all Ages enquired how much more it is.

The great *Archimedes* has brought it within the eaſieſt limits, and the beſt for common uſe.

Therefore if we ſuppoſe the Diameter to be 7, and ſo conſequently divided into

into ſeven parts, then the circumference ſhall be more than thrice ſeven, that is, more then 21 by almoſt $\frac{1}{7}$ part of the Diameter.

And hereupon the circumference, compared to the Diameter, is generally taken to be as 22 to 7: this proportion (tho' ſomewhat too great) being near enough the truth for any common uſe. To which only end this Rule was given by *Archimedes*, who could otherwiſe have proceeded nearer and nearer, to any approximation deſireable; but becauſe after all, the nature of the ſubject admits not of a Geometrical exactneſs, and equality, That great Maſter of Geometry rightly judg'd it moſt convenient, to ſtate the menſuration between the Circumference and Diameter in the eaſieſt and readieſt Terms, as 22 to 7 almoſt. So that if the Diameter be ſuppoſed to be 7, as for inſtance 7 Inches, then ſhall the circumference of the Circle be almoſt 22 Inches. And the like in Feet, Cubits, Miles, or any other meaſure.

Advertiſement.

We are laſtly to obſerve in this place, that the Equilateral Triangle, the Square, the Equilateral Pentagon and Hexagon, are the four ſimple and primitive Figures, from which all other Regular Polygons do ariſe, that are mutually with a Circle, or with one another Inſcriptible and Circumſcriptible: therefore *Euclide* has moſt accurately, in theſe four Figures, ſet forth a general method ſufficient for this kind of Subject, and applicable to all other regular Polygons. But of them in particular he makes here no mention; becauſe theſe four are only requiſite to the conſideration of the five regular *Platonick* Bodies, wherewith *Euclide* concludes his Elements. And beſides, the reſt, as they are infinite in multitude, ſo are they divided from theſe after one and the ſame manner of Conſtruction; and their Demonſtration is agreeable every way to what is here already ſet forth in theſe primitive Figures.

Firſt then, from an inſcribed Square is conſtituted an inſcribed Octagon, by the biſection of a Quadrantal Arch [Prop. 30. El. III.], and by drawing ſtrait lines from the angular points of the inſcribed Square, to the points of biſection.

Again, if there be drawn by the angular points of the inſcribed Octagon, ſtrait lines touching the Circle, then ſhall be conſtituted a circumſcribed Octagon, like as before in the Circumſcription of the Pentagon [Prop. 12. El. IV.]. Now again, if the Octagonal Arch be biſected, there may, in like manner, be inſcribed and circumſcribed, a regular Polygon of 16 ſides; and ſo forwards of 32, of 64, &c. infinitely.

Secondly, from the biſection of the Pentagonal Arch, may in like manner be inſcribed and circumſcribed a Decagon: And from the biſection of a Decagonal Arch may be inſcribed and circumſcribed a regular Polygon of 20 ſides, and ſo forward of 40, of 80, &c. infinitely.

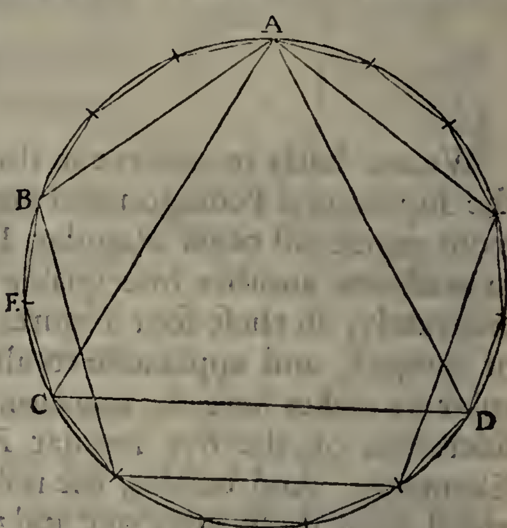
Laſtly, from the biſection of an Hexagonal Arch, may be inſcribed and circumſcribed a Duodecagon: And from the biſection of a Duodecagonal Arch, may be inſcribed and circumſcribed a regular Polygon of 24 ſides, and ſo forward of 48, of 96, &c. infinitely.

In this method therefore, by the biſection of a given Arch, there are from a Square, Pentagon, and Hexagon, conſtructed all Polygons of this kind: Only in the following Propoſition is conſtituted a Polygon of 15 ſides; which, although it be effected by the biſection of an Arch alſo; yet it is in a peculiar and different manner from the forementioned Polygons. For which reaſon it is ſubjoyn'd by *Euclide* to the precedent Propoſitions, to compleat this Element.

PROPOSITION XVI.

IN a given Circle to inſcribe an equilateral and equiangled Quindecagon, or a Figure of fifteen ſides.

Let the given Circle be ABCD. It is required in the given Circle ABCD, to inſcribe an equilateral and equiangled Quindecagon. Let AC be the ſide of an equilateral Triangle inſcribed in the Circle ABCD: and of an equilateral Pentagon let the ſide be AB; therefore of what equal parts the Circle ABCD is fifteen, of ſuch the circumference ABC, being the third part of the Circle, ſhall be five: and the circumference AB, being the fifth part of the Circle, ſhall be three; therefore the remaining Arch BC is two of thoſe equal parts. Let BC be cut into halves in the point E [by Prop. 30. El. III.]; wherefore each of the circumferences BE, EC, is the fifteenth part of the Circle ABCD. If therefore drawing the ſtrait lines BE, EC, we adapt continually, in the Circle ABCD, ſtrait lines equal to them, there ſhall be inſcribed in the ſame an equilateral and equiangled Quindecagon. Which was to be done.



In like manner, as before in the Pentagon, if by the diviſions of the Circle, we draw Tangents to the Circle, there ſhall be circumſcribed an equilateral and equiangled Quindecagon. And moreover, by the like as before ſaid in the Pentagon, we ſhall in a given Quindecagon, Equilateral, and Equiangled, inſcribe a Circle; and alſo circumſcribe.

Advertiſement.

The *Quindecagon* is the only derivative *Polygon* that *Euclide* thought neceſſary to be conſider'd, after the four *Primitive Figures*, namely, a Triangle, a Square, a Pentagon, and an Hexagon: becauſe of its peculiar manner of Conſtruction, from the inſcription of an *Equilateral Triangle*, and *Pentagon* compared together. Yet it may be ſaid, that a *Polygon* of 24 ſides might alſo have been conſtructed in the ſelf ſame manner, from the inſcription of a *Square* and *Hexagon* compared together. But we are to know, that this *Polygon* of 24 ſides ariſes more naturally from the biſection of an *Hexagon* and then of a *Duodecagon*; like as others, from the biſection of a *Square*, or of a *Pentagon*, as is obſerv'd in the foregoing Advertiſement. And therefore *Euclide* judged it inartificial to take notice of it in this place, as he hath done of a *Quindecagon*, which admits of no other way of Conſtruction.

But of the *Heptagon* and *Nonagon*, *Euclide* makes no mention, becauſe, as before, for the inſcription of the *Pentagon*, there was firſt to be inſcribed an *Equicrural Triangle*, having the angles at the baſe double to the angle at the *Vertex*; and then thoſe angles were to be biſected: So, for the inſcription of an *Heptagon*, it is firſt requiſite to inſcribe an *Equicrural Triangle*, having the angles at the baſe triple to the angle at the Vertex; and then to divide thoſe angles into three equal angles.

Again, for the inſcription of a *Nonagon*, it is firſt neceſſary to inſcribe an *Equilateral Triangle*, and then to divide every one of its angles into three equal angles, whereby to ſet forth the Arch of a *Nonagon*, for that thrice 3 is 9; wherefore theſe *Polygons* depend upon the Triſection of an angle. But HOW TO TRISECT any Angle, or Arch, as *Euclide* hath demonſtrated HOW TO BISECT, in Prop. 9. El. I. and Prop. 30. El. III. falls not within the power of the *Euclidéan plain Geometry*, whoſe inſtruments are only *a ſtrait Line*, and *a Circle*, according to the three ſimple *Poſtulata*, laid down at our entrance into the Elements. For every Angle cannot, by the help of a ſtrait line and a Circle only, be divided into three equal Angles. Yet, notwithſtanding the incapability of ſuch a Triſection, *Orontius* and many others, not having a true and full inſight into the nature of this matter, that is, not underſtanding what belongs to Plain, and what to Solid Geometry; what belongs to Magnitudes of two Dimenſions, and what to Magnitudes of three: They have after much toyl, loſt their labour and reputation, therein vexing themſelves with impoſſibilities.

THE FIFTH ELEMENT.

THIS Element depends upon none of the foregoing, but ſtands alone as an *univerſal Matheſis.* It is like Metaphyſics to Natural Philoſophy: a Tranſcendent Element of pure, and prime Mathematics, and ſo much abſtracted not only from Matter in any Subject, but alſo from every particular kind of Subjects, ſo as to be equally applicable to all the Species of Quantity; to the Sciences, Geometry, and Arithmetic; and beſides, univerſally to all other things, which are capable of compariſon, ſuch as Force, and Power in Agents; Intention, and Remiſſion in Qualities; Velocity, and Tardity in Motions; Gravity, and Levity in Ponderations; Modulation in ſounds; Value, and Eſtimation in Things; and whatſoever elſe may admit of any Gradation.

But *Euclide* in a Geometrical method purſues his courſe, and does accordingly apply this Element to Magnitudes: yet in ſuch an artificial and ſubtil Form of Demonſtration, that it might in general be made uſe of whereſoever in the nature of things, the reaſon of Man can compare one thing with an other.

This Doctrine of Proportions cannot be well explained without the uſe of Numbers; and therefore whoever intends rightly to underſtand this Element, muſt come furniſhed with a moderate skill in Arithmetic. We have therefore apply'd Numbers to the Definitions and Propoſitions, for illuſtration ſake to the younger Students.

I ſhould farther adviſe that with the Study of this Element, alſo *Euclid's* Elements of Numbers were together peruſed, eſpecially thoſe Propoſitions where Proportions are concern'd. For the Doctrine of Proportions is chiefly, or rather only explicable by Numbers: and what here is apply'd to Magnitudes, was ſecretly derived from thoſe Elements, which do much further a right underſtanding of this. It will be at firſt ſufficient for Beginners only to read the Propoſitions of thoſe Elements, and carefully to obſerve the *Expoſitions*; which may inſtruct them enough for their preſent uſe in this Element, without giving themſelves the trouble of being convinced by Demonſtrations.

DEFINITIONS.

Of Part and Multiple.

DEFINITION I.

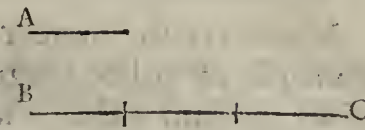
A Part is a magnitude of a magnitude, a leſs of a greater, when the leſs meaſures the greater.

DEFINITION II.

A Multiple is a greater of a leſs, when the greater is meaſured by the leſs.

A Part

A Part and Multiple are each a magnitude of a magnitude; a Part a leſs magnitude of a greater; a Multiple a greater magnitude of a leſs: both combin'd in a mutual reſpect to one another. As A is ſaid to be a part of B C, when repeated ſome certain times, as here 'tis thrice, It doth exactly meaſure, and compleat the magnitude B C.



For a Part is here to be underſtood in a peculiar ſenſe; and not as when it is ſaid that the whole is greater than its part: where a part is taken indifferently for any portion of the whole, as a leſs quantity contained in a greater.

But now in this place, by a Part is meant ſuch a portion of the whole, which repeated meaſures the whole preciſely. And again, in this reſpect the whole is call'd a Multiple of that part, either Duple, Triple, Quadruple, &c. becauſe it contains the ſame juſt ſo many times, as twice, thrice, four times, &c. and is noted thus, $\frac{2}{1}, \frac{3}{1}, \frac{4}{1}$, &c. And the Part which ſo many times repeated, meaſures the whole, is accordingly ſaid to be one half, or one third, or one fourth part, &c. of that whole or Multiple; and is noted thus, $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}$, &c.

Euclide begins with Part and Multiple as a proper foundation of the Doctrine of Proportions, becauſe theſe are in the nature of an Unite and a Number, by which only, the Meaſure, Value, and Proportion of one thing to another can be expreſſed. For as an Unite is a part and meaſure of every Number, and every Number is a Multiple of an Unite; ſo in magnitudes a part is as an Unite, the meaſure of its Multiple; and every magnitude may as a Multiple be divided by equal partitions into meaſuring parts, as Number into meaſuring Unites. Beſides, a Part and Multiple, not only anſwer to Unity and Number, but alſo to Numbers themſelves. For any Number may be a part of ſome other Numbers, and theſe again be Multiples of the ſame. For inſtance, 2 is a part of 12, becauſe 2 taken 6 times meaſures, or makes 12, and is therefore a ſixth part of 12. In like manner 3 is a fourth part, 4 a third, 6 an half of 12. And again 12 is a Multiple of each of theſe: Sextuple of 2, Quadruple of 3, Triple of 4, Duple of 6. So that 2, 3, 4, and 6; tho' each be a Number, yet in reſpect of 12, each being a part of 12, is as an Unite. For 2 is $\frac{1}{6}$, 3 is $\frac{1}{4}$, 4 is $\frac{1}{3}$, 6 is $\frac{1}{2}$ of 12.

But again, 5 is not in this ſenſe a part of 12, becauſe 5 being twice taken makes but 10, and thrice taken makes 15, and ſo added to it ſelf doth not meaſure 12, but is either under, or over it. For the ſame reaſon neither 7, 8, 9, 10, 11, are ſaid to be a part of 12. But in this caſe ſuch a portion of any Number, or Magnitude is called Parts, for that it contains ſome certain and meaſuring parts of the whole, but is not it ſelf a meaſure of that whole: As 8 meaſures not 12; yet becauſe it contains 4, a meaſure and part of 12 ſome certain times, therefore 8 is ſaid to be parts of 12, namely two thirds, or two third parts of 12. Likewiſe 5 meaſures not 12, yet becauſe it doth certain times contain 1, the Monade or Unite, which is the common part and meaſure of all Numbers, therefore 5 is properly ſaid to be parts of 12, as being five Unites of 12 the whole.

And in general *Euclide* hath demonſtrated in Prop. 4. El. VII. That *every leſs Number is of every Greater either a Part, or Parts.*

Such like Quantities both in Number, and Magnitude, are diſtinguiſhed by the names of *Quotal*, and *Quantal* Parts, uſually called *Pars Aliquota*, and *Pars Aliquanta*.

A Quotal part meaſures the whole: which is then called a Multiple of that part.

A Quantal part meaſures not the whole: but repeated is either leſs or greater than it.

From hence we may perceive that a Quotal part is either an Unite, or if a Number, yet uſed as an Unite in the menſuration of the whole. And that a Quantal part is an Aggregate of Quotal parts, which together are not a meaſure of the whole, that is, make not any Quotal part thereof.

Of Equimultiples.

Moreover in Number and Magnitude, when two or more Magnitudes are equally Multiple of other Magnitudes, that is do equally, or equal times (ἰσάκις ſays *Euclide*) contain other Magnitudes, they are then call'd *Equimultiples* of their reſpective Magnitudes, or Quotal parts. As 20 and 12 are Equimultiples of 5 and 3, for that 20 contains 5 four times, as likewiſe 12 does 3, ſo that how many *fives* are in 20, ſo many *threes* are in 12: the multitude or number of the Quotal parts being in both Multiples equal, *viz.* four in each. And therefore all Equimultiples as they equally contain their Quotal parts, ſo are they again equally diviſible into the ſame number of Quotal parts: As 20 into 4 fives, and 12 into 4 threes, which kind of Diviſion is moſt frequently uſed in the Demonſtrations of this Element.

Of Proportions, and Proportionals.

DEFINITION III.

Proportion is an habitude of two Homogeneal Magnitudes unto one another, according to Quantity.

Proportion is by *Euclide* called Λόγος, *Logos*, a word among the *Greeks* of various ſignifications, and commonly Tranſlated *Ratio*, as ambiguous a word as the *Greek*. *Cicero* therefore calls it *Proportio*, a name properly uſed where the conſideration is *what Portion* one thing is of an other.

Proportion therefore in general is an Habitude, Relation, or Compariſon of two things to one another, as of A compared to B, according to ſomething, which is common to them both, or of which they both partake, each in ſome degree of compariſon toward the other. A the firſt of the two is in the ordinary way of ſpeaking the *Antecedent*, likewiſe B the ſecond is called the *Conſequent*, unto which the Antecedent is compared. The Antecedent and the Conſequent, are ſaid to be the Terms of the Proportion, for that in them the Proportion between Antecedent and Conſequent, is bounded and terminated.

In this place Proportion is only conſidered between two Magnitudes; and therefore, as all other things comparable to one another, ſo theſe are alſo to be Homogeneal, that is, of the ſame kind: as a Line to a Line, a Superficies to a Superficies, a Solid to a Solid, is to be compared according to the Dimenſions that each do *in ſua ſpecie* partake of: and therefore the compariſon is to be made according to Quantity, that is, as far as appertains to Quantity: not in reſpect to any Quality, Power, Weight, Motion, Price (as Lead or Gold), or any other Eſtimation whatſoever.

Neither again is Quantity here taken *abſolutely*, or in a Predicamental Notion, as a *Genus* to Continual, and Diſcrete Quantity, to Magnitude, and Multitude. But it is to be underſtood *relatively*, in order to ſuch a *quantitative* Valuation of Magnitude, as where the Quantity of one Magnitude is comparatively to be eſtimated by an other.

In this ſenſe is *Euclide* to be underſtood by κατὰ πηλικότητα, according to Quantity. For the *Greeks* make a juſt diſtinction between ποσότης, and πηλικότης, between Quantity abſolute, as conſidered in its own nature, and Quantity relative, in a reſpect to Menſuration and Eſtimation. The *Latines* uſe only *Quantitas* for both, as *Rectus* for εὐθὺς and ὀρθός; but where proper words are wanting, the ſenſe muſt make out the proper meaning of ambiguous words.

For the better underſtanding this preſent matter, review the Annotations at Def. 2. El. I. concerning the application of Number to Magnitude; where the whole buſineſs about the Quantity of Magnitudes is fully explained. And further obſerve, that the Quantity of every number is ſhewn by the name of the number, as Ten ſignifies ſo many Unites collected into one number under that name: Unity,

Unity, or a Monade being the only prime conſtituent part, and therefore the common meaſure alſo of all numbers, giving the Quantity of every number by a known name in all Languages.

But the Quantity of Magnitudes doth not appear after ſuch an open manner; becauſe every Magnitude, be it never ſo great, or ſo little, is in it ſelf only *unum Integrum*, one Integral thing, and tho' diviſible into infinite parts; yet hath no prime conſtituent part for a common meaſure of Magnitudes, like an Unite in numbers. When therefore it is asked *Quantum*, or *quam Magnum*, how much or how great a thing is? The queſtion tacitely relates to ſome arbitrary meaſure commonly received amongſt us, by which Magnitudes are uſually eſtimated: as an Inch, Foot, Cubit, or the like. And when a Magnitude is ſaid to contain a certain number of ſuch, or ſuch a meaſure; That then is reputed to be the quantity of the ſame Magnitude: but in reality it is its proportion to 1. that is, to its *Geometrical Unite*, and meaſure. If an Inch be put for a general meaſure of Magnitudes, then the *Quantity of a Yard* is ſaid to be 36 Inches, becauſe it contains the ſtated meaſure ſo many times. In like manner a Foot is ſaid to be 12 Inches. But again, if a Yard as a certain length be compared to a Foot as another length, then a Yard ſhall be found to be triple of a Foot: and this is called the *Proportion of a Yard* to a Foot. Likewiſe in general, the quantity of any number is ever according to its name ſo many Unites, becauſe an Unite is the natural conſtituent of all numbers. As the *Quantity* of 12 is always 12 Unites: but in particular compariſons of number to number, the *Proportion* of 12 compared to 4, is ſaid to be triple of 4, or in a triple proportion; and compared to 3 is quadruple. The value of the Antecedent in Proportion being changeable according to the change of the Conſequent; becauſe in ſuch particular compariſons the Conſequent is as a meaſure, by which the Antecedent ought to be eſtimated. For in every Proportion is conſidered how much the Antecedent contains of the Conſequent; the πηλικότης, or *Quantuplum*, what Quantuple the Antecedent is of the Conſequent. For be the Antecedent either equal, greater or leſs than the Conſequent, it is always the *Quantum* of the Conſequent contained in the Antecedent, which gives the proportion of Antecedent to Conſequent. And as the Antecedent contains more, or leſs of its Conſequent, ſo 'tis proportionally valued in a reſpect to that Conſequent. As to give a familiar inſtance, if a Penny be made the meaſure of Mony; then the *Quantity* of a Shilling ſhall always be accounted 12 Pence. But the *Proportion* of a Shilling compared to a Groat, or to a Crown, or to a Pound, is in theſe divers compariſons of a different value; triple of a Groat, a fifth part of a Crown, a twentieth part of a Pound, as containing ſo much of each Conſequent. And in this ſenſe Proportion is ſaid to be an Habitude according to Quantity.

The Diviſion of Proportions.

Proportion is either of Equality, when the Antecedent is equal to the Conſequent; or of Inequality, when greater or leſs.

If the Antecedent be greater, then it is called *Proportion of the greater Inequality*, for that the compariſon is of the greater to the leſs.

If the Antecedent be the leſs, it is called *Proportion of the leſs Inequality*, becauſe the leſs is compared to the greater.

Moreover, becauſe there are many Homogeneal Magnitudes which are incommenſurable to one another (as the ſide of a Square and its Diameter); ſo that their mutual Proportions, or how much one contains of the other, cannot be ſet forth by any common meaſure, nor be expreſſed by any number whatſoever; therefore in magnitudes, Proportion is again divided into Effable and Ineffable, Expreſſible and Inexpreſſible by number: and commonly called proportion Rational and Irrational. This fifth Element is framed with ſuch an artifice as indifferently to comprehend both.

Proportion of Equality is always Rational (tho' the Terms be ſometimes Irrational); for that every thing may have its equal, and be to an other in a Rational account,

account, as 1 to 1. It is alſo the ground from whence all other Proportions do ariſe; and a principal Subject of the preceding Elements, tho' not under the name of Proportion: As that *Vertical Angles are equal to one another*, *the three Angles of a Triangle are equal to two Right*, &c. Beſides infinite other ſuch like Propoſitions throughout all Geometry. It is alſo of a moſt general uſe in Algebra, and the Doctrine of Equations.

Proportions of Inequality which are Rational, are diſtinguiſhed into five kinds of the greater Inequality, and into as many of the leſs.

The Varieties of Rational Proportion.

Of Multiple Proportion, and Submultiple.

The moſt ſimple Proportions of Inequality, are founded in the firſt and ſecond Definitions of Part and Multiple.

If in compariſon of the one to the other, the Multiple be Antecedent and the Part be Conſequent, then it is called *Multiple Proportion.* If the Part be Antecedent and the Multiple be Conſequent, then it is called *Submultiple Proportion.* As 12 compared to 4 is Multiple Proportion, and named triple: And 4 to 12 is Submultiple Proportion, and named Subtriple. The like appellation is uſed in all Multiple and Submultiple Proportions; as Quadruple, Subquadruple; Quintuple, Subquintuple, &c. The other Rational proportions of Inequality, are made by the various Compoſitions of Part and Multiple, as followeth.

Of Multiple Superparticular, and Submultiple Superparticular.

Firſt, if above the exact Multiple of the Conſequent, there remains in the Antecedent any *Quotal* part of the Conſequent, as an half, a third, a fourth, or a tenth part of the Conſequent, (or otherwiſe thus named, a Seſquialteral, a Seſquitertial, a Seſquiquartal, a Seſquidecimal part, &c.) then the proportion is called *Multiple Superparticular*, becauſe the overplus beſides the exact Multiple is a particular and meaſuring part of the Conſequent. As 13 to 4 is in Multiple Superparticular proportion, which is known to be ſo, by dividing the greater by the leſs; where then the Quotient $3\frac{1}{4}$ ſhews 13 to contain 4 thrice, and one fourth part of the Conſequent 4; wherefore this proportion is named triple Seſquiquartal, and is noted thus $3\frac{1}{4}$. So 10 to 4 is in Multiple ſuperparticular proportion duple Seſquialteral $2\frac{2}{4}$, that is $2\frac{1}{2}$: For where the Numerator is a part, that is, a meaſure of the Denominator, dividing the Denominator by the Numerator, and this by it ſelf, it will be brought to an Unite, and the proportion plainly appear to be ſuperparticular, as here $\frac{2}{4}$ is reducible to $\frac{1}{2}$. So in all Superparticulars the Numerator is, or may ever be reduced to an Unite: As 40 to 12 is $3\frac{4}{12}$ or $3\frac{1}{3}$, Triple Seſquitertial. Again, upon tranſverſion of the Terms, the leſs is compared to the greater, and called Submultiple Superparticular: as 13 to 4, inverted, is 4 to 13 *viz.* Subtriple ſeſquiquartal, and is noted thus $\frac{4}{13}$, which ſignifies that 4 the Antecedent, contains four parts of the Conſequent, conſiſting of 13 ſuch equal parts.

Of Multiple Superpartient, and Submultiple Superpartient.

But now, if above the exact Multiple of the Conſequent, the Surpluſage be a *Quantal* part of the Conſequent, then the proportion is called *Multiple Superpartient*, for that the Overplus is not any quotal part of the Conſequent, but ſome quotal parts taken together, which make a quantal part, that meaſures not the Conſequent: As 8 to 3 is in proportion Multiple Superpartient: for dividing the Antecedent 8 by 3 the Conſequent, the Quotient $2\frac{2}{3}$ ſhews 8 to contain 3 twice, and two thirds of the Conſequent 3: therefore this proportion is named Duple ſuperbitertial, and according to the Quotient is noted $2\frac{2}{3}$. So 22 to 8 is $2\frac{6}{8}$ or $2\frac{3}{4}$, Duple ſupertriquartal. For in all Superpartients where the Numerator and Denominator

minator happen to have a common part and meaſure, as here $\frac{6}{8}$ have 2 to each in common; then are they by Diviſion to be reduced to other numbers, which are Prime to one another, that is, have no number for a meaſure common to them both: as $\frac{6}{8}$ is reduced to $\frac{3}{4}$. And again, by tranſverſion comparing the leſs to the greater, the proportion is called Submultiple Superpartient, as 8 to 3, inverted, is 3 to 8, Subduple, and is noted thus $\frac{3}{8}$, which ſignifies, that 3 the Antecedent, contains three parts of the Conſequent, conſiſting of eight ſuch equal parts.

Superparticular and Subſuperparticular.

Now if the Antecedent be not in any manner a Multiple of the Conſequent, but contains the Conſequent *Only once*, and moreover a particular *Quotal part* of the Conſequent, then the proportion is called by the ſingle name of *Superparticular*: as 3 to 2 is in proportion Superparticular, according to the Quotient $1\frac{1}{2}$, which ſheweth 3 the Antecedent to contain 2 the Conſequent once, and one half of two; and is named proportion Seſquialteral: ſo 15 to 12 is $1\frac{3}{12}$, or $1\frac{1}{4}$. For the Numerator 3 being a quotal part of the Denominator 12, $\frac{3}{12}$ is by Diviſion reduced to $\frac{1}{4}$, and the proportion ſhewn to be Seſquiquartal $1\frac{1}{4}$. So in all Superparticular proportions the Quotient is always an Unite with a fraction, whoſe Numerator is likewiſe an Unite, or reducible to an Unite. Again, by tranſverſion comparing the leſs to the greater, as 2 to 3, the proportion is Subſuperparticular, and named Subſeſquialteral, which is thus noted $\frac{2}{3}$: ſhewing that the Antecedent 2 contains two parts of 3 the Conſequent.

Superpartient and Subſuperpartient.

Laſtly, if as before, the Antecedent be not any ways a Multiple of the Conſequent, but contains its Conſequent *Only once*, and moreover *ſome parts* (which together meaſure not the Conſequent) then the proportion is thereupon called *Superpartient*; as 8 to 5 is by the Quotient $1\frac{3}{5}$ ſhewn to be in proportion Superpartient, and particularly Supertriquintal; that is, the Antecedent 8 contains the Conſequent 5 once, and moreover three parts of the Conſequent 5. So 14 to 10 is $1\frac{4}{10}$, or $1\frac{2}{5}$, by diving 4 and 10 by their common meaſure 2, and this is named proportion Superbiquintal. So in all Superpartient proportions the Quotient is always an Unite with a fraction, whoſe Numerator is ever a number: and by this it is diſtinguiſhed from the Quotient of a Superparticular proportion, where the Numerator of the fraction is ever to be an Unite.

For further, note that in this matter of fractions, whenſoever the Numerator can meaſure the Denominator, the ſame may divide it ſelf, and the Denominator; and then ſhall that Numerator be brought to an Unite, and the fraction be Superparticular in its leaſt Terms.

And when both Numerator and Denominator can be meaſured by an other number, then each of them being divided by that common meaſure, the fraction will be Superpartient and expoſed in its ſmalleſt Terms.

Alſo in all fractions, whether of Magnitudes or Numbers, the Denominator is ever a ſuppoſed *Totum*, which conſiſts of ſo many parts, as the Number of that Denominator ſignifies.

Again, to finiſh all the Varieties of Rational proportions; the Superpartient is likewiſe by tranſverſion of its Terms in comparing the leſs to the greater, called Subſuperpartient, as 5 to 8, or $\frac{5}{8}$ is Subſupertriquintal: and 10 to 14, or $\frac{10}{14}$ is Subſuperbiquintal.

Theſe are the five kinds of Rational proportion of the greater Inequality, *Multiple*, *Superparticular*, *Superpartient*, *Multiple Superparticular*, *Multiple Superpartient*. To which anſwer as many of the leſs Inequality, ariſing from the tranſverſion of the ſame Terms, and diſtinguiſhed by adding *Sub* to the other Appellations.

Now here from the Quotients you may obſerve, that all theſe kinds of Rational proportions ariſe from Unity, Part, and Multiple. For one compared to one makes proportion of Equality: One with a part to one makes Superparticular proportion: One with parts to one makes Superpartient. Again, many to one makes Multiple proportion: Many with a part to one makes Multiple ſuperparticular: Many with parts to one makes Multiple ſuperpartient, as theſe Quotients repreſent: which are alſo the Denominators or Exponents of Proportions in the manner following.

The Species.	Superparticular.	Superpartient.	Multiple.
Exponents.	$1\frac{1}{2}$. $1\frac{1}{3}$. $1\frac{1}{4}$. $1\frac{1}{10}$.	$1\frac{2}{3}$. $1\frac{3}{4}$. $1\frac{4}{5}$ $1\frac{3}{10}$.	$\frac{2}{1}$. $\frac{3}{1}$. $\frac{4}{1}$. $\frac{10}{1}$.
Leaſt Terms.	$\frac{3}{2}$. $\frac{4}{3}$. $\frac{5}{4}$. $\frac{11}{10}$.	$\frac{5}{3}$. $\frac{7}{4}$. $\frac{9}{5}$. $\frac{13}{10}$.	$\frac{4}{2}$. $\frac{6}{2}$. $\frac{8}{2}$. $\frac{20}{2}$.

The Species.	Multiple Superparticular.	Multiple Supirpartient.
Exponents,	$2\frac{1}{2}$. $2\frac{1}{3}$. $3\frac{1}{7}$. $4\frac{1}{10}$ $5\frac{1}{4}$. $6\frac{1}{2}$.	$2\frac{2}{3}$. $2\frac{3}{4}$. $3\frac{4}{7}$. $4\frac{2}{3}$ $5\frac{3}{4}$.
Leaſt Terms.	$\frac{5}{2}$. $\frac{7}{3}$. $\frac{22}{7}$. $\frac{41}{10}$. $\frac{21}{4}$. $\frac{13}{2}$.	$\frac{8}{3}$. $\frac{11}{4}$. $\frac{25}{7}$. $\frac{14}{3}$. $\frac{23}{4}$.

Of the Exponent of Rational Proportion.

Now to diſcover unto which of theſe kinds the proportion between any two propoſed Terms (as between 45 and 40) ſhould be referred, we are to reduce thoſe Terms unto two ſuch others, which ſhall be one and the ſame Character and Exponent, common to all poſſible Terms in that proportion, and which therefore muſt neceſſarily be the leaſt and prime Terms of the ſame.

How then to find out the leaſt and prime Terms of any proportion, we are to divide the greater by the leſs; and then the Quotient gives the ſole and proper Terms of that proportion. For *as the Diviſor is to the Dividend, ſo is an Unite to the Quotient: and as the Dividend to the Diviſor, ſo the Quotient to an Unite.* Here therefore the Antecedent or Conſequent being brought to an Unite, the leaſt of Terms, the Quotient is manifeſtly the only common Exponent, and the certain ſtandard of any proportion that can be raiſed from Unity. Which alſo may be the ſame proportion in an infinite variety of ſeveral Terms. As 9 to 3, 12 to 4, 18 to 6, the Quotient 3, that is, 3 to 1, or $\frac{3}{1}$ is the common Exponent of them all. So 45 to 40, 27 to 24, 18 to 16, 9 to 8. Here between theſe ſeveral Terms the Exponent of their proportions is the common Quotient $1\frac{1}{8}$, which ſhews the proportion to be in every one Seſquioctaval. And the like infinitely in this and in all the other kinds of Rational proportions, the Quotient expounds and ſpecifies the proportion. Thus the Quotient is the Exponent of every proportion in its proper *Quantity*, *Species*, and *Name*, which therefore was by the Ancients called Πυθμὴν τοῦ λόγου, *Proportionis Fundum*, the Fundamental proportion, or the proportion in its Fundamental Terms. Whereas then the Quotient is the Exponent of a Proportion, therefore the Notation of the proportion between any two Magnitudes, as A and B, is in *Species* thus properly ſignified $\frac{A}{B}$; that is, A divided by B: which being thus noted $\frac{A}{B}$, ſignifies the Quotient, or Exponent of the proportion between A and B. So $\frac{C}{D}$ is C divided by D, and notes the proportion of C to D. And when the proportions are equal, it is thus repreſented by their Quotients, $\frac{A}{B}=\frac{C}{D}$: when unequal, the greater thus, $\frac{A}{B}>\frac{C}{D}$, and the leſs thus, $\frac{A}{B}<\frac{C}{D}$.

Of Arithmetical Proportion.

Laſtly, there is an other kind of Habitude between two Magnitudes or Numbers κατ᾽ ὑπεροχὴν, according to the *Hpyeroche*, or Exceſs of one above the other: that is, according to the difference in majority, or minority between two Magnitudes or Numbers. As in comparing 6 to 2, or 2 to 6 is conſidered, not as before how much 6 the Antecedent contains of 2 the Conſequent, or two the Antecedent contains of 6 the Conſequent; but how much 6 the greater exceeds 2 the leſs, or 2 the leſs is exceeded by 6 the greater: that is, what is the difference in majority or minority between 6 and 2, or 2 and 6. In both compariſons either of 6 the greater to 2 the leſs, or of 2 the leſs to 6 the greater, the exceſs, or difference is the ſame, namely 4. But now the proportion of 6 to 2 is triple, three to one; and of 2 to 6 is ſubtriple, one to three: And this only kind of Habitude was taken by the Ancients to be λόγος, Proportion.

But the Modern Mathematicians call the Habitude according to Exceſs, or difference by the name of Arithmetical Proportion; and the other defined by *Euclide*, is by them for diſtinction ſake called Geometrical Proportion: altho' both be applicable indifferently to Magnitudes and Numbers: Arithmetical as well to Magnitudes, as to Numbers, and Geometrical as well to Numbers, as to Magnitudes.

Arithmetical Proportion was likely ſo called, becauſe Numbers in their natural order of 1, 2, 3, &c. are not otherwiſe diſtinguiſhed than from their Exceſs, or Difference by an Unite: therefore where the Exceſs, or Difference between two Numbers, or Magnitudes is the thing conſidered, there that Habitude is ſaid to be an Arithmetical Proportion. But it might have been more properly called Proportion of Majority and Minority.

Now in Geometrical Proportion becauſe there is conſidered how much one Term contains of the other, therefore the proportion between the two Terms is ſhewn in the Quotient, by dividing the greater by the leſs.

But in Arithmetical Proportion becauſe there is conſidered the difference between the two Terms, therefore this Proportion of majority and minority, appears in the Remainder, by ſubtracting the leſs from the greater: Diviſion being made uſe of in Geometrical, and Subſtraction in Arithmetical proportions, as the different nature of theſe two kinds of Proportion do ſo require.

DEFINITION IV.

MAgnitudes are ſaid to have proportion to one another, which being multiplied can exceed each the other.

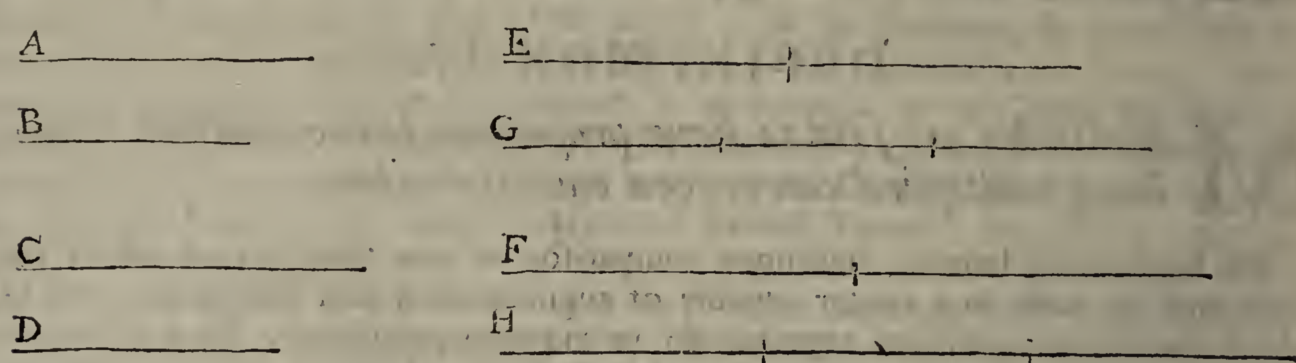
We have noted before, that upon compariſon of one thing to an other, the ſame muſt be made in a matter capable of augmentation and diminution, and of which the things compared together do in common participate. Now in Magnitudes compared according to quantity, this Definition diſcovers when Magnitudes have a participation of one anothers quantities, by this ſignal property, that upon the multiplication of themſelves they can mutually exceed each other: As the ſide of a Square doubled exceeds the Diameter: the Diameter of a Circle quadrupled exceeds the Circumference, and theſe again multiplied can exceed the others. Now by this reciprocal Exceſs, and Comprehenſion of one another, it manifeſtly appears that they fully communicate in each others Magnitudes, and therefore are capable of mutual compariſon according to Quantity. Whereas a finite ſtrait line multiplied never ſo often cannot exceed the Magnitude of an Infinite: The angle of Contact in Circles, tho' infinitely augmented, can never exceed the leaſt ſtraitlin'd angle. Theſe therefore not being to be made comprehenſive of each others Magnitudes by any poſſible multiplication, cannot be ſaid to have proportion to one another. Thus *Euclide* determines in this matter. But ſome farther imagine that he expounds in this Definition what Magnitudes ſhould be accounted Homogeneal. His words import nothing towards ſuch a ſenſe; but rather ſuppoſe that there are Homogeneal Magnitudes incapable of mutual proportion; and therefore gives us here a Touchſtone, by which to know what Homogeneal Magnitudes can be ſaid to have proportion to one another. Moreover in proportion the firſt conſideration is, that the compared Magnitudes be Homogeneal (a word here taken in that ſenſe as vulgarly underſtood): and to avoid Philoſophical diſputes (no ways ſuitable to an Elementary doctrine in Geometry) about angles, or other Magnitudes, what are Homogeneal, and what not, *Euclide* paſſes by this, (as elſewhere the like Controverſies) and without ſuch a conſideration, or determination upon this point, only ſhews in general how to diſcover when any kind of quantities are between themſelves capable of proportion; tho' the ſame be unknown: And of ſuch, on both ſides, capable and uncapable, we have now given ſome clear inſtances.

It is beſides very incongruous, and againſt this whole Elementary method to make a Definition to be prepoſterouſly an interpretation of a word uſed before.

DEFINITION V.

Magnitudes are ſaid to be in the ſame proportion, the firſt to the ſecond, and the third to the fourth, when according to any multiplication whatſoever, the Equimultiples of the firſt, and third compared to the Equimultiples of the ſecond and fourth, are either together Deficient, *or* together Equal, *or* together Exceeding *each the other.*

Let there be four Magnitudes A, B, C, D; where A is compared to B, and C to D. Then let be taken any Equimultiples whatſoever of A the firſt and of C the third, the two Antecedents. As let E be of A, and F of C any Equimultiples, each of each, either duple, triple, quadruple, decuple, centuple, &c. as here they are duple. And again, let be taken any Equimultiples of B the ſecond, and of D the fourth, the two Conſequents, namely, let G be of B, and H of D any Equimultiples, either the ſame as before of the Antecedents, or any other Equimultiples whatſoever, as here they are triple.



Now when ever it is demonſtrated that according to any Multiplication whatſoever, the Equimultiples E and F of the Antecedents A and C, compared to the Equimultiples G and H of the Conſequents B and D, each to each, E to G, and and F to H; when theſe Equimultiples I ſay are proved to be either together leſs, or equal, or Greater, E than G, and F than H; then theſe Magnitudes A, B, C, D, are ſaid to be in the ſame proportion, A to B, as C to D. And therefore when in any one particular inſtance the contrary ſhall be demonſtrated, that the Equimultiple of one Antecedent exceeds the Equimultiple of its Conſequent, and the other exceeds not, but is either equal, or leſs; then thoſe expoſed Magnitudes are not in the ſame proportion, the firſt to the ſecond as the third to the fourth; becauſe the agreement of the Equimultiples in a joint Defect, Equality, or Exceſs, ought to hold in any multiplication whatſoever.

This is a general Test, and an infallible Character of Proportional Magnitudes, Commenſurable, or Incommenſurable. But note, that Incommenſurable Magnitudes can never have their Equimultiples equal; for then they would prove to be Commenſurable to one another; therefore Incommenſurables are ſhewn to be proportional only from the *joint* Exceſs, or Defect of their Equimultiples. Whereas Commenſurable Magnitudes are capable both of a joint equality, and inequality, of their Equimultiples: And in reſpect only to Commenſurable Magnitudes the equality of the Equimultiples was here added, that the Definition might comprehends all kind of Magnitudes; and the three ſorts of Equimultiples according to Defect, Equality, or Exceſs, might anſwer to all poſſible multiplications indifferently, without conſidering whether the Magnitudes be either Commenſurable, or Incommenſurable.

The Notation of Proportionals is in *Species*, or Symbols thus commodiously signified A . B : : C . D ; That is, as A to B, so C to D. Or thus, $\frac{A}{B}=\frac{C}{D}$: that is, the Exponent $\frac{A}{B}$ is equal to the Exponent $\frac{C}{D}$, and shews the proportions to be the same.

DEFINITION VI.

Magnitudes that have the same proportion, let them be called ANALOGALS, *or Proportionals.*

After the Definition, which had now declared what Magnitudes are said to be in the same proportion; there is next a name given to them. Let them be called Ἀνάλογον, *Analogon*, says *Euclide*. This word is used adverbially, and literally translated is *Equiproportionally*: So that it is indifferent to say Magnitudes to be *in the same proportion*, or in one word to be *Equiproportionally*, or *Analogally*, the first to the second, as the third to the fourth. For Ἀνά is a Præposition of equality, and Ἀνάλογον implies ἀνὰ τὸν λόγον: In, or of equal proportion, or the same proportion repeated. From *Análogon* is the abstract *Analogia*, that is equality of proportions, which is commonly rendred *Analogy*, or *Proportionality*.

Annotations on the Fifth Definition.

This Definition all the *Greeks* received without exception, and *Archimedes* himself makes frequent use of it, as the only general and infallible sign of proportional Magnitudes. But many of the Modern Geometricians stumble at it; for want it seems of a through inspection into the bottom of this matter, which the Ancients better understood.

First, Therefore we shall examine upon what motives Men are commonly induced to except against it. *Secondly* we shall explain the nature of this Definition, and what was *Euclid's* intent, and meaning, in defining proportional Magnitudes after such a manner. *Thirdly*, we shall prove that this Definition is most suitable to the nature of Magnitude, and fully satisfactory when rightly understood.

First then the *joint*, or *Simultaneous* Defect, Equality, or Excess in the Equimultiples of Antecedents, and Consequents, which affection *Euclide* puts for the determining of proportional Magnitudes, seems to have no affinity, or conjunction with the usual, and natural notion that Men have of things, which they *account*, and *name* Proportionals: and therefore they are startled at such a strange, and unexpected Definition, so much different from the common Idea they have of this matter. For in all affairs wherein Men are conversant, whether concerning Quantities, Qualities, Powers, Actions, Motions, Value, Commerce, or other Negotiations, they do naturally judge by *Number*, and *Measure*, when things have between themselves the same proportion, and when not. Where then between things compared together they find an agreement according to *Number*, and *Measure*, there such are properly accounted Proportional to one another. As when A is as *much* of B, *as* C is of D, *Tantum*, *Quantum*, the one of the other, then A to B, and C to D, are commonly reputed to be in the same proportion. So that in Proportionals the Antecedents are conceived to be always EQUIQUANTUPLES of their Consequents: and this Equality to be set forth in number, and measure.

Wherefore Men thus rationally conceiving aforehand the condition, and state of things they call proportional, and being thus already acquainted both with the thing, and name, they are thereupon prepossessed in their judgements, and their very prænotion, or natural præcognition of things Proportional, necessarily creates in them a prejudice against *Euclid's* Definition: wherein there appears no relation to number, or measure, nor any coherence with their own Conception of Proportionals.

This seems to have been the chief ground of the Exceptions, that are made againſt *Euclid's* Definition. For the 11th. Definition of the Third Element, where *like Segments of Circles* are defin'd *by the Equality of Angles which they receive*, is as lyabie to be excepted againſt as this Definition of Proportional Magnitudes; becauſe the ſimilitude of Circular Segments appears not from the equality of thoſe Angles: but does properly conſiſt in this, that they are the ſame, or equal portions of their own Circles. But then how to ſet forth, why they are the ſame, that is, to define like Segments of Circles Eſſentially, from the immediate, and formal Cauſe of their likeneſs lyes under the ſame difficulties with the Definition of like Proportions of other Magnitudes. Wherefore in both theſe Caſes *Euclide* is conſtrained to take a ſign of the thing, inſtead of the thing it ſelf, that is, to give a Secondary, or Artificial Definition inſtead of the Prime, and Natural. For in Mathematics Definitions are not always Philoſophical, but Specially framed in order to their Geometrical uſes; as here Def. 5. El. V. is on purpoſe contrived for the Demonſtrations of Proportional Magnitudes. Yet it is to be obſerved that Def. 11. El. III. (tho' of the ſame nature with this) is received without diſpute, becauſe Men being leſs converſant with the true nature of like Segments of Circles, than of like Proportions, come not prepoſſeſſed with any other notions of their own, but do acquieſce in what *Euclide* gives them to underſtand by like Segments of Circles. Whereas they quarrel at this Definition of Proportional Magnitudes, finding themſelves diſappointed in their own thoughts and præconception of Proportionals. We muſt acknowledge the vulgar notion among Men to be moſt agreeable to the nature of Proportionality, and of all things, that can be eſteemed Proportionals. For number alone it is, by which in common Commerce the Meaſure and Value of things is moſt naturally made, and ſignified: but where Number, and Meaſure cannot be apply'd as fit inſtruments for a Mathematical proof in all ſuch things, that are really Proportionals, there for Demonſtration ſake ſome other certain, and definitive mark is to be taken: as *Euclide* hath done in his Definition of Proportional Magnitudes.

But where the Subject would bear it, *Euclide* clears this matter, and gives an other Definition of Proportional Numbers, tho' not of Proportion it ſelf; for that his former Definition of proportion given in this 5th. Element anſwers as well to Numbers, as to Magnitudes, only changing the names, and putting *Numbers* in the place of *Homogeneal Magnitudes*. For in both Numbers and Magnitudes the Habitude is *κατὰ πηλικότητα, according* to Quantity.

Now the 20th. Definition of the 7th. Element concerning proportional Numbers is very plain, and conformable to the common conception that Men have of things proportional, in theſe perſpicuous words.

The Definition of Proportional Numbers.

Numbers are Proportional, when the firſt is of the ſecond, and the third of the fourth, equally Multiple; or the ſame Part; or the ſame Parts.

This Definition is moſt natural, and plainly ſhews Numbers to be proportional from the immediate cauſes, which make them ſo. For in *equally Multiple*, or the *ſame Part*, or the *ſame Parts* is expreſly laid down wherein the Antecedents do equally contain their Conſequents, that is, are each Equiquantuples of their Conſequents. Only 'tis to be noted, that *Multiple* is here to be interpreted in a more general ſenſe than uſual, as not ſtrictly ſignifying preciſely many times, or more than once, for one number to contain an other number, as one to be exactly Duple or Triple, &c. of the other; but numbers are here underſtood to be *Equally Multiple* of others, when the Antecedents do equally, or alike contain the Quantity, that is, the numerical parts of their Conſequents, one as much, or as often as the other does in any manner, As either ſimply once in Proportion of *Equality*: or once with a Part, or Parts in proportion *Superparticular*, or *Superpartient*: or elſe more than once with a Part, or Parts in proportion *Multiple Superparticular*, or *Superpartient*. Here therefore for a farther illuſtration of this matter, take theſe examples: In which we have according to the Form of this Definition given the Varieties of four Rational Proportionals, both particularly in Numbers, and generally in *Species*.

Multiple.

6.2 :: 9.3.

3A.A :: 3B.B.

Multiple Superparticular.	*Multiple Superpartient.*
7.2 :: 21.6.	11.4 :: 33.12.
$3\frac{1}{2}$A.A :: $3\frac{1}{2}$B.B.	$2\frac{3}{4}$A.A :: $2\frac{3}{4}$B.B.
Superparticular.	*Superpartient.*
5.4 :: 15.12.	7.4 :: 21.12.
$1\frac{1}{4}$A.A :: $1\frac{1}{4}$B.B.	$1\frac{3}{4}$A.A :: $1\frac{3}{4}$B.B.
Submultiple.	*Subſuperparticular.*
3.12 :: 5.20.	9.12 :: 15.20.
$\frac{1}{4}$A.A :: $\frac{1}{4}$B.B.	$\frac{3}{4}$A.A :: $\frac{3}{4}$B.B. &c.

Thus in *Multiple*, *Part*, or *Parts*, taken ſometimes *ſingly*, and ſometimes *jointly*, are comprehended all kinds of Rational Proportions, and Proportionals.

Whereas then this Definition evidently ſets forth the nature of Proportional Numbers, which alſo very much correſponds with the common notion of Men; wherein they deem things to be proportional; it may be reaſonably demanded, why ſome ſuch like Definition might not have been accommodated to porportional Magnitudes; ſeeing that the Definition of Proportion anſwers both to Magnitudes, and Numbers? But the reaſons why ſo clear, and natural a Definition could not be apply'd to Magnitudes, are irreſiſtibly cogent, and manifeſt to a Geometrician, tho' not falling within the cognizance of the vulgar.

And to explain this matter, we are to recollect 1ſt. That Proportion is an Habitude according to Quantity. 2ly. That the Quantity of a Magnitude, or Number is only to be ſhewn in its reference, or proportion to ſome *certain*, and *known meaſure*. 3ly. That every Meaſure is ſome one ſimple thing: In Magnitude it is an Inch, or Foot, or Cubit, or any other *ſtated* meaſure: In Numbers it is an Unite; of which it *naturally* conſiſts; therefore as afore ſaid, every ſingle number carries in its very name the quantity of it ſelf; that is, its proportion to an Unite, the common Conſtituent, and natural meaſure of all Numbers: As the quantity of 9 is nine Unites; that is, nine times one, or 9 to 1: The quantity of 10 is ſo many Unites, or 10 to 1. And again, in compariſon of number to number, 9 compared to 10, is nine parts of 10, or the proportion of 9 to 10. The quantity of 5 to 10 is an half part of 10, or 5 to 10; Every leſs number compared to a greater being apparently a part, or ſome parts of the greater.

But Magnitude (tho' infinitely diviſible into parts) is not, as Number, an Aggregate of certain parts. It hath no ſuch fundamental beginning, nor any *Original Part*; *No Minimum*, or *Geometrical Unite* from which to demonſtrate in an Elementary method the equality, and inequality of Proportions; as *Euclide* hath clearly delivered the whole matter in his Elements of Numbers.

What of this kind Magnitude hath, or can have, is only arbitrary, and variable after the cuſtoms, and uſages of ſeveral Nations, and People, which therefore cannot give a natural foundation to the Geometrician, whereon to build a general doctrine of proportional Magnitudes upon the ſame grounds, or ſuch a kind of Definition from Multiple, Part, and Parts, on which that of numbers is ſettled, and Demonſtrations framed accordingly.

And altho' there be but one True, and the ſame eſſential notion of Proportionality both in Magnitude, and Number, which conſiſts in *this*, that *the Antecedents are Equiquantiples of their Conſequents*; which in numbers is plainly, and demonſtratively to be ſet forth from an Unite, *the natural Minimum* of all numbers: yet in Magnitudes, becauſe there is not a *natural Minimum*, the ſame, or like method cannot poſſibly be uſed.

Again,

Again, besides this main impediment, founded in the very nature of Magnitude, there is another more invincible obstacle of Incommensurability among Magnitudes, which renders Incommensurable Magnitudes utterly uncapable of any imaginary common Part, or that one can ever be made a Multiple of the other; and therefore that plain, and natural Definition of proportional Numbers can be no ways applicable to all Magnitudes, whereby to demonstrate that they are proportional to one another.

Euclide therefore is necessitated to seek for a more general Character of Proportionals, which may serve for all Magnitudes, Commensurable, and Incommensurable. And tho' he is in Magnitudes Incommensurable to one another deprived wholly of any helps from a common Part, or that one can ever be made a Multiple of the other; which are the fundamental instruments of demonstration in the Doctrine of Numbers: yet because all Magnitudes Commensurable, and Incommensurable, may have *any Multiples of themselves*, as duples, triples, quadruples, &c. and so by consequence *any Equimultiples*; for that any two Magnitudes may each be alike doubled, tripled, quadrupled, &c. Therefore *Euclide* has recourse to such a Multiplication, as the only, or at least the most commodious remedy left in this matter, and searches how far an Equimultiplication of the Antecedents, and an Equimultiplication of the Consequents might contribute toward a sure and certain discovery what Magnitudes (whether Commensurable or Incommensurable to one another) are proportional, and what not, when they are compared together; which proved very useful to his purpose.

§. 1. For *Euclide* found demonstratively in any four proportional numbers (and in no other but proportional numbers), that if the Antecedents be equally multiplied according to any multiplication whatsoever, and again the Consequents be equally multiplied according to any multiplication whatsoever; then shall the Equimultiples of the Antecedents be proportional to the Equimultiples of their Consequents in some one, or other proportion for ever. So those that are once Proportionals shall in this manner ever be some Proportionals.

This singular Power, and Property of four Proportionals, always to make from such a multiplication other four Proportionals, is the foundation of *Euclid's* Definition of proportional Magnitudes; as will clearly appear in the following Paragraphs.

§. 2. For because all proportions are either of equality, or of the greater, or lesser inequality; therefore in any four Proportionals if one of the Antecedents be either equal to its Consequent, or greater, or less; the other Antecedent also must accordingly be either equal, greater, or less than its Consequent. For if one Antecedent should be equal to its Consequent, and the other unequal, or the one be greater, and the other less, it is an immediate contradiction, and impossibility to suppose equals to be to one another in the same proportion with unequals; or the greater to the less to be in the same proportion with the less to the greater.

§. 3. Seeing then that of four Proportionals the Equimultiples of the Antecedents, and Consequents taken according to any multiplication whatsoever, are found to be always Proportionals (*as it is noted in the first Paragraph*): therefore the Equimultiples of the Antecedents are together equal, or together greater, or together less than the Equimultiples of the Consequents (*by the foregoing Paragraph*). When therefore it shall be demonstrated that according to any multiplication whatsoever the Equimultiples of the Antecedents [*viz.* of the first, and third Terms] are together equal, greater, or less than the Equimultiples of the Consequents [*viz.* of the second, and fourth Terms], then are the Antecedents themselves in the same proportion to their Consequents, the first to the second, as the third to the fourth, according to the 5th. Definition.

Thus we see that *Euclid's* Characteristic property of Proportional Magnitudes is but one single remove from the immediate, and apparent Cause of the same, as it hath been laid down *in the first Paragraph.* For four Magnitudes are said by *Euclide* to be Proportionals, when the Equimultiples of the Antecedents are together either equal, greater, or less, than the Equimultiples of their Consequents, according to any multiplication whatsoever; Because according to any multiplication whatsoever the Equimultiples of the Antecedents are always proportional to the Equimultiples of their Consequents. But this intermedial Proposition was concealed by *Euclide*, and in silence passed over: as Geometricians use to do, who are not obliged to give the reason of their Definitions. They are only to be privately assured of the truth of that property, which is attributed to the thing: and it lyes wholly on our part either to accept it, or to make manifest some unaptness, or insufficiency in the use of it. Now for the matter in hand, *Euclid's* Definition of Proportional Magnitudes hath stood the tryal of about two thousand years unshaken: and altho' the Antecedents, and Consequents multiplied according to any multiplication whatsoever, seem in words to make this thing perplext, and troublesome (as *Tacquet* unjustly calls it a Labyrinth) yet in use, and practice it is most plain and easie, as we shall find in the first, and last Propositions of the 6th. Element.

If it be farther demanded why *Euclide* did not define Magnitudes to be proportional, when the Equimultiples of the Antecedents were porportional to the Equimultiples of the Consequents; but rather when the Equimultiples of the Antecedents were together either equal, greater, or less than the Equimultiples of the Consequents, seeing that this is only a Result of that; the reason is apparent. For to define four Magnitudes to be proportional, when the Equimultiples of the Antecedents are proportional to the Equimultiples of the Consequents, is to define the same thing by it self, *Ignotum per æquè Ignotum*; because we are still as much to seek how to know that these Equimultiples are proportional, which was the gross oversight of *Campanus*, and *Orontius* in this matter. But how to know when the Equimultiples of the Antecedents are together either equal, greater, or less than the Equimultiples of the Consequents, each Equimultiples being taken according to any multiplication whatsoever, can be readily made out, and be at once demonstrated by some known Geometrical Proposition: As may be found in all the Ancient *Greek* Geometricians, with what facility it is apply'd upon any occasion of demonstrating Magnitudes to be Proportionals: Therefore *Euclide* gave this general Concomitant of any four Proportionals for an infallible Indicant, and the definitive Character of Proportional Magnitudes.

And for a farther Explanation of this matter, we shall here give a demonstration of the Fundamental Proposition mentioned in the first Paragraph, upon which *Euclid's* Definition of Proportional Magnitudes is grounded, and from whence it doth most naturally arise.

Of four Proportionals the Equimultiples of the Antecedents are proportional to the Equimultiples of the Consequents, according to any Multiplication whatsoever.

Let the four Proportionals be A . E : : B . C . and let N denote any number multiplying the Antecedents A, B, and let *n* denote any number multiplying the Consequents E, C. I say, that

☿ A in N . E in *n* : : B in N . C in *n*.

For let it be A . E : : B . C . and then Alternately. Prop. 13. El. VII.
☿ A . B : : E . C . Therefore
A . B : : A in N. B in N. and
E . C : : E in *n*. C in *n*. } * Prop. 17. El. VII.
☿ But A . B : : E . C. Therefore
A in N. B in N : : E in *n*. C in *n*. and Alternately
☿ A in N. E in *n* : : B in N . C in *n*. Which was to be demonstrated.

* *If a Number multiply two Numbers the Products are in the same proportion with the given Numbers.* This is next to a common notion, for who naturally sees not that as *three* to *two*, so 4 *threes* to 4 *twoes*? And upon this obvious Principle *Euclide* multiplied the Antecedents by any one number whatsoever, and again the Consequents by any one number whatsoever; from whence by one, or two easie steps he found (as we see) his Definition of Proportional Magnitudes.

Thus from the very natural Power of Proportionality to make of any four Proportionals, other four Proportionals for ever; whose Antecedents therefore shall also ever be together either equal, greater, or less than their Consequents, *Euclide* puts this necessary Adjunct for the general, and next immediate Indicant of all Proportionals.

For whether the proportions be Rational, or Irrational; the Magnitudes Commensurable, or Incommensurable, there is this agreement in all things conceived to be Proportionals, that the Antecedents do equally, or alike contain certain quantities of their Consequents; altho' this Equality, or Similitude cannot be declared, and signalized by Equimultiples, or the same Part, or the same Parts of the Consequent, as it is in Numbers; yet in regard that Proportionality is every where of the same nature, therefore one common Property is by *Euclide* justly applied to all for a standing Indicant of any four Proportionals.

For the side of one Square is the same portion of its Diameter, that the side of an other is of its Diameter, as truely and really, as if it were explicable by Part, or Parts, by Number, or some common Measure: So that in Magnitudes Incommensurable, there is something in nature, which is Equivalent, and *Tantamount*. For the Square of the side is demonstrated to be equal to half the Square of the Diameter; and therefore the sides of all Squares are manifestly Equiquantuples, and the same portions of their own Diameters, and so in the same proportion to them: the nature of Proportionality, or Equiquantipleness between Antecedents, and Consequents, being no ways altered in having the proportions either effable, or ineffable by number, and measure. For the Affections, and Properties of Proportionals, are the same in Numbers, and Magnitudes, only they cannot be demonstrated after the same manner.

And therefore to require a Definition of Proportional Magnitudes (answerable to that of Numbers) from which to demonstrate by some certain measures in what manner the Antecedents do equally contain their Consequents, in the making of equal proportions, where between the Antecedents, and Consequents there are *in nature no such measures*, is an impossibility unreasonably required of *Euclide* to make possible. But what is possible, he performs: and gives a Definition, or rather a Definitive mark, which certainly shews when Magnitudes are Proportional, tho' not *Why*, and in what particulars they are Proportional: and in this Case to demand of him a demonstration of his own Definition is *Iniquissimum mandatum*, repugnant to the doctrinal methods of all Arts, and Sciences. It had been a business proper enough for a *Scholiast*, or Commentator to illustrate, and as a Philosopher, to set forth the causes of the different Definitions given by *Euclide* of proportional Numbers, and of proportional Magnitudes, from the different Natures and Constitutions of Number, and Magnitude: but this had been improper for the Elementator himself to submit unto.

In short therefore, to summ up this whole and much controverted matter. Forasmuch as it hath been shewn, that Magnitudes are uncapable of being made known, how they are in the same proportion to one another from *any Equimultiples, or the same Part, or the same Parts of the Consequents*, according to the Definition of proportional Numbers; therefore instead of such a Primary, and Essential Definition, *Euclide* laid hold of the next immediate Property that he found to spring from the natural Power of proportionability, and might produce an infallible Character of all Proportionals: which is thus apparently deduced.

Of

Of all Proportionals, and only of Proportionals any whatſoever Equimultiples of the Antecedents, ſhall ever be Proportional to any whatſoever Equimultiples of the Conſequents.

Wherefore of all Proportionals, and only of Proportionals any whatſoever Equimultiples of the Antecedents ever are together either equal, greater, or leſs, than any whatſoever Equimultiples of the Conſequents.

And therefore where any whatſoever Equimultiples of the Antecedents are demonſtrated to be together either equal, greater, or leſs, than any whatſoever Equimultiples of the Conſequents, there the firſt expoſed Magnitudes were Proportionals: The Antecedents Equimultiples of their Conſequents, each of each Equiquantuple in any proportion whatſoever, Rational or Irrational.

Thus now this general Teſt, and property of Proportionals, that by ſome of our Moderns hath been calumniated for being ſo remote and intricate, is in brief ſhewn to be nearly conjoyn'd with the nature of whatſoever Quantities are, or can be eſteemed Proportionals, and laid open in few words.

If any thing yet remains of diſſatisfaction, it lyes unavoidably in the nature of Magnitude it ſelf, which hath no *Minimum*, and therefore no natural, or ſtanding Meaſure; where alſo incommenſurability between Magnitudes is not capable of any imaginable common Meaſure; or of any better evidence than what *Euclide* hath given in his Definition of Proportional Magnitudes: and not in any deficiency either in Geometry, or the Geometrician.

DEFINITION VII.

When of Equimultiples the Multiple of the firſt ſhall exceed the Multiple of the ſecond; and the Multiple of the third ſhall not exceed the Multiple of the fourth; then the firſt is ſaid to have to the ſecond a greater proportion, then the third hath to the fourth.

In Def. 5. Magnitudes were determined to be in the ſame proportion, when the Equimultiples of the Antecedents ſhould together be either equal, greater, or leſs than the Equimultiples of their Conſequents, according to any multiplication whatſoever; and therewithal 'twas neceſſarily implied, that when it ſhould be found otherwiſe in any one multiplication, then thoſe Magnitudes were not in the ſame proportion to one another.

Now this Definition farther declares which Antecedent hath to its Conſequent the greater proportion, and which the leſs.

As for inſtance, let A 4, be compared to B 3; and C 6, to B 5: then let be taken Equimultiples of the Antecedents A 4, and C 6; namely *quadruples* E 16, and F 24. And again, take of the Conſequents B 3, and D 5; other Equimultiples; namely *quintuples* G 15, and H 25. Here 16 the *quadruple* of the Antecedent 4 exceeds 15 the *quintuple* of the Conſequent 3: but 24 the quadruple of the Antecedent 6, exceeds not 25 the quintuple of the Conſequent 5. Therefore the Antecedent A 4, is ſaid to have to the Conſequent B 3, a greater proportion, than the Antecedent C 6, has to the Conſequent D 5: and accordingly the Quotients ſhew the ſame inequality of theſe proportions. For 4 to 3 is $1\frac{1}{3}$, or ſeſquitertial: and 6 to 5 is but $1\frac{1}{5}$, or ſeſquiquintal.

When therefore in Magnitudes, or Numbers ſuch a diſagreement is demonſtrated to be between the Equimultiples of the Antecedents, and of the Conſequents in

 equality,

equality, exceſs, or defect upon any one multiplication; then as they are not in the ſame proportion by Def. 5; ſo this Definition expounds which Antecedent hath to its Conſequent the greater proportion, and which the leſs.

But again it is to be obſerved, that Magnitudes, or Numbers may alſo have an agreement between the Equimultiples of the Antecedents and Conſequents in equality, exceſs, or defect, according to ſome multiplications; yet not according to all, as it is required in this matter of Proportionals.

For in the former inſtance take of the Antecedents 4 and 6, the quadruples 16 and 24; and of the Conſequents 3 and 5, the triples 9 and 15. Here of the Antecedents 4 and 6, their Equimultiples 16 and 24, exceed 9 and 15 the Equimultiples of the Conſequents 3 and 5, each exceeding each, 16, 9, and 24, 15: yet the proportion of 4 to 3 and of 6 to 5, is not the ſame in both, as was now before ſhewn.

Wherefore tho' in ſome multiplications of Diſproportionals, the Equimultiples of the Antecedents and Conſequents may happen to agree in a joint equality, exceſs, or defect: yet if any one multiplication ſhall be demonſtrated to diſagree, then by Def. 5. theſe Equimultiples did not ariſe from four Proportionals; for the reaſon before given that Proportionals ſhall ever make the Equimultiples of the Antecedents proportional to the Equimultiples of the Conſequents; and be therewithal either together equal, or greater, or leſs according to any multiplication whatſoever: which is the given Signal of Proportionals.

DEFINITION VIII.

A*Nalogy, or Proportionality is a ſimilitude of Proportions.*

Magnitudes, which have the ſame proportion were before in Def. 6. named *Analogals*, or Proportionals. Now again in the Abſtract is here defined Analogy. And what in the compariſon of Magnitudes was called *Equality*, that in the compariſon of proportions this Definition calls Similitude: So that to have *the ſame* proportion or *like*, or *equal*, ſignifies all one thing. Yet *Euclide* never uſes to ſay Magnitudes in *like*, or *equal* proportion; but always Magnitudes in the ſame proportion, or having the ſame proportion.

This Definition therefore ought farther to be examined, which is more ſuitable to the ſpeculations of a Philoſopher, than of any uſe to a Geometrician. For firſt, in Def. 3. *Euclide* tells us what is proportion; next in Def. 4. what Magnitudes are capable of proportion to one another; then in Def. 5. what Magnitudes are in the ſame proportion, and therewithal what are not. Again in Def. 6. to Magnitudes in the ſame proportion he gives a name; and laſtly Def. 7. ſhews when Magnitudes are not proportional, where the proportion is the greater, and where the leſs. And thus he having finiſhed his Explanations of Proportions, Proportionals, and Diſproportionals; it is manifeſtly to no purpoſe to define again Analogy to be a ſimilitude of Proportions: which bare Definition neither ſhews wherein it conſiſts, nor by any ſign to know when there is between two proportions a ſimilitude, that ſome uſe might be made thereof in the following Demonſtrations. But for this end a Sign and Character of Proportionals was given before in Def. 5. Plainly then this is a ſuperfluous Definition, a remark of ſome Scholiaſt ſhuffled into the Text without any order, and to no uſe. The next that follows is likewiſe a By-note to as little purpoſe, and both evidently interrupt the Coherence of *Euclid's* genuine Definitions.

DEFINITION IX.

A*Nalogy is in three Terms at the feweſt.*

This is ſo far from being one of *Euclid's* Definitions, that it is moſt clearly no

Definition at all. Besides in strictly speaking it is not true. For Analogy is always in four Terms, two and two in the same proportion, Antecedent to Consequent, and Antecedent to Consequent: tho' it may sometimes be only between three Magnitudes, or Numbers expressed and exposed: As let there be three Magnitudes A, B, C, in the same proportion A to B, as B to C; that is, in whatsoever proportion A is to B, in the same again is the self same B to C. Here are in reality only three Magnitudes, but between them two proportions, and therefore two Antecedents, and two Consequents: so of necessity there are always four Terms in every Analogy, the first to the second, and the third to the fourth; and therefore when the Magnitudes are only three, the middle is necessarily understood to supply the second, and third Term, and the like in Numbers. Now forasmuch as two equal proportions may sometimes be in three, sometimes in four Magnitudes, or Numbers, therefore equal proportions are distinguished into *Continual*, and *Discrete*.

Of Continual Proportion.

When Magnitudes, or Numbers in any proportion whatsoever are continued in the same proportion, by an immediate Coherence of Terms with one another, each intermedial being twice taken, first as a Consequent to the preceding, next again as an Antecedent to the following, then this is called Continual proportion: as A to B, so B to C, so C to D, and is thus noted, A, B, C, D ∺ And the Magnitudes, or Numbers are said to be ἑξῆς ἀνάλογον, *Deinceps proportionales*, continuedly Proportionals.

Of Discrete Proportion.

Again, when the intermedial Magnitudes, or Numbers are not continued, or twice taken, as a Consequent, and an Antecedent, but the proportions are as A to B, so C to D (noted thus A . B :: C . D); yet *so* is not B to C; the intermedials B and C being discontinued, and separated from each other in that proportion, then this is called Discrete proportion.

So that a Proportion is said to be continual, when the Proportions (whether two, or three, or more) are the same, and exposed in continued Terms. As A to B, so B to C, so C to D: Or in numbers, as 16 to 8, 8 to 4, 4 to 2.

And Discrete Proportion is, when the Proportions are likewise the same, but exposed in discontinued Terms. As A to B, so C to D, so E to F: Or in numbes, as 12 to 8, 6 to 4, 3 to 2.

Of Concatenate Proportions.

There are also Concatenate Proportions, ἑξῆς λόγοι, *Proportiones deinceps*; when the proportions are different (not all the same, as in Continual, and Discrete proportions) yet are exposed in continued Terms, ἐν ἑξῆς ὅροις. As 15 to 10, 10 to 6, 6 to 3, are to be called *Concatenate* Proportions.

Therefore we are carefully to distinguish between Continual Proportion, and Proportions in continued Terms. By continual Proportion is always meant the same, or equal Proportions repeated orderly in continued Terms. By the other is to be understood various, and different proportions, which likewise follow one another in continued Terms. As 4 to 3, 3 to 2 are Concatenate proportions: But 8 to 4, 4 to 2, are to one another in Continual proportion: as 8 to 4, 6 to 3, are in Discrete proportion.

It is farther worthy of observation that in Discrete proportion, because the intermedials break off one from the other, therefore an Analogy may be here between Magnitudes of different kinds, so that a Line may be to a Line, as a Superficies to a Superficies, or a Solid to a Solid. But in Continual proportion, because all the Terms from the first to the last are continued in the same proportion to one another, therefore they must all be Homogeneal, for that Heterogeneals can have no proportion one to an other: therefore in Continual proportion the

Terms are either all Lines, or all Planes, or all Solids. Likewiſe in Concatenate proportions where the proportions are put various, and different; yet are in *Continued Terms* Concatenated, as A to B in one proportion, B to C in an other, C to D in ſome other, there the Magnitudes muſt alſo for the ſame reaſon be all Homogeneal, like as they were in Continual proportion.

DEFINITION X.

WHen three Magnitudes are Proportional, the firſt is ſaid to have to the third a Duplicate Proportion of the firſt to the ſecond.

Let A, B, C, be three proportional Magnitudes, as A the firſt to B the ſecond, ſo B the ſecond to C the third. From which continuation of proportions the proportion of A the firſt to C the third is underſtood to be compounded of the two intermedial proportions, *viz*. of the proportion of A the firſt to B the ſecond, and of the proportion of B the ſecond to C the third: and therefore A the firſt is ſaid to have to C the third a Duplicate proportion of A the firſt to B the ſecond; becauſe the proportion of A to B is the ſame with that of B to C, and ſo is doubled, or twice continued between the porportion of A the firſt to C the third. But if the two intermedial proportions be not the ſame, and one of them be different from the other, as 4 to 3 in proportion Seſquitertial, and 3 to 2 in proportion Seſquialteral, then 4 the firſt cannot be ſaid to have to 2 the third a Duplicate proportion of 4 the firſt to 3 the ſecond, as in equal proportions; but yet in reference to this Definition, it is ſaid by *Euclide* at Prop. 23. El. VI. to have a proportion compounded of 4 to 3, and of 3 to 2, by which the extent of this tenth Definition, and *Euclid's* meaning is plainly declared. So that whether the intermedial proportions be equal, or unequal, the proportion of the extremes is in general underſtood by *Euclide* to be compounded of the intermedial proportions: And when the intermedial proportions are put the ſame, then the proportion of the firſt to the third, is in Special called by *Euclide* a Duplicate porportion of the firſt to the ſecond.

Now to illuſtrate this farther in Numbers, let the three Proportionals be 9, 3, 1, The proportion of 9 to 1 is ſaid to be a duplicate proportion of 9 to 3, becauſe between 9, and 1 the proportion of 9 to 3, or triple proportion is twice continued. For 9 is triple of 3, and 3 of 1: ſo that the Noncuple proportion of 9 to 1 is ſaid to be triple proportion duplicated; that is, to be compounded of two triple proportions, which continuedly intervene between 9, and 1.

DEFINITION XI.

WHen four Magnitudes are proportional, the firſt is ſaid to have to the fourth a triplicate proportion of the firſt to the ſecond: And ſo forward always more by one, as long as the Analogy ſhall be contiuued.

Let A, B, C, D, be four proportional Magnitudes as A to B, ſo B to C, ſo C to D, in continual proportion, then A the firſt is ſaid to have to D the fourth, a triplicate proportion of A the firſt to B the ſecond; becauſe between the Term A, and D, the ſame proportion of A to B is thrice iterated.

Likewiſe if there be a Series of five Terms A, B, C, D, E, the proportion of A the firſt to E the fifth, is ſaid to have a quadruplicate proportion of A the firſt to B the ſecond, and ſo forward in a continuate Chain the proportion of the extremes has its name from the number of the intermedial proportions; Duplicate from two, Triplicate from three, &c. which proportion of the extremes is alſo ſaid to be compounded of thoſe intermedial proportions.

When

When equal proportions are continued in many Terms, it is usually called Progression, or Geometrical Progression.

DEFINITION XII.

Homologal Magnitudes are Antecedents to Antecedents, and Consequents to Consequents.

When four Magnitudes as A, E, B, C, are said to be Proportionals, it is farther requisite to set forth distinctly the *Analogism*, or order of these proportional Magnitudes in their proper Terms; *that is*, to declare which are the Antecedents, and which the Consequents, and besides which Antecedent belongs to each Consequent: whether A to E as B to C, or otherwise. If it be put, as A to E so B to C, then A, and B the Antecedents are called Terms *Homologal*: and E and C the Consequents are likewise Terms *Homologal*; that is, *Comproportional*, or Terms of the same condition as *Conjugate* or *Consociate Pairs* in the order, and disposition of four Proportionals. The Antecedents being the Terms compared, and the Consequents the Terms to which is made the comparison.

Now when four Magnitudes are proportional, and the Analogism; *that is*, the *Homologal* Terms are orderly set forth, as A . E :: B . C, *Euclide* shews in the following Definitions what changes, and besides what alterations the four primary, or first proposed Proportionals are capable of; so that the changed, or altered Terms may still be proportional to one another. These variations are made five manner of ways.

DEFINITION XIII.

Alternate Proportion is a Sumption of the Antecedent to the Antecedent, and of the Consequent to the Consequent.

Here is made only a change in position of the primary Proportionals, without any alteration of the same Terms, and the Sumption, or Comparison of one Antecedent to the other as a Consequent; and of one Consequent as an Antecedent to the other Consequent is called Alternate proportion: As let the primary Proportionals be A to E, as B to C, and thus set forth.

15 . 5 :: 12 . 4.
Primary Proportionals, A . E :: B . C.
then *Alternately*, A . B :: E . C.
15 . 12 :: 5 . 4.

The Analogy between the Terms of Alternate proportion is demonstrated in Prop. 16.

DEFINITION XIV.

Inverse Proportion is a Sumption of the Consequent as Antecedent, to the Antecedent as Consequent.

Here the Proportional Terms are otherwise changed in position only, the Consequents into Antecedents, and the Antecedents into Consequents: The inward Terms into the outward, and the outward into the inward: As let there be again

15 . 5 :: 12 . 4.
Primary Proportionals, A . E :: B . C.
then *Inversly*, E . A :: C . B.
5 . 15 :: 4 . 12.

We may observe, that in Alternate proportion only one Antecedent is changed

changed into one Consequent, and one Consequent into one Antecedent. But in inverse proportion, both the Consequents are made Antecedents; and both the Antecedents are turned into Consequents, whereby proportions of the greater, and less inequality are transmuted into one another.

It is likewise Inverse proportion when the Terms are taken backward, or contrarily, as if A . E : : B . C. It is likewise between the Terms inversly C . B : : E . A. The Analogy of inverted proportion is demonstrated by a Corollary of Prop. 4.

DEFINITION XV.

Composition of proportion is a Sumption of the Antecedent together with the Consequent as one, to the same Consequent.

Here now is made an alteration of the primary, or first proposed Terms, by adding each Antecedent, and Consequent, together as one Term, and so compared to its correspondent Consequent. For let here be as before,

	15 . 5 : : 12 . 4.
Primary Proportionals	A . E : : B . C.
then by *Composition*,	A + E . E : : B + C . C.
	15 + 5 (=20) . 5 : : 12 + 4 (=16) . 4.

The Analogy between the Terms of Compound proportion is demonstrated in Prop. 18.

DEFINITION XVI.

Division of proportion is a Sumption of the excess, wherein the Antecedent exceeds the Consequent, to the same Consequent.

Here again is made another alteration of the primary Terms, by subtracting each Consequent from its Antecedent, and then each excess is assumed as an Antecedent, and compared to its proper Consequent: As let there be

	15 . 5 : : 12 . 4.
Primary Proportionals	. . . . . A . E : : B . C.
then by *Division*,	A − E . E : : B − C . C.
	15 − 5 (=10) . 5 : : 12 − 4 (= 8). 4.

The Analogy between the Terms of Divided proportion is demonstrated in Prop. 17; and 'tis manifest, that divided proportion must be of the greater inequality, for that the Consequent is to be subtracted from the Antecedent.

What is commonly translated Composition, and Division of proportion, and called by *Euclide Synthesis*, and *Diæresis*, is only an addition, and subtraction of Proportional Terms: an addition of each Consequent to its Antecedent, and a subtraction of each Consequent from its Antecedent. But Composition, of which we have made some mention in the Annotations on the 10th and 11th foregoing Definitions, is taken in a far different sense, and called Σύγκεισις, *Synkeisis*, a Composition, or Commixture of Proportions one with another, and not a *Synthesis*, or addition of the Terms to one another: As shall be farther explained at the 5th Definition of the 6th Element.

DEFINITION XVII.

Conversion, or ANASTROPHE *of proportion is a Sumption of the Antecedent to the excess, wherein the Antecedent exceeds the Consequent.*

15 . 5 : : 12 . 4.
As let it be ⎰ . . . . A . E : : B . C.
Then by *Converſion*, ⎱ A . A—E : : B . B—C.
15 . 15—5(=10) : : 12 . 12—4 (=8).

The Analogy between the Terms of *Converted* Proportion is demonſtrated by the Corollary of Prop. 19th.

Again, the Conſequent alſo might have been compared to the ſame exceſs, as E . A — E : : C . B — C . 5 . 10 : : 4 . 8. But this is only proportion of Diviſion *inverted*, and inverſion of proportion having been laid down before; it were inartificial to repeat the ſame again as a new kind of change. For it is to be noted that the ſimple, or primary Proportional Terms being either compounded, divided, or converted, may in like manner be again changed by Alternation, and Inverſion. As

by Compoſition, and Alternation, A + E . B + C : : E . C.
by Diviſion, and Alternation, A — E . B — C : : E . C. &c.

Again, if the Terms be only three in continual proportion, as A, M, E ∸, 8, 4, 2, they may in like manner be changed, and altered, for that M the middle Term is both a Conſequent, and Antecedent. As

8 . 4 : : 4 . 2.
A . M : : M . E . and
by Compoſition, and Alternation, A + M . M + E : : M . E.
by Diviſion, and Alternation, A — M . M — E : : M . E. &c.

All theſe changes, and alterations of four Proportionals may in a view be thus ſet forth.

The primary Proportional Terms.

	A . E : : B . C.	
Alternation,	A . B : : E . C.	The primary Terms changed only in poſition.
Inverſion,	E . A : : C . B.	
Compoſition,	A+E . E : : B+C . C.	The primary Terms altered by Addition, and Subtraction.
Diviſion,	A—E . E : : B—C . C.	
Converſion,	A . A—E : : B . B—C.	

Here are firſt put four primary Terms, two, and two in the ſame proportion, as A to E, ſo B to C, from which are made by five ways of variation four other proportional Terms, two, and two in ſome one, and the ſame proportion to one another.

But now beſides theſe, there may be put after another manner more Terms then four; as five, or ſix, or ſeven, &c. in two diſtinct orders, with an equal number of Terms in each order, two, and two in the ſame proportion, from which are alſo made four other Terms proportional to one another, as this following Definition declares.

DEFINITION XVIII.

Proportion of Equidiſtance (commonly called proportion of Equality) is a Sumption of the extremes by Subtraction of the intermedial Terms; when, there being many magnitudes in one Rank, and others as many in an other Rank, taken two, and two in the ſame proportion, the firſt ſhall be aſſumed to the laſt in the preceding Rank, and likewiſe the firſt to the laſt in the following Rank.

18, 3, 6, 9, 12,
Here is put a double Series ⎰ A, B, C, D, E,
of Magnitudes, as ⎱ F, G, H, I, K,
24, 4, 8, 12, 16,

Two, and two of each Rank being in the ſame proportion to one another, A to B, as F to G: and again, B to C as G to H, &c. Then ſubtracting, or paſſing over the intermedials in each Rank, if A the firſt be aſſumed to E the laſt, in one Rank; and F the firſt to K the laſt, in the other Rank, this *Sumption*, or compariſon of theſe extremes, A to E, and F to K, is called *proportion of Equidiſtance*, or of *Equality*.

Here I have adventured to give a *new Name* to this Definition from *Euclid's* very words Δι᾽ἴσȣ λόγος: It is commonly tranſlated *ex æqualitate Ratio*, or *ex æquo Ratio*, Proportion of *Equality*, or even proportion: but without any reaſon given of this interpretation to inform us what it plainly means: Δι᾽ἴσȣ λόγος, ſtrictly tranſlated, is *Ratio ex æquo*. Bnt *ex quo*? Of what? Certainly *ex æquo Intervallo*. For what can more ſuitably be underſtood by Δι᾽ἴσȣ than in taking it for δι' ἴσȣ διαςήματος, *ex æquo Intervallo*, for an equal interval, or *even* diſtance of the extreme Terms from one another. The uſe of this Definition makes this interpretation manifeſt; for that the compared extremes are always taken at the ſame diſtance from one another, by ſubtracting an equal number of the intermedials in each Series. And indifferent it is, whether the other extreme from the firſt, be either the third, the fourth, the fifth, the ſixth, or any further Term, only, for inſtance, if A the firſt be aſſumed to D the fourth in one Rank, then alſo F the firſt is aſſumed to I the fourth in the other, and the like in the reſt: For that the Equidiſtance of the extremes is conſtantly to be obſerved in both the Ranks; as *Euclid's* words imply, when in the Sumption of the extremes he puts the Magnitudes in each Rank to be equal in multitude, and ſo the aſſumed extremes, the firſt to the laſt ſhall ever be equidiſtant from one another in each of the Ranks.

The Analogy between the extreme Terms of equidiſtant proportion is demonſtrated in the 22^{d}. and 23^{d}. Propoſitions of this fifth Element.

Beſides the Phyſical reaſon why theſe extremes are Analogals, or Proportionals, is very apparent, in that the proportion of the two extremes in each Rank, *viz.* of A to E, and of F to K, is compounded of the ſame intermedial proportions, and therefore in common ſenſe thoſe extremes muſt be in the ſame proportion to one another, A to E as F to K. For upon a full conſideration of this matter, we ſhall find, that proportion *ex æquo* is nothing elſe, but a compariſon of two proportions to one another, which are each compounded of like or equal proportions, that intervene between the extreme Terms of both Ranks: which extremes are therefore eaſily perceived to be Proportionals, as alſo is hereafter demonſtrated.

And farther, this Definition of proportion *ex æquo* (a proportion of admirable uſe) is derived from the 10th. and 11th. Definitions of Duplicate, and Triplicate proportion, and agrees with them in a like Sumption of the extremes to one another: and alſo in that the proportion of the extremes is upon the ſame ground ſaid to be compounded of the intermedial proportions, expoſed in continued Terms, ἐν τοῖς ἑξῆς ὅροις, as are thoſe proportions which *Euclide* calls compounded. And the only difference between them is, that in a Series of Duplicate, and Triplicate proportion the intermediate proportions are all throughout one and the ſame: whereas in proportion *ex æquo* each Series may be of various proportions; which yet in their order are the ſame with one another, the firſt proportion of one Rank the ſame with the firſt of the other; the ſecond the ſame with the ſecond, &c. So that the proportions of the extremes in this 18th. Definition, juſt as before in the 10th. and 11th. Definitions, are alike ſaid to be compounded of equal intermedial proportions. The agreement between theſe Definitions in this point is of ſpecial remark, and whoever ſhall ſtudiouſly pierce into the depth, and ſubtil Contrivances of the 5th, 10th, and 18th, Definitions, wherein *Euclide* hath ſhewn a wonderful ſagacity in ſetting forth Proportional Magnitudes, and the compoſition of proportions, will upon a juſt conſideration find, that nothing in all theſe Elements ought to be more worthily admired, or received with greater eſteem; notwithſtanding the Cavils of ſome modern Geometricians.

Δι᾽ἴσȣ is a manner of brief expreſſion uſual with the *Greeks*, and like to *Diateſſaron*, and *Diapaſon* in Muſical Notes.

In

In this proportion *ex æquo*, or of Equidistance, there are two Cases. For the Analogal intermedial Terms of both Ranks, two, and two in the same proportion, may proceed either in order, one Term after the other, or else some of the Terms may be taken out of order, as the two following Definitions of Ordinate, and Perturbate Analogy do declare.

DEFINITION XIX.

Ordinate Analogy is, when it shall be as Antecedent to Consequent so Antecedent to Consequent: and again as Consequent to some other Magnitude, so Consequent to some other Magnitude.

	18,	3,	6,
Let the double Series	A,	B,	C,
be as before,	F,	G,	H,
	24,	4,	8,

And the Analogy be as A the Antecedent to B the Consequent, so F the Antecedent to G the Consequent: And again as B the Consequent to C *an other* Magnitude, so G the Consequent to H *an other* Magnitude in a direct order.

This orderly progress in all the Analogal Terms of both Ranks successively one after the other, is called Ordinate Analogy, wherein the extremes taken *ex æquo* A to C, and F to H, are demonstrated to be proportional to one another in Prop. 22d.

And the proportions of the extremes are each said to be compounded of the intermedial proportions taken in an Ordinate Analogy to one another.

DEFINITION XX.

Perturbate Analogy is, when in the precedent Rank it shall be as Antecedent to Consequent, so in the following, Antecedent to Consequent: and again in the Precedent, as the Consequent to some other Magnitude, so in the following, some other Magnitude to the Antecedent.

	12,	4,	2.	12 .	8 .	6 .
Let there be again a	A,	B,	C.	A .	B .	C .
double Series, as	X,	F,	G,	X .	F .	G .
	18,	9,	3,	12 .	9 .	6 .

And the Analogy be as A the Antecedent to B the Consequent, so F the Antedent to G the Consequent: And again, as B the Consequent to C an other Magnitude in a *direct* order, so X an *other Magnitude* to F the Antecedent in a *disturbed* order. From the preposterous Sumption of two Terms in one of the Ranks, as namely here of X to F, making X the Term last put to become the first extreme, this Ὕςερον Πρότερον, is called *Perturbate Analogy*: Wherein the extreme Terms tho' taken *ex æquo* A to C, and X to G, are demonstrated in Prop. 23. to be proportional to one another, as well as the extremes in Ordinate Analogy.

And the proportions of the extremes are each said to be compounded of the intermedial proportions taken in a Perturbate Analogy to one another.

For the intermedial proportions of both Ranks, either in Ordinate, or Perturbate Analogy, are alike the same (tho' not taken in a like order); and therefore the extremes being alike compounded of the same intermedial proportions, are easily conceived to be in the same proportion to one another. And this is the Physical, or Philosophical reason of this matter, which is Geometrically demonstrated in Prop. 22. and 23. El. V.

In the Definitions of Ordinate, and Perturbate Analogy, *Euclid's* words imply only three Magnitudes in each Series, whereas they might have been farther continued at pleaſure, if the matter had ſo required. But he having before fully defined Proportion *ex æquo* in general, let the Terms be never ſo many; three Terms now ſufficed to ſhew in what order, or diſorder the extremes might be *ex æquo* taken, and aſſumed to one another, and ſtill be either way found proportional to one another. For this reaſon *Euclide* puts only three Terms as moſt readily ſerving to ſet forth the two Caſes of Ordinate, and Perturbate Analogy, which may happen in this proportion of Equidiſtance, or Equal Interval of Terms.

DEFINITION XXI.

Harmonical Analogy is when there being three Magnitudes, or Numbers, it ſhall be as the firſt to the third, ſo the difference between the firſt and ſecond, to the difference between the ſecond and third.

Otherwiſe.

Harmonical Analogy is a Sumption of the extremes to the differences between the extremes, and the middle Term.

Let there be three Magnitudes, or Numbers $\overset{6,}{A}, \overset{4,}{M}, \overset{3.}{E}.$ If A be to E, as the difference between A and M to the difference between M and E, that is, if A be to E, as A — M to M — E, then this is called Harmonical Analogy, and ſignified after this manner, if A, M, E, be put as Terms of Harmonical Proportion.

$$A . E :: A - M . M - E.$$
$$6 . 3 :: 6 - 4 (=2) . 4 - 3 (=1).$$

This Analogy is called Harmonical, becauſe many Muſical Conſonancies ſuit often with the Terms of this Analogy, as 6 to 4, or proportion Seſquialteral is *Diapente*, 4 to 3, or Seſquitertial is *Diateſſaron*, 6 to 3, or Duple proportion is *Diapaſon*.

There is likewiſe Harmonical Analogy in four Terms, as A, M, N, E: 30, 12, 8, 5: when as the firſt is to the fourth, ſo is the difference of the firſt, and ſecond to the difference of the third, and fourth.

$$A . E :: A - M . N - E$$
$$30 . 5 :: 30 - 12 (=18) . 8 - 5 (=3)$$

I have added this Definition to the reſt of *Euclid's*, becauſe moſt Commentators make mention of Harmonical Proportion as worthy of Conſideration; and for that it hath ſome affinity with proportion *ex æquo*.

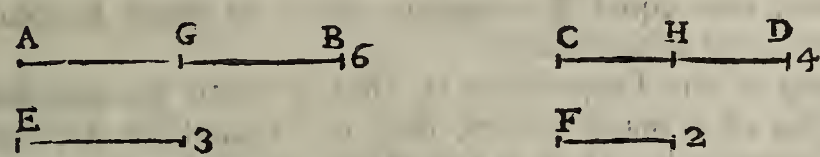
Clavius in his Comments hath abundantly ſet forth, and explained the various Properties, and mutual Correſpondencies of Proportions Geometrical, Arithmetical, and Harmonical, whom the Studious may conſult with much profit and delight.

PROPO-

PROPOSITION I.

IF there be Magnitudes how many soever Equimultiples of as many other Magnitudes, each of each; Quotuple one of the Magnitudes is of one, Totuple shall all be of all.

Let there be magnitudes how many soever AB, CD, Equimultiples of as many other magnitudes E, F, each of each. I say, that Quotuple AB is of E, Totuple shall AB, CD together, be of E, F together.



If AB . E :: CD . F. Then AB . E :: AB+CD . E+F.
6 . 3 :: 4 . 2. Then 6 . 3 :: 6 + 4 . 3 + 2.

For because AB is equimultiple of E, as CD is of F; therefore how many magnitudes are in AB equal to E, so many are in CD equal to F.

Let AB be divided into the magnitudes equal to E, namely into AG, GB; and CD into the magnitudes equal to F, namely into CH, HD.

Now then the multitude of these magnitudes CH, HD, contained in CD, shall be equal to the multitude of those magnitudes AG, GB, contained in AB [*for that* AB, CD *are by supposition Equimultiples of* E, F.]

And forasmuch as AG is equal E, and CH to F; therefore AG, CH together, are equal to E, F together [Ax. 2. If equals be added to equals, the wholes are equal.].

By the same reason GB is equal to E, and HD to F; therefore also GB, HD together, are equal to E, F together [by Ax. 2.].

Wherefore how many magnitudes are in AB equal to E, so many are in AB, CD together, equal to E, F together.

Therefore Quotuple AB is of E, Totuple shall AB, CD together, be of E, F together.

If therefore there be magnitudes how many soever Equimultiples of as many other Magnitudes, each of each, &c. Which was to be demonstrated.

ANNOTATIONS.

To this Proposition answer in Numbers the 5th. and 6th. Propositions of the seventh Element, which are thus comprised.

PROP. V, VI. El. VII.

If a number be a Part, or Parts of a number, and another be the same Part, or Parts of an other number, also both together, shall be the same Part, or Parts of both together, that one is of one.

This is differently expreſſed from our preſent Propoſition, but yet to the ſame purpoſe. For what *Euclide* demonſtrates in Magnitudes from Multiples, the ſame is in Numbers more perſpicuouſly demonſtrated from Submultiples; becauſe Numbers have a common Part, ἡ Μονὰς, The Monade, a natural Meaſure of all Numbers: from whence may be taken a certain and ſtanding beginning for the framing of demonſtrations much more clearly, than what can be made out in Magnitudes, which are wholly deſtitute of ſuch a natural foundation; therefore *Euclide* is forced to make uſe of Multiples, for that (as we have formerly noted) every Magnitude may have any Multiple of it ſelf, and all Incommenſurables may be equally multiplied. From which Equimultiplication *Euclide* moſt ingeniouſly, and with admirable Artifice manageth this Element of Proportional Magnitudes.

Foraſmuch therefore that equals added to equals make the wholes equal, He from thence proves that equal Proportions added to equal Proportions make in the ſumm the ſame equal Proportions.

And the meaning of this Propoſition is, that if there be magnitudes never ſo many, Equimultiples of as many others, that is, in the ſame Multiple proportion to as many other Magnitudes, then as one of the Antecedents is to one of the Conſequents, ſo all the Antecedents ſhall be to all the Conſequents.

2B. 2D. 2F.

As if A . B : : C . D : : E . F . &c. Then A . B : : A + C + E . B + D + F.

6 . 3 : : 4 . 2 : : 10 . 5. 6 . 3 : : 6 + 4 + 10 . 3 + 2 + 5.

Euclide here in his demonſtration gives an inſtance only in duple proportion between two and two magnitudes for brevity ſake: becauſe we may in like manner eaſily proceed to any further Multiple proportion whatſoever, triple, quadruple, &c. And alſo to any number of Magnitudes infinitely; which are to one another in the ſame Multiple proportion.

But *Euclide* expreſſeth Magnitudes that are to one another in the ſame Multiple proportion by Equimultiples, becauſe the word Equimultiples does more immediately denote the Antecedents to contain an equal number of their Conſequents, into which they may be accordingly divided: And of ſuch a diviſion *Euclide* foreſaw he was to make a ſpecial uſe in the demonſtration of this Propoſition; as likewiſe elſewhere.

Now what is here ſet forth only in Multiple proportion, the ſame is afterwards univerſally demonſtrated at Prop. 12th. in any other kind of Proportion, both Rational, and Irrational. But this was premiſed to prove other Propoſitions, which were neceſſarily required towards the demonſtration of that 12th. Propoſition. Like as the 16th. Propoſition El. I. proves the outward angle of a Triangle to be greater than either of the inward and oppoſite angles, to make way for the proof of ſome other Propoſitions, which were requiſite to demonſtrate that the ſame outward angle was equal to the two inward and oppoſite angles, Prop. 32. El. I.

PROPOSITION II.

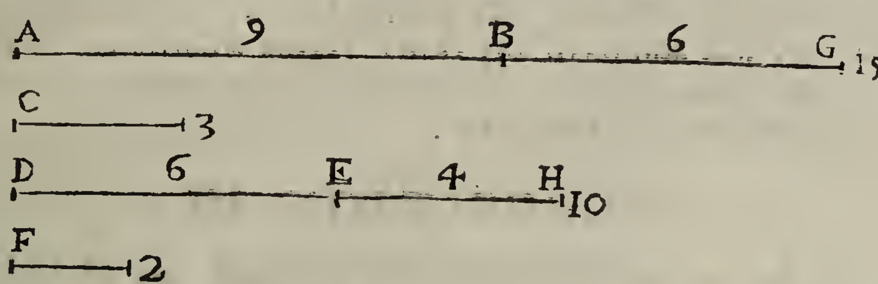
IF the firſt be Equimultiple of the ſecond, as the third is of the fourth: and there be a fifth Equimultiple of the ſecond, as a ſixth is of the fourth: then the firſt and fifth taken together ſhall be Equimultiple of the ſecond, as the third and ſixth are of the fourth.

Let AB the firſt be Equimultiple of C the ſecond, as DE the third is of F the fourth: and let there be BG a fifth Equimultiple of C the ſecond, as EH a ſixth is of F the fourth.

I ſay,

I ſay, that AG the firſt and fifth together taken, ſhall be Equimultiple of C the ſecond, as DH the third and ſixth together taken, is of F the fourth.

For becauſe AB is equimultiple of C, as DE is of F; therefore how many magnitudes are in AB equal to C, ſo many are in DE equal to F.



3C 3F 2C 2F 3C+2C 3F+2F

AB . C :: DE . F. and BG . C :: EH . F. Then AB + BG . C :: DE + EH . F.

9 . 3 :: 6 . 2. and 6 . 3 :: 4 . 2. Then 9 + 6 . 3 :: 6 + 4 . 2.

By the ſame reaſon how many are in BG equal to C, ſo many are in EH equal to F.

Wherefore how many are in the whole AG equal to C, ſo many alſo are in the whole DH equal to F.

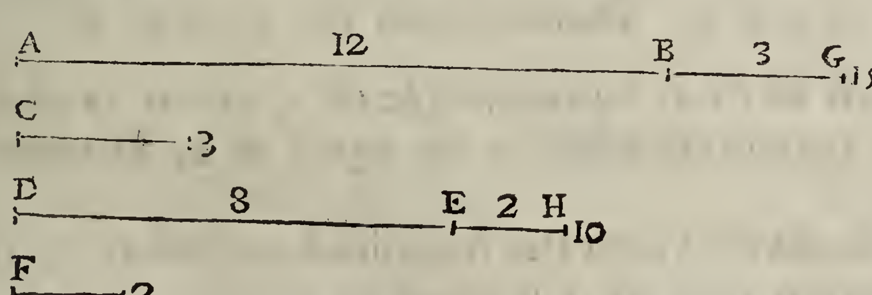
Therefore Quotuple AG is of C, Totuple DH is of F.

And therefore AG the firſt and fifth together taken, ſhall be equimultiple of C the ſecond, as DH the third and ſixth together taken, is of F the fourth.

If therefore the firſt be equimultiple of the ſecond, as the third is of the fourth, &c. Which was to be demonſtrated.

ANNOTATIONS.

This Propoſition puts the fifth and ſixth Magnitudes to be equimultiples of the ſecond, and fourth: but the proportion would hold the ſame, if the fifth were only equal to the ſecond, and the ſixth to the fourth. For in this Caſe alſo the firſt, and fifth together ſhall be equimultiple of the ſecond, as the third, and ſixth together is of the fourth.



For let AB the firſt be 12, and BG the fifth be 3, equal to the Conſequent C, 3. Alſo let DE the third be 8, and EH the ſixth be 2, equal to the Conſequent F, 2.

Then AB+BG . C :: DE+EH . F. That is,
AB+ C . C :: DE+ F . F.
12 + 3 . 3 :: 8 + 2 . 2.

But here now 'tis to be obſerved, that when the fifth and ſixth Terms, BG, EH,

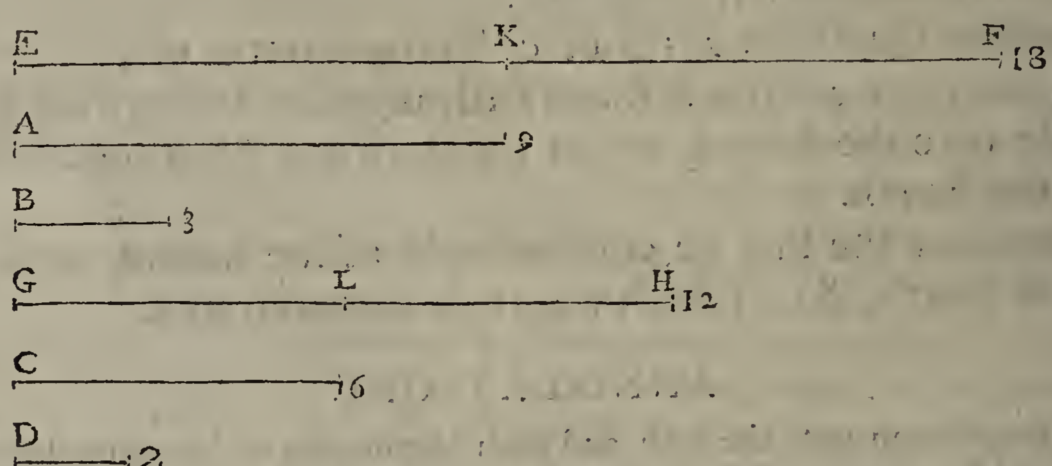
 are

are put equal to C, F, the second and fourth, only the Consequents are added to the first and third, AB, DE, the Antecedents; and then this will be the very same with that, which *Euclide* calls Composition of Proportion in Def. 14. El. V. For the Consequents C, and F are plainly added to the Antecedents, AB, DE, in that they are equal to BG, EH; the fifth and sixth Terms. And besides, this Composition of Proportion is particularly demonstrated hereafter at Prop. 18. and made a distinct notion, and of a different use from this Proposition, which is applied after a more general manner, and not confined to the addition of the Consequents to the Antecedents; but admits of any other Terms whatsoever: and not only in equimultiple Proportion as in this place, but also universally in any kind of Proportion, as is demonstrated at Prop. 24th.

PROPOSITION III.

IF the first be equimultiple of the second, as the third is of the fourth: and there be taken equimultiples of the first and third, then by Equidistance *each of the taken equimultiples shall be equimultiple of each, one of the second, the other of the fourth.*

Let A the first be equimultiple of B the second, as C the third is D the fourth: and of A and C let be taken equimultiples EF, GH. I say, that *ex æquo*, by *equidistance* [Def. 18.] EF is equimultiple of B the second, as GH is of D the fourth.



EF GH

A . B : : C . D. Then *ex æquo* 2A . B : : 2C . D

9 . 3 : : 6 . 2. Then *ex æquo* 18 . 3 : : 12 . 2.

Forasmuch as EF is equimultiple of A, as GH is of C; therefore how many magnitudes are in EF equal to A, so many are in GH equal to C.

Let EF be divided into the magnitudes equal to A, namely into EK, KF; and GH into GL, LH equal to C.

Now then the multitude of the magnitudes EK, KF shall be equal to the multitude of the magnitudes GL, LH.

And forasmuch as [by Supposition] A is equimultiple of B, as C is of D; and EK is equal to A, and GL to C; therefore EK is equimultiple of B, as GL is of D.

By the same reason KF is equimultiple of B, as LH is of D.

S Because

Because therefore EK the first, is equimultiple of B the second, as GL the third is of D the fourth: and a fifth KF is equimultiple of B the second, as a sixth LH of D the fourth; therefore EF the first and fifth together taken, is equimultiple of B the second, as GH the third and sixth together taken, is of D the fourth [by Prop. 2. El. V.].

If therefore the first be equimultiple of the second, as the third is of the fourth: and of the first and third there be taken equimultiples, then *ex æquo* each, &c. Which was to be demonstrated.

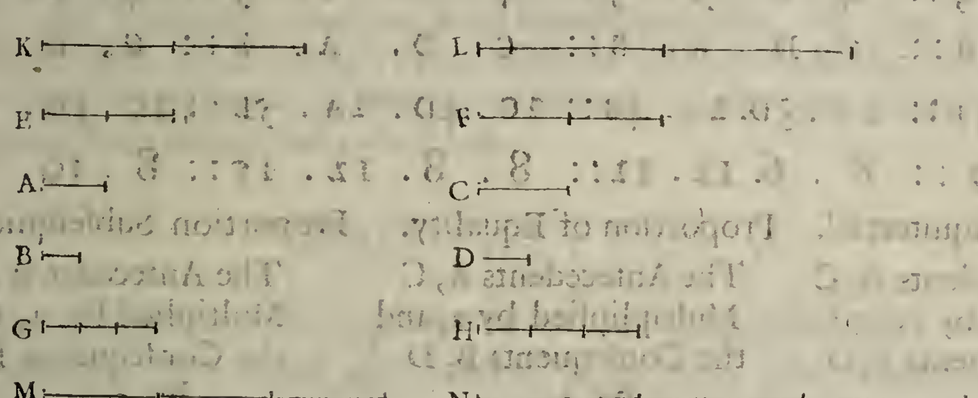
This is universally demonstrated at Prop. 22. and at present proposed only in multiple proportion to be made use of for the demonstration of the following Proposition.

PROPOSITION IV.

If the first hath to the second the same proportion, that the third hath to the fourth, then according to any multiplication whatsoever, the equimultiples of the first and third shall have to the equimultiples of the second and fourth the same proportion, being compared to one another.

Let A the first have to B the second the same proportion, that C the third hath to D the fourth: and let there be taken of A and C the first and third any equimultiples E and F; and of B and D the second and fourth any equimultiples G and H.

I say, that as E is to G, so is F to H. For again, of E and F let be taken any equimultiples K and L; and of G and H any equimultiples M and N.



A . B : : C . D. Then
E . G : : F . H.
K . M : : L . N.

Now because E is equimultiple of A, as F is of C; and of E and F hath been taken equimultiples K and L: Therefore [*ex æquo* by Prop. 3.] K is equimultiple of A, as L is of C.

Likewise because G is equimultiple of B, as H is of D; and of G and H hath been taken equimultiples M and N: Therefore *ex æquo*, M is equimultiple of B, as N is of D.

Forasmuch then that [by Supposition] as A the first is to B the ſecond, ſo C the third to D the fourth; and of A and C hath been taken any equimultiples K and L; and of B and D any equimultiples M and N: therefore [* by the fifth Definition] if K exceeds M, L exceeds H, if equal, 'tis equal, if leſs, leſs.

But K, L are equimultiples of E, F; and M, N of G, H according to any multiplication whatſoever: Therefore [* by Def. 5.] as E is to G, ſo is F to H.

If therefore the firſt hath to the ſecond the ſame proportion that the third hath to the fourth, then &c. Which was to be demonſtrated.

Corollary.
Of Inverſe Proportion.

From hence 'tis manifeſt, that if four Magnitudes be proportional, they ſhall alſo be *Inverſly* proportional.

For if E and F equimultiples of A and C, the firſt and third be greater than G and H the equimultiples of C and D, the ſecond and fourth; then on the contrary G and H ſhall be leſs than E and F: and if E and F be leſs, then ſhall G and H be greater; if equal, equal: Therefore *Inverſly*, B ſhall be to A, as D to C: the ſecond to the firſt, as the fourth to the third.

A Declaration of this fourth Propoſition in Numbers,

Shewing the equimultiples of the Antecedents to be proportional to the equimultiples of the Conſequents in ſome kind of proportion, either of equality, or of the greater, or leſs inequality.

6. 3 :: 4. 2	6. 3 :: 4. 2.	6. 3 :: 4. 2.
A. B :: C. D	A. B :: C. D.	A. B :: C. D.
2A. 3B :: 2C. 3D.	2A. 4B :: 2C. 4D.	2A. 5B :: 2C. 5D.
12. 9 :: 8. 6.	12. 12 :: 8. 8.	12. 15 :: 8. 10.
Proportion Seſquitertial.	Proportion of Equality.	Proportion Subſeſquiquartal.
The Antecedents A, C Multiplied by 2, and the Conſequents B, D by 3.	The Antecedents A, C Multipliplied by 2, and the Conſequents B, D by 4.	The Antecedents A, C Multiplied by 2, and the Conſequents B, D by 5.

ANNOTATIONS.

[* By Def. 5.] The turning of this Definition, in which lyes the chief force of *Euclid's* demonſtration may ſeem to the younger Students ſomewhat perplext. But we ought to conſider, that in all Definitions the Subject, and Prædicate are ſimply convertible, and do mutually put one another. So here there are firſt four Magnitudes propoſed to be proportional; and thereupon their equimultiples, taken according to any multiplication whatſoever, are ſuppoſed to be together either equal, greater, or leſs, as the Definition requires. Again, on the contrary, becauſe the equimultiples of four Magnitudes are found to be ſo affected, therefore thoſe four Magnitudes are concluded to be proportional. Thus here the Definition of proportional Magnitudes, and the Converſe, or *Definitum* are immediately turned upon one another in the demonſtration of this Fundamental Propoſition, the Cauſe putting the Effect, and the Effect the Cauſe.

Adver-

Advertisement.

This Propofition ought to be of a more fpecial Remark, than is commonly taken notice of. It is the principal and moft immediate property of four Proportionals, and the fole ground from whence *Euclide* raifed his Definition of proportional Magnitudes; as hath been explained in the Annotations on the 5th. Definition. Where it was alfo made manifeft, that the Primary, and Effential Conception of Proportionals confifts in this general notion, that the Antecedents do equally contain their Confequents: How much one is of one, fo much the other is of the other. As to inftance clearly and diftinctly in this main matter. If the Antecedents be *Equimultiples* of their Confequents, each of each, as if A be triple of B, and C triple of D, then in common fenfe A the firft is to B the fecond, in the fame *Multiple* proportion, that C the third is to D the fourth. And fo again, if the Antecedents be any otherwife *Equiquantuples* of their Confequents, each of each, then the firft is to the fecond in the fame proportion that the third is to the fourth: Let the Proportion be any whatfoever poffibly can be. This *Equiquantuplenefs* in Numbers *Euclide* expresfly fets forth in Def. 20. El. VII. But now in Magnitudes it cannot be expreffed from their own natural Conftitution (as in Numbers) wherein the Antecedents are equimultiples of their Confequents; for that Magnitudes have no Original meafure, or Geometrical Unite, whereby to exprefs their quantities, but only what is made by confent among our felves. And then again there are numberlefs Magnitudes which can have no common meafure at all to be made between them, whereby to exprefs their mutual proportions. Wherefore feeing that the Effential and Phyfical Definition of proportional Magnitudes cannot be made ufe of in Geometrical demonftrations (as it is in Numerical) unlefs we could fet forth wherein the Antecedents do equally contain their Confequents: therefore recourfe muft be had to fome other property, which does flow from this *Equiquantuplenefs*, that is, from the Effence of Proportionality. Now in this 4th. Propofition fuch an immediate property of four Proportionals is demonftrated of them: namely, that the equimultiples of the Antecedents fhall be proportional to the equimultiples of their Confequents, according to any multiplication whatfoever, when the four Magnitudes, whereof they are equimultiples are proportional, let the Magnitudes be Commenfurable, or Incommenfurable, the proportions Rational, or Irrational. But how again, and by what deductions from this fame property *Euclide* gave us an other infallible fign of all kinds of proportional Magnitudes in the place of a Formal, or Effential Definition of Proportionals, the Annotations on the 5th. Definition have before fully declared.

PROPOSITION V.

IF a Magnitude be equimultiple of a Magnitude, as a part detracted is of a part detracted, then fhall the Remainder be equimultiple of the Remainder, as the whole is of the whole.

Let the Magnitude AB be equimultiple of CD, as the part detracted AE is of the part detracted CF.

I fay, that the Remainder EB fhall be equimultiple of the Remainder FD, as the whole AB is of the whole CD.

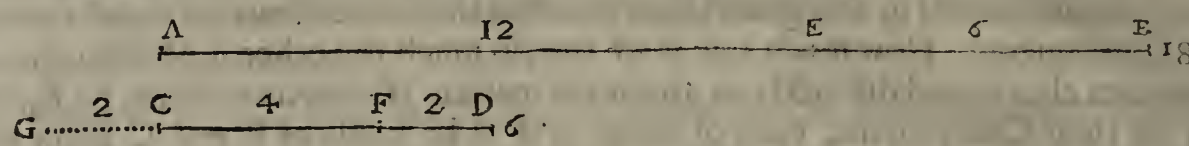
For Quotuple AE is of CF, Totuple let EB be made of an other Magnitude CG.

And becaufe AE is equimultiple of CF, as EB is of CG, therefore AE, EB together, are equimultiple of CF, CG together, that is, the whole AB is equimultiple of the whole GF, as AE is of CF [by Prop. I.

All equimultiple of all, as one of one.] But AE is alſo ſuppoſed equimultiple of CF, as AB is of CD.

Wherefore AB is equimultiple of each GF, and CD, and therefore GF is equal to CD [by Ax. 7.].

Let CF common to both be taken away, therefore the Remainder GC, is equal to the Remainder FD.



3CD 3CF 3FD 3CD
AB . CD :: AE . CF. Then EB . FD :: AB . CD.
18 . 6 :: 12 . 4. Then 6 . 2 :: 18 . 6.

Foraſmuch then that AE is equimultiple of CF, as EB is of GC, and GC is equal to FD; therefore AE ſhall be equimultiple of CF, as EB of FD.

But AE is ſuppoſed equimultiple of CF, as AB of CD; therefore the Remainder EB is equimultiple of the Remainder FD, as the whole AB is of the whole CD.

If therefore a Magnitude be equimultiple of a Magnitude, as a part detracted is of a part detracted, &c. Which was to be demonſtrated.

This here in multiple proportion is univerſally demonſtrated at Prop. 19.

To this Propoſition anſwer in Numbers the 7th. and 8th. Propoſitions of the ſeventh Element, only what is here demonſtrated of Multiple proportion, the like is there demonſtrated *è contra* of Submultiple proportion; for the reaſons before noted upon Prop. I. of this Element.

PROP. VII, and VIII. El. VII.

If a number be a Part, or Parts of a number, ſuch as a number detracted is of a number detracted; ſo alſo the Remainder ſhall be the ſame Part, or Parts of the Remainder, which the whole is of the whole.

PROPOSITION VI.

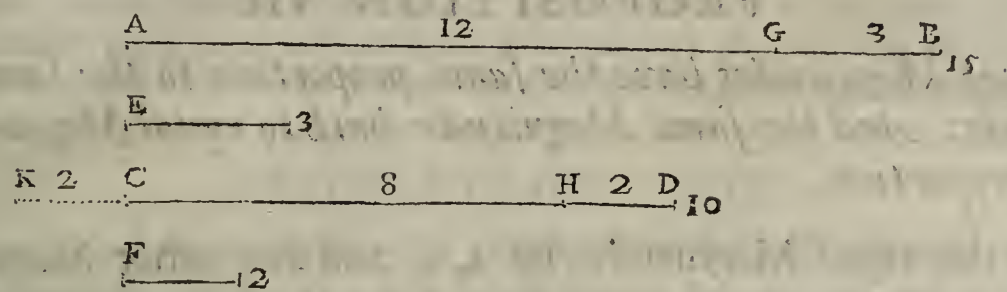
If two Magnitudes be equimultiple of two Magnitudes, and ſome parts of them detracted be equimultiple of the ſame Magnitudes, then the Remainders alſo ſhall be either equal to the ſame Magnitudes, or equimultiples of them.

Let the two Magnitudes AB, CD be equimultiples of the two Magnitudes E, F, and the parts detracted AG, CH, be ſome equimultiples of the ſame E, F.

4 I ſay,

I ſay, that the Remainders GB, HD, are either equal to E, F, or equimultiples of them.

Firſt let GB, be equal to E. I ſay, that HD is alſo equal to F; for to F let CK be put equal.



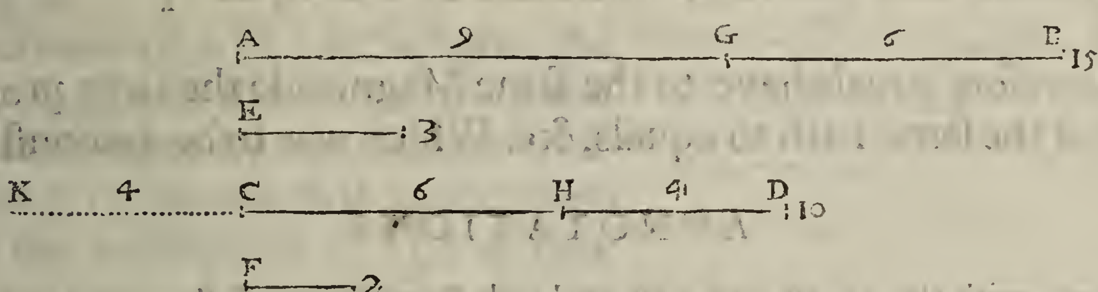
AB . E :: CD . F . and AG . E :: CH . F . and if GB = E. Then HD = F.
15 . 3 :: 10 . 2 . and 12 . 3 :: 8 . 2 . and if 3 = 3. Then 2 = 2.

And becauſe AG *the firſt* is equimultiple of E *the ſecond*, as CH *the third* is of F *the fourth*; and GB *a fifth* is equal to E *the ſecond*; *and* CK *a ſixth* is equal to F *the fourth*; Therefore AB *the firſt and fifth together*, are equimultiple of E, as KH *the third and ſixth* are of F *the fourth* [by Prop. 2.].

But AB is by Suppoſition equimultiple of E, as CD of F; therefore KH is equimultiple of F, as CD of F. Becauſe now KH, CD, each is equimultiple of F; therefore KH is equal to CD [by Ax. 6.].

Let CH common to both be detracted, therefore the Remainder KC is equal to the Remainder HD. But KC was put equal to F; therefore HD is alſo equal to F. If therefore GB be equal to E, then ſhall HD be equal to F.

In like manner ſhall be demonſtrated, that if GB be multiple of E, alſo HD ſhall be as equally multiple of F.



AB . E :: CD . F . and AG . E :: CH . F . and if GB = 2 E. Then HD = 2 F.
15 . 3 :: 10 . 2 . and 9 . 3 :: 6 . 2 . and if 6 = 6. Then 4 = 4.

If therefore two Magnitudes be equimultiple of two Magnitudes, and ſome parts of them detracted be equimultiple, &c. Which was to be demonſtrated.

This here in multiple proportion is at Prop. 24th. univerſally demonſtrated in all kinds of proportion.

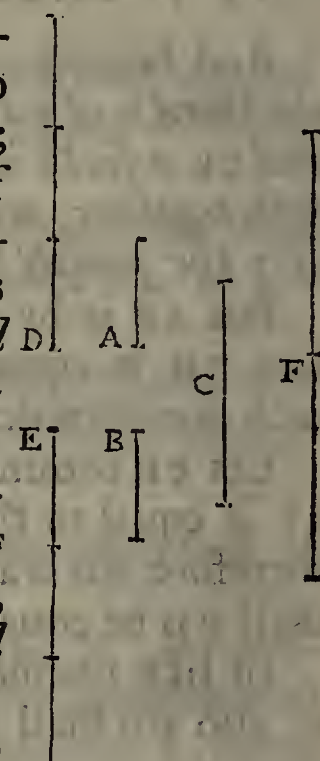
Theſe few Propoſitions, which hitherto only concern multiple proportion, are all again demonſtrated in general. But they were to be premiſed as ſubſervient to many of the following demonſtrations,

monſtrations, before the affections of Proportionals could be univerſally demonſtrated according to all proportions Rational and Irrational whatſoever.

PROPOSITION VII.

Equal Magnitudes have the ſame proportion to the ſame Magnitude: And the ſame Magnitude hath to equal Magnitudes the ſame proportion.

Let the equal Magnitudes be A, B, and any other Magnitude be C. I ſay, that each of theſe A, B, have the ſame proportion to C. And again, that C hath to each, A and B, the ſame proportion. For of A, B, let be taken any equimultiples D, E: and of C any multiple whatſoever F. Whereas then D is equimultiple of A, as E of B; and that A is equal to B, therefore D is alſo equal to E [by Ax. 6.]; wherefore if D exceeds F, alſo E ſhall exceed F; if equal, equal; if leſs, leſs. But D, E are any equimultiples of A, B [*the firſt and third*], alſo F is any multiple whatſoever of C [*the ſecond and fourth, for* C *is inſtead of two Magnitudes*]; therefore as A to C, ſo B to C [Def. 5. El. V.].



Upon the ſame conſtruction we may alike demonſtrate that D is equal to E: and therefore if F exceeds D, it alſo exceeds E; if equal, equal; if leſs, leſs. But F is any multiple of C [*the firſt and third*]; and D, E are any equimultiples of A, B [*the ſecond and fourth*]; therefore as C to A, ſo C to B.

Wherefore equals have to the ſame Magnitude the ſame proportion: and the ſame hath to equals, &c. Which was to be demonſtrated.

ANNOTATIONS.

This 7th. with the 8th, 9th, 10th, 11th, and 12th. Propoſitions following, are taken by *Tacquet* as meer Axioms, to be received without demonſtration. They are indeed evident truths, eſpecially in Numbers: but becauſe theſe are to be applied in general to Magnitudes of all ſorts, as Lines, Planes, and Solids, commenſurable, or incommenſurable indifferently; therefore being capable of a juſt demonſtration, they ought to be demonſtrated; as *Euclide* hath upon good reaſon done with an admirable ſubtility of Ratiocination uſed in theſe Propoſitions, and throughout this whole Element; which *Tacquet* might have well perceived, if his vanity had not miſlead him. And yet to help out his own method he was conſtrained to uſe theſe Propoſitions in the nature of Axioms, or common Notions: whereas *Borellus* alſo hath thought fit to demonſtrate them, as well as *Euclide* had before.

But *Borellus* likewiſe hath his failures, who is forced in his method to make his demonſtrations apart, ſome for commenſurable Magnitudes, ſome for incommenſurable: whereas Magnitudes, when compared together in this Elementary Doctrine of Proportions and Proportionals, are not propoſed in particular, ſometimes commenſurable, ſometimes incommenſurable; but as Magnitudes in general. Wherefore *Euclide* frames his demonſtrations with ſuch artifice, as at once to comprehend alike all Magnitudes commenſurable or incommenſurable, in any kind of proportion without reſtriction, or any mention of commenſurability, or incommenſurability, to be diſtinctly conſidered. And therefore in this point *Borellus* his method is defective, and comes much ſhort of *Euclid's* general way of demonſtration in this excellent Element: where the demonſtrations extend themſelves to all things in nature capable of proportion.

PROPOSITION VIII.

Of unequal magnitudes the greater hath to the ſame a greater proportion, than the leſs; And the ſame hath to the leſs a greater proportion, than to the greater.

Let the unequal Magnitudes be AB, C, and AB the greater. [AB, 5. C, 3]. Let alſo D be any other Magnitude [as 2].

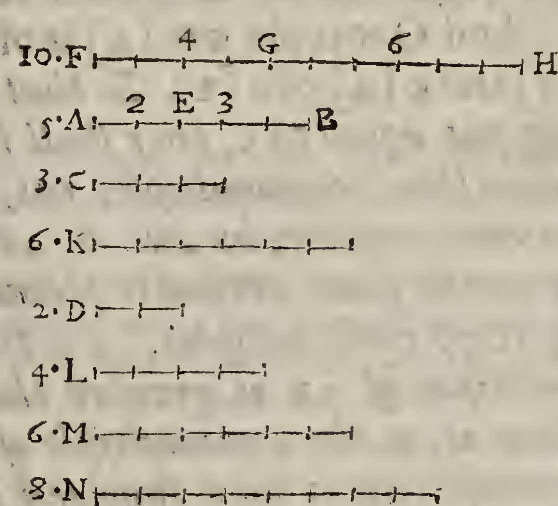
I ſay, that AB [5] hath to D [2] a greater proportion, than C [3] to D [2]. And D [2] hath to C [3] a greater proportion than to AB [5].

Whereas AB is greater than C, let in AB be put a Magnitude equal to C, namely BE [3].

Now the leſs of theſe two AE, EB, being multiplied, ſhall at length be greater than D.

Firſt, let AE (the exceſs of AB above C) be leſs than EB (which is equal to C), and multiply AE [2] till the multiple thereof be made greater than D [2], and let this multiple of AE be FG [4] greater than D [2].

Now Quotuple FG [4] is of AE [2], Totuple let GH [6] be of EB [3], and K [6] of C [3]. *So that* EB *being put equal to* C, *they ſhall likewiſe have their equimultiples* GH, K, *equal to one another*, by Ax. 6.



And now of D [2] let be taken the duple L [4], and the triple M [6], and ſo onward more by one, till the multiple of D [2] be the firſt greater than K [6] the multiple of C [3]. Let now this multiple of D be taken, and let it be N [8] the quadruple of D [2], and the firſt greater than K [6] the multiple of C [3]. So that K [6] is the firſt leſs than N [8].

Foraſmuch then as K, is the firſt leſs than N the quadruple of D; therefore K is not leſs than M the triple of D [*but either equal, or greater*].

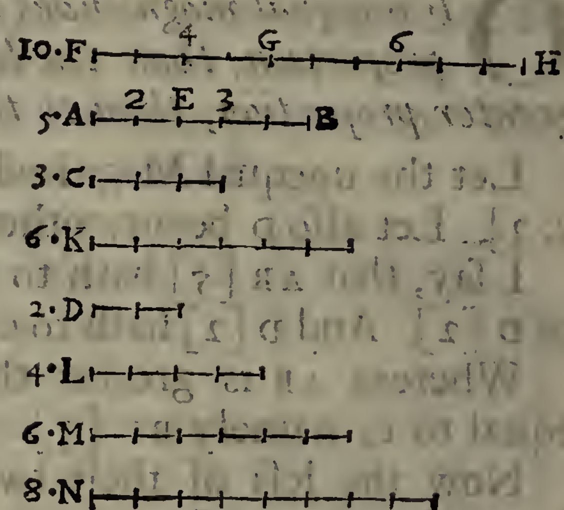
And whereas FG is equimultiple of AE, as GH of EB; therefore the whole FH is equimultiple of the whole AB, as FG of AE [by Prop. 1. El. V.]. But FG is equimultiple of AE, as K of C [by Conſtruction]; wherefore FH is equimultiple of AB, as K of C; and therefore FH and K, are equimultiples of AB, and C.

 Again,

Again, becauſe GH is equimultiple of EB, as K of C, and EB is put equal to C [by Conſtruction]; therefore GH ſhall be equal to K: [by Ax. 6.]

* But K is not leſs than M, therefore GH is not leſs than M, and FG is greater than D [by Conſtruction]; wherefore the whole FH is greater than D, M together.

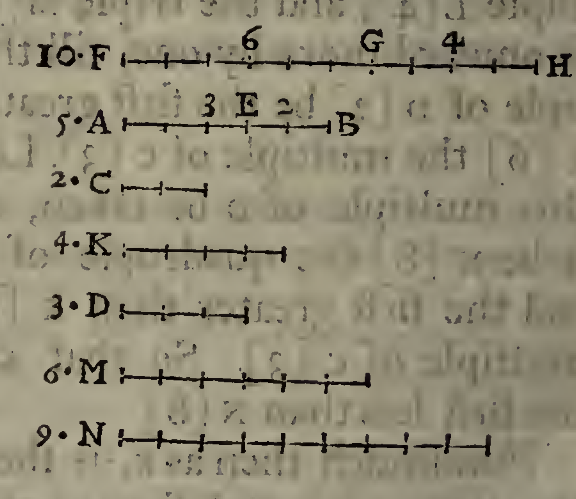
But D, M together are equal to N; therefore FH exceeds N; bnt K exceeds not N [by Conſtruction] and FH, K are equimultiples of AB, C, and N is any whatever multiple of D; therefore AB hath to D a greater proportion, than C to D [by Def. 7. El. V.].



I ſay moreover, that D hath a greater proportion to C the leſs, than D to AB the greater. For on the ſame Conſtruction we ſhall in like manner demonſtrate, that N exceeds K, but exceeds not FH; and that N is any multiple of D, and FH, K, are any whatever equimultiples of AB, C; therefore D hath to C the leſs, a greater proportion, than hath D to AB the greater.

But again, of the unequal Magnitudes AB, C, when in the greater AB there is taken a Magnitude equal to C the leſs: Let AE the exceſs of AB, above C be greater than EB, which is put equal to C [AE, 3; EB, 2]. Now EB the leſs being multiplied, ſhall at length be greater than D. Let EB [2] be multiplied, and let GH [4] the multiple of EB (equal to C) be greater than D [3].

And Quotuple GH [4] is of EB [2], Totuple let FG [6] be of AE [3] and K [4] of C [2]. *So that* EB *being put equal to* C, *they ſhall likewiſe have their equimultiples* GH, K *equal to one another, by Ax.* 6. *And alſo from the ſame Axiom it follows, that of theſe equimultiples,* * FG *the equimultiple of* AE *is greater than* GH, *that is,* K *the equimultiple of* EB, C; *becauſe* AE *is now ſuppoſed greater than* EB, *which is equal to* C. *For as of equals the equimultiples are equal, ſo of unequals the equimultiples are unequal, the greater of the greater, the leſs of the leſs.*



Now after the former manner we ſhall demonſtrate that FH [10], and K [4], are equimultiples of AB [5], and C [2].

And as before, let of D [3] be taken the multiples, till N be the firſt multiple of D, that is greater than FG [6] the multiple of AE [3]. Where-

Wherefore again, FG [6] is not less than M [6], and GH [4] is greater than D [3] by Conſtruction; therefore the whole FH [10], exceeds D,M together; that is N [9]. But K exceeds not N, becauſe FG being greater than GH which is equal to K, exceeds not N, by Conſtruction. And FH, K, are equimultiples of AB, C; and N is any whatever multiple of D; therefore AB [5], hath to D [3] a greater proportion, than C [2] to D [3]. by Def. 7. El. V.

Wherefore of unequal Magnitudes the greater hath to the ſame a greater proportion, than the leſs; And the ſame hath to the leſs a greater proportion, than to the greater. Which was to be demonſtrated.

ANNOTATIONS.

In this Propoſition there are in effect four Magnitudes, whether AB, and C be compared to D, or D to AB, and C. For in each of theſe Caſes D is twice taken: and in the firſt as a common Conſequent to AB, and C: in the latter as a common Antecedent to the ſame Magnitudes.

The ſtreſs of this ſubtil demonſtration lyes in three remarkable points. *Firſt*, of the two unequal Magnitudes AB, C, there is in the greater AB taken a part EB equal to C, the leſs Magnitude: ſo that the Remainder AE is the exceſs of AB above C.

Secondly, of theſe two parts AE, EB, That, which happens to be the leſs, is to be multiplied, till the multiple thereof is allowed to exceed D the third Magnitude.

Thirdly, D is to be multiplied, till its multiple firſt exceeds the multiple of the greater part in AB, whether it be either AE, or EB. Upon theſe Conſtructions the demonſtration chiefly depends.

Laſtly, note that of the two unequal Magnitudes, each may be either greater, or leſs than the third; or one greater, and the other leſs. As for inſtance, let a Pound, and a Crown be compared to a Shilling; or a Crown, and a Shilling to a Pound; or a Pound, and a Shilling to a Crown: And the demonſtration ſerves alike to all.

Advertiſement.

In this Propoſition are expoſed two unequal Proportions that have the ſame Conſequents: and again contrarily, two unequal proportions that have the ſame Antecedents. As $\frac{6}{2}$, and $\frac{4}{2}$, that is, 6 to 2, and 4 to 2. And contrarily $\frac{2}{6}$, and $\frac{2}{4}$. Now in this 8th. Propoſition is only demonſtrated which in each compariſon is the greater proportion. But a farther enquiry may be made, in what proportion one is greater than the other. This queſtion is, *De proportione Proportionum*, of the proportion of Proportions, for the diſcovery whereof take theſe two Rules.

1. Two unequal proportions having the ſame Conſequents, are to one another as their Antecedents. Thus $\frac{6}{2}$ is to $\frac{4}{2}$ as 6 to 4, or $1\frac{1}{2}$, in proportion Seſquialteral, one greater than the other: and thus in general ſignified, $\frac{A}{B}.\frac{E}{B}$ :: A . E . $\frac{18}{6}.\frac{12}{6}$ :: 18 . 12.

2. Two unequal proportions having the ſame Antecedents, are to one another Reciprocally as their Conſequents. Thus $\frac{2}{4}$ is to $\frac{2}{6}$, as 6 to 4; That is Subduple proportion is to Subtriple in proportion Seſquialteral, ſo that $\frac{2}{4}$ is in ſuch a proportion greater than $\frac{2}{6}$, and in general, $\frac{B}{E}.\frac{B}{A}$ :: A . E . $\frac{6}{12}.\frac{6}{18}$ :: 18 . 12. But now if the two unequal proportions have different Antecedents, and Conſequents, as 9 to 3, and 4 to 2. In this Caſe to diſcover which of them is the greater proportion, and in what proportion one is greater than the other, the Terms of theſe proportions are to be ſo changed, that the ſame unequal proportions may have the ſame Conſequents, which is called *Reduction of proportions to a common Conſequent*: and performed by this Rule.

Reduction of Proportions to a common Confequent.

Multiply the Confequents into one another; and each Antecedent into the others Confequent.

As $\frac{9}{3}$, and $\frac{4}{2}$ are brought to $\frac{18}{6}$, and $\frac{12}{6}$ by multiplying the Confequents 3 and 2 into one another: And the Antecedent 9 into 2 the others Confequent; likewife the Antecedent 4 into 3 the formers Confequent.

Now this 8th. Propofition demonftrates, that 18 hath to 6 a greater proportion than 12 to 6: and 6 to 12 a greater than 6 to 18.

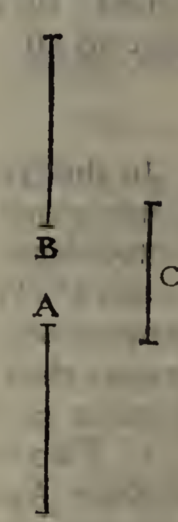
Moreover in both Cafes 'tis fhewn by the two foregoing Rules, in what proportion one proportion is greater than the other. Thus $\frac{18}{6}$ is greater than $\frac{12}{6}$, and $\frac{6}{12}$ greater than $\frac{6}{18}$, in the fame proportion of 18 to 12, or $1\frac{1}{2}$. But the proportions of $\frac{9}{3}$, and $\frac{4}{2}$ are the fame with $\frac{18}{6}$, and $\frac{12}{6}$, which is thus demonftrated: $\frac{}{3}$ multiplied by 2 make $\frac{18}{6}$, and $\frac{4}{2}$ multiplied by 3 make $\frac{12}{6}$: But if a number multiply two numbers, the Products fhall be in the fame proportion with the multiplied numbers, by Prop. 17. El. VII. Again, of the proportions $\frac{9}{3}$, and $\frac{4}{2}$ the Confequents 3 and 2 multiplied into one another, fhall make one and the fame common Confequent. For 3 into 2, or 2 into 3 make the fame number 6, by Prop. 16. El. VII. If two numbers multiply each the other, the Products fhall be equal to one another; therefore $\frac{9}{3}$, and $\frac{4}{2}$ are the fame proportions with $\frac{18}{6}$, and $\frac{12}{6}$. Thus the Rule for Reduction of Proportions to a common Confequent is demonftrated.

PROPOSITION IX.

Magnitudes which have the fame proportion to the fame magnitude, are equal to one another: And to what magnitudes the fame magnitude hath the fame proportion, they alfo are equal to one another.

Let each of the Magnitudes A, B, have to C the fame proportion. I fay, that A is equal to B. For if not, then each of thofe Magnitudes A, B, fhould not have to C the fame proportion [by Prop. 8. El. V.]. But each have; therefore A is equal to B.

Again, let C have to each of the Magnitudes A, B, the fame proportion. I fay, that A is equal to B. For if not, then C fhould not have to A and B the fame proportion [by Prop. 8.El.V.]. But it hath; therefore A is equal to B.

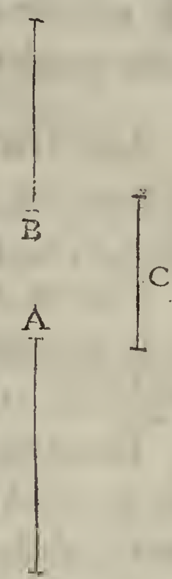


Wherefore Magnitudes which have the fame proportion to the fame Magnitude, are equal to one another. &c. Which was to be demonftrated.

PROPOSITION X.

Of Magnitudes having a proportion to the fame magnitude, that which hath the greater proportion, is the greater. And to what magnitude the fame magnitude hath a greater proportion, that is the lefs magnitude.

Let A have to C a greater proportion than B to C. I fay, that A is is greater than B: For if not, then A is either equal to B, or leſs. But A is not equal to B. For then each of the magnitudes A, B ſhould have to C the ſame proportion [by Prop. 7. El. V.]. But each of them have not; therefore A is not equal to B: neither alſo is A leſs than B. For then A ſhould have to C a leſs proportion than B to C [by Prop. 8. El. V.]. But it hath not; therefore A is not leſs than B: And it hath been demonſtrated that it is not equal; therefore A is greater than B.



Again, let C have to B a greater proportion than C to A. I ſay, that B is leſs than A. For if not: it is either equal, or greater; but B is not equal to A; for then C ſhould have to A and B the ſame proportion [by Prop. 7. El. V.]. But it hath not; therefore B is not equal to A: neither alſo is B greater than A. For then C ſhould have to B a leſs proportion than to A [by Prop. 8. El. V.]. But it hath not; therefore B is not greater than A: And it hath been demonſtrated that it is not equal: therefore B is leſs than A.

Wherefore of magnitudes having a proportion to the ſame magnitude, that which hath the greater proportion is the greater. &c. Which was to be demonſtrated.

PROPOSITION XI.

Proportions which are the ſame to the ſame proportion, are the ſame to one another.

Let A be to B as C to D: alſo E to F as C to D. I ſay, that A is to B, as E to F. For let be taken of A, C, E (*the Antecedents*) any equimultiples G, H, K: and of B, D, F (*the Conſequents*) any whatever equimultiples L, M, N. Now becauſe it is as A to B, ſo C to D; and of A, C are taken equimultiples G, H: alſo of B, D any whatever equimultiples L, M. If therefore G exceeds L, then H exceeds M; and if equal, equal; if leſs, leſs. [Def. 5. El. V.]

G . H . K
$\frac{A}{B}$: : $\frac{C}{D}$: : $\frac{E}{F}$
L . M . N

Again, becauſe E is to F as C to D: and of E, C are taken equimultiples K, H: as alſo of F, D any whatever equimultiples N, M. If therefore K exceeds N, then H exceeds M; and if equal, equal; if leſs, leſs. But if H exceeds M, then G exceeds L; and if equal, equal; if leſs, leſs [for by Suppoſition A is to B as C to D]; wherefore if G exceeds L, then K exceeds N; and if equal, equal; if leſs, leſs. But G, K are equimultiples of A, E: and L, N any whatever equimultiples of B, F; wherefore as A to B, ſo E to F [Def. 5. El. V.].

Therefore proportions which are the ſame to the ſame proportion, are the ſame to one another. Which was to be demonſtrated.

PROPOSITION XII.

If magnitudes how many ſoever be Proportionals: it ſhall be, as one of the Antecedents to one of the Conſequents, ſo all the Antecedents to all the Conſequents.

Let there be Proportional magnitudes how many ſoever, A, B, C, D, E, F; And as A to B, ſo C to D, and E to F. I ſay, that as A to B, ſo A, C, E, to B, D, F.

For of A, C, E (the Antecedents) let be taken any equimultiples G, H, K; and of B, D, F, (the Conſequents) any whatever equimultiples L, M, N.

Becauſe therefore it is as A to B, ſo C to D, and E to F: and there hath been taken of A, C, E any equimultiples G, H, K: and of B, D, F, any whatever equimultiples L, M, N.

G . H . K

$\frac{A}{B} :: \frac{C}{D} :: \frac{E}{F}$

L . M . N

If therefore G exceeds L, then H exceeds M, and K exceeds N; and if equal, equal; if leſs, leſs.

So that if G exceeds L, then G, H, K, exceeds L, M, N; and if equal, equal; if leſs, leſs.

But G, and G, H, K, are equimultiples of A, and of A, C, E. [For if there be magnitudes how many ſoever equimultiples of as many other magnitudes, each, of each, Quotuple one is of one, Totuple ſhall all be of all.]

By the ſame reaſon alſo L, and L, M, N, are equimultiples of B, and of B, D, F.

Therefore it is, as A to B, ſo A, C, E, to B, D, F.

Wherefore if magnitudes how many ſoever be proportionals: it ſhall be, as one of the Antecedents to one of the Conſequents, ſo all the Antecedents to all the Conſequents. Which was to be demonſtrated.

To this Propoſition anſwers in Numbers Prop. 12th. El. VII.

ANNOTATIONS.

What is here now univerſally proved of any kind of proportion, the ſame hath been before in the firſt Propoſition of this Element demonſtrated only of multiple proportions, that as one of the Antecedents, is to one of the Conſequents, ſo are all to all. And note, that when the Antecedents are thus annexed to Antecedents, and Conſequents to Conſequents, This is called Addition of proportions; whoſe chief uſe is declared in this 12th. Propoſition, that as one of the Antecedents is to one of the Conſequents, ſo is the ſumm of the Antecedents to the ſumm of the Conſequents. But when Antecedents are multiplied into Antecedents, and Conſequents into Conſequents, then it is ſaid to be a Compoſition of Proportions: Which is of frequent uſe in the Mathematics.

PROPO-

PROPOSITION XIII.

IF the firſt hath to the ſecond the ſame proportion, that the third hath to the fourth, and the third to the fourth hath a greater proportion, than the fifth to the ſixth; alſo the firſt hath to the ſecond a greater proportion, than the fifth to the ſixth.

Let A the firſt have to B the ſecond the ſame proportion, that C the third hath to D the fourth: and C the third a greater proportion to D the fourth, than E the fifth to F the ſixth.

I ſay, A the firſt ſhall have to B the ſecond a greater proportion, than E the fifth to F the ſixth.

Foraſmuch as C hath to D a greater proportion than E to F; therefore there are ſome equimultiples of C, E, and again ſome other whatever equimultiples of D, F, where the multiple of C exceeds the multiple of D, and the multiple of E exceeds not the multiple of F [Def. 7. El. V.]. Let them be taken, and of C, E, let G, H, be equimultiples; and of D, F any whatever equimultiples K, L; ſo that G exceeds K, but H exceeds not L.

Now Quotuple G is of C, Totuple let M be of A: And Quotuple K is of D, Totuple alſo let N be of B.

M . G . H

$\frac{A}{B} : : \frac{C}{D} > \frac{E}{F}$

N . K . L

Becauſe therefore it is as A to B, ſo C to D; and of A, C are taken equimultiples M, G; and of B, D, any whatever equimultiples N, K; then if M exceeds N, alſo G exceeds K; and if equal, equal; if leſs, leſs.

Now G exceeds K, therefore alſo M exceeds N; but H exceeds not L [by Conſtruction], and M, H are equimultiples of A, E [the Antecedents], and N, L any whatever equimultiples of B, F [the Conſequents]; therefore A hath to B a greater proportion than E to F [Def. 7. El. V.].

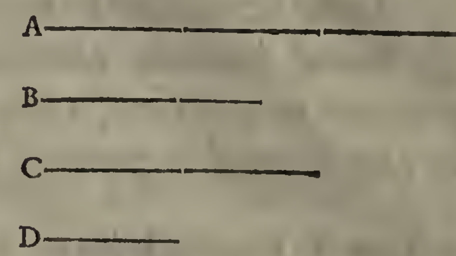
If therefore the firſt hath to the ſecond the ſame proportion, that the third hath to the fourth, and the third hath to the fourth a greater proportion, than the fifth to the ſixth, &c. Which was to be demonſtrated.

PROPOSITION XIV.

IF the firſt hath to the ſecond the ſame proportion that the third hath to the fourth; and the firſt be greater than the third; alſo the ſecond ſhall be greater than the fourth: And if equal, equal; if leſs, leſs.

Let A the firſt have to B the ſecond the ſame proportion that C the third hath to D the fourth; and let A be greater than C. I ſay, that B is greater than D. For whereas A is greater than C, and there is an other magnitude B; therefore A hath to B a greater proportion than

than C to B [Prop. 8. El. V.] But as A to B, ſo C to D; therefore alſo C hath to D a greater proportion than C to B. But to what magnitude the ſame hath a greater proportion, that is the leſs [Prop. 10. El. V.]; therefore D is leſs than B: ſo that B is greater than D. In like manner we ſhalr demonſtrate that if A be equal to C, B ſhall alſo be equal to D; and if A be leſs than C, B ſhall be alſo leſs than D.



If therefore the firſt hath to the ſecond the ſame proportion, that the third hath to the fourth; and the firſt be greater than the third, &c. Which was to be demonſtrated.

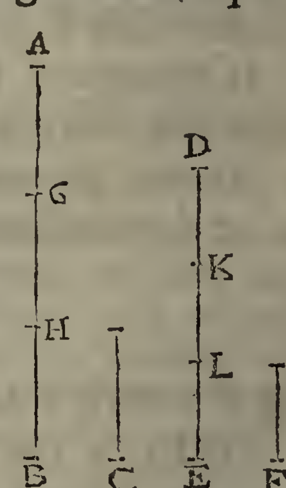
ANNOTATIONS.

Euclide thought fit to demonſtrate, that in proportionals if one Antecedent were greater, equal, or leſs than the other Antecedent; then one Conſequent ſhall be greater, equal, or leſs than the other Conſequent. But needleſs it was to prove, that if one Antecedent were greater, equal, or leſs than the Conſequent, the other likewiſe is the ſame. For that this is of it ſelf implied in the very notion of proportionals; where the proportions are always either of equality, or of the greater or leſs inequality: So that the Antecedents are either equal to their Conſequents, or greater, or leſs.

PROPOSITION XV.

Parts compared to one another have the ſame proportion with their equimultiples.

Let AB be equimultiple of C, as DE of F. I ſay, that it is as C to F, ſo AB to DE. For becauſe AB is equimultiple of C, as DE of F, therefore how many magnitudes are in AB equal to C, ſo many are in DE equal to F. Let AB be divided into the magnitudes equal to C, namely AG, GH, HB; And DE into magnitudes equal to F, as DK, KL, LE: therefore the multitude of theſe AG, GH, HB, ſhall be equal to the multitude of DK, KL, LE. Foraſmuch then as AG, GH, HB, are equal to one another: and DK, KL, LE, are alſo equal to one another; wherefore as AG to DK, ſo GH to KL, and HB to LE [Prop. 7. El. V.]. Equal magnitudes have to the ſame (*or to equal magnitudes*) the ſame proportion.



And becauſe as one of the Antecedents to one of the Conſequents, ſo all the Antecedents to all the Conſequents [Prop. 12. El. V]; therefore as AG to DK, ſo AB to DE. But AG is equal to C, and DK to F [by Conſtruction], therefore as C to F (part to part), ſo AB to DE (*equimultiple to equimultiple*).

Parts therefore compared to one another, have the ſame proportion with their equimultiples. Which was to be demonſtrated.

ANNO-

ANNOTATIONS.

To this Proposition answers in numbers Prop. 17th. El. VII. "If a number multi-"tiplying two numbers make some numbers, their Products shall have the same pro-"portion with the multiplied numbers.

For let 4 multiplying two numbers 3 and 2, make 12 and 8, then the Products 12 to 8 shall have the same proportion with 3 to 2: Both in Sesquialteral proportion, $1\frac{1}{2}$.

These Propositions are very near to common Notions, and therefore likewise ought to be as carefully remarked, in regard of their general use in proportions between magnitudes to magnitudes, and numbers to numbers. Also this Prop. 17th. El. VII. is the ground of bringing all proportions to a common Consequent, by which is discovered what proportions are greater, or less one than an other. As hath been before noted upon Proposition 8th. Which Annotations review, and fully consider with relation to these Propositions.

PROPOSITION XVI.

If four magnitudes be proportional, they shall also be alternly proportional.

Let four magnitudes A, B, C, D, be proportional, as A to B, so C to D. I say, that they shall also be alternly proportional, as A to C, so B to D. For of A, B, let be taken equimultiples E, F: and and of C, D any whatever equimultiples G, H. Forasmuch then that E is equimultiple of A, as F of B; but parts have to one another the same proportion with their equimultiples [Prop. 15. El. V.]; therefore as A to B, so E to F: but as A to B, so C to D; therefore as C to D, so E to F [Prop. 11. El. V.]

E . G
A/B : : C/D
F . H

Again, because G, H are equimultiples of C, D; therefore as C to D, so G to H: but as C to D, so E to F; therefore as E to F, so G to H. But if four magnitudes be proportional, and the first be greater than the third, the second shall be greater than the fourth, and if equal, equal; if less, less [Prop. 14. El. V.]. If therefore E exceeds G, also F exceeds H; and if equal, equal; if less, less. But E, F are equimultiples of A, B (the first, and third) also G, H are any whatever equimultiples of C, D (the second, and fourth): Therefore as A to C, so B to D [Def. 5. El. V.].

E . G
A/B : : C/D
F . H

Wherefore if four magnitudes be proportional, they shall also be alternly proportional. Which was to be demonstrated.

To this Proposition answers in numbers Prop. 13th. El. VII.

ANNOTATIONS.

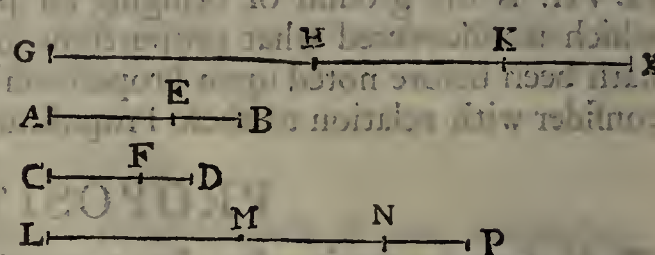
It is in the first place to be noted, that the alternation of proportions can only be used between Homogeneal magnitudes, where the four proportional Terms are either all Lines, or all Planes, or all Solids, as Def. 3. El. V. declares. For magnitudes of Heterogenal quantities cannot in nature admit of mutual comparison according to quantity.

PROPOSITION XVII.

If compounded magnitudes be proportional, they ſhall alſo divided, be proportional.

Let the Compounded proportional magnitudes be AB, BE, CD, DF; as AB to BE, ſo CD to DF. I ſay, that alſo divided they ſhall be proportional, as AE to EB, ſo CF to FD. For of AE, EB; CF, FD, let be taken equimultiples GH, HK: LM, MN. And of EB, FD, any whatever equimultiples KX, NP.

Foraſmuch then as GH is equimultiple of AE, as HK of EB; therefore GH is equimultiple of AE, as (the whole) GK of the whole AB [Prop. 1. El. V.]. But GH is equimultiple of AE, as LM of CF [by Conſtruction]; therefore GK is equimultiple of AB, as LM of CF [Prop. 11. El. V.].



Again, becauſe LM is equimultiple of CF, as MN of FD, therefore LM is equimultiple of CF, as (the whole) LN of (the whole) CD [Prop. 1. El. V.]. But LM was equimultiple of CF as GK of AB; therefore GK is equimultiple of AB, as LN of CD [Prop. 11. El. V.].

Again, foraſmuch as [by Conſtruction] HK (the firſt) is equimultiple of EB (the ſecond) as MN (the third) of FD (the fourth): And alſo KX (a fifth) is equimultiple of EB (the ſecond), as NP (a ſixth) of FD (the fourth); therefore HX (the firſt and fifth) together, is equimultiple of EB (the ſecond), as MP (the third and ſixth) together, is of FD (the fourth) [Prop. 2. El. V.].

Now becauſe it is as AB to BE, ſo CD to DF; and of AB, CD are taken equimultiples GK, LN; alſo of EB, FD any whatever equimultiples HX, MP. If therefore GK exceeds HX, alſo LN exceeds MP; and if equal, equal; if leſs, leſs [Def. 5. El. V.]. Let therefore GK exceed HX, and HK common to both being detracted, then ſhall GH exceed KX. But if GK exceeds HX, alſo LN exceeds MP, and MN common to both being detracted, then ſhall LM exceed NP: wherefore if GH exceed KX, alſo LM exceeds NP.

In like manner ſhall we prove, that if GH be equal to KX, alſo LM ſhall be equal to NP, and if leſs, leſs. But GH, LM are equimultiples of AE, CF; and KX, NP any whatever equimultiples of EB, FD; therefore as AE to EB, ſo CF, to FD.

If therefore compounded magnitudes be proportional, they ſhall alſo divided be proportional. Which was to be demonſtrated.

ANNOTATIONS.

In this Propoſition there is with great ſubtilty demonſtrated from compounded proportions, the Analogy of divided proportions, as Diviſion of proportion is by *Euclide* defined in the 16[th]. Definition.

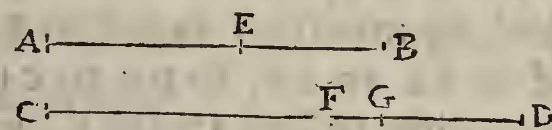
PROPO-

PROPOSITION XVIII.

IF divided magnitudes be proportional, they ſhall alſo compounded be proportional.

Let the divided magnitudes be AE, EB; CF, FD; and as AE to EB, ſo CF to FD. I ſay, that alſo compounded, they ſhall be proportional, as AB to BE, ſo CD to DF.

For if it be not as AB to BE, ſo CD to DF, then ſhall it be as AB to BE, ſo CD either to a leſs than DF, or to a greater.



Firſt, let it be to a leſs, as DG. Foraſmuch then that it is as AB to BE, ſo CD to DG; therefore theſe compound magnitudes are proportional. So that divided, they ſhall alſo be proportional [Prop. 17. El. V.]. It is therefore as AE to EB, ſo CG to GD. But by Suppoſition as AE to EB, ſo CF to FD: wherefore as CG to GD, ſo CF to FD. But CG the firſt is greater than CF the third; therefore alſo GD the ſecond is greater than FD the fourth [Prop. 14. El. V.]; but alſo 'tis leſs, which is impoſſible; therefore it is not as AB to BE, ſo CD to DG. In like manner ſhall we prove, that CD cannot be to any magnitude greater than DF: therefore it is as AB to BE, ſo CD to DF.

Wherefore if divided magnitudes be proportional, they ſhall alſo compounded be proportional. Which was to be demonſtrated.

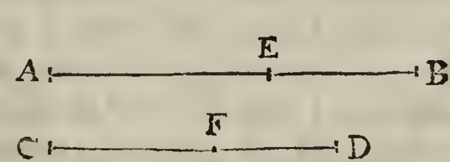
ANNOTATIONS.

In this Propoſition is demonſtrated from Divided proportions the Analogy of Compounded proportions, as Compoſition of proportion is explained in the 15th. Definition.

PROPOSITION XIX.

IF it be as the whole to the whole, ſo a part detracted to a part detracted, then the Remainder ſhall be to the Remainder as the whole to the whole.

For let it be as the whole AB to the whole CD, ſo a part detracted AE, to a part detracted CF. I ſay, that the Remainder EB ſhall be to the Remainder FD, as the whole AB to the whole CD.



Foraſmuch that it is as the whole AB to the whole CD, ſo AE to CF: therefore alternly as BA to AE, ſo DC to CF. And becauſe compounded magnitudes are proportional, therefore divided they ſhall be proportional: wherefore as BE to EA, ſo DF to FC; and therefore alternly it is as BE to DF, ſo EA to FC. But as AE to CF, ſo by Suppoſition, is the whole AB to the whole CD: therefore alſo the Remainder EB ſhall be to the Remainder FD, as the whole AB to the whole CD.

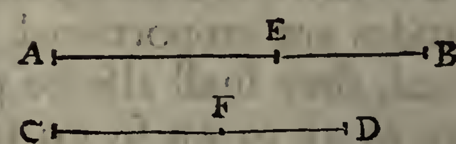
Wherefore if it be as the whole to the whole, ſo a part detracted to a part detracted, &c. Which was to be demonſtrated.

Corollary

Of Converſe Proportion.

From hence 'tis manifeſt, that if compounded magnitudes be proportional, they ſhall alſo by Converſion be proportional.

For let the compounded magnitudes be proportional as AB to BE, ſo CD to DF; therefore divided, it ſhall be as AE to EB, ſo CF to FD; And by inverſion as BE to EA, ſo DF to FC; therefore compounded as BA to AE, ſo DC to CF: which is Converſion of proportion, according to Def. 17. El. V. For AE is the exceſs of the Antecedent AB above the Conſequent BE: and CF the exceſs of the Antecedent CD above the Conſequent DF.



But this Corollary (however it hath hapned to be in this place) does more properly follow the 18th. Propoſition. As we have now ſhewn how Converſe proportion is immediately deduced from the Compoſition and Diviſion of Proportions, without any relation, or dependence on this 19th. Propoſition: where the four proportional Terms are reſtrained, and neceſſarily ſuppoſed to be all Homogeneal magnitudes. Whereas in Converſion of proportion the third and fourth Terms may be magnitudes of a different kind, from the firſt and ſecond Terms, and one proportion be in Lines, and the other correſpondent proportion in Planes, or Solids, as the following Annotations ſhall farther declare.

ANNOTATIONS.

This 19th. Propoſition and the two foregoing have a cloſe correſpondence with one another, as appears by their demonſtrations; yet there is a notable difference to be obſerved between them. For here is compared the whole to the whole, and each part of one whole to each part of the other reſpectively, ſo that theſe magnitudes are put Homogeneal, according to the Definition of proportion. But in Compoſition, Diviſion, Converſion of proportion, each whole is ſeparately compared to a part of it ſelf, and each part to the other part of the ſame whole. So that the four proportional Terms may be all of the ſame kind, either Lines, Planes, or Solids: or elſe of different: that is, the firſt and ſecond Terms, may be Lines, and the third and fourth be Planes, or Solids. As the whole AB may be a Lineal magnitude, and by conſequence the parts AE, EB are Lines. But again, the whole CD may be either a Plane, or a Solid, and by conſequence the parts CF, FD are accordingly Planes, or Solids; yet it is demonſtrated, as AB to AE, a Line to a Line, ſo is CD to CF, whether it be a Plane to a Plane, or a Solid to a Solid: Which general Analogy may be uſed in Compoſition, Diviſion, and Converſion of proportion.

But in this 19th. Propoſition, the whole AB is compared to the whole CD, and the parts to the parts: ſo that theſe are here ſuppoſed Homogeneal magnitudes, interchangeably compared to one another.

PROPOSITION XX.

IF there be three magnitudes in one Rank, and as many in an other, taken two and two in the ſame proportion: and ex æquo *the firſt be greater than the third, alſo the fourth ſhall be greater than the ſixth; and if equal, equal; if leſs, leſs.*

Let the three magnitudes be A, B, C, in one rank, and D, E, F, as many in an other, taken two and two in the ſame proportion, as A to B, ſo D to E; and as B to C, ſo E to F: and *ex æquo* let A be greater than C. I ſay, that alſo D ſhall be greater than F: and if equal, equal; if leſs, leſs.

12 . 9 . 6
A . B . C
: :   : :
D . E . F
8 . 6 . 4

Foraſmuch as A is greater than C, and there is a third magnitude B; And that the greater hath to the ſame a greater proportion than the leſs [Prop. 8. El. V.]; therefore A hath to B a greater proportion than C to B. But as A to B, ſo D to E: wherefore D hath to E a greater proportion than C to B [Prop. 13. El. V.]. But as C to B, ſo F to E by inverſion; therefore D hath to E a greater proportion than F to E. But of magnitudes having a proportion to one and the ſame, that which hath the greater proportion, is the greater [Prop. 10. El. V.]: Therefore D is greater than F.

In like manner we ſhall demonſtrate, that if A be equal to C, alſo D ſhall be equal to F; and if leſs, leſs.

Wherefore if there be three magnitudes in one rank, and as many in an other, &c. Which was to be demonſtrated.

PROPOSITION XXI.

IF there be three magnitudes in one rank, and as many in an other, taken two and two in the ſame proportion; and the Analogy be Perturbate: if then ex æquo *the firſt be greater than the third, alſo the fourth ſhall be greater than the ſixth; and if equal, equal; if leſs, leſs.*

Let the three magnitudes be A, B, C, in one rank, and D, E, F, as many in an other, taken two and two in the ſame proportion. And let the Analogy be Perturbate, as A to B, ſo E to F, and as B to C, ſo D to E: then *ex æquo* let A be greater than C. I ſay, that D ſhall be greater than F: and if equal, equal; if leſs, leſs.

18 . 12 . 4
A . B . C : : *
: :
* : : D . E . F
27 . 9 . 6

Foraſmuch as A is greater than C; and there is a third magnitude B; therefore A hath to B a greater proportion than C to B [Prop. 8. El. V.]. But as A to B, ſo E to F; wherefore E hath to F greater proportion than C to B. But becauſe it is as B to C, ſo D to E; therefore by inverſion as C to B, ſo E to D. But E hath F a greater proportion than C to B; therefore E hath to F a greater proportion than E to D. But to what magnitude the ſame hath a greater proportion, that is the leſs [Prop. 10. El. V.]; therefore F is leſs than D, and D greater than F: wherefore if A be greater than C, alſo D ſhall be greater than F.

18 . 12 . 4
A . B . C : : *
: :
* : : D . E . F
27 . 9 . 6

In like manner we ſhall demonſtrate, if A be equal to C, alſo D ſhall be equal to F; and if leſs, leſs.

Wherefore if there be three magnitudes in one rank, and as many in an other, taken two and two in the ſame proportion, &c. Which was to be demonſtrated.

PROPOSITION XXII.

IF *there be magnitudes how many ſoever in one rank, and as many in an other, taken two and two in the ſame proportion; alſo* ex æquo *they ſhall be in the ſame proportion.*

Let the magnitudes how many ſoever be A, B, C, in one rank, and D, E, F in another, taken two and two in the ſame proportion, as A to B, ſo D to E; and as B to C, ſo E to F. I ſay, that alſo *ex æquo* they ſhall be in the ſame proportion, as A to C, ſo D to F.

For of A, D let be taken equimultiples G, H. And of B, E any whatever equimultiples K, L: alſo of C, F any whatever equimultiples M, N.

G . K . M
A . B . C
: : : :
D . E . F
H . L . N

Foraſmuch then that it is as A to B, ſo D to E, and of A, D are taken equimultiples G, H: Alſo of B, E any whatever equimultiples K, L; therefore it is as G to K, ſo H to L [Prop. 4. El. V.]. By the ſame reaſon it is alſo as K to M, ſo L to N.

Foraſmuch now as there are three magnitudes in one rank, G, K, M; and as many in another, H, L, N, taken two and two in the ſame proportion; therefore *ex æquo* if G exceeds M, alſo H exceeds N; and if equal, equal; if leſs, leſs [Prop. 20. El. V.]: And G, H are equimultiples of A, D; alſo M, N any whatever equimultiples of C, F; therefore it is as A to C, ſo D to F [Def. 5. El. V.].

Wherefore if there be magnitudes how many ſoever in one rank, and as many in an other, taken two and two in the ſame proportion, &c. Which was to be demonſtrated.

ANNOTATIONS.

When there is a Concatenation of two, three, or more proportions, as of A to B, of B to C, of C to D, either in the ſame proportion, or in proportions different from one another, then in both theſe Caſes the proportion of the extremes, as of A to D, is in Def. 10th. and 11th. El. V. ſaid to be compounded of A to B, of B to C, and of C to D.

Now in this Propoſition there is put a double Series of Concatenate proportions, in each whereof the firſt Term is to the ſecond, as the firſt to the ſecond, and the ſecond to the third, as the ſecond to the third, and ſo forth, in an Ordinate Analogy: Then the extremes in each Series taken *ex æquo*, at equal diſtance, as the firſt to the third, or the firſt to the fourth, &c. are here demonſtrated to be proportional to one another, which is in effect to prove, that *two proportions compounded of equal proportions are equal to one another.*

PROPOSITION XXIII.

If there be three magnitudes in one rank, and as many in an other, taken two and two in the ſame proportion, and the Analogy be Perturbate: alſo ex æquo *they ſhall be in the ſame proportion.*

Let the three magnitudes be A, B, C, in one rank, and D, E, F, as many in another, taken two and two in the ſame proportion; and the Analogy be Perturbate, as A to B, ſo E to F; and as B to C, ſo D to E. I ſay, that *ex æquo* as A to C, ſo D to F.

G . H . L
A . B . C : : *
: :
* : : D . E . F
K . M . N

Of A, B, D, let be taken equimultiples G, H, K; Alſo of C, E, F, any whatever equimultiples L, M, N.

Foraſmuch as G, H are equimultiples of A, B; and parts have the ſame proportion with their equimultiples [Prop. 15. El. V.]; therefore it is as A to B, ſo G to H. And by the ſame reaſon, as E to F, ſo M to N. But as A to B, ſo E to F; therefore as G to H, ſo M to N [Prop. 11. El. V.].

Again, becauſe it is as B to C, ſo D to E; and of B, D are taken equimultiples H, K: alſo of C, E any whatever equimultiples L, M; therefore it is as H to L, ſo K to M [Prop. 4. El. V.].

Whereas now there are three magnitudes G, H, L, in one rank, alſo K, M, N, in an other, taken two and two in the ſame proportion, and the Analogy is Perturbate, as G to H, ſo M to N; and as H to L, ſo K to M; therefore *ex æquo* if G be greater than L, alſo K ſhall be greater than N; and if equal, equal; if leſs, leſs [Prop. 21. El. V.].

But G, K are equimultiples of A, D, and L, N, are any whatever equimultiples of C, F; therefore it is as A to C, ſo D to F [Def. 5. El. V.].

Wherefore if there be three magnitudes in one rank, and as many in an other, taken two and two in the ſame proportion, and the Analogy be perturbate: alſo *ex æquo* they ſhall be in the ſame proportion. Which was to be demonſtrated.

ANNOTATIONS.

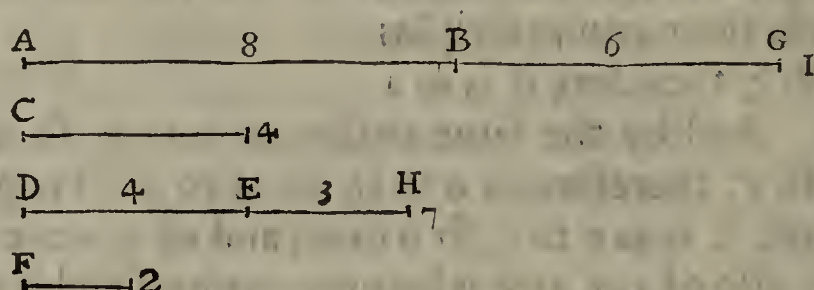
Euclide here propoſes only three magnitudes; altho' they might be *quotcunque*, like as in Prop. 22[d]. But becauſe in proportion *ex æquo* when the Analogy is Perturbate, the Geometrician never has occaſion to uſe more than three Concatenate Terms, or two proportions; therefore our Elementator puts only three magnitudes in this Propoſition: as taking ſpecial care to be no where in words or matter, defective or ſuperfluous, and ever has a proſpect to what is only and generally uſeful. Yet otherwiſe there is not any difference between Ordinate, and Perturbate Analogy in the Concluſions of theſe two Propoſitions; for as numbers, *viz.* 2, 3, 4, 5, &c. taken in any interchangeable order, and added together ſhall always give the ſame ſumm, or multiplied into one another, make the ſame Product; ſo in proportion *ex æquo*, the Terms placed either Ordinately, or Perturbately, ſhall notwithſtanding ever have their extremes compounded of the ſame intermedial proportions; and therefore the extremes taken *ex æquo*, ſhall always be proportional to one another: as here in magnitudes, and in Prop. 14[th]. and 22[d]. El. VII. is in numbers demonſtrated.

PROPOSITION XXIV.

If the firſt hath to the ſecond the ſame proportion, that the third hath to the fourth: and a fifth hath to the ſecond the ſame proportion, that a ſixth hath to the fourth: Alſo the firſt and fifth together, ſhall have to the ſecond the ſame proportion, that the third and ſixth hath to the fourth.

Let the firſt AB have to the ſecond C, the ſame proportion that the third DE hath to the fourth F. Alſo let a fifth BG have to the ſecond C, the ſame proportion, that a ſixth EH hath to the fourth F.

I ſay, that AG the firſt and fifth together, hath to C the ſecond, the ſame proportion, that DH the third and ſixth together, hath to F the fourth.



AB . C :: DE . F. and BG . C :: EH . F. Then AB + BG . C :: DE + EH . F.

8 . 4 :: 4 . 2. and 6 . 4 :: 3 . 2. Then 8 + 6 . 4 :: 4 + 3 . 2.

Foraſmuch as it is as BG to C, ſo EH to F; therefore by inverſion as C to BG, ſo F to EH.

And becauſe it is as AB to C, ſo DE to F, and as C to BG, ſo F to EH; therefore *ex æquo* as AB to BG, ſo DE to EH [Prop. 22. El. V.].

Now divided magnitudes being proportional, ſhall alſo compounded be proportional [Prop. 18. El. V.]; therefore as AG to GB, ſo DH to HE.

But alſo it is as GB to C, ſo E to F; therefore *ex æquo* as AG to C, ſo DH to F.

If therefore the firſt hath to the ſecond the ſame proportion, that the third hath to the fourth: and a fifth hath to the ſecond, &c. Which was to be demonſtrated.

ANNOTATIONS.

The 2^d^. Propoſition of this El. is the ſame with this 24^th^, excepting that it was there confined only to multiple proportions, whereas here is comprehended all kind of proportions whatſoever. The preſent Example is expoſed in Duple, and Seſquialteral proportions added to one another: As 8 the firſt in Duple proportion to 4 the ſecond; and 6 the fifth in Seſquialteral proportion to 4 the ſecond: where 8 the firſt, and 6 the fifth added together, make 14 to 4, or $\frac{14}{4}$; that is, Triple Seſquialteral proportion, or $3\frac{1}{2}$. So again, 4 the third to 2 the fourth; and 3 the ſixth to 2 the fourth, are in the ſame proportions as before 8 was to 4, and 6 to 4: And here likewiſe 4 the third, and 3 the ſixth added together, make 7 to 2, the ſame Triple Seſquialteral proportion, or $3\frac{1}{2}$. For 7 contains 2 thrice, and one half of 2. This Propoſition we have here both in *Species* and Numbers, repreſented after this manner.

8 . 4 : : 4 . 2 .
If A . B : : C . D. and
E . B : : F . D. Then
6 . 4 : : 3 . 2 .
A+E . B : : C+F . D
8+6 . 4 : : 4+3 . 2

Note farther hereupon, that the 18^th^. Propoſition is in ſome part, of the ſame nature with this 24^th^. For there the Conſequents added to the Antecedents, are as the fifth, and ſixth Terms added to the firſt, and third, making alſo four other proportionals: As thus Appears.

9 . 3 : : 6 . 2.
If A . B : : C . D. Then
A + B . B : : C + D . D
9 + 3 . 3 : : 6 + 2 . 2.

Or to ſet forth more fully the agreement between the 18^th^. Propoſition, and this 24^th^. As in this form.

If A . B : : C . D. and
B . B : : D . D. Then
A + B . B : : C+D . D.

The difference here between them is, that in the 18^th^. only the Conſequents are added to the Antecedents, and together compared to the ſame Conſequents: but in the 24^th^. any magnitudes, or numbers, which are in the ſame proportion to the given Conſequents, may be taken; and added to the given Antecedents, are together compared to the ſame Conſequents. And therefore in Prop. 18. the Conſequents being ſuppoſed as the fifth and ſixth Terms compared to themſelves as the ſecond and fourth Terms, they are always in proportion of equality: but in this 24^th^. the fifth and ſixth Terms, may be taken in any proportion whatſoever.

Addition

Addition of Proportions.

Moreover, from this 24th. Propofition, the Addition of Proportions is plainly difcovered: and how feveral proportions, which have the fame common Confequent, are united into one fingle proportion, by adding the Antecedents to one another, and making the fumm of them an Antecedent to the common Confequent: As 3 to 1, and 3 to 1, added together make 6 to 1, or $\frac{6}{1}$: That is, two Triple proportions, which have the fame common Confequent added together in this manner, do make Sextuple proportion. So again, 4 to 3, 5 to 3, 9 to 3 added together, make 18 to 3, or $\frac{18}{3}$: that is, Sextuple proportion is by addition of the Antecedents made of Sefquitertial, Sefquibitertial, and Triple proportions, when they have fuch a common Confequent.

But now when feveral proportions have feveral Confequents, then for to add thefe together, they are firft to be brought to have the fame common Confequent: As the proportions of 4 to 3, and of 3 to 2 cannot be added together, unlefs they be reduced unto a common Confequent. We have before in the Annotations upon the 8th. Propofition, given a Rule how to bring proportions to a common Confequent, like as Fractions are brought to others of the fame denomination. For the Addition, Subtraction, Multiplication, and Divifion of proportions, are performed after the fame manner as in Fractions: And proportions are as properly noted by placing the Antecedent above, and the Confequent beneath the interpofed ftrait line, as the Numerator and Denominator are ufed to be in Fractions. And therefore this particular Doctrine is to be fought after among the Arithmetical Authors.

Now in the foregoing Inftance to bring the proportions of 4 to 3, and of 3 to 2, or $\frac{4}{3}$ and $\frac{3}{2}$ unto a common Confequent, Firft multiply the Confequents 3 and 2 into one another; that is, 3 into 2, or 2 into 3, which by Prop. 16. El. VII. make the fame number 6, and this now is to be put for a common Confequent. Then multiply each Antecedent into the others Confequent, that is, 4 into 2, making 8 for one Antecedent, and 3 into 3, making 9 for the other Antecedent. Thus the proportions of $\frac{4}{3}$, and $\frac{3}{2}$ are brought unto $\frac{8}{6}$, and $\frac{9}{6}$, the very fame proportions as before, and having a common Confequent: For 4 . 3 : : 8 . 6 in proportion Sefquitertial, $1\frac{1}{3}$. and 3 . 2 : : 9 . 6 in proportion Sefquialteral, $1\frac{1}{2}$.

The reafon of this practice is manifeft, for $\frac{4}{3}$ multiplied by the fame number 2, make the Products $\frac{8}{6}$ in the fame proportion, by Prop. 17th. El. VII. or by Prop. 15. El. V. So $\frac{3}{2}$ multiplied by 3, make the fame proportion $\frac{9}{6}$.

Then of thefe two proportions $\frac{8}{6}$, and $\frac{9}{6}$, add the Antecedents 8 and 9 together, which make 17 the Antecedent to the common Confequent 6, or $\frac{17}{6}$: that is, $2\frac{5}{6}$, and named *Proportio dupla fuperquintu-partiens fextas.*

Therefore by reduction of the proportions $\frac{4}{3}$ and $\frac{3}{2}$, into a common Confequent, it now appears that thefe proportions, $1\frac{1}{3}$ and $1\frac{1}{2}$ added together make $2\frac{5}{6}$. The like practice is to be ufed in the Addition of all other proportions: but very little ufe there is made of it in the Mathematics, farther than what is fhewn in the 18th. and 24th. Propofitions of this Fifth Element. From whence we have taken occafion to touch briefly upon the Addition of proportions, and in what manner it can be only made. But the Multiplication of proportions is more remarkable: which therefore we fhall here by the way explain in a few words, for that there will be a neceffity to confider farther of this matter at the 5th. Definition of the Sixth Element: And in this place alfo to fhew the difference between the Multiplication, and Addition of proportions.

Multipli-

Multiplication of Proportions.

Multiplication of proportions is made by multiplying the Antecedents into the Antecedents, and the Consequents into the Consequents; whose Products shall make some certain proportion. As of the proportions $\frac{4}{3}$ and $\frac{3}{2}$, the Antecedents 4 and 3 multiplied into one another, and the Consequents 3 and 2 multiplied into one another, shall make $\frac{12}{6}$, or Duple proportion.

Whereas the same proportion $\frac{4}{3}$ and $\frac{3}{2}$, added to one another make $2\frac{5}{6}$; as we have before shewn. So $\frac{4}{2}$ and $\frac{3}{2}$, multiplied together make $\frac{12}{4}$ or $\frac{3}{1}$, Triple proportion. But added make $\frac{7}{2}$, or $3\frac{1}{2}$, proportion Triple Sesquialteral. Again, $\frac{3}{1}$ added to $\frac{3}{1}$, make $\frac{6}{1}$; but multiplied together make $\frac{9}{1}$. Likewise $\frac{3}{1}$ added to $\frac{2}{1}$ make $\frac{5}{1}$; but multiplied make $\frac{6}{1}$. Thus it appears, how proportions may by Addition arise to be sometimes greater, sometimes less than by Multiplication: which is a Property peculiar to proportions, and somewhat remarkable.

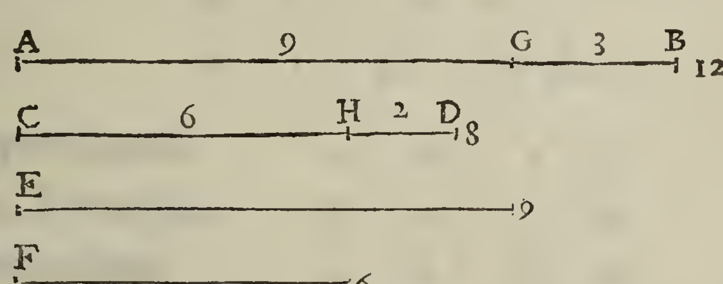
PROPOSITION XXV.

If four magnitudes be proportional, the greatest and least are greater than the other two.

Let the four proportional magnitudes be AB, CD, E, F; and as AB to CD, so E to F. Now let the greatest of them be AB, and the least F. I say, that AB, F, are greater than CD, E.

For to E let AG be put equal; and to F be put equal CH: forasmuch therefore that it is as AB to CD, so E to F: and that AG is equal to E, and CH to F; therefore as AB to CD, so AG to CH [Prop. 11. El. V.].

Also because it is as the whole AB to the whole CD, so the part detracted AG, to the part detracted CH; therefore the Remainder GB, shall be to the Remainder HD, as the whole AB to the whole CD [Prop. 19. El. V.]. But the whole AB is [by Supposition] greater than the whole CD; therefore also the Remainder GB is greater than the Remainder HD.



And forasmuch as AG is equal to E, and CH to F; therefore AG and F are equal to CH, and E. For if equals be added to equals, the wholes are equal [Ax. 2.].

But again, if to unequals be added equals, the wholes shall be unequal [Ax. 4.]: Therefore if to the unequals GB, HD be added these equals, F, AG, and E, CH, namely, F, and AG added to GB the greater, and E, and CH to HD the less; then shall AB, and F be greater than CD, and E.

If therefore four magnitudes be proportional, the greatest, and least are greater than the other two. Which was to be demonstrated.

 Adver-

Advertifement.

Here *Euclide* ends this Element: but the Moderns have (out of *Pappus*) annexed feveral other Propofitions; which becaufe they are only manifeft Confectaries arifing from the foregoing Propofitions, and not of fo frequent ufe, we have thought fit not to overcharge the younger Students with. For *Peletarius* here very well fays, *Quæ* (*propofitiones*) *per fe claræ funt, locum tantum occupant: Ingenium etiam onerant, & multitudo tædium parit.* What *Euclide* hath delivered is abundantly fufficient in this matter.

ANNOTATIONS.

This property of proportional magnitudes hath been formerly intimated in Prop. 35th. and 36th. El. I. where Parallelograms on the fame, or equal bafes in the fame Parallels, are demonftrated to be equal; that is, to be to one another in the fame proportion of Equality. And there alfo it hath been obferved, that of thofe equal Parallelograms their *Perimeters* are unequal, and of two equal Parallelograms the longeft, and fhorteft fides, are greater than the other fides. Now this Propofition demonftrates the like Affection of four proportionals univerfally in all kinds of proportion, and in all kinds of magnitude, either Lines, Planes, or Solids: and that in each of them the greateft, and leaft Terms added together, fhall always be greater than the other two.

A SYNOPSIS OF THE PRINCIPAL PROPOSITIONS OF THE FIFTH ELEMENT.

Prop. 1, and 12.	If A . B : : C . D : : E . F. Then
	A . B : : A + C + E . B + D + F.
Prop. 2, and 24.	If A . B : : C . D . and
	E . B : : F . D . Then
	A + E . B : : C + F . D.

Prop. 3, 22, and 23. Proportionals *ex æquo* Ordinate and Perturbate.

Prop. 4.	If A . B : : C . D. Then
	3A . 5B : : 3C . 5D &c.
Likewise	3A . B : : 3C . D &c.
and	A . 3B : : C . 3D. &c.
Let	Æ = A + E . and æ = a + e
Prop. 5, and 19.	If Æ . æ : : A . a. Then
	E . e : : Æ . æ.
Prop. 15.	A . B : : 3A . 3B &c.

Primary Proportionals,	A . E : : B . C.
Corol. Prop. 4. Inversion,	E . A : : C . B.
Prop. 16. Alternation,	A . B : : E . C.
Prop. 17. Composition,	A + E . E : : B + C . C.
Prop. 18. Division,	A − E . E : : B − C . C.
Corol. Prop. 19. Conversion,	A . A − E : : B . B − C.

Note farther, that Composition, Division, Conversion of proportionals, may be again, and frequently are proportionally varied by Inversion, and Alternation after this manner.

Composition and Alternation,	A + E . B + C : : E . C
Division and Alternation,	A − E . B − C : : E . C. Therefore
Composition and Division,	A + E . B + C : : A − E . B − C. by Prop. 11.

Thus

Thus hath *Euclide* finiſhed his general Doctrine of *Proportions*, and *Proportionals* in magnitudes: which we confeſs is more plainly ſet forth in his Elements of Numbers. For tho' the Properties of Proportionals be alike in Magnitudes and Numbers; yet are they demonſtrated in thoſe Elements from more natural Principles, and more obvious to the common notions of men in this matter. We have formerly expounded the different Definitions of proportional Magnitudes, and of proportional Numbers, and ſhewn the reaſons thereof for want of a natural meaſure in Magnitudes. And therefore what evidence, or what demonſtrations could in this Element be expected, or made from a certain meaſure between proportional Magnitudes (as it is in Numbers) where there is in nature no certain meaſure? Nay farther, where there cannot be put, or ſuppoſed any imaginable common meaſure. Number is only one way infinite by augmentation; but hath its indiviſible Monade for a beginning, and common meaſure, the natural inſtrument of Demonſtration in the Doctrine of proportions, and accordingly made uſe of in the Elements of Numbers. But Magnitude is both ways infinite, by Augmentation, and Diminution. It hath not a Monade, that *quid Minimum*, *unum & ſolum*, That one and ſole meaſure of all things. What foundation it hath uſeful for Demonſtration in this Subject, *Euclide* hath moſt ingeniouſly found out, and given us in the fifth Definition of proportional Magnitudes. And it is either plain Ignorance, or great Vanity in thoſe, who charge this Element of Intricacy, and Imperfection, which is framed with ſo much Art, and in as clear a Manner, as the nature of Magnitude could admit.

The *Epicheiremata*, or Attempts of thoſe Learned Geometricians *Joannes Benedictus*, *Tacquet*, *Borellus*, &c. in this matter I leave to be at large examined by the Profeſſors of Geometry in our Univerſities, and *Greſham* College.

THE

THE SIXTH

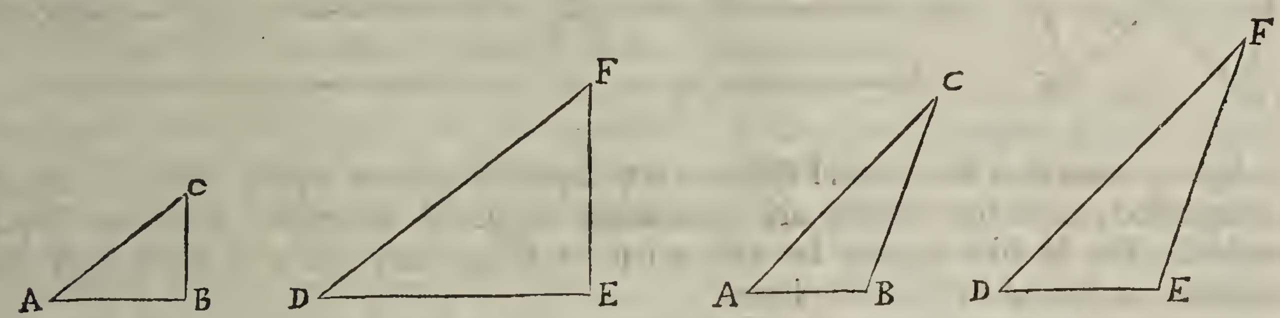
ELEMENT.

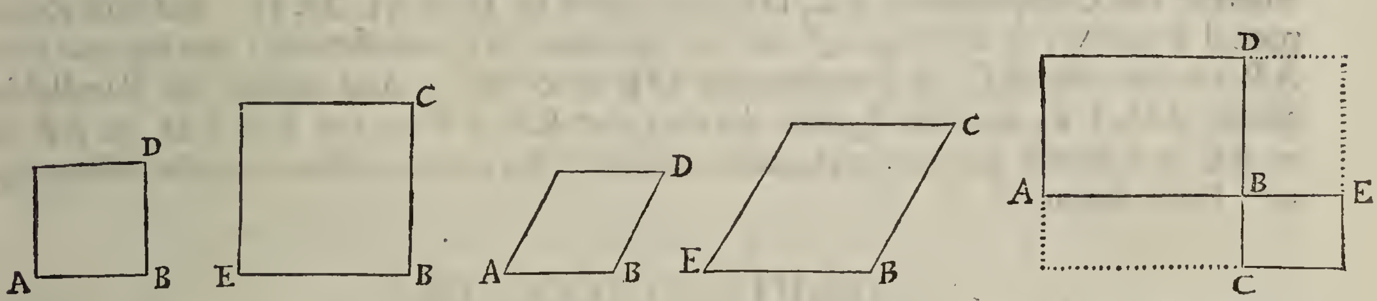
DEFINITIONS.

DEFINITION I.

Like ſtrait-lin'd Figures are thoſe, which have their ſeveral angles equal, each to each, and the ſides about the equal angles proportional.

In the Triangles A B C, D E F, if the ſeveral angles be equal, each to each, A to D, B to E, C to F, and the ſides about the equal angles proportional, A B to B C, as D E to E F; B C to C A, as E F to F D; C A to A B, as F D to D E, then theſe are ſaid to be like Triangles.



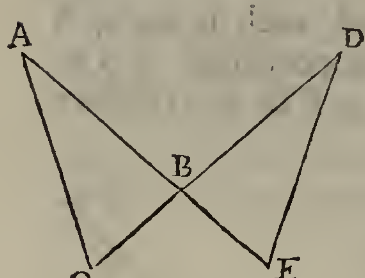


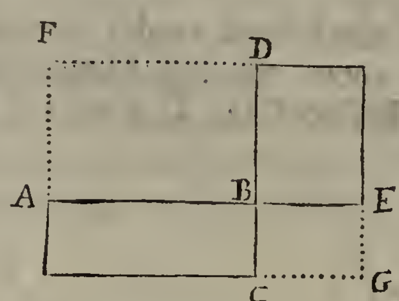
So again, in the Equiangled Parallelograms A D, E C, if A B be to B D as E B to B C, then theſe are called like Figures: And ſo forth in all Multilateral Figures, or Polygons. But Equiangled Parallelograms are not always like Figures; as it frequently happens in Right-angled Parallelograms, and Equiangled Rhomboids, which are not like Figures; unleſs they have alſo their ſides about the equal Angles proportional. But every Square is to every Square, and every Equiangled Rhombus to every Equiangled Rhombus a like Figure: becauſe they have their ſides in the ſame proportion of Equality to one another.

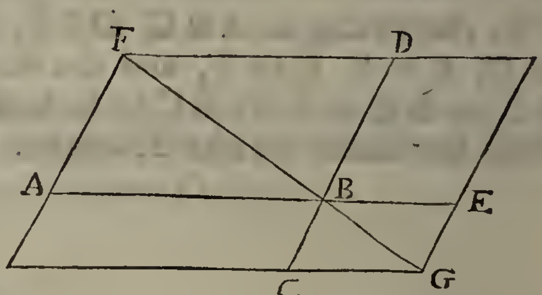
DEFINITION II.

Reciprocal Figures are, when in each of the Figures there are Terms both Antecedent and Consequent.

As in the Triangles ABC, DBE, and the Parallelograms AC, DE, if AB is to BE, as back again DB is to BC, then these are called Reciprocal Figures. In which, note for distinction between Like and Reciprocal Figures, that the extreme Terms of the two proportions in Reciprocal Figures; namely the first Antecedent, and the last Consequent, shall ever be in the same figure: whereas in like figures the first Antecedent is in one figure, and the last Consequent always in the other. Moreover, in the foregoing Definition of like Figures, all the Angles are to be equal, each to each, and all the sides about the equal Angles directly proportional to one another. But it is not necessary in all Reciprocal Figures to have the several Angles equal, but only that Angle, which is contained by those sides which are Reciprocally proportional to one another. For in the other Angles the figures may be very often unlike, and not agree in equal Angles, and sides proportional, as hereafter in Triangular Figures the 15th. Proposition will manifestly shew.







Again, note that Reciprocal Figures are supposed always equal, tho' not always Equiangled; and like figures are commonly supposed unequal, and ever Equiangled. For if like figures be also equal in Area, they are as it were only the same figure seated in several places.

Moreover, the Parallelograms FG, FG are in this matter very remarkable Schemes, wherein the Complements AC, DE, are equal by Prop. 43. El. I; and also Reciprocal Figures (as at Prop. 14th. of this Element is demonstrated) having the side AB to the side BE, as Reciprocally DB is to BC. And again, the Parallelograms AD, CE, are like figures, having the side AB to the side BD, as EB is to BC in a direct proportion to one another. As is demonstrated in the following 24th. Proposition.

DEFINITION III.

A Strait line is said to be cut in extreme and mean proportion, when it is as the whole to the greater Segment, so the greater Segment to the less.

If the line AB be divided at the point C in such a proportion, that as the whole AB is to the greater Segment AC, so is the same AC to the less Segment CB, then the line AB is said to be cut in extreme and mean proportion,

For that the whole line, and the less Segment are the two extremes, and the greater Segment is a *mean* or middle proportional Term between them. And so the whole line, the greater Segment, and the less are all three in the same continual proportion: Which by our incomparable Mr *William Oughtred*, is justly said to be *Sectio penè Divina*, and how to effect this Section is demonstrated at Prop. 30th.

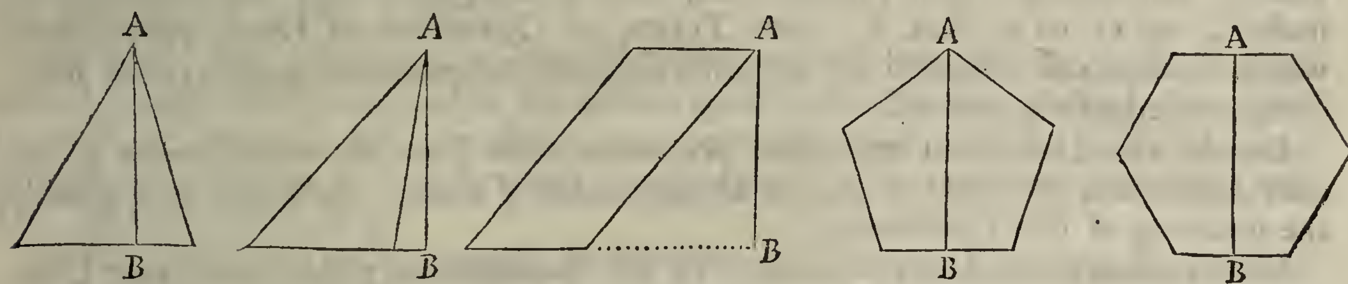
3 DEFI-

DEFINITION IV.

THe Altitude of every Figure is a strait line drawn from the Vertex perpendicular to the Base.

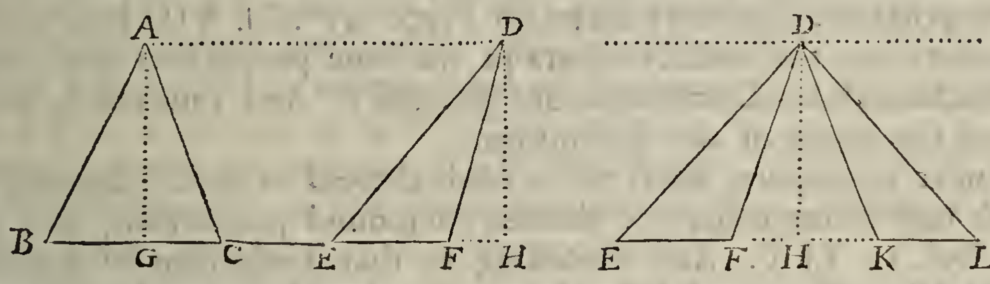
The Altitude, or Height of any thing, is vulgarly accounted to be the diſtance from the Top to the Bottom; and by diſtance is meant the ſhorteſt length between them, becauſe the ſhorteſt length is the only Singular, Certain, and Determinated Space between any two diſtant things.

So the Geometrician determines the Altitude of any Figure by a Perpendicular from the Vertex of the Figure to the oppoſite Baſe (produced if need be); as in theſe expoſed Figures their Altitudes are the Perpendiculars A B, A B, &c; for that they are in common ſenſe, and alſo demonſtratively the ſhorteſt length between the Vertex and the Baſe.



Therefore when Figures are ſaid to be of the ſame Altitude, we are to underſtand that the Perpendiculars from their Vertex to their Baſes are equal to one another, and therefore may be placed in the ſame parallels.

For let the Triangles A B C, D E F, be of the ſame Altitude, having the Perpendiculars A G, D H equal to one another. I ſay, the Triangles A B C, D E F, are in the ſame parallels. For let be drawn the ſtrait line A D: Then foraſmuch as A G, D H are equal, and at right angles to G H; therefore they are alſo parallel one to another [Prop. 28. El. I.]. But ſtrait lines equal and parallel are bounded by equals, and parallels [Prop. 33. El. I.]; therefore A D is parallel to G H, and the Triangles A B C, D E F, are in the ſame parallels.



Again, if the Triangles D E F, D K L, have a common Vertex D, ſo by conſequence a common Altitude D H, which is a Perpendicular to their continued Baſes; it is then alſo obvious, that there may be drawn by the ſame Vertical point D, a parallel to the oppoſite Baſes [by Prop. 31. El. I.].

So that in general, Triangles, and Parallelograms to be in the ſame Altitude, and in the ſame Parallels import the ſame thing.

DEFI-

DEFINITION V.

A Proportion is ſaid to be compounded of proportions, when the Quantities of the proportions multiplied into themſelves, do make ſome proportion (or ſome Quantities of a proportion).

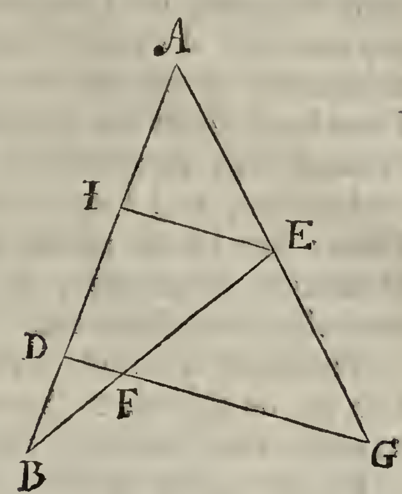
There are various Lections of this place; Some Editions have it, ποιῶσι τινὰ λόγον, *do make ſome proportion*: but the more ancient Reading is, ποιῶσι τινας, *ſcil.* πηλικότητας, that is, the Quantities of the proportions multiplied into one another *do make ſome Quantities*, to wit, *of a proportion*: which proportion being thus produced from the multiplied proportions, is ſaid to be compounded of them. As of the given proportions, let one be Seſquitertial $1\frac{1}{3}$, and the other Seſquialteral $1\frac{1}{2}$: As 4 to 3, and 3 to 2, or $\frac{4}{3}$, and $\frac{3}{2}$, which multiplied into one another, that is, Antecedents into Antecedents, and Conſequents into Conſequents, do make $\frac{12}{6}$, or 12 to 6, that is, ſome Terms, or Quantities of Duple proportion; which is here compounded of Seſquitertial, and Seſquialteral proportions, *ſcil.* $\frac{2}{1}$ compounded of $1\frac{1}{3}$, and $1\frac{1}{2}$.

For by this Definition *whatſoever proportion ariſes from the multiplication of any other proportions, the ſame is ſaid to be compounded of them*: And this is in general the meaning of this Definition.

Now farther to explain the words. By the *Quantities* of proportions our Commentators would have the Denominators to be underſtood: But it is not material whether by the *Quantities* we underſtand either the *Denominators*, or the *given, and expoſed Numbers* of the proportions, or the *leaſt Numbers* of the ſame proportions: For that each of theſe being multiplied together, do produce the ſame compound proportion, tho' in different Numbers, and alſo in ſuch, as are far enough from being the *Denominator*, or the *leaſt Numbers* of that compound proportion; as the foregoing Inſtances do apparently ſhew: only the Arithmetical operations are performed with more eaſe in the leaſt Numbers: otherwiſe the effects are ſtill the ſame. And the moſt proper uſe of Denominators is to diſcern the Species of a proportion expoſed in any Numbers whatſoever, and to denote the name. But *Euclide* in his Elements of Numbers never mentions Denominators, or the Quantities of proportions. He only ſhews in Prop. 35th. El. VII. how to bring any given Numbers into the leaſt Numbers in the ſame proportion with them: upon which Reduction all his Demonſtrations proceed: And thus much for the interpretation of the words of this Definition.

We are next to enquire, what uſe is made thereof in theſe Elements, in regard that *Euclide* had before otherwiſe defined compound proportion, as it is comprehended in Def. 10. El. V. And according to that Definition he demonſtrates in Prop. 23. El. VI. *All equiangled Parallelograms to have to one another a proportion compounded of their ſides*: And likewiſe in Prop. 5. El. VIII. *All Plain, or Superficial Numbers to have a proportion compounded of their ſides.* The demonſtrations both in Magnitudes and Numbers, are framed juſt after the ſame manner upon the 10th. Definition of the Fifth Element, without the leaſt uſe, or any mention of the multiplication of proportions according to this 5th. Definition of this Sixth Element. And ſtrange it ſeems to me, nay very abſurd, to admit that for an Elementary Definition, which is never uſed either in theſe Elements of *Euclide*, or in the *Conics* of *Apollonius*, or elſewhere in *Archimedes*: but in all theſe, and other Geometrical Authors, Compoſition of proportion is ever taken in the ſenſe, and notion of Def. 10th. El. V. It will not therefore be impertinent to examine how another uſeleſs Definition of Compound proportion, and which beſides can only relate to numbers, came to be ſo *Ungeometrically* inſerted in this place, where the proportions of Magnitudes are ſolely conſidered, and the uſe of numbers (as altogether improper, and alſo inſufficient for demonſtration in theſe 5th. and 6th. Elements) is by *Euclide* ſtudiouſly avoided.

The moſt ancient account I have met with, is in *Theon's* Commentaries on *Ptolemey's Mathematical Syntaxis*, Lib. I. cap. 12. entitled *Prolambanómena*, where, ſays *Ptolemey*, Let from the point A be produced two lines A B, A G, and from the points B, G, be drawn B E, G D, cutting each other in F; and draw E I parallel to G D. I ſay, that the proportion of G A to A E is compounded of the proportions of G D to D F, and of F B to B E.



Now to prove this, he introduces an other extraneous line, *viz.* D F taken ἔξωθεν (ſays *Ptolemy*) *extrinſecus*, or *ab extra*, as it were *from without*; and interpoſing D F for a middle Term between G D, and E I, does aſſume that the proportion of G D to E I, is compounded of the proportions of G D to D F, and of D F to E I. As a compound proportion is underſtood by *Euclide* in Def. 10. El. V.

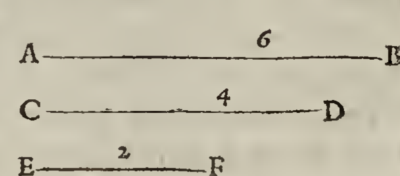
But now altho' in a Series of continued Terms, this be called by *Euclide* a compound proportion (as he might *pro arbitrio* give ſuch a name to the proportion of the extreme Terms); Yet a queſtion may be made, How doth it appear that the proportion of G D to E I, is really compounded of the Proportions of G D to D F, and of D F to E I?

In ſatisfaction to ſuch demands, and for the explication of this place, and therewithal of *Euclid's* Definition, *Theon* in his Comment, *Page* 62, gives an other Definition of Compound proportion in theſe words.

"A proportion is ſaid to be Compounded of two, or more proportions, when "the Quantities of thoſe proportions being multiplied do make ſome Quantity of "a proportion.

The uſe now that *Theon* makes of this Definition, is to explain *Euclid's* 10th. Def. El. V. and to ſhew how both agree in producing one, and the ſame Compound proportion; as ſuppoſing his own Definition to be a more natural, and clearer Notion of a Compound proportion than that of *Euclid's*; and thereby to confirm the 10th. Def. of El. V. He therefore with *Euclide* puts two proportions in a Series of three continued Terms.

For (ſays he) Let A B have to C D any given proportion (as 6 to 4) and C D to E F any proportion (as 4 to 2). I ſay, that the proportion of A B to E F, is compounded of the proportions of A B to C D, and of C D to E F: that is, if the Quantity of the proportion of AB to C D be multiplied into the Quantity of the proportion of C D to E F, it ſhall make the Quantity of the proportion of A B to E F (that is of 6 to 2.).



For let the given Proportions be 6 to 4, and 4 to 2: then $\frac{6}{4}$ multiplied into $\frac{4}{2}$, make $\frac{24}{8}$, the ſame Triple proportion with $\frac{6}{2}$, which is the proportion of the firſt Term to the third. And this *Theon* univerſally demonſtrates, *viz.* That in any three Terms (ſuppoſe A, B, C), if the proportion of A the firſt to B the ſecond, be multiplied into the proportion of B the ſecond to C the third, then the Product ſhall always make the ſame proportion with that of A the firſt to C the third. This proportion of the firſt Term to the third, *Euclide* in Def. 10th. El. V. calls a proportion Compounded of the two intermedial proportions expoſed in three continued Terms, *viz.* Of the proportion of the firſt Term to the ſecond, and of the ſecond to the third: and owns no other notion of a Compound proportion in the demonſtrations of Prop. 23d. El. VI. and Prop. 5th. El. VIII. or elſewhere.

But becauſe in *Euclid's* Series of three, four, or more continued Terms, the proportion of the firſt Term to the laſt, does not ſo immediately appear to be Compounded of the Intermedial, and Concatenate proportions; therefore *Theon* joyns the Philoſopher with the Geometrician, and gives another very natural Definition

 of

of a Compound proportion. For in *Theon's* multiplication of Antecedents into Antecedents, and Conſequents into Conſequents, there is manifeſtly made a natural Mixture, or Compoſition of proportions: which being thus involved into one another, do really produce a Compound proportion; and alſo ſuch, as he demonſtrates to be the very ſame with *Euclid's*, according to Def. 10th. El. V. which *Theon* hath from his Definition thus illuſtrated, and confirmed. So that this multiplication of proportions may well be allowed to a Commentator for a good Animadverſion, and Explication of Def. 10th. El. V. as here an occaſion was given to *Theon*: but by no means to ſtand ſo improperly, and uſeleſly for a Geometrical Definition of *Euclid's*, as it hath been by ſome *Scholiaſt* unadviſedly transferred into this Sixth Element.

The next we find is *Eutocius* in his Comment on *Archimedes*, who in Prop. 4. Lib. II. *de Sphærâ, & Cylindro*, propoſes to cut a Sphere ſo that the Segments may have to one another the ſame proportion with any given.

In the courſe of his demonſtration he uſes Compound proportion in *Euclid's* ſenſe, according to Def. 10th. El. V. and aſſumes that the proportion of RL to LQ (συνῆπται as *Archimedes* words it) is *connected by*, or (as commonly ſaid συγκεῖται) is *compounded of* the proportions, which RL hath to LD, and LD to LQ.

Hereupon ſays *Eutocius* in his Comment, 'tis manifeſt that Compoſition of proportions is taken as in the Elements (meaning Def. 10th. El. V.) in that LD is interpoſed as a middle Term between RL, and LQ. For (ſays he) if between two Numbers, or Magnitudes, be taken any middle Term, the proportion of theſe firſt Numbers or Magnitudes, is compounded of the proportion which the firſt Term hath to the middle, and of the proportion which the middle hath to the third.

But farther adds, that this is ſpoken ἀδιαρθρώτως, ſomewhat *inarticulately* (or abruptly, as if matters were not well jointed together) καὶ οὐχ οὕτως, ὥστε τὴν ἔννοιαν ἀναπληρῶσαι, *and not in ſo plain a manner, as fully to ſatisfy our underſtanding,* (*to wit*) in the proper notion of a Compound proportion. Therefore to make *Euclid's* Compoſition of proportions more perſpicuous, he cites this other Definition of a Compound proportion, as being alſo found by him in the Elements (ſo early it ſeems to have been tranſmitted into this place) and demonſtrates that

"If between two Numbers or Magnitudes, be taken any middle Term, then the "proportion of thoſe firſt Numbers or Magnitudes, to one another, ſhall be the "ſame with that Compound proportion, which is made out of the proportions "of the firſt Term to the middle, and of the middle to the third, multiplied into "one another.

Now by the demonſtration of this agreement, *Eutocius* intended to make manifeſt, that in any three continued Terms, the proportion of the firſt to the third is rightly ſaid by *Euclide* in Def. 10th. El. V. to be a Compound proportion of the firſt to the ſecond, and of the ſecond to the third.

Moreover to illuſtrate this matter, he gives ſeveral inſtances in Numbers. Let (ſays he) between 12 and 2, be interpoſed any number, as 4; then the proportion of 12 to 2, that is, Sextuple, is Compounded of the proportions of 12 to 4, and of 4 to 2, of Triple, and Duple proportions. For $\frac{12}{4}$ multiplied into $\frac{4}{2}$ makes $\frac{48}{8}$, that is $\frac{6}{1}$, the ſame proportion with $\frac{12}{2}$ the firſt given Terms.

In this inſtance of *Eutocius*, note that the two proportions $\frac{12}{4}$, and $\frac{4}{2}$, are each of the greater inequality: and therefore the proportion $\frac{48}{8}$ reſulting from their multiplication, is really *à totum Compoſitum*, a greater proportion conſiſting of two leſs proportions, as a whole of ſo many partial Components; that is, a Sextuple proportion is here rightly compounded, and made up of a Triple, and Duple proportion multiplied into one another. And every Compound proportion is always a *totum* of the like nature, when the intermedial proportions are all of the greater inequality.

But

But now on the contrary, in proportions of the leſs inequality it is to be noted, that if between 2 and 12, be again interpoſed 4; then $\frac{2}{4}$, and $\frac{4}{12}$, multiplied into one another, ſhall make $\frac{8}{48}$, or $\frac{1}{6}$, a Subſextuple proportion, which is emergent from Subduple, and Subtriple proportions; yet does not, as in the intermedial proportions of the greater inequality, conſiſt of theſe intermedials, as a whole of its parts; but contrarywiſe is as it were Subcompounded, and becomes to be a leſs proportion derived from two greater. For $\frac{1}{6}$ is a proportion leſs than either $\frac{1}{2}$, or $\frac{1}{3}$, from both which it doth deſcend; and the like will be ever found, when all the intermedial proportions are of the leſs inequality.

Again ſays *Eutocius*, Let between 9, and 6 be interpoſed 4; then the proportion of 9 to 6, that is Seſquialteral $1\frac{1}{2}$, is compounded of 9 to 4, and of 4 to 6, that is, of Duple Seſquiquartal, $2\frac{1}{4}$, and of Subſeſquialteral $\frac{4}{6}$. For $\frac{9}{4}$ multiplied into $\frac{4}{6}$ makes $\frac{36}{24}$, a proportion Seſquialteral, and the ſame with $\frac{9}{6}$, the firſt given Terms.

In this inſtance 'tis to be noted, that $\frac{36}{24}$, a proportion Seſquialteral reſulting from $\frac{9}{4}$ multiplied into $\frac{4}{6}$, is leſs than $\frac{9}{4}$, a proportion Duple Seſquiquartal, and greater than $\frac{4}{6}$, a proportion Subſeſquialteral. And the like will ever happen, that when of the intermedial proportions ſome are of the greater inequality, and ſome of the leſs, then the Compound proportion ariſing from them may be ſometimes greater, ſometimes leſs than ſome of the Components: yet it is truely made by a mixture, and as it were a temperature of all the intermedial proportions. For proportions of the leſs inequality are an allay to proportions of the greater inequality, and may ſo counterpoiſe one another, that the ſame proportion may from different mixtures be diverſly compounded.

As again to inſtance with *Eutocius* in Seſquialteral proportion. Let between 6 and 4, or $\frac{6}{4}$, a proportion Seſquialteral, be interpoſed 2, then $\frac{6}{4}$ is compounded of $\frac{6}{2}$, and $\frac{2}{4}$, that is, of a Triple, and of a Subduple proportion. For $\frac{6}{2}$ multiplied in $\frac{2}{4}$ makes $\frac{12}{8}$, a proportion Seſquialteral, and the ſame with $\frac{6}{4}$, the firſt given Terms. Thus Seſquialteral proportion is here compounded of Triple, and Subduple proportions, which in the numbers of *Eutocius* was made out of Duple Seſquiquartal, and Subſeſquialteral proportions.

There are therefore theſe three different Conſtitutions of a Compound proportion taken notice of by *Theon*, and *Eutocius*, as they ariſe from proportions, which are either all of the greater inequality, or all of the leſs, or of the greater, and leſs intermingled with one another: as the demonſtrations of *Theon*, and *Eutocius* (which may readily be found in *Clavius*) have diſtinctly comprehended.

Laſtly if all the intermedials be proportions of equality, then the proportion of the firſt Term to the laſt is ſaid to be compounded of a Duplicate, or Triplicate, &c. proportion of the firſt to the ſecond, according to the number of the intermedial proportions, as they happen to be two, three, four, five, or more indefinitely.

But theſe four ſeveral Conſtitutions of a Compound proportion (whether it ariſes from proportions of equality, or of the greater inequality, or of the leſs, or of the greater, and leſs intermixed) may in general be at once demonſtratively made evident in *Species*, or Symbolical computation.

For let there be (according to Def. 10th. El. V.) a Series of continued Terms, as A, B, C, D, E, which repreſent any Concatenate proportions whatſoever. I ſay with *Euclide*, that the proportion of the extremes of A to E, is compounded of the proportions of A to B, of B to C, of C to D, of D to E, that is, the proportions of $\frac{A}{B}$, $\frac{B}{C}$, $\frac{C}{D}$, $\frac{D}{E}$, multiplied together, Antecedents into Antecedents, and Conſequents into Conſequents, ſhall make the ſame proportion with that of A to E, as *Theon* and *Eutocius*, have expounded *Euclid's* Definition.

For let the Antecedents A, B, C, D, be multiplied together, and alſo the Conſequents B, C, D, E, after this form: Then the intermedial Antecedents B, C, D, being expunged by the intermedial Conſequents B, C, D, there only ſtands the proportion of A to E, of the firſt Term to the laſt. In this Symbolical form is laid open at one view the whole Myſtery (if any there be) of Compound proportions; which *Theon*, *Eutocius*, and *Vitellio Opticorum*, Lib. I. Prop. 13. have laboriouſly demonſtrated.

$$\frac{A\ B\ C\ D}{B\ C\ D\ E}$$

To conclude therefore this matter, when there are put three magnitudes, as in Def. 10th. El. V. in a continued Series compared to one another; *Euclide* calls the proportion of the firſt to the third a proportion compounded of the proportion of the firſt to the ſecond, and of the proportion of the ſecond to the third. Now *Theon* explains this Definition by an other, in which the nature of a Compound proportion ſeemed to be made more evident: and ſhews that in *Euclid's* Series of three continued Terms,

"If the proportion of the firſt to the ſecond be multiplied into the proportion "of the ſecond to the third, there ſhall ever be made the proportion of the firſt "to the third.

Here now *Theon* ſuppoſes his Definition to be an obvious, and natural notion of a Compound proportion, by which multiplication of proportions, and their agreement with *Euclid's* Compoſition of proportions laid down in a Series of continued Terms, he thought it a proper explanation and confirmation of *Euclid's* Definition, which indeed we readily acknowledge, as alſo that it was the firſt real, and ſecret ground of Def. 10th. El. V.

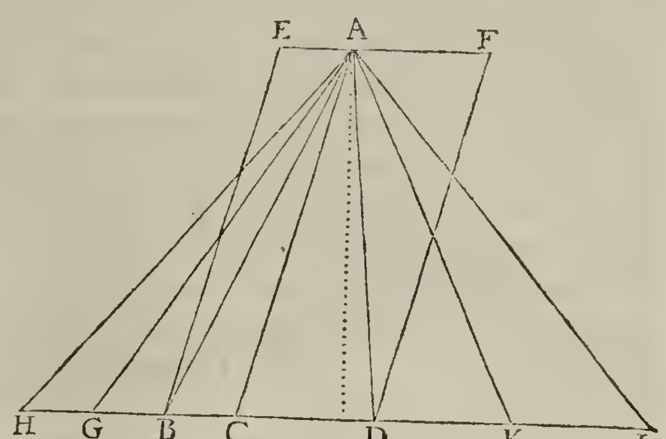
But yet *Euclide* found this kind of multiplication to be in no manner ſerviceable for demonſtrating all Compound proportions, even no more than his Definition of Proportional Numbers could be applied to Proportional Magnitudes. Therefore he ſubſtitutes an other of a more general, and uſeful form, as firm and true, tho' not ſo perſpicuous. For in Def. 10th. El. V. the Series of Concatenate proportions ſhews not ſo evidently the Geneſis, and Production of a Compound proportion ariſing from them, as *Theon's* multiplication of proportions does moſt naturally, and immediately ſuggeſt to our common underſtanding. And we muſt confeſs that *Euclid's* two eminent Definitions of Proportional Magnitudes, and of Compound proportions, lye under the like difficult circumſtances, and that both are taken up at the ſecond hand upon meer neceſſity. Yet are they ſo admirably contriv'd for general demonſtrations in thoſe concerns, that *Euclide* hath no where elſe given a more manifeſt teſtimony of an exquiſite judgement, and through inſight into all the Mathematics, than in the invention of thoſe two Definitions: which the ancient Geometricians who had ſearched this whole buſineſs to the very bottom ſaw juſt reaſon to receive, and uſe without exception, or any endeavour to amend them, or deviate from *Euclid's* method in this matter.

PROPO-

PROPOSITION I.

T*Riangles and Parallelograms of the ſame Altitude, are to one another as their Baſes.*

Let the Triangles be ABC, ACD, and the Parallelograms EC, CF, having the ſame altitude, the perpendicular drawn from A to BD. I ſay, that as the baſe BC is to the baſe CD, ſo the Triangle ABC, is to the Triangle ACD; and the Parallelogram EC to the Parallelogram CF.



Let BD be produced both ways to the points H, L; and to the baſe BC let be put equals how many ſoever BG, GH. Again, to the baſe CD, let be put equals how many ſoever DK, KL; and let be joyned AG, AH, AK, AL.

Now foraſmuch as CB, BG, GH, are equal to one another; therefore the Triangles AHG, AGB, ABC, are equal to one another [Prop. 38. El. I.]; wherefore Quotuple the baſe HC is of the baſe BC, Totuple is the Triangle AHC of the Triangle ABC. By the ſame reaſon Quotuple the baſe LC is of the baſe CD, Totuple is the Triangle ALC of the Triangle ACD.

Now if the baſe HC be equal to the baſe CL, the Triangle AHC is alſo equal to the Triangle ALC [Prop. 38. El. I.]: and therefore if the baſe HC exceeds the baſe CL, the Triangle AHC does alſo exceed the Triangle ALC: And if the baſe be leſs, the Triangle is alſo leſs [by Corol. Prop. 38. El. I.].

There being then four magnitudes the two baſes BC, CD, and the two Triangles ABC, ACD: and of the baſe BC, and of the Triangle ABC, are taken equimultiples (any whatſoever) the baſe HC, and the Triangle AHC: alſo of the baſe CD, and of the Triangle ACD, are taken other equimultiples (any whatſoever) the baſe CL, and the Triangle ALC.

And it hath been prov'd that if the baſe HC exceeds the baſe CL, the Triangle AHC does exceed the Triangle ALC: and if equal, 'tis equal; and if leſs, 'tis leſs. Therefore as the baſe BC is to the baſe CD, ſo the Triangle ABC is to the Triangle ACD [Def. 5. El. V.].

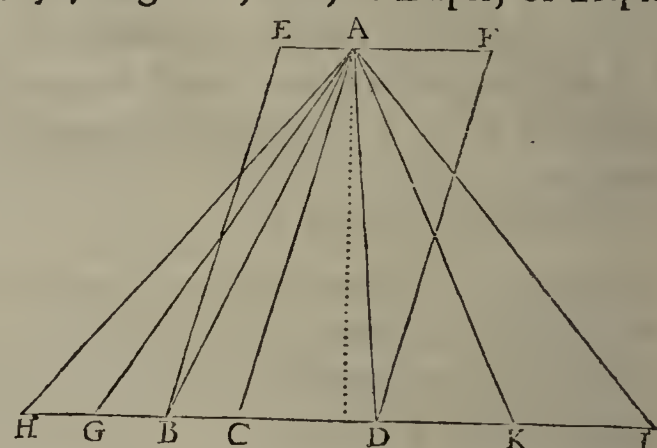
And now becauſe the Parallelogram EC is double of the Triangle ABC [Prop. 41. El. I.], and the Parallelogram FC is double of the Triangle ACD; and that parts have the ſame proportion with their equimultiples [Prop. 15. El. V.]; therefore as the Triangle ABC is to the Triangle ACD, ſo the Parallelogram EC is to the Parallelogram CF.

Becauſe therefore it has been prov'd, that as the baſe BC to the baſe CD, ſo the Triangle ABC to the Triangle ACD: And as the Triangle ABC to the Triangle ACD, ſo the Parallelogram EC to the Parallelogram CF: wherefore alſo as the baſe BC to the baſe CD, ſo the Parallelogram EC to the Parallelogram CF [Prop. 11. El. El. V.].

Therefore Triangles and Parallelograms of the ſame altitude are to one another as their baſes. Which was to be demonſtrated.

ANNOTATIONS.

On this Propoſition depends the main part of this Element, as alſo the whole doctrine of proportions in Magnitudes throughout all Geometry: and by this demonſtration it plainly appears, with what facility the 5th. Definition of El. V. is applied to Magnitudes. For what is of more eaſy Conſtruction than to Duple, or Triple, &c. the baſe BC; and then by joyning AG, AH, to Duple, or Triple, &c. the Triangle ABC, that is, the firſt Term, and the third, the Antecedents. Again, to Duple, or Triple, &c. the baſe CD, and accordingly the Triangle ACD, that is, the ſecond Term, and the fourth, the Conſequents, is the ſame obvious Conſtruction. Neither is there elſewhere in *Euclide*, or *Archimedes* any greater trouble in the multiplication of magnitudes according to Def. 5th. El. V: And upon ſuch like eaſy Conſtructions the demonſtrations do as eaſily proceed, being always readily confirmed by ſome one ſingle Propoſition. As here 'tis evident by Prop. 38th. El. I. that Triangles in the ſame parallels, (that is to ſay, of the ſame altitude) and on equal baſes are equal to one another: and therefore on unequal baſes are unequal; on the greater baſe the greater Triangle, on the leſs, the leſs. So that if the baſes be equal, greater, or leſs, one than the other; the Triangles likewiſe are the ſame in any multiplication whatſoever: therefore as baſe to baſe, ſo Triangle to Triangle, by Def. 5th. El. V. And we ſee here that there needs no tryal of various or perplexed multiplications to prove this Propoſition: but that the Analogy between the baſes, and Triangles is at once apparent by one only Geometrical Propoſition, well known aforehand, *viz.* Prop. 38th. El. I.



Now *Tacquets* confidence in this matter is very remarkable, who ſo boldly prefers his own demonſtration, which is encumbred with the diviſions of the Conſequents into *like aliquot parts*; and theſe again are to be ſubtracted from the Antecedents *quoties fieri poteſt*, &c. All which will undoubtedly ſeem to his *Mathematical Tyrones* (of whom he pretends to have a great care) a much more troubleſome, and perplexed buſineſs, than this plain Conſtruction, and Demonſtration of *Euclide*, which he unjuſtly accuſes of obſcureneſs and intricacy; whereof he himſelf is moſt guilty: as will certainly appear to whoever ſhall compare both together. But I leave *Tacquet*, and *Borellus* with the reſt, to the correction of the Geometrical Profeſſors in our Univerſities, and *Greſham* College: this ſubject being too large for Elementary Annotations intended only for the inſtruction of Younger Students.

PRO

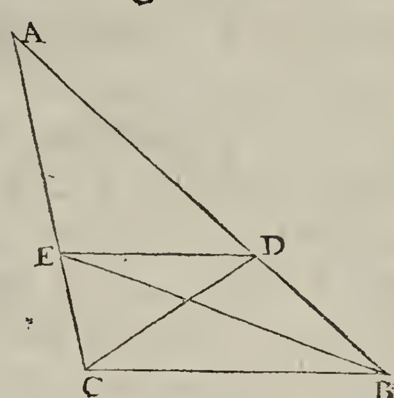
5

PROPOSITION II.

IF to one ſide of a Triangle a ſtrait line be drawn parallel; it ſhall cut the ſides of the Triangle proportionally.

And if the ſides of a Triangle be cut proportionally, the ſtrait line joyning the Sections, ſhall be parallel to the remaining ſide of the Triangle.

For in the Triangle ABC to one of the ſides BC, let DE be drawn parallel. I ſay, that as BD is to DA, ſo CE is to EA. Let be joyned BE, CD: therefore the Triangle BDE, is equal to the Triangle CDE [Prop. 37. El. I.] For they are on the ſame baſe DE, and within the ſame parallels DE, BC. But moreover there is an other Triangle ADE: and becauſe equals have to the ſame thing the ſame proportion [Prop. 7. El. V.]; therefore as the triangle BDE, is to the Triangle ADE, ſo the Triangle CDE is to the Triangle ADE: But as the Triangle BDE is to the Triangle ADE, ſo BD is to DA. For having the ſame Altitude, the perpendicular drawn from E to AB, they are to one another as their baſes [Prop. 1. El. VI.]. And by the ſame reaſon, as the Triangle CDE is to the Triangle ADE, ſo CE is to EA; therefore as BD is to DA, ſo CE is to EA [Prop. 11. El. V.].



But now of the Triangle ABC, let the ſides AB, AC be cut proportionally in the points D, E, that as BD to DA, ſo CE to EA: and let be joyned DE. I ſay, that DE is parallel to BC. For the ſame Conſtruction being made, becauſe as BD is to DA, ſo CE is to EA: and as BD is to DA, ſo the Triangle BDE is to the Triangle ADE; and as CE is to EA, ſo the Triangle CDE is to the Triangle ADE. Therefore as the Triangle BDE is to the Triangle ADE, ſo the Triangle CDE is to the Triangle ADE [Prop. 11. El. V.]: wherefore each of the Triangles BDE, CDE, have the ſame proportion to ADE; therefore the Triangle BDE is equal to the Triangle CDE [Prop. 9. El. V.]. And they are on the ſame baſe DE; but equal Triangles and on the ſame baſe, are within the ſame parallels [Prop. 40. El. I.]; therefore DE is parallel to BC.

If therefore to one ſide of a Triangle a ſtrait line be drawn parallel; it ſhall cut the ſides of the Triangle proportionally.

And if the ſides of a Triangle be cut proportionally, the ſtrait line joyning the Sections, ſhall be parallel to the remaining ſide of the Triangle. Which was to be demonſtrated.

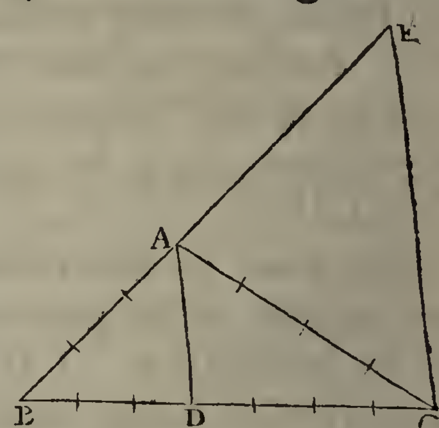
PROPO-

PROPOSITION III.

IF an angle of Triangle be cut into halves, and the ſtrait line cutting the angle does alſo cut the baſe: the Segments of the baſe ſhall have the ſame proportion with the remaining ſides of the Triangle.

And if the Segments of the baſe have the ſame proportion with the remaining ſides of the Triangle, the ſtrait line drawn from the Vertex to the Section, does cut the angle of the triangle into halves.

Let the Triangle be ABC, and let the angle BAC be cut into halves by the line AD [by Prop. 9. El. I.] I ſay, that as BD is to DC, ſo BA is to AC. For by the point C let be drawn CE parallel to DA [Prop. 31. El. I.]; and let BA produced meet with the ſame in the point E. Now foraſmuch as the ſtrait line AC falls upon the parallels AD, EC; therefore the angle ACE is equal to the alternate angle CAD [Prop. 29. El. I.]: but the angle CAD is put equal to the angle BAD; therefore the angle BAD is alſo equal to the angle ACE. Again, becauſe the ſtrait line BAE falls upon the parallels AD, EC; therefore the outward angle BAD is equal to the inward angle AEC. And it has been prov'd, that the angle ACE is alſo equal to the angle BAD; therefore the angle ACE is alſo equal to the angle AEC, ſo that alſo the ſide AE is equal to the ſide AC [Prop. 6. El. I.]. And becauſe to one ſide of the Triangle BCE, namely to EC, is drawn parallel AD; therefore proportionally as BD to DC, ſo BA to AE [Prop. 2. El. VI.]. But AE is equal to AC; therefore as BD to DC, ſo BA to AC [Prop. 7. El. V.].



But now let it be as BD to DC, ſo BA to AC: and let be joyned AD. I ſay, that the angle BAC is cut into halves by the ſtrait line AD. For the ſame Conſtruction being made, becauſe it is as BD to DC, ſo BA to AC: and as BD is to DC, ſo BA is to AE: for to one ſide of the Triangle BCE, namely to CE, is drawn AD parallel; therefore as BA is to AC, ſo BA is to AE: therefore AC is equal to AE [Prop. 9. El. V.]; ſo that alſo the angle AEC, is equal to the angle ACE [Prop. 5. El. I.]. But the angle AEC is equal to the outward angle BAD [Prop. 29. El. I.]; and the angle ACE is equal to the alternate angle CAD; wherefore the angle BAD is equal to the angle CAD; therefore the angle BAC is cut into halves by the ſtrait line AD.

If therefore an angle of a Triangle be cut into halves, and the ſtrait line cutting the angle does alſo cut the baſe: the Segments of the baſe ſhall have the ſame proportion with the remaining ſides of the Triangle.

And

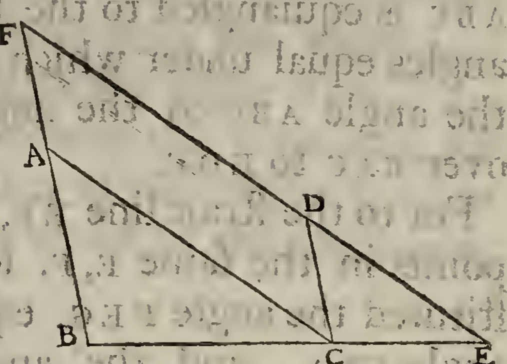
And if the Segments of the base have the same proportion with the remaining sides, &c. Which was to be demonstrated.

PROPOSITION IV.

Of equiangled Triangles the sides about the equal angles are proportional: And the sides subtended under the equal angles are Homologal.

Let the equiangled Triangles be ABC, DCE, having the angle ABC equal to the angle DCE, and the angle ACB to the angle DEC, and moreover the angle BAC to the angle CDE.

I say, that of the Triangles ABC, DCE, the sides about the equal angles are proportional, and the sides subtended under the equal angles are Homologal.



For let BC be put directly to CE, and because the angles ABC, ACB, are less than two right angles [Prop. 17. El. I.], and the angle ACB is [by Supposition] equal to the angle DEC; therefore the angles ABC, DEC, are less than two right angles: wherefore BA, ED produced shall meet [by Postulate 3.]. Let them be produced, and meet in the point F.

Now forasmuch as the angle DCE is equal to the angle ABC [by Supposition]; therefore BF is parallel to CD [Prop. 28. El. I.]. (*For that the outward angle* DCE *is equal to the inward and opposite* ABC.) Again, because the angle ACB is equal to the angle DEC [by Supposition]; therefore AC is parallel to FE [Prop. 28. El. I.]. (*For that the outward angle* ACB *is equal to the inward and opposite* DEC:) Wherefore FACD is a Parallelogram; therefore AF is equal to CD, and AC to FD [Prop. 34. El. I.]. And now because of the Triangle FBE, to one of the sides FE, the line AC is parallel; therefore it is as BA to AF, so BC to CE [Prop. 2. El. VI.]. But AF is equal to CD; as therefore BA to CD, so BC to CE [Prop. 7. El. V.]; and alternately, as AB to BC, so DC to CE.

Again, because of the same Triangle FBE, to the side BF, the line CD is parallel; therefore it is, as BC to CE, so FD to DE [Prop. 2. El. VI.]. But FD is equal to CA; therefore as BC to CE, so CA to ED: Alternately therefore as BC to CA, so CE to ED.

And forasmuch as it has been proved, that as AB to BC, so DC to CE, and as BC to CA, so CE to ED: therefore by equality, as BA to AC, so CD to DE.

Therefore of equiangled Triangles the sides about the equal angles are proportional: And the sides subtended under the equal angles are Homologal. Which was to be demonstrated.

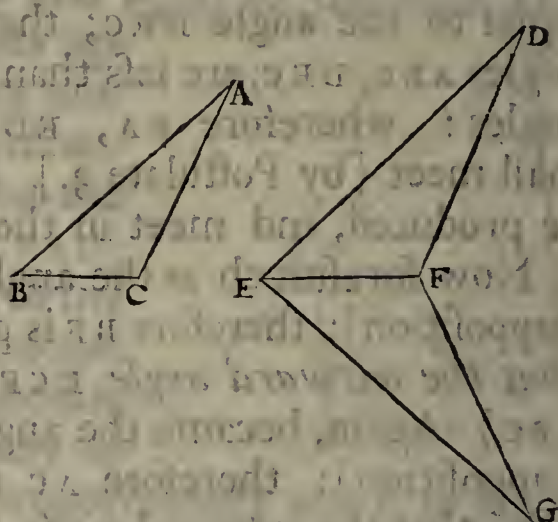
 " *Euclide*

"*Euclide* here takes no notice of the Homologal ſides, becauſe "it manifeſtly appears in the courſe of the demonſtration, that thoſe "ſides which are ſubtended under the equal angles are Homologal, "and correſpondent Antecedents and Conſequents.

PROPOSITION V.

IF two Triangles have their ſides proportional, the Triangles ſhall be equiangled: And ſhall have thoſe angles equal under which are ſubtended Homologal ſides.

Let the two Triangles be ABC, DEF, having their ſides proportional, as AB to BC, ſo DE to EF: and as BC to CA, ſo EF to FD: and moreover, as BA to AC, ſo ED to DF. I ſay, that the Triangle ABC is equiangled to the Triangle DEF: And they ſhall have thoſe angles equal under which are ſubtended Homologal ſides; namely the angle ABC to the angle DEF, and BCA to EFD, and moreover BAC to EDF.



For to the ſtrait line EF, and to the points in the ſame E, F, let be conſtituted the angle FEG, equal to the angle CBA, and the angle EFG, equal to the angle BCA [by Prop. 23. El. I.]; therefore the remaining angle BAC is equal to the remaining angle EGF: wherefore the Triangle ABC is equiangled to the Triangle GEF; therefore of the Triangles ABC, GEF, the ſides about the equal angles are proportional, and the ſides ſubtending the equal angles are Homologal: wherefore as AB is to BC, ſo GE is to EF. But as AB to BC, ſo by Suppoſition, DE is to EF; therefore as DE to EF, ſo GE to EF: wherefore each of the lines DE, GE, have the ſame proportion to EF; and therefore DE is equal to GE [Prop. 9. El. V.].

By the ſame reaſon, DF is equal to GF. Now foraſmuch as DE is equal to EG, and EF common: therefore there are the two lines DE, EF, equal to the two lines GE, EF; and the baſe DF is equal to the baſe GF; therefore the angle DEF, is equal to the angle GEF, and the Triangle DEF, is equal to the Triangle GEF, and the remaining angles equal to the remaining angles under which are ſubtended equal ſides [Prop. 8. El. I.]; therefore alſo the angle DFE is equal to the angle GFE, and EDF to EGF. And becauſe the angle DEF is equal to the angle GEF, and the angle GEF to the angle ABC; therefore the angle ABC is equal to the angle DEF. By the ſame reaſon, the angle ACB is equal to the angle DFE, and alſo the angle

angle at A, is equal to the angle at D: wherefore the Triangle ABC is equiangled to the Triangle DEF.

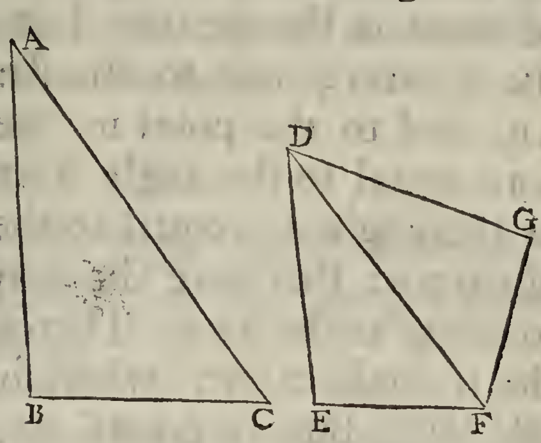
If therefore two Triangles have their ſides proportional, the Triangles ſhall be equiangled: And ſhall have thoſe angles equal under which are ſubtended Homologal ſides. Which was to be demonſtrated.

PROPOSITION VI.

IF two Triangles have one angle equal to one angle, and about the equal angles the ſides proportional; the Triangles ſhall be equiangled, and ſhall have thoſe angles equal, under which are ſubtended Homologal ſides.

Let the two Triangles be ABC, DEF, having one angle BAC equal to one angle EDF, and about the equal angles the ſides proportional, as BA to AC, ſo ED to DF. I ſay, that the Triangle ABC is equiangled to the triangle DEF: And they ſhall have the angle ABC equal to the angle DEF, and the angle ACB to the angle DFE.

For to the ſtrait line DF, and to the points in the ſame D, F, let be conſtituted the angle FDG, equal to either of the angles BAC, or EDF [Prop. 23. El. I.]; and the angle DFG equal to the angle ACB; therefore the remaining angle at B, is equal to the remaining angle at G: wherefore the triangle ABC is equiangled to the triangle DGF: it is therefore proportional as BA to AC, ſo GD to DF. But by ſuppoſition as BA to AC, ſo ED to DF; and therefore as ED to DF, ſo GD to DF: wherefore ED is equal to GD; and DF common. Therefore there are the two lines ED, DF, equal to the two lines GD, DF; and the angle EDF is equal to the angle GDF; wherefore the baſe EF is equal to the baſe GF: and the triangle DEF equal to the triangle DGF; and the remaining angles ſhall be equal to the remaining angles, each to each, under which are ſubtended equal ſides [Prop. 4. El. I.]; therefore the angle DFG is equal to the angle DFE, and the angle at G to the angle at E. But the angle DFG is equal to the angle ACB; therefore the angle ACB is equal to the angle DFE. But the angle BAC is put equal to the angle EDF; therefore alſo the remaining angle at B, is equal to the remaining angle at E: wherefore the triangle ABC is equiangled to the triangle DEF.

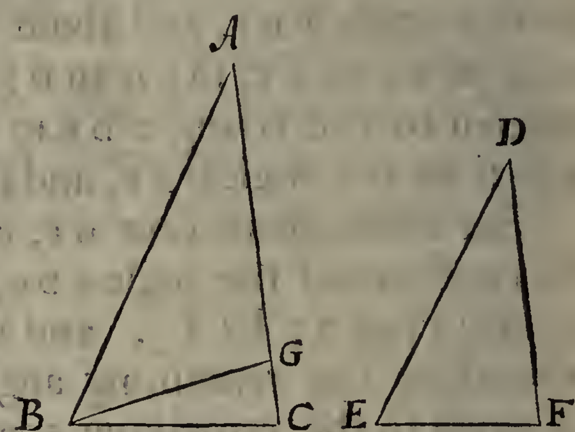


If therefore two triangles have one angle equal to one angle, and about the equal angles the ſides proportional; the triangles ſhall be equiangled, and ſhall have thoſe angles equal, under which are ſubtended Homologal ſides. Which was to be demonſtrated.

PROPOSITION VII.

If two triangles have one angle equal to one angle, and about other angles the ſides proportional; and have alſo each of the remaining angles either leſs, or not leſs than a right; the triangles ſhall be equiangled, and ſhall have thoſe angles equal, about which are the proportional ſides.

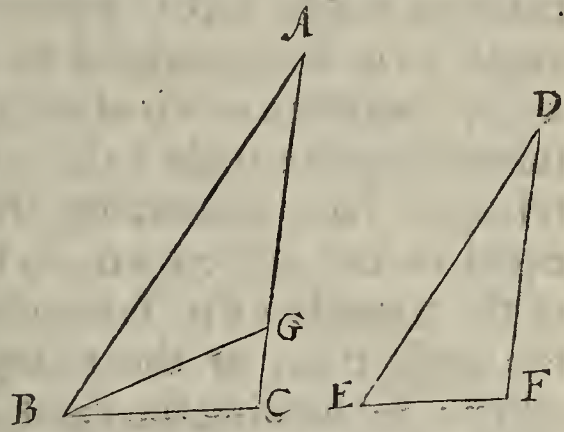
Let the two triangles be ABC, DEF, having one angle equal to one angle, the angle BAC to the angle EDF; and about other angles ABC, DEF, the ſides proportional, as AB to BC, ſo DE to EF; having alſo each of the remaining angles at C, F, firſt leſs than a right angle.



I ſay, that the triangle ABC is equiangled to the triangle DEF: and the angle ABC ſhall be equal to the angle DEF, and the remaining angle, to wit, at C, equal to the remaining angle at F. For if the angle ABC be unequal to the angle DEF, one of them is the greater. Let ABC be the greater; and to the ſtrait line AB, and to the point in the ſame B, let be conſtituted the angle ABG equal to the angle DEF [by Prop. 23. El. I.]. Now foraſmuch as the angle A is equal to the angle D, and the angle ABG to the angle DEF; therefore the remaining angle AGB is equal to the remaining angle DFE. Therefore the triangle ABG is equiangled to the triangle DEF: wherefore as AB to BG, ſo DE to EF [Prop. 4. El. VI.] But as DE to EF, ſo by Suppoſition is AB to BC: and therefore as AB to BC, ſo AB to BG: wherefore AB has the ſame proportion both to BC and BG; therefore BC is equal to BG [Prop. 9. El. V.]: ſo that alſo the angle BGC is equal to the angle BCG [Prop. 5. El. I.] But the angle at C is put leſs than a right angle; therefore alſo the angle BGC is leſs than a right angle: ſo that the conſequent angle AGB is greater than a right angle: and it hath been proved equal to the angle at F; therefore the angle at F is greater than a right angle. But it is put leſs than a right angle: which is abſurd: wherefore the angle ABC is not unequal to the angle DEF: equal therefore it is. Now alſo the angle at A is equal to the angle at D [by Suppoſition], and therefore the remaining angle at C, is equal to the remaining angle at F: wherefore the triangle ABC is equiangled to the triangle DEF.

But

But again, let each of the angles at C, F be put not leſs than a right angle. I ſay, that in this caſe alſo the triangle ABC is equiangled to the triangle DEF. For the ſame Conſtruction being made, we may in like manner demonſtrate, that BC is equal to BG; ſo that the angle at C is equal to the angle BGC. But the angle at C is put not leſs than a right angle; therefore BGC is not leſs than a right angle: wherefore of the triangle BGC there are two angles, and they not leſs than two right angles, which is impoſſible [Prop. 32. El.I.]; therefore again the angle ABC is not unequal to the angle DEF, equal therefore it is. Now alſo the angle at A is equal to the angle at D [by Suppoſition]: wherefore the remaining angle at C is equal to the remaining angle at F; therefore the triangle ABC is equiangled to the triangle DEF.

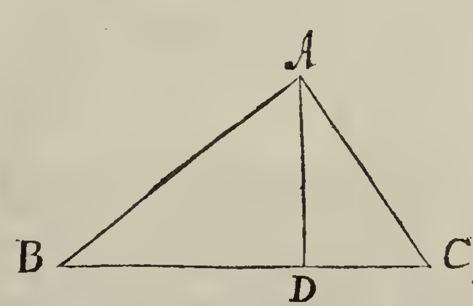


If therefore two triangles have one angle equal to one angle, and about other angles the ſides proportional; and each of the remaining angles either leſs, or not leſs than a right angle; the triangles ſhall be equiangled, and ſhall have thoſe angles equal, about which are the proportional ſides. Which was to be demonſtrated.

PROPOSITION VIII.

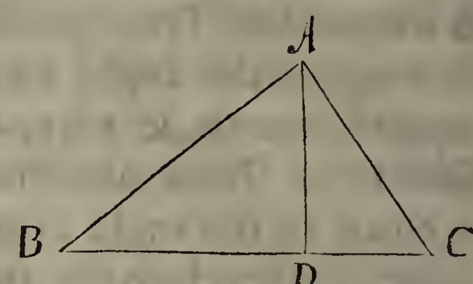
If in a right angl'd triangle from the right angle be drawn a perpendicular to the baſe, the triangles at the perpendicular are like to the whole, and to one another.

Let the right angl'd triangle be ABC, having the right angle BAC, and from the point A to the line BC, let AD be drawn perpendicular. I ſay, that each of the triangles ABD, ADC, is like to the whole triangle ABC, and alſo to one another. Foraſmuch as the angle BAC is equal to the angle ADB, for each is a right angle; and the angle at B common to the two triangles ABC, ADB; therefore the remaining angle ACB is equal to the remaining angle BAD; therefore the triangle ABC is equiangled to the triangle ABD: wherefore as BC ſubtending the right angle of the triangle ABC, is to BA ſubtending the right angle of the triangle ABD, ſo the ſame AB ſubtending the angle at C of the triangle ABC, is to BD ſubtending the angle BAD of the triangle ABD, equal to the angle at C: and alſo as AC to AD ſubtending the angle at B, common to the two triangles: wherefore the triangle ABC is equiangled to the triangle ABD, and has the ſides about the equal angles propor-



tional; therefore the triangle ABC is like to the triangle ABD [Def. 1. El. VI.]. In like manner we may demonſtrate, that alſo the triangle ADC is like to the triangle ABC: wherefore each of the triangles ABD, ADC, is like to the whole triangle ABC.

I ſay moreover, that the triangles ABD, ADC, are alſo like to one another. For becauſe the right angle BDA is equal to the right angle ADC; and alſo the angle BAD has been proved equal to the angle at C; therefore the remaining angle at B is equal to the remaining angle DAC: wherefore the triangle ABD is equiangled to the triangle ACD; therefore as BD of the triangle ABD, ſubtending the angle BAD, is to DA of the triangle ADC, ſubtending the angle at C equal to the angle BAD, ſo the ſame AD, of the triangle ABD, ſubtending the angle at B, is to DC ſubtending the angle DAC, of the triangle ADC, equal to the angle at B: and alſo BA ſubtending the right angle ADB to AC, ſubtending the right angle ADC: therefore the triangle ABD is like to the triangle ADC.



If therefore in a right angl'd triangle, from the right angle be drawn a perpendicular to the baſe; the triangles at the perpendicular are like to the whole, and to one another. Which was to be demonſtrated.

Corollary.

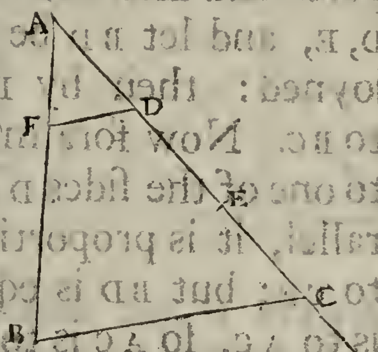
From hence 'tis manifeſt, that in a right angl'd triangle the perpendicular drawn from the right angle to the baſe, is a mean proportional between the Segments of the baſe. And moreover between the baſe and either one of the Segments, the ſide adjoyned to that Segment is a mean proportional.

PROPO-

PROPOSITION IX.

FRom a given ſtrait line to take off a demanded part.

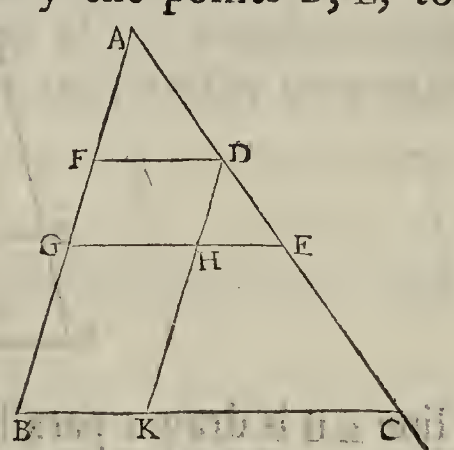
Let the given ſtrait line be AB. It is required from AB to take off a demanded part. Let a third part be demanded; and from the point A let any ſtrait line as AC be drawn, containing with AB any angle. And in AC let any point be taken as D: and to AD let DE, EC be put equal, and let be joyned BC: then by D let be drawn DF parallel to BC. Now foraſmuch as to one ſide of the triangle ABC, namely to BC, there is drawn a parallel DF; it is therefore proportionally, as CD to DA, ſo BF to FA [Prop. 2. El. VI.]. But CD is double of DA; therefore alſo BF is double of FA: wherefore BA is Triple of AF. Therefore from the given ſtrait line AB is taken off AF, a third part demanded. Which was to be done.



PROPOSITION X.

TO cut a given uncut ſtrait line like to a given cut ſtrait line.

Let the given uncut ſtrait line be AB, and the cut line be AC. It is required to cut the uncut line AB like to the cut line AC. Let AC be cut in the points D, E; and let AB, AC, be ſo put as to contain any angle, and let be joyned BC. Then by the points D, E, to BC let DF, EG, be drawn parallels, and by D to AB let DHK be drawn parallel; therefore each of the figures FH, HB, is a Parallelogram: wherefore DH is equal to FG, and HK to GB [Prop. 34. El. I.]. Now foraſmuch as of the triangle DKC to one of the ſides KC, the line HE is drawn parallel; it is therefore proportionally as CE to ED, ſo KH to HD [Prop. 2. El. VI.]. But KH is equal to BG, and HD to GF; therefore as CE to ED, ſo BG to GF. Again, becauſe of the triangle AGE, to one of the ſides EG, the line FD is drawn parallel; it is therefore proportionally as ED to DA, ſo GF to FA. But it has been proved, as CE to ED, ſo BG to GF; therefore as CE to ED, ſo BG is to GF, and as ED is to DA, ſo GF is to FA.



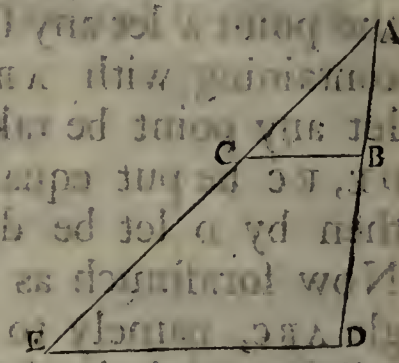
Wherefore the given uncut ſtrait line AB, is cut like to the given cut ſtrait line AC. Which was to be done.

PROPO-

PORPOSITION XI.

TWo strait lines being given to find a third proportional.

Let the given strait lines be AB, AC; and be they so put as to contain any angle. It is required unto AB, AC to find a third proportional. Let AB, AC be produced to the points D, E, and let BD be put equal to AC, and BC be joyned: then by D let DE be drawn parallel to BC. Now forasmuch as of the triangle ADE, to one of the sides DE, the line BC is drawn parallel, it is proportionally as AB to BD, so AC to CE; but BD is equal to AC; therefore as BA is to AC, so AC is to CE.

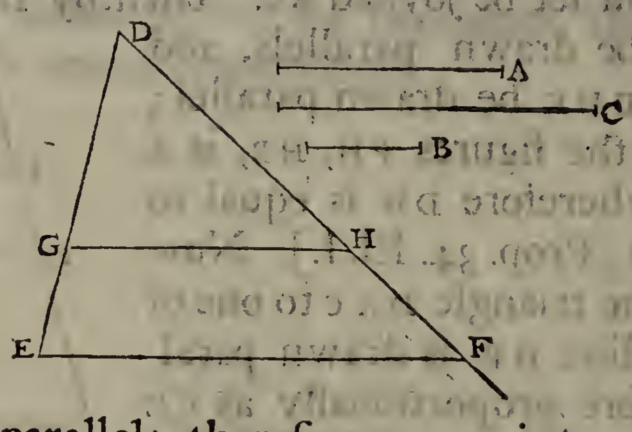


Wherefore to the two given strait lines AB, AC, is found CE a third proportional. Which was to be done.

PROPOSITION XII.

THree strait lines being given to find a fourth proportional.

Let the three given strait lines be A, B, C. It is required unto A, B, C, to find a fourth proportional. Let two strait lines DE, DF, be put, containing any angle as EDF; and to A let DG be put equal; and to B, GE, and also to C let DH be put equal: then GH being joyned, let to the same be drawn by the point E, the line EF parallel. Now forasmuch as of the triangle DEF, to one of the sides EF, the



line GH is drawn parallel; therefore as DG is to GE, so DH is to HF [Prop. 2. El. VI.]. But DG is equal to A, and GE to B, and DH to C: therefore as A is to B, so C is to HF.

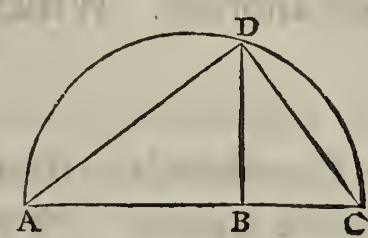
Wherefore to three given strait lines A, B, C, is found HF a fourth proportional. Which was to be done.

PRO-

PROPOSITION XIII.

TWo ſtrait lines being given to find a mean proportional.

Let the two given ſtrait lines be AB, BC. It is required unto AB, BC, to find a mean proportional. Let AB, BC be put in a direct line, and on AC let be deſcribed the Semicircle ADC; then from the point B to the ſtrait line AC, let ED be drawn at right angles, and let be joyned AD, DC. Now foraſmuch as in the Semicircle the angle ADC is a right angle [Prop. 31. El. III.], and becauſe in the right angled triangle ADC, from the right angle to the baſe the perpendicular DB is drawn; therefore DB is a mean proportional to the Segments of the baſe AB, BC [Coroll. Prop. 8. El. VI.].

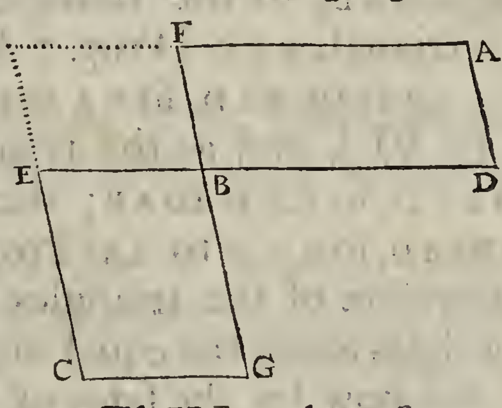


Wherefore to the two given lines AB, BC, is found DB a mean proportional. Which was to be done.

PROPOSITION XIV.

OF equal Parallelograms having one angle equal to one angle, the ſides about the equal angles are reciprocally proportional. And Parallelograms having one angle equal to one angle, and the ſides about the equal angles reciprocally proportional, are equal to one another.

Let the equal Parallelograms be AB, BC, having equal angles at B, and let DB, BE, be put directly to one another; therefore alſo FB, BG, are directly to one another. I ſay, that of the Parallelograms AB, BC, the ſides about the equal angles are reciprocally proportional, that is, as DB is to BE, ſo GB is to BF. Let the Parallelogram FE be compleated. Now foraſmuch as the Parallelogram AB is equal to the Parallelogram BC, and FE is an other; therefore as AB is to FE, ſo BC is to FE [Prop. 7. El. V.]. But as AB to FE, ſo DB to BE, [Prop. 1. El. VI.] and as BC to FE, ſo GB to BF; therefore as DB to BE, ſo GB to BF [Prop. 11. El. V.]: wherefore of the Parallelograms AB, BC, the ſides about the equal angles are reciprocally proportional.



But now let the ſides about the equal angles be reciprocally proportional, and let it be, as DB to BE, ſo GB to BF. I ſay, that the Parallelogram AB is equal to the Parallelogram BC. For becauſe as DB is to BE, ſo GB is to BF; and as DB to BE, ſo the Parallelogram AB to the Parallelogram FE [Prop. 1. El. VI.]; and as GB to BF, ſo

 the

the Parallelogram BC to the Parallelogram FE; therefore as AB is to FE, ſo BC is to FE [Prop. 11. El. V.]: wherefore the Parallelogram AB is equal to the Parallelogram BC [Prop. 9. El. V.].

Therefore of equal Parallelograms having one angle equal to one angle, the ſides about the equal angles are reciprocally proportional.

And Parallelograms having one angle equal to one angle, and the ſides about the equal angles reciprocally proportional, are equal to one another. Which was to be demonſtrated.

Animadverſion.

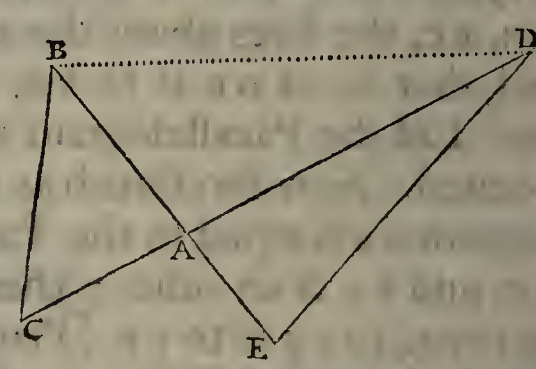
Foraſmuch as the Complements of every Parallelogram are equal to one another [Prop. 43. El. I.]; therefore by this Propoſition they have their ſides reciprocally proportional.

PROPOSITION XV.

Of equal triangles having one angle equal to one angle, the ſides about the equal angles are reciprocally proportional.

And triangles having one angle equal to one angle, and the ſides about the equal angles reciprocally proportional, are equal to one another.

Let the equal triangles be ABC, ADE, having one angle equal to one angle, namely, the angle BAC to the angle DAE. I ſay, that of the triangles ABC, ADE, the ſides about the equal angles are reciprocally proportional, that is, as CA is to AD, ſo EA is to AB. Let them be put ſo that CA be directly to AD; therefore alſo EA is directly to AB, and let be joyned BD. Now foraſmuch as the triangle ABC is equal to the triangle ADE; and ABD is an other; therefore as the triangle CAB is to the triangle BAD, ſo the triangle ADE is to the triangle BAD [Prop. 7. El. V.]. But as CAB is to BAD, ſo CA is to AD [Prop. 1. El. VI.], and as the triangle EAD is to BAD, ſo EA is to AB; therefore as CA is to AD, ſo EA is to AB [Prop. 11. El. V.]: wherefore of the triangles ABC, ADE, the ſides about the equal angles are reciprocally proportional.



But now let the ſides of the triangles ABC, ADE be reciprocally proportional, and let it be, as CA to AD, ſo EA to AB. I ſay, that the triangle ABC is equal to the triangle ADE. For again, BD being joyned; becauſe as CA is to AD, ſo EA is to AB: and as CA is to AD, ſo the triangle ABC is to the triangle BAD: and as EA is to AB, ſo the triangle EAD is to the triangle BAD; therefore as the triangle ABC is to the triangle BAD, ſo the triangle EAD is to the triangle BAD [Prop. 11. El. V.]; therefore each of the triangles ABC, EAD, have the

the ſame proportion to the triangle BAD: wherefore the triangle ABC is equal to the triangle EAD [Prop. 9. El. V.].

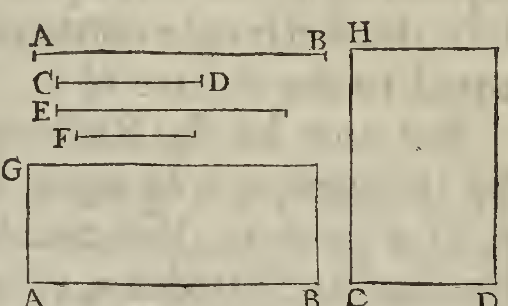
Therefore of equal triangles having one angle equal to one angle, the ſides about the equal angles are reciprocally proportional.

And triangles having one angle equal to one angle, and the ſides about the equal angles reciprocally proportional, are equal to one another. Which was to be demonſtrated.

PROPOSITION XVI.

If four ſtrait lines be proportional, the Rectangle contained by the extremes is equal to the Rectangle contained by the means. And if the Rectangle contained by the extremes be equal to the Rectangle contained by the means, the four ſtrait lines ſhall be proportional.

Let four ſtrait lines AB, CD, E, F, be proportional, as AB to CD, ſo E to F. I ſay that the Rectangle contained by AB, F, is equal to the Rectangle contained by CD, E. For from the points A, C to the ſtrait lines AB, CD, let AG, CH be drawn at right angles, and let AG be put equal to F, and CH to E, and let the Parallelograms BG, DH, be compleated. Now foraſmuch as AB is to CD, ſo E is to F; and E is equal to CH, and F to AG: therefore as AB is to CD, ſo CH is to AG: wherefore of the Parallelograms BG, DH, the ſides about the equal angles are reciprocally proportional. But equiangled Parallelograms having the ſides about the equal angles reciprocally proportional, are equal to one another [Prop. 14. El. VI.]; therefore the Parallelogram BG is equal to the Parallelogram DH. Now the Parallelogram BG is contained by AB, F; for AG is equal to F: and the Parallelogram DH, is contained by CD, E; for CH is equal to E; therefore the Rectangle contained by AB, F; is equal to the Rectangle contained by CD, E.



But now let the Rectangle contained by AB, F, be equal to the Rectangle contained by CD, E. I ſay, that the four ſtrait lines ſhall be proportional, as AB to CD, ſo E to F. For the ſame Conſtruction being made; becauſe the Rectangle under AB, F, is equal to the Rectangle under CD, E; and the Rectangle under AB, F, is BG; for AG is equal to F: and the Rectangle under CD, E, is DH: for CH is equal to E: therefore the Rectangle BG is equal to the Rectangle DH, and they are equiangled. But of equal, and equiangled Parallelograms the ſides about the equal angles are reciprocally proportional: wherefore as AB is to CD, ſo CH is to AG: but CH is equal to E, and AG to F; therefore as AB is to CD, ſo E is to F.

 If

If therefore four ſtrait lines be proportional, the Rectangle contained by the extremes is equal to the Rectangle contained by the means.

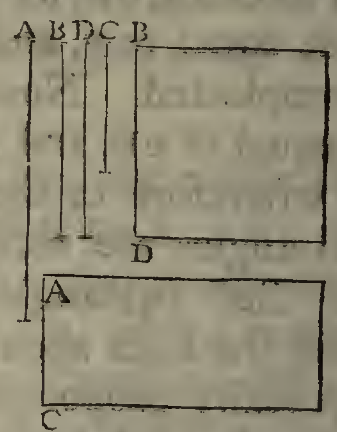
And if the Rectangle contained by the extremes be equal to the Rectangle contained by the means, the four ſtrait lines ſhall be proportional. Which was to be demonſtrated.

PROPOSITION XVII.

If three ſtrait lines be proportional, the Rectangle contained by the extremes is equal to the ſquare of the mean.

And if the Rectangle contained by the extremes be equal to the Square of the mean, the three ſtrait lines ſhall be proportional.

Let three ſtrait lines A, B, C, be proportional, as A to B, ſo B to C. I ſay, that the Rectangle contained by A, C, is equal to the Square of the mean B. Let D be put equal to B, and becauſe as A is to B, ſo B is to C; and D is equal to B, therefore as A is to B, ſo D is to C [Prop. 7. El. V.]. Now if four ſtrait lines be proportional, the Rectangle contained by the extremes is equal to the Rectangle contained by the means; therefore the Rectangle under A, C, is equal to the Rectangle under B, D. But the Rectangle under B, D, is equal to the Square of B, for B is equal to D; therefore the Rectangle contained by A, C, is equal to the Square of B.



But now let the Rectangle contained by the lines A, C, be equal to the Square of B. I ſay, that as A is to B, ſo B is to C. For the ſame Conſtruction being made: becauſe the Rectangle under A, C, is equal to the Square of B, and the Square of B is the Rectangle under B, D, for D is equal to B; therefore the Rectangle under A, C, is equal to the Rectangle under B, D. But if the Rectangle under the extremes be equal to Rectangle under the means, the four ſtrait lines are proportional [by Prop. 16. El. VI.]; therefore as A is to B, ſo D is to C. But B is equal to D; therefore as A is to B, ſo B to C [Prop. 7. El. V.].

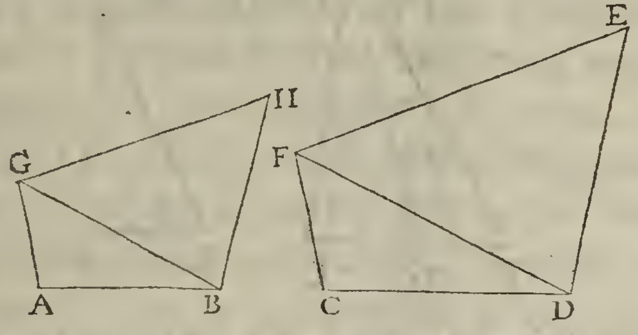
If therefore three ſtrait lines be proportional, the Rectangle contained by the extremes is equal to the Square of the mean.

And if the Rectangle contained by the extremes be equal to the Square of the mean, the three ſtrait lines ſhall be proportional. Which was to be demonſtrated.

PROPOSITION XVIII.

Upon a given ſtrait line, unto a given ſtrait-lin'd Figure to deſcribe a ſtrait-lin'd Figure like and alike ſituated.

Let the given ſtrait line be AB, and the given ſtrait-lin'd Figure CE: it is required upon the ſtrait line AB, unto the ſtrait-lin'd Figure CE, to deſcribe a ſtrait-lin'd Figure like and alike ſituated. Let be joyned DF; and to the ſtrait line AB, and to the points in the ſame A, B, let be conſtituted the angle BAG, equal to the angle at C: and the angle ABG to the angle CDF [Prop. 23. El. I.]; therefore the remaining angle CFD is equal to the remaining angle AGB: wherefore the triangle FCD is equiangled to the triangle GAB; it is therefore proportionally as FD to GB, ſo FC to GA, and CD to AB [Prop. 4. El. VI.]. Again, to the ſtrait line BG, and to the points in the ſame B, G, let be conſtituted the angle BGH equal to the angle DFE, and the angle GBH to the angle FDE: therefore the remaining angle at E is equal to the remaining angle at H: wherefore the triangle FDE is equiangled to the triangle GBH; it is therefore pro-



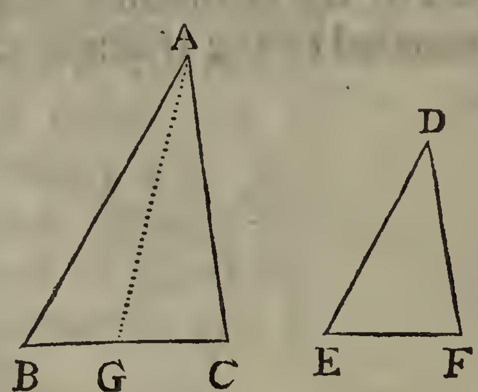
portionally, as FD to GB, ſo FE to GH, and ED to HB [Prop. 4. El. VI.]. And it has been prov'd, that alſo as FD to GB, ſo FC to GA, and CD to AB, therefore as FC to GA, ſo CD to AB, and FE to GH, and moreover ED to HB [Prop. 11. El. V.]. And becauſe the angle CFD is equal to the angle AGB, and the angle DFE to the angle BGH; therefore the whole angle CFE is equal to the whole angle AGH. By the ſame reaſon alſo the angle CDE is equal to the angle ABH. But alſo the angle at C is equal to the angle at A, and the angle at E to the angle at H: therefore AH is equiangled to CE; and hath to the ſame about the equal angles the ſides proportional. Wherefore the ſtrait lin'd Figure AH, is like to the ſtrait lin'd Figure CE [Def. 1. El. VI.].

Therefore upon the given ſtrait line AB to the given ſtrait-lin'd Figure CE, is deſcribed the ſtrait-lin'd Fignre AH, like and alike ſituated. Which was to be done.

PROPOSITION XIX.

Like triangles are to one another in a Duplicate proportion of their Homologal ſides.

Let the like triangles be ABC, DEF, having the angle at B equal to the angle at E, and let it be as AB to BC, ſo DE to EF: ſo that BC is Homologal to EF. I ſay that the triangle ABC hath to the triangle DEF a duplicate proportion of that which BC hath to EF. For to BC, EF, let be taken a third proportional BG [Prop. 11. El. VI.]: ſo that it be as BC to EF, ſo EF to BG: and let be joyned GA. Now becauſe it is as AB to BC, ſo DE to EF; therefore it is alternately as AB to DE, ſo BC to EF: but as BC to EF, ſo EF to BG. And therefore as AB to DE, ſo EF to BG [Prop. 11. El. V.]: wherefore of the triangles ABG, DEF, the ſides about the equal angles are reciprocally proportional. But triangles having one angle equal to one angle, and the ſides about the equal angles reciprocally proportional, are equal to one another [Prop. 15. El. VI.]; therefore the triangle



ABG is equal to the triangle DEF. And becauſe as BC is to EF, ſo EF is to BG, and that if three ſtrait lines be proportional, the firſt to the third is ſaid to have a duplicate proportion of that, which it hath to the ſecond [Def. 10. El. V.]; therefore BC hath to BG a duplicate proportion of that, which BC hath to EF. But as BC is to BG, ſo the triangle ABC is to the triangle ABG [Prop. 1. El. VI.]; therefore the triangle ABC hath to the triangle ABG a duplicate proportion of that, which BC hath to EF. But the triangle ABG is equal to the triangle DEF: wherefore the triangle ABC hath to the triangle DEF a duplicate proportion of that, which BC hath to EF.

Therefore like triangles are to one another in a duplicate proportion of their Homologal ſides. Which was to be demonſtrated.

Corollary

From hence 'tis manifeſt, that if three ſtrait lines be proportional; as the firſt is to the third, ſo the triangle upon the firſt is to the triangle upon the ſecond, being like and alike deſcribed: for that it hath been prov'd, that as CB is to BG, ſo the triangle ABC is to the triangle ABG, that is, to DEF.

PROPOSITION XX.

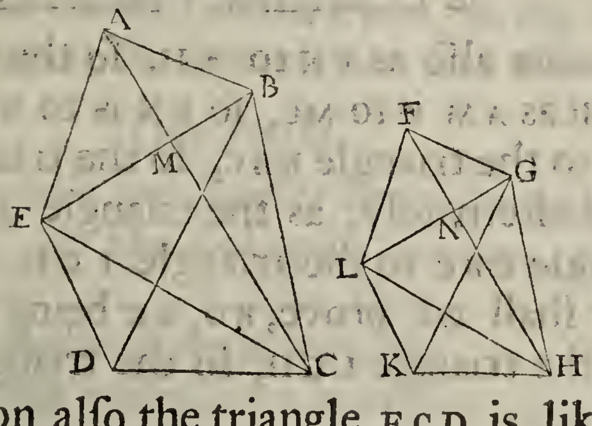
Like Polygons are divided into like triangles, equal in number: And Homologal to the wholes.

And Polygon hath to Polygon a duplicate proportion of that, which an Homologal ſide hath to an Homologal ſide.

Let the like Polygons be ABCDE, FGHKL, and let AB be Homologal to FG. I ſay, that the Polygons ABCDE, FGHKL, are divided into like triangles, equal in number, and Homologal to the wholes. And the Polygon ABCDE hath to the Polygon FGHKL a duplicate proportion of that, which AB hath to FG.

Let be joyned BE, EC, GL, LH. Now becauſe the Polygon ABCDE is like to the Polygon FGHKL; therefore the angle BAE is equal to the angle GFL; and as BA is to AE, ſo GF is to FL [Def. 1. El. VI.]. Foraſmuch therefore as there are two triangles ABE, FGL, having one angle equal to one angle, and about the equal angles the ſides proportional; therefore the triangle ABE is equiangled to the triangle FGL [Prop. 6. El. VI.], ſo that it is alſo like.

And becauſe ABE is like to FGL, therefore the angle ABE is equal to the angle FGL: But alſo the whole angle ABC is equal to the whole angle FGH, for the likeneſs of the Polygons: therefore the remaining angle EBC is equal to the remaining angle LGH. And becauſe for the likeneſs of the triangles ABE, FGL, it is as EB to BA, ſo LG to GF, and alſo for the likeneſs of the Polygons it is as AB to BC, ſo FG to GH: therefore by equality it is as EB to BC, ſo LG to GH; and theſe proportional ſides are about the equal angles EBC, LGH: therefore the triangle EBC is equiangled to the triangle LGH [Prop. 6. El. VI.]. And ſo alſo like.



By the ſame reaſon alſo the triangle ECD is like to the triangle LHK; therefore the Polygons ABCDE, FGHKL, are divided into like triangles, equal in number.

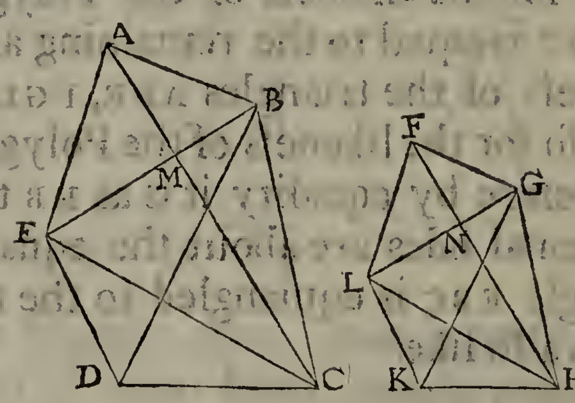
I ſay moreover, that the triangles are alſo Homologal to the wholes, that is, the triangles are proportional, and in the Polygon ABCDE, the Antecedents are ABE, EBC, ECD; and in the Polygon FGHKL, their Conſequents are FGL, LGH, LHK.

And alſo I ſay, that the Polygon ABCDE hath to the Polygon FGHKL, a duplicate proportion of that which an Homologal ſide hath to an Homologal ſide; that is, which AB hath to FG. For let be joyned AC, FH.

Now

Now becauſe for the likeneſs of the Polygons, the angle ABC is equal to the angle FGH, and as AB is to BC, ſo FG is to GH; therefore the triangle ABC is equiangled to the triangle FGH [Prop. 6. El. VI.]: wherefore the angle BAC is equal to the angle GFH, and the angle BCA to the angle GHF. And becauſe the angle BAM is equal to the angle GFN: and it hath been prov'd, that the angle ABM is equal to the angle FGN; therefore the remaining angle AMB is equal to the remaining angle FNG: wherefore the triangle AMB is equiangled to the triangle FNG.

In like manner ſhall we prove, that the triangle BMC is equiangled to the triangle GNH: proportionally therefore it is, as AM to MB, ſo FN to NG, and as MB, to MC, ſo NG to NH; ſo that by equality as AM to MC, ſo FN to NH. But as AM to MC, ſo the triangle AMB is to the triangle BMC, and the triangle AME to the triangle EMC; for they are to one another as their baſes AM, MC [Prop. 1. El. VI.]. And as one of the Antecedents to one of the Conſequents, ſo all the Antecedents to all the Conſequents [Prop. 12. El. V.]; therefore as the triangle AMB to the triangle BMC, ſo the triangle ABE to the triangle EBC. But as AMB to BMC, ſo AM to MC; and therefore as AM to MC, ſo the triangle ABE to the triangle EBC [Prop. 11. El. V.].



By the ſame reaſon alſo as FN to NH, ſo the triangle FGL to the triangle LGH. But as AM is to MC, ſo FN is to NH; and therefore as the triangle ABE to the triangle EBC, ſo the triangle FGL to the triangle LGH; and alternately, as the triangle ABE to the triangle FGL, ſo the triangle EBC to the triangle LGH.

In like manner ſhall we prove, BD, GK being joyned, that as the triangle EBC to the triangle LGH, ſo the triangle ECD to the triangle LHK.

And becauſe as the triangle ABE is to the triangle FGL, ſo the triangle EBC is to the triangle LGH, and moreover the triangle ECD to the triangle LHK; and as one of the Antecedents to one of the Conſequents, ſo all the Antecedents to all the Conſequents; therefore as the triangle ABE is to the triangle FGL, ſo the Polygon ABCDE is to the Polygon FGHKL: wherefore the like triangles and equal in number, are alſo Homologal to the wholes.

And now I ſay again, that the Polygon ABCDE, hath to the Polygon FGHKL a duplicate proportion of an Homologal ſide to an Homologal ſide.

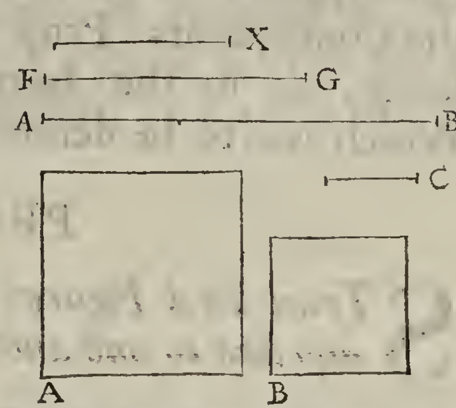
 For

For the the triangle ABE hath to the triangle FGL a duplicate proportion of that which the Homologal ſide AB hath to the Homologal ſide FG. For like triangles are in a duplicate proportion of their Homologal ſides [Prop. 19. El. VI.]. But as the triangle ABE is to the triangle FGL, ſo the Polygon ABCDE is to the Polygon FGHKL; therefore alſo the Polygon ABCDE hath to the Polygon FGHKL a duplicate proportion of that, which the Homologal ſide AB hath to the Homologal ſide FG,

Therefore like Polygons are divided into like triangles, equal in number, and Homologal to the wholes, &c. Which was to be demonſtrated.

Corollaries.

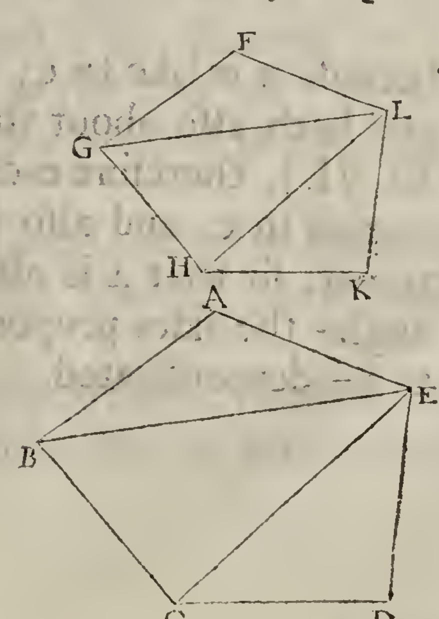
1. After the ſame manner alſo in like quadrilateral Figures ſhall be proved, that they are in a duplicate proportion of their Homologal ſides. And the ſame hath been demonſtrated in triangles. Therefore univerſally, like ſtrait lin'd Figures are to one another in a duplicate proportion of their Homologal ſides.



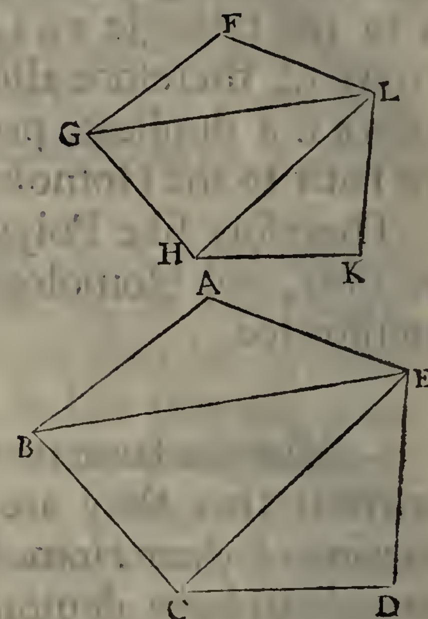
2. And if to AB, FG a third proportional X be taken, then AB is to X in a duplicate proportion of that, which AB hath to FG. But Polygon is to Polygon, and Quadrilateral figure to Quadrilateral figure in a duplicate proportion of their Homologal ſides, that is, of AB to FG; and the ſame hath been demonſtrated in triangles. So that alſo in general it is manifeſt, that if three ſtrait lines be proportional, it ſhall be as the firſt to the third, ſo the Figure upon the firſt, is to the Figure upon the ſecond, like and alike deſcribed.

Otherwiſe.

Now we will otherwiſe and ſhorter ſhew the triangles to be Homologal. For again, let the Polygons ABCDE, FGHKL, be put: and let be joyned BE, EC; GL, LH. I ſay, that as the triangle ABE is to the triangle FGL, ſo the triangle EBC is to the triangle LGH, and the triangle ECD to the triangle LHK. Now foraſmuch as the triangle ABE is like to the triangle FGL; therefore the triangle ABE hath to the triangle FGL a duplicate proportion of that, which BE hath to GL. By the ſame reaſon alſo the triangle EBC hath to the triangle LGH a duplicate proportion of that, which BE hath to GL; therefore as the triangle ABE is to the triangle FGL, ſo the triangle EBC is to the trian-



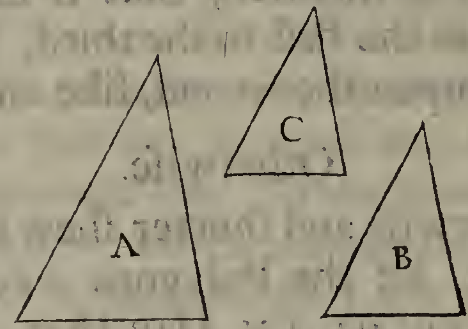
gle LGH. Again, becauſe the triangle EBC is like to the triangle LGH; therefore the triangle EBC hath to the triangle LGH, a duplicate proportion of that which CE hath to HL. By the ſame reaſon alſo, the triangle ECD hath to the triangle LHK a duplicate proportion of that which CE hath to HL; therefore as the triangle EBC is to the triangle LGH, ſo the triangle ECD is to the triangle LHK. And it hath been proved, that as EBC is to LGH, ſo ABE to FGL; therefore as the triangle ABE is to the triangle FGL, ſo the triangle EBC is to the triangle LGH, and the triangle ECD to the triangle LHK: And as one of the Antecedents to one of the Conſequents, ſo all the Antecedents to all the Conſequents [Prop. 12. El. V.], and ſo forth, as in the former demonſtration. Which was to be demonſtrated.



PROPOSITION XXI.

Strait-lin'd Figures like to the ſame Strait-lin'd Figure, are alſo like to one another.

For let each of the ſtrait-lin'd Figures A, B, be like to the ſtrait-lin'd Figure C. I ſay, that A is alſo like to B, for becauſe A is like to C; therefore it is both equiangled to the ſame, and alſo hath about the equal angles the ſides proportional [Def. I. El. VI.]. Again,



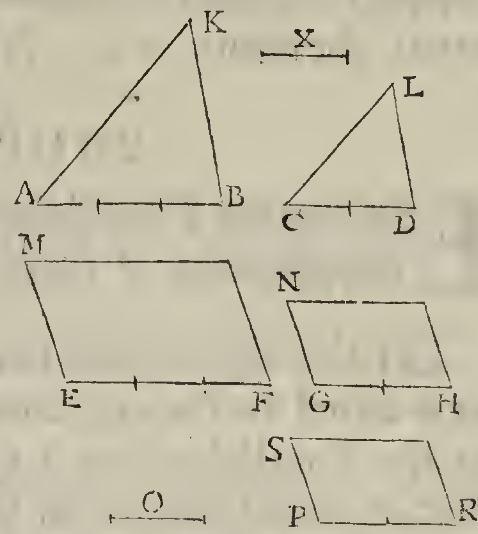
becauſe B is like to C; therefore it is both equiangled to the ſame, and hath alſo about the equal angles the ſides proportional [Def. I. El. VI.]; therefore each of the ſtrait-lin'd Figures A, B are both equiangled to C, and alſo have about the equal angles the ſides proportional: ſo that A is alſo equiangled to B, and hath about the equal angles the ſides proportional; therefore A is like to B. Which was to be demonſtrated.

PRO-

PROPOSITION XXII.

If four ſtrait lines be proportional, the ſtrait-lin'd Figures deſcribed upon them, like and alike ſituated, ſhall alſo be proportional. And if the ſtrait-lin'd Figures deſcribed upon them, like and alike ſituated be proportional; alſo the ſtrait lines ſhall be proportional.

Let four ſtrait lines AB, CD, EF, GH, be proportional, as AB to CD, ſo EF to GH, and let be deſcribed upon AB, CD, the ſtrait-lin'd Figures KAB, LCD, like and alike ſituated [by Prop. 18. El. VI.] : and upon EF, GH, the ſtrait-lin'd Figures MF, NH, like and alike ſituated. I ſay, that it is as KAB to LCD, ſo MF to NH. For to AB, CD, let be taken a third proportional X, and to EF, GH, a third proportional O [Prop. 11. El. VI.]. Now becauſe it is as AB to CD, ſo EF to GH, and as CD to X, ſo GH to O; therefore by equality as AB to X, ſo EF to O. But as AB to X, ſo KAB to LCD [Corol. 2. Prop. 20. El. VI.]: and as EF to O, ſo MF to NH; therefore as KAB is to LCD, ſo MF is to NH.



But now let it be as KAB to LCD, ſo MF to NH. I ſay, that it is as AB to CD, ſo EF to GH. For let it be made [Prop. 12. El. VI.], as AB to CD, ſo EF to PR: and upon PR let be deſcribed the ſtrait-lin'd Figure SR, like and alike ſituated to either of the Figures MF, NH. Now therefore, becauſe it is as AB to CD ſo EF to PR: and upon AB, CD are deſcribed the ſtrait-lin'd Figures KAB, LCD, like and alike ſituated: and upon EF, PR the ſtrait-lin'd Figures MF, SR, like and alike ſituated; therefore as KAB is to LCD, ſo MF is to SR. But it is alſo ſuppoſed that as KAB is to LCD, ſo MF is to NH; therefore MF hath the ſame proportion to each of the Figures NH, SR: wherefore NH is equal to SR [Prop. 9. El. V.]. And alſo it is like to the ſame and alike ſituated by the Lemma: *therefore the line* GH *is equal to the line* PR. And becauſe as AB is to CD, ſo EF is to PR, and that PR is equal to GH; therefore as AB is to CD, ſo EF is to GH.

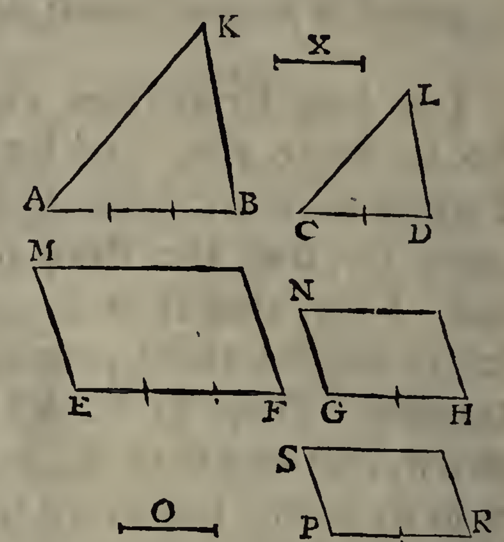
If therefore four ſtrait lines be proportional, alſo the ſtrait-lin'd Figures deſcribed upon them, like and alike ſituated, ſhall be proportional.

And if the ſtrait-lin'd Figures deſcribed upon them, like and alike ſituated be proportional; alſo the ſtrait lines ſhall be proportional. Which was to be demonſtrated.

Lemma.

For if ſtrait-lin'd Figures be equal, and like, that then their Homologal ſides are equal to one another (Antecedents to Antecedents, and Conſequents to Conſequents, each to each reſpectively) we ſhall thus demonſtrate.

Let the equal and like ſtrait-lin'd Figures be NH, SR; and let it be as HG, to GN, ſo RP to PS (ſo that HG and RP are Homologal). I ſay that RP is equal to HG.



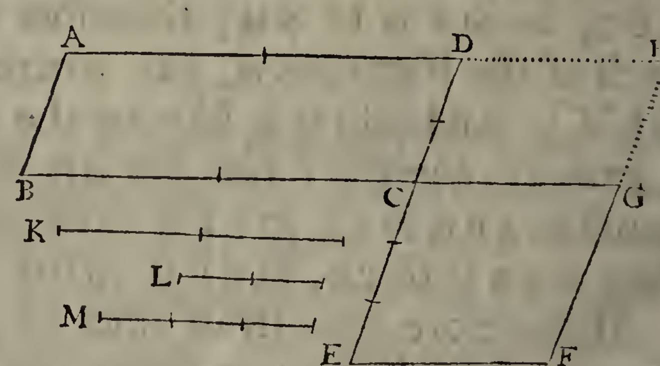
For if they be unequal, one of them is the greater. Let RP be greater than HG; and becauſe as RP is to PS, ſo HG is to GN; therefore alternately, as RP to HG, ſo PS to GN. But RP is greater than HG; therefore PS is greater than GN; ſo that alſo the Figure RS is greater than the Figure HN: but alſo it is equal, which is impoſſible: wherefore PR is not unequal to GH, equal therefore it is. Which was to be demonſtrated.

PROPOSITION XXIII.

E*Quiangled Parallelograms have to one another a proportion compounded of their ſides.*

Let the equiangled Parallelograms be AC, CF, having the angle BCD equal to the angle ECG. I ſay that the Parallelogram AC hath to the Parallelogram CF a proportion compounded of their ſides, that is, compounded of the proportions which BC hath to CG, and which DC hath to CE, (in a direct proportion, the Antecedents BC, DC, being in one Parallelogram AC, and the Conſequents CG, CE in the other Parallelogram CF.)

For let BC be put directly to CG (making one ſtrait line BG); therefore DC is alſo directly to CE [by Prop. 13, and 14. El. I.], and let be compleated the Parallelogram DG (by producing AD, FG till they meet in H).



Now let any ſtrait line be put as K: then let it be made as BC to CG, ſo K to L; and as DC to CE, ſo L to M [Prop. 12. El. VI.]; therefore the proportions of K to L, and of L to M, are the ſame with the proportions of the ſides, of BC to CG; and of DC to CE.

But the proportion of K to M is compounded of the proportions of of K to L, and of L to M [according to Def. 10. El. V.]: ſo that alſo K hath to M the proportion compounded of the ſides (of BC to CG, and of DC to CE).

And becauſe it is as BC to CG, ſo the Parallelogram AC to the Parallelogram CH [Prop. 1. El. VI.], and as BC to CG, ſo is K to L; therefore as K to L, ſo the Parallelogram AC is to the Parallelogram CH [Prop. 11. El. V.].

Again, becauſe it is as DC to CE, ſo the Parallelogram CH to the Parallelogram CF, and as DC to CE, ſo is L to M; therefore as L to M, ſo the Parallelogram CH is to the Parallelogram CF.

Now becauſe it hath been proved, that as K to L, ſo the Parallelogram AC is to the Parallelogram CH: and as L to M, ſo the Parallelogram CH to the Parallelogram CF; therefore *ex æquo* as K is to M, ſo the Parallelogram AC is to the Parallelogram CF: but K hath to M a proportion compounded of the ſides: wherefore the Parallelogram AC hath to the Parallelogram CF, a proportion compounded of their ſides.

Therefore equiangled Parallelograms have to one another a proportion compounded of their ſides. Which was to be demonſtrated.

ANNOTATIONS.

Therefore DC *is directly to* CE.] For the angles BCD, DCG, are equal to two Right [Prop. 13. El. I.] As likewiſe the angles BCE, ECG; wherefore BCD, DCG, are equal to BCE, ECG; and BCD is [by Suppoſition] equal to ECG; therefore BCE is equal to DCG. Let the angle BCD be added in common; then ſhall BCE, BCD, be equal to BCD, DCG. But BCD, DCG, are equal to two Right; therefore BCE, BCD, are alſo equal to two Right: wherefore CD is directly ſeated to CE [by Prop. 14. El. I.].

But the proportion of K *to* M *is compounded of the proportions of* K *to* L, *and of* L *to* M.] By Def. 10. El. V. where in any three continued Terms the proportion of the firſt to the third, is ſaid to be compounded of the firſt to the ſecond, and of the ſecond to the third; therefore in reference to that Definition *Euclide* reduces the two proportions of the four ſides of the Parallelograms *viz.* of BC to CG, and of DC to CE into three continued Terms, into K to L and L to M, and then he proves the proportion of the Parallelograms to be to one another as the firſt Term K is to the third M: which [in Def. 10. El. V.] is ſaid to be a proportion compounded of K to L, and of L to M. But theſe proportions were made the ſame with the proportions of the ſides of thoſe Parallelograms; and therefore the Parallelograms are to one another in a compound proportion of their ſides.

The like demonſtration *Euclide* uſes alſo in Numbers, without any mention of the multiplication of the Quantities of proportions into one another.

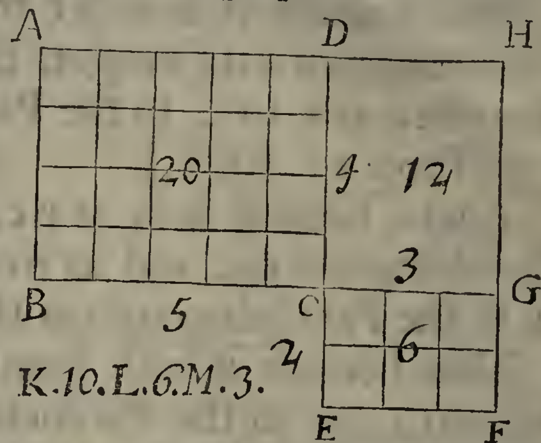
For 1ſt. in Prop. 2d. and 3d. El. VII. he ſhews how to find the greateſt common meaſure of any given numbers.

II. By the help of the greateſt common meaſure he reduces any given numbers into the leaſt numbers, having the ſame proportion with them. Prop. 37th. El. VII.

III. Proportions being given in their leaſt numbers, he ſhews how to bring them into the leaſt continued numbers in the ſame proportions. Prop. 4th. El. VIII.

IV. In the next following Prop. 5th. El. VIII. He demonſtrates Plane numbers to have a proportion compounded of their ſides, by bringing the proportions of the ſides into the leaſt continued numbers in the ſame proportions, the demonſtrations both in magnitudes and numbers proceeding upon the ſame grounds, and juſt after the ſame manner.

As the proportion of the Plane number 20 to the Plane number 6 (which is triple Seſquitertial $3\frac{1}{3}$) is proved to be compounded of the proportions of their ſides, of 5 to 3, of BC to CG; and of 4 to 2, of DC to CE. Here therefore the proportions of the ſides 5 to 3, and of 4 to 2, are firſt reduced into the leaſt continued numbers in the ſame proportions, that is, into 10 to 6, and 6 to 3, by Prop. 4th. El. VIII. and then the Plane 20 is proved to be to the Plane 6, as 10 to 3, that is $3\frac{1}{3}$. But 10 to 3 is compounded of 10 to 6, and of 6 to 3 [Def. 10th. El. V.], which proportions are the ſame with thoſe of the ſides 5 to 3, and 4 to 2, therefore the Plane number 20 is to the Plane number 6, in a proportion compounded of the ſides 5 to 3, and 4 to 2. So that neither in magnitudes, nor numbers is there any uſe of Def. 5th. El. VI. in the demonſtrations concerning compound proportions.



Moreover 'tis to be obſerved from the 19th. 20th. and 23d. Propoſitions of this VI. Element, that compound proportions are expreſt two manner of ways. For in this 23d. Prop. becauſe equiangled Parallelograms are not always like Figures, having their ſides about the equal angles proportional; therefore they are proved in general to have to one another a proportion compounded of their ſides, that is, in reality a proportion compounded of length to length, and of breadth to breadth, according to their two dimenſions in length and breadth.

But becauſe in Prop. 19th. and 20th. the Triangles, Parallelograms, and other Quadrilateral, and Multilateral Figures, are put for like Figures, where length is to length, and breadth to breadth in the ſame proportion; therefore they are proved to be to one another in a Duplicate proportion of their Homologal ſides, which is in effect to ſay, that they have to one another a proportion compounded of two equal proportions, according to length and breadth.

Now whether the Figures be put like, as in Prop. 19th. and 20th. or unlike, as in Prop. 23d. yet note, that one Plane is to an other Plane in a proportion ever compounded of length to length, and of breadth to breadth: and demonſtrated to be ſo, by reducing thoſe two proportions into three continued Terms, and proving Plane to be to Plane, as the firſt Term to the third; which by Def. 10th. El. 5. is ſaid to be compounded of the proportions of the firſt to the ſecond, and of the ſecond to the third, and in equal proportions is called a Duplicate proportion of the firſt to the ſecond.

In like manner a Solid is to a Solid in a proportion compounded of their three dimenſions, according to length, breadth, and depth: And like Solids are one to an other in a Triplicate proportion of their Homologal ſides. As from the eleventh Definition of the Fifth Element is demonſtrated in Prop. 12th, 19th. El. VIII. Prop. 33d. El. XI.

And thus much for a farther explication of the 10th, and 11th. Definitions of the Fifth Element: wherein we have had an occaſion given from the ſuppoſititious fifth Definition El. VI. to ſhew what is Natural, what Artificial, and neceſſary for demonſtration ſake upon this Subject of compound proportions.

Moreover we are to obſerve in compound proportions, that in the 19th, and 20th. Propoſitions aforegoing, *Euclide* demonſtrates all like Figures, that is, all equiangled Figures, which have their ſides about their equal angles proportional [Def. 1. El. VI.] (whether Trilateral, Quadrilateral, or Multilateral) to have to one another a Duplicate proportion of their Homologal ſides: and in this 23d. Prop. all equiangled Parallelograms to have a proportion compounded of their ſides: which is the firſt Propoſition, where in expreſs words *Euclide* names any proportion to be *Compounded*. But the thing is the very ſame, and Compoſition of proportions alike in all theſe three Propoſitions.

For

For Planes are in quantity to be compared to Planes, according to their two dimenſions of length and breadth: (One way of dimenſion being called length, and the Tranſverſe called breadth) ſo that the proportion of Plane to Plane is truely compounded of two proportions, one of length to length, the other of breadth to breadth: and theſe two proportions are here in Prop. 19th, 20th, and 23d. alike reduced (for demonſtration ſake) into *three continued Terms*, where the firſt compared to the ſecond, and the ſecond to the third, do anſwer to the two proportions of length to length, and of breadth to breadth; as upon this 23d. Propoſition *Commandinus*, and *Clavius* have demonſtrated. And *Euclide* proves the proportion of theſe Planes to be to one another as the firſt Term is to the third, in a proportion compounded of the firſt to the ſecond, and of the ſecond to the third, that is, of length to length, and of breadth to breadth.

Now in the like Figures ſpecified in the 19th, and 20th. Propoſitions, becauſe length is to length as breadth to breadth in the ſame proportion; therefore theſe two equal proportions of length and breadth, being by *Euclide* reduced into *three continued Terms*, do make thoſe continued Terms alſo *proportional* to one another, the firſt to the ſecond, as the ſecond to the third; and therefore when theſe like Figures are demonſtrated to be to one another as the firſt Term is to the third, they are ſaid to be in a Duplicate proportion of the firſt to the ſecond [Def. 10th. El. V.], which Duplicate is a proportion compounded of the two equal proportions, that anſwer to their lengths and breadths, here now ſet forth in *three continued, and proportional* Terms. And like Parallelograms, as well as like Triangles, are to one another in a Duplicate proportion of their Homologal ſides.

But becauſe *all equiangled Parallelograms* are not always *like Figures* to have the proportions of their lengths and breadths, reducible into *three proportional* Terms; yet the proportions of their ſides may be reduced into *three continued* Terms, repreſenting the two proportions of their ſides, which correſpond with the proportions of their proper lengths and breadths; and the Parallelograms are proved to be to one another as the firſt Term is to the third: but can be only ſaid in general to have a proportion compounded of the proportions of the firſt to the ſecond, and of the ſecond to the third; and not as in *like Parallelograms*, to have a Duplicate proportion of the firſt to the ſecond.

For the meaning of this whole matter is, that if three, or more magnitudes be *continuedly* compared to one another, according to quantity (that is, how much the firſt contains of the ſecond, then how much the ſecond of the third, and how much the third of the fourth, &c.), and if theſe proportions be known, or prov'd to be all the ſame, making the Terms to be *ſucceſſively proportionals* [as in Prop. 19th, and 20th. El. VI.] then the compound proportion of the firſt to the third, is ſaid to be a Duplicate proportion of the firſt to the ſecond, according to the expreſs words of Def. 10th. El. V. But if the proportions be put as unknown, or as indifferent whether the ſame, or not the ſame (as in this 23d. Propoſition), then the proportion of the firſt to the third, or of the firſt to the fourth, &c. is in general ſaid to be compounded of all the intermedial proportions.

Wherefore *Euclid's* demonſtration of this 23d. Propoſition, as alſo of all the like compounded proportions is wholly grounded on the right underſtanding, and full extent of the 10th. Definition of the Fifth Element, as it hath been before explained.

Moreover we are to obſerve, that as in Def. 3d. El. V. there is only defined proportion in magnitudes; yet the ſame ſtands for proportion in numbers, *mutatis mutandis*, without any new Definition given; ſo in the Definitions of Duplicate, and Triplicate proportions (names only proper to *equal proportions*, expoſed in *continued Terms*) there is further to be underſtood *all other compound proportions* expoſed in *continued Terms*, as plainly appears by *Euclid's* demonſtrations, who no where owns any other Definition of a compound proportion, than what is compriſed in Def. 10th. El. V.

Thus the only four rugged paſſages met withal in theſe Elements are, I hope, made plain and ſmooth, even to the weakeſt Traveller in theſe Studies, and vindicated from the exceptions of ſome capricious Geometricians: Namely, *Firſt*

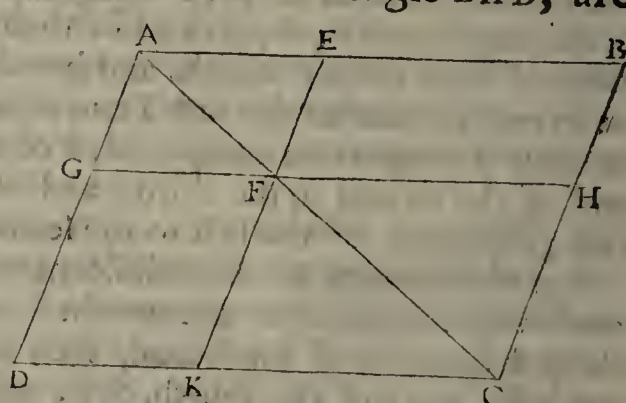
Geometrical

Geometrical *Epharmosis* used in Prop. 4th. El. I. *Secondly*, the 11th. Axiom, used in Prop. 29th. El. I. *Thirdly* and *fourthly*, the 5th, and 10th. Definitions of the Fifth Element concerning proportional magnitudes, and compounded proportions. If I be not censured for having on these matters too much enlarged.

PROPOSITION XXIV.

Of every Parallelogram, the Parallelograms about the Diameter are like to the whole, and to one another.

Let the Parallelogram be ABCD, and the Diameter AC: and about the Diameter AC, let the Parallelograms be EG, HK. I say, that each of the Parallelograms EG, HK, is like to the whole ABCD, and to one another. For because in the Triangle ABC to one of the sides BC is drawn EF parallel; therefore it is proportionally as BE to EA, so CF to FA [Prop. 2. El. VI.]. Again, because in the Triangle ACD to one of the sides CD, is drawn FG parallel; therefore it is proportionally as CF to FA, so DG to GA: but as CF to FA, so BE is prov'd to be unto EA: and therefore as BE to EA, so DG to GA [Prop. 11. El. V.]: and by Composition, as BA to EA, so DA to GA; and alternately, as BA to DA, so EA to GA; therefore of the Parallelograms ABCD, EG the sides about the common angle BAD, are proportional. And because GF is parallel to DC, therefore the angle AGF is equal to the angle ADC, and the angle GFA to the angle DCA [Prop. 29. El. I.]: also the angle DAC is common to the two Triangles ADC, AGF; therefore the Triangle ADC, is equiangled to the Triangle AGF. By the same reason also the Triangle ABC, is equiangled to the Triangle AEF: wherefore the whole Parallelogram ABCD is equiangled to the Parallelogram EG.



Proportionally therefore it is, as AD to DC, so AG to GF [Prop. 4. El. VI.], and as DC to CA, so GF to FA; and as CA to CB, so FA to FE; and moreover, as CB to BA, so FE to EA. And because it has been prov'd, that as DC to CA, so GF to FA; and as CA to CB, so FA to FE: wherefore by equality, as DC to CB, so GF, to FE; therefore of the Parallelograms ABCD, EG the sides about the equal angles are proportional: wherefore the Parallelogram ABCD, is like to the Parallelogram EG [Def. 1. El. VI.].

By the same reason also, the Parallelogram ABCD is like to the Parallelogram HK; therefore each of the Parallelograms EG, HK, is like to the Parallelogram ABCD. But Figures like to the same Figure, are like to one another [Prop. 21. El. VI.]; therefore the Parallelogram EG, is like to the Parallelogram HK.

Therefore of every Parallelogram, the Parallelograms about the Diameter,

Diameter, are like to the whole, and to one another. Which was to be demonſtrated.

ANNOTATIONS.

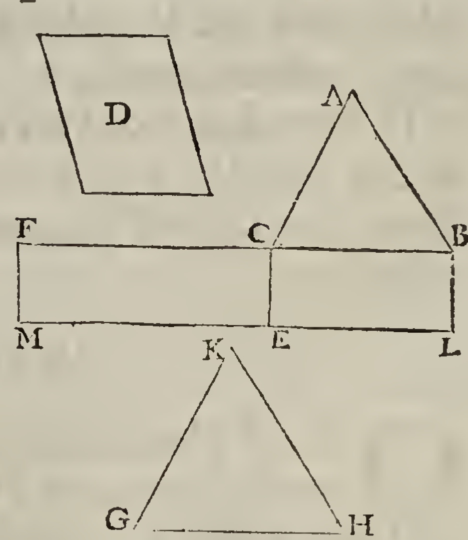
From hence 'tis obſervable, that the Parallelograms about the Diameter, are like Figures having their ſides to one another directly proportional, and the Complements are equal Parallelograms, having their ſides reciprocally proportional to one another, Prop. 43^d^. El. I. Prop. 14^th^. El. VI. Moreover either of the Complements is a mean proportional between the Parallelograms about the Diameter, by Prop. 1. El. VI. and Prop. 38^th^. El. I. which alſo are to one another in a Duplicate proportion of their Homologal ſides. Prop. 20^th^. El. VI.

Peletarius hath very well advertiſed upon the excellency of this Diagram, *Hanc ego Figuram ſoleo vocare Myſticam. Ex ea enim, velut ex locupletiſſimo promptuario, innumerabiles exeunt demonſtrationes, quod cum magna voluptate perſpiciet, qui in re Geometrica ſerio ſe exercebit.*

PROPOSITION XXV.

To a given ſtrait-lin'd Figure to conſtitute a like, and the ſame alſo equal to another given ſtrait-lin'd Figure.

Let the given ſtrait-lin'd Figure, to which a like is to be conſtituted be ABC, and that to which the ſame is to be equal let be D. It is required to ABC to conſtitute a like Figure, and the ſame equal to D. To the ſtrait line BC let be apply'd a Parallelogram BE, equal to the Triangle ABC, and to CE the Parallelogram CM equal to D, in an angle FCE, which is equal to the angle CBL [Prop. 44. El. I.]; therefore BC is direct to CF [Prop. 14. El. I.], and LE to EM. And to BC, CF let be taken a mean proportional GH [Prop. 13. El. VI.], and upon GH let be deſcribed KGH like and alike ſituated to ABC [Prop. 18. El. VI.]. Now becauſe it is as BC to GH, ſo GH to CF: and if three ſtrait lines be proportional, it is as the firſt to the third, ſo the Figure upon the firſt to the Figure upon the ſecond, like and alike deſcrib'd; therefore as BC is to CF, ſo ABC is to KGH [Corol. 2. Prop. 20. El. VI.]. But alſo as BC is to CF, ſo the Parallelogram BE is to the Parallelogram EF; therefore as ABC is to KGH, ſo the Parallelogram BE is to the Parallelogram EF [Prop. 11. El. V.]: wherefore alternately, as ABC is to the Parallelogram BE, ſo KGH is to the Parallelogram EF. But ABC is equal to the Parallelogram BE; therefore alſo KGH is equal to the Parallelogram EF: and the Parallelogram EF is equal to D: wherefore alſo KGH is equal to D; but KGH is like to ABC.



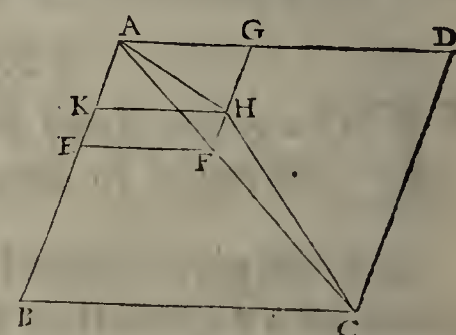
Therefore to the given ſtrait-lin'd Figure ABC, there is conſtituted KGH like; and the ſame alſo equal to an other given ſtrait-lin'd Figure. Which was to be done.

XPROPOSITION XXVI.

If from a Parallelogram be taken a Parallelogram like to the whole, and alike ſituated, having a common angle; it is about the ſame Diameter with the whole.

For from the Parallelogram ABCD, let be taken the Parallelogram AF, like to ABCD, and alike ſituated, having the angle DAB common. I ſay, that the Parallelogram ABCD is about the ſame Diameter with the Parallelogram AF.

For if not, then if poſſible, let the Diameter of ABCD be AHC, and by H to either of the lines AD, BC, let HK be drawn parallel.



Now foraſmuch as the Parallelogram ABCD is about the ſame Diameter with the Parallelogram KG [by Conſtruction]; therefore ABCD is like to KG [Prop. 24. El. VI.]: wherefore it is as DA to AB, ſo GA to AK [Def. 1. El. VI.]. But alſo for the likeneſs of the Parallelograms ABCD, EG [by Suppoſition]; it is as DA to AB, ſo GA to AE; therefore as GA to AE, ſo GA to AK [Prop. 11. El. V.]: wherefore GA hath the ſame proportion to each of the lines AK, AE; therefore AK is equal to AE [Prop. 9. El. V.], the leſs to the greater, which is impoſſible; therefore ABCD is not about the ſame Diameter with AH: wherefore the Parallelogram ABCD is about the ſame Diameter with the Parallelogram AF.

If therefore from a Parallelogram be taken a Parallelogram like to the whole, and alike ſituated, having a common angle; it is about the ſame Diameter with the whole. Which was to be demonſtrated.

PROPOSITION XXVII.

Of all Parallelogram ſpaces applyed to the ſame ſtrait line, and deficient by Parallelogram Figures, like and alike ſituated to that which is deſcribed on the half line: The greateſt is that Parallelogram, which is applyed to the half, being like to the defect (deſcribed on the other half.)

Let the ſtrait line be AB, and let it be cut into halves in C; and to AB let be applyed a Parallelogram AD, deſcribed on AC an half of AB, and deficient by the Parallelogram Figure DB, deſcribed on CB the other half of AB, like and alike ſituated to AD.

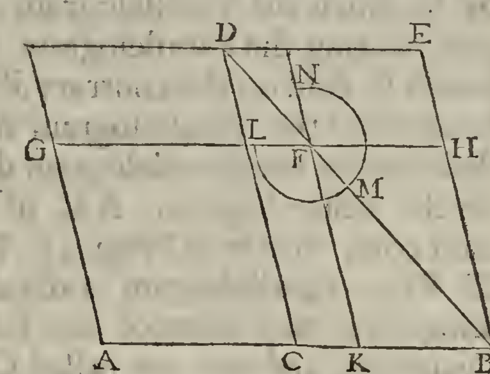
I ſay, that of all Parallelogram ſpaces applyed to AB, and deficient by Parallelogram Figures, like and alike ſituated to the defect DB, the greateſt is AD.

 For

For to the ſtrait line AB, let be applyed a Parallelogram AF, deficient by the Parallelogram Figure FB, like, and alike ſituated to DB. I ſay, that AD is greater than AF.

For becauſe the Parallelogram DB is like to the Parallelogram FB; therefore they are about the ſame Diameter [Prop. 26. El. VI.], let be drawn their Diameter DB: and the Scheme be compleated.

Now foraſmuch as CF is equal to FE [Prop. 41. El. I.], let FB be added in common; therefore the whole CH is equal to the whole KE: but CH is equal to CG, becauſe AC is equal to CB [Prop. 36. El. I.]: and therefore CG is equal to KE. Let CF be added in common; therefore the whole AF is equal to the Gnomon LMN, ſo that alſo the Parallelogram DB, that is, AD is greater than the Parallelogram AF.



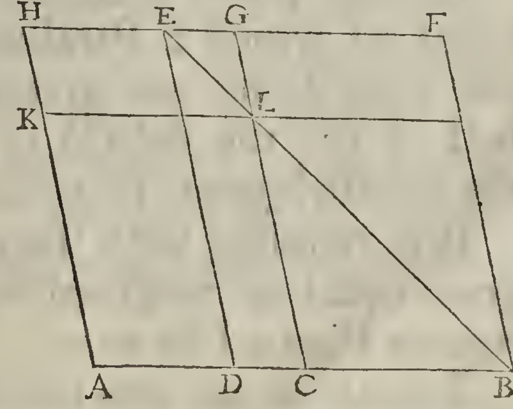
Therefore of all Parallelograms applyed to the ſame ſtrait line, and deficient by Parallelogram Figures, like and alike ſituated to that, which is deſcribed on the half line: The greateſt applyed Parallelogram is that deſcribed on the half, being like to the defect.

"By this full and general Concluſion *Euclide* ſeems to have "ended his demonſtration of this Propoſition: but becauſe it ad-"mits of two Caſes: and that only one of them is hitherto demon-"ſtrated, wherein the applyed Parallelogram is deſcribed on ſuch "a part of AB, as on AK, which is greater than the half AC; there-"fore ſtill there remains to conſider the ſame, where the Paralle-"logram is deſcribed on a part leſs than the half AC, as here on AD, "which (tho' not abſolutely neceſſary) was likely ſupplyed by *Theon*, "or ſome ancient *Scholiaſt*, as followeth.

Again, let AB be cut into halves in C; and the applyed Parallelogram be AL, deſcribed on the half AC, deficient by the Parallelogram LB. And again to the line AB, let be applyed the Parallelogram AE, deficient by the Parallelogram EB, like, and alike ſituated to the Parallelogram LB, deſcribed on CB the half of AB.

I ſay, that the applyed Parallelogram AL, deſcribed on the half of AB, is greater than the Parallelogram AE.

For becauſe the Parallelogram EB is like to the Parallelogram LB; therefore they are about the ſame Diameter. Let the Diameter be EB; and the Scheme be deſcribed.



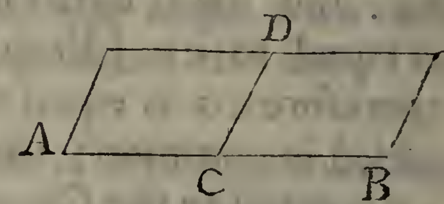
Now foraſmuch as LF is equal to LH [Prop. 36. El. I.], for FG is equal to GH; therefore FL is greater than EK. But FL is equal to LD [Prop. 43. El. I.]; therefore LD is greater

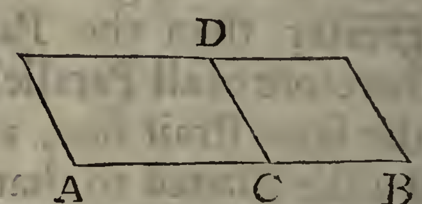
 than

than EK. Add in common KD; therefore the whole AL is greater than the whole AE. Which was to be demonſtrated.

ANNOTATIONS.

If a ſtrait line be cut into two Segments equal, or unequal, as AB into AC, CB: and to either of them, as to AC be applyed any Parallelogram as AD, then is AD ſaid to be deficient by the Parallelogram DB, which in the ſame parallels is equiangled to it, and conjoyned by a common ſide, as DC, and deſcribed on the other Segment CB. Now this Parallelogram DB is called the defect, for that by ſo much the Parallelogram AD is deficient in compleating the Parallelogram ſpace on the whole line AB. And on the contrary, if to the Segment BC be applyed the Parallelogram BD, it is ſaid to be deficient by the Parallelogram defect DA deſcribed on the other Segment AC, of the ſtrait line AB. And note, that as in Prop. 43d. El. I. and Prop. 24th. El. VI. a Parallelogram is divided into four Parallelograms, two whereof are ſaid to be about the Diameter, and two are called Complements; ſo in this Propoſition a Parallelogram is in a manner divided into two Parallelograms, and one of them is ſaid to be deficient by the other, which is called the defect. And this diſtinction rightly obſerved, makes clear *Euclid's* expreſſion of this Propoſition, that hath ſeemed to ſome of our Modern Geometricians to be obſcurely worded. But a Parallelogram *deficient*, and the *defect* is as properly here ſpoken, and as plainly to be apprehended, as Parallelograms about the Diameter, and Parallelograms the Complements.





PROPOSITION XXVIII.

Unto a given ſtrait line to apply a Parallelogram equal to a given Rectilineal ſpace, deficient by a Parallelogram Figure like to a given Parallelogram.

Now the given Rectilineal ſpace, to which the applyed Parallelogram is to be equal, ought not to be greater than that applyed Parallelogram, which is deſcribed on the half line: both defects being like to the given Parallelogram, namely the defect of the Parallelogram deſcribed on the half line, and the defect of the Parallelogram required to be applyed, equal to the given Rectilineal ſpace.

Let the given ſtrait line be AB, and the given Rectilineal ſpace, equal to which a Parallelogram is to be applyed to AB, let be C, the ſame being not greater than a Parallelogram applyed to the half of AB: the defects of theſe Parallelograms being alike; and the Parallelogram to which the defect ought to be like, let be D.

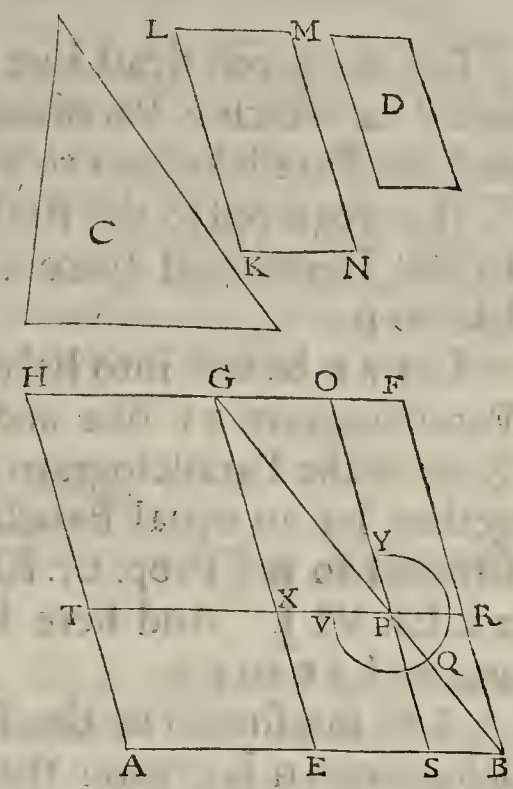
It is required unto the given ſtrait line AB to apply a Parallelogram equal to the given Rectilineal ſpace C, deficient by a Parallelogram Figure like to D.

Let AB be cut into halves in the point E, and on EB let be deſcribed EBFG, like and alike ſituated to D [Prop. 18. El. VI.]; and let be compleated the Parallelogram AG.

4 Now

Now AG is either equal to C, or greater than it, by the determination.

If AG be equal to C; then that is done which was required. For to the given ſtrait line AB is applyed the Parallelogram AG, equal to the given Rectilineal ſpace C, deficient by the Parallelogram Figure GB, being like to D.



But if AG be not equal to C; then it is greater: but AG is equal to GB [Prop. 36. El. I.]; therefore GB is greater than C.

Now by how much GB is greater than C, let a Parallelogram equal to that exceſs be conſtituted KLMN, the ſame alſo like and alike ſituated to D [Prop. 25. El. VI.]. But D is like to GB: wherefore alſo KLMN is like to GB; and here let KL be Homologal to EG, and LM to GF.

And foraſmuch as GB is equal to C, and KM together; therefore GB is greater than KM; and therefore the ſide EG is greater than the ſide KL, and GF than LM. Let now GX be put equal to KL, and GO to LM [by Prop. 3. El. I.]; and let be compleated the Parallelogram XGOP; therefore the Parallelogram GP is equal and like to the Parallelogram KM; but KM is like to GB: wherefore alſo GP is like to GB [Prop. 21. El. VI.]; therefore the Parallelogram GP is about the ſame Diameter with the whole GB. Let their Diameter be GPB and the Scheme be deſcribed.

Now foraſmuch as GB is equal to C, and KM together, and that GP is equal to KM; therefore the remaining Gnomon VQY is equal to the remaining Rectilineal ſpace C. And becauſe OR is equal to XS [Prop. 43. El. I.], let PB be added in common; therefore the whole OB is equal to the whole XB. But XB is equal to TE, becauſe the ſide AE is equal to the ſide EB [Prop. 36. El. I.]: wherefore alſo TE is equal to OB. Let XS be added in common; therefore the whole TS is equal to the whole Gnomon VQY: but the Gnomon VQY has been proved equal to C: therefore alſo TS is equal to C.

Wherefore to the given ſtrait line AB is applyed the Parallelogram TS equal to the given Rectilineal ſpace C, deficient by the Parallelogram figure PB, which is like to the given Parallelogram D, for that PB is like to GP. Which was to be done.

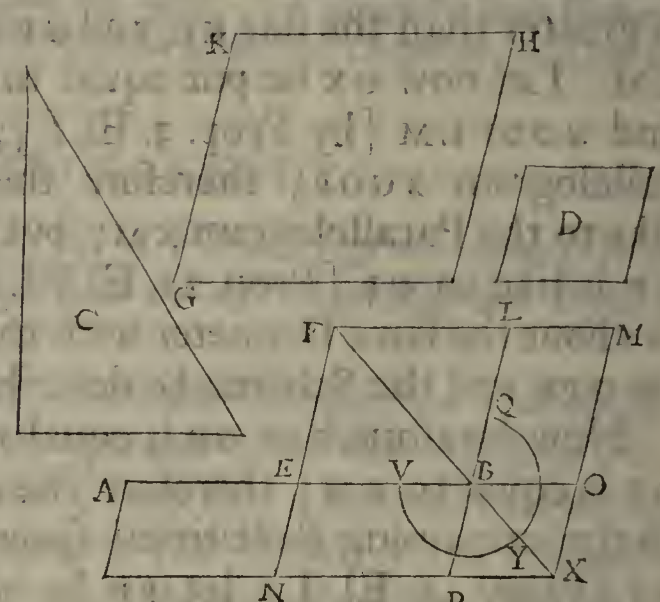
PROPOSITION XXIX.

Unto a given ſtrait line to apply a Parallelogram equal to a given Rectilineal ſpace, exceeding by a Parallelogram figure, like to a given Parallelogram.

Let the given ſtrait line be AB; and the given Rectilineal ſpace equal to which a Parallelogram is to be applyed to AB, let be C; and the Parallelogram to which the exceſs ought to be like, let be D.

It is required to the ſtrait line AB to apply a Parallelogram equal to the Rectilineal ſpace C, exceeding by a Parallelogram figure like to D.

Let AB be cut into halves in E, and on EB let be deſcribed the Parallelogram BF like and alike ſituated to D [Prop. 18. El. VI.]. Now to the Parallelogram BF, and the Rectilineal ſpace C, both together let an equal Parallelogram be conſtituted GH, like and alike ſituated to D [Prop. 25. El. VI.]; therefore GH is like to BF [Prop. 21. El. VI.]. And here let the ſide KH be Homologal to the ſide FL, and KG to FE.



And foraſmuch as the Parallelogram GH is greater than the Parallelogram BF; therefore alſo the ſide KH is greater than the ſide FL, and KG than FE. Let FL, FE be produced: and FLM be put equal to KH, and FEN to KG [by Prop. 3. El. I.], and let be compleated the Parallelogram MN; therefore MN is equal, and like to GH. But GH is like to EL: wherefore alſo MN is like to EL [Prop. 21. El. VI.]; therefore EL is about the ſame Diameter with MN [Prop. 26. El. VI.]. Let be drawn their Diameter FX, and the Scheme be deſcribed.

Now foraſmuch as GH is equal to EL, and C together, and that GH is equal to MN; therefore alſo MN is equal to EL, C. Let EL common be taken away; therefore the remaining Gnomon VYQ is equal to C. And becauſe AE is equal to EB, therefore the Parallelogram AN is equal to the Parallelogram NB [Prop. 36. El. I.]: that is to LO [Prop. 43. El. I.]. Let EX be added in common; therefore the whole AX is equal to the Gnomon VYQ. But the Gnomon VYQ is equal to C; wherefore alſo AX is equal to C.

Therefore to the given ſtrait line AB is applyed the Parallelogram AX, equal to the given Rectilineal ſpace C, exceeding by the Parallelogram

logram figure PO, which is like to D; for that EL is like to PO [Prop. 24. El. VI.]. Which was to be done.

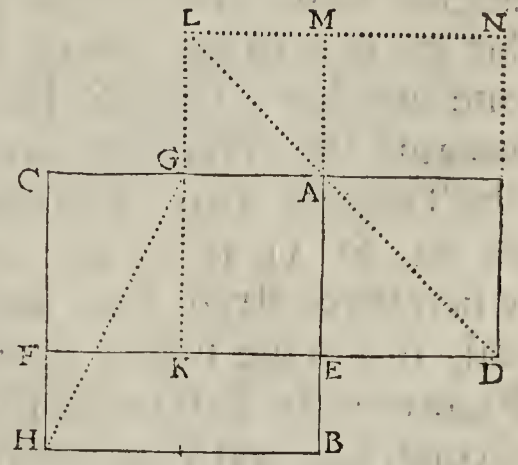
ANNOTATIONS.

These two Propositions the 28th, and 29th, are of admirable use in the Tenth Element of *Euclide*, and the *Conic* Elements of *Apollonius*, where the deficient, or *Elliptical* application made in Prop. 28th, gave occasion for the name of that Conic Section called ELLIPSIS in Prop. 13th. El. I. of *Apollonius*, and the exceeding, or *Hyperbolical* application made in Prop. 29th, occasioned the name of that Section called HYPERBOLA, in Prop. the 12th, of the same Conic Element. When therefore *Tacquet* so rashly rejects these two Propositions, as of little, or no use; 'tis manifest that he had then made but a small progress in Geometry. And thirdly, the Conic Section called PARABOLA, we have before noted at Prop. 44th. El. I. to be so named from the exact application, or Parabolism of a given Rectilineal space to the entire, and whole given line precisely, neither deficient, or exceeding. As we find it in Prop. 11th. El. I. of *Apollonius*.

PROPOSITION XXX.

TO cut a given finite strait line in Extreme and Mean proportion.

Let the given finite strait line be AB. It is required to cut the strait line AB in extreme and mean proportion. On AB let be described the Square BC [by Prop. 46. El. I.]: and to AC let be applyed the Parallelogram CD equal to BC, exceeding in Figure by AD, like to BC [Prop. 29. El. VI.]. Now BC is a Square; therefore also AD is a Square.

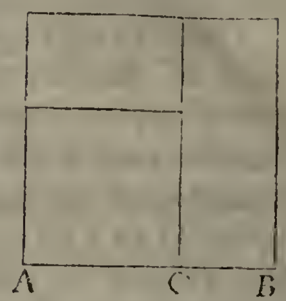


And forasmuch as BC is equal to CD; let CE common be taken away; therefore the remainder BF is equal to the remainder AD. But it is also equiangled to the same; therefore the sides of BF, AD about the equal angles are reciprocally proportional [Prop. 14. El. VI.]: wherefore as FE to ED, so AE to EB. But FE is equal to AC [Prop. 34. El. I.], that is, to AB; and ED to AE: therefore as AB to AE, so AE to EB. But AB is greater than AE: wherefore also AE is greater than EB [Prop. 14. El. V.].

Therefore the strait line AB is cut in extreme and mean proportion in the point E; and the greater Segment is AE. Which was to be done.

Otherwise.

Let the given strait line be AB. It is required to cut the strait line AB in extreme and mean proportion.



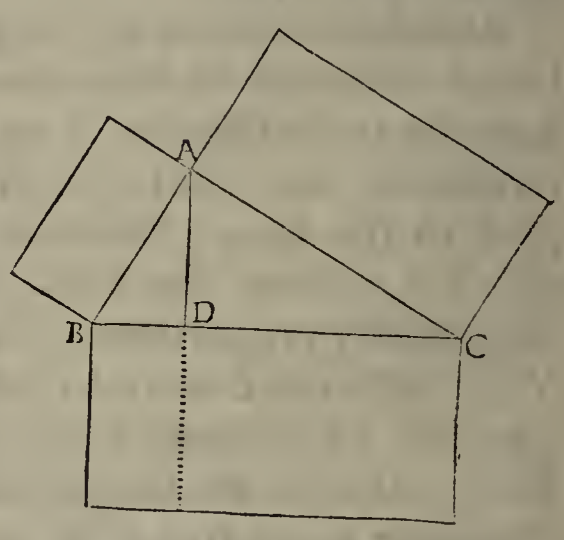
Let AB be cut in the point C, so that the Rectangle under AB, BC be equal to the Square of AC [by Prop. 11. El. II.].

Now forasmuch as the Rectangle under AB, BC is equal to the Square of AC; therefore as AB is to AC, so AC is to CB [Prop. 17. El. VI.]; AB therefore is cut in extreme and mean proportion in the point C. Which was to be done.

PROPOSITION XXXI.

IN Right-angled Triangles, the Figure constituted on the side subtending the Right angle is equal to the Figures constituted on the sides containing the Right angle, they being like and alike described.

Let the Right-angled Triangle be ABC, having the right angle BAC. I say, that the Figure upon BC is equal to the Figures on BA, AC, like and alike described. Let be drawn the perpendicular AD. Now forasmuch as in the Triangle ABC, from the right angle at A, is drawn to the base BC the perpendicular AD; therefore the Triangles ABD, ADC, at the perpendicular are like to the whole ABC, and to one another [Prop. 8. El. VI.]. And because the Triangle ABC is like to the Triangle ABD; therefore as CB is to BA, so AB is to BD: and because when three strait lines are proportional, it is as the first to the third, so the Figure on the first to the Figure on the second, like and alike described [Corol. 2. Prop. 20. El. VI.]; therefore as CB to BD, so the Figure on CB to the Figure on BA, like and alike described.



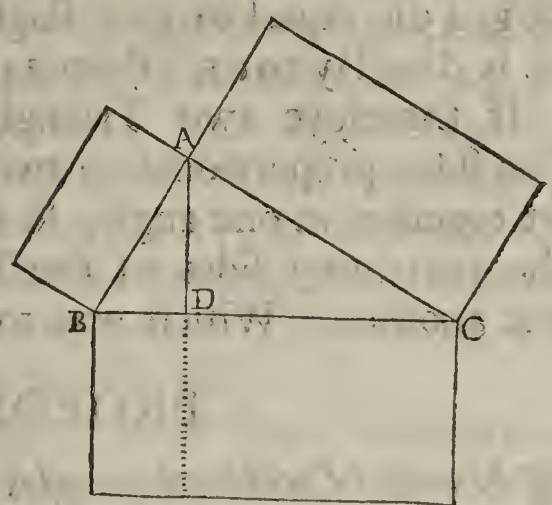
By the same reason also, as BC to CD, so the Figure on BC to the Figure on CA. So that also as BC to BD, DC, so the Figure on BC to the Figures on BA, AC, like and alike described. But BC is equal to BD, DC; therefore the Figure on BC is equal to the Figures on BA, AC.

Therefore in Right-angled Triangles the Figure on the side subtending the right angle, is equal to the Figures on the sides containing the right angle, they being like and alike described. Which was to be demonstrated.

Other-

Otherwife.

Becaufe like Figures are in a duplicate proportion of their Homologal fides [Prop. 20. El. VI.]; therefore the figure on BC hath to the Figure on BA, a duplicate proportion of that which BC hath to BA. But the Square of BC, hath to the Square of BA a duplicate proportion of that which BC hath to BA; therefore alfo as the Figure on BC to the Figure on BA, fo the Square of BC to the Square of BA [Prop. 11. El. V.]. By the fame reafon alfo as the Figure on BC to the Figure on CA, fo the Square of BC to the Square of CA: fo that alfo as the Figure on BC to the Figures on BA, AC, fo the Square of BC to the Squares of BA, AC. But the Square of BC is equal to the Squares of BA, AC [Prop. 47. El. I.]; therefore the Figure on BC is equal to the Figures on BA, AC, like and alike defcribed. Which was to be demonftrated.

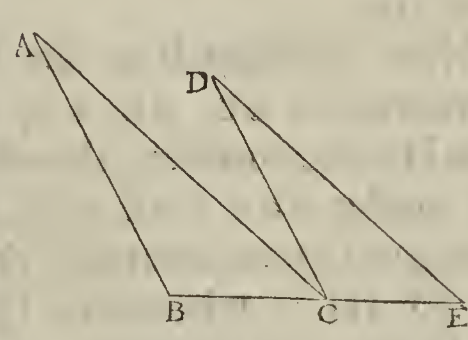


PROPOSITION XXXII.

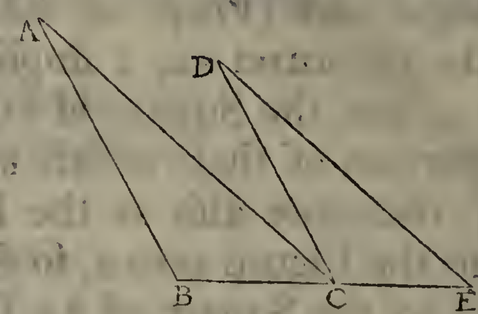
If two Triangles having two fides proportional to two fides, be fet together at one angle; fo that alfo their Homologal fides be parallel: the remaining fides of the Triangles fhall be directly fituated to one another.

Let the two Triangles be ABC, DCE, having the two fides AB, AC, proportional to the two fides DC, DE, that is, as AB to AC, fo DC to DE; and let AB be parallel to DC, and AC to DE. I fay, that BC is directly fituated to CE.

Forafmuch as AB is parallel to DC, and on them falls the ftrait line AC; therefore the alternate angles BAC, ACD, are equal to one another [Prop. 29. El. I.]. By the fame reafon alfo the angle CDE is equal to the angle ACD: fo that alfo BAC is equal to CDE. And becaufe there are two Triangles ABC, DCE, having one angle at A equal to one angle at D: and about the equal angles the fides proportional, as BA to AC, fo CD to DE; therefore the Triangle ABC is equiangled to the Triangle DCE [Prop. 6. El. VI.]: wherefore the angle ABC is equal to the angle DCE. But alfo the angle ACD has been prov'd equal to the angle BAC; therefore the whole angle ACE is equal to the two angles ABC, BAC. Let the angle ACB be added in common; therefore the angles ACE, ACB,



are equal to the angles BAC, ACB, CBA. But the angles BAC, ACB, CBA, are equal to two Right [Prop. 32. El. I.]; therefore alſo the angles ACE, ACB, are equal to two Right. Now to a certain ſtrait line AC, and to a point in the ſame C, two ſtrait lines BC, CE, not lying the ſame way make the conſequent angles ACE, ACB, equal to two Right; there- BC is directly to CE [Prop. 14. El. I.].



If therefore two Triangles having two ſides proportional to two ſides, be ſet together at one angle, ſo that their Homologal ſides be parallel: the remaining ſides of the Triangles ſhall be directly ſituated to one another. Which was to be demonſtrated.

PROPOSITION XXXIII.

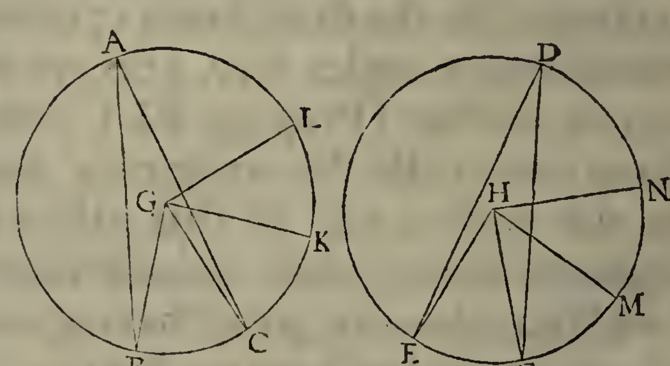
IN equal Circles, the angles have the ſame proportion with the Circumferences on which they inſiſt: whether the inſiſting angles be at the Centers, or at the Circumferences.

AND MOREOVER ALSO THE SECTORS (when conſtituted at the Centers).

Let the equal Circles be ABC, DEF, and at their Centers G, H, let the angles be BGC, EHF: and at the Circumferences, let the angles be BAC, EDF. I ſay, that as the circumference BC is to the circumference EF, ſo is the angle BGC to the angle EHF; and the angle BAC to the angle EDF: And moreover the Sector BGC, to the Sector EHF.

For to the circumference BC, let in order be put equals how many ſoever, CK, KL: and to the circumference EF, let be put alſo equals how many ſoever, FM, MN, and let be joyned GK, GL: HM, HN.

Now foraſmuch as the circumferences BC, CK, KL, are equal to one another; therefore the angles BGC, CGK, KGL, are alſo equal to one another [Prop. 27. El. III.]: wherefore Quotuple the circumference BL is of the circumference BC, Totuple is the angle BGL of the angle BGC. By the ſame reaſon Quotuple the circumference EN is of the circumference EF, Totuple is the angle EHN of the angle EHF.



If now the circumference BL be equal to the circumference EN, the angle BGL is alſo equal to the angle EHN [Prop. 27. El. III.]; and therefore if the circumference BL be greater than the circum-

ſ ference

ference EN, the angle BGL is alſo greater than the angle EHN: and if leſs, 'tis leſs.

There being then four magnitudes, the two circumferences BC, EF, and the two angles BGC, EHF: and of the circumference BC, and of the angle BGC are taken equimultiples, the circumference BL, and the angle BGL. Alſo of the circumference EF, and of the angle EHF, are taken equimultiples, the circumference EN, and the angle EHN. And it is prov'd, that if the circumference BL exceeds the circumference EN, the angle BGL does alſo exceed the angle EHN; and if equal, 'tis equal; and if leſs 'tis leſs: therefore as the circumference BC is to the circumference EF, ſo the angle BGC is to the angle EHF [Def. 5. El. V.]. But as the angle BGC is to the angle EHF, ſo the angle BAC is to the angle EDF [Prop. 15. El. V.]: for each is the double of each [Prop. 20. El. III.].

And therefore as the circumference BC is to the circumference EF, ſo the angle BGC is to the angle EHF, and the angle BAC to the angle EDF.

Therefore in equal Circles, the angles have the ſame proportion with the circumferences on which they inſiſt, whether the inſiſting angles be at the Centers, or at the circumferences. Which was to be demonſtrated.

Here *Euclide* ends the laſt Propoſition of his Sixth Element of Geometry.

THE

ADDITAMENT

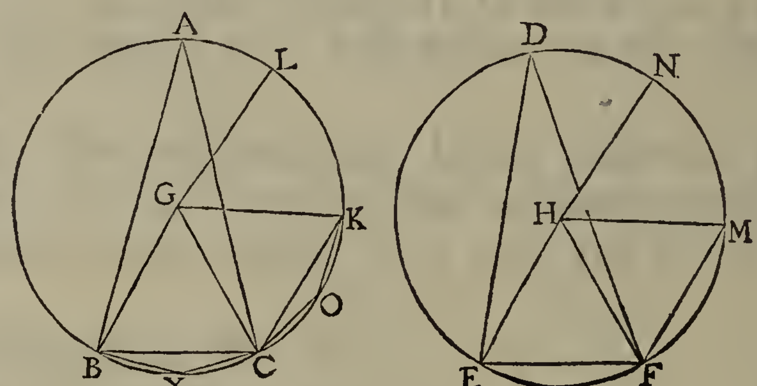
Of *THEON*,

And moreover alſo the Sectors,

That is, they have the ſame proportion with the Circumferences, on which they inſiſt.

I ſay alſo, that as the circumference BC is to the circumference EF, ſo the Sector BGC is to the Sector EHF. For let be joyned BC, CK, and in the circumferences BC, CK, the points X, O, being taken, let BX, XC, CO, OK, be alſo joyned.

Now foraſmuch as the two lines BG, GC, are equal to the two lines CG, GK, and they contain equal angles; therefore the baſe BC is equal to the baſe CK, and the Triangle GBC to the Triangle GCK [Prop. 4. El. I.].



And becauſe the circumference BC is equal to the circumference CK; therefore the remaining circumference BAC compleating the whole Circle ABC, is equal to the remaining circumference KAC, compleating the ſame Circle: ſo that alſo the angle BXC is equal to the angle COK [Prop. 27. El. III.]; therefore the Segment BXC is like to the Segment COK [Def. 10. El. III.], and they are upon equal ſtrait lines BC, CK: but like Segments of Circles upon equal ſtrait lines are equal to one another [Prop. 24. El. III.]; therefore the Segment BXC is equal to the Segment COK: and alſo the Triangle BGC is equal to the Triangle CGK; therefore the whole Sector BGC is equal to the whole Sector CGK.

By the ſame reaſon alſo the Sector GKL is equal to each of the Sectors GKC, GCB; therefore the three Sectors BGC, CGK, KGL, are equal to one another. And by the ſame reaſon the Sectors HEF, HFM, HMN, are equal to one another; therefore Quotuple the circumference LB is of the circumference BC, Totuple is the Sector GBL of the Sector GBC. By the ſame reaſon alſo Quotuple the circumference NE is of the circumference EF, Totuple is the Sector HEN of the Sector HEF.

Now if the circumference BL be equal to the circumference EN, the Sector BGL is alſo equal to the Sector EHN, and if the circumference BL exceeds the circumference EN, the Sector BGL does alſo exceed the Sector EHN, and if leſs, 'tis leſs.

There being then four magnitudes, the two circumferences BC, EF, and the two Sectors GBC, HEF, and of the circumference BC, and of the Sector GBC are taken equimultiples, the circumference BL, and the Sector GBL: alſo of the circumference EF and of the Sector HEF, are taken equimultiples, the circumference EN and the Sector HEN. And it is prov'd, that if the circumference BL exceeds the circumference EN, the Sector BGL does alſo exceed the Sector HEN; and if equal, 'tis equal; and if leſs, 'tis leſs; therefore as the circumference BC is to the circumference EF, ſo the Sector GBC is to the Sector HEF.

Corollary.

And it is manifeſt, that as the Sector is to the Sector, ſo alſo the angle is to the angle [Prop. 11. El. V.].

ANNOTATIONS.

Theon in his Commentaries on *Ptolemy's Magna Syntaxis*, takes notice of this Additament he made to *Euclid's* laſt Propoſition of the Sixth Element. Here therefore he ſays, ἔτι δὲ ϗ οἱ τομεῖς, *and moreover alſo the Sectors*; which are the only words of *Theon* added to *Euclid's* Propoſition. For what is ſubjoyned, ἅτε πρὸς τοῖς κεντροῖς συνιςάμενοι, *when conſtituted at the Centers*, muſt be ſome marginal note very abſurdly put in, as ſuppoſing there were an other kind of Sectors, beſides what are ſtated at the Center of a Circle, according to Def. 10. El. III. Indeed the Figures at the circumferences are not, as their angles are, in the ſame proportion with the Arches on which they inſiſt: but theſe Figures are not called *Sectors*, neither have they any Note or Name in Geometry to give occaſion for ſuch a needleſs caution: An overſight too great for *Theon* to be guilty of.

 Laſtly,

Laſtly, this Additament concerning Sectors mentioned by *Theon*, manifeſtly ſhews the error of *Zambertus* and others, who take the demonſtrations of theſe Elements to have been all ſupplyed by *Theon.* Yet it ſeems this miſtake went ſo far, as that *Bovillus* an Eminent Geometrician of that time, writ a Select Treatiſe about it, to vindicate the Ancient Geometricians in this matter of confirming the Propoſitions by their own Demonſtrations, and eſpecially *Euclide* for his Elements, ſo methodically diſpoſed, ſo plainly all along demonſtrated, and in the end cloſed with thoſe admirable Speculations on the five *Platonick* Bodies, thus celebrated in an Ancient *Greek* Epigram.

Σχήματα πέντε Πλάτωνος, ἃ Πυθαγόρας σοφὸς εὗρε,
Πυθαγόρας σοφὸς εὗρε, Πλάτων δ' ἀρίδηλ' ἐδίδαξεν
Εὐκλείδης ἐπὶ τοῖσι κλέος περικαλλὲς ἔτευξεν.

The five *Platonick* Bodies, ſo much fam'd,
Pythagoras firſt found, *Plato* explain'd:
Euclide on them Immortal Glory gain'd.

FINIS.

will Sadford	100
R Lucker	100
	165
Jo wap	
John Sadford	100
R [illegible]	165
	165
marlin	100
The Ad	165
wal ori	100
[illegible]	100
Hare	
mary white	065
[illegible]	100
	165
[illegible]	165
Jo pope	
John white	100
John male	050
young wi: Adam	100
old G: white	100
Jno Abbot	100
Penn	065
Poxey	165
Ross white	065
mat Homby	165
old Homby	100
John Bishop	165
old Adam	100
wife of Smyth	100
Bolter	100
R [illegible]	100
[illegible] Toogood	100
Margery Toogood	100
Eliz Bishop	100
[illegible] Skeuins	100
Robt Dame	100
Tho Banton	100
[illegible] Flowr	100
[illegible] watts	100
Giles Leg	175

helfor 150 bought home 150 more.

more for [illegible] house 85

50 pease at	25
8 flour	12
8 hosing	06
	06
	2
	2
	12
	65
	5.5

y 2 steps 3 ... — 150
24 stepps 6 foot high & 5 wide & 9 inches broad — 150
Ground worke of the Gallery 3 foot square & flowring 4 wide — 600
Round the well wainsed 3 foot high & 120 Long — 360
5 seats 6 with the Benches — 100
... — 024
8 ... of the seats — 120
the Backs of 4 seats — 157

33 ... 1 year 2 ground above woman croft & fur...
27 ~~...~~ 18 acres the 2 grounds ...
3 year ~~...~~ 18 acres Chamster croft & ~~Hunt~~
27

44 4th year sheephouse Close & Hunts ...
& Croft & furrows Close
18 5th year the 34 acres ~~...~~ & 26 acres a...
33 6th year peece above woman & 18 a...
42: 7 year 18 acres Chamster & Half...