NPS-53-88-006

NAVAL POSTGRADUATE SCHOOL

Monterey, California

B-Spline-Bezier Representation of Tau-Splines

Dieter Lasser

July 1988

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Prepared for:

Naval Postgraduate School
Monterey, CA 93943

NAVAL POSTGRADUATE SCHOOL
MONTEREY, CALIFORNIA

This report was prepared in conjunction with research conducted for the National Research Council and funded by the Naval Postgraduate School.

This report was prepared by:

UNCLASSIFIED

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE

READ INSTRUCTIONS BEFORE COMPLETING FORM

1. REPORT NUMBER NPS-53-88-006	2. GOVT ACCESSION NO.	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) B-Spline-Bezier Representation of Tau-Splines		5. TYPE OF REPORT & PERIOD COVERED Technical Report 5 88 - 7 88
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) Dieter Lasser		8. CONTRACT OR GRANT NUMBER(s)
9. PERFORMING ORGANIZATION NAME AND ADDRESS Naval Postgraduate School - Code 53 Monterey, CA 93943		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS Direct Funding
11. CONTROLLING OFFICE NAME AND ADDRESS Naval Postgraduate School - Code 53 Monterey, CA 93943		12. REPORT DATE July 1988
		13. NUMBER OF PAGES 13
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) National Research Council 2101 Constitution Avenue, N.W. Washington, D.C. 20418		15. SECURITY CLASS. (of this report) unclassified
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

Approved for public release; distribution unlimited

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

CAD, CAGD, Bezier Curves, B-Spline Curves, Nu-Splines, Tau-Splines, Geometric Continuity

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

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S/N 0102-LF-014-6601

unclassified

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

B-Spline-Bezier Representation of Tau-Splines

Dieter Lasser
Fachbereich Mathematik, AG3
Technische Hochschule Darmstadt
6100 Darmstadt, West Germany

Abstract: We present a B-spline-Bezier representation of τ splines, curvature and torsion continuous quintics which have been introduced in CAGD by *Hagen* in 1985. Explicit formulas are given for the conversion of the B-spline-Bezier representation to the τ spline representation and vice versa, and conditions and certain ranges of tension values are derived which insure the positivity of the design parameters.

0. Introduction

In 1974 *Nielson* [*Nielson 74*] gave a piecewise polynomial alternative to splines under tension [*Schweikert 66*], [*Cline 74*], the so-called v spline. The v splines are curvature continuous interpolating cubics and they are the solution of the minimization of

$$\int_{t_0}^{t_N} \|X''(t)\|^2 \, dt \quad + \quad \sum_{I=0}^{N} v_I \, \|X'(t_I)\|^2 \; , \qquad v_I \geq 0 \tag{1}$$

over the space

$$H^2 \;=\; \{\, X\colon\; X'' \in L^2[t_0, t_N],\;\; X' \textit{absolutely continuous on } [t_0, t_N] \,\}$$

subject to interpolation conditions and certain end conditions. [*Nielson 74, 86*]
In [*Boehm 85*] a B-spline-Bezier representation of v splines was given, the γ spline, and *Boehm* also pointed out the close relation to *Barsky's* β splines. [*Barsky 81*] Some details on this relation can be found in [*Fritsch 86*] especially conversion equations between β splines and v splines.

In 1985 *Hagen* [*Hagen 85*] generalized *Nielson's* approach by using

$$\int_{t_0}^{t_N} \|\mathbf{X}^{(K)}(t)\|^2 \, dt \;+\; \sum_{I=0}^{N}\sum_{L=1}^{K-1} v_{I,L} \|\mathbf{X}^{(L)}(t_I)\|^2 \, dt \;, \qquad v_{I,L} \geq 0 \tag{2}$$

($K \geq 2$), rather than (1), minimizing now with respect to

$$H^K \;=\; \{\, \mathbf{X}\text{: } \mathbf{X}^{(K)} \in L^2[t_0, t_N],\;\; \mathbf{X}^{(K-1)} \textit{absolutely continuous on } [t_0, t_N] \,\}$$

and satisfying interpolation and generalized end conditions. *Hagens's* concept of *'geometric spline curves'* includes for $K = 2$ *Nielson's* v splines and yields for $K = 3$ to curvature and torsion continuous quintics, the τ splines.

The aim of this paper is to give a B-spline-Bezier representation of τ splines. We also derive the conversion equations between the Bezier and the originally given Hermite representation in [*Hagen 85*]. We discuss positivity conditions on the design parameters of the Bezier representation and value ranges for the point weights, the v_I's.

Because we like to find a B-spline-Bezier representation of τ splines, we first introduce the Bezier representation of segmented curves. In section II we give a short discussion of *Nielson's* v splines while *Hagens's* τ splines are discussed in section III.

I. Bezier representation of segmented curves

Let $\mathbf{X}(t)$ be a planar or spatial parametrized curve defined with respect to a partition of the domain space by *'knots'*

$$t_0 \;<\; t_1 \;<\; \ldots\; t_N \, .$$

The parameter space segmentation induces a curve segmentation in Segments $\mathbf{X}_I$: $[t_I, t_{I+1}] \to \mathbb{R}^d$ $(d = 2{,}3)$. A local parameter $u \in [0, 1]$ can be introduced such that

$$I = 0, \ldots, N-1 \qquad \mathbf{X}(t) \;=\; \mathbf{X}_I(u) \qquad \textit{for } \; t \in [t_I, t_{I+1}]$$

by the linear interpolation of t_I and t_{I+1}:

$$t \;=\; (1-u)\, t_I \;+\; u\, t_{I+1} \, , \qquad \textit{where } \; u \in [0, 1].$$

The derivatives have to be calculated now by the chain rule, i.e.

$$\mathbf{X}^{(r)}(t) \;\equiv\; \frac{d^r}{dt^r}\mathbf{X}(t) \;=\; \frac{d^r u}{dt^r}\,\frac{d^r}{du^r}\mathbf{X}_I(u) \;=\; \frac{1}{\Delta_I^r}\,\frac{d^r}{du^r}\mathbf{X}_I(u)$$

where $\Delta_I \equiv t_{I+1} - t_I$.

Now the segments might be given in Bezier representation, that means

$$\mathbf{X}_I(u) \;=\; \sum_{k=0}^{n} \mathbf{b}_{nI+k}\, B_k^n(u) \tag{3}$$

where $\mathbf{b}_{nI+k} \in \mathbb{R}^d$ $(d = 2{,}3)$, $u \in [0,1]$ and

$$B_k^n(u) \;=\; \binom{n}{k} u^k \,(1-u)^{n-k}$$

are the (ordinary) **Bernstein polynomials** of degree n in u. The coefficients $\mathbf{b}_j$ are called **Bezier points.** They form in their natural ordering given by their subscripts the vertices of the so called **Bezier polygon.**

The derivatives of $X_I(u)$ with respect to u are given by

$$\frac{d^r}{du^r} X_I(u) = \frac{n!}{(n-r)!} \sum_{k=0}^{n-r} \Delta^r b_{nI+k} B_k^{n-r}(u)$$

where

$$\Delta^r b_\alpha \equiv \Delta^{r-1}(b_\alpha + 1 - b_\alpha)$$

so that in $X(t_I) = X_{I-1}(1) = X_I(0)$, the common boundary point of $X_{I-1}(u)$ and of $X_I(u)$, we have for the left sided derivatives of $X(t)$

$$X^{(r)}(t_I^-) \equiv \lim_{t \to t_I^-} X^{(r)}(t) = \frac{1}{\Delta_{I-1}^r} \frac{n!}{(n-r)!} \Delta^r b_{nI-r}$$

and for the right sided derivatives

$$X^{(r)}(t_I^+) \equiv \lim_{t \to t_I^+} X^{(r)}(t) = \frac{1}{\Delta_I^r} \frac{n!}{(n-r)!} \Delta^r b_{nI} .$$

II. Nu-splines

Nielson's ν splines are solutions of the minimization of (1) over the space H^2 subject to the interpolation conditions

$I = 0, \dots, N$ $$X(t_I) = X_I$$

and one of the following end conditions

i.) $X'(t_0) = X_0' , \quad X'(t_N) = X_N' ,$

ii.) $X''(t_0^+) = \nu_0 X'(t_0) , \quad X''(t_N^-) = \nu_N X'(t_N) ,$

iii.) $X(t_0) = X(t_N) , \quad X'(t_0) = X'(t_N) , \quad X''(t_0^+) - X''(t_N^-) = (\nu_0 + \nu_N) X'(t_0) .$

ν splines fulfill at any knot t_I $(I = 1, \dots, N-1)$ the continuity conditions

$$X(t_I^+) = X(t_I^-) \tag{4}$$

$$X'(t_I^+) = X'(t_I^-) \tag{5}$$

$$X''(t_I^+) = X''(t_I^-) + \nu_I X'(t_I^-) \tag{6}$$

what can be written in matrix form as

$$X_I^+ = A_I X_I^-$$

The $(r+1)^2$ matrix A_I is called **connection matrix** and the *'vector'* $X_I^\pm$ with $r+1$ elements of $\mathbb{R}^d$ $(d \geq r,$ here is $r = 2)$ is sometimes called the **r-jet** of X. .

Because the curvature of a planar resp. spatial curve is given by

$$\kappa = \frac{|X', X''|}{\|X'\|^3} \qquad \textit{resp.} \qquad \kappa = \frac{\|X' \times X''\|}{\|X'\|^3} \tag{7}$$

we see that a ν spline is curvature continuous.

A Bezier representation of ν splines can be derived by inserting (3) for $n = 3$ into (4) to (6). We get as continuity conditions of the Bezier representation (Figure 1)

$$(1 + q_I)\,\mathbf{b}_I = q_I\,\mathbf{b}_{3I-1} + \mathbf{b}_{3I+1}$$

$$(1 + \gamma_I q_I)\,\mathbf{b}_{3I-1} = \gamma_I q_I\,\mathbf{b}_{3I-2} + \mathbf{s}_I$$
$$(\gamma_I + q_I)\,\mathbf{b}_{3I+1} = q_I\,\mathbf{s}_I + \gamma_I \mathbf{b}_{3I+2}$$

where

$$q_I = \frac{\Delta_I}{\Delta_{I-1}},$$

and

$$\gamma_I = \frac{1}{1 + \frac{1}{1+q_I}\frac{\Delta_I}{2}\nu_I}, \tag{8}$$

$I = 1, \dots, N-1$. (8) allows the evaluation of the γ_I's of the Bezier representation of a ν spline, i.e. the evaluation of the γ_I's for given ν_I values. On the other side, the corresponding ν spline to a given curvature continuous cubic Bezier spline, a so-called γ spline, has ν_I values given by

$I = 1, \dots, n-1$
$$\nu_I = \frac{2}{\Delta_I}(1 + q_I)\left(\frac{1}{\gamma_I} - 1\right). \tag{9}$$

(9) was first given by *Boehm* in [*Boehm 85*]. He also presented a B-spline representation for curvature continuous cubics, and pointed out the relation to *Barsky's* uniformly-shaped β splines. [*Barsky 81*] The connection to general β splines, sometimes called explicit, discrete or discretely-shaped β splines [*Hoellig 86*], [*Bartels et al. 87*], was also pointed out by *Nielson* and given by *Fritsch* (see [*Fritsch 86*]).[1] In fact, looking at the connection matrices of γ splines, ν splines and (general) β splines [*Dyn et al. 85*] we see that β, γ and ν splines are nothing else than different representations of curvature continuous cubics.

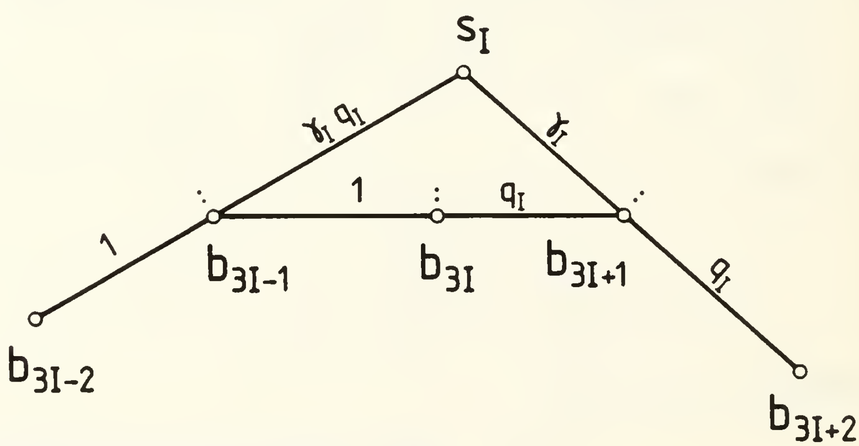

Figure 1. Construction of the Bezier polygon for ν splines

Dyn and *Micchelli* [*Dyn et al. 85*] have shown that the existence of non-negative local support basis functions which sum to one follows - for geometrically continuous spline curves, i.e. spline curves with continuous differential geometric invariants like curvature, torsion, etc. - from the **total positivity of the connection matrix**. For ν splines the total positivity of A_I has the meaning $\nu_I \geq 0$ and that is exactly the range for the tension parameters ν_I covered by the minimum norm

[1] The first local basis for GC^2 splines was developed by *Nielson* and *Lewis* in 1975. [*Lewis 75*] A local basis for uniformly-shaped β splines was given in [*Barsky 81*] and for discretely-shaped β splines in [*Bartels et al. 84*], see also [*Cohen 87*].

characterization of ν splines. [*Nielson 74, 86*] For $\nu_I \geq 0$ we obtain from (9) the γ_I range: $\gamma_I \leq 1$ (remark: $\nu_I = 0 \Leftrightarrow \gamma_I = 1$ is the usual C^2 cubic spline). But working with γ splines we know that $\gamma_I > 1$ are possible as well. Indeed, if we request **positive design parameters**, i.e. $\gamma_I > 0$, so that such important properties like the convex hull and the variation diminishing property are given, (8) yields to

$$I = 1, \dots, N-1 \qquad \nu_I > -\frac{2}{\Delta_I}(1 + q_I)\,. \tag{10}$$

Hence not only positive tension values but ν_I values in the range given by (10) guarantee positive design parameters (remark: $\nu_I < 0$ yields to $\gamma_I > 1$) and therefore properties like the two mentioned above. This result goes conform with work done by *Barsky* [*Barsky 84*] who extended the theory of ν splines by identifying certain ranges for the ν_I's that guarantee a **unique solution of the interpolation problem**. In the special case of a uniform, an equidistant parametrization, i.e. $q_I = 1$, *Barsky* gives the range $\nu_I > -4$ which is also given by (10).

III. Tau-splines

Hagens's τ splines are solutions of the minimization of (2) over the space H^K for $K = 3$ subject to the interpolation conditions

$$I = 0, \dots, N \qquad \mathrm{X}(t_I) = \mathrm{X}_I$$

and one of the following end conditions $(L = K, \dots, 2(K-1))$

i.) $\mathrm{X}^{(2K-1-L)}(t_0) = \mathrm{X}_0^{(2K-1-L)}, \qquad \mathrm{X}^{(2K-1-L)}(t_N) = \mathrm{X}_N^{(2K-1-L)},$

ii.) $\mathrm{X}^{(L)}(t_0^+) = \nu_{0,2K-1-L}\mathrm{X}^{(2K-1-L)}(t_0), \qquad \mathrm{X}^{(L)}(t_N^-) = \nu_{N,2K-1-L}\mathrm{X}^{(2K-1-L)}(t_N),$

iii.) $\mathrm{X}(t_0) = \mathrm{X}(t_N), \qquad \mathrm{X}^{(2K-1-L)}(t_0) = \mathrm{X}^{(2K-1-L)}(t_N),$

$\mathrm{X}^{(L)}(t_0^+) - \mathrm{X}^{(L)}(t_N^-) = (\nu_{0,2K-1-L} - \nu_{N,2K-1-L})\,\mathrm{X}^{(2K-1-L)}(t_0).$

τ splines fulfill at any knot t_I $(I = 1, \dots, N-1)$ the continuity conditions

$$\mathrm{X}(t_I^+) = \mathrm{X}(t_I^-) \tag{11}$$

$$\mathrm{X}'(t_I^+) = \mathrm{X}'(t_I^-) \tag{12}$$

$$\mathrm{X}''(t_I^+) = \mathrm{X}''(t_I^-) \tag{13}$$

$$\mathrm{X}'''(t_I^+) = \mathrm{X}'''(t_I^-) + \nu_{I,2}\,\mathrm{X}'(t_I^-) \tag{14}$$

$$\mathrm{X}^{iv}(t_I^+) = \mathrm{X}^{iv}(t_I^-) - \nu_{I,1}\,\mathrm{X}'(t_I^-) \tag{15}$$

Because the curvature of a spatial curve is given by (7) and the torsion by

$$\tau = \frac{|\,\mathrm{X}', \mathrm{X}'', \mathrm{X}'''\,|}{\|\mathrm{X}' \times \mathrm{X}''\|^2}$$

we see that a τ spline is curvature and torsion continuous.

The connection matrix of a general torsion continuous contact of two curve segments of $\mathbb{R}^d$ $(d \geq 3)$ of degree n $(n \geq 4)$ - we speak about **geometric C^3 continuity**, briefly **GC^3 continuity** - is given by [*Dyn et al. 85*], [*Lasser,Eck 88*]:

$$A_{GC^3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \omega_{11} & 0 & 0 \\ 0 & \omega_{12} & \omega_{11}^2 & 0 \\ 0 & \omega_{23} & \omega_{13} & \omega_{11}^3 \end{bmatrix}$$

Comparing A_{GC^3} with (11) to (14), we see that τ splines do not take advantage of all shape parameters offered by the concept of torsion continuity. They rather form a subset of the set of GC^3 continuous curves.
Furthermore we like to mention that τ splines and **visual C^3 continuous curves**, briefly **VC^3 continuous curves**, i.e. curves having **contact of order r** [*Geise 62*] with $r = 3$, are spanning two *'almost totally separated'* subsets of the set of GC^3 continuous curves. The connection matrix of two curve segments having contact of order 3 is given by [*Dyn et al. 85*], [*Lasser 88*]:

$$A_{VC^3} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & v_1 & 0 & 0 \\ 0 & v_2 & v_1^2 & 0 \\ 0 & v_3 & 3v_1v_2 & v_1^3 \end{bmatrix} .$$

and therefore, comparing A_{VC^3} with A_{GC^3}, VC^3 continuous curves form a subset of the set of GC^3 continuous curves, and, comparing A_{VC^3} with (11) to (14), we see that the only curves beeing τ splines as well as VC^3 continuous curves are the usual C^3 continuous curves.

To find a Bezier representation of τ splines we insert (3) for $n = 5$ into (11) to (15) and get as continuity conditions of the Bezier representation (Figure 2)

$$(1 + q_I)\,\mathbf{b}_I = q_I\,\mathbf{b}_{5I-1} + \mathbf{b}_{5I+1} \tag{16}$$

$$(1 + \gamma_I q_I)\,\mathbf{b}_{5I-1} = \gamma_I q_I\,\mathbf{b}_{5I-2} + \mathbf{s}_I \tag{17.1}$$

$$(\gamma_I + q_I)\,\mathbf{b}_{5I+1} = q_I\,\mathbf{s}_I + \gamma_I\,\mathbf{b}_{5I+2} \tag{17.2}$$

$$(1 + \delta_I q_I)\,\mathbf{b}_{5I-2} = \delta_I q_I\,\mathbf{b}_{5I-3} + \mathbf{e}_I^- \tag{18.1}$$

$$(\delta_I + \varepsilon_I q_I)\,\mathbf{s}_I = \varepsilon_I q_I\,\mathbf{e}_I^- + \delta_I \mathbf{e}_I^+ \tag{18.2}$$

$$(\varepsilon_I + q_I)\,\mathbf{b}_{5I+2} = q_I\,\mathbf{e}_I^+ + \varepsilon_I \mathbf{b}_{5I+3} \tag{18.3}$$

$$(1 + \rho_I q_I)\,\mathbf{b}_{5I-3} = \rho_I q_I\,\mathbf{b}_{5I-4} + \mathbf{f}_I^- \tag{19.1}$$

$$(\rho_I + \sigma_I q_I)\,\mathbf{e}_I^- = \sigma_I q_I\,\mathbf{f}_I^- + \rho_I \mathbf{t}_I \tag{19.2}$$

$$(\sigma_I + \tau_I q_I)\,\mathbf{e}_I^+ = \tau_I q_I\,\mathbf{t}_I + \sigma_I \mathbf{f}_I^+ \tag{19.3}$$

$$(\tau_I + q_I)\,\mathbf{b}_{5I+3} = q_I\,\mathbf{f}_I^+ + \tau_I \mathbf{b}_{5I+4} \tag{19.4}$$

where

$$q_I = \frac{\Delta_I}{\Delta_{I-1}}$$

and

$$\gamma_I = 1$$

and

$$\delta_I = \frac{1}{1 + \frac{1}{(1+q_I)^2} \frac{\Delta_I}{3} v_{I,2}} \tag{20}$$

$$\varepsilon_I = \frac{1}{1 + \frac{q_I}{(1+q_I)^2} \frac{\Delta_I}{3} v_{I,2}} \tag{21}$$

and

$$\rho_I = \frac{1}{3q_I - R_I} \tag{22}$$

$$\sigma_I = q_I^2 \frac{1}{T_I - 3 + (T_I - S_I)\frac{q_I}{\varepsilon_I}} \tag{23}$$

$$\tau_I = q_I \frac{1}{3 - T_I} \tag{24}$$

with

$$R_I = \frac{3q_I - 1 + \frac{1}{(1+q_I)^3} \frac{\Delta_I^3}{24} v_{I,1}}{1 + \frac{1}{(1+q_I)^3} \frac{\Delta_I}{3} v_{I,2}}$$

$$S_I = \frac{2(1-q_I) + \frac{1-q_I}{(1+q_I)^3} \frac{\Delta_I^3}{24} v_{I,1}}{1 + \frac{2q_I}{(1+q_I)^3} \frac{\Delta_I}{3} v_{I,2}}$$

$$T_I = \frac{3 - q_I + \frac{q_I}{(1+q_I)^3} \frac{\Delta_I^3}{24} v_{I,1}}{1 + \frac{q_I^2}{(1+q_I)^3} \frac{\Delta_I}{3} v_{I,2}}$$

(20) to (24) allows the evaluation of the design parameters of the Bezier representation of a τ spline and by this the Bezier representation is given. The design parameters are of course not independent of each other any longer. This is obvious because two shape parameters are given by (11) to (15) but five are given by (16) to (19). The dependences are

$$\delta_I = \frac{q_I \varepsilon_I}{1 - (1-q_I)\varepsilon_I} \quad \textit{resp.} \quad \varepsilon_I = \frac{\delta_I}{q_I + (1-q_I)\delta_I} \tag{25}$$

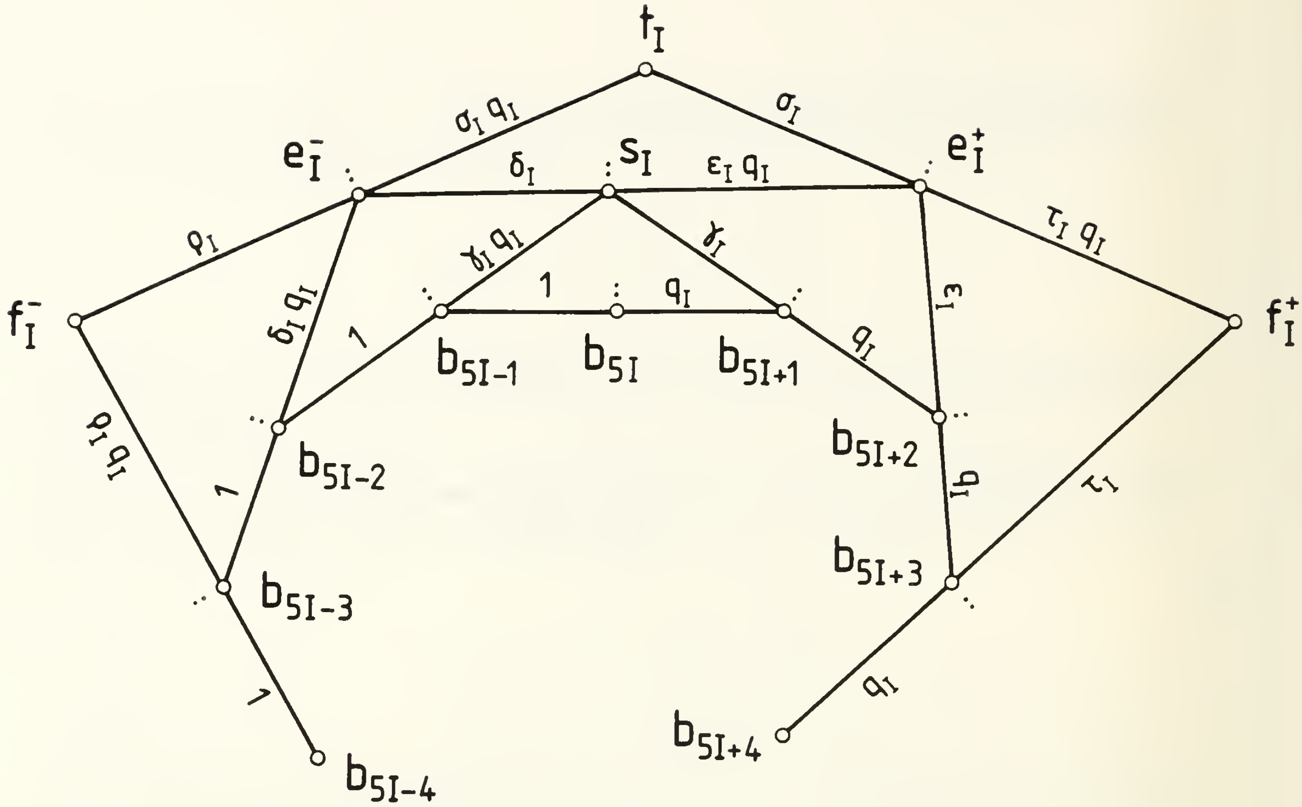

Figure 2. Construction of the Bezier polygon for a τ spline

and

$$\rho_I = \frac{1}{\delta_I}\frac{(1+q_I\delta_I)(\varepsilon_I+\delta_I)\sigma_I}{3(1+q_I)\sigma_I - (\delta_I+q_I\varepsilon_I)} \quad \textit{resp.} \quad \sigma_I = \frac{(\delta_I + q_I\varepsilon_I)\delta_I\rho_I}{3(1+q_I)\delta_I\rho_I - (1+q_I\delta_I)(\varepsilon_I+\delta_I)} \tag{26}$$

$$\sigma_I = \frac{(\delta_I + q_I\varepsilon_I)\varepsilon_I\tau_I}{3(1+q_I)\varepsilon_I\tau_I - (\varepsilon_I+q_I)(\varepsilon_I+\delta_I)} \quad \textit{resp.} \quad \tau_I = \frac{1}{\varepsilon_I}\frac{(\varepsilon_I+q_I)(\varepsilon_I+\delta_I)\sigma_I}{3(1+q_I)\sigma_I - (\delta_I+q_I\varepsilon_I)} \tag{27}$$

$$\tau_I = \frac{(\varepsilon_I+q_I)\delta_I\rho_I}{((1+q_I\delta_I)\varepsilon_I} \quad \textit{resp.} \quad \rho_I = \frac{(1+q_I\delta)\varepsilon_I\tau_I}{(\varepsilon_I+q_I)\delta_I} \tag{28}$$

That means if an interpolating τ spline is given, i.e. $v_{I,1}$ and $v_{I,2}$ values, then the design parameters of the Bezier representation of the τ spline have to be determined in such a way, that they fulfill the equations (25) to (28) And the equations (25) to (28) are valid if the design parameter are determined by (20) to (24). On the other side, the design parameters of a torsion continuous quintic Bezier spline curve have to fulfill the conditions (25) to (28) to make the quintic Bezier spline beeing an interpolating τ spline in Bezier representation. That means either δ_I or ε_I can be chosen as independent design parameter and, let's say in case of δ_I, ε_I has to be determined by (25) as $\varepsilon_I = \varepsilon_I(\delta_I)$ and in addition either ρ_I or σ_I or τ_I can be chosen as independent design parameter and, e.g. in case of ρ_I, σ_I and τ_I have to be determined by (26) and (28) as $\sigma_I = \sigma_I(\rho_I)$ and $\tau_I = \tau_I(\rho_I)$. If the design parameters of the Bezier representation are determined in this way, then the Bezier spline is equivalent to an interpolating τ spline having the property of minimizing (2) and we can calculate the $v_{I,1}$ and $v_{I,2}$ that means the point weights defining the jumps of the third and fourth derivatives in the knots of the τ spline by

$$v_{I,2} = \frac{3}{\Delta_I}(1+q_I)^2(\frac{1}{\delta_I} - 1) \qquad \textit{resp.} \qquad v_{I,2} = \frac{3}{\Delta_I}\frac{(1+q_I)^2}{q_I}(\frac{1}{\varepsilon_I} - 1)$$

and

$$v_{I,1} = \frac{24}{\Delta_I^3}(1+q_I)^2\left[(1+q_I)(1-3q_I) - (\frac{1}{\delta_I}+q_I)(\frac{1}{\rho_I}-3q_I)\right]$$

resp.

$$v_{I,1} = \frac{24}{\Delta_I^3}\frac{(1+q_I)^2}{q_I}\left[(\frac{1}{\varepsilon_I}+\frac{q_I}{\delta_I})(\frac{q_I}{\sigma_I}+q_I-2) + \frac{2}{\varepsilon_I}(1-q_I^2)\right]\frac{\varepsilon_I\,\delta_I}{\varepsilon_I+\delta_I}$$

resp.

$$v_{I,1} = \frac{24}{\Delta_I^3}\frac{(1+q_I)^2}{q_I}\left[(1+q_I)(q_I-3) - (1+\frac{q_I}{\varepsilon_I})(\frac{q_I}{\tau_I}-3)\right]$$

and so on.

The τ splines form a subset of the set of GC^3 continuous quintics. The **B-spline-Bezier representation of Tau splines** is therefore identical with the B-spline-Bezier representation of GC^3 continuous quintics which was given in [*Eck 87*], [*Lasser,Eck 88*], with the restriction that for a τ spline the design parameters $\gamma_I, \delta_I, \varepsilon_I, \rho_I, \sigma_I$ and τ_I can not be chosen independently of each other as in case of the GC^3 continuous quintic spline curve. We rather have to set $\gamma_I = 1$ and have to choose $\delta_I, \varepsilon_I, \rho_I, \sigma_I$ and τ_I according to the dependences given by (25) to (28).

Let's discuss now the **positivity of the design parameters** for τ splines. The dependences (25) to (28) imply the following:

Because of (25), positivity of $\delta_I = \delta_I(\varepsilon_I)$ needs

$0 < \varepsilon < \dfrac{1}{1-q_I}$ in case of $q_I < 1 \Leftrightarrow \Delta_I < \Delta_{I-1}$

$\varepsilon > 0$ in case of $q_I \geq 1 \Leftrightarrow \Delta_I \geq \Delta_{I-1}$

On the other side, positivity of $\varepsilon_I = \varepsilon_I(\delta_I)$ needs

$\delta > 0$ in case of $q_I \leq 1 \Leftrightarrow \Delta_I \leq \Delta_{I-1}$

$0 < \delta < \dfrac{-q_I}{1-q_I}$ in case of $q_I > 1 \Leftrightarrow \Delta_I > \Delta_{I-1}$

Because of (27) and (28), positivity of $\rho_I = \rho_I(\tau_I)$ and of $\sigma_I = \sigma_I(\tau_I)$ needs

$$\tau > \frac{(\varepsilon_I + q_I)(\varepsilon_I + \delta_I)}{3(1+q_I)\varepsilon_I} .$$

On the other side, because of (26) and (27), positivity of $\rho_I = \rho_I(\sigma_I)$ and of $\tau_I = \tau_I(\sigma_I)$ needs

$$\sigma > \frac{\delta_I + q_I\varepsilon_I}{3(1+q_I)} ,$$

and because of (26) and (28), positivity of $\sigma_I = \sigma_I(\rho_I)$ and of $\tau_I = \tau_I(\rho_I)$ needs

$$\rho > \frac{(1+q_I\delta_I)(\varepsilon_I+\delta_I)}{3(1+q_I)\delta_I}$$

The minimum norm characterization of τ splines works for non-negative tension parameter $\nu_{I,1}$ and $\nu_{I,2}$ only. As for ν splines we can extend the theory of τ splines by requesting the **positivity of the design parameters.**

Because of (20) and (21), for $\varepsilon_I > 0$ and $\delta_I > 0$ the tension parameter $\nu_{I,2}$ has to be within the range

$$\nu_{I,2} \; > \; \max \left\{ -(1+q_I)^2 \frac{3}{\Delta_I} \, , \; -\frac{(1+q_I)^2}{q_I} \frac{3}{\Delta_I} \right\} \tag{29}$$

that means

$$\nu_{I,2} \; > \; -\frac{(1+q_I)^2}{q_I} \frac{3}{\Delta_I} \qquad \text{if } q_I > 1 \;\Leftrightarrow\; \Delta_I > \Delta_{I-1}$$

$$\nu_{I,2} \; > \; -\frac{12}{\Delta_I} \qquad \text{if } q_I = 1 \;\Leftrightarrow\; \Delta_I = \Delta_{I-1}$$

$$\nu_{I,2} \; > \; -(1+q_I)^2 \frac{3}{\Delta_I} \qquad \text{if } q_I < 1 \;\Leftrightarrow\; \Delta_I < \Delta_{I-1}$$

Thus not only non-negative but also certain negative $\nu_{I,2}$ values are allowed.

Because of (22) to (24), for $\rho_I > 0$, $\sigma_I > 0$ and $\tau_I > 0$ the tension parameter $\nu_{I,1}$ has to be within the ranges given by (30).

$$\nu_{I,1} \; < \; \frac{24}{\Delta_I^3} \left[(1+q_I)^3 \; + \; q_I \Delta_I \nu_{I,2} \right] \tag{30.1}$$

$$\nu_{I,1} \; > \; -\frac{24}{\Delta_I^3} \, \frac{(1+q_I)^3 \; + \; 2(1-q_I+q_I^2) \frac{\Delta_I}{3} \nu_{I,2}}{2 \; + \; \frac{1}{1+q_I} \frac{\Delta_I}{3} \nu_{I,2}} \tag{30.2}$$

First $\nu_{I,2}$ has to be chosen such that $\varepsilon_I > 0$ and $\delta_I > 0$ is fulfilled, i.e. $\nu_{I,2}$ has to be chosen within the range given by (29), than $\nu_{I,1}$ can be chosen within the range given above.
For $q_I = 1$ (30) yields for example to

$$-4 \frac{24}{\Delta_I^3} \; < \; \nu_{I,1} \; < \; \frac{24}{\Delta_I^3} (8 + \Delta_I \nu_{I,2})$$

Barsky [*Barsky 84*] extended the theory of ν splines by identifying certain ranges for the ν_I's that guarantee a **unique solution of the interpolation problem**. This idea allows especially the consideration of different end conditions. The same can be done for τ splines, and is indeed the topic of actual research.

Furthermore the idea of [*Salkauskas 75*] and [*Foley 86,87*] of introducing interval weights can be picked up to create **interval weighted geometric spline curves**[2] minimizing

$$\sum_{I=0}^{N} p_I \int_{t_{I-1}}^{t_I} \|\mathbf{X}^{(K)}(t)\|^2 \, dt \; + \; \sum_{I=0}^{N} \sum_{L=1}^{K-1} \nu_{I,L} \|\mathbf{X}^{(L)}(t_I)\|^2 \, dt \, .$$

For $K = 3$ we get interval weighted τ splines. Actually we are working on this too.

[2] These interval weighted geometric spline curves are in general not curvature, torsion, etc. continuous.

Acknowledgment

This paper was written while the author was supported by an one year award of the *National Research Council, Washington, DC* visiting during this time at the *Naval Postgraduate School, Monterey, California.*
The author would like to thank *Professor Richard Franke* whose effort made this visit possible. And also like to thank *Professor Richard Franke* for all his help during the time in *Montery*.

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Anthony D. DeRose
Department of Computer Science, FR-35
Universitaet of Washington
Seattle, WA 98195 USA

Gerald Farin
Department of Computer Science
Arizona State University
Tempe, AZ 85287 USA

Thomas A. Foley
Department of Computer Science
Arizona State University
Tempe, AZ 85287 USA

Rida T. Farouki
Manufacturing Research Department
IBM Thomas J. Watson Research Center
Yorktown Heights, NY 10598 USA

Herbert Frank
Fachbereich Mathematik
Universitaet Dortmund
4600 Dortmund FRG

Richard Franke (5)
Department of Mathematics
Naval Postgraduate School
Monterey, CA 93943 USA

Harold M. Fredricksen
Chairman, Dept. of Mathematics
Naval Postgraduate School
Monterey, CA 93943 USA

Frederick N. Fritsch
Numerical Mathematics Group
Lawrence Livermore National Laboratory
Livermore, CA 94550 USA

Gerhard Geise
Technische Universitat Dresden
Sektion Mathematik
8027 Dresden DDR

Oswald Giering
Mathematisches Institut
Technische Universitat Munchen
8000 Munchen 2 FRG

Ronald Goldman
Department of Computer Science
University of Waterloo
Waterloo, Ontario N2L 3G1 USA

William J. Gordon
Center for Scientific
Computation & Interactive Graphics
Drexel University
Philadelphia, PA 19104 USA

John A. Gregory
Dept. of Mathematics & Statistics
Bonnel University
Uxbridge UB8 3PH England

Hans Hagen
Institut fur Graphische Datenverarbeitung
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Reinhold Klass
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7032 Sindelfingen FRG

Dieter Lasser (20)
Fachbereich Mathematik, AG3
Technische Hochschule Darmstadt
6100 Darmstadt FRG

Norbert Luscher
Department of Mathematics
Naval Postgraduate School
Monterey, CA 93943 USA

Harry W. McLauchlan
Dept. of Mathematical Sciences
Rensselaer Polytech. Inst.
Troy, NY 12181 USA

Charles Micchelli
IBM Thomas J. Watson Research Center
P.O. Box 218
Yorktown Heights, NY 10598 USA

Gregory M Nielson
Department of Computer Science
Arizona State University
Tempe, AZ 85287 USA

Horst Nowacki
Institut fuer Schiffstechnik
Technische Universitat Berlin
Salzufer 17-19
1000 Berlin FRG

Helmut Pottmann
Institut fuer Geometrie
Technische Universitat Wien
Wiedner Hauptstr. 8-10/113
A-1040 Wien Austria

Richard F. Riesenfeld
Department of Computer Science
University of Utah
Salt Lake City, UT 84112 USA

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Cambridge CB4 5BA England

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Brigham Young University 368B CB
Provo, Utah 84602 USA

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Center for Approximation Theory
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Universitat Tubingen
7400 Tubingen FRG

Andrew J. Worsey
Dept. of Mathematical Sciences
University of North Carolina
Wilmington, NC 28403-3297 USA

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